Analysis, Estimation, and Control of Partial Differential Equations Using Partial Integral Equation Representation

by

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ABSTRACT

When solving analysis, estimation, and control problems for Partial Differential Equations (PDEs) via computational methods, one must resolve three main challenges: (a) the lack of a universal parametric representation of PDEs; (b) handling unbounded differential operators that appear as parameters; and (c), enforcing auxiliary constraints such as Boundary conditions and continuity conditions.

To address these challenges, an alternative representation of PDEs called the 'Partial Integral Equation' (PIE) representation is proposed in this work. Primarily, the PIE representation alleviates the problem of the lack of a universal parametrization of PDEs since PIEs have, at most, 12 Partial Integral (PI) operators as parameters. Naturally, this also resolves the challenges in handling unbounded operators because PI operators are bounded linear operators. Furthermore, for admissible PDEs, the PIE representation is unique and has no auxiliary constraints — resolving the last of the 3 main challenges.

The PIE representation for a PDE is obtained by finding a unique unitary map from the states of the PIE to the states of the PDE. This map shows a PDE and its associated PIE have equivalent system properties, including well-posedness, internal stability, and I/O behavior. Furthermore, this unique map also allows us to construct a well-defined dual representation that can be used to solve optimal control problems for a PDE.

Using the equivalent PIE representation of a PDE, mathematical and computational tools are developed to solve standard problems in Control theory for PDEs. In particular, problems such as a test for internal stability, Input-to-Output (I/O) L_2-gain, H_∞-optimal state observer design, and H_∞-optimal full state-feedback controller design are solved using convex-optimization and Lyapunov methods for linear PDEs in one spatial dimension. Once the PIE associated with a PDE is obtained,

Lyapunov functions (or storage functions) are parametrized by positive PI operators to obtain a solvable convex formulation of the above-stated control problems. Lastly, the methods proposed here are applied to various PDE systems to demonstrate the application.

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PREFACE

This thesis covers my work in the Cybernetics Systems and Control Lab at Arizona State University under my advisor, Dr. Matthew Peet. All of the work presented here has been previously published or submitted to peer-reviewed academic conferences or journals. The chapters of this thesis can be divided into those containing original research and those containing an overview of existing results. Proofs, when especially long, have been moved to the appendices. To maintain focus on the theoretical contribution, in-depth treatment of various special cases has been omitted when such treatment can be inferred from the previous exposition.

Chapter 1

INTRODUCTION

US Energy Information Agency forecasts indicate that the global electricity energy demand will likely reach 45 trillion kilowatt hours (kWh) as reported by U.S. EIA (2023)– a 45% increase from current electricity demands. With new policies being proposed to reduce carbon emissions and a projected growth of 1.8% per year in electricity generation across all sectors, we will likely not meet future electricity demands unless there is a rapid increase in energy production from clean sources. Hence, many efforts have been directed toward finding new means of energy production, the most notable of which is nuclear fusion. Nuclear fusion is a highly tempting alternative to current methods of energy production because it has no carbon or radioactive byproducts, and the fuel required for the fusion is abundant. Hence, efforts have been directed toward harnessing the energy released during fusion reactions to generate electricity; however, to date, there has been no success.

While the research into controlled fusion in fusion reactors has been ongoing since the early 1930s, we have yet to see a large-scale reactor that can produce a break-even fusion reaction. The most recent development was the break-even fusion reaction conducted by Lawrence Livermore National Laboratory (LLNL) in December 2022, documented by Bishop (2022), that produced an excess of 1 megajoule of energy — a 1.5% gain over the energy put into the lasers that help sustain the nuclear fusion reaction. Needless to say, a fusion reactor built on this principle is not commercially viable since the projected gain in energy has to be over 2000% for the entire system to be self-sufficient and produce excess energy that can be supplied to the grid. More importantly, however, the biggest challenge in building a fusion reactor is confining the plasma in a small space so that a critical temperature is reached and a self-sustained thermonuclear fusion reaction occurs. Confining a plasma at very high temperatures requires the plasma to be suspended in a vacuum to avoid damage to the reactor. Consequently, the pressure required to confine this plasma has to be generated through indirect contact, e.g., using magnetic fields. An example of such a reactor is the Tokamak that was first conceived by Dolgov-Saveliev *et al.* (1958). However, due to the inefficiencies in the reactor's operation, the energy required to confine the plasma through magnetic force is higher than the energy produced through the fusion reaction. Consequently, there is no net energy gain. While the source of the inefficiency can be attributed to many different factors, including model inaccuracies, hardware limitations, inefficient placement of the magnetic coils, etc., there is one particular aspect of the fusion reactor system that is often ignored – **the control laws** used to do stabilize and control the magnetic fields. However, controlling the magnetic field in a Tokamak requires controlling nonlinear vector-valued Partial Differential Equations (PDEs) in 2D or 3D space (depending on the geometry considered and the model approximations).

Consequently, various efforts have been made in recent times to address the control aspect of the fusion problem — See the thesis by Witrant (2015) and the paper by Mechhoud *et al.* (2014), for example. Unfortunately, treating the control design issue of fusion reactors as a challenge specific to an application masks the more fundamental problem in the control of PDE systems. The lack of reliable methods for control of PDEs is also seen in other applications such as vibration control of beam models from Timoshenko (1921), turbulent fluid flow control methods by Alfonsi (2009), and reaction kinetics control by Chakraborty and Balakotaiah (2005) and Christofides and Chow (2002). In all such applications, efficient control techniques for a general PDE model can improve safety and operational costs. For instance: controllers designed using Euler or Timoshenko beam models can suppress seismic and wind disturbances in buildings and bridges (e.g., see designs introduced by Kannan *et al.* (1995), Ikeda (2004), and Fisco and Adeli (2011)) thereby reducing structural damage; controllers for fluid-flow models can reduce drag on aircraft wings (e.g., see the work by Quadrio (2011)) thereby reducing fuel costs; and controllers for reaction-diffusion equations can improve homogeneity (or desired stratification) of concentration and temperature in chemical reactors (e.g., see papers by Mao and Yang (2017) and Chakraborty and Balakotaiah (2005)) thereby optimizing reaction rates. One can refer to the paper by Pesch (2012) for a survey on PDE models in other applications.

Despite the significance of PDEs in the modeling and control of physical systems, the progress in the control of a general PDE model has been lacking compared to the progress in the simulation and analysis of PDEs because only a small portion of research exposition on PDEs focuses on controlling PDEs. Even when an exposition focuses on the control aspect of PDEs, it is tailored to a specific application. For instance, the Backstepping and infinite-dimensional Algebraic Riccati Equation (ARE) approaches for parabolic PDEs were presented by Deutscher (2015) and Hagen (2006), whereas the methods for hyperbolic PDEs were proposed by Krstic and Smyshlyaev (2008) and Moghadam *et al.* (2010). Furthermore, there are even more variations in control approaches when just the boundary conditions are changed while keeping the PDE dynamics the same. Despite these works on the control of PDEs, the methods presented therein are hard to extend or generalize to other systems governed by PDEs.

This does not mean that there do not exist methods that can be applied to a general PDE. Indeed, there exist control approaches that are not tailored for a specific application, e.g., early-lumping methods. However, all such methods approximate the PDE by an Ordinary Differential Equation (ODE) at some stage of the control design process – making it difficult to control the higher-frequency modes of the PDE solution, which is critical in systems sensitive to high-frequency perturbations. Furthermore, since early-lumping methods design the control for an ODE approximation of the PDE do not generally have any accuracy bounds on the solution of the ODE, there may be little or no relation between the solution of the ODE approximation and the solution of the original PDE. Thus, such methods, although generalizable, do not provide any stability guarantee or performance metrics. *Clearly, despite the significance of the PDE control problem and decades of efforts spent to resolve this problem, there is still a lack of good methods that can resolve it*, and this scarcity can be attributed to, as will be shown in this thesis, *the lack of understanding of the PDE models as a whole.*

The primary motivation behind this dissertation and its results is to revive the inquiries into the fundamental approach taken in the analysis, estimation, and control of PDEs and to propose a new approach that provides a different perspective/understanding of the challenges therein. *While the results of this work do not cover the control of nonlinear PDEs and, consequently, control of magnetic fields in a Tokamak, they are intended to be a foundation that advances the state-of-the-art approaches in control of PDEs without restricting to a specific application or a specific PDE model.*

To find a fresh perspective on the analysis and control problems for PDE systems, we first need to identify the characteristics of a PDE that complicate the solution methods of such problems. More specifically, we must identify the characteristics that make general control methods, such as Linear Matrix Inequalities (LMIs) for linear state-space ODEs, difficult/inapplicable for PDEs. Therefore, let us look at the minimal set of characteristics seen in a PDE model.

A brief history of PDEs First introduced during the 17th century in the groundbreaking works by, namely, Newton (1999) and Leibniz and Leibniz (1989), a PDE was typically characterized by a spatially-distributed differential equation used to model various physical phenomena such as heat and mass transfer. However, the central importance of Boundary Conditions (BCs) when defining a PDE model was not formally recognized until the time of Dirichlet whose work has been summarized by Fischer (1994); Also see works by Cajori (1928) and Cheng and Cheng (2005) for an overview of the history of PDEs and BCs. However, even with the inclusion of BCs, a PDE model is not complete without a restriction on the 'continuity' of the solution – spatial derivatives and boundary values must be suitably well-defined. The mathematical formalism for a continuity restriction was only established in the middle of the 20th century by Sergei Sobolev, defining what is now termed a Sobolev space and allowing for generalized functions or distributions to define weak solutions.

When the PDE, BCs, and continuity constraints are combined, we obtain what can be called a *PDE model* – a system defined by three types of constraints, none of which is individually sufficient but which, when combined, yield a well-posed map from an initial state to a unique solution. In the latter half of the 20th century, this map and its continuity properties were formalized and generalized by the notion of a C_0-semigroup, with the BCs and continuity constraints of the PDE system (now including delay systems and PDEs coupled with ODEs) being defined as the 'domain of the infinitesimal generator'; Works by Engel and Nagel (2000) and Curtain and Zwart (1995) provide a thorough introduction to the semigroup theory and its application in control theory. Today, as a consequence of almost 300 years of careful study and mathematical progress, we may conclude that a well-posed PDE model is necessarily defined by **three constraints**: a) the differential equation, or 'PDE', which constrains the spatio-temporal evolution of the solutions inside the domain, c) the continuity condition, which ensures that the solutions have sufficient regularity for the BCs to be well-defined; and c) the BCs, which may constrain the limit values or other properties of the solutions as permitted by the regularity guaranteed by the continuity constraints.

While the representation of spatially distributed systems using the *three-constraint PDE model* has a significant history and is the natural modeling framework, the presence of unbounded operators, continuity constraints, and BCs poses significant challenges to the development of a universal computational framework for analysis, control, and simulation. The recent development of efficient algorithms for optimization on the cone of positive semidefinite matrices has led to the use of Linear Matrix Inequalities (LMI) and Semi-Definite Programming (SDP) in control theory, especially for Linear state-space ODE systems. This has simplified, in terms of mathematical representation and high-level programming, the development of tools for the analysis and control of state-space linear ODEs. Unfortunately, this same simplicity has not yet been extended to PDE systems because the 3-constraint model of a PDE has certain 'undesirable' traits that are not present in state-space linear ODEs, which will be described below using a simple PDE model for demonstration.

A simple PDE model Linear state-space ODE systems have two crucial characteristics that allow the use of LMI-based methods in analysis and control: a) they are parameterized by bounded and algebraic operators, and b) they have no auxiliary constraints. In contrast, PDEs have unbounded operators, differential operators, point values in the form of BCs, and auxiliary continuity constraints — making the analysis and control via LMI-based methods quite tricky and ad hoc. We will discuss below, in detail, how these characteristics of PDEs lead to various challenges and how they are typically addressed by existing methods.

Challenges in Numerical simulation: To illustrate, consider the problem of computing the evolution of a PDE model from a given initial condition. Specifically, consider a simple transport equation $u_t = u_s$ and construct a finite-difference approximation of $u_s = \frac{u(s_{i+1})-u(s_i)}{\Delta_s}$ – yielding an finite-dimensional representation $\dot{x}(t) = \frac{1}{\Delta_s}Ax(t)$, where $x_i = u(s_i)$, $\Delta_s = s_{i+1} - s_i$ is uniform, and A is a bi-diagonal matrix of ± 1 entries. In an ideal simulation, we would desire $\Delta_s \to 0$ – which implies that an ideal ODE representation of the transport equation would have all infinitely large coefficients.

Of course, we can avoid many problems associated with discretization by constructing an explicit basis for the domain of the infinitesimal generator (bases that satisfy the continuity constraints and BCs) and projecting our solution onto this basis – an approach used in Galerkin projection. The problem, however, is that every change in the set of BCs and continuity constraints necessitates a change in the basis functions. Such changes require significant ad hoc analysis – an obstacle to the design of general/universal simulation tools.

Even if we manage to design a tool that includes a large number of variations of the 3-constraint PDE model to minimize the ad hoc steps, many problems involving PDEs remain inaccessible from a numerical perspective. To elaborate further, consider the linear transport equation used to model gas transport through a network of pipelines Baker *et al.* (2021)

$$\partial_t \rho_k + \partial_x \varphi_k = 0, \quad \delta \partial_t \varphi_k + \sigma^2 \partial_x \rho_k = \alpha \rho_k + \beta \varphi_k,$$

where ρ_k stands for the density of the gas and φ_k the flow-rate in a pipe k. If we represent the pipeline network structure using an adjacency matrix $\mathcal{E} = \{e_{ij}\}$ on a set of nodes $\mathcal{N} = \{0, \cdots, n\}$, where e_{ij} is zero if node i and j are not connected, 1 is gas flows from i to j and -1 otherwise, we can specify the boundary conditions as

$$\sum_{j \in \mathcal{N}} e_{ij} \varphi_{k(i,j)}(t, l_i) = 0, \quad \rho_0(t, l_0) = \rho_{in}(t), \quad \varphi_i(t, l_i) = w_i(t),$$

where w_i is the withdrawal of gas at node i, l_i the location of node i, and $k(i,j)$ the index of the pipe connecting nodes i and j. One of the important problems in the gas pipeline infrastructure is the operational cost, which is directly tied to regulating the compressors that control the gas pressure by changing ρ_{in}. Although many algorithms exist that discretize a transport equation to obtain an ODE and simulate given some initial and boundary conditions, they are not scalable because such gas networks typically have a large number of nodes and pipes. Therefore, one cannot use numerical or convex optimization-based methods to solve this PDE and devise a feedback control that can regulate the inlet density, ρ_{in}, under fluctuating withdrawal rates w_i.

Challenges in Computational analysis: Next, consider the problem of computational analysis and control of a PDE model. For simplicity, consider the very stable heat equation $u_t = u_{ss}$ with zero BCs, e.g. $u(t,0) = u_s(t,1) = 0$, and propose an energy metric (Lyapunov function) of the form $V(u) = \int_0^1 u(s)^2 ds$. This energy metric is uniformly decreasing with time – thus proving the stability of the PDE model. The challenge, however, is to use computation to prove this fact. By parametrizing positive operators using positive matrices, optimization-based methods for stability analysis can easily recognize that $V(u) = \langle u, u \rangle_{L_2}$ and hence V is a positive form (i.e., a valid candidate Lyapunov function) —Peet *et al.* (2009) provided a parametrization of such positive forms on L_2. However, the algorithm must also verify that $\dot{V}(u(t)) \leq 0$ for all solutions $u(t) \in W_2$ satisfying the PDE model. Unfortunately, if we differentiate $V(u(t))$ in time along solutions of the PDE model we obtain $\dot{V}(u(t)) = 2\langle u(t), \partial_s^2 u(t)\rangle = 2\int_0^1 u(t,s)u_{ss}(t,s)ds$. Because differentiation is not embedded in a *-algebra, we cannot simply parameterize a cone of positive quadratic forms involving differential operators, e.g., $\langle \partial_s u, \partial_s u\rangle$. Moreover, since the derivative operator is unbounded, the functions u and u_{ss} are independent until the continuity constraints and BCs are enforced. However, accounting for the continuity and BCs is an ad hoc process, using integration-by-parts or inequalities such as Wirtinger or Poincare.

Such ad hoc methods have been used to generate computational stability tests and input-output analysis for specific classes of PDE models (See works by Papachristodoulou and Peet (2006), Datko (1970), Fridman and Orlov (2009a), Valmorbida *et al.* (2016), Ahmadi *et al.* (2016a), and Gahlawat and Peet (2016a) for LMI-based methods, Meurer (2012) for early-lumping methods, Lasiecka and Triggiani (2000a) for late-lumping methods, and Villegas (2007) for Port-Hamiltonian methods), however, there exists no universal approach to computational analysis of PDE models.

The primary challenge in using such model-specific methods is that they become inapplicable just by introducing a simple variation to the model. For instance, if we consider heat conduction through a fin used in cooling systems with forced convection (e.g., heat-sink mounted on CPUs), we can model this phenomenon using the heat equation PDE with a minor modification. Assuming the fin to be a 1D rod, we can model the heat transfer process, as shown by Kraus *et al.* (2001), by using the PDE $\dot{u}(t,s) = c(s)u_{ss}(t,s) + d(s)w(t)$ with boundary conditions $u(t,0) = 0$ and $u_s(t,1) = g(t)$ where c, d are coefficients dependent on the cross-section of the fin, w is the forced convection heat loss and g heat flux at the tip. In such systems, where the parameters vary with space, simple Lyapunov functions of the form $V(u) = \langle u, u\rangle$ do not help prove stability or analyze input-output properties. Thus, one would need to consider parameterized quadratic Lyapunov functions, such as $V(u) = \langle u, Pu\rangle$ for some operator P. However, we must take special care to ensure P satisfies the BCs and thus vanishes at the boundary. However, depending on the BCs, finding an appropriate parametric form for P may not be trivial. Thus, such a step cannot be automated in a computational framework.

Challenges in control: The scalability issues and non-universality problems also appear when one considers the control problem of PDE systems. In this context, let us discuss the lumping-based methods for control of PDEs. Unlike PDEs, many efficient algorithms exist for optimal control of state-space ODEs, with such controllers typically obtained by solving either Riccati Equations, e.g., as proposed by Locatelli and Sieniutycz (2002) and Gerdts (2011) or LMIs, e.g., proposed by Boyd *et al.* (1994), the controller design for linear state-space systems is well-developed and considered a solved problem. Thus, the most common approach to the control of PDEs is to approximate the PDE model using a lumped state-space ODE model using methods such as projection employed in works by Bamieh *et al.* (2002), Apkarian and Noll (2020), and Collis and Heinkenschloss (2002) or finite-difference approximation methods used by Christofides and Daoutidis (1996), Ito and Ravindran (1998a), and Ito and Ravindran (1998b). Even ignoring the question of integration of lumped controllers with a PDE with distributed state, Kotsiantis and Kanellopoulos (2006) and Morris and Levine (2010) have proved that that stability and performance gains of the closed-loop state-space ODE do not necessarily translate to stability or performance of the optimal closed-loop PDE. While for specific systems for which we have an eigendecomposition and a finite number of unstable modes as shown by Prieur and Trélat (2018), projection onto these eigenfunctions can sometimes be used to obtain a stable closed-loop controller, such exceptions are rare.

If one wants to avoid reducing the PDE model to a linear state-space ODE, then there exist methods for formulating the optimal control problem in an abstract operator-theoretic state-space framework where the system solution is defined by a strongly continuous semigroup. In this context, one may formulate an operator equivalent of the Riccati Equations for controller synthesis, e.g., see the book by Lasiecka and Triggiani (2000b), and papers by Hulsing (1999) and Morris (2001). Unfortunately, however, the operators in these Riccati equations are typically unbounded. They cannot be easily parameterized so that instead of solving the equations directly, projection is gain used so that the numerical solution obtained defines the operator only when restricted to the projected subspace. This approach is often referred to as late-lumping, and applications can be found in, e.g., the paper by Moghadam *et al.* (2013). The downsides of late-lumping are: the projection requires extensive ad hoc analysis for any given PDE; the operator solutions are never obtained explicitly, only their projection onto a finite-dimensional subspace; and the closed-loop is not guaranteed to be stable for any given order of projection.

Closely related to the so-called "late-lumping" approach is backstepping, an approach thoroughly described in works by Karafyllis and Krstic (2019) and Meurer (2012), which uses the input to provide an algebraic mapping of the state to that of a nominal stable system. The advantage of backstepping is that the mapping is parameterized explicitly using integral operators and the algebraic conditions then translate to a set of PDEs on the kernels which define these integral operators. The disadvantages of backstepping are: the kernel map must be re-derived for every PDE; a parametrization of the kernels is required in order to solve the resulting PDEs numerically; and the controllers obtained are not optimal, only stabilizing.

To create more explicit parametrizations of the Lyapunov operators used in late-lumping, recently, many Control Theorists have focused on the construction of positive Lyapunov functions for use in the control of PDEs, often using positive matrices and semidefinite programming to enforce the positivity of these Lyapunov functions, e.g., Fridman and Orlov (2009b), Peet and Papachristodoulou (2010), and Gahlawat and Peet (2016b). The advantage of the approach presented by these theorists is that the resulting closed-loop controllers almost always have provable closed-loop properties. The downsides of this approach are: the assumption of specific structure on the Lyapunov function and controller adds conservatism to the problem; the use of ad hoc steps such as Poincare and Wirtinger inequalities to upper bound the derivative of the Lyapunov function by exploiting boundary conditions and continuity of the solution; and failure to resolve the bilinearity between the Lyapunov operator and the controller often renders the problem non-convex or severely limits the structure of the Lyapunov function and/or controller.

Clearly, when considering computational methods for the analysis, control, and simulation of spatially distributed phenomena, the use of a 3-constraint PDE model is inconvenient. Unfortunately, there is no direct way to eliminate these inconveniences because the well-posedness of a PDE requires these auxiliary constraints. Furthermore, one cannot avoid using such 3-constraint PDE models since the natural representation of physical phenomena such as diffusion is necessarily a 3-constraint PDE model — given the historical context and the clear physical interpretation of spatial derivatives and BCs. To summarize, the 3-constraint PDE model poses significant challenges to the development of a universal computational framework for analysis, control, and simulation. The most significant inconveniences are as follows:

1. **Non-Algebraic Structure** All computation is fundamentally algebraic – consisting primarily of a sequence of addition and multiplication operations. The PDE model formulation, however, is defined by spatial differentiation and evaluation of limit points (Dirac operations). Neither differentiation nor Dirac operators can be embedded in a *-algebra of bounded linear operators on a Hilbert space, a result proved by Segal (1947). The unbounded nature of the differential and Dirac operators complicates both simulation and analysis – resulting either in ill-conditioned ODE representations or a lack of the algebraic structure needed for parametrization and optimization.

2. **No Universality** Computational methods are traditionally centered on the 'PDE' part of the 'PDE model' and are designed for a fixed set of BCs and continuity constraints. This means every change in boundary condition or continuity constraint requires a change in the algorithm, with such changes being ad hoc and requiring significant mathematical analysis. As a result, there are no generic/universal algorithms for the analysis, control, and simulation of PDEs.

However, these limitations are primarily an artifact of the PDE modeling approach, are not inherent to spatially distributed systems, and can be remedied by using an alternative modeling framework defined by Partial Integral Equations (PIEs).

Partial Integral Equation (PIE) models describe system behaviors encountered in elasticity, mechanical fracture, etc., and were recently revived and studied by Appell *et al.* (2000) and Gil (2015). The simplest form of PIE, in which we ignore ODEs, inputs, and outputs, is defined by two Partial Integral (PI) operators, $\mathcal{T}, \mathcal{A} : L_2 \to L_2$ as $\partial_t(\mathcal{T}\mathbf{v})(t) = \mathcal{A}\mathbf{v}(t)$, where the state, $\mathbf{v}(t) \in L_2$ admits no continuity constraints or BCs. An operator $\mathcal{P}$ is said to be a 3-PI operator if there exist $R_0 \in L_\infty$ and separable functions R_1, R_2 such that

$$(\mathcal{P}\mathbf{u})(s) = R_0(s)\mathbf{u}(s) + \int_a^s R_1(s,\theta)\mathbf{u}(\theta)\, d\theta + \int_s^b R_2(s,\theta)\mathbf{u}(\theta)\, d\theta.$$

To illustrate a simple PIE, let us revisit the heat equation PDE model, $u_t = u_{ss}$ with BCs $u(t,0) = u_s(t,1) = 0$, continuity constraint $u \in W_2$ and initial condition $u(0,\cdot) = u_0 \in W_2$. A PIE representation of this PDE model is given by

$$\partial_t \left(\int_0^s \theta\, v(t,\theta)\, d\theta + \int_s^1 s\, v(t,\theta)\, d\theta \right) = -v(t,s) \tag{1.1}$$

with initial condition $v(0,\cdot) = \partial_s^2 u_0 \in L_2$. In this case, $\mathcal{T}$ is parameterized by $R_1(s,\theta) = -\theta, R_2(s,\theta) = -s$ with $R_0 = 0$, while $\mathcal{A}$ is parameterized by $R_0(s) = I$ with $R_1 = R_2 = 0$. The solution to the PIE yields a solution to the PDE model as $u(t,s) = \mathcal{T}v(t,s)$, so that $u(t,s) = -\int_0^s \theta\, v(t,\theta)d\theta - \int_s^1 s\, v(t,\theta)d\theta$. Note that since the solution of the PIE v is in L_2, we do not need any additional boundary conditions or differentiability constraints to find a unique solution. Thus, if one considers the Lyapunov function $V(u) = \langle u, \mathcal{P}u\rangle_{L_2} = \langle \mathcal{T}v, \mathcal{P}\mathcal{T}v\rangle_{L_2}$ parametrized by a PI operator $\mathcal{P}$, then the time derivative along the solutions of the PIE (1.1) is given by $\dot{V}(v(t)) = \langle v(t), (\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A})v(t)\rangle_{L_2}$ — i.e., the proof of stability requires showing that there exists a $\mathcal{P} \succ 0$ such that $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \preceq 0$. Since all the PI operators that define this PIE system are bounded, linear, integral operators on L_2 Hilbert space, and more significantly, since these PI operators form a *-algebra, one can find $\dot{V}$ without any ad-hoc manipulation such as integration-by-parts. Instead, algebraic operations on PI operators, such as addition, composition, transpose, and concatenation operations, can be performed by operating on their parameters. Furthermore, we can parameterize positive operators of this class using a basis and a positive matrix. Thus, one can solve operator-valued optimization problems involving PI operator decision variables and constraints — i.e., we can solve the proof of stability test given by the constraints $\mathcal{P} \succ 0$ $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \preceq 0$. This allows us to overcome the earlier problems in computational analysis, estimation, and control.

In short, PIE models can be considered a generalization of the integro-differential systems. A PIE model, unlike a PDE, is defined by a single integro-differential equation, is parameterized by the *-algebra of Partial Integral (PI) operators, and can be used to represent almost any well-posed PDE model. Furthermore, computational analysis of PIE systems can be performed algorithmically and without ad hoc manipulations involving boundary conditions or continuity constraints. Therefore, if we can use the properties of a PIE model of a PDE to infer the properties of the PDE, then we can resolve the challenges faced in the computational analysis of PDEs. The remainder of this thesis will focus on using the PIE representation to develop computational tools that prove the properties of a PDE.

The main contributions of this work are: a) developing an alternative representation of Linear PDE systems that is universal, defined by algebraic, bounded linear operators, called the Partial Integral Equation; b) proving the equivalence in the two representations of an infinite-dimensional system; and c) building computational/numerical tools for the analysis, estimation, and control of these PIEs inspired by the LMI-based methods called PIETOOLS. An alternative to PIETOOLS does not exist currently because, as elaborated above, a PDE model is ill-suited for developing such tools.

The proposed methodology to achieve the research goals can be divided into the following steps:

1. Express a given 3-constraint PDE model as an equivalent PIE model.
2. Develop computational tools for analysis and control of a PIE model.
3. Use these computational tools on the PIE obtained from a PDE to solve analysis and control problems for the PDE.

1.1 Overview

The content of this thesis can largely be divided into two parts: one covering the particulars of linear PDE models that admit an equivalent PIE representation, and the second leveraging the benefits of the PIE representation to formulate analysis, estimation, and control problems as solvable convex-optimization problems.

Part I: Representation and Parametrization of Linear Infinite-dimensional Systems

Here, we outline the class of PDEs for which the analysis, estimation, and control problems are addressed in this work. Specifically, in Chapter 3, We will introduce a standard parametric representation (although non-universal and not exhaustive) covering a large class of linear PDEs on one spatial dimension to aid in developing a standard computational framework. While such a parametric representation is not necessary to find an equivalent PIE representation, it is required to build a computational framework that can convert a PDE representation input to its corresponding PIE representation– i.e., a representation that is consistent, unambiguous, and standard is needed.

Chapter 4 follows the theme of Chapter 3 in specifying a standard parametric representation of a PIE system that will provide an unambiguous interpretation in a computational framework. These standard representations will allow one to build a computational tool that can solve standard problems in analysis, estimation, and control, as will be seen later in Part II.

Lastly, to wrap up Part I, in Chapter 5, we will show that under certain admissibility conditions, the standard parametric representation of a GPDE introduced in Chapter 3 has an equivalent PIE representation of the standard parametric form introduced in Chapter 4. We will show that the notions of equivalence come from the solutions for the two representations, where the solution to one representation can be used to determine the solution to the other. We will see that this equivalence of solutions automatically leads to identical properties of the systems in terms of stability, stabilizability, controllability, etc.

Part II: Analysis, Estimation, and Control of GPDEs

Here, we will focus on utilizing the new PIE representation to formulate and solve problems in the analysis, estimation, and control of PDEs as solvable LMI problems. Specifically, in Chapter 6, we will discuss the concepts of stability and dual stability, which are then used to develop optimization-based certificates for exponential stability, stabilizability, and controllability of a GPDE model. To establish dual stability, we will propose a PIE system of a specific form to be the *dual* of a given PIE system where the dual has the same stability properties as the given PIE. Furthermore, the proposed dual is chosen to have the same standard parametric representation introduced in Chapter 4. By virtue of this parametric representation, the conditions to test stability for a PIE can be extended to its dual to formulate dual stability conditions, which naturally extends to convex formulations for stabilizability criterion.

Having established internal stability properties, we include the inputs and outputs of the GPDE model in Chapter 7 to find provable input-output properties of the system, namely, H_∞-norm and passivity.

In Chapter 8, combining the duality results and input-output properties, we will formulate the H_∞-optimal observer and controller design problems to finally resolve the two important unresolved problems in the control theory of linear PDEs.

Chapter 2

BACKGROUND MATERIAL

2.1 Introduction

In this chapter, we briefly introduce the concepts of convex optimization and LMIs and then show how these techniques are used to solve convex optimization problems such as Sum-of-Squares (SOS) problems and Linear Partial Integral Inequalities (LPIs) — problems that commonly arise in control theory. Specifically in Section 2.3, we will use Lyapunov methods and show how the analysis, estimation, and control problems of Linear dynamical systems can be posed as an LMI, SOS, or LPI problem. In addition, we also discuss LPI problems in detail by formally defining the parametric form of Partial Integral (PI) Operators, proving some of their useful algebraic properties, and proposing a method for solving LPI optimization problems using LMIs.

2.1.1 Notation

Before starting, let us look at some commonly used notation and principles in naming variables. In addition to denoting the empty set, $\emptyset$ is occasionally used to denote a matrix or matrix-valued function with either zero row or column dimension and whose non-zero dimension can be inferred from context. We denote by $0_{m,n} \in \mathbb{R}^{m\times n}$ the matrix of all zeros, $0_n = 0_{n,n}$, and $I_n \in \mathbb{R}^{n\times n}$ the identity matrix. We use 0 and I for these matrices when dimensions are clear from context. $\mathbb{R}_+$ is the set of non-negative real numbers. The set of k-times continuously differentiable n-dimensional vector-valued functions on the interval $[a, b]$ is denoted by $C_k^n[a,b]$. $L_2^n[a,b]$ is the Hilbert space of n-dimensional vector-valued Lebesgue square-integrable functions on the interval $[a,b]$ equipped with the standard inner product. $L_\infty^{m,n}[a,b]$ is the Banach space of $m \times n$-dimensional essentially bounded measurable matrix-valued functions on $[a,b]$ equipped with the essential supremum singular value norm. Normal font u or $u(t)$ typically implies that u or $u(t)$ is a scalar or finite-dimensional vector (e.g. $u(t) \in \mathbb{R}^n$), whereas the bold font, $\mathbf{x}$ or $\mathbf{x}(t)$, typically implies that $\mathbf{x}$ or $\mathbf{x}(t)$ is a scalar or vector-valued function (e.g. $\mathbf{u}(t) \in L_2^n[a,b]$). For a suitably differentiable function, $\mathbf{x}$, of spatial variable s, we use $\partial_s^j \mathbf{x}$ to denote the j-th order partial derivative $\frac{\partial^j \mathbf{x}}{\partial s^j}$. For a suitably differentiable function of time and possibly space, we denote $\dot{\mathbf{x}}(t) = \frac{\partial}{\partial t}\mathbf{x}(t)$. We use W_k^n to denote the Sobolev spaces $W_k^n[a,b] = \{\mathbf{u} \in L_2^n[a,b] \mid \partial_s^l \mathbf{u} \in L_2^n[a,b] \ \forall\ l \leq k\}$ with inner product $\langle \mathbf{u}, \mathbf{v} \rangle_{W_k^n} = \sum_{i=0}^k \langle \partial_s^i \mathbf{u}, \partial_s^i \mathbf{v} \rangle_{L_2^n}$. Clearly, $W_0^n[a,b] = L_2^n[a,b]$. For given $n = \{n_0, \cdots, n_N\} \in \mathbb{N}^{N+1}$, we define the Cartesian product space $W^n = \prod_{i=0}^N W_i^{n_i}$ and for $\mathbf{u} = \{\mathbf{u}_0, \cdots, \mathbf{u}_N\} \in W^n$ and $\mathbf{v} = \{\mathbf{v}_0, \cdots, \mathbf{v}_N\} \in W^n$ we define the associated inner product as $\langle \mathbf{u}, \mathbf{v} \rangle_{W^n} = \sum_{i=0}^N \langle \mathbf{u}_i, \mathbf{v}_i \rangle_{W_i^{n_i}}$. We use $\mathbb{R}L_2^{m,n}[a,b]$ to denote the space $\mathbb{R}^m \times L_2^n[a,b]$ and for $\mathbf{x} = \begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix} \in \mathbb{R}L_2^{m,n}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \mathbf{y}_2 \end{bmatrix} \in \mathbb{R}L_2^{m,n}$, we define the associated inner product as

$$\left\langle \begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ \mathbf{y}_2 \end{bmatrix} \right\rangle_{\mathbb{R}L_2^{m,n}} = x_1^T y_1 + \langle \mathbf{x}_2, \mathbf{y}_2 \rangle_{L_2^n}.$$

Frequently, we omit the domain $[a,b]$ and simply write L_2^n, W_k^n, W^n, or $\mathbb{R}L_2^{m,n}$. For functions of time only ($L_2[\mathbb{R}_+]$ and $W_k[\mathbb{R}_+]$), we use the truncation operator

$$(P_T x)(t) = \begin{cases} x(t), & \text{if } t \leq T \\ 0, & \text{otherwise,} \end{cases}$$

to denote the extended subspaces of such functions by $L_{2e}[\mathbb{R}_+]$ and $W_{ke}[\mathbb{R}_+]$ respectively as

$$L_{2e}[\mathbb{R}_+] = \{x \mid P_T x \in L_2[\mathbb{R}_+] \; \forall \, T \geq 0\} ,$$

$$W_{ke}[\mathbb{R}_+] = \{x \mid P_T x \in W_k[\mathbb{R}_+] \; \forall \, T \geq 0\} .$$

Finally, for normed spaces A, B, $\mathcal{L}(A, B)$ denotes the space of bounded linear operators from A to B equipped with the induced operator norm. $\mathcal{L}(A) = \mathcal{L}(A, A)$. Specifically in Chapter 8, the symbol $\mathcal{L}$ may appear without an argument, which is assumed to be an operator representing the observer gains.

2.2 Convex Optimization

A convex optimization problem involves minimizing a convex function over convex sets. In general, any convex optimization problems can be written in the form of

$$\begin{aligned} &\min_{x \in X} \quad c(x), \quad s.t., && (2.1)\\ &f_i(x) \leq 0, \quad g_j(x) = 0 \qquad i \in \{0, \cdots, m\}, j \in \{0, \cdots, n\} \end{aligned}$$

where $f_i, g_i : X \to Y$ (X and Y can be the set of reals, real-valued vectors, real-valued matrices, etc.) are convex functions, $c : X \to \mathbb{R}$ is an objective function to be minimized, and x are decision variables. An $x \in X$ is said to be a *feasible* solution if x satisfies the equality and inequality constraints — $f_i(x) \leq 0$, and $g_j(x) = 0$. An $x_* \in X$ is said to be an *optimal* solution to the above problem if x_* is a feasible solution and $c(x_*) \leq c(x)$ for any $x \in X$.

Various problems in Control theory for dynamical systems can be formulated in this form for an appropriate choice of the convex functions c, f_i, and g_i. For example, given a linear ODE system in state-space representation $\dot{x}(t) = Ax(t)$, one can prove stability by proving that a feasible solution exists for the optimization problem

$$\exists \; P > 0, \quad s.t., \quad A^T P + PA \leq 0.$$

In this case, P is a matrix decision variable, $c(P) = 0$, $f_0(P) = -P$, and $f_1(P) = A^T P + PA$.

While optimization problems, in general, need not be defined only by convex functions, in this dissertation, we will restrict to the class of problems that are convex optimization problems because, in the case of convex optimization problems, any local minima is also the global minimum — i.e., if x_* and y_* are two optimal solutions to the above optimization problem in Equation (2.1) then $x_* = y_*$. This property is desirable since any method used to solve convex optimization problems, such as gradient descent and interior-point methods Boyd and Vandenberghe (2004), converges to the best possible solution in polynomial time.

Every optimization problem in which c, f_i, and g_i are linear functions is a convex optimization problem because linear functions are convex, and linear constraints (equality or inequality) define a convex region of feasible solutions. However, we are particularly interested in optimization problems where decision variables are matrices, and functions are on matrix variables — a specific class of convex optimization problems called Semi-Definite Programming (SDP), commonly encountered in Control Theory. Note that the linear ODE stability test presented above is an SDP problem. In the remaining subsections, we will focus on SDPs and their application in Control Theory.

2.2.1 Semi-definite Programming

Semi-definite Programming is a subclass of convex optimization problems that involve matrix-valued decision variables and linear sign-definite constraints on matrix-valued variables — i.e., the set of feasible solutions is described by the cone of positive semi-definite matrices. A matrix, $P \in \mathbb{R}^{n \times n}$, is said to be *positive semi-definite* if for any $x \in \mathbb{R}^n$, $x^T P x \geq 0$ (or, *positive definite* if the inequality is strict for all $x \neq 0$).

SDP problems typically take the form

$$\begin{aligned} \min_{P\in\mathbb{R}^{n\times n}} \quad & trace(C^T P), \quad s.t., \\ & P \geq 0, \quad trace(A_i^T P) \leq b_i, \quad i \in \{0, \cdots, m\}, \end{aligned}$$

for some known reals b_i, and matrices C and A_i. Such optimization problems are solved using Interior-point methods first introduced by Adler *et al.* (1989) and shown to be solvable in polynomial time by Alizadeh (1995). More significantly, as previously demonstrated in this section using the stability test for linear state-space ODEs, many problems in the control of linear state-space ODE systems lead to a class of SDP problems referred to as 'Linear Matrix Inequalities' (LMIs), a thorough study of which can be found in the book by Boyd *et al.* (1994), that take the form

$$\begin{aligned} \min_{P\in\mathbb{R}^{n\times n}} \quad & trace(C^T P), \quad s.t. \\ & P \geq 0, \quad F_i^T P G_i + G_i^T P F_i \leq 0, \quad i \in \{0, \cdots, m\}. \end{aligned}$$

While it is possible (and required) to reformulate the LMIs in the standard SDP format to use SDP solvers, many parsers, such as Yalmip by Lofberg (2004), can directly parse LMIs and convert them into the standard SDP format before linking with solvers, such as SeDuMi by Sturm (1999), Mosek by Andersen and Andersen (2000); ApS (2019), etc., to apply interior-point methods.

2.2.2 Positive Polynomials and Sum-of-Squares Polynomials

Before moving on to the particulars of formulating control problems as LMIs, we will briefly introduce optimization problems involving positive polynomials and the Sum-of-Squares approach here — a class of optimization problems that commonly appear in problems involving non-linear ODE systems.

We say, a polynomial $p(x_1, \cdots, x_n)$ is *positive* if for all $x \in \mathbb{R}^n$, we have $p(x) > 0$.

Such polynomials are useful to parametrize Lyapunov functions that will be introduced in the next section. However, for a simple demonstration, consider a dynamical system model given by a nonlinear ODE $\dot{x}(t) = f(x(t))$. We can prove the stability of this system by finding a Lyapunov function V that satisfies $V(x) > 0$ for all $x \neq 0$ and $\dot{V}(x(t)) \leq 0$ along the solutions of the dynamical system. Clearly, if we choose $V(x)$ to be a polynomial, we can parameterize the coefficients of this polynomial and search for coefficients such that V satisfies the above-stated constraints — i.e., $V(x) > 0$ and $\nabla V(x)^T f(x) \leq 0$ for all $x \in \mathbb{R}$. Thus, we can pose the question of the existence of a Lyapunov function as an optimization problem involving polynomials with positivity constraints. However, for multivariate polynomials $V(x)$, there is no practical way to verify the positivity since the problem is NP-hard.

To overcome the computational demands, we can tighten the constraints — i.e., instead of searching for a positive $V(x)$, we can search for a polynomial $V(x)$ that is a sum of squares of other polynomials. In other words, we will look for a $V(x)$ such that we can write $V(x) = \sum_{i=0}^{n} a_n f_n(x)^2$ for some polynomials $f_n(x)$ and coefficients $a_n > 0$. If such a decomposition exists, then clearly, $V(x)$ is non-negative for all x. We refer to such polynomials as *Sum-of-Squares* (SOS) polynomials. While this parametrization of Lyapunov functions is more conservative, by replacing the constraint $V(x) > 0$ with 'V is an SOS polynomial', the optimization problem becomes computationally tractable as shown by Parrilo (2000). This is because an SOS polynomial V can be written in the quadratic form

$$V(x) = Z(x)^T P Z(x),$$

for some positive semidefinite matrix P and an appropriate monomial basis vector, $Z(x)$, in independent variables x. Thus, one can replace the positivity constraints (consequently, the test for stability) on the Lyapunov function V, given by

$$V(x) > 0 \quad \dot{V}(x) = (\nabla V(x))^T f(x) \leq 0$$

with the constraints

$$P > 0, \quad Q > 0, \quad V(x) = Z(x)^T P Z(x), \quad (\nabla V(x))^T f(x) + Z(x)^T Q Z(x) = 0.$$

This problem is, in fact, an LMI optimization problem with some additional linear equality constraints — a convex optimization problem. Thus, it can be solved using standard interior-point methods in polynomial time. We should note at this point that one need not set up or extract the decision variables from the above formulation to solve the underlying SDP problem because there are many parses/libraries that aid in this process, such as SOSTOOLS by Prajna *et al.* (2005), SOSOPT by Seiler (2013), Yalmip by Lofberg (2004), etc.

While optimization problems involving SOS polynomials are not directly used in this work, as will be seen in Section 2.4, we will use this idea of sum-of-squares to parametrize positive, infinite-dimensional operators that appear in the convex formulations of the analysis, estimation, and control problems for PDEs.

2.3 Lyapunov Theory

Lyapunov Theory refers to the mathematical tools derived from the work of Lyapunov (1992) on the stability of dynamical systems where standardized concepts of stability and methods to determine stability were proposed. Using Lyapunov's first and second methods, the stability of various dynamical systems can be proved. Arguably, the more impactful result, Lyapunov's second method (or, popularly called, the *Direct* method), can be used to prove stability without constructing or finding a solution for the dynamical system. Instead, the direct method relies on proving stability properties by proving the existence of a proxy energy functional $V(x) > 0$ defined on state x of the dynamical system that decreases with time (i.e., $\dot{V}(x(t)) \leq 0$ along the solution of the dynamical system).

Since then various converse Lyapunov theorems have been proposed to show the necessity of the existence of such energy functionals of a particular form, e.g., La Salle and Lefschetz (2012) proved that for asymptotically stable linear state-space ODE systems of the form $\dot{x}(t) = Ax(t)$ there must be a quadratic Lyapunov function of the form $V(x) = x^T P x$ such that $V(x) > 0$ and $\dot{V}(x(t)) \leq 0$.

Thus, using Lyapunov's direct method and various converse Lyapunov theorems, many analysis and control problems for linear ODE systems have been formulated as optimization problems. In this work, we will particularly look at the use of Lyapunov's direct method in solving problems such as proving stability and passivity, estimating H_∞-norm, or designing H_∞-optimal observers and controllers for the system.

2.3.1 Lyapunov Methods for Analysis, Estimation, and Control

In this subsection, we will discuss two approaches to employing Lyapunov's direct method to solve different analysis, estimation, and control problems. The first approach is finding Lyapunov functions, which can then be used to verify internal stability or prove the stabilizability and detectability of a dynamical system. The second approach involves finding a storage function, which acts as an energy metric, to prove input-output properties such as input-to-output L_2-gain and passivity of a system. Applying these approaches together, one can solve problems such as H_∞-optimal estimator and controller design for dynamical systems.

To describe the first approach involving Lyapunov functions, we must first define Lyapunov functions and stability. Standard definitions for these terms are given below, first, starting with a definition for Lyapunov functions.

Definition 2.1. *Given an autonomous dynamical system*

$$\dot{x}(t) = f(x(t)), \quad f : X \to X, \quad x(0) = x_0 \in X,$$

with an equilibrium point $x = 0$, a Lyapunov function is a scalar function $V : X \to \mathbb{R}$, that is continuous and differentiable up to order 1, such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, and $\dot{V}(x(t)) = (\nabla V(x(t)))^T f(x(t)) \leq 0$ in an open-neighborhood around $x = 0$.

Next, in regards to stability, there is more than one definition of stability; however, in this work, we will consider the three commonly used notions, namely, Lyapunov, Asymptotic and Exponential stability.

Definition 2.2. *Consider an autonomous dynamical system*

$$\dot{x}(t) = f(x(t)), \quad f : X \to X, \quad x(0) = x_0 \in X,$$

with an equilibrium point $x = 0$.

1. *This equilibrium is said to be **Lyapunov stable**, if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| \leq \epsilon$ for all $t \geq 0$.*

2. *This equilibrium is said to be **Asymptotically stable**, if it is Lyapunov stable and there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\lim_{t\to\infty} \|x(t)\| = 0$.*

3. *This equilibrium is said to be **Exponentially stable** with decay rate $\alpha > 0$, if there exist M and $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| \leq M \|x(0)\| e^{-\alpha t}$ for all $t \geq 0$.*

If there is only one equilibrium point, then the stability of the equilibrium point also alludes to the system's stability. The real benefit of these definitions, however, lies in Lyapunov's 'Direct' method, wherein an equilibrium point of a dynamical system is:

1. **Lyapunov stable**, if there exists a Lyapunov function V such that $\dot{V}(x) \leq 0$ for any solution x of the system.

2. **Asymptotically stable**, if there exists a Lyapunov function V such that $\dot{V}(x) < 0$ for any solution x of the system.

3. **Exponentially stable** with decay rate α, if there exists a coercive Lyapunov function V such that $\dot{V}(x) \leq -2\alpha V(x)$ for any solution x of the system.

Therefore, one can prove stability by finding an appropriate Lyapunov function. This will be the primary approach taken in this work to prove stability, stabilizability, and detectability. Given an autonomous system, we parameterize Lyapunov function candidates $V(x)$ using quadratic forms $V(x) = \langle x, Px \rangle$ and search for a $P > 0$ while restricting $\dot{V}(x) = (2Px)^T f(x) \leq 0$ (< 0 if proving asymptotic stability, $< -2\alpha V(x)$ if exponential). Thus, the task of proving stability can be posed as an optimization problem with P as a decision variable. Likewise, as will be shown later in Chapter 6, tests for stabilizability and detectability can also be formulated as optimization problems.

The second approach requires storage functionals to prove the input-output properties of a system. We will only discuss the problems of finding the H_∞-norm and proving the passivity of a system, however, the second approach can be used for other problems, such as finding the H_2-norm. As before, we must first define the terms.

Definition 2.3. *Given an autonomous system*

$$\dot{x}(t) = f(x(t), u(t)), \quad z(t) = g(x(t), u(t)), \quad x(0) = 0$$

we say

1. *the system is passive if, for any L_2-bounded input u, any solution $\{x, z\}$ that satisfies the system also satisfies $\langle z(t), u(t) \rangle \geq 0$ for all $t \geq 0$.*

2. *$\gamma > 0$ is the H_∞-norm of the system if, for any L_2-bounded input u, any solution $\{x, z\}$ that satisfies the system also satisfies $\|z\|_{L_2} \leq \gamma \|u\|_{L_2}$.*

Using a positive storage function, one can alternatively formulate the constraints in the above definitions as

1. the system is passive if there exists a storage function $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, such that $\dot{V}(x) - 2\langle z, w\rangle \leq 0$ for all $\{x, z\}$ that satisfies the system for any input u.

2. $\gamma > 0$ is an upper bound on the H_∞-norm of the system if there exists a storage function $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, such that $\dot{V}(x) + \|z\|^2 - \gamma^2 \|u\|^2 \leq 0$ for all $\{x, z\}$ that satisfies the system for any input u.

Thus, one can, again, parameterize storage functionals V (quadratic form introduced earlier being a popular choice) and add appropriate constraints to formulate the above problems as an optimization problem. In the above case of finding H_∞-norm of a system, γ is merely an upper bound and not the exact H_∞-norm. However, one can use γ as an objective function and solve the constraints of the optimization problem while minimizing this objective to find a better estimate of the H_∞-norm of the system. In addition, one can also parameterize a feedback input $u(t) = h(x(t))$ and solve for h along with the parameters of V while minimizing γ to design H_∞-optimal observers and controllers — an approach discussed in detail in Chapter 8. In the following sections, we will divert our attention to the class of Partial Integral operators and their properties, which will prove useful later in parametrizing Lyapunov functions (or storage functions) and feedback inputs for infinite-dimensional systems such as PDEs.

2.4 Partial Integral Operators

In this section, we discuss the class of Partial Integral (PI) operators in one-spatial dimension and their properties. As noted here, and later in Chapter 4, this class of operators is a natural extension of matrices — matrices are operators on finite-dimensional vector spaces, whereas PI operators are a generalization of matrix operators on infinite-dimensional Hilbert spaces. This class of operators will be useful in parametrizing the class of Partial Integral Equations (PIEs), which is being proposed as an alternative representation of PDE models. We will also introduce the class of optimization problems with PI operator decision variables and constraints called Linear Partial-integral Inequalities (LPIs), which are a natural extension of LMI optimization problems to operator-valued optimization problems. Lastly, we will present a method to solve these LPIs using LMIs – a crucial result that will later allow us to solve analysis, estimation, and control problems for PDE systems using convex-optimization methods.

First, we start with a formal definition of PI operators and the set of PI operators, also called 'PI-algebras'. The PI-algebras are parameterized classes of bounded linear operators on $\mathbb{R}L_2^{m,n}$ (the product space of $\mathbb{R}^m$ and L_2^n). Here, we specifically denote two sub-algebras of these operators and associate a notation to them because they will be extensively used in the dissertation. The first is the algebra of 3-PI operators, which map $L_2^n \to L_2^n$, that is exclusively defined by parameters that are separable functions, which are defined below.

Definition 2.4 (Separable Function)**.** *We say $R : [a,b]^2 \to \mathbb{R}^{p\times q}$ is separable if there exist $r \in \mathbb{N}$, $F \in L_\infty^{r\times p}[a,b]$ and $G \in L_\infty^{r\times q}[a,b]$ such that $R(s,\theta) = F(s)^T G(\theta)$.*

Using separable functions as the parameters that define a 3-PI operator, we next define a standard notation for this subclass of PI operators as follows.

Definition 2.5 (3-PI operators, $\mathbf{\Pi}_3$). *Given $R_0 \in L_\infty^{p\times q}[a,b]$ and separable functions $R_1, R_2 : [a,b]^2 \to \mathbb{R}^{p\times q}$, we define the operator $\mathit{\Pi}_{\{R_i\}}$ for $\mathbf{v} \in L_2[a,b]$ as*

$$\left(\mathit{\Pi}_{\{R_i\}}\mathbf{v}\right)(s) = R_0(s)\mathbf{v}(s) + \int_a^s R_1(s,\theta)\mathbf{v}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{v}(\theta)d\theta. \tag{2.2}$$

Furthermore, we say an operator, $\mathcal{P}$, is 3-PI of dimension $p\times q$, denoted $\mathcal{P} \in [\mathbf{\Pi}_3]_{p,q} \subset \mathcal{L}(L_2^q, L_2^p)$, if there exist functions R_0 and separable functions R_1, R_2 such that $\mathcal{P} = \mathit{\Pi}_{\{R_i\}}$.

For any $p \in \mathbb{N}$, $[\mathbf{\Pi}_3]_{p,p}$ is a *-algebra, being closed under addition, composition, scalar multiplication, and adjoint. Closed-form expressions for these algebraic operations on 3-PI operators are included in the following subsections.

The algebra of 3-PI operators can be extended to $\mathcal{L}(\mathbb{R}L_2^{m,p}, \mathbb{R}L_2^{n,q})$ as follows.

Definition 2.6 (4-PI operators, $\mathbf{\Pi}_4$). *Given $P \in \mathbb{R}^{m\times n}$, $Q_1 \in L_\infty^{m\times q}$, $Q_2 \in L_\infty^{p\times n}$, and R_0, R_1, R_2 with $\mathcal{P}_{\{R_i\}} \in [\mathbf{\Pi}_3]_{p,q}$, we say $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \in \mathcal{L}(\mathbb{R}L_2^{n,q}, \mathbb{R}L_2^{m,p})$ if*

$$\left(\mathcal{P}\begin{bmatrix} u \\ \mathbf{v}\end{bmatrix}\right)(s) = \begin{bmatrix} Pu + \int_a^b Q_1(\theta)\mathbf{v}(\theta)d\theta \\ Q_2(s)u + \left(\mathit{\Pi}_{\{R_i\}}\mathbf{v}\right)(s)\end{bmatrix}. \tag{2.3}$$

Furthermore, we say $\mathcal{P}$, is 4-PI, denoted $\mathcal{P} \in [\mathbf{\Pi}_4]_{p,q}^{m,n}$, if there exist $P, Q_1, Q_1, R_0, R_1, R_2$ such that $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$.

Similar to $[\mathbf{\Pi}_3]_{p,p}$, for any $p, q \in \mathbb{N}$, $[\mathbf{\Pi}_4]_{p,p}^{q,q}$ is a *-algebra, being closed under addition, composition, scalar multiplication, and adjoint. Closed-form expressions for these algebraic operations are included in the following subsections.

Note, from here onward, we will omit the subscripts/superscripts for $\mathbf{\Pi}_3$ and $\mathbf{\Pi}_4$ when the dimensions are either evident from the context or are irrelevant.

2.4.1 Algebra of PI Operators

This subsection primarily deals with formally defining *-algebras and proving that the set of PI operators with polynomial parameters (denoted by $[\mathbf{\Pi}_3^{\mathrm{p}}]$ and $[\mathbf{\Pi}_4^{\mathrm{p}}]$) form a *-subalgebra.

Definition 2.7 (Algebra)**.** *A vector space, A, equipped with a multiplication operation, is said to be an algebra if, for every $X, Y \in A$, we have $XY \in A$.*

Definition 2.8 (Associative Algebra)**.** *An algebra, A, is said to be associative if for every $X, Y, Z \in A$*

$$X(YZ) = (XY)Z$$

where XY denotes a multiplication operation between X and Y.

Definition 2.9 (*-algebra)**.** *An algebra, A, over the $\mathbb{R}$ with an involution operation * is called a *-algebra if*

1. $(X^*)^* = X, \quad \forall X \in A$
2. $(X+Y)^* = X^* + Y^*, \quad \forall X, Y \in A$
3. $(XY)^* = Y^* X^*, \quad \forall X, Y \in A$
4. $(\lambda X)^* = \lambda X^*, \quad \forall \lambda \in \mathbb{R}, X \in A$

For the set of PI operators, $\mathbf{\Pi}_i$, we choose the addition of operators and composition of operators as the binary operations addition and multiplication, respectively. The involution is chosen to be the adjoint operation with respect to the $\mathbb{R}L_2$-inner product. Then, we can represent these algebraic operations on PI operators as a linear map of the parameters of the PI operators. Note that the scalar multiplication requirement, point (4) in the above definition, is automatically satisfied because PI operators are linear operators and hence, the proof is omitted here.

Parametric Representation of Operations on $\mathbf{\Pi}_i$: Algebraic operations on $\mathbf{\Pi}_i$ are defined by algebraic operations on the parameters that represent these operators. Specifically, corresponding to $\mathbf{\Pi}_3$ and $\mathbf{\Pi}_4$ let us associate the corresponding parameter spaces

$$[\Gamma_3]_{p,q} = \{\{R_0, R_1, R_2\} : R_i \in L_\infty^{p\times q}, R_1, R_2 \text{ are separable}\},$$

$$[\Gamma_4]^{m,p}_{n,q} = \left\{ \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \;:\; P \in \mathbb{R}^{m\times n},\, Q_1 \in L_\infty^{m\times q},\, Q_2 \in L_\infty^{p\times n},\, \{R_i\} \in [\Gamma_3]_{p,q} \right\}.$$

Then if the parametric maps $\mathbf{P}^i_\times, \mathbf{P}^i_+ : [\Gamma_i] \times [\Gamma_i] \to [\Gamma_i]$, $\mathbf{P}^i_* : [\Gamma_i] \to [\Gamma_i]$ are as defined in Lemmas 2.1 to 2.3, for any $S, T \in [\Gamma_i]$, we have

$$\mathit{\Pi}[\mathbf{P}^i_\times(S,T)] = \mathit{\Pi}[S]\mathit{\Pi}[T], \quad \mathit{\Pi}[\mathbf{P}^i_*(S)] = \mathit{\Pi}[S]^*, \quad \mathit{\Pi}[\mathbf{P}^i_+(S,T)] = \mathit{\Pi}[S] + \mathit{\Pi}[T].$$

Lemma 2.1 (Addition)**.** *For any matrices $A, L \in \mathbb{R}^{m\times p}$ and L_∞-bounded functions $B_1, M_1 : [a,b] \to \mathbb{R}^{m\times q}$, $B_2, M_2 : [a,b] \to \mathbb{R}^{n\times p}$, $C_0, N_0 : [a,b] \to \mathbb{R}^{n\times q}$, and separable functions $C_1, C_2, N_1, N_2 : [a,b]^2 \to \mathbb{R}^{n\times q}$, define a linear map $\mathbf{P}^4_+ : [\Gamma_4]^{m,p}_{n,q} \times [\Gamma_4]^{m,p}_{n,q} \to [\Gamma_4]^{m,p}_{n,q}$ such that*

$$\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] = \mathbf{P}^4_+\left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right], \left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\} \end{array}\right] \right)$$

where

$$P = A + L,\;\; Q_i = B_i + M_i,\;\; R_i = C_i + N_i.$$

If P, Q_i, R_i are as defined above, then, for any $x \in \mathbb{R}^p$ and $\mathbf{z} \in L_2^q([a,b])$

$$\mathit{\Pi}\left[\mathbf{P}_+^4\left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right], \left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\}\end{array}\right]\right)\right]\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}$$
$$= \left(\mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right] + \mathit{\Pi}\left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\}\end{array}\right]\right)\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}.$$

Proof. The proof is in the Appendix B.1. □

Lemma 2.2 (Composition)**.** *For any matrices $A \in \mathbb{R}^{m\times k}, P \in \mathbb{R}^{k\times p}$ and L_∞-bounded functions $B_1 : [a,b] \to \mathbb{R}^{m\times l}, Q_1 : [a,b] \to \mathbb{R}^{k\times q}$, $B_2 : [a,b] \to \mathbb{R}^{n\times k}, Q_2 : [a,b] \to \mathbb{R}^{l\times p}$, $C_0 : [a,b] \to \mathbb{R}^{n\times l}, R_0 : [a,b] \to \mathbb{R}^{l\times q}$, and separable functions $C_1, C_2 : [a,b]^2 \to \mathbb{R}^{n\times l}, R_1, R_2 : [a,b]^2 \to \mathbb{R}^{l\times q}$, define a linear map $\mathbf{P}_\times^4 : [\Gamma_4]_{n,l}^{m,k} \times [\Gamma_4]_{l,q}^{k,p} \to [\Gamma_4]_{n,q}^{m,p}$ such that*

$$\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\}\end{array}\right] = \mathbf{P}_\times^4\left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right], \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\right)$$

where

$$\hat{P} = AP + \int_a^b B_1(s)Q_2(s)ds, \quad \hat{R}_0(s) = C_0(s)R_0(s),$$
$$\hat{Q}_1(s) = AQ_1(s) + B_1(s)R_0(s) + \int_s^b B_1(\eta)R_1(\eta,s)d\eta + \int_a^s B_1(\eta)R_2(\eta,s)d\eta,$$
$$\hat{Q}_2(s) = B_2(s)P + C_0(s)Q_2(s) + \int_a^s C_1(s,\eta)Q_2(\eta)d\eta + \int_s^b C_2(s,\eta)Q_2(\eta)d\eta,$$
$$\hat{R}_1(s,\eta) = B_2(s)Q_1(\eta) + C_0(s)R_1(s,\eta) + C_1(s,\eta)R_0(\eta)$$
$$+ \int_a^\eta C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^s C_1(s,\theta)R_1(\theta,\eta)d\theta + \int_s^b C_2(s,\theta)R_1(\theta,\eta)d\theta,$$
$$\hat{R}_2(s,\eta) = B_2(s)Q_1(\eta) + C_0(s)R_2(s,\eta) + C_2(s,\eta)R_0(\eta)$$
$$+ \int_a^s C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_s^\eta C_2(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^b C_2(s,\theta)R_1(\theta,\eta)d\theta.$$

If $\hat{P}, \hat{Q}_i, \hat{R}_i$ are as defined above, then, for any $x \in \mathbb{R}^m$ and $\mathbf{z} \in L_2^n([a,b])$,

$$\mathcal{P}\left[\mathbf{P}^4_\times\left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right], \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\right)\right]\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}$$
$$= \mathcal{P}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right]\left(\mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}\right).$$

Proof. The proof is in the Appendix B.1. □

Lemma 2.3 (Adjoint). *For any matrices $P \in \mathbb{R}^{m\times p}$ and L_∞-bounded functions $Q_1 : [a,b] \to \mathbb{R}^{m\times q}$, $Q_2 : [a,b] \to \mathbb{R}^{n\times p}$, $R_0 : [a,b] \to \mathbb{R}^{n\times q}$, and separable functions $R_1, R_2 : [a,b]^2 \to \mathbb{R}^{n\times n}$, define a linear map $\mathbf{P}^4_* : [\Gamma_4]^{m,p}_{n,q} \to [\Gamma_4]^{p,m}_{q,n}$ such that*

$$\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\}\end{array}\right] = \mathbf{P}^4_*\left(\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\right)$$

where

$$\begin{aligned}
&\hat{P} = P^T, && \hat{Q}_1(s) = Q_2^T(s), && \hat{Q}_2(s) = Q_1^T(s), \\
&\hat{R}_0(s) = R_0^T(s), && \hat{R}_1(s,\eta) = R_2^T(\eta,s), && \hat{R}_2(s,\eta) = R_1^T(\eta,s).
\end{aligned} \tag{2.4}$$

Then, for any $\mathbf{x} \in \mathbb{R}L_2^{m,n}, \mathbf{y} \in \mathbb{R}L_2^{p,q}$, then we have

$$\left\langle \mathbf{x}, \mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\mathbf{y}\right\rangle_{\mathbb{R}L_2^{m,n}} = \left\langle \mathcal{P}\left[\mathbf{P}^4_*\left(\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\right)\right]\mathbf{x}, \mathbf{y}\right\rangle_{\mathbb{R}L_2^{p,q}}, \tag{2.5}$$

Proof. The proof is in the Appendix B.1. □

Note that the above Lemmas lead to obvious definitions for $\mathbf{P}^3_+$, $\mathbf{P}^3_\times$, and $\mathbf{P}^3_*$ involving only the 3-PI parameters. Thus, we will omit explicit definitions here.

Now that we have formally defined the binary and involution operations on the set of PI operators, the following two results are fairly straightforward and require simple algebraic manipulations, such as changing the variable or the order of integration.

Lemma 2.4. *The set $[\Pi_i]$ equipped with composition operation forms an associative algebra.*

Proof. The proof is in the Appendix B.1. □

Lemma 2.5. *The set $[\Pi_i]$ equipped with the binary operations of addition and composition and the involution operation given by the adjoint w.r.t. $\mathbb{R}L_2$ inner product is a *-algebra.*

Proof. The proof is in the Appendix B.1. □

Lastly, a trivial extension of the above result is that the set of PI operators with polynomial parameters, $\mathbf{\Pi}_i^{\mathrm{p}}$, also forms a *-subalgebra. This is because algebraic operations on the parameters of a PI operator involve addition, multiplication, integration, and transpose, all of which preserve the polynomial form of the parameters. Therefore, all algebraic operations, namely addition, composition, and adjoint, on PI operators with polynomial parameters will give another PI operator with polynomial parameters — i.e., the set $\mathbf{\Pi}_i^{\mathrm{p}}$ is a closed subalgebra of a *-algebra.

2.4.2 Positive PI Operators and Linear PI Inequalities

A convex optimization problem with PI operator decision variables and sign-definite constraints on self-adjoint PI operators is called a 'Linear PI Inequality' (LPI) optimization problem. Such optimization problems arise in various PIE systems analysis, estimation, and control problems as demonstrated in Chapter 1 with a stability test example and later in Part II.

Definition 2.10 (LPI)**.** *A Linear PI Inequality optimization problem is a convex* *optimization of the form*

$$\min_{\mathcal{P}_i \in \Pi} \quad \sum_{i=0}^{m} c_i(\mathcal{P}_i) \quad s.t.,$$

$$\mathcal{P}_i \succeq 0, \quad \mathcal{A}_i^* \mathcal{P}_i \mathcal{A}_i + \mathcal{Q}_i \preceq 0,$$

where $\mathcal{A}_i, \mathcal{Q}_i$ *are known PI operators with* $\mathcal{Q}_i$ *self-adjoint, and* $c_i : \Pi \to \mathbb{R}$ *are linear functionals on PI operators for all* $i \in \{0, \cdots, m\}$.

To solve optimization problems involving PI operators, e.g., $\mathcal{P}_i \in \Pi_4$, we need the ability to enforce/test positivity constraints $\mathcal{P}_i \succeq 0$. For this, we will use an idea identical to the SOS parametrization of positive polynomials introduced earlier in Section 2.2.2. In the case of positive PI operators, we will use a parametrization that is a linear combination of the sum-of-squares of the PI operator basis instead of the polynomial basis. Such a parametrization of PI operators can be written in a quadratic form using positive matrices as $\mathcal{P} = \mathcal{Z}^* P \mathcal{Z}$ for a fixed operator $\mathcal{Z}$ and positive semidefinite matrix $P \succeq 0$. The following theorem provides a sufficient condition for the positivity of a 4-PI operator. This result allows us to parameterize a cone of positive PI operators as positive matrices, implement LPI constraints as LMI constraints, and solve LPI optimization problems using semi-definite programming solvers such as SeDuMi by Sturm (1999), Mosek by Andersen and Andersen (2000), etc.

Theorem 2.6 (Positive PI)**.** *For any functions* $Z_1 : [a, b] \to \mathbb{R}^{d_1 \times n}$, $Z_2 : [a, b] \times$

$[a, b] \to \mathbb{R}^{d_2 \times n}$, *if* $g(s) \geq 0$ *for all* $s \in [a, b]$ *and*

$$
\begin{aligned}
&P = T_{11} \int_a^b g(s) ds, \quad R_0(s) = g(s) Z_1(s)^T T_{22} Z_1(s), \\
&Q(\eta) = g(\eta) T_{12} Z_1(\eta) + \int_\eta^b g(s) T_{13} Z_2(s, \eta) ds + \int_a^\eta g(s) T_{14} Z_2(s, \eta) ds, \\
&R_1(s, \eta) = g(s) Z_1(s)^T T_{23} Z_2(s, \eta) + g(\eta) Z_2(\eta, s)^T T_{42} Z_1(\eta) + \int_s^b g(\theta) Z_2(\theta, s)^T T_{33} Z_2(\theta, \eta) d\theta \\
&\qquad + \int_\eta^s g(\theta) Z_2(\theta, s)^T T_{43} Z_2(\theta, \eta) d\theta + \int_a^\eta g(\theta) Z_2(\theta, s)^T T_{44} Z_2(\theta, \eta) d\theta, \\
&R_2(s, \eta) = g(s) Z_1(s)^T T_{32} Z_2(s, \eta) + g(\eta) Z_2(\eta, s)^T T_{24} Z_1(\eta) + \int_\eta^b g(\theta) Z_2(\theta, s)^T T_{33} Z_2(\theta, \eta) d\theta \\
&\qquad + \int_s^\eta g(\theta) Z_2(\theta, s)^T T_{34} Z_2(\theta, \eta) d\theta + \int_a^s g(\theta) Z_2(\theta, s)^T T_{44} Z_2(\theta, \eta) d\theta.
\end{aligned} \tag{2.6}
$$

where

$$
T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \succeq 0,
$$

then the operator $\mathit{\Pi} \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array} \right]$ *as defined in Equation* (2.3) *is positive semidefinite, i.e.* $\left\langle \mathbf{x}, \mathit{\Pi} \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array} \right] \mathbf{x} \right\rangle \geq 0$ *for all* $\mathbf{x} \in \mathbb{R}^m \times L_2^n[a, b]$.

Proof. The proof is in the Appendix B.2. $\square$

Then, any LPI optimization problem of the form

$$
\min_{\mathcal{P}_i \in \Pi} \qquad \sum_{i=0}^m c_i(\mathcal{P}_i) \qquad s.t., \qquad \mathcal{P}_i \succeq 0, \quad \mathcal{A}_i^* \mathcal{P}_i \mathcal{A}_i + \mathcal{Q}_i \preceq 0,
$$

can be converted to an LMI optimization problem of the form

$$\begin{aligned}
\min_{P_i, Q_i \in \mathbb{R}^{n\times n}} \qquad & \sum_{i=0}^{m} tr(C_i^T A_i) \qquad s.t.,\\
& P_i \succeq 0, \quad P_i^D + Q_i = 0,\\
& \mathcal{P}_i = \mathcal{Z}_i^* P_i \mathcal{Z}_i, \quad \mathcal{A}_i^* \mathcal{P}_i \mathcal{A}_i = \mathcal{Z}_i^* P_i^D \mathcal{Z}_i, \quad \mathcal{Q}_i = \mathcal{Z}_i^* Q_i \mathcal{Z}_i,
\end{aligned}$$

where $\mathcal{Z}_i$ are chosen basis PI operators, C_i are known matrices, $\mathcal{A}_i$, $\mathcal{Q}_i$ are given PI operators with $\mathcal{Q}_i$ self-adjoint. Later in Part II of this dissertation, we will show that the problems in the analysis and control of PIEs, e.g., H_∞-optimal observer and controller design problems, can be formulated as LPI optimization problems.

Since all the conditions presented in this thesis are in the form of LPIs, we developed computational methods for solving LPIs (See the paper by Shivakumar and Peet (2019)). In brief, the bases used in PIETOOLS for a positive PI operator are n^{th}*-order basis* of PI operators, $\mathcal{Z}_n$, whose parameters are monomial vectors up to order n. For example, $\mathcal{P} \succeq 0$ if there exists some matrix $Q \geq 0$ such that $\mathcal{P} = \mathcal{Z}_n^* Q \mathcal{Z}_n = \mathcal{Z}_n^* Q^{\frac{1}{2}} Q^{\frac{1}{2}} \mathcal{Z}_n \succeq 0$, where the basis $\mathcal{Z}_n$ is constructed using a vector of monomials in s up to order n, Z_n, as

$$\mathcal{Z}_n \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(s) = \begin{bmatrix} x \\ Z_n(s)\mathbf{x}(s) \\ \int_a^s (Z_n(s) \otimes Z_n(\theta))\mathbf{x}(\theta) d\theta \\ \int_s^b (Z_n(s) \otimes Z_n(\theta))\mathbf{x}(\theta) d\theta \end{bmatrix}, \tag{2.7}$$

where $\otimes$ denotes the tensor product. The highest order of these monomials, n, can be used as a measure for the order of complexity of the LPIs, which will later be used to ascertain the accuracy and convergence of the solution to an LPI optimization problem.

Lastly, we will use the MATLAB toolbox that was developed to solve LPI optimization problems, PIETOOLS, because the toolbox offers convenient MATLAB functions to convert PDEs to PIE, declare PI decision variables, add LPI constraints, and solve the resulting optimization problem. We refer to the PIETOOLS User Manual by Shivakumar *et al.* (2021) for details.

2.4.3 Inverse of PI Operators

In this subsection, we address the problem of inverting PI operators of the form $\mathcal{\Pi}\left[\begin{array}{c|c} P & Q \\ \hline Q^T & \{R_i\} \end{array}\right] \in \mathbf{\Pi}_4$. Such an inverse is needed in order to reconstruct the controller gains $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$ or observer gains $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ where $\mathcal{P}, \mathcal{Z}$ are solutions obtained from an LPI optimization problem. In the special case where $\mathcal{P} = \mathcal{P}^* \succ 0$ and $R_1 = R_2$, an *analytic* expression for this inverse was given by Peet (2020a). However, the restriction $R_1 = R_2$ to integral operators with *separable* kernels introduces significant conservatism and reduces the accuracy of LPI tests for H_∞-norm bounds and H_∞-optimal control. Therefore, we need an analytical expression to construct the inverse without constraining the parameters of the PI operator. For this purpose, we use an iterative algorithm for constructing the inverse based on the generalization of a method proposed by (Gohberg *et al.*, 2013, Chapter IX.2), which is restated below in a concise form without proof.

Lemma 2.7. *Define $\mathcal{P} = \mathcal{\Pi}_{\{I,H_1,H_2\}}$ where $H_i(s,t) = -F_i(s)G_i(t)$ for some $F_i, G_i \in L_2[a,b]$. Let U and V be the unique solutions to*

$$U(s) = I + \int_a^s B(t)C(t)U(t)dt \quad \text{and} \quad V(t) = I - \int_a^t V(s)B(s)C(s)ds$$

where $C(s) = \begin{bmatrix} F_1(s) & F_2(s) \end{bmatrix}$, $B(s) = \begin{bmatrix} G_1(s) \\ -G_2(s) \end{bmatrix}$. Then $V(s)U(s) = U(s)V(s) = I$. Furthermore, if we partition

$$U(b) = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad U_{22} \in \mathbb{R}^{q\times q},$$

where q is the number of columns in F_2, then $\mathcal{P}$ is invertible if and only if U_{22} is invertible and $\mathcal{P}^{-1} = \mathit{\Pi}_{\{I,L_1,L_2\}}$ where

$$P = \begin{bmatrix} 0 & 0 \\ U_{22}^{-1}U_{21} & I_q \end{bmatrix}, \quad \begin{matrix} L_1(s,t) = C(s)U(s)(I-P)V(t)B(t), \\ L_2(s,t) = -C(s)U(s)PV(t)B(t). \end{matrix}$$

Note that, by construction, L_1 and L_2 are *separable* – i.e., there exist H_i, J_i such that $L_1(s,\theta) = H_1(s)J_1(\theta)$ and $L_2(s,\theta) = H_2(s)J_2(\theta)$. This implies that $\mathcal{P}^{-1}$ is a PI operator, albeit not necessarily with polynomial parameters. We can extend this result to a more general class of integral operators (PI operators of the form $\mathit{\Pi}_{\{R_0,R_1,R_2\}}$) as follows.

Corollary 2.8. *Given $H_0(s)$, $H_1(s,\theta) = H_{1a}(s)H_{1b}(\theta)$, and $H_2(s,\theta) = H_{2a}(s)H_{2b}(\theta)$, with H_0 invertible, define $F_i = -H_0^{-1}H_{ia}$ and $G_i = H_{ib}$. Then, if $\mathcal{P} = \mathit{\Pi}_{\{I,H_0^{-1}H_1,H_0^{-1}H_2\}}$ is invertible and $\mathcal{P}^{-1}$ is as defined in Lemma 2.7, we have $\mathit{\Pi}^{-1}_{\{H_i\}} = \mathit{\Pi}_{\{\hat{R}_i\}}$ where $\hat{R}_0 = H_0^{-1}$, $\hat{R}_1(s,\theta) = L_2(s,\theta)\hat{R}_0(\theta)$, and $\hat{R}_2(s,\theta) = L_1(s,\theta)\hat{R}_0(\theta)$.*

Proof. If H_i, $\hat{R}_i$, L_i are as defined, from the rules for composition given in Lemma 2.2, we have

$$\begin{aligned}
&\mathit{\Pi}^{-1}_{\{H_0,H_1,H_2\}} \\
&= (\mathit{\Pi}_{\{H_0,0,0\}}\mathit{\Pi}_{\{I,\hat{R}_0(s)H_1(s,\theta),\hat{R}_0(s)H_2(s,\theta)\}})^{-1} = \mathit{\Pi}^{-1}_{\{I,\hat{R}_0(s)H_1(s,\theta),\hat{R}_0(s)H_2(s,\theta)\}}\mathit{\Pi}^{-1}_{\{H_0,0,0\}} \\
&= \left(\mathit{\Pi}_{\{I,-F_1(s)G_1(\theta),-F_2(s)G_2(\theta)\}}\right)^{-1}\mathit{\Pi}_{\{\hat{R}_0,0,0\}} = \mathit{\Pi}_{\{I,L_1,L_2\}}\mathit{\Pi}_{\{\hat{R}_0,0,0\}} = \mathit{\Pi}_{\{\hat{R}_0,\hat{R}_1,\hat{R}_2\}}.
\end{aligned}$$

□

The inverse $\mathcal{P}^{-1}$ in Lemma 2.7 (and Corollary 2.8) is defined in terms of some matrix-valued functions U, V which satisfy a set of Volterra-type integral equations of the 2^{nd} kind. However, Lemma 2.7 does not provide a method for solving these equations to find U and V. Fortunately, however, this class of integral equations has been well-studied, and iterative algorithms have been proposed for solving this class of equations. Our approach to constructing the inverse is based on the method of successive approximation class of algorithms, the convergence of which has been established in the following Lemma that's proved in (Gohberg *et al.*, 2013, Chapter IX.2). Also, see approaches presented by Brunner (2017) and Rahman (2007).

Lemma 2.9. *Let $A : [a, b] \to \mathbb{R}^{n\times n}$ be Lebesgue integrable on $[a, b]$. Then, the series $I_n + \sum_{i=1}^{\infty} U_k(s)$, where $U_k = \int_a^s A(\theta)U_{k-1}(\theta)d\theta$ and $U_1(s) = \int_a^s A(\theta)d\theta$, converges uniformly on $s \in [a, b]$ to a unique function, $U : [a, b] \to \mathbb{R}^{n\times n}$, that solves $U(s) = I_n + \int_a^s A(\theta)U(\theta)d\theta$. Furthermore, for any $k \in \mathbb{N}$,*

$$\|U_k(s)\| \le \frac{1}{k!}\left(\int_a^b \|A(s)\|\, ds\right)^k, \quad s \in [a, b].$$

Lemma 2.9 can be used to formulate an algorithm for computing the $U(s)$ and $V(t)$ matrix function needed to define $\mathcal{P}^{-1}$ in Lemma 2.7 (V is found by solving for its transpose) as follows.

Algorithm 1 Approximating the inverse of
$\mathcal{R}\mathbf{x}(s) = \mathbf{x}(s) - \int_a^s F_1(s)G_1(\theta)\mathbf{x}(\theta)d\theta - \int_s^b F_2(s)G_2(\theta)\mathbf{x}(\theta)d\theta$

Given: n, ϵ, $[a, b]$, F_i, G_i. Set: $U_0 = V_0 = I, N = 0$.

for $i \in \{0, \cdots, n\}$ **do** $s_i = a + \frac{i(b-a)}{n}$

$$C(s_i) = \begin{bmatrix} F_1(s_i) & F_2(s_i) \end{bmatrix}, \; B(s_i) = \begin{bmatrix} G_1(s_i) \\ -G_2(s_i) \end{bmatrix}, \; A(s_i) = B(s_i)C(s_i)$$

end for

while $\left(\sum_{i=1}^{i=n} \|A(s_i)\|\right)^k \ge \epsilon \cdot k!$ **do** $N = N + 1$

for $i \in \{1, \cdots, n\}$ **do**

$$U_{k+1}(s_i) = \frac{(b-a)}{2n} \sum_{j=1}^{j=i} \begin{bmatrix} A(s_j) & A(s_{j-1}) \end{bmatrix} \begin{bmatrix} U_k(s_j) \\ U_k(s_{j-1}) \end{bmatrix}$$

$$V_{k+1}(s_i) = \frac{(a-b)}{2n} \sum_{j=1}^{j=i} \begin{bmatrix} V_k(s_j) & V_k(s_{j-1}) \end{bmatrix} \begin{bmatrix} A(s_j) \\ A(s_{j-1}) \end{bmatrix}$$

end for

end while

for $i \in \{0, \cdots, n\}$ **do**

$U(s_i) = \sum_{i=0}^{N} U_k(s_i) \quad V(s_i) = \sum_{i=0}^{N} V_k(s_i)$

$U(s_i) = C(s_i)U(s_i) \quad V(s_i) = V(s_i)B(s_i)$.

end for

$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = U(b) \quad P = \begin{bmatrix} 0 & 0 \\ U_{22}^{-1}U_{21} & I \end{bmatrix}$

Solve the problem

$$\min_{\alpha,\beta \in \mathbb{R}^{d+1}} \quad \sum_{i=0}^{n} \|U_p(s_i) - U(s_i)\|_2^2 + \|V_p(s_i) - V(s_i)\|_2^2$$

$$s.t. \quad U_p(s) = \alpha \mathrm{col}(1, s, \cdots, s^d), \quad V_p(s) = \beta \mathrm{col}(1, s, \cdots, s^d).$$

$L_1(s,t) = U_p(s)(I-P)V_p(t) \quad L_2(s,t) = -U_p(s)PV_p(t)$ **return** L_1, L_2

The algorithm presented above finds an approximation of the inverse of $\mathcal{R}$ at various locations $s_i \in [a,b]$ and a polynomial is fit, using least-squares regression, at the end to express $\mathcal{R}^{-1}$ as a PI operator of the form $\mathit{\Pi}_{\{I,L_1,L_2\}}$.

Corollary 2.8, unfortunately, only works for $\mathcal{P} \in \mathbf{\Pi}_3$. However, one can extend this result, with some additional assumptions, to any $\mathcal{P} \in \mathbf{\Pi}_4$ using a generalization of a standard formula for block matrix inversion.

Lemma 2.10. *Suppose* $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \in \mathbf{\Pi}_4$ *with* P *invertible, then* $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]$ *is invertible if and only if* $\mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{H_i\} \end{array}\right]$ *is invertible where* $H_0 = R_0$ *and* $H_i(s,\theta) = R_i(s,\theta) - Q_2(s)P^{-1}Q_1(\theta)$. *Furthermore, if* H_i *satisfy the conditions of Corollary 2.8 and* $\hat{R}_i$ *are as defined therein, we have that*

$$\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]^{-1} = \mathit{\Pi}\left[\begin{array}{c|c} I & -P^{-1}Q_1 \\ \hline 0 & \{I,0,0\} \end{array}\right] \mathit{\Pi}\left[\begin{array}{c|c} P^{-1} & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right] \mathit{\Pi}\left[\begin{array}{c|c} I & 0 \\ \hline -Q_2P^{-1} & \{I,0,0\} \end{array}\right].$$

Proof. Suppose P is invertible. Then

$$\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] = \overbrace{\Pi\left[\begin{array}{c|c} I & 0 \\ \hline Q_2 P^{-1} & \{I,0,0\} \end{array}\right]}^{\mathcal{M}=} \overbrace{\Pi\left[\begin{array}{c|c} P & 0 \\ \hline 0 & \{R_i\} \end{array}\right]}^{\mathcal{Q}=} \overbrace{\Pi\left[\begin{array}{c|c} I & P^{-1}Q_1 \\ \hline 0 & \{I,0,0\} \end{array}\right]}^{\mathcal{N}=}.$$

Clearly, $\mathcal{M},\mathcal{N}$ are invertible with

$$\mathcal{N}^{-1} = \Pi\left[\begin{array}{c|c} I & -P^{-1}Q_1 \\ \hline 0 & \{I,0,0\} \end{array}\right], \mathcal{M}^{-1} = \Pi\left[\begin{array}{c|c} I & 0 \\ \hline -Q_2 P^{-1} & \{I,0,0\} \end{array}\right].$$

Hence invertibility of $\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]$ is now equivalent to invertibility of $\Pi\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \{R_i\} \end{array}\right]$. Now if $\hat{R}_i$ are as defined in Corollary 2.8, we have

$$\mathcal{Q}^{-1} = \Pi\left[\begin{array}{c|c} P & 0 \\ \hline 0 & \{R_i\} \end{array}\right]^{-1} = \Pi\left[\begin{array}{c|c} P^{-1} & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right]$$

which completes the proof. □

The inversion formula for PI operators in Lemma 2.10 is expressed in terms of the composition of PI operators. This implies the inverse is a PI operator, and using the composition rules for PI operators listed above, we may obtain a precise expression for the parameters in this inverse.

$$\Pi\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\} \end{array}\right] = \Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]^{-1}$$

For convenience, these are listed in Appendix B.3.

Note that while Lemma 2.10 requires invertibility of the matrix P, such invertibility is not necessary for invertibility of the operator $\mathcal{P} = \Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]$. However, for controller reconstruction, we often require strict positivity of P to establish stability of the real part of the state – such as in the case of time-delay systems. In this case, Lemma 2.10 is necessary and sufficient. However, the invertibility of P may be avoided if, instead, $\Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_i\} \end{array}\right]$ is invertible. Such invertibility is ensured in the case of strict positivity of these operators – i.e., $R_0(s) \geq \epsilon I$. In this case, we consider generalizing the second formula for block matrix inversion.

Lemma 2.11. *Suppose* $\mathcal{P} = \Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \in \mathbf{\Pi}_4$ *with* $\Pi_{\{R_i\}}$ *invertible. Then* $\mathcal{P}$ *is invertible if and only if the matrix*

$$\hat{P} = P - \Pi\left[\begin{array}{c|c} \emptyset & Q_1 \\ \hline \emptyset & \{\emptyset\} \end{array}\right] \Pi_{\{R_i\}}^{-1} \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline Q_2 & \{\emptyset\} \end{array}\right]$$

is invertible. Furthermore, if $H_i = R_i$ *satisfy the conditions of Corollary 2.8 and* $\hat{R}_i$ *are as defined therein, we have that*

$$\mathcal{P}^{-1} = \mathcal{U}\,\Pi\left[\begin{array}{c|c} \hat{P}^{-1} & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right] \mathcal{V}$$

where

$$\mathcal{U} = \Pi\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right] \Pi\left[\begin{array}{c|c} I & 0 \\ \hline -Q_2 & \{R_i\} \end{array}\right] \qquad \mathcal{V} = \Pi\left[\begin{array}{c|c} I & -Q_1 \\ \hline 0 & \{R_i\} \end{array}\right] \Pi\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right].$$

Proof. Suppose $\Pi_{\{R_i\}}$ is invertible. Then, by defining

$$\mathcal{R} = \Pi\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right],$$

we have

$$\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] = \overbrace{\Pi\left[\begin{array}{c|c} I & Q_1 \\ \hline 0 & \{R_i\} \end{array}\right] \mathcal{R}}^{\mathcal{M}=} \overbrace{\Pi\left[\begin{array}{c|c} \hat{P} & 0 \\ \hline 0 & \{R_i\} \end{array}\right]}^{\mathcal{Q}=} \overbrace{\mathcal{R}\,\Pi\left[\begin{array}{c|c} I & 0 \\ \hline Q_2 & \{R_i\} \end{array}\right]}^{\mathcal{N}=}.$$

Clearly, $\mathcal{M}, \mathcal{N}$ are invertible with $\mathcal{N}^{-1} = \mathcal{U}$ and $\mathcal{M}^{-1} = \mathcal{V}$. Hence, the invertibility of $\mathcal{P}$ is now equivalent to the invertibility of $\hat{P}$. Now if $\hat{R}_i$ are as defined in Corollary 2.8, we have

$$\mathcal{Q}^{-1} = \Pi\left[\begin{array}{c|c} \hat{P} & 0 \\ \hline 0 & \{R_i\} \end{array}\right]^{-1} = \Pi\left[\begin{array}{c|c} \hat{P}^{-1} & 0 \\ \hline 0 & \{\hat{R}_i\} \end{array}\right]$$

which completes the proof. $\square$

Given a PI operator invertible, we may now reconstruct the inverse by using Lemma 2.11 in combination with Algorithm 1 to find the solution of the Volterra integral equations in Lemma 2.7. This will prove useful in Chapter 8 to reconstruct observer and controller gains to be implemented on the original PDE system.

2.5 PIETOOLS

Lastly, we will briefly introduce PIETOOLS, a toolbox developed to perform the operations introduced in the previous section using simple high-level programming syntax. Specifically, this toolbox allows one to define and manipulate PI operators in Matlab in a manner similar to the matrix variables. For example, one can declare a PI operator $\mathcal{P} : \mathbb{R}L_2^{2,1}[0,1] \to \mathbb{R}L_2^{1,2}[0,1]$ given by

$$\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} -1 & 2 \end{bmatrix} & 3 - s^2 \\ \hline \begin{bmatrix} 0 & -s \\ s & 0 \end{bmatrix} & \left\{ \begin{bmatrix} 1 \\ s^3 \end{bmatrix}, \begin{bmatrix} s-\theta \\ \theta \end{bmatrix}, \begin{bmatrix} s \\ \theta - s \end{bmatrix} \right\} \end{array}\right]$$

and assign parameters using the code

```
>> pvar s theta; opvar P;
>> P.P = [-1,2]; P.Q1 = (3-s^2); P.Q2 = [0,-s; s,0];
>> P.R.R0 = [1; s^3]; P.R.R1 = [s-theta; theta]; P.R.R2 = [s; theta-s];
```

Once defined, one can manipulate such PI operator variables, similar to matrices, such as

1. For addition of two PI operators: `A+B`
2. For composition of two PI operators: `A*B`
3. For adjoint of a PI operator: `A'`
4. For concatenation: `[A; B]` or `[A, B]`

5. For inversion: `inv(A)`

While the above operations simplify the process of setting up/parsing an LPI optimization problem in Matlab, most of the LPIs presented in this work have been implemented as standard functions in PIETOOLS for convenience. Thus, provided appropriate supporting libraries, such as SOSTOOLS, Multipoly toolbox by Seiler (2013), and a supported SDP solver, are installed, one can solve such LPIs just using simple function calls. While a detailed discussion about all the features offered by PIETOOLS can be found in the user manual by Shivakumar *et al.* (2021), we will mention a few key steps that will allow one to solve the numerical examples presented in this work.

A standard PIETOOLS program/script will require the following steps to be performed in proper order:

1. Define independent variables

```
>> pvar s t theta;
```

2. Define a PDE (either using the symbolic parser or the GUI). Here, we will demonstrate define a reaction-diffusion PDE using the symbolic parser. Given a PDE

$$\dot{\mathbf{x}}(t,s) = 5\mathbf{x}(t,s) + \partial_s^2\mathbf{x}(t,s) + w(t),$$
$$z(t) = \int_0^1 \mathbf{x}(t,s)ds, \quad \mathbf{x}(t,0) = \partial_s(t,1) = 0,$$

we can define the PDE in PIETOOLS as

```
>> x=state('pde'); z = state('out'); w = state('in'); pde = sys();
>> eqns = [diff(x,t)==5*x+diff(x,s,2)+w; z==int(x,s,[0,1];
          subs(x,s,0)==0; subs(diff(x,s),s,1)==0];
>> pde = addequation(pde,eqns);
```

3. Convert PDE to its PIE form

```
>> pie = convert(pde,'pie');
```

4. Set up an LPI optimization problem. For instance, to test stability, one can use the code

```
>> opts = lpisettings('heavy');
>> [sol,P]= lpisolve(pie,opts,'stability');
```

where `sol` is a structure that stores the optimization problem solution status, `P` $= \mathcal{P}$ is the PI operator in the Lyapunov function $V(\mathbf{x}) = \langle \mathbf{x}, \mathcal{P}\mathbf{x} \rangle$ that proves the stability of the PIE, and hence the PDE. Details on the proof behind these conclusions will be presented in Chapters 5 and 6.

5. Simulate the PDE for some initial conditions and input using `piesim` as

```
>> opts.pde.ic = s*(s-1); % initial condition for x
>> opts.tf = 5; % simulation time in seconds
>> opts.N = 8; % number of Chebyshev bases
>> inp.w = sin(5*t); % input disturbance w(t)
>> [solution] = piesim(pde,opts,inp);
```

where `solution` is a structure that stores the time-dependent solution $\mathbf{x}_N(t,s)$ that is an Nth-order Chebyshev approximation of the solution of the PDE, $\mathbf{x}(t,s)$.

Likewise, one can solve various analysis, estimation, and control problems by changing the third argument of the function `lpisolve()`, e.g., `'l2gain'` for H_∞-norm bound, `'hinf-estimator'` for H_∞-optimal state observer design, `'hinf-control'` for H_∞-optimal state-feedback controller synthesis. If solving an observer design or controller synthesis problem, one can construct the closed-loop system using the function call `pie_cl= closedLoopPIE(pie,K,'control');` and then simulate using `piesim()` function. The code required to run the numerical examples will not be presented in the document; however, it can be found online in GitHub repository of Shivakumar *et al.* (2020b).

The remainder of the thesis will utilize the concepts introduced in this chapter to formulate the analysis, estimation, and control problems of linear PDEs in one spatial dimension as convex LPI optimization problems that can be solved using LMIs. However, we must first express PDEs that have 3 constraints and unbounded and non-algebraic operators using PI operators that are algebraic. Furthermore, we must show that such a transformation does not affect any of the system properties that are being investigated. Therefore, Part I will focus on formally introducing the class of PDEs that admit such representations and prove the properties of the new representation.

Part I

REPRESENTATION AND PARAMETRIZATION OF LINEAR INFINITE-DIMENSIONAL SYSTEMS

Chapter 3

GENERALIZED PARTIAL DIFFERENTIAL EQUATIONS

3.1 Introduction

As briefly mentioned in Chapter 1, a universal parametric representation of linear PDEs in one spatial dimension does not exist. However, to develop a computational framework for the analysis, estimation, and control of PDEs, we need to establish some standard representation for the PDEs so that a PDE can be defined unambiguously and passed as an input to this framework. Therefore, in this chapter, we will introduce a parametric representation for linear PDE systems to encompass a large class of PDEs, such as linear parabolic PDEs with nth-order spatial derivatives, ODE coupling, integral terms, boundary values terms, and inputs all of which can be present either in the dynamics of the PDE, the outputs, or the boundary conditions. We will later derive a PIE representation before the analysis and design step for the PDEs that can be represented in the parametric form proposed here.

Before presenting the complete parametric form of a general linear PDE model considered in this work, let us illustrate a typical PDE model using a diffusion equation example to identify the various parameters needed to fully define a PDE:

$$\begin{gathered}\dot{\mathbf{x}}(t,s) = c\partial_s^2\mathbf{x}(t,s), \qquad s\in(0,1), \quad t\geq 0\\ \mathbf{x}(t,0)=0 \qquad \mathbf{x}(t,1)=0,\\ \mathbf{x}(t,\cdot)\in X=\{x\in L_2[0,1]\ :\partial_s x, \partial_s^2 x\in L_2[0,1]\}.\end{gathered}$$

The first constraint, referred to as the 'PDE dynamics', dictates the change of state $\mathbf{x}$ within the interior of the domain, $s\in(0,1)$. The two algebraic conditions, $\mathbf{x}(t,0)=0$

and $\mathbf{x}(t, 1) = 0$, are referred to as 'boundary conditions' since these dictate the value of the state $\mathbf{x}$ at the boundary. Lastly, the continuity constraints on the PDE state $\mathbf{x}$ are specified by $\mathbf{x}(t, \cdot) \in X$. In general, a PDE model is of interest only if it is well-posed. While an exhaustive set of conditions for well-posedness may be difficult to formulate, one can specify certain baseline conditions that are sensible mathematically and useful in modeling a physical phenomenon. An example of such baseline conditions is Hadamard's criteria, proposed by Hadamard (1902), for well-posedness, which are listed below:

- The problem has a solution.
- The solution is unique.
- The solution's behavior changes continuously with the initial conditions.

The first criterion, although it is the most fundamental requirement, is also the most vague requirement because the criterion itself does not specify what a **solution** is or what properties a solution must have. Typically, in the case of PDEs, the notion of a solution is obtained from the physics that is being modeled. In the heat equation PDE modeling the temperature distribution in a 1D rod, for instance, this criterion would imply that the solution must be sufficiently continuous for the spatial derivatives and boundary values to be well-defined. However, many PDEs used to model various phenomenon, if not most, do not have a solution that is differentiable in the classical sense but such PDEs have important applications and must not be excluded — e.g., transport equation used to model traffic flow. Hence, to satisfy the first criterion of Hadamard, we require the $\mathbf{x}$ to be only weakly differentiable and allow PDEs to have weakly differentiable solutions to expand the class of PDEs that can be considered well-posed.

The latter two of Hadamard's criteria are more direct in the sense they can be stated mathematically as: if $\mathbf{x}$ and $\mathbf{y}$ are solutions of the PDE then $\mathbf{x} = \mathbf{y}$, and if $\mathbf{x}(0, s)$ is perturbed by $\delta > 0$ then $\mathbf{x}(t, s)$ is perturbed by some $\epsilon > 0$. These two criteria are motivated from the perspective of application. Uniqueness is required because if a model of the system has multiple solutions, then one cannot possibly anticipate the solution trajectory that will be taken by the system — undermining the primary purpose (of analysis/control) of modeling the system. The last criterion ensures that minor errors in initial conditions, which are inevitable due to limited precision and numerical errors, do not lead to large errors later on. To summarize, any well-posed solution to the above heat equation PDE must: a) satisfy all 3 constraints, strongly or weakly; b) must be unique; and c) depend continuously on initial conditions.

Since the first criterion is the most ambiguous of the three, we will look into it more thoroughly in this chapter. Particularly, since the criterion is determined by the dynamics, BCs, and continuity constraints, we will focus on these 3 constraints. The 3 constraints are independent to some extent and can, in fact, be parameterized separately, as will be shown later. The constraints in other PDEs, with more complicated terms such as ODE coupling, integral terms, etc., also conform to this categorization. Therefore, instead of providing more illustrations of such PDEs, we will skip a beat and directly consider the exhaustive class of PDEs (with ODEs, inputs, and outputs) that will be handled in the dissertation while defining the standard parametrization.

In the following sections, we first parameterize the class of ODE-PDE models for which we may solve the analysis, estimation, and control problems. To account for the generality of the class of PDEs being considered, we will see that a large number of constraints and parameters are required in the standard representation. Furthermore, the number of these constraints and parameters will depend on the type of PDE and thus is not a fixed quantity. Therefore, to simplify the notation and analysis, we will compartmentalize the constraints of these models into two subsystems – ODE and PDE subsystems – that are interconnected; See Figure 3.3. This class of interconnected ODE-PDE models will be referred to as 'Generalized Partial Differential Equations' (GPDEs). Having separated the constraints into subsystems, we then further categorize the parameters associated with these constraints based on the type of constraint, namely, dynamics, boundary conditions, and continuity constraints. The constraints and parametrization of the ODE subsystem are defined in Section 3.2.1, whereas the constraints and the parametrization of the PDE subsystem are defined in Section 3.2.2. Finally, the subsystems are combined in Section 3.2.3 to obtain the full standard representation of a GPDE system.

3.2 Parametrization of GPDEs

In this section, we will introduce the parameters of the ODE and PDE subsystems separately. While the parameters of an ODE subsystem can be represented by a single set containing 12 matrices, the parameters of a PDE subsystem would require 3 different sets since a PDE is defined by 3 different types of constraints, namely, the dynamics, the boundary conditions, and the continuity constraints. Thus, we will need a total of 4 sets of parameters to fully define a GPDE as will be shown in the following subsections.

3.2.1 ODE Subsystem

The ODE subsystem of the GPDE model, illustrated in Figure 3.1, is a typical state-space representation with real-valued inputs and outputs. These inputs and outputs are finite-dimensional and include both the interconnection with the PDE subsystem and the inputs and outputs of the GPDE model as a whole. Specifically, we partition both the input and output signals into 3 components, differentiating these channels by function. The input channels are: the control input to the GPDE ($u(t) \in \mathbb{R}^{n_u}$), the exogenous disturbance/source driving the GPDE ($w(t) \in \mathbb{R}^{n_w}$) and the internal feedback input ($r(t) \in \mathbb{R}^{n_r}$) which is the output of the PDE subsystem. The output channels of the ODE subsystem are: the regulated output of the GPDE ($z(t) \in \mathbb{R}^{n_z}$); the sensed outputs of the GPDE ($y(t) \in \mathbb{R}^{n_y}$); and the output from the ODE subsystem which becomes the input to the PDE subsystem ($v(t) \in \mathbb{R}^{n_v}$).

Definition 3.1 (Solution of an ODE Subsystem)**.** *Given matrices A, B_{xw}, B_{xu}, B_{xr}, C_z, D_{zw}, D_{zu}, D_{zr}, C_y, D_{yw}, D_{yu}, D_{yr}, C_v, D_{vw}, D_{vu} of appropriate dimension, we say $\{x, z, y, v\}$ with $\{x(t), z(t), y(t), v(t)\} \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_v}$ satisfies the ODE with initial condition $x^0 \in \mathbb{R}^{n_x}$ and input $\{w, u, r\}$ if x is differentiable, $x(0) = x^0$ and for $t \geq 0$*

$$\left[\begin{array}{c} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{array}\right] = \left[\begin{array}{c|ccc} A & B_{xw} & B_{xu} & B_{xr} \\ \hline C_z & D_{zw} & D_{zu} & D_{zr} \\ C_y & D_{yw} & D_{yu} & D_{yr} \\ C_v & D_{vw} & D_{vu} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{array}\right]. \tag{3.1}$$

Notation: For brevity, we collect all matrix parameters from the ODE subsystem in Equation (3.1) and introduce the shorthand notation $\mathbf{G}_\mathrm{o}$ which represents the labeled tuple of such parameters as

$$\mathbf{G}_\mathrm{o} = \{A, B_{xw}, B_{xu}, B_{xr}, C_z, D_{zw}, D_{zu}, D_{zr}, C_y, D_{yw}, D_{yu}, D_{yr}, C_v, D_{vw}, D_{vu}\}. \tag{3.2}$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions. Further note that, while the size of the matrices may vary, the number of parameters in this set, $\mathbf{G}_\mathrm{o}$, remain fixed.

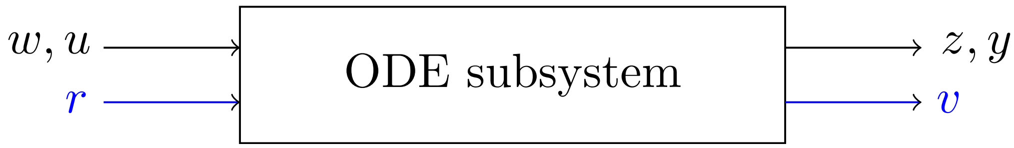


Figure 3.1: Depiction of the ODE subsystem for use in defining a GPDE. All external input signals in the GPDE model pass through the ODE subsystem and are labeled as $u(t) \in \mathbb{R}^{n_u}$ and $w(t) \in \mathbb{R}^{n_w}$, corresponding to control input and disturbance/forcing input. Likewise, all external outputs pass through the ODE subsystem and are labeled $y(t) \in \mathbb{R}^{n_y}$ and $z(t) \in \mathbb{R}^{n_z}$, corresponding to measured output and regulated output. All interaction with the PDE subsystem is routed through two vector-valued signals, where $r(t) \in \mathbb{R}^{n_r}$ is the sole output of the PDE subsystem and $v(t) \in \mathbb{R}^{n_v}$ is the sole input to the PDE subsystem.

3.2.2 PDE Subsystem

Our parametrization of the PDE subsystem is divided into three parts: the continuity constraints, the in-domain dynamics, and the BCs. The continuity constraints specify the existence of partial derivatives and boundary values for each state as required by the in-domain dynamics and BCs. The BCs are represented as a real-valued algebraic constraint subsystem that maps the distributed state and inputs to a vector of boundary values. The in-domain dynamics (or generating equation) specify the time derivative of the state, $\hat{\mathbf{x}}(t, s)$, at every point in the interior of the domain, and are expressed using integral, Dirac, and N^{th}-order spatial derivative operators. The PDE subsystem is illustrated in Figure 3.2. For simplicity, no external inputs or outputs are defined for the PDE subsystem since these external signals may be included by routing the desired signal through the ODE subsystem using the internal signals, $v(t)$ and $r(t)$.

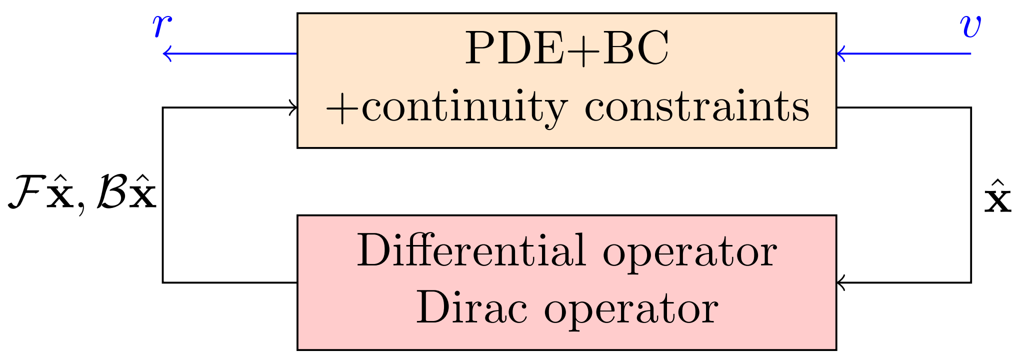


Figure 3.2: Depiction of the PDE subsystem for defining a GPDE. All interaction of the PDE subsystem with the ODE subsystem is routed through the two vector-valued signals, r and v, where $r(t) \in \mathbb{R}^{n_r}$ is an output of the PDE subsystem (and input to the ODE subsystem) and $v(t) \in \mathbb{R}^{n_v}$ is an input to the PDE subsystem (and output from the ODE subsystem). Although there are no external inputs and outputs of the GPDE, such signals can be routed to and from the PDE subsystem through the ODE subsystem using r and v.

The Continuity Constraint

The 'continuity constraint' partitions the state vector of the PDE subsystem, $\hat{\mathbf{x}}(t,\cdot)$, and specifies the differentiability properties of each partition as required for existence of the partial derivatives in the generator and limit values in the boundary condition. This partition is defined by the parameter $n \in \mathbb{N}^{N+1} = \{n_0, \cdots n_N\}$, wherein n_i specifies the dimension of the ith partition vector so that $\hat{\mathbf{x}}_i(t,s) \in \mathbb{R}^{n_i}$. The partitions are ordered by increasing differentiability so that

$$\hat{\mathbf{x}}(t,\cdot) = \begin{bmatrix} \hat{\mathbf{x}}_0(t,\cdot) \\ \vdots \\ \hat{\mathbf{x}}_N(t,\cdot) \end{bmatrix} \in W^n = \begin{bmatrix} W_0^{n_0} \\ \vdots \\ W_N^{n_N} \end{bmatrix}.$$

Given the partition defined by $n \in \mathbb{N}^{N+1}$, and given $\hat{\mathbf{x}} \in W^n$, we would like to list all well-defined partial derivatives of $\hat{\mathbf{x}}$. To do this, we first define $n_{\hat{\mathbf{x}}} = |n|_1 = \sum_{i=0}^N n_i$

to be the number of states in $\hat{\mathbf{x}}$, $n_{S_i} = \sum_{j=i}^{N} n_j \le n_{\hat{\mathbf{x}}}$ to be the total number of i-times differentiable states, and $n_S = \sum_{i=1}^{N} n_{S_i}$ to be the total number of possible partial derivatives of $\hat{\mathbf{x}}$ as permitted by the continuity constraint.

Notation: For indexed vectors (such as n or $\hat{\mathbf{x}}$) we occasionally use the notation $\hat{\mathbf{x}}_{i:j}$ to denote the components i to j. Specifically, $\hat{\mathbf{x}}_{i:j} = \mathrm{col}(\hat{\mathbf{x}}_i, \cdots, \hat{\mathbf{x}}_j)$, $n_{i:j} := \sum_{k=i}^{j} n_k$ and $n_{S_{i:j}} = \sum_{k=i}^{j} n_{S_k}$.

Next, we define the selection operator $S^i : \mathbb{R}^{n_{\hat{\mathbf{x}}}} \to \mathbb{R}^{n_{S_i}}$ which is used to select only those states in $\hat{\mathbf{x}}$ which are at least i-times differentiable. Specifically, for $\hat{\mathbf{x}} \in W^n$, we have $S^i = \begin{bmatrix} 0_{n_{S_i} \times n_{\hat{\mathbf{x}}} - n_{S_i}} & I_{n_{S_i}} \end{bmatrix}$, so that $(S^i\hat{\mathbf{x}})(s) = \begin{bmatrix} \hat{\mathbf{x}}_i(s) \\ \vdots \\ \hat{\mathbf{x}}_N(s) \end{bmatrix}$. We may now conveniently represent all well-defined ith-order partial derivatives of $\hat{\mathbf{x}}$ as $\partial_s^i S^i \hat{\mathbf{x}}$ so that

$$(\partial_s^i S^i\hat{\mathbf{x}})(s) = \begin{bmatrix} \partial_s^i \hat{\mathbf{x}}_i(s) \\ \vdots \\ \partial_s^i \hat{\mathbf{x}}_N(s) \end{bmatrix} \text{ and } (\mathcal{F}\hat{\mathbf{x}})(s) = \begin{bmatrix} \hat{\mathbf{x}}(s) \\ (\partial_s S\hat{\mathbf{x}})(s) \\ \vdots \\ (\partial_s^N S^N \hat{\mathbf{x}})(s) \end{bmatrix}$$ where $\mathcal{F}$ concatenates all the $\partial_s^i S^i \hat{\mathbf{x}}$ for $i = 0, \cdots, N$ — creating an ordered list including both the PDE state, $\hat{\mathbf{x}}$, as well as all n_S possible partial derivatives of $\hat{\mathbf{x}}$ as permitted by the continuity constraint and the vector $(\mathcal{F}\hat{\mathbf{x}})(s) \in \mathbb{R}^{n_S + n_{\mathbf{x}}}$.

This notation also allows us to specify all well-defined boundary values of $\hat{\mathbf{x}} \in W^n$ and of its partial derivatives. Specifically, we may construct $(\mathcal{C}\hat{\mathbf{x}})(s) \in \mathbb{R}^{n_S}$, the vector of all absolutely continuous functions generated by $\hat{\mathbf{x}}$ and its partial derivatives. Using $\mathcal{C}\hat{\mathbf{x}}$, we may then construct $\mathcal{B}\hat{\mathbf{x}} \in \mathbb{R}^{2n_S}$, the list all possible boundary values of $\hat{\mathbf{x}} \in W^n$.

Specifically, $\mathcal{C}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$ are defined as

$$\mathcal{C}\hat{\mathbf{x}}(s) = \begin{bmatrix} (S\hat{\mathbf{x}})(s) \\ (\partial_s S^2\hat{\mathbf{x}})(s) \\ \vdots \\ (\partial_s^{N-1} S^N\hat{\mathbf{x}})(s) \end{bmatrix} \text{ and } \mathcal{B}\hat{\mathbf{x}} = \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix}. \tag{3.3}$$

Combining $\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$, we obtain a complete list of all well-defined terms which may appear in either the in-domain dynamics or BCs.

Boundary Conditions (BCs)

Given the notational framework afforded by the continuity condition, and equipped with our list of well-defined terms ($\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$), we may now parameterize a generalized class of BCs consisting of a combination of boundary values, integrals of the PDE state, and the effect of the input signal, v. Specifically, the BCs are parameterized by the square integrable function $B_I : [a,b] \to \mathbb{R}^{n_{BC}\times(n_S+n_{\hat{\mathbf{x}}})}$ and matrices $B_v \in \mathbb{R}^{n_{BC}\times n_v}$ and $B \in \mathbb{R}^{n_{BC}\times 2n_S}$ as

$$\int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix} = 0 \tag{3.4}$$

where n_{BC} is the number of user-specified BCs. For reasons of well-posedness, we typically require $n_{BC} = n_S$. If fewer BCs are available, the continuity constraint is likely too strong – the user is advised to consider whether all the partial derivatives and boundary values are actually used in defining the PDE subsystem.

Now that we have parameterized a general set of BCs, we embed these BCs in what is typically referred to as the domain of the infinitesimal generator – which combines the BCs and continuity constraints into a set of acceptable states.

$$X_v = \left\{ \hat{\mathbf{x}} \in W^n[a,b] : \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v \\ \mathcal{B}\hat{\mathbf{x}} \end{bmatrix} = 0 \right\} \tag{3.5}$$

The set X_v is used to restrict the state and initial conditions as $\hat{\mathbf{x}}(t) \in X_{v(t)}$ and $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0 \in X_{v(0)}$.

Notation: For convenience, we collect all the parameters which define the constraint in Equation (3.4) and use $\mathbf{G}_\mathrm{b}$ to represent the labeled tuple of such parameters as

$$\mathbf{G}_\mathrm{b} = \{B,\ B_I,\ B_v\}\,. \tag{3.6}$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions.

In-Domain Dynamics of the PDE Subsystem

Having specified the continuity constraint and BCs using $\{n, \mathbf{G}_\mathrm{b}\}$, we once again use our list of well-defined terms ($\mathcal{F}\hat{\mathbf{x}}$ and $\mathcal{B}\hat{\mathbf{x}}$) to define the in-domain dynamics of the PDE subsystem and the output to the ODE subsystem. These dynamics are parameterized by the functions $A_0(s)$, $A_1(s,\theta)$, $A_2(s,\theta) \in \mathbb{R}^{n_{\hat{\mathbf{x}}}\times(n_S+n_{\hat{\mathbf{x}}})}$, $C_r(s) \in \mathbb{R}^{n_r\times(n_S+n_{\hat{\mathbf{x}}})}$, $B_{xv}(s) \in \mathbb{R}^{n_{\hat{\mathbf{x}}}\times n_v}$, $B_{xb}(s) \in \mathbb{R}^{n_{\hat{\mathbf{x}}}\times 2n_S}$, and matrices $D_{rv} \in \mathbb{R}^{n_r\times n_v}$ and $D_{rb}(s) \in \mathbb{R}^{n_r\times 2n_S}$ as follows.

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\mathbf{x}}}(t,s)\\ r(t)\end{bmatrix} &= \begin{bmatrix}A_0(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s)\\ 0\end{bmatrix} + \begin{bmatrix}B_{xv}(s) & B_{xb}(s)\\ 0 & D_{rb}\end{bmatrix}\begin{bmatrix}v(t)\\ \mathcal{B}\hat{\mathbf{x}}(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}\int_a^s A_1(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta + \int_s^b A_2(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta\\ \int_a^b C_r(\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta\end{bmatrix}\end{aligned} \tag{3.7}$$

Note: Many commonly used PDE models are defined solely by A_0. For example, if we consider $u_t = \lambda u + u_{ss}$, then $A_0 = \begin{bmatrix}\lambda & 0 & 1\end{bmatrix}$ and all other parameters are zero.

The motivation for the parameters in this representation (other than A_0) can be summarized as follows: The kernels A_1, A_2 model non-local effects of the distributed state; the function B_{xv} represents the distributed effect of the disturbance/forcing function v on the generating equation; and B_{xb} represents the distributed effect of the boundary values on the generating equation. In addition: C_r is used to model the influence of the PDE subsystem state on the dynamics and outputs of the ODE subsystem; D_{rb} is used to model the effect of boundary values of the PDE subsystem on the dynamics and outputs of the ODE subsystem.

Notation: For convenience, we collect all parameters from the in-domain dynamics of the PDE subsystem (Equation (3.7)) and use $\mathbf{G}_\mathrm{p}$ to represent the labelled tuple of such parameters as

$$\mathbf{G}_\mathrm{p} = \{A_0,\ A_1,\ A_2,\ B_{xv},\ B_{xb},\ C_r,\ D_{rb}\}\,. \tag{3.8}$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions. We may now define a notion of solution for a PDE subsystem.

3.2.3 GPDE: Interconnection of ODE and PDE Subsystems

Given the definition of ODE and PDE subsystems, a GPDE model is the mutual interconnection of these subsystems through the interconnection signals (r, v) and is collectively defined by Equations (3.1) and (3.7). This interconnection is illustrated in Figure 3.3.

Given suitable inputs w,u, for a GPDE model, parameterized by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$, we define the continuity constraint and time-varying BCs by $\{x(t), \hat{\mathbf{x}}(t)\} \in \mathcal{X}_{w(t),u(t)}$ where

$$\mathcal{X}_{w,u} = \left\{ \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathbb{R}^{n_x} \times X_v \ \mid v = C_v x + D_{vw} w + D_{vu} u \right\}. \tag{3.9}$$

Illustrative Example of the GPDE Representation

In this subsection, we illustrate the process of identifying the GPDE parameters of a given system. We begin this process by introducing a conventional PDE repre-

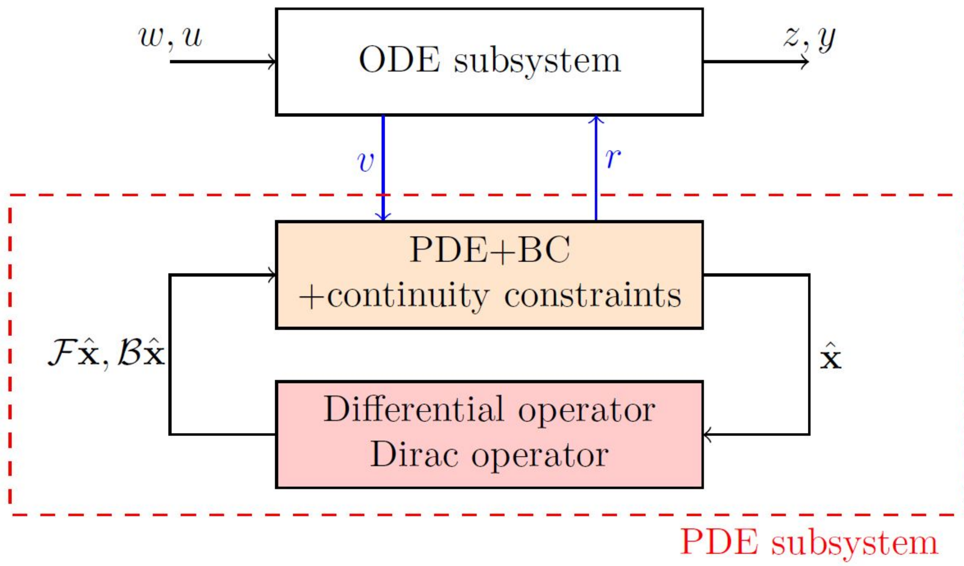


Figure 3.3: A GPDE is the interconnection of an ODE subsystem (an ODE with finite-dimensional inputs w, u, v and outputs z, y, r) with a PDE subsystem (N^{th}-order PDEs and BCs with finite-dimensional input r and output v). The BCs and internal dynamics of the PDE subsystem are specified in terms of all well-defined spatially distributed terms as encoded in $\mathcal{F}\hat{\mathbf{x}}(t)$ and all well-defined limit values as encoded in $\mathcal{B}\hat{\mathbf{x}}(t)$.

sentation. We then divide the system into ODE and PDE subsystems and focus on identifying the continuity constraint for the PDE subsystem – always the least restrictive constraint necessary for existence of the partial derivatives and boundary values. We then proceed to identify the remaining parameters.

Example 3.1. *(Damped Wave equation with delay and motor dynamics) Let us consider a wave equation*

$$\ddot{\eta}(t,s) = \partial_s^2 \eta(t,s), \ \textit{defined on the interval}\ s \in [0,1], \tag{3.10}$$

to which we apply the typical boundary feedback law $\eta_s(t,1) = -\eta_t(t,1)$, but where there is an actuator disturbance and where the control is implemented using a DC *motor and where the output from the DC motor experiences a distributed delay, so that* $\eta_s(t,1) = w(t) + \int_{-\tau}^{0} \mu(s/\tau)T(t+s)$ *where* $T(t)$ *is the output of the DC motor and* $\mu(s)$ *is a given multiplier. The delay is represented using a transport equation with distributed state* $p(t,s)$ *on the interval* $[-1,0]$ *so that*

$$\dot{p}(t,s) = \frac{1}{\tau}p_s(t,s), p(t,0) = T(t), \eta(t,1) = \int\limits_{-1}^{0} \mu(s)p(t,s)ds.$$

The DC motor dynamics relate the voltage input, $u(t)$ *to the torque* $T(t)$ *through the current,* $i(t)$ *as*

$$\dot{i}(t) = \frac{-R}{L}i(t) + u(t) \qquad T(t) = K_t i(t).$$

Finally, the sensed output is the typical feedback signal $\eta_t(1,t)$ *and the regulated output is a combination of the integral of the displacement and controller effort so that*

$$z(t) = \begin{bmatrix} \int_0^1 \eta(t,s)ds \\ u(t) \end{bmatrix}, \qquad y(t) = \eta_t(1,t).$$

Since we require all states to have first order derivatives in time and be defined on same spatial interval, we introduce the change of variables $\zeta_1 = \eta$, $\zeta_2 = \dot{\eta}$, $\zeta_3(t,s) = p(t,s-1)$. *A complete list of equations is now* $\dot{i}(t) = \frac{-R}{L}i(t) + u(t)$ *and*

$$\begin{aligned}
&\dot{\zeta}_1(t,s) = \zeta_2(t,s),\ \dot{\zeta}_2(t,s) = \partial_s^2\zeta_1(t,s),\\
&\dot{\zeta}_3(t,s) = \frac{1}{\tau}\partial_s\zeta_3(t,s),\ \zeta_1(t,0) = 0,\ \zeta_3(t,1) = K_t i(t),\\
&\partial_s\zeta_1(t,1) = w(t) + \int_0^1 \mu(s-1)\zeta_3(t,s)ds,\\
&z(t) = \begin{bmatrix} \int_0^1 \zeta_1(t,s)ds \\ u(t) \end{bmatrix},\ y(t) = \zeta_2(t,1),\ s \in [0,1], \quad t \geq 0.
\end{aligned}$$

ODE Subsystem: *We start by identifying the parameters of the ODE subsystem. Since* $i(t)$ *is the only finite dimensional state we set* $x(t) = i(t)$ *to get* $\dot{x}(t) = \frac{-R}{L}x(t) + u(t)$. *The ODE subsystem influences the PDE subsystem via signals* $w(t)$ *and* $T(t)$.

The effect of the PDE subsystem on the regulated and observed outputs (z and y, respectively) is routed through $r(t)$. The outputs, z, y and internal signals, v, r, are now defined as

$$v(t) = \begin{bmatrix} T(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} K_t \\ 0 \end{bmatrix} i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$

$$r(t) = \begin{bmatrix} \int_0^1 \zeta_1(t,s)ds \\ \zeta_2(t,1) \end{bmatrix}, \quad \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ u(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} r(t).$$

Expressing these equations in the form of Equation (3.1)*, we obtain*

$$\left[\begin{array}{c} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{array}\right] = \left[\begin{array}{c|ccc} -R/L & 0 & 1 & 0 \\ \hline \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 & 0 & \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \begin{bmatrix} K_t \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \left[\begin{array}{c} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{array}\right].$$

Extracting the submatrices of this ODE subsystem, we obtain an expression for $\mathbf{G}_\mathrm{o}$ which has the following nonzero parameters: $A = \frac{-R}{L}$, $B_{xu} = 1$, $D_{yr} = \begin{bmatrix} 0 & 1 \end{bmatrix}$,

$$D_{zu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_v = \begin{bmatrix} K_t \\ 0 \end{bmatrix}, \; D_{vw} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; D_{zr} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

PDE subsystem: *Next, we need to define n, $\mathbf{G}_\mathrm{b}$, and $\mathbf{G}_\mathrm{p}$.*

Continuity Constraint: To identify the continuity constraint, n, we consider the required partial derivatives and limit values for the three distributed states: ζ_1, ζ_2 and ζ_3. For ζ_1, $\partial_s^2\zeta_1$ appears in the in-domain dynamics and the BCs involve $\zeta_1(t,0)$ and $\partial_s\zeta_1(t,1)$. The least restrictive continuity constraint that guarantees the existence of *all three terms is* $\zeta_1 \in \hat{\mathbf{x}}_2$*. Next, no partial derivatives of* ζ_2 *are needed, but the limit value* $\zeta(t,1)$ *appears in the BCs – so we restrict* $\zeta_2 \in \hat{\mathbf{x}}_1$*. Finally,* $\partial_s \zeta_3$ *appears in the in-domain dynamics and* $\zeta_3(t,1)$ *appears in the BCs – implying* $\zeta_3 \in \hat{\mathbf{x}}_1$*. We conclude that* $n = \{n_0, n_1, n_2\} = \{0, 2, 1\}$ *and the GPDE state is*

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \emptyset \\ \begin{bmatrix} \zeta_2(t,s) \\ \zeta_3(t,s) \end{bmatrix} \\ \zeta_1(t,s) \end{bmatrix}$$

Boundary Conditions: For this definition of the continuity constraint, n*, we have* $n_{\hat{\mathbf{x}}} = 3, n_{S_0} = 3, n_{S_1} = 3, n_{S_2} = 1$ *and* $n_S = 4$ *– i.e., there are three* 0^{th}*-order, three* 1^{st}*-order and one* 2^{nd}*-order differentiable states. In addition,* $n_{\hat{\mathbf{x}}} + n_S = 7$ *indicates there are 7 possible distributed terms in* $\mathcal{F}\hat{\mathbf{x}}$ *and* $2n_S = 8$ *indicates there are 8 possible limit values in* $\mathcal{B}\hat{\mathbf{x}}$*. Specifically, recalling that* $S^i\hat{\mathbf{x}}$ *is the vector of all* i^{th} *order differentiable states, we have*

$$S^0\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_1 \end{bmatrix}, \; S^1\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \zeta_2 \\ \zeta_3 \\ \zeta_1 \end{bmatrix}, \; S^2\hat{\mathbf{x}} = \hat{\mathbf{x}}_2 = \zeta_1,$$

$$\mathcal{F}\hat{\mathbf{x}} = col(\zeta_2, \zeta_3, \zeta_1, \partial_s\zeta_2, \partial_s\zeta_3, \partial_s\zeta_1, \partial_s^2\zeta_1),$$

$$\mathcal{C}\hat{\mathbf{x}} = col(\zeta_2, \zeta_3, \zeta_1, \partial_s\zeta_1) \quad \mathcal{B}\hat{\mathbf{x}} = \begin{bmatrix} \mathcal{C}\hat{\mathbf{x}}(0) \\ \mathcal{C}\hat{\mathbf{x}}(1) \end{bmatrix}.$$

We now define the BCs. Recall that these appear in the form

$$\int_0^1 B_I(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix} = 0.$$

Checking our BCs, we note that $\zeta_1(t,0) = 0$ *can be differentiated in time to obtain*

$\zeta_2(t,0)=0$. *Collecting all the BCs, and placing these in the required form, we have*

$$\int_0^1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mu(s-1)\zeta_3(s) \end{bmatrix} ds = \begin{bmatrix} \zeta_1(0) \\ \zeta_2(0) \\ \zeta_3(1) \\ \partial_s\zeta_1(1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -v_1 \\ -v_2 \end{bmatrix}$$

Recalling the expansions of $\mathcal{F}\hat{\mathbf{x}}$ *and* $\mathcal{B}\hat{\mathbf{x}}$, *we may identify the parameters in* $\mathbf{G}_\mathrm{b}$ *as*

$$B = \begin{bmatrix} 0 & 0 & 1 & 0_{1,2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0_{1,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{1,2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0_{1,2} & 0 & 0 & 1 \end{bmatrix},\ B_v = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix},\ B_I(s) = \begin{bmatrix} 0_{3,1} & 0_{3,1} & 0_{3,5} \\ 0 & \mu(s-1) & 0_{1,5} \end{bmatrix}. \quad (3.11)$$

In-Domain Dynamics: To find the parameters $\mathbf{G}_\mathrm{p}$, *first recall that PDE dynamics have the form*

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t,s) \\ r(t) \end{bmatrix} = \begin{bmatrix} A_0(s)(\mathcal{F}\hat{\mathbf{x}}(t))(s) \\ 0 \end{bmatrix} + \begin{bmatrix} B_{xv}(s) & B_{xb}(s) \\ 0 & D_{rb} \end{bmatrix} \begin{bmatrix} v(t) \\ \mathcal{B}\hat{\mathbf{x}}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} \int_a^s A_1(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta + \int_s^b A_2(s,\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta \\ \int_a^b C_r(\theta)(\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta \end{bmatrix}.$$

Recalling the expansion of $\mathcal{F}\hat{\mathbf{x}}$, *we represent the dynamics as*

$$\dot{\hat{\mathbf{x}}}(t,s) = \begin{bmatrix} \partial_s^2\zeta_1(t,s) \\ 1/\tau\partial_s\zeta_3(t,s) \\ \zeta_2(t,s) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0_{1,3} & 0 & 0 & 1 \\ 0 & 0_{1,3} & \frac{1}{\tau} & 0 & 0 \\ 1 & 0_{1,3} & 0 & 0 & 0 \end{bmatrix}}_{A_0} (\mathcal{F}\hat{\mathbf{x}}(t))(s)$$

Likewise, from the definition of $r(t)$*, we have*

$$r(t) = \begin{bmatrix} \int_0^1 \zeta_1(t,s)ds \\ \zeta_2(t,1) \end{bmatrix} = \int_0^1 \overbrace{\begin{bmatrix} 0_{1,2} & 1 & 0_{1,4} \\ 0_{1,2} & 0 & 0_{1,4} \end{bmatrix}}^{C_r} (\mathcal{F}\hat{\mathbf{x}}(t))(\theta)d\theta + \underbrace{\begin{bmatrix} 0_{1,4} & 0 & 0_{1,3} \\ 0_{1,4} & 1 & 0_{1,3} \end{bmatrix}}_{D_{rb}} \mathcal{B}\hat{\mathbf{x}}(t)$$

Thus we have A_0, C_r, D_{rb} *– the only nonzero terms in* $\mathbf{G}_\mathrm{p}$*.*

3.3 Definition of Solutions

Although Hadamard's criteria for well-posedness specify the nature of a solution, they do not define the solution itself. In the case of GPDEs, one must specify constraints on the continuity and differentiability properties for the solution of a GPDE model while ensuring that the solution is unique. These definitions depend on the specific GPDE under consideration and cannot be standardized. However, to build a computational framework, one needs to have a general definition regardless. Therefore, in this work, we will assume the existence of a well-posed weak solution for the GPDEs and only consider the systems that satisfy this criterion. The motive behind using a weak solution lies in the fact that many PDEs do not have classical solutions or strong solutions – i.e., the solution may not be differentiable everywhere in space or time. For instance, if we consider a transport PDE, $\dot{\mathbf{x}}(t,s) = -\partial_s \mathbf{x}(t,s)$, with boundary conditions $\mathbf{x}(t,0) = 0$ and initial condition $\mathbf{x}(0,s) = f(s)$. Then, there exists a solution that transports the initial condition to the right as per the rule $\mathbf{x}(t,s) = f(s-t)$ when $s > t$ and $\mathbf{x}(t,s) = 0$ when $s \le t$. While the solution can be differentiable if f is sufficiently smooth, there may exist a solution to the transport PDE that is not differentiable. Likewise, other PDEs can have solutions that may not be differentiable at all points – an outcome contingent on the input, boundary conditions, and initial conditions. Thus, to accommodate such solutions, one must relax the differentiability criteria, which leads to the definition of a *generalized* or weak solution.

Having specified this assumption about the solution of a GPDE, we can now formally define a solution for such a system. For convenience, we define the solutions for the subsystem separately and then the full GPDE model.

Definition 3.2 (Solution of a PDE Subsystem). *For given $\hat{\mathbf{x}}^0 \in X_{v(0)}$ and $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ with $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$, we say that $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE subsystem defined by $n \in \mathbb{N}^{N+1}$ and $\{\mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ (defined in Equations (3.6) and (3.8)) with initial condition $\hat{\mathbf{x}}^0$ and input v if $r \in L_{2e}^{n_r}[\mathbb{R}_+]$, $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$, $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$, $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$, and Equation (3.7) is satisfied for almost all $t \geq 0$.*

The above definition is analogous to a weak solution of a PDE; See Evans (2022) for standard definitions of solutions. Next, we can extend this concept of weak solutions to a GPDE model by augmenting the conditions satisfied by the ODE solution and the interconnection signals.

Definition 3.3 (Solution of a GPDE model). *For given $\{x^0,\ \hat{\mathbf{x}}^0\} \in \mathcal{X}_{w(0),u(0)}$ and $w \in L_{2e}^{n_w}[\mathbb{R}_+]$, $u \in L_{2e}^{n_u}[\mathbb{R}_+]$ with $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$ and $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$, we say that $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ satisfies the GPDE defined by $\{n,\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ (See Equations (3.2), (3.6) and (3.8)) with initial condition $\{x^0,\ \hat{\mathbf{x}}^0\}$ and input $\{w,\ u\}$ if $z \in L_{2e}^{n_z}[\mathbb{R}_+]$, $y \in L_{2e}^{n_y}[\mathbb{R}_+]$, $v \in L_{2e}^{n_v}[\mathbb{R}_+]$, $r \in L_{2e}^{n_r}[\mathbb{R}_+]$, $\{x(t), \hat{\mathbf{x}}(t)\} \in \mathcal{X}_{w(t),u(t)}$ for all $t \geq 0$, x is differentiable almost everywhere on $\mathbb{R}_+$, $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$, $x(0) = x^0$, $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$, and Equations (3.1) and (3.7) are satisfied for almost all $t \geq 0$.*

Note there are many GPDEs that may not even have weak solutions. One would need to relax the differentiability requirement further to ensure the PDE is well-posed. However, such GPDEs are outside the scope of this work and typically, do not have a PIE representation.

3.4 Conclusion

In this chapter, we briefly introduced the concept of a *generalized* PDE (GPDE) parametric representation and defined a notion of weak solutions for systems in this parametric form. Specifically, we introduced a parametric representation of the PDE model class — encompassing ODEs coupled with PDEs, mixed-order spatial derivatives, integrals of the state, control inputs and disturbances, and sensed and regulated outputs. Although the set of parameters is inconsistent and highly dependent on the PDE, the type of BCs, and the continuity constraint, we managed to cover a large class of GPDEs using this parametric representation. This parametric representation was mainly achieved by utilizing a systematic approach: a) separating finite and infinite dimensional signals; b) identifying the continuity constraint parameter, n; and c) initializing all possible parameters that can appear in PDEs with such an n and assigning the non-zero parameters of a PDE in non-standard form to the appropriate initialized parameters in the standard form.

Using this systematic approach, we have ensured that any mixed-order GPDE can be represented in this form by a fixed set of parameters – a necessity when building a computational framework that takes GPDEs as input. As will be shown in Chapter 5, this standard representation also helps in identifying GPDEs that admit a PIE representation as well as obtain analytic formulae to find the PIE representation directly from the parameters of the GPDE.

Chapter 4

PARTIAL INTEGRAL EQUATIONS

4.1 Introduction

As briefly mentioned in the previous chapter Chapter 3, we will now formally introduce the class of Partial Integral Equations (PIE) and a universal parametric representation of systems governed by PIE models. While the benefits of a PIE model were discussed earlier in Chapters 1 and 2, we will briefly reiterate some of the key features of a PIE model here to summarize the motivation behind such models. The distinguishing feature of the class of PIE models is its parametrization using the *-algebras of PI operators ($\mathbf{\Pi}_i$ and $\mathbf{\Pi}_i^p$). In contrast to differential and Dirac operators that constitute a majority of the parameters defining a GPDE, all the parameters of PIE – i.e., **PI operators** – have the following properties:

1. **Algebraic Structure** The set of PI operators is a subspace of $\mathcal{L}(L_2)$ – the space of bounded linear operators on the Hilbert space L_2. PI operators form *-algebras, denoted $\mathbf{\Pi}_i$, being closed under addition, composition, and transposition (See the Lemmas presented in Appendix H of the paper by Shivakumar *et al.* (2022)). In addition, $\mathbf{\Pi}_3$ and $\mathbf{\Pi}_4$ are unital algebras – implying that these operators inherit most of the properties of matrices, including operations that preserve positivity.

2. **Parametrization by Polynomials** The subspaces of $\mathbf{\Pi}_i$ with polynomial parameters, denoted by $\mathbf{\Pi}_i^p$, also form a *-subalgebra. PIEs that represent PDE models are typically parameterized by operators in $\mathbf{\Pi}_i^p$. Because polynomials

admit a linear parametrization using coefficient vectors, and because multiplication, addition, and integration reduce to algebraic operations on these coefficient vectors, the operations involving operators in $\mathbf{\Pi}_i^p$ can typically be performed in polynomial time.

3. **Computation via PIETOOLS** Most matrix operations defined in Matlab have a $\mathbf{\Pi}_i^p$ equivalent, which is easy to compute. These operations have been embedded into an `opvar` class in the MATLAB toolbox PIETOOLS developed by Shivakumar *et al.* (2020b). This toolbox also allows one to solve Linear PI Inequality Optimization (LPIs) problems, which is a natural extension of the class of Linear Matrix Inequality (LMI) optimization problems.

In addition to the simplicity of computationally handling PI operators of a PIE model in comparison to handling unbounded differential operators of a GPDE model, the PIE models themselves offer certain benefits. The primary benefit is the absence of auxiliary constraints like BCs or continuity requirements. However, there are some secondary benefits that are stated below and must be noted, because they have significance in the context of computational analysis/control. These secondary benefits can be summarized as:

1. **Known map from GPDE model to PIE model** For the large class of well-posed linear GPDE models defined in this paper, we have explicit formulae for the construction of an associated PIE model, including the map from PIE solution to GPDE solution. In addition, most GPDE models map to PIE models parameterized by PI operators with polynomial parameters.

2. **State-Space Structure** Because PIE models are parameterized by the PI *-algebra of bounded linear operators on L_2, PIEs inherit many of the benefits of the state-space representation of linear ODEs. This implies that many

numerical methods designed for analysis, control, and simulation of ODEs in state-space form may be extended to PIEs. Specifically, many LMIs for analysis and control of ODEs have been extended to PIEs, including, stability analysis as shown by Peet (2021), L_2-gain analysis as shown by Shivakumar *et al.* (2019), H_∞-optimal estimation as shown by Das *et al.* (2019), H_∞-optimal control as shown by Shivakumar *et al.* (2020a), and robust stability/performance as shown by Das *et al.* (2020) and Wu *et al.* (2021).

3. **Universal Methods** A direct consequence of the PIE models being defined by a single differential equation with no further constraints on the state, such as BCs or continuity constraints, is that it allows us to develop universal algorithms for analysis, control, and simulation, which apply to any well-posed PIE model. Examples of such algorithms can be found in the toolbox developed by Shivakumar *et al.* (2020b).

4.2 Parametrization of PIEs

A Partial Integral Equation (PIE) is an extension of the state-space representation of ODEs (vector-valued first-order differential equations on $\mathbb{R}^n$) to spatially distributed states on the product space $\mathbb{R}L_2$. Analogous to the 9-matrix optimal control framework developed for state-space systems, a PIE system is parameterized by twelve 4-PI operators as

$$\begin{bmatrix}\mathcal{T}\dot{\underline{\mathbf{x}}}(t)\\ z(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2\\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12}\\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22}\end{bmatrix}\begin{bmatrix}\underline{\mathbf{x}}(t)\\ w(t)\\ u(t)\end{bmatrix} - \begin{bmatrix}\mathcal{T}_w\dot{w}(t)+\mathcal{T}_u\dot{u}(t)\\ 0\\ 0\end{bmatrix},$$

$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}[a,b], \tag{4.1}$$

where $z(t) \in \mathbb{R}^{n_z}$ is the regulated output, $y(t) \in \mathbb{R}^{n_y}$ is the sensed output, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance, $u(t) \in \mathbb{R}^{n_u}$ is the control input, and $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ is the internal state. Note that, unlike the 9-matrix representation of linear state-space ODEs, a PIE system has 3 additional parameters: $\mathcal{T}$ that appears on the left hand side of the equation and two parameters, $\mathcal{T}_w$ and $\mathcal{T}_u$, that appear on the right hand side. The parameters $\mathcal{T}_w$ and $\mathcal{T}_u$ are associated with the time derivative of the disturbance input w and control input u, respectively. Such terms corresponding to the time derivative of the inputs typically do not appear in the case of ODEs. However, they are quite common in the case of PIEs because inputs acting at the boundary of a GPDE model take the form of time derivatives when converted to a PIE form.

No Spatial Derivatives or Boundary Conditions: Another feature to note is that a PIE system does not permit spatial derivatives – only a first-order derivative with respect to time. Since state of the PIE system, $\underline{\mathbf{x}} \in \mathbb{R}L_2[a,b]$, is an equivalence class of functions, it is not necessarily well-defined at any given spatial point. Thus, one cannot specify BCs in the PIE framework.

Before formalizing the definition of solution for a PIE system, let us note two significant features of this definition. First, we observe that PIEs allow for the dynamics to depend on the time-derivative of the input signals: $\partial_t(\mathcal{T}_w w)$ and $\partial_t(\mathcal{T}_u u)$. **Through some slight abuse of notation, in this paper we will use expressions such as $\mathcal{T}\dot{\underline{\mathbf{x}}}$ to represent $\partial_t(\mathcal{T}\underline{\mathbf{x}})$, $\mathcal{T}_w\dot{w}$ to represent $\partial_t(\mathcal{T}_w w)$, and $\mathcal{T}_u\dot{u}$ to represent $\partial_t(\mathcal{T}_u u)$.**

Second, the internal state of the solution of a PIE system, $\underline{\mathbf{x}}$, is required to be Frechét differentiable. Recall that, from the definition of the Frechét derivative, a function $\underline{\mathbf{x}} : U \to X$ is Frechét differentiable if there exists a linear operator $D : X \to Y \subseteq X$ such that

$$\lim_{h\to 0} \frac{\|\underline{\mathbf{x}}(t+h) - \underline{\mathbf{x}}(t) - D\underline{\mathbf{x}}(t)\|_X}{h} = 0.$$

Here, D is called the derivative operator. In the case of PIEs, we will require $\mathcal{T}\underline{\mathbf{x}}$ to be Frechét differentiable (instead of just $\underline{\mathbf{x}}$) since this corresponds to the differentiability of the GPDE state $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}}$.

Notation: For brevity, we collect the 12 PI parameters that define a PIE system in Equation (4.1) and introduce the shorthand notation $\mathbf{G}_{\mathrm{PIE}}$ which represents the labeled tuple of such system parameters as

$$\mathbf{G}_{\mathrm{PIE}} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\}.$$

When this shorthand notation is used, it is presumed that all parameters have appropriate dimensions.

4.3 Definition of Solutions

Just like in the case of GPDEs, we use weak solutions of PIE systems for analysis and control in the future chapters. Therefore, we first need to define a notion of weak solutions. Unlike the case of GPDEs, the solution of a PIE does not satisfy any boundary conditions and hence is not necessarily continuous in space. However, it does satisfy an initial condition and has a time derivative almost everywhere and hence must be differentiable in time. These requirements, in fact, arise due to the invertible mapping between a GPDE solution and its associated PIE solution. To be precise, all the PIEs considered in this work are associated with a GPDE, in which case, the solution of a GPDE $\mathbf{x}$ and a PIE $\underline{\mathbf{x}}$ are related via a unitary map $\mathcal{T}$ as

$$\mathbf{x} = \mathcal{T}\underline{\mathbf{x}}.$$

Thus, if $\mathbf{x}$ is a weakly differentiable solution, $\mathcal{T}\underline{\mathbf{x}}$ has to be weakly differentiable since unitary maps are isomorphic. Therefore, to have a PIE equivalent to a GPDE with weak solutions, the solution of the PIE must also be defined in a *generalized* or weak sense. The below definition formalizes the requirements stated above to define a weak solution for a PIE.

Definition 4.1 (Solution of a PIE system)**.** *For given inputs* $u \in L_{2e}^{n_u}[\mathbb{R}_+]$*,* $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ *with* $(\mathcal{T}_u u)(\cdot, s) \in W_{1e}^{n_x+n_{\hat{\mathbf{x}}}}[\mathbb{R}_+]$ *and* $(\mathcal{T}_w w)(\cdot, s) \in W_{1e}^{n_x+n_{\hat{\mathbf{x}}}}[\mathbb{R}_+]$ *for all* $s \in [a, b]$ *and* $\underline{\mathbf{x}}^0(t) \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$*, we say that* $\{\underline{\mathbf{x}}, z, y\}$ *satisfies the PIE defined by* $\mathbf{G}_{\text{PIE}} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}\}$ *with initial condition* $\underline{\mathbf{x}}^0$ *and input* $\{w, u\}$ *if* $z \in L_{2e}^{n_z}[\mathbb{R}_+]$*,* $y \in L_{2e}^{n_y}[\mathbb{R}_+]$*,* $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}[a, b]$ *for all* $t \geq 0$*,* $\mathcal{T}\underline{\mathbf{x}}$ *is Frechét differentiable with respect to the* L_2*-norm almost everywhere on* $\mathbb{R}_+$*,* $\mathcal{T}\underline{\mathbf{x}}(0) = \mathcal{T}\underline{\mathbf{x}}^0$*, and Equation* (4.1) *is satisfied for almost all* $t \in \mathbb{R}_+$*.*

Now that we have established standard parametric representations for both GPDEs and PIEs, we next proceed with establishing the conditions under which these parametric representations are equivalent — i.e., the conditions under which the solution of a GPDE and a PIE are related by an invertible linear mapping. Furthermore, we will also find analytical expressions to obtain the parameters of a PIE from the parameters of its equivalent GPDE.

4.4 Conclusion

To summarize, in this chapter, we briefly discussed the motive behind the use of a PIE model and defined a general parametric representation of a PIE model. We also noted that, the set of parameters for a PIE is universal, and the parametric representation itself is analogous to the 9-matrix representation of linear state-space ODE systems. Furthermore, we noted that, unlike a GPDE, a PIE is exclusively described by these dynamics and has no auxiliary constraints such as boundary conditions. As will be seen later in Part II, the parametric representation proposed for a PIE, along with the algebraic properties of the PI operators, allows an almost direct extension of various LMI results used in the analysis, estimation, and control of ODEs to PIE systems. Lastly, following the idea of a weak solution for a GPDE, we defined the notion of weak solutions for a PIE.

Chapter 5

PIE REPRESENTATION OF A GPDE

In Chapter 3 and Chapter 4, we proposed parametric representations for a broad class of coupled ODE-PDEs Systems (GPDEs) and PIE Systems. Furthermore, we looked at the motivation behind the use of PI algebra and the benefits of PIE representation as an alternative modeling approach. Now, we focus on finding this 'PIE representation' for a given GPDE model. We begin this process by focusing on the conversion of the PDE subsystem to a restricted class of PIE subsystem of the form

$$\begin{bmatrix} \hat{\mathcal{T}}\dot{\hat{\underline{\mathbf{x}}}}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{A}} & \mathcal{B}_v \\ \mathcal{C}_r & \mathcal{D}_{rv} \end{bmatrix} \begin{bmatrix} \hat{\underline{\mathbf{x}}}(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} \mathcal{T}_v \dot{v}(t) \\ 0 \end{bmatrix}, \tag{5.1}$$

with initial condition $\hat{\underline{\mathbf{x}}}^0 \in L_2^m$. Such PIE subsystems are a special case of Definition 4.1 with parameter set given by

$$\mathbf{G}_{\mathrm{PIE}_s} = \left\{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_{rv}, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\right\}.$$

In this section, we will show that for any *admissible* PDE subsystem defined by $\{n, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$, there exists a corresponding PIE subsystem defined by $\{\hat{\mathcal{T}}$, $\mathcal{T}_v$, $\hat{\mathcal{A}}$, $\mathcal{B}_v$, $\mathcal{C}_{rv}$, $\mathcal{D}_{rv}\}$ such that for any suitable signal v, $\{\hat{\underline{\mathbf{x}}}, r, \emptyset\}$ is a solution of the PIE subsystem with initial condition $\hat{\underline{\mathbf{x}}}^0$ and input v if and only if $\{\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}(t) + \mathcal{T}_v v(t), r\}$ is a solution of the PDE subsystem with initial condition $(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0))$ and input v.

5.1 Admissiblity: Well-posedness of the Auxiliary Constraints

Before we proceed to prove the claim 'any admissible GPDE has a PIE form', we must define admissibility. The admissibility criterion, in a sense, imposes a notion of well-posedness on X_v, the domain of the PDE subsystem defined by the continuity constraints and the BCs. This condition ensures, e.g., that there are a correct number of independent BCs to establish a mapping between the distributed state and its partial derivatives. Without such a mapping, the solution to the PDE may not exist (too many BCs) or may not be unique (too few BCs). However, this does not explicitly guarantee a well-posed solution since a GPDE may be admissible and not have a solution in the sense defined in Definition 3.3.

First, let us recall that we first define $n_{\hat{\mathbf{x}}} = |n|_1 = \sum_{i=0}^{N} n_i$ to be the number of states in $\hat{\mathbf{x}}$, $n_{S_i} = \sum_{j=i}^{N} n_j \leq n_{\hat{\mathbf{x}}}$ to be the total number of i-times differentiable states, and $n_S = \sum_{i=1}^{N} n_{S_i}$ to be the total number of possible partial derivatives of $\hat{\mathbf{x}}$ as permitted by the continuity constraint. For indexed vectors (such as n or $\hat{\mathbf{x}}$) we occasionally use the notation $\hat{\mathbf{x}}_{i:j}$ to denote the components i to j. Specifically, $\hat{\mathbf{x}}_{i:j} = \mathrm{col}(\hat{\mathbf{x}}_i, \cdots, \hat{\mathbf{x}}_j)$, $n_{i:j} := \sum_{k=i}^{j} n_k$ and $n_{S_{i:j}} = \sum_{k=i}^{j} n_{S_k}$.

Definition 5.1 (Admissibility)**.** *Given an $n \in \mathbb{N}^{N+1}$ (with corresponding continuity constraint) and a parameter set, $\mathbf{G}_{\mathrm{b}} = \{B, B_I, B_v\}$, we say the pair $\{n, \mathbf{G}_{\mathrm{b}}\}$ (or alternatively, the PDE subsystem defined by $\mathbf{G}_{PDE_s}$ or GPDE defined by $\mathbf{G}_{PDE}$) is* ***admissible*** *if B_T is invertible where*

$$B_T = B \begin{bmatrix} T(0) \\ T(b-a) \end{bmatrix} - \int_a^b B_I(s) U_2 T(s-a) ds \in \mathbb{R}^{n_{BC} \times n_S},$$

and where T and U_2 are defined (See also Block 5.1) as

$$T_{i,j}(s) = \frac{s^{(j-i)}}{(j-i)!} \begin{bmatrix} 0_{(n_{Si}-n_{Sj})\times n_{Sj}} \\ I_{n_{Sj}} \end{bmatrix} \in \mathbb{R}^{n_{Si}\times n_{Sj}}, \tag{5.2}$$

$$T(s) = \begin{bmatrix} T_{1,1}(s) & T_{1,2}(s) & \cdots & T_{1,N}(s) \\ 0 & T_{2,2}(s) & \cdots & T_{2,N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{N,N}(s) \end{bmatrix} \in \mathbb{R}^{n_S\times n_S}, \tag{5.3}$$

$$U_{2i} = \begin{bmatrix} 0_{n_i\times n_{i+1:N}} \\ I_{n_{i+1:N}} \end{bmatrix} \in \mathbb{R}^{n_{S_i}\times n_{S_{i+1}}}, \tag{5.4}$$

$$U_2 = \begin{bmatrix} diag(U_{20},\cdots,U_{2(N-1)}) \\ 0_{n_N\times n_S} \end{bmatrix} \in \mathbb{R}^{(n_{\hat{\mathbf{x}}}+n_S)\times n_S}. \tag{5.5}$$

Since B_T must be square to be invertible, admissibility requires $n_{BC} = n_S$. One way to interpret this condition is to note that whenever we differentiate a PDE state, we lose some of the information required to reconstruct that state. As a result, if we have n_S possible partial derivatives, we need n_S BCs to relate all the partial derivatives to the original state vector. However, while the constraint $n_{BC} = n_S$ is necessary for admissibility, it is not sufficient – the BCs must be both independent and provide enough information to allow us to reconstruct the PDE state. See Subsection 3.2.2 in the paper by Peet (2021) for an enumeration of pathological cases, including periodic BCs.

Finally, note that the test for admissibility depends only on the continuity condition, $n \in \mathbb{N}^{N+1}$ and the parameters which define the boundary condition – admissibility does not depend on the dynamics.

Illustration of the Admissibility Condition

Example 5.1 (Damped wave equation with motor dynamics and delay)**.** *Let us revisit the coupled ODE-PDE from Example 3.1. Recall that for this example,* $n = \{0, 2, 1\}$*, so* $n_{S_0} = 3$*,* $n_{S_1} = 3$*,* $n_{S_2} = 1$*,* $n_S = 4$*, and* $n_{\hat{\mathbf{x}}} = 3$*. In addition,* $\mathbf{G}_{\mathrm{b}}$ *has parameters as shown in Equation* (3.11)*. Then, using Equations* (5.3) *and* (5.5)*, we compute* T*,* U_2*, and* B_T *as*

$$T_{1,1} = \begin{bmatrix} 0_{3-3,3} \\ I_3 \end{bmatrix}, \; T_{1,2} = s \begin{bmatrix} 0_{3-1,1} \\ I_1 \end{bmatrix}, \; T_{2,2} = \begin{bmatrix} 0_{1-1,1} \\ I_1 \end{bmatrix}, \; U_{20} = \begin{bmatrix} 0_{0,3} \\ I_3 \end{bmatrix}, \; U_{21} = \begin{bmatrix} 0_{2,1} \\ I_1 \end{bmatrix},$$

$$T(s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & s \\ & & & 1 \end{bmatrix}, \; U_2 = \begin{bmatrix} I_3 & 0_{3,1} \\ 0_{2,3} & 0_{2,1} \\ 0_{1,3} & 1 \\ 0_{1,2} & 0_{1,2} \end{bmatrix}, \; B_T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\int_a^b \mu(s-1)ds & 0 & 1 \end{bmatrix}.$$

B_T *is invertible for any* μ*, which implies the pair* $\{n, \mathbf{G}_{\mathrm{b}}\}$ *is admissible.*

Since analytical expressions are known to find the admissibility matrix B_T for arbitrary GPDEs defined by $\mathbf{G}_{PDE}$, we can test for admissibility using computation.

5.2 A Map from GPDE Parameters to PIE Parameters

In the following subsections, we will develop maps between the parameters of a GPDE model $\mathbf{G}_{\mathrm{PDE}}$ and the parameters of a PIE model $\mathbf{G}_{\mathrm{PIE}}$ assuming that the GPDE is admissible. However, the primary difficulty in finding $\mathbf{G}_{\mathrm{PIE}}$ from the parameters of the GPDE lies in the 'PDE' part of the GPDE. Thus, we will first handle the 'PDE' subsystem and find an equivalent PIE representation with parameters $\mathbf{G}_{\mathrm{PIE}_s}$ for the 'PDE' subsystem. Then, finding $\mathbf{G}_{\mathrm{PIE}}$ for the full GPDE becomes a matter of augmenting states and algebraic manipulations.

5.2.1 A PDE and its Associated PIE

Given an admissible pair $\{n, \mathbf{G}_\mathrm{b}\}$, we may construct a PIE subsystem which we will associate with the PDE subsystem defined by those parameters. The first step is to map $\hat{\mathbf{x}}(t) \in X_v$, the state of the PDE subsystem, to $\hat{\underline{\mathbf{x}}}(t) \in L_2$, the state of the PIE subsystem using

$$\hat{\underline{\mathbf{x}}} = \mathcal{D}\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \partial_s \hat{\mathbf{x}}_1 \\ \vdots \\ \partial_s^N \hat{\mathbf{x}}_N \end{bmatrix} \in L_2^{n_{\hat{\mathbf{x}}}}.$$

where $\mathcal{D} = \mathrm{diag}(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. The following theorem shows that this mapping is invertible, and the inverse is defined by PI operators.

Theorem 5.1. *Given an $n \in \mathbb{N}^{N+1}$, and $\mathbf{G}_\mathrm{b}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $\{\hat{\mathcal{T}},\ \mathcal{T}_v\}$ be as defined in Block 5.1, X_v as defined in Equation* (3.5) *and $\mathcal{D}$ =diag$(\partial_s^0 I_{n_0}, \cdots,$ $\partial_s^N I_{n_N})$. Then we have the following: (a) For any $v \in \mathbb{R}^{n_v}$, if $\hat{\mathbf{x}} \in X_v$, then $\mathcal{D}\hat{\mathbf{x}} \in L_2^{n_{\hat{\mathbf{x}}}}$ and $\hat{\mathbf{x}} = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}} + \mathcal{T}_v v$; and (b) For any $v \in \mathbb{R}^{n_v}$ and $\hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$, $\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v \in X_v$ and $\hat{\underline{\mathbf{x}}} = \mathcal{D}(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v)$.*

Proof. First, we generalize the Fundamental Theorem of Calculus by using Cauchy's formula for repeated integration as

Lemma 5.2. *Suppose $\mathbf{x} \in W_N^n[a, b]$ for any $N \in \mathbb{N}$. Then*

$$\mathbf{x}(s) = \mathbf{x}(a) + \sum_{j=1}^{N-1} \frac{(s-a)^j}{j!} \partial_s^j \mathbf{x}(a) + \int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!} \partial_s^N \mathbf{x}(\theta) d\theta$$

where $\partial_s^i \mathbf{x}$ is the ith classical-derivative of $\mathbf{x}$ when $i < N$ and weak-derivative for $i = N$.

This gives a map from $\partial_s^j \hat{\mathbf{x}}(a)$ and $\hat{\underline{\mathbf{x}}}$ to $\hat{\mathbf{x}}$. Next we express all possible well-defined boundary values in terms of the $\partial_s^j \hat{\mathbf{x}}(a)$ and $\hat{\underline{\mathbf{x}}}$. Applying the boundary conditions in X_v, we may now invert this map (using B_T^{-1}) to obtain an expression for the $\partial_s^j \hat{\mathbf{x}}(a)$ in terms of $\hat{\underline{\mathbf{x}}}$ and v. Substituting this expression into Lemma 5.2, we obtain the theorem statement. For details, see Appendix A.1. $\square$

For any given $v \in \mathbb{R}^{n_v}$, Theorem 5.1 provides an invertible map between the state of the PIE subsystem, $\hat{\underline{\mathbf{x}}}(t) \in L_2^{n_{\hat{\mathbf{x}}}}$ and the state of the PDE subsystem, $\hat{\mathbf{x}}(t) \in X_v$. Furthermore, this transformation is unitary, which will be proved later in this chapter. In the following subsection, we apply this mapping to the internal dynamics of the PDE subsystem in order to obtain an equivalent PIE representation of this subsystem.

5.2.2 A GPDE and its Associated PIE

Having converted the PDE subsystem to a PIE, integration of the ODE dynamics is a simple matter of augmenting the PIE subsystem (Equation (5.1)) with the differential equations which define the ODE (Equation (3.1)), followed by elimination of the interconnection signals v and r. The result is an augmented PIE system, as defined in Equation (4.1) whose parameters are 4-PI operators defined in Blocks 5.1 and 5.2.

Our first step in constructing the augmented PIE system, which will be associated with a given GPDE model, is to construct the augmented map from the GPDE state (defined on $\mathcal{X}_{w,u}$) to the associated PIE state (defined on $\mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$). Specifically, given a GPDE model $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible and state $\mathbf{x} = \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathcal{X}_{w,u}$, the associated PIE system state is $\underline{\mathbf{x}} = \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ where $\mathcal{D} = \mathrm{diag}(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. Using this definition, one can prove Corollary 5.3 that shows that if

$\{\mathcal{T},\ \mathcal{T}_w,\ \mathcal{T}_u\}$ are as defined in Block 5.2, then the map $\mathbf{x} \to \underline{\mathbf{x}}$ can be inverted as $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$.

Corollary 5.3. *Given an $n \in \mathbb{N}^{N+1}$, and $\mathbf{G}_\mathrm{b}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $\{\mathcal{T},\ \mathcal{T}_w,\ \mathcal{T}_u\}$ be as defined in Block 5.2, $\mathcal{X}_{w,u}$ as defined in Equation (3.9) and $\mathcal{D} =$diag$(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. Then for any $w \in \mathbb{R}^{n_w}$ and $u \in \mathbb{R}^{n_u}$ we have:*

(a) If $\mathbf{x} = \{x, \hat{\mathbf{x}}\} \in \mathcal{X}_{w,u}$, then $\underline{\mathbf{x}} = \{x, \mathcal{D}\hat{\mathbf{x}}\} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ and $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$.

(b) If $\underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$, then $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u \in \mathcal{X}_{w,u}$ and $\underline{\mathbf{x}} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & \mathcal{D} \end{bmatrix} \mathbf{x}$.

Proof. The proof simply applies the definitions of $\mathbf{x}$, $\underline{\mathbf{x}}$, and v – See Appendix A.3 for details. □

Thus, for any given w, u, we have an invertible transformation from $\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ to $\mathcal{X}_{w,u}$.

Illustration of the Construction of the PIE Subsystem

In this subsection, we detail the application of the formulae in Blocks 5.1 and 5.2 to a given GPDE model. Additional, less detailed examples are given in later chapters.

Example 5.2. *(The Entropy PDE) A PDE model for entropy change in a $1D$ linear thermoelastic rod clamped at both ends is given by Day (2013)*

$$\dot{\eta}(t,s) = \partial_s^2 \eta(t,s),$$

subject to the BCs

$$\eta(t,0) + \int_0^1 \eta(t,s)ds = 0, \quad \eta(t,1) + \int_0^1 \eta(t,s)ds = 0.$$

The GPDE representation of this model is defined by $n = \{0,0,2\}$, $\mathbf{G}_\mathrm{p} = \{A_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\}$, *and*

$$\mathbf{G}_\mathrm{b} = \left\{ B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B_I = -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

Using the formulae in Blocks 5.1 and 5.2, we find the PIE subsystem as follows (we neglect interconnection to the ODE subsystem as there are no ODEs, inputs, or outputs).

$$U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, U_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, T(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, Q(s) = \begin{bmatrix} s \\ 1 \end{bmatrix},$$
$$B_T = \begin{bmatrix} 2 & 1/2 \\ 2 & 3/2 \end{bmatrix}, \quad B_Q(s) = (1-s)\begin{bmatrix} \frac{s}{4} \\ -1 \end{bmatrix}, \quad G_0(s) = 0,$$
$$G_1(s,\theta) = G_2(s,\theta) + (s-\theta), \quad G_2(s,\theta) = 3s\frac{(s-1)}{4}.$$

The PIE form ($\underline{\eta} = \partial_2^2 \eta$) *of the entropy PDE is then given by*

$$\int_0^s \left(s^2 + \frac{s}{4} - \theta\right) \dot{\underline{\eta}}(t,\theta)d\theta + \int_s^1 \frac{3}{4}(s^2 - s)\dot{\underline{\eta}}(t,\theta)d\theta = \underline{\eta}(t,s).$$

In the context of optimal control framework, one would expect inputs and outputs to appear. Furthermore, such inputs and outputs maybe further classified as disturbance inputs, control inputs, regulated outputs, and observed outputs. The following example demonstrates how one can construct a PIE representation for GPDE systems while preserving this categorization of different input and output signals.

Example 5.3. *Consider the vibration suppression problem for a cantilevered Euler-*

Bernoulli beam

$$\dot{\mathbf{x}}(t,s) = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix} \partial_s^2 \mathbf{x}(t,s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$
$$\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t,0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \partial_s \mathbf{x}(t,0) = 0,$$
$$\begin{bmatrix} 0 & 1 \end{bmatrix} \partial_s^2 \mathbf{x}(t,1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \partial_s^3 \mathbf{x}(t,1) = 0,$$

where we define the state as $\mathbf{x} = col(\partial_t \eta, \partial_s^2 \eta)$ *where* η *is displacement,* w *is external disturbance and* u *is control input. To regulate a combination of vibrations and control effort we defined* $z(t) = \begin{bmatrix} \int_0^1 \eta(t,s)ds & u(t) \end{bmatrix}^T$. *The goal is to find the controller gains* $\mathcal{K} : \mathbf{x}(t) \mapsto u(t)$ *that minimizes* $\sup_{\|w\|_{L_2}=1} \|z\|_{L_2}$*. To define the values of* $\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, D_i$ *we may apply the formulae in Shivakumar* et al. *(2022). However, for illustration, we derive this representation directly. Specifically, from Cauchy's rule for repeated integration, we have*

$$\mathbf{x}(t,s) = \mathbf{x}(t,0) + s\partial_s \mathbf{x}(t,0) + \int_0^s (s-\theta)\partial_s^2 \mathbf{x}(t,\theta)d\theta.$$

Substituting the boundary conditions, we obtain the direct relationship

$$\mathbf{x}(t,s) = \int_0^s \begin{bmatrix} (s-\theta) & 0 \\ 0 & 0 \end{bmatrix} \partial_s^2 \mathbf{x}(t,\theta)d\theta + \int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & (\theta - s) \end{bmatrix} \partial_s^2 \mathbf{x}(t,\theta)d\theta.$$

Substituting this expression into the dynamics, we obtain the PIE representation

$$\partial_t \left(\begin{matrix} \int_0^s \begin{bmatrix} (s-\theta) & 0 \\ 0 & 0 \end{bmatrix} \partial_s^2 \mathbf{x}(t,\theta)d\theta \\ + \int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & (\theta - s) \end{bmatrix} \partial_s^2 \mathbf{x}(t,\theta)d\theta \end{matrix} \right)$$
$$= \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix} \partial_s^2 \mathbf{x}(t,s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

Finally, by inspection, we identify the non-zero parameters in the Partial Integral operators $\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, D_i$ *as*

$$\mathcal{T} = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{0, R_1, R_2\} \end{array}\right], \mathcal{A} = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_0, 0, 0\} \end{array}\right],$$

$$\mathcal{B}_i = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline Q_2 & \{\emptyset\} \end{array}\right], \mathcal{C} = \Pi\left[\begin{array}{c|c} \emptyset & Q_1 \\ \hline \emptyset & \{\emptyset\} \end{array}\right], D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where

$$R_1(s,\theta) = \begin{bmatrix} s-\theta & 0 \\ 0 & 0 \end{bmatrix}, \ R_2(s,\theta) = \begin{bmatrix} 0 & 0 \\ 0 & \theta - s \end{bmatrix}, \ Q_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$R_0(s) = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix}, \ Q_1(s) = \begin{bmatrix} 0 & -\frac{s^4}{12} - \frac{s^3}{6} + \frac{s^2}{2} \\ 0 & 0 \end{bmatrix}.$$

5.3 Equivalence of Representations

For finite-dimensional state-space systems, similarity transforms are used to construct equivalent representations of the input-output map. Specifically, for any invertible T, the system $G = \{A, B, C, D\}$ with internal state x may be equivalently represented as $G = \{T^{-1}AT, T^{-1}B, CT, D\}$ with internal state $\hat{x} = T^{-1}x$. In the following subsections, we will apply this approach to show equivalence between a GPDE and its PIE representation. As before, the primary difficulty in showing such an equivalence stems from the 'PDE' part of a GPDE and thus, we will first show the equivalence between a PDE subsystem and the PIE subsystem, then augment the ODE subsystem to obtain the full GPDE.

5.3.1 A PDE and its Associated PIE

Now that we have obtained an invertible transformation from $L_2^{n_{\hat{\mathbf{x}}}}$ to X_v, we apply the logic of the similarity transform to the internal dynamics of the PDE subsystem in order to obtain an equivalent PIE subsystem representation. Specifically, in Theorem 5.4, we substitute $\hat{\mathbf{x}} = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v$ in the internal dynamics of the PDE subsystem. The result is a set of equations parameterized entirely using PI operators. These PI operators, as defined in Block 5.2, specify a PIE subsystem whose input-output behavior mirrors that of the PDE subsystem and whose solution can be constructed using the solution of the PDE subsystem. Conversely, any solution of the associated PIE subsystem can be used to construct a solution for the PDE subsystem.

Theorem 5.4. *Given an $n \in \mathbb{N}^{N+1}$ and a set of PDE parameters $\{\mathbf{G}_{\mathrm{b}},\ \mathbf{G}_{\mathrm{p}}\}$ as defined in Equations* (3.6) *and* (3.8) *with $\{n, \mathbf{G}_{\mathrm{b}}\}$ admissible, suppose $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ with $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $\{\hat{\mathcal{T}},\ \mathcal{T}_v\}$ are as defined in Block 5.1 and $\{\hat{\mathcal{A}},\ \mathcal{B}_v,\ \mathcal{C}_r,\ \mathcal{D}_{rv}\}$ are as defined in Block 5.2. Define*

$$\mathbf{G}_{\mathrm{PIE}} = \left\{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_r, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\right\}.$$

Then we have the following.

1. *For any $\hat{\mathbf{x}}^0 \in X_{v(0)}$ (X_v is as defined in Equation* (3.5)*), if $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with initial condition $\hat{\mathbf{x}}^0$ and input v, then $\{\mathcal{D}\hat{\mathbf{x}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\mathcal{D}\hat{\mathbf{x}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$ and input $\{v, \emptyset\}$ where $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0 \hat{\mathbf{x}}_0, \cdots, \partial_s^N \hat{\mathbf{x}}_N)$.*

2. *For any $\hat{\underline{\mathbf{x}}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$, if $\{\hat{\underline{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ for initial condition $\hat{\underline{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$, then $\{\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v,\ r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with initial condition $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$ and input v.*

Proof. The proof is based on a partial similarity transform induced by $\hat{\mathbf{x}} = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v$ and details may be found in Appendix A.2. □

The first part of Theorem 5.4 shows that the well-posedness of the PDE subsystem guarantees the well-posedness of the associated PIE subsystem and shows that the input-output behavior of the PIE subsystem mirrors that of the PDE subsystem. The second, converse, result shows that the well-posedness of the PIE subsystem guarantees the well-posedness of the PDE subsystem and shows that the input-output behavior of the PDE subsystem mirrors that of the PIE subsystem. Because PIEs are potentially easier to numerically analyze, control, and simulate, this converse result suggests that the tasks of analysis, control, and simulation of a PDE subsystem may be more readily accomplished by performing the desired task on the PIE subsystem and then applying the result to the original PDE subsystem.

5.3.2 A GPDE and its Associated PIE

Having handled the harder task, i.e., showing the equivalence of a PDE subsystem and its associated PIE, we now define the PIE system associated with a given admissible GPDE model and prove the equivalence of their solutions. This associated PIE system is defined by 4-PI parameters as defined in Blocks 5.1 and 5.2. For convenience, we use

$$\mathbf{M} : \{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\} \mapsto \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\}$$

to represent the several formulae used to map GPDE parameters to PIE parameters.

Definition 5.2. *Given* $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}$ *where*

$$\mathbf{G}_\mathrm{b} = \{B, B_I, B_v\}\,, \quad \mathbf{G}_\mathrm{p} = \{A_0, A_1, A_2, B_{xv}, B_{xb}, C_r, D_{rb}\}$$
$$\mathbf{G}_\mathrm{o} = \{A, B_{xw}, B_{xu}, B_{xr}, C_z, D_{zw}, D_{zu}, D_{zr}, C_y, D_{yw}, D_{yu}, D_{yr}, C_v, D_{vw}, D_{vu}\}$$

we say that $\mathbf{G}_\mathrm{PIE} = \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\})$ *if* $\mathbf{G}_\mathrm{PIE} = \{\mathcal{T},\ \mathcal{T}_w,\ \mathcal{T}_u,\ \mathcal{A},\ \mathcal{B}_1,\ \mathcal{B}_2,\ \mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{D}_{11},\ \mathcal{D}_{12},\ \mathcal{D}_{21},\ \mathcal{D}_{22}\}$ *where* $\{\mathcal{T},\ \mathcal{T}_w,\ \mathcal{T}_u,\ \mathcal{A},\ \mathcal{B}_1,\ \mathcal{B}_2,\ \mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{D}_{11},\ \mathcal{D}_{12},\ \mathcal{D}_{21},\ \mathcal{D}_{22}\}$ *are as defined in Blocks 5.1 and 5.2.*

Having specified the PIE system associated with a given GPDE model, we now extend the results of Theorem 5.4 to show that the map $\mathbf{x} \mapsto \begin{bmatrix} I & 0 \\ 0 & \mathcal{D} \end{bmatrix} \mathbf{x}$ proposed in Corollary 5.3 maps a solution of a given GPDE model to a solution of the associated PIE system and that the inverse map $\underline{\mathbf{x}} \mapsto \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$ maps a solution of the associated PIE to a solution of the given GPDE model.

Corollary 5.5 (Corollary of Theorem 5.4)**.** *Given an* $n \in \mathbb{N}^{N+1}$ *and parameters* $\{\mathbf{G}_{\mathrm{o}}$, $\mathbf{G}_{\mathrm{b}}$, $\mathbf{G}_{\mathrm{p}}\}$ *as defined in Equations* (3.2)*,* (3.6) *and* (3.8) *with* $\{n, \mathbf{G}_{\mathrm{b}}\}$ *admissible, let* $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ *with* $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$*,* $u \in L_{2e}^{n_u}[\mathbb{R}_+]$ *with* $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$*. Define*

$$\mathbf{G}_{\mathrm{PIE}} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\} = \mathbf{M}(\{n, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{p}}\}.$$

Then we have the following:

1. *For any* $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{w(0),u(0)}$ *(where* $\mathcal{X}_{w,u}$ *is as defined in Equation* (3.9)*), if* $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ *satisfies the GPDE defined by* $\{n,\ \mathbf{G}_{\mathrm{o}},\ \mathbf{G}_{\mathrm{b}},\ \mathbf{G}_{\mathrm{p}}\}$ *with initial condition* $\{x^0, \hat{\mathbf{x}}^0\}$ *and input* $\{w, u\}$*, then* $\left\{ \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix},\ z,\ y \right\}$ *satisfies the PIE defined by* $\mathbf{G}_{\mathrm{PIE}}$ *with initial condition* $\begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0 \end{bmatrix}$ *and input* $\{w, u\}$ *where* $\mathcal{D}\hat{\mathbf{x}} =$ *col*$(\partial_s^0 \hat{\mathbf{x}}_0, \cdots, \partial_s^N \hat{\mathbf{x}}_N)$.

2. *For any* $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$*, if* $\{\underline{\mathbf{x}},\ z,\ y\}$ *satisfies the PIE defined by* $\mathbf{G}_{\mathrm{PIE}}$ *with initial condition* $\underline{\mathbf{x}}^0$ *and input* $\{w, u\}$*, then* $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ *satisfies the GPDE defined by* $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ *with initial condition* $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 + \mathcal{T}_w w(0) + \mathcal{T}_u u(0)$

and input $\{w, u\}$ where

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t),$$

$$v(t) = C_v x(t) + D_{vw} w(t) + D_{vu} u(t),$$

$$r(t) = \begin{bmatrix} 0_{n_{\hat{\mathbf{x}}} \times n_x} & \mathcal{C}_r \end{bmatrix} \underline{\mathbf{x}}(t) + \mathcal{D}_{rv} v(t),$$

and where $\mathcal{C}_r$ and $\mathcal{D}_{rv}$ are as defined in Block 5.2.

Proof. The proof is simply a matter of applying Theorem 5.4 to the augmented states and verifying that the definition of solution is satisfied for both the GPDE and PIE. A detailed proof can be found in Appendix A.4. □

5.4 Equivalence of System Properties

We have motivated the construction of PIE representations of GPDE models by stating that many analysis, control, and simulation tasks may be more readily accomplished in the PIE framework. However, this motivation is predicated on the assumption that the results of analysis, control, and simulation of a PIE system somehow translate to analysis, control, and simulation of the original GPDE model. For simulation, the conversion of a numerical solution of a PIE system to the numerical solution of the GPDE is trivial, as per Corollary 5.5 through the mapping $\underline{\mathbf{x}}(t) \mapsto \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t)$. In this section, we show that analysis and control of the PIE system may also be translated to the GPDE model. For input-output properties, this translation is trivial. For internal stability and control, an additional mathematical structure is required.

5.4.1 *Stability*

Unlike I/O properties, the question of the internal stability of a GPDE model is complicated because there is no universally accepted definition of stability for such models. Specifically, suppose the state-space of a GPDE model is defined to be $\mathcal{X}_{u,w}$ (a subspace of the Sobolev space W^n). In that case, the obvious norm is the Sobolev norm – implying that exponential stability requires exponential decay with respect to the Sobolev norm. However, many results on stability of PDE models use the L_2 norm as a simpler notion of size of the state.

We will see that while both notions of stability are reasonable, the use of the Sobolev norm and associated inner product confers significant advantages in terms of mathematical structure on the GPDE model and offers a clear equivalence between internal stability of the GPDE model and associated PIE system. In particular, we first show that $\mathcal{X}_{0,0}$ is a Hilbert space when equipped with the Sobolev inner product. Furthermore, the exponential stability of the GPDE model with respect to the Sobolev norm is equivalent to the exponential stability of the PIE system with respect to the L_2 norm.

Topology of $\mathcal{X}_{0,0}$ (state space of a GPDE with no inputs)

Before we begin, for $n \in \mathbb{N}^N$, let us recall the standard inner product on $\mathbb{R}^{n_x} \times W^n$

$$\left\langle \begin{bmatrix} u \\ \mathbf{u} \end{bmatrix}, \begin{bmatrix} v \\ \mathbf{v} \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x} \times W^n} = u^T v + \sum_{i=0}^{N} \langle \mathbf{u}_i, \mathbf{v}_i \rangle_{W_i^{n_i}} ,$$

$$\langle \mathbf{u}_i, \mathbf{v}_i \rangle_{W_i^{n_i}} := \sum_{j=0}^{i} \left\langle \partial_s^j \mathbf{u}_i, \partial_s^j \mathbf{u}_i \right\rangle_{L_2}$$

with associated norms $\|\mathbf{u}_i\|_{W_i^{n_i}} := \sum_{j=0}^{i} \|\partial_s^j \mathbf{x}_i\|_{L_2^{n_i}}$ and

$$\left\| \begin{bmatrix} u \\ \mathbf{u} \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} = \|u\|_{\mathbb{R}} + \sum_{i=0}^{N} \|\mathbf{u}_i\|_{W_i^{n_i}}.$$

As we will see, however, the standard inner product $\mathbb{R}^{n_x} \times W^n$ is not quite the right inner product for $\mathcal{X}_{0,0}$. For this reason, we propose a slightly modified inner product which we will denote $\langle \cdot, \cdot \rangle_{X^n}$, and show that this new inner product is equivalent to the standard inner product on W^n. Specifically, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle_{X^n} = \sum_{i=0}^{N} \left\langle \partial_s^i \mathbf{u}_i, \partial_s^i \mathbf{v}_i \right\rangle_{L_2^{n_i}} = \langle \mathcal{D}\mathbf{u}, \mathcal{D}\mathbf{v} \rangle_{L_2^{n_\times}} \tag{5.6}$$

and define the obvious extension

$$\left\langle \begin{bmatrix} u \\ \mathbf{u} \end{bmatrix}, \begin{bmatrix} v \\ \mathbf{v} \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x} \times X^n} = u^T v + \langle \mathbf{u}, \mathbf{v} \rangle_{X^n}.$$

We now show that the norms $\|\cdot\|_{\mathbb{R}^{n_x} \times W^n}$ and $\|\cdot\|_{\mathbb{R}^{n_x} \times X^n}$ are equivalent on the subspace $\mathcal{X}_{0,0}$.

Lemma 5.6. *Suppose pair $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible. Then $\|\mathbf{u}\|_{\mathbb{R}^{n_x} \times X^n} \le \|\mathbf{u}\|_{\mathbb{R}^{n_x} \times W^n}$ and there exists $c_0 > 0$ such that for any $\mathbf{u} \in \mathcal{X}_{0,0}$, we have $\|\mathbf{u}\|_{\mathbb{R}^{n_x} \times W^n} \le c_0 \|\mathbf{u}\|_{\mathbb{R}^{n_x} \times X^n}$.*

Proof. Because the map $\underline{\mathbf{x}} \to \mathbf{x}$ is a PI operator, it is bounded, which allows a bound on all terms in the Sobolev norm. See Appendix A.5.2 for a complete proof. □

Trivially, using $n_x = 0$, this result also extends to equivalence of $\|\cdot\|_{W^n}$ and $\|\cdot\|_{X^n}$ on X_0.

Next, we will show that $\hat{\mathcal{T}}$ and $\mathcal{T}$ are isometric when X_0 and $\mathcal{X}_{0,0}$ are endowed with the inner products $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n_x} \times W^n}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n_x} \times X^n}$, respectively. This implies that these spaces are complete with respect to both $\|\cdot\|_{\mathbb{R}^{n_x} \times X^n}$ ($\|\cdot\|_{X^n}$) and $\|\cdot\|_{\mathbb{R}^{n_x} \times W^n}$ ($\|\cdot\|_{W^n}$).

$\mathcal{X}_{0,0}$ is Hilbert and $\mathcal{T}$ is unitary

First, note X_0 and $\mathcal{X}_{0,0}$ are defined by $\{n, \mathbf{G}_\mathrm{b}\}$ as

$$X_0 = \left\{ \hat{\mathbf{x}} \in W^n[a,b] : B\mathcal{B}\hat{\mathbf{x}} = \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds \right\},$$

$$\mathcal{X}_{0,0} = \left\{ \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathbb{R} \times X_v \; : \; v = C_v x \right\}.$$

The sets X_0 and $\mathcal{X}_{0,0}$ are the subspaces of valid PDE subsystem and GPDE model states when $v = 0$ and when $u = 0, w = 0$, respectively. Previously, in Theorem 5.1, we have shown that $\hat{\mathcal{T}}$ is a bijective map. In Theorem 5.7 we extend this result to show that $\hat{\mathcal{T}} : L_2^{n_{\hat{\mathbf{x}}}} \to X^n$ and $\mathcal{T} : \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}} \to \mathbb{R}^{n_x} \times X^n$ are unitary in that the respective inner products are preserved under these transformations.

Theorem 5.7. *Suppose $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, $\{\hat{\mathcal{T}}, \mathcal{T}_v\}$ are as defined in Block 5.1, and $\{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u\}$ are as defined in Block 5.2 for some matrices C_v, D_{vw} and D_{vu}. If $\langle \cdot, \cdot \rangle_{X^n}$ is as defined in Equation* (5.6)*, then we have the following:*

a) for any $v_1, v_2 \in \mathbb{R}^{n_v}$ and $\underline{\hat{\mathbf{x}}}$, $\underline{\hat{\mathbf{y}}} \in L_2^{n_{\hat{\mathbf{x}}}}$

$$\left\langle \left(\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v_1 \right), \left(\hat{\mathcal{T}}\underline{\hat{\mathbf{y}}} + \mathcal{T}_v v_2 \right) \right\rangle_{X^n} = \left\langle \underline{\hat{\mathbf{x}}}, \underline{\hat{\mathbf{y}}} \right\rangle_{L_2^{n_{\hat{\mathbf{x}}}}}. \tag{5.7}$$

b) for any $w_1, w_2 \in \mathbb{R}^{n_w}$, $u_1, u_2 \in \mathbb{R}^{n_u}$, $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$,

$$\begin{aligned} &\left\langle (\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w_1 + \mathcal{T}_u u_1), \left(\mathcal{T}\underline{\mathbf{y}} + \mathcal{T}_w w_2 + \mathcal{T}_u u_2 \right) \right\rangle_{\mathbb{R}^{n_x} \times X^n} \\ &\quad = \left\langle \underline{\mathbf{x}}, \underline{\mathbf{y}} \right\rangle_{\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}}. \end{aligned} \tag{5.8}$$

Proof. The proof follows directly from the definition of the X^n inner product and the map $\mathbf{x} \mapsto \underline{\mathbf{x}}$. See Appendix A.5.1 for more details. $\square$

Corollary 5.8. *Suppose $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, $\hat{\mathcal{T}}$ is as defined in Block 5.1, $\mathcal{T}$ is as defined in Block 5.2, X_v is as defined in Equation* (3.5) *and, for any matrices C_v, D_{vw} and D_{vu}, $\mathcal{X}_{w,u}$ is as defined in Equation* (3.9). *Then X_0 is complete with respect to $\|\cdot\|_{X^n}$ and $\mathcal{X}_{0,0}$ is complete with respect to $\|\cdot\|_{\mathbb{R}^{n_x}\times X^n}$. Furthermore, $\hat{\mathcal{T}} : L_2^{n_{\hat{\mathbf{x}}}} \to X_0$ and $\mathcal{T} : \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}} \to \mathcal{X}_{0,0}$ are unitary (isometric surjective mappings between Hilbert spaces).*

Proof. From Theorem 5.1 and Corollary 5.3, we have that $\mathcal{T}$ is a bijective mapping from $\mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ to $\mathcal{X}_{0,0}$. From Theorem 5.7, we have that $\mathcal{T}$ is isometric with respect to the $\mathbb{R}^{n_x} \times X^n$ inner product. Since $\mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ is complete, we conclude that $\mathcal{X}_{0,0}$ is complete with respect to the $\mathbb{R}^{n_x} \times X^n$ norm. Completeness of X_0 follows trivially from the special case $n_x = 0$. $\square$

As a direct consequence of Corollary 5.8 and Lemma 5.6, X_0 and $\mathcal{X}_{0,0}$ are also complete with respect to $\|\cdot\|_{W^n}$ and $\|\cdot\|_{\mathbb{R}^{n_x}\times W^n}$, respectively.

As shown in Theorem 5.7, the natural definition of exponential stability of a GPDE model is with respect to the $\mathbb{R}^{n_x} \times X^n$ norm. However, as shown in Lemma 5.6, exponential stability with respect to the $\mathbb{R}^{n_x} \times X^n$ norm is equivalent to exponential stability with respect to the $\mathbb{R}^{n_x} \times W^n$ norm. Hence, we formally define stability with respect to the $\mathbb{R}^{n_x} \times W^n$ norm.

Definition 5.3 (Exponential Stability of a GPDE model)**.** *We say a GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is exponentially stable if there exist constants M, $\alpha > 0$ such that for any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$, if $\{x,\ \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0, 0\}$, then*

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \le M \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} e^{-\alpha t} \quad \text{for all } t \ge 0.$$

In contrast to the GPDE stability, the internal stability of a PIE system is with respect to the $\mathbb{R}L_2$ norm because that is natural norm on the solution space that preserves the topological structure.

Definition 5.4 (Exponential Stability of a PIE system). *We say a PIE defined by $\mathbf{G}_{\text{PIE}}$ is exponentially stable if there exist M, $\alpha > 0$ such that for any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$, if $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\text{PIE}}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{0,0\}$, then $\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \leq M \|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2} e^{-\alpha t}$ for all $t \geq 0$.*

Using these definitions of internal stability for GPDE and PIE representations, one can show that the exponential stability of a GPDE model is equivalent to the exponential stability of the associated PIE system, which is formalized below.

Theorem 5.9. *Given $\{n, \mathbf{G}_\text{o}, \mathbf{G}_\text{b}, \mathbf{G}_\text{p}\}$ with $\{n, \mathbf{G}_\text{b}\}$ admissible, the GPDE model defined by $\{n, \mathbf{G}_\text{o}, \mathbf{G}_\text{b}, \mathbf{G}_\text{p}\}$ is exponentially stable if and only if the PIE defined by $\mathbf{G}_{\text{PIE}} = \mathbf{M}(\{n, \mathbf{G}_\text{b}, \mathbf{G}_\text{o}, \mathbf{G}_\text{p}\})$ is exponentially stable.*

Proof. The proof is a direct application of the stability definitions, Theorem 5.7, and Lemma 5.6 (See Appendix A.5.3). $\square$

The results of Theorem 5.9 also imply that Lyapunov and asymptotic stability of the GPDE model in the $\mathbb{R}^{n_x} \times W^n$ norm are equivalent to Lyapunov and asymptotic stability of the associated PIE system in the $\mathbb{R}L_2$ norm. Recall that Lyapunov and asymptotic stability of GPDEs are defined as follows, which naturally lead to similar definitions of stability for PIEs but in a different normed space.

Definition 5.5 (Lyapunov Stability).

1. *We say a GPDE model defined by $\{n$, $\mathbf{G}_\text{o}$, $\mathbf{G}_\text{b}$, $\mathbf{G}_\text{p}\}$ is Lyapunov stable, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$ with*

$\left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} < \delta$, *if* $\{x,\ \hat{\mathbf{x}}, z, y, v, r\}$ *satisfies the GPDE defined by* $\{n,\ \mathbf{G}_\mathrm{o},$ $\mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ *with initial condition* $\{x^0, \hat{\mathbf{x}}^0\}$ *and input* $\{0, 0\}$*, then*

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} < \epsilon \qquad \text{for all } t \geq 0.$$

2. *We say a PIE model defined by* $\mathbf{G}_\mathrm{PIE}$ *is Lyapunov stable if for every* $\epsilon > 0$ *there exists a constant* $\delta > 0$ *such that for any* $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}$ *with* $\|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2^{m,n}} < \delta$*, if* $\{\underline{\mathbf{x}}, z, y\}$ *satisfies the PIE defined by* $\mathbf{G}_\mathrm{PIE}$ *with initial condition* $\underline{\mathbf{x}}^0$ *and input* $\{0, 0\}$*, then* $\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2^{m,n}} < \epsilon$ *for all* $t \geq 0$.

Definition 5.6 (Asymptotic Stability)**.**

1. *We say a GPDE defined by* $\{n,\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ *is asymptotically stable, if for every* $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$ *and* $\epsilon > 0$*, there exists a* $T_\epsilon > 0$ *such that if* $\{x,\ \hat{\mathbf{x}}, z, y, v, r\}$ *satisfies the GPDE defined by* $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ *with initial condition* $\{x^0, \hat{\mathbf{x}}^0\}$ *and input* $\{0, 0\}$*, then* $\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} < \epsilon$ *for all* $t > T_\epsilon$.
2. *We say a PIE model defined by* $\mathbf{G}_\mathrm{PIE}$ *is asymptotically stable, if for every* $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}$ *and* $\epsilon > 0$*, there exists a* $T_\epsilon > 0$ *such that if* $\{\underline{\mathbf{x}}, z, y\}$ *satisfies the PIE defined by* $\mathbf{G}_\mathrm{PIE}$ *with initial condition* $\underline{\mathbf{x}}^0$ *and input* $\{0, 0\}$*, then there exists* $T_\epsilon > 0$ *such that* $\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2^{m,n}} < \epsilon$ *for all* $t > T_\epsilon$.

The equivalence of Lyapunov and asymptotic stability for the two representations are formalized in the following Corollary.

Corollary 5.10. *Given* $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ *with* $\{n, \mathbf{G}_\mathrm{b}\}$ *admissible, let* $\mathbf{G}_\mathrm{PIE} = \mathbf{M}\ (\{n,$ $\mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{o},\ \mathbf{G}_\mathrm{p}\})$*. Then*

1. *The GPDE model defined by* $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ *is Lyapunov stable if and only if the PIE system defined by* $\mathbf{G}_\mathrm{PIE}$ *is Lyapunov stable.*

2. *The GPDE model defined by* $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ *is asymptotically stable if and only if the PIE system defined by* $\mathbf{G}_\mathrm{PIE}$ *is asymptotically stable.*

Proof. Based on the stability definitions, this result is a direct corollary of Theorem 5.9 (See Appendix A.5.3). □

5.4.2 Input-Output Properties

Recall the transformation of a PIE solution to the GPDE solution is limited to the internal state of the PIE whereas the inputs and outputs are unchanged. Consequently, Corollary 5.5 implies that all input-output (I/O) properties of the GPDE model are inherited by the PIE system and vice versa. As a result, we have the following Corollary, which can be trivially proved using the map between parameters of a GPDE and the parameters of its associated PIE.

Corollary 5.11 (Input-Output Properties)**.** *Given an* $n \in \mathbb{N}^{N+1}$ *and parameters* $\{\mathbf{G}_\mathrm{o},\, \mathbf{G}_\mathrm{b},\, \mathbf{G}_\mathrm{p}\}$ *as defined in Equations* (3.2)*,* (3.6) *and* (3.8) *with* $\{n, \mathbf{G}_\mathrm{b}\}$ *admissible, let* $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ *with* $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$*. Let* $\mathbf{G}_\mathrm{PIE} = \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}$*. Suppose* $\{x^0, \hat{\mathbf{x}}^0\} = \{0, 0\}$*. Then the following are equivalent.*

1. *If* $\{x,\, \hat{\mathbf{x}},\, z,\, y,\, v,\, r\}$ *satisfies the GPDE defined by* $\{n,\, \mathbf{G}_\mathrm{o},\, \mathbf{G}_\mathrm{b},\, \mathbf{G}_\mathrm{p}\}$ *with initial condition* $\{0, 0\}$ *and input* $\{w, 0\}$*, then* $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$*.*

2. *If* $\{\underline{\mathbf{x}},\ z,\ y\}$ *satisfies the PIE defined by* $\mathbf{G}_\mathrm{PIE}$ *with initial condition* 0 *and input* $\{w, 0\}$*, then* $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$*.*

Suppose $\mathcal{K} :\in L_{2e}^{n_y} \to L_{2e}^{n_u}$*. Then the following are equivalent.*

1. *If* $\{x,\, \hat{\mathbf{x}},\, z,\, y,\, v,\, r\}$ *satisfies the GPDE defined by* $\{n,\, \mathbf{G}_{\mathrm{o}},\, \mathbf{G}_{\mathrm{b}},\, \mathbf{G}_{\mathrm{p}}\}$ *with initial condition* $\{0,0\}$ *and input* $\{w, \mathcal{K}y\}$*, then* $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

2. *If* $\{\underline{\mathbf{x}},\ z,\ y\}$ *satisfies the PIE defined by* $\mathbf{G}_{\mathrm{PIE}}$ *with initial condition* 0 *and input* $\{w, \mathcal{K}y\}$*, then* $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof. Corollary 5.11 follows directly from Corollary 5.5. Since the change of internal state from GPDE state $\{x, \hat{\mathbf{x}}\}$ to the PIE state $\underline{\mathbf{x}}$ does not change the inputs or outputs and the solutions are equivalent, the input-output properties remain unaffected. □

5.4.3 A Side Note on the Conversion of a GPDE to a PIE

Because the GPDE class of model is meant to be *universal*, construction of a GPDE requires the identification of a large number of system parameters — most of which are typically zero or sparse. Furthermore, construction of the associated PIE system using the formulae in Blocks 5.1 and 5.2 can be cumbersome, requiring one to parse a rather complicated notational system. This complicated process of identification of parameters and application of formulae may thus impede the practical application of the results in this paper. For this reason, PIETOOLS versions 2021a and later include software interfaces for the construction of GPDE models, which do not require the user to understand of the notational system defined here. For example, PIETOOLS 2021b (By Shivakumar *et al.* (2020b)) includes a Graphical User Interface (GUI), which allows the user to define a GPDE data structure one term at a time. Because many GPDE models only consist of a few terms, this GUI dramatically reduces the time required to declare a GPDE model. Furthermore, this GUI automates the application of the formulae in Blocks 5.1 and 5.2 – allowing the user to construct an associated PIE system data structure that is compatible with the PIETOOLS utilities for analysis, control and simulation of PIEs. Additional details can be found in the PIETOOLS user manual by Shivakumar *et al.* (2021).

In addition to the GUI, PIETOOLS includes many tools for the analysis, control, estimation, and simulation of PIE systems in the context of simple PDE models, advanced GPDE models, and Delay Differential Equations.

5.5 Additional Examples

In this section, we present additional examples explaining the process of identification of GPDE parameters and finding PIE representation to illustrate the PIE representation for a wide variety PDE systems.

Demonstration 5.1 (ODE coupled with PDE at the Boundary)**.** *In this example, we consider a thermal reactor, $T_r(t)$, which is modeled as an ODE and is coupled to a cooling jacket, $T_c(t,s)$, which is modeled as a PDE. The dynamics of the reactor and jacket are given by*

$$\begin{aligned} \dot{T}_r(t) &= \lambda T_r(t) - C(T_r(t) - T_c(t,0)), \\ \dot{T}_c(t,s) &= k\partial_s^2 T_c(t,s), \qquad s \in (0,1), \\ T_c(t,0) &= T_r(t), \qquad \partial_s T_c(t,1) = 0 \end{aligned} \tag{5.9}$$

where λ is the reaction coefficient of the reactor, C is the specific heat of the reactor, and k is a diffusivity parameter for the coolant. In this case, we first model the ODE, where the influence of the PDE on the ODE is isolated in the signal $r(t) = T_c(t,0)$ and the influence of the PDE on the ODE is isolated in the signal $v(t) = T_r(t)$. The state of the ODE subsystem is $x(t) = T_r(t)$ with the following dynamics.

$$\begin{aligned} \dot{x}(t) = (\lambda - C)x(t) + r(t), \quad v(t) = x(t), \quad \dot{T}_c(t,s) &= k\partial_s^2 T_c(t,s), \quad s \in (0,1), \\ T_c(t,0) = v(t), \qquad \partial_s T_c(t,1) = 0 & \end{aligned} \tag{5.10}$$

Examining the ODE dynamics

$$\begin{bmatrix} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_{xw} & B_{xu} & B_{xr} \\ \hline C_z & D_{zw} & D_{zu} & D_{zr} \\ C_y & D_{yw} & D_{yu} & D_{yr} \\ C_v & D_{vw} & D_{vu} & 0 \end{array}\right] \begin{bmatrix} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{bmatrix}.$$

we may parameterize the ODE subsystem, $\mathbf{G}_\text{o}$ *as*

$$\mathbf{G}_\text{o} \; : \qquad A = \lambda - C, \qquad B_{xr} = C, \qquad C_v = 1.$$

Now, examining the PDE subsystem, we have a system similar to Illustration in Example 5.2 so that the continuity parameter is

$$n \; : \qquad n = \{0, 0, 1\} \qquad N = 2$$

with $\hat{\mathbf{x}}_2(t,s) = T_c(t,s)$. *Again, the BCs appear in the form*

$$\begin{aligned} 0 &= \int_a^b B_I(s)\mathcal{F}\hat{\mathbf{x}}(t,s)ds + \begin{bmatrix} B_v & -B \end{bmatrix} \begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t) \end{bmatrix} \\ &= \int_0^1 B_I(s) \begin{bmatrix} T_c(t,s) \\ T_{c,s}(t,s) \\ T_{c,ss}(t,s) \end{bmatrix} ds - B \begin{bmatrix} T_c(t,0) \\ T_{c,s}(t,0) \\ T_c(t,1) \\ T_{c,s}(t,1) \end{bmatrix} + B_v v(t). \end{aligned}$$

$$\begin{aligned} v(t) = x(t)\dot{T}_c(t,s) &= k\partial_s^2 T_c(t,s), \qquad & s \in (0,1), \\ T_c(t,0) &= v(t), \qquad & \partial_s T_c(t,1) = 0 \end{aligned} \tag{5.11}$$

By inspection of the BCs, we may now define the parameters for $\mathbf{G}_{\mathrm{b}}$ *as*

$$\mathbf{G}_{\mathrm{b}}\ : \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B_v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To define the parameters of the PDE dynamics, we again ignore integral terms, yielding

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t,s) \\ r(t) \end{bmatrix} = \begin{bmatrix} T_c(t,s) \\ r(t) \end{bmatrix} = \begin{bmatrix} A_0(s) \begin{bmatrix} T_c(t,s) \\ T_{c,s}(t,s) \\ T_{c,ss}(t,s) \end{bmatrix} \\ 0 \end{bmatrix} + \begin{bmatrix} B_{xv}(s) & B_{xb}(s) \\ 0 & D_{rb} \end{bmatrix} \begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t) \end{bmatrix}.$$

By inspection of Equation (5.9)*, the only non-zero parameter in this expression is*

$$\mathbf{G}_{\mathrm{p}}\ : \qquad A_0 = \begin{bmatrix} 0 & 0 & k \end{bmatrix}$$

which becomes the entire parameter set for $\mathbf{G}_{\mathrm{p}}$.

Demonstration 5.2 (Second Order Time Derivatives)**.** *For our next illustration, we consider wave motion*

$$\begin{aligned} \ddot{\eta}(t,s) &= c^2 \partial_s^2 \eta(t,s), & s \in (0,1), \\ z(t,s) &= \int_0^1 \eta(t,s) ds, \end{aligned}$$

where z *is a regulated output (the average displacement of the string) with a general form of BCs (Sturm-Liouville type BCs) given by*

$$\eta(t,0) - k\partial_s \eta(t,0) = 0, \qquad \eta(t,1) + l\partial_s \eta(t,1) = w(t),$$

where η *stands for lateral displacement,* c *is the speed of propagation of a wave in the string, and* w *is external disturbance acting on the boundary. The constants* k *and* l *represent the reflection and mirroring of the wave at the boundary.*

To rewrite this PDE model as a state-space GPDE model, we must first eliminate the second-order time-derivative. As is common in the state-space representation of ODEs, we eliminate the 2nd order time-derivative by creating a new state $\zeta_2 = \dot{\eta}$ *with* $\zeta_1 = \eta$*. This change of variable leads to a coupled PDE of the form*

$$\begin{aligned}
\dot{\zeta}_1(t,s) &= \zeta_2, \qquad\qquad s \in (0,1),\\
\dot{\zeta}_2(t,s) &= c^2\partial_s^2\zeta_1(t,s),\\
z(t,s) &= \int_0^1 \zeta_1(t,s)ds,
\end{aligned} \tag{5.12}$$

with BCs

$$\zeta_1(t,0) - k\partial_s\zeta_1(t,0) = 0, \qquad \zeta_1(t,1) + l\partial_s\zeta_1(t,1) = w(t). \tag{5.13}$$

Here we note that the ODE subsystem has the parameters related to outputs z *and inputs* w*, however, there is no ODE state. Thus, we only have parameters related to* z *and* w*. First, we include the influence of PDE on the ODE into the interconnection signal as* $r(t) = \int_0^1 \zeta_1(t,s)ds$*, whereas the influence of the ODE on the PDE is routed through* v *where* $v(t) = w(t)$*. Then, by inspection, the output* z *can be written as* $z(t) = r(t)$*. Consequently, we find that* $D_{zr} = 1$*, while the remaining parameters related to* z *are zero. Likewise, we note that* $D_{vw} = 1$ *and leave the remaining parameters of* v *as empty. This completes the definition of the ODE subsystem.*

$$\mathbf{G}_{\mathrm{o}} \;: \qquad D_{zr} = 1 \qquad D_{vw} = 1.$$

By inspecting the partial derivatives and boundary values used in Equations (5.12) *and* (5.13)*, we first define the continuity equation using* $n_0 = 1$ *so that* $\hat{\mathbf{x}}_0 = \zeta_2$ *and* $n_2 = 1$ *so that* $\hat{\mathbf{x}}_2 = \zeta_1$*.*

$$n \;: \qquad n = \{1,0,1\} \qquad N = 2.$$

For this definition of n, we have $n_{\hat{\mathbf{x}}} = n_{S_0} = 2$ and $n_{S_1} = n_{S_2} = 1$ – there are two 0^{th} order and one 1^{st} and 2^{nd} order partial derivatives. In addition, $n_S = 2$, indicating there are two possible partial derivatives. Thus

$$S^0\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \qquad S\hat{\mathbf{x}} = S^2\hat{\mathbf{x}} = \hat{\mathbf{x}}_2 = \zeta_1.$$

Next, we construct $(\mathcal{B}\hat{\mathbf{x}})$ – the vector of all possible boundary values of $\hat{\mathbf{x}}$ allowable for the given n.

$$(\mathcal{B}\hat{\mathbf{x}}) = \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(0) \\ (\mathcal{C}\hat{\mathbf{x}})(1) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}_2(0) \\ \hat{\mathbf{x}}_{2,s}(0) \\ \hat{\mathbf{x}}_2(1) \\ \hat{\mathbf{x}}_{2,s}(1) \end{bmatrix} = \begin{bmatrix} \zeta(0) \\ \zeta_{1,s}(0) \\ \zeta(1) \\ \zeta_{1,s}(1) \end{bmatrix}$$

We may now define the BCs. There is no ODE state, however, there is a disturbance w that influences the PDE via the signal v, which can be chosen as $v(t) = w(t)$. Then, the BCs appear in the form

$$\begin{bmatrix} 0 \\ v(t) \end{bmatrix} = \int_a^b B_I(s)\mathcal{F}\hat{\mathbf{x}}(t,s)ds - B(\mathcal{B}\hat{\mathbf{x}})(t)$$

$$= \int_0^1 B_I(s) \begin{bmatrix} \hat{\mathbf{x}}_1(t,s) \\ \hat{\mathbf{x}}_2(t,s) \\ \hat{\mathbf{x}}_{2,s}(t,s) \\ \hat{\mathbf{x}}_{2,ss}(t,s) \end{bmatrix} ds - B(\mathcal{B}\hat{\mathbf{x}})(t) = \int_0^1 B_I(s) \begin{bmatrix} \zeta_1(t,s) \\ \zeta_2(t,s) \\ \zeta_{1,s}(t,s) \\ \zeta_{1,ss}(t,s) \end{bmatrix} ds - B \begin{bmatrix} \zeta_1(t,0) \\ \zeta_{1,s}(t,0) \\ \zeta_1(t,1) \\ \zeta_{1,s}(t,1) \end{bmatrix}$$

By inspection of Equation (5.13), we may now define the parameters for $\mathbf{G}_{\mathrm{b}}$ and hence $X_{v(t)}$ as

$$\mathbf{G}_{\mathrm{b}} \;: \qquad B = -\begin{bmatrix} 1 & -k & 0 & 0 \\ 0 & 0 & 1 & l \end{bmatrix} \qquad B_v = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The final step is to define the parameters of the PDE dynamics. Ignoring the *integral terms for simplicity, and noting that* $\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \zeta_2 \\ \zeta_1 \end{bmatrix}$*,* $S\hat{\mathbf{x}} = S^2\hat{\mathbf{x}} = \hat{\mathbf{x}}_2 = \zeta_1$ *and* $r = v = \emptyset$*, we have*

$$\begin{bmatrix} \dot{\zeta}_2(t,s) \\ \dot{\zeta}_1(t,s) \end{bmatrix} = A_0(s) \begin{bmatrix} \zeta_2(t,s) \\ \zeta_1(t,s) \\ \zeta_{1,s}(t,s) \\ \zeta_{1,ss}(t,s) \end{bmatrix} + B_{xb}(s) \begin{bmatrix} \zeta_1(t,0) \\ \zeta_{1,s}(t,0) \\ \zeta_1(t,1) \\ \zeta_{1,s}(t,1) \end{bmatrix}.$$

By inspection of Equation (5.12)*, we have two non-zero parameters in* $\mathbf{G}_{\mathrm{p}}$*. However, the interconnection signal* r *has an integral term, which can be written as*

$$\begin{aligned} r(t) &= \int_a^b C_r(s)\mathcal{F}\hat{\mathbf{x}}(t,s)ds + D_{rb}(\mathcal{B}\hat{\mathbf{x}})(t) \\ &= \int_0^1 C_r(s) \begin{bmatrix} \hat{\mathbf{x}}_1(t,s) \\ \hat{\mathbf{x}}_2(t,s) \\ \hat{\mathbf{x}}_{2,s}(t,s) \\ \hat{\mathbf{x}}_{2,ss}(t,s) \end{bmatrix} ds + D_{rb}(\mathcal{B}\hat{\mathbf{x}})(t) \\ &= \int_0^1 C_r(s) \begin{bmatrix} \zeta_1(t,s) \\ \zeta_2(t,s) \\ \zeta_{1,s}(t,s) \\ \zeta_{1,ss}(t,s) \end{bmatrix} ds + D_{rb} \begin{bmatrix} \zeta_1(t,0) \\ \zeta_{1,s}(t,0) \\ \zeta_1(t,1) \\ \zeta_{1,s}(t,1) \end{bmatrix}. \end{aligned}$$

Clearly, only $C_{r,0} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ *is non-zero, whereas the remaining terms are zero, which gives us the final set of parameters for the PDE subsystem as*

$$\mathbf{G}_{\mathrm{p}} \;: \qquad A_{00} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_{02} = \begin{bmatrix} c^2 \\ 0 \end{bmatrix}, \qquad C_{r,0} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

This completes the definition of the GPDE.

Demonstration 5.3. *Chemical Reactor with Cooling Jacket Consider a chemical reactor with a cooling jacket as described in Karafyllis and Krstic (2019). In this model, the reactor temperature is a lumped parameter system while the coolant temperature is a distributed state that varies along the length of the pipe. Assuming conduction inside the cooling jacket to be negligible, we obtain the following coupled ODE-PDE.*

$$\begin{aligned}\dot{x}(t) &= kx(t) + \mu\int_0^1 \mathbf{x}(t,s)ds\\ \dot{\mathbf{x}}(t,s) &= -c\partial_s\mathbf{x}(t,s) - \zeta\mathbf{x}(t,s) + \zeta x(t) \qquad \mathbf{x}(t,0) = w(t)\end{aligned} \tag{5.14}$$

where x is the reactor temperature, $\mathbf{x}$ is the temperature in the cooling jacket, $w(t)$ is a disturbance, μ, c, ζ are positive constants, and k is a negative constant. In this model, the distributed state $\mathbf{x}$ has a single boundary condition and the highest spatial derivative of order 1, so $n = \{0,1\}$. In order to retain the parameter dependencies, we use the formulae in Blocks 5.1 and 5.2 to obtain the following PIE representation.

$$\begin{aligned}\dot{x}(t) &= kx(t) + \int_0^1 \mu(1-s)\hat{\underline{\mathbf{x}}}(t,s)ds,\\ \int_0^s s\dot{\hat{\underline{\mathbf{x}}}}(t,\theta)d\theta &= \zeta x(t) - c\hat{\underline{\mathbf{x}}}(t,s) - \int_0^s \zeta\hat{\underline{\mathbf{x}}}(t,\theta)d\theta - \dot{w}(t) - \zeta w(t),\end{aligned} \tag{5.15}$$

or, alternatively,

$$\begin{aligned}&\mathit{\Pi}\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \{0,s,0\}\end{array}\right]\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{\underline{\mathbf{x}}}}(t)\end{bmatrix} + \mathit{\Pi}\left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & \{0\}\end{array}\right]\dot{w}(t)\\ &= \mathit{\Pi}\left[\begin{array}{c|c} k & \mu(1-s) \\ \hline \zeta & \{-c,-\zeta,0\}\end{array}\right]\begin{bmatrix}x(t)\\ \hat{\underline{\mathbf{x}}}(t)\end{bmatrix} + \mathit{\Pi}\left[\begin{array}{c|c} 0 & 0 \\ \hline -\zeta & \{0\}\end{array}\right]w(t),\end{aligned}$$

where $\hat{\underline{\mathbf{x}}} = \partial_s\mathbf{x}$.

5.6 Conclusion

In this chapter, we proposed a test for the admissibility of a given GPDE model and proved that admissibility implies the existence of an associated Partial Integral

Equation (PIE) representation of the GPDE model. Furthermore, we found a unitary map from the state of the PIE system to the state of the GPDE model and proved that this unitary map is a PI operator. Using this unitary mapping, parameters of the GPDE model was mapped to the parameters of its associated PIE model via analytical expressions. Lastly, we proved that many properties of the GPDE model and associated PIE system are equivalent – including the existence of weak solutions, input-output properties, and internal stability.

As a side note, to aid in the practical application of the proposed GPDE models and PIE conversion formulae, we introduced an efficient open-source software (PIETOOLS) for the construction of the GPDE model, conversion to PIE system, simulation of the GPDE/PIE, and analysis/control of the GPDE/PIE. This software includes a GUI for the construction of GPDE models and conversion to an associated PIE system – a feature demonstrated on several example problems.

$$n_{\hat{\mathbf{x}}} = \sum_{i=0}^{N} n_i,\ n_{S_i} = \sum_{j=i}^{N} n_j,\ n_S = \sum_{i=1}^{N} n_{S_i}\ n_{i:j} = \sum_{k=i}^{j} n_k,\ \tau_i(s) = \frac{s^i}{i!},$$

$$Q_i(s) = \begin{bmatrix} 0 & \tau_0(s)I_{n_i} & & & \\ 0 & & \tau_1(s)I_{n_{i+1}} & & \\ & & & \ddots & \\ 0 & & & & \tau_{N-i}(s)I_{n_N} \end{bmatrix},\ Q(s) = \begin{bmatrix} Q_1(s) \\ \vdots \\ Q_N(s) \end{bmatrix},\ U_{1i} = \begin{bmatrix} I_{n_i} \\ 0_{n_{i+1:N},n_i} \end{bmatrix},$$

$$T_{i,j}(s) = \tau_{(j-i)}(s) \begin{bmatrix} 0_{(n_{Si}-n_{Sj}),n_{Sj}} \\ I_{n_{Sj}} \end{bmatrix},\ T(s) = \begin{bmatrix} T_1(s) \\ \vdots \\ T_N(s) \end{bmatrix} = \begin{bmatrix} T_{1,1}(s) & \cdots & T_{1,N}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_{N,N}(s) \end{bmatrix},$$

$$U_1 = \mathrm{diag}(U_{10}, \cdots, U_{1N}),\ U_{2i} = \begin{bmatrix} 0_{n_i,n_{i+1:N}} \\ I_{n_{i+1:N}} \end{bmatrix},\ U_2 = \begin{bmatrix} \mathrm{diag}(U_{20}, \cdots, U_{2(N-1)}) \\ 0_{n_N,n_S} \end{bmatrix},$$

$$B_T = B \begin{bmatrix} T(0) \\ T(b-a) \end{bmatrix} - \int_a^b B_I(s)U_2T(s-a)ds,\ G_v(s) = \begin{bmatrix} 0 \\ T_1(s-a)B_T^{-1}B_v \end{bmatrix},$$

$$B_Q(s) = B_T^{-1}\left(B_I(s)U_1 + \int_s^b B_I(\theta)U_2Q(\theta-s)d\theta - B \begin{bmatrix} 0 \\ Q(b-s) \end{bmatrix}\right),\ G_0 = \begin{bmatrix} I_{n_0} & \\ & 0_{(n_{\hat{\mathbf{x}}}-n_0)} \end{bmatrix},$$

$$G_2(s,\theta) = \begin{bmatrix} 0 \\ T_1(s-a)B_Q(\theta) \end{bmatrix},\ G_1(s,\theta) = \begin{bmatrix} 0 \\ Q_1(s-\theta) \end{bmatrix} + G_2(s,\theta),$$

$$\hat{\mathcal{T}} = \mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{G_i\} \end{array}\right], \qquad \mathcal{T}_v = \mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline G_v & \{\emptyset\} \end{array}\right].$$

Block 5.1: Definitions based on $n \in \mathbb{N}^{N+1}$ and the parameters of $\mathbf{G}_\mathrm{b} = \{B,\ B_I,\ B_v\}$ used in Theorem 5.1.

$$R_{D,2}(s,\theta) = U_2 T(s-a) B_Q(\theta),\ R_{D,1}(s,\theta) = R_{D,2}(s,\theta) + U_2 Q(s-\theta),$$

$$\Upsilon = \left[\begin{array}{c|c} \begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v \end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s) \end{bmatrix} \\ \hline U_2 T(s-a) B_T^{-1} B_v & \{U_1, R_{D,1}, R_{D,2}\} \end{array}\right],$$

$$\Xi = \left[\begin{array}{c|c} \begin{bmatrix} 0 & D_{rb} \end{bmatrix} & C_r \\ \hline \begin{bmatrix} B_{xv} & B_{xb} \end{bmatrix} & \{A_i\} \end{array}\right],\ \left[\begin{array}{c|c} D_{rv} & C_{rx} \\ \hline B_{xv} & \{\hat{A}_i\} \end{array}\right] = \mathbf{P}_{\times}^{4}\left(\Xi, \Upsilon\right)$$

$$\hat{\mathcal{A}} = \varPi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{\hat{A}_i\} \end{array}\right],\ \mathcal{B}_v = \varPi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline B_{xv} & \{\emptyset\} \end{array}\right],\ \mathcal{C}_r = \varPi\left[\begin{array}{c|c} \emptyset & C_{rx} \\ \hline \emptyset & \{\emptyset\} \end{array}\right],\ \mathcal{D}_{rv} = \varPi\left[\begin{array}{c|c} D_{rv} & \emptyset \\ \hline \emptyset & \{\emptyset\} \end{array}\right],$$

$$\mathcal{T} = \begin{bmatrix} I_{n_x} & 0 \\ G_v C_v & \hat{\mathcal{T}} \end{bmatrix},\ \mathcal{T}_w = \begin{bmatrix} 0 & 0 \\ G_v D_{vw} & 0 \end{bmatrix},\ \mathcal{T}_u = \begin{bmatrix} 0 & 0 \\ G_v D_{vu} & 0 \end{bmatrix},\ \mathcal{A} = \begin{bmatrix} A + B_{xr}\mathcal{D}_{rv}C_v & B_{xr}\mathcal{C}_r \\ \mathcal{B}_v C_v & \hat{\mathcal{A}} \end{bmatrix},$$

$$\mathcal{B}_1 = \begin{bmatrix} B_{xw} + B_{xr}\mathcal{D}_{rv}D_{vw} \\ \mathcal{B}_v D_{vw} \end{bmatrix},\ \mathcal{B}_2 = \begin{bmatrix} B_{xu} + B_{xr}\mathcal{D}_{rv}D_{vu} \\ \mathcal{B}_v D_{vu} \end{bmatrix},$$

$$\mathcal{C}_1 = \begin{bmatrix} C_z + D_{zr}\mathcal{D}_{rv}C_v & D_{zr}\mathcal{C}_r \end{bmatrix},\ \mathcal{C}_2 = \begin{bmatrix} C_y + D_{yr}\mathcal{D}_{rv}C_v & D_{yr}\mathcal{C}_r \end{bmatrix},\ \mathcal{D}_{11} = D_{zw} + D_{zr}\mathcal{D}_{rv}D_{vw},$$

$$\mathcal{D}_{12} = D_{zu} + D_{zr}\mathcal{D}_{rv}D_{vu},\ \mathcal{D}_{21} = D_{yw} + D_{yr}\mathcal{D}_{rv}D_{vw},\quad \mathcal{D}_{22} = D_{yu} + D_{yr}\mathcal{D}_{rv}D_{vu}.$$

Block 5.2: Definitions based on the PDE and GPDE parameters in $\mathbf{G}_{\mathrm{p}}=\{A_0,\ A_1,\ A_2,\ B_{xv},\ B_{xb},\ C_r,\ D_{rb}\}$ and $\mathbf{G}_{\mathrm{o}} = \{A,\ B_{xw},\ B_{xu},\ B_{xr},\ C_z,\ D_{zw},\ D_{zu},\ D_{zr},\ C_y,\ D_{yw},\ D_{yu},\ D_{yr},\ C_v,\ D_{vw},\ D_{vu}\}$, the Definitions from $\mathbf{G}_{\mathrm{b}}$ as listed in Block 5.1 and the map $\mathbf{P}_{\times}^{4}$ as defined in Lemma 2.2

Part II

ANALYSIS, ESTIMATION, AND CONTROL OF GPDES

Chapter 6

STABILITY, STABILIZABILITY, AND DETECTABILITY

6.1 Introduction

One of the important problems in control theory is to establish the stability of a system in the absence of inputs and to stabilize an unstable system or improve the stability of a stable system by finding feedback control.

Although there are different definitions of stability for a system, even more for infinitesimal systems due to different available choices for the norm used in the Lyapunov function, we will only look at the stability of GPDEs when the associated Lyapunov functions that prove stability are defined using the L_2-norm. Consequently, the stability of the state is also expressed in terms of L_2-norm, in the sense if we say $\lim_{t\to t_0}\mathbf{x}(t)=\mathbf{x}_0$, we imply $\lim_{t\to t_0}\|\mathbf{x}(t)-\mathbf{x}_0\|_{L_2}=0$.

In this chapter, we will introduce slightly different notions of stability for PIEs, namely asymptotic and exponential, that use a different norm than the ones presented in Definitions 5.4 to 5.6. The stability notions will be defined based on the rate of convergence of $\mathcal{T}\underline{\mathbf{x}}$ as opposed to $\underline{\mathbf{x}}$ in the previous definitions. As a result, we have a weaker stability result for the corresponding GPDE state $\mathbf{x}$ in terms of $\mathbb{R}L_2$-norm instead of the Sobolev norm. However, as will be seen later, this weaker notion is needed to maintain symmetry between a PIE and its dual representation – a representation needed for stabilizability and controller synthesis.

Using the new stability definitions, we will present associated sufficient conditions to prove the stability of a GPDE in the PIE form. Furthermore, we will look at a method to design a feedback input to stabilize unstable systems or enhance stability, e.g., use feedback to convert an asymptotically stable GPDE to an exponentially stable GPDE. For the design of such feedback inputs, we will also need a concept of dual stability for reasons explained in the following sections by using an ODE system for demonstration.

6.2 Stability and Dual Stability

Lyapunov's direct method allows us to verify/prove the properties of a dynamical system without requiring us to find a solution for the equations describing the system. The typical steps involved in Lyapunov methods are listed below:

1. Define a parameterized Lyapunov function (or a storage function) that is *strictly* positive (or positive definite for a storage function).

2. Find the time derivative of the said parametrized function along the solutions of the dynamical system.

3. Search for the parameters of this function such that the time derivative computed in Step 2 is negative (or satisfies some input-to-output bounds).

For example, to find a stabilizing state-feedback controller for an ODE given by

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A and B are matrices, we can parameterize the control signal as $u(t) = Kx(t)$. By defining a Lyapunov function $V(x) = x^T Px$ where P is a positive definite matrix, we find that the time derivative of V along the solutions of the ODE is given by $\dot{V}(t) = x(t)^T((A + BK)^T P + P(A + BK))x(t)$. Thus, if we solve an optimization problem searching for variables K and P, subject to the constraints $P > 0$ and $(A+BK)^T P+P(A+BK) \le 0$, then we can find the controller K that stabilizes the

ODE and the associated Lyapunov function V that proves the stability of the said controller.

However, as we can note, this optimization problem is bilinear and hence non-convex. Thus, although the Lyapunov method gives us solvable conditions to find a provably stable controller, K, the problem is computationally intractable. To overcome this practical limitation, we use a dual ODE representation to convexify this optimization problem as described below.

The concept of dual representation arises from the fact that A and A^T have the same eigenvalues. Therefore, $\dot{x} = Ax$ (primal ODE) is stable if and only if $\dot{y} = A^T y$ (dual ODE) is stable. Using Lyapunov's method for the primal ODE, the stability can be verified by solving the LMI optimization problem, $P > 0$ such that $A^T P + PA \leq 0$ — referred to as the 'primal stability test'. Applying the primal stability to the dual ODE, we get the LMI optimization problem $P > 0$ such that $AP + PA^T \leq 0$ — referred to as the 'dual stability test'. Since the stability of the two ODEs is equivalent, the two tests are also equivalent. Therefore, if a primal ODE $\dot{x} = Ax + Bu$ is stabilized by a state-feedback control $u = Kx$ leading to the closed-loop ODE $\dot{x} = (A + BK)x$, then the dual ODE $\dot{y} = (A + BK)^T y$ is also stable. Then, we can find the controller by solving the dual stability test: find $P > 0$ such that $(A + BK)P + P(A + BK)^T \leq 0$. The key difference, however, is the bilinearity can now be eliminated by introducing a new variable $Z = KP$ which leads to the LMI constraint $AP + BZ + (AP + BZ)^T \leq 0$.

Likewise, while finding optimal control for any ODE (*primal ODE*) with inputs and output as shown below,

$$\dot{x} = Ax + Bu, \quad z = Cx + Du,$$

one can show that there exists a *dual ODE* given by

$$\dot{x} = A^T x + C^T u, \quad z = B^T x + D^T u,$$

such that the two ODEs have the same internal stability and I/O properties. Any bilinearity that appears in the optimization problem while searching for a controller for the primal ODE using Lyapunov methods can be convexified by considering the dual ODE and the dual optimization problem.

In particular, the equivalence in the stability and I/O properties of a primal ODE and its dual ODE are crucial results that are used to reformulate optimal controller synthesis for ODEs as an LMI problem. Since we wish to reformulate optimal controller synthesis for PIEs as a convex optimization problem, we need duality results for a PIE and its dual. Therefore, before discussing the use of Lyapunov methods, we first introduce the duality concept for PIEs and then look at the use of Lyapunov methods for the stability and stabilizability of PIEs.

For any PIE system of the form in Equation (4.1), we may associate a dual PIE system, also of the same form as shown below.

Definition 6.1. *(Dual PIE) Given a PIE system of the form*

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \end{bmatrix}, \quad \underline{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}, \tag{6.1}$$

defined by PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, we define the 'dual PIE system' as

$$\begin{bmatrix} \mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & \mathcal{D}^* \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}(t) \\ \bar{w}(t) \end{bmatrix}, \quad \bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}, \tag{6.2}$$

where * represents the adjoint of an operator with respect to the $\mathbb{R} \times L_2$ inner product.

Recall that, given the polynomial parameters that define the operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, the polynomials that parameterize the dual PIE operators are easily obtained from the formula

$$\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1(s) \\ \hline Q_2(s) & \{R_0(s), R_1(s,\theta), R_2(s,\theta)\} \end{array}\right]^*$$
$$= \mathit{\Pi}\left[\begin{array}{c|c} P^T & Q_2(s)^T \\ \hline Q_1(s)^T & \{R_0(s)^T, R_2(\theta,s)^T, R_1(\theta,s)^T\} \end{array}\right].$$

In the following subsections, we will establish that a PIE system and its dual are equivalent in terms of internal stability; In particular, we consider the asymptotic stability, which is defined as follows.

Definition 6.2 (Asymptotic Stability)**.** *We say that the PIE defined by* $\{\mathcal{T}, \mathcal{A}\} \subset \mathbf{\Pi}_4$ *is **Asymptotically Stable** if for any* $\mathbf{x}_0 \in \mathbb{R}L_2$*, if* $\mathcal{T}\underline{\mathbf{x}}(0) = \mathcal{T}\mathbf{x}_0$ *and* $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t)$*, then*

$$\lim_{t\to\infty} \|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} = 0.$$

For convenience, we drop the $\|\cdot\|$ in the following theorem, however, it is implied that the metric used in the solution space is induced by the L_2-norm and limit values must be evaluated with respect to this metric.

Theorem 6.1. *Suppose* $\mathcal{T}, \mathcal{A} \in \mathcal{L}(\mathbb{R}L_2^{m,n})$ *are PI operators. Then, the following statements are equivalent.*

a) $\lim\limits_{t\to\infty} \mathcal{T}\underline{\mathbf{x}}(t) = 0$ *for any* $\mathbf{x}$ *that satisfies* $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t)$ *with initial condition* $\underline{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$*.*

b) $\lim\limits_{t\to\infty} \mathcal{T}^*\bar{\mathbf{x}}(t) = 0$ *for any* $\bar{\mathbf{x}}$ *that satisfies* $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t)$ *with initial condition* $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$*.*

Proof. To show sufficiency (i.e. a) implies b)), suppose $\underline{\mathbf{x}}$ satisfies $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t)$ with initial condition $\underline{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$ and $\lim_{t\to\infty} \mathcal{T}\underline{\mathbf{x}}(t) = 0$. Let $\bar{\mathbf{x}}$ satisfy $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t)$

with initial condition $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$. Then for any finite $t > 0$, using integration-by-parts, we get

$$\begin{aligned}\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\underline{\mathbf{x}}}(s)\rangle_{\mathbb{R}L_2} ds = \langle \bar{\mathbf{x}}(0), \mathcal{T}\underline{\mathbf{x}}(t)\rangle_{\mathbb{R}L_2} - \langle \bar{\mathbf{x}}(t), \mathcal{T}\underline{\mathbf{x}}(0)\rangle_{\mathbb{R}L_2} \\ - \int_0^t \langle \partial_s \bar{\mathbf{x}}(t-s), \mathcal{T}\underline{\mathbf{x}}(s)\rangle_{\mathbb{R}L_2} ds.\end{aligned} \tag{6.3}$$

Then, we use a change of variable ($\theta = t - s$) on the last term in Equation (6.3) to show

$$\begin{aligned}\int_0^t \langle \partial_s \bar{\mathbf{x}}(t-s), \mathcal{T}\underline{\mathbf{x}}(s)\rangle_{\mathbb{R}L_2} ds =& \int_0^t \left\langle \dot{\bar{\mathbf{x}}}(\theta), \mathcal{T}\underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta \\ =& \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta.\end{aligned}$$

Furthermore, using the same variable change on the left-hand side of Equation (6.3), we get

$$\begin{aligned}\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\underline{\mathbf{x}}}(s)\rangle_{\mathbb{R}L_2} ds &= \int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{A}\underline{\mathbf{x}}(s)\rangle_{\mathbb{R}L_2} ds \\ &= \int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \underline{\mathbf{x}}(t-\theta)\rangle_{\mathbb{R}L_2} d\theta.\end{aligned}$$

Substituting these two expressions into Equation (6.3), we have

$$\begin{aligned}\int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \underline{\mathbf{x}}(t-\theta)\rangle_{\mathbb{R}L_2} d\theta = \langle \bar{\mathbf{x}}(0), \mathcal{T}\underline{\mathbf{x}}(t)\rangle_{\mathbb{R}L_2} - \langle \bar{\mathbf{x}}(t), \mathcal{T}\underline{\mathbf{x}}(0)\rangle_{\mathbb{R}L_2} \\ + \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta.\end{aligned}$$

However, $\mathcal{A}^*\bar{\mathbf{x}}(\theta) = \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta)$ for all $\theta \in [0, t]$, and hence

$$\langle \bar{\mathbf{x}}(0), \mathcal{T}\underline{\mathbf{x}}(t)\rangle_{\mathbb{R}L_2} = \langle \bar{\mathbf{x}}(t), \mathcal{T}\underline{\mathbf{x}}(0)\rangle_{\mathbb{R}L_2}, \quad \text{for all } t > 0. \tag{6.4}$$

Since $\lim_{t\to\infty} \mathcal{T}\underline{\mathbf{x}}(t) = 0$ for any $\underline{\mathbf{x}}(0) \in \mathbb{R}L_2$, we have

$$\lim_{t\to\infty} \langle \bar{\mathbf{x}}(0), \mathcal{T}\underline{\mathbf{x}}(t)\rangle_{\mathbb{R}L_2} = \lim_{t\to\infty} \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \underline{\mathbf{x}}(0)\rangle_{\mathbb{R}L_2} = 0,$$

for any $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2$. Therefore, for any $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2$, we have $\lim_{t\to\infty} \mathcal{T}^*\bar{\mathbf{x}}(t) = 0$. Thus we have sufficiency. Since $\mathcal{T}^{**} = \mathcal{T}$, sufficiency implies necessity. □

From a computational perspective, testing asymptotic stability is difficult and controls that provide asymptotic stability are typically impractical. Hence, in the following, we consider the exponential stability and show that the primal and the dual PIE have equivalent exponential stability properties as well.

Definition 6.3 (Exponential Stability)**.** *We say that the PIE defined by* $\{\mathcal{T}, \mathcal{A}\} \subset \mathbf{\Pi}_4$ *is* ***Exponentially Stable*** *with decay rate* $\alpha > 0$ *if there exists some* $M > 0$ *such that for any* $\mathbf{x}_0 \in \mathbb{R}L_2$*, if* $\underline{\mathbf{x}}(0) = \mathbf{x}_0$ *and* $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t)$*, then*

$$\|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \leq M \|\mathbf{x}_0\|_{\mathbb{R}L_2} e^{-\alpha t} \quad \textit{for all } t \geq 0.$$

Note that this definition implies that the GPDE state ($\mathcal{T}\underline{\mathbf{x}}$) decays exponentially in the $\mathbb{R}L_2$-norm, but does not necessarily guarantee exponential stability of the PIE state ($\mathbf{x}$) unless $\mathcal{T}$ has a bounded inverse.

Corollary 6.2. *Suppose* $\mathcal{T}, \mathcal{A} \in \mathcal{L}(\mathbb{R}L_2^{m,n})$ *are PI operators. Then the following statements are equivalent:*

a) There exists $M > 0$ *and* $\alpha > 0$ *such that*

$$\|\mathcal{T}\underline{\mathbf{x}}(t)\| \leq M \|\underline{\mathbf{x}}(0)\| e^{-\alpha t}$$

for any $\mathbf{x}$ *that satisfies* $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t)$ *with initial condition* $\underline{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$*.*

b) There exists $M > 0$ *and* $\alpha > 0$ *such that*

$$\|\mathcal{T}^*\bar{\mathbf{x}}(t)\| \leq M \|\bar{\mathbf{x}}(0)\| e^{-\alpha t}$$

for any $\bar{\mathbf{x}}$ *that satisfies* $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t)$ *with initial condition* $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$*.*

Proof. To show sufficiency (i.e. a) implies b)), suppose $\bar{\mathbf{x}}$ satisfies $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t)$ for some initial condition $\bar{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}$. Then for any $t > 0$, let $\underline{\mathbf{x}}$ satisfy $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) =$

$\mathcal{A}\underline{\mathbf{x}}(t)$ with initial condition $\underline{\mathbf{x}}(0) = \mathcal{T}^*\bar{\mathbf{x}}(t)$. Then, we have from Equation (6.4) in Theorem 6.1,

$$\langle \bar{\mathbf{x}}(0), \mathcal{T}\underline{\mathbf{x}}(t)\rangle = \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \underline{\mathbf{x}}(0)\rangle = \|\mathcal{T}^*\bar{\mathbf{x}}(t)\|^2 .$$

Then, from Cauchy-Schwarz inequality,

$$\begin{aligned}\|\mathcal{T}^*\bar{\mathbf{x}}(t)\|^2 &= \langle \bar{\mathbf{x}}(0), \mathcal{T}\underline{\mathbf{x}}(t)\rangle \le \|\mathcal{T}\underline{\mathbf{x}}(t)\| \, \|\bar{\mathbf{x}}(0)\| \\ &\le M \|\underline{\mathbf{x}}(0)\| e^{-\alpha t} \|\bar{\mathbf{x}}(0)\| = M \|\mathcal{T}^*\bar{\mathbf{x}}(t)\| e^{-\alpha t} \|\bar{\mathbf{x}}(0)\| ,\end{aligned}$$

which implies

$$\|\mathcal{T}^*\bar{\mathbf{x}}(t)\| \le M \|\bar{\mathbf{x}}(0)\| e^{-\alpha t}.$$

Thus we have sufficiency. Since $\mathcal{T}^{**} = \mathcal{T}$, sufficiency implies necessity. □

Now, we can use the Lyapunov approach introduced earlier to formulate sufficient conditions for the stability of a PIE, as shown in the following subsections.

6.2.1 LPI for Testing Stability

In the following theorem, we propose primal and dual LPI tests for exponential stability and use Corollary 6.2 to show that the feasibility of either implies exponential stability of both the primal and dual systems.

Theorem 6.3. *Suppose that* ***either*** *of the two statements hold for some $\alpha > 0$ and bounded linear operator $\mathcal{P} = \mathcal{P}^* \succeq \eta I$ with $\eta > 0$.*

a) $\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \preceq -2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T}$

b) $\mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \preceq -2\alpha\mathcal{T}\mathcal{P}\mathcal{T}^*$

Then the PIEs defined by $\{\mathcal{T}, \mathcal{A}\} \subset \mathbf{\Pi}_4$ and $\{\mathcal{T}^*, \mathcal{A}^*\} \subset \mathbf{\Pi}_4$ are ***Exponentially Stable*** *with decay rate α.*

Proof. Suppose a) holds. Define $V(\underline{\mathbf{x}}) = \langle \mathcal{T}\underline{\mathbf{x}}, \mathcal{P}\mathcal{T}\underline{\mathbf{x}}\rangle_{\mathbb{R}L_2}$. Since $\mathcal{P}$ is bounded,

$$\eta \left\|\mathcal{T}\underline{\mathbf{x}}\right\|^2 \leq V(\underline{\mathbf{x}}) \leq \left\|\mathcal{P}\right\|_{\mathcal{L}(\mathbb{R}L_2)} \left\|\mathcal{T}\underline{\mathbf{x}}\right\|^2.$$

Suppose $\underline{\mathbf{x}}(t)$ satisfies $\underline{\mathbf{x}}(0) = \mathbf{x}_0$ and $\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t)$. Differentiating $V(\underline{\mathbf{x}}(t))$ with respect to time, t, we obtain

$$\begin{aligned}\dot{V}(\underline{\mathbf{x}}(t)) &= \langle \mathcal{T}\underline{\mathbf{x}}(t), \mathcal{P}\mathcal{A}\underline{\mathbf{x}}(t)\rangle + \langle \mathcal{A}\underline{\mathbf{x}}(t), \mathcal{P}\mathcal{T}\underline{\mathbf{x}}(t)\rangle \\ &= \langle \underline{\mathbf{x}}(t), (\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T})\underline{\mathbf{x}}(t)\rangle \leq -2\alpha V(\underline{\mathbf{x}}(t)).\end{aligned}$$

Therefore, we conclude $\dot{V}(\underline{\mathbf{x}}(t)) \leq -2\alpha V(\underline{\mathbf{x}}(t))$ for all t and, from Gronwall-Bellman inequality, $V(\underline{\mathbf{x}}(t)) \leq V(\underline{\mathbf{x}}(0))e^{-2\alpha t}$. Let $\beta = \left\|\mathcal{T}\right\|_{\mathcal{L}(\mathbb{R}L_2)}$ and $\zeta = \left\|\mathcal{P}\right\|_{\mathcal{L}(\mathbb{R}L_2)}$. Then

$$\begin{aligned}\left\|\mathcal{T}\underline{\mathbf{x}}(t)\right\|^2 &\leq \frac{1}{\eta}V(\underline{\mathbf{x}}(t)) \leq \frac{1}{\eta}V(\underline{\mathbf{x}}(0))e^{-2\alpha t} \\ &\leq \frac{1}{\eta}\zeta \left\|\mathcal{T}\underline{\mathbf{x}}(0)\right\|^2 e^{-2\alpha t} \leq \frac{\zeta\beta^2}{\eta} \left\|\underline{\mathbf{x}}(0)\right\|^2 e^{-2\alpha t}.\end{aligned}$$

By taking square root on both sides,

$$\left\|\mathcal{T}\underline{\mathbf{x}}(t)\right\| \leq M \left\|\underline{\mathbf{x}}(0)\right\| e^{-\alpha t}$$

where $M = \sqrt{\frac{\zeta}{\eta}}\beta$. This implies the PIE defined by $\{\mathcal{T}, \mathcal{A}\} \subset \mathbf{\Pi}_4$ is **Exponentially Stable** with decay rate α. Then, from Corollary 6.2, the PIE defined by $\{\mathcal{T}^*, \mathcal{A}^*\} \subset \mathbf{\Pi}_4$ is **Exponentially Stable** with decay rate α.

The proof similarly establishes exponential stability for b) with $\mathcal{T} \mapsto \mathcal{T}^*$ and $\mathcal{A} \mapsto \mathcal{A}^*$. $\square$

Both a) and b) in Theorem 6.3 imply exponential stability of both primal and dual using the definition of exponential stability in Definition 6.3, $\left\|\mathcal{T}\underline{\mathbf{x}}(t)\right\|_{\mathbb{R}L_2} \leq M \left\|\mathbf{x}_0\right\|_{\mathbb{R}L_2} e^{-\alpha t}$ where the upper bound is defined using the L_2-norm of the PIE initial condition (which is equivalent to the Sobolev norm of the PDE initial condition). This slightly stronger norm is needed to preserve the symmetry of the primal and dual.

However, we also note that from the proof of Theorem 6.3, a) implies exponential stability of the primal and b) implies exponential stability of the dual using an upper bound of the form $\|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \leq M\|\mathcal{T}\mathbf{x}_0\|_{\mathbb{R}L_2} e^{-\alpha t}$. Practically, however, there is no difference between these definitions of exponential stability since we always assume that $\mathbf{x}_0 \in \mathbb{R}L_2$.

It is worth noting that, although we do not explicitly provide a condition to test asymptotic stability, we can use the conditions of Theorem 6.3 with $\alpha = 0$ to establish asymptotic stability. Asymptotic stability is considered inferior in practice since it does not provide any practical metric to quantify the time a system takes to reach equilibrium. Therefore, moving forward, we will exclusively consider the case of exponential stability. On the note of practicality, in Theorem 6.3, we used a Lyapunov function of the form $V(\underline{\mathbf{x}}) = \langle \underline{\mathbf{x}}, \mathcal{T}^*\mathcal{P}\mathcal{T}\underline{\mathbf{x}}\rangle$ where $\mathcal{P}$ is a bounded linear operator. However, if one carefully inspects the proof, one does not necessarily need $\mathcal{P}$ to be bounded in the Lyapunov function V to be positive as long as $\mathcal{P}\mathcal{T}$ is bounded; Boundedness of $\mathcal{P}$ is used only in the last step

$$\frac{1}{\eta}V(\underline{\mathbf{x}}(0))e^{-2\alpha t} \leq \frac{1}{\eta}\zeta\,\|\mathcal{T}\underline{\mathbf{x}}(0)\|^2 e^{-2\alpha t} \leq \frac{\zeta\beta^2}{\eta}\|\underline{\mathbf{x}}(0)\|^2 e^{-2\alpha t}.$$

The conclusion will hold when $\mathcal{P}\mathcal{T}$ is bounded even if $\mathcal{P}$ is not — i.e., $\frac{1}{\eta}V(\underline{\mathbf{x}}(0))e^{-2\alpha t} \leq M\|\underline{\mathbf{x}}(0)\|^2 e^{-2\alpha t}$ when $\mathcal{P}\mathcal{T}$ is bounded. Therefore, one can relax this boundedness constraint on $\mathcal{P}$, while computationally searching for $\mathcal{P}$, to obtain a less conservative test for stability as shown in the following results.

Corollary 6.4. *Suppose there exist $\alpha > 0, \eta > 0$ and PI operator $\mathcal{P} = \mathcal{P}^* \succeq \eta I$, such that either (a) or (b) is satisfied:*

$$(a)\ \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \preceq -2\alpha\mathcal{T}^*\mathcal{P}\mathcal{T}$$

$$(b)\ \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \preceq -2\alpha\mathcal{T}\mathcal{P}\mathcal{T}^*$$

Then:

1. *There exist PI operators $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that*

$$(c)\ \mathcal{T}^*\mathcal{Q} = \mathcal{Q}^*\mathcal{T} = \mathcal{R}, \quad \mathcal{Q}^*\mathcal{A} + \mathcal{A}^*\mathcal{Q} + \eta(\mathcal{T}^*\mathcal{A} + \mathcal{A}^*\mathcal{T}) \preceq -2\alpha\mathcal{R}$$

if (a) is satisfied. Otherwise, if (b) is satisfied, then there exist PI operators $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that

$$(d)\ \mathcal{T}\mathcal{Q} = \mathcal{Q}^*\mathcal{T}^* = \mathcal{R}, \quad \mathcal{Q}^*\mathcal{A}^* + \mathcal{A}\mathcal{Q} + \eta(\mathcal{T}\mathcal{A}^* + \mathcal{A}\mathcal{T}^*) \preceq -2\alpha\mathcal{R}.$$

2. *For any $\mathbf{x}_0 \in L_2$, the PIE systems (A) and (B)*

$$(A)\ \mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t),\ \underline{\mathbf{x}}(0) = \mathbf{x}_0,$$

$$(B)\ \mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t),\ \bar{\mathbf{x}}(0) = \mathbf{x}_0,$$

are exponentially stable with a decay rate α.

Proof. The proof is similar to the proof of Theorem 6.3. If conditions of (a) are satisfied, then we see that for $\mathcal{Q} = \mathcal{P}\mathcal{T} + \eta\mathcal{T}$, the conditions of 1c) are automatically satisfied. Likewise, one can verify that (b) implies conditions 1d) are satisfied with $\mathcal{Q} = \mathcal{P}\mathcal{T}^* + \eta\mathcal{T}^*$. Furthermore, from Theorem 6.3, we automatically get the 2A) and 2B). $\square$

6.3 Stabilizability and Detectability

So far, we have introduced the concepts of stability and a proof of stability by finding Lyapunov functions. However, in practice, one would be more interested in stabilizing an unstable system via a feedback input or improving the stability by increasing the speed at which the system moves to the equilibrium point. However, such a feedback input may not always exist, e.g., a car cannot be made to fly.

Therefore, we need a test to determine if a system can be stabilized using a feedback input; If such a feedback control exists, then the system is said to be *stabilizable*.

Another concept somewhat similar to stabilizability is *detectability*. We say a system is detectable if all unstable states can be inferred by using output measurements. In the case of ODEs, the concepts of stabilizability and detectability are duals – i.e., given a state-space ODE (A, B, C, D), the pair $(A, B, -)$ is said to be stabilizable if and only if the pair $(A^T, -, B^T)$ is also detectable.

This dual relationship between stabilizability and detectability can be extended to PIEs defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, D\}$ of the form

$$\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}} + \mathcal{B}w \quad z(t) = \mathcal{C}\underline{\mathbf{x}}(t) + Dw(t).$$

In case of PIEs, we can show that $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, -\}$ is stabilizable if and only if $\{\mathcal{T}^*$, $\mathcal{A}^*$, $-$, $\mathcal{B}^*\}$ is detectable. This is a direct consequence of the choice of parametrization of the dual PIE for a given PIE system. However, to prove these claims, we need to define the stabilizability and detectability of a PIE formally.

Definition 6.4. *If there exists a PI operator $\mathcal{K}$ such that the PIE defined by $\{\mathcal{T}, \mathcal{A} + \mathcal{B}\mathcal{K}\}$ is asymptotically stable, then we say PIE system defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, D\}$ is stabilizable. If $\{\mathcal{T}, \mathcal{A} + \mathcal{B}\mathcal{K}\}$ is exponentially stable with some decay rate $\alpha > 0$, then we say the PIE is exponentially stabilizable.*

Definition 6.5. *If there exists a PI operator $\mathcal{L}$ such that the PIE defined by $\{\mathcal{T}, \mathcal{A} + \mathcal{L}\mathcal{C}\}$ is asymptotically stable, then the PIE system defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, D\}$ is detectable. If $\{\mathcal{T}, \mathcal{A} + \mathcal{L}\mathcal{C}\}$ is exponentially stable with some decay rate $\alpha > 0$, then we say the PIE is exponentially detectable.*

Using the definitions above, we can now arrive at the duality relationship between stabilizability and detectability of a PIE system.

Theorem 6.5. *Given* $=\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C}\in\mathbf{\Pi}_4$*, and a matrix* D *the following two statements are equivalent:*

1. *The PIE defined by* $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{C},D\}$ *is stabilizable.*

2. *The PIE defined by* $\{\mathcal{T}^*,\mathcal{A}^*,\mathcal{C}^*,\mathcal{B}^*,D^T\}$ *is detectable.*

Proof. The proof simply follows form the definition of stabilizability and detectability. Let the PIE defined by $\{\mathcal{T}^*,\mathcal{A}^*,\mathcal{C}^*,\mathcal{B}^*,D^T\}$ be detectable. Then, there exists a PI operator $\mathcal{L}$, such that $\{\mathcal{T}^*,\mathcal{A}^*+\mathcal{L}\mathcal{B}^*\}$ is asymptotically stable. Then, from Theorem 6.1, we have that the PIE defined by

$$\{(\mathcal{T}^*)^*,(\mathcal{A}^*+\mathcal{L}\mathcal{B}^*)\}=\{\mathcal{T},\mathcal{A}+\mathcal{B}\mathcal{L}^*)\}$$

is also asymptotically stable. Therefore, there exists $\mathcal{K}=\mathcal{L}^*$, a PI operator, such that $\{\mathcal{T},\mathcal{A}+\mathcal{B}\mathcal{K}\}$ is asymptotically stable. Likewise, one can prove the converse implication using the same approach. □

Remark 6.6. *Note that since exponentially stability of a PIE and its dual are also equivalent, the above dual relationship also holds for exponential stabilizability and exponential detectability of a PIE system.*

6.3.1 LPI for Stabilizability and Detectability

Now that we have the dual stability test for the PIEs, we can use the change of variable trick to eliminate the bilinearity that appears in the controller synthesis problem. Given a PIE of the form,

$$\mathcal{T}\dot{\underline{\mathbf{x}}}(t)+\mathcal{T}_u\dot{u}(t)=\mathcal{A}\underline{\mathbf{x}}(t)+\mathcal{B}u(t)$$

we present the LPI that is used to find a stabilizing state-feedback controller of the form $u(t)=\mathcal{K}\mathbf{x}(t)$ where $\mathcal{K}:\mathbb{R}L_2^{m,n}[a,b]\to\mathbb{R}^q$ is a 4-PI operator. By Definition 6.4, this LPI will also be a test for stabilizability of PIEs.

Since the presence of $\mathcal{T}_u$ term introduces a quadratic nonlinearity in the optimization problems, we will treat the two cases ($\mathcal{T}_u = 0$ and $\mathcal{T}_u \neq 0$) separately.

Stabilizability of PIEs with $\mathcal{T}_u = 0$

When GPDE systems that do not have inputs at the boundary are converted to PIEs, we always obtain a PIE with dynamics of the form

$$\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t) + \mathcal{B}u(t).$$

Since the $\mathcal{T}_u$ term is zero in the above case, we can directly employ the dual LPI for stability along with the change of variable trick to derive an LPI to find a stabilizing controller which gives us the following result.

Theorem 6.7. *Suppose there exist $\alpha > 0$, $\eta > 0$, and bounded linear operators $\mathcal{P} \succeq \eta I$ and $\mathcal{Z}$, such that*

$$(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})^* \leq -2\alpha\mathcal{T}\mathcal{P}\mathcal{T}^*. \tag{6.5}$$

Then, for $u(t) = \mathcal{K}\underline{\mathbf{x}}(t)$, where $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, the PIE defined by $\{\mathcal{T}, \mathcal{A} + \mathcal{B}\mathcal{K}\} \subset \mathbf{\Pi}_4$ is exponentially stable with a decay rate of α.

Proof. Let $\mathcal{Z} = \mathcal{K}\mathcal{P}$. Define a Lyapunov candidate as $V(y) = \langle \mathcal{T}^* y, \mathcal{P}\mathcal{T}^* y\rangle_{\mathbb{R}L_2}$. Then

$$\eta \left\|\mathcal{T}^* y\right\|_{\mathbb{R}L_2}^2 \leq V(y) \leq \|\mathcal{P}\| \left\|\mathcal{T}^* y\right\|_{\mathbb{R}L_2}^2.$$

The time derivative of $V(y)$ along the solutions of the PIE

$$\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t) + \mathcal{K}^*\mathcal{B}^*\bar{\mathbf{x}}(t), \qquad y(0) \in \mathbb{R}L_2^{m,n}[a,b]$$

is given by

$$\begin{aligned}\dot{V}(\bar{\mathbf{x}}(t)) &= \left\langle \mathcal{T}^*\bar{\mathbf{x}}(t), \mathcal{P}\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t)\right\rangle + \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(t), \mathcal{P}\mathcal{T}^*\bar{\mathbf{x}}(t)\right\rangle \\ &= \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \mathcal{P}\mathcal{A}^*\bar{\mathbf{x}}(t)\rangle + \langle \mathcal{A}^*\bar{\mathbf{x}}(t), \mathcal{P}\mathcal{T}^*\bar{\mathbf{x}}(t)\rangle \\ &\quad + \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \mathcal{P}\mathcal{K}^*\mathcal{B}^*\bar{\mathbf{x}}(t)\rangle + \langle \mathcal{K}^*\mathcal{B}^*\bar{\mathbf{x}}(t), \mathcal{P}\mathcal{T}^*\bar{\mathbf{x}}(t)\rangle \\ &= \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathcal{P}\mathcal{A}^*\bar{\mathbf{x}}(t)\rangle + \langle \bar{\mathbf{x}}(t), \mathcal{A}\mathcal{P}\mathcal{T}^*\bar{\mathbf{x}}(t)\rangle \\ &\quad + \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathcal{Z}^*\mathcal{B}^*\bar{\mathbf{x}}(t)\rangle + \langle \bar{\mathbf{x}}(t), \mathcal{B}\mathcal{Z}\mathcal{T}^*\bar{\mathbf{x}}(t)\rangle \\ &\le -\epsilon \left\|\mathcal{T}^*\bar{\mathbf{x}}(t)\right\|_{\mathbb{R}L_2} \le -\frac{\epsilon}{\beta}V(\bar{\mathbf{x}}(t)).\end{aligned}$$

Then, by using Gronwall-Bellman Inequality, there exists constants M and k such that

$$V(\bar{\mathbf{x}}(t)) \le V(\bar{\mathbf{x}}(0))Me^{(-kt)}.$$

As $t \to \infty$, $V(\bar{\mathbf{x}}(t)) \to 0$ which implies $\|\mathcal{T}^*\bar{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \to 0$. Then, from Theorem 6.1, $\|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \to 0$ where $\underline{\mathbf{x}}$ satisfies the equation

$$\mathcal{T}\dot{\underline{\mathbf{x}}}(t) = \mathcal{A}\underline{\mathbf{x}}(t) + \mathcal{B}\mathcal{K}\underline{\mathbf{x}}(t) = \mathcal{A}\underline{\mathbf{x}}(t) + \mathcal{B}u(t)$$

for any $\underline{\mathbf{x}}(0) \in \mathbb{R}L_2^{m,n}[a,b]$. □

Similar to the case of the stability test, we can relax the boundedness requirement of $\mathcal{P}$ in the above theorem to obtain a less conservative LPI condition for the stabilizability of a PIE, as shown below.

Corollary 6.8. *Suppose there exist $\alpha > 0$, $\eta > 0$, and bounded linear operators $\mathcal{P} \succeq \eta I$ and $\mathcal{Z}$, such that*

$$(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})^* \le -2\alpha\mathcal{T}\mathcal{P}\mathcal{T}^*. \tag{6.6}$$

Then, there exist bounded linear operators $\mathcal{Q}$, $\mathcal{Z}_Q$, and $\mathcal{R} \succeq 0$ such that

$$\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z}_Q + (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z}_Q)^* + \eta(\mathcal{T}\mathcal{A}^* + \mathcal{A}\mathcal{T}^*) \le -2\alpha\mathcal{R}. \tag{6.7}$$

Then, for $u(t) = \mathcal{K}\underline{\mathbf{x}}(t)$, where $\mathcal{K} = \mathcal{Z}_Q\mathcal{Q}^{-1}$, the PIE defined by $\{\mathcal{T}, \mathcal{A}+\mathcal{B}\mathcal{K}\} \subset \mathbf{\Pi}_4$ is exponentially stable with a decay rate of α.

Proof. The proof for the above corollary is similar to the proof of Corollary 6.4. We can show that $\mathcal{Q} = \mathcal{P}\mathcal{T}^* + \eta\mathcal{T}^*$ and $\mathcal{Z}_Q = \mathcal{Z}\mathcal{T}^*$ satisfy Equation (6.7) if Equation (6.5) is satisfied. □

Stabilizing control of PIEs with $\mathcal{T}_u \neq 0$

In the case of GPDE systems with inputs at the boundary, i.e., $B_v D_{vu} \neq 0$, the PIE representation of the GPDE is of the form

$$\mathcal{T}\dot{\underline{\mathbf{x}}}(t) + \mathcal{T}_u\dot{u}(t) = \mathcal{A}\underline{\mathbf{x}}(t) + \mathcal{B}u(t).$$

For such PIEs (with $\mathcal{T}_u \neq 0$), the following Corollary can be used to find a stabilizing state-feedback controller of the form $u(t) = \mathcal{K}\mathbf{x}(t)$ where $\mathcal{K} : \mathbb{R}L_2^{m,n}[a,b] \to \mathbb{R}^q$ is a 4-PI operator, however, note that the constraints are non-convex and quadratic in the decision variable $\mathcal{K}$.

Corollary 6.9. *Suppose there exist bounded linear operators $\mathcal{P} : \mathbb{R}L_2^{m,n}[a,b] \to \mathbb{R}L_2^{m,n}[a,b]$, such that $\mathcal{P}$ is self-adjoint, coercive and*

$$\begin{aligned}&(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{K}\mathcal{P})(\mathcal{T} + \mathcal{T}_u\mathcal{K})^* + (\mathcal{T} + \mathcal{T}_u\mathcal{K})(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{K}\mathcal{P})^* \\ &\leq -\epsilon(\mathcal{T} + \mathcal{T}_u\mathcal{K})(\mathcal{T} + \mathcal{T}_u\mathcal{K})^*. \qquad (6.8)\end{aligned}$$

Then, for $u(t) = \mathcal{K}\underline{\mathbf{x}}(t)$, any $\underline{\mathbf{x}} \in \mathbb{R}L_2^{m,n}[a,b]$ that satisfies the system

$$\mathcal{T}\dot{\underline{\mathbf{x}}}(t) + \mathcal{T}_u\mathcal{K}\dot{u}(t) = \mathcal{A}\underline{\mathbf{x}}(t) + \mathcal{B}u(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}L_2^{m,n}[a,b]$$

also satisfies $\lim_{t\to\infty} \|\mathcal{T}\underline{\mathbf{x}}(t)\| = 0$.

Proof. The proof is similar to the proof for Theorem 6.7. Replace $\mathcal{T}$ by $\mathcal{T} + \mathcal{T}_u\mathcal{K}$ in all the steps. □

To eliminate the quadratic terms in the inequality Equation (6.8), we use the result, Young's relation for matrices Zemouche *et al.* (2016), and extend it to PI operators.

Lemma 6.10. *For any* $\mathcal{Z}_1 : \mathbb{R}L_2^{m,n} \to \mathbb{R}L_2^{p,q}$, $\mathcal{Z}_2 : \mathbb{R}L_2^{m,n} \to \mathbb{R}L_2^{l,k}$ *and* $\mathcal{P} : \mathbb{R}L_2^{m,n} \to \mathbb{R}L_2^{m,n}$, *such that* $\mathcal{P} \succ 0$,

$$\langle x, \mathcal{Z}_1 \mathcal{P} \mathcal{Z}_2^* y \rangle + \langle y, \mathcal{Z}_2 \mathcal{P} \mathcal{Z}_1^* x \rangle \preccurlyeq \langle x, \mathcal{Z}_1 \mathcal{P} \mathcal{Z}_1^* x \rangle + \langle y, \mathcal{Z}_2 \mathcal{P} \mathcal{Z}_2^* y \rangle$$

where $x \in \mathbb{R}L_2^{p,q}$ *and* $y \in \mathbb{R}L_2^{l,k}$.

Proof. Suppose $\mathcal{P}$ is coercive. Then, the following sequence of inequalities holds.

$$\begin{aligned} 0 &\preccurlyeq \langle (\mathcal{Z}_1^* x - \mathcal{Z}_2^* y), \mathcal{P}(\mathcal{Z}_1^* x - \mathcal{Z}_2^* y) \rangle \\ &= \langle x, \mathcal{Z}_1 \mathcal{P} \mathcal{Z}_1^* x \rangle - \langle y, \mathcal{Z}_2 \mathcal{P} \mathcal{Z}_1^* x \rangle - \langle x, \mathcal{Z}_1 \mathcal{P} \mathcal{Z}_2^* y \rangle + \langle y, \mathcal{Z}_2 \mathcal{P} \mathcal{Z}_2^* y \rangle \end{aligned}$$

Therefore, by rearranging the terms in the inequality,

$$\langle x, \mathcal{Z}_1 \mathcal{P} \mathcal{Z}_1^* x \rangle + \langle y, \mathcal{Z}_2 \mathcal{P} \mathcal{Z}_2^* y \rangle \succcurlyeq \langle y, \mathcal{Z}_2 \mathcal{P} \mathcal{Z}_1^* x \rangle + \langle x, \mathcal{Z}_1 \mathcal{P} \mathcal{Z}_2^* y \rangle .$$

□

Similar to Schur's complement for matrices, we can also find the Schur's complement for the 4-PI operators.

Lemma 6.11. *Suppose* $\mathcal{E}$, $\mathcal{F}$, *and* $\mathcal{G}$ *are 4-PI operators where* $\mathcal{G}$ *is positive definite. Then*

$$\begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{G} \end{bmatrix} \succ 0 \tag{6.9}$$

if and only if $\mathcal{E} - \mathcal{F}\mathcal{G}^{-1}\mathcal{F}^* \succ 0$.

Proof. Suppose $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are as stated above. Then $\mathcal{G}^{-1}$ exists and is positive definite (refer Theorem 2 of Peet (2020a)). Then the following inequalities are equivalent.

$$\begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{G} \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} I & -\mathcal{F}\mathcal{G}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{G} \end{bmatrix} \begin{bmatrix} I & -\mathcal{F}\mathcal{G}^{-1} \\ 0 & I \end{bmatrix}^* \succ 0,$$

$$\begin{bmatrix} \mathcal{E} - \mathcal{F}\mathcal{G}^{-1}\mathcal{F}^* & 0 \\ 0 & \mathcal{G} \end{bmatrix} \succ 0,$$

$$\mathcal{E} - \mathcal{F}\mathcal{G}^{-1}\mathcal{F}^* \succ 0.$$

□

Using the above Lemmas, we can derive the LPI to find a stabilizing controller for PIEs with $\mathcal{T}_u \neq 0$ as follows.

Theorem 6.12. *Suppose there exists a $\mathcal{P} \succ 0$ and $\mathcal{Z}$, such that*

$$\begin{bmatrix} \mathcal{P}_H & \mathcal{T}_u\mathcal{Z} & \mathcal{B}\mathcal{Z} \\ \mathcal{Z}^*\mathcal{T}_u^* & -\mathcal{P} & 0 \\ \mathcal{Z}^*\mathcal{B}^* & 0 & -\mathcal{P} \end{bmatrix} \preccurlyeq 0 \tag{6.10}$$

where

$$\mathcal{P}_H = \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* + \begin{bmatrix} \mathcal{T}_u\mathcal{Z} & \mathcal{B}\mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{A}^* \\ \mathcal{T}^* \end{bmatrix} + \begin{bmatrix} \mathcal{A} & \mathcal{T} \end{bmatrix} \begin{bmatrix} \mathcal{Z}^*\mathcal{T}_u^* \\ \mathcal{Z}^*\mathcal{B}^* \end{bmatrix}.$$

Then the system,

$$(\mathcal{T} + \mathcal{T}_u\mathcal{K})^*\dot{\mathbf{x}}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K})^*\mathbf{x}(t)$$

is Lyapunov stable for $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$.

Proof. Define a quadratic Lyapunov functional,

$$V(\mathbf{x}(t)) = \langle (\mathcal{T}+\mathcal{T}_u\mathcal{K})^*\mathbf{x}(t), \mathcal{P}(\mathcal{T}+\mathcal{T}_u\mathcal{K})^*\mathbf{x}(t)\rangle_{\mathbb{R}L_2}$$

where $\mathcal{P} \succ 0$. Since $\mathcal{P} \succ 0$, $V(\mathbf{x}(t)) > 0$ for all $t > 0$ such that $\mathbf{x}(t) \neq 0$. Suppose there exists a $\mathcal{Z}$ such that $\mathcal{P}$ and $\mathcal{Z}$ satisfy the LPI Equation (6.10). If $\mathbf{x}$ satisfies the equation

$$(\mathcal{T}+\mathcal{T}_u\mathcal{K})^*\dot{\mathbf{x}}(t) = (\mathcal{A}+\mathcal{B}\mathcal{K})^*\mathbf{x}(t),$$

then the time derivative of $V(t)$ is

$$\begin{aligned}
\dot{V}(t) &= \langle \mathbf{x}(t), (\mathcal{T}+\mathcal{T}_u\mathcal{K})\mathcal{P}(\mathcal{A}^*+\mathcal{K}^*\mathcal{B}^*)\mathbf{x}(t)\rangle + \langle \mathbf{x}(t), (\mathcal{A}+\mathcal{B}\mathcal{K})\mathcal{P}(\mathcal{T}^*+\mathcal{K}^*\mathcal{T}_u^*)\mathbf{x}(t)\rangle \\
&= \langle \mathbf{x}(t), (\mathcal{T}\mathcal{P}\mathcal{A}^*+\mathcal{A}\mathcal{P}\mathcal{T}^*)\mathbf{x}(t)\rangle + \langle \mathbf{x}(t), (\mathcal{T}_u\mathcal{K}\mathcal{P}\mathcal{A}^*+\mathcal{B}\mathcal{K}\mathcal{P}\mathcal{T}^*)\mathbf{x}(t)\rangle \\
&\quad + \langle \mathbf{x}(t), (\mathcal{A}\mathcal{P}\mathcal{K}^*\mathcal{T}_u^*+\mathcal{T}\mathcal{P}\mathcal{K}^*\mathcal{B}^*)\mathbf{x}(t)\rangle + \langle \mathbf{x}(t), (\mathcal{T}_u\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{B}^*+\mathcal{B}\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{T}_u^*)\mathbf{x}(t)\rangle \\
&= \langle \mathbf{x}(t), \mathcal{P}_H\mathbf{x}(t)\rangle + \langle \mathbf{x}(t), (\mathcal{T}_u\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{B}^*+\mathcal{B}\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{T}_u^*)\mathbf{x}(t)\rangle.
\end{aligned}$$

However, using Lemma 6.10, if we choose $\mathcal{Z}_1 = \mathcal{T}_u\mathcal{K}$ and $\mathcal{Z}_2 = \mathcal{B}\mathcal{K}$ then

$$\mathcal{T}_u\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{B}^*+\mathcal{B}\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{T}_u^* \preccurlyeq \mathcal{T}_u\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{T}_u^*+\mathcal{B}\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{B}^*. \tag{6.11}$$

Then,

$$\begin{aligned}
\dot{V}(t) &\leq \langle \mathbf{x}(t), \mathcal{P}_H\mathbf{x}(t)\rangle + \langle \mathbf{x}(t), (\mathcal{T}_u\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{T}_u^*+\mathcal{B}\mathcal{K}\mathcal{P}\mathcal{K}^*\mathcal{B}^*)\mathbf{x}(t)\rangle \\
&= \langle \mathbf{x}(t), \mathcal{P}_H\mathbf{x}(t)\rangle + \left\langle \mathbf{x}(t), (\mathcal{T}_u\mathcal{Z}\mathcal{P}^{-1}\mathcal{Z}^*\mathcal{T}_u^*+\mathcal{B}\mathcal{Z}\mathcal{P}^{-1}\mathcal{Z}^*\mathcal{B}^*)\mathbf{x}(t)\right\rangle
\end{aligned} \tag{6.12}$$

where we have substituted $\mathcal{K}\mathcal{P} = \mathcal{Z}$. Let $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ be 4-PI operators defined as follows.

$$\begin{aligned}
&\mathcal{E} = (\mathcal{T}\mathcal{P}\mathcal{A}^*+\mathcal{A}\mathcal{P}\mathcal{T}^*+\mathcal{T}_u\mathcal{Z}\mathcal{A}^*+\mathcal{B}\mathcal{Z}\mathcal{T}^*+\mathcal{A}\mathcal{Z}^*\mathcal{T}_u^*+\mathcal{T}\mathcal{Z}^*\mathcal{B}^*), \\
&\mathcal{F} = \begin{bmatrix}\mathcal{T}_u\mathcal{Z} & \mathcal{B}\mathcal{Z}\end{bmatrix}, \qquad \mathcal{G} = -\begin{bmatrix}\mathcal{P} & 0\\ 0 & \mathcal{P}\end{bmatrix}.
\end{aligned}$$

Then, the inequality in Equation (6.12) can be compactly written as

$$\dot{V}(t) \le \left\langle \mathbf{x}(t), (\mathcal{E} - \mathcal{F}\mathcal{G}^{-1}\mathcal{F}^*)\mathbf{x}(t) \right\rangle_{\mathbb{R}L_2}. \tag{6.13}$$

Furthermore, the LPI Equation (6.10) can be written in a compact form as

$$\begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{G} \end{bmatrix} \prec 0, \tag{6.14}$$

where $\mathcal{G} \prec 0$ and invertible because $\mathcal{P}$ is coercive (refer Theorem 2 in Peet (2020a)). Then, by Lemma 6.11,

$$(\mathcal{E} - \mathcal{F}\mathcal{G}^{-1}\mathcal{F}^*) \prec 0. \tag{6.15}$$

Combining the inequalities, Equations (6.13) and (6.14) we get

$$\dot{V}(t) \le \left\langle \mathbf{x}(t), (\mathcal{E} - \mathcal{F}\mathcal{G}^{-1}\mathcal{F}^*)\mathbf{x}(t) \right\rangle_{\mathbb{R}L_2} \le 0, \ \forall \mathbf{x}(t) \in \mathbb{R}L_2^{m,n}.$$

Hence the system is Lyapunov stable. $\square$

Unfortunately, unlike Theorem 6.3 and Corollary 6.4, one cannot obtain a less conservative version of Theorem 6.12 because setting $\mathcal{P}\mathcal{T}^* = \mathcal{Q}$ would require inversion of $\mathcal{T}^*$ during the reconstruction of a controller. Although this can be done using the inverse formulae presented in Section 2.4, the inverse of $\mathcal{T}^*$ happens to be a differential operator if the PIE represents a GPDE and thus unbounded — leading to both numerical and well-posedness issues when trying to find the closed-loop GPDE.

6.4 Numerical Examples

All the numerical tests in this section are performed using PIETOOLS toolbox in MATLAB. The standard process of using PIETOOLS includes: a) defining the GPDE using the parser; b) conversion of GPDE to its PIE representation; and c) setting up and solving the LPI optimization problem for the PIE (specifically, stability and stabilizability using `lpisolve()` function). Furthermore, all of the following tests were performed using `lpisettings('heavy')`, which is typically passed to the `lpisolve` function in the form `lpisolve(PIE, lpisettings('heavy'),lpi-test-type)`. For more details on the PIETOOLS functions and settings, refer Shivakumar *et al.* (2021).

We apply the primal and dual LPIs for the exponential decay rate in Corollary 6.2 to a linear delay-differential equation and a PDE reaction-diffusion equation to obtain the maximum lower bound on the exponential decay rate, α. To maximize α, we observe that the LPIs in Corollary 6.2 are convex in α for a fixed $\mathcal{P}$ – which implies a bisection search on α can be used to maximize the lower bound on exponential decay rate. Note that these LPIs have been implemented as a standard function in PIETOOLS and are accessed through `lpisolve` function.

Example 6.1 (Exponential Stability of a Linear Time-Delay System)**.** *Consider the following autonomous linear delay-differential equation from, e.g. Mondie and Kharitonov (2005).*

$$\dot{x}(t) = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0 \\ 4 & 0.1 \end{bmatrix} x(t-0.5)$$

The formulae for conversion of a delay-differential equation to a PIE can be found in Peet (2020b) and is automated in PIETOOLS. The primal and dual LPIs obtained lower bounds on the exponential decay rate of $\alpha_p = \alpha_d = 1.1534$*. These are similar to the estimate of* $\alpha = 1.153$ *as reported in Mondie and Kharitonov (2005).*

Example 6.2 (Exponential Stability of a reaction-diffusion PDE)**.** *Consider the following PDE model of a reaction-diffusion equation.*

$$\dot{\mathbf{x}}(t,s) = 2\mathbf{x}(t,s) + \partial_s^2 \mathbf{x}(t,s), \quad \mathbf{x}(t,0) = \partial_s \mathbf{x}(t,1) = 0.$$

Using PIETOOLS, we find the relevant parameters of the PIE representation of this PDE to be

$$\mathcal{T} = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{0, -\theta, -s\} \end{array}\right], \mathcal{A} = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{1, -\lambda\theta, -\lambda s\} \end{array}\right].$$

Using the primal and dual LPIs in Corollary 6.2, the primal and dual lower bounds on exponential decay rate are $\alpha_p = \alpha_d = 0.4674$. *One can find an analytical solution to the above PDE (by performing a change of variable* $\mathbf{y}(t,s) = e^{-2t}\mathbf{x}(t,s)$ *and using the method of separation of variables) and see that the largest eigenvalue of the solution is* -0.4674 *– validating the lower bound obtained from the LPIs.*

Next, we look at an unstable PDE and test if the PDE is stabilizable using the LPI presented in Theorem 6.7.

Example 6.3. *Consider the reaction-diffusion equation given by*

$$\dot{\mathbf{x}}(t,s) = 10\mathbf{x}(t,s) + \partial_s^2\mathbf{x}(t,s), \qquad s \in [0,1], t > 0,$$
$$\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0, \mathbf{x}(0,s) = \mathbf{x}_0 \in W_2[0,1].$$

Then, one can show analytically that the system is unstable. To stabilize the system, we introduce an in-domain control input leading to an altered dynamics

$$\dot{\mathbf{x}}(t,s) = 10\mathbf{x}(t,s) + \partial_s^2\mathbf{x}(t,s) + u(t)$$

where $u(t) = \int_0^1 K(s)\partial_s^2\mathbf{x}(t,s)ds$ *is the control input. Converting this PDE to a PIE, we get*

$$\mathcal{T} = \Pi_{\{0, s\theta-\theta, s\theta-s\}}, \mathcal{A} = \Pi_{\{1, 10(s\theta-\theta), 10(s\theta-s)\}}, \mathcal{B} = 1.$$

Then, by solving the LPI in Theorem 6.7, we can prove that the PDE is stabilizable. Furthermore, using the inversion technique presented in Section 2.4.3, we find that the operator $\mathcal{K}\mathbf{x} = \int_0^1 K(s)\partial_s^2\mathbf{x}(t,s)ds$ *stabilizes the system where*

$$K(s) = 0.29s^5 - 1.01s^4 + 0.95s^3 + 0.16s^2 - 0.51s + 0.98$$

6.5 Conclusions

In this chapter, we introduced slightly weaker notions of stability for PIEs and showed that under these notions, there exists a dual PIE with the same stability properties as the primal PIE. Using Lyapunov approach, we formulated primal and dual LPI optimization formulations to test for the internal stability of a PIE. Furthermore, we used the primal LPI formulation to formulated detectability of a PIE system. We proved the duality between the notions of stabilizability and detectability that allowed use to obtain an LPI formulation of the stabilizability problem using the dual LPI test for internal stability.

Lastly, using the numerical examples, we verified that there is no conservatism in the bounds on the exponential decay rate obtained by using LPIs in Theorem 6.3 and corollary 6.4.

Chapter 7

INPUT-OUTPUT PROPERTIES

7.1 Introduction

While Chapter 6 focused on internal stability in the absence of external inputs, dynamical systems rarely are isolated from surroundings and hence, in this chapter, we look at systems with inputs and outputs. While stability is an important property of interest, one should consider the impact of external inputs on the system to determine the system behavior under non-ideal conditions. For this purpose, some of the standard properties to investigate are the impact of an input of unit energy on the equilibrium state of the system (input-to-state stability) or the output of the system (input-to-output stability) when the state cannot be measured completely – referred to as the H_∞-norm of the system. Additionally, passivity of the system is another input-output property that is commonly investigated as it allows one to identify components of a dynamical system that follow Thermodynamic laws and do not produce energy.

7.2 H_∞-norm and Passivity

Here, we briefly recall the LMI approach to bound the H_∞-norm of an ODE system. For an ODE system represented in traditional state-space representation Equation (7.1),

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \; x(0) = 0 \\ y(t) &= Cx(t) + Dw(t) \end{aligned} \tag{7.1}$$

the following LMI condition by Boyd *et al.* (1994), established using bounded-real lemma, can be used to find a bound on H_∞-norm.

Theorem 7.1. *Define:*

$$G(s) = C(sI - A)^{-1}B + D.$$

If there exists a positive definite matrix P, such that

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \leq 0, \tag{7.2}$$

then $\|G\|_{H_\infty} \leq \gamma$.

In the following subsections, we generalize this LMI to a general class of infinite-dimensional systems - replacing the matrices A, B, C, D with operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and the positive matrix variable P with an operator variable $\mathcal{P}$.

The second input-to-output property that we consider is the passivity. Recall that the ODE Equation (7.1) is passive if for any input $w \in L_2$, we have $y \in L_2$ and $\langle w, y\rangle_{L_2} \geq 0$. For ODEs, an LMI test for passivity can be formulated as follows.

Theorem 7.2. *If there exists a positive definite matrix P such that*

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} \leq 0 \tag{7.3}$$

then for any $w \in L_2$ and $y \in L_2$ which satisfy Equation (7.1) *for some x,* $\langle w, y\rangle_{L_2} \geq 0$.

In the upcoming subsections, we will look at the LPI formulations of these analysis problems for PDEs, however, first we will establish the duality results for input-output properties of PIEs similar to the duality results for stability of PIEs to allow dual LPI formulations. As will be seen in Chapter 8, we will use these dual formulations in H_∞-optimal controller design.

Theorem 7.3. *(Dual L_2-gain) Suppose $\mathcal{T}, \mathcal{A} \in \mathcal{L}(\mathbb{R}L_2^{m,n})$, $\mathcal{B} \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}L_2^{m,n})$, and $\mathcal{C} \in \mathcal{L}(\mathbb{R}L_2^{m,n}, \mathbb{R}^r)$ are PI operators and $D \in \mathbb{R}^{r\times p}$ is a matrix. Then the following statements are equivalent.*

a) For $\underline{\mathbf{x}}(0) = 0$ and any $w \in L_2^p[0,\infty)$, if $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{m,n}$ and $z(t) \in \mathbb{R}^r$ satisfy

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \end{bmatrix}, \tag{7.4}$$

then $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

b) For $\bar{\mathbf{x}}(0) = 0$ and any $\bar{w} \in L_2^r[0,\infty)$, if $\bar{\mathbf{x}}(t) \in \mathbb{R}L_2^{m,n}$ and $\bar{z}(t) \in \mathbb{R}^p$ satisfy

$$\begin{bmatrix} \mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}(t) \\ \bar{w}(t) \end{bmatrix}, \tag{7.5}$$

then $\|\bar{z}\|_{L_2} \le \gamma \|\bar{w}\|_{L_2}$.

Proof. To show sufficiency (i.e. a) implies b)), let $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{m,n}$ and $z(t) \in \mathbb{R}^r$ satisfy Equation (7.4) for $\underline{\mathbf{x}}(0) = 0$ and some $w \in L_2^p[0,\infty)$. Then, $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$. Let $\bar{\mathbf{x}}(t) \in \mathbb{R}L_2^{m,n}$ and $\bar{z}(t) \in \mathbb{R}^p$ satisfy Equation (7.5) for $\bar{\mathbf{x}}(0) = 0$ and some $\bar{w} \in L_2^r[0,\infty)$. Then, by using Equation (6.3) in Theorem 6.1 and substituting initial conditions, we find

$$\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\underline{\mathbf{x}}}(s)\rangle_{\mathbb{R}L_2} ds = \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta. \quad (\star)$$

Furthermore, by using the variable change $\theta = t - s$ on the left-hand side of the above equation,

$$\begin{aligned}
&\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\underline{\mathbf{x}}}(s)\rangle_{\mathbb{R}L_2} ds && (\#)\\
=&\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{A}\underline{\mathbf{x}}(s)\rangle_{\mathbb{R}L_2} ds + \int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{B}w(s)\rangle_{\mathbb{R}L_2} ds\\
=&\int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \underline{\mathbf{x}}(t-\theta)\rangle_{\mathbb{R}L_2} d\theta + \int_0^t \mathcal{B}^*\bar{\mathbf{x}}(\theta)^T w(t-\theta) d\theta.
\end{aligned}$$

Combining the two Eqns. ($\star$) and (#), we obtain

$$\begin{aligned}&\int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta \\ &= \int_0^t \left\langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta + \int_0^t \mathcal{B}^*\bar{\mathbf{x}}(\theta)^T w(t-\theta)d\theta.\end{aligned}$$

However, $\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) - \mathcal{A}^*\bar{\mathbf{x}}(t) = \mathcal{C}^*\bar{w}(t)$. Then

$$\begin{aligned}&\int_0^t \left\langle \mathcal{C}^*\bar{w}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta \\ &= \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta) - \mathcal{A}^*\bar{\mathbf{x}}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta \\ &= \int_0^t \mathcal{B}^*\bar{\mathbf{x}}(\theta)^T w(t-\theta)d\theta.\end{aligned}$$

Since $z = \mathcal{C}\underline{\mathbf{x}} + Dw$, we obtain

$$\begin{aligned}&\int_0^t \bar{w}(\theta)^T z(t-\theta)d\theta - \int_0^t \bar{w}(\theta)^T (Dw(t-\theta))d\theta \\ &= \int_0^t \bar{w}(\theta)^T (\mathcal{C}\underline{\mathbf{x}}(t-\theta))d\theta = \int_0^t \left\langle \mathcal{C}^*\bar{w}(\theta), \underline{\mathbf{x}}(t-\theta)\right\rangle_{\mathbb{R}L_2} d\theta \\ &= \int_0^t \mathcal{B}^*\bar{\mathbf{x}}(\theta)^T w(t-\theta)d\theta.\end{aligned}$$

Likewise, we know $\bar{z} = \mathcal{B}^*\bar{\mathbf{x}} + D^T\bar{w}$. Hence

$$\begin{aligned}&\int_0^t \bar{z}(\theta)^T w(t-\theta)d\theta - \int_0^t D^T\bar{w}(\theta)^T w(t-\theta)d\theta \\ &= \int_0^t \mathcal{B}^*\bar{\mathbf{x}}(\theta)^T w(t-\theta)d\theta \\ &= \int_0^t \bar{w}(\theta)^T z(t-\theta)d\theta - \int_0^t \bar{w}(\theta)^T (Dw(t-\theta))d\theta.\end{aligned}$$

We conclude that for any $t > 0$, if z and w satisfy the primal PIE and $\bar{z}$ and $\bar{w}$ satisfy the dual PIE, then

$$\int_0^t \bar{z}(\theta)^T w(t-\theta)d\theta = \int_0^t \bar{w}(\theta)^T z(t-\theta)d\theta. \tag{7.6}$$

For any $\bar{w} \in L_2$, let $\bar{z}$ solve the dual PIE for some $\bar{\mathbf{x}}$. For any fixed $T > 0$, define $w(t) = \bar{z}(T-t)$ for $t \leq T$ and $w(t) = 0$ for $t > T$. Then $w \in L_2$ and for this input, let z solve the primal PIE for some $\underline{\mathbf{x}}$. Then, if we define the truncation operator P_T, we have

$$\begin{aligned}
\|P_T\bar{z}\|_{L_2}^2 &= \int_0^T \bar{z}(s)^T\bar{z}(s)ds = \int_0^T \bar{z}(s)^T w(T-s)ds \\
&= \int_0^T \bar{w}(s)^T z(T-s)ds \le \|P_T\bar{w}\|_{L_2}\|P_T z\|_{L_2} \\
&\le \|P_T\bar{w}\|_{L_2}\|z\|_{L_2} \le \gamma\|P_T\bar{w}\|_{L_2}\|w\|_{L_2} \\
&= \gamma\|P_T\bar{w}\|_{L_2}\|P_T w\|_{L_2} = \gamma\|P_T\bar{w}\|_{L_2}\|P_T\bar{z}\|_{L_2}.
\end{aligned}$$

Therefore, we have that $\|P_T\bar{z}\|_{L_2} \le \gamma\|P_T\bar{w}\|_{L_2}$ for all $T \ge 0$. We conclude that $\|\bar{z}\|_{L_2} \le \gamma\|\bar{w}\|_{L_2}$. Since $\mathcal{T}^{**} = \mathcal{T}$ and

$$\begin{bmatrix} (\mathcal{A}^*)^* & (\mathcal{B}^*)^* \\ (\mathcal{C}^*)^* & (D^T)^T \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix}^* = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}^{**} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}$$

we have that sufficiency implies necessity. □

Remark 7.4. *Note the relationship between primal and dual mappings $w \mapsto z$ and $\bar{w} \mapsto \bar{z}$ as given in Equation* (7.6) *of the proof:*

$$\int_0^t \bar{z}(\theta)^T w(t-\theta)d\theta = \int_0^t \bar{w}(\theta)^T z(t-\theta)d\theta.$$

If one were to define a Laplace transform for these inputs $(\hat{w}, \hat{z}, \hat{\bar{w}}, \hat{\bar{z}})$ and transfer function for the systems ($\hat{z}(s) = G(s)\hat{w}(s)$ and $\hat{\bar{z}}(s) = G_d(s)\hat{\bar{w}}(s)$), then this equation would imply $\hat{\bar{z}}(s)^T\hat{w}(s) = \hat{\bar{w}}(s)^T\hat{z}(s)$ or $\hat{\bar{w}}(s)^T G_d(s)^T\hat{w}(s) = \hat{\bar{w}}(s)^T G(s)\hat{w}(s)$ so that $G_d(s)^T = G(s)$ — which is precisely the standard interpretation of the dual transfer function for ODEs. In addition, we note that while Theorem 7.3 assumes input-output stability of the primal and dual, the relationship in Equation (7.6) *holds for any finite time, t, and hence does not require the primal or dual to be input-output stable.*

The duality relation, Equation (7.6), between input and outputs is a crucial requirement in proving the equivalence in I/O properties of a PIE and its dual. This relation can be verified numerically for any PIE and its dual. To perform the numerical verification, we simulate various PIE systems of the form Equation (7.4) and the corresponding dual systems Equation (7.5) using a MATLAB library, PIESIM (See Peet and Peet (2020)). For each example, the simulations are performed with zero initial conditions and L_2-bounded disturbance inputs, $w(t) = \sin(5t)\exp(-2t)$ for the primal representation of a PIE and $\bar{w}(t) = (t - t^2)\exp(-t)$ for the corresponding dual PIE. The simulation is performed for a total time $t = 5$ and the outputs from these simulations, z and $\bar{z}$, are used to measure the error, if any, in the duality relation Equation (7.6) by using the quantity, err, defined as

$$err(t) = \int_0^t \left(\bar{z}(\theta)^T w(t-\theta) - \bar{w}(\theta)^T z(t-\theta)\right) d\theta.$$

The PIE examples used in the simulations are obtained by converting the following PDEs to PIEs:

(E1) $\dot{\mathbf{x}}(t,s) = \partial_s^2 \mathbf{x}(t,s) + w(t)$, $\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0$, $z(t) = \int_0^1 \mathbf{x}(t,s)ds$.

(E2) $\dot{\mathbf{x}}(t,s) = -\partial_s \mathbf{x}(t,s) + w(t)$, $\mathbf{x}(t,0) = 0$, $z(t) = \mathbf{x}(t,1)$.

(E3) $\dot{\mathbf{x}}(t,s) = 3\mathbf{x}(t,s) + \partial_s^2 \mathbf{x}(t,s) + w(t)$, $\mathbf{x}(t,0) = \partial_s \mathbf{x}(t,1) = 0$, $z(t) = \mathbf{x}(t,1)$.

For each PDE listed above, we use the formulae presented in (Shivakumar *et al.*, 2022, Block 4 and 5)) to find the parameters $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ that define the PIEs Equation (7.4) and Equation (7.5). The results of the simulation are tabulated in Table 7.1 which indicates that $err(t)$ is close to numerical zero for all t and for all examples.

Note that Example (E3) is unstable and hence its primal and dual PIE representation is likewise unstable. However, as mentioned in Remark 7.4, the intertwining relationship in Equation (7.6) does not require stability – an assertion verified by the numerical analysis in Table 7.1.

Example	(E1)	(E2)	(E3)
$err(1.0)$	1.2e-07	-3.2e-06	2.1e-06
$err(2.5)$	-4.7e-07	2.8e-05	-2.3e-05
$err(5.0)$	-2.2e-07	1.6e-05	-2.3e-04

Table 7.1: In this table, we list the error in the intertwining relationship given by the quantity $err(t) = \int_0^t \left((\bar{z}(\theta))^T w(t-\theta) - (\bar{w}(\theta))^T z(t-\theta)\right) d\theta$ obtained by numerical integration, where $z, \bar{z}$ are obtained from simulation of different examples of primal PIE Equation (7.4) and its dual Equation (7.5) under disturbance $w = \sin(5t)\exp(-2t)$ and $\bar{w} = (t - t^2)\exp(-t)$.

Remark 7.5. *Finally, we remark that the significance of Theorems 6.1 and 7.3 is not simply that a dual representation exists but that it has the same parametrization as the primal (making the primal and dual interchangeable). In addition, the proofs of Theorems 6.1 and 7.3 do not utilize the algebraic structure of the PI algebra – implying that the duality result (and intertwining relationship) holds for any class of well-posed systems parameterized by a set of bounded operators on a reflexive Hilbert space which is closed under adjoint.*

7.2.1 LPI for Upper-bounding H_∞-norm

In the following theorem, we propose LPI generalizations of the primal and dual versions of the KYP Lemma and use Theorem 7.3 to show that the solution of either proves a bound on the L_2-gain of both the primal and dual systems.

Note that the LPI conditions in Theorem 7.6 are expressed using an extension of block matrices to block PI operators – The formal definition of concatenation of PI operators can be found in Appendix B.1.1. However, because the domain and range of PI operators of the form given in Definition 2.6 are an ordered concatenation of $\mathbb{R}$ and L_2, the arrangement of the blocks of the operators in the proposed LPI conditions are slightly different from that in the tradition formulations of the KYP Lemma for state-space ODEs.

Theorem 7.6. *Suppose that* ***either*** *of the two statements hold for some $\gamma > 0$ and bounded linear operator $\mathcal{P} = \mathcal{P}^* \succeq 0$.*

a) $$\begin{bmatrix} -\gamma I & D & \mathcal{C} \\ D^T & -\gamma I & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{C}^* & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \preceq 0$$

b) $$\begin{bmatrix} -\gamma I & D^T & \mathcal{B}^* \\ D & -\gamma I & \mathcal{C}\mathcal{P}\mathcal{T}^* \\ \mathcal{B} & \mathcal{T}\mathcal{P}\mathcal{C}^* & \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \end{bmatrix} \preceq 0$$

Then, for any $w \in L_2$, if z satisfies either

$$\begin{bmatrix} \mathcal{T}\dot{\mathbf{x}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \tag{7.7}$$

or

$$\begin{bmatrix} \mathcal{T}^*\dot{\mathbf{x}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ w(t) \end{bmatrix}, \tag{7.8}$$

for some $\mathbf{x}(t)$ with $\mathbf{x}(0) = 0$, then $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof. Suppose a) holds. Define $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x}\rangle_{\mathbb{R}L_2}$. For any $w \in L_2$, suppose z satisfies Equation (7.7) for some $\mathbf{x}$ with $\mathbf{x}(0) = 0$. Differentiating $V(\mathbf{x}(t))$ with respect to time, t, we obtain

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \langle \mathcal{T}\mathbf{x}(t), \mathcal{P}\left(\mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t)\right)\rangle + \langle (\mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t)), \mathcal{P}\mathcal{T}\mathbf{x}(t)\rangle \\ &= \left\langle \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle. \end{aligned}$$

Now let $v(t) = \frac{1}{\gamma} z(t)$. Then we have

$$\dot{V}(\mathbf{x}(t)) - \gamma \left\| w(t) \right\|_{\mathbb{R}}^2 + \frac{1}{\gamma} \left\| z(t) \right\|_{\mathbb{R}}^2 = \dot{V}(\mathbf{x}(t)) - \gamma \left\| w(t) \right\|^2 - \frac{1}{\gamma} \left\| z(t) \right\|^2 + \frac{2}{\gamma} \left\| z(t) \right\|^2$$

$$= \dot{V}(\mathbf{x}(t)) - \gamma \left\| w(t) \right\|^2 - \gamma \left\| v(t) \right\|^2 + v(t)^T z(t) + z(t)^T v(t)$$

$$= \left\langle \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} -\gamma I & D & \mathcal{C} \\ D^T & -\gamma I & \mathcal{B}^* \mathcal{P} \mathcal{T} \\ \mathcal{C}^* & \mathcal{T}^* \mathcal{P} \mathcal{B} & \mathcal{T}^* \mathcal{P} \mathcal{A} + \mathcal{A}^* \mathcal{P} \mathcal{T} \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle \le 0.$$

Integrating this inequality in time, we obtain

$$V(\mathbf{x}(T)) - V(\mathbf{x}(0)) \le \gamma \int_0^T \left\| w(t) \right\|^2 dt - \frac{1}{\gamma} \int_0^T \left\| z(t) \right\|^2 dt.$$

Now, since $\mathbf{x}(0) = 0$ and $V(\mathbf{x}(T)) \ge 0$ for all $T \ge 0$, we obtain $\left\| z \right\|_{L_2}^2 \le \gamma^2 \left\| w \right\|_{L_2}^2$. Furthermore, Theorem 7.3 implies the same bound hold if z and $\mathbf{x}$ satisfy Equation (7.8).

Since $\mathcal{T}^{**} = \mathcal{T}$ and

$$\begin{bmatrix} (\mathcal{A}^*)^* & (\mathcal{B}^*)^* \\ (\mathcal{C}^*)^* & (D^T)^T \end{bmatrix} = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & D^T \end{bmatrix}^* = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}^{**} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & D \end{bmatrix}$$

we have that b) likewise implies the same bounds. $\square$

Before applying the results of Theorem 7.6 to controller synthesis, we note that while the operator variable, $\mathcal{P}$, that is used to parameterize storage function $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x} \rangle$ in Theorem 7.6 is not required to be *strictly* positive we will require strict positivity of this operator during observer design and controller synthesis in order to ensure boundedness of $\mathcal{P}^{-1}$.

As with the case of stability LPIs, we can relax the boundedness constraint on $\mathcal{P}$ by changing the parametric form of the storage function V to obtain a less conservative LPI, as shown below.

Corollary 7.7. *Suppose there exist $\gamma > 0$ and PI operator $\mathcal{P}$ with $\mathcal{P} = \mathcal{P}^* \succeq 0$, such that either (a) or (b) is satisfied:*

a) $$\begin{bmatrix} -\gamma & D^T & \mathcal{B}^*\mathcal{P}\mathcal{T} \\ D & -\gamma & \mathcal{C} \\ \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \preceq 0$$

b) $$\begin{bmatrix} -\gamma & D & \mathcal{C}\mathcal{P}\mathcal{T}^* \\ D^T & -\gamma & \mathcal{B}^* \\ \mathcal{T}\mathcal{P}\mathcal{C}^* & \mathcal{B} & \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \end{bmatrix} \preceq 0$$

Then, we have the following two results:

1. *If (a) is satisfied, there exist PI operators $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that $\mathcal{T}^*\mathcal{Q} = \mathcal{Q}^*\mathcal{T} = \mathcal{R}$ and*
$$\begin{bmatrix} -\gamma & D^T & \mathcal{B}^*\mathcal{Q} \\ D & -\gamma & \mathcal{C} \\ \mathcal{Q}^*\mathcal{B} & \mathcal{C}^* & \mathcal{Q}^*\mathcal{A} + \mathcal{A}^*\mathcal{Q} \end{bmatrix} \preceq 0.$$
Otherwise, if (b) is satisfied, then there exist PI operators $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that $\mathcal{T}\mathcal{Q} = \mathcal{Q}^*\mathcal{T}^* = \mathcal{R}$ and
$$\begin{bmatrix} -\gamma & D & \mathcal{C}\mathcal{Q} \\ D^T & -\gamma & \mathcal{B}^* \\ \mathcal{Q}^*\mathcal{C}^* & \mathcal{B} & \mathcal{Q}^*\mathcal{A}^* + \mathcal{A}\mathcal{Q} \end{bmatrix} \preceq 0.$$

2. *For any $w \in L_2$, if z satisfies either Equation (7.7) or Equation (7.8) for some $\mathbf{x}$, then $\|z\| \leq \gamma \|w\|$.*

Proof. The proof is trivial and follows directly from the assumptions of the Corollary statement. One can show that $\mathcal{Q} = \mathcal{P}\mathcal{T}$ and $\mathcal{R} = \mathcal{T}^*\mathcal{P}\mathcal{T} \succeq 0$ satisfy (1) if (a) is satisfied. Likewise, for (b) we have $\mathcal{Q} = \mathcal{P}\mathcal{T}^*$ and $\mathcal{R} = \mathcal{T}\mathcal{P}\mathcal{T}^*$.

The second statement, (2), follows directly from Theorem 7.6. □

7.2.2 LPI for Passivity

In the following theorem, we propose LPI generalizations of the primal and dual versions of the Positive-real Lemma to show that the solution of either proves the passivity of both the primal and dual systems.

Theorem 7.8. *Suppose that **either** of the two statements hold for some bounded linear operator $\mathcal{P} = \mathcal{P}^* \succeq 0$.*

a) $$\begin{bmatrix} -(D + D^T) & \mathcal{T}^*\mathcal{P}\mathcal{B} - \mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T} - \mathcal{C} & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \preceq 0$$

b) $$\begin{bmatrix} -(D + D^T) & \mathcal{T}\mathcal{P}\mathcal{C}^* - \mathcal{B} \\ \mathcal{C}\mathcal{P}\mathcal{T}^* - \mathcal{B}^* & \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \end{bmatrix} \preceq 0$$

Then, for any $w \in L_2$, if z satisfies either Equation (7.7) or Equation (7.8) for some $\mathbf{x}(t)$ with $\mathbf{x}(0) = 0$, then $\langle w, z \rangle_{L_2} \geq 0$.

Proof. The proof for this is trivial and follows the same steps as the proof of Theorem 7.6. Assuming a) holds, one can define $V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}\mathcal{T}\mathbf{x} \rangle_{\mathbb{R}L_2}$ and show that $\dot{V}(\mathbf{x}(t)) - \langle w(t), z(t) \rangle - \langle z(t), w(t) \rangle \leq 0$. Again, integrating forward in time, using Gronwall-Bellman inequality, and initial conditions, one can show that the system is passive. Likewise, the converse can be proven either using the symmetry argument or using $\mathcal{T}^*$ instead of $\mathcal{T}$ in V. $\square$

Mirroring the previous subsection, we also have a less conservative LPI for the Positive-real Lemma, as shown below.

Corollary 7.9. *Suppose there exists $\mathcal{P}$ with $\mathcal{P} = \mathcal{P}^* \succeq 0$, such that either (a) or (b) is satisfied:*

a) $\begin{bmatrix} -(D+D^T) & \mathcal{T}^*\mathcal{P}\mathcal{B}-\mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T}-\mathcal{C} & \mathcal{T}^*\mathcal{P}\mathcal{A}+\mathcal{A}^*\mathcal{P}\mathcal{T} \end{bmatrix} \preceq 0$

b) $\begin{bmatrix} -(D+D^T) & \mathcal{T}\mathcal{P}\mathcal{C}^*-\mathcal{B} \\ \mathcal{C}\mathcal{P}\mathcal{T}^*-\mathcal{B}^* & \mathcal{T}\mathcal{P}\mathcal{A}^*+\mathcal{A}\mathcal{P}\mathcal{T}^* \end{bmatrix} \preceq 0$

Then:

1. *If (a) is satisfied, there exist PI operators $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that $\mathcal{T}^*\mathcal{Q} = \mathcal{Q}^*\mathcal{T} = \mathcal{R}$ and*

$$\begin{bmatrix} -(D+D^T) & \mathcal{Q}^*\mathcal{B}-\mathcal{C}^* \\ \mathcal{B}^*\mathcal{Q}-\mathcal{C} & \mathcal{Q}^*\mathcal{A}+\mathcal{A}^*\mathcal{Q}\mathcal{T} \end{bmatrix} \preceq 0.$$

   Otherwise, if (b) is satisfied, then there exist PI operators $\mathcal{Q}$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that $\mathcal{T}\mathcal{Q} = \mathcal{Q}^*\mathcal{T}^* = \mathcal{R}$ and

$$\begin{bmatrix} -(D+D^T) & \mathcal{Q}^*\mathcal{C}^*-\mathcal{B} \\ \mathcal{C}\mathcal{Q}-\mathcal{B}^* & \mathcal{Q}^*\mathcal{A}^*+\mathcal{A}\mathcal{Q} \end{bmatrix} \preceq 0.$$

2. *For any $w \in L_2$, if z satisfies either Equation (7.7) or Equation (7.8) for some $\mathbf{x}(t)$ with $\mathbf{x}(0) = 0$, then $\langle w, z \rangle_{L_2} \geq 0$.*

Proof. The proof is trivial and follows directly from the assumptions of the Corollary statement. One can show that $\mathcal{Q} = \mathcal{P}\mathcal{T}$ and $\mathcal{R} = \mathcal{T}^*\mathcal{P}\mathcal{T} \succeq 0$ satisfy (1) if (a) is satisfied. Likewise, for (b) we have $\mathcal{Q} = \mathcal{P}\mathcal{T}^*$ and $\mathcal{R} = \mathcal{T}\mathcal{P}\mathcal{T}^*$.

The second statement, (2), follows directly from Theorem 7.8. □

7.3 Numerical Examples

All the numerical tests in this section are performed using PIETOOLS toolbox in MATLAB. The standard process of using PIETOOLS includes: a) defining the GPDE using the parser; b) conversion of GPDE to its PIE representation; and c) setting up and solving the LPI optimization problem for the PIE (specifically, H_∞-norm and passivity using `lpisolve()` function). Unless stated otherwise, all of the following tests were performed using `lpisettings('heavy')`, which is typically passed to the `lpisolve` function in the form

```
lpisolve(PIE, lpisettings('heavy'),lpi-test-type).
```

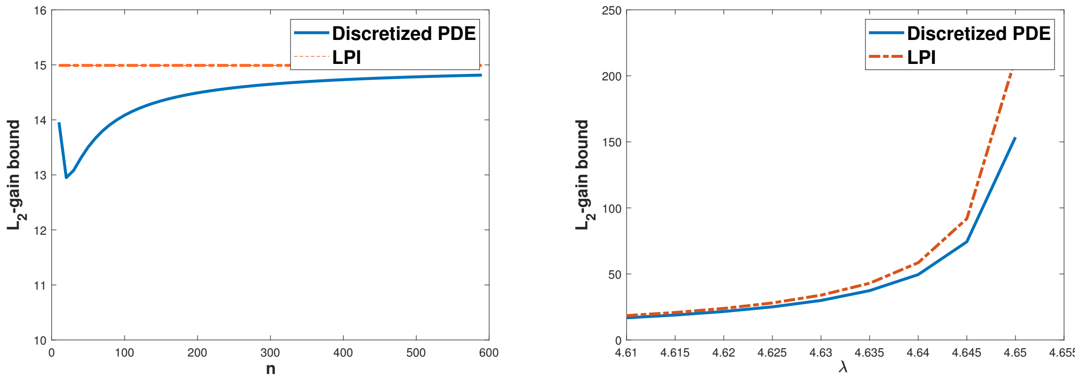
For more details on the PIETOOLS functions and settings, refer to the user manual by Shivakumar *et al.* (2021).

In this section, we perform various numerical tests to find conservatism, scalability, and accuracy of the LPIs proposed in Theorem 7.6 and corollary 7.7 to find bounds on the H_∞-norm of a GPDE system using the PIETOOLS toolbox and finite difference discretization method. We compare the estimate of H_∞ norm bound obtained using a numerical discretization (2nd-order central difference approximation is used for spatial derivatives to obtain an ODE approximation of PDE) with the estimate obtained using LPIs in Theorem 7.6 and corollary 7.7 implemented in PIETOOLS.

Example 7.1. *Consider the system shown below. In Peet (2018), it was shown to be stable for* $\lambda < 4.65$.

$$
\begin{aligned}
u_t(t,s) &= A_0(s)u(t,s) + A_1(s)u_s(t,s) + A_2(s)u_{ss}(t,s) + w(t)\\
z(t) &= \int_0^1 u(t,s)ds, \quad u(t,0)=0, \quad u_s(t,1)=0\\
A_0(s) &= (-0.5s^3 + 1.3s^2 - 1.5s + 0.7 + \lambda)\\
A_1(s) &= (3s^2 - 2s), \quad A_2(s) = (s^3 - s^2 + 2)
\end{aligned}
$$

Figure 7.1a shows the variation of an estimate of the L_2 *gain obtained from spatial discretization while varying mesh size. At a mesh size of 600, we had an* L_2



(a) (b)

Figure 7.1: For the PDE system in Example 7.1, we approximate the PDE by an ODE, which is obtained using a central difference scheme of 2nd order on spatial derivatives. Then, an estimate L_2-gain bounds for the obtained ODE is found using MATLAB `hinfnorm` function. The above plots show: (a) Mesh size vs L_2-gain obtained, (b) value of the parameter, λ vs L_2-gain obtained

gain of 14.82 (LPI bound was 14.99). Although this example obtained the largest residual gap of all examples at 3%, this residual is likely due to our naive method of discretization and not conservatism in Theorem 7.6. Figure 7.1b shows the bounds obtained when the system parameter λ is varied. Using higher degree polynomials shows minor change in the L_2-gain bound, typically of the order 10^{-6}. This suggests that relatively low-degree polynomials give tight bounds.

Example 7.2. *For the PDE systems listed below, we compare the L_2-gain bounds obtained by our algorithm and finite difference discretization method in Example 7.2.*

B.1: Following PDE is stable for $\lambda \leq \pi^2$.

$$u_t(t,s) = \lambda u(t,s) + u_{ss}(t,s) + w(t)$$
$$z(t) = \int_0^1 u(t,s)ds, \quad u(t,0) = 0, \quad u(t,1) = 0.$$

B.2: Following PDE is stable for $\lambda \leq 2.467$.

$$u_t(t,s) = \lambda u(t,s) + u_{ss}(t,s) + w(t)$$
$$z(t) = \int_0^1 u(t,s)ds, \quad u(t,0) = 0, \quad u_s(t,1) = 0.$$

B.3: The following coupled PDE was shown to be stable for $R < 21$ in Ahmadi et al. *(2016b).*

$$u_t(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} u(t,s) + \frac{1}{R} u_{ss}(t,s) + w(t)$$
$$z(t) = \int_0^1 u(t,s)ds, \quad u(t,0) = 0 \quad u(t,1) = 0$$

	LPI from Theorem 7.6a	Discretization approach	Parameter
B.1	8.214	8.253	$\lambda = 0.98\pi^2$
B.2	12.03	12.31	$\lambda = 2.4$
B.3	3.9738	3.9708	$R = 20$

Table 7.2: A bound on L_2 gain using different methods.

Example 7.3. *Consider,*

$$u_{t,i}(t,s) = \lambda u_i(t,s) + \sum_{k=1}^{i} u_{ss,k}(t,s) + w(t)$$
$$z(t) = \int_0^1 u(t,s)ds, \quad u(t,0) = 0 \quad u(t,1) = 0.$$

This example was tailored to test the time complexity of the algorithm proposed. We use the value $\lambda = 0.5\pi^2$ for all i. CPU time of the algorithm for different number of coupled PDEs is tabulated in Table II.

i	1	2	3	4	5	10	20
CPU time(s)	0.60	1.45	5.22	13.7	36.5	2317	27560

Table 7.3: This table shows the computational cost scaling of the L_2-gain primal LPI for an increasing number of PDEs in a coupled PDE system. The CPU runtime on a standard Intel i7 (4 CPUs) with 16GB RAM laptop is shown for a system of i coupled PDEs. Refer Example 7.3

Example 7.4. *Next, to verify the claim that results of Theorem 7.6 are indeed more conservative in practice than Corollary 7.7, we will apply both the results to the examples in the PDE library of the PIETOOLS as well as the some examples that are not present in the PDE library; All the PDEs are listed below. The results are tabulated in the tables that follow (See Example 7.4). Note that for most of these PDE examples, an analytical value of H_∞-norm is not known.*

A.1 $\dot{\mathbf{x}}(t,s) = A_0(s)\mathbf{x}(t,s) + A_1(s)\partial_s\mathbf{x}(t,s) + A_2(s)\partial_s^2\mathbf{x}(t,s) + w(t)$,

$\mathbf{x}(t,0) = \partial_s\mathbf{x}(t,1) = 0$, *and* $z(t) = \mathbf{x}(t,1)$

where

$$A_0(s) = -0.5s^3+1.3s^2-1.5s+0.7+4.6, \quad A_1(s) = 3s^2-2s, \quad A_2(s) = s^3-s^2+2.$$

A.2 $\dot{\mathbf{x}}_i(t,s) = 0.99\pi^2\mathbf{x}_i(t,s) + \sum_{k=1}^{i}\partial_s^2\mathbf{x}_k(t,s) + w(t)$, $\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0$, *and* $z(t) = \int_0^1 \mathbf{x}(t,s)ds$.

A.3 $\dot{\mathbf{x}}(t,s) = C_m(s)\mathbf{x}(t,s) + R\partial_s^2\mathbf{x}(t,s) + w(t)$,

$\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0$, *and* $z(t) = \int_0^1 \begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(t,s)ds$

where

$$C_m(s) = \begin{bmatrix}1 & 1.5\\ 5 & 0.2\end{bmatrix}, \quad R = 2.6.$$

A.4 $\dot{\mathbf{x}}(t,s) = C_m(s)\mathbf{x}(t,s) + R\partial_s^2\mathbf{x}(t,s) + w(t)$,

$\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0$, *and* $z(t) = \int_0^1 \mathbf{x}(t,s)ds$

where

$$C_m(s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix}, \quad R = \frac{1}{20}.$$

A.5 $\dot{\mathbf{x}}(t,s) = \partial_s^2\mathbf{x}(t,s) + w(t)$,

$\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0$, *and* $z(t) = \int_a^b \mathbf{x}(t,s)ds$.

A.6 $\ddot{\mathbf{x}}(t,s) = \partial_s^2\mathbf{x}(t,s) + w(t)$,

$\mathbf{x}(t,0) = \mathbf{x}(t,1) = 0$, *and* $z(t) = \partial_s\mathbf{x}(t,1)$.

A.7 $\dot{\mathbf{x}}(t,s) = \partial_s^2\mathbf{x}(t,s) + w(t)$,

$\mathbf{x}(t,0) = \partial_s\mathbf{x}(t,1) = 0$,*and* $z(t) = \int_a^b \mathbf{x}(t,s)ds$.

7.4 Conclusions

In this chapter, we developed results analogous to Bounded-real and Positive-real Lemma for systems governed by PIEs. We also established that the duality relationship, established in Chapter 6 on the equivalence of internal stability of a PIE and its dual PIE, also extends to the input-output relationship. Thus, we showed that there exist dual formulations of the Bounded-real and Positive-real Lemmas for PIE systems.

Using numerical examples, we showed that the bounds on H_∞-norm obtained using the LPI optimization-based approach were accurate when compared against the traditional discretization approach. Lastly, using numerical examples, we also showed that the bounds obtained using the 'modified' LPIs presented in Corollary 7.7 are less conservative in comparison to those obtained using LPIs in Theorem 7.6.

Bound on H_∞-norm				
Settings	"heavy"		"veryheavy"	
LPI	Th. 7.6a (b)	Cor. 7.7c (d)	Th. 7.6a (b)	Cor. 7.7c (d)
Ex. A.1	23.73 (*)	23.70 (24.85)	23.70 (23.70)	23.70 (23.70)
Ex. A.2	9.333 (*)	8.21 (8.21)	8.21 (10.05)	8.21 (8.21)
Ex. A.3	0.81 (13.25)	0.81 (0.81)	0.81 (3.501)	0.81 (0.81)
Ex. A.4	2.145 (*)	2.145 (2.157)	2.145 (9.427)	2.145 (2.145)
Ex. A.5	0.083 (8.00*)	0.0833 (0.0834)	0.083 (6.89*)	0.0833 (0.0833)
Ex. A.6	10.26* (18.3*)	0.5 (0.5)	4.23* (8.71*)	0.5 (0.5)
Ex. A.7	0.333 (3.93*)	0.33 (0.33)	0.33 (3.49*)	0.33 (0.33)

Table 7.4: This table lists the bounds on H_∞-norm for the PDEs (from PIETOOLS examples library and the examples listed in this subsection) obtained by solving the LPI in Theorem 7.6a and Corollary 7.7c (value in parentheses correspond to bound obtained by solving LPI in Theorem 7.6b and Corollary 7.7d) for different LPI settings. The value '*' indicates that the LPI was not successfully solved due to numerical errors/infeasibility.

Chapter 8

H_∞-OPTIMAL OBSERVER AND CONTROLLER DESIGN FOR GPDES

8.1 Introduction

In this chapter, we will apply the various results derived in Chapters 6 and 7 to formulate problems of H_∞-optimal observer design and controller synthesis as LPI problems. For the case of optimal observer design, we can simply use the primal LPI in Theorem 7.6 on the PIE representation of the observer error dynamics to obtain an LPI for observer synthesis. Although the optimization problem now involves two decision variables, the parameters in the storage function $\mathcal{P}$ and observer gains $\mathcal{L}$, the optimization problem is still be convexified using a simple invertible variable change. However, for controller synthesis, as previously mentioned in Section 6.2, the problem is bilinear and non-convex. Thus, we must use the dual LPI from Theorem 7.6 to overcome the non-convexity and obtain convex solvable LPI conditions.

8.2 State Observers

First we will solve the estimation problem since it is convex and thus easily solved. Moreover, in practice, we find an observer to estimate the state $\underline{\mathbf{x}}$ because, typically, full information of the state is not available to perform state-feedback control and the state must be estimated using sensor measurements y. If one considers a PIE model of the form

$$\begin{aligned} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) &= \mathcal{A}\underline{\mathbf{x}} + \mathcal{B}_1 w(t), \qquad \underline{\mathbf{x}}(0) = \mathbf{x}_0 \\ z(t) &= \mathcal{C}_1\underline{\mathbf{x}}(t) + D_{11}w(t), \quad y(t) = \mathcal{C}_2\underline{\mathbf{x}}(t) + D_{21}w(t), \end{aligned} \tag{8.1}$$

where z is the output to be regulated and $\underline{\mathbf{x}}$ is the state to be measured using sensor measurements y, we can design a *Luenberger observer* with dynamics

$$\begin{aligned} \mathcal{T}\dot{\underline{\mathbf{x}}}_o(t) &= \mathcal{A}\underline{\mathbf{x}}_o + \mathcal{L}(y_o(t) - y(t)), \qquad \underline{\mathbf{x}}_o(0) = \mathbf{x}_0 \\ z_o(t) &= \mathcal{C}_1\underline{\mathbf{x}}_o(t), \quad y_o(t) = \mathcal{C}_2\underline{\mathbf{x}}_o(t), \end{aligned} \tag{8.2}$$

where $\underline{\mathbf{x}}_o$ is the observer's estimation of the state, z_o, y_o are estimated outputs, and $\mathcal{L}$ is the observer gain that drives the error between the estimate and the actual state $\underline{\mathbf{x}}_o - \underline{\mathbf{x}}$ to zero via a feedback input $\mathcal{L}(y_o(t) - y(t))$.

Then, one can see that the dynamics of the observer error $\mathbf{e} = \underline{\mathbf{x}} - \underline{\mathbf{x}}_o$ is given by

$$\begin{bmatrix} \mathcal{T}\dot{\mathbf{e}}(t) \\ z_o(t) - z(t) \end{bmatrix} = \begin{bmatrix} (\mathcal{A} + \mathcal{L}\mathcal{C}_2) & -(\mathcal{B}_1 + \mathcal{L}D_{21}) \\ \mathcal{C}_1 & -D_{11} \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ w(t) \end{bmatrix}, \qquad \mathbf{e}(0) = 0. \tag{8.3}$$

Note that Equation (8.3) is in the standard form Equation (7.7), and we can use Theorem 7.6a to formulate an optimization problem that searches for observer gains $\mathcal{L}$ such that input-to-output gain $\frac{\|z-z_o\|}{\|w\|}$ is minimized.

8.2.1 LPI for H_∞-optimal Observer Gains

Now, we look at the LPI to find the H_∞-optimal observer, $\mathcal{L}$, for PIEs of the form Equation (8.1).

Theorem 8.1. *Suppose there exist $\gamma > 0$ and PI operators $\mathcal{P}, \mathcal{Z}$ with $\mathcal{P} \succeq \eta I$, such that*

$$\begin{bmatrix} -\gamma I & -D_{11}^T & -(\mathcal{B}_1^*\mathcal{P} + D_{21}^T\mathcal{Z}^*)\mathcal{T} \\ -D_{11} & -\gamma I & \mathcal{C}_1 \\ -\mathcal{T}^*(\mathcal{P}\mathcal{B}_1 + \mathcal{Z}D_{21}) & \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 \end{bmatrix} \preceq 0.$$

Then, for any $w \in L_2$, if $z_o - z$ satisfies Equation (8.3) for some $\mathbf{e}$ and $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$, then $\|z_o - z\| \le \gamma \|w\|$.

Proof. Let $\mathcal{P}$, $\mathcal{Z}$, and $\mathcal{L}$ satisfy the corollary statement. Then, $\mathcal{Z} = \mathcal{P}\mathcal{L}$, and

$$\begin{bmatrix} -\gamma I & -D_{11}^T & -(\mathcal{B}_1^*\mathcal{P} + D_{21}^T\mathcal{Z}^*)\mathcal{T} \\ -D_{11} & -\gamma I & \mathcal{C}_1 \\ -\mathcal{T}^*(\mathcal{P}\mathcal{B}_1 + \mathcal{Z}D_{21}) & \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 \end{bmatrix}$$
$$= \begin{bmatrix} -\gamma I & -D_{11}^T & -(\mathcal{B}_1^* + D_{21}^T\mathcal{L}^*)\mathcal{P}\mathcal{T} \\ (\cdot)^* & -\gamma I & \mathcal{C}_1 \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}^*\mathcal{P}(\mathcal{A} + \mathcal{L}\mathcal{C}_2) \end{bmatrix} \preceq 0.$$

Thus, from Theorem 7.6 (statement a), for $z_o - z, w, \mathbf{e}$ as in the Theorem statement, we have that $\|z_o - z\|_{L_2} \leq \gamma \|w\|_{L_2}$. □

As is customary from previous chapters, we also have the following less conservative LPI for H_∞-optimal observer design which is obtained by using a different parametrization of the quadratic storage functions.

Corollary 8.2. *Suppose there exist $\gamma > 0$ and PI operators $\mathcal{P}, \mathcal{Z}$ with $\mathcal{P} \succeq \eta I$, such that*

$$\begin{bmatrix} -\gamma I & -D_{11}^T & -(\mathcal{B}_1^*\mathcal{P} + D_{21}^T\mathcal{Z}^*)\mathcal{T} \\ -D_{11} & -\gamma I & \mathcal{C}_1 \\ -\mathcal{T}^*(\mathcal{P}\mathcal{B}_1 + \mathcal{Z}D_{21}) & \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{C}_2^*\mathcal{Z}^*\mathcal{T} + \mathcal{T}^*\mathcal{Z}\mathcal{C}_2 \end{bmatrix} \preceq 0$$

Then:

1. *There exist PI operators $\mathcal{Q}, \mathcal{Z}_Q$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that $\mathcal{T}^*\mathcal{Q} = \mathcal{Q}^*\mathcal{T} = \mathcal{R}$ and*

$$\begin{bmatrix} -\gamma I & -D_{11}^T & -(\mathcal{B}_1^*\mathcal{Q} + D_{21}^T\mathcal{Z}_Q^*) \\ -D_{11} & -\gamma I & \mathcal{C}_1 \\ -(\mathcal{Q}^*\mathcal{B}_1 + \mathcal{Z}_Q D_{21}) & \mathcal{C}_1^* & \mathcal{Q}^*\mathcal{A} + \mathcal{A}^*\mathcal{Q} + \mathcal{C}_2^*\mathcal{Z}_Q + \mathcal{Z}_Q^*\mathcal{C}_2 + \eta(\mathcal{T}^*\mathcal{A} + \mathcal{A}^*\mathcal{T}) \end{bmatrix} \preceq 0.$$

2. For any $w \in L_2$, if $z_o - z$ satisfies Equation (8.3) *for some* $\mathbf{e}$ *and* $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$*, then* $\|z_o - z\| \leq \gamma \|w\|$.

Proof. The proof follows from the proof of Corollary 7.7. □

Having devised a way to estimate the state $\underline{\mathbf{x}}$, we can now look toward design state-feedback controllers for PIEs.

8.3 Full State-Feedback Controllers

In this section, we return to the state-feedback controller synthesis problems. Specifically, given a PIE system

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C} & D_1 & D_2 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \\ u(t) \end{bmatrix},$$

our goal is to synthesize state-feedback controllers of the form $u(t) = \mathcal{K}\underline{\mathbf{x}}(t)$, where $\underline{\mathbf{x}}$ is the state of the PIE and the controller gain, $\mathcal{K}$, is a PI operator. To do this, we apply Corollary 6.2 and Theorem 7.6b to the closed-loop system

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}_2\mathcal{K} & \mathcal{B}_1 \\ \mathcal{C} + D_2\mathcal{K} & D_1 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \end{bmatrix}. \tag{8.4}$$

The resulting operator inequality then includes the term $\mathcal{KP}$ which is bilinear in the decision variables $\mathcal{K}$ and $\mathcal{P}$. However, as described in the introduction, and following the approach used for SS ODEs, we then construct an equivalent LPI by making the invertible variable substitution $\mathcal{KP} \to \mathcal{Z}$. An iterative algorithm for the inversion of this variable substitution is presented in Section 2.4. Finally, we address the PDE implementation of both the stabilizing and H_∞-optimal controllers.

8.3.1 LPI for H_∞-optimal Full State-Feedback Controller Gains

Next, we provide an LPI to find the H_∞-optimal state-feedback controller, $\mathcal{K}$, for PIEs with inputs and outputs of the form Equation (8.4). Here, we use $(\cdot)^*$ notation to represent the symmetric adjoint/transpose of the block operators.

Theorem 8.3. *Suppose there exist bounded linear operators $\mathcal{Z}$, $\mathcal{P} = \mathcal{P}^* \succeq \eta I$ with $\eta > 0$, and $\gamma > 0$ such that*

$$\begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ D_1 & -\gamma I & (\mathcal{C}\mathcal{P} + D_2\mathcal{Z})\mathcal{T}^* \\ \mathcal{B}_1 & \mathcal{T}(\mathcal{C}\mathcal{P} + D_2\mathcal{Z})^* & (\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} \preceq 0.$$

Then if $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, for any $w \in L_2$, if z satisfies

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}_2\mathcal{K} & \mathcal{B}_1 \\ \mathcal{C} + D_2\mathcal{K} & D_1 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \end{bmatrix}, \tag{8.5}$$

for some $\underline{\mathbf{x}}$ with $\underline{\mathbf{x}}(0) = 0$, then $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

Proof. Let $\mathcal{P}$, $\mathcal{Z}$, and $\mathcal{K}$ satisfy the corollary statement. Then, $\mathcal{Z} = \mathcal{K}\mathcal{P}$, and

$$\begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ (\cdot)^* & -\gamma I & (\mathcal{C}\mathcal{P} + D_2\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} = \begin{bmatrix} -\gamma I & D_1^T & \mathcal{B}_1^* \\ (\cdot)^* & -\gamma I & (\mathcal{C} + D_2\mathcal{K})\mathcal{P}\mathcal{T}^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}\mathcal{P}(\mathcal{A} + \mathcal{B}_2\mathcal{K})^* \end{bmatrix} \preceq 0.$$

Thus, from Theorem 7.6 (statement b), for $z, w, \underline{\mathbf{x}}$ as in the Theorem statement, we have that $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$. □

Given a PDE with associated PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, D_i\}$, Theorem 8.3 provides a controller gain $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$ such that $u(t) = \mathcal{K}\underline{\mathbf{x}}(t)$ achieves a closed-loop performance bound of $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$. Note that this controller does not necessarily imply internal exponential stability unless $\mathcal{P}, \mathcal{Z}$ also satisfy the LPI in Theorem 6.7.

Lastly, we also have a less conservative formulation of the H_∞-optimal state-feedback control problem, as shown below.

Corollary 8.4. *Suppose there exist $\gamma > 0$ and PI operators $\mathcal{P}, \mathcal{Z}$ with $\mathcal{P} \succeq \eta I$, such that*

$$\begin{bmatrix} -\gamma & D_1 & (\mathcal{C}\mathcal{P} + D_2\mathcal{Z})\mathcal{T}^* \\ D_1^T & -\gamma & \mathcal{B}_1^* \\ \mathcal{T}(\mathcal{Z}^* D_2^T + \mathcal{P}\mathcal{C}^*) & \mathcal{B}_1 & \mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* + \mathcal{B}_2\mathcal{Z}\mathcal{T}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}_2^* \end{bmatrix} \preceq 0$$

Then:

1. *(Less conservative LPI) There exist PI operators $\mathcal{Q}, \mathcal{Z}_Q$ and $\mathcal{R}$ with $\mathcal{R} \succeq 0$, such that $\mathcal{T}\mathcal{Q} = \mathcal{Q}^*\mathcal{T}^* = \mathcal{R}$ and*

$$\begin{bmatrix} -\gamma & D_1 & \mathcal{C}\mathcal{Q} + D_2\mathcal{Z}_Q \\ D_1^T & -\gamma & \mathcal{B}_1^* \\ \mathcal{Q}^*\mathcal{C}^* + \mathcal{Z}_Q^* D_2^T & \mathcal{B}_1 & \mathcal{Q}^*\mathcal{A}^* + \mathcal{A}\mathcal{Q} + \mathcal{B}_2\mathcal{Z}_Q + \mathcal{Z}_Q^*\mathcal{B}_2^* + \eta(\mathcal{T}\mathcal{A}^* + \mathcal{A}\mathcal{T}^*) \end{bmatrix} \preceq 0.$$

2. *For any $w \in L_2$, if z satisfies Equation (8.5) for some $\underline{\mathbf{x}}$ and $u = \mathcal{Z}_Q\mathcal{Q}^{-1}\underline{\mathbf{x}} = \mathcal{Z}\mathcal{P}^{-1}\underline{\mathbf{x}}$, then $\|z\| \leq \gamma \|w\|$.*

Proof. The proof follows from the proof of Corollary 7.7. □

Following suit from section 6.3, we will again use Young's Lemma for PI operators to handle the case of boundary control – i.e., when $\mathcal{T}_u \neq 0$.

H_∞-optimal control of PIEs with $\mathcal{T}_u \neq 0$

ODE-PDE with inputs at the boundary, necessarily, have the PIE form given by

$$\begin{aligned} \mathcal{T}\dot{\mathbf{v}}(t) + \mathcal{T}_u\dot{u}(t) &= \mathcal{A}\mathbf{v}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \\ z(t) &= \mathcal{C}_1\mathbf{v}(t) + \mathcal{D}_{11}w(t) + \mathcal{D}_{12}u(t), \end{aligned} \tag{8.6}$$

where z is the regulated output, w is the disturbance, and u is the input at the boundary. If a state-feedback controller of the form $u(t) = \mathcal{K}\mathbf{v}(t)$ is used then the system is written in the form

$$\begin{aligned}(\mathcal{T} + \mathcal{T}_u\mathcal{K})\, \dot{\mathbf{v}}(t) &= (\mathcal{A} + \mathcal{B}_2\mathcal{K})\, \mathbf{v}(t) + \mathcal{B}_1 w(t),\\ z(t) &= (\mathcal{C}_1 + \mathcal{D}_{12}\mathcal{K})\, \mathbf{v}(t) + \mathcal{D}_{11} w(t).\end{aligned} \tag{8.7}$$

The dual PIE for Equation (8.7) is then given by

$$\begin{aligned}(\mathcal{T} + \mathcal{T}_u\mathcal{K})^*\, \dot{\bar{\mathbf{v}}}(t) &= (\mathcal{A} + \mathcal{B}_2\mathcal{K})^*\, \bar{\mathbf{v}}(t) + (\mathcal{C}_1 + \mathcal{D}_{12}\mathcal{K})^*\, \bar{w}(t),\\ z(t) &= \mathcal{B}_1^*\bar{\mathbf{v}}(t) + \mathcal{D}_{11}^*\bar{w}(t).\end{aligned} \tag{8.8}$$

Theorem 8.5. *(LPI for H_∞ Optimal Boundary Controller Synthesis:) Suppose there exist $\epsilon > 0, \gamma > 0$, bounded linear operators $\mathcal{P} : \mathbb{R}L_2^{m,n}[a,b] \to \mathbb{R}L_2^{m,n}[a,b]$ and $\mathcal{Z}$: $\mathbb{R}L_2^{m,n}[a,b] \to \mathbb{R}^p$, such that $\mathcal{P}$ is self-adjoint, coercive and*

$$\begin{bmatrix} -\Gamma & \mathbf{D} & \mathbf{C} \\ (\cdot)^* & -\mathbf{P} & \mathbf{Z} \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + (\mathcal{T}\mathcal{P}\mathcal{A}^* + +\mathcal{T}_u\mathcal{Z}\mathcal{A}^* + \mathcal{T}\mathcal{Z}^*\mathcal{B}_2^*) \end{bmatrix} \preccurlyeq 0 \tag{8.9}$$

$$\Gamma = \begin{bmatrix} \gamma & -\mathcal{D}_{11}^* \\ -\mathcal{D}_{11} & \gamma \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathcal{B}_1^* \\ \mathcal{C}_1\left(\mathcal{P}\mathcal{T}^* + \mathcal{Z}^*\mathcal{T}_u^*\right) + \mathcal{D}_{12}\mathcal{Z}\mathcal{T}^* \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0\; 0 & 0 \\ 0\; 0 & \mathcal{D}_{12}\mathcal{Z} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \sqrt{2}\mathcal{Z}^*\mathcal{T}_u^* \\ \mathcal{Z}^*\mathcal{B}_2^* \\ 0 \end{bmatrix}.$$

Then, for any $w \in L_2$, for $u(t) = \mathcal{K}\underline{\mathbf{x}}(t)$ where $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, any $\mathbf{v}$ and z that satisfy the PIE Equation (8.7) *also satisfy $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.*

Proof. This can be proved by defining a Lyapunov function

$$V(t) = \left\langle (\mathcal{T} + \mathcal{T}_u\mathcal{K})^*\, x(t), \mathcal{P}\, (\mathcal{T} + \mathcal{T}_u\mathcal{K})^*\, x(t) \right\rangle_H$$

and calculating the time derivative $\dot{V}(t)$ along the solutions of the PIE Equation (8.7). Finally, using Lemmas 6.10 and 6.11, by substituting $\mathcal{KP} = \mathcal{Z}$ and $\dot{V}$, we can show that the inequality Equation (8.9) implies

$$\dot{V}(t) - \gamma \left\| w(t) \right\|^2 + \frac{1}{\gamma} \left\| z(t) \right\|^2 < 0.$$

The above inequality can be integrated with respect to t to prove the claims of the Theorem.

□

8.3.2 A Note on Boundary Control

When the control input enters the dynamics of a PDE through the boundary conditions (e.g., $\mathbf{x}(t,0) = u(t)$), novel questions arise that are not readily apparent in the PDE representation but are made explicit when using the PIE framework. These questions arise because PDEs with distributed states are partly rigid – i.e., they are constrained by the continuity properties (e.g., $\mathbf{x}(t,\cdot) \in W_2$) necessary for boundary values to be well defined. The simplest illustration of this is the heat equation with boundary conditions $\mathbf{x}(t,0) = u_1(t)$ and $\partial_s \mathbf{x}(t,0) = u_2(t)$. In this case, we have the relationship

$$\mathbf{x}(t,s) = u_1(t) + s u_2(t) + \int_0^s (s-\eta) \partial_s^2 \mathbf{x}(t,\eta) d\eta$$

which implies that the effect of the input is felt *immediately* throughout the distributed state and is NOT filtered through the dynamics (as is the case in ODEs or in-domain control). If we integrate this type of *semi-algebraic* relationship into the dynamics, we obtain a unitary PIE representation of the heat equation as follows.

$$\partial_t \left(\int_0^s (s-\eta) \partial_s^2 \mathbf{x}(t,\eta) d\eta \right) = \partial_s^2 \mathbf{x}(t,s) - \dot{u}_1(t) - s \dot{u}_2(t)$$

In this representation, the partially algebraic nature of the boundary conditions is made explicit in that the dependence is not on u_1, u_2, but on their time-derivatives.

This type of dependence is allowed in the parametrization of PIE but is not included in detail since the resulting controller synthesis problem is either nonconvex or conservative. One reason is that there is a valid argument to be made that such types of control are unrealistic in that they do not account for the inertia of the beam (or whatever the distributed state happens to be, assuming it has inertia) and, hence, such inputs would be better modeled by filtering through an ODE which represents the dynamics of the actuator. The other reason is that if we are searching for an H_∞-optimal controller, then we are trying to minimize the gain from $\|w\|_{L_2}$ to $\|z\|_{L_2}$ and, if we were to include the derivative $\dot{w}$, this implies that the natural norm for w is the Sobolev norm – an approach taken in, e.g., (Curtain and Zwart, 1995, Thm. 3.3).

Therefore, to account for the case of inputs at the boundary, we will assume that the actual disturbance or input signal is not w or u, but rather $\dot{w}, \dot{u}$ which we can relabel as w_d, u_d. This approach allows us to take any PIE optimal control problem involving time derivatives of the inputs and reformulate it as a PIE free of such derivatives. Specifically, if we are given a PIE representation of the form

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C} & D_1 & D_2 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \\ u(t) \end{bmatrix} + \mathcal{B}_{1d}\dot{w}(t) + \mathcal{B}_{2d}\dot{u}(t),$$

then we will augment the state $\underline{\mathbf{x}}(t) \to \begin{bmatrix} w(t) \\ u(t) \\ \underline{\mathbf{x}}(t) \end{bmatrix}$ and redefine the PIE system as

$$\begin{bmatrix} \overbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{T} \end{bmatrix}}^{\hat{\mathcal{T}}} \begin{bmatrix} \dot{w}(t) \\ \dot{u}(t) \\ \dot{\underline{\mathbf{x}}}(t) \end{bmatrix} \\ z(t) \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{A} \end{bmatrix}}^{\hat{\mathcal{A}}} & \overbrace{\begin{bmatrix} I \\ 0 \\ \mathcal{B}_{1d} \end{bmatrix}}^{\hat{\mathcal{B}}_1} & \overbrace{\begin{bmatrix} 0 \\ I \\ \mathcal{B}_{2d} \end{bmatrix}}^{\hat{\mathcal{B}}_2} \\ \overbrace{\begin{bmatrix} D_1 & D_2 & \mathcal{C} \end{bmatrix}}^{\hat{\mathcal{C}}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} w(t) \\ u(t) \\ \underline{\mathbf{x}}(t) \end{bmatrix} \\ w_d(t) \\ u_d(t) \end{bmatrix}$$

which is now of the form in Equation (4.1) using the parameters $\hat{\mathcal{T}}$, $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}_1$, etc. Numerical examples of such boundary control problems are included in Section 8.4 as Examples 8.2 and 8.3.

8.4 Numerical Examples

All the numerical tests in this section are performed using PIETOOLS toolbox in MATLAB. The standard process of using PIETOOLS includes: a) defining the GPDE using the parser; b) conversion of GPDE to its PIE representation; and c) setting up and solving the LPI optimization problem for the PIE (specifically, H_∞-optimal controller in this case using the `lpisolve()` function). Unless stated otherwise, all of the following tests were performed using `lpisettings('heavy')`. For more details on the PIETOOLS functions and settings, refer Shivakumar *et al.* (2021).

We apply the LPI from Theorem 8.3 to find H_∞-optimal state-feedback controllers for 3 PDEs: 1) the Euler-Bernoulli beam equation with in-domain actuation; 2) a reaction-diffusion PDE with actuation at the boundary; and 3) the wave equation with actuation at the boundary.

Table 8.1 summarizes the achievable closed-loop L_2-gain (H_∞-norm) for monomial bases of order $n = 1, 2, 3, 4$ where the order is as defined in Equation (2.7). This table also includes the computation time required to obtain the resulting controllers where the solutions were obtained on a desktop computer with Intel Core *i7-5960X* CPU and 64GB DDR4 RAM. In addition, each example includes the PI operators defining the associated PIE representation (as determined by PIETOOLS) and the resulting controller gains (as calculated for order $n = 3$). For examples 19) and 20), the closed-loop response was simulated for a test disturbance using PIESIM Peet and Peet (2020) to verify the closed-loop L_2-gain bound is satisfied. Finally, in each case, the achievable L_2-gains were compared to those achievable using standard LMIs for state-feedback as applied to an ODE approximation of the PDE. In each case, this ODE was obtained using a simple finite difference (FD) approximation scheme where a 2^{nd}-order central difference approximation was used for 2^{nd}-order spatial derivatives and a first-order forward difference was used for time derivatives.

Example 8.1 (Euler-Bernoulli Beam equation)**.** *In Example 5.3, we formulated the problem of optimal control of an Euler-Bernoulli (EB) beam model as follows:*

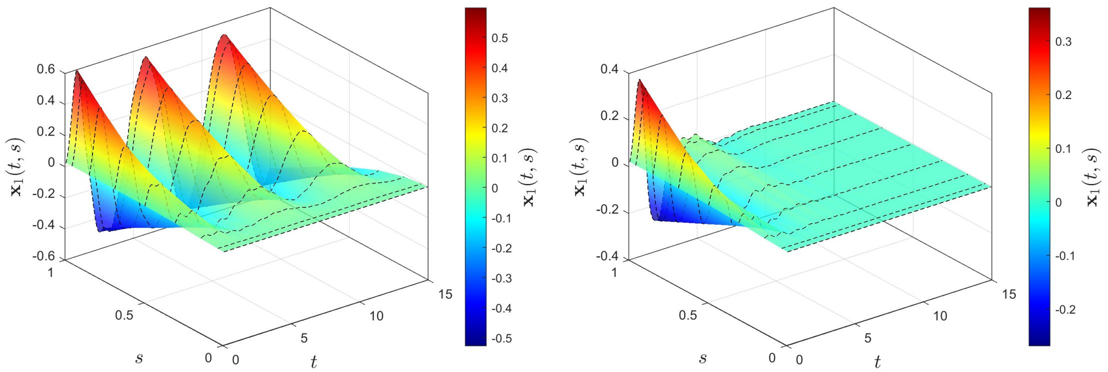
$$\dot{\mathbf{x}}(t,s) = \begin{bmatrix} 0 & -0.1 \\ 1 & 0 \end{bmatrix} \partial_s^2 \mathbf{x}(t,s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

$$z(t) = \begin{bmatrix} u(t) \\ \int_0^1 \begin{bmatrix} 0 & (0.5(1-s-s^2) \end{bmatrix} \mathbf{x}(t,s) ds \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t,0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \partial_s \mathbf{x}(t,0) = 0,$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \partial_s^2 \mathbf{x}(t,1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \partial_s^3 \mathbf{x}(t,1) = 0.$$

The parameters of the PIE representation associated with this PDE are given in Example 5.3. Solving the LPI in Corollary 8.4, we find the H_∞-optimized state-feedback



(a) Response with $u(t) = 0$ (b) Response with $u(t) = \mathcal{K}\mathbf{x}(t)$

Figure 8.2: Ex. 8.1: Surface plot of open loop (a) and closed-loop response (b) of $\mathbf{x}_1(t, s)$, as defined in Ex. 8.1 with disturbance $w(t) = \sin(3t)exp(-t)$ and control input $u(t) = \mathcal{K}\mathbf{x}(t)$ with $\mathcal{K}$ defined in Example 8.1.

controller to be $u(t) = \mathcal{K}\mathbf{x}(t)$*, where*

$$\mathcal{K}\mathbf{x} = \int_0^1 \begin{bmatrix} Q_a(s) & Q_b(s) \end{bmatrix} \partial_s^2 \mathbf{x}(s) ds,$$

$$Q_a(s) = 6.61s^5 - 16.6s^4 + 14.5s^3 - 7.43s^2 + 3.99s - 1.47,$$

$$Q_b(s) = 0.68s^5 - 1.72s^4 + 1.59s^3 - 0.87s^2 + 0.04s + 0.003.$$

The upper bound on the H_∞*-norm of the corresponding closed-loop PDE obtained from the LPI in Corollary 8.4 is* 0.73*. The simulated* L_2*-gain under disturbance* $w(t) = \sin(3t)exp(-t)$ *is* 0.1312 *which verifies the bound. The closed-loop* H_∞*-norm bound using an ODE approximation of the PDE is* 0.1030*. In Figure 8.2a, Figure 8.2b and Figure 8.3, we plot the system response for a disturbance* $w(t) = \sin(3t)exp(-t)$ *with the zero initial conditions.*

Example 8.2. *Consider the following optimal boundary control problem of a reaction-*

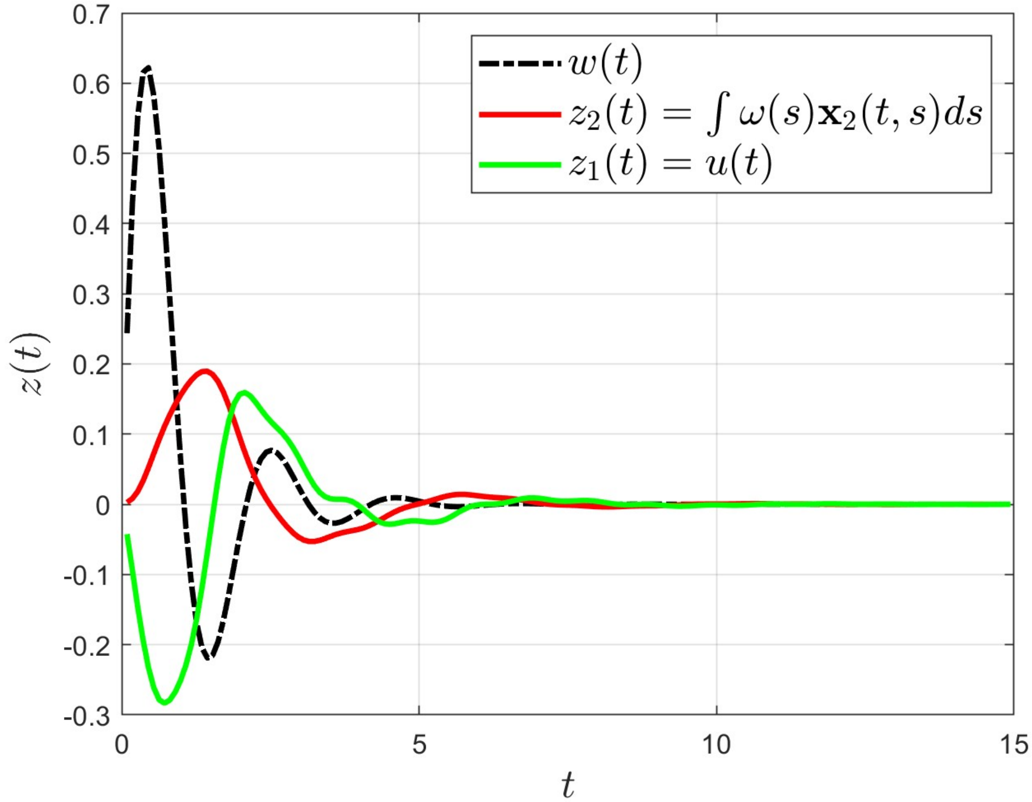


Figure 8.3: Ex. 8.1: Plot of output $z_2(t) = \int_0^1 0.5(1 - s - s^2)\mathbf{x}_2(t,s)ds$ and $z_1 = u(t)$ against t under the disturbance $w(t) = \sin(3t)exp(-t)$ is presented above.

diffusion PDE:

$$\dot{\mathbf{x}}(t,s) = 5\mathbf{x}(t,s) + \partial_s^2\mathbf{x}(t,s) + w(t), \;\; \dot{x}(t) = u(t),$$
$$z(t) = \begin{bmatrix} x(t) \\ \int_0^1 \mathbf{x}(t,s)ds \end{bmatrix}, \;\; \mathbf{x}(t,0) = 0, \;\; \partial_s\mathbf{x}(t,1) = x(t),$$

The parameters of the PIE representation associated with this PDE are

$$\mathcal{T} = \mathit{\Pi}\left[\begin{array}{c|c} 1 & 0 \\ \hline s & \{0, -\theta, -s\} \end{array}\right], \mathcal{A} = \mathit{\Pi}\left[\begin{array}{c|c} 0 & 0 \\ \hline 5s & \{1, -5\theta, -5s\} \end{array}\right],$$

$$\mathcal{B}_1 = \mathit{\Pi}\left[\begin{array}{c|c} 0 & \emptyset \\ \hline 1 & \{\emptyset\} \end{array}\right], \mathcal{B}_2 = \mathit{\Pi}\left[\begin{array}{c|c} 1 & \emptyset \\ \hline 0 & \{\emptyset\} \end{array}\right],$$

$$\mathcal{C} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} & \begin{bmatrix} 0 \\ 0.5s^2 - s \end{bmatrix} \\ \hline \emptyset & \{\emptyset\} \end{array}\right].$$

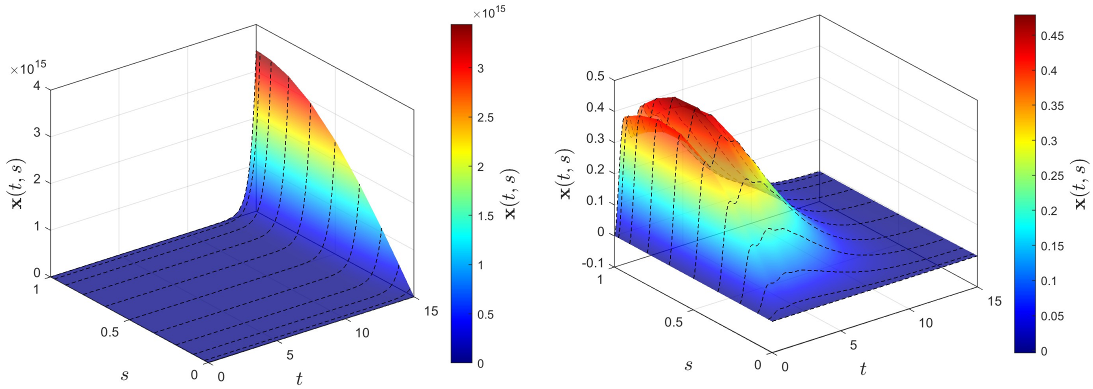
Solving the LPI in Corollary 8.4, we find the H_∞-optimized state-feedback controller to be $u(t) = \mathcal{K}\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(t)$, where

$$\mathcal{K}\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} = -6.71x + \int_0^1 K(s)\partial_s^2 \mathbf{x}(s)ds,$$

$$K(s) = (-11.68s^8 + 44.23s^7 - 65.93s^6 + 49.38s^5 - 19.82s^4 + 4.27s^3 - 0.46s^2 + 0.02s - 0.0002) \cdot 10^3.$$

The upper bound on the H_∞-norm of the corresponding closed-loop PDE obtained from the LPI in Corollary 8.4 is 4.99. *The simulated L_2-gain under disturbance $w(t) = \sin(3t)exp(-t)$ is* 1.8905 *which verifies the bound. The closed-loop H_∞-norm bound using an ODE approximation of the PDE is* 3.441. *In Figures 8.4a, 8.4b and 8.5, we plot the system response for a disturbance $w(t) = \sin(5t)exp(-t)$ with the zero initial conditions.*

Example 8.3 (Wave equation)**.** *Consider the following optimal boundary control*



(a) Response with $u(t) = 0$

(b) Response with $u = \mathcal{K}\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}$

Figure 8.4: Ex. 8.2: Surface plot of open loop (a) and closed-loop response (b) of $\mathbf{x}(t, s)$, as defined in Example 8.1 with disturbance $w(t) = \sin(5t)exp(-t)$ and control input $u(t) = \mathcal{K}\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(t)$ with $\mathcal{K}$ defined in Example 8.2.

problem of a wave equation:

$$\ddot{\eta}(t,s) = \partial_s^2\eta(t,s) + w(t),\;\; \dot{x}(t) = u(t)$$
$$z(t) = \begin{bmatrix} x(t) \\ \int_0^1 \eta(t,s)ds \end{bmatrix},\;\; \eta(t,0) = 0,\;\; \partial_s\eta(t,1) = x(t).$$

We change the state variable to $\mathbf{x} = col(\eta, \dot{\eta})$ *to obtain*

$$\dot{\mathbf{x}}(t,s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x}(t,s) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\partial_s^2\mathbf{x}(t,s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}w(t),\;\; \dot{x}(t) = u(t),$$
$$z(t) = \begin{bmatrix} x(t) \\ \int_0^1 \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(t,s)ds \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}(t,0) \\ \mathbf{x}(t,1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}x(t).$$

We omit the parameters of the PIE associated with this PDE. Solving the LPI in Corollary 8.4, we find the H_∞*-optimized state-feedback controller to be* $u(t) = \mathcal{K}\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(t)$,

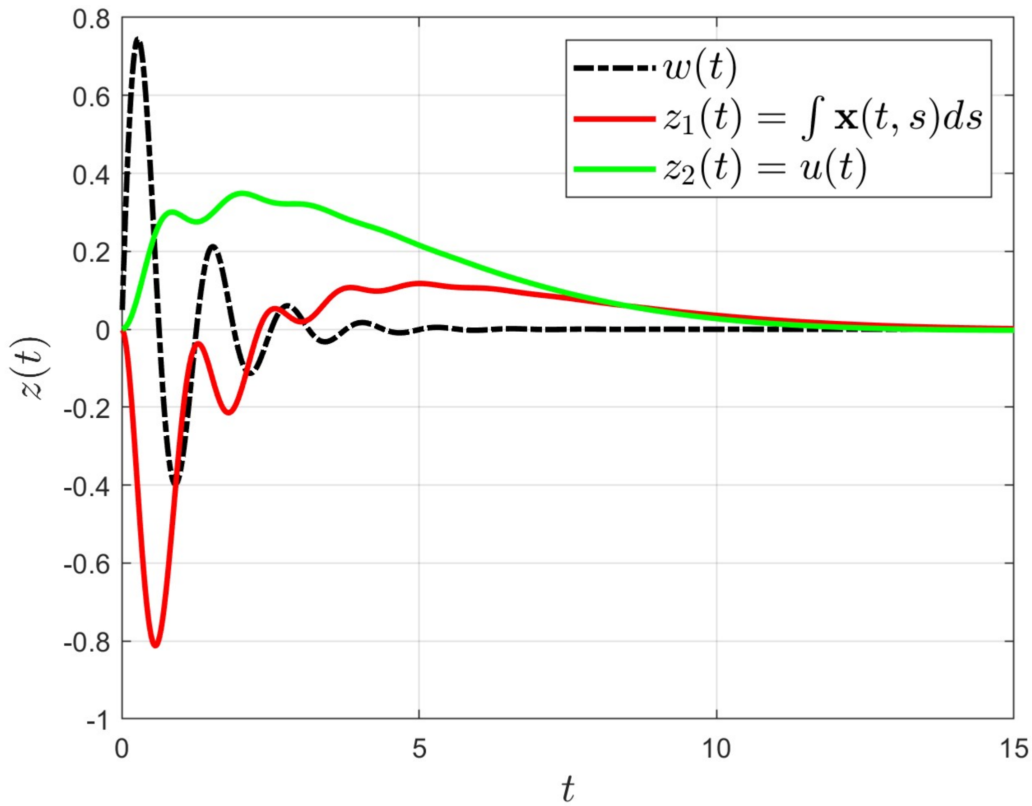


Figure 8.5: Ex. 8.2: Outputs $z_1(t) = \mathbf{x}(t,1)$ and $z_2(t) = u(t)$ under the disturbance $w(t) = \sin(5t)exp(-t)$ are shown above.

where

$$\mathcal{K}\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} = -0.17x + 10^{-2}\int_0^1 \begin{bmatrix} Q_1(s) & Q_2(s) \end{bmatrix} \begin{bmatrix} \partial_s^2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(s)ds$$

$$Q_1(s) = 0.5s^8 - 2s^7 + 3s^6 - 2s^5 - 30s^4 + 60s^3 - 70s^2 + 20s - 0.8,$$

$$Q_2(s) = 0.2s^8 - 0.7s^7 + 0.06s^6 + 0.6s^5 - 5s^4 - 20s^3 + 80s^2 - 2s - 40.$$

The upper bound on the H_∞-norm of the corresponding closed-loop PDE obtained

Degree, n	1	2	3	4
Ex. 8.1	3.29 (6.2)	0.89 (15)	0.73 (30.5)	0.66 (67.4)
Ex. 8.2	7.86 (4.6)	5.11 (5.5)	4.59 (9.5)	4.25 (13.9)
Ex. 8.3	0.65 (10.9)	0.64 (27.1)	0.64 (49.5)	0.639 (87.5)

Table 8.1: In this table, we list the lower bound on H_∞-norm for examples in Section 8.4 by solving the LPI in Theorem 8.3 whose decision variables are parameterized as $\mathcal{P} = \mathcal{Z}_n^* Q_p \mathcal{Z}_n$, $\mathcal{Z} = Q_z \mathcal{Z}_n$, and $Q_p \geq 0$, Q_z are matrices. The values within parentheses correspond to the total CPU runtime, in seconds, for solving the H_∞-optimal state-feedback problem — i.e., time for setting up the LPIs, solving the LPIs, and controller reconstruction.

from the LPI in Corollary 8.4 is 0.64*.*

Lastly, from the estimates in Table 8.1, we conclude that the upper bound on the H_∞-norm of the controller has converged since increasing the order n does not cause a significant change in the bound.

In the next example, we use Theorem 6.12 to find a boundary control law for a PDE. However, note that, due to the conservatism the optimization problem was solvable only when the stability parameter λ of the reaction-diffusion PDE was close to stable values. For larger values of λ the LPI was unable to find a feasible solution for low-order polynomial parametrization of the decision variables $\mathcal{P}$ and $\mathcal{Z}$.

Example 8.4. *Consider the reaction-diffusion PDE*

$$\begin{aligned} &\dot{\mathbf{x}}(t,s) = \lambda \mathbf{x}(t,s) + \partial_s^2 \mathbf{x}(t,s) \\ &\mathbf{x}(t,0) = 0, \quad \mathbf{x}(t,1) = u(t), \quad z(t) = \begin{bmatrix} u \\ \int_0^1 \mathbf{x}(t,s) ds \end{bmatrix}. \end{aligned}$$

λ	Filtered Boundary Control	Unfiltered Boundary control
10	0.9334	8.023
12	5.696	27.687
15	2.83×10^4	∞
30	∞	∞

Table 8.2: In this table for the reaction-diffusion PDE in Example 8.4, we list the H_∞-norm bounds for controllers obtained by two different methods as the PDE becomes increasingly unstable, i.e., λ increases. The second column shows the filtered controller approach where the control input is fed through an ODE at the boundary, whereas the third column shows a boundary controller designed using the LPI in Theorem 8.5. The cases listed as ∞ are those for which the optimization problem did not yield a feasible solution.

This time, we use an unfiltered boundary input to stabilize the system instead of the filtered boundary controller used in the previous examples. The goal of this exercise was to determine if the conservative LPI derived in Theorem 8.5 can perform better than the dynamic boundary controller approach. We will try to find a control law $u(t) = \int_a^b K(s)\partial_s^2\mathbf{x}(t,s)ds$ *by solving the LPI in Theorem 8.5. Preliminary tests indicate that the conservatism of Young's inequality LPI is too high. A stabilizing static boundary controller could not be found for the PDE for large* λ *as documented below.*

8.5 Conclusions

In this chapter, we have used the set of duality results to solve the H_∞-optimal state observer and state-feedback controller synthesis for GPDEs. Using the primal LPI formulations of the stability and L_2-gain, we solve the H_∞-optimal state observer design problem for GPDEs. Likewise, using the dual LPI formulations of the stability and L_2-gain LPIs, we solved the problem of stabilizing and H_∞-optimal state-feedback controller synthesis. Finally, numerical testing is used to verify the theorems and obtain observers/controllers with provable H_∞-norm bounds. The numerical results show no apparent sub-optimality in the resulting observer/controller gains or H_∞ bounds.

Although we presented a possible remedy to the non-convexity in the optimal boundary feedback problem using Young's inequality, numerical tests indicate that the conservatism introduced by the use of Young's inequality is significant even in the case of a simple reaction-diffusion equation, however, alternative findings on time-delay systems indicate that transport equation-type PDEs do not experience high conservatism. However, more analysis is required for an in-depth insight and to make provable claims.

Chapter 9

CONCLUSIONS

Motivated by the challenges in computational analysis, estimation, and control posed by the 3-constraint representation of PDE systems, we proposed an alternative class of systems to represent PDE models called the class of Partial Integral Equations – a class of systems parameterized by the $*$-algebra of Partial Integral operators. We showed how PIEs are a natural extension of linear state-space ODE representation to infinite dimensional systems and show that such a representation overcomes all the challenges of the 3-constraint PDE model. The main contributions and insights of this thesis can be divided into two parts that coincide with the Parts of this thesis.

In Part I, **Representation and Parametrization of Linear Infinite dimensional Systems**, we considered a generalized class of coupled ODE-PDE models (GPDEs) which can be used to define analysis, simulation, and optimal control/estimation problems. This generalized class allows for inputs and outputs which enter through the limit values of the GPDE model through the in-domain dynamics of the PDE subsystem and a coupled ODE. The GPDE class allows for integral constraints on the PDE state. Additionally, we may model integrals of the PDE state acting: on the PDE dynamics, on a coupled ODE, or the outputs of the system. Finally, this class includes PDE models with n^{th}-order spatial derivatives. The GPDE model unifies several existing classes of PDE models in a single parameterized framework. Despite that, we showed that the parameter set for GPDEs changes with the type of PDE, order of differentiation, and boundary conditions– unsuitable for building algorithms for computational analysis and control.

Having parameterized a broad class of coupled ODE-PDE models, we proposed a test for the admissibility of a given GPDE model. We showed that admissibility implies the existence of an associated Partial Integral Equation (PIE) representation of the GPDE model with a unitary map from the state of the PIE system to the state of the GPDE model. Furthermore, we have shown that the unitary map from PIE to GPDE state is a PI operator. This is analogous to the invertible coordinate transforms used on linear state-space ODE systems. Using this analogy, we have shown that many properties of the GPDE model and associated PIE system are equivalent – including the existence of solutions, input-output properties, internal stability, and controllability.

In Part II, **Analysis, Estimation, and Control of GPDEs**, we presented notions of stability, stabilizability, and detectability for the class of PIE systems. Using these definitions, we successfully replicated the duality properties seen in linear state-space ODE systems. Specifically, we showed that every PIE system has a dual PIE system with equivalent properties such as internal stability and input-output L_2-gain. Using this duality property, we proposed primal and dual formulations of a test for internal stability, Bounded-real Lemma, and Positive-real Lemma for PIEs. Then, using PI operators, we reformulated these tests as convex PI operator-valued optimization problems called LPIs.

Similar to the idea of tightening positive polynomial constraints by using SOS polynomial constraints, we proposed a quadratic form to parametrize positive PI operators using positive matrices and PI operator bases. Consequently, the solution set of the LPI optimization problems can be tightened to a set defined by the cone of positive PI operators and solved using semidefinite programming.

We proposed an iterative method to invert positive PI operators, thus enabling the reconstruction of H_∞-optimal observer and controller gains from the solution of an LPI optimization problem. In addition, we looked at less conservative versions of the various proposed LPIs and demonstrated their benefit in practice through numerical examples.

Lastly, to aid in applying the proposed GPDE models and PIE conversion formulae, we built efficient open-source software (PIETOOLS) for constructing the GPDE model, conversion to the PIE system, simulation of the GPDE/PIE, and analysis/control of the GPDE/PIE. This software includes a GUI for the construction of GPDE models and conversion to an associated PIE system – a feature demonstrated on several example problems.

We want to note that although this framework has addressed many unresolved computational problems in the control theory of PDEs, as seen in Section 8.4, the boundary control problem for PDEs is still unresolved because the LPIs proposed for the boundary control problem seem to be very conservative and infeasible in practice. Furthermore, possible extensions of the results in this work to GPDEs that do not admit a PIE representation are unclear and an open question—an example of such inadmissible systems is a PDE with periodic boundary conditions. Furthermore, it is unclear if PDEs with less regular solutions have an equivalent PIE representation. Even if they do, the equivalence of solutions and other system properties may not hold.

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APPENDIX A

EQUIVALENCE IN REPRESENTATION OF A GPDE AND ITS ASSOCIATED PIE

In these appendices, we provide proofs for all theorems, lemmas, and corollaries stated in the paper along with several intermediate lemmas. In Appendix A.1, the goal is to prove Theorem 5.1 - the map between the domain of the PIE subsystem and the domain of the PDE subsystem. In Appendix A.2, we prove equivalence of solutions for the PIE subsystem and PDE subsystem. In Appendix A.3 we construct the map between the domain of the GPDE and associated PIE representation. In Appendix A.4, we prove equivalence of solutions of the GPDE and associated PIE system. In Appendices A.5.1 to A.5.3, we prove that map from PIE to GPDE state is unitary and that internal stability of PIE and GPDE model is equivalent. Finally, in Appendices B.1 and B.1.1 we show that the Partial Integral (PI) operators form a $*$-algebra and provide formulae for composition, adjoint, and concatenation of PI operators.

A.1 Bijective Map Between PIE and PDE States

To find a map between the *fundamental* state (state of the PIE) and the *primal* state (state of the PDE subsystem), we will use the Fundamental Theorem of Calculus (FTC) and the BCs. First, we recall the FTC and extend it to vector-valued functions on the interval $[a, b]$ as shown below.

Lemma 5.2. *Suppose $\mathbf{x} \in W_N^n[a,b]$ for any $N \in \mathbb{N}$. Then*

$$\mathbf{x}(s) = \mathbf{x}(a) + \sum_{j=1}^{N-1} \frac{(s-a)^j}{j!} \partial_s^j \mathbf{x}(a) + \int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!} \partial_s^N \mathbf{x}(\theta) d\theta$$

where $\partial_s^i \mathbf{x}$ is the ith classical-derivative of $\mathbf{x}$ when $i < N$ and weak-derivative for $i = N$.

Proof. We prove this using the principle of induction. Suppose the lemma is true for some N and $\mathbf{x} \in C_{N+1}^n[a,b]$. Because the lemma is true for N, we have

$$\mathbf{x}(s) = \mathbf{x}(a) + \sum_{j=1}^{N-1} \frac{(s-a)^j}{j!} (\partial_s^j \mathbf{x})(a) + \int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!} (\partial_s^N \mathbf{x})(\theta) d\theta. \tag{A.1}$$

Now, by the FTC, we have

$$\partial_s^N \mathbf{x}(s) = (\partial_s^N \mathbf{x})(a) + \int_a^s (\partial_s^{(N+1)} \mathbf{x})(\theta) d\theta.$$

Next, we substitute the above identity into Equation (A.1), and using the integral identity

$$\int_a^b \int_a^\theta f(\theta,\eta) d\eta d\theta = \int_a^b \int_\eta^b f(\theta,\eta) d\theta d\eta$$

we have

$$\mathbf{x}(s) = \mathbf{x}(a) + \sum_{j=1}^{N-1} \frac{(s-a)^j}{j!} \partial_s^j \mathbf{x}(a) + \int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!} \left(\partial_s^N \mathbf{x}(a) + \int_a^\theta \partial_s^{(N+1)} \mathbf{x}(\eta) d\eta \right) d\theta.$$

While the first two terms are close to the required form, the last term (the integral term) is not and can be simplified by using integration by parts. Then, from the integral term we get

$$\begin{aligned}
&\int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!}\left(\partial_s^N \mathbf{x}(a) + \int_a^\theta \partial_s^{(N+1)}\mathbf{x}(\eta)d\eta\right)d\theta \\
&= \left(\int_a^s \frac{(s-\theta)^{N-1}}{(N-1)!}d\theta\right)\partial_s^N \mathbf{x}(a) + \int_a^s\int_a^\theta \frac{(s-\theta)^{N-1}}{(N-1)!}\partial_s^{(N+1)}\mathbf{x}(\eta)d\eta\, d\theta \\
&= \frac{(s-a)^N}{N!}\partial_s^N \mathbf{x}(a) + \int_a^s\left(\int_\eta^s \frac{(s-\theta)^{N-1}}{(N-1)!}d\theta\right)\partial_s^{(N+1)}\mathbf{x}(\eta)d\eta \\
&= \frac{(s-a)^N}{N!}\partial_s^N \mathbf{x}(a) + \int_a^s \frac{(s-\eta)^N}{N!}\partial_s^{(N+1)}\mathbf{x}(\eta)d\eta.
\end{aligned}$$

Finally, by substituting the above terms back into the equation we get,

$$\mathbf{x}(s) = \mathbf{x}(a) + \sum_{j=1}^{N}\frac{(s-a)^j}{j!}\partial_s^j \mathbf{x}(a) + \int_a^s \frac{(s-\eta)^N}{N!}\partial_s^{(N+1)}\mathbf{x}(\eta)d\eta.$$

Therefore, if the statement of the lemma is true for N, then it is also true for $N+1$. Clearly, the lemma is true for $N=1$. $\square$

We can extend Lemma 5.2 to obtain an expression for the derivatives of $\mathbf{x} \in C_N^n$ in terms of $\partial_s^N \mathbf{x}$ and of a given set of *core* boundary values of $\mathbf{x}$.

Lemma A.1. *Suppose $\mathbf{x} \in W_N^n$. Then, for any $i < N$, we have*

$$(\partial_s^i \mathbf{x})(s) = \sum_{j=i}^{N-1}\tau_{j-i}(s-a)(\partial_s^j \mathbf{x})(a) + \int_a^s \tau_{N-i-1}(s-\theta)(\partial_s^N \mathbf{x})(\theta)d\theta$$

where $\tau_i(s) = \frac{s^i}{i!}$.

Proof. First note that $\tau_i(0) = 0$ for any $i > 0$, $\partial_s \tau_0(s) = 0$ and

$$\tau_i(s) = \frac{s^i}{i!} \qquad \rightarrow \qquad \partial_s \tau_i(s) = i\frac{s^{i-1}}{i!} = \tau_{i-1}(s)$$

and suppose the formula holds for $i-1 \geq 0$. Then

$$(\partial_s^{i-1}\mathbf{x})(s) = \sum_{j=i-1}^{N-1}\tau_{j-i+1}(s-a)(\partial_s^j \mathbf{x})(a) + \int_a^s \tau_{N-i}(s-\theta)(\partial_s^N \mathbf{x})(\theta)d\theta.$$

and hence, since $\partial_s \tau_0(s) = 0$, we have

$$\begin{aligned}
&(\partial_s^i \mathbf{x})(s) = \partial_s (\partial_s^{i-1}\mathbf{x})(s) \\
&= \sum_{j=i-1}^{N-1} (\partial_s \tau_{j-i+1}(s-a))\,(\partial_s^j \mathbf{x})(a) + \tau_{N-i}(0)(\partial_s^N \mathbf{x})(s) + \int_a^s (\partial_s \tau_{N-i}(s-\theta))\,(\partial_s^N \mathbf{x})(\theta) d\theta. \\
&= \sum_{j=i}^{N-1} (\partial_s \tau_{j-i+1}(s-a))\,(\partial_s^j \mathbf{x})(a) + \tau_{N-i}(0)(\partial_s^N \mathbf{x})(s) + \int_a^s (\partial_s \tau_{N-i}(s-\theta))\,(\partial_s^N \mathbf{x})(\theta) d\theta. \\
&= \sum_{j=i}^{N-1} \tau_{j-i}(s-a)(\partial_s^j \mathbf{x})(a) + \int_a^s \tau_{N-i-1}(s-\theta)(\partial_s^N \mathbf{x})(\theta) d\theta.
\end{aligned}$$

By lemma 5.2, the result holds for $i = 0$, which completes the proof. □

We now propose a mixed-order version of Lemma 5.2

Corollary A.2. *Suppose* $\mathbf{x} \in \prod_{i=0}^N C_i^{n_i}$ *and define*

$$J_{i,j} = \begin{bmatrix} 0_{n_{i:j-1} \times n_{j:N}} \\ I_{n_{j:N}} \end{bmatrix} \in \mathbb{R}^{n_{i:N} \times n_{j:N}}, \quad \tau_i(s) = \frac{s^i}{i!}, \quad T_{i,j}(s) = \tau_{j-i}(s) J_{i,j} \quad j \geq i,$$

$$\mathcal{C}\mathbf{x} = \begin{bmatrix} S\mathbf{x} \\ \partial_s S^2 \mathbf{x} \\ \vdots \\ \partial_s^{N-1} S^N \mathbf{x} \end{bmatrix},$$

we have

$$\begin{aligned}
\begin{bmatrix} \mathbf{x}_1(s) \\ \vdots \\ \mathbf{x}_N(s) \end{bmatrix} &= \begin{bmatrix} T_{1,1}(s-a) & T_{1,2}(s-a) & \cdots & T_{1,N}(s-a) \end{bmatrix} (\mathcal{C}\mathbf{x})(a) \\
&\quad + \int_a^s \begin{bmatrix} \tau_0(s-\theta) I_{n_1} & & \\ & \ddots & \\ & & \tau_{N-1}(s-\theta) I_{n_k} \end{bmatrix} \begin{bmatrix} \partial_\theta \mathbf{x}_1(\theta) \\ \vdots \\ \partial_\theta^N \mathbf{x}_N(\theta) \end{bmatrix} d\theta \\
&= T_1(s-a)(\mathcal{C}\mathbf{x})(a) + \int_a^s Q_1(s-\theta) \begin{bmatrix} \mathbf{x}_0(\theta) \\ \vdots \\ \partial_\theta^N \mathbf{x}_N(\theta) \end{bmatrix} d\theta. \qquad \text{(A.2)}
\end{aligned}$$

Proof. For convenience, let us denote $P_{i,j} \in \mathbb{R}^{n_i \times n_{S_j}}$ to be the uniquely defined 0-1 matrix so that $\mathbf{x}_i(s) = P_{i,j} S^j \mathbf{x}(s)$ and which is given by

$$P_{i,j} = \begin{bmatrix} 0_{n_i \times (n_{S_j} - n_{S_{i-1}})} & I_{n_i} & 0_{n_i \times n_{S_{i+1}}} \end{bmatrix} = \begin{bmatrix} 0_{n_i \times n_{j:i-1}} & I_{n_i} & 0_{n_i \times n_{i+1:N}} \end{bmatrix}.$$

We now use $P_{i,j}$ and the identity from Lemma 5.2 to write $\mathbf{x}_i$ in terms of $(\mathcal{C}\mathbf{x})(a)$ and $\partial_s^i \mathbf{x}_i$. Specifically, if $\mathbf{x}_k \in C_k^{n_k}[a,b]$, then

$$\begin{aligned}
&\mathbf{x}_i(s)\\
&= \sum_{j=0}^{i-1} \tau_j(s-a)\partial_s^j \mathbf{x}_i(a) + \int_a^s \tau_{i-1}(s-\theta)\partial_s^i \mathbf{x}_i(\theta)d\theta\\
&= \sum_{j=0}^{i-1} \tau_j(s-a)P_{i,j+1}\partial_s^j S^{j+1}\mathbf{x}(a) + \int_a^s \tau_{i-1}(s-\theta)\partial_s^i \mathbf{x}_i(\theta)d\theta\\
&= \begin{bmatrix}\tau_0(s-a)P_{i,1} & \cdots & \tau_{i-1}(s-a)P_{i,i}\end{bmatrix} \begin{bmatrix} S\mathbf{x}(a) \\ \vdots \\ \partial_s^{i-1}S^i\mathbf{x}(a)\end{bmatrix} + \int_a^s \tau_{i-1}(s-\theta)\partial_s^i \mathbf{x}_i(\theta)d\theta\\
&= \begin{bmatrix}\tau_0(s-a)P_{i,1} & \cdots & 0_{n_i\times n_{S_{i+1:N}}}\end{bmatrix} \begin{bmatrix} S\mathbf{x}(a) \\ \vdots \\ \partial_s^{N-1}S^N\mathbf{x}(a)\end{bmatrix} + \int_a^s \tau_{i-1}(s-\theta)\partial_s^i \mathbf{x}_i(\theta)d\theta\\
&= \begin{bmatrix}\tau_0(s-a)P_{i,1} & \cdots & \tau_{i-1}(s-a)P_{i,i} & 0_{n_i\times n_{S_{i+1:N}}}\end{bmatrix} (\mathcal{C}\mathbf{x})(a) + \int_a^s \tau_{i-1}(s-\theta)\partial_s^i \mathbf{x}_i(\theta)d\theta.
\end{aligned}$$

Now, we can concatenate the $\mathbf{x}_i$'s to get,

$$\begin{aligned}
&\begin{bmatrix}\mathbf{x}_1(s)\\ \vdots \\ \mathbf{x}_N(s)\end{bmatrix}\\
&= \begin{bmatrix}\tau_0(s-a)P_{1,1} & 0 & 0\\ \vdots & \ddots & 0\\ \tau_0(s-a)P_{N,1} & \cdots & \tau_{N-1}(s-a)P_{N,N}\end{bmatrix}(\mathcal{C}\mathbf{x})(a) + \int_a^s \begin{bmatrix}\tau_0(s-\theta)\partial_s \mathbf{x}_1(\theta)\\ \vdots \\ \tau_{N-1}(s-\theta)\partial_s^N\mathbf{x}_N(\theta)\end{bmatrix}d\theta\\
&= \begin{bmatrix}\tau_0(s-a)\begin{bmatrix}P_{1,1}\\ P_{2,1}\\ \vdots \\ P_{N,1}\end{bmatrix} & \tau_1(s-a)\begin{bmatrix}0\\ P_{2,2}\\ \vdots \\ P_{N,2}\end{bmatrix} & \cdots & \tau_{N-1}(s-a)\begin{bmatrix}0\\ 0\\ \vdots \\ P_{N,N}\end{bmatrix}\end{bmatrix}(\mathcal{C}\mathbf{x})(a)\\
&\qquad + \int_a^s \begin{bmatrix}\tau_0(s-\theta)\partial_s \mathbf{x}_1(\theta)\\ \vdots \\ \tau_{N-1}(s-\theta)\partial_s^N\mathbf{x}_N(\theta)\end{bmatrix}d\theta\\
&= [\tau_0(s-a)J_{1,1} \quad \tau_1(s-a)J_{1,2} \quad \cdots \quad \tau_{N-1}(s-a)J_{1,N}]\,(\mathcal{C}\mathbf{x})(a)\\
&\qquad + \int_a^s \begin{bmatrix}\tau_0(s-\theta) & & \\ & \ddots & \\ & & \tau_{N-1}(s-\theta)\end{bmatrix}\begin{bmatrix}\partial_\theta \mathbf{x}_1(\theta)\\ \vdots \\ \partial_\theta^N \mathbf{x}_N(\theta)\end{bmatrix}d\theta\\
&= [T_{1,1}(s-a) \quad \cdots \quad T_{1,N}(s-a)]\,(\mathcal{C}\mathbf{x})(a)\\
&\qquad + \int_a^s \begin{bmatrix}\tau_0(s-\theta) & & \\ & \ddots & \\ & & \tau_{N-1}(s-\theta)\end{bmatrix}\begin{bmatrix}\partial_\theta \mathbf{x}_1(\theta)\\ \vdots \\ \partial_\theta^N \mathbf{x}_N(\theta)\end{bmatrix}d\theta
\end{aligned}$$

where we have used the fact that for any i

$$\begin{bmatrix} P_{i,i} \\ P_{i+1,i} \\ \vdots \\ P_{N,i} \end{bmatrix} = \begin{bmatrix} [0_{n_i \times n_{i:i-1}} & I_{n_i} & 0_{n_i \times n_{i+1:N}}] \\ [0_{n_{i+1} \times n_{i:i}} & I_{n_{i+1}} & 0_{n_{i+1} \times n_{i+2:N}}] \\ & \vdots & \\ [0_{n_N \times n_{i:N-1}} & I_{n_N} & 0_{n_N \times n_{N+1:N}}] \end{bmatrix} = \begin{bmatrix} I_{n_i} & & & \\ & I_{n_{i+1}} & & \\ & & \ddots & \\ & & & I_{n_N} \end{bmatrix} = I_{n_{i:N}}$$

and hence

$$\begin{bmatrix} 0_{n_{1:i-1} \times n_{i:N}} \\ P_{i,i} \\ \vdots \\ P_{N,i} \end{bmatrix} = \begin{bmatrix} 0_{n_{1:i-1} \times n_{i:N}} \\ I_{n_{i:N}} \end{bmatrix} = J_{1,i}.$$

□

We conclude that it is possible to express any continuously differentiable function $\mathbf{x} \in \prod_{i=1}^{N} C_i^{n_i}$ using left boundary values (at $s = a$) of the continuous partial derivatives ($\mathcal{C}\mathbf{x}$) and the fundamental state $\hat{\underline{\mathbf{x}}} = \mathrm{col}(\mathbf{x}_0, \cdots, \partial_s^N \mathbf{x}_N)$. Since we require a map from the fundamental state $\hat{\underline{\mathbf{x}}}$ to the primal state $\mathbf{x}$, we need to eliminate the left boundary values $(\mathcal{C}\mathbf{x})(a)$. The first step in this direction is to express $\mathcal{C}\mathbf{x}$ in terms of $\hat{\underline{\mathbf{x}}}$ and $(\mathcal{C}\mathbf{x})(a)$.

Corollary A.3. *Suppose* $\mathbf{x} \in \prod_{i=0}^{N} C_i^{n_i}$. *Then, for* T *and* Q *are as defined in Block 5.2, and*

$$(\mathcal{C}\mathbf{x}) = \begin{bmatrix} S\mathbf{x} \\ \partial_s S^2 \mathbf{x} \\ \vdots \\ \partial_s^{N-1} S^N \mathbf{x} \end{bmatrix}, \qquad \hat{\underline{\mathbf{x}}} = \begin{bmatrix} \mathbf{x}_0 \\ \partial_s^1 \mathbf{x}_1 \\ \vdots \\ \partial_s^N \mathbf{x}_N \end{bmatrix},$$

we have

$$(\mathcal{C}\mathbf{x})(s) = T(s-a)(\mathcal{C}\mathbf{x})(a) + \int_a^s Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta) d\theta.$$

Proof. We will use the identity from corollary A.2 and lemma A.1 to find $\partial_s^{i-1} S^i \mathbf{x}$ for all $1 \leq i \leq N$ and concatenate them vertically to obtain ($\mathcal{C}\mathbf{x}$). First, we need to find an expression for $\partial_s^{i-1} S^i \mathbf{x}$. By definition, we have

$$\partial_s^{i-1} S^i \mathbf{x}(s) = \begin{bmatrix} \partial_s^{i-1} \mathbf{x}_i(s) \\ \partial_s^{i-1} \mathbf{x}_{i+1}(s) \\ \vdots \\ \partial_s^{i-1} \mathbf{x}_N(s) \end{bmatrix}.$$

By lemma A.1,

$$(\partial_s^i \mathbf{x})(s) = \sum_{j=i}^{N-1} \tau_{j-i}(s-a)(\partial_s^j \mathbf{x})(a) + \int_a^s \tau_{N-i-1}(s-\theta)(\partial_s^N \mathbf{x})(\theta) d\theta$$

i which can be generalized for $\mathbf{x}_k \in \mathcal{C}_k$ with $k < N$ as

$$(\partial_s^i \mathbf{x}_k)(s) = \sum_{j=i}^{k-1} \tau_{j-i}(s-a)(\partial_s^j \mathbf{x}_k)(a) + \int_a^s \tau_{k-i-1}(s-\theta)(\partial_s^k \mathbf{x}_k)(\theta) d\theta.$$

To find the $(i-1)^{th}-$ derivative for each component of the vector we just perform concatenation to get

$$\begin{aligned}
\partial_s^{i-1} S^i \mathbf{x}(s) &= \begin{bmatrix} \partial_s^{i-1}\mathbf{x}_i(s) \\ \partial_s^{i-1}\mathbf{x}_{i+1}(s) \\ \vdots \\ \partial_s^{i-1}\mathbf{x}_N(s) \end{bmatrix} \\
&= \begin{bmatrix} (\partial_s^{i-1}\mathbf{x}_i)(a) \\ (\partial_s^{i-1}\mathbf{x}_{i+1})(a) + \tau_1(s-a)(\partial_s^i \mathbf{x}_{i+1})(a) \\ \vdots \\ (\partial_s^{i-1}\mathbf{x}_N)(a) + \sum\limits_{j=i}^{N-1} \tau_{j-i+1}(s-a)(\partial_s^j \mathbf{x}_N)(a) \end{bmatrix} + \int_a^s \begin{bmatrix} (\partial_s^i \mathbf{x}_i)(\theta) \\ \tau_1(s-\theta)(\partial_s^{i+1}\mathbf{x}_{i+1})(\theta) \\ \vdots \\ \tau_{(N-i)}(s-\theta)(\partial_s^N \mathbf{x}_N)(\theta) \end{bmatrix} d\theta.
\end{aligned}$$

The matrices $J_{i,j}$ for $j > i$ are used to select the elements from $(\partial_s^{j-1} S^j \mathbf{x})(a) \in \mathbb{R}^{n_{j:N}}$ (jth part of $(\mathcal{C}\mathbf{x})(a)$) which appear in the $(j-i)$th to $(N-i)$th components of $(\partial_s^{i-1} S^i \mathbf{x})(s) \in \mathbb{R}^{n_{i:N}}$ (ith part of $(\mathcal{C}\mathbf{x})(s)$). Specifically, for $j \geq i$, we will see that $\partial_s^{i-1} S^i \mathbf{x}$ is the combination of terms of the form

$$\begin{aligned}
&\begin{bmatrix} 0_{n_{i:j-1}\times 1} \\ (\partial_s^{j-1}\mathbf{x}_j)(s) \\ \vdots \\ (\partial_s^{j-1}\mathbf{x}_N)(s) \end{bmatrix} = \begin{bmatrix} 0_{n_{i:j-1}\times n_{j:N}} \\ I_{n_{j:N}} \end{bmatrix} \begin{bmatrix} (\partial_s^{j-1}\mathbf{x}_j)(s) \\ \vdots \\ (\partial_s^{j-1}\mathbf{x}_N)(s) \end{bmatrix} = \begin{bmatrix} 0_{n_{i:j-1}\times n_{j:N}} \\ I_{n_{j:N}} \end{bmatrix} (\partial_s^{j-1} S^j \mathbf{x})(s) \\
&= J_{i,j}(\partial_s^{j-1} S^j \mathbf{x})(s).
\end{aligned}$$

By exploiting the $J_{i,j}$ notation, we can represent the first term in the expression for

$\partial_s^{i-1} S^i \mathbf{x}$ as

$$\begin{aligned}
&\begin{bmatrix} (\partial_s^{i-1}\mathbf{x}_i)(a) \\ (\partial_s^{i-1}\mathbf{x}_{i+1})(a) + \tau_1(s-a)(\partial_s^{i}\mathbf{x}_{i+1})(a) \\ \vdots \\ (\partial_s^{i-1}\mathbf{x}_N)(a) + \sum_{j=i}^{N-1} \tau_{j-i+1}(s-a)(\partial_s^{j}\mathbf{x}_N)(a) \end{bmatrix} \\
&= \tau_0(s-a)\begin{bmatrix} \partial_s^{i-1}\mathbf{x}_i(a) \\ \partial_s^{i-1}\mathbf{x}_{i+1}(a) \\ \vdots \\ \partial_s^{i-1}\mathbf{x}_N(a) \end{bmatrix} + \cdots + \tau_{j-i}(s-a)\begin{bmatrix} 0_{n_{i:j-1}\times 1} \\ (\partial_s^{j-1}\mathbf{x}_j)(a) \\ \vdots \\ (\partial_s^{j-1}\mathbf{x}_N)(a) \end{bmatrix} + \cdots \\
&\quad + \tau_{N-i}(s-a)\begin{bmatrix} 0_{n_i} \\ 0_{n_{i+1}} \\ \vdots \\ \partial_s^{N-1}\mathbf{x}_N(a) \end{bmatrix} \\
&= \tau_0(s-a)(\partial_s^{i-1}S^i\mathbf{x})(a) + \cdots + \tau_{j-i}(s-a)J_{i,j}(\partial_s^{j-1}S^j\mathbf{x})(a) + \cdots \\
&\quad + \tau_{N-i}(s-a)J_{i,N}(\partial_s^{N-1}S^N\mathbf{x})(a) \\
&= \begin{bmatrix} \tau_0(s-a)J_{i,i} & \cdots & \tau_{N-i}(s-a)J_{i,N} \end{bmatrix} \begin{bmatrix} \partial_s^{i-1}S^i\mathbf{x}(a) \\ \vdots \\ \partial_s^{N-1}S^N\mathbf{x}(a) \end{bmatrix} \\
&= \begin{bmatrix} 0_{n_{i:N}\times n_{S_{1:i-1}}} & \tau_0(s-a)J_{i,i} & \cdots & \tau_{N-i}(s-a)J_{i,N} \end{bmatrix} \begin{bmatrix} S\mathbf{x}(a) \\ \vdots \\ \partial_s^{i-1}S^i\mathbf{x}(a) \\ \partial_s^{i-1}S^{i+1}\mathbf{x}(a) \\ \vdots \\ \partial_s^{N-1}S^N\mathbf{x}(a) \end{bmatrix} \\
&= T_i(s-a)(\mathcal{C}\mathbf{x})(a).
\end{aligned}$$

The second vector with the integral terms can be written as

$$\begin{aligned}
&\int_a^s \begin{bmatrix} \partial_s^{i}\mathbf{x}_i(\theta) \\ \tau_1(s-\theta)\partial_s^{i+1}\mathbf{x}_{i+1}(\theta) \\ \vdots \\ \tau_{N-i}(s-\theta)\partial_s^{N}\mathbf{x}_N(\theta) \end{bmatrix} d\theta \\
&= \int_a^s \begin{bmatrix} 0 & I_{n_i} & & & \\ 0 & & \tau_1(s-\theta)I_{n_{i+1}} & & \\ \vdots & & & \ddots & \\ 0 & & & & \tau_{N-i}(s-\theta)I_{n_N} \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \partial_s\mathbf{x}_1(\theta) \\ \vdots \\ \partial_s^{N}\mathbf{x}_N(\theta) \end{bmatrix} d\theta \\
&= \int_a^s Q_i(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta
\end{aligned}$$

Therefore, combining both the terms we get

$$\partial_s^{i-1} S^i \mathbf{x}(s) = T_i(s-a)(\mathcal{C}\mathbf{x})(a) + \int_a^s Q_i(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta.$$

Since $\partial_s^{i-1} S^i \mathbf{x}(s)$ can be uniquely determined using $(\mathcal{C}\mathbf{x})(a)$ and $\hat{\underline{\mathbf{x}}}$, we can now generalize this to all of $(\mathcal{C}\mathbf{x})$ by using concatenation of $\partial_s^{i-1} S^i \mathbf{x}(s)$ over all $i \in n$ as

$$\begin{aligned}(\mathcal{C}\mathbf{x})(s) = \begin{bmatrix} S\mathbf{x}(s) \\ \vdots \\ \partial_s^{N-1} S^N \mathbf{x}(s) \end{bmatrix} &= \begin{bmatrix} T_1(s-a) \\ T_2(s-a) \\ \vdots \\ T_N(s-a) \end{bmatrix} \begin{bmatrix} S\mathbf{x}(a) \\ \partial_s S^2 \mathbf{x}(a) \\ \vdots \\ \partial_s^{N-1} S^N \mathbf{x}(a) \end{bmatrix} + \int_a^s \begin{bmatrix} Q_1(s-\theta) \\ \vdots \\ Q_N(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta)d\theta \\ &= T(s-a)(\mathcal{C}\mathbf{x})(a) + \int_a^s Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta.\end{aligned}$$

□

Next, we use the map from $(\mathcal{C}\mathbf{x})(a)$ and $\hat{\underline{\mathbf{x}}}$ to $(\mathcal{C}\mathbf{x})$ to obtain the following list of identities.

Corollary A.4. *Suppose $\hat{\mathbf{x}} \in W^n$ for some $v \in \mathbb{R}^q$. Define*

$$(\mathcal{F}\hat{\mathbf{x}}) = \begin{bmatrix} \hat{\mathbf{x}} \\ \partial_s S\hat{\mathbf{x}} \\ \vdots \\ \partial_s^N S^N \hat{\mathbf{x}} \end{bmatrix}, \qquad (\mathcal{C}\hat{\mathbf{x}}) = \begin{bmatrix} S\hat{\mathbf{x}} \\ \partial_s S^2\hat{\mathbf{x}} \\ \vdots \\ \partial_s^{N-1} S^N \hat{\mathbf{x}} \end{bmatrix}, \qquad \hat{\underline{\mathbf{x}}} = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \partial_s^1 \hat{\mathbf{x}}_1 \\ \vdots \\ \partial_s^N \hat{\mathbf{x}}_N \end{bmatrix}.$$

Then we have the following.

(a) For U_i, T, and Q as defined in Block 5.2, we have

$$\begin{aligned}(\mathcal{F}\hat{\mathbf{x}})(s) &= U_1\hat{\underline{\mathbf{x}}}(s) + U_2(\mathcal{C}\hat{\mathbf{x}})(s) \\ &= U_2 T(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + U_1\hat{\underline{\mathbf{x}}}(s) + \int_a^s U_2 Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta.\end{aligned}$$

(b) Given a set of parameters $\mathbf{G}_\mathrm{b}$, if $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, then for B_Q as defined in Block 5.2

$$B\begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} - \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds = B_T(\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_T B_Q(s)\hat{\underline{\mathbf{x}}}(s)ds.$$

(c) Given a set of parameters $\mathbf{G}_\mathrm{b}$, if $\hat{\mathbf{x}} \in X_v$ and $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, then

$$(\mathcal{C}\hat{\mathbf{x}})(a) = \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + B_T^{-1}B_v v.$$

(d) Given a set of parameters $\mathbf{G}_{\mathrm{b}}$*, if* $\hat{\mathbf{x}} \in X_v$ *and* $\{n, \mathbf{G}_{\mathrm{b}}\}$ *is admissible, then*

$$\begin{bmatrix} \begin{bmatrix} v \\ (\mathcal{B}\hat{\mathbf{x}}) \end{bmatrix} \\ (\mathcal{F}\hat{\mathbf{x}})(\cdot) \end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v \end{bmatrix} & \begin{bmatrix} 0_{n_r \times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s) + Q(b-s) \end{bmatrix} \\ \hline U_2 T(s-a) B_T^{-1} B_v & \{U_1, R_{D,1}, R_{D,2}\} \end{array}\right] \begin{bmatrix} v \\ \underline{\hat{\mathbf{x}}} \end{bmatrix}.$$

Proof. Let $\hat{\mathbf{x}} \in X_v$ for some $v \in \mathbb{R}^q$.

For (a), we examine the terms $\partial_s^i S^i \hat{\mathbf{x}}$ in the vector $(\mathcal{F}\hat{\mathbf{x}})$. These terms may be divided into those from $\underline{\hat{\mathbf{x}}}$ and those from $(\mathcal{C}\hat{\mathbf{x}})$. Specifically, we define the permutation matrix $U = [U_1 \quad U_2]$ so that

$$\underbrace{\begin{bmatrix} \hat{\mathbf{x}}(s) \\ \partial_s S \hat{\mathbf{x}}(s) \\ \vdots \\ (\partial_s^N S^N \hat{\mathbf{x}})(s) \end{bmatrix}}_{(\mathcal{F}\hat{\mathbf{x}})} = U \begin{bmatrix} \hat{\mathbf{x}}(s) \\ \begin{bmatrix} S\hat{\mathbf{x}}(s) \\ \vdots \\ \partial_s^{N-1} S^N \hat{\mathbf{x}}(s) \end{bmatrix} \end{bmatrix} = [U_1 \quad U_2] \begin{bmatrix} \underline{\hat{\mathbf{x}}}(s) \\ (\mathcal{C}\hat{\mathbf{x}})(s) \end{bmatrix}.$$

To justify our expression for the permutation matrix, U, first note that

$$\begin{aligned} \partial_s^i S^i \hat{\mathbf{x}} &= \begin{bmatrix} \partial_s^i \hat{\mathbf{x}}_i \\ \partial_s^i \hat{\mathbf{x}}_{i+1} \\ \vdots \\ \partial s^i \hat{\mathbf{x}}_N \end{bmatrix} = \begin{bmatrix} \underline{\hat{\mathbf{x}}}_i \\ \partial_s^i S^{i+1} \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \underline{\hat{\mathbf{x}}}_i \\ (\mathcal{C}\hat{\mathbf{x}})_{i+1} \end{bmatrix} \\ &= \left[\begin{bmatrix} I_{n_i} \\ 0_{n_{i+1:N} \times n_i} \end{bmatrix} \quad \begin{bmatrix} 0_{n_i \times n_{i+1:N}} \\ I_{n_{i+1:N}} \end{bmatrix} \right] \begin{bmatrix} \underline{\hat{\mathbf{x}}}_i \\ (\mathcal{C}\hat{\mathbf{x}})_{i+1} \end{bmatrix} = [U_{1,i} \quad U_{2,i}] \begin{bmatrix} \underline{\hat{\mathbf{x}}}_i \\ (\mathcal{C}\hat{\mathbf{x}})_{i+1} \end{bmatrix} \end{aligned}$$

which holds for $i < N$. For $i = N$, we simply have

$$\partial_s^N S^N \hat{\mathbf{x}} = \partial_s^N \hat{\mathbf{x}}_N = \underline{\hat{\mathbf{x}}}_N = \underbrace{I_{n_N}}_{U_{1,N}} \underline{\hat{\mathbf{x}}}_N.$$

Clearly, then

$$\begin{aligned} &(\mathcal{F}\hat{\mathbf{x}}) \\ &= \begin{bmatrix} \partial_s^0 S^0 \hat{\mathbf{x}} \\ \vdots \\ \partial_s^N S^N \hat{\mathbf{x}} \end{bmatrix} = \underbrace{\begin{bmatrix} U_{1,0} & & \\ & \ddots & \\ & & U_{1,N} \end{bmatrix}}_{U_1} \underbrace{\begin{bmatrix} \underline{\hat{\mathbf{x}}}_0 \\ \vdots \\ \underline{\hat{\mathbf{x}}}_N \end{bmatrix}}_{\underline{\hat{\mathbf{x}}}} + \underbrace{\begin{bmatrix} U_{2,0} & & \\ & \ddots & \\ & & U_{2,(N-1)} \\ 0_{n_N \times n_{1:N}} & \cdots & 0_{n_N \times n_{N:N}} \end{bmatrix}}_{U_2} \underbrace{\begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})_1 \\ \vdots \\ (\mathcal{C}\hat{\mathbf{x}})_N \end{bmatrix}}_{(\mathcal{C}\hat{\mathbf{x}})} \\ &= U_1 \underline{\hat{\mathbf{x}}} + U_2 (\mathcal{C}\hat{\mathbf{x}}). \end{aligned}$$

Finally, by Corollary A.3, we write

$$\begin{aligned} (\mathcal{F}\hat{\mathbf{x}})(s) = U_1 \underline{\hat{\mathbf{x}}}(s) + U_2 (\mathcal{C}\hat{\mathbf{x}})(s) &= U_1 \underline{\hat{\mathbf{x}}}(s) + U_2 T(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^s U_2 Q(s-\theta) \underline{\hat{\mathbf{x}}}(\theta) d\theta \\ &= U_2 T(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + U_1 \underline{\hat{\mathbf{x}}}(s) + \int_a^s U_2 Q(s-\theta) \underline{\hat{\mathbf{x}}}(\theta) d\theta \end{aligned}$$

Now, suppose we are given $\mathbf{G}_\mathrm{b} = \{B, B_I, B_v\}$ such that B_T is invertible where

$$B_T = B \begin{bmatrix} I_{n_S} \\ T(b-a) \end{bmatrix} - \int_a^b B_I(s) U_2 T(s-a) ds.$$

For (b), we temporarily partition B as $B = [B_l \quad B_r]$ where both B_l and B_r have equal number of columns. Then, we look at the expression $B \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} - \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds$. Clearly, we need an expression for $(\mathcal{C}\hat{\mathbf{x}})(b)$ which can be obtained from corollary A.3 (by substituting $s = b$) as

$$(\mathcal{C}\hat{\mathbf{x}})(b) = T(b-a)(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^b Q(b-s)\underline{\hat{\mathbf{x}}}(s)ds.$$

Replacing $(\mathcal{C}\hat{\mathbf{x}})(b)$ and $(\mathcal{F}\hat{\mathbf{x}})$ in the expression for $B \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} - \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds$, we get

$$\begin{aligned}
&\underbrace{[B_l \quad B_r]}_{B} \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} - \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds \\
&= (B_l + B_r T(b-a))(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^b B_r Q(b-s)\underline{\hat{\mathbf{x}}}(s)ds \\
&- \left(\int_a^b B_I(s)U_2 T(s-a)ds\right)(\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b \left(B_I(s)U_1 + \int_s^b B_I(\theta)U_2 Q(\theta - s)d\theta\right)\underline{\hat{\mathbf{x}}}(s)ds \\
&= \underbrace{\left(B_l + B_r T(b-a) - \int_a^b B_I(s)U_2 T(s-a)ds\right)}_{B_T}(\mathcal{C}\hat{\mathbf{x}})(a) \\
&+ \int_a^b \underbrace{\left(B_r Q(b-s) - B_I(s)U_1 - \int_s^b B_I(\theta)U_2 Q(\theta - s)d\theta\right)}_{B_T B_Q(\theta)}\underline{\hat{\mathbf{x}}}(s)ds \\
&= B_T(\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_T B_Q(s)\underline{\hat{\mathbf{x}}}(s)ds,
\end{aligned}$$

which proves the second statement of the corollary.

For (c), we have the additional constraint that $\hat{\mathbf{x}} \in X_v$. Then, we know that

$$B \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} - \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s)ds - B_v v = 0.$$

Therefore, from second statement of the corollary, we have

$$B_T(\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_T B_Q(s)\underline{\hat{\mathbf{x}}}(s)ds - B_v v = 0,$$

and since B_T is invertible, we can conclude that

$$(\mathcal{C}\hat{\mathbf{x}})(a) = \int_a^b B_Q(s)\hat{\underline{\mathbf{x}}}(s)ds + B_T^{-1}B_v v.$$

For (d), we know that $(\mathcal{F}\hat{\mathbf{x}})$ and $(\mathcal{C}\hat{\mathbf{x}})(a)$ (from steps (a) and (b)) can be expressed as

$$\begin{aligned}
(\mathcal{F}\hat{\mathbf{x}})(s) &= U_2T(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + U_1\hat{\underline{\mathbf{x}}}(s) + \int_a^s U_2Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta, \\
(\mathcal{C}\hat{\mathbf{x}})(a) &= \int_a^b B_Q(s)\hat{\underline{\mathbf{x}}}(s)ds + B_T^{-1}B_v v.
\end{aligned}$$

Thus, by substituting $(\mathcal{C}\hat{\mathbf{x}})(a)$ in the expression for $(\mathcal{F}\hat{\mathbf{x}})$, we get

$$\begin{aligned}
&(\mathcal{F}\hat{\mathbf{x}})(s) \\
&= U_2T(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + U_1\hat{\underline{\mathbf{x}}}(s) + \int_a^s U_2Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta \\
&= U_2T(s-a)\left(\int_a^b B_Q(s)\hat{\underline{\mathbf{x}}}(s)ds + B_T^{-1}B_v v\right) + U_1\hat{\underline{\mathbf{x}}}(s) + \int_a^s U_2Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta \\
&= \mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, U_2(T(s-a)B_Q(\theta)+Q(s-\theta)), U_2T(s-a)B_Q(\theta)\} \end{array}\right]\begin{bmatrix} v \\ \hat{\underline{\mathbf{x}}}(\cdot) \end{bmatrix} \\
&= \left(\mathit{\Pi}\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, R_{D,1}, R_{D,2}\} \end{array}\right]\begin{bmatrix} v \\ \hat{\underline{\mathbf{x}}} \end{bmatrix}\right)(s).
\end{aligned}$$

where we define the variables

$$R_{D,1}(s,\theta) = R_{D,2}(s,\theta) + U_2Q(s-\theta), \quad R_{D,2}(,s\theta) = U_2T(s-a)B_Q(\theta).$$

Now, since $\hat{\mathbf{x}} \in X_v$ for all $t \geq 0$, by Corollary A.3, we have

$$(\mathcal{C}\hat{\mathbf{x}})(s) = T(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^s Q(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta.$$

Furthermore, since B_T is invertible, from Corollary A.4, we have

$$(\mathcal{C}\hat{\mathbf{x}})(a) = \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + B_T^{-1}B_v v,$$

and hence we can express $(\mathcal{B}\hat{\mathbf{x}})$ in terms of $\hat{\mathbf{x}}$ and v as

$$\begin{aligned}
(\mathcal{B}\hat{\mathbf{x}}) &= \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} = \begin{bmatrix} I \\ T(b-a) \end{bmatrix}(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^b \begin{bmatrix} 0 \\ Q(b-\theta) \end{bmatrix}\hat{\underline{\mathbf{x}}}(\theta)d\theta \\
&= \begin{bmatrix} I \\ T(b-a) \end{bmatrix}\left(\int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + B_T^{-1}B_v v\right) + \int_a^b \begin{bmatrix} 0 \\ Q(b-\theta) \end{bmatrix}\hat{\underline{\mathbf{x}}}(\theta)d\theta \\
&= \mathit{\Pi}\left[\begin{array}{c|c} \begin{bmatrix} I \\ T(b-a) \end{bmatrix}B_T^{-1}B_v & \begin{bmatrix} I \\ T(b-a) \end{bmatrix}B_Q + \begin{bmatrix} 0 \\ Q \end{bmatrix} \\ \hline \emptyset & \{\emptyset\} \end{array}\right]\begin{bmatrix} v \\ \hat{\underline{\mathbf{x}}} \end{bmatrix}.
\end{aligned}$$

To get an expression for the combined v, $(\mathcal{B}\hat{\mathbf{x}})$ and $(\mathcal{F}\hat{\mathbf{x}})$, we can just concatenate them vertically to get

$$\begin{bmatrix}\begin{bmatrix} v \\ (\mathcal{B}\hat{\mathbf{x}}) \end{bmatrix} \\ (\mathcal{F}\hat{\mathbf{x}})(\cdot)\end{bmatrix} = \varPi\left[\begin{array}{c|c} \begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v \end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s) \end{bmatrix} \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, R_{D,1}, R_{D,2}\} \end{array}\right]\begin{bmatrix} v \\ \underline{\hat{\mathbf{x}}} \end{bmatrix}.$$

□

Now, from a), we have a map from $\{(\mathcal{C}\hat{\mathbf{x}})(a),\ \underline{\hat{\mathbf{x}}}\}$ to the vector of all well-defined terms, $\mathcal{F}\hat{\mathbf{x}}$. Furthermore, from c), when the BCs are admissible we have a map from $\{\underline{\hat{\mathbf{x}}},\ v\}$ to $(\mathcal{C}\hat{\mathbf{x}})(a)$. This allows us to express the left boundary values, $(\mathcal{C}\hat{\mathbf{x}})(a)$ in terms of $\{\underline{\hat{\mathbf{x}}},\ v\}$ – yielding a map from $\underline{\hat{\mathbf{x}}}$ to $(\mathcal{F}\hat{\mathbf{x}})$. Extending this result, we can use corollary A.2 to obtain a map from $\{\underline{\hat{\mathbf{x}}},\ v\}$ to $\hat{\mathbf{x}}$.

Theorem 5.1. *Given an $n \in \mathbb{N}^{N+1}$, and $\mathbf{G}_\mathrm{b}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $\{\hat{\mathcal{T}},\ \mathcal{T}_v\}$ be as defined in Block 5.1, X_v as defined in Equation (3.5) and $\mathcal{D}$ =diag$(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$. Then we have the following: (a) For any $v \in \mathbb{R}^{n_v}$, if $\hat{\mathbf{x}} \in X_v$, then $\mathcal{D}\hat{\mathbf{x}} \in L_2^{n_{\hat{\mathbf{x}}}}$ and $\hat{\mathbf{x}} = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}} + \mathcal{T}_v v$; and (b) For any $v \in \mathbb{R}^{n_v}$ and $\underline{\hat{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$, $\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v \in X_v$ and $\underline{\hat{\mathbf{x}}} = \mathcal{D}(\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v)$.*

Proof. **Proof of Part 1.** Let $\hat{\mathbf{x}} \in X_v$ for some $v \in \mathbb{R}^q$. Clearly, by definition of X_v, $\partial_s^i \hat{\mathbf{x}}_i \in L_2^{n_i}$. Therefore, $\mathcal{D}\hat{\mathbf{x}} \in L_2^{n_{\hat{\mathbf{x}}}}$. Next we need to express $\hat{\mathbf{x}}$ in terms of $\underline{\hat{\mathbf{x}}} = \mathcal{D}\hat{\mathbf{x}}$ and v. For that, we will first express $(\mathcal{C}\hat{\mathbf{x}})(a)$ solely in terms of $\underline{\hat{\mathbf{x}}}$ and v. From corollary A.4, we know that if $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible, then

$$(\mathcal{C}\hat{\mathbf{x}})(a) = \int_a^b B_Q(\theta)\underline{\hat{\mathbf{x}}}(\theta)d\theta + B_T^{-1}B_v v.$$

Now that we have an expression for $(\mathcal{C}\hat{\mathbf{x}})(a)$, we simply substitute this into the expression for $\hat{\mathbf{x}}$ from Corollary A.2 to obtain

$$\begin{aligned}\hat{\mathbf{x}}_{1:N}(s) &= T_1(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^s Q_1(s-\theta)\underline{\hat{\mathbf{x}}}(\theta)d\theta \\ &= \int_a^b T_1(s-a)B_Q(\theta)\underline{\hat{\mathbf{x}}}(\theta)d\theta + \int_a^s Q_1(s-\theta)\underline{\hat{\mathbf{x}}}(\theta)d\theta + T_1(s-a)B_T^{-1}B_v v.\end{aligned}$$

Adding on the somewhat incongruous $\hat{\mathbf{x}}_0$ term, we obtain

$$\begin{aligned}
\hat{\mathbf{x}}(s) &= \begin{bmatrix} \hat{\mathbf{x}}_0(s) \\ \hat{\mathbf{x}}_{1:N}(s) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0_{n_x-n_0} \end{bmatrix}}_{G_0} \hat{\underline{\mathbf{x}}}(s) + \int_a^b \underbrace{\begin{bmatrix} 0_{n_0\times n_{\hat{\mathbf{x}}}} \\ T_1(s-a)B_Q(\theta) \end{bmatrix}}_{G_2(s,\theta)} \hat{\underline{\mathbf{x}}}(\theta)d\theta + \int_a^s \underbrace{\begin{bmatrix} 0_{n_0\times n_{\hat{\mathbf{x}}}} \\ Q_1(s-\theta) \end{bmatrix}}_{G_1(s,\theta)-G_2(s,\theta)} \hat{\underline{\mathbf{x}}}(\theta)d\theta \\
&\quad + \underbrace{\begin{bmatrix} 0_{n_0\times n_v} \\ T_1(s-a)B_T^{-1}B_v \end{bmatrix}}_{G_v(s)} v \\
&= G_0\hat{\underline{\mathbf{x}}}(s) + \int_s^b G_2(s,\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + \int_a^s G_1(s,\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + G_v(s)v \\
&= (\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}})(s) + (\mathcal{T}_v v)(s).
\end{aligned} \tag{A.3}$$

□

Proof. **Proof of Part 2.** Let $v \in \mathbb{R}^q$ and $\hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$ be arbitrary. Our first step is to show that $\mathcal{D}\left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v\right) = \hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$. By the definition of $\hat{\mathcal{T}}$ and $\mathcal{T}_v$, Eq. (A.3) at the end of the proof of Part 1 shows that for any $\hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$ and $v \in \mathbb{R}^q$,

$$\begin{aligned}
&(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}})(s) + (\mathcal{T}_v v)(s) \\
&= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^b \begin{bmatrix} 0 \\ T_1(s-a)B_Q(\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta)d\theta + \int_a^s \begin{bmatrix} 0 \\ Q_1(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta)d\theta \\
&\quad + \begin{bmatrix} 0 \\ T_1(s-a)B_T^{-1}B_v \end{bmatrix} v.
\end{aligned}$$

Thus, we may group the terms with $T_1(s-a)$ together and apply the $\mathcal{D}$ operator to obtain

$$\begin{aligned}
\mathcal{D}\left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v\right)(s) &= \mathcal{D}\begin{bmatrix} 0 \\ T_1(s-a) \end{bmatrix} \left(\int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + B_T^{-1}B_v v\right) \\
&\quad + \mathcal{D}\left(\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0_{n_0\times n_{\hat{\mathbf{x}}}} \\ Q_1(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta)d\theta\right).
\end{aligned}$$

Now, examining the first term we have

$$\begin{aligned}
\mathcal{D}\begin{bmatrix} 0 \\ T_1(s-a) \end{bmatrix} &= \begin{bmatrix} I_{n_0} & & \\ & \ddots & \\ & & \partial_s^N I_{n_N} \end{bmatrix} \begin{bmatrix} 0 \\ T_1(s-a) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \left(\begin{bmatrix} \partial_s I_{n_1} & & \\ & \ddots & \\ & & \partial_s^N I_{n_N} \end{bmatrix} T_1(s-a)\right) \end{bmatrix}
\end{aligned}$$

and since $\partial_s^i \tau_j(s) = 0$ for any $j > i$, we have

$$
\begin{aligned}
&\left(\begin{bmatrix}\partial_s I_{n_1} & & \\ & \ddots & \\ & & \partial_s^N I_{n_N}\end{bmatrix} T_1(s-a)\right)\\
&= \begin{bmatrix}\partial_s I_{n_1} & & \\ & \ddots & \\ & & \partial_s^N I_{n_N}\end{bmatrix} \begin{bmatrix}\tau_0(s-a)J_{1,1} & \tau_1(s-a)J_{1,2} & \cdots & \tau_{N-1}(s-a)J_{1,N}\end{bmatrix}\\
&= \begin{bmatrix}\partial_s I_{n_1} & & \\ & \ddots & \\ & & \partial_s^N I_{n_N}\end{bmatrix} \begin{bmatrix}\tau_0(s-a)I_{n_{1:N}} & \begin{bmatrix}0\\ \tau_1(s-a)I_{n_{2:N}}\end{bmatrix} & \cdots & \begin{bmatrix}0\\ \tau_{N-1}(s-a)I_{n_N}\end{bmatrix}\end{bmatrix}\\
&= 0.
\end{aligned}
$$

Hence the first term in our expression for $\mathcal{D}\left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v\right)$ is zero. Now, consider the second term in the expression for $\mathcal{D}\left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v\right)$,

$$
\begin{aligned}
&\mathcal{D}\left(\begin{bmatrix}I_{n_0} & 0\\ 0 & 0\end{bmatrix}\hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix}0_{n_0\times n_x}\\ Q_1(s-\theta)\end{bmatrix}\hat{\underline{\mathbf{x}}}(\theta)d\theta\right)\\
&= \mathcal{D}\left(\begin{bmatrix}I_{n_0} & 0\\ 0 & 0\end{bmatrix}\hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix}0_{n_0} & 0 & & 0\\ 0 & \tau_0(s)I_{n_1} & & \\ \vdots & & \ddots & \\ 0 & & & \tau_{N-1}(s)I_{n_N}\end{bmatrix}\hat{\underline{\mathbf{x}}}(\theta)d\theta\right).
\end{aligned}
$$

For this term, we use an inductive approach. Specifically, we factor $\mathcal{D}$ into first-order derivative operators as

$$
\mathcal{D} = \begin{bmatrix}I & & & \\ & \partial_s I & & \\ & & \ddots & \\ & & & \partial_s^N I\end{bmatrix} = \prod_{i=1}^{N} \underbrace{\begin{bmatrix}I_{n_{0:i-1}} & 0\\ 0 & I_{n_{i:N}}\partial_s\end{bmatrix}}_{\mathcal{D}_i}.
$$

Now, since $\partial_s \tau_i(s) = \tau_{i-1}(s)$ for $i \geq 1$, $\tau_i(0) = 0$ for $i > 0$ and $\tau_0(0) = 1$, we have that for any $i < N$,

$$
\begin{aligned}
&\mathcal{D}_i \begin{bmatrix}0_{n_{0:i-1}\times n_x}\\ Q_i(s-\theta)\end{bmatrix} = \begin{bmatrix}I_{n_{0:i-1}} & 0\\ 0 & I_{n_{i:N}}\partial_s\end{bmatrix} \begin{bmatrix}0_{n_{0:i-1}\times n_{0:i-1}} & & & \\ & \tau_0(s)I_{n_i} & & \\ & & \ddots & \\ & & & \tau_{N-i}(s)I_{n_N}\end{bmatrix}\\
&= \begin{bmatrix}0_{n_{0:i}\times n_{0:i}} & & & \\ & \tau_0(s)I_{n_{i+1}} & & \\ & & \ddots & \\ & & & \tau_{N-i-1}(s)I_{n_N}\end{bmatrix} = \begin{bmatrix}0_{n_{0:i}\times n_x}\\ Q_{i+1}(s-\theta)\end{bmatrix}
\end{aligned}
$$

and $\mathcal{D}_N \begin{bmatrix} 0_{n_{0:N-1}\times n_x} \\ Q_N(s-\theta) \end{bmatrix} = 0$. Additionally, for $i \geq 0$, we have

$$\begin{bmatrix} 0_{n_{0:i-1}\times n_x} \\ Q_i(0) \end{bmatrix} = \begin{bmatrix} 0_{n_{0:i-1}\times n_{0:i-1}} & & \\ & I_{n_i} & \\ & & 0_{n_{i+1:N}\times n_{i+1:N}} \end{bmatrix}.$$

We conclude that

$$\begin{aligned}
&\mathcal{D}_i \left(\begin{bmatrix} I_{n_{0:i-1}} & \\ & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0_{n_{0:i-1}\times n_x} \\ Q_i(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta) d\theta \right) \\
&= \begin{bmatrix} I_{n_{0:i-1}} & \\ & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \begin{bmatrix} 0_{n_{0:i-1}\times n_x} \\ Q_i(0) \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0_{n_{0:i}\times n_x} \\ Q_{i+1}(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta) d\theta \\
&= \begin{bmatrix} I_{n_{0:i}} & \\ & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0_{n_{0:i}\times n_x} \\ Q_{i+1}(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta) d\theta.
\end{aligned}$$

Applying this inductive step to each of the $\mathcal{D}_i$ operators in $\mathcal{D}\hat{\mathbf{x}}$, we have

$$\begin{aligned}
&\mathcal{D} \left(\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0 \\ Q_1(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta) d\theta \right) \\
&= \mathcal{D}_N \cdots \mathcal{D}_1 \left(\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0 \\ Q_1(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta) d\theta \right) = \hat{\underline{\mathbf{x}}}(s).
\end{aligned}$$

Combining these results, we conclude that for any $\hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$

$$\begin{aligned}
&\left(\mathcal{D} \left(\hat{\mathcal{T}} \hat{\underline{\mathbf{x}}} + \mathcal{T}_v v \right) \right)(s) \\
&= \mathcal{D} \left(\int_a^b G_2(s,\theta) \hat{\underline{\mathbf{x}}}(\theta) d\theta + G_v(s) v \right) + \mathcal{D} \left(\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix} \hat{\underline{\mathbf{x}}}(s) + \int_a^s \begin{bmatrix} 0 \\ Q_1(s-\theta) \end{bmatrix} \hat{\underline{\mathbf{x}}}(\theta) d\theta \right) \\
&= \hat{\underline{\mathbf{x}}}(s).
\end{aligned}$$

Finally, we need to show that for any $\hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$, $\hat{\mathcal{T}} \hat{\underline{\mathbf{x}}} + \mathcal{T}_v v \in X_v$. Let $\hat{\mathbf{x}} = \hat{\mathcal{T}} \hat{\underline{\mathbf{x}}} + \mathcal{T}_v v$. Clearly, since $\mathcal{D}\hat{\mathbf{x}} = \hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$, we have $\hat{\mathbf{x}} \in W^n$. To show that $\hat{\mathbf{x}} \in X_v$, however, we must now show that the BCs are satisfied. For this part, we have that if $\hat{\mathbf{x}} = \hat{\mathcal{T}} \hat{\underline{\mathbf{x}}} + \mathcal{T}_v v \in W^n$, then by Corollary A.4,

$$B \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(a) \\ (\mathcal{C}\hat{\mathbf{x}})(b) \end{bmatrix} - \int_a^b B_I(s)(\mathcal{F}\hat{\mathbf{x}})(s) ds - B_v v = B_T \left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(s) \hat{\underline{\mathbf{x}}}(s) ds - B_T^{-1} B_v v \right).$$

Since $\hat{\mathbf{x}} \in W^n$ and B_T is invertible, we have that $\hat{\mathbf{x}} \in X_v$ if and only if

$$(\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(s) \hat{\underline{\mathbf{x}}}(s) ds - B_T^{-1} B_v v = 0.$$

Recall from the beginning of the proof of Part 2 that

$$\begin{aligned}\hat{\mathbf{x}}(s) &= \begin{bmatrix}\hat{\mathbf{x}}_0(s)\\ \hat{\mathbf{x}}_{1:N}(s)\end{bmatrix}\\ &= \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\hat{\underline{\mathbf{x}}}(s) + \int_a^b \begin{bmatrix}0\\ T_1(s-a)B_Q(\theta)\end{bmatrix}\hat{\underline{\mathbf{x}}}(\theta)d\theta + \int_a^s \begin{bmatrix}0\\ Q_1(s-\theta)\end{bmatrix}\hat{\underline{\mathbf{x}}}(\theta)d\theta\\ &\quad + \begin{bmatrix}0\\ T_1(s-a)B_T^{-1}B_v\end{bmatrix} v\end{aligned}$$

and hence

$$\hat{\mathbf{x}}_{1:N}(s) = \int_a^b T_1(s-a)B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + \int_a^s Q_1(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + T_1(s-a)B_T^{-1}B_v v.$$

In addition, from Corollary A.3, we have

$$\hat{\mathbf{x}}_{1:N}(s) = T_1(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^s Q_1(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta.$$

Substituting this identity in the previous equation, we get

$$\begin{aligned}&T_1(s-a)(\mathcal{C}\hat{\mathbf{x}})(a) + \int_a^s Q_1(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta\\ &= \int_a^b T_1(s-a)B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + \int_a^s Q_1(s-\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta + T_1(s-a)B_T^{-1}B_v v,\end{aligned}$$

which implies

$$T_1(s-a)\left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v\right) = 0.$$

We will use induction to show that the above equality holds when T_1 is replaced by T_i. First, suppose

$$T_i(s-a)\left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v\right) = 0.$$

Then, since

$$\partial_s \left[0_{n_{i+1:N}\times n_i} \quad I_{n_{i+1:N}}\right] T_i(s-a) = T_{i+1}(s-a),$$

we have the relation

$$\begin{aligned}&\partial_s \left[0_{n_{i+1:N}\times n_i} \quad I_{n_{i+1:N}}\right] T_i(s-a)\left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v\right)\\ &= T_{i+1}(s-a)\left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v\right) = 0.\end{aligned}$$

Since the equality is true for $i = 1$, by induction we can conclude, for any $i \geq 1$,

$$T_i(s-a)\left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v\right) = 0.$$

By stacking all T_i's and using $T = \mathrm{col}(T_1, \cdots, T_N)$, for any $s \in [a, b]$ we have

$$T(s-a)\left((\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v\right) = 0.$$

and since $T(0) = I_{n_S}$, we have that

$$(\mathcal{C}\hat{\mathbf{x}})(a) - \int_a^b B_Q(\theta)\hat{\underline{\mathbf{x}}}(\theta)d\theta - B_T^{-1}B_v v = 0.$$

which completes the proof. $\square$

A.2 Equivalence of PIE and PDE Subsystems

Now that we have established a PI map from L_2 to X_v, we will obtain the PIE associated with a PDE subsystem by replacing $\hat{\mathbf{x}}$ in the PDE subsystem with $\hat{\mathbf{x}} = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v$. Because we have shown that this PI map is a bijection, we will then conclude that existence of a solution for the PIE subsystem guarantees the existence of a solution for the PDE subsystem. This proof is split into two parts.

Theorem 5.4. *Given an $n \in \mathbb{N}^{N+1}$ and a set of PDE parameters $\{\mathbf{G}_\mathrm{b},\ \mathbf{G}_\mathrm{p}\}$ as defined in Equations (3.6) and (3.8) with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, suppose $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ with $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $\{\hat{\mathcal{T}},\ \mathcal{T}_v\}$ are as defined in Block 5.1 and $\{\hat{\mathcal{A}},\ \mathcal{B}_v,\ \mathcal{C}_r,\ \mathcal{D}_{rv}\}$ are as defined in Block 5.2. Define*

$$\mathbf{G}_{\mathrm{PIE}} = \left\{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_r, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\right\}.$$

Then we have the following.

1. *For any $\hat{\mathbf{x}}^0 \in X_{v(0)}$ (X_v is as defined in Equation (3.5)), if $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathbf{x}}^0$ and input v, then $\{\mathcal{D}\hat{\mathbf{x}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\mathcal{D}\hat{\mathbf{x}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$ and input $\{v, \emptyset\}$ where $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0\hat{\mathbf{x}}_0, \cdots, \partial_s^N\hat{\mathbf{x}}_N)$.*

2. *For any $\hat{\underline{\mathbf{x}}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$, if $\{\hat{\underline{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ for initial condition $\hat{\underline{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$, then $\{\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v,\ r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$ and input v.*

Proof. Suppose $\{\hat{\mathbf{x}},\ r\}$ satisfies the PDE Equation (3.7) defined by $n \in \mathbb{N}^{N+1}$ and $\{\mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial conditions $\hat{\mathbf{x}}^0$ and input v. Then by Definition 3.2: a) $r \in L_{2e}^{n_r}[\mathbb{R}_+]$; b) $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$; c) $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$; d) Equation (3.7) is satisfied for almost all $t \geq 0$; and e) $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$.

Let $\underline{\hat{\mathbf{x}}} = \mathcal{D}\hat{\mathbf{x}}$, $\underline{\hat{\mathbf{x}}}^0 = \mathcal{D}\hat{\mathbf{x}}^0$, $n = n_{\hat{\mathbf{x}}}$ and $m = 0$. Our goal is to show that for $\mathbf{G}_{\text{PIE}}$ as defined above, $\{\underline{\hat{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{PIE}$ for initial condition $\underline{\hat{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$. For this, we must show that: 1) $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ and $(\mathcal{T}_v v)(\cdot, s) \in W_{1e}^{n_{\hat{\mathbf{x}}}}[\mathbb{R}_+]$ for all $s \in [a,b]$; 2) $\underline{\hat{\mathbf{x}}} : \mathbb{R}_+ \to \mathbb{R}L_2^{0,n_{\hat{\mathbf{x}}}}[a,b]$ and $r \in L_{2e}^{n_r}[\mathbb{R}_+]$; 3) $\underline{\hat{\mathbf{x}}}^0 \in \mathbb{R}L_2^{0,n_{\hat{\mathbf{x}}}}[a,b]$ and $\underline{\hat{\mathbf{x}}}(0) = \underline{\hat{\mathbf{x}}}^0$; 4) $\underline{\hat{\mathbf{x}}}$ is Frechét differentiable with respect to the $\hat{\mathcal{T}}$-norm almost everywhere on $\mathbb{R}_+$; and 5) Equation (4.1) is satisfied for almost all $t \in \mathbb{R}_+$.

For 1), $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ from the theorem statement and by the definition of $\mathcal{T}_w$, $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$ implies

$$(\mathcal{T}_w w)(s) = (\mathcal{T}_v v)(s) = \begin{bmatrix} 0 \\ T_1(s-a) \end{bmatrix} B_T^{-1} B_v v \in W_{1e}^{n_{\hat{\mathbf{x}}}}[\mathbb{R}_+].$$

For 2), from Theorem 5.1a we have that $\hat{\mathbf{x}}(t) \in X_{v(t)}$ implies $\underline{\hat{\mathbf{x}}}(t) = \mathcal{D}\hat{\mathbf{x}}(t) \in RL_2^{0,n_{\hat{\mathbf{x}}}} = L_2^{n_{\hat{\mathbf{x}}}}$ for all $t \geq 0$. Furthermore, from the definition of solution of the PDE, $r \in L_{2e}^{n_r}[\mathbb{R}_+]$.

For 3), from Theorem 5.1a we have that $\hat{\mathbf{x}}^0 \in X_{v(0)}$ implies $\underline{\hat{\mathbf{x}}}^0 = \mathcal{D}\hat{\mathbf{x}}^0 \in RL_2^{0,n_{\hat{\mathbf{x}}}} = L_2^{n_{\hat{\mathbf{x}}}}$. Furthermore, since $\underline{\hat{\mathbf{x}}}(t) = \mathcal{D}\hat{\mathbf{x}}(t)$ for all $t \geq 0$, we have $\underline{\hat{\mathbf{x}}}(0) = \mathcal{D}\hat{\mathbf{x}}(0) = \mathcal{D}\hat{\mathbf{x}}^0 = \underline{\mathbf{x}}^0$.

For 4), because $\hat{\mathbf{x}}$ is Frechét differentiable almost everywhere on $\mathbb{R}_+$, the limit of $\frac{\hat{\mathbf{x}}(t+h)-\hat{\mathbf{x}}(t)}{h}$ as $h \to 0^+$ exists for any $t \geq 0$ when convergence is defined with respect to the L_2 norm. This, and the fact that $\mathcal{T}_v v \in W_{1e}^{n_v}$ implies that

$$\lim_{h\to 0^+} \frac{\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}(t+h) - \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}(t)}{h} = \lim_{h\to 0^+} \frac{\hat{\mathbf{x}}(t+h) - \hat{\mathbf{x}}(t)}{h} - \lim_{h\to 0^+} \frac{\mathcal{T}_v v(t+h) - \mathcal{T}_v v(t)}{h}$$

similarly exists for all $t \geq 0$. Thus, $\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}$ is Frechét differentiable with respect to L_2-norm, and hence, $\underline{\hat{\mathbf{x}}}$ is Frechét differentiable with respect to $\hat{\mathcal{T}}$-norm.

Lastly, for 5), since $\hat{\mathbf{x}}(t)$ satisfies (3.4)-(3.7) for almost all $t \geq 0$, we have

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t,s) \\ r(t) \end{bmatrix} = \sum_{i=0}^{N} \begin{bmatrix} A_0(s) + \int_a^s A_1(s,\cdot) + \int_s^b A_2(s,\cdot) \\ \int_a^b C_r(\cdot) \end{bmatrix} (\mathcal{F}\hat{\mathbf{x}})(t,\cdot) \tag{A.4}$$

$$+ \begin{bmatrix} B_{xv}(s) & B_{xb}(s) \\ 0 & D_{rb} \end{bmatrix} \begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t) \end{bmatrix}. \tag{A.5}$$

Since $\hat{\mathbf{x}}(t) \in X_{v(t)}$ and $\underline{\hat{\mathbf{x}}}(t) = \mathcal{D}\hat{\mathbf{x}}(t)$ for all $t \geq 0$, from Theorem 5.1, we have that

$$\hat{\mathbf{x}}(t) = \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}(t) + \mathcal{T}_v v(t) \qquad \text{which implies} \qquad \dot{\hat{\mathbf{x}}}(t) = \hat{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t) + \mathcal{T}_v \dot{v}(t).$$

We can substitute this into Eq. A.5 and re-write Eq. A.5 using the PI operator notation to get the compact relation

$$\begin{bmatrix} r(t) \\ \hat{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t) + \mathcal{T}_v \dot{v}(t) \end{bmatrix} = \varPi\left[\begin{array}{c|c} [0 \quad D_{rb}] & C_r \\ \hline [B_{xv} \quad B_{xb}] & \{A_i\} \end{array}\right] \begin{bmatrix} \begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t) \end{bmatrix} \\ (\mathcal{F}\hat{\mathbf{x}})(t) \end{bmatrix}. \tag{A.6}$$

where we define

$$(\mathcal{F}\hat{\mathbf{x}})(t) = U_1\underline{\hat{\mathbf{x}}}(t) + U_2(\mathcal{C}\hat{\mathbf{x}})(t),\ (\mathcal{B}\hat{\mathbf{x}})(t) = \begin{bmatrix}(\mathcal{C}\hat{\mathbf{x}})(t,a)\\(\mathcal{C}\hat{\mathbf{x}})(t,b)\end{bmatrix}\ (\mathcal{C}\hat{\mathbf{x}})(t) = \begin{bmatrix}(S\hat{\mathbf{x}})(t)\\(\partial_s S^2\hat{\mathbf{x}})(t)\\\vdots\\(\partial_s^{N-1}S^N\hat{\mathbf{x}})(t)\end{bmatrix}.$$

We know from Corollary A.4d that when B_T is invertible, $\hat{\mathbf{x}}(t) \in X_{v(t)}$ and $\underline{\hat{\mathbf{x}}}(t) = \mathcal{D}\hat{\mathbf{x}}(t)$, we have the relation

$$\begin{bmatrix}\begin{bmatrix}v(t)\\(\mathcal{B}\hat{\mathbf{x}})(t)\end{bmatrix}\\(\mathcal{F}\hat{\mathbf{x}})(t)\end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c}\begin{bmatrix}I\\P_b\end{bmatrix} & \begin{bmatrix}0\\Q_b\end{bmatrix}\\\hline U_2T(s-a)B_T^{-1}B_v & \{\{U_1,R_{D,1},R_{D,2}\}\}\end{array}\right]\begin{bmatrix}v(t)\\\underline{\hat{\mathbf{x}}}(t)\end{bmatrix}.$$

Using the above expression for $\begin{bmatrix}v(t)\\(\mathcal{B}\hat{\mathbf{x}})(t)\\(\mathcal{F}\hat{\mathbf{x}})(t)\end{bmatrix}$, we now expand Eq. A.6 to obtain

$$\begin{aligned}\begin{bmatrix}r(t)\\\hat{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t)+\mathcal{T}_v\dot{v}(t)\end{bmatrix} &= \mathit{\Pi}\left[\begin{array}{c|c}[0\quad D_{rb}] & C_r\\\hline [B_{xv}\quad B_{xb}] & \{A_i\}\end{array}\right]\begin{bmatrix}\begin{bmatrix}v(t)\\(\mathcal{B}\hat{\mathbf{x}})(t)\end{bmatrix}\\(\mathcal{F}\hat{\mathbf{x}})(t)\end{bmatrix}\\
&= \mathit{\Pi}\left[\begin{array}{c|c}[0\quad D_{rb}] & C_r\\\hline [B_{xv}\quad B_{xb}] & \{A_i\}\end{array}\right]\mathit{\Pi}\left[\begin{array}{c|c}\begin{bmatrix}I\\P_b\end{bmatrix} & \begin{bmatrix}0\\Q_b\end{bmatrix}\\\hline U_2T(s-a)B_T^{-1}B_v & \{\{U_1,R_{D,1},R_{D,2}\}\}\end{array}\right]\begin{bmatrix}v(t)\\\underline{\hat{\mathbf{x}}}(t)\end{bmatrix}\\
&= \mathit{\Pi}\left[\begin{array}{c|c}D_{rv} & C_{rx}\\\hline B_{xv} & \{\hat{A}_i\}\end{array}\right]\begin{bmatrix}v(t)\\\underline{\hat{\mathbf{x}}}(t)\end{bmatrix}\end{aligned}$$

where

$$\begin{bmatrix}D_{rv} & C_{rx}\\B_{xv} & \hat{A}_i\end{bmatrix} =$$
$$\mathbf{P}^4_\times\left(\begin{bmatrix}[0\quad D_{rb}] & C_r\\ [B_{xv}\quad B_{xb}] & A_i\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}I_{n_v}\\B_T^{-1}B_v\\T(b-a)B_T^{-1}B_v\end{bmatrix} & \begin{bmatrix}0_{n_r\times n_x}\\B_Q(s)\\T(b-a)B_Q(s)+Q(b-s)\end{bmatrix}\\ U_2T(s-a)B_T^{-1}B_v & U_1,R_{D,1},R_{D,2}\end{bmatrix}\right)$$

which shows that $\{\underline{\hat{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_{\text{PIE}}$ for initial condition $\underline{\hat{\mathbf{x}}}^0$ and input $\{v,\emptyset\}$. $\square$

Theorem 5.4. *Given an $n \in \mathbb{N}^{N+1}$ and a set of PDE parameters $\{\mathbf{G}_\text{b},\ \mathbf{G}_\text{p}\}$ as defined in Equations (3.6) and (3.8) with $\{n, \mathbf{G}_\text{b}\}$ admissible, suppose $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ with $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $\{\hat{\mathcal{T}},\ \mathcal{T}_v\}$ are as defined in Block 5.1 and $\{\hat{\mathcal{A}},\ \mathcal{B}_v,\ \mathcal{C}_r,\ \mathcal{D}_{rv}\}$ are as defined in Block 5.2. Define*

$$\mathbf{G}_{\text{PIE}} = \left\{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_r, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\right\}.$$

Then we have the following.

1. *For any $\hat{\mathbf{x}}^0 \in X_{v(0)}$ (X_v is as defined in Equation (3.5)), if $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathbf{x}}^0$ and input v, then $\{\mathcal{D}\hat{\mathbf{x}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition $\mathcal{D}\hat{\mathbf{x}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$ and input $\{v, \emptyset\}$ where $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0 \hat{\mathbf{x}}_0, \cdots, \partial_s^N \hat{\mathbf{x}}_N)$.*

2. *For any $\hat{\underline{\mathbf{x}}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$, if $\{\hat{\underline{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ for initial condition $\hat{\underline{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$, then $\{\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v,\ r\}$ satisfies the PDE defined by $\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$ and input v.*

Proof. Suppose $\{\hat{\underline{\mathbf{x}}}, r, \emptyset\}$ satisfies the PIE Equation (4.1) defined by the set of parameters $\mathbf{G}_\mathrm{PIE}$ for initial conditions $\hat{\underline{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$. Then we have: a) $r \in L_{2e}^{n_r}[\mathbb{R}_+]$; b) $\hat{\underline{\mathbf{x}}}(t, \cdot) \in \mathbb{R}L_2^{m,n}[a, b]$ for all $t \geq 0$; c) $\hat{\underline{\mathbf{x}}}$ is Frechét differentiable with respect to the $\mathcal{T}$-norm almost everywhere on $\mathbb{R}_+$; d) Equation (4.1) is satisfied for almost all $t \in \mathbb{R}_+$; and e) $\hat{\underline{\mathbf{x}}}(0, \cdot) = \hat{\underline{\mathbf{x}}}^0$. Let

$$\hat{\mathbf{x}}(t) = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}(t) + \mathcal{T}_v v(t), \quad \hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0).$$

Then, our goal is to show that, $\{\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v,\ r\}$ satisfies the PDE Equation (3.7) defined by $n \in \mathbb{N}^{N+1}$ and $\{\mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial conditions $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$ and input v. For this, we must show: 1) $r \in L_{2e}^{n_r}[\mathbb{R}_+]$; 2) $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$; 3) $\hat{\mathbf{x}}^0 \in X_{v(0)}$ and $\hat{\mathbf{x}}(0, \cdot) = \hat{\mathbf{x}}^0$; 4) $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$; and 5) Equation (3.7) is satisfied for almost all $t \geq 0$.

For 1), $r \in L_{2e}^{n_r}[\mathbb{R}_+]$ holds immediately by the definition of solution of the PIE.

For 2), Theorem 5.1b states that for any $v(t) \in \mathbb{R}$ and $\hat{\underline{\mathbf{x}}}(t) \in L_2^{n_{\hat{\mathbf{x}}}}$, we have $\hat{\mathbf{x}}(t) = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}(t) + \mathcal{T}_v v(t) \in X_{v(t)}$.

For 3), Theorem 5.1b states that for any $v(0) \in \mathbb{R}$ and $\hat{\underline{\mathbf{x}}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$, we have $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0) \in X_{v(0)}$. In addition, $\hat{\mathbf{x}}(0) = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}(0) + \mathcal{T}_v v(0) = \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}^0 + \mathcal{T}_v v(0) \in X_{v(0)}$.

For 4), we know $\hat{\underline{\mathbf{x}}}$ is Frechét differentiable with respect to $\mathcal{T}$-norm. This implies that $\lim_{h\to 0^+} \frac{\mathcal{T}\hat{\underline{\mathbf{x}}}(t+h) - \mathcal{T}\hat{\underline{\mathbf{x}}}(t)}{h}$ exists when the convergence is with respect to L_2-norm. Since $\mathcal{T}_v v \in W_{1e}^{n_v}$, we conclude that

$$\lim_{h\to 0^+} \frac{\hat{\mathbf{x}}(t+h) - \hat{\mathbf{x}}(t)}{h} = \lim_{h\to 0^+} \frac{\mathcal{T}\hat{\underline{\mathbf{x}}}(t+h) - \mathcal{T}\hat{\underline{\mathbf{x}}}(t)}{h} + \lim_{h\to 0^+} \frac{\mathcal{T}_v v(t+h) - \mathcal{T}_v v(t)}{h}$$

exists for all $t \geq 0$. Thus, $\hat{\mathbf{x}}$ is Frechét differentiable with respect to L_2-norm.

For 5), since $\hat{\mathbf{x}}$ is Frechét differentiable and $\hat{\underline{\mathbf{x}}}$ satisfies the PIE, we have

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathcal{T}}\dot{\hat{\underline{\mathbf{x}}}}(t) + \mathcal{T}_v \dot{v}(t) = \hat{\mathcal{A}}\hat{\underline{\mathbf{x}}}(t) + \mathcal{B}_v v(t)$$

and furthermore, $r(t) = \mathcal{C}_r \hat{\underline{\mathbf{x}}}(t) + \mathcal{D}_{rv} v(t)$. Combining these expressions, we obtain

$$\begin{bmatrix} r(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{rv} & \mathcal{C}_r \\ \mathcal{B}_v & \hat{\mathcal{A}} \end{bmatrix} \begin{bmatrix} v(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix} = \mathit{\Pi}\left[\begin{array}{c|c} D_{rv} & C_{rx} \\ \hline B_{xv} & \{\hat{A}_i\} \end{array}\right] \begin{bmatrix} v(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix}.$$

Now, we use the relation from Block 5.2

$$
\begin{aligned}
&\Pi\left[\begin{array}{c|c} D_{rv} & C_{rx} \\ \hline B_{xv} & \{\hat{A}_i\}\end{array}\right] \\
&= \Pi\left[\mathbf{P}^4_\times\left(\left[\begin{matrix}[0 \quad D_{rb}] & C_r \\ [B_{xv} \quad B_{xb}] & A_i\end{matrix}\right], \left[\begin{matrix}\begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v\end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s)\end{bmatrix} \\ U_2T(s-a)B_T^{-1}B_v & U_1, R_{D,1}, R_{D,2}\end{matrix}\right]\right)\right] \\
&= \Pi\left[\begin{array}{c|c} [0 \quad D_{rb}] & C_r \\ \hline [B_{xv} \quad B_{xb}] & \{A_i\}\end{array}\right] \Pi\left[\begin{array}{c|c}\begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v\end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s)\end{bmatrix} \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, R_{D,1}, R_{D,2}\}\end{array}\right],
\end{aligned}
$$

to obtain

$$
\begin{aligned}
&\begin{bmatrix} r(t) \\ \dot{\hat{\mathbf{x}}}(t)\end{bmatrix} = \Pi\left[\begin{array}{c|c} D_{rv} & C_{rx} \\ \hline B_{xv} & \{\hat{A}_i\}\end{array}\right]\begin{bmatrix} v(t) \\ \hat{\underline{\mathbf{x}}}(t)\end{bmatrix} = \\
&\Pi\left[\begin{array}{c|c} [0 \quad D_{rb}] & C_r \\ \hline [B_{xv} \quad B_{xb}] & \{A_i\}\end{array}\right] \Pi\left[\begin{array}{c|c}\begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v\end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s)\end{bmatrix} \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, R_{D,1}, R_{D,2}\}\end{array}\right]\begin{bmatrix} v(t) \\ \hat{\underline{\mathbf{x}}}(t)\end{bmatrix}.
\end{aligned}
$$

We need to eliminate $\hat{\underline{\mathbf{x}}}(t)$ from the right hand side to get an expression solely in terms of $\hat{\mathbf{x}}$. For this purpose, we use Theorem 5.1b, which gives us the relation $\hat{\underline{\mathbf{x}}}(t) = \mathcal{D}\hat{\mathbf{x}}(t)$. Defining now

$$
(\mathcal{F}\hat{\mathbf{x}})(t) = \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \partial_s S\hat{\mathbf{x}}(t) \\ \vdots \\ \partial_s^N S^N\hat{\mathbf{x}}(t)\end{bmatrix}, \qquad (\mathcal{C}\hat{\mathbf{x}})(t) = \begin{bmatrix} S\hat{\mathbf{x}}(t) \\ \partial_s S^2\hat{\mathbf{x}}(t) \\ \vdots \\ \partial_s^{N-1} S^N\hat{\mathbf{x}}(t)\end{bmatrix} \qquad (\mathcal{B}\hat{\mathbf{x}}) = \begin{bmatrix} (\mathcal{C}\hat{\mathbf{x}})(t,a) \\ (\mathcal{C}\hat{\mathbf{x}})(t,b)\end{bmatrix}.
$$

Using Corollaries A.3 and A.4d, these definitions now imply

$$
\begin{bmatrix}\begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t)\end{bmatrix} \\ (\mathcal{F}\hat{\mathbf{x}})(t,\cdot)\end{bmatrix} = \Pi\left[\begin{array}{c|c}\begin{bmatrix} I_{n_v} \\ B_T^{-1}B_v \\ T(b-a)B_T^{-1}B_v\end{bmatrix} & \begin{bmatrix} 0_{n_r\times n_x} \\ B_Q(s) \\ T(b-a)B_Q(s)+Q(b-s)\end{bmatrix} \\ \hline U_2T(s-a)B_T^{-1}B_v & \{U_1, R_{D,1}, R_{D,2}\}\end{array}\right]\begin{bmatrix} v(t) \\ \hat{\underline{\mathbf{x}}}(t)\end{bmatrix}.
$$

Then, we can re-write the expressions for r and $\dot{\hat{\mathbf{x}}}$ as

$$
\begin{aligned}
&\begin{bmatrix} r(t) \\ \dot{\hat{\mathbf{x}}}(t)\end{bmatrix} = \Pi\left[\begin{array}{c|c} [0 \quad D_{rb}] & C_r \\ \hline [B_{xv} \quad B_{xb}] & \{A_i\}\end{array}\right]\begin{bmatrix}\begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t)\end{bmatrix} \\ (\mathcal{F}\hat{\mathbf{x}})(t,\cdot)\end{bmatrix} \\
&= \sum_{i=0}^{N}\begin{bmatrix} \int_a^b C_r(\cdot) \\ A_0(s) + \int_a^s A_1(s,\cdot) + \int_s^b A_2(s,\cdot)\end{bmatrix}(\mathcal{F}\hat{\mathbf{x}})(t,\cdot) + \begin{bmatrix} 0 & D_{rb} \\ B_{xv}(s) & B_{xb}(s)\end{bmatrix}\begin{bmatrix} v(t) \\ (\mathcal{B}\hat{\mathbf{x}})(t)\end{bmatrix}.
\end{aligned}
$$

Thus we conclude that $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE Equation (3.7) with initial condition $\hat{\mathbf{x}}^0$ and input v. □

A.3 Bijective Map between PIE and GPDE States

We now construct the map between the domain of the GPDE and associated PIE representation and show this is a bijection.

Corollary 5.3. *Given an* $n \in \mathbb{N}^{N+1}$*, and* $\mathbf{G}_\mathrm{b}$ *with* $\{n, \mathbf{G}_\mathrm{b}\}$ *admissible, let* $\{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u\}$ *be as defined in Block 5.2,* $\mathcal{X}_{w,u}$ *as defined in Equation* (3.9) *and* $\mathcal{D} = diag(\partial_s^0 I_{n_0}, \cdots, \partial_s^N I_{n_N})$*. Then for any* $w \in \mathbb{R}^{n_w}$ *and* $u \in \mathbb{R}^{n_u}$ *we have:*

(a) If $\mathbf{x} = \{x, \hat{\mathbf{x}}\} \in \mathcal{X}_{w,u}$*, then* $\underline{\mathbf{x}} = \{x, \mathcal{D}\hat{\mathbf{x}}\} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ *and* $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$*.*

(b) If $\underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$*, then* $\mathbf{x} = \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u \in \mathcal{X}_{w,u}$ *and* $\underline{\mathbf{x}} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & \mathcal{D} \end{bmatrix} \mathbf{x}$*.*

Proof. **Proof of Part 1.** Let $\begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathcal{X}_{w,u}$ for some $w \in \mathbb{R}^p, u \in \mathbb{R}^q$. Clearly, by definition of $\mathcal{X}_{w,u}$, $\hat{\mathbf{x}} \in X_v$ with $v = C_v x + D_{vw} w + D_{vu} u$ for arbitrary matrices C_v, D_{vw}, and D_{vu}. Therefore, from theorem 5.1a, $\mathcal{D}\hat{\mathbf{x}} \in L_2^{n_{\hat{\mathbf{x}}}}$ and hence $\{x, \mathcal{D}\hat{\mathbf{x}}\} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$. Furthermore, for $\hat{\mathcal{T}}$ and $\mathcal{T}_v$ as defined in Block 5.1, we have

$$\hat{\mathbf{x}} = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}} + \mathcal{T}_v v = \begin{bmatrix} \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} + \mathcal{T}_v \begin{bmatrix} D_{vw} & D_{vu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$

Then, by concatenating x and $\hat{\mathbf{x}}$ and by using the definitions of $\mathcal{T}$, $\mathcal{T}_w$, $\mathcal{T}_u$, we have

$$\begin{aligned}
\begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} &= \begin{bmatrix} x \\ \begin{bmatrix} \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} + \mathcal{T}_v \begin{bmatrix} D_{vw} & D_{vu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} x \\ \begin{bmatrix} \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{T}_v D_{vw} \end{bmatrix} w + \begin{bmatrix} 0 \\ \mathcal{T}_v D_{vu} \end{bmatrix} u \\
&= \begin{bmatrix} I & 0 \\ \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{T}_v D_{vw} \end{bmatrix} w + \begin{bmatrix} 0 \\ \mathcal{T}_v D_{vu} \end{bmatrix} u = \mathcal{T} \begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} + \mathcal{T}_w w + \mathcal{T}_u u.
\end{aligned}$$

□

Proof. **Proof of Part 2.** Let $w \in \mathbb{R}^p$, $u \in \mathbb{R}^q$ and $\underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ be arbitrary.

Let $\begin{bmatrix} x \\ \hat{\underline{\mathbf{x}}} \end{bmatrix} = \underline{\mathbf{x}}$ where $x \in \mathbb{R}^{n_x}$ and $\hat{\underline{\mathbf{x}}} \in L_2^{n_{\hat{\mathbf{x}}}}$.

By substituting the definitions of $\mathcal{T}$, $\mathcal{T}_w$ and $\mathcal{T}_u$,

$$\begin{aligned}
\mathcal{T} \begin{bmatrix} x \\ \hat{\underline{\mathbf{x}}} \end{bmatrix} + \mathcal{T}_w w + \mathcal{T}_u u &= \begin{bmatrix} I & 0 \\ \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x \\ \hat{\underline{\mathbf{x}}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{T}_v D_{vw} \end{bmatrix} w + \begin{bmatrix} 0 \\ \mathcal{T}_v D_{vu} \end{bmatrix} u \\
&= \underbrace{\begin{bmatrix} x \\ \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v \left(\begin{bmatrix} C_v & D_{vw} & D_{vu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \right) \end{bmatrix}}_{\begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} =}.
\end{aligned}$$

Clearly, from theorem 5.1b, defining $\hat{\mathbf{x}}$ as $\hat{\mathbf{x}} = \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}+\mathcal{T}_v\left([C_v \quad D_{vw} \quad D_{vu}]\begin{bmatrix}x\\w\\u\end{bmatrix}\right)$ implies that $\hat{\mathbf{x}} \in X_v$ with $v = [C_v \quad D_{vw} \quad D_{vu}]\begin{bmatrix}x\\w\\u\end{bmatrix}$. Therefore, by definition of $\mathcal{X}_{w,u}$, $\begin{bmatrix}x\\\hat{\mathbf{x}}\end{bmatrix} \in \mathcal{X}_{w,u}$.

Our next step is to show that $\begin{bmatrix}I & 0\\0 & \mathcal{D}\end{bmatrix}(\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u) = \underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x,n_{\hat{x}}}$. Earlier, we defined $\begin{bmatrix}x\\\underline{\hat{\mathbf{x}}}\end{bmatrix} = \underline{\mathbf{x}}$ and showed that if we define $(\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u) = \begin{bmatrix}x\\\hat{\mathbf{x}}\end{bmatrix}$ for some $\hat{\mathbf{x}} \in X_v$ with $v = C_v x + D_{vw}w + D_{vu}u$. Thus, from theorem 5.1b, we have

$$\mathcal{D}\hat{\mathbf{x}} = \mathcal{D}\left(\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v\right) = \underline{\hat{\mathbf{x}}}.$$

Therefore,

$$\begin{bmatrix}I & 0\\0 & \mathcal{D}\end{bmatrix}(\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u) = \begin{bmatrix}I & 0\\0 & \mathcal{D}\end{bmatrix}\begin{bmatrix}x\\\hat{\mathbf{x}}\end{bmatrix} = \begin{bmatrix}x\\\mathcal{D}\hat{\mathbf{x}}\end{bmatrix} = \begin{bmatrix}x\\\underline{\hat{\mathbf{x}}}\end{bmatrix} = \underline{\mathbf{x}}.$$

□

A.4 Equivalence of PIE and GPDE

The equivalence of solutions between a GPDE model and associated PIE is a straightforward extension of Theorem 5.4). This proof is split into two parts.

Corollary 5.5 (Corollary of Theorem 5.4)**.** *Given an $n \in \mathbb{N}^{N+1}$ and parameters $\{\mathbf{G}_\text{o}$, $\mathbf{G}_\text{b}$, $\mathbf{G}_\text{p}\}$ as defined in Equations (3.2), (3.6) and (3.8) with $\{n, \mathbf{G}_\text{b}\}$ admissible, let $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ with $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $u \in L_{2e}^{n_u}[\mathbb{R}_+]$ with $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$. Define*

$$\mathbf{G}_\text{PIE} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\} = \mathbf{M}(\{n, \mathbf{G}_\text{b}, \mathbf{G}_\text{o}, \mathbf{G}_\text{p}\}.$$

Then we have the following:

1. *For any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{w(0),u(0)}$ (where $\mathcal{X}_{w,u}$ is as defined in Equation (3.9)), if $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_\text{o}, \mathbf{G}_\text{b}, \mathbf{G}_\text{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{w, u\}$, then $\left\{\begin{bmatrix}x\\\mathcal{D}\hat{\mathbf{x}}\end{bmatrix}, z, y\right\}$ satisfies the PIE defined by $\mathbf{G}_\text{PIE}$ with initial condition $\begin{bmatrix}x^0\\\mathcal{D}\hat{\mathbf{x}}^0\end{bmatrix}$ and input $\{w, u\}$ where $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0\hat{\mathbf{x}}_0, \cdots, \partial_s^N\hat{\mathbf{x}}_N)$.*
2. *For any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{x}}}$, if $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_\text{PIE}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{w, u\}$, then $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE*

defined by $\{n, \mathbf{G}_\text{o}, \mathbf{G}_\text{b}, \mathbf{G}_\text{p}\}$ *with initial condition* $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 + \mathcal{T}_w w(0) + \mathcal{T}_u u(0)$ *and input* $\{w, u\}$ *where*

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t),$$
$$v(t) = C_v x(t) + D_{vw} w(t) + D_{vu} u(t),$$
$$r(t) = [0_{n_{\hat{\mathbf{x}}}\times n_x} \quad \mathcal{C}_r]\, \underline{\mathbf{x}}(t) + \mathcal{D}_{rv} v(t),$$

and where $\mathcal{C}_r$ *and* $\mathcal{D}_{rv}$ *are as defined in Block 5.2.*

Proof of Part 1. Suppose $\{x,\, \hat{\mathbf{x}},\, z,\, y,\, v,\, r\}$ satisfies the GPDE defined by $\{\mathbf{G}_\text{o}, \mathbf{G}_\text{b}, \mathbf{G}_\text{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{w, u\}$. Then, we have: a) $x \in W_{1e}^{n_x}[\mathbb{R}_+]$, $z \in L_{2e}^{n_z}[\mathbb{R}_+]$, $y \in L_{2e}^{n_y}[\mathbb{R}_+]$, $v \in L_{2e}^{n_v}[\mathbb{R}_+]$, $r \in L_{2e}^{n_r}[\mathbb{R}_+]$; b) $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$; c) x is differentiable almost everywhere on $\mathbb{R}_+$, $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$; d) Equations (3.1) and (3.7) are satisfied for almost all $t \geq 0$; and e) $x(0) = x^0$, $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$ and $\hat{\mathbf{x}}^0 \in X_{v(0)}$.

Now, from above points, since $\hat{\mathbf{x}}^0 \in X_{v(0)}$ and $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$, $r \in L_{2e}^{n_r}[\mathbb{R}_+]$, $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$, $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$, and Equation (3.7) is satisfied for almost all $t \geq 0$, we have that $\{\hat{\mathbf{x}}, r\}$ satisfies the PDE defined by $n \in \mathbb{N}^{N+1}$ and $\{\mathbf{G}_\text{b}, \mathbf{G}_\text{p}\}$ with initial conditions $\hat{\mathbf{x}}^0$ and input v. Furthermore, since

$$v(t) = C_v x(t) + D_{vw} w(t) + D_{vu} u(t),$$

we have that $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ with $B_v v \in W_{1e}^{2n_S}[\mathbb{R}_+]$. Thus, by Theorem 5.4, $\{\mathcal{D}\hat{\mathbf{x}},\ r\}$ is a solution to the PIE defined

$$\mathbf{G}_{\text{PIE}_s} = \left\{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_r, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\right\}$$

with initial condition $\mathcal{D}\hat{\mathbf{x}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$. Therefore, if we define $\hat{\underline{\mathbf{x}}}(t) = \mathcal{D}\hat{\mathbf{x}}(t)$ and $\hat{\underline{\mathbf{x}}}^0 = \mathcal{D}\hat{\mathbf{x}}^0$, we have that: f) $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ and $(\mathcal{T}_v v)(\cdot, s) \in W_{1e}^{n_{\hat{\mathbf{x}}}}[\mathbb{R}_+]$ for all $s \in [a, b]$; g) $\hat{\underline{\mathbf{x}}} : \mathbb{R}_+ \to \mathbb{R}L_2^{0, n_{\hat{\mathbf{x}}}}[a, b]$ and $r \in L_{2e}^{n_r}[\mathbb{R}_+]$; h) $\hat{\underline{\mathbf{x}}}^0 \in \mathbb{R}L_2^{0, n_{\hat{\mathbf{x}}}}[a, b]$ and $\hat{\underline{\mathbf{x}}}(0) = \hat{\underline{\mathbf{x}}}^0$; i) $\hat{\underline{\mathbf{x}}}$ is Frechét differentiable with respect to the $\hat{\mathcal{T}}$-norm almost everywhere on $\mathbb{R}_+$; and j) Equation (4.1) (defined by $\mathbf{G}_{\text{PIE}_s}$) is satisfied for almost all $t \in \mathbb{R}_+$, i.e.

$$\begin{bmatrix} r(t) \\ \hat{\mathcal{T}}\dot{\hat{\underline{\mathbf{x}}}}(t) + \mathcal{T}_v \dot{v}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{rv} & \mathcal{C}_r \\ \mathcal{B}_v & \hat{\mathcal{A}} \end{bmatrix} \begin{bmatrix} v(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix}.$$

Now, let $\underline{\mathbf{x}}^0 = \begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0 \end{bmatrix} = \begin{bmatrix} x^0 \\ \hat{\underline{\mathbf{x}}}^0 \end{bmatrix}$ and $\underline{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix}$ for all $t \geq 0$. Our goal is to show that $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_\text{PIE}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{w, u\}$, which means we need to show that: 1) $(\mathcal{T}_w w)(\cdot, s), (\mathcal{T}_u u)(\cdot, s) \in W_{1e}^{n_x + n_{\hat{\mathbf{x}}}}$ for all $s \in [a, b]$; 2) $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}[a, b]$ for all $t \geq 0$; 3) $\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}^0$ and $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$; 4) $\underline{\mathbf{x}}$ is Frechét differentiable with respect to the $\mathcal{T}$-norm almost everywhere on $\mathbb{R}_+$; and 5) Equation (4.1) (defined by $\mathbf{G}_\text{PIE}$) is satisfied for almost all $t \in \mathbb{R}_+$.

For 1), $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$ and hence by the definition of $\mathcal{T}_w$, we have

$$(\mathcal{T}_w w(\cdot))(s) = \begin{bmatrix} 0 \\ T_1(s-a) \end{bmatrix} B_T^{-1} B_v D_{vw} w(\cdot) \in W_{1e}^{n_x+n_{\hat{\mathbf{x}}}}[\mathbb{R}_+].$$

Likewise, $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$ implies

$$(\mathcal{T}_u u(\cdot))(s) = \begin{bmatrix} 0 \\ T_1(s-a) \end{bmatrix} B_T^{-1} B_v D_{vu} u(\cdot) \in W_{1e}^{n_x+n_{\hat{\mathbf{x}}}}[\mathbb{R}_+].$$

For 2), since $\hat{\underline{\mathbf{x}}}(t) \in L_2^{0,n_{\hat{\mathbf{x}}}}[a,b]$ and $x(t) \in \mathbb{R}^{n_x}$, we have $\underline{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ for all $t \geq 0$.

For 3), since $\hat{\underline{\mathbf{x}}}^0 \in L_2^{0,n_{\hat{\mathbf{x}}}}[a,b]$ and $x^0 \in \mathbb{R}^{n_x}$, we have $\underline{\mathbf{x}}^0 = \begin{bmatrix} x^0 \\ \hat{\underline{\mathbf{x}}}^0 \end{bmatrix} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$. Furthermore, $\underline{\mathbf{x}}(0) = \begin{bmatrix} x(0) \\ \hat{\underline{\mathbf{x}}}(0) \end{bmatrix} = \begin{bmatrix} x^0 \\ \hat{\underline{\mathbf{x}}}^0 \end{bmatrix} = \underline{\mathbf{x}}^0$.

For 4), by definition of $\mathcal{T}$-norm and definitions of $\mathcal{T}$ and $\underline{\mathbf{x}}$, there exists a $k > 0$ such that

$$\begin{aligned} \|\underline{\mathbf{x}}(t)\|_{\mathcal{T}} = \|\mathcal{T}\underline{\mathbf{x}}(t)\|_{L_2} &= \left\| \begin{bmatrix} I & 0 \\ G_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix} \right\|_{L_2} = \left\| \begin{bmatrix} x(t) \\ G_v C_v x(t) + \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}(t) \end{bmatrix} \right\|_{L_2} \\ &= \|x(t)\|_{L_2} + \left\| G_v C_v x(t) + \hat{\mathcal{T}}\hat{\underline{\mathbf{x}}}(t) \right\|_{L_2} \leq k\, \|x(t)\|_{\mathbb{R}} + \|\hat{\underline{\mathbf{x}}}(t)\|_{\hat{\mathcal{T}}}. \end{aligned}$$

Since $\hat{\underline{\mathbf{x}}}(t)$ is Frechét differentiable with respect to the $\hat{\mathcal{T}}$ norm and $x \in W_{1e}^{n_x}$ is differentiable, we have that $\underline{\mathbf{x}}(t)$ is Frechét differentiable with respect to the $\mathcal{T}$ norm.

Finally, for 5), we need to show that

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \\ u(t) \end{bmatrix} - \begin{bmatrix} \mathcal{T}_w \dot{w}(t) + \mathcal{T}_u \dot{u}(t) \\ 0 \\ 0 \end{bmatrix}$$

is satisfied for all $t \geq 0$.

Since x, z, y, v satisfy the GPDE, we have

$$\left[\begin{array}{c} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{array}\right] = \left[\begin{array}{c|ccc} A & B_{xw} & B_{xu} & B_{xr} \\ \hline C_z & D_{zw} & D_{zu} & D_{zr} \\ C_y & D_{yw} & D_{yu} & D_{yr} \\ C_v & D_{vw} & D_{vu} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{array}\right]. \tag{A.7}$$

Furthermore, as stated above,

$$\begin{bmatrix} \hat{\mathcal{T}}\dot{\hat{\underline{\mathbf{x}}}}(t) + \mathcal{T}_v \dot{v}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{A}} & \mathcal{B}_v \\ \mathcal{C}_r & \mathcal{D}_{rv} \end{bmatrix} \begin{bmatrix} \hat{\underline{\mathbf{x}}}(t) \\ v(t) \end{bmatrix}. \tag{A.8}$$

These two identities are all that are required to conclude the proof. Specifically, extracting expression for v, r and $\mathcal{T}_v \dot{v}$, we obtain

$$v(t) = [C_v \quad D_{vw} \quad D_{vu}] \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix},$$

$$r(t) = \mathcal{C}_r \hat{\underline{\mathbf{x}}}(t) + D_{rv} v(t) = [\mathcal{D}_{rv} C_v \quad \mathcal{C}_r \quad \mathcal{D}_{rv} D_{vw} \quad \mathcal{D}_{rv} D_{vu}] \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ w(t) \\ u(t) \end{bmatrix},$$

$$\begin{aligned} (\mathcal{T}_v \dot{v}(t))(s) &= G_v(s) \, [C_v \quad D_{vw} \quad D_{vu}] \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \\ \dot{u}(t) \end{bmatrix} \\ &= G_v(s) C_v \dot{x}(t) + G_v(s) D_{vw} \dot{w}(t) + G_v(s) D_{vu} \dot{u}(t). \end{aligned}$$

Substituting these expressions back into Eq. (A.8) yields

$$\begin{aligned} &\hat{\mathcal{T}} \dot{\hat{\underline{\mathbf{x}}}}(t) + G_v(s) C_v \dot{x}(t) + G_v(s) D_{vw} \dot{w}(t) + G_v(s) D_{vu} \dot{u}(t) \\ &= \hat{\mathcal{A}} \hat{\underline{\mathbf{x}}}(t) + \mathcal{B}_v \, [C_v \quad D_{vw} \quad D_{vu}] \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} &\begin{bmatrix} G_v(s) C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{\underline{\mathbf{x}}}}(t) \end{bmatrix} + G_v(s) D_{vw} \dot{w}(t) + G_v(s) D_{vu} \dot{u}(t) \\ &= \begin{bmatrix} \mathcal{B}_v C_v & \hat{\mathcal{A}} & \mathcal{B}_v D_{vw} & \mathcal{B}_v D_{vu} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ w(t) \\ u(t) \end{bmatrix}. \end{aligned}$$

Appending the above equation to the system of equations in Eq. (A.7) and omitting the equation for v yields

$$\left[\begin{array}{c}\begin{bmatrix} I & 0 \\ G_v(s)C_v & \hat{\mathcal{T}}\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{\underline{\mathbf{x}}}}\end{bmatrix} + \mathcal{T}_w\dot{w}(t) + \mathcal{T}_u\dot{u}(t) \\ \hline z(t) \\ y(t)\end{array}\right]$$

$$= \left[\begin{array}{cc|cc} A & 0 & B_{xw} & B_{xu} \\ \mathcal{B}_v C_v & \hat{\mathcal{A}} & \mathcal{B}_v D_{vw} & \mathcal{B}_v D_{vu} \\ \hline C_z & 0 & D_{zw} & D_{zu} \\ C_y & 0 & D_{yw} & D_{yu}\end{array}\right]\left[\begin{array}{c} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ \hline w(t) \\ u(t)\end{array}\right] + \begin{bmatrix}\begin{bmatrix} B_{xr} \\ 0\end{bmatrix} \\ D_{zr} \\ D_{yr}\end{bmatrix} r(t)$$

$$= \left[\begin{array}{cc|cc} A & 0 & B_{xw} & B_{xu} \\ \mathcal{B}_v C_v & \hat{\mathcal{A}} & \mathcal{B}_v D_{vw} & \mathcal{B}_v D_{vu} \\ \hline C_z & 0 & D_{zw} & D_{zu} \\ C_y & 0 & D_{yw} & D_{yu}\end{array}\right]\left[\begin{array}{c} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ \hline w(t) \\ u(t)\end{array}\right]$$

$$+ \begin{bmatrix}\begin{bmatrix} B_{xr} \\ 0\end{bmatrix} \\ D_{zr} \\ D_{yr}\end{bmatrix}\begin{bmatrix}\mathcal{D}_{rv}C_v & \mathcal{C}_r & \mathcal{D}_{rv}D_{vw} & \mathcal{D}_{rv}D_{vu}\end{bmatrix}\begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ w(t) \\ u(t)\end{bmatrix}$$

$$= \left[\begin{array}{c|cc}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22}\end{array}\right]\left[\begin{array}{c}\underline{\mathbf{x}}(t) \\ \hline w(t) \\ u(t)\end{array}\right].$$

We conclude that

$$\begin{bmatrix}\mathcal{T}\dot{\underline{\mathbf{x}}}(t) \\ z(t) \\ y(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22}\end{bmatrix}\begin{bmatrix}\underline{\mathbf{x}}(t) \\ w(t) \\ u(t)\end{bmatrix} - \begin{bmatrix}\mathcal{T}_w\dot{w}(t) + \mathcal{T}_u\dot{u}(t) \\ 0 \\ 0\end{bmatrix}$$

which implies that $\{\begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}}\end{bmatrix}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\text{PIE}}$ with initial condition $\begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}\end{bmatrix} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ and input $\{w, u\}$. □

We now proceed with Part 2 of the proof – starting with a restatement of the Corollary.

Corollary 5.5 (Corollary of Theorem 5.4)**.** *Given an* $n \in \mathbb{N}^{N+1}$ *and parameters* $\{\mathbf{G}_{\text{o}}, \mathbf{G}_{\text{b}}, \mathbf{G}_{\text{p}}\}$ *as defined in Equations* (3.2), (3.6) *and* (3.8) *with* $\{n, \mathbf{G}_{\text{b}}\}$ *admissible, let* $w \in L_{2e}^{n_w}[\mathbb{R}_+]$ *with* $B_v D_{vw} w \in W_{1e}^{2n_S}[\mathbb{R}_+]$, $u \in L_{2e}^{n_u}[\mathbb{R}_+]$ *with* $B_v D_{vu} u \in W_{1e}^{2n_S}[\mathbb{R}_+]$*. Define*

$$\mathbf{G}_{\text{PIE}} = \{\mathcal{T}, \mathcal{T}_w, \mathcal{T}_u, \mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}\} = \mathbf{M}(\{n, \mathbf{G}_{\text{b}}, \mathbf{G}_{\text{o}}, \mathbf{G}_{\text{p}}\}.$$

Then we have the following:

1. *For any* $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{w(0),u(0)}$ *(where* $\mathcal{X}_{w,u}$ *is as defined in Equation* (3.9)*), if* $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ *satisfies the GPDE defined by* $\{n,\ \mathbf{G}_{\mathrm{o}},\ \mathbf{G}_{\mathrm{b}},\ \mathbf{G}_{\mathrm{p}}\}$ *with initial condition* $\{x^0, \hat{\mathbf{x}}^0\}$ *and input* $\{w, u\}$*, then* $\left\{\begin{bmatrix} x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix}, z, y\right\}$ *satisfies the PIE defined by* $\mathbf{G}_{\mathrm{PIE}}$ *with initial condition* $\begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0 \end{bmatrix}$ *and input* $\{w, u\}$ *where* $\mathcal{D}\hat{\mathbf{x}} = col(\partial_s^0 \hat{\mathbf{x}}_0, \cdots, \partial_s^N \hat{\mathbf{x}}_N)$*.*

2. *For any* $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$*, if* $\{\underline{\mathbf{x}},\ z,\ y\}$ *satisfies the PIE defined by* $\mathbf{G}_{\mathrm{PIE}}$ *with initial condition* $\underline{\mathbf{x}}^0$ *and input* $\{w, u\}$*, then* $\{x,\ \hat{\mathbf{x}},\ z,\ y,\ v,\ r\}$ *satisfies the GPDE defined by* $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ *with initial condition* $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 + \mathcal{T}_w w(0) + \mathcal{T}_u u(0)$ *and input* $\{w, u\}$ *where*

$$\begin{aligned}
\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} &= \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t), \\
v(t) &= C_v x(t) + D_{vw} w(t) + D_{vu} u(t), \\
r(t) &= [0_{n_{\hat{\mathbf{x}}} \times n_x} \quad \mathcal{C}_r]\, \underline{\mathbf{x}}(t) + \mathcal{D}_{rv} v(t),
\end{aligned}$$

and where $\mathcal{C}_r$ *and* $\mathcal{D}_{rv}$ *are as defined in Block 5.2.*

Proof of Part 2. In this proof, we will use definitions in Block 5.2 using the parameters contained in $\{\mathbf{G}_{\mathrm{o}},\ \mathbf{G}_{\mathrm{b}},\ \mathbf{G}_{\mathrm{p}}\}$.

Now, suppose $\{\underline{\mathbf{x}},\ z,\ y\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{w, u\}$. Then, by definition of solution of a PIE: a) $z \in L_{2e}^{n_z}[\mathbb{R}_+]$, $y \in L_{2e}^{n_y}[\mathbb{R}_+]$; b) $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}[a, b]$ for all $t \geq 0$; c) $\underline{\mathbf{x}}$ is Frechét differentiable with respect to the $\mathcal{T}$-norm almost everywhere on $\mathbb{R}_+$; d) $\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}^0$; and e) The equation

$$\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) + \mathcal{T}_w \dot{w}(t) + \mathcal{T}_u \dot{u}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}}(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{A.9}$$

is satisfied for almost all $t \in \mathbb{R}_+$.

For $\underline{\mathbf{x}}(t) \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ we define $\hat{x}(t) \in \mathbb{R}^{n_x}$ and $\hat{\underline{\mathbf{x}}}(t) \in L_2^{n_{\hat{\mathbf{x}}}}$ by $\begin{bmatrix} \hat{x}(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix} = \underline{\mathbf{x}}(t)$. Similarly, we define the elements $\begin{bmatrix} \hat{x}^0 \\ \hat{\underline{\mathbf{x}}}^0 \end{bmatrix} = \underline{\mathbf{x}}^0$. Now, by the definitions of $\mathcal{T}$, $\mathcal{T}_w$ and $\mathcal{T}_u$, we have

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ G_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ G_v D_{vw} \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ G_v D_{vu} \end{bmatrix} u(t)$$

and hence $x(t) = \hat{x}(t)$. Similarly, $x^0 = \hat{x}^0$. Hence we have $\underline{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \end{bmatrix}$ and $\underline{\mathbf{x}}^0 = \begin{bmatrix} x^0 \\ \hat{\underline{\mathbf{x}}}^0 \end{bmatrix}$.

Now, using the definitions of r and v and examining the right hand side of Eq. (A.9), we have

$$\begin{aligned}
\left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}\right] \left[\begin{array}{c} \underline{\mathbf{x}}(t) \\ \hline w(t) \\ u(t) \end{array}\right] &= \left[\begin{array}{cc|cc} A & 0 & B_{xw} & B_{xu} \\ \hat{B}_{xv}C_v & \hat{\mathcal{A}} & \hat{B}_{xv}D_{vw} & \hat{B}_{xv}D_{vu} \\ \hline C_z & 0 & D_{zw} & D_{zu} \\ C_y & 0 & D_{yw} & D_{yu} \end{array}\right] \left[\begin{array}{c} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ \hline w(t) \\ u(t) \end{array}\right] \\
&\quad + \begin{bmatrix} \begin{bmatrix} B_{xr} \\ 0 \end{bmatrix} \\ D_{zr} \\ D_{yr} \end{bmatrix} \begin{bmatrix} \mathcal{D}_{rv}C_v & \mathcal{C}_r & \mathcal{D}_{rv}D_{vw} & \mathcal{D}_{rv}D_{vu} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ w(t) \\ u(t) \end{bmatrix} \\
&= \left[\begin{array}{cc|cc} A & 0 & B_{xw} & B_{xu} \\ \mathcal{B}_vC_v & \hat{\mathcal{A}} & \mathcal{B}_vD_{vw} & \mathcal{B}_vD_{vu} \\ \hline C_z & 0 & D_{zw} & D_{zu} \\ C_y & 0 & D_{yw} & D_{yu} \end{array}\right] \left[\begin{array}{c} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ \hline w(t) \\ u(t) \end{array}\right] + \begin{bmatrix} \begin{bmatrix} B_{xr} \\ 0 \end{bmatrix} \\ D_{zr} \\ D_{yr} \end{bmatrix} r(t) \\
&= \left[\begin{array}{cc|ccc} A & 0 & B_{xw} & B_{xu} & B_{xr} \\ \mathcal{B}_vC_v & \hat{\mathcal{A}} & \mathcal{B}_vD_{vw} & \mathcal{B}_vD_{vu} & 0 \\ \hline C_z & 0 & D_{zw} & D_{zu} & D_{zr} \\ C_y & 0 & D_{yw} & D_{yu} & D_{yr} \end{array}\right] \left[\begin{array}{c} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{array}\right].
\end{aligned}$$

Likewise, if we substitute the definitions of the PI operators $\mathcal{T}$, $\mathcal{T}_w$, and $\mathcal{T}_u$ in the left hand side of Eq. (A.9), we get

$$\begin{aligned}
&\begin{bmatrix} \mathcal{T}\dot{\underline{\mathbf{x}}}(t) + \mathcal{T}_w\dot{w}(t) + \mathcal{T}_u\dot{u}(t) \\ z(t) \\ y(t) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} I & 0 \\ G_v(s)C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{\underline{\mathbf{x}}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ G_v(s)D_{vw} \end{bmatrix} \dot{w}(t) + \begin{bmatrix} 0 \\ G_v(s)D_{vu} \end{bmatrix} \dot{u}(t) \\ z(t) \\ y(t) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \hat{\mathcal{T}}\dot{\hat{\underline{\mathbf{x}}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{T}_v\dot{v}(t) \end{bmatrix} \\ z(t) \\ y(t) \end{bmatrix}.
\end{aligned}$$

Adding the definition of v, we conclude that

$$\begin{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \hat{\mathcal{T}}\dot{\hat{\underline{\mathbf{x}}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{T}_v\dot{v}(t) \end{bmatrix} \\ z(t) \\ y(t) \\ v(t) \end{bmatrix} = \left[\begin{array}{cc|ccc} A & 0 & B_{xw} & B_{xu} & B_{xr} \\ \mathcal{B}_vC_v & \hat{\mathcal{A}} & \mathcal{B}_vD_{vw} & \mathcal{B}_vD_{vu} & 0 \\ \hline C_z & 0 & D_{zw} & D_{zu} & D_{zr} \\ C_y & 0 & D_{yw} & D_{yu} & D_{yr} \\ C_v & 0 & D_{vw} & D_{vu} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ \hat{\underline{\mathbf{x}}}(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{array}\right].$$

Therefore,

$$\left[\begin{array}{c} \dot{x}(t) \\ \hline z(t) \\ y(t) \\ v(t) \end{array}\right] = \left[\begin{array}{c|ccc} A & B_{xw} & B_{xu} & B_{xr} \\ \hline C_z & D_{zw} & D_{zu} & D_{zr} \\ C_y & D_{yw} & D_{yu} & D_{yr} \\ C_v & D_{vw} & D_{vu} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ \hline w(t) \\ u(t) \\ r(t) \end{array}\right]$$

and

$$\begin{bmatrix} \hat{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t)+\mathcal{T}_v\dot{v}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{A}} & \mathcal{B}_v \\ \mathcal{C}_r & \mathcal{D}_{rv} \end{bmatrix} \begin{bmatrix} \underline{\hat{\mathbf{x}}}(t) \\ v(t) \end{bmatrix}.$$

Thus, we conclude: f) Since $\underline{\mathbf{x}}(t)$ is Frechét differentiable, $x(t)$ and $\underline{\hat{\mathbf{x}}}(t)$ are Frechét differentiable; g) Since $x(t)$ is Frechét differentiable, $x \in W_{1e}^{n_x}$ and $w \in L_{2e}^{n_w}$, $u \in L_{2e}^{n_u}$ from the theorem statement, thus $v \in L_{2e}^{n_v}$; h) Since $\underline{\hat{\mathbf{x}}}(t)$ is Frechét differentiable and $v \in L_{2e}^{n_v}$, we have $r \in L_{2e}^{n_r}$; i) Since $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$, we have $x^0 \in \mathbb{R}^{n_x}$, $\underline{\hat{\mathbf{x}}}^0 \in L_2^{n_{\hat{\mathbf{x}}}}$; and j) For all $t \geq 0$

$$\begin{bmatrix} r(t) \\ \hat{\mathcal{T}}\dot{\underline{\hat{\mathbf{x}}}}(t)+\mathcal{T}_v\dot{v}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{rv} & \mathcal{C}_v \\ \mathcal{B}_v & \hat{\mathcal{A}} \end{bmatrix} \begin{bmatrix} v(t) \\ \underline{\hat{\mathbf{x}}}(t) \end{bmatrix}.$$

Thus, $\{\underline{\hat{\mathbf{x}}}, r\}$ (as defined above), satisfies the PIE defined by

$$\mathbf{G}_{\mathrm{PIE}_s} = \{\hat{\mathcal{T}}, \mathcal{T}_v, \emptyset, \hat{\mathcal{A}}, \mathcal{B}_v, \emptyset, \mathcal{C}_v, \emptyset, \mathcal{D}_{rv}, \emptyset, \emptyset, \emptyset\}$$

for initial condition $\underline{\hat{\mathbf{x}}}^0$ and input $\{v, \emptyset\}$. Thus, from theorem 5.4, $\{\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}} + \mathcal{T}_v v,\ r\}$ satisfies the PDE defined by n and $\{\mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$ and input v. Since

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t) + \mathcal{T}_w w(t) + \mathcal{T}_u u(t) = \begin{bmatrix} x(t) \\ \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{T}_v v(t) \end{bmatrix}$$

and since similarly $\hat{\mathbf{x}}^0 = \hat{\mathcal{T}}\underline{\hat{\mathbf{x}}}^0 + \mathcal{T}_v v(0)$, by the definition of solution of a PDE in 3.2, we have: k) $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$; l) $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm almost everywhere on $\mathbb{R}_+$; m) Equation (3.7) is satisfied for almost all $t \geq 0$; and n) $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}^0$.

Reviewing all the above steps, we conclude that: 1) $z \in L_{2e}^{n_z}[\mathbb{R}_+]$ and $y \in L_{2e}^{n_y}[\mathbb{R}_+]$ by definition of solution of the PIE defined by $\mathbf{G}_\mathrm{PIE}$; 2) $v \in L_{2e}^{n_v}[\mathbb{R}_+]$ and $r \in L_{2e}^{n_r}[\mathbb{R}_+]$ since r, v satisfy the PDE; 3) $\hat{\mathbf{x}}(t) \in X_{v(t)}$ for all $t \geq 0$ since $\hat{\mathbf{x}}$ satisfies the PDE; 4) $x \in W_{1e}^{n_x}$ since $\hat{\mathbf{x}}$ is Frechét differentiable; 5) $\hat{\mathbf{x}}$ is Frechét differentiable with respect to the L_2-norm since $\hat{\mathbf{x}}$ satisfies the PDE; $\begin{bmatrix} x(0) \\ \hat{\mathbf{x}}(0) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(0) + \mathcal{T}_w w(0) + \mathcal{T}_u u(0) = \mathcal{T}\underline{\mathbf{x}}^0 + \mathcal{T}_w w(0) + \mathcal{T}_u u(0)$ by definition of x and $\hat{\mathbf{x}}$; and 6) Equations (3.1) and (3.7) are satisfied for almost all $t \geq 0$ as shown above.

We conclude that $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by n and $\{\mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix}$ and input $\{w, u\}$. $\square$

A.5 Equivalence of Internal Stability

Having proven the equivalence between solutions of GPDE model and associated PIE, we now prove that these models have the same internal stability properties. Specifically, when $u = w = 0$, the solution to associated PIE is stable if and only if the solution to GPDE model is internally stable. We do this in three parts. First, we show that the map $\underline{\mathbf{x}} \to \mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w + \mathcal{T}_u u$ is an isometric map between inner product spaces L_2 and X^n. Next, we show that the W^n and X^n (defined in Equation (5.6)) norms are equivalent. Finally, we show equivalence of internal stability in the respective norms.

A.5.1 Proof of Theorem 5.7

Theorem 5.7. *Suppose* $\{n, \mathbf{G}_\mathrm{b}\}$ *is admissible,* $\{\hat{\mathcal{T}}, \mathcal{T}_v\}$ *are as defined in Block 5.1, and* $\{\mathcal{T},\ \mathcal{T}_w,\ \mathcal{T}_u\}$ *are as defined in Block 5.2 for some matrices* C_v*,* D_{vw} *and* D_{vu}*. If* $\langle\cdot,\cdot\rangle_{X^n}$ *is as defined in Equation* (5.6)*, then we have the following:*

a) for any $v_1, v_2 \in \mathbb{R}^{n_v}$ *and* $\hat{\underline{\mathbf{x}}},\ \hat{\underline{\mathbf{y}}} \in L_2^{n_{\hat{\mathbf{x}}}}$

$$\left\langle \left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v_1\right), \left(\hat{\mathcal{T}}\hat{\underline{\mathbf{y}}} + \mathcal{T}_v v_2\right) \right\rangle_{X^n} = \left\langle \hat{\underline{\mathbf{x}}}, \hat{\underline{\mathbf{y}}} \right\rangle_{L_2^{n_{\hat{\mathbf{x}}}}}. \tag{5.7}$$

b) for any $w_1, w_2 \in \mathbb{R}^{n_w}$*,* $u_1, u_2 \in \mathbb{R}^{n_u}$*,* $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$*,*

$$\begin{aligned}&\left\langle (\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w_1 + \mathcal{T}_u u_1), \left(\mathcal{T}\underline{\mathbf{y}} + \mathcal{T}_w w_2 + \mathcal{T}_u u_2\right) \right\rangle_{\mathbb{R}^{n_x} \times X^n} \\ &\quad = \left\langle \underline{\mathbf{x}}, \underline{\mathbf{y}} \right\rangle_{\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}}.\end{aligned} \tag{5.8}$$

Proof. Let $\hat{\underline{\mathbf{x}}}, \hat{\underline{\mathbf{y}}} \in L_2^{n_{\hat{\mathbf{x}}}}$ and $v_1, v_2 \in \mathbb{R}^p$. Then, from Theorem 5.1, we have

$$\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v_1 \in X_{v_1}, \quad \hat{\mathcal{T}}\hat{\underline{\mathbf{y}}} + \mathcal{T}_v v_2 \in X_{v_2}.$$

Therefore, by definition Equation (5.6) and the result in Theorem 5.1b,

$$\begin{aligned}\left\langle \left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v_1\right), \left(\hat{\mathcal{T}}\hat{\underline{\mathbf{y}}} + \mathcal{T}_v v_2\right) \right\rangle_{X^n} &= \left\langle \mathcal{D}\left(\hat{\mathcal{T}}\hat{\underline{\mathbf{x}}} + \mathcal{T}_v v_1\right), \mathcal{D}\left(\hat{\mathcal{T}}\hat{\underline{\mathbf{y}}} + \mathcal{T}_v v_2\right) \right\rangle_{L_2^{n_{\hat{\mathbf{x}}}}} \\ &= \left\langle \hat{\underline{\mathbf{x}}}, \hat{\underline{\mathbf{y}}} \right\rangle_{L_2^{n_{\hat{\mathbf{x}}}}}.\end{aligned}$$

For b), let $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ and $w_1, w_2 \in \mathbb{R}^p$, $u_1, u_2 \in \mathbb{R}^q$. Then, from Corollary 5.3, we have

$$\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w_1 + \mathcal{T}_u u_1 \in \mathcal{X}_{w_1, u_1}, \quad \mathcal{T}\underline{\mathbf{y}} + \mathcal{T}_w w_2 + \mathcal{T}_u u_2 \in \mathcal{X}_{w_2, u_2}.$$

Since $\mathbb{R}^{n_x} \times X^n$ inner product is just sum of $\mathbb{R}$ and X^n inner products, using definitions of $\mathcal{T}$, $\mathcal{T}_w$, and $\mathcal{T}_u$ and the result in Corollary 5.3b, we have

$$\begin{aligned}&\left\langle (\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w_1 + \mathcal{T}_u u_1), \left(\mathcal{T}\underline{\mathbf{y}} + \mathcal{T}_w w_2 + \mathcal{T}_u u_2\right) \right\rangle_{\mathbb{R}^{n_x} \times X^n} \\ &= \left\langle \begin{bmatrix} I & 0 \\ 0 & \mathcal{D} \end{bmatrix} (\mathcal{T}\underline{\mathbf{x}} + \mathcal{T}_w w_1 + \mathcal{T}_u u_1), \begin{bmatrix} I & 0 \\ 0 & \mathcal{D} \end{bmatrix} \left(\mathcal{T}\underline{\mathbf{y}} + \mathcal{T}_w w_2 + \mathcal{T}_u u_2\right) \right\rangle_{\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}} \\ &= \left\langle \underline{\mathbf{x}}, \underline{\mathbf{y}} \right\rangle_{\mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}}.\end{aligned}$$

□

Using this result, we conclude that when $v = 0$, the PI map ($\hat{\mathcal{T}}$) is unitary. Since $\hat{\mathcal{T}}$ is a unitary map from L_2 to X_v, the space X_v is complete under the X-norm because L_2 is complete.

A.5.2 Proof of Lemma 5.6

Next, we prove that the $\mathbb{R}X$ norm is equivalent to the W^n-norm on the subspace $\mathbb{R} \times X$.

Lemma 5.6. *Suppose pair $\{n, \mathbf{G}_\mathrm{b}\}$ is admissible. Then $\|\mathbf{u}\|_{\mathbb{R}^{n_x}\times X^n} \le \|\mathbf{u}\|_{\mathbb{R}^{n_x}\times W^n}$ and there exists $c_0 > 0$ such that for any $\mathbf{u} \in \mathcal{X}_{0,0}$, we have $\|\mathbf{u}\|_{\mathbb{R}^{n_x}\times W^n} \le c_0 \|\mathbf{u}\|_{\mathbb{R}^{n_x}\times X^n}$.*

Proof. Suppose $\mathcal{X}_{0,0}$ is as defined in Equation (3.9). Then, for any $\begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \in \mathcal{X}_{0,0}$, we have, $x \in \mathbb{R}^{n_x}$ and $\hat{\mathbf{x}} \in X_{C_v x}$ for some matrix C_v and hence, from Theorem 5.1, we have $\hat{\mathbf{x}} = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}} + \mathcal{T}_v C_v x$ where $\hat{\mathcal{T}}$ and $\mathcal{T}_v$ are as defined in Block 5.1.

Let the space $\mathcal{X}_{0,0}$ be equipped with two different inner products $\mathbb{R}^{n_x} \times X^n$ and $\mathbb{R}^{n_x} \times W^n$. Then

$$\begin{aligned}\left\| \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \right\|^2_{\mathbb{R}^{n_x}\times W^n} &= \|x\|^2_{\mathbb{R}} + \sum_{i=0}^{N}\sum_{j=0}^{i} \left\|\partial_s^j \hat{\mathbf{x}}_i\right\|^2_{L_2} = \|x\|^2_{\mathbb{R}} + \sum_{i=0}^{N} \left\|\partial_s^i \hat{\mathbf{x}}_i\right\|^2_{L_2} + \sum_{i=0}^{N}\sum_{j=0}^{i-1} \left\|\partial_s^j \hat{\mathbf{x}}_i\right\|^2_{L_2} \\ &\ge \|x\|^2_{\mathbb{R}} + \sum_{i=0}^{N} \left\|\partial_s^i \hat{\mathbf{x}}_i\right\|^2_{L_2} = \|x\|^2_{\mathbb{R}} + \|\hat{\mathbf{x}}\|^2_{X^n} \ge \left\| \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \right\|^2_{\mathbb{R}^{n_x}\times X^n}.\end{aligned}$$

For the reverse inequality we try to find an upper bound on $\|\cdot\|_{\mathbb{R}^{n_x}\times W^n}$ as follows.

$$\left\| \begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix} \right\|^2_{\mathbb{R}^{n_x}\times W^n} = \|x\|^2_{\mathbb{R}} + \sum_{i=0}^{N}\sum_{j=0}^{i} \left\|\partial_s^j \hat{\mathbf{x}}_i\right\|^2_{L_2} = \|x\|^2_{\mathbb{R}} + \|(\mathcal{F}\hat{\mathbf{x}})\|^2_{L_2},$$

where $(\mathcal{F}\hat{\mathbf{x}}) = \mathrm{col}(\partial_s^0 S^0 \hat{\mathbf{x}}, \cdots, \partial_s^N S^N \hat{\mathbf{x}})$. Recall from Corollary A.4,

$$(\mathcal{F}\hat{\mathbf{x}}) = \mathcal{T}_D \begin{bmatrix} v \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix}, \text{ for any } \hat{\mathbf{x}} \in X_v \text{ and } \{n, \mathbf{G}_\mathrm{b}\} \text{ admissible}$$

where $\mathcal{T}_D = \Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline U_2 T(s-a) B_T^{-1} B_v & \{U_1, R_{D,1}, R_{D,2}\} \end{array}\right]$ is a bounded PI operator. Substituting $v = C_v x$ specifically, we have

$$\begin{aligned}\|(\mathcal{F}\hat{\mathbf{x}})\|^2_{L_2} &= \left\| \mathcal{T}_D \begin{bmatrix} C_v x \\ \mathcal{D}\hat{\mathbf{x}} \end{bmatrix} \right\|^2_{L_2} \le \|\mathcal{T}_D\|^2_{\mathcal{L}(\mathbb{R}L_2)} \left((b-a)^2 \|C_v x\|^2_{\mathbb{R}} + \|\mathcal{D}\hat{\mathbf{x}}\|^2_{L_2}\right) \\ &= K_0 \|\mathcal{D}\hat{\mathbf{x}}\|^2_{L_2} + K_1 \|x\|^2_{\mathbb{R}}\end{aligned}$$

where $K_0 = \|\mathcal{T}_D\|^2_{\mathcal{L}(\mathbb{R}L_2)}$ and $K_1 = K_0 (b-a)^2 \bar{\sigma}(C_v)^2$. Recall from Theorem 5.7, for any $\hat{\mathbf{x}} \in L_2$ and $v \in \mathbb{R}$, $\left\|\hat{\mathcal{T}}\hat{\mathbf{x}} + \mathcal{T}_v v\right\|_{X^n}^2 = \|\hat{\mathbf{x}}\|_{L_2}^2$. Then

$$\begin{aligned}
&\left\|\begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n}^2 \\
&= \|x\|_{\mathbb{R}}^2 + K_0 \|\mathcal{D}\hat{\mathbf{x}}\|_{L_2}^2 + K_1 \|x\|_{\mathbb{R}}^2 = (1+K_1)\|x\|_{\mathbb{R}}^2 + +K_0 \left\|\hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}} + \mathcal{T}_v C_v x\right\|_{X^n}^2 \\
&\le (1+K_1)\|x\|_{\mathbb{R}}^2 + K_0 \|\hat{\mathbf{x}}\|_{X^n}^2 \le (1+K_0+K_1)\left\|\begin{bmatrix} x \\ \hat{\mathbf{x}} \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n}^2.
\end{aligned}$$

□

A.5.3 Proof of Theorem 5.9

Now that we have established equivalence of the X^n and W^n norms, we may prove that a GPDE model is internally (exponential, Lyapunov, or asymptotically) stable if and only if the associated PIE is internally stable.

Theorem 5.9. *Given $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, the GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is exponentially stable if and only if the PIE defined by $\mathbf{G}_\mathrm{PIE} = \mathbf{M}(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\})$ is exponentially stable.*

Proof. Suppose GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is exponentially stable. Then, there exist constants M, $\alpha > 0$ such that for any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$, if $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0, 0\}$, we have

$$\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} \le M \left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} e^{-\alpha t} \qquad \text{for all } t \ge 0.$$

For any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$, let $\{\underline{\mathbf{x}}, z, y\}$ satisfy the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ and input $\{0, 0\}$. Then, from Corollary 5.5, $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 \in \mathcal{X}_{0,0}$ and input $\{0, 0\}$ for some v and r where $\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t)$. Then, by the exponential stability of the GPDE, we have

$$\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} \le M \left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} e^{-\alpha t} \qquad \text{for all } t \ge 0.$$

Since $\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \in \mathcal{X}_{0,0}$ and $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \in \mathcal{X}_{0,0}$, from lemma 5.6, we have

$$\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} \quad \text{and} \quad \left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} \le c_0 \left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n}.$$

By theorem 5.7, for any $\mathbf{x} \in \mathbb{R}L_2$ we have $\|\mathbf{x}\|_{\mathbb{R}L_2} = \|\mathcal{T}\mathbf{x}\|_{\mathbb{R}^{n_x}\times X^n}$. Thus, we have the following:

$$\begin{aligned}
&\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2}\\
&= \|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}^{n_x}\times X^n} = \left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} \le M\left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} e^{-\alpha t}\\
&\le c_0 M\left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n} e^{-\alpha t} = c_0 M\left\|\mathcal{T}\underline{\mathbf{x}}^0\right\|_{\mathbb{R}^{n_x}\times X^n} e^{-\alpha t} = c_0 M\left\|\underline{\mathbf{x}}^0\right\|_{\mathbb{R}L_2} e^{-\alpha t}.
\end{aligned}$$

Therefore, the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ is exponentially stable.

Suppose the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ is exponentially stable. Then, there exist constants M, $\alpha > 0$ such that for any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{m,n}$, if $\underline{\mathbf{x}}$ satisfies the PIE defined by $\{\mathbf{G}_{\mathrm{PIE}}\}$ with initial condition $\underline{\mathbf{x}}^0$ and input $\{0,0\}$, we have

$$\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \le M\left\|\underline{\mathbf{x}}^0\right\|_{\mathbb{R}L_2} e^{-\alpha t} \qquad \text{for all } t \ge 0.$$

For any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$, let $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfy the GPDE defined by n and $\{\mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0,0\}$. Then, from Corollary 5.5, $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ and input $\{0,0\}$ where

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t)\end{bmatrix}, \qquad \underline{\mathbf{x}}^0 = \begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0\end{bmatrix}.$$

Since $\hat{\mathbf{x}}(t) \in X_{C_v x(t)}$, from theorem 5.1, we have $\hat{\mathbf{x}}(t) = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}}(t) + \mathcal{T}_v C_v x(t)$. Therefore,

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix} = \begin{bmatrix} x(t) \\ \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}}(t) + \mathcal{T}_v C_v x(t)\end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathcal{T}_v C_v & \hat{\mathcal{T}}\end{bmatrix}\begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t)\end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t).$$

Similarly, we have $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0$.

By the exponential stability of the PIE, we have

$$\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} \le M\left\|\underline{\mathbf{x}}^0\right\|_{\mathbb{R}L_2} e^{-\alpha t} \qquad \text{for all } t \ge 0.$$

Again, from lemma 5.6, we have

$$\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} \le c_0\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n}, \quad \left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n}$$

and, from theorem 5.7, $\|\mathcal{T}\mathbf{x}\|_{\mathbb{R}^{n_x}\times X^n} = \|\mathbf{x}\|_{\mathbb{R}L_2}$ for any $\mathbf{x} \in \mathbb{R}L_2$ which implies

$$\begin{aligned}
\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} &\le c_0\left\|\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t)\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n} = c_0\|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}^{n_x}\times X^n} = c_0\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2}\\
&\le c_0 M\left\|\underline{\mathbf{x}}^0\right\|_{\mathbb{R}L_2} e^{-\alpha t} = c_0 M\left\|\mathcal{T}\underline{\mathbf{x}}^0\right\|_{\mathbb{R}^{n_x}\times X^n} e^{-\alpha t}\\
&= c_0 M\left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times X^n} e^{-\alpha t} \le c_0 M\left\|\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0\end{bmatrix}\right\|_{\mathbb{R}^{n_x}\times W^n} e^{-\alpha t}.
\end{aligned}$$

Therefore, the GPDE defined by n and $\{\mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is exponentially stable. $\square$

Corollary 5.10. *Given $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with $\{n, \mathbf{G}_\mathrm{b}\}$ admissible, let $\mathbf{G}_\mathrm{PIE} = \mathbf{M}\left(\{n, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{p}\}\right)$. Then*

1. *The GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is Lyapunov stable if and only if the PIE system defined by $\mathbf{G}_\mathrm{PIE}$ is Lyapunov stable.*

2. *The GPDE model defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is asymptotically stable if and only if the PIE system defined by $\mathbf{G}_\mathrm{PIE}$ is asymptotically stable.*

Proof. **Proof of part 1.** Suppose GPDE defined by $\{n, \mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ is Lyapunov stable. For any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$, let $\{\underline{\mathbf{x}}, z, y\}$ satisfy the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ and input $\{0,0\}$. Then, from Corollary 5.5, $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by n and $\{\mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 \in \mathcal{X}_{0,0}$ and input $\{0,0\}$ for some v and r where $\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t)$. Suppose $\epsilon > 0$, then by the Lyapunov stability of the GPDE, there exists δ such that

$$\left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \delta \implies \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \epsilon \qquad \text{for all } t \geq 0.$$

Since $\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \in \mathcal{X}_{0,0}$ and $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \in \mathcal{X}_{0,0}$, from lemma 5.6, we have

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} \leq \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \quad \text{and} \quad \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \leq c_0 \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n}.$$

Let $\|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2} < \frac{\delta}{c_0}$. By theorem 5.7, for any $\underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ we have $\|\underline{\mathbf{x}}\|_{\mathbb{R}L_2} = \|\mathcal{T}\underline{\mathbf{x}}\|_{\mathbb{R}^{n_x}\times X^n}$. Thus, we have the following:

$$\left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \leq c_0 \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} = c_0 \left\|\mathcal{T}\underline{\mathbf{x}}^0\right\|_{\mathbb{R}^{n_x}\times X^n} = c_0 \left\|\underline{\mathbf{x}}^0\right\|_{\mathbb{R}L_2} < \delta, \quad \text{and}$$

$$\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} = \|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}^{n_x}\times X^n} = \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} \leq \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \epsilon.$$

Therefore, the PIE defined by $\mathbf{G}_\mathrm{PIE}$ is Lyapunov stable.

Suppose the PIE defined by $\mathbf{G}_\mathrm{PIE}$ is Lyapunov stable. For any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$, let $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfy the GPDE defined by n and $\{\mathbf{G}_\mathrm{o}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{p}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0,0\}$. Then, from Corollary 5.5, $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_\mathrm{PIE}$ with initial condition $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ and input $\{0,0\}$ where

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t) \end{bmatrix}, \qquad \underline{\mathbf{x}}^0 = \begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0 \end{bmatrix}.$$

Since $\hat{\mathbf{x}}(t) \in X_{C_v x(t)}$, from theorem 5.1, we have $\hat{\mathbf{x}}(t) = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}}(t) + \mathcal{T}_v C_v x(t)$. Therefore,

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}}(t) + \mathcal{T}_v C_v x(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t).$$

Similarly, we have $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0$. Again, from lemma 5.6, we have

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \le c_0 \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n}, \qquad \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n}$$

and, from theorem 5.7, $\|\mathcal{T}\mathbf{x}\|_{\mathbb{R}^{n_x}\times X^n} = \|\mathbf{x}\|_{\mathbb{R}L_2}$ for any $\mathbf{x} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$. Let $\epsilon > 0$. Then, by the Lyapunov stability of the PIE, there exists δ such that

$$\|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2} < \delta \implies \|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} < \frac{\epsilon}{c_0} \qquad \text{for all } t \ge 0.$$

For any initial condition for the GPDE such that $\left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \delta$, we have

$$\|\underline{\mathbf{x}}^0\|_{\mathbb{R}L_2} = \|\mathcal{T}\underline{\mathbf{x}}^0\|_{\mathbb{R}^{n_x}\times X^n} = \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\| \begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \delta, \quad \text{and}$$

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} \le c_0 \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} = c_0 \|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}^{n_x}\times X^n} = c_0 \|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} < \epsilon.$$

Therefore, the GPDE defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is Lyapunov stable.

Proof of part 2. Suppose GPDE defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is asymptotically stable. For any $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$, let $\{\underline{\mathbf{x}}, z, y\}$ satisfy the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ and input $\{0,0\}$. Then, from Corollary 5.5, $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfies the GPDE defined by n and $\{\mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with initial condition $\begin{bmatrix} x^0 \\ \hat{\mathbf{x}}^0 \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}^0 \in \mathcal{X}_{0,0}$ and input $\{0,0\}$ for some v and r where $\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t)$. Suppose $\epsilon > 0$, then by the asymptotic stability of the GPDE, there exists T_0 such that

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \epsilon \qquad \text{for all } t \ge T_0.$$

Since $\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \in \mathcal{X}_{0,0}$, from lemma 5.6, we have

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n}.$$

By theorem 5.7, for any $\underline{\mathbf{x}} \in \mathbb{R}L_2^{n_x,n_{\hat{\mathbf{x}}}}$ we have $\|\underline{\mathbf{x}}\|_{\mathbb{R}L_2} = \|\mathcal{T}\underline{\mathbf{x}}\|_{\mathbb{R}^{n_x}\times X^n}$. Thus, for any $t > T_0$, we have,

$$\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} = \|\mathcal{T}\underline{\mathbf{x}}(t)\|_{\mathbb{R}^{n_x}\times X^n} = \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times X^n} \le \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x}\times W^n} < \epsilon.$$

Therefore, the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ is asymptotically stable.

Suppose the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ is asymptotically stable. For any $\{x^0, \hat{\mathbf{x}}^0\} \in \mathcal{X}_{0,0}$, let $\{x, \hat{\mathbf{x}}, z, y, v, r\}$ satisfy the GPDE defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ with initial condition $\{x^0, \hat{\mathbf{x}}^0\}$ and input $\{0, 0\}$. Then, from Corollary 5.5, $\{\underline{\mathbf{x}}, z, y\}$ satisfies the PIE defined by $\mathbf{G}_{\mathrm{PIE}}$ with initial condition $\underline{\mathbf{x}}^0 \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$ and input $\{0, 0\}$ where

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t) \end{bmatrix}, \qquad \underline{\mathbf{x}}^0 = \begin{bmatrix} x^0 \\ \mathcal{D}\hat{\mathbf{x}}^0 \end{bmatrix}.$$

Again, we know $\hat{\mathbf{x}}(t) \in X_{C_v x(t)}$, and hence from theorem 5.1, we have $\hat{\mathbf{x}}(t) = \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}}(t) + \mathcal{T}_v C_v x(t)$. Therefore,

$$\begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \hat{\mathcal{T}}\mathcal{D}\hat{\mathbf{x}}(t) + \mathcal{T}_v C_v x(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathcal{T}_v C_v & \hat{\mathcal{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \mathcal{D}\hat{\mathbf{x}}(t) \end{bmatrix} = \mathcal{T}\underline{\mathbf{x}}(t).$$

Again, from lemma 5.6, we have

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} \le c_0 \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times X^n},$$

and, from theorem 5.7, $\|\mathcal{T}\mathbf{x}\|_{\mathbb{R}^{n_x} \times X^n} = \|\mathbf{x}\|_{\mathbb{R}L_2}$ for any $\mathbf{x} \in \mathbb{R}L_2^{n_x, n_{\hat{\mathbf{x}}}}$. Let $\epsilon > 0$. Then, by the asymptotic stability of the PIE, there exists T_0 such that

$$\|\underline{\mathbf{x}}(t)\|_{\mathbb{R}L_2} < \frac{\epsilon}{c_0} \qquad \text{for all } t \ge T_0.$$

Then, for any $t \ge T_0$, we have

$$\left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times W^n} \le c_0 \left\| \begin{bmatrix} x(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \right\|_{\mathbb{R}^{n_x} \times X^n} = c_0 \left\| \mathcal{T}\underline{\mathbf{x}}(t) \right\|_{\mathbb{R}^{n_x} \times X^n} = c_0 \left\| \underline{\mathbf{x}}(t) \right\|_{\mathbb{R}L_2} < \epsilon.$$

Therefore, the GPDE defined by $\{n, \mathbf{G}_{\mathrm{o}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{p}}\}$ is asymptotically stable.

$\square$

APPENDIX B

PI ALGEBRAS, POSITIVE PI OPERATORS, AND INVERSION OF PI OPERATORS

B.1 Set Of PI Operators Forms A *-Algebra

In this section, we prove that a set of PI operators, when parameterized by L_∞-bounded functions, forms a *-algebra, i.e., closed algebraically. Furthermore, the formulae provided here will act as a guideline to perform the binary operations (addition, composition, and concatenation) of PI operators since various formulae in the paper were presented using such binary operation notation. First, we provide a formal definition of the list of properties a set must satisfy to be a *-algebra. A *-algebra must be an associative algebra with an involution operation. Since the definition of *-algebra depends on definitions of an associative algebra, we introduce those definitions first.

Definition B.1 (Algebra)**.** *A vector space, A, equipped with a multiplication operation, is said to be an algebra if, for every $X, Y \in A$, we have $XY \in A$.*

Definition B.2 (Associative Algebra)**.** *An algebra, A, is said to be associative if for every $X, Y, Z \in A$*

$$X(YZ) = (XY)Z$$

where XY denotes a multiplication operation between X and Y.

Definition B.3 (*-algebra)**.** *An algebra, A, over the $\mathbb{R}$ with an involution operation * is called a *-algebra if*

1. $(X^*)^* = X, \quad \forall X \in A$
2. $(X+Y)^* = X^* + Y^*, \quad \forall X, Y \in A$
3. $(XY)^* = Y^* X^*, \quad \forall X, Y \in A$
4. $(\lambda X)^* = \lambda X^*, \quad \forall \lambda \in \mathbb{R}, X \in A$

To prove that the set of PI operators $\Pi_{q,q}^{p,p}$ satisfy all the above properties, we prove that $\Pi_{q,q}^{p,p}$ satisfies the requirements of each of the above definitions where

$$\Pi_{q,q}^{p,p} := \left\{ \begin{array}{c} \mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_0, R_1, R_2\} \end{array}\right] \mid P \in \mathbb{R}^{p\times p},\ Q_1(s), Q_2(s)^T \in \mathbb{R}^{p\times q}, \\ R_0(s), R_1(s,\theta), R_2(s,\theta) \in \mathbb{R}^{q\times q},\ \text{and } Q_i,\ R_i \in L_\infty, \\ R_1,\ R_2 \text{ are separable} \end{array} \right\}.$$

To prove that the set is an algebra, we need to define two binary operations, addition and multiplication, which in case of $\Pi_{q,q}^{p,p}$ will be given the addition of PI operators (as defined in Lemma 2.1) and composition of PI operators (as defined in Lemma 2.2). For the set to be a *-algebra, we also need an involution operation, which is given by the adjoint with respect to $\mathbb{R} \times L_2$ inner-product (as defined in Lemma 2.3).

Lemma 2.1 (Addition)**.** *For any matrices $A, L \in \mathbb{R}^{m\times p}$ and L_∞-bounded functions $B_1, M_1 : [a,b] \to \mathbb{R}^{m\times q}$, $B_2, M_2 : [a,b] \to \mathbb{R}^{n\times p}$, $C_0, N_0 : [a,b] \to \mathbb{R}^{n\times q}$, and separable* *functions* $C_1, C_2, N_1, N_2 : [a,b]^2 \to \mathbb{R}^{n\times q}$*, define a linear map* $\mathbf{P}^4_+ : [\Gamma_4]^{m,p}_{n,q} \times [\Gamma_4]^{m,p}_{n,q} \to [\Gamma_4]^{m,p}_{n,q}$ *such that*

$$\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] = \mathbf{P}^4_+ \left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right], \left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\} \end{array}\right] \right)$$

where

$$P = A + L,\ Q_i = B_i + M_i,\ R_i = C_i + N_i.$$

If P, Q_i, R_i *are as defined above, then, for any* $x \in \mathbb{R}^p$ *and* $\mathbf{z} \in L_2^q([a,b])$

$$\begin{aligned} &\mathit{\Pi} \left[\mathbf{P}^4_+ \left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right], \left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\} \end{array}\right] \right) \right] \begin{bmatrix} x \\ \mathbf{z} \end{bmatrix} \\ &= \left(\mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right] + \mathit{\Pi}\left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\} \end{array}\right] \right) \begin{bmatrix} x \\ \mathbf{z} \end{bmatrix}. \end{aligned}$$

Proof. Let $x \in \mathbb{R}^p$ and $\mathbf{y} \in L_2^q[a,b]$ be arbitrary. Then

$$\begin{aligned} &\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \begin{bmatrix} x \\ \mathbf{y} \end{bmatrix}(s) = \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{y}(s)ds \\ Q_2(s)x + \mathit{\Pi}_{\{R_i\}}\mathbf{y}(s) \end{bmatrix} \\ &= \begin{bmatrix} (A+L)x + \int_a^b (B_1 + M_1)(s)\mathbf{y}(s)ds \\ (B_2 + M_2)(s)x + (\mathit{\Pi}_{\{C_i+N_i\}})\mathbf{y}(s) \end{bmatrix} \\ &= \begin{bmatrix} Ax + \int_a^b B_1(s)\mathbf{y}(s)ds \\ B_2(s)x + \mathit{\Pi}_{\{C_i\}}\mathbf{y}(s) \end{bmatrix} + \begin{bmatrix} Lx + \int_a^b M_1(s)\mathbf{y}(s)ds \\ M_2(s)x + \mathit{\Pi}_{\{N_i\}}\mathbf{y}(s) \end{bmatrix} \\ &= \mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right] \begin{bmatrix} x \\ \mathbf{y} \end{bmatrix}(s) + \mathit{\Pi}\left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\} \end{array}\right] \begin{bmatrix} x \\ \mathbf{y} \end{bmatrix}(s) \\ &= \left(\mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right] + \mathit{\Pi}\left[\begin{array}{c|c} L & M_1 \\ \hline M_2 & \{N_i\} \end{array}\right] \right) \begin{bmatrix} x \\ \mathbf{y} \end{bmatrix}(s). \end{aligned}$$

□

Lemma 2.2 (Composition)**.** *For any matrices* $A \in \mathbb{R}^{m\times k}, P \in \mathbb{R}^{k\times p}$ *and* L_∞*-bounded functions* $B_1 : [a,b] \to \mathbb{R}^{m\times l}, Q_1 : [a,b] \to \mathbb{R}^{k\times q}$*,* $B_2 : [a,b] \to \mathbb{R}^{n\times k}, Q_2 : [a,b] \to \mathbb{R}^{l\times p}$*,* $C_0 : [a,b] \to \mathbb{R}^{n\times l}, R_0 : [a,b] \to \mathbb{R}^{l\times q}$*, and separable functions* $C_1, C_2 : [a,b]^2 \to \mathbb{R}^{n\times l}, R_1, R_2 : [a,b]^2 \to \mathbb{R}^{l\times q}$*, define a linear map* $\mathbf{P}^4_\times : [\Gamma_4]^{m,k}_{n,l} \times [\Gamma_4]^{k,p}_{l,q} \to [\Gamma_4]^{m,p}_{n,q}$ *such that*

$$\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\} \end{array}\right] = \mathbf{P}^4_\times \left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right], \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \right)$$

where

$$\hat{P} = AP + \int_a^b B_1(s)Q_2(s)ds, \quad \hat{R}_0(s) = C_0(s)R_0(s),$$
$$\hat{Q}_1(s) = AQ_1(s) + B_1(s)R_0(s) + \int_s^b B_1(\eta)R_1(\eta,s)d\eta + \int_a^s B_1(\eta)R_2(\eta,s)d\eta,$$
$$\hat{Q}_2(s) = B_2(s)P + C_0(s)Q_2(s) + \int_a^s C_1(s,\eta)Q_2(\eta)d\eta + \int_s^b C_2(s,\eta)Q_2(\eta)d\eta,$$
$$\hat{R}_1(s,\eta) = B_2(s)Q_1(\eta) + C_0(s)R_1(s,\eta) + C_1(s,\eta)R_0(\eta)$$
$$+ \int_a^\eta C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^s C_1(s,\theta)R_1(\theta,\eta)d\theta + \int_s^b C_2(s,\theta)R_1(\theta,\eta)d\theta,$$
$$\hat{R}_2(s,\eta) = B_2(s)Q_1(\eta) + C_0(s)R_2(s,\eta) + C_2(s,\eta)R_0(\eta)$$
$$+ \int_a^s C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_s^\eta C_2(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^b C_2(s,\theta)R_1(\theta,\eta)d\theta.$$

If $\hat{P}, \hat{Q}_i, \hat{R}_i$ are as defined above, then, for any $x \in \mathbb{R}^m$ and $\mathbf{z} \in L_2^n([a,b])$,

$$\mathit{\Pi}\left[\mathbf{P}^4_\times\left(\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right], \left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\right)\right]\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}$$
$$= \mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right]\left(\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}\right).$$

Proof. Let $\{A,\ B_i,\ C_i\}$, $\{P,\ Q_i,\ R_i\}$ and $\{\hat{P},\ \hat{Q}_i,\ \hat{R}_i\}$ be such that

$$\mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right]\left(\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\begin{bmatrix} x_1 \\ x_2\end{bmatrix}\right)(s) = \left(\mathit{\Pi}\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}\}\end{array}\right]\begin{bmatrix} x_1 \\ x_2\end{bmatrix}\right)(s),$$

for any $x_1 \in \mathbb{R}^p$ and $x_2 \in L_2^q[a,b]$. Since PI operators are bounded operators on $\mathbb{R} \times L_2$, we define

$$\begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix} = \left(\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R\}\end{array}\right]\begin{bmatrix} x_1 \\ x_2\end{bmatrix}\right)(s).$$

Then, by definition of a PI operator,

$$y_1 = Px_1 + \int_a^b Q_1(s)x_2(s)ds$$
$$y_2(s) = Q_2(s)x_1 + R_0(s)x_2(s) + \int_a^s R_1(s,\eta)x_2(\eta)d\eta + \int_s^b R_2(s,\eta)x_2(\eta)d\eta.$$

Likewise, let us also define

$$\begin{bmatrix} z_1 \\ z_2(s)\end{bmatrix} = \left(\mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right]\begin{bmatrix} y_1 \\ y_2\end{bmatrix}\right)(s) = \mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\}\end{array}\right]\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\begin{bmatrix} x_1 \\ x_2\end{bmatrix}(s),$$

which gives us the equations

$$\begin{aligned}
z_1 &= Ay_1 + \int_a^b B_1(s)y_2(s)ds \\
z_2(s) &= B_2(s)y_1 + C_0(s)y_2(s) + \int_a^s C_1(s,\eta)y_2(\eta)d\eta + \int_s^b C_2(s,\eta)y_2(\eta)d\eta.
\end{aligned}$$

We will try to find a direct map between x_i and z_i by substituting y_i in the above equation. However, we will perform the substitution by taking one term at a time. First,

$$\begin{aligned}
&\int_a^b B_1(s)y_2(s)ds \\
&= \int_a^b B_1(s)\Big(Q_2(s)x_1 + R_0(s)x_2(s) + \int_a^s R_1(s,\eta)x_2(\eta)d\eta + \int_s^b R_2(s,\eta)x_2(\eta)d\eta\Big)ds.
\end{aligned}$$

Then

$$\begin{aligned}
z_1 &= Ay_1 + \int_a^b B_1(s)y_2(s)ds \\
&= APx_1 + \int_a^b AQ_1(s)x_2(s)ds + \int_a^b B_1(s)\Big(Q_2(s)x_1 + R_0(s)x_2(s) \\
&\qquad + \int_a^s R_1(s,\eta)x_2(\eta)d\eta + \int_s^b R_2(s,\eta)x_2(\eta)d\eta\Big)ds \\
&= \hat{P}x_1 + \int_a^b \hat{Q}_1(s)x_2(s)ds.
\end{aligned}$$

Next, we substitute y_i in the map from y_i to $z_2(s)$ to get

$$\begin{aligned}
&z_2(s) \\
&= B_2(s)Px_1 + \int_a^b B_2(s)Q_1(\eta)x_2(s)d\eta + C_0(s)Q_2(s)x_1 + C_0(s)R_0(s)x_2(s) \\
&+ \int_a^s C_0(s)R_1(s,\eta)x_2(\eta)d\eta + \int_s^b C_0(s)R_2(s,\eta)x_2(\eta)d\eta + \int_a^s C_1(s,\eta)Q_2(\eta)x_1d\eta \\
&+ \int_a^s C_1(s,\eta)R_0(s)x_2(s)d\eta + \int_a^s\int_a^\eta C_1(s,\eta)R_1(\eta,\beta)x_2(\beta)d\beta d\eta \\
&+ \int_a^s\int_\eta^b C_1(s,\eta)R_2(\eta,\beta)x_2(\beta)d\beta d\eta + \int_s^b C_2(s,\eta)Q_2(\eta)x_1d\eta \\
&+ \int_s^b C_2(s,\eta)R_0(\eta)x_2(\eta)d\eta + \int_s^b\int_a^\eta C_2(s,\eta)R_1(\eta,\beta)x_2(\beta)d\beta d\eta \\
&+ \int_s^b\int_\eta^b C_2(s,\eta)R_2(\eta,\beta)x_2(\beta)d\beta d\eta.
\end{aligned}$$

Next, we separate the terms by factoring x_1. Then, we change the order of integration in the double integrals (and swap the variable $\beta \leftrightarrow \eta$) to get

$$
\begin{aligned}
&z_2(s)\\
&= \Big(B_2(s)P + C_0(s)Q_2(s) + \int_a^s C_1(s,\eta)Q_2(\eta)d\eta \int_s^b +C_2(s,\eta)Q_2(\eta)d\eta\Big)x_1\\
&+ C_0(s)R_0(s)x_2(s) + \int_a^b B_2(\eta)Q_1(s)x_2(\eta)d\eta + \int_a^s C_0(s)R_1(s,\eta)x_2(\eta)d\eta\\
&+ \int_s^b C_0(s)R_2(s,\eta)x_2(\eta)d\eta + \int_s^b C_2(s,\eta)R_0(\eta)x_2(\eta)d\eta + \int_a^s C_1(s,\eta)R_0(s)x_2(s)d\eta\\
&+ \int_a^s \left(\int_a^\eta C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^s C_1(s,\theta)R_1(\theta,\eta)d\theta + \int_s^b C_2(s,\theta)R_1(\theta,\eta)d\theta\right) x_2(\eta)d\eta\\
&+ \int_s^b \left(\int_a^s C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_s^\eta C_2(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^b C_2(s,\theta)R_1(\theta,\eta)d\theta\right) x_2(\eta)d\eta\\
&= \hat{Q}_2(s)x_1 + \hat{S}(s)x_2(s) + \int_a^s \hat{R}_1(s,\eta)x_2(\eta)d\eta + \int_s^b \hat{R}_2(s,\eta)x_2(\eta)d\eta.
\end{aligned}
$$

This completes the proof. □

Lemma 2.3 (Adjoint). *For any matrices $P \in \mathbb{R}^{m\times p}$ and L_∞-bounded functions $Q_1 : [a,b] \to \mathbb{R}^{m\times q}$, $Q_2 : [a,b] \to \mathbb{R}^{n\times p}$, $R_0 : [a,b] \to \mathbb{R}^{n\times q}$, and separable functions $R_1, R_2 : [a,b]^2 \to \mathbb{R}^{n\times n}$, define a linear map $\mathbf{P}_*^4 : [\Gamma_4]_{n,q}^{m,p} \to [\Gamma_4]_{q,n}^{p,m}$ such that*

$$
\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\} \end{array}\right] = \mathbf{P}_*^4\left(\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]\right)
$$

where

$$
\begin{aligned}
&\hat{P} = P^T, && \hat{Q}_1(s) = Q_2^T(s), && \hat{Q}_2(s) = Q_1^T(s),\\
&\hat{R}_0(s) = R_0^T(s), && \hat{R}_1(s,\eta) = R_2^T(\eta,s), && \hat{R}_2(s,\eta) = R_1^T(\eta,s).
\end{aligned}
\tag{2.4}
$$

Then, for any $\mathbf{x} \in \mathbb{R}L_2^{m,n}, \mathbf{y} \in \mathbb{R}L_2^{p,q}$, then we have

$$
\left\langle \mathbf{x}, \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \mathbf{y} \right\rangle_{\mathbb{R}L_2^{m,n}} = \left\langle \mathit{\Pi}\left[\mathbf{P}_*^4\left(\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]\right)\right] \mathbf{x}, \mathbf{y} \right\rangle_{\mathbb{R}L_2^{p,q}},
\tag{2.5}
$$

Proof. To prove this, we use the fact that for any scalar a we have $a = a^\top$. Let

$\mathbf{x}(s) = \begin{bmatrix} x_1 \\ \mathbf{x}_2(s) \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \mathbf{y}_2(s) \end{bmatrix}$. Then

$$\begin{aligned}
&\left\langle \mathbf{x}, \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \mathbf{y} \right\rangle_{\mathbb{R}L_2^{m,n}} \\
&= x_1^\top P y_1 + \int_a^b x_1^\top Q_1(s)\mathbf{y}_2(s)ds + \int_a^b \mathbf{x}_2^\top(s) Q_2(s) y_1 ds \\
&\quad + \int_a^b \mathbf{x}_2(s)^\top R_0(s)\mathbf{y}_2(s)ds + \int_a^b \int_a^s \mathbf{x}_2^\top(s) R_1(s,\eta)\mathbf{y}_2(\eta)d\eta ds \\
&\quad + \int_a^b \int_s^b \mathbf{x}_2^\top(s) R_2(s,\eta)\mathbf{y}_2(\eta)d\eta ds \\
&= y_1^\top P^\top x_1 + \int_a^b y_1^\top Q_2^\top(s)\mathbf{x}_2(s)ds + \int_a^b \mathbf{y}_2(s) Q_1^\top(s) x_1 ds \\
&\quad + \int_a^b \mathbf{y}_2^\top(s) R_0^\top(s)\mathbf{x}_2(s)ds + \int_a^b \int_a^s \mathbf{y}_2^\top(s) R_2^\top(\eta,s)\mathbf{x}_2(\eta)d\eta ds \\
&\quad + \int_a^b \int_s^b \mathbf{y}_2^\top(s) R_1^\top(\eta,s)\mathbf{x}_2(\eta)d\eta ds \\
&= y_1^\top \hat{P} x_1 + \int_a^b y_1^\top \hat{Q}_1(s)\mathbf{x}_2(s)ds + \int_a^b \mathbf{y}_2^\top(s) \hat{Q}_2(s) x_1 ds \\
&\quad + \int_a^b \mathbf{y}_2^\top(s) \hat{R}_0(s)\mathbf{x}_2(s)ds + \int_a^b \int_a^s \mathbf{y}_2^\top(s) \hat{R}_1(s,\eta)\mathbf{x}_2(\eta)d\eta ds \\
&\quad + \int_a^b \int_s^b \mathbf{y}_2^\top(s) \hat{R}_2(s,\eta)\mathbf{x}_2(\eta)d\eta ds \\
&= \left\langle \mathbf{y}, \mathit{\Pi}\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\} \end{array}\right] \mathbf{x} \right\rangle_{\mathbb{R}L_2^{p,q}} = \left\langle \mathit{\Pi}\left[\begin{array}{c|c} \hat{P} & \hat{Q}_1 \\ \hline \hat{Q}_2 & \{\hat{R}_i\} \end{array}\right] \mathbf{x}, \mathbf{y} \right\rangle_{\mathbb{R}L_2^{p,q}}
\end{aligned}$$

where,

$$\begin{aligned}
&\hat{P} = P^\top, && \hat{Q}_1(s) = Q_2^\top(s), && \hat{Q}_2(s) = Q_1^\top(s), \\
&\hat{R}_0(s) = R_0^\top(s), && \hat{R}_1(s,\eta) = R_2^\top(\eta,s), && \hat{R}_2(s,\eta) = R_1^\top(\eta,s).
\end{aligned}$$

This completes the proof. □

Now that we have formally defined the binary and involution operations on the set of PI operators, we show that $\Pi_{q,q}^{p,p}$ when equipped with these operations forms a *-algebra.

Lemma 2.4. *The set $[\Pi_i]$ equipped with composition operation forms an associative algebra.*

Proof. Suppose $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right], \mathit{\Pi}\left[\begin{array}{c|c} A & B_1 \\ \hline B_2 & \{C_i\} \end{array}\right] \in \Pi_{q,q}^{p,p}$. From Lemma 2.2, we have that $\Pi\left[\begin{array}{c|c}\hat{P} & \hat{Q}_1\\ \hline \hat{Q}_2 & \{\hat{R}_i\}\end{array}\right] = \Pi\left[\begin{array}{c|c}A & B_1\\ \hline B_2 & \{C_i\}\end{array}\right] \Pi\left[\begin{array}{c|c}P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right]$ with

$$
\begin{aligned}
&\hat{P} = AP + \int_a^b B_1(s)Q_2(s)ds,\ \hat{R}_0(s) = C_0(s)R_0(s),\\
&\hat{Q}_1(s) = AQ_1(s) + B_1(s)R_0(s) + \int_s^b B_1(\eta)R_1(\eta,s)d\eta + \int_a^s B_1(\eta)R_2(\eta,s)d\eta,\\
&\hat{Q}_2(s) = B_2(s)P + C_0(s)Q_2(s) + \int_a^s C_1(s,\eta)Q_2(\eta)d\eta + \int_s^b C_2(s,\eta)Q_2(\eta)d\eta,\\
&\hat{R}_1(s,\eta) = B_2(s)Q_1(\eta) + C_0(s)R_1(s,\eta) + C_1(s,\eta)R_0(\eta)\\
&+ \int_a^\eta C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^s C_1(s,\theta)R_1(\theta,\eta)d\theta + \int_s^b C_2(s,\theta)R_1(\theta,\eta)d\theta,\\
&\hat{R}_2(s,\eta) = B_2(s)Q_1(\eta) + C_0(s)R_2(s,\eta) + C_2(s,\eta)R_0(\eta)\\
&+ \int_a^s C_1(s,\theta)R_2(\theta,\eta)d\theta + \int_s^\eta C_2(s,\theta)R_2(\theta,\eta)d\theta + \int_\eta^b C_2(s,\theta)R_1(\theta,\eta)d\theta.
\end{aligned}
$$

Since B_i, C_i, Q_i, R_i are all L_∞ we have $\hat{Q}_i, \hat{R}_i \in L_\infty$. Thus, the composition of any two PI operators in $\Pi_{q,q}^{p,p}$ is a PI operator in the same set.

Similarly, by using composition formulae from Lemma 2.2, we can show that for any 3 PI operators $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \Pi_{q,q}^{p,p}$ we have $(\mathcal{P}\mathcal{Q})\mathcal{R} = \mathcal{P}(\mathcal{Q}\mathcal{R})$. The steps are omitted here since the proof is a straightforward arithmetic exercise. Thus $\Pi_{q,q}^{p,p}$ is an associative algebra. □

So far, we have shown that the set $\Pi_{q,q}^{p,p}$ is closed algebraically, i.e., the binary and involution operations on PI operators also result in PI operators. In the following Lemma, we conclude that $\Pi_{q,q}^{p,p}$ is a $*$-algebra.

Lemma 2.5. *The set $[\Pi_i]$ equipped with the binary operations of addition and composition and the involution operation given by the adjoint w.r.t. $\mathbb{R}L_2$ inner product is a *-algebra.*

Proof. To prove this, we first show that $\Pi_{q,q}^{p,p}$ when equipped with the adjoint operator satisfies the requirements of a *-algebra. Since PI operators are operators on a Hilbert space $\mathbb{R} \times L_2$, from Propositions 2.6 and 2.7 in (Conway, 2019, p .32), we know that for any two such operators $\mathcal{P}$ and $\mathcal{Q}$

- $(\mathcal{P}^*)^* = \mathcal{P}$
- $(\lambda\mathcal{P})^* = \lambda\mathcal{P}^*$
- $(\mathcal{P} + \mathcal{Q})^* = \mathcal{P}^* + \mathcal{Q}^*$
- $(\mathcal{P}\mathcal{Q})^* = \mathcal{Q}^*\mathcal{P}^*$.

Therefore, since $\Pi_{q,q}^{p,p}$ is a Banach algebra with an involution * that satisfies all the properties in the definition of a *-algebra, $\Pi_{q,q}^{p,p}$ is a *-algebra. □

B.1.1 Concatenation of PI Operators

The results presented in this subsection are specific to the notational convenience granted by the concatenation of PI operators. In this subsection, we assume that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}L_2$ are identical if a permutation matrix P exists such that $\mathbf{x} = P\mathbf{y}$. This assumption is made to accommodate for the notational convenience that concatenation of PI operators provide because any $\Pi_{n,q}^{m,p}$ PI operator requires inputs to be completely segregated with finite-dimensional part of the vector to be on the top while infinite-dimensional part at the bottom. However, since concatenation of such vectors is likely to lose such a segregation, we think of the vector $\mathbf{x}, \mathbf{y} \in \mathbb{R}L_2$ as ordered pairs $(x, \mathbf{x}_1), (y, \mathbf{y}_1)$ with $x, y \in \mathbb{R}$ and $\mathbf{x}_1, \mathbf{y}_1 \in L_2$ with concatenation of two such vectors being performed individually on each element of the ordered pair. This allows us to retain the convenient segregation of the vector's finite and infinite dimensional parts and use the concatenation notation of PI operators.

Lemma B.1 (Horizontal concatenation)**.** *Suppose $A_j \in \mathbb{R}^{m\times p_j}$ and $B_{1,j} : [a,b] \to \mathbb{R}^{m\times q_j}$, $B_{2,j} : [a,b] \to \mathbb{R}^{n\times p_j}$, $C_{0,j} : [a,b] \to \mathbb{R}^{n\times q_j}$, $C_{i,j} : [a,b]\times[a,b] \to \mathbb{R}^{n\times q_j}$, for $i \in \{0,1,2\}$, $j \in \{1,2\}$, are bounded functions. If we define P, Q_1, Q_2 and R_k, for $k \in \{0,1,2\}$ as*

$$P = [A_1 \quad A_2]\,, \; Q_i = [B_{i,1} \quad B_{i,2}]\,, \; R_i = [C_{i,1} \quad C_{i,2}]\,,$$

then

$$\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] = \left[\mathit{\Pi}\left[\begin{array}{c|c} A_1 & B_{1,1} \\ \hline B_{2,1} & \{C_{i,1}\} \end{array}\right] \quad \mathit{\Pi}\left[\begin{array}{c|c} A_2 & B_{1,2} \\ \hline B_{2,2} & \{C_{i,2}\} \end{array}\right]\right].$$

Proof. We will prove this identity by a series of equalities. Let $x_1 \in \mathbb{R}^{p_1}$, $y_1 \in \mathbb{R}^{p_2}$, $\mathbf{x}_2 \in L_2^{q_1}[a,b]$, and $\mathbf{y}_2 \in L_2^{q_2}[a,b]$ be arbitrary. Next, we define $z_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{p_1+p_2}$ and $\mathbf{z}_2 \in L_2^{q_1+q_2}$. Then the following series of equalities hold. We can substitute $\{z_1, \; \mathbf{z}_2\}$

in terms of $\{x_1,\ y_1,\ \mathbf{x}_2,\ \mathbf{y}_2\}$ and perform matrix multiplication to get

$$
\begin{aligned}
&\mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] \begin{bmatrix} z_1 \\ \mathbf{z}_2 \end{bmatrix}(s) \\
&= \begin{bmatrix} Pz_1 + \int_a^b Q_1(s)\mathbf{z}_2(s)ds \\ Q_2(s)z_1 + R_0(s)\mathbf{z}_2(s) + \int_a^s R_1(s,\theta)\mathbf{z}_2(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{z}_2(\theta)d\theta \end{bmatrix} \\
&= \begin{bmatrix} P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \int_a^b Q_1(s)\begin{bmatrix} \mathbf{x}_2(s) \\ \mathbf{y}_2(s) \end{bmatrix} ds \\ Q_2(s)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + R_0(s)\begin{bmatrix} \mathbf{x}_2(s) \\ \mathbf{y}_2(s) \end{bmatrix} + \int_a^s R_1(s,\theta)\begin{bmatrix} \mathbf{x}_2(\theta) \\ \mathbf{y}_2(\theta) \end{bmatrix} d\theta + \int_s^b R_2(s,\theta)\begin{bmatrix} \mathbf{x}_2(\theta) \\ \mathbf{y}_2(\theta) \end{bmatrix} d\theta \end{bmatrix} \\
&= \begin{bmatrix} A_1x_1 + \int_a^b B_{0,1}(s)\mathbf{x}_2(s)ds \\ B_{2,1}(s)x_1 + C_{0,1}(s)\mathbf{x}_2(s) + \int_a^s C_{1,1}(s,\theta)\mathbf{x}_2(\theta)d\theta + \int_s^b C_{2,1}(s,\theta)\mathbf{x}_2(\theta)d\theta \end{bmatrix} \\
&\quad + \begin{bmatrix} A_2y_1 + \int_a^b B_{0,2}(s)\mathbf{y}_2(s)ds \\ B_{2,2}(s)y_1 + C_{0,2}(s)\mathbf{y}_2(s) + \int_a^s C_{1,2}(s,\theta)\mathbf{y}_2(\theta)d\theta + \int_s^b C_{2,2}(s,\theta)\mathbf{y}_2(\theta)d\theta \end{bmatrix} \\
&= \mathcal{P}\left[\begin{array}{c|c} A_1 & B_{1,1} \\ \hline B_{2,1} & \{C_{i,1}\} \end{array}\right] \begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}(s) + \mathcal{P}\left[\begin{array}{c|c} A_2 & B_{1,2} \\ \hline B_{2,2} & \{C_{i,2}\} \end{array}\right] \begin{bmatrix} y_1 \\ \mathbf{y}_2 \end{bmatrix}(s) \\
&= \begin{bmatrix} \mathcal{P}\left[\begin{array}{c|c} A_1 & B_{1,1} \\ \hline B_{2,1} & \{C_{i,1}\} \end{array}\right] & \mathcal{P}\left[\begin{array}{c|c} A_2 & B_{1,2} \\ \hline B_{2,2} & \{C_{i,2}\} \end{array}\right] \end{bmatrix} \begin{bmatrix} x_1 \\ \mathbf{x}_2 \\ y_1 \\ \mathbf{y}_2 \end{bmatrix}(s).
\end{aligned}
$$

By rearranging the vector $\mathrm{col}(x_1, \mathbf{x}_2, y_1, \mathbf{y}_2)$ we can obtain $\{z_1, \mathbf{z}_2(s)\}$. Thus, the horizontal concatenation of two PI maps gives rise to another uniquely defined PI map. $\square$

Note that in the last equality, permutation of vector rows is needed to obtain $\{z_1,\ \mathbf{z}_2\}$ back. However, that does not affect the conversion formulae for PIE since states can be arranged in any order based on convenience.

Lemma B.2 (Vertical concatenation)**.** *Suppose $A_j \in \mathbb{R}^{m_j\times p}$ and $B_{1,j} : [a,b] \to \mathbb{R}^{m_j\times q}$, $B_{2,j} : [a,b] \to \mathbb{R}^{n_j\times p}$, $C_{0,j} : [a,b] \to \mathbb{R}^{n_j\times q}$, $C_{i,j} : [a,b]\times[a,b] \to \mathbb{R}^{n_j\times q}$, for $i \in \{0,1,2\}$, $j \in \{1,2\}$, are bounded functions. If we define P, Q_1, Q_2 and R_k, for $k \in \{0,1,2\}$ as*

$$
P = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ Q_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \end{bmatrix},\ R_i = \begin{bmatrix} C_{i,1} \\ C_{i,2} \end{bmatrix},
$$

then

$$
\mathcal{P}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right] = \begin{bmatrix} \mathcal{P}\left[\begin{array}{c|c} A_1 & B_{1,1} \\ \hline B_{2,1} & \{C_{i,1}\} \end{array}\right] \\ \mathcal{P}\left[\begin{array}{c|c} A_2 & B_{1,2} \\ \hline B_{2,2} & \{C_{i,2}\} \end{array}\right] \end{bmatrix}.
$$

Proof. Similar to horizontal concatenation, we will prove this identity through equalities. Let $x_1 \in \mathbb{R}^p$ and $\mathbf{x}_2 \in L_2^q[a,b]$ be arbitrary. Then the following series of equalities hold. We can substitute $\{P,\ Q_i,\ R_i\}$ in terms of $\{A_j,\ B_{i,j},\ C_{i,j}\}$ and perform matrix multiplication to get

$$
\begin{aligned}
&\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]\begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}(s) \\
&= \begin{bmatrix} Px_1 + \int_a^b Q_1(s)\mathbf{x}_2(s)ds \\ Q_2(s)z_1 + R_0(s)\mathbf{x}_2(s) + \int_a^s R_1(s,\theta)\mathbf{x}_2(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}_2(\theta)d\theta \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x_1 + \int_a^b \begin{bmatrix} B_{1,1}(s) \\ B_{1,2}(s) \end{bmatrix}\mathbf{x}_2(s)ds \\ \begin{bmatrix} B_{2,1}(s) \\ B_{2,2}(s) \end{bmatrix} x_1 + \begin{bmatrix} C_{0,1}(s) \\ C_{0,2}(s) \end{bmatrix}\mathbf{x}_2(s) + \int_a^s \begin{bmatrix} C_{1,1}(s,\theta) \\ C_{1,2}(s,\theta) \end{bmatrix}\mathbf{x}_2(\theta)d\theta + \int_s^b \begin{bmatrix} C_{2,1}(s,\theta) \\ C_{2,2}(s,\theta) \end{bmatrix}\mathbf{x}_2(\theta)d\theta \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x_1 + \int_a^b \begin{bmatrix} B_{1,1}(s) \\ B_{1,2}(s) \end{bmatrix}\mathbf{x}_2(s)ds \\ \begin{bmatrix} B_{2,1}(s) \\ B_{2,2}(s) \end{bmatrix} x_1 + \begin{bmatrix} C_{0,1}(s) \\ C_{0,2}(s) \end{bmatrix}\mathbf{x}_2(s) + \int_a^s \begin{bmatrix} C_{1,1}(s,\theta) \\ C_{1,2}(s,\theta) \end{bmatrix}\mathbf{x}_2(\theta)d\theta + \int_s^b \begin{bmatrix} C_{2,1}(s,\theta) \\ C_{2,2}(s,\theta) \end{bmatrix}\mathbf{x}_2(\theta)d\theta \end{bmatrix}.
\end{aligned}
$$

By rearranging the rows of the above vector, we get

$$
\begin{aligned}
&\begin{bmatrix} \begin{bmatrix} A_1x_1 + \int_a^b B_{0,1}(s)\mathbf{x}_2(s)ds \\ B_{2,1}(s)x_1 + C_{0,1}(s)\mathbf{x}_2(s) + \int_a^s C_{1,1}(s,\theta)\mathbf{x}_2(\theta)d\theta + \int_s^b C_{2,1}(s,\theta)\mathbf{x}_2(\theta)d\theta \end{bmatrix} \\ \begin{bmatrix} A_2x_1 + \int_a^b B_{0,2}(s)\mathbf{x}_2(s)ds \\ B_{2,2}(s)x_1 + C_{0,2}(s)\mathbf{x}_2(s) + \int_a^s C_{1,2}(s,\theta)\mathbf{x}_2(\theta)d\theta + \int_s^b C_{2,2}(s,\theta)\mathbf{x}_2(\theta)d\theta \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \mathit{\Pi}\left[\begin{array}{c|c} A_1 & B_{1,1} \\ \hline B_{2,1} & \{C_{i,1}\} \end{array}\right]\begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}(s) \\ \mathit{\Pi}\left[\begin{array}{c|c} A_2 & B_{1,2} \\ \hline B_{2,2} & \{C_{i,2}\} \end{array}\right]\begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}(s) \end{bmatrix} \\
&= \begin{bmatrix} \mathit{\Pi}\left[\begin{array}{c|c} A_1 & B_{1,1} \\ \hline B_{2,1} & \{C_{i,1}\} \end{array}\right] \\ \mathit{\Pi}\left[\begin{array}{c|c} A_2 & B_{1,2} \\ \hline B_{2,2} & \{C_{i,2}\} \end{array}\right] \end{bmatrix}\begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}(s).
\end{aligned}
$$

Thus, the vertical concatenation of two PI maps gives rise to another uniquely defined PI map. □

B.2 Parametrization of Positive PI Operators

In the section, we provide sufficient conditions for the positivity of a PI-operator. We assume that the square root of a PI-operator is also a PI-operator. Additionally, we consider the case where its components, Q_i and R_i, are matrix-valued polynomial functions. This leads to the following result, where a test for the positivity of a PI-operator can be equivalently formulated as a test for the positivity of a matrix. This allows converting operator-valued constraints to LMI constraints, which can be solved using an SDP solver.

Theorem 2.6 (Positive PI)**.** *For any functions $Z_1 : [a,b] \to \mathbb{R}^{d_1\times n}$, $Z_2 : [a,b]\times[a,b] \to \mathbb{R}^{d_2\times n}$, if $g(s) \geq 0$ for all $s \in [a,b]$ and*

$$\begin{aligned}
&P = T_{11}\int_a^b g(s)ds, \quad R_0(s) = g(s)Z_1(s)^T T_{22} Z_1(s),\\
&Q(\eta) = g(\eta)T_{12}Z_1(\eta) + \int_\eta^b g(s)T_{13}Z_2(s,\eta)ds + \int_a^\eta g(s)T_{14}Z_2(s,\eta)ds,\\
&R_1(s,\eta) = g(s)Z_1(s)^T T_{23}Z_2(s,\eta) + g(\eta)Z_2(\eta,s)^T T_{42}Z_1(\eta) + \int_s^b g(\theta)Z_2(\theta,s)^T T_{33}Z_2(\theta,\eta)d\theta\\
&\qquad + \int_\eta^s g(\theta)Z_2(\theta,s)^T T_{43}Z_2(\theta,\eta)d\theta + \int_a^\eta g(\theta)Z_2(\theta,s)^T T_{44}Z_2(\theta,\eta)d\theta,\\
&R_2(s,\eta) = g(s)Z_1(s)^T T_{32}Z_2(s,\eta) + g(\eta)Z_2(\eta,s)^T T_{24}Z_1(\eta) + \int_\eta^b g(\theta)Z_2(\theta,s)^T T_{33}Z_2(\theta,\eta)d\theta\\
&\qquad + \int_s^\eta g(\theta)Z_2(\theta,s)^T T_{34}Z_2(\theta,\eta)d\theta + \int_a^s g(\theta)Z_2(\theta,s)^T T_{44}Z_2(\theta,\eta)d\theta. \qquad (2.6)
\end{aligned}$$

where

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14}\\ T_{21} & T_{22} & T_{23} & T_{24}\\ T_{31} & T_{32} & T_{33} & T_{34}\\ T_{41} & T_{42} & T_{43} & T_{44}\end{bmatrix} \succeq 0,$$

then the operator $\varPi\left[\begin{array}{c|c} P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right]$ as defined in Equation (2.3) *is positive semidefinite, i.e. $\left\langle \mathbf{x}, \varPi\left[\begin{array}{c|c} P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right]\mathbf{x}\right\rangle \geq 0$ for all $\mathbf{x} \in \mathbb{R}^m \times L_2^n[a,b]$.*

Proof. Let $T = \begin{bmatrix} T_1 & T_2\\ T_2^T & T_3\end{bmatrix} \geq 0$ where $T_1 = T_{11}$, $T_2 = [T_{12} \quad T_{13} \quad T_{14}]$, and $T_3 = \begin{bmatrix} T_{22} & T_{23} & T_{24}\\ T_{32} & T_{33} & T_{34}\\ T_{42} & T_{43} & T_{44}\end{bmatrix}$. Then, there exists a U such that $T = U^T U$ where $U = \begin{bmatrix} U_1 & U_2\\ U_2^T & U_3\end{bmatrix}$. We have

$$\begin{aligned}
&\varPi\left[\begin{array}{c|c} P & Q_1\\ \hline Q_2 & \{R_i\}\end{array}\right]\\
&= \varPi\left[\begin{array}{c|c} I & 0\\ \hline 0 & \{Z_i\}\end{array}\right]^* \varPi\left[\begin{array}{c|c} T_1 & T_2\\ \hline T_2^T & \{T_3,0,0\}\end{array}\right] \varPi\left[\begin{array}{c|c} I & 0\\ \hline 0 & \{Z_i\}\end{array}\right]\\
&= \varPi\left[\begin{array}{c|c} I & 0\\ \hline 0 & \{Z_i\}\end{array}\right]^* \varPi\left[\begin{array}{c|c} U_1 & U_2\\ \hline U_2^T & \{U_3,0,0\}\end{array}\right]^* \varPi\left[\begin{array}{c|c} U_1 & U_2\\ \hline U_2^T & \{U_3,0,0\}\end{array}\right] \varPi\left[\begin{array}{c|c} I & 0\\ \hline 0 & \{Z_i\}\end{array}\right]\\
&= (\mathcal{T})^*(\mathcal{T})
\end{aligned}$$

where $\mathcal{T} = \varPi\left[\begin{array}{c|c} U_1 & U_2\\ \hline U_2^T & \{U_3,0,0\}\end{array}\right] \varPi\left[\begin{array}{c|c} I & 0\\ \hline 0 & \{Z_i\}\end{array}\right]$, $\bar{Z}_0(s) = \begin{bmatrix}\sqrt{g(s)}Z_1(s)\\ 0\\ 0\end{bmatrix}$, $\bar{Z}_1(s,\theta) = \begin{bmatrix} 0\\ \sqrt{g(s)}Z_2(s,\theta)\\ 0\end{bmatrix}$ and $\bar{Z}_2(s,\theta) = \begin{bmatrix} 0\\ 0\\ \sqrt{g(s)}Z_2(s,\theta)\end{bmatrix}$. $\square$

B.3 Formulae for PI Operator Inversion

We provided formulae to calculate the inverse of a PI operator $\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$ (refer Lemmas 2.10 and 2.11), wherein the inverse was presented as a composition of multiple PI operators.

Unlike Lemmas 2.10 and 2.11, in this section, we avoid the composition notation and directly specify the parameters of the inverse PI operator to simplify the inverse computation. Consequently, we have two sets of formulae to compute the inverse of $\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$ dependent on the invertibility of either P or $\Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_i\}\end{array}\right]$. The choice of formulae depends on the PI being inverted and the application. For example, in a PDE with a dynamic controller, if the stability of the controller state is not necessary, then P is not necessarily invertible, and the formulae in Lemma B.4 should be used. However, if the stability of the controller state is necessary and the asymptotic/exponential stability of the PDE state is not required, then the inverse in Lemma B.3 should be used. We have omitted the proof here since the proof was presented in Section 2.4.3.

Lemma B.3. *Suppose* $\mathcal{P} = \Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \in \mathbf{\Pi}_4$ *with* P *invertible, then* $\mathcal{P}$ *is invertible if and only if* $\Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{H_i\}\end{array}\right]$ *is invertible where* $H_0 = R_0$ *and* $H_i(s,\theta) = R_i(s,\theta) - Q_2(s)P^{-1}Q_1(\theta)$. *Furthermore, if* H_i *satisfy the conditions of Cor. 2.8 and* $\hat{R}_i$ *are as defined therein, we have that*

$$\left(\Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]\right)^{-1} = \Pi\left[\begin{array}{c|c} \tilde{P} & \tilde{Q}_1 \\ \hline \tilde{Q}_2 & \{\tilde{R}_i\}\end{array}\right],$$

where $\tilde{P}, \tilde{Q}_i, \tilde{R}_i$ *are defined as* $\tilde{R}_i = \hat{R}_i$,

$$\tilde{P} = P^{-1} + P^{-1}\left(\int_a^b \underline{Q}(s)ds\right)P^{-1},$$

$$\underline{Q}(s) = Q_1(s)\hat{R}_0(s)Q_2(s) + \int_s^b Q_1(\theta)\hat{R}_1(\theta,s)Q_2(s)d\theta + \int_a^s Q_1(\theta)\hat{R}_2(\theta,s)Q_2(s)d\theta,$$

$$\tilde{Q}_1(s) = -P^{-1}\left(Q_1(s)\hat{R}_0(s) - \int_s^b Q_1(\theta)\hat{R}_1(\theta,s)d\theta - \int_a^s Q_1(\theta)\hat{R}_2(\theta,s)d\theta\right),$$

$$\tilde{Q}_2(s) = \left(-\hat{R}_0(s)Q_2(s) - \int_a^s \hat{R}_1(s,\theta)Q_2(\theta)d\theta - \int_s^b \hat{R}_2(s,\theta)Q_2(\theta)d\theta\right)P^{-1}.$$

Lemma B.4. *Suppose* $\mathcal{P} = \Pi\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \in \mathbf{\Pi}_4$ *with* $\Pi\left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & \{R_i\}\end{array}\right]$ *invertible. Then* $\mathcal{P}$ *is invertible if and only if the matrix*

$$\hat{P} = P - \int_a^b Q_1(s)\hat{R}_0(s)Q_2(s)ds - \int_a^b\int_a^s Q_1(s)\hat{R}_1(s,\theta)Q_2(\theta)d\theta ds$$
$$- \int_a^b\int_s^b Q_1(s)\hat{R}_2(s,\theta)Q_2(\theta)d\theta ds$$

is invertible. Furthermore, if $H_i = R_i$ satisfy the conditions of Cor. 2.8 and $\hat{R}_i$ are as defined therein, we have that

$$\mathit{\Pi}\left[\begin{array}{c|c} \tilde{P} & \tilde{Q}_1 \\ \hline \tilde{Q}_2 & \{\tilde{R}_i\} \end{array}\right] = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\} \end{array}\right]^{-1}$$

where $\tilde{P} = \hat{P}^{-1}$, $\tilde{R}_0 = \hat{R}_0$,

$$\begin{aligned}
\tilde{Q}_1(s) &= -\tilde{P}\left(Q_1(s)\hat{R}_0(s) - \int_s^b Q_1(\theta)\hat{R}_1(\theta,s)d\theta - \int_a^s Q_1(\theta)\hat{R}_2(\theta,s)d\theta\right), \\
\tilde{Q}_2(s) &= \left(-\hat{R}_0(s)Q_2(s) - \int_s^b \hat{R}_1(\theta,s)Q_2(\theta)d\theta - \int_a^s \hat{R}_2(\theta,s)Q_2(\theta)d\theta\right)\tilde{P}, \\
\tilde{R}_1(s,\theta) &= \hat{R}_1(s,\theta) + \int_a^s \hat{R}_1(s,\eta)Q_2(\eta)\tilde{Q}_1(\theta)d\eta + \int_s^b \hat{R}_2(s,\eta)Q_2(\eta)\tilde{Q}_1(\theta)d\eta, \\
\tilde{R}_2(s,\theta) &= \hat{R}_2(s,\theta) + \int_a^\theta \hat{R}_1(s,\eta)Q_2(\eta)\tilde{Q}_1(\theta)d\eta + \int_\theta^b \hat{R}_2(s,\eta)Q_2(\eta)\tilde{Q}_1(\theta)d\eta.
\end{aligned}$$