Mathematical Models of Opinion Dynamics

by

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ABSTRACT

This dissertation consists of three papers about opinion dynamics. The first paper is in collaboration with Prof. Lanchier while the other two papers are individual works. Two models are introduced and studied analytically: the Deffuant model and the Hegselmann-Krause (HK) model. The main difference between the two models is that the Deffuant dynamics consists of pairwise interactions whereas the HK dynamics consists of group interactions. Translated into graph, each vertex stands for an agent in both models.

In the Deffuant model, two graphs are combined: the social graph and the opinion graph. The social graph is assumed to be a general finite connected graph where each edge is interpreted as a social link, such as a friendship relationship, between two agents. At each time step, two social neighbors are randomly selected and interact if and only if their opinion distance does not exceed some confidence threshold, which results in the neighbors' opinions getting closer to each other. The main result about the Deffuant model is the derivation of a positive lower bound for the probability of consensus that is independent of the size and topology of the social graph but depends on the confidence threshold, the choice of the opinion space and the initial distribution.

For the HK model, agent i updates its opinion x_i by taking the average opinion of its neighbors, defined as the set of agents with opinion at most ϵ apart from x_i. Here, $\epsilon > 0$ is a confidence threshold. There are two types of HK models: the synchronous and the asynchronous HK models. In the former, all the agents update their opinion simultaneously at each time step, whereas in the latter, only one agent is selected uniformly at random to update its opinion at each time step. The mixed model is a variant of the HK model in which each agent can choose its degree of stubbornness and mix its opinion with the average opinion of its neighbors. The main results of this dissertation about HK models show conditions under which the asymptotic stability holds or a consensus can be achieved, and give a positive lower bound for the probability of consensus and, in the one-dimensional case, an upper bound for the probability of consensus. I demonstrate the bounds for the probability of consensus on a unit cube and a unit interval.

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Chapter 1

INTRODUCTION

This dissertation consists of three papers on the common topic of opinion dynamics. The first paper is about the Deffuant model:

Nicolas Lanchier and Hsin-Lun Li. Probability of consensus in the multivariate Deffuant model on finite connected graphs. Electron. Commun. Probab. 25 (2020) no. 79, 12 pp.

The other two papers are concerned with the Hegselmann-Krause (HK) model:

Hsin-Lun Li. Mixed Hegselmann-Krause Dynamics. To appear in Discrete Continuous Dyn Syst Ser B.

Hsin-Lun Li. Probability of Consensus of Hegselmann-Krause Dynamics. Submitted to ALEA Lat. Am. J. Probab. Math. Stat.

Both models consist of a finite set of interacting agents that are characterized by their opinion, and the dynamics depends on a confidence threshold $\epsilon > 0$. The main difference between the two models is that agents in the Deffuant model interact by pairs, whereas agents in the HK model interact in groups. Translated into graph, each vertex stands for an agent in both models.

For the Deffuant model, two graphs are combined: the social graph and the opinion graph. The social graph is assumed to be a general finite connected graph where each edge is interpreted as a social link, such as a friendship relationship, between two agents. Individuals are characterized by their opinion, where the opinion space is a bounded convex subset Δ of a normed vector space. At each time step, two social neighbors are randomly selected and interact if and only if their opinion distance does not exceed some confidence threshold, which results in the neighbors' opinions getting closer to each other. More precisely, given that vertices/agents x and y are selected, the mechanism is as follows:

$$\xi_{t+1}(x) = \xi_t(x) + \mu(\xi_t(y) - \xi_t(x))\mathbb{1}\{\|\xi_t(x) - \xi_t(y)\| \leq \tau\}$$

$$\xi_{t+1}(y) = \xi_t(y) + \mu(\xi_t(x) - \xi_t(y))\mathbb{1}\{\|\xi_t(x) - \xi_t(y)\| \leq \tau\}$$

where

$\tau > 0$ is a confidence threshold,

$\mu \in [0, 1/2]$ is a convergence parameter,

$\xi_t(x) =$ opinion of vertex x at time t.

Namely, if the interaction happens, the two neighbors' opinions get closer to each other equally. We first show that the opinion limits exist and that, in the limit, the opinion distance between any neighbors is either zero or larger than τ. In particular, $\lim_{t\to\infty} \max_{x,y\in\mathscr{V}} \|\xi_t(x) - \xi_t(y)\|$ exists and we are interested in the event

$$\mathscr{C} = \left\{ \lim_{t\to\infty} \max_{x,y\in\mathscr{V}} \|\xi_t(x) - \xi_t(y)\| = 0 \right\},$$

the collection of all sample points that lead to a consensus. Assume that all initial opinions are independent and identically distributed random variables. Let X stands for the random variable. Let

$$\mathbf{r} = \inf\{r > 0 : \Delta \subset B(c, r) \text{ for some } c \in \Delta\}$$

where $B(c, r) = \{a \in \mathbb{R}^n : \|a - c\| \leq r\}$ and fix $\mathbf{c} \in \Delta$ such that $\Delta \subset B(\mathbf{c}, \mathbf{r})$. Our main result gives the following lower bound for the probability of consensus.

Theorem 1.1 (probability of consensus)**.** For all $\tau > \mathbf{r}$,

$$P(\mathscr{C}) \geq 1 - \frac{E\,\|X - \mathbf{c}\|}{\tau - \mathbf{r}}.$$

The steps of the proof are as follows:

1. find a bounded supermartingale,
2. define a stopping time,
3. find a subset $\mathscr{A} \subset \mathscr{C}$.

From the above, we can apply the optional stopping theorem to get an upper bound for a certain expected value, whereas restricting the expectation to the complement of $\mathscr{A}$ produces a lower bound. In particular, we obtain a nontrivial lower bound for the probability of consensus.

In the second paper, I consider a variant of the HK model called the mixed HK model. There are two types of HK models: the synchronous HK model and the asynchronous HK model. In the former, all the agents update their opinion simultaneously at each time step, whereas in the latter, only one agent is selected uniformly at random to update its opinion at each time step. In the mixed model, each agent can choose its degree of stubbornness and mix its opinion with the average opinion of its neighbors, defined as the set of agents whose opinion is "close" to the opinion of the agent under consideration. The degree of the stubbornness of agents can be different and/or vary over time. The mechanism is as follows:

$$x_i(t+1) = \alpha_i(t)x_i(t) + \frac{1-\alpha_i(t)}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) \tag{1.1}$$

where

$x_i(t)$ = opinion of vertex i at time t,

$\alpha_i(t) \in [0, 1]$ is the degree of stubbornness of agent i,

$N_i(t) = \{j \in [n] : \|x_i(t) - x_j(t)\| \leq \epsilon\}$ is the collection of all neighbors of agent i.

The larger α_i is, the more stubborn agent i is. Note that (1.1) reduces to

- the synchronous HK model if $\alpha_i(t) = 0$ for all $i \in [n]$ and $t \geq 0$,
- the asynchronous HK model if exactly one $\alpha_i(t) = 0$ for all $t \geq 0$ and for some $i \in [n]$.

In particular, (1.1) covers both the synchronous and the asynchronous HK models. The goal is to find conditions under which the asymptotic stability holds or a consensus can be achieved. The synchronous model has some properties, such as finite-time convergence, that do not hold for the mixed model. The steps of the proof are:

1. study properties of the mixed model,
2. find a monotone bounded function,
3. utilize lemmas such as the Cheeger's inequality.

Define

$$\beta_t := \max_{i,j \in [n], \alpha_i(t) \geq \alpha_j(t)} \left(\alpha_i(t) - \frac{\alpha_i(t) - \alpha_j(t)}{n} \right).$$

A *Profile* at time t is an undirected graph $\mathscr{G}(t)$ with vertex set and edge set

$$\mathscr{V}(t) = [n] \quad \text{and} \quad \mathscr{E}(t) = \{ij : i \neq j \text{ and } \|x_i(t) - x_j(t)\| \leq \epsilon\}.$$

A profile $\mathscr{G}(t)$ is δ*-trivial* if any two of its vertices are at distance at most δ apart. The main results of the second paper are:

Theorem 1.2. Assume that $\limsup_{t\to\infty} \beta_t < 1$ and that $\mathscr{G}(t)$ is ϵ-trivial. Then,

$$\lim_{t\to\infty} \max_{i,j \in [n]} \|x_i(t) - x_j(t)\| = 0.$$

Theorem 1.3. Define $d_t^i = \max_{j\in N_i(t)} \|x_i(t) - x_j(t)\|$. If

$$\sum_{t=0}^{\infty} (1-\alpha_i(t))\left(1 - \frac{1}{|N_i(t)|}\right)d_t^i < \infty, \text{ then } x_i(t) \to x_i \in \mathbf{R^d} \text{ as } t \to \infty.$$

Theorem 1.4. Assume that $\limsup_{t\to\infty} \max_{i\in[n]} \alpha_i(t) < 1$. Then, for any $\delta > 0$, every component of a profile is δ-trivial in finite time, i.e.,

$$\tau_{\alpha,\delta} := \inf\{t \geq 0 : \text{every component of } \mathscr{G}(t) \text{ is } \delta\text{-trivial}\} < \infty.$$

Corollary 1.1. Assume that $\sup_{t\in\mathbf{N}} \max_{i\in[n]} \alpha_i(t) < 1$. Then, $\tau_{\alpha,\delta}$ is bounded from above. Also, letting $\tau_m = \tau_{\alpha,\epsilon/m}$ for $m \geq 4$, there is no interactions between any two components of $\mathscr{G}(t)$ at the next time step for some $M \geq 4$ and for all $t \geq \tau_M$, i.e.,

$$\mathscr{G}(t) = \mathscr{G}(\tau_M) \quad \text{for some} \quad M \geq 4 \quad \text{and for all} \quad t \geq \tau_M.$$

Hence, x is asymptotically stable.

The main objective of the third paper is to study the probability of consensus of the synchronous HK model. Because it is difficult to keep track of the entire dynamics' trajectory, I focus on the initial opinions. Assume that all initial opinions are independent and identically distributed random variables with a convex support of positive Lebesgue measure and a probability density function f. Because the Lebesgue measure m has some properties such as completion, one can prove that a convex set is measurable. The main results are separated into two parts: the general case and the one dimensional case. In general, the probability of consensus has a positive lower bound that only depends on the initial conditions. In the one-dimensional case, the probability of consensus has an upper bound that only depends on the initial conditions. I respectively prove the bounds for the probability of consensus on a unit cube and a unit interval. The main results of the third paper are:

Theorem 1.5.

$$P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is connected}) \text{ for } 1 \leq n \leq 4.$$

In general,

$$\begin{aligned} P(\mathscr{C}) &\geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) \\ &\geq P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) \\ &= \int_{\mathbf{R^d}} f(x_1)\left(\int_{B(x_1,\epsilon/2)} f(x)dm(x)\right)^{n-1} dm(x_1) > 0 \text{ for } n \geq 1. \end{aligned}$$

In particular, the probability of consensus is positive.

Corollary 1.2. Assume that $S = [0,1]^d$ and that $x_i(0) = \text{Uniform}([0,1]^d)$. Then,

$$P(\mathscr{C}) \geq \left(\left(\frac{\epsilon}{2}\right)^d m(B(0,1))\right)^{n-1} (1-\epsilon)^d = \left(\left(\frac{\epsilon}{2}\right)^d \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\right)^{n-1} (1-\epsilon)^d$$

for all $i \in [n]$ and $\epsilon \in (0,1)$.

$x_{(i)}$ is the i-th smallest number among $(x_k)_{k=1}^n$. For $n \geq 4$, let $m = \lfloor \frac{n-4}{3} \rfloor$ and $k = n - m - 1$. Say $\mathscr{G}(t)$ satisfies $(*)$ if

$$((m+2),(k)) \in \mathscr{E}(t) \text{ and } (x_{(n)} - x_{(k)} + x_{(m+2)} - x_{(1)})(t) \leq \epsilon.$$

Say $\mathscr{G}(t)$ satisfies $(**)$ if

$$\max\left((x_{(n)} - x_{(n-i-1)})(t),\ (x_{(n-i-1)} - x_{(i+2)})(t),\ (x_{(i+2)} - x_{(1)})(t)\right) \leq \frac{\epsilon}{2} \text{ for some } 0 \leq i \leq m.$$

Theorem 1.6 ($d = 1$)**.**

$$P(\mathscr{C}) = P(\mathscr{G}(0) \text{ is connected}) \text{ for } 1 \leq n \leq 4.$$

$$P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ satisfies } (*)) \text{ for } 5 \leq n \leq 7.$$

In general,

$$P(\mathscr{G}(0) \text{ is connected}) \geq P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial or satisfies } (**)) \text{ for } n \geq 1.$$

Corollary 1.3. Let $S = [0,1]$, $d = 1$, $\epsilon \in (0,1)$ and $x_i(0) = \text{Uniform}([0,1])$ for all $i \in [n]$. Then, for $n = 2$,

$$P(\mathscr{C}) = \epsilon(2-\epsilon)$$

for $n = 3$,

$$P(\mathscr{C}) = \begin{cases} 6\epsilon^2(1-\epsilon) & \epsilon \in (0,\frac{1}{2}) \\ 1 - 2(1-\epsilon)^3 & \epsilon \in [\frac{1}{2},1) \end{cases}$$

for $n = 4$,

$$P(\mathscr{C}) = \begin{cases} 24\epsilon^3(1-3\epsilon) + 36\epsilon^4 & \epsilon \in (0,\frac{1}{3}) \\ 19\epsilon^4 - 4\epsilon^3(1-2\epsilon) + (1-2\epsilon)^4 - 6\epsilon^2(3\epsilon-1)^2 & \\ -4\epsilon(1-2\epsilon)^3 + 12\epsilon^3(1-2\epsilon) + 12\epsilon^2(1-2\epsilon)^2 & \epsilon \in [\frac{1}{3},\frac{1}{2}) \\ \epsilon^4 + 4\epsilon^3(1-\epsilon) + 6\epsilon^2(1-\epsilon)^2 + 4\epsilon(1-\epsilon)^3 - 2(1-\epsilon)^4 & \epsilon \in [\frac{1}{2},1) \end{cases}$$

for $n \geq 1$,

$$P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) = \epsilon^{n-1}[n-(n-1)\epsilon]$$

$$P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) = \frac{2}{n}\epsilon^n(1-\frac{1}{2^n}) + \epsilon^{n-1}(1-\epsilon)$$

In general, $P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) = \epsilon^{n-1}[n-(n-1)\epsilon]$ for $n \geq 1$.

Chapter 2

PROBABILITY OF CONSENSUS IN THE MULTIVARIATE DEFFUANT MODEL ON FINITE CONNECTED GRAPHS

Nicolas Lanchier and Hsin-Lun Li

Abstract. The Deffuant model is a spatial stochastic model for the dynamics of opinions in which individuals are located on a connected graph representing a social network and characterized by a number in the unit interval representing their opinion. The system evolves according to the following averaging procedure: at each time step, two neighbors are randomly chosen and interact if and only if the distance between their opinions does not exceed a certain confidence threshold, with each interaction resulting in the neighbors' opinions getting closer to each other. Most of the analytical results established so far about this model assume that the individuals are located on the integers. In contrast, we study the more realistic case where the social network can be any finite connected graph. In addition, we extend the opinion space to any bounded convex subset of a normed vector space where the norm is used to measure the level of disagreement or distance between the opinions. Our main result gives a lower bound for the probability of consensus. Our proof leads to a universal lower bound that depends on the confidence threshold, the opinion space (convex subset and norm) and the initial distribution, but not on the size or the topology of the social network.

2.1 Introduction

This paper is concerned with opinion dynamics on connected graphs. The first and most popular stochastic model in this topic is the voter model, introduced independently in [9, 22]. The main mechanism in the voter model is social influence, the tendency of individuals to become more similar when they interact. More precisely, individuals located on the vertex set of a connected graph (traditionally the d-dimensional integer lattice) are characterized by one of two competing opinions, and update their opinion at rate one by simply mimicking one of their neighbors chosen uniformly at random. Using a duality relationship between the voter model and a system of coalescing random walks, it can be proved that the process on the infinite square lattice clusters in one and two dimensions whereas opinions coexist at equilibrium in higher dimensions [22]. While mathematicians studied analytically various aspects of the model such as the asymptotics for the cluster size in one and two dimensions [6, 11], the spatial correlations at equilibrium in higher dimensions [5], and the occupation time of the process [10], social scientists and statistical physicists developed and studied numerically more realistic models of opinion dynamics. We refer to [26, 34] for reviews of the main results about the voter model, and to [7] for a review of more recent stochastic models of opinion dynamics introduced by applied scientists.

Apart from social influence, an important component of opinion dynamics is homophily, the tendency to interact more frequently with individuals who are more similar. The most popular spatial model that includes social influence and homophily is probably the Axelrod model [1] where individuals are now characterized by a vector of cultural features, and interact with their neighbors at a rate proportional to the number of features they share (homophily), which results in the two neighbors having one more feature in common (social influence). For a mathematical treatment of the Axelrod model, we refer to [24, 27, 28, 31, 33]. Other spatial stochastic models of opinion dynamics include homophily in the form of a confidence threshold: individuals interact with their neighbors on the graph if and only if the level of disagreement between the two individuals before the interaction does not exceed a certain threshold. The simplest such model is the constrained voter model [37], the voter model with three opinions (leftist, centrist and rightist) where leftists and rightists do not interact. Extensions of this model where the opinion space takes the form of a finite connected graph and the level of disagreement is measured using the geodesic distance on this graph were introduced and studied analytically in [30, 36]. The Deffuant model [12] and the Hegselmann-Krause model [18] are two other important spatial stochastic models that include social influence and homophily in the form of a confidence threshold.

In the original version of the Deffuant model [12], individuals are located on a general finite connected graph representing a social network and characterized by opinions that are initially chosen independently and uniformly at random in the unit interval. At each time step, an edge is chosen at random and the two neighbors connected by this edge interact if and only if the distance between their opinions before the interaction does not exceed a confidence threshold τ (homophily), which results in the two neighbors' opinions getting closer to each other after the interaction (social influence). Because [12] is purely based on numerical simulations, the authors only considered specific graphs: the complete graph and the two-dimensional torus. Their simulations on large graphs suggest the following conjecture for the infinite system obtained by assuming that pairs of neighbors are now chosen in continuous time at rate one: the process exhibits a phase transition at the critical threshold one-half in that a consensus is reached when $\tau > 1/2$ whereas disagreements persist in the long run when $\tau < 1/2$. This conjecture was first established for the process on the integers in [25] using probabilistic and geometric techniques while a slightly stronger result was proved shortly after in [15] using a different approach. The existence of a phase transition along with lower and upper bounds for the critical threshold were also proved for variants of the model: a multivariate version where the opinion space is a (subset of a) finite-dimensional vector space and certain metrics are used to quantify the disagreement between individuals [14, 20, 21], and a discrete version called the vectorial Deffuant model also introduced in [12] where the opinion space is the hypercube and the disagreement between individuals is quantified using the Hamming distance [29].

In this paper, we study a version of the model where both the opinion space and the social network are fairly general. The opinion space is a bounded convex subset of a finite-dimensional normed vector space (where the norm is used to measure the disagreements). Under the averaging procedure [12], convexity is a necessary assumption because future opinions must be on the segment connecting past opinions, but we also point out that an extension of the model has been introduced in [14] where the opinion space is a general path-connected set, the opinion distance is measured by the length of some geodesics connecting the opinions and each update displaces the opinions along these geodesics. More importantly, while all the previous analytical results assume that the inviduals are located on the integers, with the notable exception of [16] where the process is also studied on the d-dimensional lattice and even the infinite bond percolation cluster, we assume more realistically that the individuals are located on a general finite connected graph, meaning any possible real-world social networks. But unlike [12] that relies on simulations for specific graphs, our results apply to all possible finite connected graphs. Due to the finiteness of the graph, the existence of a phase transition at a specific critical threshold no longer holds, and we instead derive a general lower bound for the probability of consensus. While our bound depends on the choice of the opinion space (convex subset and norm), it is uniform in all possible choices of the social network.

2.2 Model description and main results

The two key components of the model studied in this paper are the social network on which the individuals are located and the opinion space. To define these two components,

- we let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a finite connected graph and
- we let $\Delta \subset \mathbb{R}^n$ be a bounded convex subset and $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

The multivariate Deffuant model is a discrete-time Markov chain whose state at time t is a configuration of opinions on the graph:

$$\xi_t : \mathscr{V} \to \Delta \quad \text{where} \quad \xi_t(x) = \text{opinion at vertex } x \text{ at time } t.$$

Following all the previous works in this topic, we assume that the process starts from a constant product measure, meaning that the initial opinions $\xi_0(x)$, $x \in \mathscr{V}$, are independent and identically distributed, and we let X be the random variable with distribution

$$P(X \in B) = P(\xi_0(x) \in B) \quad \text{for all } x \in \mathscr{V} \text{ and all Borel subsets } B \subset \Delta.$$

The evolution rules are based on two parameters: the confidence threshold $\tau > 0$ and the convergence parameter $\mu \in (0, 1/2]$. At each time step, an edge is chosen uniformly at random, which results in a potential update of the system at the two vertices connected by this edge. More precisely, assuming that edge $(x, y) \in \mathscr{E}$ is selected at time t, we let

$$\begin{aligned} \xi_t(x) &= \xi_{t-1}(x) + \mu\,(\xi_{t-1}(y) - \xi_{t-1}(x))\ \mathbf{1}\{\|\xi_{t-1}(x) - \xi_{t-1}(y)\| \leq \tau\} \\ \xi_t(y) &= \xi_{t-1}(y) + \mu\,(\xi_{t-1}(x) - \xi_{t-1}(y))\ \mathbf{1}\{\|\xi_{t-1}(x) - \xi_{t-1}(y)\| \leq \tau\} \end{aligned}$$

while the opinions at the other vertices remain unchanged. In words, the two neighbors that are selected interact if and only if their opinion distance or level of disagreement before the interaction does not exceed the confidence threshold τ, which results in a partial averaging of their opinions by a factor μ, called the convergence parameter.

Our main result gives a lower bound for the probability of consensus that applies to any finite connected graph, any opinion space (convex set and norm), and any initial distribution with value in the opinion space. To state this result, we let

$$\mathbf{r} = \inf\{r > 0 : \Delta \subset B(c, r) \text{ for some } c \in \Delta\} \quad \text{where} \quad B(c, r) = \{a \in \mathbb{R}^n : \|a - c\| \leq r\}$$

and fix $\mathbf{c} \in \Delta$ such that $\Delta \subset B(\mathbf{c}, \mathbf{r})$. Note that, by definition of $\mathbf{r}$, which we call the radius of the opinion space, and because the opinion space is convex, such a point $\mathbf{c}$ indeed exists.

Theorem 2.1 (probability of consensus)**.** – For all $\tau > \mathbf{r}$,

$$P(\mathscr{C}) \geq 1 - \frac{E\,\|X - \mathbf{c}\|}{\tau - \mathbf{r}} \quad \text{where} \quad \mathscr{C} = \left\{ \lim_{t \to \infty} \max_{x, y \in \mathscr{V}} \|\xi_t(x) - \xi_t(y)\| = 0 \right\}.$$

Recall that the simulations in [12] suggest that, when the individuals are located on an infinite connected graph and the initial opinions are chosen uniformly at random in the unit interval, the

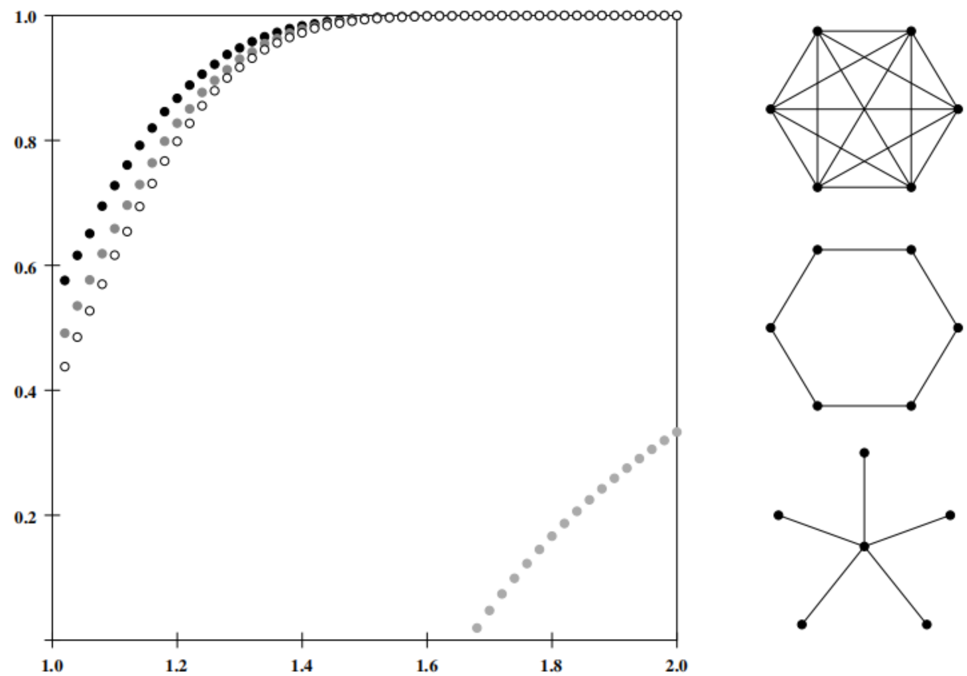

Figure 2.1: Simulation results for the probability of consensus as a function of τ for the process on the complete graph (black dots), the ring (grey dots) and the star (white dots) with six vertices depicted on the right. The convergence parameter $\mu = 1/4$ while the opinion space is the unit Euclidean ball equipped with the Euclidean norm in two dimensions. Each simulation point is obtained from the average of 100,000 realizations of the process with confidence threshold ranging from the radius to the diameter of the opinion space. The grey dots at the bottom right show the universal lower bound (valid for all finite connected graphs) derived from Theorem 2.1.

process exhibits a phase transition from coexistence to consensus at the critical threshold $\tau = 1/2$, a result that was proved rigorously in some particular cases. In view of this conjecture, it is reasonable to believe that, again for infinite connected graphs and uniformly distributed initial opinions but a more general opinion space, there is now a phase transition at the critical value $\tau =$ the radius of the opinion space. In particular, we conjecture that, for infinite connected graphs, we have almost sure consensus under the assumption of our theorem: $\tau > \mathbf{r}$. The reason why the probability of consensus when $\tau > \mathbf{r}$ is strictly less than one when switching to finite graphs is simply due to the presence of strong random fluctuations on finite graphs. The law of large numbers used in [15] no longer operates. For the same reasons, while the condition $\tau < \mathbf{r}$ should lead to coexistence for infinite graphs, the probability of consensus on finite graphs is strictly positive. Indeed, it follows from the argument of convexity in the proof of Lemma 2.5 below that, when all the opinions are initially in the same ball with radius $\tau/2$, which occurs with probability at least the ratio of the Lebesgue measure of this ball to the Lebesgue measure of the opinion space raised at the power the number of vertices, consensus occurs. Note however that, in contrast with the lower bound in our theorem, the lower bound above strongly depends on the size of the graph.

The key to proving the theorem is to study a collection of auxiliary processes (see (2.1) below) that keep track of the cumulative disagreement between a fixed opinion $c \in \Delta$ and the opinions at each of the vertices at time t. Using a triangle-type inequality (Lemma 2.1), we first prove that all these auxiliary processes are almost surely nonincreasing, meaning that, for all c, the averaging procedure can only decrease the overall level of disagreement between an observer with fixed opinion c and the population (Lemma 2.2). Almost sure monotonicity implies two important results:

1. The opinion model converges almost surely to a (random) limiting configuration.

   In addition, due to the evolution rules, each limiting configuration is characterized by a partition of the graph into connected components such that all the individuals in the same component share the same opinion and the distance between opinions in two adjacent components exceeds the confidence threshold τ (Lemma 2.5).

2. All the auxiliary processes are bounded supermartingales.

   In particular, one may apply the optional stopping theorem to these supermartingales and a certain stopping time (Lemma 2.6) to obtain a lower bound for the probability that the random partition above consists of only one set, meaning that all the individuals in the limiting configuration share the same opinion and consensus occurs.

Our proof leads to a lower bound that depends on the confidence threshold, the opinion space (convex set and norm) and the initial distribution, but not on the size and/or the topology of the social network. In particular, our lower bound is universal in the sense that it is uniform over all possible choices of the network, but we point out that, as shown in Figure 2.1, the (exact) probability of consensus should depend on the network. Indeed, our simulations suggest for instance that the complete graph promotes consensus more than the star graph.

The rest of the paper is devoted to proofs. In the next section, we show that the opinion model converges almost surely to a (random) limiting configuration in which neighbors either share the same opinion or disagree too much to interact. Then, we use the optional stopping theorem for supermartingales to derive the universal lower bound for the probability of consensus.

2.3 Limiting configurations

The objective of this section is to prove that, regardless of the initial configuration, the process converges almost surely to a limiting configuration in which any two neighbors either share the same opinion or disagree too much to interact, i.e.,

(P1) $\lim_{t \to \infty} \xi_t(x) = \xi_\infty(x)$ exists for all $x \in \mathscr{V}$

(P2) $\|\xi_\infty(x) - \xi_\infty(y)\| \notin (0, \tau]$ for all edges $(x, y) \in \mathscr{E}$.

From now on, we let $(X_t(c))$ be the process defined by

$$X_t(c) = \sum_{x \in \mathscr{V}} \|\xi_t(x) - c\| \quad \text{for all} \quad c \in \mathbb{R}^n. \tag{2.1}$$

That is, the process keeps track of the cumulative disagreement between a fixed opinion c possibly outside Δ and the opinions at each of the vertices. In particular, this collection of processes is somewhat reminiscent of the concept of energy in [15] in the sense that both can be viewed as measures of the overall disorder in the process that is expected to decrease under the influence of the averaging procedure. To shorten the notation, we let

$$\phi : \Delta \times \Delta \to \Delta \quad \text{defined as} \quad \phi(a, b) = (1 - \mu)a + \mu b = a + \mu(b - a).$$

In particular, whenever a vertex x that has opinion a interacts with a vertex y that has a compatible opinion $b \in B(a, \tau)$, the opinion at x becomes $\phi(a, b)$ and the opinion at y becomes $\phi(b, a)$. Although the details are somewhat more complicated, the basic idea to prove the two properties above is to show that the processes $(X_t(c))$ converge almost surely. To begin with, we prove the following lemma which is illustrated in Figure 2.2 and gives two variants of the triangle inequality.

Lemma 2.1 (triangle inequalities)**.** – For all $a, b \in \Delta$ and all $c \in \mathbb{R}^n$,

$$\begin{aligned} \|\phi(a, b) - c\| + \|\phi(b, a) - c\| &\leq \|a - c\| + \|b - c\| \\ \|\phi(a, b) - c\| + \|\phi(b, a) - c\| &\leq \|a - c\| + \|b - c\| - 2\|\phi(a, b) - a\| + \|a + b - 2c\|. \end{aligned}$$

Proof. Using the triangle inequality and absolute homogeneity, we get

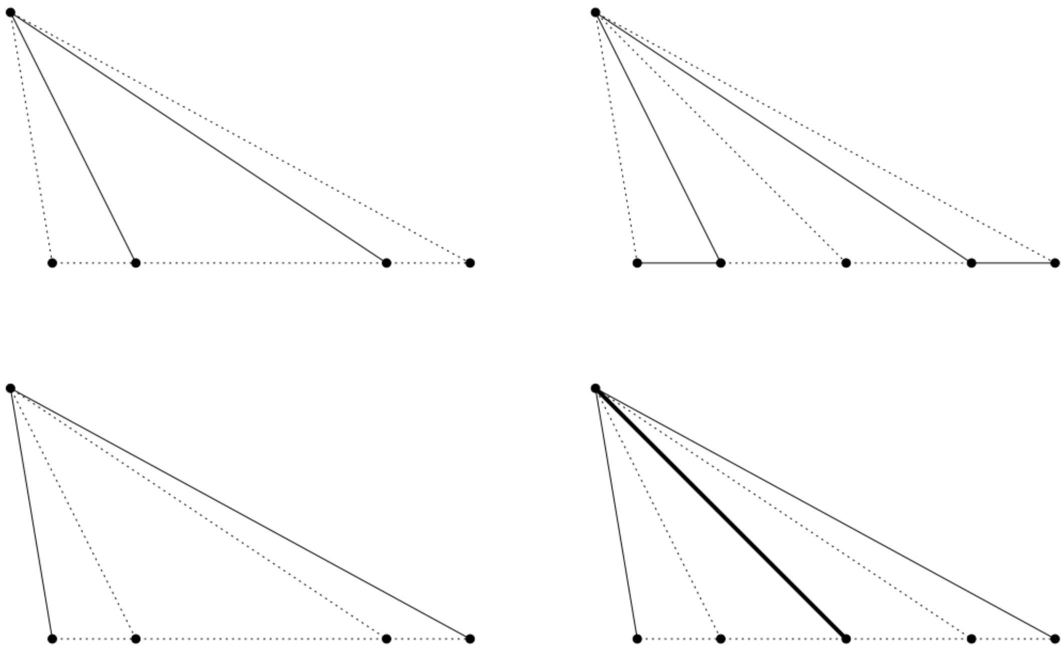

Figure 2.2: Illustration of Lemma 2.1. The lemma simply states that the sum of the norms of the vectors in solid lines is larger for the pictures at the bottom than for the pictures at the top, where the median in thick line in the bottom right picture is counted twice.

$$
\begin{aligned}
&\|\phi(a,b)-c\|+\|\phi(b,a)-c\| = \|(1-\mu)a+\mu b-c\|+\|(1-\mu)b+\mu a-c\| \\
&\quad = \|(1-\mu)(a-c)+\mu(b-c)\|+\|(1-\mu)(b-c)+\mu(a-c)\| \\
&\quad \leq \|(1-\mu)(a-c)\|+\|\mu(b-c)\|+\|(1-\mu)(b-c)\|+\|\mu(a-c)\| \\
&\quad = \|a-c\|+\|b-c\|
\end{aligned}
$$

which proves the first inequality. Now, because $0<\mu\leq 1/2$, the opinions

$$a, \quad \phi(a,b), \quad c_0=(a+b)/2, \quad \phi(b,a), \quad b$$

all lie on the segment line $[a,b]$ in this specific order going from point a to point b, therefore using again the triangle inequality and absolute homogeneity, we obtain

$$
\begin{aligned}
&\|\phi(a,b)-c\|+\|\phi(b,a)-c\| \leq \|\phi(a,b)-c_0\|+\|c_0-c\|+\|\phi(b,a)-c_0\|+\|c_0-c\| \\
&\quad = \|\phi(a,b)-\phi(b,a)\|+2\|c_0-c\| \\
&\quad = \|a-b\|-\|\phi(a,b)-a\|-\|\phi(b,a)-b\|+\|2(c_0-c)\| \\
&\quad \leq \|a-c\|+\|b-c\|-2\|\phi(a,b)-a\|+\|a+b-2c\|
\end{aligned}
$$

which proves the second inequality. This completes the proof. $\square$

In the next lemma, we use the first inequality in Lemma 2.1 to prove that, for all $c \in \Delta$, the processes $(X_t(c))$ are almost surely nonincreasing.

Lemma 2.2 (monotonicity)**.** – For all $c \in \Delta$,

$$0 \leq X_t(c) \leq X_s(c) \leq 2\mathbf{r} \cdot \operatorname{card}(\mathscr{V}) \quad \text{for all} \quad s \leq t.$$

Proof. At each update of the processes, say at time s,

$$\xi_s(x) = \phi(\xi_{s-1}(x), \xi_{s-1}(y)) \quad \text{and} \quad \xi_s(y) = \phi(\xi_{s-1}(y), \xi_{s-1}(x)) \quad \text{for some} \quad (x, y) \in \mathscr{E}.$$

In particular, applying Lemma 2.1 with $a = \xi_{s-1}(x)$ and $b = \xi_{s-1}(y)$, we get

$$\begin{aligned} X_s(c) - X_{s-1}(c) &= \|\xi_s(x) - c\| + \|\xi_s(y) - c\| - \|\xi_{s-1}(x) - c\| - \|\xi_{s-1}(y) - c\| \\ &= \|\phi(a, b) - c\| + \|\phi(b, a) - c\| - \|a - c\| - \|b - c\| \leq 0. \end{aligned}$$

In addition, because $c \in \Delta$ and $\Delta \subset B(\mathbf{c}, \mathbf{r})$, we have

$$0 \leq X_t(c) = \sum_{x \in \mathscr{V}} \|\xi_t(x) - c\| \leq \sum_{x \in \mathscr{V}} \Big(\|\xi_t(x) - \mathbf{c}\| + \|c - \mathbf{c}\| \Big) \leq \sum_{x \in \mathscr{V}} 2\mathbf{r} = 2\mathbf{r} \cdot \operatorname{card}(\mathscr{V}) < \infty.$$

This completes the proof. $\square$

Note that Lemma 2.2 implies that the processes $(X_t(c))$ are bounded supermartingales, which will be used later with the optional stopping theorem to derive our universal lower bound for the probability of consensus. By the martingale convergence theorem, each of these processes converges almost surely to a finite random variable, which suggests almost sure convergence of the interacting particle system. The main difficulty to prove this result is that whenever two vertices with compatible opinions a and b interact, the process $(X_t(c))$ does not "see the update" when a, b, c are aligned in this order. For some norms, the lack of alignment is not even a sufficient condition for the process to see the change of opinions so it is not clear how to deduce convergence of the system. To prove this result, we now use Lemma 2.2 and the second inequality in Lemma 2.1 to show that the process keeps slowing down in the sense that the jumps at each vertex get smaller and smaller.

Lemma 2.3 (slow-down)**.** – For all $\epsilon > 0$, there is $S = S(\epsilon)$ almost surely finite such that

$$\|\xi_s(x) - \xi_{s-1}(x)\| < \epsilon \quad \text{for all} \quad s \geq S \text{ and } x \in \mathscr{V}.$$

Proof. Assume by contradiction that, for some $\epsilon > 0$ and $x \in \mathscr{V}$, the opinion at x jumps by more than ϵ infinitely often with positive probability, and let (s_i) be the times of these updates:

$$\|\xi_{s_i}(x) - \xi_{s_i-1}(x)\| \geq \epsilon \quad \text{for all} \quad i > 0.$$

Letting $y_i \in \mathscr{V}$ be the vertex that interacts with x at time s_i, setting

$$a_i = \xi_{s_i-1}(x), \quad b_i = \xi_{s_i-1}(y) \quad \text{and} \quad c_i = (a_i + b_i)/2,$$

and applying the second inequality in Lemma 2.1 with $a = a_i$ and $b = b_i$, we get

$$\begin{aligned} X_{s_i}(c) - X_{s_i-1}(c) &= \|\xi_{s_i}(x) - c\| + \|\xi_{s_i}(y) - c\| - \|\xi_{s_i-1}(x) - c\| - \|\xi_{s_i-1}(y) - c\| \\ &= \|\phi(a_i, b_i) - c\| + \|\phi(b_i, a_i) - c\| - \|a_i - c\| - \|b_i - c\| \\ &\leq -2\|\phi(a_i, b_i) - a_i\| + \|a_i + b_i - 2c\| = -2\|\xi_{s_i}(x) - \xi_{s_i-1}(x)\| + 2\|c_i - c\| \\ &\leq -2\epsilon + 2\|c_i - c\| \leq -\epsilon \end{aligned} \tag{2.2}$$

for all $c \in B(c_i, \epsilon/2)$. Now, observe that there exists $\epsilon' > 0$ such that

$$B(c, \epsilon/2) \cap \Delta(\epsilon') \neq \varnothing \quad \text{for all} \quad c \in \Delta \quad \text{where} \quad \Delta(\epsilon') = \Delta \cap (\epsilon'\mathbb{Z})^n \tag{2.3}$$

and where $(\epsilon'\mathbb{Z})^n$ is a grid with mesh size ϵ' in n dimensions. It follows from the Pythagorean theorem that the Euclidean distance from c to the grid is bounded by

$$\sqrt{(\epsilon'/2)^2 + \cdots + (\epsilon'/2)^2} = \sqrt{n(\epsilon'/2)^2} = \sqrt{n}\,(\epsilon'/2)$$

which implies that (2.3) holds for $\epsilon' < \epsilon/\sqrt{n}$. This and the equivalence of the norms in finite dimensions imply that, for each norm, there indeed exists $\epsilon' > 0$ such that (2.3) holds. In addition, because the opinion space Δ is bounded, and again the dimension is finite,

$$\operatorname{card}(\Delta(\epsilon')) < \infty \quad \text{for all} \quad \epsilon' > 0. \tag{2.4}$$

Combining (2.3) and (2.4), we deduce that

$$\Delta'(\epsilon') = \{c \in \Delta(\epsilon') : \operatorname{card}\{i : c \in B(c_i, \epsilon/2)\} = \infty\} \neq \varnothing.$$

In particular, there exists

$$c' \in \Delta'(\epsilon') \quad \text{such that} \quad I = \{i \in \mathbb{N} : c \in B(c_i, \epsilon/2)\} \text{ is infinite.}$$

This, together with (2.2) and Lemma 2.2, implies that

$$\lim_{t \to \infty} X_t(c') \leq X_0(c') + \sum_{i \in I} (X_{s_i}(c') - X_{s_i-1}(c')) = X_0(c') + \sum_{i \in I} (-\epsilon) = -\infty,$$

which contradicts the fact that $(X_t(c'))$ is positive. $\square$

The next lemma shows that the jumps getting smaller and smaller implies that, for large times, neighbors must either be incompatible or have almost the same opinion.

Lemma 2.4 (clustering)**.** – For all $0 < \epsilon < \tau$, there is $T = T(\epsilon)$ almost surely finite such that

$$\|\xi_s(x) - \xi_s(y)\| \notin [\epsilon, \tau] \quad \text{for all} \quad s \geq T \text{ and } (x, y) \in \mathscr{E}.$$

Proof. Assume by contradiction that there exist $\epsilon > 0$ and $(x, y) \in \mathscr{E}$ such that the opinion distance along the edge belongs to $[\epsilon, \tau]$ infinitely often, meaning that

$$\xi_{s_i}(x) - \xi_{s_i}(y) \in [\epsilon, \tau] \quad \text{for an increasing sequence} \quad (s_i) \subset \mathbb{N}.$$

Letting A_i be the event that edge (x, y) is selected at time $s_i + 1$, because the edge selected at each time step is chosen uniformly at random and independently of everything else,

$$\sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} \frac{1}{\operatorname{card}(\mathscr{E})} = \infty \quad \text{and} \quad \text{the events } (A_i) \text{ are independent.}$$

In particular, it follows from the second Borel-Cantelli lemma that

$$P\Big(\limsup_{i \to \infty} A_i\Big) = P(\operatorname{card}\{i \geq 1 : A_i \text{ occurs}\} = \infty) = 1. \tag{2.5}$$

In addition, on the event A_i,

$$\|\xi_{s_i+1}(x) - \xi_{s_i}(x)\| = \|\phi(\xi_{s_i}(x), \xi_{s_i}(y)) - \xi_{s_i}(x)\| = \|\mu(\xi_{s_i}(x) - \xi_{s_i}(y))\| \geq \mu\epsilon. \tag{2.6}$$

Combining (2.5) and (2.6), we deduce that, with probability one, the opinion at x jumps by more than $\mu\epsilon$ infinitely often, which contradicts Lemma 2.3. This completes the proof. □

To deduce almost sure convergence of the particle system from the previous lemma, the last step is to prove that neighbors who almost totally agree cannot randomly oscillate together, which follows from an argument of convexity. The proof of the next lemma shows in fact a little bit more: there is a partition of the graph into connected components such that all the opinions in the same component are eventually trapped in a fixed ball with arbitrarily small radius while opinions in two adjacent components are incompatible, which implies in particular (P1) and (P2).

Lemma 2.5 (convergence)**.** – Properties (P1) and (P2) hold.

Proof. Let $N = \operatorname{card}(\mathscr{V})$ and $0 < \epsilon < \tau/N$. According to Lemma 2.4, there exists a random but almost surely finite time T such that

$$\|\xi_s(x) - \xi_s(y)\| \notin [\epsilon/N, \tau] \quad \text{for all} \quad s \geq T \text{ and } (x, y) \in \mathscr{E}$$

and we write $x \leftrightarrow y$ if there exist $x_0 = x, x_1, \ldots, x_j = y$ all distinct such that

$$(x_i, x_{i+1}) \in \mathscr{E} \text{ and } \|\xi_s(x_i) - \xi_s(x_{i+1})\| < \epsilon/N \quad \text{for all} \quad 0 \leq i < j \text{ and } s \geq T.$$

In particular, by the triangle inequality,

$$\|\xi_T(x) - \xi_T(y)\| \leq \sum_{i=0}^{j-1} \|\xi_T(x_i) - \xi_T(x_{i+1})\| < \frac{j\epsilon}{N} \leq \epsilon. \tag{2.7}$$

The relationship $\leftrightarrow$ defines an equivalence relationship so it induces a partition of the vertex set into equivalence classes $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_k$ that correspond to connected components of the graph. In addition, by (2.7) and the definition of $\leftrightarrow$, there exist $c_1, c_2, \ldots, c_k \in \Delta$ such that

(a) for all $i = 1, 2, \ldots, k$, we have $\xi_T(x) \in B(c_i, \epsilon)$ for all $x \in \mathscr{V}_i$ and

(b) whenever $\mathscr{V}_i$ and $\mathscr{V}_j$ are connected by $(x, y) \in \mathscr{E}$, we have $\|\xi_T(x) - \xi_T(y)\| > \tau$.

Assume that properties (a) and (b) hold from time T to time $s > T$ and that edge (x, y) is selected at time $s + 1$. Then, either $x \leftrightarrow y$, say $x, y \in \mathscr{V}_i$, in which case

$$[\xi_{s+1}(x), \xi_{s+1}(y)] = [(1 - \mu)\, \xi_s(x) + \mu\, \xi_s(y), (1 - \mu)\, \xi_s(y) + \mu\, \xi_s(x)] \subset [\xi_s(x), \xi_s(y)] \subset B(c_i, \epsilon)$$

by convexity of $B(c_i, \epsilon)$, or edge (x, y) connects two different classes in which case

$$\|\xi_s(x) - \xi_s(y)\| > \tau \quad \text{therefore} \quad \xi_{s+1}(x) = \xi_s(x) \text{ and } \xi_{s+1}(y) = \xi_s(y).$$

In either case, properties (a) and (b) remain true after the interaction. Because $\epsilon > 0$ can be chosen arbitrarily small, this proves that properties (P1) and (P2) hold. □

2.4 Stopping time and consensus event

This section is devoted to the proof of Theorem 2.1. As mentioned after the proof of Lemma 2.2, the processes $(X_t(c))$ are bounded supermartingales so the idea is to apply the optional stopping theorem. Before proving the theorem, we define a suitable stopping time and show how the consensus event relates to the configuration of the system at this stopping time. Let

$$T_* = \inf\{t : \|\xi_t(x) - \xi_t(y)\| \notin [\tau/2, \tau] \text{ for all } x, y \in \mathscr{V}\}.$$

Note that time T_* is a stopping time for the natural filtration of the process. Time T_* is also almost surely finite according to Lemma 2.4, so we have the following result.

Lemma 2.6. – Time T_* is an almost surely finite stopping time.

We now identify a collection of configurations at the stopping time T_* that always lead the population to consensus eventually. More precisely, we let

$$\mathscr{A} = \bigcup_{x \in \mathscr{V}} \left\{ \sup_{c \in \Delta} \|\xi_{T_*}(x) - c\| < \tau \right\}$$

be the event that, at the stopping time, there is (at least) one "centrist" individual whose opinion is within distance τ of all other possible opinions. Then, we have the following inclusion showing that the event $\mathscr{A}$ plays the role of an attractor in the sense that, whenever this event occurs, the process will almost surely evolve to a consensus.

Lemma 2.7 (attractor)**.** – We have the inclusion $\mathscr{A} \subset \mathscr{C}$.

Proof. The definition of T_* implies that

$$\xi_{T_*}(y) \in B(\xi_{T_*}(x), \tau) \;\Rightarrow\; \xi_{T_*}(y) \in B(\xi_{T_*}(x), \tau/2). \tag{2.8}$$

In addition, by the proof of Lemma 2.5 (convexity argument),

$$\xi_{T_*}(y) \in B(c, \tau/2) \text{ for all } y \in \mathscr{V} \;\Rightarrow\; \xi_s(y) \in B(c, \tau/2) \text{ for all } y \in \mathscr{V} \text{ and } s > T_*. \tag{2.9}$$

This, together with Lemma 2.5 itself, gives the implications

$$\begin{aligned}
&\sup\nolimits_{c \in \Delta} \|\xi_{T_*}(x) - c\| < \tau \text{ for some } x \in \mathscr{V} \\
&\quad \Rightarrow\; (\xi_{T_*}(y) \in B(\xi_{T_*}(x), \tau) \text{ for all } y \in \mathscr{V}) \text{ for some } x \in \mathscr{V} \\
&\quad \Rightarrow\; (\xi_{T_*}(y) \in B(\xi_{T_*}(x), \tau/2) \text{ for all } y \in \mathscr{V}) \text{ for some } x \in \mathscr{V} \quad \text{(by (2.8))} \\
&\quad \Rightarrow\; (\xi_s(y) \in B(c, \tau/2) \text{ for all } y \in \mathscr{V} \text{ and } s > T_*) \text{ for some } c \in \Delta \quad \text{(by (2.9))} \\
&\quad \Rightarrow\; \lim\nolimits_{s \to \infty} \|\xi_s(y) - \xi_s(z)\| = 0 \text{ for all } y, z \in \mathscr{V} \quad \text{(by (P2) and choice of } \tau/2).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 2.1. According to Lemma 2.2, for all $c \in \Delta$, the processes $(X_t(c))$ is bounded and almost surely nonincreasing. In particular, the process is a bounded supermartingale with respect to the natural filtration of the opinion model. According to Lemma 2.6, we also have that the random time T_* is an almost surely finite stopping time with respect to the same filtration. In particular, it follows from the optional stopping theorem that, for all $c \in \Delta$,

$$E(X_{T_*}(c)) \leq E(X_0(c)) = E\bigg(\sum_{x \in \mathscr{V}} \|\xi_0(x) - c\| \bigg) = \operatorname{card}(\mathscr{V}) \cdot E\,\|X - c\|. \tag{2.10}$$

Now, on the complement of $\mathscr{A}$,

$$\text{for all } x \in \mathscr{V}, \quad \text{there exists } c_x \in \Delta \quad \text{such that} \quad \|\xi_{T_*}(x) - c_x\| \geq \tau.$$

This and the triangle inequality imply that

$$\|\xi_{T_*}(x) - \mathbf{c}\| \geq \|\xi_{T_*}(x) - c_x\| - \|c_x - \mathbf{c}\| \geq \tau - \mathbf{r} \quad \text{for all} \quad x \in \Delta.$$

This gives the following bound for the conditional expectation:

$$E(X_{T_*}(\mathbf{c}) \,|\, \mathscr{A}^c) = E\bigg(\sum_{x \in \mathscr{V}} \|\xi_{T_*}(x) - \mathbf{c}\| \,\bigg|\, \mathscr{A}^c\bigg) \geq (\tau - \mathbf{r}) \cdot \operatorname{card}(\mathscr{V}). \tag{2.11}$$

Combining (2.10) with $c = \mathbf{c}$ and (2.11), we deduce that

$$(\tau - \mathbf{r})(1 - P(\mathscr{A})) \leq \frac{E(X_{T_*}(\mathbf{c}) \,|\, \mathscr{A}^c)\, P(\mathscr{A}^c)}{\operatorname{card}(\mathscr{V})} \leq \frac{E(X_{T_*}(\mathbf{c}))}{\operatorname{card}(\mathscr{V})} \leq E\,\|X - \mathbf{c}\|$$

which, together with Lemma 2.7, implies that

$$P(\mathscr{C}) \geq P(\mathscr{A}) \geq 1 - \frac{E\,\|X - \mathbf{c}\|}{\tau - \mathbf{r}} \quad \text{for all} \quad \tau > \mathbf{r}.$$

This completes the proof of the theorem. $\square$

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Chapter 3

MIXED HEGSELMANN-KRAUSE DYNAMICS

Hsin-Lun Li

Abstract. The original Hegselmann-Krause (HK) model consists of a set of n agents that are characterized by their opinion, a number in $[0, 1]$. Each agent, say agent i, updates its opinion x_i by taking the average opinion of all its neighbors, the agents whose opinion differs from x_i by at most ϵ. There are two types of HK models: the synchronous HK model and the asynchronous HK model. For the synchronous model, all the agents update their opinion simultaneously at each time step, whereas for the asynchronous HK model, only one agent chosen uniformly at random updates its opinion at each time step. This paper is concerned with a variant of the HK opinion dynamics, called the mixed HK model, where each agent can choose its degree of stubbornness and mix its opinion with the average opinion of its neighbors at each update. The degree of the stubbornness of agents can be different and/or vary over time. An agent is not stubborn or absolutely open-minded if its new opinion at each update is the average opinion of its neighbors, and absolutely stubborn if its opinion does not change at the time of the update. The particular case where, at each time step, all the agents are absolutely open-minded is the synchronous HK model. In contrast, the asynchronous model corresponds to the particular case where, at each time step, all the agents are absolutely stubborn except for one agent chosen uniformly at random who is absolutely open-minded. I first show that some of the common properties of the synchronous HK model, such as finite-time convergence, do not hold for the mixed model. I then investigate conditions under which the asymptotic stability holds, or a consensus can be achieved for the mixed model.

3.1 Introduction

The Hegselmann-Krause (HK) model is a popular opinion dynamics model describing the interactions among a population of agents. In the standard HK model, there are n agents and each agent updates its opinion by taking the average opinion of its neighbors. More precisely, let

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) \quad \text{where} \quad x_i(t) \in \mathbf{R}^{\mathbf{d}}$$

represents the opinion of agent i at time $t \in \mathbf{N}$, and let

$$N_i(t) = \{j \in [n] : \|x_i(t) - x_j(t)\| \leq \epsilon\} \quad \text{where} \quad [n] = \{1, 2, \ldots, n\}$$

be the set of agents whose opinion differs from the opinion of agent i by at most ϵ, that I call the neighbors of agent i at time t. Here, $\| \ \|$ refers to the Euclidean norm and ϵ is a positive number that represents a confidence bound. The authors of [17] considered the one-dimensional modified HK model as follows:

$$x_i(t+1) = \alpha_i \, x_i(t) + \frac{(1-\alpha_i)}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) \quad \text{where} \quad x_i(t) \text{ and } \alpha_i \in [0,1].$$

In words, the convex combination indicates that agent i mixes its opinion with the average opinion of its neighbors, with the parameter α_i measuring the degree of stubbornness of agent i. In this paper, I extend the modified HK model to higher dimensional sets of opinions and allow the degree of stubbornness α_i to vary over time. The resulting model can be expressed in matrix form as

$$x(t+1) = \text{diag}(\alpha(t)) \, x(t) + (I - \text{diag}(\alpha(t))) \, A(t) \, x(t) \tag{3.1}$$

where $A(t) \in \mathbf{R^{n \times n}}$ is row stochastic with

$$A_{ij} = \mathbb{1}\{j \in N_i(t)\}/|N_i(t)|$$

and where

$$\begin{aligned} x(t) &= (x_1(t), x_2(t), \ldots, x_n(t))' \ = \text{transpose of } (x_1(t), x_2(t), \ldots, x_n(t)), \\ \alpha(t) &= (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))' = \text{transpose of } (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)). \end{aligned}$$

In particular, agent i is absolutely stubborn when $\alpha_i(t) = 1$ and absolutely open-minded when $\alpha_i(t) = 0$. Observe also that (3.1) reduces to

- the synchronous HK model if $\alpha(t) = \vec{0}$ for all $t \geq 0$ and
- the asynchronous HK model if $\alpha(t) = (\mathbb{1}\{j \neq i(t)\})_{j=1}^n$ for all $t \geq 0$ and for some $i(t) \in [n]$ chosen uniformly at random.

Our main objective is to study the strategies the agents should play so that the asymptotic stability holds, or a consensus can be achieved. Some of the common properties of the synchronous HK model do not hold for the mixed HK model. Before going into the details, I need the following definitions.

Definition 3.1. An *opinion profile* at time t or simply a *profile* at time t is an undirected graph $\mathscr{G}(t)$ with the vertex set and edge set

$$\mathscr{V}(t) = [n] \quad \text{and} \quad \mathscr{E}(t) = \{ij : i \neq j \text{ and } \|x_i(t) - x_j(t)\| \leq \epsilon\}.$$

Apart from [2], the opinion profile is simple.

Definition 3.2. The *termination time* of n agents, T_n, is the maximum number of iterations in (3.1) by reaching a steady state over all initial profiles, i.e.,

$$T_n = \inf\{t \geq 0 : x(t) = x(s) \text{ for all } s \geq t\}.$$

Definition 3.3. The *convex hull* generated by $v_1, v_2, \ldots, v_n \in \mathbf{R^d}$ is the smallest convex set containing $v_1, v_2, \ldots, v_n$, i.e.,

$$C(\{v_1, v_2 \ldots, v_n\}) = \{v : v = \sum_{i=1}^{n} \lambda_i v_i \text{ where } (\lambda_i)_{i=1}^{n} \text{ is stochastic}\}.$$

Definition 3.4. A profile $\mathscr{G}(t)$ is δ-*trivial* if any two of its vertices are at a distance of at most δ apart. In particular, $\mathscr{G}(t)$ is complete if it is ϵ-trivial.

Definition 3.5. For $\delta > 0$, $x(t)$ in (3.1) is a δ-*equilibrium* if there is a partition

$$\{G_1, G_2, \ldots, G_m\} \text{ of the set } \{x_1(t), x_2(t), \ldots, x_n(t)\}$$

such that the following two conditions hold:

$$\text{dist}(C(G_i), C(G_j)) > \epsilon \text{ for all } i \neq j \quad \text{and} \quad \text{diam}(C(G_i)) \leq \delta \text{ for all } i \in [m].$$

Definition 3.6. A *merging time* is a time t that two agents with different opinions at time $t-1$ have the same opinion at time t, i.e.,

$$x_i(t) = x_j(t) \quad \text{and} \quad x_i(t-1) \neq x_j(t-1) \quad \text{for some} \quad i, j \in [n].$$

The following are some properties distinct from the synchronous HK model.

Property 3.1. The termination time is not finite.

Example 3.1. Assume that $n = 2,\ d = 1$,

$$x_1(0) = 0,\ x_2(0) = \epsilon \text{ and } \alpha_1(t) = \alpha_2(t) = 1/2 \text{ for all } t \geq 0.$$

Then, at each time step, x_1 and x_2 get closer to each other. However, never do they reach a steady state in finite time.

Property 3.2. Agents merging at time t may depart at time $t+1$. In particular, $\mathscr{G}(t)$ ϵ-trivial may not imply that $x(t+1)$ in (3.1) is a steady state.

Example 3.2. Assume that $n = 3,\ d = 2$,

$$
\begin{aligned}
x_1(0) &= (0,0), & \alpha_1(0) &= 0, & \alpha_1(1) &= 1/3,\\
x_2(0) &= (\epsilon,0), & \alpha_2(0) &= 0, & \alpha_2(1) &= 1/2,\\
x_3(0) &= (\epsilon/2,\epsilon).
\end{aligned}
$$

Then, x_1 and x_2 merge at time $t = 1$ but depart at time $t = 2$.

Property 3.3. A δ-equilibrium may not exist for all $0 < \delta \leq \epsilon$.

Example 3.3. Assume that $n = 3,\ d = 2$,

$$x_1(0) = (0,0),\ x_2(0) = (\epsilon,0),\ x_3(0) = (\epsilon/2,\epsilon) \text{ and } \alpha_1(t) = \alpha_2(t) = 1/2$$

for all $t \geq 0$. Then, x has no δ-equilibrium for all $0 < \delta \leq \epsilon$. Note that vertex 3 of the profile is isolated all the time.

The following lemma plays an important role in the proof of the main theorems.

Lemma 3.1. Let $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ with $\sum_{i=1}^n \lambda_i = 0$ and $x_1, \ldots, x_n \in \mathbf{R^d}$. Then, for

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n,$$

the terms with positive coefficients can be matched with the terms with negative coefficients in the sense that

$$\sum_{i=1}^n \lambda_i x_i = \sum_{i, c_i \geq 0, j, k \in [n]} c_i (x_j - x_k) \quad \text{and} \quad \sum_i c_i = \sum_{j, \lambda_j \geq 0} \lambda_j.$$

Proof. I prove the result by induction on n. Without loss of generality, I may assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $n = 2$, $\lambda_1 + \lambda_2 = 0$ implies that

$$\lambda_2 = -\lambda_1 \text{ and } \lambda_1 \geq 0 \quad \text{therefore} \quad \lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 (x_1 - x_2),$$

which proves the result for $n = 2$. Now, assume that $n > 2$. Because the λ_i's add up to 0, I have $\lambda_n \leq 0$. Define

$$\lambda_n = -\lambda \quad \text{and} \quad i = \min\left\{m \in \mathbf{Z}^+ : \sum_{k=1}^m \lambda_k \geq \lambda\right\}.$$

Then, $\lambda_k \geq 0$ for all $1 \leq k \leq i$ so

$$\sum_{k=1}^{n} \lambda_k x_k = \sum_{k=1}^{i-1} \lambda_k (x_k - x_n) + \Big(\lambda - \sum_{k=1}^{i-1} \lambda_k\Big)(x_i - x_n) + \Big(\sum_{k=1}^{i} \lambda_k - \lambda\Big) x_i + \sum_{k=i+1}^{n-1} \lambda_k x_k.$$

Now, observe that $\lambda - \sum_{k=1}^{i-1} \lambda_k \geq 0$, $\sum_{k=1}^{i} \lambda_k - \lambda \geq 0$ and

$$\Big(\sum_{k=1}^{i} \lambda_k - \lambda\Big) + \sum_{k=i+1}^{n-1} \lambda_k = \sum_{k=1}^{n-1} \lambda_k - \lambda = \sum_{k=1}^{n-1} \lambda_k + \lambda_n = \sum_{k=1}^{n} \lambda_k = 0.$$

By the induction hypothesis,

$$\Big(\sum_{k=1}^{i} \lambda_k - \lambda\Big) x_i + \sum_{k=i+1}^{n-1} \lambda_k x_k = \sum_{\ell, c_\ell \geq 0, j,k \in [n-1]-[i-1]} c_\ell \,(x_j - x_k),$$

$$\sum_{\ell} c_\ell = \Big(\sum_{k=1}^{i} \lambda_k - \lambda\Big) + \sum_{k \in [n-1]-[i], \lambda_k \geq 0} \lambda_k.$$

Hence, $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ can be written as

$$\sum_{\ell, \hat{c}_\ell \geq 0, j,k \in [n]} \hat{c}_\ell (x_j - x_k)$$

where the sum of the coefficients $\hat{c}_\ell$ is given by

$$\begin{aligned}\sum_{\ell} \hat{c}_\ell &= \sum_{k=1}^{i-1} \lambda_k + \Big(\lambda - \sum_{k=1}^{i-1} \lambda_k\Big) + \sum_{\ell} c_\ell = \lambda + \sum_{\ell} c_\ell \\ &= \lambda + \sum_{k=1}^{i} \lambda_k - \lambda + \sum_{k \in [n-1]-[i], \lambda_k \geq 0} \lambda_k = \sum_{k \in [n-1], \lambda_k \geq 0} \lambda_k = \sum_{k \in [n], \lambda_k \geq 0} \lambda_k.\end{aligned}$$

This completes the proof. □

This result allows us to observe the interactions among the agents and derive a better upper bound. For any $x, y \in C(\{v_1, ..., v_n\})$,

- the coefficients of all v_i's in $x - y$ add up to zero and
- the sum of the positive coefficients of the v_i's in $x - y$ is at most one.

In particular, by Lemma 3.1 and the triangle inequality,

$$\|x - y\| \leq \max_{i,j \in [n]} \|v_i - v_j\| \leq \operatorname{diam}(C(\{v_1, \ldots, v_n\}))$$

therefore $\operatorname{diam}(C(\{v_1, \ldots, v_n\})) = \max_{i,j \in [n]} \|v_i - v_j\|$.

Lemma 3.2. I have

$$\operatorname{diam}(C(\{v_1,\ldots,v_n\})) = \max_{i,j\in[n]} \|v_i - v_j\| \text{ for all } v_i \in \mathbf{R^d}.$$

In contrast with the synchronous HK model, $\mathscr{G}(t)$ ϵ-trivial may not imply that a consensus is reached at the next time step. However, $\mathscr{G}(t+1)$ is again ϵ-trivial. Observe that

$$x_i(t+1) \in C(\{x_1(t),\ldots,x_n(t)\}) \quad \text{for all} \quad i \in [n]$$

and according to Lemma 3.2,

$$\max_{i,j\in[n]} \|x_i(t+1) - x_j(t+1)\| \le \max_{i,j\in[n]} \|x_i(t) - x_j(t)\|.$$

Lemma 3.3 (δ-trivial-preserving)**.** For any $\delta > 0$, if

$$\mathscr{G}(t) \text{ is } \delta\text{-trivial, then } \mathscr{G}(t+1) \text{ is } \delta\text{-trivial.}$$

Indeed, I can derive a better upper bound for $\|x_i(t+1) - x_j(t+1)\|$ by re-organizing the terms of $x_i(t+1) - x_j(t+1)$.

Lemma 3.4. Assume that $\mathscr{G}(t)$ is ϵ-trivial. Then,

$$\begin{aligned}&\max_{i,j\in[n]} \|x_i(t+1) - x_j(t+1)\| \\ &\qquad \le \max_{i,j\in[n],\alpha_i(t)\ge\alpha_j(t)} \left(\alpha_i(t) - \frac{\alpha_i(t)-\alpha_j(t)}{n}\right) \max_{i,j\in[n]} \|x_i(t) - x_j(t)\|.\end{aligned}$$

Proof. Let $x = x(t)$, $x' = x(t+1)$ and $\alpha = \alpha(t)$. For any $i, j \in [n]$ with $\alpha_i \ge \alpha_j$,

$$x_i' - x_j' = \left(\alpha_i - \frac{\alpha_i - \alpha_j}{n}\right)x_i - \left(\alpha_j + \frac{\alpha_i - \alpha_j}{n}\right)x_j - \frac{\alpha_i - \alpha_j}{n} \sum_{k\in[n]-\{i,j\}} x_k.$$

Observe that

$$\alpha_i - \frac{\alpha_i - \alpha_j}{n} \ge \alpha_i - \frac{\alpha_i}{n} \ge 0, \quad \alpha_j + \frac{\alpha_i - \alpha_j}{n} \ge 0 \quad \text{and} \quad \frac{\alpha_i - \alpha_j}{n} \ge 0,$$

showing that x_i is the only term with nonnegative coefficient, whereas the other terms have non-positive coefficients. Because $x_i' \in C(\{x_1, x_2, \ldots, x_n\})$ for all $i \in [n]$, it follows from Lemma 3.1 that

$$x_i' - x_j' = \left(\alpha_j + \frac{\alpha_i - \alpha_j}{n}\right)(x_i - x_j) + \frac{\alpha_i - \alpha_j}{n} \sum_{k\in[n]-\{i,j\}} (x_i - x_k)$$

and the coefficients of the terms $x_i - x_k$ for $k \in [n] - \{i\}$ add up to $\alpha_i - \frac{\alpha_i - \alpha_j}{n}$. Thus, by the triangle inequality,

$$\begin{aligned}
\|x_i' - x_j'\| &\le \left(\alpha_j + \frac{\alpha_i - \alpha_j}{n}\right)\|x_i - x_j\| + \frac{\alpha_i - \alpha_j}{n} \sum_{k \in [n]-\{i,j\}} \|x_i - x_k\| \\
&\le \left(\alpha_i - \frac{\alpha_i - \alpha_j}{n}\right) \max_{k \in [n]-\{i\}} \|x_i - x_k\| \\
&= \left(\alpha_i - \frac{\alpha_i - \alpha_j}{n}\right) \max_{k \in [n]} \|x_i - x_k\| \\
&\le \max_{i,j \in [n], \alpha_i \ge \alpha_j} \left(\alpha_i - \frac{\alpha_i - \alpha_j}{n}\right) \max_{i,k \in [n]} \|x_i - x_k\|.
\end{aligned}$$

If $\alpha_i \le \alpha_j$, then exchanging the roles of i and j, I get

$$\begin{aligned}
\|x_j' - x_i'\| &\le \max_{j,i \in [n], \alpha_j \ge \alpha_i} \left(\alpha_j - \frac{\alpha_j - \alpha_i}{n}\right) \max_{j,k \in [n]} \|x_j - x_k\| \\
&= \max_{i,j \in [n], \alpha_i \ge \alpha_j} \left(\alpha_i - \frac{\alpha_i - \alpha_j}{n}\right) \max_{i,k \in [n]} \|x_i - x_k\|.
\end{aligned}$$

In conclusion,

$$\max_{i,j \in [n]} \|x_i' - x_j'\| \le \max_{i,j \in [n], \alpha_i \ge \alpha_j} \left(\alpha_i - \frac{\alpha_i - \alpha_j}{n}\right) \max_{i,k \in [n]} \|x_i - x_k\|.$$

This completes the proof. □

Observe that

$$\beta_t := \max_{i,j \in [n], \alpha_i(t) \ge \alpha_j(t)} \left(\alpha_i(t) - \frac{\alpha_i(t) - \alpha_j(t)}{n}\right) \le 1.$$

Therefore, $\mathscr{G}(t)$ ϵ-trivial implies $\mathscr{G}(s)$ ϵ-trivial for all $s \ge t$. Hence,

$$\max_{i,j \in [n]} \|x_i(s+1) - x_j(s+1)\| \le \beta_s \max_{i,j \in [n]} \|x_i(s) - x_j(s)\| \quad \text{for all} \quad s \ge t.$$

Theorem 3.1. Assume that $\limsup_{t \to \infty} \beta_t < 1$ and that $\mathscr{G}(t)$ is ϵ-trivial. Then,

$$\lim_{t \to \infty} \max_{i,j \in [n]} \|x_i(t) - x_j(t)\| = 0.$$

Proof. Define

$$d_s = \max_{i,j \in [n]} \|x_i(s) - x_j(s)\|.$$

According to Lemma 3.4,

$$\mathscr{G}(t) \ \epsilon\text{-trivial} \quad \Longrightarrow \quad d_{s+1} \le \beta_s d_s \text{ for all } s \ge t.$$

Since $\limsup_{t \to \infty} \beta_t < 1$, there exists $(t_i)_{i=1}^{\infty} \subset \mathbf{N}$ strictly increasing with $t_1 \ge t$ such that $\beta_{t_i} \le \delta < 1$ for some δ and for all $i \ge 1$. For any $s > t_1$, I have $t_{i_s} < s \le t_{i_s+1}$ for some $i_s \in \mathbf{Z}^+$ therefore

$$d_s \le \beta_{s-1}\beta_{s-2} \cdots \beta_{t_1} d_{t_1} \le \delta^{i_s} d_{t_1}.$$

As $s \to \infty$, $i_s \to \infty$. Thus,

$$\limsup_{s\to\infty} d_s \leq 0,$$

showing that the limit exists. This completes the proof. $\square$

In an ϵ-trivial profile, agents need not be open-minded all the time. As long as there are infinitely many β_t with an upper bound less than one, eventually will the population reach a consensus. The next theorem shows that, even though the profile is not ϵ-trivial, still can the agents' opinions converge.

Theorem 3.2. Define $d_t^i = \max_{j\in N_i(t)} \|x_i(t) - x_j(t)\|$. If

$$\sum_{t=0}^{\infty} (1-\alpha_i(t))\left(1 - \frac{1}{|N_i(t)|}\right) d_t^i < \infty, \text{ then } x_i(t) \to x_i \in \mathbf{R^d} \text{ as } t \to \infty.$$

Proof. By Lemma 3.1 and the triangle inequality,

$$\|x_i(t) - x_i(t+1)\| = \|(1-\alpha_i(t))\left(1 - \frac{1}{|N_i(t)|}\right)x_i(t) - \frac{1-\alpha_i(t)}{|N_i(t)|}\sum_{j\in N_i(t)-\{i\}} x_j(t)\|$$
$$= \frac{1-\alpha_i(t)}{|N_i(t)|}\left\|\sum_{j\in N_i(t)-\{i\}}[x_i(t) - x_j(t)]\right\| \leq (1-\alpha_i(t))\left(1 - \frac{1}{|N_i(t)|}\right)d_t^i,$$

from which it follows that

$$\sum_{t=0}^{\infty} \|x_i(t) - x_i(t+1)\| < \infty.$$

This shows that $(x_i(t))_{t=0}^{\infty}$ is a Cauchy sequence in $\mathbf{R^d}$. Hence, $x_i(t)$ converges to some x_i in $\mathbf{R^d}$ as t goes to infinity. This completes the proof. $\square$

The assumption of Theorem 3.2 is difficult to check because it depends on the entire dynamics' trajectory. However, since $\alpha_i(t)$ is controllable and

$$\left(1 - \frac{1}{|N_i(t)|}\right) d_t^i$$

is bounded, the assumption holds if the sum of $1-\alpha_i(t)$ over time is finite. For instance, given $a > 1$, if

$$1 - \alpha_i(t) = O\left(\frac{1}{t^a}\right), \text{ then } x_i(t) \text{ converges to some } x_i \in \mathbf{R^d} \text{ as } t \to \infty.$$

Next, I study several conditions under which every component of a profile is δ-trivial in finite time or under which the asymptotic stability holds. The following definition and lemmas will lead us to these conditions.

Definition 3.7. A symmetric matrix M is called a *generalized Laplacian* of a graph $G = (V, E)$ if for $x, y \in V$, the following two conditions hold:

$$M_{xy} = 0 \text{ for } x \neq y \text{ and } xy \notin E \quad \text{and} \quad M_{xy} < 0 \text{ for } x \neq y \text{ and } xy \in E.$$

Let $d_G(x) =$ degree of x in G, let $V(G) =$ vertex set of G, and let $E(G) =$ edge set of G. Then, the *Laplacian* of G is defined as $\mathscr{L} = D_G - A_G$ where

$$D_G = \operatorname{diag}((d_G(x))_{x \in V(G)}) \quad \text{and} \quad A_G = \text{ the adjacency matrix.}$$

In particular, $(A_G)_{xy} = \mathbb{1}\{xy \in E(G)\}$ when the graph G is simple.

Note that there is no restrictions on the diagonal entries of the matrix M. Also, the Laplacian of G is clearly a generalized Laplacian.

Lemma 3.5 (Perron-Frobenius for Laplacians [4])**.** Assume that M is a generalized Laplacian of a connected graph. Then, the smallest eigenvalue of M is simple and the corresponding eigenvector can be chosen with all entries positive.

Lemma 3.6 (Courant-Fischer Formula [23])**.** Assume that Q is a symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and corresponding eigenvectors $v_1, v_2, \ldots, v_n$. Let S_k be the vector space generated by $v_1, v_2, \ldots, v_k$ and $S_0 = \{0\}$. Then,

$$\lambda_k = \min\{x'Qx : \|x\| = 1, x \in S_{k-1}^{\perp}\}.$$

Lemma 3.7 (Cheeger's Inequality [3])**.** Assume that $G = (V, E)$ is an undirected graph with the Laplacian $\mathscr{L}$. Define

$$i(G) = \min\left\{\frac{|\partial S|}{|S|} : S \subset V, 0 < |S| \leq \frac{|G|}{2}\right\}$$

where $\partial S = \{uv \in E : u \in S, v \in S^c\}$. Then,

$$2i(G) \geq \lambda_2(\mathscr{L}) \geq \frac{i^2(G)}{2\Delta(G)} \quad \text{where} \quad \Delta(G) = \text{ maximum degree of } G.$$

Lemma 3.8. Let $Z(t) = \sum_{i,j \in [n]} \|x_i(t) - x_j(t)\|^2 \wedge \epsilon^2$. Then, Z is nonincreasing with respect to t. In particular,

$$Z(t) - Z(t+1) \geq 4 \sum_{i=1}^{n} \left(1 + |N_i(t)| \frac{\alpha_i(t)}{1 - \alpha_i(t)} \mathbb{1}\{\alpha_i(t) < 1\}\right) \|x_i(t) - x_i(t+1)\|^2.$$

Proof. Let $N_i = N_i(t)$, $N_i^* = N_i(t+1)$, $\alpha = \alpha(t)$, $x = x(t)$, $x^* = x(t+1)$ and $x_i' = \frac{1}{|N_i|}\sum_{k\in N_i} x_k$ for all $i \in [n]$. Via the Cauchy-Schwarz inequality, I obtain

$$
\begin{aligned}
Z(t) - Z(t+1) &= \sum_{i,j\in[n]} (\|x_i - x_j\|^2 \wedge \epsilon^2 - \|x_i^* - x_j^*\|^2 \wedge \epsilon^2) \\
&= \sum_{i=1}^n \Bigg[\sum_{j\in N_i \cap N_i^*} (\|x_i - x_j\|^2 - \|x_i^* - x_j^*\|^2) + \sum_{j \in N_i - N_i^*} (\|x_i - x_j\|^2 - \epsilon^2) \\
&\qquad + \sum_{j\in N_i^* - N_i} (\epsilon^2 - \|x_i^* - x_j^*\|^2) \Bigg] \\
&\geq \sum_{i=1}^n \sum_{j\in N_i} (\|x_i - x_j\|^2 - \|x_i^* - x_j^*\|^2) \\
&= \sum_{i=1}^n \sum_{j\in N_i} (\|x_i - x_j\|^2 - \|x_i^* - x_j\|^2 + \|x_i^* - x_j\|^2 - \|x_i^* - x_j^*\|^2) \\
&= \sum_{i=1}^n \sum_{j\in N_i} (\|x_i - x_i^* + x_i^* - x_j\|^2 - \|x_i^* - x_j\|^2 + \|x_i^* - x_j^* + x_j^* - x_j\|^2 \\
&\qquad - \|x_i^* - x_j^*\|^2) \\
&= \sum_{i=1}^n \sum_{j\in N_i} (\|x_i - x_i^*\|^2 + 2 < x_i - x_i^*, x_i^* - x_j > + \|x_j^* - x_j\|^2 \\
&\qquad + 2 < x_i^* - x_j^*, x_j^* - x_j >) \\
&= \sum_{i=1}^n |N_i| (\|x_i - x_i^*\|^2 + 2 < x_i - x_i^*, x_i^* - x_i' >) + \sum_{j=1}^n \sum_{i\in N_j} \|x_j^* - x_j\|^2 \\
&\quad + 2 \sum_{j=1}^n \sum_{i\in N_j} < x_i^* - x_i + x_i - x_j^*, x_j^* - x_j > \\
&= \sum_{i=1}^n |N_i| \left(\|x_i - x_i^*\|^2 + \frac{2\alpha_i}{1-\alpha_i} \mathbb{1}\{\alpha_i < 1\} \|x_i - x_i^*\|^2 \right) + \sum_{j=1}^n |N_j| \|x_j^* - x_j\|^2 \\
&\quad + 2 \sum_{j=1}^n \sum_{i\in N_j} < x_i^* - x_i, x_j^* - x_j > + 2 \sum_{j=1}^n \sum_{i\in N_j} < x_i - x_j^*, x_j^* - x_j > \\
&= \sum_{i=1}^n |N_i| \left(1 + \frac{2\alpha_i}{1-\alpha_i} \mathbb{1}\{\alpha_i < 1\} \right) \|x_i - x_i^*\|^2 + \sum_{i=1}^n |N_i| \|x_i^* - x_i\|^2 \\
&\quad + 2 \sum_{j=1}^n < x_j^* - x_j, x_j^* - x_j > + 2 \sum_{j=1}^n \sum_{i\in N_j - \{j\}} < x_i^* - x_i, x_j^* - x_j > \\
&\quad + 2 \sum_{j=1}^n |N_j| < x_j' - x_j^*, x_j^* - x_j >
\end{aligned}
$$

$$
\geq \sum_{i=1}^n |N_i| \left(2 + \frac{2\alpha_i}{1-\alpha_i} \mathbb{1}\{\alpha_i < 1\} \right) \|x_i - x_i^*\|^2 + 2 \sum_{j=1}^n \|x_j^* - x_j\|^2
$$

$$
\begin{aligned}
&-2\sum_{j=1}^{n}\sum_{i\in N_j-\{j\}}\|x_i^*-x_i\|\|x_j^*-x_j\|+2\sum_{j=1}^{n}|N_j|\frac{\alpha_j}{1-\alpha_j}\mathbb{1}\{\alpha_j<1\}\|x_j^*-x_j\|^2\\
=&\sum_{i=1}^{n}|N_i|\left(2+\frac{4\alpha_i}{1-\alpha_i}\mathbb{1}\{\alpha_i<1\}\right)\|x_i-x_i^*\|^2+2\sum_{j=1}^{n}\|x_j^*-x_j\|^2\\
&+\sum_{j=1}^{n}\sum_{i\in N_j-\{j\}}\left[(\|x_i^*-x_i\|-\|x_j^*-x_j\|)^2-\|x_i^*-x_i\|^2-\|x_j^*-x_j\|^2\right]\\
\geq&\sum_{i=1}^{n}|N_i|\left(2+\frac{4\alpha_i}{1-\alpha_i}\mathbb{1}\{\alpha_i<1\}\right)\|x_i-x_i^*\|^2+2\sum_{j=1}^{n}\|x_j^*-x_j\|^2\\
&-\sum_{i=1}^{n}\sum_{j\in N_i-\{i\}}\|x_i^*-x_i\|^2-\sum_{j=1}^{n}\sum_{i\in N_j-\{j\}}\|x_j^*-x_j\|^2\\
=&\sum_{i=1}^{n}|N_i|\left(2+\frac{4\alpha_i}{1-\alpha_i}\mathbb{1}\{\alpha_i<1\}\right)\|x_i-x_i^*\|^2+2\sum_{j=1}^{n}\|x_j^*-x_j\|^2\\
&-\sum_{i=1}^{n}(|N_i|-1)\|x_i^*-x_i\|^2-\sum_{j=1}^{n}(|N_j|-1)\|x_j^*-x_j\|^2\\
=&\sum_{i=1}^{n}4\left(1+|N_i|\frac{\alpha_i}{1-\alpha_i}\mathbb{1}\{\alpha_i<1\}\right)\|x_i-x_i^*\|^2
\end{aligned}
$$

This completes the proof. □

Lemma 3.9. Assume that Q is a real square matrix and that V is invertible such that the matrix $VQ=\mathscr{L}$ is the Laplacian of some connected graph. Then, 0 is a simple eigenvalue of $Q'Q$ corresponding to the eigenvector $\mathbb{1}=(1,1,\dots,1)'$. In particular, I have

$$\lambda_2(Q'Q)=\min\{x'Q'Qx:\|x\|=1\text{ and }x\perp\mathbb{1}\}.$$

Proof. To begin with, observe that

$$Q'Qx=0\iff Qx=0\iff\mathscr{L}x=0.$$

Recall that a real symmetric matrix is diagonalizable, and that its algebraic multiplicity = its geometric multiplicity. Since $\mathscr{L}$ is positive semi-definite and has an eigenvalue 0 corresponding to the eigenvector $\mathbb{1}$, by Lemma 3.5, 0 is a simple eigenvalue of $\mathscr{L}$. Hence, by the above relation between $\mathscr{L}$ and $Q'Q$, the matrix $Q'Q$ has a simple eigenvalue 0 corresponding to the eigenvector $\mathbb{1}$. Since in addition

$$x'Q'Qx=\|Qx\|^2\geq 0,$$

the matrix $Q'Q$ is positive semi-definite. Finally, applying Lemma 3.6, I get

$$\lambda_2(Q'Q)=\min\{x'Q'Qx:\|x\|=1\text{ and }x\perp\mathbb{1}\}.$$

This completes the proof. $\square$

Now, I am ready to investigate several conditions under which, for any $\delta > 0$, every component of a profile is δ-trivial in finite time.

Theorem 3.3. Assume that $\limsup_{t\to\infty} \max_{i\in[n]} \alpha_i(t) < 1$. Then, for any $\delta > 0$, every component of a profile is δ-trivial in finite time, i.e.,

$$\tau_{\alpha,\delta} := \inf\{t \geq 0 : \text{every component of } \mathscr{G}(t) \text{ is } \delta\text{-trivial}\} < \infty.$$

Proof. If every component of $\mathscr{G}(t)$ is δ-trivial, I am done. Now, assume that $\mathscr{G}(t)$ has a δ-nontrivial component. Without loss of generality, I may assume that $\mathscr{G}(t)$ is connected; if not, I can restrict to a δ-nontrivial component. For $\mathbb{1} \in \mathbf{R^n}$ and $W = \text{Span}(\{\mathbb{1}\})$, $\mathbf{R^n} = W \oplus W^\perp$. Then, write

$$x(t) = [c_1\mathbb{1} \,|\, c_2\mathbb{1} \,|\, \cdots \,|\, c_d\mathbb{1}] + \left[\hat{c}_1 u^{(1)} \,|\, \hat{c}_2 u^{(2)} \,|\, \cdots \,|\, \hat{c}_d u^{(d)}\right]$$

where c_i and $\hat{c}_i$ are constants and $u^{(i)} \in \mathbb{1}^\perp$ is a unit vector for all $i \in [d]$.

$$\text{Claim:} \quad \sum_{k=1}^d \hat{c}_k^2 > \frac{\delta^2}{2}.$$

Assume by contradiction that this is not the case. Then, for any $i, j \in [n]$,

$$\begin{aligned}\|x_i(t) - x_j(t)\|^2 &= \sum_{k=1}^d \hat{c}_k^2 (u_i^{(k)} - u_j^{(k)})^2 \\ &\leq \sum_{k=1}^d \hat{c}_k^2 \; 2((u_i^{(k)})^2 + (u_j^{(k)})^2) \leq 2\sum_{k=1}^d \hat{c}_k^2 \leq \delta^2,\end{aligned}$$

contradicting the δ-nontriviality of $\mathscr{G}(t)$. Let $B(t) = \text{diag}(\alpha(t)) + (I - \text{diag}(\alpha(t)))A(t)$. Then,

$$x(t) - x(t+1) = (I - B(t))\, x(t) = \left[\hat{c}_1(I - B(t))u^{(1)} \,|\, \cdots \,|\, \hat{c}_d(I - B(t))u^{(d)}\right],$$

from which it follows that

$$\sum_{i=1}^n \|x_i(t) - x_i(t+1)\|^2 = \sum_{j=1}^d \hat{c}_j^2 \|(I - B(t))u^{(j)}\|^2.$$

Now, observe that

$$I - B(t) = (I - \text{diag}(\alpha(t)))(I + D(t))^{-1}\mathscr{L}$$

where $\mathscr{L}$ is the Laplacian of $\mathscr{G}(t)$ and $D(t)$ is diagonal with $D_{ii}(t) = d_i(t)$, the degree of vertex i. Assume that $\alpha_i(t) < 1$ for all $i \in [n]$. Then, $I - \text{diag}(\alpha(t))$ is invertible, and according to Lemmas 3.7

and 3.9,

$$\begin{aligned}
\|(I-B(t))u^{(j)}\|^2 &= u^{(j)'}(I-B(t))'(I-B(t))u^{(j)} \geq \lambda_2((I-B(t))'(I-B(t)))\\
&= \lambda_2\bigg(\mathscr{L}\ \mathrm{diag}\bigg(\bigg(\bigg(\frac{1-\alpha_i(t)}{1+d_i(t)}\bigg)^2\bigg)_{i=1}^n\bigg)\mathscr{L}\bigg)\\
&\geq \bigg(\frac{1-\max_{i\in[n]}\alpha_i(t)}{n}\bigg)^2\lambda_2(\mathscr{L}^2) = \bigg(\frac{1-\max_{i\in[n]}\alpha_i(t)}{n}\bigg)^2\lambda_2^2(\mathscr{L})\\
&> \frac{4(1-\max_{i\in[n]}\alpha_i(t))^2}{n^8}
\end{aligned}$$

where I used that

$$\lambda_2(\mathscr{L}) \geq \frac{i^2(\mathscr{G}(t))}{2\Delta(\mathscr{G}(t))} > \frac{(2/n)^2}{2n} = \frac{2}{n^3}.$$

In particular, I obtain

$$\sum_{i=1}^n \|x_i(t)-x_i(t+1)\|^2 > \frac{2\delta^2(1-\max_{i\in[n]}\alpha_i(t))^2}{n^8}.$$

Since $\limsup_{t\to\infty}\max_{i\in[n]}\alpha_i(t) < 1$, there exists $(t_k)_{k\geq 1}\subset \mathbf{N}$ strictly increasing such that

$$\max_{i\in[n]}\alpha_i(t_k) \leq \gamma < 1 \quad \text{for some} \quad \gamma \quad \text{and for all} \quad k\geq 1.$$

Now, let $\tau = \tau_{\alpha,\delta}$. By Lemma 3.8, for all $m \geq 1$,

$$\begin{aligned}
n^2\epsilon^2 &> Z(0) \geq Z(0)-Z(m) = \sum_{t=0}^{m-1}(Z(t)-Z(t+1))\\
&\geq \sum_{t=0}^{m-1}\sum_{i\in[n],\alpha_i(t)<1} 4\bigg(1+|N_i(t)|\frac{\alpha_i(t)}{1-\alpha_i(t)}\bigg)\|x_i(t)-x_i(t+1)\|^2\\
&\geq 4\sum_{t=0}^{m-1}\sum_{i\in[n],\alpha_i(t)<1}\|x_i(t)-x_i(t+1)\|^2. \qquad (*)
\end{aligned}$$

Now, assume by contradiction that $\tau=\infty$. Letting $m\to\infty$, I get

$$\begin{aligned}
n^2\epsilon^2 &\geq 4\sum_{t=0}^{\infty}\sum_{i\in[n],\alpha_i(t)<1}\|x_i(t)-x_i(t+1)\|^2\\
&\geq 4\sum_{t\geq 0,\max_{i\in[n]}\alpha_i(t)<1}\frac{2\delta^2(1-\max_{i\in[n]}\alpha_i(t))^2}{n^8}\\
&\geq \sum_{k\geq 1}\frac{8\delta^2(1-\max_{i\in[n]}\alpha_i(t_k))^2}{n^8} \geq \sum_{k\geq 1}\frac{8\delta^2(1-\gamma)^2}{n^8} = \infty,
\end{aligned}$$

a contradiction. This completes the proof. □

From Theorem 3.3, if $\limsup_{t\to\infty}\max_{i\in[n]}\alpha_i(t) < 1$, then $\tau_{\alpha,\delta}<\infty$. Thus, if $\mathscr{G}(\tau_{\alpha,\delta})$ is connected for some $0<\delta\leq\epsilon$, then by Theorem 3.1, a consensus is reached eventually. The main parts of the proof of Theorem 3.3 resemble the ones in the proof of Theorem 2 in [2]. The similarities between the proofs consist in the derivation of a lower bound for

$$\sum_{i=1}^{n} \|x_i(t) - x_i(t+1)\|^2$$

by restricting to a δ-nontrivial component and then choosing a bounded function to construct an inequality involving the sum. The main difference is that Theorem 3.3 assumes that

$$\limsup_{t\to\infty} \max_{i\in[n]} \alpha_i(t) < 1 \tag{3.2}$$

to ensure that the smallest eigenvalue of $(I - B(t))'(I - B(t))$ is simple, but Theorem 2 in [2] has no such assumptions since (3.2) automatically holds if $\alpha(t) = \vec{0}$ for all $t \geq 0$. Theorem 2 in [2] states that the termination time of the synchronous HK model is independent of d and bounded from above. In fact, the result is a special case of the following corollary.

Corollary 3.1. Assume that $\sup_{t\in\mathbf{N}} \max_{i\in[n]} \alpha_i(t) < 1$. Then, $\tau_{\alpha,\delta}$ is bounded from above. Also, letting $\tau_m = \tau_{\alpha,\epsilon/m}$ for $m \geq 4$, there is no interactions between any two components of $\mathscr{G}(t)$ at the next time step for some $M \geq 4$ and for all $t \geq \tau_M$, i.e.,

$$\mathscr{G}(t) = \mathscr{G}(\tau_M) \quad \text{for some} \quad M \geq 4 \quad \text{and for all} \quad t \geq \tau_M.$$

Hence, x in (3.1) is asymptotically stable.

Proof. Because

$$\limsup_{t\to\infty} \max_{i\in[n]} \alpha_i(t) \leq \sup_{t\in\mathbf{N}} \max_{i\in[n]} \alpha_i(t) < 1,$$

it follows from Theorem 3.3 that $\tau < \infty$. For $\tau \geq 1$, setting $m = \tau$ in $(*)$, I get

$$\begin{aligned} n^2\epsilon^2 &> 4\sum_{t=0}^{\tau-1} \sum_{i\in[n],\alpha_i(t)<1} \|x_i(t) - x_i(t+1)\|^2 \\ &= 4\sum_{t=0}^{\tau-1}\sum_{i=1}^{n} \|x_i(t) - x_i(t+1)\|^2 \geq 4\sum_{t=0}^{\tau-1} \frac{2\delta^2(1-\max_{i\in[n]}\alpha_i(t))^2}{n^8} \\ &\geq \frac{8\tau\delta^2(1-\sup_{t\in\mathbf{N}}\max_{i\in[n]}\alpha_i(t))^2}{n^8}, \end{aligned}$$

from which it follows that

$$\tau < \frac{n^{10}}{8(1-\sup_{t\in\mathbf{N}}\max_{i\in[n]}\alpha_i(t))^2}\left(\frac{\epsilon}{\delta}\right)^2.$$

Hence, τ is bounded from above. To show the asymptotic stability of x, I first observe that τ_m is finite and nondecreasing with respect to m. For all $0 < \delta \leq \epsilon/4$ and $t \geq 0$, assume that every component of $\mathscr{G}(t)$ is δ-trivial. Then, the following three conditions are equivalent:

1. Some component of $\mathscr{G}(t+1)$ is δ-nontrivial.

2. Some components of $\mathscr{G}(t)$ interact at time $t+1$.

3. Some component of $\mathscr{G}(t+1)$ is $\epsilon/2$-nontrivial.

It is clear that $1 \Rightarrow 2$ and $3 \Rightarrow 1$; therefore I show $2 \Rightarrow 3$.

Proof of 2 ⇒ 3. Let the convex hull of a component G be

$$Cv(G) = C(\{x_j : j \in V(G)\}).$$

The fact that some components of $\mathscr{G}(t)$ interact at time $t+1$ implies that there exist

$$i, j \in [n] \quad \text{with} \quad ij \in \mathscr{E}(t+1), \qquad i \in V(G_{\tilde{i}}) \quad \text{and } j \in V(G_{\tilde{j}})$$

for some distinct components $G_{\tilde{i}}$ and $G_{\tilde{j}}$ of $\mathscr{G}(t)$. Therefore,

$$x_i(t+1) \in Cv(G_{\tilde{i}}) \quad \text{and} \quad x_j(t+1) \in Cv(G_{\tilde{j}}).$$

Hence,

$$\begin{aligned}
\epsilon &< \|x_i(t) - x_j(t)\| \\
&\leq \|x_i(t) - x_i(t+1)\| + \|x_i(t+1) - x_j(t+1)\| + \|x_j(t+1) - x_j(t)\| \\
&\leq \delta + \|x_i(t+1) - x_j(t+1)\| + \delta = \|x_i(t+1) - x_j(t+1)\| + 2\delta.
\end{aligned}$$

This implies that

$$\|x_i(t+1) - x_j(t+1)\| > \epsilon - 2\delta \geq \epsilon - 2 \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2} \quad \text{for all} \quad 0 < \delta \leq \frac{\epsilon}{4}$$

so the component of $\mathscr{G}(t+1)$ containing ij is $\epsilon/2$-nontrivial. □

Let

$$A_m = \{t \in [\tau_m, \tau_{m+1}) : \text{some component of } \mathscr{G}(t) \text{ is } \epsilon/m\text{-nontrivial}\}$$

and $t_m = \inf A_m$.

$$\text{Claim:} \quad \text{the set } \mathscr{A} := \{t_k : A_k \neq \varnothing\} \text{ is finite.}$$

For $t_m \in \mathscr{A}$, since some component of $\mathscr{G}(t_m)$ is ϵ/m-nontrivial and all components of $\mathscr{G}(t_m - 1)$ are ϵ/m-trivial, by 1⇒ 3, some component of $\mathscr{G}(t_m)$ is $\epsilon/2$-nontrivial. Using $(*)$ and letting $m \to \infty$, I get

$$\begin{aligned} n^2\epsilon^2 &\geq 4\sum_{t\geq 0}\sum_{i\in[n],\alpha_i(t)<1}\|x_i(t)-x_i(t+1)\|^2 = 4\sum_{t\geq 0}\sum_{i=1}^{n}\|x_i(t)-x_i(t+1)\|^2 \\ &\geq 4\sum_{t\in\mathscr{A}}\frac{2(\epsilon/2)^2(1-\max_{i\in[n]}\alpha_i(t))^2}{n^8} \\ &\geq |\mathscr{A}|\,\frac{8(\epsilon/2)^2(1-\sup_{t\geq 0}\max_{i\in[n]}\alpha_i(t))^2}{n^8}, \end{aligned}$$

from which it follows that

$$|\mathscr{A}| \leq \frac{n^{10}}{2(1-\sup_{t\geq 0}\max_{i\in[n]}\alpha_i(t))^2}.$$

Hence, the set $\mathscr{A}$ is finite. By the fact that $\mathscr{A}$ is finite and that 2⇒ 1, there is no interactions between any two components of $\mathscr{G}(s)$ at the next time step for some $M \geq 4$ and for all $s \geq \tau_M$. Hence, I deduce that every component of $\mathscr{G}(\tau_M)$ is an independent system. Since in addition

$$\limsup_{t\to\infty}\beta_t \leq \sup_{t\in\mathbf{N}}\beta_t \leq \sup_{t\in\mathbf{N}}\max_{i\in[n]}\alpha_i(t) < 1,$$

by Theorem 3.1, x in (3.1) is asymptotically stable. □

Note that the upper bound for τ is independent of d, and (3.1) reduces to the synchronous HK model if $\alpha(t) = \vec{0}$ for all $t \geq 0$. Since $\sup_{t\geq 0}\max_{i\in[n]}\alpha_i(t) < 1$ automatically holds if $\alpha(t) = \vec{0}$ at all times, $\mathscr{G}(s) = \mathscr{G}(\tau_M)$ for some $M \geq 4$ and for all $s \geq \tau_M$. This shows that $\mathscr{G}(\tau_M + 1)$ is a steady state and that the termination time of the synchronous HK model is bounded from above.

3.2 Conclusion

The mixed HK model covers both the synchronous and the asynchronous HK models, and is therefore more general and more complicated. At each time step, each agent can choose its degree of stubbornness and mix its opinion with the average opinion of its neighbors. Agents with the same opinion may depart later, depicting the changeability of agents, which is closer to real world circumstances. Given the givens, make it more difficult to reach asymptotic stability or a steady state. However, under some conditions, not only does the asymptotic stability hold, but also a consensus can be achieved.

Chapter 4

PROBABILITY OF CONSENSUS OF HEGSELMANN-KRAUSE DYNAMICS

HSIN-LUN LI

Abstract. The original Hegselmann-Krause (HK) model comprises a set of n agents characterized by their opinion, a number in $[0, 1]$. Agent i updates its opinion x_i via taking the average opinion of its neighbors whose opinion differs by at most ϵ from x_i. In the article, the opinion space is extended to $\mathbf{R^d}$. The main result is to derive bounds for the probability of consensus. In general, I derive a positive lower bound for the probability of consensus and demonstrate a lower bound for the probability of consensus on a unit cube. In particular for one dimensional case, I derive an upper bound and a better lower bound for the probability of consensus and demonstrate them on a unit interval.

4.1 Introduction

The original Hegselmann-Krause (HK) model consists of a set of n agents characterized by their opinion, a number in $[0, 1]$. Agent i updates its opinion x_i via taking the average opinion of its neighbors whose opinion differs by at most ϵ from x_i for a confidence bound $\epsilon > 0$. In this essay, the opinion space is extended to $\mathbf{R^d}$. The aim is to derive a lower bound for the probability of consensus for the synchronous HK model as follows:

$$x(t+1) = A(t)x(t) \text{ for } t \in \mathbf{N}, \tag{4.1}$$

$$A_{ij}(t) = \mathbb{1}\{j \in N_i(t)\}/|N_i(t)|,$$

$$x(t) = (x_1(t), \ldots, x_n(t))' = \text{transpose of } (x_1(t), \ldots, x_n(t))$$

for $[n] := \{1, 2, \ldots, n\}$, $N_i(t) = \{j \in [n] : \|x_j(t) - x_i(t)\| \leq \epsilon\}$ the collection of agent $i's$ neighbors at time t and $\| \ \|$ the Euclidean norm. [8] gives an overview of HK models. [19, 35] elaborate that (4.1) has finite-time convergence property. [13] further illustrates that the *termination time*

$$T_n = \inf\{t \geq 0 : x(t) = x(s) \text{ for all } s \geq t\}$$

is bounded from above. Finite-time convergence property is enough to imply

$$\lim_{t\to\infty} \max_{i,j\in[n]} \|x_i(t) - x_j(t)\| \text{ exists.}$$

Let the initial opinions $x_i(0)$ be independent and identically distributed random variables with a convex support $S \subset \mathbf{R^d}$ of positive Lebesgue measure and a probability density function f, where $P(x_i \in B) = \int_B f(x_i) dm(x_i)$ for all $i \in [n]$, B a Borel set and m the Lebesgue measure. Here, Say a function or a set is measurable if it is Lebesgue measurable. A *profile* at time t is an undirected graph $\mathscr{G}(t) = (\mathscr{V}(t), \mathscr{E}(t))$ with the vertex set and edge set

$$\mathscr{V}(t) = [n] \text{ and } \mathscr{E}(t) = \{(i,j) : i \neq j \text{ and } \|x_i(t) - x_j(t)\| \leq \epsilon\}.$$

A profile $\mathscr{G}(t)$ is δ*-trivial* if any two vertices are at a distance of at most δ apart. Observe that a consensus is reached at time $t+1$ if $\mathscr{G}(t)$ is ϵ-trivial.

4.2 Main results

Define

$$\mathscr{C} = \{\lim_{t\to\infty} \max_{i,j\in[n]} \|x_i(t) - x_j(t)\| = 0\},$$

the collection of all sample points that lead to a consensus.

Theorem 4.1.

$$P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is connected}) \text{ for } 1 \leq n \leq 4.$$

In general,

$$\begin{aligned} P(\mathscr{C}) &\geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) \\ &\geq P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) \\ &= \int_{\mathbf{R^d}} f(x_1) \left(\int_{B(x_1,\epsilon/2)} f(x) dm(x) \right)^{n-1} dm(x_1) > 0 \text{ for } n \geq 1. \end{aligned}$$

In particular, the probability of consensus is positive.

Corollary 4.1. Assume that $S = [0,1]^d$ and $x_i(0) = \text{Uniform}\,([0,1]^d)$. Then,

$$P(\mathscr{C}) \geq \left((\frac{\epsilon}{2})^d m(B(0,1)) \right)^{n-1} (1-\epsilon)^d = \left((\frac{\epsilon}{2})^d \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \right)^{n-1} (1-\epsilon)^d$$

for all $i \in [n]$ and $\epsilon \in (0,1)$.

Define

$$(1) = \arg\min_{k\in[n]} x_k \text{ and } (i) = \arg\min_{k\in[n]-\{(j)\}_{j=1}^{i-1}} x_k \text{ for } i \geq 2.$$

Namely $x_{(i)}$ is the i-th smallest number among $(x_k)_{k=1}^n$. For $n \geq 4$, let $m = \lfloor \frac{n-4}{3} \rfloor$ and $k = n-m-1$. Say $\mathscr{G}(t)$ satisfies $(*)$ if

$$((m+2),(k)) \in \mathscr{E}(t) \text{ and } (x_{(n)} - x_{(k)} + x_{(m+2)} - x_{(1)})(t) \leq \epsilon.$$

Say $\mathscr{G}(t)$ satisfies $(**)$ if

$$\max\big((x_{(n)} - x_{(n-i-1)})(t),\ (x_{(n-i-1)} - x_{(i+2)})(t),\ (x_{(i+2)} - x_{(1)})(t)\big) \leq \frac{\epsilon}{2}$$

for some $0 \leq i \leq m$.

Theorem 4.2 $(d = 1)$**.**

$$P(\mathscr{C}) = P(\mathscr{G}(0) \text{ is connected) for } 1 \leq n \leq 4.$$

$$P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ satisfies } (*)) \text{ for } 5 \leq n \leq 7.$$

In general,

$$P(\mathscr{G}(0) \text{ is connected}) \geq P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial or satisfies } (**)) \text{ for } n \geq 1.$$

Corollary 4.2. Let $S = [0,1]$, $d = 1$, $\epsilon \in (0,1)$ and $x_i(0) = \text{Uniform}([0,1])$ for all $i \in [n]$. Then, for $n = 2$,

$$P(\mathscr{C}) = \epsilon(2-\epsilon)$$

for $n = 3$,

$$P(\mathscr{C}) = \begin{cases} 6\epsilon^2(1-\epsilon) & \epsilon \in (0, \frac{1}{2}) \\ 1 - 2(1-\epsilon)^3 & \epsilon \in [\frac{1}{2}, 1) \end{cases}$$

for $n = 4$,

$$P(\mathscr{C}) = \begin{cases} 24\epsilon^3(1-3\epsilon) + 36\epsilon^4 & \epsilon \in (0, \frac{1}{3}) \\ 19\epsilon^4 - 4\epsilon^3(1-2\epsilon) + (1-2\epsilon)^4 - 6\epsilon^2(3\epsilon-1)^2 & \\ -4\epsilon(1-2\epsilon)^3 + 12\epsilon^3(1-2\epsilon) + 12\epsilon^2(1-2\epsilon)^2 & \epsilon \in [\frac{1}{3}, \frac{1}{2}) \\ \epsilon^4 + 4\epsilon^3(1-\epsilon) + 6\epsilon^2(1-\epsilon)^2 + 4\epsilon(1-\epsilon)^3 - 2(1-\epsilon)^4 & \epsilon \in [\frac{1}{2}, 1) \end{cases}$$

for $n \geq 1$,

$$P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) = \epsilon^{n-1}[n - (n-1)\epsilon]$$

$$P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) = \frac{2}{n}\epsilon^n(1 - \frac{1}{2^n}) + \epsilon^{n-1}(1-\epsilon).$$

In general,

$$P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) = \epsilon^{n-1}[n-(n-1)\epsilon] \text{ for } n \geq 1.$$

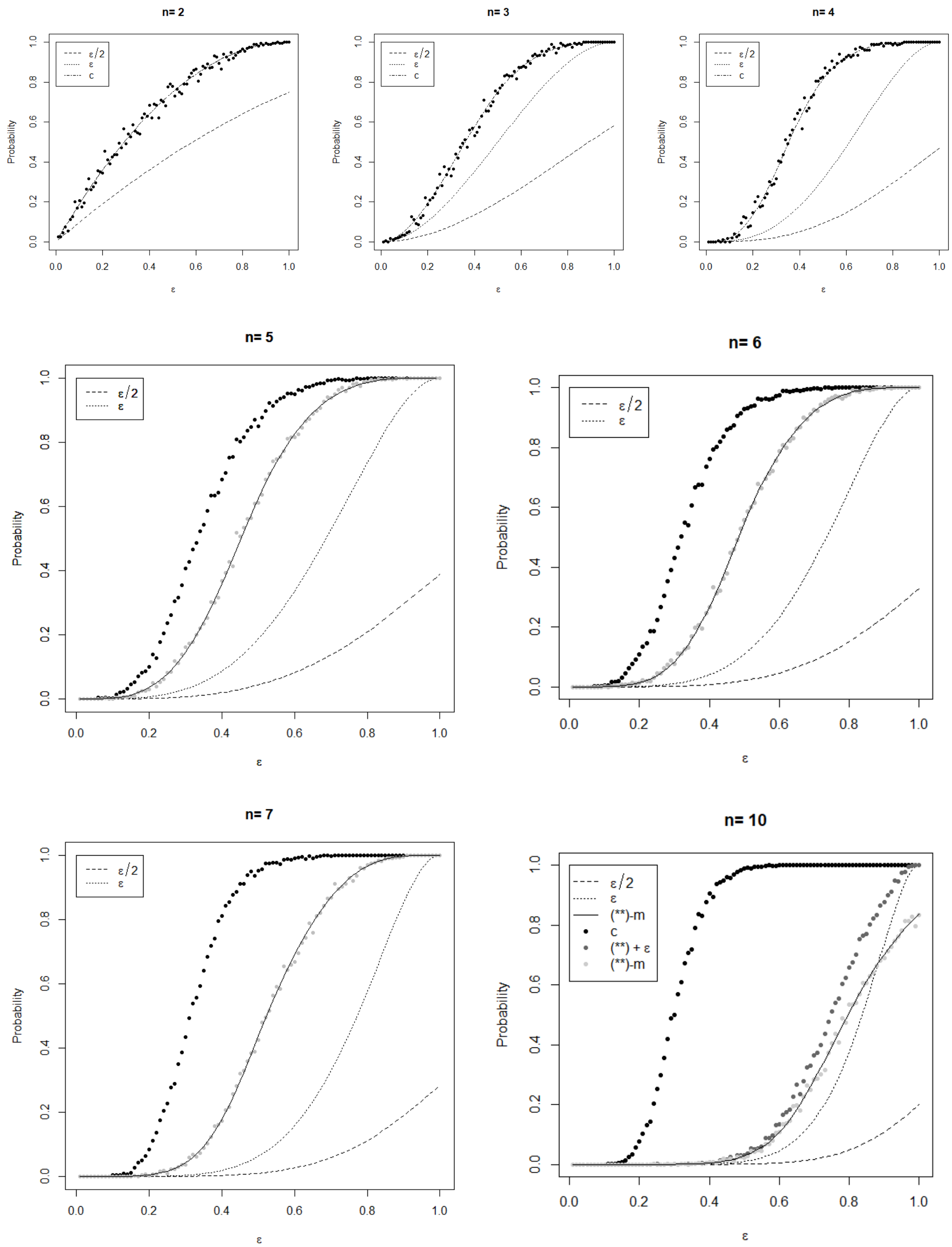


Figure 4.1: Bounds for $P(\mathscr{C})$

From Figure 4.1, labels $\epsilon/2$, ϵ and c denote respectively the lines of

$$P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]),\ P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) \text{ and } P(\mathscr{G}(0) \text{ is connected}).$$

The black points are simulations for the probability of consensus. For $n = 2$,

$$P(\mathscr{C}) = P(\mathscr{G}(0) \text{ is connected}) = P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial})$$

so ϵ and c-lines overlap. Observe that the points for the probability of consensus are around c-line for $2 \leq n \leq 4$, which meets the theory. For $5 \leq n \leq 7$, the gray points and the solid line are respectively simulations and numerical integrals of $P(\mathscr{G}(0)$ satisfies $(*))$, suggesting that theoretically $P(\mathscr{G}(0)$ satisfies $(*))$ is a better lower bound for $P(\mathscr{C})$ than $P(\mathscr{G}(0)$ is ϵ-trivial). For $n = 10$, the dark gray points are simulations of $P(\mathscr{G}(0)$ is ϵ-trivial or satisfies $(**))$, and the points and solid line are respectively simulations and numerical integrals of $P(\mathscr{G}(0)$ satisfies $(**)$ and $i = (m))$. Suggest that

$$P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) \vee P(\mathscr{G}(0) \text{ satisfies } (**) \text{ and } i = (m))$$

is a better lower bound than each of the two for the probability of consensus.

4.3 Probability of consensus

To derive a better lower bound for the probability of consensus, I study properties other than ϵ-triviality that leads to a consensus. If a profile is connected-preserving, then a consensus can be achieved in finite time. I illustrate that any profile $\mathscr{G}$ is connected-preserving for $1 \leq n \leq 4$ and some profile $\mathscr{G}$ of some configuration x fails to remain connected for $n > 4$. Thus $P(\mathscr{C}) \geq P(\mathscr{G}(0)$ is connected) for $1 \leq n \leq 4$. It is straightforward in general, $P(\mathscr{G}(0)$ is ϵ-trivial) is a lower bound for $P(\mathscr{C})$ but it is uneasy to calculate in high dimensions. Therefore I provide an easier calculated lower bound for the probability of consensus and also depict that the probability of consensus is positive.

Lemma 4.1 is the key to depict that any profile $\mathscr{G}$ is connected-preserving for $1 \leq n \leq 4$.

Lemma 4.1 ([32])**.** Given $\lambda_1, ..., \lambda_n$ in $\mathbf{R}$ with $\sum_{i=1}^{n} \lambda_i = 0$ and $x_1, ..., x_n$ in $\mathbf{R^d}$. Then for $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$, the terms with positive coefficients can be matched with the terms with negative coefficients in the sense that

$$\sum_{i=1}^{n} \lambda_i x_i = \sum_{i, c_i \geq 0, j, k \in [n]} c_i (x_j - x_k) \text{ and } \sum_i c_i = \sum_{j, \lambda_j \geq 0} \lambda_j.$$

From Lemma 4.1, I derive a good upper bound for $\|x_i(t+1) - x_j(t+1)\|$ for any $(i,j) \in \mathscr{E}(t)$.

Lemma 4.2. Assume that $(i,j) \in \mathscr{E}(t)$ and that $|N_i(t)| \le |N_j(t)|$. Then,

$$\|x_i(t+1) - x_j(t+1)\| \le \epsilon \left(3 - |N_i(t) \cap N_j(t)|(\frac{2}{|N_j(t)|} + \frac{1}{|N_i(t)|})\right).$$

Proof. Let $x = x(t)$, $x' = x(t+1)$ and $N_i = N_i(t)$ for any $i \in [n]$. Via Lemma 4.1, for any $i, j \in [n]$,

$$\begin{aligned} x'_i - x'_j &= \frac{1}{|N_i|}\sum_{k\in N_i} x_k - \frac{1}{|N_j|}\sum_{k\in N_j} x_k \\ &= (\frac{1}{|N_i|} - \frac{1}{|N_j|})\sum_{k\in N_i\cap N_j} x_k + \frac{1}{|N_i|}\sum_{k\in N_i - N_j} x_k - \frac{1}{|N_j|}\sum_{k\in N_j - N_i} x_k \\ &= \sum_{p\in N_i\cap N_j, q\in N_j - N_i} a_r(x_p - x_q) + \sum_{p\in N_i - N_j, q\in N_j - N_i} b_r(x_p - x_q) \end{aligned}$$

where $a_r, b_r \ge 0$, $\sum_r a_r = (\frac{1}{|N_i|} - \frac{1}{|N_j|})|N_i \cap N_j|$ and $\sum_r b_r = |N_i - N_j|/|N_i|$. Thus by the triangle inequality,

$$\begin{aligned} \|x'_i - x'_j\| \le &\sum_{p\in N_i\cap N_j, q\in N_j - N_i} a_r(\|x_p - x_j\| + \|x_j - x_q\|) \\ &+ \sum_{p\in N_i - N_j, q\in N_j - N_i} b_r(\|x_p - x_i\| + \|x_i - x_j\| + \|x_j - x_q\|) \\ &\le (\frac{1}{|N_i|} - \frac{1}{|N_j|})|N_i \cap N_j|(\epsilon + \epsilon) + \frac{|N_i - N_j|}{|N_i|}(\epsilon + \epsilon + \epsilon) \\ &= (\frac{1}{|N_i|} - \frac{1}{|N_j|})|N_i \cap N_j|2\epsilon + (1 - \frac{|N_i \cap N_j|}{|N_i|})3\epsilon \\ &= \epsilon\left(3 - |N_i \cap N_j|(\frac{2}{|N_j|} + \frac{1}{|N_i|})\right). \end{aligned}$$

□

Thus $\epsilon\left(3 - |N_i(t) \cap N_j(t)|(\frac{2}{|N_j(t)|} + \frac{1}{|N_i(t)|})\right) \le \epsilon$ implies $(i,j) \in \mathscr{E}(t+1)$. For the following lemmas, assume $x = x(t)$, $x' = x(t+1)$ and $N_i = N_i(t)$ for any $i \in [n]$ without specifying.

Lemma 4.3. Assume that $(i,j) \in \mathscr{E}(t)$ with $|N_i(t)| \le |N_j(t)|$ and that

$$|N_i(t) \cap N_j(t)|(\frac{2}{|N_j(t)|} + \frac{1}{|N_i(t)|}) \ge 2.$$

Then, $(i,j) \in \mathscr{E}(t+1)$.

Proof. By Lemma 4.2,

$$\|x'_i - x'_j\| \le \epsilon\left(3 - |N_i \cap N_j|(\frac{2}{|N_j|} + \frac{1}{|N_i|})\right) \le \epsilon(3-2) = \epsilon.$$

Thus $(i,j) \in \mathscr{E}(t+1)$. □

Observe that $|N_i(t) \cap N_j(t)| \geq 2$ for $(i, j) \in \mathscr{E}(t)$ and that the inequality $\frac{2}{|N_j(t)|} + \frac{1}{|N_i(t)|} \geq 1$ automatically holds for $1 \leq n \leq 3$. It is not straight forward to see Lemma 4.3 works for $n = 4$. However, categorizing the degrees of the pair $(i, j) \in \mathscr{E}(t)$, a profile $\mathscr{G}$ remains connected for $n = 4$.

Lemma 4.4 (connected-preserving)**.** For $1 \leq n \leq 4$, a profile $\mathscr{G}$ is connected-preserving.

Proof. Since $i, j \in N_i \cap N_j$ for $(i, j) \in \mathscr{E}(t)$ and $|N_i| \leq n$ for all $i \in [n]$,

$$|N_i \cap N_j|(\frac{2}{|N_j|} + \frac{1}{|N_i|}) \geq 2(\frac{2}{n} + \frac{1}{n}) \geq 2(\frac{2}{3} + \frac{1}{3}) = 2 \text{ for } 1 \leq n \leq 3.$$

From Lemma 4.3, $(i, j) \in \mathscr{E}(t + 1)$. Thus any edge in $\mathscr{E}(t)$ remains in $\mathscr{E}(t + 1)$. Hence a profile is connected-preserving for $1 \leq n \leq 3$.

For $n = 4$, let $d_i = d_i(t)$=the degree of vertex i at time t. For $(i, j) \in \mathscr{E}(t)$ and $d_i \leq d_j$, i is either a leaf or a non-leaf, and i and j can not be both leaves. So the cases of (d_i, d_j) are as follows:

$$\begin{bmatrix} d_i & 1 & 1 & 2 & 2 & 3 \\ d_j & 2 & 3 & 2 & 3 & 3 \end{bmatrix}.$$

Thus the cases of corresponding $(|N_i|, |N_j|)$ are

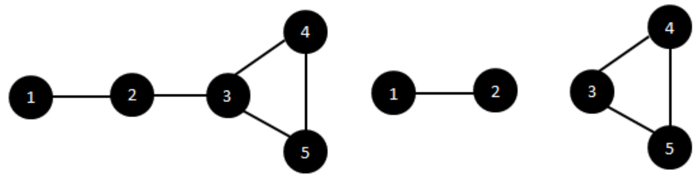
$$\begin{bmatrix} |N_i| & 2 & 2 & 3 & 3 & 4 \\ |N_j| & 3 & 4 & 3 & 4 & 4 \end{bmatrix}.$$

From Lemma 4.3, if $\frac{2}{|N_j|} + \frac{1}{|N_i|} \geq 1$ or $|N_i \cap N_j|(\frac{2}{|N_j|} + \frac{1}{|N_i|}) \geq 2$, then $(i, j) \in \mathscr{E}(t + 1)$. I check if each case meets one of the two conditions:

$$\begin{aligned}
(2, 3) &: \frac{2}{3} + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = 1 \\
(2, 4) &: \frac{2}{4} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 \\
(3, 3) &: \frac{2}{3} + \frac{1}{3} = 1 \\
(3, 4) &: 3(\frac{2}{4} + \frac{1}{3}) = \frac{3}{2} + 1 > 1 + 1 = 2 \\
(4, 4) &: 4(\frac{2}{4} + \frac{1}{4}) = 2 + 1 = 3 > 2.
\end{aligned}$$

Since each case satisfies one of the two conditions, $(i, j) \in \mathscr{E}(t + 1)$ for each case above, so a profile is connected-preserving for $n = 4$. □

Lemma 4.3 does not work for $n = 5$ even by categorizing the degrees of the pair $(i, j) \in \mathscr{E}(t)$. But indeed some profile $\mathscr{G}$ of some configuration x fails to remain connected.



(a) (b)

Figure 4.2

Lemma 4.5. For $n \geq 5$, some profile $\mathscr{G}$ of some configuration x is not connected-preserving.

Proof. Need only show that there is a configuration x with the profile $\mathscr{G}(0)$ connected but $\mathscr{G}(1)$ disconnected for $n = 5$ and $d = 1$. Consider

$$\epsilon = 1,\ x_1(0) = -1,\ x_2(0) = 0,\ x_3(0) = 1 \text{ and } x_4(0) = x_5(0) = 2.$$

Then, $\mathscr{G}(0)$ as Figure 4.2(a) is connected,

$$x_1(1) = -0.5,\ x_2(1) = 0,\ x_3(1) = 1.25 \text{ and } x_4(1) = x_5(1) = \frac{5}{3}.$$

So $\mathscr{G}(1)$ as Figure 4.2(b) is disconnected. For $n > 5$, let the new added vertices whose opinion be -1 or 2. Then, at the next time step, opinion 0 goes much closer to -1 or opinion 1 goes much closer to 2, and so $\mathscr{G}(0)$ connected but $\mathscr{G}(1)$ disconnected. This completes the proof. □

Hence a profile is connected-preserving for $n \leq 4$, and some profile $\mathscr{G}$ of some configuration x fails to remain connected for $n > 4$. I can estimate the probability of consensus via the initial profiles $\mathscr{G}(0)$. Lemmas 4.6-4.9 indicate the probability of consensus is positive.

Lemma 4.6. Let $g > 0$ be a measurable function on a measurable set A with $m(A) > 0$. Then, $\int_A g dm > 0$.

Proof. Let $E_k = \{g > \frac{1}{k}\}$. Then, $A = \cup_{k \geq 1} E_k$. Suppose by contradiction that $\int_A g dm = 0$. Then,

$$\frac{1}{k} m(E_k) \leq \int_{E_k} g dm \leq \int_A g dm = 0 \text{ for all } k \geq 1.$$

Thus via the subadditivity of a measure,

$$m(A) \leq \sum_{k \geq 1} m(E_k) = 0, \text{ a contradiction.}$$

□

Lemma 4.7. (i) The intersection of convex sets is convex. (ii) The closure of a convex set is convex.

Proof. (i) Let $Q_\alpha's$ be convex sets. If $\cap_\alpha Q_\alpha = \emptyset$, then clearly it is convex. Else, for $a, b \in \cap_\alpha Q_\alpha$, $a, b \in Q_\alpha$ for all α. So by convexity of convex sets, any point on the segment $\overline{\text{ab}}$ is in Q_α for all α. Thus any point on the segment $\overline{\text{ab}}$ is in $\cap_\alpha Q_\alpha$.

(ii) Let V be a convex set. For $v \in \overline{V}$, there exists $(v_n)_{n\geq 1} \subset V$ with $v_n \to v$ as $n \to \infty$. For $u, v \in \overline{V}$,

$$tu + (1-t)v = \lim_{n,m\to\infty} [tu_n + (1-t)v_m]$$

where $u_n, v_m \in V$ for all $n, m \geq 1$ and $t \in (0,1)$. By convexity of V, $tu_n + (1-t)v_m \in V$, so $tu + (1-t)v \in \overline{V}$. □

A *convex hull* generated by $v_1, \ldots, v_k \in \mathbf{R^d}$, denoted by $C(\{v_1, \ldots, v_k\})$, is the smallest convex set containing $v_1, \ldots, v_k$, *i.e.*,

$$C(\{v_1, \ldots, v_k\}) = \{v : v = \sum_{i=1}^{k} a_i v_i, (a_i)_{i=1}^{k} \text{ is stochastic}\}.$$

Lemma 4.8. A convex set in $\mathbf{R^d}$ is measurable.

Proof. Let $\mathscr{L}$ be the collection of all Lebesgue sets in $\mathbf{R^d}$ and $V \subset \mathbf{R^d}$ be a convex set.

$$\text{Claim: } m(\partial V) = 0.$$

For $V^\circ = \emptyset$, if $m(\overline{V}) > 0$, then $\overline{V}$ is uncountable, and there exist $d+1$ distinct points, $v_1, v_2, ..., v_{d+1}$, in $\overline{V}$ not in any hyperplane in $\mathbf{R^d}$. By convexity of $\overline{V}$, $C(\{v_1, ..., v_{d+1}\}) \subseteq \overline{V}$ with its interior nonempty, a contradiction.

For $V^\circ \neq \emptyset$, since measurability is shift-invariant, may assume zero vector $\vec{0} \in V^\circ$. Then $B(0,r) \subset V$ for some $0 < r < 1$. For $n \in \mathbf{Z}^+$, let $A_n = B(0,n) \cap V$ then by Lemma 4.7, A_n is bounded and convex. For $q \in \partial A_n$, by convexity of $\overline{V}$ and $A_n \supset B(0,r)$,

$$p = s \cdot q + (1-s) \cdot \vec{0} \in A_n^\circ \text{ for all } s \in (0,1).$$

Thus $q \in \frac{1}{s} A_n^\circ$. Since $\frac{1}{s} A_n^\circ \supset A_n^\circ$,

$$\begin{aligned} m(\partial A_n) &\leq m(\frac{1}{s} A_n^\circ - A_n^\circ) = m(\frac{1}{s} A_n^\circ) - m(A_n^\circ) \\ &= (\frac{1}{s})^d m(A_n^\circ) - m(A_n^\circ) \to 0 \text{ as } s \to 1. \end{aligned}$$

Since $\cup_{n\geq 1}\partial A_n \supset \partial V$,

$$m(\partial V) \leq m(\cup_{n\geq 1}\partial A_n) \leq \sum_{n\geq 1} m(\partial A_n) = 0.$$

Thus ∂V is a null set. By the completion of Lebesgue measure $\partial V \cap V \in \mathscr{L}$. Hence

$$V = V^\circ \cap (\partial V \cap V) \in \mathscr{L}.$$

□

Lemma 4.9. Let $V \subset \mathbf{R^d}$ be a convex set with $m(V) > 0$. Then,

$$m(V \cap B(x,r)) > 0 \text{ for any } x \in \overline{V} \text{ and } r > 0.$$

Proof. From the proof of Lemma 4.8, $m(\partial V) = 0$ so

$$m(V^\circ) = m(V) - m(\partial V \cap V) = m(V) > 0.$$

Thus V° is uncountable and $V^\circ = \cup_{u\in V^\circ} B(u, r_u)$ for some $r_u > 0$. For $x \in \overline{V}$ and $r > 0$, by the convexity of $\overline{V}$, there exists $y \in V^\circ$ with $\|y - x\| < \frac{r}{2}$, so by the triangle inequality, $B(y, \frac{r}{2}) \subset B(x, r)$. Hence

$$V \cap B(x,r) \supset B(y, r_y) \cap B(y, \frac{r}{2}) = B(y, r_y \wedge \frac{r}{2}), \text{ so } m(V \cap B(x,r)) \geq m(B(y, r_y \wedge \frac{r}{2})) > 0.$$

This completes the proof. □

Proof of Theorem 4.1. From Lemma 4.4, $\{\mathscr{G}(0) \text{ is connected}\} \subset \mathscr{C}$ for $1 \leq n \leq 4$ so

$$P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is connected}) \text{ for } 1 \leq n \leq 4.$$

Observe that $\mathscr{C} \supset \{\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}\} \supset \{x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]\}$ so

$$\begin{aligned}
P(\mathscr{C}) &\geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) \\
&\geq P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) \\
&= \int_{\mathbf{R^d}} \int_{B(x_1,\epsilon/2)} \cdots \int_{B(x_1,\epsilon/2)} \prod_{i=1}^{n} f(x_i) dm(x_n) \ldots dm(x_1) \\
&:= \int_{\mathbf{R^d}} (\int_{B(x_1,\epsilon/2)})^{n-1} \prod_{i=1}^{n} f(x_i) dm(x_n) \ldots dm(x_1) \\
&= \int_{\mathbf{R^d}} f(x_1) \left(\int_{B(x_1,\epsilon/2)} f(x) dm(x) \right)^{n-1} dm(x_1).
\end{aligned}$$

Observe that $f > 0$ on the convex set S and $m(B(x_1, \epsilon/2) \cap S) > 0$ for all $x_1 \in S$ from Lemma 4.9. Hence via Lemma 4.6,

$$\begin{aligned} P(\mathscr{C}) &\geq \int_{\mathbf{R^d}} f(x_1) \left(\int_{B(x_1,\epsilon/2)} f(x)dm(x) \right)^{n-1} dm(x_1) \\ &= \int_{S} f(x_1) \left(\int_{B(x_1,\epsilon/2)\cap S} f(x)dm(x) \right)^{n-1} dm(x_1) > 0. \end{aligned}$$

□

Proof of Corollary 4.1. From theorem 4.1,

$$\begin{aligned} P(\mathscr{C}) &\geq P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) \\ &\geq \int_{[\epsilon/2, 1-\epsilon/2]^d} dm(x_1) \left(\int_{B(x_1,\epsilon/2)} 1 dm(x) \right)^{n-1} \\ &= \int_{[\epsilon/2, 1-\epsilon/2]^d} m(B(x_1, \frac{\epsilon}{2}))^{n-1} dm(x_1) \\ &= \left((\frac{\epsilon}{2})^d m(B(0,1)) \right)^{n-1} (1-\epsilon)^d = \left((\frac{\epsilon}{2})^d \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \right)^{n-1} (1-\epsilon)^d. \end{aligned}$$

□

4.4 One dimensional probability of consensus

In this section, I focus on the one dimensional HK model. Apart from higher dimensions, opinions in one dimension are ordered by $\leq$. I demonstrate that opinions are order-preserving and profiles are disconnected-preserving. Hence $P(\mathscr{C}) = P(\mathscr{G}(0)$ is connected) for $1 \leq n \leq 4$ and in general $P(\mathscr{G}(0)$ is connected) is an upper bound for the probability of consensus. Furthermore, I demonstrate the probability of consensus on $[0, 1]$.

Lemma 4.10 (order-preserving)**.** For d=1, if $x_i(t) \leq x_j(t)$ then $x_i(t+1) \leq x_j(t+1)$.

Proof. Let $x = x(t), x' = x(t+1)$, and $N_i = N_i(t)$ for all $i \in [n]$. From Lemma 4.1,

$$\begin{aligned} x'_j - x'_i &= (\frac{1}{|N_j|} - \frac{1}{|N_i|}) \sum_{k \in N_i \cap N_j} x_k + \frac{1}{|N_j|} \sum_{k \in N_j - N_i} x_k - \frac{1}{|N_i|} \sum_{k \in N_i - N_j} x_k \\ &= \begin{cases} \displaystyle\sum_{k,p \in N_j - N_i, q \in N_i} a_k(x_p - x_q) & \text{if } |N_j| \geq |N_i| \\ \displaystyle\sum_{k,p \in N_j, q \in N_i - N_j} a_k(x_p - x_q) & \text{else,} \end{cases} \end{aligned}$$

where $a_k \geq 0$ for all k. I claim that

1. If $a \in N_i$ and $b \in N_j - N_i$, then $x_a < x_b$.

2. If $a \in N_i - N_j$ and $b \in N_j$, then $x_a < x_b$.

Proof of Claim 1. Assume by contradiction that there exist $a \in N_i$ and $b \in N_j - N_i$ such that $x_a \geq x_b$. Then, $x_j - \epsilon \leq x_b < x_i - \epsilon$, a contradiction. □

Proof of Claim 2. Assume by contradiction that there exist $a \in N_i - N_j$ and $b \in N_j$ such that $x_a \geq x_b$. Then, $x_i + \epsilon \geq x_a > x_j + \epsilon$, a contradiction. □

Either way, $x'_j - x'_i \geq 0$. This completes the proof. □

Lemma 4.11 (disconnected-preserving)**.** For $d = 1$, if $\mathscr{G}(t)$ is disconnected then $\mathscr{G}(t+1)$ is disconnected.

Proof. Assume $x_1(t) \leq x_2(t) \leq ... \leq x_n(t)$. Since $\mathscr{G}(t)$ is disconnected,

$$x_{i+1}(t) - x_i(t) > \epsilon \text{ for some } i \in [n-1].$$

Since vertices i and $i+1$ have respectively no neighbors on its right and left at time t,

$$x_i(t+1) \leq x_i(t) \text{ and } x_{i+1}(t) \leq x_{i+1}(t+1).$$

Hence $x_{i+1}(t+1) - x_i(t+1) > \epsilon$. From Lemma 4.10,

$$x_1(t+1) \leq x_2(t+1) \leq ... \leq x_n(t+1).$$

Thus $\mathscr{G}(t+1)$ is disconnected. □

Next, I consider several circumstances under which a profile is connected at the next time step. Let $M \subset \mathbf{R}$ be a finite nonempty set and $\overline{M} = \frac{\sum_{x \in M} x}{|M|}$ be the average on M. It is clear that

$$\overline{a + M} > \overline{M} \iff a > \overline{M}. \tag{4.2}$$

Lemma 4.12. For $4 \leq n \leq 7$, if $\mathscr{G}(t)$ satisfies $(*)$, then so does $\mathscr{G}(t+1)$.

Proof. Let $x = x(t)$, $x' = x(t+1)$, $\epsilon_1 = x_{(m+2)} - x_{(1)}$, $\epsilon_2 = x_{(n)} - x_{(k)}$, $y_1 = x_{(1)}$, $y_2 = x_{(m+2)}$, $y_3 = x_{(k)}$, $y_4 = x_{(n)}$. Since $x'_i s$ are order-preserving, need only consider $y'_3 - y'_2$, $y'_2 - y'_1$, and $y'_4 - y'_3$. By

(4.2),

$$\max_{(x_i)_{i=1}^n - \{y_2,y_3\}} (y_3' - y_2') = \frac{y_2 + y_3 + (y_3+\epsilon_2) + [sy_2 + (k-m-3-s)y_3] + m(y_3+\epsilon_2)}{k}$$
$$- \frac{(y_2-\epsilon_1) + y_2 + y_3 + m(y_2-\epsilon_1) + [sy_2 + (k-m-3-s)y_3]}{k}$$

for some $0 \le s \le k-m-3$,

$$= \frac{(y_3-y_2) + \epsilon_2 + \epsilon_1 + m(y_3-y_2) + m(\epsilon_2+\epsilon_1)}{k}$$
$$= \frac{(m+1)(y_3-y_2) + (m+1)(\epsilon_1+\epsilon_2)}{k}$$
$$\le \frac{2(m+1)}{k}\epsilon < \epsilon \text{ so } ((m+2),(k)) \in \mathscr{E}(t+1).$$

$$\max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') = \frac{y_1 + y_2 + (n-m-2)(y_2+\epsilon) + [sy_1 + (m-s)y_2]}{n}$$
$$- \frac{y_1 + y_2 + [sy_1 + (m-s)y_2]}{m+2} \text{ for some } 0 \le s \le m.$$

Since

$$\partial_s \max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') = \frac{y_1-y_2}{n} - \frac{y_1-y_2}{m+2} = (y_1-y_2)(\frac{1}{n} - \frac{1}{m+2}) \ge 0, \text{ set } s = m,$$

$$\max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1')$$
$$= \frac{(m+2)[(m+1)y_1 + (n-m-1)y_2 + (n-m-2)\epsilon] - n[(m+1)y_1 + y_2]}{n(m+2)}$$
$$= \frac{(m+1)(n-m-2)(y_2-y_1) + (m+2)(n-m-2)\epsilon}{n(m+2)}.$$

By symmetry,

$$\max_{(x_i)_{i=1}^n - \{y_3,y_4\}} (y_4' - y_3') = \frac{(m+1)(n-m-2)(y_4-y_3) + (m+2)(n-m-2)\epsilon}{n(m+2)}.$$

Hence

$$\max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') + \max_{(x_i)_{i=1}^n - \{y_3,y_4\}} (y_4' - y_3')$$
$$= \frac{(m+1)(n-m-2)(y_2-y_1+y_4-y_3) + (m+2)(n-m-2)\epsilon}{n(m+2)}$$
$$\le \frac{(n-m-2)(2m+3)}{n(m+2)}\epsilon \le \epsilon \text{ for } 4 \le n \le 7.$$

So $y_4' - y_3' + y_2' - y_1' \le \epsilon$ for $4 \le n \le 7$. This completes the proof. □

Observe that a profile is connected-preserving if it satisfies $(*)$. It is clear that an ϵ-trivial profile satisfies $(*)$ and there exists an ϵ-nontrivial profile satisfies $(*)$. Thus $\{\mathscr{G}(0) \text{ satisfies } (*)\} \supsetneq \{\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}\}$.

Lemma 4.13. For any $0 \le i \le m$ and $n \ge 4$, assume that

$$\max\left((x_{(n)} - x_{(n-i-1)})(t),\ (x_{(n-i-1)} - x_{(i+2)})(t),\ (x_{(i+2)} - x_{(1)})(t)\right) \le \frac{\epsilon}{2}.$$

Then,

$$\max\left((x_{(n)} - x_{(n-i-1)})(t+1),\ (x_{(n-i-1)} - x_{(i+2)})(t+1),\ (x_{(i+2)} - x_{(1)})(t+1)\right) < \frac{\epsilon}{2}.$$

Proof. Let $x = x(t),\ x' = x(t+1),\ y_1 = x_{(1)},\ y_2 = x_{(i+2)},\ y_3 = x_{(n-i-1)},\ y_4 = x_{(n)}$. By the assumption, the neighborhood of $(i+2)$ is the same as that of $(n-i-1)$, so $y_3' - y_2' = 0$. Via (4.2),

$$\begin{aligned}
&\max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') \\
&= \frac{y_1 + y_2 + [s_1 y_1 + (i - s_1) y_2] + [s_2 y_2 + (n - 2i - 3 - s_2)(y_2 + \frac{\epsilon}{2})] + (i+1)(y_2 + \epsilon)}{n} \\
&- \frac{y_1 + y_2 + [s_1 y_1 + (i - s_1) y_2] + [s_2 y_2 + (n - 2i - 3 - s_2)(y_2 + \frac{\epsilon}{2})]}{n - i - 1}
\end{aligned}$$

for some $0 \le s_1 \le i$ and $0 \le s_2 \le n - 2i - 3$. Since $3m + 4 \le n \le 3m + 6$, $0 \le i \le m$,

$$\partial_{s_1} \max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') = (y_1 - y_2)\left(\frac{1}{n} - \frac{1}{n-i-1}\right) \ge 0,$$

$$\partial_{s_2} \max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') = [y_2 - (y_2 + \frac{\epsilon}{2})]\left(\frac{1}{n} - \frac{1}{n-i-1}\right) \ge 0,$$

set $s_1 = i$ and $s_2 = n - 2i - 3$,

$$\begin{aligned}
&\max_{(x_i)_{i=1}^n - \{y_1,y_2\}} (y_2' - y_1') \\
&= \frac{(i+1)y_1 + (n-i-1)y_2 + (i+1)\epsilon}{n} - \frac{(i+1)y_1 + (n-2i-2)y_2}{n-i-1} \\
&= \frac{(n-i-1)[(i+1)y_1 + (n-i-1)y_2 + (i+1)\epsilon] - n[(i+1)y_1 + (n-2i-2)y_2]}{n(n-i-1)} \\
&= \frac{(i+1)^2(y_2 - y_1) + (n-i-1)(i+1)\epsilon}{n(n-i-1)} \le \frac{\frac{(i+1)^2}{2} + (n-i-1)(i+1)}{n(n-i-1)}\epsilon \\
&= \frac{1}{2}\frac{(i+1)(2n-i-1)}{n(n-i-1)}\epsilon \le \frac{1}{2}\frac{(m+1)(2n-1)}{n(n-m-1)}\epsilon \le \frac{1}{2}\frac{(m+1)(6m+11)}{(3m+4)(2m+3)}\epsilon < \frac{\epsilon}{2}.
\end{aligned}$$

By symmetry,

$$\max_{(x_i)_{i=1}^n - \{y_3,y_4\}} (y_4' - y_3') = \frac{(i+1)^2(y_4 - y_3) + (n-i-1)(i+1)\epsilon}{n(n-i-1)} < \frac{\epsilon}{2}.$$

This completes the proof. ∎

Observe that an ϵ-trivial profile may not satisfies the assumption of Lemma 4.13. Consider $n > 1,\ x_1(t) = 0,\ x_i(t) = \epsilon$ for $i > 1$ then $\mathscr{G}(t)$ is ϵ-trivial but does not satisfy the assumption of Lemma 4.13. Observe that $\mathscr{G}(s)$ satisfies $(**)$ for all $s \geq t$.

Proof of Theorem 4.2. From Lemmas 4.4 and 4.11, $\mathscr{G}(t)$ is connected-preserving and disconnected-preserving for $1 \leq n \leq 4$ so $\mathscr{C} = \{\mathscr{G}(0) \text{ is connected}\}$. Thus

$$P(\mathscr{C}) = P(\mathscr{G}(0) \text{ is connected}) \text{ for } 1 \leq n \leq 4.$$

Since $\mathscr{G}(t)$ is disconnected-preserving for all $n \geq 1$ and $\mathscr{G}(t)$ is connected-preserving if it satisfies $(**)$ for all $n \geq 4$,

$$\{\mathscr{G}(0) \text{ is connected}\} \supset \mathscr{C} \supset \{\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}\} \cup \{\mathscr{G}(0) \text{ satisfies } (**)\}.$$

Hence

$$P(\mathscr{G}(0) \text{ is connected}) \geq P(\mathscr{C}) \geq P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial or satisfies } (**)).$$

□

Proof of Corollary 4.2. For $n = 2$,

$$P(\mathscr{C}) = 2! \int_{[0,1]} dx_1 \int_{[x_1, x_1+\epsilon]\cap[0,1]} dx_2.$$

$$\begin{aligned}
&\int_{[0,1]} dx_1 \int_{[x_1,x_1+\epsilon]\cap[0,1]} dx_2 = \int_{[0,1-\epsilon]+[1-\epsilon,1]} \left[(x_1+\epsilon) \wedge 1 - x_1\right] dx_1 \\
&= \int_{[0,1-\epsilon]} \epsilon dx_1 + \int_{[1-\epsilon,1]} 1 - x_1 dx_1 \\
&= \epsilon(1-\epsilon) - \left[\frac{(1-x_1)^2}{2}\right]_{1-\epsilon}^{1} = \epsilon(1-\epsilon) + \frac{1}{2}\epsilon^2 = \epsilon(1-\frac{\epsilon}{2}).
\end{aligned}$$

Thus

$$P(\mathscr{C}) = \epsilon(2-\epsilon).$$

For $n = 3$,

$$P(\mathscr{C}) = 3! \int_{[0,1]} dx_1 \int_{[x_1,x_1+\epsilon]\cap[0,1]} dx_2 \int_{[x_2,x_2+\epsilon]\cap[0,1]} dx_3.$$

(i) $\epsilon \in [\frac{1}{2}, 1)$

$$\int_{[0,1]} dx_1 \int_{[x_1,x_1+\epsilon]\cap[0,1]} dx_2 \int_{[x_2,x_2+\epsilon]\cap[0,1]} dx_3$$
$$= \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \int_{[x_2,(x_2+\epsilon)\wedge 1]} dx_3$$
$$= \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} (x_2+\epsilon)\wedge 1 - x_2 dx_2$$
$$= \int_{[0,1]} dx_1 \left(\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-\epsilon]} \epsilon dx_2 + \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]} 1 - x_2 dx_2 \right)$$
$$= \int_{[0,1-\epsilon]} dx_1 \left(\int_{[x_1,1-\epsilon]} \epsilon dx_2 + \int_{[1-\epsilon,x_1+\epsilon]} 1 - x_2 dx_2 \right) + \int_{[1-\epsilon,1]} dx_1 \int_{[x_1,1]} 1 - x_2 dx_2$$
$$= \int_{[0,1-\epsilon]} \epsilon(1-\epsilon-x_1) - \left[\frac{(1-x_2)^2}{2} \right]_{x_2=1-\epsilon}^{x_1+\epsilon} dx_1 + \int_{[1-\epsilon,1]} - \left[\frac{(1-x_2)^2}{2} \right]_{x_2=x_1}^{1} dx_1$$
$$\frac{-\epsilon(1-\epsilon-x_1)^2}{2}\Big|_0^{1-\epsilon} + \frac{1}{2}\int_{[0,1-\epsilon]} \epsilon^2 - (1-\epsilon-x_1)^2 dx_1 + \frac{1}{2}\int_{[1-\epsilon,1]} (1-x_1)^2 dx_1$$
$$= \frac{1}{2}\left\{ \epsilon(1-\epsilon)^2 + \epsilon^2(1-\epsilon) + \left[\frac{(1-\epsilon-x_1)^3}{3} \right]_0^{1-\epsilon} - \left[\frac{(1-x_1)^3}{3} \right]_{1-\epsilon}^{1} \right\}$$
$$= \frac{1}{2}\left\{ \epsilon(1-\epsilon) - \frac{(1-\epsilon)^3}{3} + \frac{\epsilon^3}{3} \right\} \text{ so}$$
$$P(\mathscr{C}) = 3\epsilon(1-\epsilon) - (1-\epsilon)^3 + \epsilon^3 = \epsilon^3 + (1-\epsilon)^3 + 3\epsilon(1-\epsilon) - 2(1-\epsilon)^3$$
$$= \epsilon^2 - \epsilon(1-\epsilon) + (1-\epsilon)^2 + 3\epsilon(1-\epsilon) - 2(1-\epsilon)^2$$
$$= \epsilon^2 + 2\epsilon(1-\epsilon) + (1-\epsilon)^2 - 2(1-\epsilon)^3 = 1 - 2(1-\epsilon)^3.$$

(ii) $\epsilon \in (0, \frac{1}{2})$

$$\begin{aligned}
&\int_{[0,1]} dx_1 \int_{[x_1,x_1+\epsilon]\cap[0,1]} dx_2 \int_{[x_2,x_2+\epsilon]\cap[0,1]} dx_3 \\
&= \int_{[0,1]} dx_1 \left(\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-\epsilon]} \epsilon dx_2 + \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]} 1 - x_2 dx_2 \right) \\
&= \int_{[0,1-2\epsilon]} dx_1 \left(\int_{[x_1,x_1+\epsilon]\cap[0,1-\epsilon]} \epsilon dx_2 + \int_{[x_1,x_1+\epsilon]\cap[1-\epsilon,1]} 1 - x_2 dx_2 \right) \\
&+ \int_{[1-2\epsilon,1-\epsilon]} dx_1 \left(\int_{[x_1,x_1+\epsilon]\cap[0,1-\epsilon]} \epsilon dx_2 + \int_{[x_1,x_1+\epsilon]\cap[1-\epsilon,1]} 1 - x_2 dx_2 \right) \\
&+ \int_{[1-\epsilon,1]} dx_1 \left(\int_{[x_1,1]\cap[0,1-\epsilon]} \epsilon dx_2 + \int_{[x_1,1]\cap[1-\epsilon,1]} 1 - x_2 dx_2 \right) \\
&= \int_{[0,1-2\epsilon]} \epsilon^2 dx_1 + \left(\int_{[1-2\epsilon,1-\epsilon]} \epsilon(1-\epsilon-x_1) - \left[\frac{(1-x_2)^2}{2} \right]_{x_2=1-\epsilon}^{x_1+\epsilon} dx_1 \right) \\
&+ \int_{[1-\epsilon,1]} - \left[\frac{(1-x_2)^2}{2} \right]_{x_2=x_1}^{1} dx_1 \\
&= \epsilon^2(1-2\epsilon) + \int_{[1-2\epsilon,1-\epsilon]} \epsilon(1-\epsilon-x_1) \\
&+ \frac{1}{2} \left[\epsilon^2 - (1-\epsilon-x_1)^2 \right] dx_1 + \frac{1}{2} \int_{[1-\epsilon,1]} (1-x_1)^2 dx_1 \\
&= \epsilon^2(1-2\epsilon) - \left[\frac{\epsilon(1-\epsilon-x_1)^2}{2} \right]_{1-2\epsilon}^{1-\epsilon} + \frac{1}{2} \left\{ \epsilon^3 + \left[\frac{(1-\epsilon-x_1)^3}{3} \right]_{x_1=1-2\epsilon}^{1-\epsilon} \right\} \\
&- \frac{1}{2} \left[\frac{(1-x_1)^3}{3} \right]_{1-\epsilon}^{1} \\
&= \epsilon^2(1-2\epsilon) + \frac{1}{2}(\epsilon^3 + \epsilon^3 - \frac{\epsilon^3}{3} + \frac{\epsilon^3}{3}) = \epsilon^2(1-2\epsilon) + \epsilon^3 = \epsilon^2(1-\epsilon) \text{ so} \\
&P(\mathscr{C}) = 6\epsilon^2(1-\epsilon).
\end{aligned}$$

Thus

$$P(\mathscr{C}) = \begin{cases} 6\epsilon^2(1-\epsilon) & \epsilon \in (0, \frac{1}{2}) \\ 1 - 2(1-\epsilon)^3 & \epsilon \in [\frac{1}{2}, 1) \end{cases}$$

For $n = 4$,

$$P(\mathscr{C}) = 4! \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \int_{[x_2,(x_2+\epsilon)\wedge 1]} dx_3 \int_{[x_3,(x_3+\epsilon)\wedge 1]} dx_4.$$

(i) $\epsilon \in [\frac{1}{2}, 1)$

$$\int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \int_{[x_2,(x_2+\epsilon)\wedge 1]} dx_3 \int_{[x_3,(x_3+\epsilon)\wedge 1]} dx_4$$

$$= \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \int_{[x_2,(x_2+\epsilon)\wedge 1]} (x_3+\epsilon)\wedge 1 - x_3 dx_3$$

$$= \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \left(\int_{[x_2,(x_2+\epsilon)\wedge 1]\cap[0,1-\epsilon]} \epsilon dx_3 + \int_{[x_2,(x_2+\epsilon)\wedge 1]\cap[1-\epsilon,1]} 1 - x_3 dx_3 \right)$$

$$= \int_{[0,1]} dx_1 \left[\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-\epsilon]} \left(\int_{[x_2,1-\epsilon]} \epsilon dx_3 + \int_{[1-\epsilon,x_2+\epsilon]} 1 - x_3 dx_3 \right) \right.$$

$$\left. + \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]} dx_2 \left(\int_{[x_2,1]} 1 - x_3 dx_3 \right) \right]$$

$$= \int_{[0,1]} dx_1 \left(\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-\epsilon]} \epsilon(1-\epsilon-x_2) - \left[\frac{(1-x_3)^2}{2} \right]_{1-\epsilon}^{x_2+\epsilon} dx_2 \right.$$

$$\left. + \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]} - \left[\frac{(1-x_3)^2}{2} \right]_{x_2}^{1} dx_2 \right)$$

$$= \int_{[0,1-\epsilon]} dx_1 \left(\int_{[x_1,1-\epsilon]} \epsilon(1-\epsilon-x_2) + \frac{1}{2}[\epsilon^2 - (1-\epsilon-x_2)^2] dx_2 \right.$$

$$\left. + \frac{1}{2} \int_{[1-\epsilon,x_1+\epsilon]} (1-x_2)^2 dx_2 \right) + \int_{[1-\epsilon,1]} dx_1 \left(\frac{1}{2} \int_{[x_1,1]} (1-x_2)^2 dx_2 \right)$$

$$= \int_{[0,1-\epsilon]} - \left[\frac{\epsilon(1-\epsilon-x_2)^2}{2} \right]_{x_1}^{1-\epsilon} + \frac{1}{2} \left(\epsilon^2(1-\epsilon-x_1) + \left[\frac{(1-\epsilon-x_2)^3}{3} \right]_{x_1}^{1-\epsilon} \right)$$

$$- \frac{1}{2} \left[\frac{(1-x_2)^3}{3} \right]_{1-\epsilon}^{x_1+\epsilon} dx_1 + \int_{[1-\epsilon,1]} -\frac{1}{2} \left[\frac{(1-x_2)^3}{3} \right]_{x_1}^{1} dx_1$$

$$= \frac{1}{2} \left(\int_{[0,1-\epsilon]} \epsilon(1-\epsilon-x_1)^2 + \epsilon^2(1-\epsilon-x_1) - \frac{(1-\epsilon-x_1)^3}{3} \right.$$

$$\left. + \frac{[\epsilon^3 - (1-\epsilon-x_1)^3]}{3} dx_1 + \int_{[1-\epsilon,1]} \frac{(1-x_1)^3}{3} dx_1 \right)$$

$$= \frac{1}{2} \left\{ - \left[\frac{\epsilon(1-\epsilon-x_1)^3}{3} \right]_0^{1-\epsilon} - \left[\frac{\epsilon^2(1-\epsilon-x_1)^2}{2} \right]_0^{1-\epsilon} \right.$$

$$\left. + \frac{1}{3} \left(\epsilon^3(1-\epsilon) + \left[\frac{2(1-\epsilon-x_1)^4}{4} \right]_0^{1-\epsilon} - \frac{1}{3} \left[\frac{(1-x_1)^4}{4} \right]_{1-\epsilon}^{1} \right) \right\}$$

$$= \frac{1}{2} \left\{ \frac{\epsilon(1-\epsilon)^3}{3} + \frac{\epsilon^2(1-\epsilon)^2}{2} + \frac{1}{3} \left[\epsilon^3(1-\epsilon) - \frac{2(1-\epsilon)^4}{4} \right] + \frac{\epsilon^4}{12} \right\}$$

$$= \frac{1}{2} \left(\frac{\epsilon(1-\epsilon)^3}{3} + \frac{\epsilon^2(1-\epsilon)^2}{2} + \frac{\epsilon^3(1-\epsilon)}{3} - \frac{(1-\epsilon)^4}{6} + \frac{\epsilon^4}{12} \right)$$

$$= \frac{1}{24} \left(\epsilon^4 + 4\epsilon^3(1-\epsilon) + 6\epsilon^2(1-\epsilon)^2 + 4\epsilon(1-\epsilon)^3 - 2(1-\epsilon)^4 \right) \text{ so}$$

$$P(\mathscr{C}) = \epsilon^4 + 4\epsilon^3(1-\epsilon) + 6\epsilon^2(1-\epsilon)^2 + 4\epsilon(1-\epsilon)^3 - 2(1-\epsilon)^4.$$

(ii) $\epsilon \in [\frac{1}{3}, \frac{1}{2})$

$$\int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \int_{[x_2,(x_2+\epsilon)\wedge 1]} dx_3 \int_{[x_3,(x_3+\epsilon)\wedge 1]} dx_4$$

$$= \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} dx_2 \left(\int_{[x_2,(x_2+\epsilon)\wedge 1]\cap[0,1-\epsilon]} \epsilon dx_3 + \int_{[x_2,(x_2+\epsilon)\wedge 1]} 1 - x_3 dx_3 \right)$$

$$= \int_{[0,1]} dx_1 \left[\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-2\epsilon]} dx_2 \left(\int_{[x_2,x_2+\epsilon]} \epsilon dx_3 \right) \right.$$

$$+ \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-2\epsilon,1-\epsilon]} dx_2 \left(\int_{[x_2,1-\epsilon]} \epsilon dx_3 + \int_{[1-\epsilon,x_2+\epsilon]} 1 - x_3 dx_3 \right)$$

$$\left. + \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]} dx_2 \int_{[x_2,1]} 1 - x_3 dx_3 \right]$$

$$= \int_{[0,1]} dx_1 \left(\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-2\epsilon]} \epsilon^2 dx_2 \right.$$

$$+ \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-2\epsilon,1-\epsilon]} \epsilon(1-\epsilon-x_2) - \left[\frac{(1-x_3)^2}{2} \right]_{1-\epsilon}^{x_2+\epsilon} dx_2$$

$$\left. + \int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]} - \left[\frac{(1-x_3)^2}{2} \right]_{x_2}^{1} dx_2 \right)$$

$$= \int_{[0,1-2\epsilon]} dx_1 \left(\int_{[x_1,1-2\epsilon]} \epsilon^2 dx_2 + \int_{[1-2\epsilon,x_1+\epsilon]} \epsilon(1-\epsilon-x_2) \right.$$

$$\left. + \frac{1}{2}[\epsilon^2 - (1-\epsilon-x_2)^2] dx_2 \right)$$

$$+ \int_{[1-2\epsilon,1-\epsilon]} dx_1 \left(\int_{[x_1,1-\epsilon]} \epsilon(1-\epsilon-x_2) + \frac{1}{2}[\epsilon^2 - (1-\epsilon-x_2)^2] dx_2 \right.$$

$$\left. + \frac{1}{2} \int_{[1-\epsilon,x_1+\epsilon]} (1-x_2)^2 dx_2 \right) + \frac{1}{2} \int_{[1-\epsilon,1]} dx_1 \int_{[x_1,1]} (1-x_2)^2 dx_2$$

$$= \int_{[0,1-2\epsilon]} \epsilon^2(1-2\epsilon-x_1) - \left[\frac{\epsilon(1-\epsilon-x_2)^2}{2} \right]_{1-2\epsilon}^{x_1+\epsilon}$$

$$+ \frac{1}{2} \left(\epsilon^2(x_1+3\epsilon-1) + \left[\frac{(1-\epsilon-x_2)^3}{3} \right]_{1-2\epsilon}^{x_1+\epsilon} \right) dx_1$$

$$+ \int_{[1-2\epsilon,1-\epsilon]} - \left[\frac{\epsilon(1-\epsilon-x_2)^2}{2} \right]_{x_1}^{1-\epsilon} + \frac{1}{2} \left(\epsilon^2(1-\epsilon-x_1) + \left[\frac{(1-\epsilon-x_2)^3}{3} \right]_{x_1}^{1-\epsilon} \right)$$

$$- \frac{1}{2} \left[\frac{(1-x_2)^3}{3} \right]_{1-\epsilon}^{x_1+\epsilon} dx_1 - \frac{1}{2} \int_{[1-\epsilon,1]} \left[\frac{(1-x_2)^3}{3} \right]_{x_1}^{1} dx_1$$

$$= \int_{[0,1-2\epsilon]} \epsilon^2(1-2\epsilon-x_1) + \frac{1}{2}[\epsilon^3 - \epsilon(1-2\epsilon-x_1)^2]$$

$$+ \frac{1}{2} \left(\epsilon^2(x_1+3\epsilon-1) + \frac{1}{3}[(1-2\epsilon-x_1)^3 - \epsilon^3] \right) dx_1 + \int_{[1-2\epsilon,1-\epsilon]} \frac{\epsilon(1-\epsilon-x_1)^2}{2}$$

$$+\frac{1}{2}\left(\epsilon^2(1-\epsilon-x_1)-\frac{(1-\epsilon-x_1)^3}{3}\right)+\frac{1}{6}[\epsilon^3-(1-\epsilon-x_1)^3]dx_1$$
$$+\frac{1}{2}\int_{[1-\epsilon,1]}\frac{1}{3}(1-x_1)^3dx_1$$
$$=-\left[\frac{\epsilon^2(1-2\epsilon-x_1)^2}{2}\right]_0^{1-2\epsilon}+\frac{1}{2}\left(\epsilon^3(1-2\epsilon)+\left[\frac{\epsilon(1-2\epsilon-x_1)^3}{3}\right]_0^{1-2\epsilon}\right)$$
$$+\frac{1}{2}\left[\left[\frac{\epsilon^2(x_1+3\epsilon-1)^2}{2}\right]_0^{1-2\epsilon}+\frac{1}{3}\left(-\left[\frac{(1-2\epsilon-x_1)^4}{4}\right]_0^{1-2\epsilon}-\epsilon^3(1-2\epsilon)\right)\right]$$
$$+\frac{1}{2}\left\{-\left[\frac{\epsilon(1-\epsilon-x_1)^3}{3}\right]_{1-2\epsilon}^{1-\epsilon}-\left[\frac{\epsilon^2(1-\epsilon-x_1)^2}{2}\right]_{1-2\epsilon}^{1-\epsilon}+\left[\frac{(1-\epsilon-x_1)^4}{12}\right]_{1-2\epsilon}^{1-\epsilon}\right.$$
$$\left.+\frac{1}{3}\left(\epsilon^4+\left[\frac{(1-\epsilon-x_1)^4}{4}\right]_{1-2\epsilon}^{1-\epsilon}\right)\right\}-\frac{1}{6}\left[\frac{(1-x_1)^4}{4}\right]_{1-\epsilon}^{1}$$
$$=\frac{\epsilon^2(1-2\epsilon)^2}{2}+\frac{1}{2}[\epsilon^3(1-2\epsilon)-\frac{\epsilon(1-2\epsilon)^3}{3}]+\frac{1}{4}[\epsilon^4-\epsilon^2(3\epsilon-1)^2]+\frac{1}{24}(1-2\epsilon)^4$$
$$-\frac{1}{6}\epsilon^3(1-2\epsilon)+\frac{1}{6}\epsilon^4+\frac{1}{4}\epsilon^4-\frac{1}{24}\epsilon^4+\frac{1}{6}\epsilon^4-\frac{1}{24}\epsilon^4+\frac{1}{24}\epsilon^4$$
$$=\frac{19}{24}\epsilon^4-\frac{1}{6}\epsilon^3(1-2\epsilon)+\frac{1}{24}(1-2\epsilon)^4-\frac{1}{4}\epsilon^2(3\epsilon-1)^2-\frac{1}{6}\epsilon(1-2\epsilon)^3$$
$$+\frac{1}{2}\epsilon^3(1-2\epsilon)+\frac{1}{2}\epsilon^2(1-2\epsilon)^2 \text{ so}$$
$$P(\mathscr{C})=19\epsilon^4-4\epsilon^3(1-2\epsilon)+(1-2\epsilon)^4-6\epsilon^2(3\epsilon-1)^2-4\epsilon(1-2\epsilon)^3$$
$$+12\epsilon^3(1-2\epsilon)+12\epsilon^2(1-2\epsilon)^2.$$

(iii) $\epsilon\in(0,\frac{1}{3})$

$$\int_{[0,1]}dx_1\int_{[x_1,(x_1+\epsilon)\wedge 1]}dx_2\int_{[x_2,(x_2+\epsilon)\wedge 1]}dx_3\int_{[x_3,(x_3+\epsilon)\wedge 1]}dx_4$$
$$=\int_{[0,1]}dx_1\left(\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[0,1-2\epsilon]}\epsilon^2dx_2+\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-2\epsilon,1-\epsilon]}\epsilon(1-\epsilon-x_2)\right.$$
$$\left.-\left[\frac{(1-x_3)^2}{2}\right]_{1-\epsilon}^{x_2+\epsilon}dx_2+\int_{[x_1,(x_1+\epsilon)\wedge 1]\cap[1-\epsilon,1]}-\left[\frac{(1-x_3)^2}{2}\right]_{x_2}^{1}dx_2\right)$$
$$=\int_{[0,1-3\epsilon]}dx_1\int_{[x_1,x_1+\epsilon]}\epsilon^2dx_2+\int_{[1-3\epsilon,1-2\epsilon]}dx_1\left(\int_{[x_1,1-2\epsilon]}\epsilon^2dx_2\right.$$
$$\left.+\int_{[1-2\epsilon,x_1+\epsilon]}\epsilon(1-\epsilon-x_2)+\frac{1}{2}[\epsilon^2-(1-\epsilon-x_2)^2]dx_2\right)$$
$$+\int_{[1-2\epsilon,1-\epsilon]}dx_1\left(\int_{[x_1,1-\epsilon]}\epsilon(1-\epsilon-x_2)+\frac{1}{2}[\epsilon^2-(1-\epsilon-x_2)^2]dx_2\right.$$
$$\left.+\frac{1}{2}\int_{[1-\epsilon,x_1+\epsilon]}(1-x_2)^2dx_2\right)+\frac{1}{2}\int_{[1-\epsilon,1]}dx_1\int_{[x_1,1]}(1-x_2)^2dx_2$$

$$
\begin{aligned}
&= \int_{[0,1-3\epsilon]} \epsilon^3 dx_1 + \int_{[1-3\epsilon,1-2\epsilon]} \epsilon^2(1-2\epsilon-x_1) - \left[\frac{\epsilon(1-\epsilon-x_2)^2}{2}\right]_{1-2\epsilon}^{x_1+\epsilon} \\
&+ \frac{1}{2}\left(\epsilon^2(x_1+3\epsilon-1) + \left[\frac{(1-\epsilon-x_2)^3}{3}\right]_{1-2\epsilon}^{x_1+\epsilon}\right) dx_1 \\
&+ \int_{[1-2\epsilon,1-\epsilon]} - \left[\frac{\epsilon(1-\epsilon-x_2)^2}{2}\right]_{x_1}^{1-\epsilon} + \frac{1}{2}\left(\epsilon^2(1-\epsilon-x_1) + \left[\frac{(1-\epsilon-x_2)^3}{3}\right]_{x_1}^{1-\epsilon}\right) \\
&- \frac{1}{2}\left[\frac{(1-x_2)^3}{3}\right]_{1-\epsilon}^{x_1+\epsilon} dx_1 + \frac{1}{2}\int_{[1-\epsilon,1]} - \left[\frac{(1-x_2)^3}{3}\right]_{x_1}^{1} dx_1 \\
&= \epsilon^3(1-3\epsilon) + \int_{[1-3\epsilon,1-2\epsilon]} \epsilon^2(1-2\epsilon-x_1) + \frac{1}{2}[\epsilon^3 - \epsilon(1-2\epsilon-x_1)^2] \\
&+ \frac{1}{2}\epsilon^2(x_1+3\epsilon-1) - \frac{1}{6}[\epsilon^3 - (1-2\epsilon-x_1)^3]dx_1 \\
&+ \frac{1}{2}\int_{[1-2\epsilon,1-\epsilon]} \epsilon(1-\epsilon-x_1)^2 + \epsilon^2(1-\epsilon-x_1) - \frac{(1-\epsilon-x_1)^3}{3} \\
&+ \frac{1}{3}[\epsilon^3 - (1-\epsilon-x_1)^3]dx_1 + \frac{1}{6}\int_{[1-\epsilon,1]} (1-x_1)^3 dx_1 \\
&= \epsilon^3(1-3\epsilon) - \left[\frac{\epsilon^2(1-2\epsilon-x_1)^2}{2}\right]_{1-3\epsilon}^{1-2\epsilon} + \frac{1}{2}\epsilon^4 + \frac{1}{2}\left[\frac{\epsilon(1-2\epsilon-x_1)^3}{3}\right]_{1-3\epsilon}^{1-2\epsilon} \\
&+ \frac{1}{2}\left[\frac{\epsilon^2(x_1+3\epsilon-1)^2}{2}\right]_{1-3\epsilon}^{1-2\epsilon} - \frac{1}{6}\epsilon^4 - \frac{1}{6}\left[\frac{(1-2\epsilon-x_1)^4}{4}\right]_{1-3\epsilon}^{1-2\epsilon} \\
&\frac{1}{2}\left(-\left[\frac{\epsilon(1-\epsilon-x_1)^3}{3}\right]_{1-2\epsilon}^{1-\epsilon} - \left[\frac{\epsilon^2(1-\epsilon-x_2)^2}{2}\right]_{1-2\epsilon}^{1-\epsilon} + \frac{1}{3}\left[\frac{(1-\epsilon-x_1)^4}{4}\right]_{1-2\epsilon}^{1-\epsilon}\right. \\
&\left. + \frac{1}{3}\epsilon^4 + \frac{1}{3}\left[\frac{(1-\epsilon-x_1)^4}{4}\right]_{1-2\epsilon}^{1-\epsilon}\right) - \frac{1}{6}\left[\frac{(1-x_1)^4}{4}\right]_{1-\epsilon}^{1} \\
&= \epsilon^3(1-3\epsilon) + \frac{1}{2}\epsilon^4 + \frac{1}{2}\epsilon^4 - \frac{1}{6}\epsilon^4 + \frac{1}{4}\epsilon^4 - \frac{1}{6}\epsilon^4 + \frac{1}{24}\epsilon^4 + \frac{1}{6}\epsilon^4 + \frac{1}{4}\epsilon^4 - \frac{1}{24}\epsilon^4 \\
&+ \frac{1}{6}\epsilon^4 - \frac{1}{24}\epsilon^4 + \frac{1}{24}\epsilon^4 \\
&= \epsilon^3(1-3\epsilon) + \frac{3}{2}\epsilon^4 \text{ so}
\end{aligned}
$$

$$P(\mathscr{C}) = 24\epsilon^3(1-3\epsilon) + 36\epsilon^4.$$

For $n \geq 2$,

$$
\begin{aligned}
&\int_{[0,1]} dx_1 \int_{[x_1,x_1+\epsilon]\cap[0,1]} dx_n \left(\int_{[x_1,x_n]} dx_n\right)^{n-2} = \int_{[0,1]} dx_1 \int_{[x_1,(x_1+\epsilon)\wedge 1]} (x_n-x_1)^{n-2} dx_n \\
&= \int_{[0,1]} \frac{(x_n-x_1)^{n-1}}{n-1}\Big|_{x_n=x_1}^{(x_1+\epsilon)\wedge 1} dx_1 \\
&= \frac{1}{n-1}\int_{[0,1]} [(x_1+\epsilon)\wedge 1 - x_1]^{n-1} dx_1 \\
&= \frac{1}{n-1}\left\{\int_{[0,1-\epsilon]} \epsilon^{n-1} dx_1 + \int_{[1-\epsilon,1]} (1-x_1)^{n-1} dx_1\right\}
\end{aligned}
$$

$$= \frac{1}{n-1}\left\{\epsilon^{n-1}(1-\epsilon) - \left[\frac{(1-x_1)^n}{n}\right]_{1-\epsilon}^{1}\right\} = \frac{1}{n-1}\left\{\epsilon^{n-1}(1-\epsilon) + \frac{1}{n}\epsilon^n\right\}$$
$$= \frac{\epsilon^{n-1}}{n-1}(1-\epsilon+\frac{\epsilon}{n}) = \frac{\epsilon^{n-1}}{n-1}\left[1-(1-\frac{1}{n})\epsilon\right] \text{ so}$$
$$P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) = \epsilon^{n-1}\left[n-(n-1)\epsilon\right].$$

Observe that $\epsilon^{n-1}\left[n-(n-1)\epsilon\right] = 1$ for $n = 1$. So

$$P(\mathscr{G}(0) \text{ is } \epsilon\text{-trivial}) = \epsilon^{n-1}\left[n-(n-1)\epsilon\right] \text{ for } n \geq 1.$$

$$P(x_i(0) \in B(x_1(0), \epsilon/2) \text{ for all } i \in [n]) = \int_{[0,1]} dx_1 \left(\int_{[x_1-\frac{\epsilon}{2}, x_1+\frac{\epsilon}{2}]\cap[0,1]} dx\right)^{n-1}$$
$$= \int_{[0,1]} \left[(x_1+\frac{\epsilon}{2})\wedge 1 - (x_1-\frac{\epsilon}{2})\vee 0\right]^{n-1} dx_1$$
$$= \int_{[0,\frac{\epsilon}{2}]} (x_1+\frac{\epsilon}{2})^{n-1}dx_1 + \int_{[\frac{\epsilon}{2},1-\frac{\epsilon}{2}]}[(x_1+\frac{\epsilon}{2})-(x_1-\frac{\epsilon}{2})]^{n-1}dx_1$$
$$+ \int_{[1-\frac{\epsilon}{2},1]}[1-(x_1-\frac{\epsilon}{2})]^{n-1}$$
$$= \frac{(x_1+\frac{\epsilon}{2})^n}{n}\Big|_0^{\frac{\epsilon}{2}} + \epsilon^{n-1}(1-\epsilon) - \left[\frac{(1+\frac{\epsilon}{2}-x_1)^n}{n}\right]_{1-\frac{\epsilon}{2}}^{1}$$
$$= \frac{1}{n}[\epsilon^n - (\frac{\epsilon}{2})^n] + \epsilon^{n-1}(1-\epsilon) + \frac{1}{n}[\epsilon^n - (\frac{\epsilon}{2})^n]$$
$$= \frac{2}{n}\epsilon^n(1-\frac{1}{2^n}) + \epsilon^{n-1}(1-\epsilon)$$

□

In conclusion, the probability of consensus is positive and is bounded from below by the one that an initial profile is connected for $1 \leq n \leq 4$. In particular for one dimension, the disconnected-preserving property for a profile engenders an upper bound $P(\mathscr{G}(0)$ is connected) for the probability of consensus, and therefore $P(\mathscr{C}) = P(\mathscr{G}(0)$ is connected) for $1 \leq n \leq 4$.

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