What is Instantaneous Rate of Change? An Investigation of Students' Conceptions and Learning of Instantaneous Rate of Change

by

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ABSTRACT

This dissertation reports on three studies about students' conceptions and learning of the idea of instantaneous rate of change. The first study investigated 25 students' conceptions of the idea of instantaneous rate of change. The second study proposes a hypothetical learning trajectory, based on the literature and results from the first study, for learning the idea of instantaneous rate of change. The third study investigated two students' thinking and learning in the context of a sequence of five exploratory teaching interviews.

The first paper reports on the results of conducting clinical interviews with 25 students. The results revealed the diverse conceptions that Calculus students have about the value of a derivative at a given input value. The results also suggest that students' interpretation of the value of a rate of change is related to their use of covariational reasoning when considering how two quantities' values vary together.

The second paper presents a conceptual analysis on the ways of thinking needed to develop a productive understanding of instantaneous rate of change. This conceptual analysis includes an ordered list of understandings and reasoning abilities that I hypothesize to be essential for understanding the idea of instantaneous rate of change. This paper also includes a sequence of tasks and questions I designed to support students in developing the ways of thinking and meanings described in my conceptual analysis.

The third paper reports on the results of five exploratory teaching interviews that leveraged my hypothetical learning trajectory from the second paper. The results of this teaching experiment indicate that developing a coherent understanding of rate of change using quantitative reasoning can foster advances in students' understanding of

instantaneous rate of change as a constant rate of change over an arbitrarily small input interval of a function's domain.

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CHAPTER 1

INTRODUCTION & PROBLEM STATEMENT

This manuscript reports on the results of three studies that investigated undergraduate students' conceptions and learning of instantaneous rate of change in a Calculus course. The goals of these studies were to: (1) characterize students' understandings of instantaneous rate of change; and (2) hypothesize a learning trajectory for student construction of a productive understanding of instantaneous rate of change. This study achieves these goals by employing clinical interviews (1) and a teaching experiment (2).

Calculus is the mathematics of how quantities change. In particular, the main idea of Calculus 1 can be productively summed up as "You know how much of a quantity you have at all times and want to know how fast that quantity is changing at all times." From as early as the ancient Egyptians (Kline, 1990) to Isaac Newton (one of the many people credited with inventing Calculus), philosophers struggled to comprehend the nature of motion. The idea of motion seemed like a paradox: if, at any particular moment, zero time elapses (and therefore, at any particular moment an object does not move), then how is motion possible? The creation of Calculus offered a set of tools for addressing this paradox by mathematizing continuous change. For Newton, he wanted to describe the speed of a falling object. However, he knew of no existing mathematical explanation for why a falling object's speed increases every second. On his journey in formulating the laws of motion, Newton developed the idea of Calculus and its focus on rate of change (Calinger, 1999).

Since Newton's time, Calculus's applications have been integral in the realm of mathematics, physics, engineering, and more. Engineering draws heavily on Calculus because engineers need to model quantities and analyze how they change together. For example, civil engineers often deal with fluid mechanics that model how fluids behave in motion. Mechanical engineers often use Calculus via Newton's law of cooling in HVAC design (Heating, ventilation, and air conditioning). In the social sciences, Calculus is used to make economic predictions, characterize trends in birth/death rates, and predict patterns in disease outbreaks. Many fields rely on Calculus as a tool for describing quantitative relationships between covarying quantities. As a result, Calculus is often characterized as a gateway to pursuing degrees in the sciences, making it crucial that Calculus is accessible and meaningful to students.

Despite the importance of Calculus in STEM fields, researchers have shown that even high-performing students demonstrate impoverished understandings of key Calculus concepts (Selden et al., 2000; Carlson et al., 2002). Recent studies have shown that the workforce demand for STEM majors has been increasing since 1971, yet the number of STEM majors has not increased in response to the increase in demand (Carnevale et al., 2011; Hurtado et al., 2010). Ellis et al. (2014) argue that a 10% increase in the current number of STEM majors is needed to meet the workforce demands; otherwise, there may be economic implications. According to the Higher Education Research Institute, less than 40% of STEM majors complete their degree (Hurtado et al., 2010), and Calculus is often cited as the reason for losing these STEM majors (Seymour & Hewitt, 1997; Seymour, 2006). This is not a new problem, as Calculus has long been cited as a filter course (Steen, 1988). In recent years, researchers have analyzed student demographics of

Calculus courses (Rasmussen et al., 2013; Ellis et al., 2013), the characteristics of successful Calculus Programs at universities across the United States (Törner et al., 2014; Bressoud et al., 2012; Bressoud et al., 2013; Bressoud et al., 2015; Apkarian et al., 2017), and effects of the composition of the faculty on student retention and classroom culture in Calculus courses (Reed-Rhoads et al., 2005; Robst et al., 1998; Bressoud et al., 2012). These studies reveal that many factors contribute to the success of a Calculus program, such as usage of active-learning strategies, coordination of instruction, having a diverse faculty, and engaging course content. The findings from these studies portray a holistic picture of the multi-faceted issues that impact Calculus education in the United States.

The teaching of Calculus is also an often-cited reason for STEM majors switching out of their majors. Seymour and Hewitt (1997) found that students in the U.S. who left STEM degrees attribute their departure to the emphasis on rote memorization of procedures rather than conceptual understanding. A decade later, Seymour (2006) reported that students are continuing to leave STEM majors because of lackluster instruction in their mathematics courses, with Calculus as the main cited course. Rasmussen and Ellis (2013) extended this finding in their reporting that students' decision to not move onto Calculus 2 was due to the difficulties and overall negative experiences they encountered when taking Calculus 1. Conversely, the authors reported that STEM Calculus students who experienced engaging pedagogy had better persistence through the Calculus sequence. The teaching of Calculus is not the only contributing factor to STEM persistence. Other studies have analyzed the effects of homework types (Ellis et al., 2015; Lenz, 2010; Halcrow & Dunnigan, 2012), the impact of Calculus course variation types (e.g., Calculus for Engineering, Brief Calculus, and Calculus with

Analytic Geometry) on DFW rates (“D,” “F,” and Withdrawal) (Rasmussen & Ellis, 2013; Voigt et al., 2017), and student placement into mathematics courses (Rasmussen et al., 2014).

Due to the staggering number of STEM majors dropping out and other findings that the learning of Calculus is complex, mathematics research educators have studied different aspects of the learning and teaching of Calculus to improve Calculus education. According to Rasmussen et al. (2014), research on Calculus has followed a pattern of (1) identifying student difficulties, (2) investigating how students learn particular concepts, (3) classroom studies that investigate the effects of curricular or pedagogical innovations on student learning and (4) research on teacher knowledge. According to the authors, no single research endeavor can be expected to solve Calculus education issues; instead, the authors suggest that research should build upon each other to build knowledge and inform practice.

Following the research pattern identified by Rasmussen et al., this manuscript involves three papers focused on one topic of Calculus (instantaneous rate of change) by building upon existing work and addressing gaps in the research literature. These papers focus on the learning and teaching of derivative as instantaneous rate of change. Derivatives as instantaneous rate of change are a key concept in Calculus, yet the literature reveals that the learning of derivative is difficult (Zandieh, 2000; Oehrtman, 2002; Monk, 1994; Park, 2013; Ubuz, 2007; Yu, 2020).

A derivative in Calculus describes how one quantity changes with respect to another quantity [See CONCEPTUAL ANALYSIS for further explanation]. It then follows that the idea of derivative is relevant for STEM majors, particularly those studying physics and engineering. Many physics applications require the use of the idea of derivative, including thermodynamics, electromagnetism, modeling vibrations for mechanics, and fluid dynamics. Since the derivative concept is used in so many diverse applications in mathematics, science, and engineering, it follows that building a productive meaning for derivative should be a primary concern.

One of the seemingly paradoxical issues that students have to resolve is interpreting the derivative at a given input value (e.g., $f'(3) = 7$) as regarding a single instance (at the input of 3), yet derivatives are about change, so how can you talk about change if only one instance is involved? If Isaac Newton, one of the inventors of Calculus, struggled to comprehend how an object could be moving when no time was passing, then it is not surprising that students will also find this idea to be challenging. Furthermore, students are often introduced to function as a correspondence (Sfard, 1992) and see one input mapped to one output. It should be natural then that as this notion is rarely challenged, students' conception of function has the property of being concerned with a single value. This conception of function is likely hindering students from conceptualizing a derivative function with an image of quantities changing dynamically. Typical teaching of derivative involves a sliding secant line receding to a tangent line. However, despite this imagery of motion, Zandieh and Knapp (2006) indicated that students tend to recall the finished static product of the tangent line rather than the dynamics of considering smaller and smaller input intervals. Zandieh (2000) also noted that students hold multiple disconnected meanings about the idea of derivative. For example, a student may say that a derivative is the slope of a tangent line or an instantaneous rate of change but may fail to see the connections between these ideas.

While students may associate the words “instantaneous rate of change” with derivative, researchers have shown that students have impoverished meanings for rate of change (Byerley et al., 2012; Thompson, 1994; Rasmussen & King, 2000; Castillo-Garsow, 2010). Rate of change is a crucial topic in higher mathematical courses as well as engineering courses. However, researchers have observed that students are exiting Calculus with weak meanings for the idea of rate of change. Rasmussen and King (2000) reported that students in a differential equations course conflated the number of fish in a pond with the rate of change of the number of fish in the pond with respect to the time elapsed. Prince et al. (2012) reported that their engineering students struggled to distinguish a rate of heat transfer with an amount of heat transfer. Flynn et al. (2018) reported that their engineering students confused rate of change and accumulation processes when engaging in hydrology contexts. Ibrahim and Robello (2012) indicated that even when students demonstrated a strong understanding of rate of change in motion contexts, these understandings did not transfer in non-motion contexts such as work (The physics term of a measure of energy transfer). These studies reveal that having a robust understanding of rate of change is vital for STEM majors and that the current Calculus curriculum in the US is not effective in supporting students in building productive meanings for rate of change.

With these concerns in mind, this manuscript reports on three studies organized around the learning and teaching of derivative as an instantaneous rate of change. The papers follow a logical progression that mimics the first two steps of the research pattern indicated by Rasmussen et al. (2014). First, I describe students’ current understandings of instantaneous rate of change and how they reason covariationally. Next, based on the first study’s results and literature on the learning of derivative, I articulate a productive meaning for instantaneous rate of change by explaining the mathematical concepts and foundational ways of reasoning involved. Finally, I provide results from a teaching experiment (Steffe & Thompson, 2000) that describes how students’ understanding of rate of change shifted throughout the instructional sequence.

The studies described in this manuscript differ from other existing studies by demonstrating the impact of students’ meanings for rate of change on how they reason covariationally and how that relates to students’ interpretation of the derivative at a given input value (Paper 1). The findings of the first paper indicate that the current covariational reasoning framework proposed by Carlson et al. (2002) does not fully capture the ways in which students imagine two quantities’ values as covarying, nor does it explain how a student might interpret the value of a rate of change. The first study's findings contribute to the field by providing new insights into students’ covariational reasoning and why individual students might reason at a particular level.

The second study provides a Hypothetical Learning Trajectory on the idea of derivative as instantaneous rate of change, focusing on quantitative and covariational reasoning. This study differs from other teaching studies (e.g., Ely & Samuels, 2019; Soares & Borba, 2014; Jones & Watson, 2017) by supporting students in reasoning quantitatively about what a rate of change measures about a situation. The third study employs this Hypothetical Learning Trajectory in the context of a teaching experiment to track how individual students’ meaning for rate of change shift throughout each teaching session. The findings of this study provide qualitative data that support the claim that helping students build a coherent understand of rate of change is beneficial to developing a meaning for derivative as instantaneous rate of change. Additionally, this study's findings implicate that students engaging with an applet that can provide a visual demonstration of their mathematical symbolization can aid them in developing fluency in using mathematical symbols and operations to represent the quantities they imagine.

Before addressing my research questions for each paper, I will describe my theoretical perspective and present my review of the current literature on the learning and teaching of derivative.

CHAPTER 2

THEORETICAL PERSPECTIVE

According to Cobb (2007), mathematics education can be productively interpreted "as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (p. 3). However, learning processes are not always concerned with just individual students. It could also concern classroom communities or school districts. Thompson (1991) argues then that a researcher's choice of theory informs the grain size of analysis and the types of questions the researcher is interested in and is able to pursue. Therefore, making salient our theoretical perspective helps others understand the theories we put forth and the rationale of our choices. Differing perspectives offer different interests to those researching mathematics education. Cobb (2007) suggested that as researchers, we should become "pragmatic realists" and attend to the utility that each theoretical perspective offers rather than view them as superior to another.

The studies presented in this manuscript focus on students' meaning for and the learning of the idea of derivative, emphasizing individual students' cognitive activity. Cobb (2007) explained that cognitive psychology theories account for differences in individual students' reasoning and are used to model and explain students' mathematical actions. This perspective places individuals' reasoning as the focus rather than considering the collective of students and their social interactions. Taking a cognitive perspective then informs the researcher's grain of analysis focuses on understanding an individual's learning. As such, I adopt a cognitive perspective due to my research interest in individual student thinking and learning.

Radical Constructivism

This section describes my theoretical perspective for this study, Radical Constructivism (Glasersfeld, 1995), and explains the theoretical assumptions that stem from this stance. Radical Constructivism is an epistemological theory that describes a way of knowing and learning based on Piaget's (1971) Genetic Epistemology. In particular, Radical Constructivism articulates how an individual comes to know something and characterizes what that knowledge consists of. An individual's knowledge does not consist of a direct representation of something called "Reality." Instead, it is a set of schemes that become more viable through the individual's experiences.

Knowledge arises from a need for *equilibration* (Piaget & Inhelder, 1969), which is the self-regulatory process by which individuals adapt to external stimuli. One mechanism for this adaptation is called *assimilation*, which is the process in which an individual attempts to fit an experience by recognizing features that are analogous to previous experiences. Glasersfeld (1995) clarifies that assimilation is treating a new experience "as an instance of something known" (p. 62).

Constructivists argue that individuals cannot come to know an objective "Reality"; instead, individuals construct their own reality through their experiences. Piaget (1971) posits that cognition is the tool for adaption in which an individual interacts with their conceptual model of reality by continually fitting stimulus to their previous experiences. For example, suppose an individual experiences something they may term as "eating a cherry pie." In that case, it is because their mind has constructed knowledge about "eating a cherry pie" based on that individual's previous experiences. It is only through the individual's knowledge, perhaps supported by their sight, smell, and taste, that what the individual is experiencing is consistent with previously perceived experiences of “eating a cherry pie.”

Since Radical Constructivists take the stance that verifying an objective “Reality” is impossible, they instead characterize knowledge in terms of its viability (Glasersfeld, 1995). We can evaluate knowledge as being more viable if the outcomes of the individual’s actions produce anticipated results. If an unanticipated outcome occurs, an individual has likely failed to assimilate an experience which would likely cause that individual to be in a state of *disequilibrium*. The individual can then reflect on their actions and the associated outcome and then adapt their knowledge to better match their experience, making their knowledge more viable. For example, if an individual perceived through their sight and smell that they were experiencing a “cherry pie,” they would expect to taste something that resembles their previous taste experiences. However, if they bite into the pie and taste something that they perceive to be an “apple,” the individual may adapt their perception of the “pie.”

The tenet that knowledge lies in an individual’s mind rather than in the outside world has clear implications in the realm of mathematics education. One implication is that the words and symbols that an individual perceives do not inherently contain information. Instead, an individual attributes an interpretation based on their previous experiences. For example, one student may think about “multiplication” as a number of objects in equally sized groups such as “$3 * 4$” as 3 groups of 4 (or vice versa). Another student may be thinking about “multiplication” as the relative size between 2 quantities such as “$3 * 4$” as measuring out a unit of 3, 4 times such that the new quantity is 4 times as large as the unit of 3.

Choosing the Radical Constructivist perspective has clear implications on the researcher's view of data collection and analysis and the researcher's role in this process. Since knowledge is constructed in an individual's mind, it cannot be directly accessed by another. If we could simply access someone else's knowledge, there would be little need for research into how someone learns. Instead, researchers attempt to form an explanatory model of students' thinking about mathematical ideas (Steffe & Thompson, 2000). The purpose of building the model is to explain why students produce the responses that they do. A model is considered viable when the student's utterances, produced work, and actions are explained by the model. Additionally, these models can predict how a student may respond when given a situation and can reveal the difficulties that students face when learning a particular concept. By developing models of students' mathematics, we can have in mind the potential ways that other students may be reasoning about a concept. These models can then serve as a guide for teaching or curricular design to help students construct productive meanings for a mathematical concept.

Taking the Radical Constructivist stance also informs the types of research questions that I investigate for the learning and teaching of instantaneous rate of change. For example, Paper One investigates students' interpretations of the derivative evaluated at a given input value. By taking the Radical Constructivist perspective, I assume that the understanding I have for derivative may differ from my subjects. Therefore my research goal is to characterize students' interpretations by building models of their thinking. Further, analyzing my data would then involve testing and refining these models by analyzing students' utterances and produced work. In each paper's individual sections, I will elucidate the methodology and data analysis informed by taking a Radical Constructivist perspective.

In the following sections, I explain two other theoretical frameworks that have guided my thinking and framed how I analyzed my data.

Quantitative Reasoning

Smith and Thompson's (2007) theory of quantitative reasoning examines the thinking involved in conceptualizing a situation and its quantities. Thompson (1990) defines *quantitative reasoning* as analyzing a situation in terms of quantities and the relationships among them. A *quantity* is a conceived attribute of an object that an individual envisions having a measurement. Thompson (2011) defines *quantification* as the process by which one assigns numerical values to an attribute they have conceptualized. Additionally, quantification entails a unit of measure and the attribute's measure being in a proportional relationship with that unit. Thompson argues that for an individual to conceptualize a quantity, the individual must have an image of an object and attributes of the object that can be measured.

I leverage quantitative reasoning to i) examine how students are reasoning about quantities changing (Paper 1), ii) describe a productive meaning for instantaneous rate of change in terms of reasoning about quantities varying (Paper 2), iii) to inform the design of an instructional intervention to support student learning about instantaneous rate of change (Paper 3).

Piaget (1952) classifies two kinds of quantities, intensive and extensive. Nunes et al. (2003) elaborate on Piaget's idea and define *intensive quantities* as "quantities that are measured by a relation between two variables" (p. 652). *Extensive quantities* are measured by one number that expresses the number of times the measurement unit can be applied to the quantity. For example, when we say that a person is 5.7 feet tall, we mean that the unit of measurement, foot, can be applied 5.7 times to the person's height. To Piaget (1952), this 5.7 feet would be the numerical value of the extensive quantity of height since it is a quantity that is "susceptible to actual addition." By comparison, an intensive quantity usually measures the intensity of a relationship between 2 quantities. For example, if another person was 4.9 feet tall (another extensive quantity), we could measure the *ratio* between the 5.7-foot-tall person with the 4.9-foot-tall person and obtain a ratio of $\frac{5.7}{4.9} \approx 1.163$. This ratio is an intensive quantity because it measures a relationship between 2 quantities and is not susceptible to addition. Adding two ratios together would not accurately yield a new third ratio. I hastily add that the criterion for an intensive quantity is not whether you can add quantities together; instead, the primary importance is that an intensive quantity is conceived by comparing 2 quantities. This notion of an intensive quantity is akin to what Thompson (1990) called a *quantitative operation*, which is the "conception of two quantities being taken to produce a new quantity" (p. 9). The distinction to note is that some quantities are conceived by considering a relationship between 2 quantities, such as a *difference* representing the additive comparison between 2 values of a quantity. Researchers have shown that conceiving of intensive quantities is difficult due to several factors, such as students relying on perceptual reasoning rather than actively comparing 2 quantities (Singer et al., 1997) or the lack of classroom experiences to connect intensive quantities with numbers (Nunes et al., 2003).

In particular to my research, derivatives are about *rate of change* which is a multiplicative comparison between 2 varying quantities. It should be clear then that a rate is an intensive quantity, and thus one of the potential issues in the learning of derivative may be due to students' failure to conceive of a rate as a comparison of 2 quantities. Thus, each of the papers focuses on discussing rate of change as a multiplicative comparison as fundamental to learning derivative productively.

Covariational Reasoning

When conceptualizing a situation, a student may imagine the situation as composed of quantities, but they may also be attentive to how the quantities change together. A student may then coordinate the two quantities' variations, which some researchers term as engaging in *covariational reasoning*.

To be clear, I use the term "imagine" to mean more than having a visual image. Rather, when an individual "imagines" they engage in the act of *re-presentation* (Glasersfeld, 1991), where the individual recalls an experience which may include an image, thought, action, or interpretation.

Covariational reasoning as a theoretical construct is used to explain how someone conceptualizes two quantities changing. However, there is some disagreement over what covariational reasoning entails. Confrey (1991,1992) characterized covariation as coordinating two variables' values as they change. Confrey & Smith (1994) emphasized that covariation involved seeing a function as the juxtaposition of two sequences and generating a correspondence rule. Thompson (1993) characterized covariation as conceptualizing individual quantities' values as varying and then conceptualizing two or more quantities as varying simultaneously. Thompson's characterization was an extension of his quantitative reasoning framework where he was concerned with ways students might conceive of situations as composed of quantities and relationships among quantities whose values vary and the ways students conceive of rate of change (Thompson, 1993). Carlson (1998) conducted a cross-sectional investigation of College Algebra and Calculus 2 students that included tasks prompting the students to construct a graph to represent how the values of two quantities change together in a dynamic event. From this, Carlson et al. (2002) developed a covariation framework that specified the mental actions and reasoning students used to make sense of how quantities change together in dynamic non-linear contexts. Unlike the Confrey and Thompson characterizations of covariation, the Carlson et al. (2002) framework characterized how students reasoned about the rate of change of one quantity with respect to another over successive fixed intervals of change in the independent variable. I leverage the Carlson and Thompson definition since it better aligns with the perspective of how someone is reasoning about quantities (Carlson et al., 2002; Thompson & Carlson, 2017). In particular, I will be using the Carlson et al., (2002) framework over the Thompson and Carlson (2017) framework since the former entails the mental actions an individual engages in rather than the how they envision how quantities change (if it is smooth or chunky).

The beginnings of covariation reasoning first require the mental action of associating two quantities together. For example, an individual may note that in one instance, the height of water in a water bottle is 5 inches tall; they may also observe that in the same instance, the amount of water in the water bottle is 13 fluid ounces. To associate these two quantities together, the individual couples them as exemplified by the statement, “When the water’s height is 5 inches tall, the amount of water in the water bottle is 13 fluid ounces”. When someone has linked together two attributes into a singular object, they have constructed a *multiplicative object*, an object that simultaneously combines the attributes of 2 conceived quantities (Saldanha & Thompson, 1998; Thompson, 2011). An example of the construction of a multiplicative object in mathematics is when an individual perceives an ordered pair (x, y) as a single object that simultaneously represents the values of the variables x and y. Higher-level covariational reasoning involves linking two quantities as they continuously vary by coordinating how one quantity changes with continuous changes in another quantity.

In Carlson et al.’s (2002) framework (Figure 1), the authors provide descriptions of the mental actions that students might evidence in coordinating how two quantities change. The top-level of this framework, Mental Action 5 (MA5), describes someone coordinating the instantaneous rate of change of a function with continuous changes in

Mental Actions of the Covariation Framework

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 1: Carlson et al.'s Covariation Framework

the input variable. A student engaging in MA5 coordinates how two quantities change together, including an awareness that the instantaneous rate of change comes from choosing smaller and smaller intervals in calculating average rates of change around a particular input value. This way of thinking about instantaneous rate of change is similar to how students are often introduced to the derivative at a point through the limit definition of derivative with receding secants lines (whose slopes represent average rates of change). MA5 also describes how someone is coordinating changes in two quantities over a function's domain and thus, students exhibiting this thinking can reason about inflection points and how/where the rate of change changes.

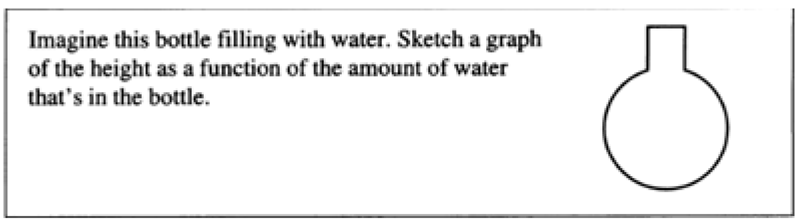


Figure 2:The Bottle Problem

One part that is absent in this framework is how students interpret the value of a rate of change. It is not abundantly apparent in Carlson et al.'s framework how a student would reason about what it means for a car's speedometer to read 43 miles per hour at a particular time (instantaneous rate of change). In Carlson et al.'s study, one of the tasks the researchers used was the bottle problem (Figure 2), and a student exhibiting MA5 would reason that in the bottom rounded half of the bottle, the rate at which the height

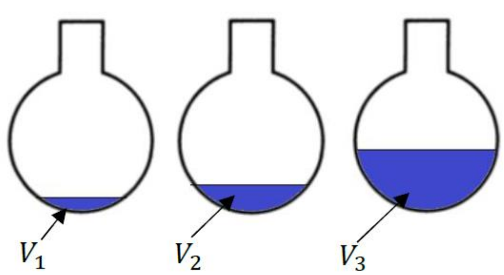


Figure 3: The Bottle Problem Part 2

changes with respect to volume is decreasing. However, there was no discussion on how a student with MA5 (or any of the other mental actions) interprets the rate at a particular volume of water in this task. For example, in Figure 3, a student engaging in MA5 would reason that the rate at V_1 is higher than the rate at V_2 which is higher than the rate at V_3, but it is unclear in the framework how someone thinks about quantities changing at the instance when the volume of the water is V_1.

Table 1 describes each level of the existing covariation framework and how a student reasoning at each level may interpret the value of an instantaneous rate of change.

Table 1: Carlson et al.'s (2002) Covariation Framework and Derivative Examples

Level	Description	Example of a student reasoning about $f'(3) = 6$
Mental Action 1 (MA1) - *Coordinating Quantities*	The student coordinates the value of one quantity with changes in the other.	The student may believe that the value of the output quantity changed by 6 and then subsequently that the value of the input quantity changed from 3 to 4. (The student is not describing how the input and output quantities change together; instead, they observe that both quantities changed)
Mental Action 2 (MA2) - *Directional Coordination of Values*	The student conceptualizes that one quantity varies as another quantity varies, but in a gross variation manner by not considering specific values.	The student interprets that the output value is increasing as the input increases.
Mental Action 3 (MA3) - *Coordination of Values*	The student coordinates the amount of change of one quantity with changes in the amount of the other quantity.	A student may consider the current input and output values $(3, f(3))$ and anticipate that a change in the input (usually a 1-unit change), results in new values for the input quantity and output quantity, e.g., $(4, f(4))$.

		Generally, a student interprets the value of 6 as the change in the output value for a 1-unit change in the input value, e.g., $f(4) = f(3) + 6$.
Mental Action 4 (MA4) – *Coordinating Average Rates of Change*	The student coordinates the average rate of change of the function with uniform increments of change in the input variable.	A student may consider the current values of the input and output quantities $(3, f(3))$ and anticipate that for some change in the input, Δx, the output value will vary 6 times as much. However the student does not verbalize an awareness that the rate of change varies within this Δx interval.
Mental Action 5 (MA5) – *Coordinating Instantaneous Rate of Change*	The student coordinates the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.	A student may consider the current values of the input and output quantities $(3, f(3))$ and anticipate that for some change in the input, Δx, the output will vary 6 times as much. The student verbalizes an awareness that the value of the rate of change will vary in this Δx interval, but for small Δx values, the actual change in the output will be *essentially* 6 times as large. A student may consider continuous changes in the independent variable and anticipate that the values of the associated changes in the dependent variable will vary.

The following example on linear approximation (Figure 4) illustrates how someone engaging in quantitative and covariational reasoning utilizes an instantaneous rate of change as the multiplicative relationship in varying quantities in an arbitrarily small input interval.[1]

Let $P(t)$ represent the weight of a fish, in ounces, when the fish is t months old. If $P\ (3)\ =\ 15$ and $P'(3)\ =\ 6$ estimate the value of $P\ (3.05)$.

Figure 4: The Fish Task - A Problem on Linear Approximation

First, it is essential to discuss what a productive way of processing the question entails. As a student reads the question, they should imagine the values of the quantities of the *fish's age* and the *weight of the fish* as varying. It is then helpful if a student interprets $P(t)$ as the coupling of these two quantities as well as interpreting $P(3) = 15$ as "when the fish is three months old, the fish weighs 15 ounces." Then a student should interpret $P'(3) = 6$ as the *instantaneous rate of change of the fish's weight with respect to the fish's age*. This entails imagining that if the fish's age were to vary a tiny bit from the age of 3 months, the variation in the fish's weight would *essentially* be six times as large (this is to say that the rate of change is 6 ounces per month). Lastly, the student interprets "estimate the value of $P(3.05)$" as approximating the fish's weight at 3.05 months.

The conceptualization of the problem is foundational to how a student engages in solving this task. One productive conceptualization of the situation and the associated mathematical symbolization is illustrated in Table 2.

Conceptualized Quantity	**Unit of Measurement and Frame of Reference** (if	**Symbolic Representation**	**Relation to Other Quantities**

[1] In this task, I am assuming that the function is continuous and differentiable.

	none is stated, then it is from a numerical 0)		
The age of the fish	Number of months since the fish was born	t	
The age of a fish at a particular instance	Number of months since the fish was born at some instance	t_1, t_2, t_3 ...	
Weight of the fish (at a given time, t)	Number of ounces	$P(t)$	At a given time, t, there is an associated weight of the fish.
Instantaneous Rate of Change of the fish (at a given time, t)	Ounces per month	$P'(t)$	Relates variations in the age of the fish, Δt, with variations in the weight of the fish. $\Delta P(t)$
Variation in the age of the fish	Number of months (that have elapsed) since a chosen age value	Δt	Variation in the age of a fish. $t_2 - t_1 = \Delta t$ Or in this case $t_1 + \Delta t = t_2$
Variation in the weight of a fish (between two times)	Number of ounces (that changed) from a different weight value	$\Delta P(t)$	Variation in the weight of the fish. $P(t_2) - P(t_1)$

Table 2: A Conceptualization of Quantities in the Fish Task

In solving the fish task, a student needs to think about the fish's age and weight varying from a particular reference point. In this case, at the age of 3 months, the fish already weighs 15 ounces. Then by using the value of the instantaneous rate of change, the student imagines that the variation in the fish's weight will essentially be six times as large as the variation in the age of the fish (symbolized as $\Delta P(t) = 6 * \Delta t$). To calculate this, the student needs to measure the variation in the fish's age as $\Delta t = 3.05 - 3$. Then the student engages in multiplication $[6 * (3.05 - 3)]$ to determine the variation in the weight of the fish. Then the student adds the estimated variation in the fish's weight of 0.3 ounces with the current known weight of 15 ounces to produce an estimated weight at the age value of 3.05 months. Table 3 illustrates this productive way of thinking in solving the fish task.

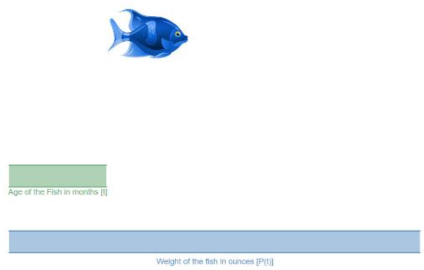
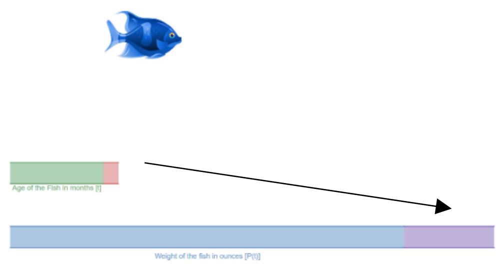
$P(3) = 15$ The length of the green bar represents the fish’s age of 3 months. The length of the blue bar represents the fish’s weight of 15 ounces	
The length of the orange bar represents the variation of the fish’s age of $(3.05 - 3) = 0.05$ months	
The length of the purple bar represents the corresponding variation in the fish’s weight of $6 * (3.05 - 3) = 0.3$ ounces	
The purple bar's length measured by the length of the orange bar represents the relative size of the variation in the fish’s weight of $6 * (3.05 - 3)$ as being 6 times as large as the variation in the fish’s age.	
The combined lengths of the blue and purple bar represent the fish’s estimated weight at 3.05 months of $15 + 6(3.05 - 3)$	

Table 3: Solving the Fish Task

CHAPTER 3

LITERATURE REVIEW

In this chapter, I discuss the existing literature on research in Calculus education. To align with the 3 papers, I organize the literature review in the following way.

I. A general overview of what has been studied in Calculus education and Zandieh's (2000) Derivative Framework.

II. Literature on Students' Understandings of the Idea of Derivative

III. Literature on Pre-requisite Understandings of Derivative

IV. Literature on the Teaching of Derivative

General Overview

Research on Calculus education has been extensive and covers a wide range of topics. On the one hand, much quantitative research (Rasmussen et al., 2013) has focused on Calculus curriculums, classroom settings, and student demographics. On the other side, a focus on particular topics of Calculus (Zandieh, 2000; Asiala et al., 1997; Aspinwall et al., 1997., Borji et al., 2018; Ubuz, 2007), prerequisite knowledge (Byerley et al., 2012; Carlson et al., 2015; Monk & Nemirovsky, 1994), teachers' mathematical meanings (Eichler & Erens, 2014; Yoon, 2019), and teaching studies (Ely & Samuels, 2019; Soares & Borba, 2014).

Since derivative is a central concept in Calculus, there is also a large body of research on the learning and teaching of derivatives that investigated the many issues and factors involved. Early research focused on students' difficulties in understanding derivatives (Orton, 1983; Ferrini-Mundy & Graham, 1991; Monk, 1994). More recently, works have focused on different representations or aspects of derivative such as student understandings of the derivative as a graph (Baker et al., 2000; Aspinwall et al., 1997; Asiala et al., 1997; Borgi et al., 2018; Ubuz, 2007), alternative instructional approaches to derivative (Tall, 2013; Ely & Samuels, 2019; Marrongelle et al., 2003, Thompson & Ashbrook, 2019; Dray & Manogue, 2010), or reasoning abilities of Calculus students (Oehrtman, 2009; Carlson et al., 2015; Petersen et al., 2014).

Zandieh's Derivative Framework

One of the significant works that has impacted Calculus research was Zandieh's (2000) theoretical framework for exploring student understanding of the derivative concept. Her framework provides an instrument to organize thinking on the teaching and learning of derivatives but does not describe how students might learn the idea of derivative.

In examining the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)}$, Zandieh describes "the derivative of f, f', is a function whose value at any point is defined as the limit of a ratio" (p. 106). Zandieh identifies ratio, limit, and function as the three aspects of the limit definition of derivative as the three layers of her framework. In the formal definition of derivative, each aspect can be conceived in two ways: a dynamic process or a static object. Zandieh argues that the concept of derivative involves the three aspects as process-object pairs in a chain in the following manner:

- The ratio *process* takes two differences and acts by division, then the result of this division is an *object* that is used by the limiting *process*
- The limiting *process* iterates through infinitely many ratios to determine the limiting value. [The ratios here are *objects* since one is imagining the value of the ratio and not determining the value of the ratio by performing division]

- The value of the limit is treated as an *object* to define each value of the derivative function.
- The derivative function involves the *process* of passing through many input values to determine an output given by the limit.
- Lastly, the derivative function can be considered an *object* by thinking about an associated ordered pair rather than the process that produced the output value.

Zandieh utilizes these process-object pairs to identify students' understanding of derivative in each layer. For example, suppose a student describes the derivative function as a process that gives the speed at each point. In that case, the student understands the function as a process but is not engaged in the underlying limit or ratio processes used in producing the speed values. Zandieh's framework is a valuable lens for identifying how individuals conceive each process-object pair in different contexts. The impact of Zandieh's derivative framework can be seen in research that has leveraged her work to forward Calculus education research (e.g., Park, 2013; Petersen et al., 2014; Jones & Watson, 2017; Zandieh & Knapp, 2006).

While Zandieh's work has proven to be useful for other Calculus education researchers, one area that Zandieh states that her framework does not explain is "how or why students learn as they do, nor to predict a learning trajectory" (p. 103). Since one of my research goals is to conjecture how a student may come to construct a productive meaning for instantaneous rate of change, I deviate from her framework and focus on how students reason about quantities and how to leverage their meanings to support them in understanding instantaneous rate of change. This research interest aligns with the call made by Thompson and Harel (2021) that it would be profitable to research students' difficulties in Calculus due to their meanings and ways of thinking about variables, functions, and rates of change.

Literature on Students' Understandings of the Idea of Derivative

Student Understandings of the Idea of Derivative as Ratio/Rate

Students are often told that the derivative as a function is a "rate of change function"; therefore, students understanding of rate of change will likely impact the meaning they construct for derivative. It is not surprising then that many studies indicate that students' ability to reason about dynamic relationships is fundamental for understanding Calculus's major concepts (Carlson et al., 2003; Carlson et al., 2001; Cottrill et al., 1996; Kaput & West, 1994; Zandieh, 2000). However, investigations of students' meanings for functions have revealed that students struggle with modeling function relationships involving rate of change of one quantity as it continuously varies with respect to another quantity (Byerley et al., 2012; Carlson, 1998; Monk & Nemirovsky, 1994; Thompson & Thompson, 1994). Hence it is essential to discuss the meanings and issues students have about rate of change because of how central the concept of rate of change is to the understanding of derivative that students construct.

Orton (1983) studied students' understanding of differentiation by examining students' responses to various tasks. He reported that in students' responses in tasks that involved interpreting a rate of change, they confounded the output value of a function with the value of the rate of change. Additionally, he indicated that many students did not demonstrate a robust understanding of what composes the structure of a ratio. Many of his students expressed an average rate of change as either adding values (e.g., $\frac{y_2+y_1}{x_2+x_1}$) or as a singular value for the numerator or denominator (e.g., $\frac{y_2-y_1}{x_1}, \frac{y}{x}$). Carlson et al. (2002) indicated that many high-performing Calculus students struggled with tasks involving average and instantaneous rate of change. These students described rates of change as additive changes in the output rather than expressing rates as a multiplicative comparison of changes in two quantities. These researchers indicate that many students' understandings of ratio (and rate) as an *object* lack the underlying division process as the relative size measurement between two quantities' values.

Several researchers demonstrated that many undergraduate students confuse rate quantities with amount quantities (Byerley et al., 2012; Flynn et al., 2018; Castillo-Garsow, 2010; Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006). Byerley et al. (2012) investigated calculus students' understanding of division and found that many students employed additive reasoning when interpreting the value of a rate. The researchers reported that the students who engaged in additive reasoning described a constant rate of change as equally spaced intervals in the dependent quantity. Similarly, in another study, one student stated that if a rate function "was just 2, then you'd be saying that you only *added* 2 pounds of salt for the whole time" (Rasmussen & Marrongelle, 2006) (p. 408). Students in these studies communicated that they interpreted the value of a rate to be an amount to add instead of interpreting a rate as a multiplicative relationship between 2 varying quantities. Similarly, Thompson (1994) reported that students confused "changing" with "rate of change" and also "amount and change in the amount." We can interpret from these studies that if students are reasoning additively about rates of change, this will be an obstacle to understanding derivative as a rate of change function.

Student Understandings of the Limit[2] Aspect of Derivative

The process of evaluating a Limit entails examining the behavior of a multitude of values and determining what these values converge to. A student can also conceive of a Limit as an *object* as the result of the limiting process. For example, students are often introduced to the limit definition of derivative with a graphical depiction of receding secant lines that converge to a tangent line. The limit process involves examining the values of the slopes of the secant lines, whereas the end product is the value of the slope of the tangent line.

Researchers have reported that the concept of limit is paramount in understanding calculus and developing rigorous mathematical thinking beyond calculus (Tall, 1992). Limit is central to many concepts such as derivative, integration, and infinite series, but studies have shown that many students have misconceptions about limits. Some issues occur because students consider a finite process to solve limit problems rather than conceive an infinite process. Thus, these students will often confuse the limit with the value of the function (Cottrill et al., 1996; Roh, 2008). Other issues about limits occur from seeing the limit as the infinite process rather than the result of the infinite process (Vinner, 2002).

Roh (2008) classified different types of thinking about limit that students held when evaluating the limit of a sequence; *limit as asymptote, limit as cluster points*, and *imaging true limit points*. *Limit as asymptote* is a conception of limit where students believe the sequence will get close to but never attain the limit value. This type of

[2] For 'Limit' I will be including studies about infinitesimals, differentials, or some similar idea. The Limit portion of the limit definition of derivative entails the idea of infinitesimals or "picking a really small window" or being "essentially equal".

reasoning explains why students believe that sequences can have more than one limit value since the sequence can be approaching two different values, e.g.: $\{a_n\} = (-1)^n + \frac{1}{n}$ as approaching -1 and 1. *Limit as cluster points* is the type of thinking where students determine the limit value by noticing many output values within a reasonable margin around that limit value. This is another kind of thinking that could allow students to believe that a limit could have more than one value if they see two or more clusters of points. Unlike thinking about limit as asymptote, students who view limit as cluster points allow the sequence's values to be the value of the limit. The last way of thinking is *imagining true limit points* where students consider both infinitely many points near or at the limit value and finitely many values outside of such a cluster. Students engaging in imaging true limit points do not believe there can be more than one limit value.

Zandieh and Knapp's (2006) work on metonymy with calculus students revealed that students often forget the dynamic imagery of receding secant lines and instead only recall the finished product of the tangent line. This is similar to the research conducted on student understanding of limits (Roh, 2008; Davis & Vinner, 1986; Cornu, 1991; Przenioslo, 2004), where students tend to fixate on a single value and forget the dynamic imagery of values converging (Williams, 1991). Research on student meanings for limit has shown that students think of the word "approaching" and "getting close to" as synonymous with "asymptotic," where limiting values can get close to but not touch the limit value. This issue would potentially affect how students view the limit of the difference quotient, where the values of the average rates of change get close to the value of the instantaneous rate of change value but never the actual value of the instantaneous rate of change. If students do not have the dynamic imagery of values converging when they work with limits, then when students are presented with receding secant lines, the tangent line as their finished product (Zandieh & Knapp, 2006) becomes a static image for their meaning for derivative.

In the context of a derivative, limits are about *differentials* since they indicate infinitesimally small changes. One issue with the teaching of differentials is the lack of agreement on what a differential is. McCarty and Sealey (2019) interviewed seven mathematicians about differential and reported no instances of total agreement, only common recurring themes. Even among researchers, there is disagreement about the definition of *differential*. Some descriptions of the idea of differential include; An arbitrarily small change or changes, a shorthand for limit, and an infinitesimal of the hyperreal numbers (Dray & Manogue, 2010). Nevertheless, many researchers argue that a productive way of learning Calculus entails discussing the idea of *differential* early on (Thompson & Dreyfus, 2016; Dray & Manogue, 2010; Kouropatov & Dreyfus, 2013).

Thompson and Dreyfus (2016) argue that the current approaches to Calculus that utilize concepts of limits or differentials fail to address the issue that students tend to conceive of variables statically. Many researchers support this claim by attributing students' weaknesses in modeling dynamic situations to their static conceptions of variables (Carlson et al., 2002; Moore & Carlson, 2012; Trigueros & Jacobs, 2008). Conversely, the benefits of employing differentials focusing on variables varying are present in the work of Thompson (2019), Ely and Samuels (2019), and Tall (2013).

<u>Student Understandings of the Derivative as a Function</u>

Function as a *process* involves interpreting the function's defining formula (in this case, the limit of the difference quotient) as describing how input values are used to produce output values. Function as *object* focuses on the ordered pair, $(x, f'(x))$. Existing studies on derivative are more focused on this aspect since they investigate how students use derivatives in various contexts.

Some studies about the derivative as a function (as *process*) address the covariation of quantities involved. Oehrtman et al. (2008) argue that students need to go beyond viewing function as an entity that takes in an input to produce an output. Instead, the authors argue for moving to a function conception that enables reasoning about quantities varying.

Other studies investigated students' understandings of a derivative as a graph (Baker et al., 2000; Aspinwall et al., 1997; Asiala et al., 1997; Borgi et al., 2018; Ubuz, 2007). These researchers report that students have trouble relating the graph of the function with the graph of the derivative function. Aspinwall et al. (1997) argue that students' graphical image of a function can prevent them from understanding derivative. In their study, they described a student's image of a quadratic function as having vertical asymptotes. This student's image led to them to reasoning that the function's derivative is shaped like a cubic function. Similarly, Baker et al. (2000) reported that students consistently struggled with explaining a derivative when a function has a cusp, vertical tangent, or removable discontinuity. These interpretations and issues with graphs are consistent with Oehrtman et al.'s (2008) assertion that many students' meaning for function lacks the imagery of variables varying.

Monk (1994) and Park (2013) explored students' understanding of the derivative as a function versus their understanding of the derivative at a point. Monk (1994) noted that students coming into Calculus understood functions as if they were a table of values where they were associating particular values of the input to particular output values. Monk indicated that interpreting a function this way does not lend students to understanding the crucial issues of Calculus, such as: "How does change in one variable lead to change in others? How is the behavior of the output variables influenced by variation in the input variable?" In one example, students determined the correct shape of the graph of a function, f, given f', but at the same time, they were unable to estimate the value of $f'(2)$ or determine when $f'(x) = 0$. Monk hypothesized that these students were likely memorizing a procedure instead of understanding the derivative as a function. Similarly, Park (2013) reported that students' thinking about derivative as a function was not fully developed. Park noted that many of her students inconsistently mixed up the derivative function with the equation of a tangent line.

We can draw from these studies' results that students tend to interpret a function as an object by thinking of ordered pairs or engaging in physical features of the graph of a function. These static interpretations lack the dynamic imagery of quantities changing and often result in students interpreting derivatives as tangent lines or the reading on a speedometer.

Literature on Student Reasoning with Derivative

Several researchers investigated students' reasoning abilities when utilizing and interpreting derivatives (Oehrtman, 2009; Ubuz, 2007; Marongelle, 2004). Oehrtman (2009) investigated students' use of metaphorical reasoning in understanding and resolving problematic situations. Metaphorical reasoning is similar to what Thompson et al. (2014) call a "way of thinking," which is the "habitual anticipation of using specific meanings or ways of thinking in reasoning." In the context of Calculus, Oehrtman (2002) discusses that $\frac{1}{3}$ of his students engaged in a collapsing metaphor to understand the limit definition of derivative. Students were reasoning in such a way that involved collapsing the difference in two values of the independent variable to 0. Ubuz (2007) also noted that students were using specific examples as ‘cognitive reference points’ to interpret new information and judge examples to see how they may (or may not) fit into these referential examples. These researchers claim that these ways of thinking implicate how Calculus concepts should be presented and that teachers should attend to how students are reasoning. Marongelle (2004) classified how some students utilized ideas of kinematics when reasoning about derivative. One classification was *contextualizers* who blended two contexts and thought of them as equivalent. Another was *example-users* who used kinematic examples to make sense of a problem, followed by removing the context to consider the mathematical ideas. Other students in the study either mixed physics and mathematics while problem-solving or did not use kinematics when reasoning about derivatives.

Overall, a survey of the literature indicates that students experience multiple obstacles in learning the idea of derivative. This is not surprising since understanding derivative relies on multiple ideas, including ideas of rate, limit, and function. With these issues in mind, I examined students’ understanding of derivative as a rate of a change function and kept in the background how students were interpreting the Ratio and Limit layers of the derivative.

Literature on Pre-requisite Understandings of Derivative

Thompson and Harel (2021) indicated that it would be beneficial for the research community to investigate students’ difficulties in calculus by examining their pre-requisite meanings for variable, function, and rate of change. Carlson (1998) and Frank and Thompson (2021) found that the US Pre-Calculus curricula fail to support students in developing robust meanings for variation, function notation, and average rate of change that would be productive for students in understanding the important ideas of Calculus. Similarly, Toh (2021) demonstrated that students experience a disconnect between their early mathematics and their Calculus learning. These findings indicate a need to study the pre-requisite ideas necessary for understanding derivatives.

Since the idea of derivative can be interpreted as an *instantaneous rate of change*, a discussion on student understandings of rate pertains to the learning of derivative. Rate of change is not a concept that solely exists in the context of Calculus. Ideas of rate of change are utilized in Algebra classes when working with slope as a graphical representation of constant rate of change or comparing differences in tables (Confrey & Smith, 1995). Within the realm of mathematics education researchers, the definition of *rate of change* differs from researcher to researcher. For some, a rate of change can be a directly perceived quantity, such as your walking speed, or a mathematical relationship between 2 quantities (Noble et al., 2001). In my research, I define rate of change as the multiplicative relationship between changes in 2 varying quantities. I clarify this meaning in the following sections.

Thompson and Thompson (1994) provided a conceptual curriculum for speed that I leverage to articulate the productive ways of thinking for rate of change (Figure 5). Put together in one statement, a *rate of change* quantifies a *multiplicative relationship* between 2 *varying quantities*.

1. Rate of change is a **quantification of variations**
2. Rate of change relates variations in **two varying quantities**
3. Rate as a quantification of variations in two quantities is made by a **multiplicative comparison** of these variations
4. To say that rate of change of quantity Y with respect to quantity X is “m” is to mean that the variation in quantity Y (Δy) is m times as large as the variation in quantity X (Δx), i.e., $\Delta y = m\Delta x$

Figure 5: Productive Ways of Thinking for Rate of Change

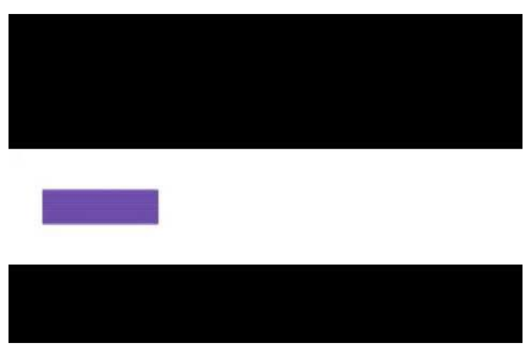
<u>*Quantification of Variations*</u>

A rate of change involves quantities varying, but what quantities are we talking about? If we examine a typical way that constant rate of change is presented (in the context of slope), $m = \frac{y_2 - y_1}{x_2 - x_1}$, we notice that there is a *difference* in both the numerator and denominator. Thompson (1993) calls a “quantitative difference” to mean “a quantity constituted by an additive comparison of two quantities.” Thompson stressed that students need to conceive of a difference as a new quantity rather than just the result of subtraction. For example, if we imagine some quantity such as “the amount of water (measured in fluid ounces) in a water bottle,” we can define a variable to represent the value of a varying quantity, call it x, to represent the quantity's value at some particular moment. We can measure the value of the quantity at different moments, e.g.: $x_1 = 4$, and $x_2 = 19$. We can then find the difference in the amount of water between these two moments by making an additive comparison, $x_2 - x_1 = 15$. If we examine the value of this difference, 15, it is not the same as saying that $x = 15$. Therefore, since this 15 is not “the amount of water in a water bottle,” it must represent another quantity in this situation, namely “*the variation* in the amount of water between two moments.” An individual must mentally distinguish that 15 measures a different quantity representing a relationship between 2 measurements of the initially conceived quantity. While this distinction may seem trivial, researchers have shown that many students do not think about the expression $x_2 - x_1$ as representing a single quantity or "difference," instead, they tend to think about "difference" as a subtraction symbol (Musgrave et al., 2015; Orton, 1983; Thompson, 1994). It is essential then that a student regards a variation in a quantity as a newly conceived quantity distinct from the initial quantity. If a student is conceiving of $x_2 - x_1$ as a variation, this will aid them in constructing a meaningful understanding that slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$, quantifies a ratio of variations in quantities rather than thinking of slope as "rise over run," which may not have productive meanings for students (Nagle et al. 2013, Stanton & Moore-Russo 2012).

<u>*Variables Vary*</u>

A rate of change is about quantities changing; in other words, the variables involved have to be varying! Thompson and Carlson (2017) explain that if students view variables statically, they cannot imagine expressions as representations of relationships among varying quantities (e.g., a rate). For a variable to vary, a student imagines that if the value of a variable changes from $x = 2$ to $x = 3$, then the variable smoothly changes by going through all the values between 2 and 3 instead of jumping from the value of 2 directly to the value of 3. Videos 1 and 2 illustrate the differences between these two

Video 1 : Jumping from 1 Value to Another



Video 2:Smoothly Changing from 2 to 3

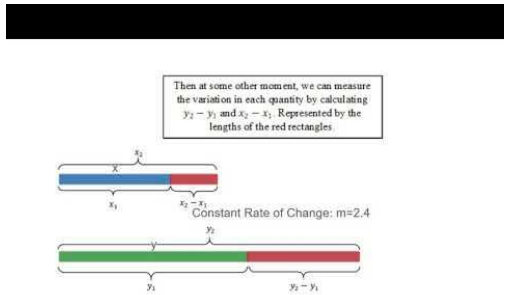
conceptions. If students view variables statically, they will not interpret a rate of change with the imagery of two covarying quantities; instead, they still only consider the division operation on two static variations.

Multiplicative Comparison

To calculate a constant rate of change ($m = \frac{y_2 - y_1}{x_2 - x_1}$), division is used on two variations to produce a value of m. Similar to subtraction, division is not just an operation used to calculate a value; rather, the result from dividing represents the measuring of the size of one variation ($y_2 - y_1$) in terms of the size of another variation ($x_2 - x_1$) (Thompson & Saldanha, 2003). Division as a quantitative operation quantifies the multiplicative relationship between 2 quantities. Understanding this relationship of $m = \frac{y_2 - y_1}{x_2 - x_1}$ entails the following

- $y_2 - y_1 = m(x_2 - x_1)$: "The variation in y is m times as large as the variation in x"
- $(x_2 - x_1) = (y_2 - y_1) * \frac{1}{m}$: "The variation in x is $\frac{1}{m}$ times as large as the variation in y"
- $m = \frac{y_2 - y_1}{x_2 - x_1}$: "The variation in y measured in the size of the variation of x is m"

The ways of thinking involved with multiplication and division entail the *relative size* between 2 quantities' values. In processing the statement of $y_2 - y_1 = m(x_2 - x_1)$, one productive way of imagining the multiplication involved ($m * (x_2 - x_1)$) involves picturing the size of the variation in x (the value of $(x_2 - x_1)$), measured out m times. For division $\frac{y_2 - y_1}{x_2 - x_1}$, the mental actions involved require determining the sizes of the variations and then measuring the value of $y_2 - y_1$ using the value of $x_2 - x_1$ as a unit of measure. Video 3 illustrates the associated imagery.



Video 3:Measuring a Variation with another Variation

When an individual conceptualizes this relationship between the two variations for only the two moments that the quantities were measured (i.e., At one moment, the values of the quantities were x_1 and y_1, at the other moment, the values of the quantities were x_2 and y_2), we can say that the individual has a multiplicative understanding of *ratio*. Only when this ratio is generalized to represent the multiplicative relationship of the covarying quantities can we consider it to be an understanding of rate of change.

<u>Ratio versus Rate</u>

Since the idea of ratio is related to the idea of rate, how students conceive of ratio likely affects their construction of a rate of change. Among mathematics educators, there are two popular contrasting conceptions of ratio (Johnson, 2015): *ratio as identical groups* (Simon, 2006) and *ratio as measure* (Simon & Blume, 1994; Thompson & Thompson, 1994). *Ratio as identical groups* is when a ratio is conceived of as two quantities that are associated together. However, the ratio itself is not a single quantity that the student is conceiving. *Ratio as measure* is the conception that the ratio describes a relationship between the two quantities. For example, a student conceiving of *ratio as* *identical groups* would likely interpret a ratio of $\frac{1}{4}$ donuts per dollar as associating 1 donut with 4 dollars. A student conceiving of *ratio as measure* would likely interpret the ratio of $\frac{1}{4}$ donuts per dollar as the relationship between the number of donuts and the number of dollars, and it describes a whole set of covarying quantities. These two contrasting conceptions may explain why some students interpret the value of a rate as an amount to add because those students may have a conception of ratio as *identical groups*. These students may be thinking of the value of the rate as the amount of change for 1 quantity for a 1-unit change in the other quantity (e.g.: $f'(3) = 6$ is interpreted as a change of 6 units in the output quantity for a 1-unit change in the input quantity).

What distinguishes a rate from a ratio is that a ratio describes the multiplicative relationship in a static comparison. In contrast, a rate is a reconceived ratio that applies to an entire class of covarying quantities. To explain this, I use Lamon's (2007) example on the understanding of a fraction as a number itself (rather than the numbers that compose the fraction). If we consider the fraction $\frac{1}{4}$, this number refers to the same relevant amount in each of the pictures in Figure 6. This way of thinking about a fraction as a rational number is the same for interpreting the value of a rate of change. The value of a constant rate of change describes the invariant ratio between the variations in a dependent and independent quantity. The value of the rate is not just a ratio of a single situation (just like $\frac{3}{12}$ might be for the top right picture in Figure 6), but instead, it describes a characteristic of a whole set of covarying quantities (Thompson, 1994).

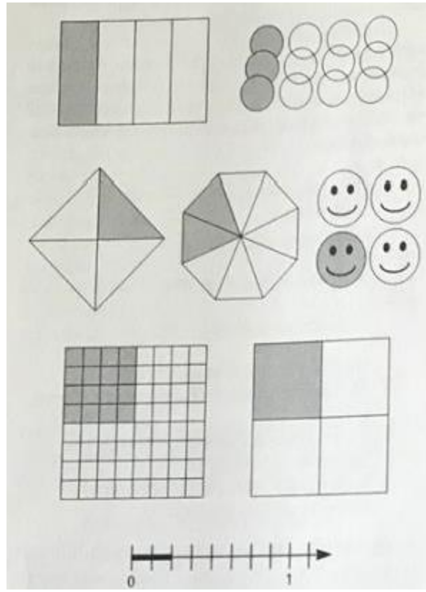
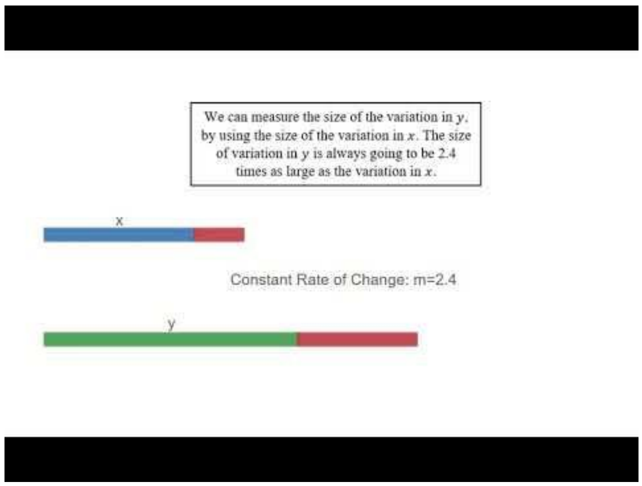


Figure 6: Picture from Lamon (2007); 1/4 as representing the relationship in each

To conceive of a constant rate of change entails understanding that the multiplicative relationship between the variations stays in constant ratio as the quantities covary. Another way of saying this is first discerning a multiplicative relationship between two quantities (a ratio) and then extending that relationship to other pairs (e.g.: $x_4 - x_3$ and $y_4 - y_3$). Discerning the multiplicative relationship may usually begin with what we might term *proportional correspondence,* where a student is using the initial variations ($x_2 - x_1$ and $y_2 - y_1$) as a reference when describing the proportionality between other pairs of quantities. For example, given the statement, “If Jack walks at a constant speed and walks 7 feet in 3 seconds”, a student may recognize that if Jack travels 4 feet, then he has traveled $\frac{4}{7}$ of the 7 feet, so it will take him $\frac{4}{7}$ of the 3 seconds. At this stage, the student recognizes a multiplicative relationship between the two quantities, but they have not yet determined that multiplicative relationship's numerical nature. Framing this in terms of quantitative reasoning, we can say this student has not yet conceived of this relationship as an attribute of something that can be measured. The next mental action involves extending this relationship to other pairs of quantities and anticipating that if you keep letting the quantities vary, the ratio is anticipated to be maintained. When a student utilizes the value of a constant rate of change (m), they imagine the quantities covarying such that as the input quantity varies, the output quantity simultaneously varies m times as much. Video 4 illustrates the associated imagery of this understanding of constant rate of change.



Video 4:Rate of Change

<u>Literature on the Teaching of Derivative</u>

In many of the textbooks used in a standard Calculus 1 course (e.g., Stewart (2013) and Larson (2007)), the introduction to derivatives is through a “special type of limit” or as a slope of a tangent line (Figure 7). While there is mention of instantaneous rate of change, the descriptions in textbooks usually describe sliding a secant line to obtain the desired slope of the tangent line. Reading these standard texts reveals little attention to what a rate of change means and reasoning about how quantities change together. As a result, research has shown that Calculus students have difficulties

2 DERIVATIVES

In this chapter we study a special type of limit, called a derivative, that occurs when we want to find a slope of a tangent line, or a velocity, or any instantaneous rate of change.

2.1 DERIVATIVES AND RATES OF CHANGE

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object involve finding the same type of limit, which we call a *derivative*.

Figure 7: Stewart's (2013) Introduction to Derivatives

interpreting the derivative as a rate of change function. Orton (1983) noticed that in tasks that involved interpreting a rate of change, students confounded the output value with the rate of change. Ferrini-Mundy and Graham (1991) reported that students struggled to make sense of a sliding secant line and its relationship to the rate of change on a small interval. Weber et al. (2012) propose that student issues with the sliding line stem from typical calculus textbook presentations of the limit definition of derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, where x is fixed, and h varies to 0. They hypothesize that fixing x does not allow one to visualize the derivative function being generated for all the values of x. To put it in terms of Zandieh's (2000) framework, this presentation of $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ through the imagery of the sliding secant line does not seem to be effective in allowing students to conceptualize the limit as an object used to define each value of the derivative function.

On the learning of derivative, many researchers agree that students should engage with the idea of derivative in multiple contexts (e.g., Graphical, symbolic, verbal, physical) (Jones & Watson, 2018; Roorda et al., 2009). The National Council of Teachers of Mathematics (2000) indicates that understanding the derivative concept involves a broad base of examples such as the average velocity of a car, the average air temperature over a day, or the average slope while climbing a hill. Jones and Watson (2018) demonstrated that students who worked with contexts that involved different aspects of the derivative concept (ratio, limit, and function) potentially allowed for better mapping to novel derivative contexts. One early example of the benefit of engaging in multiple contexts is the work done by Asiala et al. (1997). They offered a framework based in APOS theory of a genetic decomposition of the derivative. The framework explicated different kinds of constructions students might make for learning derivative. For example, the graphical path involved calculating the slopes of secant lines as the 2 points get closer and closer. An analytical path consists of calculating average rates of change over smaller and smaller intervals. Students in their study were reported to have had more success in constructing a graphical understanding of derivative than their traditional peers.

Due to the relevance of derivative in the fields of science and engineering, a few Calculus researchers have studied how students comprehend and apply the derivative concepts to contexts outside of a pure mathematics setting (Jones, 2017; Roorda et al., 2007, 2010). The majority of other Calculus researchers focus on derivative in the context of position, velocity, and acceleration. Jones (2017) argues that teaching derivative as velocity may not allow students to reason about rate of change since velocity can be an intuitive topic that students already understand. Petersen et al. (2014) study of Calculus students' reasoning revealed that students could not effectively engage in ideas of velocity and instead utilized other ideas such as physical features of a graph to reason about derivatives. However, other researchers have demonstrated the benefits of using examples involving velocity to develop deeper understandings of calculus concepts. Berry and Nyman (2003) utilized graphing calculators and Calculator Based Ranger (CBR) to have students develop their intuitive understanding of derivative. The CBR recorded the displacement of a student for pre-set intervals of time and this data was transferred to the calculator. Students in the study engaged in “CBR walks” to physically model motion and then subsequently reason about velocity graphs. These students demonstrated a stronger connection between a graphical representation of a function and its derivative.

Some researchers argue for alternative ways to learn derivative over the standard limit approach used in typical U.S. classes. For example, Tall (2009, 2013) argued for a local linearity approach by having students zoom into a graph until it appeared linear. The slope of the supposed “line” that students saw could be considered the derivative value, and students could see that the slopes would be different for different input values. Ely and Samuels (2019) support Tall’s finding through their alternative teaching lessons that delay the teaching of limit and instead have students focus on zooming in until the function is essentially linear. When interviewed, these students actively recalled the process of these informal limit ideas when describing finding the slope of the line. We can infer from these findings that the current standard calculus curriculum focused on formal learning on limits and limit notation is not a prerequisite for understanding the derivative as the limit of average rates of change.

Other researchers argue for a focus on a rate of change as conceptualizing the covariation between two varying quantities (Thompson et al., 2013; Carlson et al., 2002).

Thompson and Ashbrook (2019) developed a Calculus course that puts the Fundamental Theorem of Calculus as central in order to have students develop richly connected meanings for rate-of-change function and accumulation functions. This course was designed to support students in overcoming difficulties that were identified by research on students' calculus learning, such as students thinking that variables do not vary, believing that Calculus is a set of rules and procedures, and a derivative is a slope of a tangent rather than about a rate of change (Thompson et al., 2013). Thompson (2019) compared students' performance in this course with students in traditional Calculus courses using an 11-item Calculus 1 concept inventory focused on variation, covariation, and rate of change understandings.[3] He reported that students who took his conceptual Calculus course scored higher on average than the other students (Table 4). I leverage Thompson's work on a conceptual approach to Calculus to conjecture how a student in a standard Calculus 1 course can build productive meanings for instantaneous rate of change functions (derivatives). Carlson et al.'s (2001) study further evidence the benefits of focusing on covariational reasoning in a Calculus course. Carlson et al. investigated the effect of a covariational curriculum in developing ideas of limit and accumulation. The results of their study indicate that their students had developed flexible and productive covariational reasoning abilities and were able to apply this reasoning in limit and accumulation tasks.

[3] It should be noted that according to Thompson (2019), "a committee of five people, two teaching traditional Calculus 1, two teaching DIRACC Calculus 1, and the department's director of STEM education, constructed" the concept inventory and that "no item was included without unanimous agreement that it assessed a central idea in Calculus and addressed it acceptably" (p. 10)

Both Thompson and Carlson's studies demonstrate that a calculus curriculum focused on reasoning covariationally can help students understand the important ideas of Calculus. While Thompson has provided a conceptual analysis for the Fundamental Theorem of Calculus (1994) and quantitative data on the benefits of his conceptual calculus course (2019), what is currently absent in his work is qualitative data that describes how his students are building productive meanings for each Calculus topic. Carlson et al. (2001) provided some qualitative data on the learning of limit and accumulation. However, I know of no papers describing how a curriculum focused on

<table>
<tr><th></th><th colspan="2">PreTest</th><th colspan="2">PostTest</th></tr>
<tr><th></th><th>Mean</th><th>StdDev</th><th>Mean</th><th>StdDev</th></tr>
<tr><td>Traditional
$n = 248$</td><td>3.18</td><td rowspan="2">1.53</td><td>4.89</td><td rowspan="2">2.45</td></tr>
<tr><td>DIRACC
$n = 149$</td><td>2.98</td><td>7.90</td></tr>
</table>

Table 4:Thompson's (2019) Pre-Post Comparison of Traditional versus his Conceptual Calculus (DIRACC). Possible total score of 11.

reasoning covariationally aids students in building a productive meaning for instantaneous rate of change. The studies in this manuscript will add to Thompson and Carlson's work by adding qualitative research on student learning of the idea of derivative as instantaneous rate of change.

CHAPTER 4

INTRODUCTION TO THE THREE PAPERS

Zandieh's (2000) study of calculus students' understanding of derivative led to her developing a derivative framework that described the conceptual complexities of understanding the idea of derivative. She described students conceptualizing process-object pairs on three layers of representation: ratio, limit, and rate. She further elaborates what a student might be conceptualizing as they progress through and move between the three layers. There is currently no research describing the interventions that are successful in supporting students in understanding the idea of instantaneous rate of change. This dissertation is a response to this gap in the literature by taking a multi-pronged approach to researching and identifying students' difficulties (Paper 1), elucidating a productive meaning for instantaneous rate of change (Paper 2), and investigating the mechanisms for supporting student learning on the idea of derivative (Paper 3).

Paper One: What are Students' Meanings for the Derivative at a Given Input Value?

Research Question: *How do students interpret the derivative at a specific input value in an instantaneous rate of change context? In this context, how do the students attend to quantities' values changing?*

In order to support student learning on the idea of derivative, one has to first determine what ways of thinking and understandings students have about derivative and related ideas. This study explored Calculus 1 students' meaning for the derivative at a specific value of a function's input variable. The study engaged 25 students in individual clinical interviews in which they were asked to explain their thinking when responding to tasks aimed at revealing their meaning for derivative. During these clinical interviews, I focused on how students attended to the values of the input and output quantities of a function as changing and any other quantity they may have described during the interview. This paper presents results from analyzing the clinical interview data and characterizes the thinking exhibited by students.

Paper Two: A Productive Meaning for Instantaneous Rate of Change and Relevant Prerequisite Ideas

Research Question: *What is a productive meaning for instantaneous rate of change? What understandings are foundational for understanding the concept of derivative?*

While researchers (Zandieh, 2000) have presented the results of a mathematical analysis of the derivative concept, I know of no papers that attend to the mental actions involved in coordinating the values in two varying quantities when interpreting the value of an instantaneous rate of change. In this paper, I synthesize relevant literature about derivative, leverage the results from the clinical interviews described in the first study, and describe a hypothetical learning trajectory for developing a productive understanding of instantaneous rate of change. A productive meaning is one that is useful for future mathematical learning and in this manuscript, I provide two examples to demonstrate how this meaning would be coherent and useful in related topics involving instantaneous rate of change.

Paper Three: A Teaching Experiment on Instantaneous Rate of Change

Research Question: What understanding of the concept of derivative do individual students develop during an instructional sequence designed to support students in using quantitative and covariational reasoning?

Much of the current literature on the learning of derivative is focused on exposing students to multiple representations of derivative (Hähkiöniemi, 2004; Jones & Watson, 2018) that utilizes Zandieh's (2000) derivative framework. However, many of these works are not focused on the types of reasoning students are employing. Instead, this paper builds on the work on quantitative and covariational reasoning (Carlson et al., 2002; Thompson et al., 2013) to foster productive understandings of rate of change. In this study, I model two students' thinking and understanding of rate of change during a teaching experiment that emphasized conceptualizing quantities and reasoning about how two quantities' values are related and vary together.

CHAPTER 5

PAPER 1: WHAT ARE STUDENTS' MEANINGS FOR THE DERIVATIVE AT A GIVEN INPUT VALUE?

INTRODUCTION

The National Council of Teachers of Mathematics (1998, 2000) has consistently recommended that students develop the ability to analyze patterns of change in various contexts. They suggest that students should understand how changes in quantities can be mathematically represented (2000). However, many students do not have the opportunity to construct these productive understandings of changes (variation). One reason is that many students conceptualize variables as static and as placeholders for specific values instead of imagining variables as varying (White & Mitchelmore, 1996; Frank & Thompson, 2021). Thompson and Carlson (2017) argue that if a student has a static interpretation of variables, then the student cannot imagine mathematical expressions as representing relationships among varying quantities. This static conception of variables would then lead students to focus on symbolic manipulations instead of what the mathematical operations they employ might represent in a context.

Many researchers agree that imagining quantities' values as varying is essential for understanding rate of change, functions, and graphs (Johnson, 2015; Thompson, 1994; Carlson et al., 2002; Moore, 2010). However, researchers indicate that many undergraduate students struggle to model functional relationships of problems that involve the rate of change of one variable as it continuously varies with another variable (Carlson, 1998; Carlson et al., 2002; Monk & Nemirovsky, 1994). Further, research indicates that the ability to model a dynamic relationship between two varying quantities is foundational for understanding the key ideas of Calculus (Kaput & West, 1994; Thompson, 1994; Zandieh, 2000). Therefore it is imperative to understand how to aid students in developing an understanding of dynamic functional relationships. Thompson (1994) argues that the concept of rate is central to this development which involves the coordination between the values of two quantities as they covary together.

In this study, I report my findings on students' understandings of the derivative evaluated at a given input value in the context of instantaneous rate of change. I explored their understanding of derivative by conducting clinical interviews (Clement, 2000) with 25 students who had taken Calculus 1. As a result of this study, I argue that attending to a student's interpretation of a rate of change can provide valuable insight into their covariational reasoning.

Research Question: *How do students interpret the derivative at a specific input value in an instantaneous rate of change context. In this context, how do the students attend to quantities' values changing?*

METHODOLOGY

By taking the Radical Constructivist stance (Thompson, 2000), I assume that it is impossible to know another's thinking. Therefore, investigating student thinking aims to build models of students' mathematics (Steffe & Thompson, 2000) that may explain why students produce specific responses. Using this approach to build an explanatory model of students' thinking can help describe the process of developing a productive meaning for derivative at a point. The word 'meaning' will be used in the way that Thompson (2013) uses it to describe mathematical meaning. It is the organization of an individual's experiences with an idea that determines how the individual will act. Meanings are personal, and they might be incoherent, procedural, robust, or productive. However, these meanings are used by individuals to respond to mathematics tasks and make sense of and access mathematical ideas. For example, a person's meaning for derivative might only be associated with calculating the limit of the difference quotient. At the same time, another's meaning for derivative could involve the slope of a tangent line. Since meanings are personal, if one student writes a response similar to another student, it cannot be assumed that they both have the same meaning.

To address the research question, I report on the results of conducting a single clinical interview with 25 students (Clement, 2000). These were *generative* clinical interviews in that I did not originally have a coding scheme before the interviews took place. According to Clement, *generative* clinical interviews should "generate new observation categories and models of mental processes that give plausible explanations for the observed behavior" (p. 10). In comparison, a *convergent* clinical interview involves a coded analysis that focuses the observations on predefined categories.[4]

Clinical interview methodology consists of an interviewer, a single student, and a camera to record each interview. During these interviews, the interviewer asks the student to engage in a mathematical task. These tasks intend to explore how students think and reason about the tasks. As the interview progresses, the interviewer creates models of the student's understandings and tests their hypotheses by asking questions and probing their

[4] Other papers (Yu, 2019, 2020, 2021) describe the findings of these generative clinical interviews. There were no predefined categories used from the Covariational Framework (Carlson et al., 2002) when analyzing these clinical interviews in those papers. In this paper, I applied the Covariational Framework (and my modifications for it) retrospectively as a lens to examine how students coordinated the variations in two quantities' values.

thinking. In this interview, there is no teaching component as the goal of a clinical interview is to build models of student understanding.

The interviews were semi-structured (Zazkis & Hazzan, 1998) in that the interview was planned in advance, but follow-up tasks would differ based on the interviewee's responses. The semi-structured interviews allowed for unplanned follow-up questions and variations on the planned questions. Semi-structured interviews allow the interviewer to test their model of the students' thinking by presenting potential tasks based on how they respond during the interview. For example, if a student stated that they interpreted the derivative at a point as the slope of the tangent line, the interview might follow up by asking the student to draw a graph with the tangent line they are thinking of.

The main tasks for these interviews were as follows

Task 0: What does the word 'derivative' mean to you?

Figure 8:Task 0 - The Immediate Meaning

Given that $P(t)$ represents the weight (in ounces) of a fish when it is t months old, a.) Explain the meaning of $P'(3) = 6$ b.) If $P\ (3)\ =\ 15$ and $P'(3)\ =\ 6$ estimate the value of $P\ (3.05)$ and say what this value represents.

Figure 9: Task 1 - The Fish Task

The purpose of Task 0 (Figure 8) was to elicit the students' spontaneous meaning for derivatives. Task 1 (Figure 9) probed students' interpretation and use of a derivative value in an applied context. More specifically, I was interested in investigating the following questions:

1) Does a student associate a derivative with an "instantaneous rate of change"?
2) How does a student interpret the value of the derivative at a given input value?

3) How does a student utilize the value of a derivative to solve a linear approximation problem?
4) Does a student recognize that the linear approximation they performed in part b is an approximation because the value of the rate of change would likely change in that input interval?

I anticipated that many students would verbalize the derivative as representing an instantaneous rate of change. I used this task to explore how they would interpret the value of an instantaneous rate of change in an applied context. Even if a student did not verbalize the derivative as an instantaneous rate of change, I used this task to investigate how a student might conceptualize the problem context and how they used their meaning for derivative to solve a linear approximation problem.

Based on the student's written work and the student's utterances, I followed up with additional tasks and questions to test my model of the students' understanding of the derivative concept. The types of questions I asked focused on assessing how the student interpreted the derivative at a point, how they thought about 'rate of change' (assuming they articulated derivative as a rate of change), and how they conceptualized the problem. Figure 10 provides a sample of the interview protocol for this task.

Task 1 – Interview Protocol
Let student work on tasks uninterrupted, only answer questions to clarify symbols or readings of the question, otherwise let the student answer it as is. *After the student has finished - (Give the student a different color pen (Green) for any clarifications, drawings, or additions they may add as the interviewer asks questions)* *For Part a.* - ***If they wrote something that appears to have a different meaning than what they said in Task 0, ask them if those are the same to them, if not ask them what is different about what they said.***

- *If they wrote "Instantaneous rate of change" ask what they mean by instantaneous, if unclear ask them to give an example.*
- *If they wrote "Slope of the tangent line at 3" ask them what they mean by slope as well as tangent line. If they are unable to describe it outside of a geometric/graphical setting, ask them to say what that means in relation to the fish.*
- *If they wrote something along the lines of "How fast the fish is growing at three months" probe them about their units. Ask whether they mean "In 3 months the fish grew 6 ounces", or "In the next 3 months the fish will grow 6 ounces" or "At 3 months the fish grew 6 ounces". Basically, probe about what students mean when they say this, I anticipate that they might flip-flop between a few of these.*
- *If they wrote "The speed of the fish", ask them what speed means in this context for them.*

Make sure to check if they wrote units or not, if not ask them what the units would be. If students articulate change in their response, ask "If I were to take a picture of this fish at exactly 3 months, then are you saying that the fish is growing at this rate in this frozen picture? is change happening in this picture or is there no change happening in this picture?" I suspect that students will say something along the lines of the fish is projected to grow at this rate, that if some amount of time were to go by then this would be the rate of change, again this means that change is not happening yet for them.

For Part b.: This part is less of a focus, but this part will allow for further evidence of students' meanings in part a.

- *If students wrote down a calculation, ask them what each part of their calculation means and why they did it.*
    - *If they did "6*0.05" ask them why they did this, and what this represents*
        - *Pay attention to if they articulate the rate of change in this elapsed time interval*
    - *If they included "15" ask them why they wrote this and what this represents to them*
- *If they found a value and didn't explain it as an approximation, ask them if this is the value of P(3.05).*
- *If blank, ask them what they were thinking as they attempted to solve this problem, what was confusing, was it not possible for them to do?*
- ***If they answered somewhat correctly, ask them why we were able to use the derivative to get this value they found***
    - *I suspect the language the students use here will most likely be the most useful as evidence of students' meanings for the derivative at a point from part a.*
    - *I suspect most students will say something along the lines of "If you know how much time has gone by, then you know how much change is*

supposed to happen", this will be in line with a student thinking about anticipated change for the derivative at a point.

Figure 10: Task 1 Sample Protocol

The protocol in Figure 10 includes hypothetical student responses based on Zandieh's (2000) derivative framework. Anticipating the potential types of responses allowed me to prepare specific questions and follow-up tasks to investigate further a student's understanding of the derivative concept. I provide the following two optional tasks as examples of how I adapted my interviews based on how the student responded to the Fish Task.

Optional Task 1 – The Tangent Line Task

Here is the graph of the function, P, and the tangent line at the point is included.

Potential Questions (depending on what the student does in Task 1)
- Is the tangent line a graph? Is it the graph of a function?
- What does this tangent line represent with respect to the previous problem?
- Why does the derivative give us the slope of the tangent line?

Figure 11: Optional Task 1 - The Tangent Line Task

Optional Task 1 (Figure 11) was used if a student associated the derivative value with the slope of a tangent line. I used this task to explore the student's understanding of a tangent line and how the tangent line was related to the Fish Task. In particular, I investigated the student's interpretation of the derivative value in the context of slope and their conception of what a point on the tangent line would represent.

Optional Task 2 – The Car Task

Suppose at 9:30am there are 3 cars and each speedometer reads a different number. Can you describe what is different about each car?

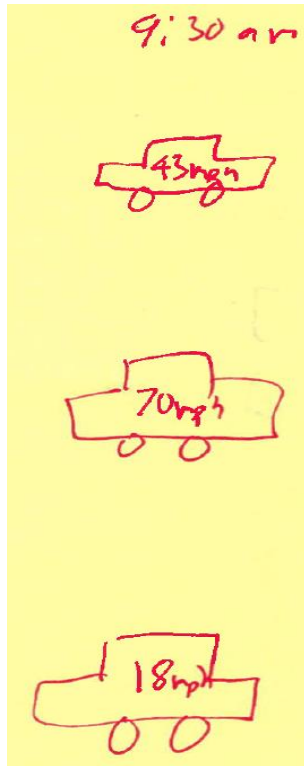


Figure 12: Optional Task 2 - The Car Task

Optional Task 2 (Figure 12) was used if a student associated the derivative value with a rate of change or a speed. The Fish Task was designed to explore a student's meaning for rate of change and ability to apply a rate value in a context. In this task, I continued to explore a student's understanding of the value of a rate of change by having students compare different car speeds. Students were asked to describe the difference between 3 different cars traveling at different speeds. I investigated how a student interpreted a speed value, what quantities they would associate a speed with, and how they were coordinating changes between those quantities.

Grounded Theory and Generative Clinical Interviews

Since the purpose of these *generative* clinical interviews was to generate new elements of a theoretical model in the form of mental actions and processes, the data analysis of the interviews was conducted in the spirit of Grounded Theory (Strauss & Corbin, 1990). Grounded theory is a qualitative research methodology in which the researcher is grounded in empirical data. Conducting research with Grounded Theory means developing theory that is "grounded" in the data instead of applying or verifying existing theories[5]. Lining up with Radical Constructivism, Grounded Theory is used by researchers to develop theories to explain observations and hypothesize models of students' thinking. With respect to clinical interview methodology (Clement, 2000), grounded theory aligns with *generative* clinical interviews, where the researcher attempts to generate new elements of a theoretical model in the form of mental actions or processes that can explain the data. I intended to interview students to continually generate models of students' thinking until viable models were developed to explain patterns in observations. As each interview was conducted, I constructed conjectures of the student's interpretation of the derivative at a point based on their utterances, written responses, and gestures. I asked follow-up questions to test these conjectures and updated

[5] It may seem strange to say that I am not applying existing theories (quantitative reasoning and covariational reasoning). First, I would say that Quantitative Reasoning is a lens I am using to understand and frame my students' responses (rather than using it to code the responses). And second, when I conducted these interviews I did not go into these interviews knowing what theoretical framework to apply. Instead, after re-watching the interviews, I found that characterizing the students' conceptualization of quantities varying was productive for explaining my students' responses.

my model of the student's thinking as the interview progressed. This process allowed me to generate models of a student's thinking that emerged both during and after the interview.

Participant Selection

I interviewed 25 students over a 2.5-year time frame, beginning in the Summer of 2017 and ending in the Fall semester of 2019. The subjects were students enrolled in a Calculus 1 or Calculus 2 course at a large southwestern university. Twelve of these students were enrolled in Calculus 2, and thirteen students were enrolled in a Calculus 1 course. There were four rounds of interviews, each conducted at the end of the semester over the duration of the study. Even though students were interviewed at different times throughout the study's duration, the main tasks were the same for all 25 students.

Data Analysis

The analysis involved Open and Axial Coding (Strauss & Corbin, 1990) for moment-by-moment coding of students' responses and interpretations. Using the codes from each student, I conducted a thematic analysis (Clarke & Braun, 2013) across moments within each student's interviews and across different students' moments. This thematic analysis aimed to identify and analyze the patterns of student responses to model the types of thinking that students were engaging in.

I conducted my data analysis in the following ways. During the interview, I developed an in-the-moment model of students' thinking grounded in their actions and utterances. After the interview, I documented my conjectured model of the student's thinking by making notes alongside the students' written work. Later, I transcribed the interview recording and coded the student's responses by describing how the student was likely interpreting something or what mental actions they were engaging in. After coding the transcript, I looked for commonalities in the codes to build and refine my model of the student's interpretation of the derivative at a point. After each round of interviews at the end of each semester, I looked for patterns across different students and similarities in the codes between students (thematic analysis). From these patterns, I generated categories of student thinking that captured different types of student responses to these tasks and described the associated ways of thinking involved in each category. After coding around 20 interviews, no new reasoning patterns emerged, and I was able to use already defined codes to characterize a student's thinking. After refining the codes, I re-analyzed each interview to check for viability and consistency in the analysis.

After the coding was completed (Yu, 2019, 2020, 2021), a follow-up analysis was conducted by examining the data using the Carlson et al. (2002) covariation framework. I compared my set of codes and categories with the descriptions and Mental Actions described in the original framework. I then observed several nuances and ways of thinking not sufficiently described in the original framework. These results led to me extending the covariation framework by including a level that characterizes how students reason about the value of a rate of change and the connections to their covariational reasoning. I hypothesized that attending to how students interpreted the value of a rate would reveal the potential mental obstacles that prevented them from reasoning at higher covariational reasoning levels. For example, suppose students were reasoning about a rate of change as an amount to add. In that case, students' covariational reasoning would be limited to MA3 since they will be coordinating amounts of change instead of conceptualizing the multiplicative relationship between two varying quantities. Table 5 describes each level of the modified covariation framework and my hypothesis for how a student at various developmental levels will reason about the value of a rate of change. I modify Carlson et al.'s (2002) covariation framework by incorporating MA0, where an individual imagines variation in one quantity (in other words, they are not attending to how two quantities vary together). I also added MA3+, where students' intuitive understandings of quantities varying involve smooth and continuous changes but are limited in how they coordinate variations in each quantity's value because of their interpretation of a rate value.

Level	Description	Example of a student reasoning about $f'(3) = 6$
Mental Action 0 (MA0) - *No Coordination*	The student focuses on the variation in the value of one quantity only. The student has no image of quantities varying together.	The student may interpret the "6" as the output value of f changing or changed by 6. Alternatively, the student may interpret the "6" as the output value of f. In either case, there is no mention of the input quantity varying.
Mental Action 1 (MA1) - *Coordinating Quantities*	The student coordinates the value of one quantity with changes in the other.	The student may believe that the value of the output quantity changed by 6 and then subsequently that the value of the input quantity changed from 3 to 4. (The student is not describing how the input and output quantities change together; instead, they observe that both quantities changed)
Mental Action 2 (MA2) - *Directional Coordination of Values*	The student conceptualizes that one quantity varies as another quantity varies, but in a gross variation manner by not considering specific values.	The student interprets that the output value is increasing as the input increases. The 6 does not necessarily measure something; instead, it is like the reading on a speedometer.

Mental Action 3 (MA3) - *Coordination of Values*	The student coordinates the amount of change of one quantity with changes in the amount of the other quantity.	A student may consider the current input and output values $(3, f(3))$ and anticipate that a change in the input (usually a 1-unit change) results in new values for the input quantity and output quantity, e.g., $(4, f(4))$. For example, a student interprets the value of 6 as the change in the output value for a 1-unit change in the input value, e.g., $f(4) = f(3) + 6$.
Mental Action 3+ (MA3+) -*Coordination of Values+*	The student has an image of the value of the rate of change varying while coordinating the amount of change of one quantity with changes in the amount of the other quantity by assuming a constant rate of change.	A student verbalizes that the value of a rate of change should vary as the input quantity's value varies. However, they consider "6" as the change (or the approximated change) in the output quantity for a 1-unit change in the input quantity. For example: "If the rate of change stays constant, then the output value will change by 6 as the input value changes from 3 to 4."
Mental Action 4 (MA4) – *Coordinating Average Rates of Change*	The student coordinates the average rate of change of the function with uniform increments of change in the input variable.	A student may consider the current values of the input and output quantities $(3, f(3))$ and anticipate that for some change in the input, Δx, the output value will vary 6 times as much. However, the student does not verbalize an awareness that the value of the rate of change varies within this Δx interval.
Mental Action 5 (MA5) – *Coordinating Instantaneous Rate of Change*	The student coordinates the instantaneous rate of change of the function with continuous changes in the independent variable for	A student may consider the current values of the input and output quantities $(3, f(3))$ and anticipate that for some change in the input,

	the entire domain of the function.	Δx, the output will vary 6 times as much. The student verbalizes an awareness that the value of the rate of change will vary in this Δx interval, but for small Δx values, the actual change in the output will be *essentially* 6 times as large. A student may consider continuous changes in the independent variable and anticipate that the values of the associated changes in the dependent variable will vary.

Table 5: An Updated Covariational Reasoning Framework

Results

The following section provides examples of each level of covariational reasoning and examples of how students at each level interpreted the value of a rate of change. I focus on describing and comparing MA3 versus MA3+ due to most students reasoning at these levels.

Mental Action 0 (MA0) – *No Coordination*

Researchers have indicated that many students confuse amount functions with rate of change functions (Flynn et al., 2018; Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006: Ibrahim & Robello, 2012). One potential reason for this is that if students are not coordinating how two quantities' values covary. I utilize Thompson and Carlson's (2017) construct of *No Coordination* (MA0) to classify this type of reasoning. A distinctive marker of this level of reasoning is conflating the value of a rate of change of a quantity with the amount of that quantity or how that quantity changed with no attention to the input quantity varying. It is important to note that this does not always mean that a student reasoning at MA0 does not think about the input quantity. Instead, they might think about the input quantity's value as a way to distinguish the instance the output quantity was measured. What characterizes MA0 reasoning is the lack of attention to the input quantity varying and its relation to how the output quantity varies.

<u>Examples of MA0 reasoning</u>

Gemma was a student who interpreted $P'(3) = 6$ in the fish task as an amount of weight (Table 6). Throughout the task, Gemma only mentioned time once when explicating her interpretation of the value of 6. She appeared to have used a time value to tag a point in time instead of mentioning how time is also varying [Line 3]. Additionally, Gemma discussed 6 as "the fish weight had changed by 6" [Lines 5-6], which furthers the notion that Gemma was primarily tracking the value of the weight since she never articulated a reference point of where she measured from. In a follow-up task on interpreting a speedometer reading of 54 mph, she explained that 54 was "how many miles the car's distance had changed" but again, never discussed time as varying. I classify this interpretation as MA0 because she interpreted the rate value as an amount of weight, and her lack of attention to the input quantity varying in her explanations.

Table 6: Gemma's Explanation for Instantaneous Rate of Change

1	Gem:	Cause if I know that if derivative is like at an instance… I
2		don't know that's just the same as P(3). The fish is 15 ounces,
3		but at 3 months it's growing at 6 ounces.
4	Int:	Can you say a little more about what you mean by that?
5	Gem:	Yeah, like growing at 6 ounces is like the fish weight had
6		changed by 6…umm... like I know that the fish is 15 ounces
7		but like the 6 is like how the weight has changed.

Similarly, Leah was a student who interpreted $P'(3) = 6$ as an amount of weight gained by the fish (Table 7). While her initial writing of "from 0 to 3 months, the fish gained 6 ounces" might indicate MA3 reasoning (Figure 13), her explanation revealed that she used the time values to distinguish between different measured instances of the fish's weight. Her choice of "then at 3 months" and "by that third month" supported the idea that she probably was not imagining time changing continuously [Lines 5&8]. She continued to reason in this manner after being asked to clarify whether 6 was the weight of the fish or how the weight had changed. She explicated that she thought of the 6 as if she "looked at the fish at 0 months" and then "look(ed) at 3 months" [Lines 11-13]. Since Leah's explanation consistently used language that evidenced her thinking about two different points in time rather than an interval of time, this corroborates the claim that she was not attending to variations in time. Instead, Leah utilized specific times to help her refer to which instance of weight she had in mind but never demonstrated that she was coordinating both weight and time as varying together.

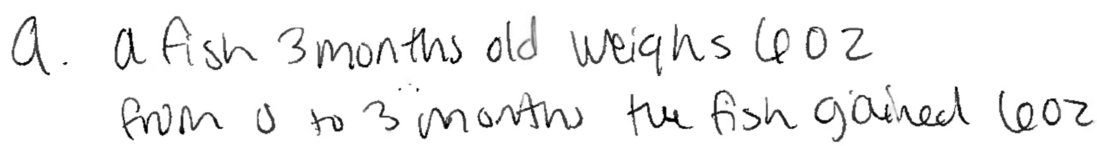


Figure 13: Leah's Interpretation of $P'(3) = 6$

Table 7: Leah's Explanation for Instantaneous Rate of Change

1	Int:	So can you explain what you wrote and what that means to
2		you?
3	Leah:	Yeah, like at 3 months the fish weighs 6 ounces, and that
4		would be like at I'm guessing when the fish was born so like 0
5		ounces and then at 3 months the fish gained 6 ounces.
6	Int:	So 6 here is the fish's weight at 3 months?
7	Leah:	Umm yeah? Like it is also what the fish gained… the weight
8		increased by 6 by that third month.
9	Int:	Wait so is 6 what the fish weighs at 3 months or how much
10		weight the fish gained by then?
11	Leah:	I guess both? Like well if we looked at the fish at 0 months…
12		the fish weighs 0? But like I now look at 3 months the fish
13		weighs 6 ounces so the fish gained 6 ounces by the 3rd month.

Mental Action 1 (MA1) – *Coordinating Quantities*

A student reasoning at MA1 notices variations in two quantities' values but may not realize that these variations happen simultaneously. So when a student engaging in MA1 considers the value of a rate of change, they will likely interpret the value as an amount of change in the output quantity and a subsequent change in the input quantity. Thompson and Thompson's (1994) construct of a speed-length is a prime example of MA1 reasoning where a student considers the value of a speed as an amount of distance for a given amount of time or that "traveling a distance at some constant speed will produce an amount of time" (p. 5)

Example of MA1 reasoning from the study

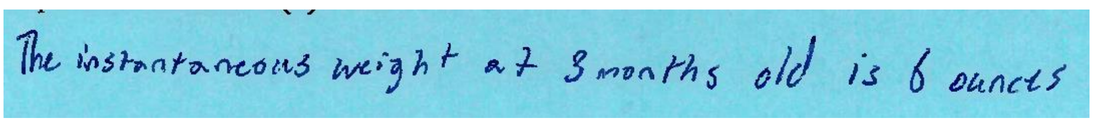


Figure 14: Keenan's Interpretation of $P'(3) = 6$

Table 8: Keenan's Explanation for Instantaneous Rate of Change

Int:	So can you explain what you wrote down?
Kee:	Yes, so the instantaneous weight being 6 ounces is the instantaneous change at 3 months is like 6 ounces.
Int:	So what are you imagining when you say this?
Kee:	Uh. Like if I looked at the fish at 2 months then the fish at 3 months the fish's weight gained 6 ounces, uh yeah changed by 6 ounces… like in that one month of time.
Int:	Okay so like the difference between the fish's weight at 2 months versus at 3 months would be 6 ounces?
Kee:	Yeah, like you know P(2) would be 6 less than P(3).

Keenan initially wrote $P'(3) = 6$ as the "instantaneous weight at 3 months is 6 ounces" (Figure 14), and while this may look similar to the MA0 examples, Keenan's explanation revealed that he noticed time passing (Table 8). However, as Keenan explained his interpretation, time did not seem to be the central focus of what a rate of change entailed to him. When discussing two measurements of the fish's weight, Keenan mentioned two different points in time as he stated, “I looked at the fish at 2 months then at 3 months the fish’s weight gained 6 ounces” [Lines 5-6]. Keenan’s language indicated that he primarily focused on the fish’s weight changing since it “gained 6 ounces” and “changed by 6 ounces”, and it was not until he paused for a moment (as indicated by the ‘…’ in the transcript) that he noticed that time had changed as well [Lines 5-7]. Keenan primarily associated the value of a rate of change with the output quantity due to his consistent response of discussing the 6 as a number of ounces [Lines 2-3, 6, 10]. Even though Keenan eventually associated a month with the 6 ounces, he mainly coordinated the value of the fish’s weight and later noticed time as elapsing; therefore, I classify his explanation as engaging in MA1.

Mental Action 2 (MA2) – *Directional Coordination of Values*

MA2 marks the beginning of simultaneously coordinating the variations in two quantities’ values. A student reasoning at MA2 recognizes that two quantities vary together, yet they will likely talk about non-specific amounts of change. They will likely interpret the value of a rate like a reading on a speedometer. This would mean that the value of the rate, for example, 6 ounces per month, does not entail 6 of something; instead, the student utilizes the value to compare to other rates (e.g., 6 ounces per month is *slower* than 8 ounces per month).

Examples of MA2 reasoning from the study

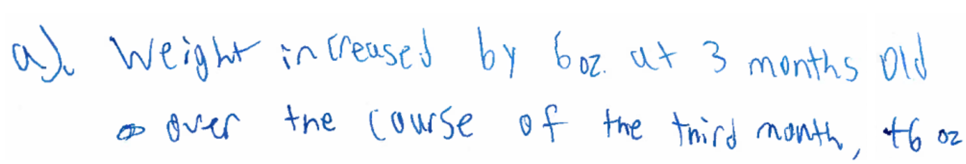


Figure 15: Bob’s Interpretation of $P'(3) = 6$

Table 9: Bob's Explanation for Instantaneous Rate of Change

Bob:	It would be like at the exact moment it is changing by 6. So I guess I mean I was thinking about this the other day. I mean I guess it wouldn't really just be like at between 2 and 3 months he's adding 6 pounds, I don't know how I think about that actually. I just kind of see it as a number to throw out there.

Bob initially explained that he interpreted a rate as the weight increase over the third month (Figure 15). As he continued to explain, he attended to both weight and time as varying, but his description lacked the specificity of what the 6 represented (Table 9). Bob coordinated both time and weight as changing as he verbalized that it would not be "like at between 2 and 3 months he's adding 6 pounds"; instead, he saw the 6 as "a number to throw out there" [Lines 3-5]. Later, when the interviewer probed him about his choice of units, Bob said that he chose 'ounces' because that was how the fish's weight was changing, but he also verbalized that when he "usually read these (rates), I kind of think of a unitless number." Throughout his explanation, Bob demonstrated that he was attending to time and weight changing simultaneously, and he was coordinating the variations in a unitless manner. Later in the interview, Bob was presented with Optional Task 2 (Figure 12), where Bob was asked to explain the difference between three different cars traveling at different speeds. Bob explained that one of the cars would be traveling faster, which meant that the car would travel further as time passed. Bob stated that "that car would obviously go farther than the other two cars, but like I can't really say exactly how much further it would travel." His statement revealed that Bob did not seem to attribute the value of a speed as quantifying something. Instead, he could only use the value to compare the distance traveled between each car in a gross variation manner. Bob's explanation of rate in a unitless manner, yet still as entailing how two quantities' values vary simultaneously, is consistent with MA2 reasoning.

Mental Action 3 (MA3) – *Coordination of Values*

A student exhibiting MA3 coordinates specific amounts of variation between the values in two quantities. Students engaging at MA3 will likely interpret 6 ounces per month as the amount of change in weight for a 1-unit change in time.

Examples of MA3 reasoning from the study

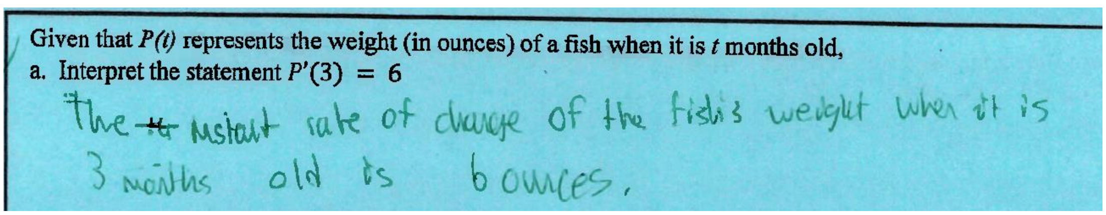

Given that $P(t)$ represents the weight (in ounces) of a fish when it is t months old,
a. Interpret the statement $P'(3) = 6$
The instant rate of change of the fish's weight when it is 3 months old is 6 ounces.

Figure 16*: Will's Interpretation of* $P'(3) = 6$

Table 10: Will's Explanation for Instantaneous Rate of Change

1	Int:	So what does it mean to you that "the instant rate of change of the
2		fish's weight when it is 3 months old is 6 ounces?"
3	Will:	So it's at 3 months and to say that's it's growing by 6 ounces. So
4		like in that entire third month it gained 6 ounces.
5	Int:	Like that happened at the end of the month or something else?
6	Will:	No, that change was over the entire month. *Draws calendar*

Will was a student that interpreted $P'(3) = 6$ as "the instant rate of change of the fish's weight when it is 3 months old is 6 ounces" (Figure 16). He explained that $P'(3) = 6$ as the change in weight for a 1-unit change in time (Table 10). Will described that "it's

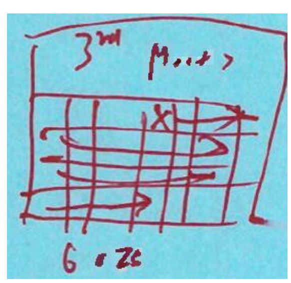


Figure 17: Will's drawing of 6 ounces over the entire 3rd month

growing by 6 ounces" meant that "in that entire third month it gained 6 ounces" [Lines 3-4]. He later clarified his imagery by drawing a picture of a calendar and drawing an arrow through the dates (Figure 17) to demonstrate his awareness of time passing and his coordination of the overall change in the weight of the fish [Line 6]. Will interpreted the value of 6 in an additive fashion since he coordinated specific amounts of variations between weight and time (1 entire month and 6 ounces of weight); therefore, I consider Will to be engaging in MA3 reasoning in this excerpt.

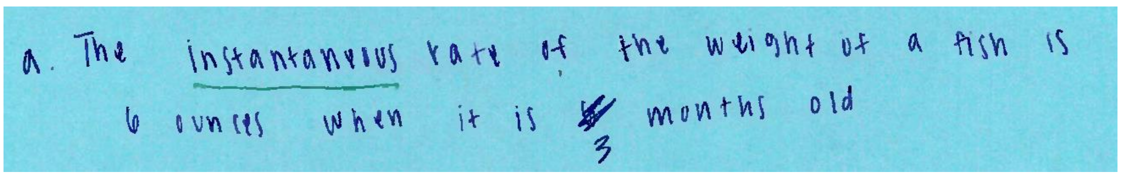


Figure 18: Lucy's Interpretation of $P'(3) = 6$

Table 11: Lucy's Explanation for Instantaneous Rate of Change

1	Int:	Can you explain what the means to you?
2	Lucy:	So the instantaneous rate is there's a certain rate over a period
3		of time so every like 3 months it is going to go up 6 ounces.
4	Int:	Like every 3 months the fish will gain 6 ounces?
5	Lucy:	Yeah like since $P(3) = 15$, then $P(6)$ would be 6 more, 21.
6		Oh wait, no it should be that $P(4)$ would be 6 more.
7	Int:	Oh? And why do you say that?
8	Lucy:	I like mixed it up with the time we are at.
9	Int:	Okay, so can you restate what you wrote down means?
10	Lucy:	Yeah, like the instantaneous rate here tells us that in one month
11		the fish's weight will go up 6 ounces.

Lucy was another student who exhibited MA3 reasoning during the entire interview (Tables 11&12). Lucy wrote that $P'(3) = 6$ was "the instantaneous rate of the weight of a fish is 6 ounces when it is 3 months old" (Figure 18) and initially explained it as the amount of change in weight over 3 months (Table 11). She eventually changed her explanation to say that it was a change in the weight for the next month [Lines 6, 10-11]. In both cases, her language indicated that she interpreted the 6 as an amount to add since she used phrases such as “go up,” “would be 6 more,” and “go up 6 ounces” [Lines 3,5-6,10-11]. Lucy’s explanation suggested that she was coordinating the amounts of change between time and weight together since she explained that it was “a certain rate over a period of time” [Lines 2-3], and she continually associated the change in the input value with a corresponding change in the output value [Lines 3, 5-6, 10-11].

Table 12: Lucy’s Explanation for her calculation in part b

1	Int:	So how did you get 15.25?
2	Lucy:	I was estimating because 3 would be 15 and you with that have
3		to divide and it is probably wrong.
4	Int:	So what did you want to do?
5	Lucy:	You would do 6 divided by 0.05 **writes it down in green**
6	Int:	And so why did you do that?
7	Lucy:	To get the 0.05 rate of change to add to this *points to 15*, like
8		if this like 4 it would be one more than this so it would be plus
9		6, but because it is 0.05 of 1, we want 0.05 of 6.

In part b of the Fish Task, Lucy continued to reason that the value of a rate was an amount of change for a 1-unit change in the input quantity (Table 12). Lucy explained that she was trying to find “the 0.05 rate of change to add” to the initial value of 15 ounces [Line 7]. She then articulated that if the change in the number of months were one, she would add 6 and then deduced that since she had 0.05 of 1, the fish would gain 0.05 of 6 [Lines 7-9]. While Lucy employed proportionality in her explanation, it is important to highlight that to Lucy, 6 was not describing the multiplicative relationship between how the age and weight of the fish would covary. Instead, it was the change in

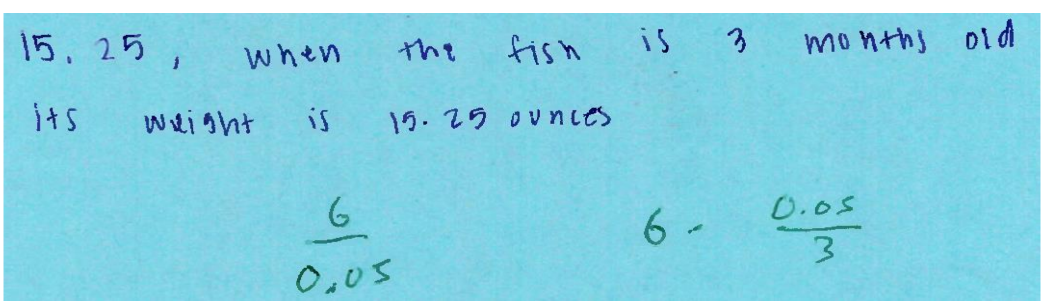


Figure 19: Lucy’s written work for part b of the Fish Task

the weight for 1 month of time, and she wanted to find 0.05 of that 6-ounce change. Additionally, Lucy struggled to mathematically represent what she explained since she initially did not write a calculation and instead estimated it [Lines 2-3]. When prompted, she tried several incorrect calculations, which indicated a lack of procedural fluency in using the value of a rate of change (Work written in green in Figure 19).

Many other students also engaged in proportional correspondence in part b of the Fish Task by using 6 as the reference amount for a 1-month change in time and then setting up equivalent ratios to find a proportional amount of change. One student, April, explained how she solved part b by thinking of 6 as “how much it will change in a month” (Table 13). She later described that her calculation involved finding “0.05 of that” because she wanted 5% of 6 [Lines 7&10] (Figure 20). April seemed to be coordinating specific amounts of variations for particular times; however, she did not articulate how both quantities covaried together.

Table 13: April’s Explanation for Instantaneous Rate of Change

1	Int:	Can you tell me what this ‘15’ represents to you?
2	Apr:	Uhh, yeah that was how much the fish weighed at 3 months.
3	Int:	Okay, so what does this ‘$0.05 * 6$’ mean?
4	Apr:	That was the change in the fish’s weight
5	Int:	So why does ‘$0.05 * 6$’ represent that?
6	Apr:	Well like… 6 is how much it will change in a month and I
7		wanted to find like 0.05 of that.
8	Int:	Sorry, can you say again what 0.05 meant to you?
9	Apr:	Yeah, like 0.05 was like how much of the change in 6 ounces I
10		wanted to find, like…ummm like I wanted 5% of 6

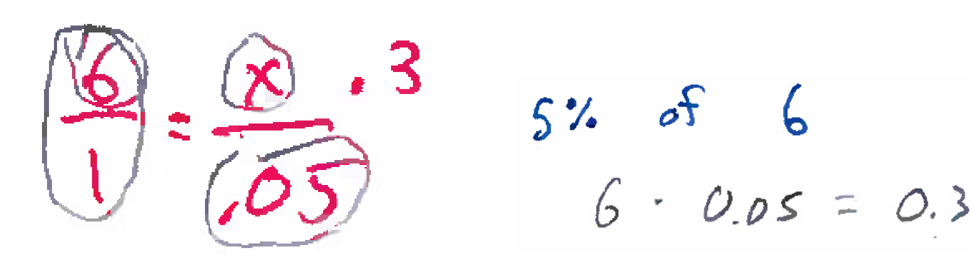


Figure 20: Examples of MA3 reasoning

Similarly, other students who explicated 6 as an amount of change of weight in a month set up equivalent fractions (Figure 20) because they looked for a proportional amount of change. For example, a student named Anu described her calculation as “like if I split the rate up into little pieces like 20ths.” Anu’s description revealed that she thought of 6 as an amount of change in the fish’s weight and that she could subdivide the 6 into 20 equal pieces and could find the corresponding amount of change in the fish’s weight for a 0.05 change in fish’s age. Many other students also described that 6 was an amount of change for a 1-month change in time and that their calculation was finding a portion of that change (Figure 21). Similar to Anu, these students engaged in proportional correspondence by finding the corresponding amount of change in the weight that would keep the proportion of 6 ounces to 1 month.

The students’ actions and explanations for estimating the change in the fish’s weight for a 0.05 change in the amount of time suggested that they were imagining variations between weight and time using chunky reasoning (Castillo-Garsow, 2010). In other words, they imagined 6 as a discrete amount of change in the weight of the fish and took actions to find a smaller-sized chunk of change that maintained the 6:1 ratio. What was absent in the interpretations from all the students who exhibited MA3 reasoning was that the quantities of the weight of the fish and the age of the fish would vary continuously and smoothly. This is supported by the students' interpretation that 6 was a completed change in the number of ounces after some elapsed amount of time instead of a value that quantified the relative size of a varying amount of time and a varying fish weight (in ounces) since the fish hatched.

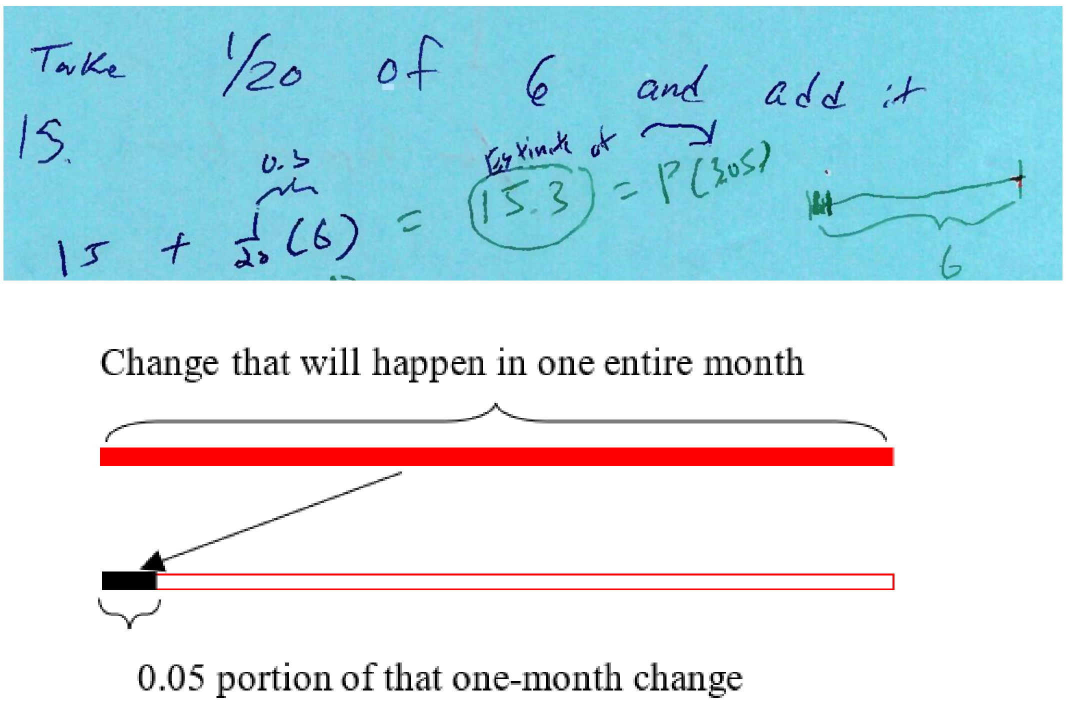


Figure 21: Additional Example of MA3 Reasoning

Mental Action 3+ (MA3+) – *Coordination of Values+*

MA3+ is similar to MA3, except that a student is cognizant that the value of the rate of change is also varying. While verbalizing an awareness of how the instantaneous rate of change of a function continually varies as the input variable varies is an indication of MA5 reasoning, MA3+ is different in that a student is limited to coordinating discrete amounts of changes between quantities instead of them varying continuously and smoothly. I argue here that a student's meaning for the value of a rate of change is one of the potential obstacles that hinder them from reasoning at MA5. If a student interprets the value of a rate of change additively, they will likely reason about variation happening in discrete chunks, which may prevent them from understanding what it means for a rate of change to vary. A student reasoning at MA3+ experiences a disconnect between their intuitive understanding that quantities vary smoothly and continuously versus their interpretation that a rate refers to a fixed amount of change. This new classification of MA3+ is necessary since some students will demonstrate an awareness of how the instantaneous rate of change of a function continually varies as the input variable varies. However, their behaviors are limited to MA3 due to their conception of rate of change.

Examples of MA3+ reasoning

Table 14: Max's Explanation for $P'(3) = 6$

1	Max:	It gained 6 ounces in weight when it was 3 months old
2	Int:	So are you saying that after 3 months, the change in weight is 6
3		ounces. So like from 0 months to 3 months the fish gained 6
4		ounces?
5	Max:	No… not really
6	Int:	So what are you trying to describe? Are you saying in the third
7		month it gained 6 ounces?
8	Max:	No because…I do not have enough information to give that
9		number. At 3 months he is in a sense projected to grow at that
10		rate that he is growing at. Like in that month he should gain 6
11		ounces.
12	Int:	Okay, so what are the units on this derivative? You wrote that
13		the units of $P(t)$ is ounces, what are the units for $P'(t)$?
14	Max:	Change in ounces?... Yeah cause change in ounces is still an
15		ounce so the unit is ounces.

Max displayed MA3+ reasoning as he explained his interpretation of $P'(3) = 6$ as an amount of change in the fish's weight (Table 14). Similar to students who exhibited MA3 reasoning, Max also articulated that 6 was the amount the fish was "projected to grow" and that "in that month he should gain 6 ounces" [Lines 9-10]. Max coordinated an amount of change in the fish's weight with an amount of change in time; however, he also consistently qualified his language to indicate his awareness that the value of the rate of change would likely vary. Even though the interviewer asked if Max meant that “in the third month it gained 6 ounces”, Max quickly denied that because he did “not have enough information” [Line 8]. His explanation included words such as “projected” and “should” to indicate that the 6 was not the exact amount of change in a month. Instead, it meant that if the rate stayed the same, the fish’s weight would gain 6 ounces [Lines 6-11]. Although Max was cognizant that the rate at which the fish was growing was varying, it seemed that his meaning for rate of change as a “change in ounces” [Line 14] prevented him from fully engaging in MA5 reasoning and instead limited him to coordinating amounts of variations between the two quantities (MA3).

Table 15: Fred’s Explanation for his solution to part b

1	Fred:	At $P(3.05)$… uh 15+0.3 ounces… assuming the rate at which
2		it grows is the same or very close to the same oh okay… so
3		this is under the assumption that over the span of the third
4		month they’re growing at 6 ounces so I just took a small
5		portion of that.
6	Int:	Over the entire third month you said?
7	Fred:	Yeah until the third month finishes.
8	Int:	So the entire month finishes it’s growing at a rate of 6?
9	Fred:	Yeah so I took a portion of that, like assuming the entire 3rd
10		month is going to grow 6 ounces I took a portion of that like
11		0.05 and found the associated change for that time.

Fred was another example of an MA3+ reasoner when he explained how he used the derivative value to estimate $P(3.05)$ (Table 15). Fred explained that 6 was the number of ounces the fish will grow “until the third month finishes” and repeated this later as “the entire third month [the fish] is going to grow 6 ounces” [Lines 3-4 & 9-10]. Like Max, Fred consistently justified his estimation with language such as “assuming the rate at which it grows is the same” and that he was “under the assumption that over the span of the third month they’re growing at 6 ounces” [Lines 1-3 & 9-10]. His word choice demonstrated that he was aware that the value of the rate of change might not be constant but using the value of an instantaneous rate of change involved making that assumption. Again, like Max, Fred's meaning for a rate of change entailed a change in the fish's weight of 6 ounces [Lines 4&10] which led him to engage in coordinating amounts of variations since he wanted to find a "portion" of the 6 ounces for the "associated change for that time" [Lines 4-5 &10-11]. I claim that had both Fred and Max conceptualized rate of change as a multiplicative relationship between two varying quantities instead of a specific amount of variation, they would reason at higher levels of covariation reasoning.

Mental Action 4 (MA4) – ***Coordination of Average Rates of Change***

Engaging in MA4 and higher requires recognizing that a rate of change entails a multiplicative relationship between the variations in the values of two quantities. In contrast to MA3, a student at MA4 would not utilize equivalent ratios or resize a one-unit change; instead, they conceptualize the value of a rate of change as describing how many times as large the variation in one quantity will be with respect to another.

Examples of MA4 reasoning

Randy explanation of instantaneous rate of change was consistent with MA4 reasoning (Table 16). Randy described instantaneous rate of change as "how much it's (the fish's weight) changing by over a process of time," and as he said this, he slid his right hand away from his other hand to indicate the motion that went with his verbal description [Lines 1-3]. As Randy continued to explain, he articulated that the 6 described how the weight would change "from there to there it would keep changing by like 6 ounces per month" [Lines 2 & 6-7] and that they vary together because "it (the weight) is not changing if time isn't changing" [Lines 12-13]. Due to his gestures and how he attempted to describe weight and time changing together, Randy evidenced that he thought of a rate of change as describing how the quantities vary together smoothly and continuously.

Table 16: Randy's Explanation for Instantaneous Rate of Change

Ran:	Like the instantaneous rate of change, so like… that's how much it is changing by over a process of time. *Slides his hands to motion*
Int:	So like over the first three months it gained 6 ounces?
Ran:	No like…let's say like from…. like 2.9 to 3.1, like the average rate of change is like 6, like from there to there it would keep changing by like 6 ounces per month
Int:	So why did you pick 2.9 and 3.1? Does it have to be those numbers?
Ran:	Nah like that was just something close to 3, we could have picked like from 2.85 to 3.15 that average rate would be 6, I mean I'm just trying to explain it cause it (the weight) is not changing if time isn't changing.

Although Randy never explicitly described 6 as representing the relative size of the change in weight compared to the change in time, his actions suggested that this rate of change entailed the simultaneity of weight and time covarying together. Additionally, he verbalized that he was thinking about average rates of change over small intervals and that the weight would be changing at a rate of 6 [Lines 5-7 & 10-13]. This suggested that he was not engaging in MA3 by thinking of 6 as a change in weight; rather, he attempted to articulate that the 6 described how fast the weight would change during the time interval. Randy never demonstrated an awareness that the rate of change would vary, and in fact, he used more definitive language such as "that's how much it is changing by" and "it would keep changing," which implied that he thought about the rate being constant over those small intervals [Lines 1-2 & 6-7]. Therefore I deem his interpretation and explanation as engaging in MA4 reasoning.

Table 17: Winnie's Explanation for her solution to part b

1	Win:	So I multiplied 6 by 0.05 for some reason
2	Int:	And what were you trying to represent?
3	Win:	I think it would represent the amount that it is changing in that
4		small interval that it's defined up to 3.05… so we're using the
5		fish's instantaneous rate of change from 3, yeah that's what I
6		did.
7	Int:	So how did you get 0.05?
8	Win:	Because you know that $P(3)$ is 15… so I multiplied by 0.05
9		since that's what the amount that's after that. *Slides her right
10		hand as she describes this*
11	Int:	So why does 0.05 times 6 get a change in weight?
12	Win:	Uhh… because like the instantaneous rate of change is 6 and
13		we know that the time only progresses after the interval for
14		0.05, and the weight changes with it so I multiplied those to get
15		out the change.

Another student, Winnie, exhibited MA4 reasoning as she explained her solution to part b of the Fish Task (Table 17). Winnie initially struggled to articulate why she multiplied 6 by 0.05, and it is only in the latter portion of the interview where she described the 6 as being related to "the time only progresses after the interval…and the weight changes with it so I multiplied those…" [Lines 13-15]. Similar to Randy, Winnie never demonstrated that she interpreted 6 as an amount of weight; instead, she explained that the 6 had something to do with how weight and time varied together [Lines 12-15]. Additionally, she also utilized a similar hand gesture when attempting to explain her

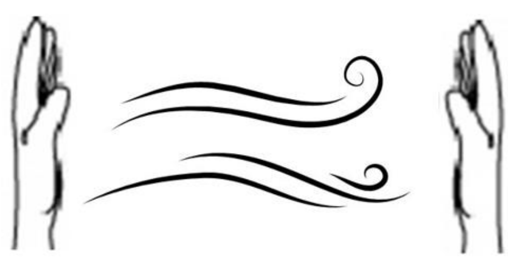


Figure 22: Depiction of Winnie's gesture as she explained her interpretation of $P'(3) = 6$

calculation. She slid one of her hands from her other stationary hand (Figure 22) to describe what she imagined [Lines 8-10]. Her explanation for her calculation and her gestures suggested that she imagined weight and time varying together smoothly and continuously. Winnie's actions suggested to her, a rate of change entailed how two quantities would vary together, however, she did not communicate that she interpreted the 6 as a relative size measurement between variations in the fish's weight and the age of the fish. Lastly, Winnie never explicated an awareness that the value of the rate of change would vary, which would preclude her from being classified as MA5; therefore, I classify her reasoning as MA4.

Mental Action 5 (MA5) – *Coordination of Instantaneous Rates of Change*

MA5 includes all of MA4 with the added distinction of recognizing that the value of the rate of change varies as the input quantity also varies. A student engaging at MA5 will consistently qualify the amount of change in a quantity with "if the rate stays the same…". This is further evidenced when a student anticipates that for some input, a, and for some change from the input, Δx, the output value will vary $f'(a)$ times as much, in other words $f'(a) * \Delta x \approx f(a + \Delta x) - f(a)$. The student also verbalizes an awareness that the rate of change will vary in this Δx interval, but for small Δx values, the actual change in the output will *essentially* be 6 times as large. It is important to note that among the 25 students, I only observed one instance that had sufficient evidence for being classified at MA5. This does not mean that there were no other instances where students were reasoning at MA5; instead, there was insufficient evidence to support such a claim. However, since most students interpreted the value of a rate as an amount of change, it is likely that these students were not engaging in MA5 reasoning.

Example of MA5 reasoning

Cyrus demonstrated MA5 reasoning as he explained his solution to part b of the Fish Task (Table 18). Throughout the entire interview, Cyrus never indicated that he interpreted 6 as an amount of change; instead, he always employed examples where he would use the 6 and multiply it by some amount of time. While Cyrus never explicitly stated he interpreted a rate as a ratio between changes in two quantities, he only utilized the 6 to employ multiplication to discuss how time and the fish's weight varied together [Lines 2 & 11-13]. Cyrus described the 6 as the fish was "changing at 6 ounces per month" and explained that as how the fish's weight was "changing" and not as an amount of change [Lines 6-9]. As Cyrus explained his calculation, he consistently verbalized that he assumed a constant rate since "it probably is not going to be changing very much faster or very much less" and that his estimation was "somewhere close, but I know that's not the correct value" [Lines 3&7-10]. This evidenced his awareness that the rate of change would vary even in the small interval between 3 and 3.05 and that even if the rate did vary, it would not change drastically unless it "hit a growth spurt right before or after," which meant that his estimation was close enough [Lines 7-10]. Altogether, Cyrus demonstrated that he was coordinating how time and the fish's weight covaried together smoothly and continuously as well as coordinating the instantaneous rate of change of the function with continuous changes in the independent variable.

Table 18: Cyrus' explanation for his solution to part b

1	Int:	So what did you try doing here?
2	Cyr:	0.05 * 6 and got 0.3, so I estimated 15.3. I know it's
3		somewhere close, but I know that's not the correct value.
4	Int:	So why did you do this part over here 0.05 * 6?
5	Cyr:	So that was the estimation, it was 3 and 0.05 months, at 3
6		months it was changing at 6 ounces per month, and at 3.05
7		months it probably is not going to be changing very much
8		faster or very much less but that's an estimation you could
9		have hit a growth spurt right before or after, that's why I did it,
10		the rate of change probably won't change much between 3 and
11		3.05. So I multiplied the rate of change which was 6, times the
12		value added on to 3 when the rate of change was 6 and I
13		multiplied those two numbers together.

Discussion

Based on the results of these clinical interviews, each student's explanation of the value of an instantaneous rate of change revealed how they might have conceptualized how two quantities' values covaried. In Carlson et al.'s (2002) study, students could exhibit MA5 reasoning with the bottle problem if they coordinated that equal changes in water would result in decreasing (then increasing) changes in height. I argue that some of these students leveraged their intuitive understanding but may have struggled to demonstrate MA5 if they had to attend to the values of a rate of change at a given volume. In this study, some of the students demonstrated an awareness that the instantaneous rate of change of the fish's weight varies as the age of the fish varies. However, it was apparent that their interpretation of a value of a rate of change limited them to coordinating specific amounts of change, which was demonstrative of MA3 reasoning.

To recap, I highlight two major insights from the results of this study

1) Attending to how a student interprets the value of an instantaneous rate of change can provide insight into how they reason covariationally. Further, it is likely that a student's meaning for rate of change is what causes them to reason at a particular level of covariational reasoning.
2) New categories of MA0 and MA3+, and an updated description to MA4 and MA5 to further describe several nuances in student thinking regarding covariational reasoning that were not originally described in the original Covariational Reasoning Framework as proposed by Carlson et al., (2002).

Conclusion

Overall, 16 of the 25 students did not exhibit beyond MA3/3+ reasoning when utilizing the value of an instantaneous rate of change. The results of this study indicate that many Calculus students have impoverished understandings of instantaneous rate of change and that one potential source for this issue is their mathematical understanding of rate of change. Many of the students in this study were limited to reasoning at MA3/3+ due to their conception of rate of change as an amount to add to the function's output value for a one-unit change in the input. Students in this sample that had an additive conception of rate of change took actions to suggest they were thinking about completed changes instead of quantities varying smoothly and continuously. Therefore it stands to reason that supporting students in constructing a productive understanding of rate of change can be beneficial for their understanding of derivative as instantaneous rate of change.

Many students in this sample struggled to interpret the value of a rate of change as entailing the multiplicative comparison between two varying quantities and instead employed additive reasoning. Even the few students who evidenced MA4 or MA5 reasoning could not articulate the underlying reason for employing multiplication when using the value of a rate of change. It is not surprising then that students with an additive conception will struggle to adapt this understanding to their other STEM courses, such as physics, differential equations, and various engineering courses (Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006; Ibrahim & Robello, 2012). While developing a robust understanding of rate of change should be seeded early on (e.g., Thompson & Thompson, 1994), the findings of this study suggest that Calculus instructors should attend to what a rate of change entails in order to support their students in understanding the derivative as a rate of change function.

CHAPTER 6

PAPER 2: A CONCEPTUAL ANALYSIS FOR THE IDEA OF INSTANTANEOUS RATE OF CHANGE

Based on the previous study's findings, many Calculus students interpret the value of an instantaneous rate of change as an amount to add to the output value for a 1-unit change in the input. From my perspective, this way of thinking will be a conceptual barrier for some students and will thus be an obstacle to their understanding of ideas of Calculus, such as accumulation. Therefore, an appropriate follow-up question is, "What is a productive meaning for instantaneous rate of change, and how might we support students in constructing that meaning?" In this paper, I propose a *conceptual analysis* (Thompson, 2008) of the derivative concept rooted in relevant literature and the results of my clinical interviews (Yu, 2019, 2020, 2021). This conceptual analysis includes a Hypothetical Learning Trajectory (Simon & Tzur, 2004) that I conjecture would be propitious for students in understanding the idea of instantaneous rate of change.

Research Question: *What is a productive meaning for instantaneous rate of change? What understandings are foundational for understanding the concept of derivative?*

Conceptual Analysis

According to Glasersfeld (1995), a *conceptual analysis* entails a detailed description of what is involved in understanding a particular concept. This description consists of explicating the mental operations that might explain why individuals think the way they do. Thompson (2008) elaborates that conceptual analyses can effectively express productive ways of thinking for learning an idea.

As a student is reasoning about 'derivative,' as describing the *instantaneous rate of change of one quantity* (output quantity) *with respect to another quantity* (input quantity). What this entails is that a function, f, having an instantaneous rate of change value of $f'(a)$ at the input value a means that for small variations in the input quantity, the variation in the output quantity will *essentially* be $f'(a)$ times as large as the variation in the input. In contrast, typical Calculus textbooks (e.g., Stewart (2013) and Larson (2007)) discuss derivatives as the slope of a tangent line or as velocity with little attention to describing how quantities are changing. Zandieh's (2000) research evidenced that many students interpret the derivative as the slope of a tangent line or like the reading on a speedometer, but researchers have demonstrated that students' weak meanings for derivative as rate of change will be an obstacle for related mathematical topics (Byerley et al., 2012; Flynn et al. 2018; Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006; Ibrahim & Robello, 2012). In this paper, I explicate the ways of thinking and mental actions involved for derivative as instantaneous rate of change that I conjecture would be productive and coherent for future mathematical learning.[6]

It may seem paradoxical to talk about a desired meaning for a concept while taking the stance of Radical Constructivism. In Radical Constructivism, each individual constructs knowledge through their own experiences, and it is impossible to verify if one person's knowledge is the same as another's. Therefore it might be strange to describe a desired meaning with no way of verifying that a student holds this exact meaning. Instead, I argue that explicating what is involved in learning a topic entails describing the

[6] To be clear, I am not claiming that this is the only meaning a student should have for "derivative." Rather, I argue that when interpreting a derivative as an "instantaneous rate of change" that it would be useful if it entails a multiplicative comparison.

productive mental actions that a student may be engaging in. Additionally, I will describe behaviors that would evidence that a student is engaging in these mental actions.

Productive Meaning

Thompson (2016) defines a *productive meaning* in learning mathematics as a meaning that is useful for future mathematical learning. Additionally, Thompson states that a productive meaning is a meaning that allows students to see different mathematical ideas as being coherent and interconnected. What one researcher believes to be a productive meaning may be deemed unproductive by another. Researchers use the term *productive meaning* to explicate or identify the ways of thinking about a mathematical idea that they believe will be useful in their conception of future mathematical learning. For example, many physics education researchers (Roundy et al., 2015; Dray et al., 2019) argue that calculating the slope between 2 carefully chosen values is a derivative. In their conception of derivative, these researchers believe that students thinking about derivative in this manner is more consistent with how physicists and engineers utilize derivatives. Conversely, some mathematics education researchers (Thompson & Dreyfus, 2016; Rogers, 2005) argue that *differentials* should be taught in place of limits with a focus on quantities varying. For Thompson, thinking about differentials as a variable whose value varies through a small interval is a productive meaning since it coherently fits his conception of mathematical thinking based on quantitative reasoning (2011) and how quantities covary with one another (Thompson & Carlson, 2017). This comparison illustrates that a researcher uses the term productive meaning to describe a meaning the researcher believes will be useful in a variety of situations and is coherent in their conception of mathematical thinking.

One clarification that needs to be made regards the issue that meanings are personal and the Radical Constructivist stance that another's knowledge cannot be known to another. Therefore it is necessary to explain how what a researcher believes to be a productive meaning would be useful for another's conception of mathematical thinking. I claim that articulating a productive meaning can be used as part of *conceptual analysis* to "describe ways of knowing that might be propitious for students' mathematical learning" and "in analyzing the coherence, or fit, of various ways of understanding a body of ideas" (Thompson, 2008). By elucidating a productive meaning, a researcher can leverage their meaning to hypothesize how students may come to build such a meaning. When someone hypothesizes a goal for a student, they also must envision the potential paths that would lead to that goal and the pitfalls that would prevent students from attaining it. An example of leveraging a productive meaning to anticipate how students may come to learn an idea can be seen in the works of Thompson and Thompson (1994, 1996). The authors describe a productive meaning for speed as (1) a quantification of motion, (2) completed motion involves distance traveled and the amount of time required to travel that distance, (3) a multiplicative relationship between distance and time traveled, and (4) a proportional relationship between distance and time traveled. They demonstrated that if a student was thinking of a speed as a length (e.g., a student would attempt to fit a number of speed-lengths into a total distance), the student would encounter difficulties when answering questions about finding the needed constant speed to travel a distance in a given number of seconds. To combat this way of thinking, Thompson and Thompson made deliberate teaching moves that allowed their student to conceive of motion as entailing distance and time simultaneously. By considering what a conceptual understanding for motion entailed, they identified the ways of thinking that the students needed to engage in and the actions the instructor needed to take in order for the student to build such a meaning.

One Productive Meaning for Instantaneous Rate of Change

In a typical Calculus 1 course, instantaneous rate of change is introduced to students via the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)}$ (Stewart, 2013; Larson et al., 2006). One productive interpretation of this limit is the multiplicative relationship (the value of $f'(x)$) between a variation in a function's output ($f(x+\Delta x) - f(x)$) and a variation in the input ($(x+\Delta x)-(x)$) so long as the variation from the input value x is arbitrarily small ($\lim_{\Delta x \to 0}$). The convergence of this limit is what we call "instantaneous rate of change," which represents the multiplicative relationship between two varying quantities with respect to one another's relative variation size. To use a derivative value as a rate of change having some value m, one must imagine the input quantity varying while simultaneously imagining the output quantity varying m times as much as the input quantity's variation. This is the same meaning we might attribute to an average rate of change over a small interval, in that someone imagines the constant rate of change needed to achieve the same accrual in one quantity with respect to the accrual size of the other quantity. Additionally, a student must recognize that the variation in the output value will be *essentially equal* to the actual variation since the quantity is not changing at a constant rate of change. In this case, what it means to be *essentially equal* is that the approximation of the variation in the output by assuming a constant rate of change will be so close to the actual variation that the difference between the two is imperceptible.

In the next section, I provide an example of the usefulness of the meaning I described for instantaneous rate of change.

A Differential Equations Example

The first example was about linear approximation (in the Theoretical Perspective Chapter), which is generally taught right after teaching the derivative as a function. However, for the meaning to be productive, the meaning should be useful for future mathematical learning. I outline the usefulness of this meaning for instantaneous rate of change in a topic that is centrally about derivatives, *differential equations*. According to Rasmussen and Keene (2019), students need to develop increasingly sophisticated ways of reasoning about rate of change in order to predict solutions to autonomous differential equations. One way of reasoning that they highlight as significant is coordinating a rate of change of a function that depends explicitly on the dependent variable (where the dependent variable is not time). Below is an example involving a differential equation that demonstrates the usefulness of attending to a rate of change as a quantity that measures something about a situation. This way of reasoning about a rate of change as a function is aligned with researchers findings on supporting students in differential equations courses (Rasmussen & Keene, 2019; Donovan, 2007; Jones and Kluster, 2021).

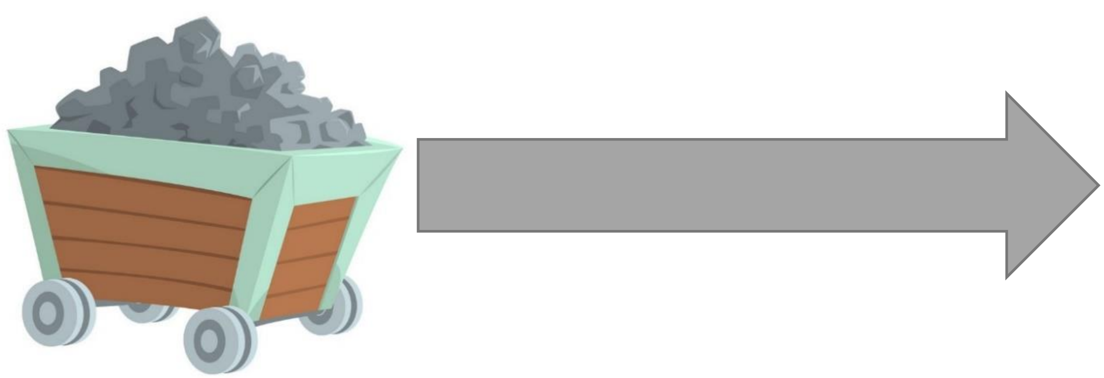


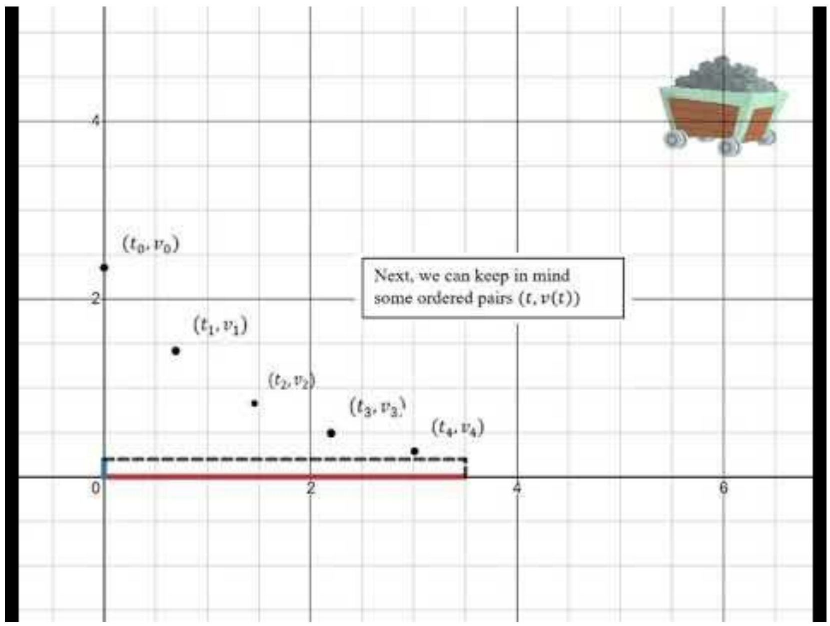
Figure 23: A Cart Rolling to a Stop

Imagine a cart that is moving and is rolling to a stop (Figure 23). The differential equation gives the acceleration of the cart: $v' = -k * v$, where v represents the velocity of the cart[7], v' is the rate at which the velocity changes, and k is the drag coefficient[8]. Suppose that we are given that $k = 0.7$ and that we can measure the velocity of the cart at any time t with an initial starting value $v(0) = 2.3$. Typically in a differential equations class, students are taught various techniques to determine an explicit function that defines $v(t)$. In this case, since the derivative of v is -0.7 times as large as the original velocity function, then the function is of the form $e^{-0.7t}$, where $v(0) = 2.3$[9]. Solving this gives that $v(t) = 2.3e^{-0.7t}$. This procedure builds upon students' experience with rules for differentiation, but it does not seem to help students think about quantities changing. I offer the following way of thinking that employs quantitative reasoning to understand this differential equations problem.

First, when one is reading the statement, $v' = -k * v$, a student can imagine that at higher velocity values, the value of the rate at which the velocity is changing is more negative. Note that in this case, a student is not imagining that v is increasing in this situation. Instead, the student imagines that as the cart slows down (v is decreasing), the rate at which the velocity is changing is increasing (becoming less negative). Next, the student imagines that v' represents how the velocity is changing with respect to time for small changes in time. Then the student can manually find different coordinate pairs (t, v) and coordinate how the quantities' values are changing. Afterward, the student can

[7] Implied by this situation is that v is a function of time.
[8] One way to make sense of the entire equation is that "the faster the cart is moving, the more it will slow down".
[9] Recall that the derivative of e^{-kt} with respect to t is $-ke^{-kt}$.

imagine that the velocity changes at a constant rate between each consecutive input value. Graphing these piecewise lines together creates a hypothesized velocity graph. Then the student can compare that graph to the graph of the proposed solution $2.3e^{-0.7t}$ and can then see that they are very close to one another. Video 5 demonstrates this way of thinking.



Video 5: Coordinating Velocity Values

Lastly, a student can further verify this solution by examining ordered pairs of (v, v') and note that v' would change at a constant rate of -2.3 with respect to v. This way of reasoning aligns with the findings of Jones and Kuster (2021). They identified that simultaneously reasoning about a rate of change as a variable output and relating to two other variables was important in understanding differential equation variables and functions.

To be clear, I am not arguing that students will approach every differential equation with this way of thinking. Instead, I argue for having students discover the solution and why the solution makes sense quantitatively. Similar to the limit definition of derivative, we do not expect students to continually engage in evaluating the limit of average rates of change to calculate a derivative function. Instead, we stress how the derivative function is derived so that students will associate the imagery of tiny variations in quantities with the derivative.

Hypothetical Learning Trajectory

Based on the ideas outlined in the previous section and the literature on derivative and prerequisite topics (see Introduction and Literature Review), I propose that the following foundational ideas that are essential to learning derivative (Table 19). This ordering is based on Simon & Tzur's (2004) hypothetical learning trajectory (HLT). An HLT includes an ordered list of learning goals (Table 19), tasks intended to promote these learning goals (Table 20), and hypotheses about how students may learn from these tasks. Table 19 outlines a proposed list of learning goals for a student in Calculus 1 for the learning of derivative[10]. The usage of an HLT aligns with Thompson's (2008) description the usage of conceptual analyzes to describe the ways of knowing that might be propitious for students' mathematical learning and to provide imagistically-grounded descriptions of mathematical cognition.

Proposed Order of Instruction	**Foundational Idea for the Learning of Derivative**	**Desired Student Understanding**

[10] I am making assumption that students understand Limit as imagining the value of an expression varying as the limiting variable varies towards the limiting value ($\lim_{x \to a}$). I hasten to note that the learning of limit is non-trivial (Roh, 2008; Tall & Vinner, 1981; Przenioslo, 2004). In my experience in teaching Calculus 1, students with diverse conceptions of limit such as convergence of values or as asymptotes does not seem to have a significant impact on how students conceptualize rate of change. As such, I omit learning goals about Limit in this HLT. Further, the tasks will not employ limit notation, and will instead utilize intuitive understandings of picking smaller and smaller sized intervals.

1	Variation: Change (noun) as a difference. A completed change, $\boldsymbol{\Delta x}$	A change from one value of Quantity A, x_1, to another value of Quantity A, x_2, is conceptualized as a difference in the values through an additive comparison. $x_2 - x_1 = \Delta x$ Δx is how much x changed from instance 1 to instance 2.
2	Ratio	A ratio between two quantities is conceptualized as measuring the value of one quantity in terms of another quantity's value. The resultant number represents how many times larger the value of one quantity is in terms of the value of another. This is describing the multiplicative relationship between the two quantities at a particular instance.
3	Rate as a Reconceptualized Ratio	A rate between two quantities is conceptualized as imagining that as the quantities' values vary, the ratio between the two quantities remains constant.
4	Constant Rate of Change	For x and y, (the values of the varying Quantities A and B respectively), if the variation of the value of Quantity B (Δy) is always m times as large as the variation in Quantity A (Δx), then Quantity Y changes at a *constant rate of change* with respect to Quantity X. Imagining that x and y are both varying, and coordinating the variations in them as maintaining the multiplicative relationship of $\Delta y = m * \Delta x$
5	Average Rate of Change	A student determines the variations in two quantities (over the same two instances), and then finds a ratio between them. Then the student uses the value of the ratio, m, to imagine how the values of the quantities will vary if they were to vary at a constant rate, m.
6	The Difference Quotient $\frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-x}$	The average rate of change of a function, f, over the interval of an input value, a, and some variation, Δx, from that input value.
7	The Limit of the Difference Quotient (at a given input value of a) $\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}$	The average rate of change of the function, f, at the input value of a, over the input interval a and $a + \Delta x$, as the value of Δx varies towards 0. Imagining the variation in x as becoming smaller and smaller, and coordinating the value of the average rate of change of the function over that interval.
8	Instantaneous Rate of Change (at a point) $f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}$	A function having an instantaneous rate of change of value $f'(a)$ at the input value a means that for small variations in the input quantity, Δx, the variation in the output value of the function, Δf will *essentially* be $\Delta f = f'(a) * \Delta x$

Table 19: Learning Goals for the Learning of Derivative

According to Radical Constructivists, learning happens when students reflect on their actions and the results of those actions. To reflect on their ways of thinking, students need to experience perturbations in their current ways of thinking. I designed the following task sequence with these ideas in mind (Table 20). Table 20 describes a set of tasks that I envision would be useful for leading up to the idea of derivative. As a preface to the sequencing of tasks, the following list outlines the key ideas that I conjecture will be productive in assisting a student to build a meaning for instantaneous rate of change as *a constant rate of change in an arbitrarily small interval*.

I. A (Constant) Rate of Change is a quantity that measures the multiplicative relationship between variations in the values of two varying quantities ($\frac{\Delta y}{\Delta x} = m$).

II. An Average Rate of Change is a hypothetical constant rate of change over a function's input interval that achieves the same change in the output quantity over the input interval, from x_1 to x_2, on which the average rate of change is determined. ($\frac{f(x_2)-f(x_1)}{x_2-x_1} = m$, where m is the value of the imagined constant rate of change).

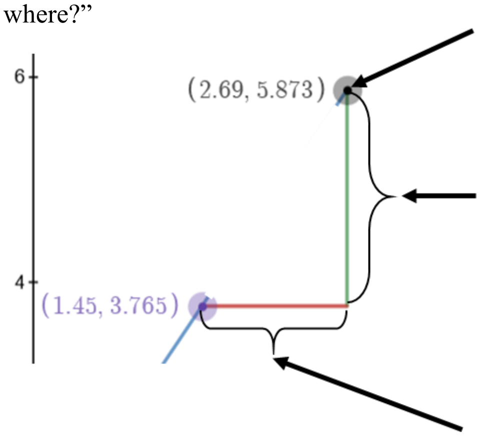
III. One can obtain better approximations of the rate of change at a given value of the input variable by determining average rates of change on smaller and smaller intervals that include that value of the input variable.

IV. An Instantaneous Rate of Change is a hypothetical constant rate of change in a small enough interval such that imagining the output quantity's value changing at this constant rate of change will be *imperceptibly different* from how the output quantity's value actually varies.

The tasks in this instructional sequence involve kinematic problems where time is the input quantity. While of course, not all Calculus problems involve time, Hitt and Dufour (2021) demonstrated the benefits of employing kinematic problems to motivate Calculus concepts and as a way to deemphasize symbolic manipulation. Hitt and Dufour also observed that their students struggled to construct coherent mathematical representations for these problems but conversing with other students (or with the instructor) enabled successful solutions to the problems. It is important to note that it is non-trivial to help students understand that not all rates are time-dependent and supporting students to consider a time-invariant rate of change function entails additional mental obstacles (Rasmussen & Keene, 2019).

Table 20: A Proposed Instructional Sequence

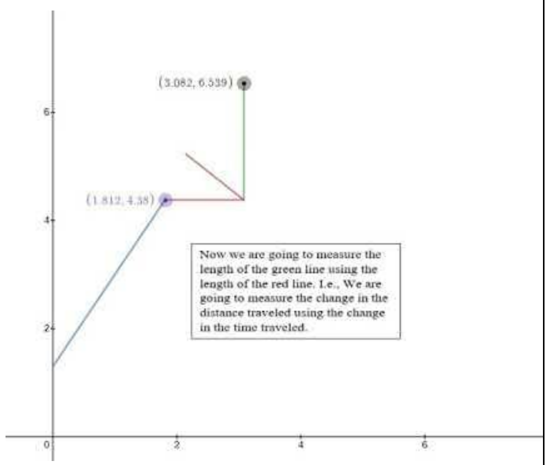
Proposed Order of Tasks	**Task**	**Rationale and Additional Protocol**
Task 1: Constant Rate of Change		
1a	"I will be walking away from the wall at a constant rate of 1.7 meters per second. What does it mean to have a constant speed of 1.7 meters per second?" If students say, "For every 1 second, you will travel 1.7 meters", then follow up on the conversation, asking what happens if I travel for 0.4 seconds? "How does how you defined constant speed connect to your previous answer?"	This task is meant to elicit students' current ways of understanding about constant rate of change. In general, I expect that students will say something like, "For every 1 second, you will travel 1.7 meters". I will explore if students' conception of a constant rate of change is similar to that of *ratio as identical groups* in that a student interprets the value of the constant rate of change as a 1.7 change in the output value and a change of 1 in the input value.
1b	We are going to keep track of my distance from the wall with this graph [This is done with a Desmos Animation, https://www.desmos.com/calculator/on4zt70oj3].	This portion of the task introduces students to the Desmos Animation that will facilitate the conversation

	What does the ordered pair displayed on the graph represent? Does 1.7 meters per second show up here? If so, where?	about the idea of constant rate of change. I will explore whether students recognize that the value of a constant rate of change describes how two quantities' values vary together ($\Delta y = m * \Delta x$). In particular, whether they recognize that the ratio $\frac{\Delta y}{\Delta x}$, remains constant as the two quantities' value change together; and what aspects of the animation the student uses to explain this relationship.
1c	Suppose we fix the current distance and time values in mind. Next, we are going to let the quantities vary from there and keep track of the changes. What does each of the following represent? Then ask "Does 1.7 show up somewhere? If so, where?" 	This portion of the task is to familiarize students with the Desmos animation and have them attend to variations in distance and time. I intend for students focus on what the parts of the animation are representing in this context. In particular, I want students to interpret the length of the red line as representing a variation in time from a chosen input value, and the length of the green line as representing the variation in distance traveled. By asking "Does 1.7 show up somewhere", I will explore if a student recognizes that 1.7 describes the multiplicative relationship between the variations in the input and output values ($\Delta y = 1.7 * \Delta x$), or if they still think of 1.7 as an amount to add to the output (or to go up 1.7 units). If a student has not exhibited reasoning about 1.7 as an invariant relationship that is restricted by a constant ratio between the variations, I built the animation to have the values vary (by animating the Desmos animation) and that the 1.7 should describe something about this situation. The intent is to move students who are thinking of 1.7 in the sense of *ratio as identical groups* towards a meaning of *ratio as measure*.

<table>
<tr>
<td>1d</td>
<td>Ask students to measure the length of the green line in units of the length of the red line.

Repeat with different time values and changes in the time value.

Video 6 illustrates the animation that students will be engaging with (not the pacing or the questions asked)



Video 6: The Constant Rate of Change Graph</td>
<td>This portion is to have students engage in reasoning about division (or multiplication) as “how many times as large” the value of one quantity is in terms of another.

I will explore if students now recognize (or continue to recognize) that no matter the size of the change in the input value (Δx) the size of the associated change in the output value (Δy) is always 1.7 times as large ($\frac{\Delta y}{\Delta x} = 1.7$).

A student (who did not previously articulate an understanding of rate as a multiplicative relationship between 2 varying quantities) can reflect on the repeated activity of measuring variations in two quantities and conclude that the 1.7 does not describe a fixed change in the output value, instead it describes the invariant ratio between the variations in the quantities’ values. Additionally, I will explore what aspects of this Desmos activity the student uses to articulate this meaning for a constant rate of change.</td>
</tr>
<tr>
<td colspan="3">**Summary of Task 1:** The intention of Task 1 is first to draw out the student’s current understanding of constant rate of change. In doing so, I hope to bring about cognitive conflict in what the value of “1.7” means. I assume that most students will be engaging in *ratio as identical groups* in that they are associating 1.7 meters with 1 second. I intend that students will abandon this way of reasoning and instead start to conceive that 1.7 is describing a relationship between the two varying quantities (*ratio as measure*).</td>
</tr>
<tr>
<td colspan="3">**Task 2: Constant Rate of Change as Quantifying a Multiplicative Relationship**</td>
</tr>
<tr>
<td>2a</td>
<td>Suppose I was walking away from the wall at a constant rate of 4.1 meters per second. How far would I travel in 0.8 seconds? (What do we have 4.1 of?) [Measure out 4.1 of 0.8s]

How far would I travel between 2.2 seconds and 3.9 seconds?

What about between times a and b?</td>
<td>This portion of the task is to have students engage in using the value of a constant rate of change to coordinate changes in two quantities’ values.

To build off the previous task, I intend to support students in reasoning about the value of a constant rate of change as describing the multiplicative relationship between the changes in 2 varying quantities ($\Delta y = m * \Delta x$). Additionally, the activity is designed to have students reflect on “what we</td>
</tr>
</table>

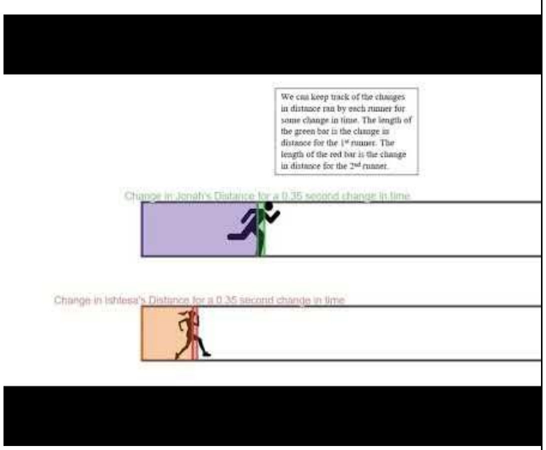
		have 4.1 of" since 4.1 is not a number of meters. Students may reflect on the activity-effect relationship of this task to conclude that 4.1 measures something about this situation (*ratio as measure*). I will explore whether having students attend to what 4.1 measures will perturb any *ratio as identical groups* conceptions of constant of rate. The last part of this task does not provide a numerical value for the student to work with. Instead, I will investigate whether a student has generalized $\Delta y = m * \Delta x$. A student doing so might either indicate that the change in distance would be 4.1 times as large as the change in time, or that they would find the variation in time (b-a) and multiply that by 4.1.
2b	Suppose you are driving at a constant speed. Your app tells you that your entire trip will take 38 minutes. If you have traveled $\frac{4}{5}$ of the total distance, how much more time will it take until you reach your destination? What portion of the total time is left to travel? Suppose you are driving at a constant speed. Your app tells you that your entire trip will take 17 minutes. If you have traveled $\frac{1}{3}$ of the total distance, how much more time will it take until you reach your destination? What portion of the total time is left to travel? Suppose you are driving at a constant speed. Your app tells you that your entire trip will take 17 minutes. If you have traveled $\frac{129}{293}$ of the total distance, how much more time will it take until you reach your destination? What portion of the total time is left to travel? Suppose you are driving at a constant speed. Your app tells you that your entire trip will take x minutes. If you have traveled $\frac{2}{7}$ of the total distance, how much more time will it take until you reach your destination? Suppose you are driving at a constant speed. If you have traveled $\frac{1}{3}$ of the total time, what would you do to determine how much distance was left if the total distance was x meters?	The purpose of this task is to have students engage in reasoning multiplicatively about the relationship between distance and time traveled when traveling at a constant rate of change. I have chosen "not nice" numbers so that students will be discouraged from trying to calculate the time traveled and then subtract that from the total. This task intends to promote students to reflectively abstract that a constant rate of change entails a proportional relationship between 2 quantities. In particular, I intend for students to reflect on this activity to realize that traveling at a constant speed entails proportionality between total distance traveled and total time traveled. That is to say, if you go some portion of the total distance, then the same portion of total time has also elapsed. Ex: Going one-third of the total distance also means to go one-third of the total time. The last problem is a "mental run" task (Simon et al. 2010), where a student is asked to narrate their solution to an activity without actually engaging in writing it down. The intent of this is to help students

	Suppose you are given a total distance and a portion of the total time traveled. Describe what you would do to determine how much distance there was left to travel.	think about their previous actions and the results of those actions.
2c	Suppose a water bottle fills up in such a way that the height of water in the bottle changes at a constant rate with respect to volume at a rate of 0.037 inches per cubic inch of water. Let V represent the volume of water in the water bottle (in cubic inches) and let h represent the height of water in the water bottle (in inches). Suppose V changes from $V = 9.5$ to $V = 13.4$. How much has the volume changed (ΔV) between from $V = 9.5$ to $V = 13.4$ What was the change in the height of the water bottle (Δh) from $V = 9.5$ to $V = 13.4$? Over this interval, the change in the height of water (Δh) is how many times as large as the change in the volume of water (ΔV)? [Repeat with a different set of numbers] Suppose I know how much the Volume has changed (ΔV). What would you do to find how much the height of the water has changed (Δh)?	The purpose of this task is 3-fold 1) Have students actively work with variations in quantities 2) Familiarize students with Δ as representing a variation in a quantity's value 3) Continue having students engage in reasoning with constant rates of change as the multiplicative relationship between changes in two quantities' values. This task introduces the Δ notation as a variation in a given quantity's value. I intend for students to actively measure out a variation by making an additive comparison between two instances of a quantity's value ($\Delta V = V_2 - V_1$). I will explore whether students recognize that a constant rate of change entails a proportional relationship between 2 varying quantities ($\Delta V = 0.037 * \Delta h$). The last portion of the task is to have students reflect on the activity-effect relationship of this task and to imagine the result of the activity as if the action of measuring the variations (and subsequently measuring out 0.037 of that variation) was already performed.

Summary of Task 2: The intent of Task 2 is to have students build a meaning for rate of change that is similar to Thompson and Thompson's (1994a) conceptual curriculum for speed.

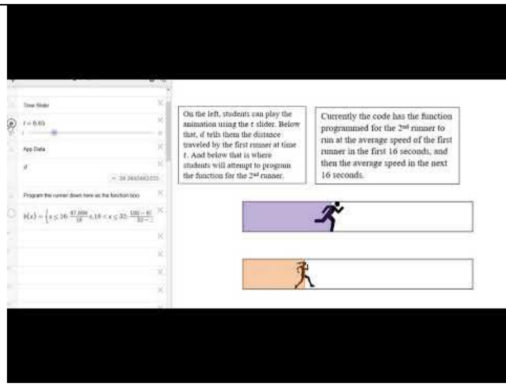
- Rate of change is a **quantification of variations**
- Rate of change relates variations in **two varying quantities**
- Rate as a quantification of variations in two quantities is made by a **multiplicative comparison** of these variations
- To say that rate of change of quantity Y with respect to quantity X is "m" is to mean that the variation in quantity Y (Δy) is m times as large as the variation in quantity X (Δx), i.e., $\Delta y = m\Delta x$

I do not intend for these two tasks to be an HLT for the idea of constant rate of change. Instead, I work within slightly more realistic confines that students come into Calculus with some ideas about Constant Rate of Change. The work here will help them envision that a rate of change regards a multiplicative relationship between variations in 2 varying quantities.

Task 3: Average Rate of Change		
3a	Jonah is running on a racetrack [https://www.desmos.com/calculator/bpdeilsrsb] Is he running at a constant speed? How do you know? Let $s(t)$ represent the distance ran by the first runner, in meters, after he ran for t seconds. If he finished the 100-meter race in 32 seconds, what was his average speed? How would you represent his average speed in the first 23 seconds of his run? Between 4 seconds and 8.9 seconds?	This portion of the task is to elicit students' understandings about average rate of change and how to calculate it. During this portion, the instructor asks students what they are writing and what they believe is being represented. This is to have students attend to variations in quantities and engage in using division to represent a ratio. I will explore whether a student interprets "average speed" as "adding all the speeds and then dividing" or if they recognize that an average speed is an imagined constant speed needed to travel the same distance in the same amount of time.
3b	Suppose we wanted to run the same race as Jonah. We want to travel the same total distance and use the same amount of time as Jonah did. However, we want to run at a constant speed. What constant speed would that have to be? [Students can put their answer into the Desmos file and run the animation to check] Video 7 illustrates what might be shown in the animation.  *Video 7: The Runner Task*	This portion of the task is to have students continue to engage in reasoning about constant rate of change. I will explore whether a student can apply what they did in Tasks 1 and 2 to determine a constant speed by recognizing that the desired constant speed in this task involves the ratio between the completed distance traveled and total time traveled ($\frac{s(32)-s(0)}{32-0}$).
3c	The calculation you did for finding the constant speed is the same as for the average speed of the 1st runner over the entire 32 seconds. Why is that? What does average speed mean?	This part is to discuss connecting average rate of change with constant rate of change. I will explore if a student demonstrates an awareness that an

		average speed is the constant speed that the 1st runner would have to run at in order to travel the same distance in the same amount of time. If not, I will use the optional portion of this task to help perturb their understanding on the meaning of the word "average".
(3d) Optional	Suppose out of 5 quizzes (graded out of 10 points) you earned a 7, 9, 10, 4, 9. What is your average test score? If you earned the same total score as the above and scored the same score on each quiz, what score would that have to be? How is "average" here similar to "average" in average speed?	Students may not have strong quantitative meanings for average outside of "add up everything and divide." I provide additional examples to help students realize that "average" is about a replacement with a constant. I designed this optional task to perturb students who rely on calculating an average as "add up everything and divide" by having them reconcile the meaning of "average" in 1 context with another. I intend for students to reflect on their meaning of average in Task 3 and consider that an average involves a replacement of values. (For the previous task, one meaning may be to replace all of the 1st runner's speeds with 1 speed that would have him travel the same distance in the same amount of time).
Summary of Task 3: The intent of this task is to draw out students' meanings for average rate of change and perturb understands of average as "add up and divide". The goal is to have students interpret "average" to mean a replacement of values with a constant one. In the context of Average Rate of Change, a student is determining variations in two quantities and then finds a ratio between them. Then the student uses the value of the ratio to imagine how the values of the quantities will vary if they were to vary that constant rate. Again I want to be quick to make the disclaimer that I do not intend this task to be an HLT on average rate of change. I constrain myself to imagine that 1 set of tasks would take 1 class period and is the amount of time I would allot in teaching average rate of change in a Calculus 1 course.		
Task 4: Average Rate of Change Over Smaller and Smaller Intervals		
4a	Suppose the 2nd runner now runs the first 16 seconds by running at the average speed of the 1st runner in this 16 seconds. Write a function that represents the 2nd runner's distance after traveling for t seconds at this speed. [Students will be using https://www.desmos.com/calculator/gaiwt9fjwm to check their solution] In the next 16 seconds, the 2nd runner will run at the average speed of the 1st run in this time. Rewrite your function to include this.	This task's overarching goal is to have students engage in using average rates of change over smaller and smaller intervals to introduce the idea of instantaneous rate of change. Students are presented with the goal that we want to know about the 1st runner's speed at any time. Since students do not have direct access to the values of the function, s, they will have to use function

		notation to represent variations in the distance in order to represent an average speed. I will explore if students recognize that $s(t)$ represents the distance the 1st runner has ran after running for t seconds, without needing an explicit function definition to determine the value at a given input value. If students struggle with this, then I will investigate whether having the Desmos functionality of typing in $s(3)$ and Desmos showing the value of it, aids students in building this understanding of function notation. (For example, if a student decides to represent the average speed in the first 16 seconds using $\frac{s(16)-s(0)}{16-0}$ instead of finding the value of $s(16)$ I would say that a student evidences this desired understanding of function notation). The 2nd portion of this task is challenging because students have to consider that the amount of time the runner is running at this constant speed for is not the value of the variable t; it is $t-16$. I will closely follow students' work and ask them about what they are trying to represent and attempt to perturb student thinking. Ex: If a student is writing $\frac{s(32)-s(16)}{32-16} * t$, then I may ask that if $t=17$, what happens? Additionally, the Desmos interface will help with visualizing the issue. I will investigate if students are able to utilize the value of the imagined constant rate of change to determine a projected change in the output value for a change in the input value.
4b	Repeat activity but with more intervals. 4, then 10 intervals. Video 8 illustrates the task that students will be interacting with and the result of what they might see.	The purpose of having students repeat this process is to 1) Have them repeatedly reason about average rates and constant rates 2) Have them experience that doing this for too many intervals makes it

	 *Video 8: The Racer Problem*	tedious, so there is a necessity for simplifying the process 3) This exercise is similar to how some students may learn about accumulation functions (integrals), so having students engage in connecting an accumulated distance with a rate of change function fits into making the rest of Calculus coherent for the student. 4) When students are using the animation to test their solutions, they can also see that picking more and more intervals makes the two runners line up more and more.
4c	What do you expect will happen if we did it for 100 intervals? If you had to do it, what do you think the process would be like? Is the average speed in that interval the speed the 1st runner runs at in that interval?	This is the part of the task that would be too tedious to do. Instead, this task allows students to reflect on their previous work and imagine the results of their activity as if the actions were performed. Again this question is akin to a "mental run" (Simon et al., 2010) to have students actively recall the actions they engaged in to allow them to reflect on their actions and results of those actions. A student can reflect on this activity (and the result of "running" the animation) to imagine that they can keep choosing smaller and smaller intervals ($\lim_{\Delta x \to 0}$), and that doing so makes the 2 runners line up more and more.
Summary of Task 4: The purpose of Task 4 is to have students keep engaging in using average rates of change, the varying quantities involved, and creating a necessity for limits. Additionally, the student may start to consider that an average rate in a small time window is good enough to model someone's speed in that time window.		
Task 5: Average Rate of Change Over Smaller and Smaller Intervals		
5a	[This task uses https://www.desmos.com/calculator/psagu3btkq] Let us look at the graph of the 1st runner. Let $s(t)$ represent the distance ran by the first runner, in meters, after he ran for t seconds. Use the Desmos File and use two intervals to model the 2nd runner running at the average speed of the 1st runner in each interval. (Then 4, then 10) [Before a student plays the Time slider, ask the student what they think will appear on the screen and why]	The purpose of this part of the task is for students to connect what they did in Task 4 to a graphical context. This task has the student once again program the average speeds by calling on the values of the given function to define a linear change in each interval. The student can determine the number of intervals

		and can use the time slider to check their solution. In this portion of the task, I will investigate if a student recognizes that the constant speed in each interval will be the average speed of the 1st runner. That is to say that the average speed of the 1st runner is the imagined constant speed for the 2nd runner to run the same distance in the same amount of time.
5b	Describe what you would do if you had to do 50 intervals? What do you think that graph will look like? [The student will then have access to https://www.desmos.com/calculator/ltnnrgqfww which pre-programs each portion. The student can slide the n slider to see what the graph of the 2nd runner would look like for any number of intervals]	The student will then be asked to perform another "mental run." This allows the student to reflect on how average rates of change are imagined constant rates of change over a given interval. The latter portion of this task has the student predict what happens as they let the number of intervals increase (or letting the size of the variation decrease, $\lim_{\Delta x \to 0}$). Additionally, this illustrates the tedious amount of work needed to calculate the average rate of change for a large number of intervals, motivating the need to encapsulate the process. I will investigate: 1) if a student demonstrates an understanding that average rates of change over small intervals is *essentially equal* to the speed over those time intervals, and 2) the aspects of this animation the student uses to articulate this understanding.
5c	Suppose you are the engineer on a film set. The director wants to shoot a scene where a car moves along a road and wants the camera to run alongside the car. The director asks you to program the track that the camera will run on. The only information you have access to is that you know exactly how far the car has traveled after any amount of time since the scene started. [Student will use https://www.desmos.com/calculator/u7ldxbssf2] to answer the problem Depending on how the student approaches this, the instructor can prompt questions (What	This task aims to have students repeatably reason about modeling motion using average rates of change by applying the methods they used in the previous tasks to this task. The Desmos interface in this task does not provide the built-in interface to program piecewise linear functions that was in the previous task. Instead, the student will be prompted to discuss what approach they might take to model the camera's movement and how they would employ mathematics to do so. Based on their response, the instructor can

	approach do you think will work? What information are you trying to represent? Is this similar to the previous questions?) The instructor can also provide the Desmos framework that was provided in the previous tasks. It is omitted in this one to not push the student to make the association that the tasks are similar.	assist the student in providing the previous interface to build piecewise linear functions, or explain the commands available in Desmos. Again, I will assess if students reason that using an average rate of change over small intervals will be *essentially equal* to the actual speed of the car.
Summary of Task 5: The purpose of this task is to have students continually utilize average rates of change and to consider that average rate of change in a small enough interval is good enough to model speed. Without bringing up the formal limit definition of derivative, $\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}$, students will be imagining the variation in x as becoming smaller and smaller, and coordinating the value of the average rate of change of the function over that interval.		
Task 6: Utilizing Average Rates of Change		
6a	[Students will use the function they made for the previous task] For the camera, how much distance would they travel between (pick two values, the values depend on how they defined their function. Have the interval be in the middle of the endpoints on an interval they chose) How much does the car travel in the same time period? Are you surprised by your answers? Repeat with different intervals.	This part of the task is to have students use an average rate of change to determine how a quantity would change by utilizing the value of the rate of change. The purpose of this portion of the task is to have students numerically compare how determining the car's average speed over a small interval can be used to determine the car's accumulated distance. I will investigate if students are able to represent a projected change in distanced traveled by utilizing an imagined constant rate of change over a given input interval. When implementing this task I will repeatedly explore if a student recognizes that using an average rate of change over small intervals will be *essentially equal* to the actual speed of the car.
6b	Students will fill out a table that they will pick the interval size. What would you need to do if you wanted the error between them to be less than 0.01? 0.003? [Repeat as needed]	The Desmos applet in this portion of the task automates the process from the previous problem for the student. All the student has to do is decide an interval size, and they can compare the approximated distance traveled with the actual distance traveled. This allows the student to focus on reasoning about the results of the

		calculations instead of having to calculate them. I will explore whether a student recognizes that they can get better and better approximations because they can choose interval sizes as small as they want ($\lim_{\Delta x \to 0}$).
6c	The film director wants to make sure that the car is not going too fast for legal reasons. She thinks that at the end of the scene (t=30) that it might be too fast. She asks you "at that time what speed is the car traveling at?" What would you do if she asked for the speed at t=10? At t=2.98? If the director gave you a particular time that she wanted to know the speed for, describe what you would do to determine that for her.	This task requires students to use and compute the average speed to internalize how speed at a point is computed. By repeatedly reasoning about it, the student can generalize the process, which will allow them not to have to go through the process to determine the speed. Instead, the student will then have the basis for building a rate of change function.
Summary of Task 6: The two major points of this task is for students to reason in the following ways • Reason about an *Average Rate of Change* as imagining how two quantities would change together at a constant rate (if we wanted them to start and end at the same values) • Reason about *Speed at a Point* as an average rate of change in a small enough interval and that this process is repeatable for other input values.		
Task 7: Instantaneous Rate of Change		
7a	It appears that if we know about how much of a quantity we have at all times (distance the car has traveled), $s(t)$, we can determine how fast that quantity is changing at all times. Given a value t we could find an average rate of change in a small enough interval, and that was our "speed at a point." Since we can do this for any t value, let us call the "speed of the car at time t" as $s'(t)$. This function will be called the *derivative of s(t),* which is "how fast the distance the car has traveled is changing" or "*the*	This part of the task is to introduce the derivative notation s'. Just as students have (hopefully) encapsulated the process of a function mapping an input to an output, the students can then encapsulate the previous process they engaged in.

	instantaneous rate of change of distance with respect to time." [The student will use a Desmos activity] Find the speed of the car at $t = 2.3$. Then type "$s'(2.3)$" what do you notice? Use the value you found to estimate what the distance traveled by the car would be for $t = 2.5$. Type "s(2.5)" to check your answer, what do you notice? Are you surprised? Repeat for other values	
7b	[Students will be using https://www.desmos.com/calculator/nidqma1dya] Let us look at the graph of a function f. Suppose we want to know how fast the output value of the function changes at a particular input value. (Student will be given the option to choose an input value of their choice, and then follow the instructions on the Desmos activity where they will zoom in until the graph looks linear) (Student will then be instructed to find the value of f' at the associated input value) What does this value tell you? (Students can move the slider pick a small change in x and the associated change in the value of the function) How much has the output changed given the change in your input? Are you surprised? What does the value of the derivative at the input tell you? Repeat with other input values.	This portion of the task is adapted from works of Tall (2009, 2013) and Ely and Samuels (2019) where students will be "zooming in" to see that the function is *essentially* changing at a constant rate of change, and the value of the derivative is this constant rate of change. I will explore if a student demonstrates an understanding that A function having an instantaneous rate of change of value $f'(a)$ at the input value a means that for small variations in the input quantity, Δx, the variation in the output value of the function, Δf will *essentially* be $\Delta f = f'(a) * \Delta x$. I will also investigate what aspects of this Desmos activity the student uses to express this understanding.
Learning Goal for Task 7: Students will interpret the value of a derivative as the value of the *instantaneous rate of change* at a given input value. This means that if they were to look at a small enough interval, the quantity they are imagining is changing at *essentially* a constant rate of change. The understanding of instantaneous rate of change that a student will hopefully construct is "A function having an instantaneous rate of change of value $f'(a)$ at the input value a means that for small variations in the input quantity, Δx, the variation in the output value of the function, Δf will *essentially* be $\Delta f = f'(a) * \Delta x$"		

Animations in the Context of Productive Learning

Desmos animations and applets are used during the instructional sequence. The animations and applets were designed to facilitate a mathematical conversation about quantities varying. What a student understands from an animation or applet depends on how the teacher manages the conversation around the mathematical ideas the teacher wishes the students to learn (Thompson, 2002). According to researchers, animations should have an *enabling function* to help reduce the cognitive load (Mayer, 2001) and also support reflective discourse by focusing the conversation on students' understandings (Cobb et al., 1997). One way to enable reflective discourse involves having students anticipate what they will see before the animation plays and then explain what they see in the animation (Schnotz & Rasch, 2005; Thompson, 2019).

I hypothesize that one vital aspect for the success of this hypothetical learning trajectory is that students must come to realize that variables vary. Therefore, the animations and applets play an essential role in supporting students in imagining a quantity's value varying continuously and smoothly. Many Calculus reform efforts leverage technology (e.g., Swidan, 2020; Thompson et al., 2013, Tall, 2009) and have demonstrated the benefits of illustrating dynamic images in teaching mathematics. Further, the usage of the Desmos applet allows for an objective mediator of the mathematical expressions that students will employ. Since the various Desmos activities will visually demonstrate the mathematical expressions that the students type in, the Desmos applet acts as a neutral third party that validates the work the students produce. This neutrality affords students a safe space to reflect upon their mathematical thinking and is beneficial when they are perturbed by the animation.

The following study in this manuscript reports on the results of leveraging the hypothetical learning trajectory in the context of a teaching experiment (Steffe & Thompson, 2000). This teaching experiment leveraged the findings from the previous study (See Paper 1) by supporting students in developing a coherent understanding of rate of change that they can leverage for instantaneous rate of change.

CHAPTER 7

PAPER 3: RESULTS OF A TEACHING EXPERIMENT ON TWO STUDENTS' UNDERSTANDING OF RATE OF CHANGE

INTRODUCTION

This chapter discusses the results of a teaching experiment (Steffe & Thompson, 2000) designed to support students in constructing a productive understanding of instantaneous rate of change. In my first study (see Paper 1), I reported that it is common for students to interpret the value of an instantaneous rate of change as an amount to add to the output value of a function for a one-unit change in the input quantity. This result that many students have an additive conception of rate of change is corroborated by findings reported by Byerley et al. (2012) and Castillo-Garsow (2010). As a result, I developed a Hypothetical Learning Trajectory and designed tasks to support students in confronting and overcoming this conception (see Paper 2). I leveraged my HLT to conduct a teaching experiment to support students in conceptualizing a rate of change of one quantity with respect to another as the relative size measurement of the two quantities' values as they vary together or as a proportional relationship between the changes in two quantities' values. Additionally, the teaching experiment aimed to support students in constructing a coherent understanding of rate of change across multiple contexts in an effort to prevent students from building disconnected meanings for derivative. Jones and Watson (2018) observed the benefits of having students apply their derivative understanding in multiple applied contexts to promote comprehension of the ratio-limit-function layers of derivative from Zandieh's (2000) derivative framework.

METHODOLOGY

This study was conducted by engaging 3 students in individual teaching experiments (Steffe & Thompson, 2000). The teaching experiment involved six sessions (1 pre-interview and 5 teaching sessions) that focused on characterizing and advancing students' ways of thinking about rate of change.

According to Steffe and Thompson (2000), a *teaching experiment* involves a sequence of teaching sessions that include a student, a teacher, a witness, and a camera to record each episode. A teaching experiment's primary purpose is to identify a start and ending point of student progress and how students construct knowledge as they progress through each teaching episode. The goal of a teaching experiment then is to hypothesize and test models of student thinking.

Initially, a teacher has in mind a set of tasks and questions they believe will help students progress. As each teaching episode progresses, the teacher builds models of each student's thinking and tests them through questioning and tasks. As the teacher tests and refines their model, they adapt the lesson to help them determine the reasoning a student is engaging in and the potential mental actions involved. Since the witness is not active in a teaching role, they aid the teacher-researcher by offering additional conjectures about the student's reasoning and may suggest questions to the teacher-researcher to test their observation. The teacher is the one who takes the active teaching role by interacting with the student and making teaching moves, whereas the witness does not interact with the student. Additionally, between each session, the teacher and witness debrief to review what they observed and analyze each student's actions and behaviors. During this debriefing, they discuss the student's current understandings while determining goals and questions for the following interview. The teacher (researcher) then cycles through the building, studying, and refining their models of student thinking to hypothesize a progression for learning the concept.

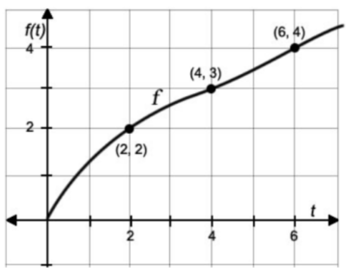
During the Fall 2021 semester, students were recruited from a MAT 265 course (Calculus 1 for Engineers). Students were asked to take a Pre-Study survey, and based on my analysis of their responses, 10 students were selected to participate in a pre-interview. From those 10 students, 3 students were selected for the teaching experiment. In the following sections, I describe the Pre-Study survey, the pre-interview, and how students were selected to participate in the study.

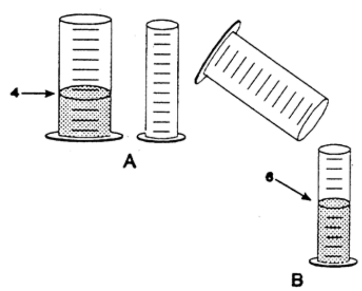
Pre-Study Survey

The Pre-Study survey contained 9 multiple-choice questions designed to assess students' ability to imagine a quantity's value varying, function notation, proportional reasoning, and rate of change (Table 21). Five of the nine questions were adaptations from the Precalculus Concept Assessment (Carlson et al., 2010)

Table 21:Pre-Study Survey Questions

Survey Question	**Question Rationale**
1) Lily walks to school every weekday morning. Let $f(t)$ represent the distance (in miles) that Lily has walked on a particular morning and let t represent the number of minutes elapsed since Lily left home. Which of the following expressions represents the number of miles Lily traveled between 3 minutes and 7 minutes after she left home? a) $f(7-3)$ b) $f(7)$ c) $f(7)-f(3)$ d) $f^{-1}(7)-f^{-1}(3)$ e) $\frac{1}{f^{-1}(4)}$	• Conceptualizing function notation as representing the simultaneous value of a function's input and output • How to use function notation to represent a change in the value of the function as the input quantity changes from one value to another.
2) Two boys, Brother A and Brother B, each have a little sister, Sister A and Sister B.	• Conceptualize a difference as how much larger one quantity is than another.

The two boys argue about how much taller they are than their respective sisters. After measuring, it turns out that the difference in Brother A and Sister A's height is 17cm more than the difference in Brother B and Sister B's height. If Brother A was 186cm tall, Sister A was 87cm tall, and Brother B was 193 cm tall. How tall was Sister B? a) 111 b) 107 c) 77 d) 82 e) 176	• Conceptualize a difference as a measured quantity that can be compared with another quantity.
3) (PCA) The graph of f gives the number of liters of water, $f(t)$ that have flowed into a water tank t seconds since the water began to flow into the tank. Evaluate $f(4)$ and explain its meaning.  a) $f(4) = 4$; Four seconds after the pipe was turned on, four liters of water had been poured into the water tank. b) $f(4) = 3$; Four seconds after the pipe was turned on, three liters of water had been poured into the water tank. c) $f(4) = 3$; Three seconds after the pipe was turned on, four liters of water had been poured into the water tank. d) $f(4) = 6$; Six seconds after the pipe was turned on, four liters of water had been poured into the water tank. e) $f(4) = 6$; Four seconds after the pipe was turned on, six liters of water had been poured into the water tank.	• Conceptualizing a point as a *multiplicative object* (Saldanha & Thompson, 1998) where a point simultaneously represents the value of a function's input and output. • Conceptualizing function notation as representing the simultaneous value of a function's input and output
4) (PCA) Given the function g, defined as $g(x) = 2x^2 - 3x + 7$, find $g(x + a)$ a) $g(x + a) = 2x^2 + 2a^2 - 3x - 3a + 7$ b) $g(x + a) = 2x^2 - 6xa + 2a^2 - 3x + 7$ c) $g(x + a) = 2(x + a)^2 - 3(x + a) + 7$ d) $g(x + a) = 2(x + a)^2 - 3x + 7$ e) $g(x + a) = 2x^2 - 3x + 7 + a$	• Representing a function's output for a variable argument. • Coordinating a function's argument with its function definition.
5) (PCA) The weight of a fish is modeled by the formula $w = 1.24x + 0.31$ where w is the weight	• Understanding constant rate of change as the relative size of the changes in the

of the fish in pounds in terms of the number of years x since the fish was born. Which of the following describes what 1.24 conveys in the context of this situation? I. For any change in the age of the fish, Δx, the change in the weight of the fish is $(1.24) * (\Delta x)$ II. The weight of the fish increases by 124% every year III. The fish gains 1.24 pounds every year a) I only b) II only c) III only d) I and II only e) I and III only	values between two varying quantities. (Constant rate of change as $\Delta w = 1.24 * \Delta x$) • A constant rate of change entails coordinating any size change in the input, Δw, with the corresponding change in the output. A constant rate of change is not limited to only 1-unit changes in the input.
6) (PCA) To the right are drawings of a wide and a narrow cylinder. The cylinders have equally spaced marks on them. Water is poured into the wide cylinder up to the 4th mark (see A). This water rises to the 6th mark when poured into the narrow cylinder (see B). Both cylinders are emptied, and water is poured into the narrow cylinder up to the 11th mark. How high would this water rise if it were poured into the empty wide cylinder?  a) To the 7 ½ mark b) To the 9th mark c) To the 8th mark d) To the 7 ⅓ mark e) To the 11th mark	• A proportional relationship between two quantities entails that the values of the two quantities maintain a constant ratio.
7) (PCA) A candle has been burning at a constant rate of change of 2.5 inches per hour. The candle has been burning for 4 hours and is 3.5 inches tall. What was the length of the candle before it was lit? a) 3.5 inches b) 6 inches c) 10 inches d) 13.5 inches e) 16.5 inches	• Coordinating changes in the values between quantities by utilizing the value of a constant rate of change.

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8) Suppose you are driving your car on the freeway. Below is a table that has some data points on the distance you have traveled and the associated times.

Time since start of trip (in hours)	Distance Traveled on Trip (in miles)
1	23
1.03	25.2
1.5	53.1
2	72.8

Which of the following is the best estimate of your car's speed at 1 hour since you started driving?

a) $\frac{25.2-23}{1.03-1}$

b) $72.8-23$

c) $\frac{23+25.2+53.1+72.8}{4}$

d) $\frac{23}{1}$

e) Cannot be determined

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- Conceptualizing speed as a multiplicative comparison between change in distance and change in time.
- Rate of change as involving how involving two quantities varying together (instead of describing how the output value will change).

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9) Suppose you are driving your car on the freeway. At 4pm your speedometer reads 51mph. Which of the following best represents the meaning of the 51?

a) In the next hour, your change in distance will be 51 miles
b) Since you started driving up until 4pm, your average speed was 51 miles per hour
c) You have traveled a total of 51 miles by 4pm
d) The steepness of the road you are traveling on is 51
e) In the next minute, you will travel approximately $51 * \frac{1}{60}$ of a mile

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- Conceptualizing speed as a multiplicative comparison between change in distance and change in time.
- Conceptualizing an instantaneous rate of change as a constant rate of change over a small period of time.
- A constant rate of change entails coordinating any size change in the input, Δw, with the corresponding change in the output. A constant rate of change is not limited to only 1-unit changes in the input.

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Students who took the survey were classified as "weak," "average," or "strong" based on their responses to the survey items (Table 22). I considered a "strong" student as one who demonstrated productive understandings of variation as a quantity, function notation, and rate of change as involving two quantities. An "average" student demonstrates productive understandings of variation as a quantity, function notation, and may consider a rate of change as an amount to add to an output quantity's value. A "weak" student is one who failed to demonstrate understandings of variation as a quantity, function notation, and rate of change. Table 22 displays the qualifications for each category. Out of the 39 students who took the survey, 10 students were selected for the Pre-Interview (5 “Average” and 5 “Strong”).

Table 22:Criteria for "Strong", "Average", and "Weak" students

	Strong	Average	Weak
Student Understandings Categorization	❖ Demonstrates pre-requisite understandings of variation and function notation ❖ Demonstrates an understanding of rate of change as involving changes in 2 quantities.	❖ Mostly demonstrates pre-requisite understandings of variation and function notation ❖ Demonstrates an understanding of constant rate of change as 1-unit changes in the output ❖ Demonstrates some understanding of rate of change as involving two quantities	❖ Fails to demonstrate pre-requisite understandings of variation and function notation. ❖ May demonstrate some understandings of rate of change
Possible answers on survey	❖ Correctly Answers Questions 1-4, and at least 3 of 5-9	❖ At most, incorrectly answers only 1 of Questions 1-4. ❖ At least 3 of the following occurs ➢ Chooses answer C for Question 5 ➢ Chooses answer D for Question 6 ➢ Incorrectly answers Question 7 ➢ Chooses answer B or D for Question 8 ➢ Chooses Answer A for Question 9	❖ Incorrectly answers at least 2 of Questions 1-4. ❖ Incorrectly answers at least 3 of Questions 5-9

Pre-Interviews

The pre-interviews consisted of a single *exploratory teaching interview* (Castillo-Garsow, 2010; Moore, 2010). Exploratory teaching interviews consist of a one-on-one interview including an interviewer/teaching agent, a single student, and a video camera to record the single episode. During these interviews, the interviewer asked the student to complete a problem designed to produce a perturbation in the student’s thinking. As the interview progressed, the interviewer attempted to create a model of the student’s thinking and understandings. As the model was being created, the interviewer tested their hypotheses by asking questions targeted to verify their conjectures. Additionally, the interviewer suggested a productive way of interpreting a problem when the interviewer determined that the student could not make further progress or was pursuing an unproductive path of reasoning. The researcher sometimes attempted to perturb the student by suggesting an alternative way of thinking. However, the interviewer primarily focused on assessing the student's understanding and reasoning when answering the survey questions. The suggestions made by the interviewer were the only teaching components of the interview; this teaching component is what distinguishes an exploratory teaching interview from a clinical interview (Clement, 2000).

During the interviews, students were asked to verbalize the thinking they used in the survey to discuss how they chose their answers. Students were also presented with other tasks to reveal the students' thinking about and understandings of ideas of variation and rate of change. Additionally, the researcher used these interviews to gauge how well students could communicate their thinking aloud. The types of questions the interviewer employed were similar to the following:

- What led you to choose the response that you did?
- What was it about the problem that prompted you to do that (calculation, ruled out an answer, etc.)
- If you did so, how did you eliminate incorrect choices? How did you know that they were not correct?
- You chose this… would that still be true if…?

After each interview, the researcher documented their model of the student's understanding. Afterward, the researcher engaged in retrospective analysis (Steffe & Thompson, 2000) by reviewing the videos to determine if their model of the student's thinking was supported by how the student acted (this included utterances, gestures, and drawings). After conducting all 10 interviews, each student was ranked according to their understanding of pre-requisite topics, engagement with the pre-interview tasks, and ability to express their thinking.

Three students were selected for the teaching experiment, one "strong" and two "average." A "strong" student was selected due to the research interest in exploring how someone with robust pre-requisite understandings of variation and function notation might develop ideas about derivatives. Since derivative is a complex topic with multiple layers (Zandieh, 2000), choosing a "strong" student reduces the time needed to re-teach pre-requisite concepts and notation. Two "average" students were chosen to explore how they develop understandings of rate of change. No "weak" students were selected because of the amount of time needed to re-teach ideas of variation, function notation, and rate of change. This paper discusses two of these students and their progress through the teaching sessions.

The 2 Students

<u>Scott</u>

Scott was a first-year computer science major classified as an "average" student. During the Pre-interview, Scott demonstrated a productive understanding of function notation and was able to identify a difference as a quantity[11]. However, when discussing rate of change (on question 8 in particular), he consistently referenced a slope. When probed to describe what the slope of a line represented, he said, "slope is like the speed

[11] I say, "identify a difference" (instead of "conceptualizing a difference as a quantity") since he would verbalize that $f(7) - f(3)$ was a "difference", but his language was largely rooted in describing the operation of subtraction. This is further explained in the analysis of the first teaching session.

like one value is distance and one value is time, it's like how much we go over in each direction." Based on his responses, Scott was likely focused on the algebraic structure and operations involved in a constant rate of change and was accustomed to imagining a rise over run process in a graphical context. Further, when asked why he could use multiplication (Questions 5 and 9) or employ equivalent fractions (Question 6), Scott said that he did not know how to explain it and that "that was what I was taught to do for these kinds of problems." Since Scott struggled to describe the underlying reason for using multiplication when utilizing the value of a constant rate of change and mainly focused on the procedures involved, the researcher classified Scott as having a weak conceptual understanding of constant rate of change. Scott was selected because he understood foundational ideas for understanding rate of change but had a weak meaning for rate of change. Therefore, he was a good candidate for characterizing how his meaning for rate of change might shift during the teaching experiment.

Hans

Hans was a first-year Aerospace engineering major who was also classified as an "average" student. Similar to Scott, Hans demonstrated productive understandings of function notation. When discussing rate of change, he often thought of an amount of change or a set of "different ratios, but same proportion." For example, Hans explained that $\frac{3}{4}$ was a different ratio than $\frac{6}{8}$ because a ratio is for one instance, and "they can't be the same ratio, because their denominators are not equal." He also explained that they were the "same proportion" if the fractions could be "reduced down to the same ratio." Hans also discussed speed as "distance and time," but he was not specific about the distance and time he referenced, nor did he indicate how he compared them. Due to his weak meaning for rate of change and inability to express what a rate of change entails, there was potential for observing shifts in Han's meaning for rate of change during the teaching experiment.

THE TEACHING EXPERIMENT

The following table describes the key ideas and understandings that the researcher attempted to support their subjects in constructing when interacting with them during the teaching experiment. The researcher conjectured that these key ideas would be productive in assisting students in constructing a coherent understanding of instantaneous rate of change at a particular input value as a *hypothetical constant rate of change on an arbitrarily small interval of a function's domain that includes that input value*.

I. A (Constant) Rate of Change is a quantity that measures the multiplicative relationship between variations in the values of two varying quantities ($\frac{\Delta y}{\Delta x} = m$).

II. An Average Rate of Change is a hypothetical constant rate of change over a function's input interval that achieves the same change in the output quantity, as achieved by the function, over the input interval, from x_1 to x_2, on which the average rate of change is determined. ($\frac{f(x_2)-f(x_1)}{x_2-x_1} = m$, where m is the value of the imagined constant rate of change).

III. One can obtain better approximations of the rate of change at a given value of the input variable by determining average rates of change on smaller and smaller intervals that include that value of the input variable.

IV. An Instantaneous Rate of Change is a hypothetical constant rate of change in a small enough interval such that imagining the output quantity's value changing at this constant rate of change will be *imperceptibly different* from how the output quantity's value actually varies.

Foundational Understanding 1 – What is Rate of Change

The first teaching session focused on supporting students in building a productive meaning for constant rate of change; in particular, a (constant) rate of change is a quantity that measures the multiplicative relationship between variations in the values of two varying quantities ($\frac{\Delta y}{\Delta x} = m$). This session was designed to detect whether a student possessed an additive conception of rate of change; and if so, engage the student in a sequence of tasks to support them in confronting and overcoming this conception (Byerley et al., 2012; Castillo-Garsow, 2010). In order to support students in developing a productive meaning for constant rate of change, the session focused on developing four foundational understandings of what rate of change entails.

1. Rate of change is a **quantification of variations**: This entails that a completed variation in a quantity's value is a quantity itself that must be distinguished from the original quantity.
2. Rate of change relates variations in **two varying quantities**: A rate of change results from a *quantitative operation* (Thompson, 1990) of comparing variations in two varying quantities. One important nuance is that the quantities' values *vary* since a rate does not entail a static comparison.
3. Rate of change is a quantification of variations in two quantities made by a **multiplicative comparison** of these variations: The quantitative operation entails

the relative size between a change in the value of one quantity with respect to a change in the value of another quantity.

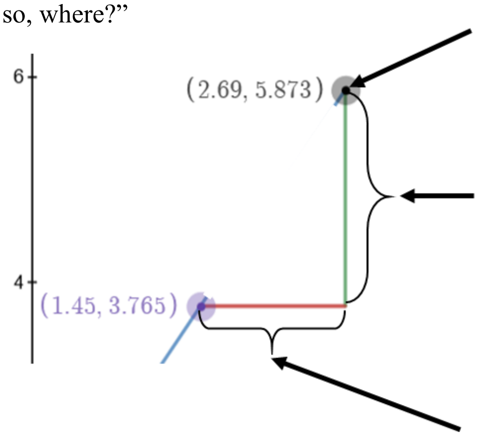
4. To say that a rate of change of quantity Y with respect to quantity X is "m" is to mean that the variation in quantity Y (Δy) is m times as large as the variation in quantity X (Δx), i.e., $\Delta y = m\Delta x$ and $m = \frac{\Delta y}{\Delta x}$.

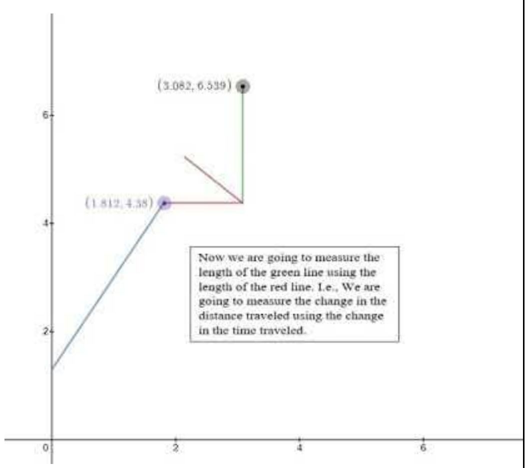
Summary of Session 1

Session 1 (Table 23) introduces a situation where a person is walking away from a wall at a constant speed. The interviewer asks the student to explain what it means to have a constant speed of 1.7 meters per second. The interviewer then poses questions to assess how the student interprets the value of 1.7. Most students will likely be engaging in *ratio as identical groups* (Johnson, 2015) in that they are associating 1.7 meters with 1 second. The remainder of the task is designed to support students in abandoning this way of reasoning and instead conceive that 1.7 describes a relationship between the two varying quantities of distance traveled and time elapsed. The Desmos portion of this task involves an animation that displays a horizontal length (representing the change in time) being used to measure a vertical length (representing the change in distance) (Video 6).

Table 23:Session 1 Tasks and Rationale

Session 1: Constant Rate of Change		
Order of Instruction	**Task**	**Task Rationale**
1a	"I will be walking away from the wall at a constant rate of 1.7 meters per second. What does it mean to have a constant speed of 1.7 meters per second?" If students say, "For every 1 second, you will travel 1.7 meters", then follow up on the conversation, asking what happens if I travel for 0.4 seconds? "How does how you defined	This task is meant to elicit students' current ways of understanding about constant rate of change. In general, I expect students to say something like, "For every 1 second, you will travel 1.7 meters". I will explore if students' conception of a constant rate of change is similar to

	constant speed connect to your previous answer?”	that of *ratio as identical groups* in that a student interprets the value of the constant rate of change as a 1.7 change in the output value and a change of 1 in the input value.
1b	We are going to keep track of my distance from the wall with this graph [This is done with a Desmos Animation, https://www.desmos.com/calculator/on4zt70oj3]. What does the ordered pair displayed on the graph represent? Does 1.7 meters per second show up here? If so, where?	This portion of the task introduces students to the Desmos Animation that will facilitate the conversation about the idea of constant rate of change. I will explore whether students recognize that the value of a constant rate of change describes how two quantities’ values vary together ($\Delta y = m * \Delta x$). In particular, whether they recognize that the ratio $\frac{\Delta y}{\Delta x}$, remains constant as the two quantities’ value change together; and what aspects of the animation the student uses to explain this relationship.
1c	Suppose we fix the current distance and time values in mind. Next, we are going to let the quantities vary from there and keep track of the changes. What does each of the following represent? Then ask “Does 1.7 show up somewhere? If so, where?” 	This portion of the task is to familiarize students with the Desmos animation and have them attend to variations in distance and time. I intend for students focus on what the parts of the animation are representing in this context. In particular, I want students to interpret the length of the red line as representing a variation in time from a chosen input value, and the length of the green line as representing the variation in distance traveled. By asking “Does 1.7 show up somewhere”, I will explore if a student recognizes that 1.7 describes the multiplicative relationship between the variations in the input and output values ($\Delta y = 1.7 * \Delta x$), or if they still think of 1.7 as an amount to add to the output (or to go up 1.7 units). If a student has not exhibited reasoning about 1.7 as an invariant relationship that is restricted by a constant ratio between the variations, I built the animation to have the values vary (by animating the Desmos animation) and that the 1.7 should describe something about this situation. The intent is to move students who are thinking of 1.7

		in the sense of *ratio as identical groups* towards a meaning of *ratio as measure*.
1d	Ask students to measure the length of the green line in units of the length of the red line. Repeat with different time values and changes in the time value. Video 6 illustrates the animation that students will be engaging with (not the pacing or the questions asked)  *Video 6: The Constant Rate of Change Graph*	This portion is to have students engage in reasoning about division (or multiplication) as "how many times as large" the value of one quantity is in terms of another. I will explore if students now recognize (or continue to recognize) that no matter the size of the change in the input value (Δx) the size of the associated change in the output value (Δy) is always 1.7 times as large ($\frac{\Delta y}{\Delta x} =$ 1.7). A student (who did not previously articulate an understanding of rate as a multiplicative relationship between 2 varying quantities) can reflect on the repeated activity of measuring variations in two quantities and conclude that the 1.7 does not describe a fixed change in the output value, instead it describes the invariant ratio between the variations in the quantities' values. Additionally, I will explore what aspects of this Desmos activity the student uses to articulate this meaning for a constant rate of change.
Summary of Task 1: The purpose of Task 1 is to reveal the student's current understanding of constant rate of change. In doing so, I hope to reveal and advance the student's conception of what the value of 1.7 means. I anticipate that most students will be engaging in *ratio as identical groups* in that they are associating 1.7 meters with 1 second. I intend that students will abandon this way of reasoning and instead start to conceive that 1.7 describes a relationship between the two varying quantities (*ratio as measure*).		

Teaching Session 1 - Scott

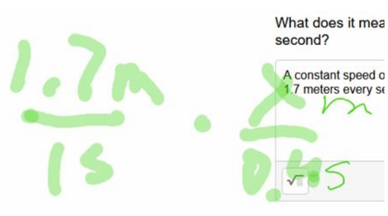


Figure 24:Scott's Written Work for Determining a Change in Distance for a 0.4-second Change in Time

In the first session, Scott explained his interpretation of 1.7 meters per second as "a constant speed of 1.7 meters per second is a change in distance of 1.7 meters every second." Afterward, he was asked to determine the distance traveled in 0.4 seconds, and he set up equivalent fractions to determine the number of meters traveled (Figure 24)[12]. When asked to consider the animation of the walking man and why his definition reflected that he could utilize equivalent fractions, Scott replied that he was trying to find the "partial amount of distance" for those 0.4 seconds. However, Scott was perturbed when the interviewer provided an example where the man's speed varied as he walked (very slowly at first and then sped up) such that after one second, the distance traveled was still 1.7 meters. The interviewer then asked why that would not be the same as his written definition. Scott said that the man in this situation did not walk at a constant speed but did not know how to modify his previous description that "a constant speed of 1.7 meters per second is a change in distance of 1.7 meters every second."

When discussing the graphical portion of the task, Scott appeared to conceptualize a variation distinct from the initial quantity (Figure 25). When asked if he saw 1.7 show up in the graph, he indicated that he did not see 1.7 of anything but that it could be "found by dividing the change in distance traveled by the change in time elapsed" (Figure 26) and that even if we moved the second point around (and therefore the values of the changes would be different), the division would still end up as 1.7. Up to this point, it seemed that Scott's understanding of constant speed entailed the structure of "change in distance divided by change in time," and the value of the speed could determine the change in distance of 1.7 meters for a 1-second change in time. This response suggested

[12] As a reminder from the Pre-Interview, Scott stated that did not know how to explain why he should employ equivalent fractions and that "that was what I was taught to do for these kinds of problems."

that Scott lacked a quantitative understanding of division as the relative size between the changes in two quantities' values.

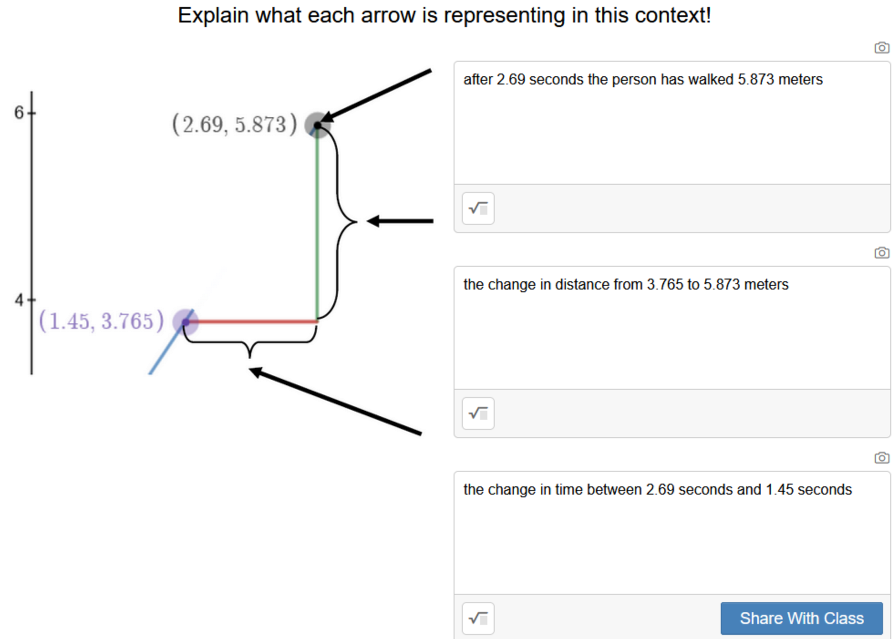


Figure 25: Scott's Interpretation of Various Features of a Graph

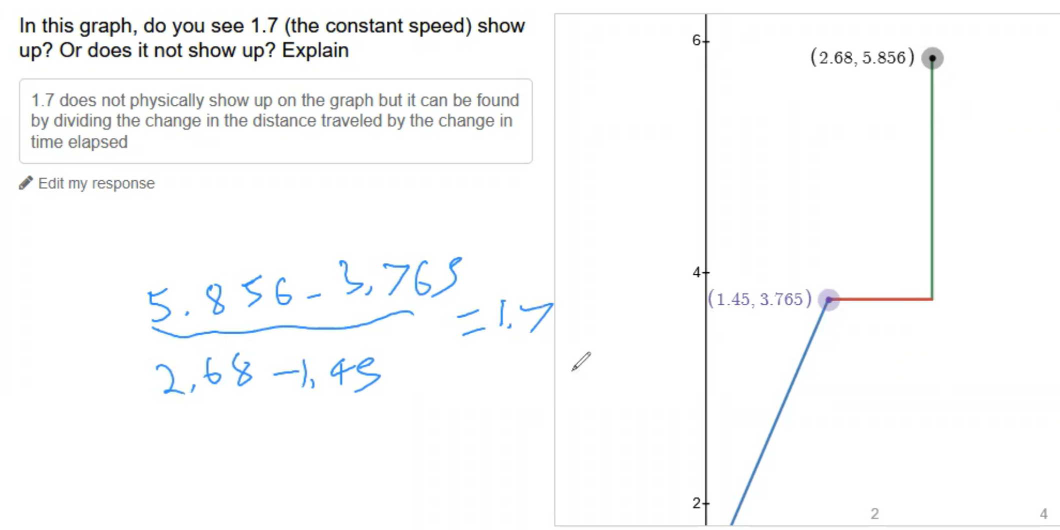


Figure 26: Scott's Interpretation of where he sees 1.7 show up in the Desmos applet

When playing the animation, Scott initially attended to the value of the change in time (the length of the red line) when discussing what the length of the vertical line up to the first tick mark measured (Figure 27). Due to this, he had trouble interpreting what quantities were displayed in the animation since he was fixated on determining the length's magnitude. However, dragging the slider on the applet to display different amounts of variation in the input quantity supported him in attending to a change in time as a unit of measure; “I guess it’s going about the same distance as what’s on the x-axis, so that is like a one-to-one ratio.” As Scott continued working with the applet, he sometimes shifted back to discussing the specific values involved. However, with minimal suggestions, he observed that “we can use the slider to end up with different numbers (values of the change in time).” This observation allowed him to conceptualize the change in time as a unit of measure: “oh, the change, that’s the change in time we use to measure the other change.” After working through questions posed in the context of

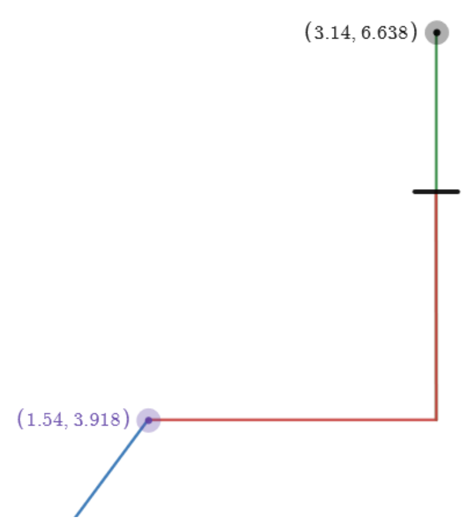


Figure 27:Screencapture of the Desmos Applet where one horizontal line segment (representing the value of the change in time) is marked off on the vertical line segment (representing the value of the change in distance)

the applet, Scott summarized that "we took 1.7 of the change in time, and that was the change in distance no matter how big they [the changes] were."

During the retrospective analysis of the teaching session, the interviewer hypothesized that Scott's initial difficulty stemmed from his inability to conceive of a variation as a measured quantity that could be used as a unit of measure. During the pre-interview, Scott had no difficulty identifying that an expression of the form $x_2 - x_1$ was a change in a quantity's value. Therefore, what likely explains Scott's focus on specific values was that his conception of $x_2 - x_1$ was a calculation to be performed. However, the expression did not represent a quantity that he could operate with until he had computed its value. This claim is corroborated by the fact that it was not until the end of the session that he recognized that "oh, the change, that's the change in time." What likely allowed Scott to shift from thinking about specific values to unitizing the change in the input quantity was the applet's functionality to show the continuous variations in the two quantities' values. Initially, when Scott interpreted what was shown in Figure 27, he took actions to calculate the magnitude of the horizontal line. However, as Scott used the sliders on the applet to vary the amount of variation in the independent quantity, he discontinued his efforts to calculate the value of the variation. Instead, he considered what was being marked off on the vertical line segment that represented the change in distance. This claim is supported by his statement that "it's going about the same distance as what's on the x-axis." Scott's shift from taking actions to calculate a value to considering that the indicated length between the tick marks represented one of something likely supported Scott in measuring the size of the change in distance relative to an arbitrary change in time. As he moved away from calculating the magnitude of the length, he then stated, “oh, the change, that’s the change in time we use to measure the other change.”

After working through the activity, Scott was asked what 1.7 represented in this situation, and he said, “the change in distance is 1.7 times the change in time at any given point.” In the following teaching session, when asked what he remembered about the first lesson, Scott recalled that “I said that [rate of change] was the change in this for one second, and then we broke it down into how the changes vary with each other.” Moreover, he also remembered the Desmos animation as he used his hands to mimic the animation where the length of the red horizontal line (change in time) was used to measure the length of the vertical green line (change in distance) (Figure 27). Scott was cognizant that his original meaning focused on a one-second change in time (and the associated change in distance) and that his new meaning could attend to any change in time. His recollection supports the hypothesis that attending to the relative size between an arbitrary amount in the change in time and the associated change in distance supported him in reconceiving what a rate of change entails.

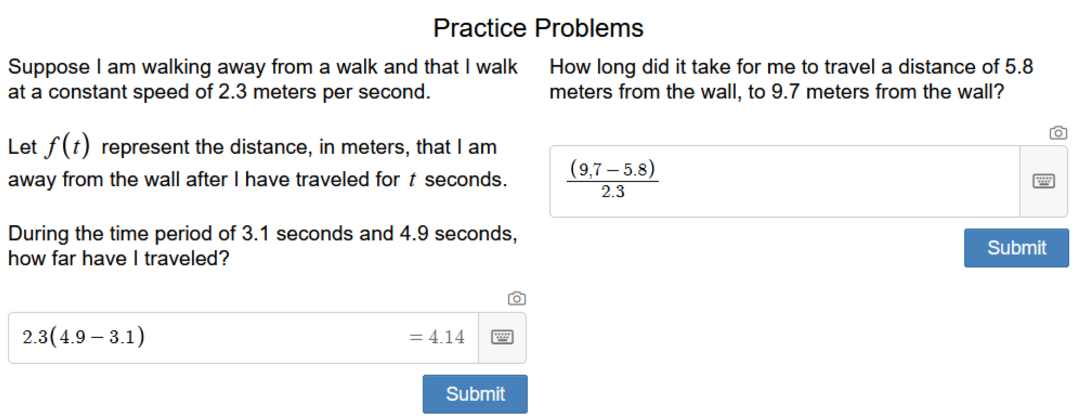


Figure 28: Scott's Usage of the Value of a Constant Rate of Change

When asked to use the value of a constant rate of change, Scott appeared to leverage his conception of rate of change as a relative size measurement when constructing his response (Figure 28). His calculations in Figure 28 are contrary to his previous actions during the pre-interview and the beginning of this teaching session, where he utilized an equivalent fractions method (Figure 24). In determining how far he would travel at a constant speed of 2.3 meters per second between 3.1 and 4.9 seconds, Scott explained that he typed in $(4.9 - 3.1)$ to represent the change in time and then multiplied by 2.3 because "for any change in time, the change in the distance would be 2.3 times as much." In subsequent teaching episodes, Scott consistently used "times as much" phrasing when talking about a rate of change. For example, in the context of the height of water in a jar changing at a constant rate of 8.34 inches per gallon, Scott explained it "as the amount of water or however many gallons is added, the height of the water increases will be 8.34 times that". His shifts in language and mathematical representations of rate of change as a relative size measurement indicate that he spontaneously conceptualized a variation as a quantity without calculating its value and that he was able to use proportional reasoning to determine the change in height of the water for an arbitrary amount of change in the volume of water. This suggests that he was fluent in considering a rate of change as entailing a multiplicative relationship between the amounts of variation in the independent and dependent quantities' values.

Teaching Session 1 - Hans

During the pre-interview, Hans repeatedly utilized the value of a rate of change by employing equivalent fractions. For example, he wrote $\frac{4}{6} = \frac{a}{11}$ and described it as "different ratios, but same proportion," and they "can't be the same the ratio, because the one on this (denominator) is not equal.” In the first teaching session, Hans explained a constant speed of 1.7 meters per second as "every second that passes, I move 1.7 meters.” When asked to explain his statement, he said that a constant rate is a constant ratio between distance and time[13] that is maintained. He clarified that a ratio is “something to simplify down to,” such as 8:6 or $\frac{8}{6}$ would be the same as 4:3 or $\frac{4}{3}$ (Figure 29). Based on

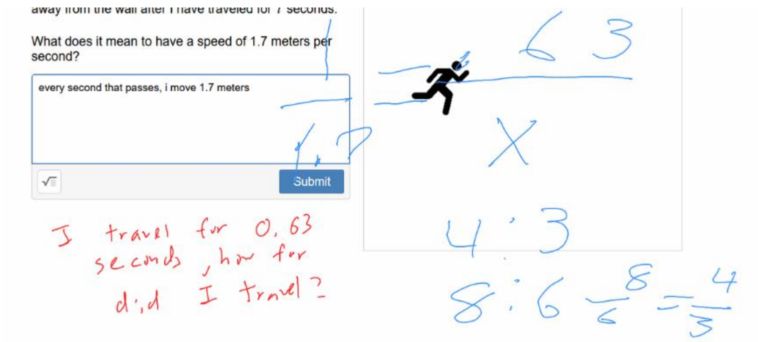


Figure 29: Hans' Interpretation of "Rate of Change" and "Ratio"

his utterances and written work, Hans appeared to be interpreting the value of a rate of change as a simplified ratio that defined what other fractions needed to be equivalent to. When asked to determine how far he would travel if he traveled for 0.63 seconds, Hans set up equivalent fractions, $\frac{1}{1.7} = \frac{0.63}{x}$, and determined what value of x would make the statement true. Hans explained that because “1 second equals to 1.7 meters, that’s a constant ratio that is proven, so if I were to plug in another number for 1, it would have the same ratio just with different values.” Moreover, as he says, “plug in another number for 1,” he used the cursor to point to the .63 he had written. Then he stated that he wanted to find the value of x that would keep the ratio intact.

[13] Throughout this first teaching session, Hans was consistently vague about which “distance” and “time” he was thinking about. Based on his responses throughout this session, he was likely thinking about a rate as involving a quantity called “distance” and a quantity called “time” but did have a specific distance and time in mind.

Upon retrospective analysis of this portion of the teaching session, Hans appeared to interpret the value of a rate of change as describing the equivalent amounts between two different quantities' values. For example, he described two fractions with the same numerical value as "different ratios, but same proportion," and "1 second equals 1.7 meters." Even though Hans verbalized that the 1.7 meters per second described a ratio between distance and time being maintained, he did not articulate 1.7 as describing the relative size between the change in distance and the change in time. Instead, he thought about the number of meters he needed to associate 1 second with, which is supported by his statements that "every second that passes, I move 1.7 meters" and "1 second equals to 1.7 meters, that's a constant ratio that is proven." His actions also suggested that he interpreted the value of a rate of change as the number that described the fraction that other "ratios" had to be equivalent to.

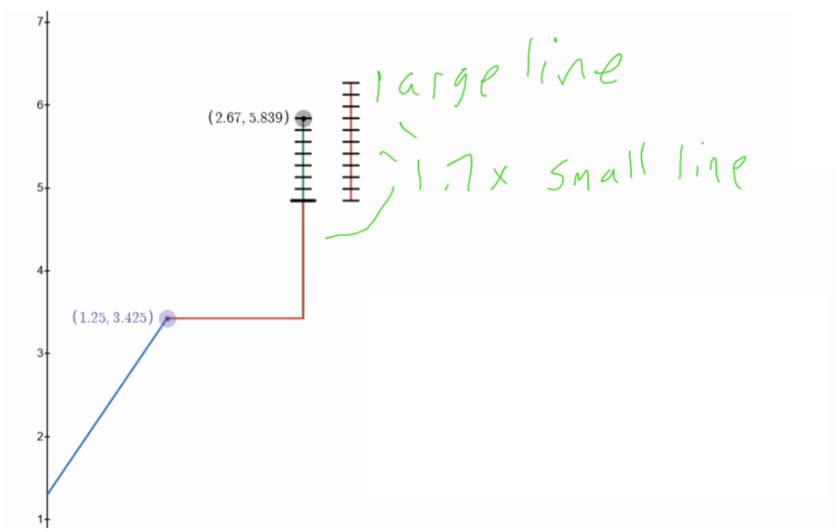


Figure 30: Hans' observation that the length of the vertical line segment was 1.7 times the length of the horizontal line segment

When discussing the graphing portion of the task, Hans continually referred to x, y, and physical features of the graph with little or no mention of the quantities he had in mind. His language only shifted when he was prompted to talk about what the features of the graph represented in the walking context. While the animation was playing with the red line marking off one change in time on the change in distance segment, Hans commented that it “was like seconds but in relation to the meter line.” However, when he explained what we had 1.7 of, he referred to the length of the red line as 1 second instead of as the change in the input quantity. He stated the length of the red line as 1 second several times, and the interviewer intervened by dragging the slider to demonstrate that the length of the red line could take on different values. While Hans eventually described the length of the red line as representing a change in time, it was apparent that Hans struggled to distinguish between the variation in a quantity and the unit of that quantity (Hans never said the length of the red line was the amount of time elapsed, rather, he indicated that it was one second of time). When asked to explain what he saw in the animation, Hans described that the length of the green line was 1.7 times as large as the red line or that it was 1.7 small lines (Figure 30). When asked to talk about the quantities represented by the length of the lines, Hans first stated that they were “x and y values.” Then when asked to refer to the quantities in the walking context, Hans was vague about which quantities he had in mind (since he said “time” and “distance”). It took prompting for him to specify the lengths of the lines as representing a change in time and a change in distance from a specific reference point. Through consistent prompting to refer to the quantities in the walking context, Hans summarized that we had a “1.7 change in time for the change in meters” (Figure 31).

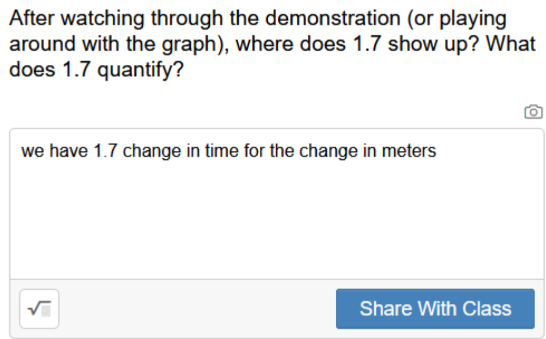


Figure 31: Hans' interpretation of 1.7 after working through the Desmos Applet

Through the retrospective analysis of this session, Hans likely was not accustomed to reasoning quantitatively or talking conceptually about mathematics. Throughout the pre-interview and first teaching session, Hans' language about mathematics was procedurally oriented and lacked specification on what quantities he had in mind. For example, when asked to describe what we had 1.7 of in the animation, Hans was comfortable explaining that the length of the vertical green line was 1.7 times as large as the small red line (Figure 30). However, he repeatedly struggled to discuss what quantities in the walking context were represented and compared in the Desmos animation. As previously mentioned, Hans went back and forth between describing the length of the red line as representing 1 second versus an arbitrary change in the amount of time. He also talked about the length of the green line as a change in y or a distance (instead of representing a change in distance). Hans required consistent prompting to talk about which distance he was thinking about. During the second teaching session, when asked what he recalled about the animation, he stated that "rate as a change in y-value with the change in x-value." Even though Hans discussed "change in" as part of his recollection, it is worth noting that he chose to talk about "x" and "y" values instead of distance or time.

During the analysis of this first teaching episode, the researcher conjectured that Hans could articulate the multiplicative relationship between changes in output values (y-values) with respect to changes in input values (x-values) but was not yet fluent in speaking quantitatively. This hypothesis resulted in adapting the later teaching sessions to include more prompts for Hans to reference the quantities represented by symbols and graphs. The interviewer hypothesized that these prompts would support Hans in consistently reasoning quantitively about rate of change.

Foundational Understanding 2 – Average Rate of Change

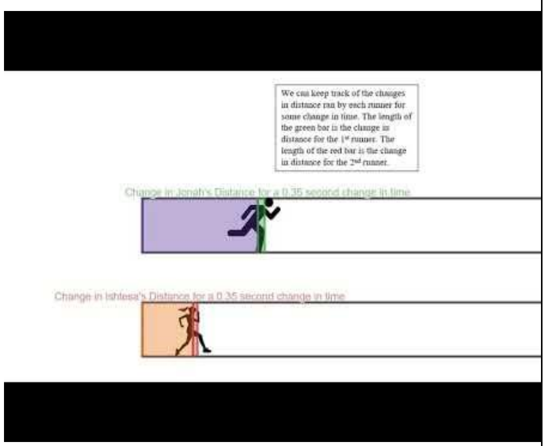
Teaching sessions 2 and 3 focused on developing students' understanding of average rate of change as a constant rate of change over a function's input interval that achieves the same change in the output quantity over the input interval, from x_1 to x_2, on which the average rate of change is determined ($\frac{f(x_2)-f(x_1)}{x_2-x_1} = m$, where m is the value of the constant rate of change). Thompson (1994) indicated that developing this understanding is nontrivial since he noticed that undergraduate students "did not have operational schemes for average rate of change" (p.49), likely because their image of constant rate of change did not entail a proportional relationship. Frank and Thompson (2021) indicated that while students could correctly identify $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ as computing an average rate of change, their interpretations of the value hindered them from developing a quantitative interpretation as an estimate for the rate of change of the output quantity, $f(x)$, with respect to the input quantity, x. Dorko and Weber (2013) explain that one reason for students' weak meanings for average rate of change is due to the "lexical ambiguity because of its uses in statistics and everyday language" (p.386). Since the previous teaching session focused on developing students' understanding of rate of change as a multiplicative comparison, teaching sessions 2 and 3 focused on developing students' meanings for average rate of change as also entailing a multiplicative comparison between the variations in the amounts of two quantities' values.

Summary of Session 2

Session 2 (Table 24) involves an animation showing a runner, Jonah, running a 100-meter race in 32 seconds (not at a constant speed). A second racer, Ishtesa, runs the same distance in the same amount of time at a constant speed. Students are asked to compare their calculations for Jonah's average speed over the entire duration of the race and Ishtesa's constant speed. This task probes students for their meanings for average rate of change and potentially perturb their understandings of average as "add up and divide." This session aims to support students in interpreting an "average" to mean a replacement of values with a constant value. The instructor guides the student to consider the similarities between both runners (they travel the same amount of distance in the same amount of time) and the calculations for an average speed and a constant speed (Video 7).

Table 24: Session 2 Tasks and Rationale

Session 2: Average Rate of Change – The Runner Task		
Order of Instruction	**Task**	**Task Rationale**
3a	[https://www.desmos.com/calculator/bpdeilsrsb] Jonah is running on a racetrack that is 100 meters long and his distance from the starting line is given by the function s, where $s(t)$ represents his distance in meters from his starting spot after running for t seconds. Jonah finishes running the 100 meters in 32 seconds. a) Does Jonah run at a constant rate of change (with respect to time)? How do you know? (What would you do to verify your answer)	This portion of the task is to elicit students' understandings about average rate of change and how to calculate it. During this portion, the instructor asks students what they are writing and what they believe is being represented. This is to have students attend to variations in quantities and engage in using division to represent a ratio.

	b) What was his average speed? How would you represent his average speed in the first 23 seconds of his run? Between 4 seconds and 8.9 seconds?	I will explore whether a student interprets “average speed” as “adding all the speeds and then dividing” or if they recognize that an average speed is an imagined constant speed needed to travel the same distance in the same amount of time.
3b	Suppose we wanted to run the same race as Jonah. We want to travel the same total distance and use the same amount of time as Jonah did. However, we want to run at a constant speed. What constant speed would that have to be? [Students can put their answer into the Desmos file and run the animation to check] Video 7 illustrates what might be shown in the animation.  *Video 7: The Runner Task*	This portion of the task is to have students continue to engage in reasoning about constant rate of change. I will explore whether a student can interpret their calculation in Tasks 1 and 2 to determine a constant speed by recognizing that the desired constant speed involves the ratio between the completed distance traveled and total time traveled ($\frac{s(32)-s(0)}{32-0}$).
3c	The calculation you did for finding the constant speed is the same as for the average speed of the 1st runner over the entire 32 seconds. Why is that? What does average speed mean?	The portion promotes that an average rate of change involves a constant rate of change (not necessarily the entire definition of an average rate of change, instead this task provides an opportunity for students to make a connection that an average speed entails a constant speed) I will explore if a student demonstrates an awareness that an average speed is the constant speed that the 1st runner would have to run in order to travel the same distance in the same amount of time as the 2nd runner. If not, I will use the optional portion of this task to help

		perturb their understanding on the meaning of the word "average".
(3d) Optional	Suppose out of 5 quizzes (graded out of 10 points) you earned a 7, 9, 10, 4, 9. What is your average test score? If you earned the same total score as the above and scored the same score on each quiz, what score would that have to be? How is "average" here similar to "average" in average speed?	Students may not have strong quantitative meanings for average outside of "add up everything and divide." I provide additional examples to help students realize that "average" is about a replacement with a constant. I designed this optional task to perturb students who rely on calculating an average as "add up everything and divide" by having them reconcile the meaning of "average" in 1 context with another. I intend for students to reflect on their meaning of average in Task 3 and consider that an average involves a replacement of values. (For the previous task, one meaning may be to replace all of the 1st runner's speeds with 1 speed that would have him travel the same distance in the same amount of time).
Summary of Session 2: The intent of this task is to explore students' meanings for average rate of change and potentially perturb their understandings of average as "add up and divide". The goal is to support students in interpreting "average" to mean a replacement of values with a constant value. In the context of Average Rate of Change, a student is determining variations in two quantities and then finds a ratio between them. Then the student uses the value of the ratio to imagine how the values of the quantities will vary if they were to vary at that constant rate. Again I want to be clear to that I do not intend this task to be an HLT on average rate of change. I constrain myself to imagine that 1 set of tasks would take 1 class period and is the amount of time I would allot in teaching average rate of change in a Calculus 1 course.		

The third teaching session was designed to continue developing students' understanding of average rate of change. A second goal was to support students in recognizing a need for and then choosing smaller and smaller input intervals ($\lim_{\Delta x \to 0}$) when representing the average rates of change ($\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)}$). This session was designed to engage students in developing fluency in using and explaining the idea of average rate of change. This session began by asking students to use average rates of change to model a hypothetical runner running at Jonah's average speed over smaller and smaller time intervals. As students choose smaller and smaller time intervals, the Desmos applet provides a visual representation that an average speed over a small enough interval will be visually indistinguishable from the actual speed of the original runner.

Summary of Session 3

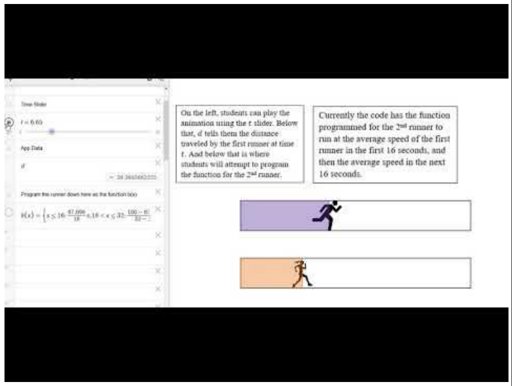
Session 3 (Table 25) builds off of the Runner Task in Session 2 by creating the goal of modeling Jonah's motion (his distance traveled as time elapses) using mathematics. The task involves students programming a function that will model the second runner's distance with respect to time elapsed. Students are instructed to use Jonah's average speed as the second runner's constant speed over each specified time interval (Video 8). This task aims to support students in reasoning quantitatively about the numbers and operations they employ. In this session, students need to symbolize and distinguish between how the runner's distance will vary as time varies (a rate of change multiplied by a change in time) and the amount of distance already traveled by the start of the time interval (See 3a in Table 25). The Desmos applet provides a visual representation of the second runner's distance traveled based on the students' mathematics, and this can provide opportunities for the student to be perturbed if their expectation does not match what happens in the animation. For example, if a student does not include the distance accrued by the start of an interval, the animation will show the runner instantly jumping back to the start of the race when $x = 16$.

Table 25: Session 3 Tasks and Rationale

Session 3: Average Rate of Change Over Smaller and Smaller Intervals		
Order of Instruction	**Task**	**Task Rationale**
3a	Suppose the 2nd runner now runs the first 16 seconds by running at the average speed of the 1st runner in this 16 seconds. Write a	This task's overarching goal is to have students engage in using average rates of change over smaller and smaller

	function that represents the 2nd runner's distance after traveling for t seconds at this speed. [Students will be using https://www.desmos.com/calculator/gaiwt9fjwm to check their solution] In the next 16 seconds, the 2nd runner will run at the average speed of the 1st run in this time. Rewrite your function to include this.	intervals to introduce the idea of instantaneous rate of change. Students are presented with the goal that we want to know about the 1st runner's speed at any time. Since students do not have direct access to the values of the function, s, they will have to use function notation to represent variations in the distance in order to represent an average speed. I will explore if students recognize that $s(t)$ represents the distance the 1st runner has ran after running for t seconds, without needing an explicit function definition to determine the value at a given input value. If students struggle with this, then I will investigate whether having the Desmos functionality of typing in $s(3)$ and Desmos showing the value of it, aids students in building this understanding of function notation. (For example, if a student decides to represent the average speed in the first 16 seconds using $\frac{s(16)-s(0)}{16-0}$ instead of finding the value of $s(16)$ I would say that a student evidences this desired understanding of function notation). The 2nd portion of this task is challenging because students have to consider that the amount of time the runner is running at this constant speed for is not the value of the variable t; it is $t-16$. I will closely follow students' work and ask them about what they are trying to represent and attempt to perturb student thinking. Ex: If a student is writing $\frac{s(32)-s(16)}{32-16} * t$, then I may ask that if $t = 17$, what happens? Additionally, the Desmos interface will help with visualizing the issue. I will investigate if students are able to utilize the value of the imagined constant rate of change to determine a projected change in the output value for a change in the input value.
3b	Repeat activity but with more intervals. 4, then 10 intervals.	The purpose of having students repeat this process is to

<table>
<tr>
<td></td>
<td>Video 8 illustrates the task that students will be interacting with and the result of what they might see.



Video 8: The Racer Problem</td>
<td>1) Have them repeatedly reason about average rates and constant rates
2) Have them experience that doing this for too many intervals makes it tedious, so there is a necessity for simplifying the process
3) This exercise is similar to how some students may learn about accumulation functions (integrals), so having students engage in connecting an accumulated distance with a rate of change function fits into making the rest of Calculus coherent for the student.
4) When students are using the animation to test their solutions, they can also see that picking more and more intervals makes the two runners line up more and more.</td>
</tr>
<tr>
<td>3c</td>
<td>What do you expect will happen if we did it for 100 intervals? If you had to do it, what do you think the process would be like?

Is the average speed in that interval the speed the 1st runner runs at in that interval?</td>
<td>This is the part of the task that would be too tedious to do. Instead, this task allows students to reflect on their previous work and imagine the results of their activity as if the actions were performed. Again this question is akin to a “mental run” (Simon et al., 2010) to have students actively recall the actions they engaged in to allow them to reflect on their actions and results of those actions.

A student can reflect on this activity (and the result of “running” the animation) to imagine that they can keep choosing smaller and smaller intervals ($\lim_{\Delta x \to 0}$), and that doing so makes the 2 runners line up more and more.</td>
</tr>
<tr>
<td colspan="3">Summary of Session 3: The purpose of this session is to have students keep engaging in using average rates of change, the varying quantities involved, and creating a necessity for limits. Additionally, the student may start to consider that an average rate in a small time window is good enough to model someone’s speed in that time window.</td>
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<u>Teaching Session 2 - Scott</u>

During the Runner Task, Scott stated that $\frac{s(32)-s(0)}{32-0}$ was Jonah’s average speed since it was a change in distance over a change in time. However, he also verbalized that an average is “every different speed that he’s moving at, add them all together and divide

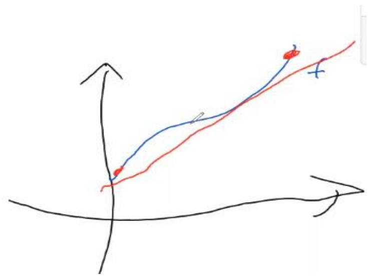

Figure 32: Scott's drawing of a secant line with an average slope

and call that the average.” Further, he stated that “like two different points from where he is running and we find the average slope,” and he drew a line between two points on a graph (Figure 32). While Scott’s explanations contradicted each other, his interpretation represented a common understanding of “average” across different contexts. Scott expressed three inconsistent meanings for average rate of change i) a procedural meaning as a change in output values divided by a change in input values, ii) a process meaning as “add up and divide,” and iii) an “average slope” obtained by determining the slope of a line between two points on a function’s graph. As Scott discussed his understanding of average in different contexts, it did not seem like Scott recognized the inconsistencies in these conceptions. Scott had disconnected understandings of average rate of change which parallels how some students develop disconnected understandings of derivative (Zandieh, 2000)[14]. Hypothetically, it stands to reason that if these inconsistencies were not addressed, Scott might have compartmentalized different understandings for derivative: procedurally as the limit of the difference quotient, a process of calculating limits or sliding a secant line, and graphically as the slope of a tangent line.

[14] As a reminder, Zandieh (2000) indicated that there are several (often disconnected) representations of derivatives that students tend to recall: graphically as the slope of the tangent line, verbally as instantaneous rate of change, physically as speed, and symbolically via the limit definition of derivative, yet some students are often unaware of how and why these ideas are related to each other.

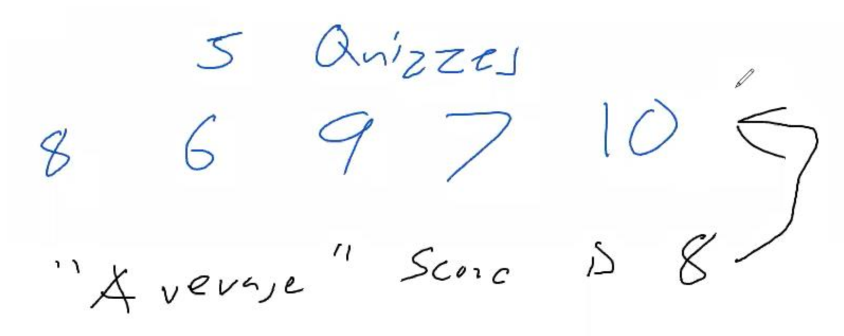

Figure 33: The Quiz Problem

To perturb Scott's way of thinking, the interviewer introduced a question about an average quiz score and asked what the average quiz score would describe in that situation (Figure 33). Scott replied, "I have no idea what to say because average is one of those words that I use in a lot of other situations…like here it might be like the middle score". The interviewer then asked Scott to consider two of his meanings for average simultaneously by writing out the calculations for an average for the Runner Task and the quiz problem (Figure 34). Next, the interviewer provided a hypothetical second student who scored 8 on all five quizzes and highlighted that both students had the same total quiz score (Figure 35). Scott remarked, "it's kind of clicking for the other one (The Runner Task), I guess like they both run the same total they just run different speeds throughout that total distance… like they both run the same total distance for the same total time with different speeds."

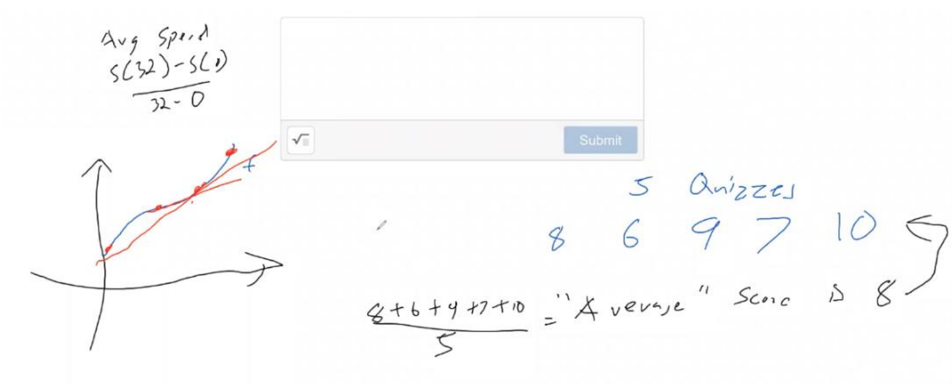

Figure 34:Representation of "Average" in the Runner Task and Quiz Problem

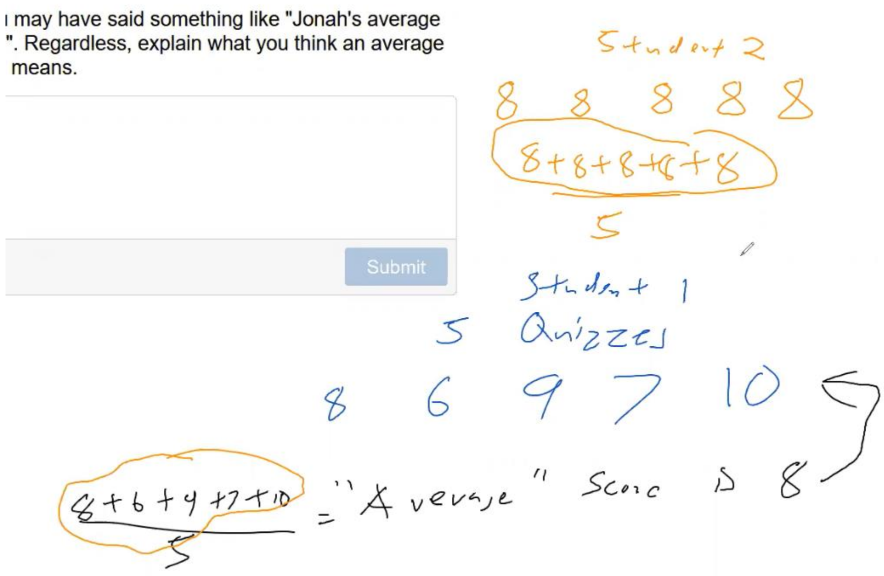

Figure 35: Average Quiz Score as the Constant Score needed for all 5 quizzes to have the same total quiz score across 5 quizzes

When presented with the quiz score problem, Scott verbalized his awareness that he had different meanings for average depending on the context. Having Scott consider the juxtaposition between the average speed of the runner and the average quiz score supported Scott in realizing the inconsistencies across his meanings for average. Prompting Scott to compare the runner and quiz situations caused him to consider an average as involving the same net changes over the same input interval in each situation. For example, in Figure 34, Scott wrote $\frac{s(32)-s(0)}{32-0}$ to represent Jonah's average speed and $\frac{8+6+9+7+10}{5}$ to represent the average quiz score, and seeing both as representing an "average" deterred him from explaining an average as a particular procedure. Instead, Scott noticed that "they both run the same total for the same total time in the runner task."

At the start of the third teaching session, Scott discussed his recollection of the second session. Scott mentioned the two runners, one running at a constant speed and the other running at a varying speed, but both finished running 100 meters in the same amount of time. He then verbalized that the runner who had an average speed of $\frac{100}{32}$ meters per second meant that "if every value of speed were to be the same, they would still travel the 100 meters in 32 seconds." Scott's recollection indicated a shift in his understanding of average speed as entailing the same total distance and the same amount of time as someone who would travel at a constant speed of $\frac{100}{32}$ meters per second for 32 seconds.

Teaching Session 2 – Hans

In the Runner Task, Hans' interpretation of average speed was similar to Scott's. Hans also recognized that $\frac{s(32)-s(0)}{32-0}$ was "Jonah's average constant speed" since it was a change in distance divided by a change in time (Figure 36a). When asked to explain what he meant by average, he struggled to verbalize his meaning. He eventually drew a graph of the situation (Jonah's distance traveled with respect to time elapsed) and a dashed blue secant line and then indicated that the whole line was the average (Figure 36b).

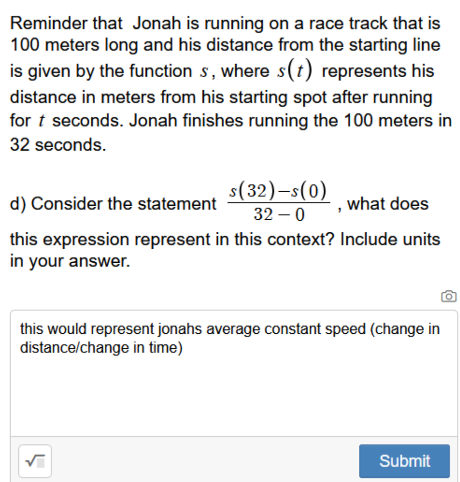

Figure 36a: Hans' Interpretation of $\frac{s(32)-s(0)}{32-0}$

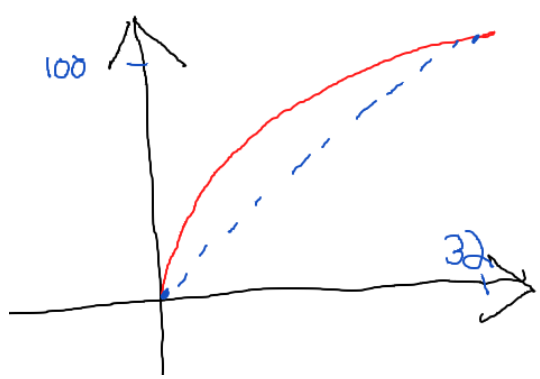

Figure 36b: Hans' Drawing of a Secant line as the Average

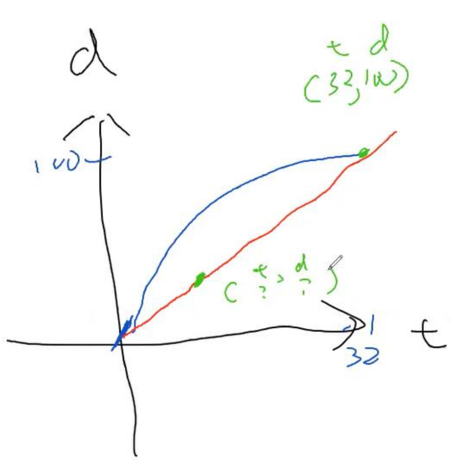

Figure 37a: What does a point on the secant line represent in this context?

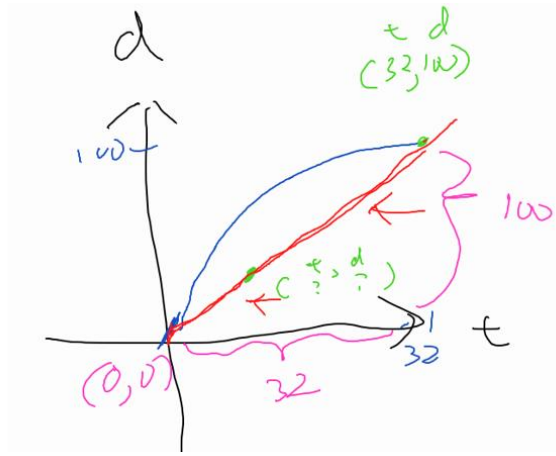

Figure 37b: Hans' observation of the Total Distance and Total Time in the Runner Task

Unlike Scott, Hans never mentioned that his interpretation of average speed included a process of adding up and dividing. However, Hans' interpretation of the secant line as representing the average speed was reminiscent of his lack of associating quantities with features of a graph in the first teaching session. The interviewer then recreated Hans' drawing and asked him what a point on his secant line represented in the runner context (Figure 37a). After determining that a point on the secant line would be an ordered pair of time elapsed and distance traveled, Hans said that the average speed would be the constant speed over the time period between 0 and 32 seconds. He then drew the associated total distance and time on the graph to match what he said (Pink lines in Figure 37b). After drawing, he paused and noted that the average speed of $\frac{100}{32}$ meters per second would mean that "Jonah (the first runner) will end at the same distance and time as Ishtesa (the constant speed runner)"; however, he said that he still did not know how to explain what $\frac{100}{32}$ quantified about Jonah.

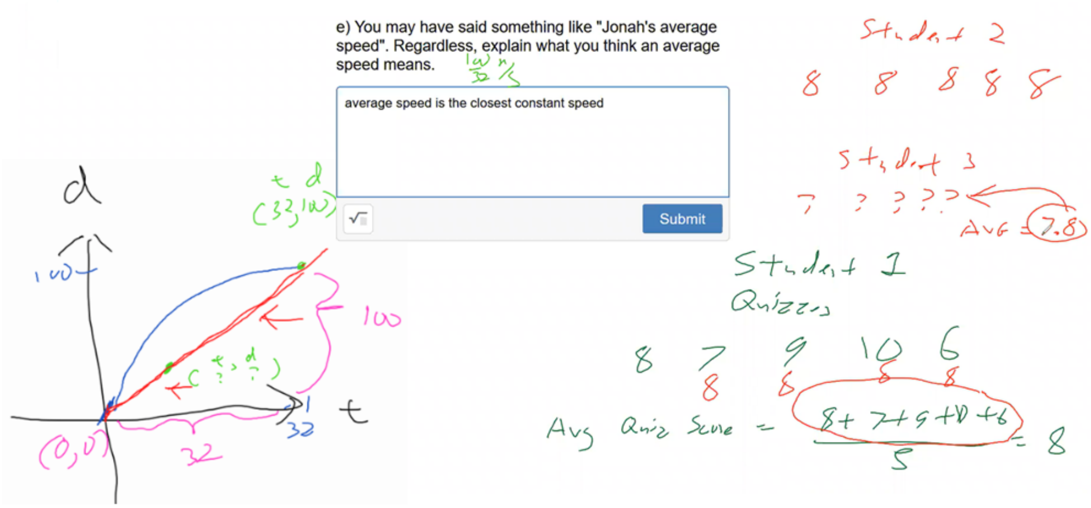

Figure 38: Hans Comparing the Runner Task and Quiz Problem

To support Hans in developing a meaning for the value of an average speed, the interviewer employed the same average quiz score situation used in Scott's teaching session. After working through the example and considering both the runner and quiz score scenarios (Figure 38), Hans indicated that "they [the situations] were similar because like the totals were the same, like on the left [the runner situation], Jonah has the same ratio of distance and time, and on the right [quiz score situation] the students have the same ratio of scores and tests." Hans was then prompted to explain what Jonah's average speed of $\frac{100}{32}$ meters per second meant, and he updated his description to "Jonah having an average speed of $\frac{100}{32}$ means that if his speed fluctuates, he will end at the same time and distance as if he were going at a constant rate."

After conducting the retrospective analysis of this session, it is likely that having Hans attend to the quantities that composed an average (the total distance and total time in the runner task, and the total quiz score and total number of quizzes in the quiz problem) helped him articulate a productive understanding for average rate of change. As the session progressed, Hans' thinking about an average shifted from a secant line connecting two points on a graph (Figure 36b) to interpreting Jonah's average speed as involving the same total distance and total time (Figure 38), and finally, that the value of an average speed involved a hypothetical constant rate of change. What supported Hans' actions was the prompting to associate numbers and features of a graph with quantities. For instance, Hans initially thought the secant line represented the average speed. After being asked what a point on that secant line represented (Figure 37a), he realized the average speed involved the total distance and time traveled (Figure 37b). Later, Hans noticed the similarities across the runner and quiz score problems by taking note of the same ratios in each context. He then connected how 'average' referred to this similarity by updating his description of Jonah's average speed to entail the constant speed that resulted in the runner traveling the same total distance over the same period of time. This description provided evidence of a shift in Han's understanding of average speed.

Teaching Session 3 – Scott

During the third teaching session, Scott was given a Desmos applet and was asked to program a second runner (that would run alongside Jonah) that ran at Jonah's average speed over the entire 32-second race using 2, 4, and then his choice on the number of time intervals. When working on the 2-interval portion of the task, Scott had no difficulty determining an expression that represented the distance traveled by the runner after running for x seconds for the first time interval. He typed $s(16)$ into the applet to determine the distance Jonah had traveled by 16 seconds and used this in his average rate of change expression, $\frac{67.098}{16}x$. However, for the next time interval, he wrote $\frac{100-67.098}{32-16}(x-16)$. Before continuing, the interviewer asked what he attempted to represent with his second expression. Scott replied that $\frac{100-67.098}{32-16}$ was Jonah's average speed in the second half of the race, and the expression $(x - 16)$ represented an amount of time. Scott clarified that he wrote $(x - 16)$ because "we have to worry about this interval and so total time, x, that means we are taking away the first interval of time." His explanation suggests that he was using x to represent the number of seconds since the start of the race and that he conceptualized $(x - 16)$ as another quantity, the number of seconds elapsed after the first 16 seconds of the race (Figure 39).

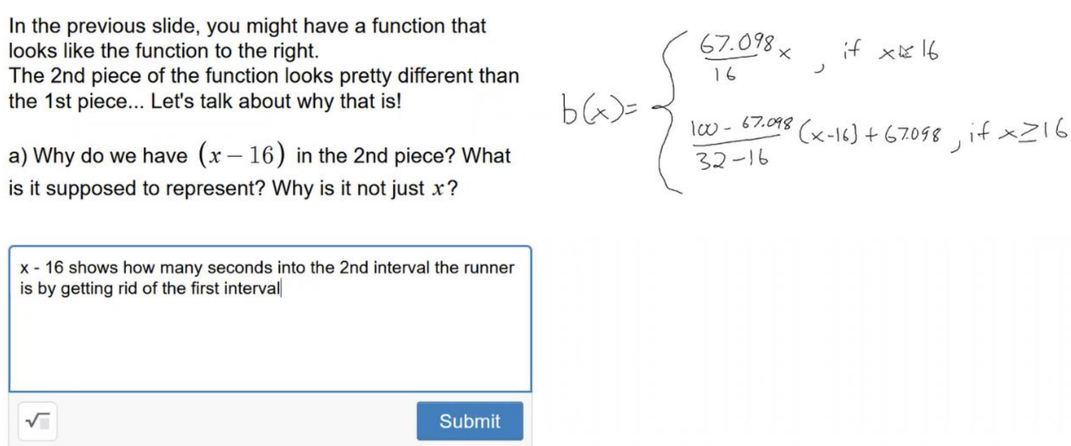

Figure 39: Scott's explanation for $(x - 16)$

What was absent from Scott's second expression was the amount of distance traveled by the runner during the first 16 seconds of the race. The interviewer decided to press play on the animation to demonstrate what Scott's expression represented in this situation. When the time elapsed hit the 16-second mark, the runner jumped back to the starting position. Scott was surprised when he saw the runner jump back, and he paused to look at his expression, $\frac{100-67.098}{32-16}$ $(x - 16)$. The interviewer then asked him what he attempted to represent and let the animation rerun. After watching the animation a second time, Scott remarked that he was missing the distance the runner had already traveled and proceeded to add 67.098 to his expression. In the subsequent tasks that prompted Scott to determine Jonah's average rate of change over four different time intervals, Scott multiplied one of the average speed values by a value of a change in time and then added the previously accumulated distance. As Scott completed each part of the task, he continued to check his work by playing the animation to verify that the runner was running according to his expectation.

Throughout this teaching session, the Desmos applet played a pivotal role in supporting Scott in connecting his mathematical expressions with how he imagined distance accruing. The Desmos animation provided a visual representation of his mathematical expressions, and when the runner jumped back to the start of the race, Scott was perturbed by what he saw. Since Scott's expectation of what would happen did not align with what he observed, this prompted Scott to reconsider what quantities his symbols represented and then make adjustments to accurately represent the quantitative relationships he intended. Having the Desmos animation display the runner's movement and distance from the starting line provided Scott with immediate feedback by displaying the quantities being represented by his expression. Seeing his programmed runner make an instantaneous jump back to the starting point prompted Scott to reconsider how to represent the distance he wanted to model (the distance of the programmed runner from the starting line as a function of time elapsed since the start of the race); in particular, he recognized that he needed both an expression to represent how that runner's distance would vary as time varied, $\frac{100-67.098}{32-16}\ (x-16)$, and the initial distance already accrued (67.098 meters) after 16 seconds since the start of the race.

<u>Teaching Session 3 – Hans</u>

Due to Hans' previous struggles in associating quantities with a mathematical representation, the interviewer anticipated that Hans would encounter difficulties in programming the average speed runner. Initially, during the 2-interval portion of the task, Hans only wrote the average speed, $\frac{67.098-0}{16-0}$, and then verbalized that he did not know what he needed to write to represent time passing. The interviewer then leveraged Hans' understanding of how to use a speed and an amount of time traveled to determine a distance traveled by asking how far the runner would travel if he ran for 1.2 seconds. The interviewer also drew on the animation and swept the mouse cursor over the associated distance to support Hans in thinking about how distance smoothly varied with time (Figure 40). Hans replied that he would use the average speed value and multiply it by 1.2. He then mentioned that he needed to "add time" to his previous expression, in which he multiplied an x to his existing expression. He repeated this for the second interval by typing in $\frac{100-67.098}{32-16}x$, but he did not attend to x as representing the total amount of time elapsed; he also failed to include the distance the runner had traveled during the first 16 seconds (Figure 41).

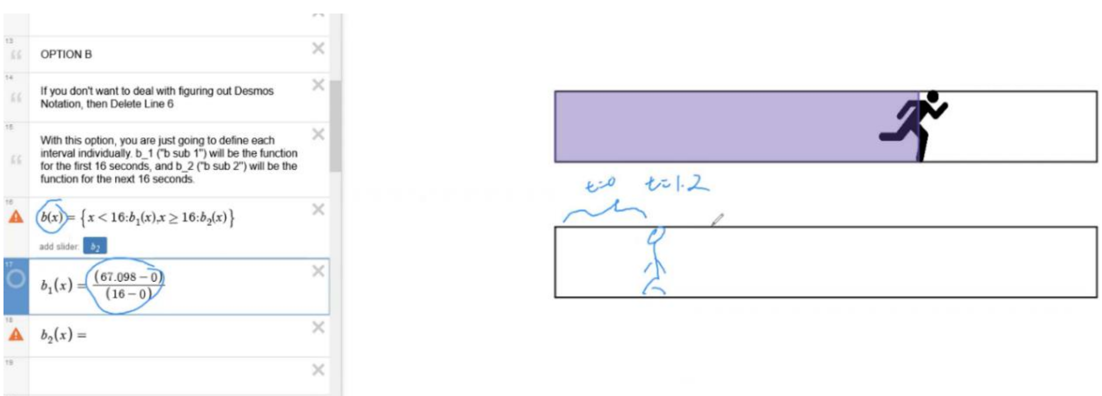

Figure 40: Hans' first attempt at programming the average speed runner

$$b_1(x) = \frac{(67.098 - 0)}{(16 - 0)} x$$

$$b_2(x) = \frac{100 - 67.098}{32 - 16} x$$

Figure 41: Hans' mathematical expressions for the average speed runner

For the first issue, the interviewer hypothesized that Hans was thinking “rate multiplied with time equals distance” and was not yet distinguishing between total distance and distance varying within an interval (similar with total time and the amount of time elapsed in the interval). The interviewer then made the following teaching moves to aid Hans in conceptualizing the difference between total distance and a change in the distance (Figure 42).

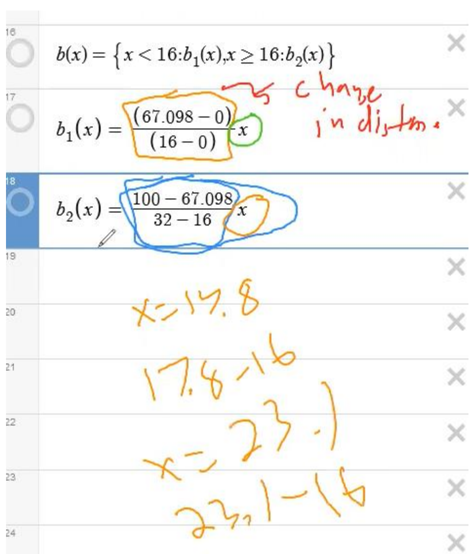

Figure 42: The Interviewer's Annotations on Hans' Expressions

1. The interviewer referred back to the first expression and asked Hans to review what each piece of his expression represented and what he was trying to represent.

[Hans identified the average speed and the amount of time elapsed and then noted that all together, it represented a "change in distance"]

2. The interviewer mentioned that for $x = 17.8$ we are not saying that the runner will run at the speed of $\frac{100-67.098}{32-16}$ for 17.8 seconds because the first expression accounted for the first 16 seconds of the race.
3. The interviewer then asked how long the runner had run at the speed of $\frac{100-67.098}{32-16}$ when $x = 17.8$ and then for $x = 23.1$. [Hans replied that he needed to determine $17.8 - 16 = 1.8$ since the runner was running at a different speed during the first 16 seconds. He had a similar response when answering the question when the total time elapsed was 23.1 seconds]
4. The interviewer asked Hans how he might incorporate his response into his expression to represent the runner's distance traveled with respect to the number of seconds since the race started. [Hans said that "we need $(x - 16)$ to account for the first 16 seconds of our race" and that "$(x - 16)$ is our change in time from 16 seconds"]

To aid Hans in conceptualizing the distance already traveled in the first 16 seconds, the interviewer played the animation, and the runner jumped back to the starting line after the runner had traveled for 16 seconds. Unlike Scott, Hans initially thought to add the expression $\frac{67.098-0}{16-0}x$ to $\frac{100-67.098}{32-16}x$. He appeared to conceptualize the expression $\frac{67.098-0}{16-0}x$ as a completed distance traveled that could be determined by multiplying a rate times a time (Table 26). The interviewer then prompted Hans to talk about the quantities he conceptualized and attempted to represent [Lines 4-6] while probing the conception of the starting point for a quantity’s measurement [Lines 7-9]. Eventually, he noted that we still needed “the first distance” and proceeded to add 67.098 to his expression [Line 11]. Hans’ statement that he needed “the first distance” suggested that he shifted to making a distinction between different distances in the situation since he chose to label the distance accrued within the first 16 seconds as the “first distance.”

Table 26: Teaching Session 3 – Hans working through the Runner Task

Hans:	So would I… add the top?
Int:	So you want to add this? *Highlights $\frac{67.098-0}{16-0}x$*
Hans:	Yeah
Int:	Remember x is going to vary (Hans: oh yeah) so what quantity do you want to add? So think about what you’re trying to represent. This over here *highlights $\frac{100-67.098}{32-16}(x-16)$*, what is this again?
Hans:	Um that’s my change in distance.
Int:	From what?
Hans:	From $x-16$… like change in distance from… from 16 seconds
Int:	Okay so what part are we missing?
Hans:	The beginning part of the race… the first distance

A retrospective analysis revealed that focusing Hans’ attention on the quantities to be represented appeared to support him in constructing the accurate expressions for those quantities. It is also noteworthy that Hans did not recognize how to symbolize a value for time when its value varied. As reported in other studies (e.g., Moore & Carlson, 2008), an interaction that led to him conceptualizing the specific quantities in a contextual problem (Figure 40) enabled him to use symbols meaningfully. In particular, he transitioned to conceptualizing and defining x as representing the runner’s distance from the starting line and constructing an accurate expression to represent the time elapsed after the first 16 seconds (Figure 42).

The Desmos animation also supported Hans in developing fluency in applying an average rate of change in a context and connecting his mathematical expressions with the quantities he attempted to represent. Similar to Scott, Hans was perturbed when the runner instantly jumped back to the starting line, which prompted him to reflect on what quantities his symbols represented. Upon seeing the runner jump back to the starting point, Hans realized that he needed the distance traveled by the runner during the first 16 seconds of the race, and this led to a conversation (Table 26) that supported Hans in making adjustments to accurately represent the quantitative relationships he conceptualized.

Foundational Understanding 3 – Average Rate of Change in a Small Interval

The third key idea is that we can obtain better approximations of a function's rate of change at a given value of the input variable by finding average rates of change on smaller and smaller intervals that include that value of the input variable. In the third teaching session, the students programmed the average speed runner using more and more time intervals. They could see in the animation that as they chose smaller and smaller time intervals, their programmed runner's motion aligned more and more with the original runner's motion. An important nuance of the last portion of the teaching session was that students had the agency to determine how many intervals to use (in other words, they could choose how small Δx could be). Both students mentioned they could improve their approximations by making the time intervals as small as they wanted.

Summary of Session 4

Teaching session 4 (Table 27) involves a similar problem to the Runner Task, where a student is prompted to model the motion of an object (this time without deliberate priming to use average rates of change). This session also included questions on how they might determine the speed at a particular time value and approximate future values of a function using that speed.

Table 27: Session 4 Tasks and Rationale

<table>
<tr><th colspan="3">Session 4: Average Rate of Change Over Smaller and Smaller Intervals – The Camera Problem</th></tr>
<tr><th>Order of Instruction</th><th>Task</th><th>Task Rationale</th></tr>
<tr><td>4a</td><td>Suppose you are the engineer on a film set. The director wants to shoot a scene where a car moves along a road and wants the camera to run alongside the car. The director asks you to program the track that the camera will run on. The only information you have access to is that you know exactly how far the car has traveled after any amount of time since the scene started.

Student will use https://www.desmos.com/calculator/u7ldxbssf2] to answer the problem

Depending on how the student approaches this, the instructor can prompt questions (What approach do you think will work? What information are you trying to represent? Is this similar to the previous questions?) The instructor can also provide the Desmos framework that was provided in the previous tasks. It is omitted in this one to not push the student to make the association that the tasks are similar.</td><td>This task aims to have students repeatably reason about modeling motion using average rates of change by applying the methods they used in the previous tasks to this task. The Desmos interface in this task does not provide the built-in interface to program piecewise linear functions that was in the previous task. Instead, the student will be prompted to discuss what approach they might take to model the camera's movement and how they would employ mathematics to do so. Based on their response, the instructor can assist the student in providing the previous interface to build piecewise linear functions, or explain the commands available in Desmos.

Again, I will assess if students reason that using an average rate of change over small intervals will be essentially equal to the actual speed of the car.</td></tr>
<tr><td colspan="3">Summary of the Camera Problem: The purpose of this task is to have students continually utilize average rates of change and to consider that average rate of change in a small enough interval is good enough to model speed. Without bringing up the formal limit definition of derivative, $\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}$, students will be imagining the variation in x as becoming smaller and smaller, and coordinating the value of the average rate of change of the function over that interval.</td></tr>
<tr><td>4b</td><td>[Students will use the function they made for the previous task]
For the camera, how much distance would they travel between (pick two values, the values depend on how they defined their function. Have the interval be in the middle of the endpoints on an interval they chose)

How much does the car travel in the same time period? Are you surprised by your answers?

Repeat with different intervals.</td><td>This part of the task is to have students use an average rate of change to determine how a quantity would change by utilizing the value of the rate of change.

The purpose of this portion of the task is to have students numerically compare how determining the car's average speed over a small interval can be used to determine the car's accumulated distance. I will investigate if students are able to represent a projected change in distanced traveled by utilizing an imagined constant rate of change over a given input interval.

When implementing this task I will repeatedly explore if a student</td></tr>
</table>

		recognizes that using an average rate of change over small intervals will be *essentially equal* to the actual speed of the car.
4c	Students will fill out a table that they will pick the interval size. What would you need to do if you wanted the error between them to be less than 0.01? 0.003? [Repeat as needed]	The Desmos applet in this portion of the task automates the process from the previous problem for the student. All the student has to do is decide an interval size, and they can compare the approximated distance traveled with the actual distance traveled. This allows the student to focus on reasoning about the results of the calculations instead of having to calculate them. I will explore whether a student recognizes that they can get better and better approximations because they can choose interval sizes as small as they want ($\lim_{\Delta x \to 0}$).
4d	The film director wants to make sure that the car is not going too fast for legal reasons. She thinks that at the end of the scene (t=30) that it might be too fast. She asks you "at that time what speed is the car traveling at?" What would you do if she asked for the speed at t=10? At t=2.98? If the director gave you a particular time that she wanted to know the speed for, describe what you would do to determine that for her.	This task requires students to use and compute the average speed to internalize how speed at a point is computed. By repeatedly reasoning about it, the student can generalize the process, which will allow them not to have to go through the process to determine the speed. Instead, the student will then have the basis for building a rate of change function.
Summary of Task 6: The two major points of this task is for students to reason in the following ways • Reason about an *Average Rate of Change* as imagining how two quantities would change together at a constant rate (if we wanted them to start and end at the same values) • Reason about *Speed at a Point* as an average rate of change in a small enough interval and that this process is repeatable for other input values.		

Teaching Session 4 – Scott

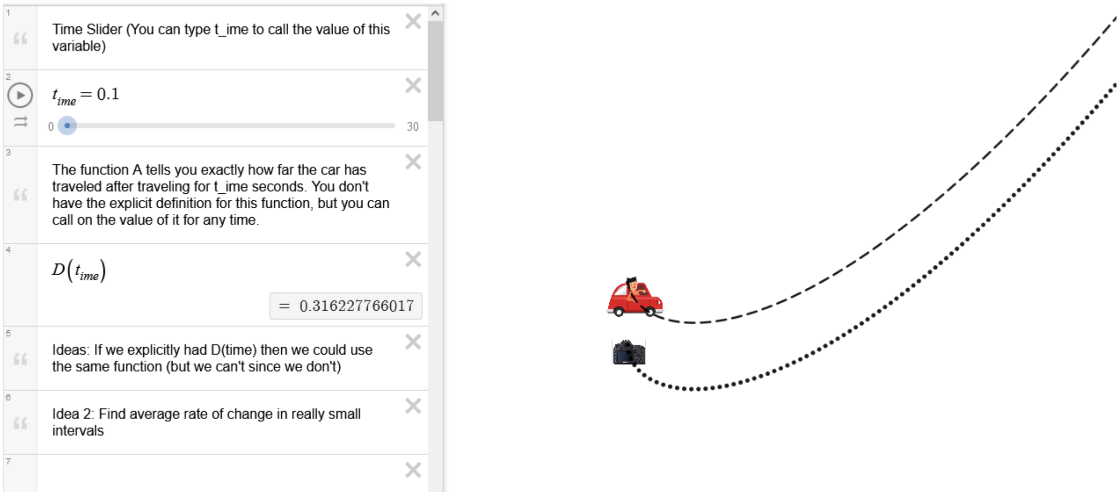

Figure 43: The Camera Problem and Scott's initial ideas

The camera problem prompted students to program a camera's movement on a track to mirror the movement of a car on a parallel track. Scott initially stated that he wanted to determine the equation of the line that would create the same path the camera was on[15]. However, he realized that having the function whose graph matched the physical shape of the track path would not model how the camera's distance traveled along the track as time varied. Scott then suggested using the given distance function of the car, D, but then recognized he could not do this since he did not have the function definition for D (Figure 43). Scott then decided that we could "find the average rate of change in really small intervals where it would be pretty close but not exact." As Scott started to define a function, C, where he used the car's average speed and multiplied it by an amount of time, he verbalized that he wanted to use the same method as the previous

[15] In other words, Scott wanted a function where the physical shape of that function's graph matched the shape of the track he saw on the Desmos applet.

session (the Runner Task)[16]. The interviewer then provided the prior interface to aid Scott in defining piecewise linear functions. Scott’s choice to utilize average rates of change is important to note since the questions that accompanied the Desmos applet did not have any prompts to define piecewise linear functions or use average rates of change to model the camera’s motion. Scott decided to break up the 30-second time interval into one-second intervals (Figure 44). As he defined his piecewise linear function, he checked his work by playing the animation to ensure that the camera moved according to his expectation.

$t_{IntNumber}$	m	d_{car}
1	$\frac{D(1)-D(0)}{1}$	0
2	$\frac{D(2)-D(1)}{1}$	$D(1)$
3	$\frac{D(3)-D(2)}{1}$	$D(2)$
4	$\frac{D(4)-D(3)}{1}$	$D(3)$
5	$\frac{D(5)-D(4)}{1}$	$D(4)$
6	$\frac{D(6)-D(5)}{1}$	$D(5)$
7	$\frac{D(7)-D(6)}{1}$	$D(6)$
	19 more rows Show all	
27	$\frac{D(27)-D(26)}{1}$	$D(26)$
28	$\frac{D(28)-D(27)}{1}$	$D(27)$
29	$\frac{D(29)-D(28)}{1}$	$D(28)$
30	$\frac{D(30)-D(29)}{1}$	$D(29)$

Figure 44: Scott programming the camera's distance traveled over time using the car's average speed in each one-second interval

[16] Scott stated that he wanted to use the previous interface, since the previous interface was easier to navigate through. Desmos does not have a user-friendly interface for programming a piecewise function, and the interface from the Runner task removes having to deal with Desmos coding syntax when creating piecewise functions.

After programming the function to model the camera's distance traveled along the track with respect to time elapsed, Scott was asked a series of questions on determining the car's speed at a particular value of time (Figure 45). Scott stated that he would "find the change in distance between 2 really small intervals of time and divide it by the change in time." He later generalized this statement as "I would just take 2 values very close to that time, and I would calculate the rate of change between them." There are several aspects of Scott's worth response worth highlighting:

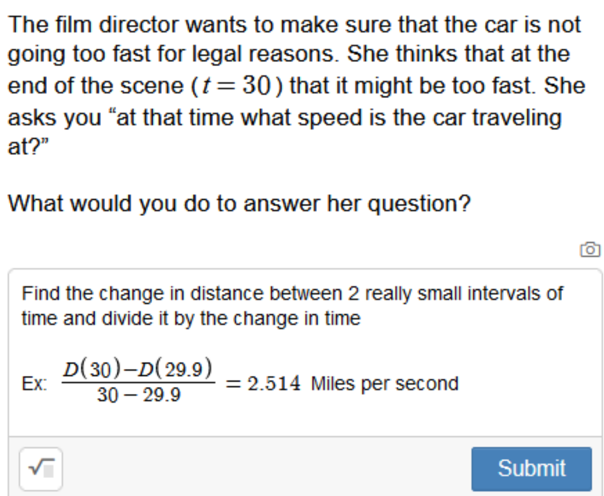

Figure 45: Scott's response to finding the speed of the car at a given input value

1. The sizes of the time intervals he chose to determine the speed at $t = 30$ were different from the size he used to program the camera's motion. For programming the camera's distance traveled, he used the car's average speed in 1-second intervals, and for the speed at $t = 30$, he used the car's average speed in a 0.1-second interval. This suggests that Scott's choice in the "2 values very close to that time" was arbitrary and that there was no specific size of an interval needed.
2. During the start of this session, Scott verbalized that using average rates of change over a small time interval "would be pretty close but not exact" to modeling the

car’s motion as time passed. As he answered each of the prompts about determining the car's speed at a particular time, he never verbalized a need to justify that they were close approximations. It is likely that because the animation displayed the motion of the camera and the car was essentially the same, he might have believed that his approximations were good enough and that the difference between the actual and the approximated distance functions was insignificant.

3. Compared to the first teaching session, Scott had no issue talking about a change in distance without calculating it. He comfortably leveraged function notation to represent a change in distance in modeling the camera’s distance traveled (Figure 44) and when he discussed how to calculate the speed at a given time value (Figure 45). This suggests that Scott could consistently conceptualize a variation as a distinct quantity.

<u>Teaching Session 4 – Hans</u>

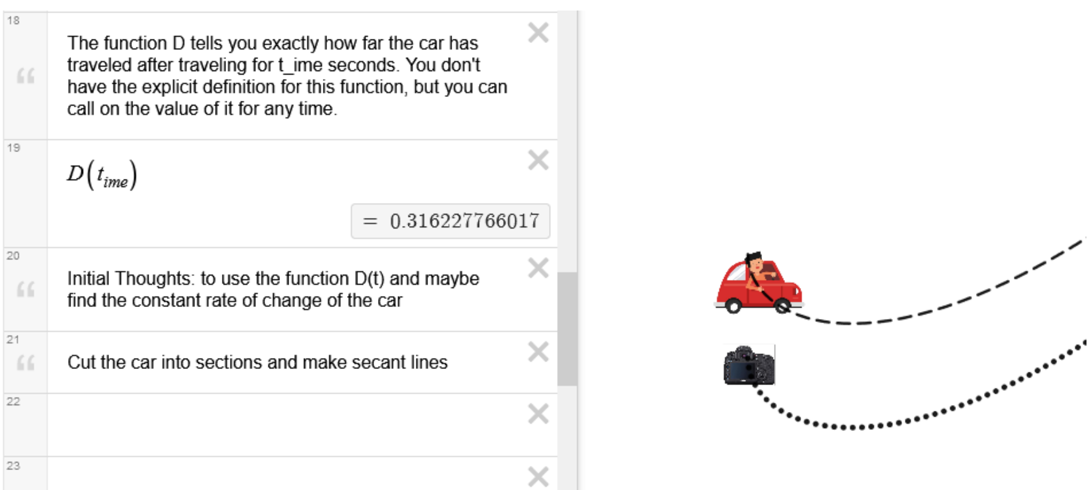

Figure 46: The Camera Problem and Hans' Initial Ideas

Like Scott, Hans’ initial approach to the camera problem was to determine the equation of the line whose graph matched the path of the road. He then suggested finding “the constant rate of change of the car.” However, after playing the animation, Hans recognized that the car was moving at varying speeds, so he proposed that we “cut the car into sections and make secant lines” (Figure 46). Hans verbalized that this was similar to the Runner Task, and he wanted to build a piecewise linear function using the car’s average speed over small time intervals to model the camera’s motion.

$t_{IntNumber}$	m	d_{car}
1	$\frac{(D(3)-D(0))}{3-0}$	0
2	$\frac{D(6)-D(3)}{6-3}$	$D(3)$
3	$\frac{D(9)-D(6)}{9-6}$	$D(6)$
4	$\frac{D(12)-D(9)}{12-9}$	$D(9)$
5	$\frac{D(15)-D(12)}{15-12}$	$D(12)$
6	$\frac{D(18)-D(15)}{18-15}$	$D(15)$
7	$\frac{D(21)-D(18)}{21-18}$	$D(18)$
8	$\frac{D(24)-D(21)}{24-21}$	$D(21)$
9	$\frac{D(27)-D(24)}{27-24}$	$D(24)$
10	$\frac{D(30)-D(27)}{30-27}$	$D(27)$

Figure 47: Hans’ programming the camera's distance traveled over time using the car's average speed in each three-second interval

Hans chose to employ 3-second intervals (Figure 47) and did not play the animation until he finished defining the piecewise function. After playing the animation, he noticed that the camera and the car’s movements did not align adequately in the first few seconds. He then suggested that “we could do better if I made it like 1 second [the interval size] instead.” Rather than having Hans rewrite his piecewise function, the interviewer asked hypothetical questions such as “suppose we wanted to do 1-second intervals, what would we need to change?”. Hans replied that he would have to change everything since “you wouldn’t be able to use like 3 seconds…like how these are done. I would have to re-do them to resize these rates of change." After this round of questioning, Hans demonstrated that he could obtain better approximations of the car's distance traveled with respect to time elapsed by using average rates of change over a smaller time interval.

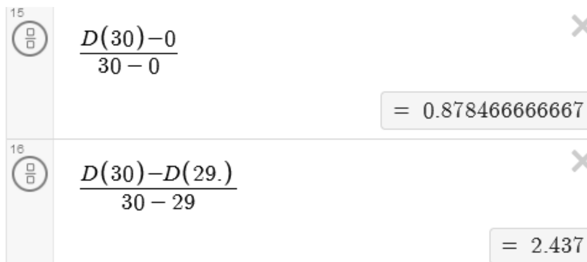

Figure 48: Hans moving from $\frac{D(30)}{30}$ *to* $\frac{D(30)-D(29)}{30-29}$

The interviewer then asked Hans how he might determine the speed at $t = 30$. Hans initially stated "$\frac{D(30)}{30}$ since it is the rate of change for 30… err… it's just a rate, not a rate of change." He later clarified that $\frac{D(30)}{30}$ "is a rate [and not a rate of change] because it remains a constant value of 0.878 like it would be the constant rate at 30." The interviewer then wrote $\frac{D(30)}{30}$ as $\frac{D(30)-0}{30-0}$ and asked what $\frac{D(30)-0}{30-0}$ would represent in this context. Hans identified that $\frac{D(30)-0}{30-0}$ would be the car's average speed between 0 and 30 seconds, and then quickly realized that $\frac{D(30)}{30}$ would not be the car's speed at 30 seconds. Afterward, he suggested we could determine the average speed between 29 and 30 seconds since "it's in the region that we want it in like 30, so it's close enough to the speed" (Figure 48). On the other tasks that prompted Hans to determine the speed at a given time value, Hans' responses indicated that he would determine the average rate of change over a small interval near the requested time (Figure 49).

There are several aspects of Hans' responses that are important to discuss.

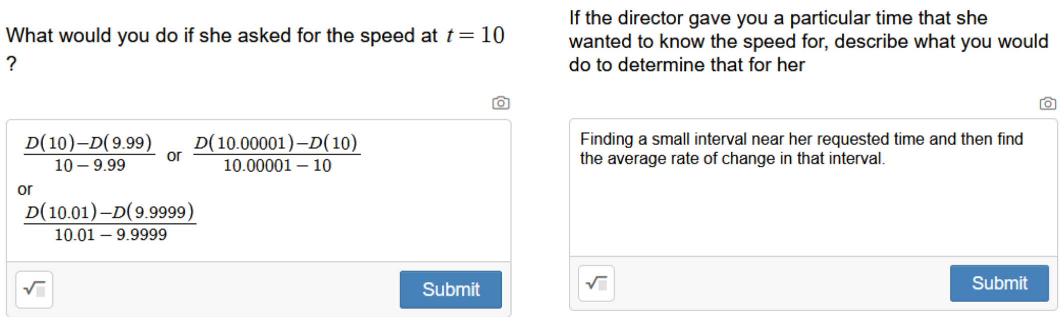

Figure 47: Hans' explanation of how to find a speed at a given input value

1. Hans thought about $\frac{D(30)}{30}$ as a "rate not a rate of change." In his pre-interview, his conception of rate was as "different ratio, same proportion" and would set up equivalent fractions. His expression, $\frac{D(30)}{30}$, suggested that a fraction with a singular distance divided by a singular time would determine a "rate," and something of the form $\frac{y_2-y_1}{x_2-x_1}$ would determine a "rate of change" since the numerator and denominator represented a change in one quantity's value. It was likely through reconceiving $\frac{D(30)}{30}$ as being equivalent to $\frac{D(30)-D(0)}{30-0}$ (which he identified as the car's average speed between 0 and 30 seconds) that supported Hans in conceiving $\frac{D(30)}{30}$ as the car's average speed over 30 seconds.
2. Another possibility for his response of $\frac{D(30)}{30}$ as the car's speed at 30 seconds is that he considered that a rate was "distance over time," and so he used the distance at 30 seconds, $D(30)$ and divided it by the time value of 30 seconds. In either case, what supported Hans in shifting his conception of $\frac{D(30)}{30}$ was the prompting to explain what each portion of his expression represented. Additionally, writing an equivalent expression in the form that he was familiar

with (e.g., recognizing that $\frac{D(30)-D(0)}{30-0}$ would represent an average speed) aided him in reflecting on the quantities he attempted to represent.

3. Like Scott, Hans also demonstrated an understanding of the arbitrariness of choosing how small an interval had to be when determining the speed at a given input value. When determining how he might determine the speed at $t = 10$, Hans initially wrote $\frac{D(10)-D(9.99)}{10-9.99}$ but also noted that he could choose other sized intervals like the subsequent ones he wrote in Figure 49.

Foundational Understanding 4 – Instantaneous Rate of Change

The final waypoint involves encapsulating the process of calculating average rates of change in arbitrarily small intervals into a single expression. This is essentially the limit definition of derivative, $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)} = f'(x)$. The next step in conceiving of a quantity we call "instantaneous rate of change" is to interpret the rate value as a constant rate of change over a small enough input interval such that imagining the output quantity's value changing at this constant rate of change will be *imperceptibility different* from how the output quantity's value will actually vary.

Teaching Session 5 (Table 28) introduces derivative notation and has students practice interpreting and utilizing the value of an instantaneous rate of change in an applied context. This task prompts students to compare changes in a function's output value with approximated changes using a derivative value (linear approximation) and a zooming-in task to display that most continuous functions behave linearly in a small enough input interval.

Table 28: Session 5 Tasks and Rationale

Session 5: Instantaneous Rate of Change		
Order of Instruction	**Task**	**Task Rationale**
5a	It appears that if we know about how much of a quantity we have at all times (distance the car has traveled),$D(t)$, we can determine how fast that quantity is changing at all times. Given a value t we could find an average rate of change in a small enough interval, and that was our "speed at a point." Since we can do this for any t value, let us call the "speed of the car at time t" as $D(t)$. This function will be called the *derivative of* D, which is "how fast the distance the car has traveled is changing" or "*the instantaneous rate of change of distance* with respect to time." [The student will use a Desmos activity] Find the speed of the car at $t = 2.3$. Then type "$D'(2.3)$" what do you notice? Use the value you found to estimate what the distance traveled by the car would be for $t = 2.5$. Type "D(2.5)" to check your answer, what do you notice? Are you surprised? Repeat for other values	This part of the task is to introduce the derivative notation s'. Just as students have encapsulated the process of a function mapping an input to an output, the students can then encapsulate the previous process they engaged in.
5b	[Students will be using https://www.desmos.com/calculator/nidqma1dya] Let us look at the graph of a function f. Suppose we want to know how fast the output value of the function changes at a particular input value. (Student will be given the option to choose an input value of their choice, and then follow the instructions on the Desmos activity where they will zoom in until the graph looks linear) (Student will then be instructed to find the value of f' at the associated input value) What does this value tell you? (Students can move the slider pick a small change in x and the associated change in the value of the function) How much has the output changed given the change in your input? Are you surprised? What does the value of the derivative at the input tell you? Repeat with other input values.	This portion of the task is adapted from works of Tall (2009, 2013) and Ely and Samuels (2019) where students will be "zooming in" to see that the function is *essentially* changing at a constant rate of change, and the value of the derivative is this constant rate of change. I will explore if a student demonstrates an understanding that a function having an instantaneous rate of change of value $f'(a)$ at the input value a means that for small variations in the input quantity, Δx, the variation in the output value of the function, Δf will *essentially* be $\Delta f = f'(a) * \Delta x$. I will also investigate what aspects of this Desmos

		activity the student uses to express this understanding.
Learning Goal for Task 5: Students will interpret the value of a derivative as the value of the *instantaneous rate of change* at a given input value. This means that if they were to look at a small enough interval, the quantity they are imagining is changing at *essentially* a constant rate of change. The understanding of instantaneous rate of change that a student will hopefully construct is “A function having an instantaneous rate of change of value $f'(a)$ at the input value a means that for small variations in the input quantity, Δx, the variation in the output value of the function, Δf will *essentially* be $\Delta f = f'(a) * \Delta x$”		

Teaching Session 5 – Scott[17]

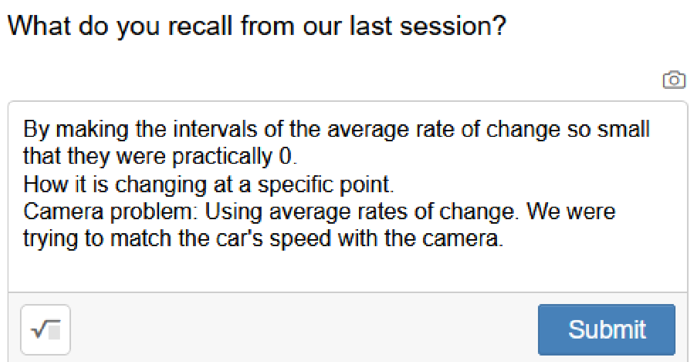

Figure 50: Scott's recollection of Session 4

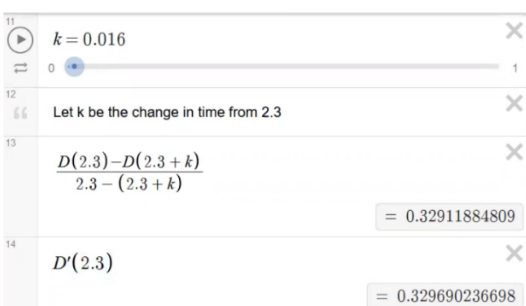

Figure 51a: Desmos Applet displaying the average rate of change near $t = 2.3$ and the instantaneous rate of change at $t = 2.3$

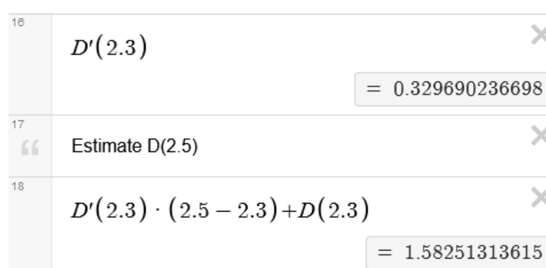

Figure 51b: Scott using the value of the instantaneous rate of change to estimate a future output value of the function.

The teaching session began with Scott recalling how he approached the camera problem. Scott recalled that he made “the intervals of the average rate of change so small

[17] Hans and Scott had a very similar experience with Teaching Session 5. For the purpose of brevity, only Scott’s is discussed here.

they were practically 0"[18] and that the goal of the camera problem was "to match the car's speed with the camera" (Figure 50). The interviewer then introduced derivative notation to explain that $D'(t)$ would represent "the speed of the car at the time, t, or *the instantaneous rate of change* of distance with respect to time at time t." The interviewer used the Desmos applet to demonstrate how calculating average rates of change around $t = 2.3$ would converge to the value of $D'(2.3)$ (Figure 51a). Scott was then prompted to estimate $D(2.5)$ and he wrote $D'(2.3) * (2.5 - 2.3)$ because "that $[D'(2.3)]$ is the rate, and we have our change in time, so I multiplied rate times time to get a change in distance." Then he added $D(2.3)$ to his expression to represent "the distance we already traveled by 2.3" (Figure 51b). In a later task, Scott was asked to interpret the value of a derivative, and he employed relative size language such as "times as large as" when describing the value of a rate of change (Figure 52). His interpretation demonstrated his understanding of instantaneous rate of change which is described in the following table.

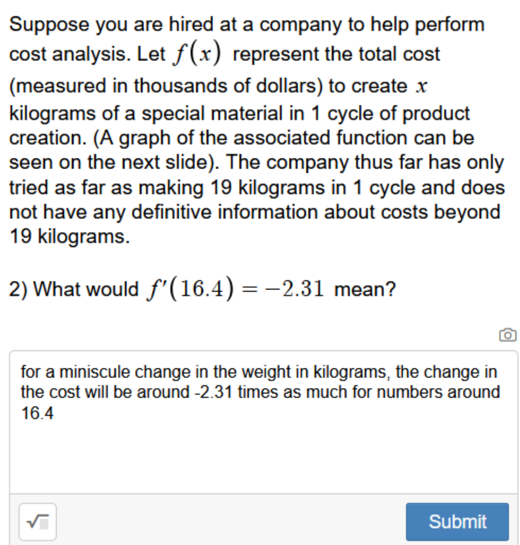

Figure 52: Scott's explanation for the derivative at a given input value

[18] Scott clarified that "practically 0" referred to the size of the input interval not the value of the average rates of change.

I. Scott attended to “ a minuscule change” in the input quantity ($\lim_{\Delta x \to 0}$), which suggested that he was aware that an instantaneous rate of change involves small amounts of changes between two quantities’ values.

II. His interpretation of a rate entailed that for some change in the input quantity, the output quantity would be $f'(16.4)$ “times as much” ($\Delta f = f'(a) * \Delta x$). Moreover, Scott’s attention to the input and output quantities (weight and cost of the special material) suggested that Scott’s interpretation of a rate of change entailed distinguishing which two quantities were covarying together.

III. He qualified his statement with “the change in the cost will be *around* -2.31 times as much *for numbers around* 16.4”. His statement demonstrated his understanding that using the value of an instantaneous rate of change over a small enough interval would approximate the actual amount of change in the output quantity so long as we were sufficiently close to the specified input value.

Through the retrospective analysis of this session, Scott appeared to have developed a strong foundation for understanding instantaneous rate of change as an average rate of change over an arbitrary small input interval of a function’s domain. His recollection of the camera problem from Session 4 indicated that he could use the car’s average rate of change over a small interval to program the camera to match the car’s motion (Figure 50). In Session 4, Scott verbalized an awareness that an average speed over a small time interval “would be pretty close but not exact” to the car’s actual speed in that time interval. In this session, Scott described $f'(16.4)$ as how the value of the output quantity would essentially vary for minuscule variations in the input quantity around the input value of 16.4 (Figure 52). This evidenced a stabilization of his understanding that the value of the derivative for a specific value of the input quantity is essentially equal to an average rate of change over a small enough input interval containing that input value. Lastly, his actions indicated that his interpretation of the value of a rate of change (whether it was a constant, average, or in this case, an instantaneous rate of change) entailed the relative size of the change in two quantities as the quantities' values change together.

DISCUSSION

Overall, the combination of prompting students to reason about quantities and the usage of the Desmos applets supported these students in advancing their understanding of instantaneous rate of change as an average rate of change over an arbitrarily small input interval. The interviewer prompted the subjects to conceptualize that a number quantified or measured something about a given context. For example, in the first teaching session, both students were perturbed when they had to reconcile their written definition for a constant speed of 1.7 meters per second with an example of someone walking 1.7 meters in 1 second but at a varying speed. For both students, puzzling about what the value of a constant speed represented prompted them to reflect on their meanings for constant rate of change. Additionally, attending to the quantities in each context supported the students in representing these quantities with appropriate mathematical symbolization and distinguishing between a quantity and a variation in that quantity's value (Sessions 2 and 3). As the two students advanced their understanding of constant and average rate of change, they were able to construct a quantitative understanding of instantaneous rate of change (Sessions 4 and 5).

The Desmos applet played a pivotal role in providing a visual representation of the problem context and immediate feedback about the quantities each student was trying to represent with their mathematical expressions. The following list outlines the aspects of the Desmos applet that I consider as crucial for helping these students construct their meaning for instantaneous rate of change.

1. The Desmos applet provided a visual aid to enable students to imagine quantities' values varying instead of only making static comparisons between quantities. For Scott, he initially fixated on calculating the value of the variation in a quantity. However, the ability to vary the value in the variation in the input quantity through a slider on the Desmos applet prompted Scott to reconsider his conception of a difference outside of calculating its value. For Hans, he initially conflated an arbitrary change in time with 1 second of time. It was not until he envisioned the value of the variation in time as varying (by seeing in the Desmos applet the length of the horizontal line as varying) that he conceived of an arbitrary change in time as a unit of measure (Session 1).
2. The Desmos applet displayed a visual representation of the quantities in each student's mathematical expressions in the Runner and Camera problems (Sessions 2-4). When the animation displayed a quantitative relationship that differed from their expectations, both students appeared perturbed by what they saw. This was followed by both students examining their symbols and comparing them with what was displayed in the animation. For example, when constructing a function to model Jonah's motion in the Runner Task (Session 3), both students failed to include the distance traveled during the first 16 seconds of the race. This resulted

in the runner instantly moving back to the beginning of the race after 16 seconds had elapsed. This perturbed both students and provided the basis for discussing what they attempted to represent and distinguishing between how the runner's distance would vary as time varied versus the runner's total distance traveled.

3. As each student progressed through the Runner Task (using an increasing number of time intervals), they noticed in the Desmos animation that the motion of their programmed runner aligned more and more with the motion of the runner they were trying to model. In the Camera Task (Session 4), students spontaneously chose to use average rates of change over small time intervals to model the camera's distance traveled with respect to time elapsed. Both students verbalized that they could choose smaller-sized input intervals if they did not see the camera and car's movements aligned in the animation. As the student progressed in completing the tasks in the Desmos applets, they made advances in conceptualizing "the average rate of change in really small intervals where it would be pretty close (to the actual rate of change) but not exact."

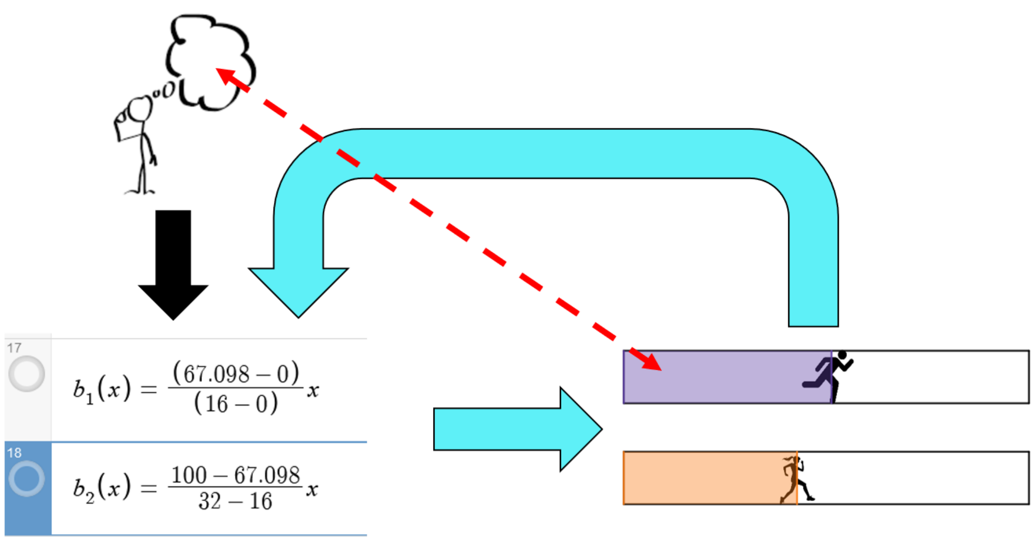

Figure 53: Overall Flow of the Discussions that led to Shifts in Student Thinking

In each teaching session, the Desmos applets played a significant role in the students' conceptualization of the quantities in each problem context and their construction of the symbols that were personally meaningful to them. Figure 53 displays how I leveraged Desmos to support a conversation to support shifts in student thinking.

Figure 54a: Students have an image of the situation in their mind

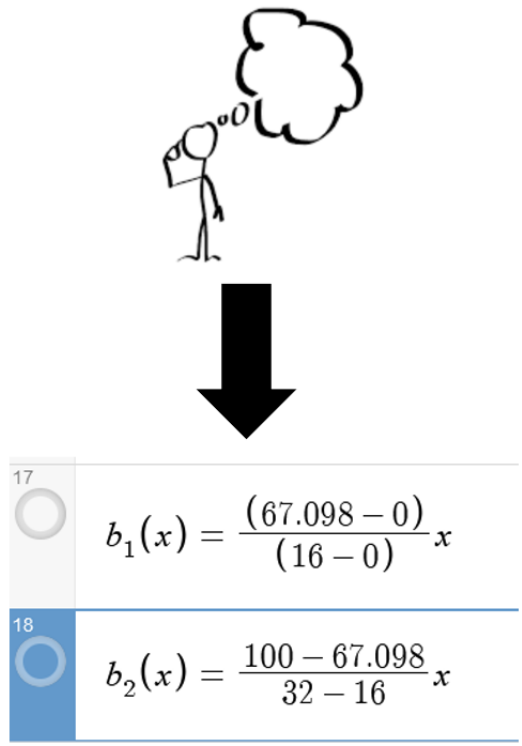

Figure 54b: A student's mathematical symbolization of a quantity within that situation

In each teaching session, the tasks were situated within a context (e.g., The Runner Task, The Camera Problem, etc.). Initially, a student has an image of the problem context (Figure 54a) and some mathematical symbols that they believe represent a quantity or quantities within that situation (Figure 54b). One role that the applets played in these teaching sessions was to use each student's symbols and display what their mathematics would represent in that situation (Figure 55). Before the animation plays, a student already has an expectation of what they would see in that animation because of

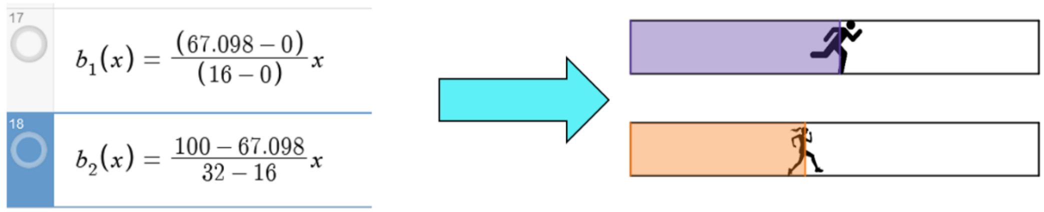

Figure 55:Desmos using each students' symbols to display what would happen

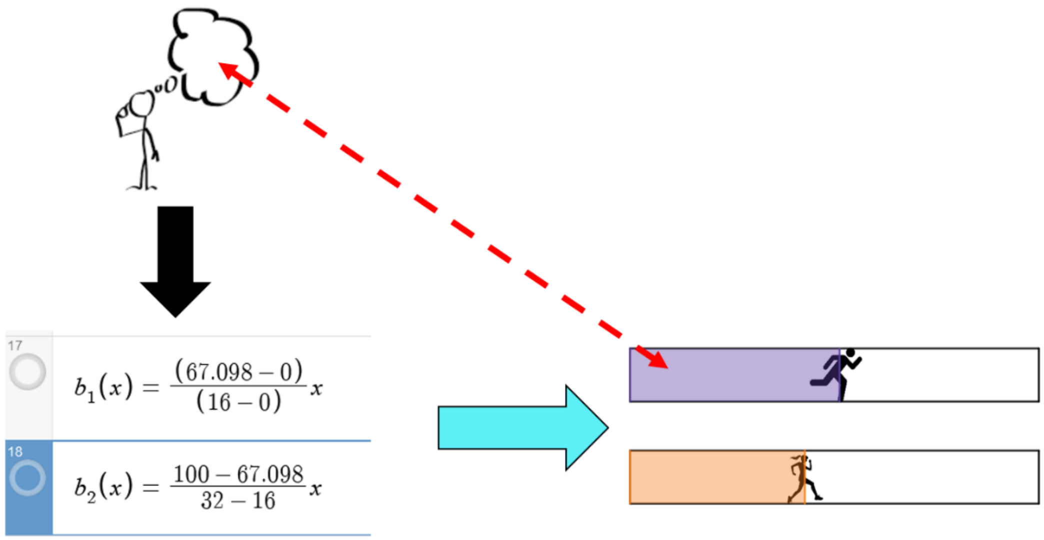

Figure 56:A student anticipates what they might see

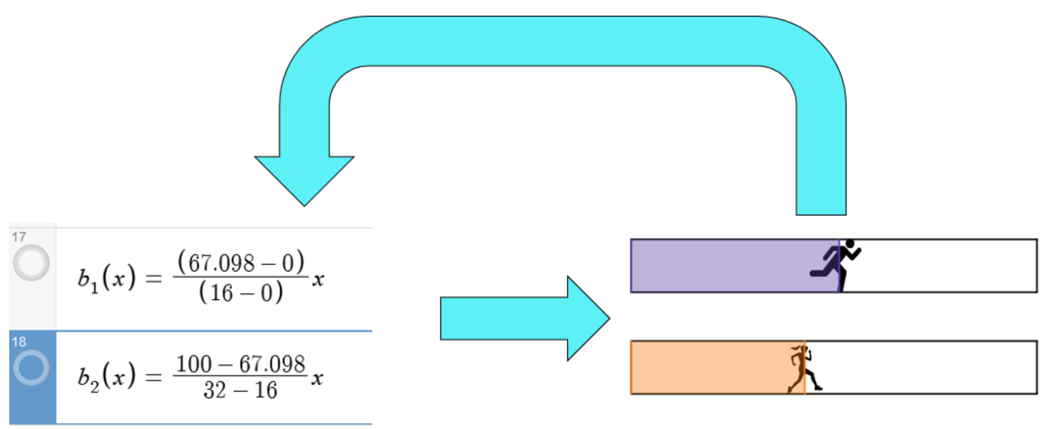

Figure 57:Students reflect on what they initially symbolized

their initial image of the situation (Figure 56). However, if what each student saw in the animation did not align with what they imagined, they then experienced a perturbation.

This perturbation allowed students to reflect on what they initially symbolized (Figure 57). The instructor used this opportunity to facilitate a conversation where the instructor posed questions to aid the student in reasoning about quantities and how they would represent those quantities. This discussion led students to reconsider what they had symbolized and how to make refinements to represent the quantities they had in mind.

According to Radical Constructivists, students need to experience perturbations to catalyze self-reflection on their actions. In this teaching experiment, the Desmos applets enabled opportunities for students to experience these perturbations. The results of this study suggest how instructors can leverage applets and animations to help students reason about quantities and how to mathematical symbolize those quantities.

Limitations and Future Directions

While the results of this teaching experiment suggest the benefits of an instructional intervention on rate of change, the limited sample size does not provide a basis for extending these findings to every student. The conclusions can only be known to hold true for the students in this sample. However, during the pre-interview and first teaching session, both students demonstrated a weak understanding of constant rate of change and inexperience in reasoning quantitatively. These issues are reminiscent of what the students from the first study exhibited and were likely the sources of preventing these students from articulating a productive understanding of instantaneous rate of change (see Paper 1). Therefore, it stands to reason that an instructional intervention focused on developing students' meanings for rate of change can support students in developing a productive understanding of instantaneous rate of change. One avenue for future research is to explore implementing the tasks in this teaching experiment in a classroom setting.

Another limitation is that almost all of the tasks involved relative motion where time was the input quantity. One future direction is to develop items that involve rates of change where time is not the independent quantity and investigate how students may reason about quantities varying where time is parametrized in the context. While Jones and Watson (2018) evidenced the benefits of reasoning about derivatives in multiple applied and representational contexts, this study's findings suggest that students should also reason about rate of change in different representational and applied contexts. The idea of rate of change is used to describe quantitative relationships in many applied and mathematical contexts. However, researchers have reported that many students leave Calculus with weak meanings for rate of change (Flynn et al., 2018; Castillo-Garsow, 2010; Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006) and struggle to apply their understanding of derivative outside of kinematic situations (Marrongelle, 2004; Ibrahim & Robello, 2012; Jones, 2017). Therefore, students would benefit from developing a coherent and robust understanding of derivatives that they can leverage in a variety of contexts.

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APPENDIX A

CONSENT FORM FOR PRE-INTERVIEWS

CONSENT FORM FOR PRE-INTERVIEWS

Title of research study: An Instructional Sequence on The Learning of Instantaneous Rate of Change

Investigator: *Franklin Yu*

Why am I being invited to take part in a research study?

We invite you to take part in a research study because you are enrolled in Pre-Calculus,18 years or older, and are willing to participate.

Why is this research being done?

We would like to study ways in which course materials can be developed to support students in learning meaningful mathematics.

How long will the research last?

We expect that individuals will spend 1 to 1.5 hours participating in a pre-interview and if selected, an additional 8-10 hours over 3 weeks.

How many people will be studied?

We expect about 10 people will participate in the pre-interviews and 3 to participate in the instructional sequence.

What happens if I say yes, I want to be in this research?

You will meet with the investigator and participate in 1 pre-***interview session*** and if selected, an additional 8-10 one to two hour sessions. You are free to decide whether you wish to participate in this study. You will be compensated $15 per hour for this research.

What happens if I say yes, but I change my mind later?

You can leave the research at any time; it will not be held against you.

Is there any way being in this study could be bad for me?

Your participation will entail working through course materials and discussing your thinking aloud. Your participation will be video recorded.

Will being in this study help me in any way?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include developing a deeper understanding of the mathematics involved in calculus.

What happens to the information collected for the research?

Copies of your recorded interviews will be stored on a password protected folder on a personal laptop (a cloud storage service). Any physical data will be stored in a locked filing cabinet/desk drawer in a locked office. We will analyze the data to try to describe how your mathematical meanings changed over the duration of the intervention. The data may be used in reports or publications. Raw video footage, however, may be shown during research presentations.

Who can I talk to?

If you have questions, concerns, or complaints, talk to the research team at fhyu1@asu.edu or Marilyn.carlson@asu.edu.

This research has been reviewed and approved by the Social Behavioral IRB. You may talk to them at (480) 965-6788 or by email at research.integrity@asu.edu if:

- Your questions, concerns, or complaints are not being answered by the research team.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You have questions about your rights as a research participant.
- You want to get information or provide input about this research.

Signature Block for Capable Adult

Your signature documents your permission to take part in this research.

Signature of participant Date

Printed name of participant

Signature of person obtaining consent Date

APPENDIX B

ASU IRB APPROVAL

ASU IRB APPROVAL

ASU Knowledge Enterprise Development

EXEMPTION GRANTED

Marilyn Carlson
CLAS-NS: Mathematics and Statistical Sciences, School of (SMSS)
480/452-7435
MARILYN.CARLSON@asu.edu

Dear Marilyn Carlson:

On 7/21/2021 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	A Dissertation Study on the Learning of Instantaneous Rate of Change
Investigator:	Marilyn Carlson
IRB ID:	STUDY00014124
Funding:	Name: Mathematics and Statistical Sciences, School of - Graduate Students
Grant Title:	
Grant ID:	
Documents Reviewed:	• IRB Social Behavioral Protocol Franklin Yu.docx, Category: IRB Protocol; • recruitment_methods_consentform$600+_19-07-2021.pdf, Category: Consent Form; • recruitment_methods_consentform_19-07-2021.pdf, Category: Consent Form; • recruitment_methods_email_17-06-2021.pdf, Category: Recruitment Materials; • recruitment_methods_surveyconsentform_19-07-2021.pdf, Category: Consent Form; • supoorting documents 15-07-2021.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Yu,Franklin Harris_Block Grant Award Signed 2018.pdf, Category: Sponsor Attachment;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings, (2) Tests, surveys, interviews, or observation on 7/21/2021.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

If any changes are made to the study, the IRB must be notified at research.integrity@asu.edu to determine if additional reviews/approvals are required. Changes may include but not limited to revisions to data collection, survey and/or interview questions, and vulnerable populations, etc.

Sincerely,

IRB Administrator

cc: Franklin Yu
Franklin Yu
Marilyn Carlson