Sameness of Representation of Mathematical Entities

by

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ABSTRACT

This dissertation is on the topic of sameness of representation of mathematical entities from a mathematics education perspective. In mathematics, people frequently work with different representations of the same thing. This is especially evident when considering the prevalence of the equals sign (=). I am adopting the three-paper dissertation model. Each paper reports on a study that investigates understandings of the identity relation.

The first study directly addresses function identity: how students conceptualize, work with, and assess sameness of representation of function. It uses both qualitative and quantitative methods to examine how students understand function sameness in calculus contexts. The second study is on the topic of implicit differentiation and student understanding of the legitimacy of it as a procedure. This relates to sameness insofar as differentiating an equation is a valid inference when the equation expresses function identity. The third study directly addresses usage of the equals sign ("="). In particular, I focus on the notion of symmetry; equality is a symmetric relation (truth-functionally), and mathematicians understand it as such. However, results of my study show that usage is not symmetric. This is small qualitative study and incorporates ideas from the field of linguistics.

Each study is at a different point in the journey of becoming a self-contained journal article. Portions of the first study have been published in two separate conference proceedings (Mirin, 2018, 2020b). The second dissertation study is already published in a journal (Mirin & Zazkis, 2020). Copyright of the journal paper is held by For the Learning of Mathematics Publishing Association (https://flm-journal.org/). The third

study is earliest in this process; no empirical aspects of it have been published in any proceedings or journals. However, preliminary results were presented at the Northeastern RUME (Research in Undergraduate Mathematics Education) conference (Mirin & Dawkins, 2020), and theoretical contributions are reported in Mirin (2019) and Mirin (2020a).

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GENERAL INTRODUCTION

The importance of the identity relation in mathematics cannot be overstated. Four of Euclid's five Common Notions deal with equality, and the fifth deals with inequality (Euclid, 300 B.C.E./2013).We see the identity relation regularly with any equation; "a=b" tells us that a and b are identical. We use the equals sign to express that the object represented by "a" and the object represented by "b" are in fact the same object. These identity statements allow us to make inferences about mathematical objects and are essential to most, if not all, subfields of mathematics. Leibniz' law of indiscernibles states that two objects x and y are the same object if and only if they share all properties (Noonan & Curtis, 2014). This means that when x=y, we can infer that whatever properties x has, y also has, and vice versa. This law allows us to make powerful inferences. Consider the following examples from a variety of subfields of mathematics. Numerical equations such as "2(2)=3+1" tell us that the object represented by "2(2)" is the same as the object (the number four) represented by "3+1", and that therefore 3+1 is even. In other words, because 2(2) has the property of being even and 3+1 is identical to 2(2), we can conclude using Leibniz' law that 3+1 is even. Similarly, the equation "$e^{i\pi} = -1$" tells us that $e^{i\pi}$ is in fact a real number. These inferences are not limited to statements about numbers. For example, if p is an element of a Boolean algebra, then $p \cdot \overline{p} = 0$, and therefore $p \cdot \overline{p}$ is an additive identity (since it is identical to 0). The permutation equation "(1 3 2) = (1 3) (1 2)" tells us that the cycle (1 3 2) is in fact even. Similarly, a set theoretic proof that the singleton {x} exists for any set x might establish that {x,x} exists (by the Pairing Axiom), and since {x,x}={x} by (Extensionality Axiom), {x} exists (Devlin & Devlin, 1993). An instance of the fundamental theorem of calculus tells us a statement of function identity: that the function defined by $y = \int_2^x 3t^2 dt$ is identical to the function defined by $y=x^3-8$, and therefore $y=x^3-8$ represents an accumulation function corresponding to the rate of change function defined by $y = 3x^2$. Proofs in mathematics sometimes involve long strings of equalities, which consist of multiple identity statements; for example, in my Master's Thesis, I include a string of six equalities to show that a particular unary operation distributes over a particular binary operation in a relation algebra (Mirin, 2013). As the above examples illustrate, identity statements (often in the form of equations) are indispensable to many subfields in mathematics.

The notion of identity is closely linked to the notion of representation. While this issue is elaborated on in a more philosophically rigorous manner later, it is important to emphasize that this dissertation is a mathematics education (not philosophy) dissertation. Thus, it is helpful to continue to keep psychological considerations in mind. To continue with the examples above, (1 3 2) = (1 3) (1 2) because "(1 3 2)" and "(1 3) (1 2)" are different representations of the same thing (a particular permutation). However, each representation is likely to bring different properties to mind. For example, (1 3) (1 2) emphasizes that the permutation at hand is even. When we make identity claims, such as equations, what makes them informative is that the representations differ; "2(2)=3+1" is informative because "2(2)" and "3+1" differ in their representations. The representation "2(2)" emphasizes that the number at hand is a perfect square and is even, whereas the representation "3+1" emphasizes that the number at hand is one more than a prime number. The differences in representation underscore the fact that we are always thinking of an object in a particular way or with respect to certain properties. How someone conceptualizes an object is going to depend on the individual person as well as on the way the object is represented. We are never talking about objects independently of how we conceive of them.

Before addressing identity in specific aspects of mathematics, I discuss more generally its meaning. It might seem strange to discuss something as quotidian in mathematics as identity, but identity is a surprisingly difficult concept. Characterizing the identity relation is a longstanding theme in intellectual history; Plato addressed identity in Parmenides in the fourth century, Leibniz addressed it in *Discourse on Metaphysics* in the 17th century, Frege addressed it in *On Concept and Object* (and several other works) in the 19th century, and Williamson addressed it in *Identity and Discrimination* recently in the late 20th century (Dejnozka, 1981; Frege, 1879/1967; Leibniz, 1846/1992; Williamson, 1990/2013). The fact that identity has been discussed in such depth by many intellectuals underscores its significance. My intent in bringing up this issue is *not* to solve a long-standing problem in philosophy, nor to provide a comprehensive overview of identity in intellectual history. Instead, I am stressing the nontriviality of understanding the identity relation. When we evaluate student understanding of identity, it helps to keep in mind that students are not missing something trivial and straightforward. The philosophical and mathematical theory is relevant for sensitizing the reader and the researcher to think carefully about meaning, equality, and identity to provide a sound conceptual basis prior to studying student conceptions of these ideas. Looking to intellectual history to situate mathematics education research is a longstanding theme in the mathematics education field -- see, for example, Sfard (1992), Thompson and Carlson (2017), and Harel et al. (2009). Sfard (1992) takes the perspective that an individual's development of a particular concept might parallel the historical intellectual development. Additionally, examining our own views might help us reflect more on our research. It also provides a starting point for thinking about how one might conceptualize identity – philosophers and mathematicians are people with these ideas, so students could have similar intuitions or rationales. We have seen this in at least one other case; Antonini and Mariotti (2008) observe that some students are unconvinced by non-constructive proofs, an idea shared by intuitionists such as Brouwer (Iemhoff, 2019).

Identity is a tricky concept to put into words. Consider the sentence "A and B are identical". The "are" in that previous sentence is a red flag; if A is indeed identical to B, then there is only one object we are talking about, yet the use of "are" suggests the presence of more than one. So, I can more precisely say "A is identical to B". That sentence has an easy fix, but this is not the case for other sentences. Suppose I want to ask someone "what does it mean for two functions to be the same?" If the functions really are the same, then there certainly aren't two of them, but asking "what does it mean for a function to be the same as itself?" does not make too much sense either. I could rephrase it to "what does it mean for a function f to be identical to a function g," but then I've introduced unnecessary naming. Perhaps "f" and "g" have already been used in that conversation to name particular functions. Then, I must find other letters, and I then have an abundance of letters. In short, there is some deliberate imprecision for the sake of readability.

As mentioned earlier, the normative meaning of the equals sign is that of identity; the equals sign expresses identity, so "a=b" is synonymous with "a and b are identical". Yet, as alluded to above, discussing identity is tricky. It is so tricky that people appear to avoid making identity statements, and instead they use other words or phrases that appear to convey the idea of identity. These phrases include:

- Can be thought of as
- Is equivalent to (without defining an equivalence relation)
- Can be described as
- Can be represented by/as
- Is essentially
- Can be written as
- Can be expressed by
- Represents

This becomes especially evident when we look at wording of the fundamental theorem of arithmetic. When Googling "the fundamental theorem of arithmetic," two out of the first three results use the phrase "can be written as" or "can be represented as" rather than simply "is" or "equals".[1] I did a search of Velleman's 2nd Edition of *How to Prove It*, the standard textbook used for MAT300 ("Mathematical Structures", a transition-to-proofs course) here at Arizona State University (ASU) and commonly used in transition-to-proofs courses at US universities (David & Zazkis, 2019). The phrase "can be written as" occurs eleven times in this text. Nine of these occurrences appear to convey the idea of identity, such as "x can be written as a product of two smaller integers" (p. 6).

Another example of this phenomenon is in situations where notation is described

[1] Googled on 10/23/2019. First three results are:
https://en.wikipedia.org/wiki/Fundamental_theorem_of_arithmetic
https://www.mathsisfun.com/numbers/fundamental-theorem-arithmetic.html
https://brilliant.org/wiki/fundamental-theorem-of-arithmetic/

or introduced. I encountered the following wording as part of the College Algebra MAT117 homework series here at Arizona State University: “C(t) represents the number of cases of Ebola t days after May 1, 2014”. Technically, C(t) actually is the number of cases of Ebola t days after May 1, 2014, whereas “C(t)” represents it.

Before going further, it is important that I stress what I mean by “identity” and “equality”. As alluded to above, “a=b” is true if and only if a and b are identical. This means that “a=b” is true if and only if {a,b} has cardinality 1. I use “a is the same as b”, “a=b”, “a equals b”, and “a is identical to b” synonymously. Occasionally, I use “a is b” as well, which comes with some ambiguities; sometimes I mean the “is” of predication (as in “Socrates is mortal”), but other times I mean the “is” of identity (as in “Paris is the capital of France”). It is important to note that by “identity”, I mean true identity, not some sort of weaker equivalence relation. I make this point in Mirin (2019), from which I include an excerpt below:

> Equality represents true identity, not merely an equivalence relation: a=b if and only if a *is the same thing* as b. It does not suffice for a to be *equivalent* or *isomorphic* to b. Taking an example from algebra, it is not the case that $Z/2Z = Z_2$. Mathematicians might casually refer to them as “the same group,” but they are actually different groups (members of $Z/2Z$ are *sets* of integers, whereas members of Z_2 are integers). $Z/2Z$ and Z_2 are of the same *isomorphism class*, but they are not *equal* to each other. This is not to say that they are unequal simply because we write members of $Z/2Z$ one way and members of Z_2 another way; indeed, we can have two different names for the same thing. For example, we can write *the same* group with additive or multiplicative notation; we have the same group, not merely

isomorphic groups. Similarly, we can call the same function both “f” and “g”. So long as the set for the group, together with its operation, are identical (despite different names), the groups are identical.

To elaborate on my point: while we want to be explicit that equivalence (in this case, isomorphism) is not always equality, we also ought not think that different names always name different objects. Consider the following group tables:

G_1

*	0	1
0	0	1
1	1	0

G_2

+	a	b
a	a	b
b	b	a

a=0
b=1

G_3

⋆	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

[0]={0,2,-2,4,-4,…}
[1]={0,1,-1,3,-3,…}

Figure 1.1. Groups of Order Two.

How many groups have I defined above in Figure 1.1? All three group tables certainly look different visually, and a group theorist might say that they are all just Z_2. However, if we look carefully, we can see that there are two distinct groups. G_1 and G_2 are identical. They have the same elements, and their group operation is identical. Notice that their group operation is identical because the operation agrees on all elements – the notion of identical function(s) is discussed later. Since a group is a set together with a particular operation that satisfies certain axioms, by the nature of what defines a group, $G_1 = G_2$. This is despite the fact that the elements in the group table for G_1 are written differently than the elements in the group table for G_2; note that we know that $G_1 = G_2$ because we know (by stipulation) that the element a is identical to 0 and b is identical to 1. If a or b were anything else, this would not be the case. It is clear that G_2 and G_3 are different groups; elements of G_2 are integers, and elements of G_3 are infinite sets.[2] So, among G_1, G_2, and G_3, there is one isomorphism class, two groups, and three names of groups.

While I and the modern mathematical community use "=" to express identity, it bears mentioning that the equals sign has not always been used this way. In the 19th century, there was a widespread debate about whether in mathematics "=" represents true identity. Some thinkers "posited some weaker form of 'equality' such that the numbers 4(2) and 11-3 would be said to be equal in number or equal in magnitude without thereby constituting one and the same thing" (Klement, 2019). People with such viewpoints might disagree that the fundamental theorem of arithmetic tells us that every non-prime number is (identical to) a product of primes, while at the same time they might claim that every non-prime number equals a product of primes.

1.1 Frege and Identity

I now turn to a discussion of philosophy. Since there is so little mathematics education literature on identity, it makes sense for me to include what literature there happens to be on identity (and the bulk of that literature is philosophical). Since the topic of identity is what ties my dissertation papers together, it is worth exploring more

[2] I acknowledge that some people might name an equivalence class with just an element. That is, it is not uncommon to see someone name the set of even integers "0". However, when people do this, they are usually explicit that they are adopting this convention. Furthermore, I'm not adopting this convention; I defined the group G_1 to have the integers 0 and 1 as members.

generally the meaning of identity. Another reason I include this discussion is an effort of intellectual honesty; my own interest in how people understand identity is motived by my background in philosophy. The existence of philosophical literature on the topic also underscores the nontriviality of understanding identity.

How the issue of identity manifests in non-mathematical contexts can inform how we approach them in mathematical contexts. Accordingly, we begin with one of Frege's puzzles of identity. This involves a story. The Ancient Greeks observed a dull white glowing sphere in the sky during sunrise. They called this "The Morning Star (Phosphorus)". They also observed a sphere in the sky during sunset and called this sphere "The Evening Star (Hesperus)". It wasn't until around the sixth century BCE, through empirical observation, that the Greeks discovered that The Morning Star and The Evening Star are in fact the same celestial body: the planet Venus (Frege, 1879/1967; Makin, 2010). This is despite the fact that there was an *experiential* difference viewing Venus as The Morning Star versus as The Evening Star. With this in mind, consider the following sentences:

(1) The Morning Star is The Evening Star.

(2) Venus is Venus.

Observe that (1) is informative, whereas (2) is not. Frege puzzled over what the identity relation is on: names, or objects (the things named by names). Early Frege (1879/1967) proposed a theory of meaning in which the referent (*bedeutung*) of a name is the meaning of a name (specifically, that all noun phrases do is refer to objects). In order to account for the informativeness of sentences like (1), Frege initially rejected the idea that the identity relation is between objects (referents). He thus concluded that it must be a relation between names. With the identity relation between names, it became a less trivial-seeming relation that expresses information. “The Morning Star is the Evening Star” was not just saying that Venus was itself; instead, it was saying that the names ‘The Morning Star’ and ‘The Evening Star’ refer to the same object, so that “*The Morning Star*” = “*The Evening Star*” rather than *The Morning Star* = *The Evening Star*. Under this ultimately rejected conception, identity is then no longer truly identity, but some sort of equivalence relation on names.

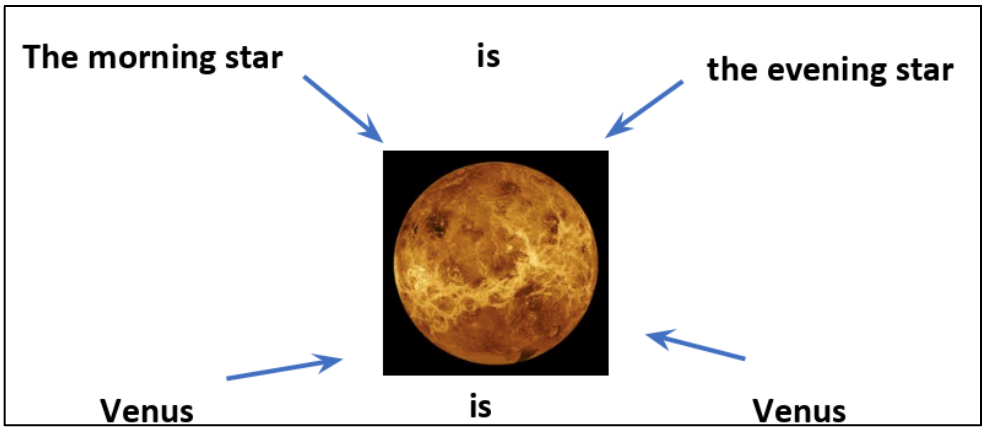


Figure 1.2. Venus and its Representations.

Frege later rejected this view on the grounds that meaning should involve more than arbitrary linguistic conventions (names) and instead express “objective knowledge” about the world (Frege, 1879/1967; Makin, 2010). Thus, he ultimately decided that the identity relation is between objects rather than names. In order to deal with the problem of informativeness (“Venus=Venus” isn’t an informative statement), he modified his original theory by creating the construct of sense (which he also calls “cognitive value”) to complement that of referent. Sense is what leads you to think of the planet Venus when you hear “The Morning Star” and Joe Biden when you hear “the president of the United States inaugurated in 2021”. Frege refers to a sense as a “mode of presentation” and is something that we “grasp” (Dejnozka, 1981; Frege, 1892/1948; Makin, 2010). The sentence “The Morning Star = The Evening Star” tells us that the senses picked out by “The Morning Star” and “The Evening Star” point to the same referent (Venus, see Figure 1.2). The expressions “2+3” and “4+1” express different senses but have the same referent (the number five). Frege never defines “sense” precisely, but in this example the sense of “2+3” involves thinking of the addition function (arguably, the sentence “2+3=4+1” is informative in that it says something about addition).

The above story illustrates that in order to really understand identity statements, we have to think of more than just the referents or objects named in the statements; we have to think of how they are named. This is one reason that identity is tricky to talk about; if we think we are only talking about objects, identity statements become tautological and uninformative. We have to re-orient ourselves and realize that we are not talking just about objects existing independently of how we conceive of them. In a way, every identity statement can be thought of as a story – when we see “A=B” we can think to ourselves “Once upon a time, I thought of A, and I thought of B without knowing whether or not they are the same. Later, I found out that they are the same”. Perhaps Hodges (1997) is expressing a similar sentiment when he writes “name the elements of the structure first, then decide how they should behave” (p.2).

This careful attention to identity statements tells us not only about how to understand identity statements, but also the importance of representation. I would argue that there is no such thing as a fully transparent representation; that is, there is no way that any object can be represented in a way that shows all of its properties. Think of an object – any object. There is some aspect of it that you’re missing. Perhaps it is an aspect that you haven't even thought of yet. If you were thinking of Venus, maybe you had the image in Figure 1.2 in your mind. Were you thinking about the fact that it is the second planet from the sun? There are many ways to represent a single object, but there's no reason to think that there is some ultimate transparent representation. This brings us back to the point I emphasized earlier: we are always thinking of the object *in some way or with respect to certain properties*. Thus, when we talk about an object or ask students about an object, we cannot assume that we are simply *giving* them an object – we are thinking about the object in some way, and they are thinking about the object in some way. We were never really talking just about a planet (Venus), but how we conceived of it. I return to this topic later when discussing mathematics education literature on identity.

All this discussion about objects raises an issue; what *thing* does a name of a mathematical entity (e.g., a number) refer to? Clearly, "mathematical entity" is the answer to my question. But what is that? Is it even an object? There is no physical object of a number that we can simply point to, and there is a longstanding historical discussion about what an abstract mathematical object even is, if anything (Horsten, 2016). In the sentence $\binom{100}{3} = \binom{100}{97}$, the terms on each side of the equals sign refers to the "object", the number 161700, perhaps out in the Platonic heavens (Figure 1.3).

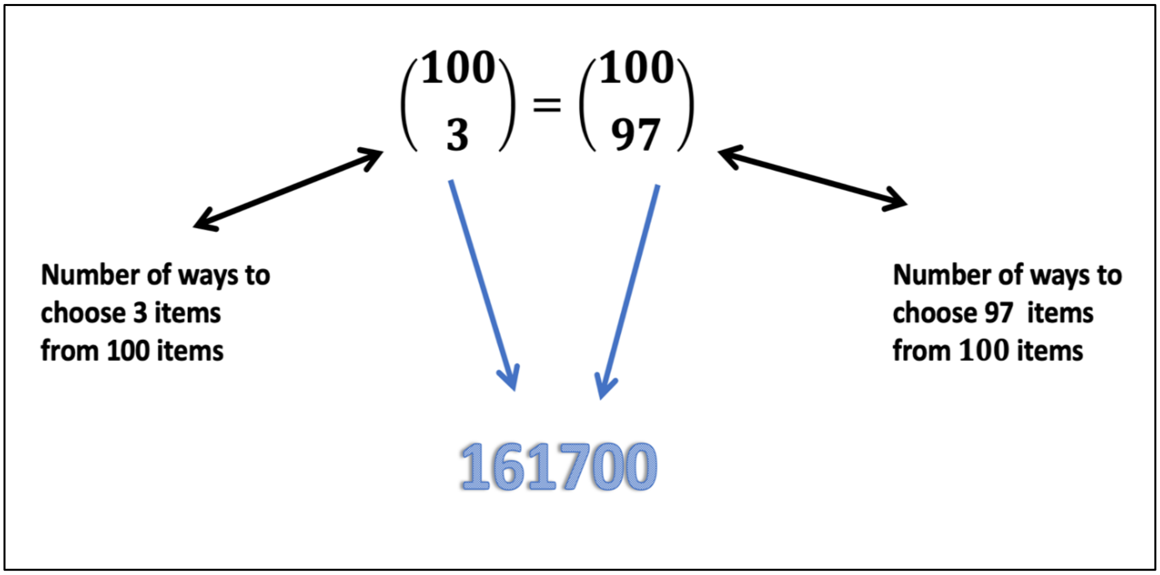


Figure 1.3. Multiple Representations of the Same Number.

Yet, mathematicians go on to do mathematics and use identity statements without solving this problem. Identity statements can give us a way of sidestepping some of this messy ontology and provide great inferential power. Consider the example of Zermelo-Fraenkel (ZF) set theory. The axiom of extensionality says that for sets, S=T if and only if S and T have the same elements. Mathematicians work within the system of axiomatized set theory with "set" being an undefined notion. This is partially because the notion of sameness of sets captures some of the notion of what a set is: a collection of objects. I discuss this idea, that sameness captures the essence of what an object is, in further detail later as it has applicability well beyond set theory.

1.2 Mathematical Identity in Education: A General Literature Review

Having reviewed some of the philosophical literature on identity, I now move to a discussion of the mathematics education literature on identity. Literature informing the topic of mathematical identity largely falls into two categories. The first category consists of literature on student understanding of the equals sign in numerical equations. The second category includes literature that, although is not purported to be about identity, includes information about how students might view identity. I turn to a review of this literature more broadly. Literature more specific to each individual study is addressed in the chapters pertaining to each study.

1.2.1 Equals Sign Literature

This literature largely centers on the relational versus operational dichotomy, although authors are not always explicit about this dichotomy. While a relational understanding is a normative or productive understanding, an operational understanding of the equals sign is any non-normative or unproductive understanding. I first provide literature reviews on the characterizations of relational and operational views of the equals sign.

1.2.1.1 "Relational" Equals. Mathematics education authors tend to define the "relational" (normative) meaning of the equals sign in a way that is problematic. Some authors give a nominal view of the equals sign as expressing a relation between names or signs, some define it narrowly and limited to subtopic of study, and some treat it as expressing intersubstitutability. Often, authors are unclear precisely on what sort of objects (names versus numbers or mathematical entities) the equals sign expresses a relation and use the word "equivalence" without specifying an equivalence relation. In Mirin (2019), I discuss some of these characterizations. I use the philosophy discussed earlier to frame these characterizations. My main finding is that authors appear to give a precise definition of a relational understanding of "=", but closer investigation reveals that the authors are not successful. Most of the characterizations are ambiguous, narrow, or philosophically problematic. For example, several authors appear to characterize "="

as expressing sameness of attributes of expressions (rather than sameness of referent), and several other authors focus on nominal sameness (which, as discussed earlier, Frege had reasons to reject). Now, these authors are not philosophers writing philosophy papers, so there is no reason to expect that they even *should* solve a longstanding philosophical problem of the meaning of "=". Yet, it appears that the authors purport to characterize a meaning of "=". My main conclusion in Mirin (2019) is that a crucial commonality throughout all the characterizations of a relational meaning of "=" is that it is important that students understand "=" in a way that is tantamount to expressing an equivalence relation.

At this point, it is worth providing a link between the present discussion on the meaning of the equals sign and the prior discussion of the relationship between what defines a class of objects and the identity criteria within that class of objects. This is difficult in the case of numbers, because it is difficult to imagine asking someone if a and b are the same number and having them get the question wrong for reasons other than a computational error (although it is possible, if someone conflates 'number' with 'numeral', as I discuss in more detail in Mirin (2020a) with the example of "D" in Behr et al. (1980)). The discussion from Mirin (2020a) about Behr et al. (1980) and equinumerosity provides some insight. One way to view a (natural) number is as an equivalence class of equinumerous sets; the number 3 represents all sets of three objects (a philosophical statement), and someone begins to understand the idea of "3" by experiencing various sets of three objects (the psychological counterpart). This philosophical statement is the view of both Frege and Russell (Russell, 1993) and is endorsed by variation theorists (discussed below) as well as several psychologists (Izard et al., 2014). This is the viewpoint that Behr et al. (1980) appear to be taking in the following quote "the most basic meaning is an abstraction of the notion of *sameness*. This is an intuitive notion of equality which arises from experience with equivalent sets of objects. This is the notion of equality which we would hope children would exhibit" (p.13). It appears that the authors are endorsing that a number is an equivalence class of objects, and that therefore numbers are the same if and only if they are the same equivalence class.[3]

1.2.1.2 "Operational" Equals. In Mirin (2020a), I provide an operational counterpart to Mirin (2019). Specifically, I include a literature review and analysis of the various "operational" characterizations of the equals sign. For a more detailed literature review of the meaning of the "operational" classification, please consult Mirin (2020a). Below is an excerpt from Mirin (2020a), pp. 805-806.

> Many students struggle with accepting equations of the form (i) "5=2+3", (ii) "5=5", and (iii) "3+2=4+1", preferring equations like (i') "2+3=5", (ii') "5+0=5", and (iii') 2+3=5+1=6, respectively (Behr et al., 1980; Byrd et al., 2015; Denmark et al., 1976; McNeil et al., 2006; Oksuz, 2007; Sáenz-Ludlow & Walgamuth, 1998). The equations (i), (ii), and (iii) can be described as "rule violations" and characterize operational understandings of the equals sign (Oksuz, 2007). The idea behind this terminology is that students with operational understandings are accustomed to seeing the equals sign in contexts like "2+3=5", where "2+3" is an arithmetic problem to which "5" is the answer. Such a student might have in mind

[3] For this interpretation to make sense, there has to be a way of making sense of numerical operations as operations on classes of objects. This is not far-fetched. We can imagine conceiving of "2+3" as the cardinality of the set resulting from forming the union of two disjoint sets, one of cardinality two and the other of cardinality three.

certain rules about how equations should look. In particular, the equations in Table [4] violate the rule that to the left of the equals sign is an arithmetic problem on the right of which is a single numeral as an answer. A common explanation posited for such understandings is that students view the equals sign as a command to perform an operation. These understandings of the equals sign that involve arithmetic, problems, answers, and calculations are characterized as "operational".

Table 1. Equations that students frequently reject accompanied by preferred alternatives.

	Rule Violation	Preferred Equation(s)	
(i)	5=2+3	(i') 2+3=5	
(ii)	2+3=4+1	(ii') 2+3=5+1=6	2+3=5 / 4+1=5
(iii)	5=5	(iii') 5+0=5	

1.2.2 Other Literature Pertaining to Identity

Although there are no papers explicitly about student understanding of identity, the idea of identity is not completely ignored. Specifically, some work has implications for how students might understand identity. At this point, it is worth reminding ourselves

[4] Table 1 refers to the first table in the reference being cited.

that our more general topic of investigation is on people's use and understanding of identity and the ideas and symbols associated with it. Although the equals sign expresses identity, the above research shows that many students do not think of it as such. Hence, when we investigate such student's understanding of the equals sign, we are not really investigating their understanding of identity exactly (hence why the topic of investigation includes "ideas and symbols associated with it"). The point I am trying to emphasize is that the existence of students with such conceptions indicates that investigating student understanding of identity is not always the same thing as investigating student understanding of the equals sign. This discrepancy highlights the need to discuss not only understanding of the equals sign, but of the concept of identity and sameness of representation in general. How someone understands identity statements is entangled with how someone understands the objects (if they even think of objects as being involved, see Thompson & Sfard, 1994); as alluded to above, how a student understands "a=b" or "a is identical to b" is inextricably tied to how they understand "a" and "b". Accordingly, the discussion herein is a review of the mathematics education literature that, while not ostensibly on the topic of identity, still relates to identity and how students might understand identity statements.

I revisit the issue of representation discussed earlier. Recall that identity statements involve both sameness and difference. A statement such as "The Morning Star is identical to The Evening Star" is a statement about sameness, and this statement is informative because it appeals to *different representations* (rather than simply stating "Venus is identical to Venus"). Hence, the idea of *different representations* is closely related to the idea of identity.

Various mathematics education researchers have considered that there are different representations of mathematical objects. Lesh et al. (1987) introduce the idea of *transparent* and *opaque* representations. They describe *transparent* representations as those that have no more meaning than the thing that is being represented, and *opaque* representations as emphasizing some aspects but not others of the thing that is being represented. This distinction is problematic. As alluded to earlier in the philosophical discussion on Frege, it seems problematic to claim that there is such a thing as a truly transparent representation that embodies an object without any mediation of language. Zazkis and Gadowsky (2001) accordingly adapt the framework of Lesh et al. (1987) by characterizing representations of numbers as transparent or opaque *with respect to a particular property*. For example,"28^2"is a transparent representation of a number with respect to the property of being a perfect square but opaque with respect to the property of being divisible by 98. We can reframe the discussion in the introduction of making inferences about numbers by using identity statements. When we observe that $e^{i\pi} = -1$ and therefore conclude that $e^{i\pi}$ is a real number, we are able to make this conclusion because "-1" is a transparent representation of $e^{i\pi}$ with respect to the property of being real.

There is little research in how students view equations (using the equals sign) between things other than numbers. One of the first times students encounter equations of functions is in the context of differential equations; a differential equation is a (particular kind) of equation between function(s), and a solution to that equation is a function that satisfies it. Rasmussen (2001) observes that students' experience with "solving" involves only numerical solutions, making the notion of a solution as a function novel to some students. He found that students often did not view a solution to a differential equation *as a function*. This was especially the case with constant functions. While this research is not purported to be about the equals sign, it seems to suggest that student might not view differential equations as expressing function identity.

Some mathematics education researchers discuss the idea of multiple *registers* (types) of representations of functions (e.g., graphical, verbal, analytic) and the translation between these registers. Overall, the literature suggests that college students tend to struggle translating between representation type, yet it does not address the idea of identity between representations (Chinnappan & Thomas, 2001; Even, 1998; Gagatsis et al., 2004). I discuss more of this representation literature below in the specific literature review for the paper on function sameness (chapter 2). Chinnappan and Thomas (2001) describe a teacher who said she did not view the algebraic representations as representations of functions and claimed that she associates functions with graphs rather than "algebra". While the authors do not address identity between representations, the fact that the teacher considers an algebraic representation to not be of a function suggests that she would not view it as being identical to the graphical representation. These researchers do not consider the possibility that a student could view identity as lost in translation when moving from one representation type to another. While to us it might seem obvious that some sort of identity is maintained when we perform certain transformations or changes in representation, this might not be the case for students. Consider the case of Mindi, described in Thompson (2013b): when given the equation "$w/3 = 11$", she claims "$w/3$" stands for a number, but in order to know what that number is, she would need to find what number "w" is first. In a way, Mindi was implicitly allowing the possibility that the " $w/3$" in the given equation refers to a different number than " $w/3$" after finding the value of w and dividing by 3. Thompson explains that Mindi was thinking procedurally: "The meaning of an equation, for Mindi, was that it was a symbolic form that she was expected to act on to end with another form x=*number*" (p. 66). We can imagine that college students might perform a similar syntactic manipulation from one representation to another without attending to meaning or the link between the prompt (problem) and the result (answer), as well as without viewing an expression as maintaining its identity throughout the problem.

Recall earlier the discussion of set theory. The axiom of extensionality says that a set A is identical to a set B if and only if A and B have the same elements. There is a sense in which this axiom captures the essence of what a set is intuitively (a collection of objects); collections of objects are identical if and only if they contain the same objects. Another example from set theory is that of the formalization of ordered pairs. The ordered pair (a,b) is defined to be the set $\{\{a\},\{a,b\}\}$. The justification for this formalization is that under this definition, (a,b)=(c,d) if and only if a=c and b=d (see, for example, Devlin and Devlin, 1993; Enderton, 1977). Observe that the criteria for equality of ordered pairs is what is used to justify the definition of the ordered pairs; this definition of ordered pair works because two (one) ordered pairs are the same if and only if they have the same elements in the same order.

These examples illustrate the relationship between identity criteria within a class of things and the defining features of that class. Said informally: what makes an A an A is closely related to how we determine when two A's are actually the same A (e.g., what makes a set a set is closely related to how we determine when two sets are actually the same set). There is a clear psychological corollary to this observation: a person's criteria for identity within a category closely relates to that person's conception of the defining features of that category. For example, if someone views $\{2,3\}$ as not the *same* set as $\{3,2\}$, then perhaps they view sets as *ordered* collections of objects rather than just

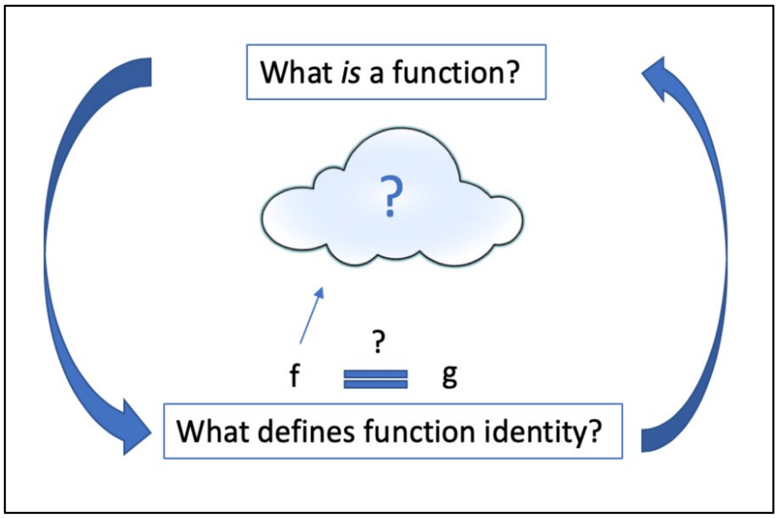


Figure 1.4. An Illustration of the Relationship Between a Category and Identity within that Category.

collections of objects. We can see how this psychological corollary manifests itself in other areas of mathematics. If a student views the group $\mathrm{Z}/2\mathrm{Z}$ as identical to Z_2, then perhaps they do not view a group as a set together with a binary operation and instead view a group as what we think of as an isomorphism class of groups (since $\mathrm{Z}/2\mathrm{Z}$ and Z_2 are indeed isomorphic). If a student views the polar points $(1, \pi/4)$ and $(-1,5\pi/4)$ as the same point, then they might view points as locations rather than ordered pairs.

Some mathematics education researchers have leveraged this psychological corollary for the concept of function. That is, they have exploited student conception of sameness of function as a way of getting at students' understanding of the defining features of the function category (Mirin, 2017; Novotná et al., 2006; Sfard, 1992). I discuss this literature in more detail in the literature review specific to my paper on function identity (chapter 2). My more general point here is emphasizing the relationship between sameness within a category and the defining features of a category.

1.2.2.1 Variation theory in mathematics education. The importance of sameness in mathematics education has been expressed by variation theorists. While I do not use variation theory per se, I include it as an example of mathematics education literature that addresses issues of sameness. For this reason, it is placed here in the literature review rather than as a theoretical perspective.

Variation theory focuses on difference rather than sameness. The general idea is that seeing difference helps one see sameness. Note that the focus on difference is not as antithetical to my topic of sameness as it first appears; recall the discussion earlier about different representations of the same thing. Variation theory is about how people learn through difference/sameness, rather than what people think about sameness itself. For this reason, it is not central to my work. Nevertheless, because of its peripheral relevance, I choose to address it. The ideas illustrated in Figure 1.4 (relationship between identity within a category and defining features of that category) highlight its connection to my topic of research. I discuss this connection below.

Variation theory is based on the idea that humans learn through discernment of differences and invariants. It involves the following principles:

Principle 1: Varying the nonessential features of a concept can help one see the critical features of a concept (instances of a concept can help one understand the concept);

Principle 2: Non-instances of a concept can help one understand a concept,[5] with emphasis on Principle 2. Suppose someone wants to understand the concept of

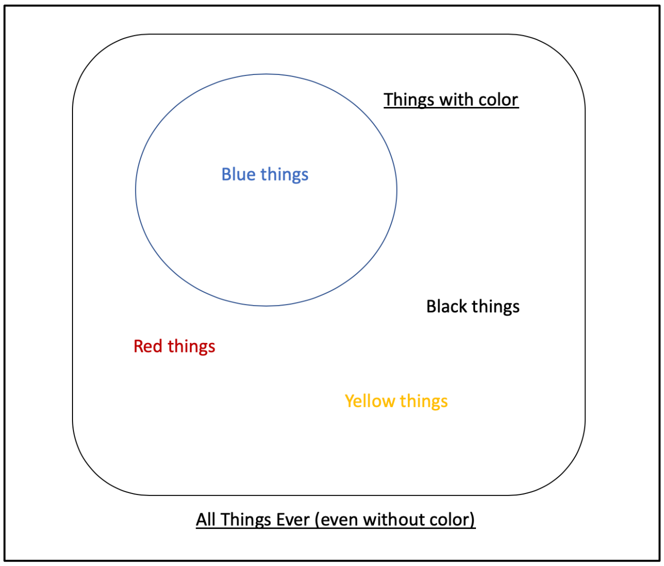


Figure 1.5. Variation Theory for "Blue".

blueness. Seeing various blue-colored objects helps someone see what blueness is (Principle 1). The shape of something does not determine whether or not it is blue, so exposing someone to blue objects of various shapes can help them discern the essential aspect (blueness) from the nonessential aspect (shape). The general idea of this principle is that seeing different instances of a concept can help one distinguish what is essential to that concept. Seeing something of a color other than blue can help someone see what blueness is (Principle 2), because being able to discern when things are not blue is closely

[5] These Principles are a reframing of the *Contrast, Separation,* and *Generalization* principles described in Marton, Runesson, & Tsui (2004)

related to the notion of being able to discern when things are blue. Kullberg et al. (2017) give the example of the concept of linear function. In order to understand what a linear function is, someone should be exposed to multiple examples of linear functions (Principle 1) as well as non-linear functions (Principle 2).

In the context of variation theory, there always seems to be some unstated larger universe of discourse. This is needed to invoke Principle 2. For example, in the context of the concept of *blue*, we assume that we are in the universe of discourse of color or things that have color. In the case of *linear function* described above, the authors assume that we are in the universe of discourse of *function*. In the case of function, we might assume that we are in the universe of discourse of binary relations. It is not always easy to discern what the universe of discourse is, but it appears to involve a larger category, and members of this larger category might not be immediately obviously in the smaller category (the "concept"). For example, if someone is still learning the concept of blue, the larger category (which includes the non-instances) will not include things that have nothing to do with color (say, mathematical objects, or perhaps sound waves). This idea is illustrated in Figure 1.5. If someone is learning the concept of function, cars will not be in the larger category since it is immediately obvious that cars are not functions.

It is important to keep in mind that the larger class of things that we have in mind may not be the same as that of the student. For example, we might ask "is this a function?" and think of *this* as a binary relation, whereas the student might just see it as symbols. When we ask "does *this* equal *that*" and think that *this* and *that* refer to numbers, the student might be thinking of them as problems or as processes.

There's a relationship between Figure 1.4 and variation theory. Through the lens of variation theory, we can view the "concept" to be a category of representations of a particular function. How someone understands what makes a representation of a function in that particular category closely relates to how they discern whether a representation of a function is not in that category. At first glance it might appear that for the category of functions, I just reiterated variation theory when discussing the relationship between the defining features of a category and identity within a category. However, *function* is not the appropriate category. Instead, the category is *(representation of) a particular function*. The thing that is the same is a particular mathematical object (rather than a class of mathematical objects), and its representation varies. The diagram below (Figure 1.6) illustrates the idea of instances and non-instances of three categories ("concepts") related to that of function: *function* itself, *function identity*, and a *particular function*. The leftmost diagram is an illustration of variation theory (specifically, Principles 1 and 2) applied to the concept of function. It illustrates that f, g, m, and k are all functions (instances of the concept of function), whereas f-inverse and $\{(x,y): x=y^4\}$ are both non-functions (non-instances of the concept of function). The diagram on the right portion of Figure 1.6 is an analogue of the "function sameness" concept; rather than a larger mathematical category being varied (like is typical in variation theory), a representation of an individual function (*the function that squares natural numbers*) is being varied, and we have instances of representations of it. The thing that's the same is the specific function. Now turn to the middle diagram, which is an application of variation theory to the concept of *function identity* (rather than simply to the concept of *function*). There are varying instances of it (inside the cloud) and non-instances of it (outside the cloud, three shown). All three diagrams can be viewed as variation theory, but on different “concepts” (each of which is represented by a cloud).[6]

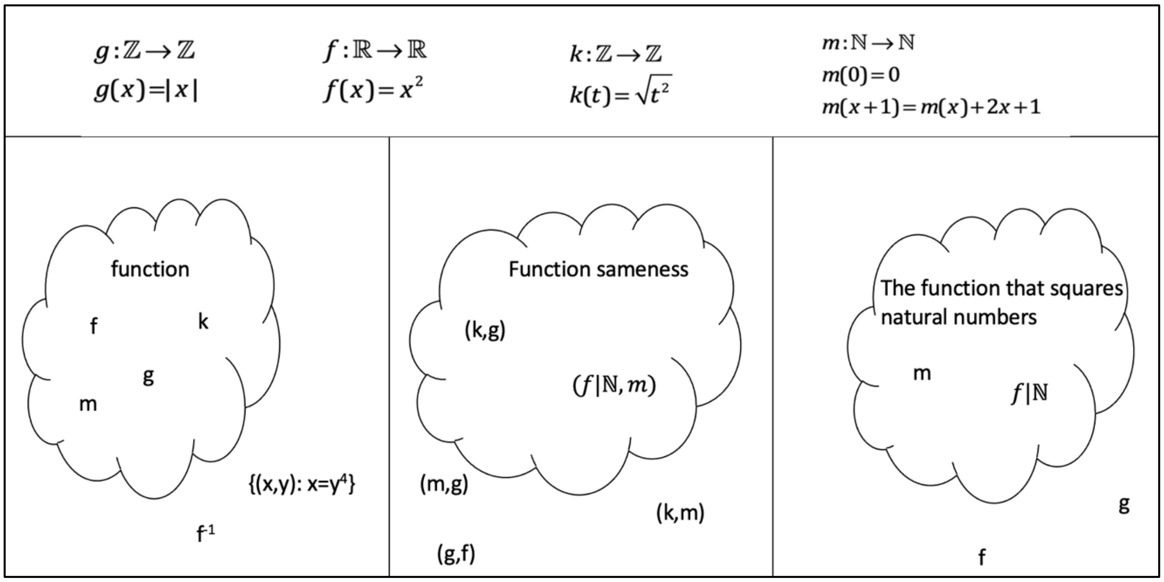


Figure 1.6. Varying Concepts Related to Function.

1.3 Epistemology and Theoretical Perspective

I have been discussing the notion of “multiple representations” of (identical) mathematical objects. Yet, I never defined what exactly representation means. When we define words, we run into a conundrum; we use more words, and must we define those words too? “Explanations come to an end somewhere” (Wittgenstein, 1953/2009). Yet, this does not mean that definitions or explanations of words are useless. Ideally, they should describe the word using other words that are easier to understand, so that the topic of interest is more transparent to the reader. The definitions of “representation” that I have found do not appear to bring a clearer picture than simply saying “representation”.

However, the following characterization is accurate: “any concept of

[6] The function sameness cloud assumes that functions are identical if and only if they have the same graph and thus uses the Ordered Pairs definition of function (Mirin et al., 2020)

representation must involve two related but functionally separate entities. We call one entity the representing world and the other the represented world" (p.23, Kaput, 1987, citing Palmer, 1977). In the topic of interest here, the *represented world* constitutes mathematical objects (e.g., numbers, functions), however the thinker conceives of them. Defining the *representing world* (the representation itself) is a bit trickier. Is it marks on a page? If so, then does a verbal description of those same marks on a page constitute the same representation? Is it the marks themselves that are the representation, or the way in which the marks refer to an object? Thompson might argue that discussing marks on a page, or representations in general, is problematic in some contexts (Thompson & Sfard, 1994). If a representation is just marks on a page, then is a task about a representation just a task about marks on a page? Thompson (1982) would not consider marks on a page, or soundwaves in the air, to even be a task. It is also difficult to even characterize what constitutes different representations to us. Consider a verbal description of a function, say "the cubing function". Does the visual display that you just read constitute a different representation than if you were to hear it spoken? I treat "representation" to mean something like Fregeian senses (discussed earlier). Representations are different when they bring something different to mind. There are some cases in which we might say that some representations are *different* from each other (e.g., a function defined by a single equation versus a function defined piecewise, as I describe in the first dissertation paper (chapter 2) and in Mirin, 2018). Various mathematics education researchers have considered that there are different representations of mathematical objects. "2(3)" and "4+2" are different representations of the number six, as they bring different things to mind and are transparent and opaque with respect to different properties (see Zazkis &

Gadowsky, 2001). I remain agnostic on whether “6”, “six”, and “SIX” are the same representation. So, while we do not have a complete characterization of what makes representations different, we have some idea of what different representations might be.

Thompson has a detailed debate with Sfard about the issue of representation of mathematical objects (Thompson & Sfard, 1994). He offers a general critique of the multiple representation literature: “My criticism is of people using ‘representation’ too loosely, without mentioning a person to whom some sign, symbol, or expression has some meaning” (p.10). When describing representations, we ought to be diligent about not mis-ascribing an understanding to students. What are two representations of the same thing to us may be completely unrelated to students. Hence, when we refer to “multiple representations” in the context of mathematics education research, it is important to ask ourselves “multiple representations of *what* and *to whom*?”. If we are to talk about how students understand “graphical and verbal representations of a function,” we must be clear, both in our minds and in our writing, where the notion of representation resides - often it is in the mind of the researcher, not the student. If we are to say “such-and-such student does not understand multiple representations of a function,” we need to clarify -- does such-and-such student even view those “multiple representations” as of the same function? As even being of functions? As anything beyond a representation itself?

Unfortunately, this approach brings us to a dilemma: can we really say that this research is about “sameness” if students to not view anything as being the same? This dilemma applies to almost any subject of inquiry - how can we say that we are investigating students’ understanding of the concept of anything, when a concept is by its very definition internal to a student? Are we really investigating students’ understanding of (for example) the concept of function when we give students tasks that we as mathematicians view to be about functions but the student views to be about, say, symbolic manipulation? The straight-forward way is to simply avoid the paradoxical notion of "investigating students understanding of the concept of ______" and instead talk about students' understandings of words. This appears to be the approach in variation theory – in variation theory, a "concept" appears to just be a word describing a class of things. We could talk about students' understanding of the word "function". This makes Tall and Vinner's (1981) notion of concept-image and concept-definition especially appealing - research grounded in this framework is inherently about students' association with a word or phrase. The research question is then reduced to a question about word meaning (e.g., "what do students think the word _______ means, and how do their stated meanings differ from their hidden meanings?"). Although this creates a nice way of framing a topic of investigation, this limits us to talking about only words rather than mathematical concepts or something deeper.

While I use the construct of concept-image in my work, I do not limit myself to students' understanding of the meaning of words. My work is about sameness of representation, while remaining sensitive to the fact that students (research subjects) might not interpret their tasks to be about sameness at all. My areas of investigation are consistent with constructivism, so a constructivist methodology of model creation is appropriate (Thompson, 1982). A guiding aspect of my research is to not assume that what is a representation of an abstract mathematical object to us is also viewed as an abstract mathematical object by a student (Thompson & Sfard, 1994). Similarly, what is the same to us might not be the same to students. Thompson (1982) provides an excellent explanation of the constructivist approach to such questions:

> The constructivist asks: "What is the problem that this student is solving, given that I have attempted to communicate to him the problem I have in mind?" This is a legitimate research question to a constructivist; to an environmentalist it most assuredly is not (p. 153).

It is worth noting that data collection, data analysis, and theoretical perspective cannot be separated; how you collect data has to do with how you analyze it and the data analysis techniques that are available to you, and the way you want to analyze your data determines the way you collect it. Similarly, your research question and methodologies are closely related to your theoretical perspective. The type of data you collect as a behaviorist might be different from the type of data you collect as a constructivist since your research question might be more about behaviors and tendencies than about ways of thinking (Cobb, 2007; Thompson, 1982).

Steffe and Thompson (2000) provide useful terminology. They use the phrase “students’ mathematics” to refer to the students’ mathematical realities, and “mathematics of students” to refer to interpretations (models) of students’ mathematics. This parallel language emphasizes the fact that we apply the same epistemology to ourselves as researchers as we do to students (model-builders).

I now elaborate on what I mean by “model”. Thoughts and mental actions are not directly observable in the sense that thoughts are not directly sensible (e.g., viewable, smellable, touchable). This is not simply a matter of not having the proper brain imaging technology or a fundamental flaw in the notion of studying cognition. Consider the following quote by Leibniz (Jorati, n.d.):

> If we imagine that there is a machine whose structure makes it think, sense, and have perceptions, we could conceive it enlarged, keeping the same proportions, so that we could enter into it, as one enters into a mill. Assuming that, when inspecting its interior, we will only find parts that push one another, and we will never find anything to explain a perception (Monadology, Section 17).

Now imagine that "students' mathematics" were there in place of "perception". Even if we were to see all the parts of the brain, we still could not see the thoughts themselves. Thus, we model them. Modeling is a form of abduction, distinct from induction or deduction. It involves creating explanations in order to account for observed events and is necessary for understanding things to which we do not have direct observation (Jorati, n.d.). It is also an essential aspect of scientific discovery that is not unique to studying abstract notions like thought (Clement, 2000; Schickore, 2018). For example, we can see the use of abduction in medicine (a diagnosis is a model that best explains someone's symptoms).

The idea of model-building is grounded in constructivism as an epistemology. The following quote characterizes ideas underlying radical constructivism: "(1) Knowledge is not passively received but actively built up by the cognizing subject; (2) the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (Glasersfeld, 1989 p.114, found in Thompson, 2013a). The epistemology that we apply to the student and the researcher is the same; neither have direct access to ontological reality and therefore must construct models of it.

The term “first order models” refers to the models that students have (students’ mathematics), and the term “second order models” refers to the models that researchers create of the student models (mathematics of students). The data collected through interacting with students are used to create these second-order models (Steffe and Thompson, 2000).

Building models of *individual* students might be viewed as a limitation (like any qualitative research, a limitation is that it’s not quantitative). The researcher might need to spend several hours interacting with a student in order to build a robust model of that one student’s mathematics. Despite all this careful time, some people view qualitative research as not legitimate due to not being generalizable on the grounds that few subjects are involved (Kvale, 1994). However, there is a general assumption that a way a student thinks will be shared by other students. Reframing this in a constructivist way, this means that the researcher will continue to experience similar things, just as the student experiences patterns (Steffe, 1991). A model of knowing that might generalize beyond a particular student is referred to as an “epistemic subject” (Thompson, 2013a). An epistemic subject can encompass many students who have similar ways of thinking (Thompson, 2013a). Thus, studying an individual student can have the utility of generalizing to other students.

This sort of potential generalizability is not unique to mathematics education. We tend to assume that some categories of things have some sort of regularity. These categories are often called “natural kinds” (Bird & Tobin, 2018).[7] This assumption of

[7] “To say that a kind is *natural* is to say that it corresponds to a grouping that reflects the structure of the natural world” (Bird & Tobin, 2018).

regularity in nature occurs in other fields as well. For example, a biologist might dissect a pigeon under the assumption that other pigeons will share similar characteristics; they are learning something about pigeons in general, not just that particular pigeon. Consider the study of human anatomy. The cadavers that medical students and researchers dissect only represent a convenience sample. Yet, there is an underlying assumption that the phenomena observed in those bodies will extend to other bodies as well. This is true even when an abnormality is found; there might be a general assumption that this abnormality (perhaps a disease) exists in other cadavers. This is similar to how we handle studying students' minds. Although we might have a convenience sample, we assume that minds tend to resemble each other. If we discover a way of thinking in one student, we tend to believe that it might exist in some other students as well. This regularity might lead to generalizations amongst multiple students and instances, in what Clement (2000) describes as "convergent studies".

WHERE WE SEE ONE FUNCTION, THEY SEE TWO

Multiple representations of functions play an important role in mathematics and mathematics education. There is a body of literature addressing college students' difficulties linking multiple representations of functions, and some studies suggest that post-secondary students struggle translating between different representations (Chinnappan & Thomas, 2001; Even, 1998; Gagatsis et al., 2004). The literature on multiple representations tends to focus on translation between multiple types of representations (e.g., graphic, analytic, and verbal), rather than multiple representations of the same type.

However, working with multiple analytic representations of a function is also a crucial part of mathematics. This occurs prominently in differential equations; a differential equation is a particular type of equation that asserts identity of functions (see, for example, Boyce & DiPrima, 2009). This means that each side of a differential equation is a representation of the same function. We also see function identity in the context of implicit differentiation and related rates problems; when we differentiate an equation, each side of the equation is a representation of the same function (included as Section 4, published as Mirin & Zazkis, 2020). Specifically, what allows us to "differentiate both sides" of an equation is to understand that equation as asserting that two different representations of the same function are indeed the same function and therefore have the same derivative. Hence, being able to assess when two functions are actually the same function can enable powerful inferences.

The notion of two different analytic representations of the same function appears also in the fundamental theorem of calculus. Viewed as a statement of function identity, the fundamental theorem of calculus asserts that, given a differentiable function f and a number a in the domain of f, the function g defined on the domain of f by $g(x) = \int_a^x f'(t)dt$ is the same as the function h defined by h(x)=f(x)-f(a). This is arguably the manner in which Newton conceived of the fundamental theorem of calculus (Thompson & Silverman, 2008). The prevalence of the fundamental theorem of calculus, combined with the prevalence of the procedure of differentiating both sides of equations, underscores the importance of function sameness to calculus learning. As discussed in Section 4, differentiating both sides of an equation is legitimate because the equation is serving to assert function sameness.

This study investigates the following research question: *How do calculus students understand multiple analytic representations of the same function?* More specifically, I address: How do students assess when two analytic representations of the same function are indeed the same function? Is sameness of graph enough for students to infer sameness of derivative? Do students view instances of the fundamental theorem of calculus as about function sameness?

One might wonder: after the discussion (in the introduction to this document) about identity being equality, why did I not say function "equality" rather than "function sameness" or "function identity"? Recall earlier the research on student understanding of the equals sign, which suggested that many students do not view the equals sign as expressing identity or sameness. While these studies were done primarily on younger children (elementary and middle school), it is possible that older students hold similar conceptions of the equals sign. Hence, to such students, identity and equality might not be the same. So, investigating student understanding of function "equality" might be different from investigating student understanding of function "identity" or "sameness".

2.1 Theoretical Background

I follow Thompson's (1982) constructivist approach. Thompson makes the point that, when referring to representations of something, we ought to be clear about *to whom these are representations* of whatever "something" is (Thompson & Sfard, 1994). So, we ought to be sensitive to the fact that a student might agree with the assertion that two representations of the same function share a derivative, but these students might have non-standard understandings of what "same function" is. In fact, this is precisely the sort of reasoning a particular student used to determine that sharing a graph was not sufficient for sameness of functions; she concluded that two particular representations of functions share the same graph but do not share a derivative, leading her to conclude that, to be the same function, having the same ordered pairs on the graph is not sufficient (Mirin, 2017). Hence, a fundamental assumption of this study is that students might not understand the tasks to be about sameness of function.

I adopt the constructs described in Tall and Vinner (1981): A student's concept image is "the total cognitive structure that is associated with the concept, which includes all the mental attributes and associated properties and processes" (p.152). One component of a student's concept image is their *concept definition*, which is their stated definition of a concept. This study involves investigating student concept definitions for function sameness. A student's concept definition is just one aspect of their concept image and does not comprise it entirely. Hence, investigating a student's concept definition is on its own insufficient. This study therefore includes tasks concerning function sameness that go beyond students' stated concept definitions.

2.2 Literature Review

Since this study is about sameness of different representations of functions in a calculus context, the literature review covers three related topics: multiple representations of function, multiple representations of derivative, and sameness of representation of function and graph.

2.2.1 Multiple Representation of Function Literature

There is a significant body of literature explicitly on the concept of multiple representation of functions (Chang et al., 2015; Delos Santos & Thomas, 2003, 2001; Even, 1998; Gagatsis et al., 2004; Zandieh, 2000; Zazkis, 2016). Such literature focuses on multiple types (registers) of representations - e.g., graphical, analytic, verbal, and physical. These studies tend to focus on how students, in particular preservice teachers, struggle with linking multiple types of representations of the same function. For example, students tend to have trouble linking the graph of a function with its equation. Even (1998) reports a study in which 152 prospective secondary mathematics teachers, majoring in mathematics, were surveyed via an open-ended questionnaire. Ten of these students were subsequently interviewed about their answers. It was found that students struggled to move flexibly between one kind of representation and another, even with familiar functions, such as quadratics. Gagatsis et al. (2004) describe a study on the relationship between students' ability to translate between representations and to solve problems. One hundred ninety-five students studying education at a university in Cyprus were enrolled in this study. The students took two assessments, one to measure their ability to translate from one representation to another (verbal, graphical, analytic), and another to measure their problem-solving ability. Problems on the translation test involved giving students a function in a verbal, analytic (equation of the form "y =_____"), or graphical form and having them present it in one of the other two forms. The problem-solving tasks involved having students fill in missing entries in tables, solving word problems, and sketching graphs. The researchers found a positive correlation between success on the problem-solving tasks and success on the translation assessment. This correlation highlights the importance of coordinating multiple representations of the same function. Chinnappan and Thomas (2001) report on a study in which four preservice math teachers were the subjects of a free-response interview on the topic of functions and how to teach them. The authors found that the teachers gravitated towards graphical representations. At least one student expressed that she considered a function to be a graph rather than "algebra" and struggled to link graphs with algebraic representations. In general, the teachers showed weakness in linking a graph of a function to an analytic (e.g., polynomial equation) form and tended to be fixated on visual representations independent of equation.

2.2.2 Multiple Representation Derivative Literature

There is also research on multiple representations of derivative. Like the multiple representation literature on function, this literature also focuses on representation type. Zandieh (2000) uses Sfard's (1992) process-object distinction as a basis for a theoretical framework for the multiple representations of derivative. The kinds of representations of derivative that Zandieh addresses are graphical (derivative as slope of tangent line), verbal (instantaneous rate of change), physical (velocity), and symbolic (limit of difference quotients). These representations share a similar three-layer conceptual structure: the ratio layer, the limit layer, and the function layer. Zandieh (2000) summarizes the relationship between these layers as: "A derivative is a function [third layer] whose value at any point is the limit [second layer] of a ratio [first layer]" (p.106). A "process-object" pair composes each layer.

The three-layer process-object structure applies to the many representations of derivative. For example, in the graphical interpretation, the ratio is the "rise over run" of the secant line, the limit is the slope of the tangent line, and the function is a visual graph. In the physical interpretation, the ratio is an average velocity, the limit is the instantaneous velocity, and the function is a pairing of each instantaneous velocity with corresponding time. For the ratio layer, the process is division, and the reified object is a ratio. The limiting process involves "passing through" infinitely many of these ratios while approaching the limit, which is the reified object. The function layer is viewed operationally as a mapping process, and structurally as a set of ordered pairs. Zandieh interviewed nine students about their understandings of derivative and found that they tended to mention the graphical (slope of a tangent line) interpretation most often. Notice that this bias toward a graphical interpretation is consistent with Chinnappan and Thomas (2001).

An important aspect of Zandieh's framework is that an object part of a process-object pair is not simply the result of a process, but a reification of the process itself. Zandieh uses the word "pseudo-object" to describe when someone views something as an object without attending to its underlying process structure. Consider the ratio layer. In the symbolic representation, a student may view a ratio as a pseudo-object by a numerical limit of the difference quotient as a single number or magnitude rather than relating to division or ratio. For the graphical interpretation, a student may view the slope of a tangent line simply as *slantiness* without considering secant lines or the ratio of rise to run (Byerley & Thompson, 2017). For the physical interpretation, a student may view the derivative as representing *instantaneous velocity* without considering instantaneous velocity to be a multiplicative quantity composed of accumulated time and distance (Thompson et al., 2013). A student with a pseudo-structural conception of the ratio layer would be failing to attend to what I called the *locality* of the derivative operator. Attending to the locality of the derivative operator involves considering the points in a neighborhood around x when determining the derivative of a function at x. This parallels the idea in Thompson and Dreyfus (2016) of "all variation is blurry" (p. 357). Zandieh does not discuss what is most pertinent to my study: how students link the various representations to each other.

Delos Santos and Thomas (2001) investigated how students understand different representations of derivative. Thirty-two 16-17-year-old students at a top performing girls school in New Zealand were given a 10-question task sheet that included problems that involved interpreting the meaning of "dy/dx" and translating between different representational forms of the derivative. Exactly one student solved a problem of forming a graphical interpretation of $f'(5) = 1$, and exactly three students, given values of a function in tabular form, symbolically represented an average rate of change. Additionally, whether students correctly gave a graphical interpretation of the symbol "dy/dx" depended on the equation in which it appeared. In other words, students lacked the representational fluency to flexibly and consistently use and move between different kinds of representations of function and derivative.

2.2.3 Representational Sameness Literature

Although the multiple representation literature generally does not discuss sameness of representation, a few authors do address this topic. For example, Moore and Thompson (2015) stress the importance of seeing different visual displays of the same graph (e.g., the same graph in different coordinate systems) as representing the same quantitative relationship. Additionally, as alluded to in the introduction of this document, there is some mathematics education work that leverages the relationship between student conception of *function identity* and student conception of *function*. In order to assess student understanding of the concept of binary operation, Novotná et al. (2006) designed and used tasks that ask whether particular binary operations are the same. Sfard (1988) explains that, because students acknowledged that the function f defined on the natural numbers by $f(x)=x^2$, and the function g defined on the natural numbers by $g(0)=0$, $g(x+1)=g(x)+2x+1$ are "equivalent" yet would not describe them as "the same", these students had a mathematically non-normative concept of function. In Mirin (2017), I present a case study to illustrate the relationship between student conception of function identity and of function. I describe a student, Jane, who thinks of functions as processes and therefore thinks that for functions to be the same they must be the same process. For example, Jane claimed that the function defined by $|x|$ and the function defined by $\sqrt{x^2}$ are different functions because they "describe different mathematical processes". Mirin (2017) illustrates how one's concept of sameness-of-representation-of-function and the function concept itself are interlinked. If a student views a derivative as operating on a function, then their concept of function is inextricably tied to their concept of derivative. For example, their criteria for determining whether two function representations share a derivative might be influenced by their criteria for determining whether those representations refer to the same function. Indeed, this is what happened with Jane. She initially assessed two functions as being merely *equivalent* rather than *the same* on the grounds that, despite having the same graph, they did not have the same derivative. Interestingly, this brings us back to Leibniz' laws of indiscernibles; Jane concluded that two functions are not identical on the grounds that they do not share the same set of properties.

It bears mentioning that mathematicians do not agree on the notion that same graph implies same function. Mirin et al. (2020) explains how in both the mathematics education and the mathematics communities, there are two conflicting definitions of function. One definition is a univalent set of ordered pairs, and so two functions are the same if and only if they have the same graph (set of ordered pairs). The other definition is the Bourbaki Triple; under this definition, a function is a triple (a Bourbaki Triple) (X, Y, F) where X is the domain, Y is the codomain, and F is a univalent and total set of ordered pairs on X. Recall the earlier discussion about the relationship between sameness within a category and the defining features of a category. Here, this relationship manifests itself in the sense that the criteria for sameness of function is dependent on which definition of *function* is being used – the Bourbaki Triple definition or the Ordered Pair definition. Consider, for example, the function f: $R \rightarrow R$ defined by $f(x)=x^2$ and the function g: $R \rightarrow R^+$ defined by $g(x)=x^2$. Under the Bourbaki Triple definition, f and g are different functions, since they do not have the same codomain and are hence different Bourbaki

Triples. Under the Ordered Pairs definition, f and g are the same function, since they have (are) the same set of ordered pairs.

As discussed, the fundamental theorem of calculus can be viewed as a statement about function identity. For this reason, a task about the fundamental theorem is used as part of this study (see Fig 2.3). Unfortunately, there is little literature on how students understand the fundamental theorem. Thompson (1994) finds that students' issues grasping the theorem are grounded in underdeveloped understandings of rate of change and covariation. Orton (1983) reports the types of mistakes students make in doing problems with definite integrals. He focuses on how students understand definite integrals as limits. However, his study does not address integrals in the context of the fundamental theorem or as functions. Thompson and Silverman (2008) make the point that an integral as a function is conceptually different from a definite integral as a number. That is, conceptualizing $g(x) = \int_a^x f'(t)dt$ as a function is different than conceptualizing $g(x) = \int_a^b f'(t)dt$ for a particular number b, in the same way that conceptualizing the squaring function is different from conceptualizing a particular number being squared. In this study, I situate the fundamental theorem as a statement about function identity, and hence also a statement about functions. The literature does not address whether students conceptualize integrals as functions and generally does not pose tasks to students where an integral is represented as a function. However, there is some hint at the idea of an integral as a function in Jones (2013). He explains that many students have a "function matching" conception of integral, which he describes as follows: "The function inside of the integration is a *derivative*. The purpose of the integral is to match it back to the *original function* from whence it came" (p. 130). It appears that the students in this particular study understood the integral's "purpose" as being to find an original function, but it's unclear if such students understand the integral as representing such a function (the author does not address this issue, only the "purpose" of the integral). Hence, we do not know how students understand integrals as functions, and therefore the extent to which students view the fundamental theorem as even involving functions is an open question.

2.3 Overview of Methods, Task Design, and Timeline

This study is based on three tasks, all of which concern how calculus students understand sameness of representation. The first task presents students with a function f defined by $f(x)=x^3$ if $x \neq 2$, $f(x)=8$ if $x=2$ and asks them to evaluate f'(2). Observe that f is merely a piecewise-defined version of the cubing function. I refer to this task as "the cubing function task". The second task also involves two analytic representations of the same function: p defined by $p(x)=\int_2^x 3t^2 dt$ and $q(x)=x^3-8$. Students are presented with p and q and asked to evaluate whether p and q are the same or different functions. I hereafter refer to this task as the "fundamental theorem task". The third task asks students to give their concept definition for function sameness. I hereafter refer to this task as the "concept definition task". The data were collected in three stages (Table 2.1). Stage 1 consists of the students' written work collected from the open-ended cubing function task. Stage 2 consists of the interview data from a subset of the students who participated in Stage 1. Stage 3 consists of the data from an entire written quiz given to a new group of students. This quiz includes a multiple-choice version of the cubing function task together with two additional related prompts - the fundamental theorem task, and the concept definition task.

The cubing function task is what inspired this study. A very similar task that uses a piecewise-defined version of the squaring function was given in Mirin (2017); this is the task that inspired Jane, the participant, to decide that having the same graph was not sufficient for two functions to be the same function. She assessed the piecewise version as having a different derivative than the standard version and therefore concluded that they might be different functions. As discussed in section four (Mirin & Zazkis, 2020), being able to reason that "two" functions share a derivative on the grounds that they have the same graph is paramount for robustly understanding implicit differentiation and related rates problems.

The design of the cubing function task was inspired by an anecdote in Harel and Kaput (1991): when prompted to differentiate the function g defined piecewise by $g(x) = \sin x$ if $x \neq 0$ and $g(x) = 1$ if $x = 0$, respondents answered with $g'(x) = \cos x$ if $x \neq 0$ and $g'(x) = 0$ if $x = 0$, appearing to use the constant rule. To these students, the only aspect of the representation as relevant for determining the value of $g'(0)$ is the second line of the piecewise function definition. It seems reasonable to believe that, if the definition of g were modified to instead have $g(x) = 0$ if $x = 0$ (resulting in a nonstandard representation of the sine function), students would answer identically. However, given the anecdotal nature of Harel and Kaput's claim, there is no data available to substantiate how common such errors are or why they occur. My study began by undertaking the task of studying this phenomenon more systematically (Stages 1 and 2). The other tasks (Stage 3) complement this task by investigating the same topic of how calculus students understand function sameness.

In the cubing function task, students encounter a nonstandard analytic representation (piecewise definition) of the cubing function, whether they recognize it as such or not. There are at least two ways a student might reason about multiple representations to come to the correct answer that f'(2)=12. After graphing the piecewise-defined function, a student might recognize that the resulting graph is the same as that of the cubing function and conclude that they have the same derivative. This kind of reasoning does not require that the student have a strong understanding of derivative, but only an understanding that derivative is a property of the graph of a function. This could be accomplished with a view of derivative as anything having to do with the tangent line (e.g., slope of tangent line, *slantiness* of tangent line, or even the tangent line itself, (Byerley & Thompson, 2017) . In this situation, the student would be coordinating visual and analytic representations. Alternatively, a student could, after noticing that $f(2) = 8 = 2^3$, make the same determination without using a visual graph, by linking the piecewise representation with the standard analytic representation ($h(x) = x^3$).

First, I discuss the data collection process for the open-ended cubing function task (Stages 1 and 2), and then I move to discuss the multiple-choice cubing function task along with the fundamental theorem task and the concept definition task (Stage 3). Initially, the open-ended cubing function task -- exactly as pictured in Figure 2.1 below – was given to 240 introductory calculus students during the last week of the semester at Anonymous State University (ASU) (Stage 1).

Let f be the function defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$$

Evaluate $f'(2)$, and provide an explanation of your answer.

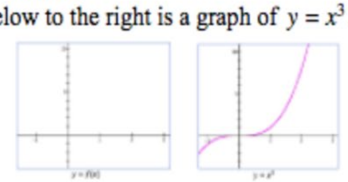
Figure 2.1. The Open-Ended Cubing Function Task, Stage 1 (written) and Stage 2 (interview).

Stage 1 was administered in an exam environment by course instructors, where students were required to work silently and independently. Stage 2 involved interviewing a subset of 8 Stage 1 students, and a preliminary analysis of such interview data informed the analysis and classification of students' answers to Stage 1 as well as the design of Stage 3.

Stage 2 involved eight students from the original Stage 1 cohort. An initial analysis of the interview data helped form the design of Stage 3. In particular, it informed the design of the multiple-choice cubing function task that appeared in Stage 3 (Figure 2.2). For example, it was not immediately clear that students were providing two answers in Stage 1. After interviewing students, it became more evident that they indeed were. Following analysis of the Stage 3 data, the Phase 2 interview data were re-analyzed to create a more in-depth picture of student thinking.

Stage 3 consisted of three parts: the revised (multiple-choice version) version of the cubing function task, the fundamental theorem task, and the concept definition task. The task sheet, is included as Appendix A. It was given to different introductory calculus students at the same institution at the same relative time in the semester and was administered in the same manner as Stage 1. One-hundred two (102) students participated in Stage 3. One purpose of Stage 3 was to gain more insight into how students understand the cubing function task. One hypothesis I had about students' answers on the open-ended version (Stage 1) was that many students gave the incorrect answer due to inattention or carelessness. Perhaps they had not even noticed that y=f(x) and $y=x^3$ define the same graph. In other words, perhaps they would have gotten the answer correct had they noticed that the graphs were the same, by observing that f and the cubing function agree at x=2. To remedy this issue, I designed the multiple-choice version of the cubing function task. Before being asked to evaluate f'(2), students were prompted to calculate 2^3. The purpose of this task was to orient students toward noticing that the function f agrees with the cubing function at x=2. Students were also provided a graph of $y=x^3$ and prompted to graph y=f(x) next to it. This was to orient students to compare the graph of y=f(x) with $y=x^3$. In other words, the purpose was to ensure that I was not tricking students and to provide them an opportunity to recognize that f and the cubing function are indeed the same. Figure 2.2 below shows a visually condensed version of the multiple-choice cubing function task.

Suppose *f* is the function defined by $f(x) = x^3$ if $x \neq 2$, $f(x) = 8$ if $x = 2$.
1. Calculate 2^3
2. Below to the right is a graph of $y = x^3$. On the blank axes provided, graph $y = f(x)$ for the function *f* defined above.



3. Evaluate $f'(2)$ for f defined above. Please provide a 1-3 sentence explanation of your answer. (a)0, (b)8 (c)12, (d)undefined, (e)0 if $x = 2$, and 12 if $x \neq 2$, (f) 0 if $x = 2$, and undefined if $x \neq 2$.

Figure 2.2. The Multiple-Choice Cubing Function Task, Stage 3.

I now discuss the remainder of Stage 3. While the cubing function task is interesting in its own right, this investigation is not just about a single task. More generally, the goal is to learn more about how calculus students understand sameness of representation of function. One major finding of the results of the cubing function task is that where we (as mathematicians) see one function, students see two. It is natural to ask whether this result extends to other contexts such as with the fundamental theorem of calculus (see Figure 2.3 below).

4. Let p be the function defined on all real numbers by

$$p(x) = \int_2^x 3t^2 dt$$

and let q be the function defined on all real numbers by

$$q(x) = x^3 - 8$$

(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

ii. p and q are not the same function.

(b) Provide an explanation for your answer for 4(a).

Figure 2.3. The Fundamental Theorem Question, Stage 3.

Additionally, students' concept definitions of function sameness provide us insight into how students understand sameness of representation of function (see Figure 2.4 below). These tasks, together, help provide insight into the guiding research question about function sameness.

Suppose g is a function and h is a function. What does it mean for g and h to be the same function? Explain.

Figure 2.4. The Concept Definition Question, Stage 3.

To summarize, the main task driving this study is the cubing function task. Extensive data were collected on this task, including qualitative data. This task was initially given in open-ended form (Stage 1), and eight students were interviewed (Stage 2). An initial analysis of the interview data informed the design of the multiple-choice expanded version of the cubing function task. This multiple-choice version, along with other tasks concerning function sameness (the fundamental theorem question and the concept definition question), were given in the form of a quiz to a new group of students in order to learn more about students' responses to this task and, more generally, students' understanding of function sameness (Stage 3). Table 2.1 below summarizes these stages.

Table 2.1. Stages of Data Collection.

	Stage 1	Stage 2	Stage 3
Tasks	• Open-ended Cubing function task (Fig. 2.1)	• Open-ended cubing function task[8] (Fig. 2.1)	• Multiple-choice cubing function task (Fig. 2.2)

[8] The interview itself included other tasks (Fig 2.5), but the purpose of the interview was to learn about how students understand the open-ended cubing function task.

			• Fundamental theorem question (Fig. 2.3) • Concept definition question (Fig. 2.4)
Data type	Written work	Interview recording	Written work
Population	240 Calc I students	8 Calc I students (from the 240 in Stage 1)	102 Calc I students (different students from Stages 1 and 2)

2.4 Interview Protocol and Data Analysis Methods

First, I discuss the open-ended cubing function task (Stages 1 and 2), and then I discuss the data analysis methods for Stage 3 (which includes the multiple-choice cubing function task, the concept definition question and the fundamental theorem question).

2.4.1 Interview

The interviews (Stage 2) were semi-structured, task-based, and lasted 60-80 minutes each. They were screen and audio recorded using *Notability* on an iPad. The interviews were exploratory in nature, and the tasks evolved slightly over the course of the study. They operated according to clinical interview methodology (in the sense of Clement, 2000) and served as establishing students' rationale for their responses to the open-ended cubing function task. If the student answered differently than they had in their open-ended written version, they were questioned about their change of answer. Additionally, students were given similar problems, as well as asked to graph the function f, and asked to illustrate, using their graph of f, the rationale behind their answer to the cubing function task. The other administered tasks, as well as the follow-up questions, were closely related to the cubing function task. These included questions about function sameness, derivative, and graph. One of the purposes of these tasks was to see if students believed that same points on the graph in the neighborhood of a particular number implied same derivative at that number. A selection of the tasks used are shown in Figure 2.5 below.

$$h(x) = \begin{cases} x^3 & \text{if } x \neq 5 \\ x^2 + 100 & \text{if } x = 5 \end{cases}$$

Evaluate $h'(5)$

$$g(x) = \begin{cases} \frac{cos(x)-1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Evaluate $g'(0)$

Suppose we have some function f. What does $f(1) = 3$ tell us about the graph of f? Does it tell us anything about f's derivative? What does $f(x) = 3$ when $x = 1$ tell us about f?

Suppose we have some function f and we know that $f'(1) = 0$. What does that tell us about the graph of f?

$$k(x) = \begin{cases} x^3 & \text{if } 4.99 \leq x \leq 5.01 \\ 0 & \text{if otherwise} \end{cases}$$

Evaluate $k'(5)$

Suppose $f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$

and $h(x) = x^3$ for all x. Are h and f the same function?

Figure 2.5. A Selection of Interview Tasks (Stage 2).

As I interviewed the students, questions I had in mind were "What aspect of a function's representation does a student view as relevant for determining its derivative at a point?" and "What does a student think determines a function's derivative at a point?" Although these might seem like different research questions than my stated research question about function sameness, as alluded to earlier, they are closely related. Consider the Cubing Function Task, and let f denote the piecewise definition of the cubing function. Suppose a student correctly believes that only a function's inputs and outputs (ordered pairs) determine its derivative at a point. Then such a student would think that f and the cubing function have the same derivative. Conversely, a student who believes that f and the cubing function have the same derivative might believe that the points on a function's graph determine its derivative. When we vary a function's representation, we ascertain what aspect of that representation a student views as relevant to its derivative. Using a similar idea, I investigated how students understood the locality of the derivative operator; for example, do students believe that functions that are the same in the neighborhood of x=a necessarily have the same derivative at x=a? To investigate this question, I presented various functions with the same graph near x=a (but different graphs elsewhere).

An illustration of ways of thinking to account for three different answer types is provided in Section 2.5. These ways of thinking are *epistemic students* (see Thompson, 2013a) and were developed based on the student interviews (Stage 2) together with student written work from both the open-ended cubing function task (Stage 1) and the multiple-choice version (Stage 3). Note that the idea of epistemic students is grounded in constructivism, which I discuss in further detail in the Epistemology and Theoretical Perspective section of this document. In analyzing the interviews, my goal was to provide viable models (in the sense of Clement, 2000) of how individual students were thinking. This, together with the rest of the data, provides the basis for me to construct these more general epistemic students. These epistemic students then informed my coding of students' written data from Phase 1 (the written work of 240 students on the open-ended cubing function task).

2.4.2 Coding

Stage 1 and Stage 3, each consisting of quantitative data, were both coded. I first discuss the coding in Stage 1, and then I move to Stage 3. Students' answers to the open-ended cubing function task (Stage 1) were coded as if they had taken a multiple-choice test; that is, when I coded a student as answering "12", I did so if I believed that that is what they would have bubbled in had they been given a multiple-choice question. This means that the strength or coherence of students' justifications was not considered, and many students were coded as answering correctly (the answer "12") even if their justification indicated a severe misunderstanding. This allowed me to compare the open-ended answers in Stage 1 with the multiple-choice answers in Stage 3.

Stage 2 informed the Stage 1 coding. That is, the coding of the open-ended cubing function task was influenced by the student interviews. Consider students 110, 138, 157, and 178 from the open-ended cubing function task (Stage 1), whose answers are below in Figure 2.6.

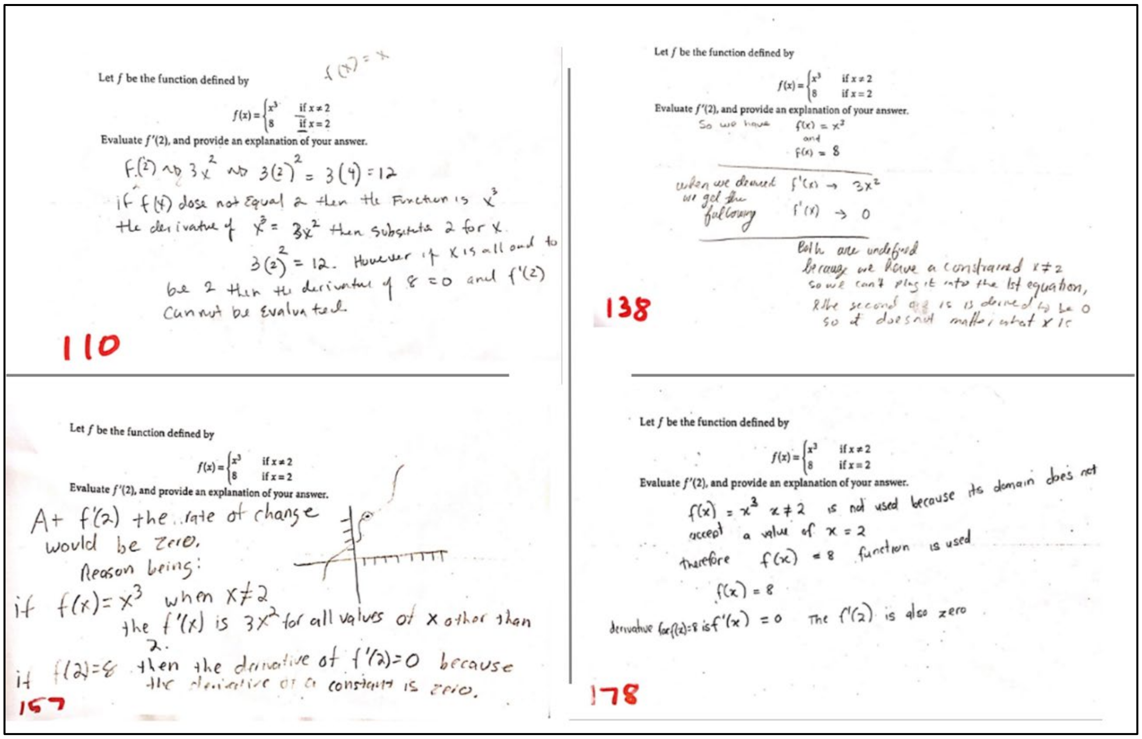


Figure 2.6. Students' Answer to the Open-ended Cubing Function Task, Stage 1.

When I interviewed Student 157, I found that he meant to answer "12" where he wrote "$3x^2$". When I gave him the same problem during Stage 2 (the interview), he did the exact same thing but substituted "2" for "x" and explained that he meant to do that initially when posed with the open-ended task. A similar occurrence happened with Student 138 and Student 178. Thus, whenever a student wrote "$3x^2$", I coded their answer as if "12" were written in place of "$3x^2$". For example, Student 157 and Student 138 were coded as answering both 0 and 12. As discussed in Section 2.3, the interviews illuminated when students were giving two answers, which included students such as 110 who answered both "undefined" and "0".

I now discuss data analysis methods for Stage 3: written results from the multiple-choice cubing function task (Figure 2.2), the concept definition task (Figure 2.4), and the fundamental theorem task (Figure 2.3). Coding the results of the multiple-choice cubing function task (Figure 2.2, Problem 3) was, of course, straightforward. Coding the students' graphs (Figure 2.2, Problem 2) was more involved. A student's graph was considered to be "correct" if and only if it appeared to have all the correct points (ordered pairs) on it. This classification aligns with conventional mathematics. Hence, graphs that were not visually identical to the provided graph of $y=x^3$ were coded as correct. This included graphs with prominent dots on them, so long as those dots lined up with points that satisfy $y=x^3$. This also included graphs that had an open "hole" at (2,8) with a dot inside, but with the hole not completely filled in (see the middle graph in Figure 2.7). While one might think that the space surrounding the hole indicates that certain points were meant to be excluded from the graph, the interviews revealed that this was not the case – the surrounding space was for a different purpose (discussed in Section 2.5). Students who provided an open circle (with no closed dot inside of it and just a hole) at (2,8) were not coded as having a "correct" graph.

Normatively, two graphs (of functions) are the same if and only if they consist of the same ordered pairs. It seems reasonable to believe that some students might not have this criterion for sameness of graph. Indeed, Moore and Thompson (2015) report on undergraduate students who view two graphs as different despite having the same ordered pairs. Such graphs were visually different in terms of the displays of the coordinate system. In the study reported here, the coordinate system does not vary, but we still have the notion of students distinguishing between graphs with the same points. The interviews informed the resulting categorization; interviews suggested that some students viewed a graph of $y=x^3$ with an extra "dot" placed at (2,8) as different from a graph of

$y=x^3$ without one. Some students referred to the point (2,8) as “separate”. Accordingly, a sub-category (category B) of “correct” was created: mathematically normative graphs that highlighted (2,8) in the sense that they had a dot (closed circle) on (2,8) that was more prominent than any other dots. The remaining “correct” graphs were grouped together as Category A. In other words, Category A graphs indicated nothing special about the point (2,8), whereas Category B graphs did. Included in Category B were graphs that have a dot at (2,8). Excluded from Category B (and instead in Category A) were graphs that have a dot on (2,8) but also have at least one other dot of equal or greater prominence. I interpreted these dots to be dots that students used to help them draw the graph, rather than attaching any significance to the point (2,8). See Figure 2.7 below for a sample of Category A and Category B graphs. The remaining graph categorizations are described in Section 2.5

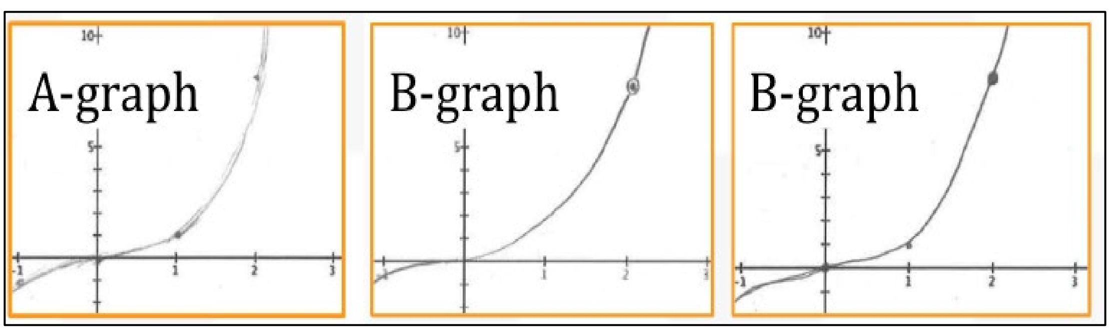


Figure 2.7. Category A versus Category B Graphs from Stage 3 (see Figure 2.2).

The function sameness (Figure 2.4) categorization was performed in a similar manner to that of graph categorization; students’ function sameness definitions were “correct” when they were extensional. This includes the characterization of function identity as same graph, same ordered pairs, or same output for every input. Statements such as "g and h are the same when g(x)=h(x)" were not coded as "correct"; this is because in the absence of quantifiers, students could view "g(x)=h(x)" to mean that g(x) and h(x) are identical as equations or that g(x) transforms to h(x) under certain rules (Mirin, 2017; Sfard, 1988). Additionally, students might not view g(x) as representing a number or value of a dependent variable and instead view it as a name of a function (Musgrave & Thompson, 2014; Thompson, 1994, 2013b). Not one student brought up the topic of codomain, so there was no need to distinguish between functions as sets of ordered pairs and functions as Bourbaki Triples.

Due to the multiple-choice nature of the fundamental theorem question (see Fig 2.2 above as it was presented to students), coding the results of that question was straightforward; either students selected option i (the same) or option ii (not the same) to assess whether the functions p and q are the same. Two students did not answer the question, nor did they provide an explanation or complete the concept definition task. For this reason, they are excluded from the analysis of both the fundamental theorem task and the concept definition task, leaving us with a convenient sample size of 100 for these two tasks.

2.5 Results

Recall that the guiding research question is about students' understanding of function sameness. One of the tasks is a concept definition, which asks students their meaning for function sameness (Tall & Vinner, 1981). I first discuss the results of that task. Recall that concept definition is just one facet of a student's concept image (Tall & Vinner, 1981), so it makes sense to consider concept definition not just in isolation, but also in relation to other facets of students' concept image. Hence, I revisit the results of the concept definition task when discussing the results of the other tasks.

2.5.1 The Concept Definition Task

As discussed above, students' concept definitions were coded as "correct" when they were extensional. Thirty-five students' answers were placed into the "correct" category (particulars of the criteria for this category are included in Section 2.4.2) Examples of this category include "all inputs will yield the same outputs for entire function", "for all x values they get the same output y", and "for every value in their domain, g and h have the same value". The "incorrect" answers vary. Included are blank answers. A common theme with the "incorrect" answers is the inclusion of other properties of functions. In particular, eleven students mentioned sameness of derivative in their concept definitions of function sameness. For example, one student wrote "both have the same derivative, so same function" while another wrote " g=h, g'=h', 3''=3'' " (it is unknown why this particular student mentioned 3''). This inclusion of sameness of derivative is especially interesting in light of my general investigation. As discussed, one of the reasons identity is so important is for inferences about sameness using Leibniz' law of indiscernibles – that if two objects are the same, then they have the same properties. In particular, this is what allows us to take the derivative of both sides of an equation, which I discuss in Section 4. We can also think of this sort of inference relating to the cubing function task – f and the cubing function are the same, and therefore they share a derivative. The results of the cubing function task, as well as the third dissertation paper, show that this is a nontrivial inference for students. Interestingly, the results of the concept definition task seem to indicate that for a number of students, the converse is true. That is, some students seemed to infer that functions are the same due to sharing a derivative. In the subsequent sections on the cubing function task and the fundamental theorem task, I address whether there is a relationship between extensional concept definitions and success on those two tasks.

2.5.2 The Cubing Function Task

I first discuss the quantitative data (Stages 1 and 3) regarding the cubing function task, and then I turn to the interview data (Stage 2) to provide a more in-depth account of student thinking. Recall that this task asks students to evaluate f'(2) for the cubing function, but represented in a nonstandard manner (piecewise). For the open-ended cubing function task (Stage 1, Figure 2.1), the majority (56.3%) of students claimed that the answer was 0, while many (41.2%) explicitly cited the constant rule. These results are consistent with the anecdote of Harel and Kaput (1991), who asked students to differentiate the function g defined piecewise by $g(x) = \sin x$ if $x \neq 0$ and $g(x) = 1$ if $x = 0$; students typically answered that $g'(x) = \cos x$ if $x \neq 0$ and $g'(x) = 0$ if $x = 0$. Harel and Kaput posit that students were not considering the neighborhood of the function around x=0 and were only looking at the function at precisely x=0 (they refer to this approach as "pointwise"). They explain that students were applying differentiation as an algorithm to the formula at this point. Although the authors do not explicitly claim that the students use the constant rule with differentiating, their description of using the "formula" suggests that this is what they understand students to be doing.

Now that I have established that there is a larger phenomenon, the next natural question to ask is, "why"? It is possible that some students erred due to inattention or carelessness, rather than a major misconception. That is, they might have simply seen the "8" and applied the constant rule out of habit or simply not realized that 2^3 is 8 and that the given function is in fact continuous. This would explain why some students answered "undefined," and it is also consistent with some of the graphs that students volunteered (graphs with removable discontinuities). Further, it might not have occurred to students to compare the graph of f with that of the cubing function - as discussed earlier, the piecewise-defined f is a representation of the cubing function to us, but perhaps not to students.

Hence, I turn to the multiple-choice version of the cubing function task (Stage 3, Figure 2.2). Recall that the multiple-choice version of this task has prompts to encourage students to compare f to the standard cubing function; students are asked to calculate 2^3, and to graph y=f(x) alongside a provided graph of $y=x^3$, all prior to evaluating f'(2). Since the multiple-choice version in some sense primed the students to compare f with the cubing function, I expected that these students would have a higher rate of correctness than the original Stage 1 group. However, it was necessary to consider that despite being primed, students still might not have a normative conception of function sameness; so, while students might observe that f and the cubing function have the same ordered pairs, they still might not understand them as being "the same". Conversely, it makes sense that students who have a normative (extensional) criterion of function sameness could reason that, because f and the cubing function are the same function, they share the same derivative. Note that this is an application of Leibniz's laws of indiscernibles under the assumption that having a particular derivative is a property of a function. Even without a robust understanding of function sameness, students still might be able to use multiple representations of function to reason that f and the cubing function share a derivative. They could use a formal definition of derivative or perhaps conclude that the graphs share a tangent line at x=2. Hence, I formed four hypotheses regarding the results of the multiple-choice cubing function task (Stage 3):

(1) Overall, students should perform significantly better on the multiple-choice version (Stage 3) than on the open-ended version (Stage 1), leaving open the possibility that inattention or carelessness could account for students' tendency to do poorly on the cubing function task in isolation (Stage 1). Students might, because of the prompting in the multiple-choice version, be more likely to compare f to that of the cubing function.

(2) Of those students who answered 12, those who did so in response to the multiple-choice question (as compared to an open-ended question of Stage 1) would be more likely to provide a justification involving the comparison of f with the cubing function.

(3) Students who provided a mathematically normative definition of function sameness would be more likely to answer "12" on the multiple-choice cubing function task than students who did not (Stage 3)

(4) Students who provided a correct graph of y=f(x) would be more likely to answer correctly on the multiple-choice cubing function task (Stage 3).

The data (see Table 2.2) reveal no evidence to support that inattention could account for student responses in Stage 1. Although there was a slight improvement in the correctness rate from the open-ended version (Stage 1) to the multiple-choice version (Stage 3), this improvement was not statistically significant (χ^2=1.21, p>.05), contrary to

(1). In other words, prompting students to compare the graph of $y = f(x)$ to that of $y = x^3$ did not appear to cause improvement, suggesting that Stage 1 students did not err simply due to inattention to the function's graph. Moreover, the Stage 3 students who answered "12" were no more likely than the Stage 1 students who answered "12" to draw an explicit comparison between f and the cubing function (4.2% of Stage 1 students who answered 12 did so, whereas only 2.9% of Stage 3 did so), contrary to (2). Also, the students who provided a mathematically normative definition of function sameness in Stage 3 were no more likely to answer "12" than those who did not, contrary to (3).

Table 2.2. Responses to the Cubing Function Task

	0	8	12	Undef	Multiple answers	Other	Blank	Total
Open-Ended (Stage 1) (%,n)	56.3% 135	4.6% 11	18.3% 44	5.8% 14	8.3%, 20 0&12: 5.8%,14 0&undef: 2.5%, 6	5.4% 13	1.3% 3	100% 240
Multiple Choice (Stage 3) (%,n)	40.2% 41	7.8% 8	23.5% 24	9.8% 10	18.7%, 19 e (0&12): 12.8%,13 f (0&undef): 5.9%, 6	N/A	0% 0	100% 102

These results suggest that contrary to three of my hypotheses, prompting students to compare f to the cubing function did not appear to encourage them to infer that f and the cubing function share a derivative at 2. This naturally led to the emergent question: if inattention to the graph of f does not account for students' tendency to answer incorrectly, then why are students answering the way they are answering? To address this question, we turn to the student graphs (Stage 3) together with the student interviews

(Stage 2). As expected, the results of Stage 3 confirmed hypothesis (4): students who provided a correct graph were statistically more likely to get the multiple-choice cubing function task correct than those who did not provide a correct graph. Indeed, this makes sense; there might be some students who understand a graph as determining a derivative: 32% of Stage 3 'correct graph' students answered the multiple-choice cubing function task correctly, whereas only 10.3% of Stage 3 'incorrect graph' students did so ($\chi^2 = 6.1824$, $p < .05$).

Recall the earlier discussion about student graphs: The "correct" graphs in Stage 3 were partitioned into two subcategories, Category A and Category B. Category B graphs were the mathematically normative graphs that highlighted (2,8) in the sense that they had a dot on (2,8) that was more prominent than any other dots. The other "correct" graphs – those that were correct but indicated nothing special about (2,8) - were grouped together as Category A. The remaining graphs were classified as follows: those with a single dot at (2,8) (2.0%) (C-graphs), those with just a graph of $y = 8$ (6.9%) (D-graphs), those with a removable discontinuity at x=2 (6.9%) (E-graphs), those that were blank (4.9%) (F-graphs), those whose graphs included both $y = 8$ and $y = x^3$ on a nontrivial interval (2.9%) (G graphs), and other (8.8%) (O-graphs). See Figure 2.8 below for a sample of these graphs.

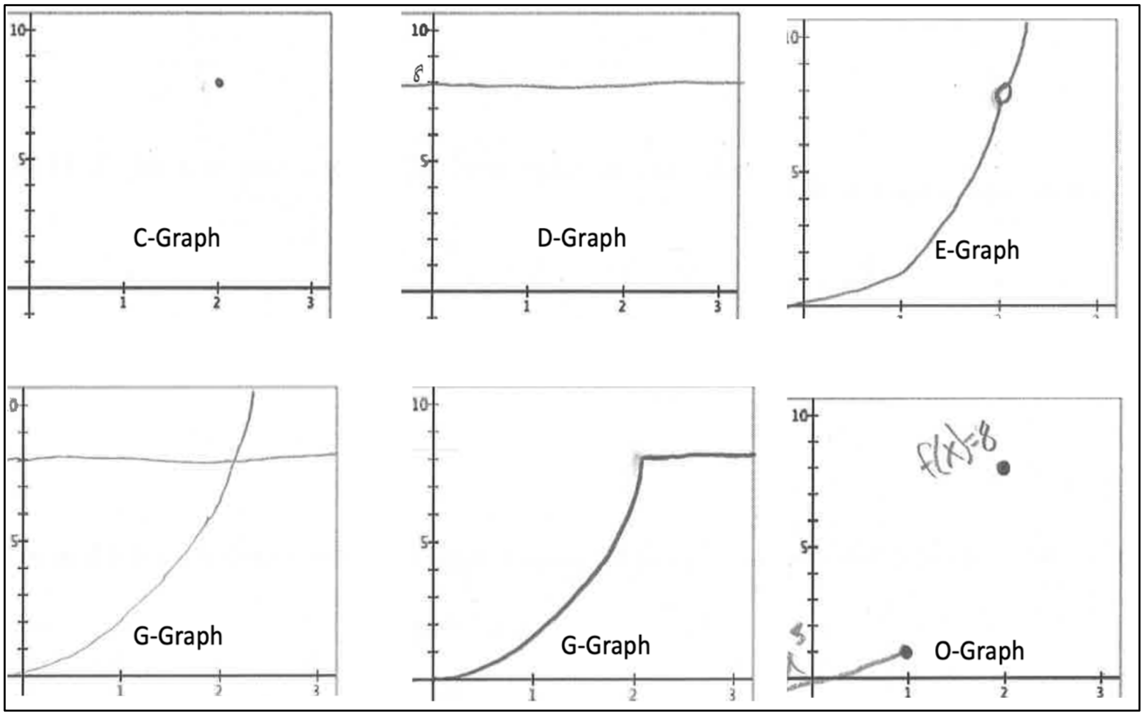


Figure 2.8. Categories for Students' Graphs of the Cubing Function (Stage 3).

Among the Stage 3 correct graphs (A and B), 'graph A students' were more likely than 'graph B students' to answer "12" (χ^2=3.932, p<.05), suggesting some sort of difference (in some of the minds of the 'graph B students') between f and the cubing function. Figure 2.9 depicts the relationship between correctness on the multiple cubing function task and graph type for Stage 3 students. Observe that the green color represents a correct answer, while the red colors represent various incorrect answers. Notice that the Category A bar has a larger percentage green than the Category B bar, which reflects the observation that students who seemed to think that the point (2,8) is special were less likely to get the cubing function task correct (this is discussed in further detail below). Notice, also, the relationship between the middle red color (which represents the "undefined" answer) and E-graphs (removeable discontinuities).

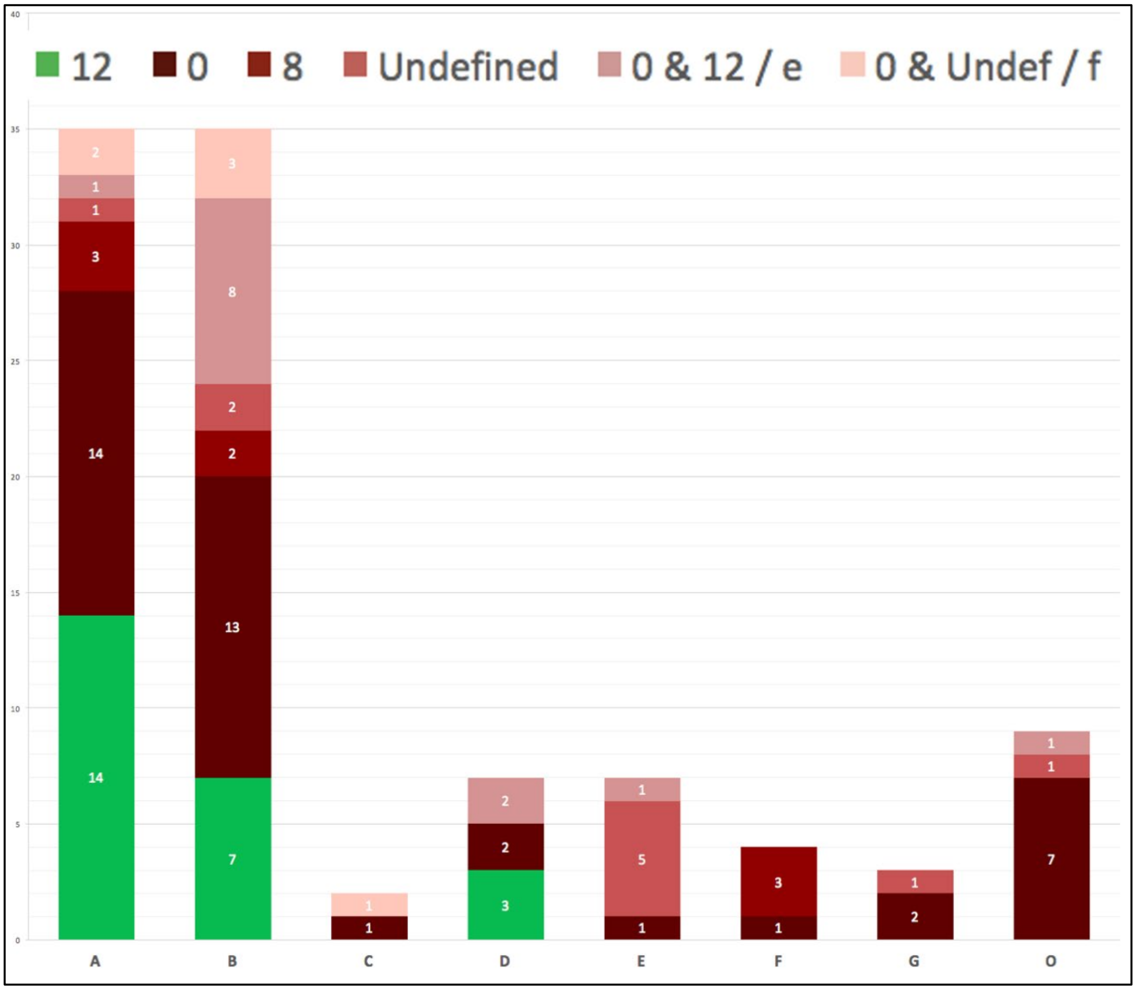


Figure 2.9. Results of the (Stage 3) Multiple-Choice Cubing Function Task Organized by Graph Type (see Figures 2.7 and 2.8).

Now we construct models to explain students approaches to the cubing function task. This analysis is guided by the interviews of the eight students who completed the open-ended cubing function task (Stage 2). Musgrave and Thompson's (2014) construct of "function notation as idiom" was useful in accounting for student responses. A student views function notation idiomatically when he or she views "$f(x)$" in its entirety as a name for a function (Musgrave & Thompson, 2014). Such a student might view "$f(x)$" as no more than another name for "y" (Thompson, 2013c). It appeared that many students thought in this manner when evaluating f'(2), as they seemed to view "8" and "x^3" as names of functions, with "$f(x)$" referring to each of these functions. Another common theme, appearing both on the written quizzes and in the interviews, was the viewpoint that the "if $x \neq 2$" served as a restriction on the domain rather than as a condition. I use this construct to explain how students arrived at select incorrect answers on the cubing function task. Each answer type is described individually below. These descriptions should be viewed as illustrations of student thinking that explain their answers. These descriptions each begin with a direct, written quote from a student, which provides a concise summary of their way of thinking. I also discuss how, for the students, the point (2,8) was special and the way students made sense of their graphs. Additionally, I discuss how students' ways of thinking are reflected in their responses to the interview prompt to find h'(5) for the function h defined by h(x)=x^3 if x ≠5, h(x)=x^2+100 if x=5.

2.5.2.1 Students Who Answered "0". "When the graph is at the point x=2, the function is determined by the piecewise part '8'. So, f(x) itself equals 8. When 8 is derived, it becomes 0" [Pete, 'multiple choice student', emphasis added]. The rationale summarized by Pete appears to exemplify a common way of thinking amongst students who answered "0". For these students, the "f'(2)" tells them that they are in the situation "x = 2," which serves as an instruction to use the function "8". Here, the "8" serves as a name of a function rather than a particular output, suggesting an idiomatic conception. Many of these students provided a category B graph of f (graph of $y = x^3$ but a special dot at (2,8)) and found no issue with the fact that they couldn't "see" that f'(2) = 0 in their graph; when asked to explain graphically, they would provide a graph of y = 8 and explain why its derivative at 2 is 0. This rationale is summarized in Figure 2.10 below.

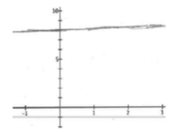
$f'(2) \rightarrow x = 2 \rightarrow y = 8 \rightarrow$  $\rightarrow y' = 0$

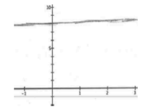
Figure 2.10. Student Rationale for Answering "0" on the Cubing Function Task.

Interviewed students extended this way of thinking to evaluating h'(5) for the function h defined by $h(x) = x^3$ if $x \neq 5$, $h(x) = x^2 + 100$ if x=5. It was common for students to answer "10" by evaluating the derivative of $x^2 + 100$ as 2x and substituting x=5 to result in 10, with the rationale that "Um I used this part, the part that makes the parabola [$y = x^2 + 100$]. Because we're interested in the time when x equals 5. And that's kind of the rule here, when x equals 5 to use the parabola" [Jennifer, 'open-ended student']. She elaborated: "The derivative of h when x equals 5 is gonna be 2x um…..if x were to equal some number other than 5, you would use this (underlines x^3) function up here, but because x is 5 we use this one." Jennifer's rationale exemplifies the way of thinking that led students to answer "f'(2) = 0": viewing the conditions on a piecewise-defined function as instructions for which function to use, and a piecewise- defined function as involving two different function.

Many of these students (Pete included) provided a graph of f that was like $y = x^3$ but with a dot at (2,8) (category B graph). It seems that students viewed the dot at (2,8) as separate or independent from the rest of the graph. For example, one student recreated his graph during the interview, explaining his reasoning as follows: "At the point (2,8) I draw a circle to show there is an opening there, there's a gap. I'm excluding that point from what it is we are talking about in this point in time.” He elaborated: “So the two...they’re existing on the same coordinate system but existing independent of each other”.

2.5.2.2 Students who answered both “12” and “0”. “If f(x) does not equal 2, the function is x^3. The derivative of x^3 equals $3x^2$, then substitute 2 for x, $3(2)^2=12$. However, if x is allowed to be 2, then the derivative of 8=0” [Carlos, ‘open-ended student’]. The case of Carlos illustrates how a student can reason idiomatically to get the answers 0 and 12. In the interview he reiterated his reasoning: “If x isn’t 2 then the function is x^3. The derivative of x^3 is $3x^2$. Then substitute 2 for x here and you get 12. However, if x is allowed to be 2, then the derivative of 8 is 0”. For Carlos, the “if x=2” condition told him that he was in the case in which “the function” is the function “$f(x) = 8$,” and that the “if $x \neq 2$” condition told him he was in the case in which “the function” is x^3. Carlos did not even make the connection that the “2” in “$f(2)$” told him he was in the case where “$x = 2$”; for him, the “$f(x)$” was just a shorthand for “y”. When prompted to graph f, he provided a graph of (what he thought was) y=8 as well as a graph of $y = x^3$, indicating that he viewed himself as graphing two separate functions. When asked how $f'(2)$ can be 12 while he had said prior that it was 0, he explained: “this is an entirely different function”, indicating that the conditions on the piecewise function were instructions about which function to use. This rationale is summarized in Figure 2.11 below.

Two cases: $x = 2,\ x \neq 2$

$x = 2 \rightarrow y = 8 \rightarrow$  $\rightarrow y' = 0 \rightarrow f'(2) = 0$ if $x = 2$

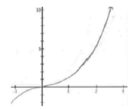
$x \neq 2 \rightarrow y = x^3 \rightarrow$  $\rightarrow y' = 12 \rightarrow f'(2) = 12$ if $x \neq 2$

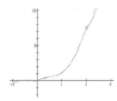
Figure 2.11. Student rationale for answering "0 if x=2, 12 if x ≠ 2" on the cubing function task.

Carlos' way of thinking was confirmed when he was asked to calculate h'(5) when h is defined by $h(x) = x^3$ if x≠5, $h(x) = x^2 + 100$ if x=5. He graphed $y = x^3$ and $y = x^2 + 100$ on the same axes. When prompted to find the value of h'(5), he differentiated x^3 and plugged in 5 to get 75, and then he differentiated $x^2 + 100$ and plugged in 5 to get 10. When asked which was the value of h'(5), he exclaimed confidently, "both! 75 and 10!".

2.5.2.3 Students Who Answered Both "0" and "undefined". "If just looking at $f(x) = 8$, the derivative of a constant would make $f'(2) = 0$. If just looking at $f(x) = x^3$, the derivative would be undefined because $f(2)$ is not on the graph of x^3. There is a hole at x=2" [Eric, 'multiple-choice student']. Eric's reasoning exemplifies how students could have come to select choice "f" in the multiple-choice quiz. A different student, Sarah, explained her reasoning in detail in the interview. Sarah initially answered that "both" are undefined, but during the interview, she revealed that she interprets "0" to mean the same thing as "undefined" (which was a common trend in student responses). Like Carlos, she viewed two functions as being involved, which was again confirmed when she was asked about the piecewise-defined function "h". She appeared to reason about two different functions, and calculated $f'(2)$ by treating the first function as "$y =$

x^3, $x \neq 2$", and the second function as "$y = 8$, $x = 2$". She interpreted the "$x \neq 2$" as a restriction on the first function, and the "$x = 2$" a clarification that such a restriction did not exist on the second function. Thus, for the first function, f'(2) is undefined, and for the second function, f'(2) equals 0. Figure 2.12 below provides a summary of this rationale.

Two cases, two functions

$y = x^3, x \neq 2 \rightarrow$  $\rightarrow f'(2)$ undefined if $x \neq 2$.

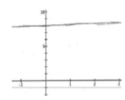
$y = 8, x = 2 \rightarrow y = 8$ and x can be 2 $\rightarrow$  $\rightarrow f'(2) = 0$ if $x = 2$.

Figure 2.12. Student Rationale for Answering "0 if x=2, Undefined if x ≠ 2".

The interpretation of the "first function" conforms with typical secondary mathematics. Consider the problems that ask students to "find" the domain of a function and then graph it. A typical problem of this type would be "find the domain of y= (x-2)·x^3/(x-2)." A student would typically solve this problem by setting the denominator equal to 0 and writing "$x \neq 2$". In light of these exercises, it makes sense that students would see the "f(x)=x^3 if x≠ 2" as a function on its own whose domain does not include 2. In this case, the graph is an E-graph (Figure 2.8), and f'(2) is undefined.

2.5.2.4. Discussion of the Cubing Function Task. The results of the open-ended cubing function task (Stage 1) demonstrated that Harel and Kaput's (1991) anecdote is indeed indicative of a larger phenomenon: many students appeared to differentiate a piecewise function formally by differentiating each expression as a separate function. The results of the multiple-choice version (Stage 3) confirmed that this phenomenon cannot be attributed merely to inattention or unawareness that the piecewise-defined function agrees with the cubing function. This data, together with the interview data (Stage 2), suggest that several students have a non-normative understanding of piecewise function notation, stemming from a view of function notation as idiom and the conditions on the domain as either instructions or as restrictions.

The results of this task show that students do not view the same function, represented in two different analytic ways, as sharing a derivative at a particular value. However, this last sentence was ambiguous; when I say “a function”, I am not being clear if students view these function representations as referring to the same function. Students might, for example, consider it possible for two distinct functions to share a graph, and we can ask: do students believe that same graph implies same derivative? The answer to this appears to be “no,” as many students provided normative graphs of f yet did not evaluate $f'(2)$ correctly. Yet, there is another ambiguity: what students view as “same graph” might not be consistent with the normative notion of “same graph,” as suggested by students’ insistence that the point (2,8) being highlighted and that it is “separate”, together with the discrepancy between Category A and Category B graphs.

The results of this task highlight how students think about function notation, independently of how they think about derivative. To illustrate this point, consider the way of thinking that accounted for many students answering “0.” It arose from a misconception of function notation: no matter how strong of a meaning the student had of “derivative”, the student was still keying on the graph of “$y = 8$”, leading to an answer of “0”.

As discussed earlier, it seemed reasonable to hypothesize that students who provided normative definitions of what it means for functions g and h to be the same (the concept definition task) would be more likely to correctly evaluate f'(2) ; this is because it seems these students would be more likely to assess the piecewise-defined f and the cubing function as "the same," positioning them to infer that f and the cubing function share a derivative. In light of the interviews and students' ways of thinking, the counter-intuitive result – that this hypothesis did not hold – makes sense. This is because, to students, f was not a function in the same way that the cubing function is; instead, f was two functions. Having a strong criteria for sameness of functions did not help many students evaluate f'(2) because f was not in the category of "functions" to which function sameness can apply.

Recall that the topic of investigation concerns function sameness. To mathematicians, f and the cubing function are just one function, since they are the same function. However, the analysis of the results of the cubing function task suggests that students do not view these two particular representations as referring to the same function. Additionally, the results show that some students might not view the piecewise representation as referring to a function at all, but instead to two functions. Both of these situations can be thought of as students seeing two functions where we mathematicians see one function.

2.5.3 The Fundamental Theorem Task

A natural question to ask is whether this phenomenon of students seeing two functions where we see one applies to other situations. We now move to Phase 3 to address this issue. In mathematics, students often see functions represented as integrals, such as in the fundamental theorem of calculus. The fundamental theorem task is described in Figure 2.3 and was answered by 100 students. Of those, 61 chose option (i) (that p and q are the same function), and 39 chose option (ii) (that p and q are not the same function). One thing to note is that the students were not asked to “evaluate” the integral, that is, put it in closed form (e.g., as a polynomial, in this case). This means that there is a possibility that some students might have evaluated the integral incorrectly and assessed p and q as different for that reason. Of the 100 students, 46 attempted to evaluate the integral (in writing – it is possible that others evaluated but did not write their work), and 29 did so correctly. Unsurprisingly, there is a strong correlation between those who evaluated the integral correctly and those who answered that p and q are the same function, with 27 out of 29 (93%) who evaluated the integral correctly also claiming that p and q are the same function, and 8 out of 17 (47%) who evaluated the integral incorrectly claiming that p and q are not the same function (χ^2=12.4883, p<.05).

The nature of students’ incorrect evaluations was illuminating and not due to computational errors. In fact, only two students who incorrectly evaluated the integral did so in such a way that it was a function of x (e.g., writing $p(x)=x^3+12$). Instead, 14 out of 17 (82%) included a “+C” in their evaluation of the integral. Of those who included a +C, four (28.6%) wrote an expression with t rather than x as an integral evaluation. More generally, students’ inscriptions suggested misunderstanding of function notation. Eighteen students misused function notation in some way on the fundamental theorem problem. Of those, 17 (17%) had a variable mismatch (e.g. $p(t)=x^3-8$), while one student wrote $p'=3x^2$. As discussed in the section on the cubing function task, such variable mismatch suggested an idiomatic understanding of function notation (Musgrave & Thompson, 2014).

Students' explanations provide insight to student thinking. Their explanations seem to suggest that some might have viewed the integral as representing a string of symbols. This is consistent with Musgrave and Thompson's (2014) and Sfard's (1992) findings suggesting that some students think of a function as a string of symbols. To many of the students who evaluated the integral as involving a C (e.g., x^3+C), it would make sense that these students would not think of x^3+C as being the same as x^3-8, as these are different strings of symbols. For example, one student explains "the -8 in q is not shown in the equation for p." Similarly, the students who evaluated the integral correctly tended to find that the resulting string of symbols (x^3-8) was identical to that in the definition of q, and therefore q and p are identical: "once calculated, the integral in p(x) becomes the same expression as q(x)". A summary of these results is included in Table 2.3.

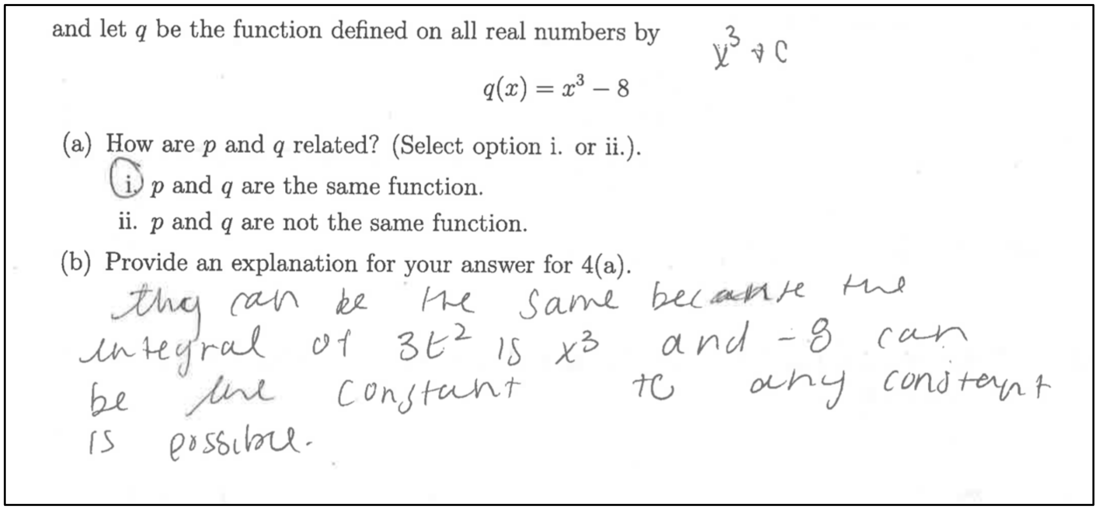
Table 2.3 A Summary of Results of the Fundamental Theorem Task

	Same	Not Same
Correct Integral Evaluation (29)	27	2
Incorrect Integral Evaluation (17)	9	8
No Integral Evaluation	54	

There's a sense in which 36 out of 46 gave consistent responses; they either (1) evaluated the integral correctly and wrote that p and q are the same function, or (2) evaluated the integral incorrectly and wrote that p and q are different functions. This is consistent with thinking of a function as a string of symbols; if a student evaluates the integral correctly, they observe that the resulting string of symbols is the same as x^3-8, and if they evaluate it incorrectly, they observe that the resulting string of symbols is different from x^3-8 (discussed above). The remaining 10 students had mixed responses. Those students' written explanations in part b provide some insight into their understanding of function identity. For example, some students included a +C for the integral yet assessed p and q as the same on the grounds that they share a derivative. Relatedly, some students wrote that p and q are the same function while also stating that they had a different constant. For these students, sameness of derivative was sufficient for sameness of function. Additionally, 15 students justified their assessment of p and q being the same by explaining that p and q share a derivative. This justification was expressed in a few different ways. These ways included explanations such as "they have the same slope at any given x", "derivatives are the same", "$p'(x)=3x^2$, $q'(x)=3x^2$", and "If you take the derivative of them they both come out to the same function". This rationale makes sense in light of the fact that several (15) students listed sameness of derivative as a criterion for function sameness in the concept definition question.

It bears mentioning that not all the students who understood p(x) as involving a constant assessed p and q as the same. While, as discussed above, some students rationalized that the constant indicated that p and q differ by comparing symbols, others took a different approach. This included students who did not necessarily evaluate the integral but still referred to the constant in their explanations. In some situations, this took the form of arguing that $p(x)=x^3+C$ shares a derivative with $q(x)=x^3$, as discussed above.

In other situations, students treated C as an unknown or undetermined (but fixed) number. Sometimes, this led to students assessing that p and q are the same because they could be the same (in the sense that C could be -8). See Figure 2.13 below for an example of such a student.


and let q be the function defined on all real numbers by $x^3 + C$

$$q(x) = x^3 - 8$$

(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

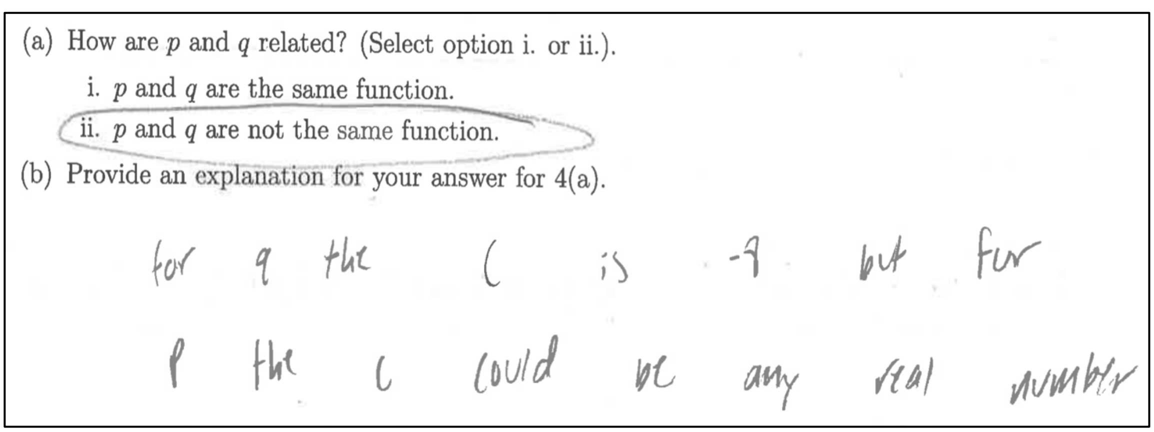
ii. p and q are not the same function.

(b) Provide an explanation for your answer for 4(a).

they can be the same because the integral of $3t^2$ is x^3 and -8 can be the constant tc any constant is possible.

Figure 2.13. A Students' Answer to the Fundamental Theorem Task.

Other times this led to students assessing that p and q are not the same because they could be different (in the sense that C could be some number other than -8). See Figure 2.14 below for an example of such an explanation.


(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

ii. p and q are not the same function.

(b) Provide an explanation for your answer for 4(a).

for q the C is -8 but for p the c could be any real number

Figure 2.14. A Different Student's Answer to the Fundamental Theorem Task (cf. Fig. 2.13).

There were six students of the former type, while there were five who gave the latter argument.

I had originally hypothesized that there would be a correlation between students who give extensional function sameness concept definitions (discussed in Section 2.5.1) and those who answer that p and q are the same function. This is because I expected students with other, non-normative understandings of function identity to claim that p and q are different. This was indeed the case with at least two students, who asserted that p and q differ because one represents an area under a curve, and the other does not. However, a chi square analysis revealed no such correlation (χ^2=0.337, p>.05, see Table 2.4). It seems that because p could be expressed in closed form, students' assessment of sameness of p and q was primarily about how they calculated the integral. This allowed students to assess that p and q are the same on the grounds that they are expressed by the same equation, rather than requiring a robust understanding of function sameness. This resulted in the possibility that students who understand functions as strings of symbols answered that p and q are the same function.

Table 2.4. Results of the Concept Definition Task in Relation to the Fundamental Theorem Task

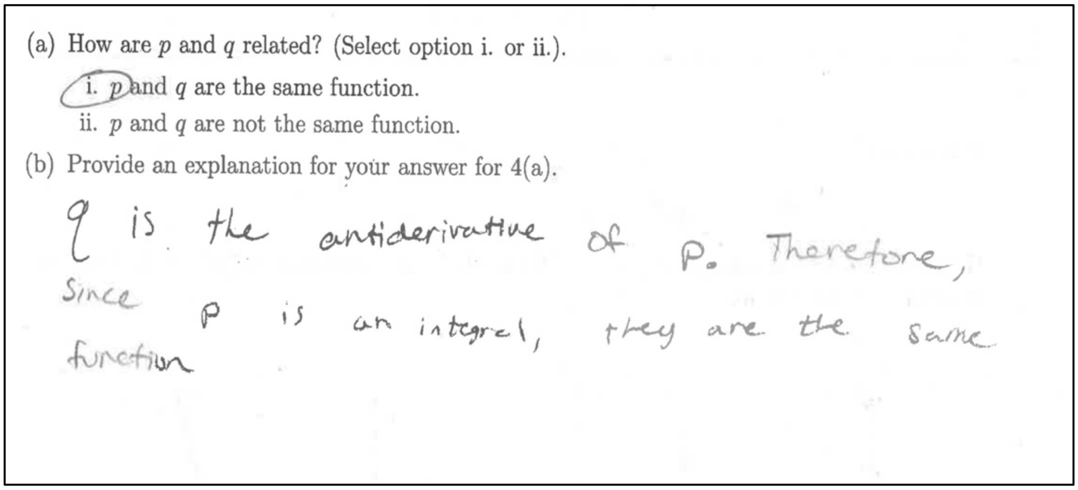
	Fun. Thm Task Correct	Fun. Thm Task Incorrect
Concept Def. Correct	20	15
Concept Def. Incorrect	41	24

That so many students evaluated the integral with a "+C" is especially revealing. This might suggest that, despite the function notation p(x) being used and the quiz explicitly telling them that p is a function, these students might not have viewed p as a function (perhaps, as one student above put it, "a formula"). This leaves open the possibility that, when these students were asked if p and q are the same function, they were not viewing p as a function at all. This is consistent with the results of the cubing function task, in which students appeared to not think of a particular piecewise function as a function. It is possible that such students just do not understand integrals as functions; perhaps they understand an integral as a command to anti-differentiate.

Such an understanding would be consistent with other mathematics education literature. For example, Hall (2010) reports that, when asked about the meaning of definite integrals, students tended to discuss only how to evaluate them. While it is unclear if students understood the meaning as being about calculation (rather than answering a question that was not being asked), there is other literature to suggest that students truly do understand some notation as instructions to calculate. For example, as discussed in the general introduction to this document, young students understand "=" as a command to calculate rather than expressing a relation. Unpublished data concerning a Calculus I Concept Inventory suggests that some students understand function notation as instructions to substitute and calculate. In light of these results, it would be unsurprising if some students viewed the integral sign similarly.

A close look at the data reveals that this could be the case with students. Although students are told that p is a function, they might have seen the integral sign as a command to anti-differentiate, and the only function that's provided to them in this command is in the integrand. For example, one student assessed p and q as different and explained that "the antiderivative of p(x) equals t^3+c and q(x) is x^3-8. So the derivative of p equals q. p does not equals q”. Observe that this student viewed the antiderivative of p as a cubic expression, suggesting that p is the integrand, and then used this reasoning to conclude that p and q differ. This interpretation conforms with the fact that 23 students equated p or p(x) with the integrand. Some students explicitly wrote an equation (e.g., $p(x)=3t^2$, also a use of idiomatic function notation), while others indicated in other ways that they were equating the function p with the integrand. For example, several students explained “q is the antiderivative of p”. With some students, an integral as an action accompanied such explanations, such as “when taking the integral of p(x), the final answer does not equal q(x)”

Although several students identified p with the integrand, they did not identify p *uniquely* with the integrand; that is, there are students who appeared to use the same notation to represent two different things (a cubic function as well as its derivative, the integrand in the definition of p). In some sense, all the students who identified p with the integrand did this, since p(x) is defined to be an integral. However, I consider only the students who themselves referred to p as the integral function (sometimes expressed by writing $p(x)=x^3-8$). Eleven students used the same symbol (p(x), p(t), or p) to denote both the integrand and a cubic function or integral (see Fig. 2.15 below).



(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

ii. p and q are not the same function.

(b) Provide an explanation for your answer for 4(a).

q is the antiderivative of p. Therefore, Since p is an integral, they are the same function

Figure 2.15. A Student's Dual Usage of Notation.

Some of these students additionally wrote that $p(x)=x^3-8$ while also equating p with the integrand. This is especially interesting, since this shows us that a decent portion of students understand the same name to refer to two different things. This violates a fundamental norm in mathematics that within a context, a name can denote only one object (e.g., functions must be "well-defined"). Students' wording provides some insight into how this can happen; five students use the word "becomes" in their explanation, such as the student who wrote "they are the same function by p(x) being the derivative but by taking the integral of it it now becomes q(x)" (emphasis added). The use of the word "becomes" suggests that they might understand "p" as denoting one thing at one point in time and another thing at another point in time. In some sense, these students are seeing two functions where we see one; we understand "p" as denoting a single function, while they understand it as denoting both a quadratic (the integrand) and a cubic function (the integral).

2.6 Discussion

The results of this study inform us that many students have a non-normative understanding of function sameness. As discussed, being able to recognize sameness of functions is essential to mathematics and allows for powerful inferences. When two things, say x and y, are the same thing, then we know that {x,y} has cardinality 1. Conversely, if x and y are different, then {x,y} has cardinality 2. In this study, x and y are functions. We have learned that there are situations in which mathematicians understand the functions x and y to be the same (they are seeing {x,y} as having cardinality 1), and where students understand x and y to be different. The notion that students see two functions where we see one is tied to the notion of students seeing functions as different when we see them as the same. However, there is an extra layer of complication here. Recall that in the cubing function task, the issue was not just that students saw the piecewise-defined function f as different from the cubing function – they saw it as two functions rather than a singular function. Where we were seeing an object x, they were not. This is another sense in which students saw two functions when we saw one – f was not a "function" but instead "functions" and not a unified entity in its own right. This way of thinking also occurred in the fundamental theorem task; while we see p as denoting one function, some students appeared to use it to denote two different functions (a cubic function and a quadratic function).

Why is it that, when we see two function representations as denoting one and the same function, students see something different? It is important to answer this question so that we can see what barriers or potential obstacles students face when developing a robust and mathematically normative conception of function sameness, not only because of the importance of function sameness, but also because of the importance of the concept of function more generally. As discussed in the general introduction to this dissertation, there is a close relationship between a person's conception of sameness within a category and the defining features of that category. Hence, the importance of students' conceptions of function sameness is tied to their conception of function more generally.

The fact that students did far better on the fundamental theorem task than the cubing function task is notable (23.5% versus 61%, Stage 3 students). One difference might be familiarity; translating between representations involved is a procedure that students have done before in the fundamental theorem task but likely have not done in the cubing function task. That is, introductory students have evaluated integrals before, but they have not rewritten piecewise-defined functions as single equations. This discrepancy is reflected in the types of reasoning students tended to give for the fundamental theorem task; students performed a translation exercise (evaluation p) which gave them a representation ($p(x)=x^3-8$) to compare with, symbol by symbol, the representation $q(x)=x^3-8$. With the cubing function task, there is no translation exercise that the students perform that results in the ability to compare two representations symbol-by-symbol. Additionally, students are in some sense being prompted to make a translation that allows for such a comparison by encountering the integral symbol (as discussed above, it seems that students viewed the integral symbol as a command to antidifferentiate). In other words, students were prompted to translate p to a different analytic representation and then compare to q, which happened to be the same representation as the result of that translation. In some sense, the multiple-choice cubing function task can be viewed as a translate-and-compare task. This is because students were indeed prompted to produce a graphical representation of the piecewise-defined f alongside a graphical representation of the standard cubing function. In theory, students could have performed this translation and then observed that the graphs were identical. However, as discussed, most students did not do this. Recall that the results suggested that the graphs that students produced that we would view as identical to the cubing function graph might not have been viewed as identical from the students' perspective. Additionally, there is the possibility that students do not view a graph as uniquely representing a function. This possibility opens up various questions about how we should understand the translation tasks in multiple representation literature (discussed in the literature review to this chapter). While this literature discusses how students translate between multiple translations, it does not address how or if students understand identity as being maintained throughout these translations or the uniqueness of such translations.

This study was intended to be about assessing when students saw two functions as actually being the same function. There was an implicit assumption that we were working within the category of "function"; students were asked about representations that the prompts referred to as "functions". I had only intended to investigate students' concepts of function sameness *within the category of function*. However, the results of this study suggest that students did not view certain objects labeled as "functions" to actually be functions (or even objects, for that matter). This occurred in the cubing function task – the piecewise-defined function f was not viewed as a function, but as two functions with instructions about when to use each function. This also appears to have occurred in the fundamental theorem task – although p was referred to as a "function" in the written task itself and written with function notation (as p(x)= $\int_2^x 3t^2 dt$), it seems that several students did not view it as a function (or at the very least, not a function of x). If one does not view a particular thing as a function, then they would be hard-pressed to say that this particular thing is identical to a thing that they do view as a function (following Leibniz' law of indiscernibles, if x is not a function and y is, then x and y cannot be the same). Going back to our question about why students do not see function sameness where we mathematicians do, one reason is that students might not see something as a function when a mathematician does (despite being told that it is). As a constructivist, this result is unsurprising. In the cubing function task and the fundamental theorem task, objects that we as mathematicians understand as functions are not even functions to many students. This is despite the fact that, from our mathematical perspective, these representations do refer to functions that students recognize as functions when represented differently (e.g., transparently as polynomials).

The results of this study bring up foundational issues about notation and denotation in a few different ways to suggest that notation is one barrier to students having a normative understanding of function sameness. First, we have the issue of viewing function notation idiomatically (Musgrave & Thompson, 2014). Many students did not view "f(x)=8 if x=2" as meaning the same thing as "f(2)=8". Similarly, in the fundamental theorem task, several students evaluated the integral with a "t" and wrote equations with mismatched variables such as $p(x)=t^3$, suggesting, again, an idiomatic view of function notation. Second, we have the idea that the thing we call a function might be something to the student that we would not call a function. This issue is described in the preceding paragraph; some students do not view “f” as denoting a single function, and students seem not to view an integral as denoting a function. Third, we have the idea of a symbol being used to refer to two different things in the same context (a homonym or an instance of polysemy). This occurs in the cubing function task with the students who gave multiple answers; some students viewed f’(2) as being both 0 and 12. This also occurs with the fundamental theorem task, in which students refer to both the integrand and the integral as p (or p(x) or p(t)). While mathematics does allow for different names for the same thing (as discussed, this occurs in statements of identity such as a=b), the mathematical community does not allow for using the same name to denote different things within a context. For example, we care that functions are “well-defined” so that “g(t)” cannot name two different things. In the field of logic, referring to two things with the same name is considered such a significant fallacy that it has a name: “equivocation” (Hansen, 2020). In some sense, it appears that students were performing this fallacy. However, further investigation is needed. As a constructivist, it is important that what looks like a logical fallacy might be a much more nuanced understanding that is consistent from the students’ perspective.

With written analytic definitions of functions, function notation appears. One essential take-away from this study is that students’ understanding of notation has bearing on their understanding of a written representation of functions, and therefore further investigation of student understanding of function representations should carefully investigate how such students understand (de)notation.

INFERENCES AND KNOWLEDGE TYPES: SITUATING IMPLICIT DIFFERENTIATION

Chapter four (Mirin & Zazkis, 2020) is an already published paper, and as such, it cannot be modified. This chapter (Chapter 3) provides an expanded discussion of the topic discussed in Chapter 4, which is a result of an expanded literature search and analysis. It also situates Chapter 4 in relation to the general dissertation topic (sameness of representation). Specifically, it addresses the relationship between implicit differentiation and sameness as well as how my work on implicit differentiation relates to the topic of deep procedural knowledge (Star, 2005).

Like chapter two, Mirin and Zazkis (2020) is related to sameness insofar as it involves function identity. Specifically, it addresses implicit differentiation and, more generally, differentiating equations. The guiding research questions for Chapter 4 are:

1. What does it mean to understand the legitimacy of differentiating equations (e.g., in implicit differentiation problems)?
2. What difficulties might a student encounter when constructing such an understanding?

I answer the first question in terms of function sameness: When it is legitimate to differentiate each side of an equation (related rates problems, implicit differentiation problems), that legitimacy is grounded in the fact that the equation being differentiated is a statement about function identity (sameness). This is another instance of Leibniz' law of indiscernibles; when two functions f and g are the same function, they share all properties. Hence, since being a derivative is a derivative of a function, f and g share a derivative. In other words, f and g being the same function acts as a warrant (Toulmin,

1969) for claiming that they share a derivative. It follows that students' understanding of function sameness is paramount to understanding differentiating equations (and hence implicit differentiation) as a legitimate procedure. It is important to note that I am not assuming that students view differentiating equations in this way. In fact, I do not even assume that students understand differentiating equations as an inference or as involving mathematical argumentation. Instead, my paper gives a conceptual analysis (Thompson, 2008) of how students could come to understand differentiating equations in this way and what difficulties students might encounter in doing so. Chapter two of this dissertation also provides some insight into answering the second question by explaining the various obstacles student encounter in assessing function sameness. In particular, the results of the previous study suggest that inferring that two functions share a derivative on the grounds that they share a graph is nontrivial for several students.

This topic is not unique to implicit differentiation. For example, suppose we are working on some sort of word problem, and to solve that word problem, we set up the equation "x+3=9". We might apply a procedure (e.g., subtracting 3 from each side) to obtain "x=6". Why is this procedure legitimate in this situation? Well, if a=b, then a-3=b-3. Reframing in terms of Leibniz' law of indiscernibles, we can say that if a and b are the same, then they share the same properties; in particular, since subtracting 3 from a results in a-3, we can conclude that subtracting 3 from b also results in b-3, and hence a-3=b-3. That is; we can conceptualize subtracting 3 from each side as an inference. This illustrates what I mean when I say "understanding the legitimacy of a procedure". It involves understanding that there is an inference involved, what exactly this inference is, and why this is a valid inference. The conceptual analysis in Mirin and Zazkis (2020), i.e., Chapter four of this dissertation, explains *how a student can reconceptualize the procedure of implicit differentiation as a valid inference*. This is just a more technical way of saying *how a student can come to understand why the procedure of implicit differentiation works*. In answering this question, the conceptual analysis discusses the conceptualizations entailed in doing so.

In a quest to better characterize my investigation, I turned to the literature on procedural vs conceptual knowledge. In Mirin and Zazkis (2020), I characterize my investigation to be about "deep procedural knowledge" in the sense of Star (2005). Due to space restrictions for that particular journal, I was unable to delve fully into how and if my topic of investigation is about deep procedural knowledge. For this reason, I delve into the literature more deeply here. Star (2005) believes that there is a history in mathematics education of a procedural/conceptual dichotomy being conflated with a shallow/deep dichotomy. He traces this conflation to Hiebert and Lefevre (1986), in which the authors define conceptual knowledge as knowledge that is rich in relationships, and procedural knowledge being knowledge about algorithmic procedures for solving problems. Star argues that this definition seems to conflate knowledge type (conceptual versus procedural) with knowledge quality (deep/interconnected versus shallow). Star's main point is that knowledge type is in some sense distinct from knowledge quality to allow for deep (quality) procedural (type) knowledge. "Knowledge type" is what the knowledge is about. If the knowledge is about *concepts*, then the knowledge is *conceptual knowledge*. If the knowledge is about *procedures*, then the knowledge is *procedural knowledge*. Knowledge quality actually qualifies that knowledge; it is about how deep or shallow that procedural knowledge is. This leaves room for "deep procedural knowledge," which Star describes as deep knowledge "about procedures". With Star's construct, it is still unclear how to classify the knowledge of why a procedure is valid. Arguably, we could say that knowledge of why a procedure works is "about procedures" and classify it under "deep procedural knowledge". On the other hand, Star does not give this type of knowledge as an example of deep procedural knowledge. Additionally, my topic of investigation includes all the background knowledge (conceptualizations) that a student might need in order to understand this the differentiating procedure as a valid procedure, which also involves conceptual knowledge about functions.

Baroody et al. (2007) appear to refute Star (2005). They characterize Star as claiming (a) that knowledge type and quality are purely independent; and (b) that the idea of deep procedural knowledge has been ignored. There are a few things wrong with Baroody et al.'s characterization and rebuttal to Star's claim of (a). Baroody et al. are not convinced that knowledge type (procedural – conceptual) and knowledge quality (shallow – deep) can be disentangled, and they view this as an empirical question. This particular claim is strange; even if knowledge type and knowledge quality are not independent in real life, they can still be seen as independent measures that warrant different labels. Conceptually we can disentangle physical strength and muscle size in the human body, even though they are interlinked in real life. For this reason, it's strange that Baroody et al. view these aspects as potentially dependent as being inconsistent with treating them as different measures. Furthermore, Star (2005) never actually claims that they are independent, so it seems that Baroody et al. have mis-characterized his work. However, Baroody et al. do make some interesting points regarding (a) that might be relevant to my work. They claim that “although conceptual knowledge is not necessary for the former, it is unclear how substantially deep comprehension of a procedure can exist without understanding its rationale (e.g., the conceptual basis for each of its steps).” Here, it seems that Baroody et al. are claiming that understanding a procedure’s rationale (e.g., understanding why implicit differentiation works) is an instance of conceptual knowledge, at least insofar as they are using the phrase “conceptual basis”. It is unclear if the authors understand “conceptual basis” to mean “why a procedure works”. To elaborate on (b), Baroody et al. give examples of mathematics education literature to refute Star’s (2005) claim that the mathematics education literature has since largely ignored procedural knowledge. I will address this literature in more detail after I address Star’s response to Baroody et al.

Star (2007) concedes to Baroody et al. (2007) in acknowledging that the modern mathematics education research community has not completely ignored procedural knowledge. However, as Star points out, the work on procedural knowledge has been purely theoretical and not yet operationalized. He claims that there is a small group of distinguished mathematics education people who have a nuanced view of procedural knowledge but emphasizes that this group is very small. As evidence of his claim, he explains that there are rarely in-depth interviews about students’ understanding of procedures, and instead only of students’ understanding of concepts. Star (2007) notably disagrees with what he refers to as Baroody et al.’s (2007) “premise” that procedures learned without concepts are necessarily rote. He makes the point that there is a teleological view of procedures. One can perform procedures with a particular goal in mind and be especially skilled at pursuing this goal. This is how a skilled programmer might function. Such a programmer might copy a pre-written section of code and use it in a larger program. In that instance, they know that the bit of code does a particular thing, and they use that knowledge skillfully and in a goal-oriented way, without being concerned with how or why that bit of code works.

I now return to the relationship between this literature and my investigation in Mirin and Zazkis (2020). Thankfully, Star (2007) does provide additional insight into what he means by “deep procedural knowledge”. The fact that he argues that procedural knowledge can be disentangled from conceptual knowledge by discussing the idea of skilled (teleological and efficient) use of algorithms suggests that he is putting the skilled use of algorithms in the “purely procedural camp”. The fact that he uses this as an example, rather than rationale for a procedure, suggests that he might be putting rationale for a procedure in the “conceptual” camp. Yet, he never clarifies.

I now turn to the literature on “adaptive expertise”, a closely related topic to that of “deep procedural knowledge”. Hatano (2003) characterizes “adaptive expertise” by contrasting it with “routine expertise”. While routine expertise involves the successful execution of routines, adaptive expertise is characterized by being able to use procedures adaptively in new situations. Essentially, adaptive expertise is whatever skill the student has that allows for transfer (Baroody, 2003). Hatano suggests that to have adaptive expertise, students must understand “why a procedure works”. He does not clarify what he means by “why”. It could mean “why a procedure gets you closer to your goal” – that is, why subtracting 3 from both sides of $-x+3=7$ gets you closer to isolating x (not relevant to Mirin & Zazkis, 2020). Alternatively, it could mean “why a procedure is valid” (very relevant to Mirin & Zazkis, 2020).

Baroody (2003) provides a more detailed account of adaptive expertise. He describes students who had learned why it makes sense to check subtraction problems by adding. He attributes the students' ability to come up with the idea to check division problems by multiplying to the fact that students understand the rationale for doing the analogous thing with subtraction problems. This sort of reasoning involves not just the skillful use of procedures toward a goal, but also having some knowledge of why the procedures work the way they do. In this view, one might classify the investigation in Mirin and Zazkis (2020) as being about adaptive expertise. Baroody (2003) characterizes adaptive expertise as involving an integration of procedural and conceptual knowledge. Interestingly, Baroody (2003) describes Hiebert and Lefevre's (1986) distinction between procedural and conceptual knowledge as follows: "Hiebert and Lefevre (1986) defined procedural knowledge (skills) as knowing how-to and conceptual knowledge (understanding or concepts) as knowing why" (p. 11). Under this particular definition, it seems that my topic of investigation in Mirin and Zazkis (2020) is about why – at least, it's certainly not "how-to". Hence, under this characterization, my topic of investigation is actually not about procedural knowledge.

To summarize, my topic of investigation in Mirin and Zazkis (2020) (Chapter four) is about how students can come to understand implicit differentiation as a warranted inference from function identity. The analysis of the literature here reveals no definitive answers about whether understanding a procedure as a warranted inference counts as procedural knowledge or conceptual knowledge. Star's (2005, 2007) characterization of procedural knowledge as being knowledge about procedures suggests that indeed I am investigating procedural knowledge – knowledge about the validity of a procedure is indeed about a procedure. On the other hand, Star's (2007) examples of procedural knowledge only involve discussions of skilled teleological uses of procedures. Interestingly, Baroody's (2003) characterization of Hiebert and Lefevre (1986) describes procedural knowledge as "how-to" and conceptual knowledge as "why". Under this classification, it appears that my topic of investigation aligns more with "conceptual knowledge". I am not investigating if students know how to implicitly differentiate. On the other hand, perhaps I am investigating if students know how to justify implicit differentiation. While the construct of adaptive expertise (Baroody, 2003) keeps procedural and conceptual knowledge intertwined, this construct appears to be more about transfer and flexible adaptation of procedures to varying contexts than it is about understanding the validity or legitimacy of procedures. One aspect of adaptive expertise is knowing when a particular procedure might be useful – this is arguably related to the idea of when a particular procedure is valid in the sense that they both concern the question "when should we use a particular procedure?" We can argue that the answer to "when should I use implicit differentiation?" is "when you have a statement of function equality" and is hence closely related to why implicit differentiation is legitimate. Yet, at the same time, students might have a way of superficially understanding when to use implicit differentiation without considering functions or function equality at all. Furthermore, "when you have a statement of function equality" does not answer why it is okay to take the derivative of both sides of an equation. Additionally, Mirin and Zazkis (2020) discusses not only the legitimacy of implicit differentiation in isolation, but also the conceptualizations involved in understanding this legitimacy. Some of these conceptualizations (e.g., robust understandings of function notation) do not directly reference any procedures and are therefore clearly not procedural knowledge. So, while Mirin and Zazkis (2020) does discuss procedures, its main contribution is a conceptual analysis, not a procedural analysis.

FUNCTION SAMENESS: BRINGING COHERENCE TO IMPLICIT DIFFERENTIATION

John is doing well in his introductory calculus class. He has learned how to generate a new function, a derivative, from a given function. He has completed several implicit differentiation problems and can apply differentiation rules fluently. He is excited to participate in a mathematics education interview. After he successfully differentiates an equation, the interviewer asks him "Why was it OK to take the derivative of both sides?". John thinks for a moment back to his experience in secondary school algebra and replies "If I have x=1, I multiply by 2 and get 2x=2, it would be the same thing." Next, the interviewer asks John what happens if he differentiates both sides of x=1. John writes "1=0" on his paper and feels confused. Later in the interview John differentiates an equation without hesitation. When asked why it was OK for him to perform this differentiation, he says "because math teacher said so" (with a chuckle).

The vignette centers around (a lack of) what Star (2005) calls deep procedural knowledge. This involves knowledge of when and why a procedure works. As mathematics educators, we consider it important that students not only know how to apply a procedure, but also why a procedure works. Specifically, we

(1) Describe a way that introductory calculus students could understand not only the 'how' but also the 'why' of implicit differentiation.

(2) Outline the difficulties that students might have in coming to this understanding.

In doing (1) and (2), we present concerns that could guide calculus teachers. In particular, (1) can help them with their initial presentation of the subject, and (2) can sensitize them to the difficulties students might encounter.

Mathematics education literature (e.g., Engelke, 2004; Hare & Phillippy, 2004) tends to treat differentiating equations as an unproblematic application of previously learned rules. We could find only two articles, Thurston (1972) and Staden (1989), that address the legitimacy of differentiating both sides of an equation. While both Thurston and Staden acknowledge that the justification for differentiating equations requires explanation, neither author considers the conceptualizations that might be involved for a student to come to understand when and why differentiating an equation is legitimate.

We therefore provide an approach that entails a justification for implicit differentiation that coheres with the rest of introductory calculus. We hope that this discussion of implicit differentiation will sensitize the reader to the mathematics involved in relation to their own understanding as well as guide them in presenting the topic to their students.

Our approach to addressing the above begins with a conceptual analysis (Thompson, 2008) of what it means for an introductory calculus student to understand (the legitimacy of) implicit differentiation. The conceptual analysis addresses point (1) by answering the question ‘what does it mean to understand the legitimacy of implicit differentiation in a way that is consistent with introductory calculus?’. Answering this question provides a lens for point (2). Specifically, it helps us begin to answer ‘what struggles might a student encounter in constructing such an understanding?’ We address this second question by consulting the relevant literature and presenting novel student data. Educators can use this conceptual analysis, together with the discussion of potential student struggles, to assist them in helping students understand the 'why' of implicit differentiation.

4.1 A Conceptual Analysis

To guide our conceptual analysis, we consider a classic implicit differentiation problem, which we call *Ladder Problem 1* (see, for example, Rogawski, 2011; Stewart, 2006):

A 3-meter ladder is sliding down a vertical wall. Find the rate of change of the height of the ladder's top with respect to the distance of the ladder's bottom from the wall. A typical written solution to Ladder Problem 1 involves designating y as the height of the ladder's top and x the distance of the ladder's bottom from the wall and performing the following computation:

$$
\begin{aligned}
&x^2 + y^2 = 9 \\
&2x + 2y\left(\frac{dy}{dx}\right) = 0 \\
&2x = -2y\left(\frac{dy}{dx}\right) \\
&\frac{dy}{dx} = \frac{-x}{y}
\end{aligned}
$$

Computation A

While the solution procedure in Computation A is correct, there is no explanation for why concluding the second line from the first line is a valid inference; that is, each side is differentiated, but there is no justification for *why* this is ok. We therefore solve Ladder Problem 1 in a way that not only elucidates the 'why' of implicit differentiation, but also does so in a way that is coherent with the rest of introductory calculus. To clarify, we are not claiming that the line of reasoning described in the conceptual analysis below reflects what a student could come up with on their own. Rather, it forms a trajectory of what we believe might be possible given appropriate instructional support.

4.1.1 Working Through the Ladder Problems

We solve Ladder Problem 1 in a way that reflects a trajectory that an introductory calculus student such as John could be guided through. As before, we have

$$(*)\ x^2+y^2=9,\ y>0.$$

The first crucial insight John might have, is that any x-value between 0 and 3 has a corresponding y-value that makes (*) true. He might then relate this insight to the notion of function and observe that any value of x determines a unique value of y. So, it might make sense to switch to a notation that indicates that type of relationship. Accordingly, we re-write equation (*) as

$$(**)\ x^2+(f(x))^2=9$$

where f(x) is that unique value of y determined by x in (*). Next, John might be guided toward the insight that f is not the only function involved. Namely, x^2, 9, and $x^2+(f(x))^2$ can all be thought of as dependent on x. Hence, (**) is a statement about function equality. Accordingly, we consolidate some of the functions involved by calling the function defined on the left side of equation (**) m and the function on the right side r. So, for $0<x<3$, $m(x)=x^2+(f(x))^2$ and $r(x)=9$. Equation (**) tells us that m(x) and r(x) are equal on this interval. The final insight involves inferring that because m(x) and r(x) are equal on this interval, they therefore share a derivative on this interval. The inference is central to why differentiating both sides works. Returning to the problem, a student such as John then might feel like he understands why he can differentiate both sides.

Differentiating both sides of equation (**) yields f'(x)=-x/f(x). We summarize this line of reasoning in Computation B.

$$
\begin{aligned}
&(*)\ x^2 + y^2 = 9, y > 0 \\
&f(x) = \text{the value of } y \text{ that makes } (*) \text{ true} \\
&m(x) = x^2 + \left(f(x)\right)^2 \\
&r(x) = 9 \\
&m(x) = r(x) \\
&m'(x) = r'(x) \\
&f'(x) = \frac{-x}{f(x)}
\end{aligned}
$$

Computation B

The Conceptual Steps[9] involved in making the inference of taking the derivative of both sides are as follows:

1. Defining f by using (*).
2. Viewing both sides of the equation as functions.
3. Recognizing that the functions defined on the left side and the right side are equal.
4. Inferring that, since the functions are equal, their respective derivatives are equal.

Note that we are not claiming that first-year calculus students could generate the above line of reasoning on their own. However, we believe that, like Thompson's conceptualization of integration as an accumulation (Thompson & Silverman, 2008), it can form the basis of instruction aimed at student understanding. In particular, the

[9] This footnote is not part of the original paper. This conceptual analysis addresses implicit differentiation only in contexts in which the relation defined by the equation actually is a function and each side of the equation represents a function. Possible areas of expansion for this conceptual analysis include addressing situations in which a non-function relation is being defined by the equation (e.g., without the constraint of y>0) in which implicit differentiation is valid, as well as situations in which implicit differentiation is not valid (e.g., the equation defining a discrete relation that is not differentiable over an interval).

Conceptual Steps cohere with introductory calculus insofar as derivatives are of functions and the equation being differentiated is an equation about functions. Both Thurston (1972) and Staden (1989) suggested that the legitimacy of differentiating equations is rooted in function equality. However, as we discussed, much of the current education research on related rates and implicit differentiation problems overlooks the legitimacy of the procedure.

We wish to emphasize an aspect of our conceptual analysis. The conceptual analysis was grounded in the fact that the standard introductory calculus curriculum that precedes implicit differentiation treats derivatives as being of functions. Therefore, how an introductory calculus student understands implicit differentiation should involve derivatives as being of functions (rather than of, say, expressions) and differentiation rules being applied to functions. This is consistent with how differentiation rules are introduced in many textbooks; for example, the sum rule states that the derivative of the function f+g is the derivative of the function f plus the derivative of the function g. Thus, for example, when students apply the sum rule, power rule, and chain rule to x^2+y^2, they must think of these rules as applying to functions. This approach is supported by research that suggests that some students need to see equations explicitly written with standard function notation before differentiating (Engelke, 2008). This involves viewing each side of (*) as representing a function, and therefore viewing (*) as expressing function identity.

4.1.2 When does the Equation Serve as a Function Definition?

Typically, in calculus textbooks, implicit differentiation and related rates problems are introduced with little distinction between the two. Accordingly, we illustrate these differences by briefly reformulating the problem at hand as a related rates problem. We keep the previous ladder situation, but this time we specify that the top of the ladder is sliding down the wall at 0.1 m/s and ask the student to find the speed of the bottom of the ladder. We call this new problem *Ladder Problem 2*, a related rates problem. Procedurally, solving Ladder Problem 2 is very similar to solving Ladder Problem 1; it involves differentiating the same equation ($x^2+y^2=9$) but with respect to t (time) instead of x.

$$
\begin{aligned}
&x^2 + y^2 = 9\\
&2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0\\
&2x\left(\frac{dx}{dt}\right) + 2y(-.1) = 0\\
&2x\left(\frac{dx}{dt}\right) = 2y(.1)\\
&2x\left(\frac{dx}{dt}\right) = .2y\\
&\left(\frac{dx}{dt}\right) = \frac{y}{10x}
\end{aligned}
$$

Computation C

We use Computation C to stress the procedural similarity between Ladder Problem 1 (Computation A) and Ladder Problem 2. This similarity may contribute to the common conflation of implicit differentiation with any differentiation of equations using Leibniz notation. Consider, for example, Hare and Phillippy (2004), who explain “Implicit differentiation must be used whenever the differentiation variable differs from the variable in the algebraic expression (p.9).” Thus, the authors appear to be conflating implicit differentiation with use of the chain rule. That is, when we ‘take d/dt’ of $x^2+y^2=9$, we have to use the chain rule with x and y. Similarly, when we ‘take d/dx’ of the same equation, we have to use the chain rule with y. So, viewed procedurally (without attending to the legitimacy of the procedure), these problems are almost identical.

However, as we now illustrate, the conceptual operations entailed in Ladder Problem 2 are not identical to those in Ladder Problem 1. In fact, Ladder Problem 2 is not truly an implicit differentiation problem. In order to provide this comparison, we solve Ladder Problem 2 in a way that an introductory calculus student such as John might understand.

We begin as before with the equation $x^2+y^2=9$. Unlike in Ladder Problem 1, where we conceptualized y as a function of x, we conceptualize y and x as functions of t: for all t, $(x(t))^2+(y(t))^2=9$. Similar to our earlier discussion, if we give labels to the functions on the left and right sides of the equation, say $m(t)=(x(t))^2+(y(t))^2$ and $r(t)=9$, then the equation $x^2+y^2=9$ simply asserts that m(t) and r(t) are equal for all t. As with Ladder Problem 1, this statement of function equality implies that m'(t)=r'(t). So: $2x(t)x'(t)+2y(t)y'(t)=0$, which, since we know $y'(t)=-0.1$, yields: $x'(t)=0.1y(t)/x(t)$. We summarize this reasoning in Computation D.

$$
\begin{aligned}
&(*) x^2 + y^2 = 9, y > 0 \\
&m(t) = (x(t))^2 + (y(t))^2 \\
&r(t) = 9 \\
&m(t) = r(t) \\
&m'(t) = r'(t)
\end{aligned}
$$

Computation D

Unlike with Ladder Problem 1, solving Ladder Problem 2 does not involve Conceptual Step 1, as there was no function of t implicitly defined by the equation. This difference is what distinguishes related rates from implicit differentiation problems. Instead, the equation $x^2+y^2=9$ describes a relationship between yet unspecified functions of t. We carefully contrasted the two types of problems to emphasize that, while these problems have similar procedural solutions, they differ in Conceptual Step 1 when attending to the legitimacy of differentiating the equation.

4.1.3 Summary of the Conceptual Analysis

We are addressing an aspect of deep procedural knowledge, specifically, how introductory calculus students can understand the legitimacy of differentiating each side of an equation. In order for such a student to see the legitimacy of this procedure for differentiating an equation, they must understand the equation as asserting a statement of function equality (Conceptual Step 3). Doing so requires viewing each side of the equation as defining a function (Conceptual Step 2). Viewing each side of the equation this way in Ladder Problem 1 (an implicit differentiation problem) requires a significant conceptualization (Conceptual Step 1) that Ladder Problem 2 (a related rates problem) does not. Yet, when viewed as symbol manipulation exercises, Ladder Problems 1 and 2 are nearly indistinguishable, which could explain why some educators appear to conflate related rates problems with implicit differentiation problems. Importantly, the conceptual analyses described above present implicit differentiation (and related rates) problems in a manner that makes the role of functions more transparent and foregrounds the reasons for the legitimacy of the procedure.

4.2 Potential Student Struggles: Insight from Previous Studies

Viewing an equation as implicitly defining a function (Conceptual Step 1) might be problematic for students. Notice that defining f takes the form of ‘f(x) is the y such that the proposition P(x,y) is true.’ Conceiving of a function definition that involves outputs according to whether or not a proposition is true requires a process conception of function, which many students have not yet developed (Breidenbach, Dubinsky, Hawks, & Nichols, 1992).

As discussed above, a key aspect of understanding the legitimacy of differentiating an equation is conceptualizing that equation as expressing function equality (Conceptual Steps 2 and 3). Doing so requires not thinking of the equation as merely expressing numerical equality. For example, thinking of $x^2+y^2=9$ as referring to fixed specific values of x and y is antithetical to thinking of it as a statement of function equality. Knowing when an equation does or does not express only numerical equality seems to be difficult for students (White & Mitchelmore, 1996; Engelke, 2004). Engelke (2004) argues that a major student impediment in solving related rates problems involves difficulty in viewing equations and problem situations *covariationally* (in the sense of Confrey & Smith, 1995). Engelke found that students tended to label their diagrams with constants when they should have been using variables. These observations suggest that students struggle with Conceptual Steps 2 and 3 – if students are viewing x and y as constants, then they are not viewing $x^2+y^2=9$ as representing a statement about function equality. However, while viewing equations covariationally might be necessary for understanding the legitimacy of differentiating equations, it is not sufficient. As we will illustrate with John's clinical interview, a student might think of an equation as expressing a relationship between varying quantities while not considering the role of functions. Students cannot think of function equality if they are not even thinking of functions, impeding Conceptual Step 3.

Even if a student views functions as being involved (Conceptual Step 2), they still might struggle with the notion of function equality (Conceptual Step 3). Mirin (2018) suggests that a strong understanding of function equality may be absent in a number of calculus students. Specifically, students struggle with inferring that sameness of graph (pointwise equality) implies sameness of function. Thankfully, students need not have a fully developed sense of function equality in order to achieve Conceptual Step 4; they need only reason that because the function on the left side and the function on the right side agree on all inputs, their respective derivatives agree on all inputs (that $m(x)=r(x)$ for all x implies $m'(x)=r'(x)$ for all x). However, Mirin (2018) reports that this inference might be especially problematic for students. When presented with a piecewise-defined version of the function defined by $y=x^3$, only 32% of the first-semester calculus students who assessed it as sharing a graph with the function $y=x^3$ believed that their derivatives agreed at a particular point. So, not only do some students struggle to infer function equality, but some students did not use equal graphs on an interval to infer equal derivatives at a point on that interval. Hence, even if students were to consider equations that they differentiate as statements of function equality, it is not clear that they would infer that the derivatives of those functions are also equal. Consequently, they would not understand why taking the derivative of both sides of an equation is ever a valid procedure (Conceptual Step 4).

The literature discussed above provided insights into how students might understand the Conceptual Steps. Specifically, we used our conceptual analysis as a framework for investigating the literature to delineate where students might struggle with the necessary conceptualizations for understanding the legitimacy of implicit differentiation. Although the existing literature provides guidance regarding where students might struggle, it does not directly address student understanding of the legitimacy of differentiating equations. Our conceptual analysis is itself novel and provides a lens for incorporating information from previous studies, and as we will see in the next section, it will also provide a lens for making sense of how students understand (the legitimacy of) implicit differentiation.

4.3 A Clinical Interview

Equipped with our conceptual analysis, we return to John's interview from the opening vignette. At the time of the interview, John was enrolled in second-semester introductory calculus at Anonymous State University (ASU). He had, the semester prior, taken first-semester introductory calculus. John was a successful calculus student in that he earned a 'B' in his first-semester calculus course. John had learned about derivatives as functions and being of functions (we reviewed videos of his lectures). In the first-semester calculus course, John had learned to take the derivative of both sides of an equation in solving 'implicit differentiation problems' and 'related rates problems' in a similar procedural way as illustrated in Computations A and C, without an explanation for why this procedure works.

4.3.1 Interview Protocol

The interview was an hour-long semi-structured clinical interview aimed at discovering and identifying the student's mental structures (Ginsburg, 1981). Throughout the interview, John was asked to think about ideas regarding implicit differentiation and function equality that he had perhaps not reflected on before. John might have never considered these matters and might therefore have improvised explanations. Four prompts guided the interview, as shown in Figure 4.1 below. Only the most pertinent highlights of the interview are reported here.

Prompt 1. What is your meaning for implicit differentiation? How do you interpret the word "implicit" in this situation?

Prompt 2. Find $\frac{dy}{dx}$ for $x^2 + y^2 = 1$ when $y > 0$.

Prompt 3. A 10-foot ladder leans against a wall; the ladder's bottom slides away from the wall at a rate of 1.3 ft/sec after a mischievous monkey kicks it. Suppose $h(t) =$ the height (in feet) of the top of the ladder at t seconds, and $g(t) =$ the distance (in feet) the bottom of the ladder is from the wall at t seconds. Then $(h(t))^2 - 100 = -(g(t))^2$. How fast is the ladder sliding down the wall?

Prompt 4. True or false: Suppose $f(x) = g(x)$ for all values of x. Then $f'(x) = g'(x)$.

Figure 4.1. Four Prompts that Guide the Student Interview.

4.3.1.1 Prompt 1. In response to Prompt 1, John expressed that he did not remember exactly what the procedure of implicit differentiation was, but that it was something that must be done when there is no function (due to failure of the vertical line test). He did not have an idea of what the implicit referred to in implicit differentiation, suggesting a difficulty with Conceptual Step 1.

4.3.1.2 Prompt 2. John did not have an idea of how to approach Prompt 2, so the interviewer reminded him that $x^2+y^2=1$, $y>0$ defines the top half of a circle and that a particular procedure was done in his first-semester calculus class: replacing y with f(x) before differentiating the equation. The interviewer then asked him to elaborate on what $x^2+(f(x))^2=1$ means. He explained that 1 is "the radius", and having f(x) (in place of y) "makes the computation easier". He was then asked explicitly what it means for the right-hand side of $x^2+(f(x))^2=1$ to equal the left hand side, and he responded "It's a circle. I just see a circle." When prompted to explain what the circle has to do with the equation, he graphed two parabolas on the same axes: a sideways parabola (representing y^2) and an upright parabola (representing x^2) and asked “how is that a circle?”. In this situation, it seems that John was not thinking of y (or f(x)) as a function of x. Instead, he seemed to be thinking of y^2 as denoting the parabola associated with the equation $x=y^2$. This association indicates that he was engaging in what Moore and Thompson (2015) call ‘shape thinking’ (associating shapes with symbols), rather than understanding the equation as a statement of function equality (Conceptual Step 2).

After reasoning with a graph was unhelpful to John, he began considering specific values of x and y, observing that “as they change together, in this equation here, they have to change together in such a way that it always equals 1.” It seems that here, John began thinking covariationally, but it was still not clear how John’s approach related to his understanding of the legitimacy of the differentiation procedure.

As discussed in the opening vignette, John justified the procedure of implicit differentiation by drawing an analogy to algebra. He subsequently related the procedure of taking d/dx to inferring equal rates of change: “if you take the rate of change of this [left side], it is the rate of change of this [right side]. They’re equal to each other, so the change in one is gonna be the change in the other.” Since John believed the inference of equal rate of change came from something being equal, to get at what that something was, the interviewer asked him what happens if he differentiates each side of $x=1$. As discussed in the vignette, John noticed that it results in $0=1$, which he said did not make sense. It appears that John was struggling with Conceptual Steps 2, 3, and 4; he was not viewing the equation as an equation of functions (Conceptual Step 2 and 3) and, despite being explicitly prompted, did not justify use of the differential operator (Conceptual Step 4).

4.3.1.3 Prompt 3. Prompt 3 is a related rates problem, like Ladder Problem 2. In this prompt, function notation is provided explicitly in order to encourage the student to talk about functions, and an animation of the problem situation is provided in order to give the student a context to refer to (Engelke (2004) suggests that having a dynamic image of the problem situation is helpful for helping students reason about related rates problems). John was reminded that he could take the derivative of both sides of the equation, and he did so. He explained that the ladder's distance from the wall, g(t), and the ladder's distance from the floor, h(t), "change together". When pushed, he did not say why taking the derivative of both sides is a valid procedure. Instead, John continued to express an understanding of the two distances as changing together with time and did not mention each side of the equation as representing a function:

> *"We take the derivative of both sides because [pause] you need to have the two rates change together, in order for this scenario to work. Because if they don't with respect to each other, then uh [pause] it just doesn't hold true. So we do it on both sides in order to have the scenario change together and everything stay true to itself [pause] maybe."*

Even when asked *what* exactly is being differentiated on the left hand side, John talked about only h(t) as a function and did not seem to consider the entire left hand side as representing a function. This suggests that John was struggling with Conceptual Step 2. The only further justification he gave for the legitimacy of differentiating both sides was covered in the opening vignette: "because math teacher says so".

4.3.1.4. Prompt 4. Since John was not using the language of functions on his own, the interviewer decided to move to Prompt 4 in order to see if he would relate taking the derivative of each side of an equation to an inference from function equality. John almost immediately provided what he viewed as a counterexample to the assertion that if two functions are equal for all inputs, then their derivatives are also equal for all inputs. By misapplying the quotient rule, he argued that $f(x)=x$ and $g(x)=2x/2$ are equal for all values of x but have different derivatives, which is the antithesis of Conceptual Step 4. John continued his explanation that, if he were to simplify g(x) prior to differentiating it, he would end up with the same derivative as that of f(x). However, he noted that simplification before finding derivatives is not permitted in his calculus class. John's response highlights that he had a fundamental misunderstanding of how the derivatives of equal functions relate, a key aspect in understanding the legitimacy of applying the differentiation operator. This shows us that, for John, Conceptual Step 4 is problematic. Even if he had viewed equations he was differentiating as statements of function equality, he still would have the obstacle regarding understanding the differentiation inference. In other words, John not only struggled to understand *why* it was acceptable to differentiate equations of functions, but he also misunderstood *that* it was acceptable to differentiate such equations.

4.3.2 Interview Results: Discussion

The fact that John reasoned covariationally, yet still struggled with Conceptual Steps 1- 4, indicates that understanding the legitimacy of differentiating equations (and hence implicit differentiation) is a significant challenge for John. Specifically, it provides an existence proof that there is more to understanding implicit differentiation than correct mathematization, covariational reasoning, and algorithm implementation. The analysis of

John's work also serves to demonstrate the utility of our Conceptual Steps framework for highlighting which components of John's knowledge could benefit from reinforcement.

4.4 Discussion

Being aware of the difficulties students might encounter in developing the conceptualizations described in the conceptual analysis (the Conceptual Steps) can be useful to calculus instructors when working with their students. We wish to emphasize that the conceptual analysis was helpful in delineating which conceptualizations impeded John when formulating his explanations. Importantly, our work reformulates the topic of implicit differentiation in a way that coheres with typical calculus curriculum. Developing the topic in this way can serve to enrich the connections students make between implicit differentiation and the differentiation that precedes it. Two natural questions emerge from this work. First, how might the conceptual analysis presented here inform the creation of implicit differentiation and related rates units? Adequately addressing this question involves both the development of such units and studying their implementation. Second, what can we discover from analogous work in alternative instructional paradigms such as infinitesimal calculus? The conceptual analysis used in this study was predicated on derivatives being of functions, which is the dominant calculus instructional paradigm.

IS EQUALITY REALLY SYMMETRIC?

In the general literature review earlier, I discussed at length the body of mathematics education research that addresses mostly elementary school students' understanding of =[10]. One thing I addressed is the matter of symmetry; although = is a symmetric relation, it seems that many children do not view it as such. This is most evident with Rule Violation (i) (Table 1 of Mirin, 2020a), in which students reject equations of the form 5=2+3 and accept equations of the form 2+3=5. It seems obvious that most mathematicians would not take exactly this viewpoint. In fact, I would suspect that any mathematician would not hesitate to say that a=b [11] and b=a are truth-functionally equivalent. However, do mathematicians actually *use* the equals sign in a way that is symmetric? Do they feel that a=b and b=a truly mean the same thing? The fact that experts tend to interpret mathematical texts differently from novices (Veel, 1999) suggests that exploring the meaning of the equals sign amongst experts is a worthwhile endeavor.

Recall the earlier philosophical discussion about Frege and equality. Frege spent a long time trying to dissect the *meaning* (not just criterion of truth) of a=b. One thing Frege never appeared to address was the issue of symmetry. In both his early and later writings, Frege did not distinguish between the meanings of a=b and b=a. It seems reasonable to suspect that meaning is something more than just truth-functional value. Indeed, it was the meaning of "a=b" that Frege puzzled over, *not* the criterion for truth.

[10] There are several situations in which I am making use-vs-mention errors by omitting quotation marks. For example, there are several equations (e.g., "2+3=5") that are being mentioned (I am not asserting that two plus three equals five) where, for the sake of readability, I often omit quotation marks. This is the same convention that Ernest (2008) follows and describes.

[11] This is not an exponent. I am using "a" and "b" as schematic variables in the sense of "a" and "b" to be stand-ins for any terms/nouns.

Furthermore, Weber and Alcock (2005) found that mathematicians attend to more than just truth functions, at least in the case of the material conditional.

Throughout this document, I have been using the word "meaning" without defining it. This has been somewhat intentional – thoroughly defining what "meaning" means is a longstanding philosophical issue that I do not intend to solve here (Gasparri & Marconi, 2019). Thompson (2013b) discusses the very paradoxical and recursive nature of discussing the meaning of the word "meaning". Consistent with a constructivist perspective, this study takes the approach that the meaning of a word or symbol (in this case, =) is not something that is objectively "out there". Instead, there are two considerations when addressing meaning: usage and understanding. The former (usage) can be thought of as external to an individual's mind – words have meanings within a community of practice. The meaning of a word (or phrase) is tied to its usage. People give words meaning based on how they understand and communicate with words and what usages of words they accept or contest from others. This is consistent with the description Wittgenstein gives of word meaning in a community of practice (Wittgenstein, 1953/2009). Here, the relevant community of practice is the mathematics community. The latter (understanding) can be thought of as internal to an individual's mind. I take the perspective that meaning is closely tied to understanding; how someone understands a word (or phrase) is essentially their meaning of a word. This approach is consistent with radical constructivism and is described in more detail in Thompson (2013b).

Equipped with this operationalized characterization of meaning, I can now characterize in further detail what this study is about. It starts by addressing the question

(1) Do mathematicians use and understand the equals sign symmetrically?

This topic is not worth studying in-depth if the answer to (1) is a straightforward "yes". In this study, I show that the answer is "no". Establishing the existence of this asymmetry paves the way to learn more about mathematicians' asymmetrical usage of the equals sign. Accordingly, the main emphasis of this study is on the following question:

(2) In what ways is the equals sign used asymmetrically? What rules and expectations govern the ordering of terms in equations?

Research question (2) can be rephrased as: "what are the norms that govern the ordering of terms in equations?[12]". It is worth elaborating *why* I hypothesized that the answer (1) is "no". A quick informal search of textbooks shows that when f is defined as a homomorphism, the equation written is almost always $f(a+b)=f(a)+f(b)$ rather than $f(a)+f(b)=f(a+b)$. Rules for derivatives are almost always written from left-to-right as $(g+f)'=g'+f'$. Similarly, when long computations are presented, it seems that one tends to work left-to-right from *known* (or perhaps *given*) to *unknown* (or perhaps *derived*) results. My study (1) establishes the existence of ordering norms, as well as (2) provides evidence regarding what these ordering norms are.

5.1 A Discussion of Literature

As alluded to earlier, there is evidence that mathematicians do not attend only to truth-functional value. There are potentially other interests that govern human utterances. Van der Henst et al. (2002) argue that the Gricean maxim of truthfulness is not the only interest governing utterances. Instead, truthfulness and accuracy must be balanced with

[12] This particular study concerns the English-speaking mathematics community. It does not address issues of asymmetry regarding equations embedded in texts in other language, such as those that read right to left. While the issue of other languages is interesting, it is beyond the scope of this dissertation.

relevance (to the listener). The authors report on an empirical study that suggests that relevance motivates people to round when giving the time. It bears mentioning that Ernest (2008b) also identifies relevance as an important factor when communicating mathematics. We can see how relevance might influence the order in which an equation is written. For example, due to the fact that we read left to right, the distributive law written as x(y+z)=xy+xz might be more relevant to someone who is doing a mathematics problem that requires distribution than the same law written as xy+xz=x(y+z) would. This is because such a person might want to substitute the term x(y+z) with the term xy+xz. Similarly, the same law written as xy+xz=x(y+z) might be more relevant for a student whose task is to factor. Indeed, the results demonstrate that this was a concern for the participants. While at first this discussion of relevance could appear to conflict with Gricean pragmatics, the authors explain that "human communication involves the attribution of mental states by the interlocutors to one another" (p.465). This attribution is closely related to the ideas of constructivism discussed in the introductory chapter of this document; the utterer is essentially working with second-order models; the utterer considers second order models to anticipate the relevance of the claim to the listener. What this tells us is that a mathematician might consider the mental state of the listener or reader, so the mathematician's beliefs about the reader might influence their decision to present an equation in a certain order. Indeed, the interview data show that this is the case.

Attending to issues other than truthfulness helps us begin to answer question (2) above; *if* the equals sign is understood asymmetrically then why might someone use one ordering over another? Clearly the answer is not "one is truer than the other," so there must be other motivations at play. This is a big “if”, which is why Research Question (1) is listed separately. Now I move to the topic of linguistics in considering how and why order might affect the meaning of equations.

When we read a sentence, we are constrained by our language, space, and time. That is -- some words have to come before other words. Ernest (2008b) explains “in any form of representation, there is always an ordering present and this structures the access and role of readers” (p.44). Consider the various ways one could read “a=b”. One could read it as, for example, “a equals b”, “b equals a,” “a and b are equal”, or “b and a are equal” (note that something like this is the case for any equivalence relation, not just equality). While these four sentences are equivalent and perfectly acceptable ways of reading aloud “a=b”, they might have slightly different meanings or connotations. The first sentence, “a equals b”, seems to emphasize a over b – a is the subject of the sentence, and b is not. Similarly, “a and b are equal” also seems to emphasize a but perhaps less so; a and b are both included in the subject, but a comes first.

Halliday’s *Systemic Functional Linguistics (SFL)* informs my perspective. (Schleppegrell, 2004, 2007; Veel, 1999). The notions of theme and rheme highlight the fact that in a sentence, some words come before others, and hence a reader experiences some words before others. For example, as discussed above, in the sentence “a=b”, the fact that “a” comes before “b” might give the impression of greater importance to “a”. The theme is whatever object comes first in a clause: “*theme* can be identified as the elements up to and including the first experiential element at the beginning of a clause” (Schleppegrell, 2004, p.68). The theme of a sentence starts at the first part of a clause and ends after the first noun has been mentioned. For example, the theme is underlined in the following clauses:

Actually, the number three is a factor of six.

At the start of an expression is often a parenthesis.

Six has a factor of three.

Three is a factor of six.

A is equal to B.

The numbers A and B are equal

A and B are equal.

The quadratic equation was solved by the student.

We now have terminology for illustrating a potential difference of meaning between “a and b are equal” and “b and a are equal” and the accompanying asymmetry in equation meaning.

Accompanying the idea of *theme* is that of *rheme*. The *rheme* of a clause is the portion that is not the theme (Schleppegrell, 2004). Halliday uses the notions of theme and rheme to discuss how information in text is structured; for example, the rheme of a clause often becomes the theme of the following clause. The idea that part of the rheme becomes the theme is part of what makes a text effective and coherent. This theme/rheme structure supports the hypothesis that mathematicians might use the equals sign asymmetrically in a text; if “b” is the rheme of a clause and the author is to next claim that b and a are equal, then “b=a” (or “b and a are equal”) rather than “a=b” fits the theme/rheme structure just described. This is because “b” starts as the rheme and becomes the theme. Similarly, we can see the theme/rheme structure in running equations such as “a=b=c”. If we interpret this to mean “a=b and b=c”, then we observe that “a=b=c” is a sentence with two clauses: “a=b” and “b=c”. In the first clause, “b” is part of the rheme, and in the sentence clause it is the theme.

The shift from theme to rheme often involves *grammatical metaphor* through nominalization (Schleppegrell, 2004). More generally, grammatical metaphor involves a shift in function of a word or idea. Veel (1999) assesses that mathematical text is, in particular, dense with grammatical metaphors. One function that grammatical metaphor serves is to turn processes into objects, which is called *nominalization* (Schleppegrell, 2004). Consider, for example the role of the words “invented” and “invention” in the following: “The telephone was invented. The invention of telephone created many opportunities for enhanced communication” (Schleppegrell, 2004, p. 73). Observe that the term “invent” shifts from being a verb “invented” to a noun “the invention” while simultaneously shifting from theme to rheme. Like the ideas of theme and rheme, grammatical metaphor serves to structure information in texts. This process-object duality is familiar in mathematics education (Sfard, 1992). While normatively it would appear that on either side of the equals sign is a noun that represents an object, I leave open the possibility that one side could represent a process. For example, in Mirin, (2020a), I discuss evidence that in equations like 2+3=5, some students view the 2+3 as a process rather than as a number.

I use the idea of nominalization to consider potential asymmetry. Consider, for example, an instance of the sum rule for derivatives: (f+g)’=f’+g’. It is possible that the instance of the prime symbol (derivative) on the left-hand side serves more as a verb than a noun. That is, the prime symbol might be understood as referring to the process of differentiation - whereas, on the right-hand side, the prime symbol might refer to the derivative function – an object, rather than a process. Thus, the idea of grammatical metaphor (specifically, nominalization) could account for the order in which equations are written. Gray and Tall's (1994) description of *procept* captures some similar ideas as the construct of nominalization. A procept is the idea of thinking of something as both a process (which we associate with verbs) and as an object (which we associate with nouns). We might understand 2+3 as both a number (an object) as well as the addition process, and which understanding we are using in any moment might be context dependent. Thus, thinking of 2+3 as a process and then as an object is a psychological parallel to the notion of nominalization. For the case of derivatives, the idea of procept accounts for how one can understand one side of the equation as representing a differentiation process and the other as representing a derivative function. The literature (e.g. Behr et al., 1980; Falkner et al., 1999 -- this is discussed at length in the general literature review) seems to suggest that young students tend to view the left hand side of equations as representing processes (e.g., problems to be performed) and the right hand side as representing objects (answers or results of processes). As we see in the results of this study, this idea extends to some mathematicians as well.

From Halliday's SFL is the construct of *relational clause*, namely a clause that expresses a relationship between two objects (Veel, 1999). Mathematical texts are dense with such clauses (Veel, 1999). There are two types of relational clauses: *attributive* and *identifying*. An *attributive* relational clause involves an asymmetric relation, whereas an *identifying* clause is a statement about identity and typically involves some conjugation of "to be". Our main concern initially appears to be with identifying clauses: "In an identifying clause, the process (often the verb *to be*) is the linguistic equivalent of the equals sign" (Veel, 1999, p. 196). However, part of what I am investigating is if experts truly do understand identifying relational clauses as symmetrical. Veel (1999) also includes a discussion of relational clauses (in the context of mathematical texts) as functioning to bridge something new to the students/readers with something they already know. Veel (1999) gives examples that involve the new information or term being introduced first, such as "The mean, or average, score is the sum of the scores divided by the number of scores (p.195)". This example highlights a degree of potential asymmetry of equations, which are symbolic relational clauses; the verbiage preceding "is" (which can be thought of as the left side of the sentence) involves an unfamiliar or more technical term ("the mean, or average score"), whereas the subsequent language (right side of sentence) involves terms familiar to the readers ("the sum of the scores divided by the number of scores"). This idea echoes how Frege discusses the informativeness of statements of identity; the reader is being informed that two different representations, in this case one familiar and the other unfamiliar, are in fact referring to the same thing. The results of this study, discussed later, demonstrate that mathematicians understand equations in this way.

Mathematics tends to be *multi-modal* (Veel, 1999; Schleppegrell, 2007; Ernest, 2008b). That is, the semiotics of mathematics involves various modes of presentation such as symbols, verbiage, and pictures. In other words, the written symbol "=", the written word "equals", and the spoken word "equals" can all be viewed as different modalities (Schleppegrell, 2007). In a mathematics class, a teacher frequently plays the role of mediator between symbolic and verbal forms (Schleppegrell, 2007). Schleppegrell

(2007) additionally makes the claim that a verbalized equation is more object-oriented than a written equation with the reasoning that a verbalized equation involves noun-dense phrases. While it is unclear what her grammatical analysis of written equations is that leads her to this conclusion, the role of translation from written to verbal is clearly important. As discussed earlier, there is more than one way that one might verbalize a=b, and some of these ways might not even use the word “equals”. Schleppegrell (2007) illustrates her point about verbalizations involving more object-oriented language by discussing the following equation: $a^2+(a+2)^2=340$. She translates it as “the sum of the squares of two consecutive positive even integers is 340” without justifying this translation (p.144). It is unclear why she chooses to, for example, refer to “two consecutive positive even integers” and neglect to mention that one of these integers is named “a”. Notice that in her translation, there are indeed noun-dense phrases (“the sum of the squares of two numbers”). It is unclear how one would do such an analysis of an equation in symbolic form. One way is to translate (excluding the parentheses) the symbols word by word “a squared plus a plus two squared equals 340”. This translation does not have such a complicated noun-dense phrase, and one might argue (relatedly) that it is less object-oriented by claiming that “plus” suggests an action (seeing as it is a verb) whereas “the sum” does not (seeing as it is a noun). As discussed earlier, the dual nature of viewing a symbol or idea as both an action and a process (a procept) is not foreign to mathematics education. Notice, additionally, that these two translations have different theme-rheme structures – the former has “the sum” as its theme, and the latter has “a” as its theme (and relatedly, the former appears to be about sums, whereas the latter is about the number a). Indeed, as I discuss later, the results of this study suggest that this is an asymmetry perceived by mathematicians.

Another idea from SFL has to do with the frequency with which certain linguistic items occur. Halliday (1985) states: “A speaker of a language has a fairly clear idea of the probabilities attached to stored items; he ‘knows’ (in other words it is a property of the system) how likely a particular word or group or phrase is to occur, both in the language as a whole and in any given register of the language” (p. xxii). To take a nonmathematical example, consider the norms for ordering of adjectives in the verbal register of English. The term “nice new house” sounds more natural and likely occurs more frequently than “new nice house” (Murphy, 2012). An English speaker has a general idea that “nice new house” occurs more frequently than “new nice house” but might not know why. As discussed in my introductory section to this study, certain equations seem to appear to be ordered consistently throughout textbooks. Observe that Halliday refers to “any given register” – in this case, we are concerned with the symbolic register. What the quote of Halliday tells us is that others might also perceive such consistency. Viewing a=b and b=a as different linguistic items in the symbolic register, a speaker might have a sense of which occurs more frequently and hence have a sense of what ordering norms or traditions exist. It is reasonable to believe that a participant has some sense of which orderings are typical and thus has a sense of when ordering norms are breached.

5.2 Methodology: A Breaching Experiment

I return to the research questions guiding this study. As discussed earlier and in Mirin (2020a), there is a body of literature establishing that students tend to view equality as asymmetrical. Since students might understand equations asymmetrically, this study's research questions follow naturally: do experts understand the equals sign asymmetrically, and, if, so, what are the norms that govern this asymmetry?

One aspect of the studies discussed earlier stands out as especially relevant to this study; Behr et. al (1980) describe six children who read aloud a sentence of the form 5=2+3 as "two plus three equals five" (differently than how it was written symbolically). One student asked the interviewer "do you read backwards?" Translating between the symbolic and the verbal modalities suggests that these children viewed 5=2+3 as a rule violation and assumed that it must have been an error. By breaking the "rule" that the "answer" should be to the right of the problem, the researchers gained data to suggest that this was indeed a rule for these children.

The technique of breaking rules in order to confirm their existence is known as a breaching experiment (Rafalovich, 2006). A breaching experiment is a technique in sociology research that involves breaking a purported social rule (without the subjects knowing that this is the intention of the researcher) and observing the subjects' reactions. The idea is that the subjects respond in such a way that reveals that they felt that there was a rule broken. Some mathematics educators use this research technique to confirm and explore the social norms governing classroom mathematics activity (see, for example, Chazan et al., 2012; Weiss et al., 2009). In this study, I use a breaching experiment as one technique within the context of individual task-based interviews (discussed further in Subjects and Methods)

As discussed above, breaching experiments are socially situated. Ernest (2008a) explains how mathematical texts (and hence equations) take place within a social context:

"mathematical signs or texts always have a human or social context" (p.5). This perspective is consistent with Halliday's SFL; one of Halliday's meta-functions[13] of text is the *interpersonal*, which encompasses the ways that interpersonal communication is reflected in the language of a text (Halliday, 1985). When someone is reading a text, a social interaction is taking place – one between the author and the reader. Additionally, in an interview, there is a social relationship between the interviewer and the interviewee; if I have an interviewee read a written equation that is "wrong" somehow, then there are potentially at least two social rules being breached: that the interviewer presents the interviewee with mathematics that conforms to the norms of mathematics as a whole, and that the text itself conforms to such a norm.

5.3 Subjects and Methods

All participants currently teach or have taught mathematics at a university. Nine participants were enrolled (Jacob, Larry, Kevin, Warren, Ben, Edgar, Patrick, Ming, Xena), all of whom have graduate degrees in mathematics and experience teaching mathematics at a university. Five (Jacob, Kevin, Patrick, Edgar, and Ben) are tenured mathematics professors, two of whom (Patrick and Ben) perform mathematics education research. Five (Ming, Jacob, Kevin, Patrick, and Edgar) have doctoral degrees in mathematics, and two (Warren and Ben) have doctoral degrees in mathematics education. Warren is a recent mathematics education Ph.D. who teaches mathematics at a community college, and Larry is a current mathematics Ph.D. student who works as a teaching assistant at a university. Xena is a mathematics instructor at a university. Edgar

[13] Halliday identifies three metafunctions of text: ideational, interpersonal, and textual. The constructs of "theme" and "rheme", discussed earlier, are part of the textual metafunctions.

is a recent mathematics Ph.D. who works in the tech industry and has formerly worked as a teaching assistant at a university and a tutor of graduate-level mathematics courses. Pseudonyms reflect the participants' perceived gender; all participants present as cisgender men, with the exception of Xena, who is a cisgender woman.

Each participant took part in a 60-95-minute-long individual semi-structured task-based clinical interview (in the sense of Clement, 2000). Following the interviews were an adaptation of member-checking interviews (Creswell, 2012), which I call "member checking emails". I wrote a narrative summary (details provided in the "Data Analysis Methods" section) of each individual participant. I then contacted them to ask if they were interested in reading their summary. If they answered "yes", I then emailed them the summary and asked if it conformed with their understanding of themselves. Four interviewees – Kevin, Patrick, Edgar, and Ben – participated in member-checking and confirmed that my narrative summaries were consistent with their understanding of themselves.

5.3.1 Tasks and Data Collection

Recall that this study seeks to 1) document the existence of norms for ordering and 2) learn about the nature of the norms by eliciting explanations for why one ordering is preferable. I now explain the interviews, associated tasks, and how they relate to the research questions. I discuss only a subset of tasks here – the remaining tasks are in the Appendix. The tasks were created based on hypotheses and intuition about the nature of these norms, the efficacy of each task in breaching a norm was not itself part of the study design. As such, no single task was essential for answering the research questions. Thus, not every task was done with every participant (see Tables 5.1 and 5.2). My goal was about establishing that there are norms for term ordering in equations and to explain these norms; my goal was not to count norms or provide a behaviorist experiment. Therefore, I selected tasks to provide opportunities to reveal and describe such norms. I chose tasks based on what came naturally in the conversation within the interview, the mathematical background of the interviewee, and the fruitfulness of the task in prior interviews. For example, while *SetTheory* did produce some interesting data (discussed in "Results" section), administering the task took much longer than expected; this is because it introduced new notation, including notation (the big-U symbol) that I did not realize would be unfamiliar to the participants. In some situations, the interview had simply gone on too long, so not all tasks could be completed. Sometimes (e.g., with Edgar) tasks that were intended as breaching tasks ended instead being given later in the interview within the context of explicit discussions regarding ordering. In some cases, like with Edgar in *Induction*, the participant did not recognize a breach, but I still wanted to learn more about how they understood ordering. In this case, I informed them that other participants found there to be a breach and asked them their thoughts.

It bears mentioning that the interviewees were not told that this study is about equations or symmetry. The interviewees were instead told that the study is about "how experts read, write, and interpret mathematics". Withholding this information is necessary for performing a breaching experiment.

The interview can be roughly partitioned into three sections. The first involves open-ended questions; these tasks all have a label starting with "O". A subset of such tasks are in Figure 5.1, and all the tasks are in Appendix B. Every open-ended task was completed with every participant. These are tasks that prompt the interviewee to write an equation. The purpose of this exercise was to establish that certain norms or habits about symmetry with respect to the equals sign do exist amongst the mathematical community. This means that experts (mathematicians) hold certain expectations about the order of terms in equations. I hypothesized that, for example, all of the subjects would write the sum rule for derivatives with the derivative of the sum on the left side of the equals sign (indeed, this was the case, as I discuss in the Results section). The interviewee was asked to read the text out loud, carry out the task, and then explain their answer. Follow-up questions included prompts for elaboration, such as "could you explain your answer?", "could you say more about that?", "how would you explain to someone what this equation says in other words?" and "what does this equation mean?". A selection of open-ended tasks is below. Working with *OEuler* was somewhat more involved, since not all participants were aware of which Euler's formula I was referring to. In this situation, I asked the participant to guess. If they requested more guidance, then I informed them that it was a formula with "sines, cosines, and an exponential".

OSumRule	OEuler
State the sum rule for derivatives, and write it down.	Write down and state Euler's formula.

OMice

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. Using t to denote time, $p(t)$ to denote the population, and r to represent the growth rate, write a differential equation expressing this relationship.

OIdentity

Suppose $\langle S, \star \rangle$ is a binary algebraic structure. What does it mean for an element $e \in S$ to be a left identity element?

Figure 5.1. Selection of Open-ended Tasks.

Since textbooks appear to traditionally write the sum rule for derivatives with the derivative of the sum on the left, Euler's formula with the exponential term on the left, the definition of an identity element having the operation on the left, and differential equations with the derivative on the left, I wanted to see if these norms extend to my participants.

The second portion of the interview is the portion in which the breaching experiment takes place. A sample of the tasks involved is shown in Figure 5.2, and Table 5.1 shows which tasks were done with which participants. All tasks are included in Appedix C. An asterisk indicates that for the particular participant, the task was not done as a breaching task. Instead, the task was visited later in the interview and served as a comparison task in which order and asymmetry were explicitly inquired about.

Table 5.1. Breaching Tasks Completed by Each Participant.

	DifferenceQuotient	Homomorphism	Exponents	Induction
Jacob	X	X	X	X
Larry	X	X	X	X
Warren	X	X		X
Edgar	X	X		X
Patrick	X	X		X
Xena	X	X		X
Kevin	X	X	X	
Ben	X	X		X
Ming	X	X		X

	SetTheory	Product Rule	Idempotent	Proof	Distributive
Jacob	X	X	X	X	X
Larry		X	X	X	X
Warren		X	X*	X	X
Edgar	X	X	X	X	X
Patrick		X	X	X	X*
Xena		X	X	X	X
Kevin	X	X	X		
Ben		X	X	X	X
Ming	X	X	X	X	X

These tasks involve having the interviewee read mathematics text that includes an equation. That equation is reversed from the way in which it typically appears in textbooks, with the intention of violating an order norm. I chose equations that seemed wrong or atypical when presented in the reversed ordering. I then discussed and confirmed these task designs with another mathematician. While I had not predicted every ordering norm that these tasks violated, I began with some hypotheses about what ordering norms might be breached while creating these tasks. Thus, hypotheses about order norms together with intuition guided me in task design. The hypotheses that guided the task design are as follows: *ordering should abide by tradition, equations should go left to right from complex to simple, calculations go left-to-right-top-to-bottom, when proving x=y one should start with x and end with y, some equations are rules for calculation.* As discussed in "Results", all of these norms with the exception of "when proving x=y one should start with x and end with y" were evoked within the interviews.

For example, the task *Idempotent* (Figure 5.2) has $p=p\star p$, whereas $p\star p=p$ appears to be the equation texts typically use. This violates the potential norm that an operation should be on the left side, or that a simplified expression be on the right. In *Homomorphism*, the equation is written with $\phi(x\star y)$ on the right, whereas it usually appears on the left. While tradition was the only norm I was attempting to violate with this particular task, the results reveal that there were other norms violated. Similarly, while *Distributive* was designed only with the idea of violating tradition in mind, implementing the task revealed norms regarding substitution. The task *Exponents* was designed with the simplification heuristic in mind; there's a sense in which one might consider a^{x+y} to be more simplified than $a^x a^y$, but this particular norm was not evoked in this task.

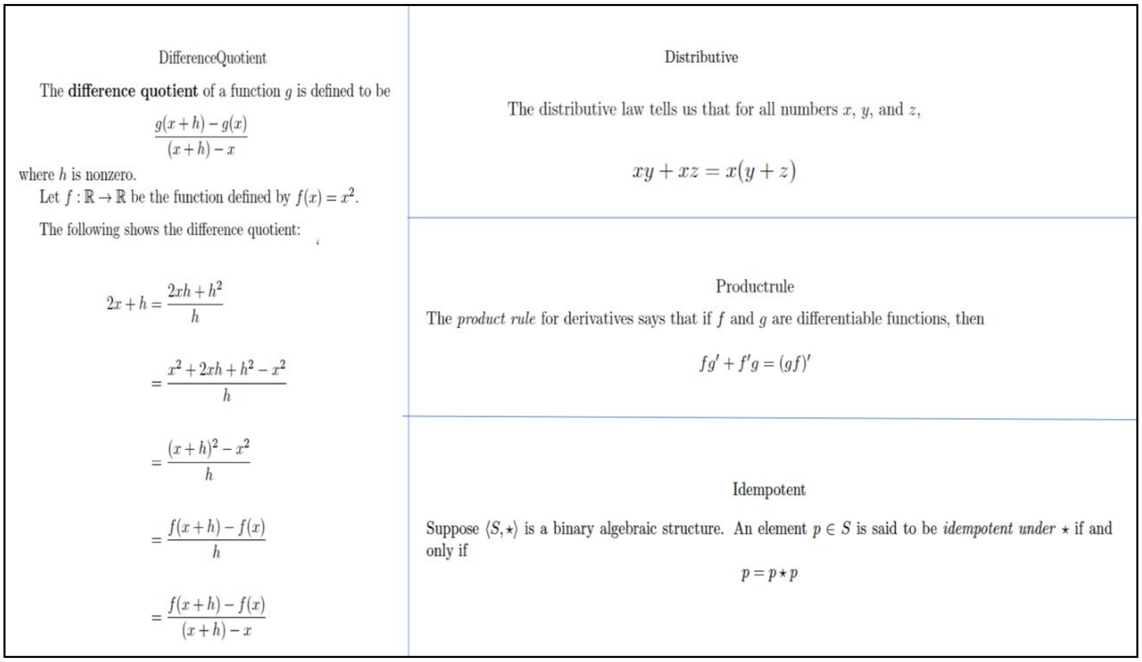
Other tasks were designed to reveal the norm that calculations go left-to-right-top-to-bottom: that is, that on the left or beginning of a running equation is something that is given or presented to the problem-solver and on the right is something derived or calculated. For example, *DifferenceQuotient* includes a string of equations that starts with 2x+h and ends with (f(x+h)-f(x))/(x+h)-x. Similarly, *Induction* includes a string of equations that starts with $(n^3-n)+3n(n+1)$ and ends with $(n+1)^3-(n+1)$. This string of equalities was obtained by taking and reversing the a string of equalities that appears in a textbook (Stankova et al., 2008). As the results discuss, these particular tasks revealed several norms and not just those surrounding the idea of calculation. The task *SetTheory* was designed to violate two different norms. It was taken from a set theory textbook (Enderton, 1977) that shows that the set $\bigcup a^+$ is equal to the set a by starting with $\bigcup a^+$ and working to a. My version of the tasks reverses the computation; it still purports to show the same equality but it does so by starting with a and ending with $\bigcup a^+$. This was intended to violate the norm that when proving x=y, one should start with x and end with y, as well as the simplification heuristic; there's a sense in which $\bigcup a^+$ computes or simplifies to a. Similarly, the task *ProductRule* was designed to violate not only the tradition that the product of the sum typically occurs on the left in its presentation, but to also evoke the asymmetrical meaning that the equation is a rule for calculation.

One breaching task, *Proofs*, is somewhat different from the others. For the others, I simply reversed equations, so that a=b was changed to b=a and a=b=c=d=e was changed to e=d=c=b=a. With *Proofs*, I change a=b=c=d=e (the form of the proof presented in a textbook, Fraleigh, 2003) to d=b=e=a=c. In other words, the ordering of terms was not simply reversed; it was jumbled. My goal with this task was to verify the following related norms; that mathematicians do *not* read a=b=c=d=e as "a, b, c, d and e are all equal to each other" but instead read it as a conjunction of equations, as well as the norm that mathematicians are concerned with not only truth but also deducibility and inference when reading a proof. In other words, my goal was to confirm that mathematicians read it as "a=b and b=c and c=d and d=e, hence a=e". My purpose for introducing such a long string of equations is to explore the idea that the reader experiences the left side of the equation first – the separation between the left (start) and right (end) of the string of equalities is larger with several terms. Furthermore, results of this task help highlight the expected finding that mathematicians care about more than just truthfulness.

The purpose of these tasks is to see if the interviewee reads the equations from right to left (like the child did in Behr et al., 1980) or remarks that the equations are reversed in some way. It also provides an opportunity for the interviewees to discuss their thoughts about the way the equations are ordered.

Like the first portion, the interviewee is prompted to read the text aloud and is asked follow-up questions for elaboration. They are additionally sometimes asked their opinion on the equation, such as "what do you think of this equation?" and "would you write it differently?". These latter questions are included for the interviewees who do not mention breaches on their own. It seems reasonable to believe that some mathematicians might notice a breach but just not mention it – they might instead only focus on mathematical correctness. Just because you would have done it differently or find it unconventional doesn't mean you would necessarily remark on that observation. Indeed, I found evidence of this with at least one participant.

In the cases in which the participant mentioned a breach (in particular, in the tasks *DifferenceQuotient* and *Induction*), I made attempts to repair the breach and then inquired further. For example, in *DifferenceQuotient*, some participants explained that they imagined the context in which students are learning how to compute difference quotients, and therefore the 2x+h should not be mentioned right away (discussed in more detail in "Results"). In these cases, I subsequently modified the task to be in the context in which the student has already worked with derivatives and is proving that the derivative of $f(x)=x^2$ is $f'(x)=2x$. Similarly, in *Induction*, several participants remarked that the term 3n(n+1) appears to "come out of nowhere". To repair this breach, I changed the task to move the sentence "Since either n or n+1 is even, 3n(n+1) is divisible by 6" to appear before the string of equations; this way, the 3n(n+1) in the string of equations no longer "came out of nowhere". In other words, this investigated whether the breach could be amended without reversing the order of the equations, or whether the breach was fundamentally tied to the equation order. This provided the opportunity to learn more about what, exactly, was being breached, which therefore revealed more information about reasons for asymmetry.



DifferenceQuotient

The **difference quotient** of a function g is defined to be

$$\frac{g(x+h)-g(x)}{(x+h)-x}$$

where h is nonzero.

Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=x^2$.

The following shows the difference quotient:

$$2x+h=\frac{2xh+h^2}{h}$$

$$=\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\frac{(x+h)^2-x^2}{h}$$

$$=\frac{f(x+h)-f(x)}{h}$$

$$=\frac{f(x+h)-f(x)}{(x+h)-x}$$

Distributive

The distributive law tells us that for all numbers x, y, and z,

$$xy+xz=x(y+z)$$

Productrule

The *product rule* for derivatives says that if f and g are differentiable functions, then

$$fg'+f'g=(gf)'$$

Idempotent

Suppose $\langle S,\star\rangle$ is a binary algebraic structure. An element $p\in S$ is said to be *idempotent under* $\star$ if and only if

$$p=p\star p$$

Figure 5.2. Selection of Breaching Tasks.

The third portion of the interview involves explicit discussions about ordering of equations. This involves tasks (which I call "Comparison Tasks") in which the interviewee is asked to explicitly compare equations (Figure 5.3). All these tasks revisit equations that are included as either breaching or open-ended tasks. Table 5.2 shows which tasks were done with which participants. These tasks are in Appendix D.

Table 5.2 Comparison Tasks Completed by Each Participant.

	CIdentity	CProofs	CEuler	CSumRule	CMVT
Jacob	X	X	X	X	X
Larry	X	X	X		X
Warren	X	X	X	X	X
Edgar		X		X	X
Patrick	X	X		X	

Xena	X	X		X	X
Kevin	X		X		
Ben	X	X	X	X	X
Ming	X	X			X

Some of the tasks are revisited from earlier in the interview, and the interviewee is prompted to compare different ways of writing the same equation (Figure 5.3). For example, assuming the interviewee answered with (f+g)'=f' +g' to *OSumRule* (Figure 5.1), the interviewee is presented with an equation written the other way (*CSumRule*) and asked to compare it to theirs.

Probing interview questions include "I noticed that earlier you wrote the equation differently", "I have found that textbooks usually present the equation this way. Do you think there is a reason for that?", "Is there a difference in meaning between these equations?", "Which way do you prefer?", "Is there an advantage to writing the equation one way over the other?", and "Can you give an example where writing it this way would be preferable and explain why?".

MVTCompare

There are various ways that textbooks state the mean value theorem.

Theorem 1. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 2. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Theorem 3. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f'(c)(b - a) = f(b) - f(a)$$

Theorem 4. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a)$$

Figure 5.3. A Comparison Task Used in the Third Portion of the Interview.

5.3.2 Data Analysis Methods

There is not one standard method that I can cite to capture the entirety of my data analysis. For this reason, I discuss some of the details of how I handled my data and the variety of techniques I used to reach my conclusions.

Some degree of data analysis took place during the interviews themselves. I adapted to participants' responses instead of having a fully deterministic interview. This involved interpreting my participants' meanings to make in-the-moment decisions. Like with my other studies in this dissertation, this is consistent with constructivist epistemology (elaborated upon more in the general introduction); as a researcher, I am forming second-order models of my participants' meanings (Steffe & Thompson, 2000). This, of course, makes sense; I had to interpret the interviewee's responses and then respond accordingly. For example, when a participant explained that *DifferenceQuotient* was “backwards”, I had to interpret the participant as pointing out a breach and then take that opportunity to inquire further.

After each interview, I wrote up brief notes of my overall impression of the participant. These notes were done based on memory as well as any field notes I had jotted down. Specifically, I took notes of any situation in which the participants answered the open-ended tasks in atypical ways, any situation in which they pointed out a breach, and a list of considerations regarding asymmetry (e.g., “this participant seemed to focus on the idea that terms shouldn’t come out of nowhere”).

Once all the interviews were completed, I took notes on each individual interview. This involved re-watching the interview recording and transcribing the portions that pertained to ordering or asymmetry. I transcribed the situations in which participants explained a breach, as well as their responses in parts three and four of the interviews (comparison tasks and explicit questions). In the second portion of the interview, I took note of situations in which the participant mentioned the right-hand side of the equation first in their explanation (e.g., explaining *Idempotent* by saying “p multiplied by itself results in p”). Notice that mentioning of a breach and reading things from right to left parallels the anecdote in Behr et al. (1980) in which a child reads 5=2+3 as “two plus three equals five” and refers to it as “backwards”. Relatedly, I took note of language that appeared to entail asymmetry, such as “e times x becomes x” (emphasis added). In order to have the data necessary for cataloging and describing reasons for asymmetry, I transcribed any portion of the interview in which the participant explained why they wanted an equation to be written in a certain order.

Some portions I did not transcribe. For example, in the open-ended tasks, I only noted the equation given by each participant -- not their exact description of the meaning of the equation. I did not transcribe every description or explanation that participants gave of equations; often, they explained aspects that did not pertain to order (for example, explaining a derivative rule in terms of rate of change functions).

These notes were organized by task, and each task was organized by participant. This is the point at which my coding began. I used a hybrid of constant comparison inquiry (Creswell, 2012) with a narrative approach (Creswell, 2012) from a constructivist perspective (Clement, 2000). I began with an initial list of codes to account for asymmetry, that I created from hypothesizing about norms for asymmetry together with initial observations made when entering the data. I then applied these codes to the interview data and took note of any reason for asymmetry that was not captured by these codes. All the reasons that I had expected people to have for ordering did appear at some point, but reviewing the interview notes led to additional codes. I then expanded these codes to include the additional reasons for asymmetry that were not in the original list. At this point, the level of analysis was somewhat low-level. I was only coding the reasons that people gave me. This happened due to my own naïve understanding that coding was a straightforward process; I expected that there would be little interpretation involved and that I could simply write down the reasons for orderings given to me (e.g., "the operations should be on the left side") and categorize them based on the words used.

This initial approach led to a rather incomplete interpretation of what was happening; if two different words were used to describe the same idea, then two different codes were created for the same idea. For example, Jacob used the words "mysterious" and “complicated” synonymously to describe a mathematical idea that would be relatively new to his envisioned reader (for example, he mentioned that in *MVTCompare*, f’(c) is more “mysterious” because the amount of mathematics needed to know what it denotes is greater than the amount of mathematics needed to know what the difference quotient denotes). Similarly, sometimes the same word was used to describe different ideas. Unlike with Jacob, Larry used “mysterious” to mean “new to the conversation” or “whose purpose for being mentioned in this context is unclear”. Careful word interpretation was also paramount in understanding the use of the words “new” and “given”. Sometimes, “new” meant “new to the conversation”. For example, in *DifferenceQuotient*, the term 3n(n+1) is “new” in the sense that it has not occurred elsewhere in the proof or in the statement of the theorem – it is new to the reader. Contrast this with $e^{i\theta}$ in *CEuler*; it is “new” in the sense that the reader might be less familiar with an imaginary exponent than with isin(θ)+cos(θ). Thus, while the claims “the new thing should be on the left in the equation” and “the new thing should be on the right in the equation” might appear to be contradictory, they are instead using different notions of the word “new”. Similarly, in some situations, participants used the phrase “given” to mean “posed to you” or “given to work with”; in the task *Productrule*, (fg)’ is “given” in this sense. On the other hand, some participants used the word “given” to mean “given information”. In the task *Productrule*, the given information might be information about the functions f, g, f’, and g’. Notice that these two interpretations of “given” are at odds with each other.

The above examples illustrate why this was not just a straight-forward task of coding via categorizing words and why I refined my approach. I delved into the meanings of each participant to understand how they were using words and the various contexts that they were envisioning. In order to interpret a participant's description for a reason for ordering a certain way, the context of a task was important – was it a proof? Was it a theorem? How did the participant frame or understand the context of the mathematical activity? These issues are where the notion of constructivism and abduction come in (discussed in more detail in the general introduction to this dissertation). Simon (2019)'s critique of coding as purely categorization aligns with my need for greater interpretation than simple sorting of words given by participants: "categorization is generally an inductive process (sorting the data as observed), not the multilevel process needed to gain new insights and work toward abduction of new theoretical constructs" (p. 114). I am not merely categorizing data, but I am interpreting data in a way that contributes to the categorization. This is not a purely inductive process.

In order to interpret each participant's motivations for asymmetry, I looked at each person's responses individually. This provided me the opportunity to carefully consider each participant's word meanings and focus on how each individual person was thinking. I analyzed each interview in a way that is consistent with the methods described in Simon (2019). This involved three layers of analysis. Although presented as separate levels, as Simon (2019) notes, there is an interaction; I first considered local portions of the interview transcript and noted reasons given for ordering. This was done on a line-by-line or sentence-by-sentence basis, so, at this point, the analysis was not unlike the initial analysis I performed across tasks. This, as Simon (2019) puts it, stays "close to the data". The second level involved more interpretation; as I read the data, I made hypotheses about how the individual participant was thinking about the given task or the asymmetry being discussed. The third level involved making an overall assessment about how the individual understood the meanings of equations. In order to develop a robust interpretation, I moved back and forth between these levels; while the results of the first level informed the second and third levels, I still returned to the "lower" levels to check for consistency after doing the higher-level analysis. The product of the individual analysis was a narrative summary together with a list of themes capturing the various reasons for ordering. Below is the narrative I wrote for Larry:

> Like Jacob, Larry uses "starting" and "on the left" synonymously. He continues to talk about "showing what's going on", where he appears to be talking about clarity or transparency to the reader. Like Jacob, he believes that things should not come out of nowhere. This idea extends to the notion that we should "start with" the thing we are proving something about – this is what the prover themself does, so this should be reflected on the page to the reader ("showing what's going on"). He uses transformational language throughout, wherein a=b means something like "we start with a, and we get b", fitting the general notion of time mentioned in my first sentence of this paragraph. Like Jacob, he mentions tradition and verification of truth as important things. Unlike Jacob, he mentions something almost like theme-rheme structure: that if you end with something about b, it makes sense to start the next sentence with something about b – he links this to the notion of something being "given". Where he diverges significantly from Jacob is his reasoning on the *Identity*. Recall that, for Jacob, the more "complicated" or "sophisticated" thing being on the left was why $e\star s$ should be on the left. For Larry, $e\star s$ should be on the left because the idea of operating and getting a result

is part of the *meaning* of left identity. However, he can see some utility for having it the other way: for example, in a proof where you end up writing $\phi(e \star s) = \phi(s)$. This response suggests that the way something is written as a law should reflect the order in which it is used in a proof or computation. For the distributive law, he says "we want to see how to distribute something"; some action is performed to get from a to b in a=b in the sense that *we work with a to get b*. In this sense, an equation is an instruction for a transformation. So far, throughout the interviews, we have a general theme that the right-hand side gives you information about the left-hand side, and you are interested in learning something about the left-hand side. This can be in the context of a proof (e.g., we are proving something about the left-hand side, as in how the proof in *Induction* is conventionally given), or it can be in the context of a calculation (the right-hand side tells us how to compute the left-hand side, as in *OEuler*).

As these analyses took place, I organized and described the various themes. Thus, when I moved to analyze a new participant, I had ideas in mind from previous participants. For example, when analyzing Jacob's interview, I began to see that the notion that the left-hand side of the equation should be "the thing you want to know more about" is linked to the general maxim that "things shouldn't come out of nowhere" (I discuss these links in more detail in my results section).

5.4 Results

I begin by answering research question (1): Do mathematicians use and understand the equals sign symmetrically? In doing so, I establish that there are indeed norms surrounding the usage of the equals sign that suggest asymmetry. This is accomplished primarily with the first two portions of the interviews: the open-ended tasks and the breaching tasks. I focus first on establishing that there are norms for ordering (research question 1) and then later discuss specifically what these norms are (research question 2).

5.4.1 Evidence of Asymmetry (Research Question 1)

The results of the open-ended tasks suggest that there are norms for writing equations in certain orders. In all the tasks, participants gave relatively consistent answers. In *OMice*, all nine participants wrote the equation the same way, with the derivative on the left-hand side. In *OEuler*, eight participants wrote the exponential term on the left side, with the exception of Xena, who wrote the formula incorrectly as $\sin(i\pi) = e^{i\pi} + 1$. In other words, all eight of the participants who wrote a correct answer put the exponential term on the left. It is notable that even Xena's incorrect answer has the more concise or less expanded term on the left; as I discuss later, evidence supports the claim that it is a norm to put the shorter or less expanded term on the left. Similarly, all eight of the participants who wrote the correct equation for *OMVT* put f'(c) on the left. In *OSumRule*, all nine participants wrote their equation with the derivative of the sum on the left. There were very strong norms evoked by this task and the related task *ProductRule*, which I elaborate on later. In *OIdentity*, every participant wrote the identity element on the left. To summarize, participants ordered their equations consistently on the open-ended tasks, the only exceptions being Larry on *OMVT* and Xena on *OEuler*.

The results of the breaching tasks also confirmed that there are norms regarding asymmetry. However, this portion of the interview was not as straightforward as expected. With some participants, it was revealed later in the interview that in some tasks, they perceived a breach but did not say anything. They explained that this is because they viewed the breach as either unproblematic or relatively unproblematic compared to a different breach.

In *Proof*, all eight of the participants mentioned an order breach. This is unsurprising, given that the equation was not simply reversed but instead had terms in a jumbled order. In both *Induction* (eight people interviewed) and *DifferenceQuotient* (nine people interviewed), all participants except Edgar mentioned an order breach. However, I revisited both of these tasks toward the end of the interview with Edgar and explained to him that other participants found there to be an order breach. He explained that he did not "like" the presentation *DifferenceQuotient* but had just not bothered to mention it, and that he could understand why others would take issue with *Induction*. In *Idempotent*, all eight of the interviewees did not mention an order breach. Warren encountered the task only as a comparison task (not as a breaching task) in the context of an explicit discussion of ordering and explained that he did indeed have a preference for writing it the other way. In *Homomorphism*, only Kevin and Warren mentioned an order breach. In *Exponents*, two out of three (Jacob and Larry) mentioned an order breach. In *Distributive*, three (Warren, Larry, and Ben) out of the seven people who encountered it as a breaching task mentioned an order breach. All three took issue not with the equation itself, but with calling the law the "distributive" law rather than the "factoring" law; in other words, they found the name of the law to be incongruent with the way in which the law was written, suggesting an asymmetry (discussed further in 5.4.1). In *ProductRule*, six out of nine participants (everyone except Edgar, Ming, and Xena) mentioned an order breach. With Xena, I returned to the task at the very end of the interview and explained the design of the task. At that point, she had already explained that she realized the true purpose of the interview. She explained that she "didn't mention it [the order breach] then, but was so focused on the lack of variables". She was focused on the fact that the equation used f' to name a derivative rather than f'(x). While Ming and I did not return to the task, he appeared to be distracted by the same issue; he explained that many students might have difficulty with an equality of functions rather than of numbers. Edgar did not mention an order breach during this task, and we did not return to the task later in the interview.

Table 5.3 below summarizes the above results of the breaching experiment. As I discuss later, a participant not seeing an order breach does not mean that the participant did not understand there to be differing meanings of an equation based on ordering. In some cases, both orderings were acceptable but had different meanings. See, for example, Kevin's explanation in "*The Topic Goes on the Left*". This especially occurred with *Idempotent* and *Distributive*; the ordering affected the meaning. I discuss this in more detail when answering research question (2).

Table 5.3 Results of breaching experiment by task and participant.

	DifferenceQuotient	Homomorphism	Exponents	Induction
Jacob	Y	N	Y	Y
Larry	Y	N	Y	Y
Warren	Y	Y		Y
Edgar	NY	N		NY
Patrick	Y	N		Y
Xena	Y	N		Y
Kevin	Y	Y	NN	

Ben	Y		N		Y
Ming	Y		N		Y
Total	Y(8), NY(1)		Y(2), N(7)	Y(2), NN(1)	Y(7), NY(1)
	SetTheory	Product Rule	Idempotent	Proof	Distributive
Jacob	N	Y	N	Y	N
Larry		Y	N	Y	Y
Warren		Y	*Y	Y	Y
Edgar	N	N	NN	Y	N
Patrick		Y	N	Y	NN
Xena		NY	N	Y	N
Kevin	N	Y	N		
Ben		Y	N	Y	Y
Ming	N	NY	N	Y	N
Total	N(4)	Y(6), N(1), NY(2)	*Y(1),N(7), NN(1)	Y(8)	Y(3), N(4), NN(1)

Note. Y means they encountered it as a breach task and mentioned an order breach. N means they encountered it as a breach task and did not mention an order breach. NN means they encountered it as a breach task and a comparison task and did not find there to be an order breach with either. NY means they encountered it as a breach task but didn't find there to be an order breach, but encountered it again as a comparison task and did find there to be an order breach. *Y means they did not encounter it as a breach task, did encounter it as comparison task, and found there to be an order breach as a comparison task (when asked explicitly about ordering).

In breaching tasks, it was common for participants to explain the meaning of the equation from right to left. Notice that this parallels the story in Behr et al. (1980), in which young students read 2+3=5 as 5=2+3. For example, in *Idempotent*, seven of the eight participants explained the meaning of the equation by mentioning the right-hand side of the equation first. This is despite the fact that none of them mentioned an order breach. In *Idempotent*, participants had explanations that were close to “if you take this element and you apply it to itself, you obtain the original element” (Larry). In this case, the language was action-based and actually mirrors the students’ language in Behr et al. (1980) regarding performing an operation and obtaining a result (a discussion of operations-produces-results follows in a subsequent section). Mentioning the right side first occurred also in *ProductRule*, *Homomorphism*, and *Exponents*. In *ProductRule*, six of the nine people interviewed mentioned the right-hand side first in their explanation of the equation. This includes the three people, Xena, Ming, and Edgar, who did not mention an order breach. Similarly, in *Homomorphism*, out of nine participants interviewed, Larry and Peter mentioned the right-hand side first when explaining the equations but did not mention an order breach. For the first and third equations in *Exponents* (the equations that breach the supposed norm), Jacob explained from right to left but did not mention an order breach. I interpret the existence of these right-to-left explanations as responses to order breaches in a similar way that Behr et al. (1980) does; the fact that participants in some sense read “a=b” as “b=a” suggests that “b=a” better reflects their understanding of what such an equation is trying to express and that "a=b" violates an order expectation. This interpretation is not inconsistent with the fact that several participants did not mention an order breach. As discussed previously, some participants later revealed that they noticed an order breach but had not bothered to say anything. Additionally, people might in some sense correct a breach without even noticing that there is a breach.

5.4.2 Reasons for Asymmetry (Research Question 2)

Now that I have established that there is asymmetry, I move to answering Research Question 2 in establishing what makes equations asymmetric. I characterize the various reasons for asymmetry given by participants. These are the resulting "codes" or "themes". These codes are not intended to represent a strict partition of the reasons given; that is, there are some overlapping codes. This makes sense considering the close relationships between the various codes, elucidating a possible context-dependence of the grammatical preferences and influences of norms that induce an asymmetry in the equations.

5.4.2.1 Texts Should be Coherent. Our first overarching theme is textual coherence. This code captures the idea that mathematical text, like any text, ought to be structured in a way that is coherent, cohesive, and organized. This is, perhaps, the most common reason for asymmetry throughout all the interviews. It occurred in multiple tasks with every participant. Within this textual coherence umbrella were various other common ideas or rules for ordering. These rules, although listed separately, are interrelated.

5.4.2.1.1 Ordering Should be Consistent and Match Expectation. The notions of consistency of order appeared in a few situations. Both consistently across contexts and consistency within a context were suggested as reasons for particular orderings. The former occurred when participants discussed the frequency with which a particular ordering occurs in other contexts as a reason for using that ordering. For example, some participants expressed that the way that an equation is written in a theorem should reflect the way that it appears most often in proofs or computations, and some participants cited "tradition" as a reason for certain orderings. Ben explained that he had no strong preference for ordering in *Idempotent* because he hasn't "used it enough times to have a sense of the frequency with which each [ordering] occurs".

Consistency within a context was also given as a reason for particular orderings. For example, in *CProofs*, some participants explained that because the definition of left identity has the operations on the left, the proof should mirror this same structure. A similar norm appeared in *Homomorphism*; some participants explained that the "from" and "to" language (a homomorphism is a function "from" something "to" something) suggested that the equation should be ordered a certain way. Warren explained that the equation in *Homomorphism* should be reversed:

> *For the simple fact that the homomorphism maps S to S', it's directional, we start with S and we want to get S'. And, also, the two algebraic structures are presented in that order, S then S'. And then when we write the equation we're actually kind of, reading left to right anyway, we're actually going backwards from the way everything is presented.*

Some participants mentioned the idea of mirroring an implication structure; when proving P implies Q, the proof should start with P and end with Q. I had expected a similar response in *SetTheory*. That is, I had expected that the fact that the equation in the theorem was written one way as a=b, that starting with b and ending with a would be perceived as a breach. However, no participants mentioned this breach. This might be because only four participants were interviewed, and three of them expressed unfamiliarity with the field of set theory and the notation within the task. I hypothesize that they were too busy making sense of the symbols to perceive an order breach. Unfortunately, there were no other tasks to test this particular norm

5.4.2.1.2 Theme-Rheme Structure Should Be Respected. The idea of theme and rheme from Systemic Functional Linguistics (SFL) occurred as an instance of textual coherence. Participants used this as a reason for ordering in the task *CProofs*; specifically, some participants explained that they preferred the fourth proof because it starts with s', which is what the previous sentence ended with. In other words, the components of the rheme of one sentence ("…such that ϕ(s)=s' ") become the theme of the next sentence ("s'= ϕ(s)"). The notions of theme and rheme also help account for why participants tended to dislike the second proof and would rewrite it. The first four equations in this proof are of the form (stacked) "a=b, e=d, c=b, d=c", and participants tended to take objection with this presentation and wanted to rewrite as a string of equalities with a first, b second, c third, d fourth, and e last. This came in various forms; some participants preferred it written as "a=b=c=d=e" horizontally as one string. Others preferred it vertically stacked, either as "a=b [new line] =c [new line] =d [new line] =e [new line]" or "a=b [new line] b=c [new line] c=d [new line] d=e". These preferences suggest that viewing each equation as a sentence, participants wanted to impose the theme-rheme structure so that the theme of one sentence (equation) was in the rheme of the previous sentence (equation). This occurred with other tasks as well, such as *CMVT* and *Induction*. In *CMVT*, one reason participants cited for preferring f'(c) on the left is that the sentence above ended with c, and so it therefore made sense to start the next sentence (equation) with c. Xena explained "If I say there exists a point c such that, I'd wanna then say something about c." Participants gave similar theme-rheme explanations in the task *Induction*. For example, Ben remarked that he wanted the string of equations to end with $(n^3\text{-}n)+3n(n+1)$ because it's the first thing mentioned in the subsequent sentence. Edgar suggested a similar theme-rheme structure when I provided him a modified version of *DifferenceQuotient* in which the text "The following shows the difference quotient" is replaced with "The following is a proof that the difference quotient of f is 2x+h". In this situation, he preferred the subsequent string of equalities to start with 2x+h, since the previous sentence had ended with 2x+h. Through the lens of SFL, such a structure enhances textual coherence.

5.4.2.1.3 Things Shouldn't Come Out of Nowhere. Perhaps the strongest norm is the rule that *things should not come out of nowhere*. The tasks *Induction* and *DifferenceQuotient* evoked these responses very strongly. Recall *Induction* consists of a proof of the inductive step in showing that $k^3\text{-}k$ is divisible by six for all k and begins with a string of equations starting with $(n^3\text{-}n)+3n(n+1)$ and ending with $(n+1)^3\text{-}(n+1)$. All participants except Edgar disliked this presentation on the grounds that 3n(n+1) was introduced out of nowhere. For example, Warren remarked that he disliked that it was "summoned out of thin air", and Jacob referred to it as "pulled out of a hat". Some participants seemed to have an emotional reaction. Larry explained “we have this mysterious n^3-n. I don’t know where this is coming from. Then the 3n, that haunts me as well”, and Ben said “I’m annoyed”. Similar responses occurred with *DifferenceQuotient.* Recall that *DiffenceQuotient* first states the meaning of “difference quotient” and then “shows the difference quotient” of $f(x)=x^2$ in a string of equations starting with 2x+h and ending with (f(x+h)-f(x))/((x+h)-x). In a sense, the 2x+h comes out of nowhere. All participants except for Edgar disliked this presentation and pointed out an order breach. When reading, Larry asked “where does 2x+h come from?” and Ben remarked “It’s backwards in exactly the same sense that the *Induction* proof was backwards”. It bears mentioning that this particular norm is related to the notion of theme-rheme. Starting a sentence with the previous sentence’s rheme ensures that such a sentence begins with something that has been mentioned prior and is thus not “out of nowhere”.

5.4.2.1.4 The Reader Should Know Why a Term is Being Introduced. The rule that “things shouldn’t come out of nowhere” overlaps with other aspects of textual coherence. Generally speaking, participants wanted it to be evident to the reader why a term is being introduced. In the context of proofs, this means that it needed to be clear how introducing a term contributed to the proof at hand. This occurred regularly in *Induction* and *DifferenceQuotient*. For example, in *DifferenceQuotient*, participants explained that they should know why the term 2x+h is introduced to begin with. It was unclear to the readers what 2x+h had to do with the topic of difference quotients. Edgar explains his reaction in the following exchange:

> Edgar: *I did not like that. Like, I kinda got to when it said “the following shows the difference quotient” and I’m like, where are you going with this? What*

are you gonna tell me, right? And then you start with 2x plus… and what I had to do is I read 2x plus h and I was like, I don't even care about the steps, I'm like what's the last step? I jumped directly from there to the last step to figure out what in the hell you were talking about. I didn't make a big deal out of it, but maybe I should have. I didn't like it.

Alison: *OK, tell me more about how you were feeling. Like, what you didn't like about it. What was the issue?*

Edgar: *Well it's like, what is your point? Like, what are you trying to tell me?*

Alison: *Was it clear once you finished?*

Edgar: *Yes, but then I was annoyed at having had to like, go around your presentation. Like, literally I went around it. I jumped from 2x plus h to the bottom to see what the hell you were talking about.*

Notice that Edgar gave more of a reason than simply 2x+h coming out of nowhere. He actually mentioned that he was bothered that he did not see why the term was being introduced and what role it plays in the proof ("where are you going with this?"). Responses were similar in *Induction*. For example, Ben explained that he did not like the 3n(n+1) in *Induction* and the 2x+h in *DifferenceQuotient*: "Who is this? I don't know her". When I attempted to repair the breach in *DifferenceQuotient* by changing the prompt to say "the following shows that the difference quotient is equal to 2x+h" (this way, 2x+h does not come out of nowhere), he remarked "ok, so I do know her, but I still don't know why I should". This response suggests that the issue was not just about familiarity with the term, but also about wanting to know the role that the term played in the context of the proof.

5.4.2.1.5 The Topic Goes on the Left. Another norm surrounding textual coherence is that *the topic should go on the left*. This norm was evoked in several tasks and is perhaps the norm that came up most frequently. The general idea is that the equation a=b, as compared to the equation b=a, is more about a. One reason participants gave for preferring f'(c) to be on the left hand side in *CMVT* is that the Mean Value Theorem is a theorem about derivatives. Jacob explains: "the f'(c) is almost like, the topic of this theorem". Participants gave similar reasons for why they preferred the identity element in *CIdentity* to be on the left-hand side – the thing being discussed is the identity element, so it should be on the left. This idea also occurred in *DifferenceQuotient*. Jacob explained "They're trying to learn about the difference quotient. They should start with the difference quotient". Notice that this idea is not disjoint from the general rule that "things shouldn't come out of nowhere". If a topic of discussion is established (e.g., difference quotients), then a new seemingly unrelated term being introduced without explanation for why it is being introduced would appear to "come out of nowhere". In *DifferenceQuotient*, I attempted to repair the "things coming from nowhere" breach with Edgar. Unlike Ben, he was satisfied with this repair and explained: "that would be much more preferable, because now you have changed the topic". Some participants made comparisons to the grammatical notion of a subject of a sentence. For example, in the *CProofs* task, Edgar explained why he preferred to write $\phi(e) \star' s'$ on the very left of the string of equalities:

> *But uh, the thing that you are saying something about, that you feel like you're talking about should be on the left-hand side. And are we making a statement about $\phi(e) \star' s'$ or are we making a statement about s'? And I feel like because*

we're trying to establish that $\phi(e)$ is the identity element, that's the subject of the sentence, so it should be on the left.

Similarly, in *OMice*, Kevin explains that p'=rp translates to "the rate of change turns out to be a multiple of p" while rp=p' translates to "a multiple of p turns out to be the rate of change". A key distinction is that the subject of the sentence switches, and the subject reflects the topic of conversation. Patrick uses several tasks to explain the idea of the topic being on the left:

There is some subjective sense of what the object of inquiry is. So with the difference quotient thing you have on this page [DifferenceQuotient], it really seems that the object of inquiry is what is the average rate of change of f from x to x plus h. And then we calculate that, and it turns out to be 2x plus h. With the mean value theorem [CMVT], the average rate of change from a to b, the average rate of change of f from a to b is a static thing, and I feel like it's not the reason for the mean value theorem existing. When we use the mean value theorem in calculus or analysis, usually we are using it to say that there is some specific point in the domain where the derivative is equal to the function's average rate of change. I feel like that sort of centers the derivative or the existence of a point where the derivative or certain value as the object or the interesting thing in the inquiry there. With the differential equation [OMice] one, I feel like what's happening there when I state the rule p'(t) equals r times p(t), so what I'm saying there is I'm making a statement that I perceive to be principally about the rate of change of the population. Uh, I mean it certainly has to do with the other side of the equation. It has to do with the fact that uh the rate of change is proportional

> *to the value of the population at a given time…but I feel like in each of those situations there is a sense of what I'm most interested in studying, and I'm usually putting that thing first. Same with e star s for that matter. I think maybe the reason I put e star s first and why everybody puts e star s first is that whatever statement we're making is really a statement about what it means to be an identity.*

It bears mentioning that, although this particular study is about equations and the equals sign, the idea of the topic being on the left-hand side or first in a sentence is likely not limited to equations. Compare the statements $2>x$ and $x<2$, which are clearly equivalent. Arguably, the first statement is principally about 2, whereas the second statement is principally about x. The notion is that the topic of conversation should be the subject of the sentence and appear on the left side.

5.4.2.2 The Right Side Explains. Since the left side is the topic of interest, it makes sense that the right-hand side would be explanatory; it should give information about the topic. The fact that this norm could easily apply to non-equations (e.g., inequalities) highlights this asymmetry. Consider, again, the inequality $x<2$. While x can be viewed as the topic of discussion (which happens to be the theme), the right side (which happens to be the rheme) gives information of the topic of discussion; $x<2$ tells us that x (left side) has the property of being less than 2 (right side), whereas $2>x$ tells us that 2 has the property of being greater than x. We can interpret equations similarly. The equation $a=b$ can be interpreted as "a has the property of being b". This interpretation lends asymmetry in meaning to relational clauses; in this sense, such a clause can be understood as attributive. This interpretation is consistent with a lot of the asymmetrical language used throughout the interviews. Many participants used phrases such as "ends up being", "happens to be", and "turns out to be". For example, in *CEuler*, Warren explained that "$e^{i\theta}$ turns out to be isin(θ)+cos(θ)". Under this umbrella of the right side being explanatory, there are several closely interrelated norms for ordering:

(a) *unknown→known:* an unknown thing is on the left while a known is on the right

(b) *sophisticated → less sophisticated*: the more mathematically sophisticated thing is on the left.

(c) *question → answer*: a question is on a left, whose answer is on the right.

(d) *less expanded →more expanded:* the left is shorter or less expanded than the right.

(e) *defined→definition*: the concept being defined goes on the left, and its definition goes on the right.

Recall the idea that the left side is the topic of discussion or interest, while the right side explains what is on the left. The categorization listed above fits into this general frame. Why might something be a topic of discussion? One reason is that there are aspects of it that are unknown. We might discuss something because we want to learn about it, or it might be a topic in the textbook because it is being taught to students. Hence, it makes sense that an unknown thing would be on the left (as the topic of discussion), and the known thing on the right. In *OMice*, Kevin explained that if he knew the value of p' then he would write the equation as rp=p', whereas if he knew the value of rp, he would write the equation as p'=rp. To him, this was linked to the topic of discussion; the topic of discussion is something unknown that we want to find out about (which is on the left), and we have information that tells us something about it (which is on the right). Notice that this idea also appeared in *Induction*; $(n+1)^3-(n+1)$ is what we (the problem solver) want to know about (prove something about), so in this sense, it is unknown. Contrast this with $(n^3-n)+3n(n+1)$. Before we've proven that $(n^3-n)+3n(n+1)$ is actually equal to $(n+1)^3-(n+1)$, we know less information about it.

The notion of the right side being more known than the left side parallels the idea that the right side is more understandable or mathematically less sophisticated than the left-hand side. If something is less understood, then it is in some sense less "known". In the task *CEuler*, Larry explained that isin(θ)+cos(θ) is easier to understand ("you see what's going on"): "You want to find real and imaginary parts, so decomposing in that way makes life simpler. You see what's going on". In *CMVT*, Ming explained his rationale for having f'(c) on the left side: "formulas often have the form something we don't understand equals something we understand"

Closely related to the *unknown*→ *known* and the *more sophisticated* → *less sophisticated* norms is the *question*→ *answer* norm. We ask questions about things that we do not fully understand and our answers should be easier to understand or more known than the question. Hence, it makes sense that the notion of question and answer closely parallels the ideas of unknown and known as well less understandable and more understandable. It bears mentioning that an analogous *question*→ *answer* norm appears in the literature on young students; as I discuss in Mirin (2019), Denmark et al. (1976) characterizes young students as understanding the equals sign as "a one-directional operator separating a problem from its answer" (p.31). With mathematicians, the notion of problem and answer is less of a strict rule and more generally relates to the norm that the right side explains the left side. In *CEuler*, Ming explains the connection between *unknown* → *known, more understood* → *less understood*, and *question/problem* → *answer*:

> *It's like, this thing we don't understand in terms of this thing we could understand. The assumption being that E to a complex number being something most people didn't understand. And so I think there is just like a cultural, bias from reading left to right. Like, we begin with the thing that might pique your interest, then end with the thing that, you know, gives you the answer.*

In some cases, the "problem" at hand is to compute or evaluate something. In several instances, participants explained that derivative rules were instructions for how to compute the left side.

Jacob frames ordering in terms of "more sophisticated" and "less sophisticated" and links this framing to the notion of question and answer. He uses the word "simple" to mean "mathematically less sophisticated" or "easier to understand". For example, in *CMVT*, he explains: "the left-hand side is the more mysterious quantity, and here we're giving, it's like the question, what is this mysterious quantity? And the answer is the right-hand side. As opposed to the simple quantity equals the more complicated thing, which is the way that the second one describes it". While in general, Jacob's responses about this question-answer format were representative of the other participants, there is a notable aspect on which he potentially diverges; for him, one thing that indicates sophistication is the number of stipulations needed for the object in question to exist. In *CMVT* and *OMice*, he explains that the stipulations needed for the left side (the side with the derivative) to exist (which are not needed for the right side) indicate that the left side is more "sophisticated". This could relate to the notion of theme-rheme structure in the sense that he is following the norm that the stipulations just mentioned should be used right after they were mentioned. This reasoning overlaps with participants' explanation that the proof in *CProofs* should start with s' because s' was just mentioned and starting with s' uses the stipulation that ϕ is onto.

Some participants explained that the "newer" object should be on the left. Recall the discussion in 5.3 about the two different notions of "new". Here, we are concerned with "cognitively new" rather than "new to the conversation". This notion closely overlaps with the idea of the unknown or mathematically more sophisticated thing being on the left; the reader might be learning about something that is new, and hence unknown, to them. In *CEuler*, Kevin uses "new" and "unknown" synonymously: "[the left side] is the unknown or new expression I wanna make a statement about it so I want to say the ROC is or turns out to be a multiple of p."

Such an idea might be newer or unknown because it is more mathematically sophisticated or more difficult to understand. Jacob links the notions of "new" an "unknown" when explaining why the sum rule is written with the derivative of the sum on the right: "if you're starting with these two differentiable functions, and you already understand f' and g' in some sense, f+g is a new function (…) this is like the new thing, and now you wonder about its derivative. Like new or in some sense more complicated and expressing in terms of things you previously know". In *ProductRule*, Kevin used the ideas of "unknown", "new", and "problem" being on the left:

> Kevin: *I would typically start with, um, what you call word problems, applications, geometric problems. And uh, then the question is um, if we knew how fast each of the quantities grows or function um, how could we uh,*

get some information about the um, uh, growth rate of or the rate of change of the product

Alison*: So you're saying this would be in a problem where this (points to (fg)') is* --

Kevin: *Something that I'm interested in, yes.*

Alison: *Then what role would the other side (points to fg'+f'g) play?*

Kevin: *Um, that it's basically using things that we already know about uh so we, I mean, the easiest problem's always in terms of area. Uh but, area of a rectangle, so but uh I'd like to have changes um where we may have um, where we change a quantity how many things we buy and they get smaller or larger or so on. So I would have a variety of examples of it's not purely geometric and the typical question is we know the rates of change of f and g and not we're interested in the rate of change of the product*

Alison*: So you're saying the known part is on the right, and the thing we are trying to find out about is on the left? So why do you think that is?*

Kevin: *That's just how we read things from left to right. It similar to when I write a computer program and I make an assignment or I make a definition that usually the new object is written on the left-hand side*

Notice that Kevin related the notion of "new" to that of "definition", specifically in a programming context. In *Homomorphism*, Kevin explained that someone defining the operation ⋆' would need to put $\phi(x)\star' \phi(y)$ on the left. Similarly, in *CEuler*, he explained that because Euler's formula is actually a definition of irrational exponents, the exponential term must be on the left. Because the idea of an asymmetrical equals sign of definition is already discussed explicitly in the mathematical community (and even symbolized as :=), I had not intended to include it as a topic investigation. However, participants still mentioned the issue.

Some participants used the notion of defining as a metaphor for this more general idea. For example, Xena explained that in *ProductRule*, the equals sign is similar to a definition because the "quick symbol" is on the left. Closely related to the idea of the definition being on the left is the norm that the right-hand side is more expanded or verbose than the left. When an equals sign is used to define something, then the term being defined is on the left, and a definition tends to be longer than the thing being defined. This links to our overarching norm of the right side being explanatory of the left or giving information about the left; like with definitions, an explanation tends to be more verbose than the thing being explained. In the context of *Productrule*, Patrick links the notion of question/answer with the notion of defining by explaining how the right side is expanded and explanatory:

> *The convention is going to be let's put the thing being evaluated first, and then how to evaluate it or the formula for evaluating it over on the right. I feel like that's a consistent convention across a lot of mathematical writing. Like, if you state a definition. You know sometimes occasionally you'll see a definition as if this object has this and this property blah-blah-blah-blah-blah then we call it and then thing in italics, thing being term being defined. But usually you see it the other way. We say that blah-blah-blah is term in italics if blah-blah-blah has the following properties. So like the less expanded form first and then more expanded form later seems to be the unwritten rule of mathematical writing.*

Observe that the idea of a new/unknown thing being introduced on the left with known information detailed on the right is an ideal that Veel (1999) discusses as part of SFL to account for how relational clauses in mathematics "bridge" something new with something known.

Recall the discussion of Leibniz' law of indiscernibles given in the general introduction to this dissertation; that objects a and b are identical if and only if they have the same properties. In this section, I had discussed how identity allows us to make inferences; if we know that a=b, then whatever properties one of them has, so does the other one. This links to the notion of transparency of representation with particular properties; if b (or a) is transparent with respect to some property, then we can claim that a (or b) also has this property. Although Leibniz' law of indiscernibles is symmetric, what the results of my study suggests is that it is applied somewhat asymmetrically; with a=b, we tend to conclude that a has whatever properties b has, rather than that b has whatever properties a has. This fits with the general idea that the left side is the topic of inquiry while the right side gives information or properties about the left side. For example, if we want to conclude that the cycle (1 3 2) is even, then this interview data suggests that the equation we use to conclude that is more likely to be (1 3 2)= (1 3)(1 2) rather than (1 3) (1 2)=(1 3 2). It is therefore perhaps unsurprising that of the examples I gave, four out of five involve using properties of the right side to make conclusions about the left side.

5.4.2.2 Transformations and Substitution: a Produces or Becomes b. Our next category concerns transformation. This is a cluster of norms surrounding the general idea that a=b means that a transforms to b. This category is in some sense related to the idea that the left side is the topic of discussion, while the right side gives information about the left; the right side of the sentence (the rheme) is giving information about a - that it has the property of transforming to b. Ernest (2008b) explains that transformations induce directionality (and hence asymmetry); “The transformation of signs in semiotic systems is directional” (p.43). Transformations tend to take place in problem-solving contexts (Ernest, 2008b). In our case, the notion of transformation occurs additionally in theorem/rule tasks (e.g., *ProductRule*); however, this occurs because the participants are envisioning a problem-solving context (discussed below). Generally speaking, transformative meanings for a=b can be translated as “a turns into b” or “a becomes b”; something is done to a (the left side) in order for it to become or produce the right side. There are several ways this can happen.

The first is that *operations produce a result*, and therefore the *operations go on the left while the result goes on the right*. Since the operations *produce* a result, they must precede it and therefore go on the left. Warren, in the context of *Idempotent*, explains his reasoning for this norm: “I start with the operation, then a second later, if you will, I have a result. With our thinking, that’s what happens. So, when you’re reading left to right, that should mimic as it happens in your brain too”. The tasks *Idempotent*, (O and C) Identity, and *CProofs* evoked this norm. As discussed earlier, participants tended to use action-based language to suggest the idea that operations produce a result. For example, in *OIdentity*, Luke explained: “for any element, if we apply that particular element to that we obtain the original one that we started with”. For at least some participants, the notion of operation producing a result was an essential aspect of the meaning of a left identity element. Their explanation in *CProofs*, in favor of First Proof ($\phi(e) \star' s' = \ldots = s'$) over

Fourth Proof (s'=…= ϕ(e) ⋆' s'), was that the first proof shows that (ϕ(e) ⋆' s') ends up resulting in or producing s' (notably, Edgar actually uses the word "transformation" in his description). Xena explains: "like if I could do it starting with the star operation with the identity element and show it doesn't do anything, then that would be my preference. The other order, although equivalent, seems less natural".

A related meaning for an equation as a transformation is that *what is given is on the left, whereas the goal or the result is on the right*. This is a givens → goals format. Observe that this closely mirrors the question → answer format discussed prior. However, there is a subtle difference; rather than the right-hand side being just an answer to the left-hand side, the left-hand side actually becomes (transforms to) the right-hand side via some actions. In this case, "given" does not apply to a fact or a proposition; it is not information that is "given". Instead, a particular representation or string of symbols is given as something that you are to transform toward a particular goal. In *DifferenceQuotient*, participants conceptualized (f(x+h)-f(x))/((x+h)-x) as "given" with 2x+h as the "goal". Luke explained that the proof in the difference quotient should instead show "the result once you reduce everything possible", and Patrick explained that the presentation has "the result of the calculation first". As mentioned above, this idea of transformation also occurred in theorem contexts. However, they occurred in theorem contexts in the sense that participants were envisioning how that theorem could be used in a transformation context. In the context of *ProductRule*, Edgar explains that "what you *have* is on the left, and what you *can have if you want* is on the right", and Ben similarly explains "the thing I got already is on the left, and the rule is telling me what I should do with the thing". In other words, theorems of the form "a=b" suggest that, in problem-solving contexts, one might want to transform a to b. Ben illustrated this idea in the context of *Distributive* (the distributive law) by explaining “we’re presenting rules for operating on symbols” and using an arrow in place of the equals sign to indicate transformation. His point was that xy+xz=x(y+z) suggests a different transformation than x(y+z)= xy+xz:

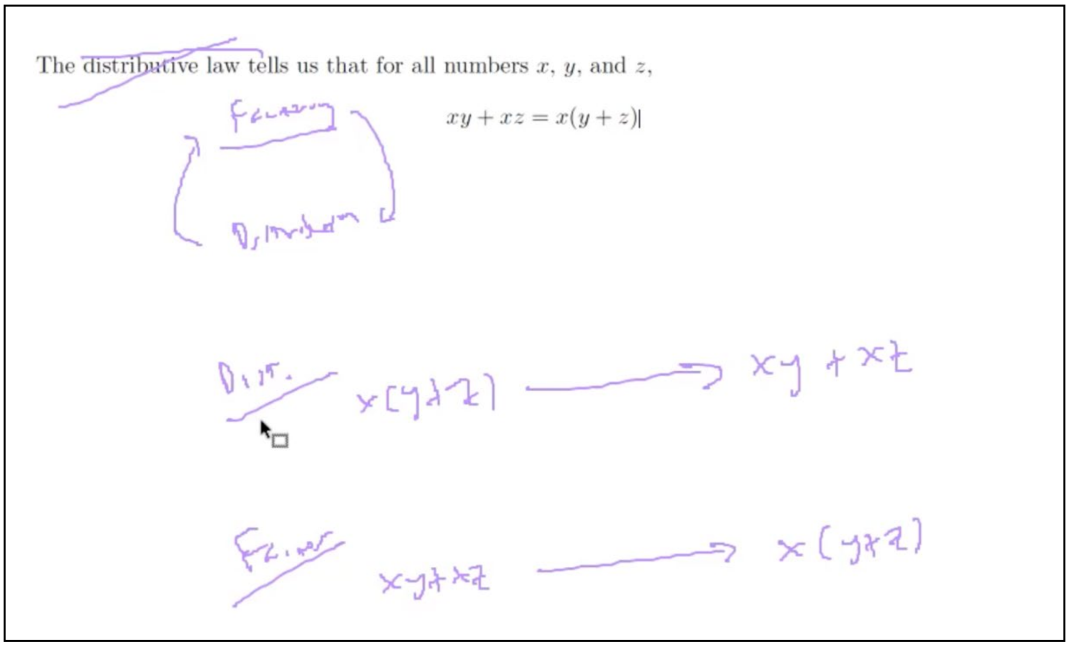


Figure 5.4. Ben’s Illustration of his Meaning for the Distributive/Factoring Law(s).

The purpose of performing such transformations is to move from what is given to whatever your goals are. This notion generalizes to the idea of substitution; that in an equation, *the thing on the left is substituted with the thing on the right*. Observe that the transformations discussed are a special case of substitution. Starting with or being “given” a term a (say x(y+z)), and ending with a term b (say xy+xz) is effectively the same thing as replacing the term a with the term b. In other words, while sometimes we transform a to b, other times we transform P(a) to P(b), where P(x) is some string of symbols. This occurred in both problem-solving and in theorem contexts. For example, both Ben and Luke explained that in *CProofs*, they wanted the equations to be written in such a way that entails a substitution. Ben explains: "my favorite thing about the equals sign is that it literally means that the two things on either side of it are the same thing, and therefore when you see one of them come up later you can replace it with what's on the other side of the equals sign…like if I see $\phi(s)$ and I know $\phi(s)$ is s', then I can write s' whenever I see $\phi(s)$".

In the case of theorems and rules, participants explained that an ordering a=b is preferable to b=a if there are more situations in which we would want to replace a with b (or transform a to b). In other words, a=b suggests the transformation of replacing a with b (in this sense, a *becomes* b), and this transformation works toward a goal of obtaining a certain expression. For example, in *CEuler*, Edgar explains that in a problem-solving process, if you go from that (points to $e^{i\pi}$) to -1 you've now greatly simplified things", and that this transformation is a reason for presenting Euler's formula with the exponential term on the left. This idea of frequency of substitution and transformation was enough to, in some sense, overwrite the other norm about operations producing a result. In both *Idempotent* and *CIdentity*, several participants explained that the fact that there are situations in which one would want to replace p with p star p or replace e with e star x are enough grounds for finding the presentation with the operations on the right acceptable. In *CIdentity*, Ben explained that he was only "slightly" annoyed and generally okay with writing the operation on the right because of such substitutions: "it comes up often enough in an abstract algebra proof that you're gonna make an identity appear and then do some shit to it".

We can understand this notion of substitution in terms of Leibniz' law of indiscernibles (discussed in the general introduction to this dissertation). Recall that Leibniz' law of indiscernibles states that objects x and y are identical (equal) if and only if they share the same properties. In symbolic logic, this principle is framed as $\forall F(Fx \leftrightarrow Fy) \rightarrow x=y$ for all properties F. This rule allows someone to deduce Fy from Fx under the assumption that x=y; syntactically, this allows the prover to substitute/replace "x" with "y". While Leibniz' law is formed symmetrically (one can also deduce Fx from Fy), the results of my study here suggest that substitution is not understood fully symmetrically.

Not all transformations are created equal. Clearly, provided there is some sort of goal, transformations that move toward a particular goal (givens → goals) are preferable to those that do not move toward a particular goal. This is not the only sense in which transformations are unequal. A common theme expressed by participants is that *simplification is preferable to the opposite*, and simplification occurs as a transformation from left to right. Participants use "simplification" in two different senses. One is the sense in which the simplified thing (the thing on the right) is easier to work with or more understandable. This idea overlaps with the idea of substitution; if b is "simpler" than a in the sense of "easier to work with", then one might want to replace a with b (this is what Edgar seems to suggest in *CEuler*, discussed above). It also overlaps with the idea of the expression on the right being more understandable (simpler) than the one on the left. Here, we are concerned with simplification as a transformation that someone performs, not simply a comparison of sides of an equation for which is "simplest". Ernest (2008b) refers to an "implicit heuristic of simplification" which "seeks to reduce the complexity of terms in an equation *en route* to a solution" (p. 40). Ernest (2008b) explains how simplification suggests directionality and, hence, asymmetry:

> *Note that the simplification heuristic described above plays a central role in operationalizing directionality in mathematical tasks. That is, a significant part of the appropriation of directionality is associated with the implicit understanding of the simplification heuristic as a technique for goal-directed activity. (p.46)*

Kevin was the only participant who defined the notion of "simplification", which he described as "try(ing) to write this in as few symbols as possible". A key characteristic of simplification shared by the other participants seems to be the reduction in total number of symbols, although they did not state it explicitly. For example, in *DifferenceQuotient*, (f(x+h)-f(x))/((x+h)-x) simplifies to 2x+h. Participants explained that simplification is preferable to the opposite because it is easier to perform. I adopt Xena's word "messification" to refer to the opposite or reverse procedure of simplification. Two reasons were given for simplification being easier than messification: the first is that, through school mathematics, people are trained to perform simplification procedures; the other reason is that simplification is more deterministic than messification. In the context of *DifferenceQuotient*, Warren uses both of these reasons to explain why simplification is easier:

> *Students are mathematically trained to simplify expressions. They're not trained to complicate expressions in order to fit a pre-defined structure. And, the steps of factoring, reducing, simplifying, cancelling, whatever in more simple terms, these are all directions students are comfortable with. And, the other direction, you know when you say, why would I want to change 2/3 to 4/6 or change it to 2x/3x?*

There's multiples of those steps, where you're complicating in the dark not knowing what steps you need to do so that the chain of applications leads to the difference quotient. That is not, there's no series of, there's no sequence of instructions available to a student to know how to do that backwards I think, especially when there's multiple steps.

In various other instances, participants explained that adding more symbols is more difficult and less deterministic than simplification. Ming explained that messification is "inspired", rather than simplification or cancellation which is more automatic. The general idea is that, if we were given 2+3, we might immediately in our head think "that's 5". However, if we were given 5, there are a number of other representations that could come to mind; 2+3, 4+1, etc.

5.4.2.3 Proofs Should Reflect the Prover's Process for Creating the Proof. The tasks *DifferenceQuotient* and *Induction* both evoked the norm that a proof or a problem's solution should, when possible, communicate how the prover came up with the proof. In this sense, a proof should be a record of a problem-solving process. For example, if the prover performed a process to transform a to b, then the equation a=b should be in the proof rather than b=a. The participants read both *DifferenceQuotient* and *Induction* as being a record of mental processes and transformations. For example, in *Induction*, participants explained that at first they found the equation $(n^3-n)+(3n^2+3n) =(n^3+3n^2+3n+1)-(n+1)$ to be "clever" because it suggested that the prover performed a transformation of adding 1, subtracting 1, and regrouping (as discussed, simplification was considered to be an easier transformation than messification). Participants took issue with this presentation on the grounds that the prover (the person who wrote the proof) did not perform this transformation themselves in the sense that the prover did not perform the action of manipulating $(n^3-n)+(3n^2+3n)$ to get $(n^3+3n^2+3n+1)-(n+1)$. This norm was reflected in two different ways. The first way was that participants initially seemed to assume that the prover came up with the proof in the order in which it was presented. This was reflected in remarks about the proofs in *Induction* and *DifferenceQuotient* being "clever". Similarly, in *Induction*, Xena explained: "I would have done this from the other way. I would have started here (underlines $(n+1)^3-(n+1)$) and expanded it and gotten something else and then tried to figure out how to prove it". Her remarks indicate that she conflated the presentation containing the proof with how the prover came up with the proof. The second way this norm was expressed was through explicit disbelief that the proof reflected the way the prover came up with it, together with objection of the order of the presentation on those grounds. In *DifferenceQuotient*, Ben remarked "no human would go this way".

Generally speaking, in both *DifferenceQuotient* and *Induction*, participants objected on the grounds that the prover had actually come up with the proof in the opposite order in which it was presented. Consider Jacob's response to *Induction*:

> *I think it would be better if the sequence of equations was reversed. Because the very first thing is totally out of the blue. This thing that's been pulled of a hat. I mean, why that? Whereas we know we want to know about the very last thing, and that would already be very motivated to the reader.* ***And really, to write the proof this way somebody did it in the order I'm suggesting and then reorganized it.*** *It's not that they just studied the n^3-n and then realized this is the magic thing to add.*

The bolded portion indicates that Jacob preferred the proof to be written in the way that the prover came up with the proof. Observe also that for Jacob, this norm was closely tied to the idea of textual coherence ("totally out of the blue") and the notion that simplification is an easier transformation than the opposite ("the magic thing to add"). It bears mentioning that there is potential overlap between this category and *the reader should know why a term is being introduced*". Making a reasoning process transparent might often communicate why a term is introduced; if, in *Induction* I communicate that I am going to fiddle with $(n+1)^3-(n+1)$ to get a statement with n^3-n, that then conveys why I mentioned $(n+1)^3-(n+1)$. This makes sense if we assume that people's thoughts are interconnected. However, it is conceivable that the converse holds; I might present a proof without explaining how I came up with this proof. Recall earlier Ben's issue with epsilon-delta proofs in that they do not reflect the order in which the prover came up with them. It is conceivable, however, that when reading an epsilon-delta proof, the reader knows why each term is being introduced. When I am reading an epsilon-delta proof, and I see a particular delta introduced, I might not be following the particular reasoning process of how the prover came up with that particular delta. However, I do have some knowledge of why that particular delta is introduced; this value of delta does the job. Hence, while there is some overlap between ideas of textual coherence and the idea of making a reasoning process transparent, this overlap is not necessarily absolute.

5.4.2.4 Ordering Should be Pedagogically Optimal. Another rule governing ordering is that *equations should be ordered in such a way that is pedagogically optimal.* This is unsurprising considering that the participants were all currently teaching mathematics or had in the recent past taught mathematics. Recall the discussion in the literature review of this study about a speaker attributing mental states to the listener or reader; the fact that participants had pedagogical concerns suggests that mathematicians do consider the mental state of their reader or student. The results of this study complement the study described in Lai and Weber (2014), which describes mathematicians' pedagogical concerns for presenting proofs. Notably, Lai and Weber (2014) also observe that mathematicians consider the role of their audience when presenting a proof.

This norm frequently overlapped with the belief that the proof should reflect the prover's reasoning process; a proof or solution to a problem communicates to the student a solution or reasoning process that the student should learn about. This norm appeared prominently in both *Induction* and *DifferenceQuotient*. In *Induction*, participants objected on the grounds that students would not be able to produce the proof in the order that it is presented. Ming explained why there are pedagogical reasons for presenting a proof in the way that the prover came up with it: "Part of teaching math isn't teaching theorems that are true, it's teaching students how they could have done it themselves". Part of the purpose for demonstrating the reasoning process is so that students can apply this reasoning process to other contexts. For example, *Induction* is an opportunity to teach students that in induction proofs, it is useful to fiddle with the expression involving n+1. In *DifferenceQuotient*, participants similarly explained that the equation should be presented with the difference quotient form first because it was communicating to the student a way of dealing with difference quotients that could be generalized to other functions; a particular problem-solving process is being demonstrated and hence communicated to students. Jacob explains his pedagogical reasons for preferring the standard ordering in *DifferenceQuotient*:

> *There's like a process going on which then can be generalized about how to deal with other derivatives. Which If you try to mimic the style here would be extremely hard, whereas if you just write them in the other order it's relatively easy to follow.*

It bears mentioning that not all participants thought that this norm of reflecting the prover's reasoning process is pedagogically optimal in every situation. Ben mentioned that there are cases, such as epsilon-delta proofs in analysis, where we do present a proof in the opposite order in which we created in (we often start with a statement in epsilon and "work backwards" to find an appropriate delta). However, he explains that we have "culturally decided not to share this scratchwork" (tradition allows breaching the norm). This is consistent with Herbst et al.'s (2011) characterization of norms in which they are not always inviolable rules, but default expectations whose breaches require note and some justification. The fact that he is quickly aware of sanctioned breaches shows the role of the norm, even though it is broken systematically in particular contexts. Xena explained that the preferable order of the equations in *DifferenceQuotient* depends on the pedagogical goal. If the goal is to learn about difference quotients, then it should start with (f(x+h)-f(x))/((x+h)-x). However, she explained that if the goal is to aid students in developing algebraic solving techniques, then it might help students to learn non-routine techniques such as adding and subtracting a number.

Some of the norms under the textual coherence umbrella also occurred as pedagogical concerns. For example, several participants explained that students might be confused if something came out of nowhere. Furthermore, if a reasoning process is transparent in a presentation of a proof, then it is clear why a term is being introduced. In both *Induction* and *DifferenceQuotient*, these norms were intertwined with pedagogical concerns. Students should know why a term (e.g., 3n(n+1)) is being introduced and what role it plays in the proof, hence gaining access to the provers reasoning process. If a term is the topic of discussion (e.g., $(n+1)^3$-(n+1)), then the student understands why that term is being introduced, and that term does not appear "out of nowhere".

5.4.2.5 Inferences and Ease of Verification of Truth Are Important. Another common order norm is that ordering in a proof should be written in such a way as to make it as easy as possible to verify truth and make inferences. In proofs, the goal is not just to say true things, but to make statements (which are often inferences) that are verifiable to the reader. In proof tasks, participants explained that they verified each equation pairwise. Strings of equations, such as "a=b=c=d=e" were interpreted as "a=b, b=c, c=d, d=e, *and therefore* a=e" rather than "a, b, c, d, and e are all the same". This norm was especially evident in participants' responses to *Proofs*, since this particular task violated this norm. Ben illustrated how he read the equation string from left to right, attempting to verify each equation pairwise:

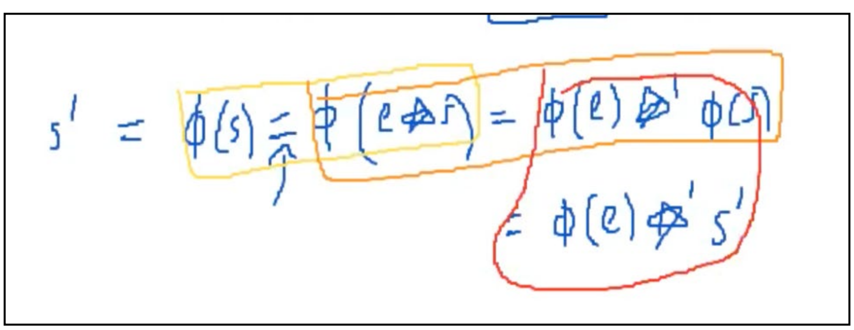


Figure 5.5 Ben's Illustration of How he Reads a String of Equations in a Proof.

In deciding which of the proofs in *CProofs* was preferable, participants explained that they had two concerns: the first is that each statement is deducible or verifiable from previous statements, and the second is that each statement is as easy to verify as possible. Ming explains: “a string of equalities suggests that each equality is followable or deducible and you start and end with the things you want to show are equal”. For example, participants took issue with the Third Proof on the grounds that it was not of the form “a=b=c=d=e”; while each individual equation was true, it was not inferable from the previous equations. This task was somewhat unique in that it is the only task in which participants described the presentation as mathematically incorrect or wrong, rather than having a preference or an expectation for a different ordering. Participants felt less strongly about the second norm: that a proof should be presented in such a way that each statement is not only inferable from the previous, but as easily inferable as possible. This is a reason that participants took issue with the Second Proof, which is of the form “e=d, a=b, c=d, b=c, and therefore e=a”. Verifying the truth of e=a using the other steps, participants explained, requires too much work on the reader’s part. Reading the proof aloud, Ben exclaimed “therefore by equations (1) through (4), we sprinkle magic dust on all of our stuff and get the conclusion that we want”, while Edgar objected “why have like three completely different independent facts and then have me just sort of need to have to ram them together at the end? It’s like, yeah it’s true, but you left all the work on the table!”.

5.5 Discussion and Future Directions

Overall, the results of this study suggest that mathematicians do not use the equals sign symmetrically. This is evident by the consistent responses to the open-ended tasks.

The norms surrounding ordering of terms in equations are so strong that the vast majority of the participants (eight out of nine) pointed out order breaches. In other words, as expected, the answer to research question (1) is “no, mathematicians do not use and understand the equals sign symmetrically”. The third portion of the interview gives us more insight into research question (2); we learned about what, specifically, the norms that govern the ordering of terms in equations are, as well as some reasons for having these norms.

During the breaching experiment, participants confirmed the existence of ordering norms in two ways; the first is in outright claiming that there was a breach or that something was “wrong”, while the other way involved reading right to left. By reading right to left, the participants were repairing the breach. This leaves open the possibility that what is written is not wrong, but at the same time not consistent with ordering norms.

There are broad, major norms concerning order. The first is that texts should be coherent. Within this textual coherence umbrella are the following norms: ordering should be consistent within and across contexts, theme-rheme structure should be respected, terms shouldn’t come out of nowhere, the reader should know why a term is being introduced/what role it plays in the proof, and that the left side is the topic of conversation or inquiry. The second major norm follows naturally from the notion that the left side is the topic: *the right side explains the left side*. Within this umbrella are the interconnected ideas of moving left to right from unknown to known, from sophisticated to less sophisticated, from question to answer, from less expanded to more expanded, and from defined to definition. In other words, while the left side is the topic of inquiry, the right side gives information about the topic of inquiry. Under this interpretation, a=b roughly means “a has the property of being b”. The third major norm involves interpreting “a=b” as “a becomes b”; it suggests that *equations can represent transformations, and these transformations occur in time from left to right in the sense that the result of the transformation goes on the right*. For example, operations producing a result represents a transformation as does the idea of simplification (which participants explained is an easier transformation to perform than its opposite, messification). One important transformation is substitution; a=b allows someone to transform any statement or term with “a” in it to the same statement with “b” in place of “a”. Our fourth broad norm is that *proofs should, when possible, represent a record of the prover’s thought process and mathematical activity*. This brings us to our fifth norm – that *ordering should be pedagogically optimal.* One reason to show the prover’s reasoning process is to help students mimic such a process on their own and generalize to other problem-solving contexts. Our sixth and final norm concerns proofs; *ordering in proofs should occur in such a way so that each statement can be understood as an inference from previously established facts, and ordering of equations should be done in such a way that these inferences are as easy to make as possible*.

In summary, I described the various contexts that evoke asymmetrical usage of the equals sign as well as participants’ understandings of these contexts. It bears mentioning that contexts and norms surrounding ordering are interlinked. Participants brought context to the problems. For example, in *DifferenceQuotient*, participants tended to imagine a Calculus I setting in which students are learning about difference quotients or the definition of derivative. Additionally, participants envisioned contexts outside the particular tasks at hand that had bearing on how they understood the ordering of the terms in the equations. This occurred in the various situations in which participants mentioned other contexts that they viewed as parallel or analogous to their envisioned context. For example, in *DifferenceQuotient*, participants explained that they were picturing other problems or tasks of the same type. In *Identity* and similar tasks, participants considered how often various transformations might occur. The relationship between the invoked ordering norms and the wider context pictured by the participants is worth studying in future investigations.

As discussed, despite presenting these order norms separately, they were closely related. Common to all these ordering norms is the underlying fact that mathematicians care not just about truth-function, but also communication, which takes place in/over time (both imagined and experiential time). Ernest (2008b) explains:

> *While there is no universal timepiece ticking away in semiotic space, nevertheless individual and group engagement in mathematical activity is always over time (Mason et al., 2007). What this means is that accessing mathematical texts always has a sequential nature. (p. 43)*

For example, the textual coherence norms can be explained by reading the left side of equations first. Similarly, the transformation norms can be explained by performing a transformation from start (the left) to finish (the right). Participants were reading left to right and thus equated "left" with "first", and several participants even explicitly stated this fact. For example, Ming explained "I think there is just like a cultural, bias from reading left to right." Since not all cultures read left to right, it would be worth comparing ordering norms in other cultures.

The results of this study are interesting in light of the equals sign literature centering on children. As discussed in the general introduction to this dissertation, the equals sign literature tends to focus on children's deficits with regard to equals sign understanding. One common deficit concerns the property of symmetry; children tend to understand the equals sign asymmetrically. Oksuz (2007), for example, explains that students find equations of the form "5=2+3" as "rule violations", and Denmark et al. (1976) explain that students view an equals sign as expressing an asymmetric relation between problem and answer. Consider, additionally, the anecdote in Behr et al. (1976) that inspired my breaching experiment; a child read "5=2+3" aloud as "2+3=5". A major finding of my study is that it suggests that children might not differ so much from experts. Of course, the experts (mathematicians) in this study (and experts in general) know that truth-functionally, the equals sign expresses a symmetric relation. The study reported herein tells us that regarding concerns beyond truth-function (e.g., meaning), experts have asymmetric usages and understandings of the equals sign. Interestingly, there is some overlap between experts' asymmetrical meanings of the equals sign and childrens'. Consider, for example, the operations-produces-result transformational norm (that on the left is an operation, to which on the right is the result). All the "rule violations" in Oksuz (2007) do not have the operations on the left. McNeil and Alibali (2005) report that even college students have this norm of operations being on the left (they call this an "operational pattern"). In other words, we know that young children, college students, and mathematics experts all have meanings of the equals sign that suggest that the operations should be on the left. Similarly, consider the operational idea that the left is a problem and the right is an answer. My study shows that a similar idea exists amongst experts with the question→ answer norm (under the transformation umbrella). Note that, since experts understand the equals sign as truth-functionally symmetric, these asymmetrical norms are context-dependent amongst experts and not necessarily rigid – writing $x=e \star x$ is not considered "wrong" by experts, but connotatively different and in some contexts less preferable than writing $e \star x=x$. This is somewhat unsurprising when we consider that even Frege believed there to be more to meaning than truth-function (Frege, 1892/1948). In light of experts' views, it makes sense to revisit the equals sign literature about students. While the major take-away from this work is generally about student misunderstanding of the equals sign, we should consider that students' understanding might be more subtle and less rigid than expected. In my study, experts distinguished between "wrong" and stylistically not preferable. It seems possible that, like experts, students might understand certain equations as stylistically not preferable and "reject" such equations by calling them "wrong" or "false". Future research should consider exploring how (and if) students distinguish between finding equations to be a breach of expectation while having the symmetric view of equality is important for learning algebra (Byrd et al., 2015), that does not take away from the communicative aspect of equality that imposes some asymmetry. We should not belittle children noting communicative breaches that may be appropriate insights if our only goal is to add other ways of reasoning that are useful in other ways. These experts show how the same person can hold both interpretations in tandem or stylistically unpleasant versus "wrong" or "false".

One interesting related avenue to explore in the future concerns the other "rule violations" described by Oksuz (2007). Perhaps experts also have a dislike for equations of the form “a=a”, despite understanding the equals sign as expressing a reflexive relation. When we consider mathematical writing as a communicative, social act – as we have been throughout this study – we must consider what utility mathematicians might perceive in asserting “a=a” and how this utility (or lack of utility) aligns with Gricean pragmatics regarding informativeness.

CONCLUDING REMARKS

This dissertation investigates people's understandings and usages of ideas associated with the identity relation. There is a close link between the notion of identity and representation; often, in mathematics, we work with multiple representations of the same object. A fundamental assumption of my work is that what might be the same for one person, such as a mathematician, might not be the same for others, such as students. This assumption is grounded in constructivism, the underlying epistemology that guides my work (Thompson, 1982). One reason that sameness is important is that it allows us to make powerful mathematical inferences. These inferences can often be framed in terms of Leibniz' law of indiscernibles; two objects are identical if and only if they share the same properties. Hence, when a=b, it follows that a and b share the same set of properties. Leibniz' law is relevant for each individual study, which I discuss below.

The first paper directly addresses function identity: how students conceptualize, work with, and assess sameness of representation of function. Portions of this study are reported in Mirin (2018) and Mirin (2020b). It discusses the results of three tasks: a function sameness concept definition task, a task in which students assess sameness of functions in the fundamental theorem of calculus (the fundamental theorem task), and a task in which students evaluate the derivative of a piecewise-defined version of the cubing function (the cubing function task). A total of 360 students participated in this study, which included both qualitative (interviews) and quantitative (statistical) data. A key result of this study is that students did not appear to believe that sameness of graph was sufficient for sameness of derivative. Many students understand graphs with highlighted points as essentially different than graphs without highlighted points – i.e., a function's graph is not determined by its points. Other key findings suggest more foundational issues with notation and denotation. Students were presented with functions that were labeled as "functions" yet did not seem to understand each label as referring to a singular function. This occurred both in the cubing function task, in which several students viewed "f" (the piecewise defined version of the cubing function) as denoting two functions, as well as in the fundamental theorem task, in which several students used "p" to denote both an integrand and an integral. How a student assesses sameness of function will impact how they make inferences in accordance with Leibniz' law of indiscernibles. For example, f and the cubing function are the same and therefore share the property of having a derivative of 12 at x=2. However, results of this study suggest that this is a nontrivial inference for students. There are various possible barriers to making this inference: assessing f and the cubing function as the same function, understanding having a particular derivative as a property of functions, as well as denotation issues regarding whether f even is a function. While inferences regarding the fundamental theorem were not directly explored in this study, the fact that some students do not understand the function p defined using an integral to even be a function is a potential barrier to making inferences about sameness.

The second paper (Mirin & Zazkis, 2020) concerns implicit differentiation, and more generally, how students can come to understand the legitimacy of differentiating both sides of an equation. It also provides a case study together with a description of the obstacles that students might face when constructing such an understanding. The main contribution of this paper is that understanding implicit differentiation requires having a robust understanding of function sameness; it is valid to differentiate both sides of an equation because each side of the equation is a representation of the same function, and therefore they share a derivative. The steps outlined in the conceptual analysis discuss the conceptualizations necessary for making such an inference. In this case, Leibniz' law of indiscernibles tells us that when two functions are identical, they must share a derivative. Notably, the interviewee in this study explicitly claimed that two functions agreeing on every input does not necessitate that they share a derivative. This tells us that making the inference of same derivative from same function is nontrivial for at least some students. Perhaps the interviewee did not understand having a derivative as being a property of a function. Further investigation is needed to assess how students can come to understand differentiating each side of an equation as an inference from function equality.

The third paper concerns the equals sign directly. The equals sign expresses when two objects are indeed the same object. As discussed in the Introduction, philosophical accounts of equality (identity) address only symmetrical meanings. However, my investigation here considers asymmetrical meanings of the equals sign. Asymmetrical understandings of the equals sign are reported in the literature on children. I show that experts also use the equals sign asymmetrically. Specifically, I use Systemic Function Linguistics as well as Gricean pragmatics to consider mathematical writing as a communicative and not purely truth-functional act. One major finding is that it appears that mathematicians do not generally use Leibniz' laws symmetrically; given the equation $x=y$, it is more common to conclude that y has the same properties as x as opposed to x having the same properties as y. This observation provides an interesting link between the philosophy of identity (e.g., the work of Leibniz) as well as linguistic concerns (Halliday's Systemic Functional Linguistics). Another major finding is that mathematicians do use the equals sign asymmetrically in ways that overlap with the understandings of children. This suggests that we ought to revisit the literature on children with a more critical eye.

As a constructivist, I must consider that the mistakes students make concerning sameness (e.g., using the same symbol to represent two things, having asymmetrical understandings of the equals sign) might not just be straightforward logical fallacies. Second-order models of students' mathematical meanings are necessarily non-judgmental; they are based on the assumption that students construct their mathematical meanings in ways that are sensible and coherent to them. A student's conception of a mathematical idea serves the purpose of organizing the student's experience and is thus endowed with a personal, non-objective, rationality (Tallman, 2021).

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APPENDIX A

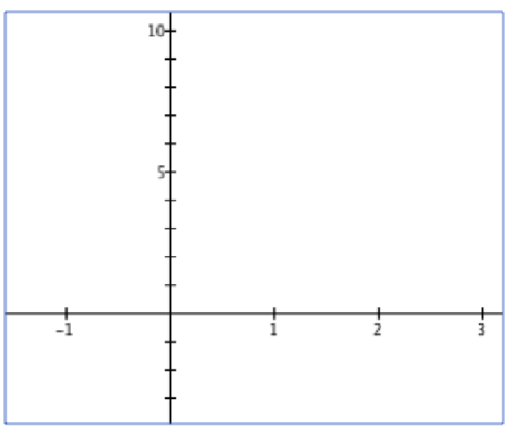
PHASE THREE QUIZ FOR CHAPTER TWO: ITEMS GIVEN TO 102 CALCULUS STUDENTS

Suppose f is the function defined by

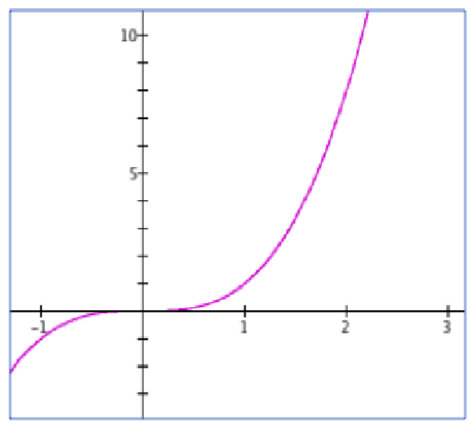
$$f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$$

1. Calculate 2^3

2. Below to the right is a graph of $y = x^3$. On the blank axes provided, graph $y = f(x)$ for the function f defined above.



$y = f(x)$



$y = x^3$

Recall that

$$f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$$

3. Evaluate $f'(2)$ for the function f defined above. Please provide a 1-3 sentence explanation of your answer.

   (a) 0
   (b) 8
   (c) 12
   (d) undefined
   (e) 0 if $x = 2$, and 12 if $x \neq 2$
   (f) 0 if $x = 2$, and undefined if $x \neq 2$

4. Let p be the function defined on all real numbers by

$$p(x) = \int_2^x 3t^2 dt$$

and let q be the function defined on all real numbers by

$$q(x) = x^3 - 8$$

(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

ii. p and q are not the same function.

(b) Provide an explanation for your answer for 4(a).

5. Suppose g is a function and h is a function. What does it mean for g and h to be the same function? Explain.

APPENDIX B

OPEN-ENDED TASKS ON EXPERTS' USE OF THE EQUALS SIGN

OSumRule

State the sum rule for derivatives, and write it down.

OEuler

Write down and state Euler's formula.

OMice[1]

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. Using t to denote time, $p(t)$ to denote the population, and r to represent the growth rate, write a differential equation expressing this relationship.

OIdentity[2]

Suppose $\langle S, \star \rangle$ is a binary algebraic structure. What does it mean for an element $e \in S$ to be a left identity element?

OMVT

Finish the following statement of the Mean Value Theorem by writing an equation:

Theorem 1. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that...*

[1] This task was borrowed from Boyce and Deprima (2009)

[2] The wording for this task (e.g. "binary algebraic structure") and other related abstract algebra tasks is from Fraleigh (2003).

APPENDIX C

BREACHING TASKS USED ON EXPERTS' USE OF THE EQUALS SIGN

DifferenceQuotient

The **difference quotient** of a function g is defined to be

$$\frac{g(x+h)-g(x)}{(x+h)-x}$$

where h is nonzero.

Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=x^2$. The following shows the difference quotient:

$$\begin{aligned}
2x+h &= \frac{2xh+h^2}{h} \\
&= \frac{x^2+2xh+h^2-x^2}{h} \\
&= \frac{(x+h)^2-x^2}{h} \\
&= \frac{f(x+h)-f(x)}{h} \\
&= \frac{f(x+h)-f(x)}{(x+h)-x}
\end{aligned}$$

Homomorphism

The following is a portion of an introductory Abstract Algebra text.

Let $\langle S,\star\rangle$ and $\langle S',\star'\rangle$ be binary algebraic structures. A **homomorphism from** $\langle S,\star\rangle$ **to** $\langle S',\star'\rangle$ is a function $\phi: S\to S'$ such that for all $x,y\in S$,

$$\phi(x)\star'\phi(y)=\phi(x\star y)$$

Exponents

The following is a portion of a Precalculus text.

Recall the Properties of Exponents:

$$b^{x+y} = b^x \cdot b^y$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$b^{xy} = (b^x)^y$$

Induction

The following is a portion of a proof by induction that for all natural numbers k, $k^3 - k$ is divisible by 6. At this point in the proof, it has been assumed that $n^3 - n$ is divisible by 6, and it is being shown that $(n+1)^3 - (n+1)$ is therefore also divisible by 6.

$$(n^3 - n) + 3n(n+1) = (n^3 - n) + (3n^2 + 3n)$$

$$= (n^3 + 3n^2 + 3n + 1) - (n+1)$$

$$= (n+1)^3 - (n+1)$$

Since either n or $n+1$ is even, $3n(n+1)$ is divisible by 6. By assumption, $n^3 - n$ is divisible by 6. Hence, $(n^3 - n) + 3n(n+1)$ is divisible by 6, and therefore $(n+1)^3 - (n+1)$ is divisible by 6.

SetTheory[1]

The following is a proof in a set theory textbook that if a is a transitive set, then $\bigcup(a^+) = a$. Note that a transitive set is defined to be a set a such that all members of a are subsets of a, and a^+ is defined to be $a \cup \{a\}$.

Proof.

$$a = (\bigcup a) \cup a$$

$$= (\bigcup a) \cup (\bigcup \{a\})$$

$$= \bigcup (a \cup \{a\})$$

$$= \bigcup (a^+)$$

[1]This proof is a reversed version of the proof given in Enderton (1977).

Productrule

The *product rule* for derivatives says that if f and g are differentiable functions, then

$$fg' + f'g = (fg)'$$

Proof

Theorem 1. *Suppose $\langle S, \star\rangle$ and $\langle S', \star'\rangle$ are binary algebraic structures, and ϕ is an isomorphism from $\langle S, \star\rangle$ onto $\langle S', \star'\rangle$. Further suppose that e is a left identity element in $\langle S, \star\rangle$. Then $\phi(e)$ is a left identity element in $\langle S', \star'\rangle$.*

Proof. Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(s) = \phi(e) \star' \phi(s) = s' = \phi(e) \star' s' = \phi(e \star s)$$

□

Distributive

The following is a portion of a precalculus text.
The distributive law tells us that for all numbers x, y, and z,

$$xy + xz = x(y + z)$$

APPENDIX D

COMPARISON TASKS ON EXPERTS' USE OF THE EQUALS SIGN

CIdentity

An element $e \in S$ is a *left identity element* for $\langle S, \star \rangle$ if and only if for all $x \in S$

$$x = e \star x$$

CProofs

Theorem 1. *Suppose $\langle S, \star \rangle$ and $\langle S', \star' \rangle$ are binary algebraic structures, and ϕ is an isomorphism from $\langle S, \star \rangle$ onto $\langle S', \star' \rangle$. Further suppose that e is a left identity element in $\langle S, \star \rangle$. Then $\phi(e)$ is a left identity element in $\langle S', \star' \rangle$.*

Proof. **First Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(e) \star' s' = \phi(e) \star' \phi(s) = \phi(e \star s) = \phi(s) = s'$$

by the properties of homomorphism and the fact that e is a left identity element in $\langle S, \star \rangle$ □

Proof. **Second Proof** Suppose $s' \in S'$. Since ϕ is onto, there exists $s \in S$ such that

$$s' = \phi(s) \tag{1}$$

By (1) we also know that

$$\phi(e) \star' s' = \phi(e) \star' \phi(s) \tag{2}$$

Since e is a left identity element,

$$\phi(e \star s) = \phi(s) \tag{3}$$

and since ϕ is a homomorphism,

$$\phi(e) \star' \phi(s) = \phi(e \star s). \tag{4}$$

Therefore, by equations (1)-(4),

$$s' = \phi(e) \star' s'. \tag{5}$$

□

Proof. **Third Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(s) = \phi(e) \star' \phi(s) = s' = \phi(e) \star' s' = \phi(e \star s)$$

□

Proof. **Fourth Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$s' = \phi(s) = \phi(e \star s) = \phi(e) \star' \phi(s) = \phi(e) \star' s'$$

by the properties of homomorphism and the fact that e is a left identity element in $\langle S, \star \rangle$ □

CEuler

Euler's formula states that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

CSumRule

The *sum rule* for derivatives says that if f and g are differentiable functions, then

$$f' + g' = (f+g)'$$

CMVT

There are various ways that textbooks state the mean value theorem.

Theorem 1. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 2. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Theorem 3. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f'(c)(b - a) = f(b) - f(a)$$

Theorem 4. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a)$$