Graphs of Sets of Reduced Words

by

Jennifer E. Elder

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved July 2021 by the
Graduate Supervisory Committee:

Susanna Fishel, Chair
Nancy Childress
Andrzej Czygrinow
Julien Paupert
Oscar Vega

ARIZONA STATE UNIVERSITY

August 2021

ABSTRACT

Any permutation in the finite symmetric group can be written as a product of simple transpositions $s_i = (i\ i+1)$. For a fixed permutation $\sigma \in \mathfrak{S}_n$ the products of minimal length are called reduced decompositions or reduced words, and the collection of all such reduced words is denoted $R(\sigma)$. Any reduced word of σ can be transformed into any other by a sequence of commutation moves or long braid moves. One area of interest in these sets are the congruence classes defined by using only braid moves or only commutation moves. This document will present work towards a conjectured relationship between the number of reduced words and the number of braid classes.

The set $R(\sigma)$ can be drawn as a graph, $G(\sigma)$, where the vertices are the reduced words, and the edges denote the presence of a commutation or braid move between the words. This paper will present brand new work on subgraph structures in $G(\sigma)$, as well as new formulas to count the number of braid edges and commutation edges in $G(\sigma)$.

The permutation σ covers τ in the weak order poset if the length of τ is one less than the length of σ, and there exists a simple transposition s_i such that $\sigma = \tau s_i$. This paper will cover new work on the relationships between the size of $R(\sigma)$ and $R(\tau)$, and how this creates a new method of writing reduced decompositions of σ as products of permutations α and β, where both α and β have a length greater than one.

Finally, this thesis will also discuss how these results help relate the number of reduced words and the number of braid classes in certain cases.

ACKNOWLEDGMENTS

First I would like to thank my advisor, Susanna Fishel. The last five years have had many ups and downs, and you have offered your time, support, and advice through all of it. Thank you for the comments on my work; they were never mean, and they always improved what I had. Thank you for your help during this last pandemic year, supporting me through my research and job search.

To my committee, Nancy Childress, Andrzej Czygrinow, Julien Paupert, and Oscar Vega, thank you for all your help through my doctoral degree. The classes I took with each of you communicated so much knowledge and passion for mathematics, and helped remind me why I love the subject so much. You all offered me encouragement when I needed it, attended my seminar talks, and chatted with me at AWM lunches.

To Samantha Dahlberg, thank you for the tea, for proof reading my work, and for offering advice on how to improve it.

To Terri Miller, thank you for the coffee, the fan in your office to help cool off, and always being up for a dinner where we could relax while enjoying good food.

To my friends, David Polletta, Wendy Caldwell, Danielle Brager, Katie Wifvat, Mela Hardin, and many others. Thank you for the office chats, the "Jim from The Office" glances in meetings, and the non-math nerd discussions that kept me sane.

To Allison Beemer, Carmen Caprau, and Elaina Aceves, thank you for supporting me through the course of my degree(s) and preparation for life afterward, even when we were no longer working at the same university.

Finally, to my parents Carl and Dana, and sister Katherine, thank you for everything. From helping me move out here, to the trips home where you always made sure I knew how happy you were to see me, to the fact that Katherine and I take turns with the brain cell, I could not have done this without you.

TABLE OF CONTENTS

Page

LIST OF FIGURES iv

CHAPTER

1 INTRODUCTION 1

1.1 The Symmetric Group 2

1.2 Graphs of Sets of Reduced Words 6

1.3 Young Tableaux and Reduced Words 7

1.4 Existing Results 11

2 SUBGRAPH STRUCTURES 14

2.1 Subgraphs and the Number of Edges in $G(\sigma)$ 14

2.2 The Weak Order Lattice 22

3 RATIOS OF SETS OF REDUCED WORDS 26

3.1 Examples and Difficulties 26

3.2 Fully Commutative Permutations 30

3.3 Vexillary Permutations 34

3.4 Other Permutations 37

4 BRAID CLASSES AND SETS OF REDUCED WORDS 43

4.1 General Work 44

4.2 A Special Case 45

4.3 Permutation Shuffling 47

4.4 Cases for the Lower Bound for $|B(\sigma)|$ 58

5 CONCLUSION 64

REFERENCES 66

LIST OF FIGURES

Figure Page

1.1 Two Representations of $W(\mathfrak{S}_4)$ 4

1.2 The Graph of the Set of Reduced Words of $\sigma = [42315]$ 6

1.3 The Quotient Graph $G_c(\sigma)$ 7

1.4 The Quotient Graph $G_b(\sigma)$ 7

1.5 A Young tableau of shape $\lambda = (3, 3, 1)$ 8

1.6 A Standard Young Tableau of Shape $\lambda = (3, 3, 1)$ 8

1.7 Example 1.24 (A): Step One in the Edelman-Greene Correspondence . . 9

1.8 Example 1.24 (A): Step Two in the Edelman-Greene Correspondence . . 10

1.9 Example 1.24 (A): Step Three in the Edelman-Greene Correspondence . 10

1.10 Example 1.24 (B): Step One in the Edelman-Greene Correspondence . . 10

1.11 Example 1.24 (B): Step Two in the Edelman-Greene Correspondence . . 11

1.12 Example 1.24 (B): Step Three in the Edelman-Greene Correspondence . 11

2.1 The Graph $G(w_0)$ for $w_0 \in \mathfrak{S}_4$ 17

2.2 When $i, i+1 \in D(\sigma)$, We Have This Sublattice In $W(\mathfrak{S}_n)$ 19

2.3 Proposition 2.15 Part 1 23

2.4 Proposition 2.15 Part 2 24

3.1 The Young Diagram for $\lambda = (3, 2, 1)$ 39

3.2 The Young Diagrams for $\lambda_1 = (2, 2, 1)$, $\lambda_2 = (3, 1, 1)$, and $\lambda_3 = (3, 2)$ . . . 40

4.1 The Graph $G([2431])$ 47

4.2 The Graph $G([32154])$ 48

4.3 Part of the Graph $G([3215476])$ 53

Chapter 1

INTRODUCTION

The topic of sets of reduced words is popular in the field of combinatorics. We look at permutations in the symmetric group on n objects, and consider all possible ways to write that permutation under certain conditions. Then we can study the structure of the set of all possible ways to write the same permutation. This can be done using a variety of different methods, including graphical representations of the set, or looking at the congruence classes in the set.

This research began with what appeared to be a simple question about the relative size of sets of congruence classes in a set of reduced words, and the size of the set itself. As we attempted to answer this question, we discovered that the problem was more complicated than originally thought. When we attempted to apply tools from the existing literature to this problem, we discovered that we could not answer our questions for arbitrary permutations. We would need to create new tools in order to move forward.

In this chapter, we will discuss the basic definitions of the symmetric group, as well as the existing results and tools we will use in our research. In Chapter 2 we will cover our new results for subgraph structures in graphs for sets of reduced words. In Chapter 3 we will discuss new bounds on the relative sizes of sets of reduced words for permutations that are closely related in the weak order poset. In Chapter 4 we will cover our work generalizing which subgraphs we can consider, and our work on the relative size of the sets of congruence classes and the size of the related sets of reduced words. And in Chapter 5 we will discuss what our results mean, and how the research may progress in the future.

1.1 The Symmetric Group

In this section, we rely heavily on the definitions and conventions in [2] and [15].

Definition 1.1. Let A be a finite set containing n elements. We define $\mathfrak{S}_A$ as the set of bijections from A to itself. For simplicity's sake, we write this as $A = \{1, 2, \ldots, n\}$, or $A = [n]$, and use the notation $\mathfrak{S}_n$. This is a group under composition of functions, and the elements in this set are called permutations. The identity for the group is denoted e.

Definition 1.2. For $n \geq 2$, we define the *generators* of $\mathfrak{S}_n$ as follows: $s_i = (i\ i{+}1)$ is called a simple transposition. Each s_i swaps the elements i and $i{+}1$, where $1 \leq i < n$. These generators also have specific relationships:

$$s_i^2 = e \tag{1.1}$$

$$s_i s_j = s_j s_i, \text{ for } |i - j| > 1 \tag{1.2}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ for } |i - j| = 1 \tag{1.3}$$

Definition 1.3. The second relation, labeled (1.2), is called a *commutation relation*, while the third relation, labeled (1.3), is called a *braid relation*.

The braid relation is sometimes referred to as a Yang-Baxter relation.

Definition 1.4. Let $\sigma \in \mathfrak{S}_n$. We will write σ in one line notation as

$$\sigma = [a_1 \ldots a_n]$$

where $\sigma(i) = a_i$.

Using the generators in Definition 1.2, we will also write

$$\sigma = s_{x_1} s_{x_2} \ldots s_{x_k}$$

On occasion, we may also represent this permutation as the word

$$x_1x_2\dots x_k.$$

Definition 1.5. Let $\sigma = [a_1 \dots a_n]$, and $p = p_1 \dots p_m$, for $m \leq n$ and $p_i \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq m$. The permutation σ *contains the pattern* p if there exist indices $i_1 < \dots < i_m$ such that $a_{i_1} \dots a_{i_m}$ are in the same order as $p_1 \dots p_m$. That is, $p_j < p_k$ if and only if $a_{i_j} < a_{i_k}$. If σ does not contain p, then σ is *p pattern avoiding.*

For example, $\sigma = [321]$ is 312 pattern avoiding, while the permutation $\omega = [53421]$ contains the pattern 312.

Definition 1.6. For $\sigma \in \mathfrak{S}_n$, written in one line notation as $\sigma = [a_1 \dots a_n]$, we define the *descent set of* σ as follows: $D(\sigma) = \{i \in [n] \mid a_i > a_{i+1}\}$.

For any element $\sigma \in \mathfrak{S}_n$, there is some minimum number of s_i's required for a product that will produce σ. For example, in $\mathfrak{S}_3$, the cycle $\sigma = (1\ 3)$ can be written as the product of length 3: $s_2s_1s_2 = (2\ 3)(1\ 2)(2\ 3)$. We denote this word of σ as 212, since $\sigma = s_2s_1s_2$. Furthermore, this is a reduced word for σ since it is of minimal length.

Definition 1.7. Let $\sigma \in \mathfrak{S}_n$. Suppose that $s_{i_1} \dots s_{i_k}$ is a reduced decomposition of σ. Then all reduced decompositions will be k elements long, so the *length of* σ is denoted as $\ell(\sigma) = k$.

Once we have the minimum length of the word σ as a product of s_i's, we look at the set of all of the reduced words of σ. For example, if $\sigma = (1\ 3)$, we have two reduced words: $\{212, 121\}$.

Definition 1.8. For $\sigma \in \mathfrak{S}_n$, we will denote the set of all reduced words as $R(\sigma)$.

The generating relations in Definition 1.2 can be used to produce the lattice for the weak order poset of $\mathfrak{S}_n$.

Definition 1.9. The *right weak order poset* is a partial order defined on $\mathfrak{S}_n$, denoted $W(\mathfrak{S}_n)$. $W(\mathfrak{S}_n)$ is a bounded lattice for all $n \geq 2$.

The cover relations are defined on the addition or removal of a single simple transposition on the right hand side. That is, $\tau \lessdot \sigma$ if and only if $\tau s_i = \sigma$ for some $i \in D(\sigma)$. Then $\ell(\tau) = \ell(\sigma) - 1$.

In general, if $\tau \leq \sigma$, then there exists a collection of simple transpositions such that $\tau s_{i_1} \ldots s_{i_k} = \sigma$, where $\ell(\tau) = \ell(\sigma) - k$.

There is a similar definition for the left weak order poset, but we will only consider the right weak order in this paper.

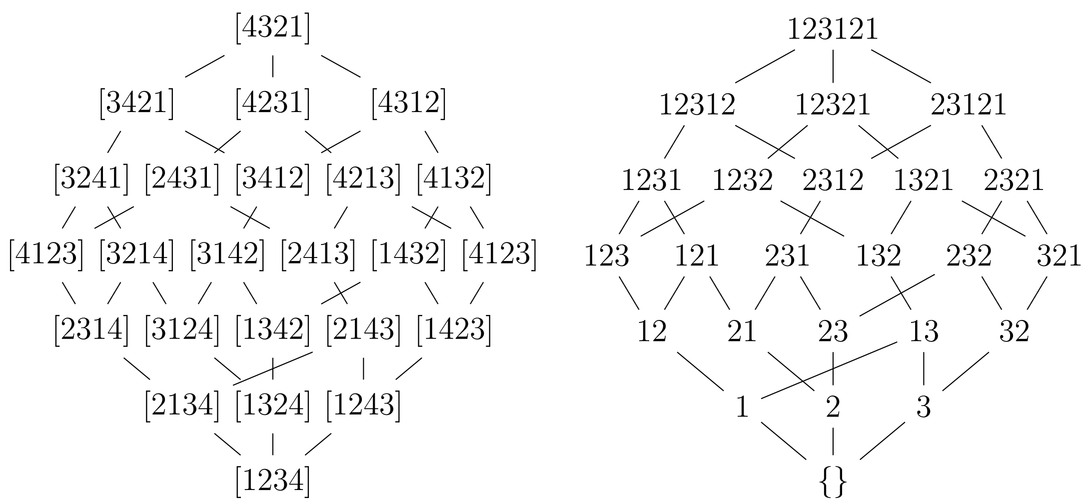

Figure 1.1: Two Representations of $W(\mathfrak{S}_4)$.

In Figure 1.1, we see two ways to represent $W(\mathfrak{S}_4)$. On the left, the vertices are labeled by the one line notation of the permutations. On the right, the vertices are labeled by one of the reduced words for each permutation.

Throughout this document, we will use the cover relations in $W(\mathfrak{S}_n)$ extensively. For $\sigma \in \mathfrak{S}_n$, the set $D(\sigma)$ gives us all the elements that σ covers in $W(\mathfrak{S}_n)$. The set of permutations covered by σ, $\{\sigma s_i \mid i \in D(\sigma)\}$, will give us a way to partition the set $R(\sigma)$ into smaller sets. We will discuss this in more detail in Chapter 2.

Remark 1.10. As we move forward, we will use Greek letters like σ to denote a permutation, while we will use standard lower case letters like w to denote a reduced word of a permutation, with the exception of the following permutation.

Definition 1.11. For any $n \in \mathbb{N}$, the longest element in $\mathfrak{S}_n$ is denoted w_0, where in one line notation we have

$$w_0 = [n\ n-1\ \dots\ 3\ 2\ 1]$$

The length of this permutation is $\ell(w_0) = \binom{n}{2}$.

Because $w_0 \in \mathfrak{S}_n$ is well studied, there are a few reduced decompositions we can immediately write down once we have selected our n. For example,

$$s_1 s_2 \dots s_{n-1} s_1 s_2 \dots s_{n-2} \dots s_1 s_2 s_3 s_1 s_2 s_1$$

and

$$s_{n-1} s_{n-2} s_{n-1} s_1 s_2 \dots s_{n-1} s_1 s_2 \dots s_{n-2} \dots s_1 s_2 s_3$$

are both reduced decompositions for w_0.

Definition 1.12. Let $r_1, r_2 \in R(\sigma)$ for some $\sigma \in \mathfrak{S}_n$. We say that r_1 and r_2 are in the same commutation class if we can perform a series of commutation moves on r_1 to produce r_2, and vice versa. We denote the collection of commutation classes of σ as $C(\sigma)$.

Definition 1.13. Let $r_1, r_2 \in R(\sigma)$ for some $\sigma \in \mathfrak{S}_n$. We say that r_1 and r_2 are in the same braid class if we can perform a series of braid moves on r_1 to produce r_2, and vice versa. We denote the collection of braid classes of σ as $B(\sigma)$.

1.2 Graphs of Sets of Reduced Words

If we wish to study a set of reduced words, and the congruence classes in each set, simply looking at the set as a list may not be the best method. We will now discuss how to build a graph for our set of reduced words, with the focus on the commutation and braid relations between words.

Definition 1.14. Let $\sigma \in \mathfrak{S}_n$. We produce an undirected graph whose vertex set is $R(\sigma)$. If two words are associated by commutation moves, we give a solid edge in the graph, and if they are associated by braid moves, we give a dashed edge. We call this graph $G(\sigma)$.

Example 1.15. Consider a permutation $\sigma \in \mathfrak{S}_5$, where $\sigma = [42315]$. Figure 1.2 shows the graph $G(\sigma)$.

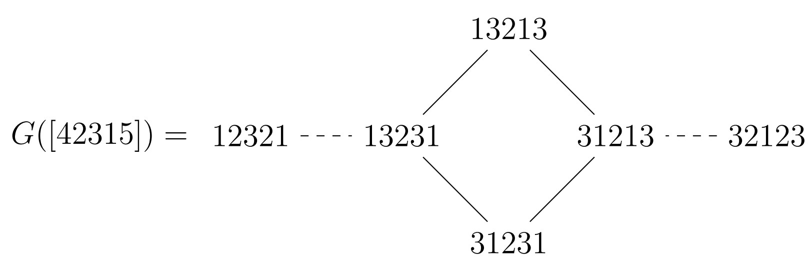

Figure 1.2: The Graph of the Set of Reduced Words of $\sigma = [42315]$

Theorem 1.16 ([2], Theorem 3.3.1)**.** *The graph $G(\sigma)$ is connected.*

Definition 1.17. If we contract all the commutation edges in $G(\sigma)$, the vertices of this quotient graph will be $C(\sigma)$, the commutation classes of $R(\sigma)$. We will denote this as

$$G_c(\sigma) = G(\sigma)/ < \text{commutation moves} > .$$

If we delete the braid edges in $G(\sigma)$, the connected components that remain will be $C(\sigma)$. We will denote this graph as $G'_c(\sigma)$.

Definition 1.18. If we contract all the braid edges in $G(\sigma)$, the vertices of this quotient graph will be $B(\sigma)$, the braid classes of $R(\sigma)$. We denote this as

$$G_b(\sigma) = G(\sigma)/ < \text{braid moves} > .$$

If we delete the commutation edges in $G(\sigma)$, the connected components that remain will be $B(\sigma)$. We will denote this graph as $G'_b(\sigma)$.

Using $\sigma = [42315]$, Figure 1.3 shows the graph of the commutation classes of $R(\sigma)$, while Figure 1.4 shows the graph of braid classes of $R(\sigma)$.

$G_c([42315]) =$ 12321 - - - - - - - - {13231, 31231, 13213, 31213} - - - - - - - - 32123

Figure 1.3: The Quotient Graph $G_c(\sigma)$

$G_b([42315]) =$

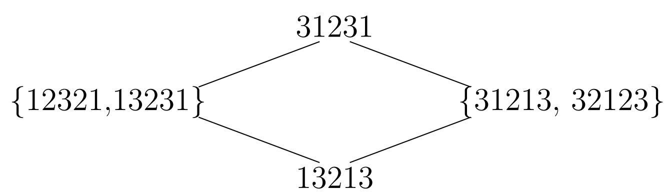

Figure 1.4: The Quotient Graph $G_b(\sigma)$

Theorem 1.19 ([3], Theorem 3.1)**.** *The graphs $G_c(\sigma)$ and $G_b(\sigma)$ are bipartite.*

Commutation moves and the quotient graph $G_c(\sigma)$ have been studied extensively. For example, see [3], [6], [16], or [17]. Less is known about $G_b(\sigma)$. This is where we will focus most of our efforts in later sections.

1.3 Young Tableaux and Reduced Words

Definition 1.20. We say that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a *partition of the number m into r parts* if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$ and $m = \lambda_1 + \ldots + \lambda_r$. We denoted this as $\lambda \vdash m$.

Definition 1.21. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of the number m. A *Young tableau* of shape λ is an array of m boxes having r left-justified rows where row i has λ_i boxes, $1 \leq i \leq r$, and the boxes contain the numbers from the set $[m]$.

We will be using what is commonly referred to as English notation for our tableaux, where λ_1 is the largest part of our partition, and the remaining parts/rows are non-increasing. Consider the tableau in Figure 1.5.

1	2	3
3	4	5
4		

Figure 1.5: A Young tableau of shape $\lambda = (3, 3, 1)$

Definition 1.22. Let $\lambda \vdash m$. We will have a *standard Young tableau* of shape λ if there is a bijection between $[m]$ and the numbers in the Young diagram. Additionally, the rows and columns must be increasing from left to right, and from top to bottom.

The example in Figure 1.5 is not a standard Young tableau, but the example in Figure 1.6 is:

1	2	5
3	6	7
4		

Figure 1.6: A Standard Young Tableau of Shape $\lambda = (3, 3, 1)$

Definition 1.23. Let $\lambda \vdash m$. We define f^λ as the number of standard Young tableaux of shape λ.

Young tableaux are widely studied for representation theory, as well as just for their own sake. We used [8] and [11] as resources for the basics of Young tableaux.

We study standard Young tableaux as a representation of the symmetric group. A well known correspondence between the sets of objects is the Robinson-Schensted-Knuth Correspondence [12].

Edelman and Greene generalized this correspondence in [5]. They took reduced words for σ, and generated pairs of tableaux (P, Q). They use a variation on a standard bumping algorithm to generate the pairs of tableaux. Q will be a standard Young tableau that keeps track of the order in which the boxes of our diagrams were created. P will have the letters of a reduced word of σ, and will not necessarily be standard.

There are formulas that will allow us to calculate how many standard Young tableaux of shape λ exist. If we wish to study the sizes of sets of reduced words, this correspondence gives us another way to count the number of elements in $R(\sigma)$. We will not formally define the Edelman Greene bumping algorithm here, but we will look at an example of how to use it to generate the tableaux pairs.

Example 1.24. Consider $\sigma = s_1 s_2 s_1$. We note that $R(\sigma) = \{121, 212\}$, so we will look at the tableau for each of these words.

(A) Consider first $w = 121$. We look at the word from right to left. We create the first boxes in our tableaux pair. Into the tableau on the left we will place the first letter 1, and on the right we will also place a 1 to signify that this was the first box created.

$$\boxed{1} \quad \boxed{1}$$

Figure 1.7: Example 1.24 (A): Step One in the Edelman-Greene Correspondence

Next we create another box for each of our tableaux. Because the next letter in our word, 2, is greater than 1, we create the second box in the same row as the first.

We place the letter 2 in the new box in the left tableau. We will also place a 2 in the equivalent box on the right as the second box created.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

Figure 1.8: Example 1.24 (A): Step Two in the Edelman-Greene Correspondence

We finish by placing the remaining letter 1 in the box held by 2 in the left tableau. However, the bumping algorithm will not allow repeated numbers in the same row, so we increase it to a 2 as well. The bumped 2 will now sit in the second row, in the new box created for the tableaux. In the right tableau, we place a 3 for the third box created. We note that the tableau on the right is a standard Young tableau.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Figure 1.9: Example 1.24 (A): Step Three in the Edelman-Greene Correspondence

(B) We repeat this process with the second word, $w = 212$. Again reading from right to left, we create the first boxes, and place our first letter 2 in the left diagram, and the number 1 in the right.

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

Figure 1.10: Example 1.24 (B): Step One in the Edelman-Greene Correspondence

We move on to the next letter. Since 1 is less than 2, the letter 1 will bump the 2 from the first row down to the second, which is where our new box will sit in both tableaux. Because of this bumping, we have two copies of the same tableau:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

Figure 1.11: Example 1.24 (B): Step Two in the Edelman-Greene Correspondence

And finally, we place the last letter 2 at the end of the first row and get the final pair of tableaux in Figure 1.12.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Figure 1.12: Example 1.24 (B): Step Three in the Edelman-Greene Correspondence

The tableau on the left is the same tableau that we got for the first reduced word. The tableau on the right is the other standard Young tableau for this staircase shape. In fact, there are only two standard Young tableau for this staircase shape, and only two reduced words for σ. This is not a coincidence, and we will discuss this more in the next section.

1.4 Existing Results

The following results are existing bounds on the various congruence classes in $R(\sigma)$.

Proposition 1.25 ([7], Proposition 2.12)**.** $|B(\sigma)| = 1$ *if and only if* σ *has a reduced word of the form* $i(i+1)i$ *or of the form* $i(i+\epsilon)(i+2\epsilon)\cdots(i+k\epsilon)$ *for some fixed* $\epsilon \in \{\pm 1\}$ *and* $k \geq 0$*.*

Theorem 1.26 ([7], Theorem 3.6)**.** *For any permutation* σ*,*

$$|B(\sigma)| + |C(\sigma)| - 1 \leq |R(\sigma)| \leq |B(\sigma)| \cdot |C(\sigma)|$$

We have some recursions for the size of the sets we care about, such as the following result by Elnitsky.

Theorem 1.27 ([6], Proposition 2.3). *If $\sigma = [w_1 \dots w_n]$ is the identity permutation, then $|C(\sigma)| = 1$. Otherwise,*

$$|C(\sigma)| = \sum_{\emptyset \neq A \subset D(\sigma),\ |i-j|>1 \forall i,j \in A} (-1)^{|A|+1} \left| C\left(\sigma \prod_{i \in A} s_i\right) \right|$$

where $D(\sigma) = \{i \mid w_i > w_{i+1}\}$ is the standard descent set for the word σ.

The following recursions are well known results for reduced words and braid classes.

Theorem 1.28. *For $\sigma \in \mathfrak{S}_n$,*

$$|R(\sigma)| = \sum_{i \in D(\sigma)} |R(\sigma s_i)|$$

Theorem 1.29. *For $\sigma \in \mathfrak{S}_n$,*

$$|B(\sigma)| = \left(\sum_{i \in D(\sigma)} |B(\sigma s_i)|\right) - \sum_{i,i+1 \in D(\sigma)} |B(\sigma s_i s_{i+1} s_i)|$$

The following result by Stanley tells us how the Edelman Greene Correspondence can be used to calculate the size of $R(\sigma)$.

Theorem 1.30 ([14], Corollary 3.1). *Let $\sigma \in \mathfrak{S}_n$. Then there exist integers $a_\lambda \geq 0$ such that*

$$|R(\sigma)| = \sum_{\lambda \vdash \ell(\sigma)} a_\lambda f^\lambda$$

If σ is vexillary*, that is, 2143 pattern avoiding, there is a $\lambda \vdash \ell(\sigma)$ such that*

$$|R(\sigma)| = f^\lambda$$

This last statement is what we noted in example 1.24, since $\sigma = s_1 s_2 s_1$ is vexillary.

Lastly, we will also use the following result from Reiner on the number of braid edges in $G(w_0)$.

Theorem 1.31 ([9], Theorem 1). *Let $d_b(v)$ be the number of braid edges incident to the vertex v in $G(\sigma)$. For $\sigma = w_0 \in \mathfrak{S}_n$,*

$$\sum_{v \in G(\sigma)} d_b(v) = |R(\sigma)|$$

Theorem 1.31 was used by Tenner in [17] to find the average number of commutation edges incident to any vertex in $G(w_0)$. And in [13], Schilling et. al. used this result to find the average number of braid edges that are incident to the connected components in $G'_c(w_0)$.

Chapter 2

SUBGRAPH STRUCTURES

As n becomes larger, and we consider longer permutations $\sigma \in \mathfrak{S}_n$, the set $R(\sigma)$ can become extremely large. This means that we would like to be able to break up the set $R(\sigma)$ into smaller, more manageable pieces. If we knew how to consider those smaller pieces, we would be able to make general statements about the larger set. For this reason, we will consider induced subgraphs inside $G(\sigma)$.

We have not found much information in the existing literature about relationships between the sets $R(\sigma)$ and $R(\tau)$ when $\tau \lessdot \sigma$ in the weak order. So we will begin by considering these graphs, $G(\sigma)$ and $G(\tau)$, and the relationships between them.

2.1 Subgraphs and the Number of Edges in $G(\sigma)$

Definition 2.1. An *induced subgraph* H of G is such that $V(H) \subset V(G)$ and $E(H)$ is defined as the subset of $E(G)$ that have both end points in $V(H)$. We will denote this subgraph relation in the standard way: $H \leq G$.

Proposition 2.2. *Let $\sigma, \tau \in \mathfrak{S}_n$, $n \geq 2$ such that $\tau \lessdot \sigma$ in the weak order lattice. Then $G(\tau)$ is an induced subgraph of $G(\sigma)$.*

Proof. If $\tau \lessdot \sigma$ in the weak order lattice, then there is some $i \in D(\sigma)$ such that $\tau s_i = \sigma$. Let $t \in R(\tau)$. Then $ti \in R(\sigma)$. This is how we will consider $R(\tau)$ as a subset of $R(\sigma)$.

In fact, this is also how we can consider $G(\tau)$ as a subgraph of $G(\sigma)$. If there is a commutation edge from t_1 to t_2 in $G(\tau)$, then there is a commutation edge from $t_1 i$ to $t_2 i$ in $G(\sigma)$. The same applies for braid edges in $G(\tau)$. This is how we map edges from $G(\tau)$ to $G(\sigma)$.

Additionally, we can look at $V(G(\tau))$ as a subset of $V(G(\sigma))$, and consider an edge in $G(\sigma)$ incident to two vertices contained in $V(G(\tau))$. This edge must already be contained in $G(\tau)$, because any commutation or braid move is happening between words in $R(\tau)$, since these vertices are of the form $\tau_1 s_i$ and $\tau_2 s_i$.

Therefore, $G(\tau)$ is an induced subgraph of $G(\sigma)$. $\square$

Corollary 2.3. *Let $\sigma, \tau \in \mathfrak{S}_n$, $n \geq 2$ such that $\tau \leq \sigma$ in the weak order lattice. Then $G(\tau)$ is an induced subgraph of $G(\sigma)$.*

Proof. Consider the chain in the weak order lattice

$$\tau \lessdot \tau_1 \lessdot \tau_2 \lessdot \ldots \lessdot \tau_k \lessdot \sigma$$

where each element in the chain covers the previous element. We know such a chain exists because $\tau \leq \sigma$. From the previous proposition, we can construct the following chain of induced subgraphs:

$$G(\tau) \leq G(\tau_1) \leq G(\tau_2) \leq \ldots \leq G(\tau_k) \leq G(\sigma)$$

So we have that $G(\tau) \leq G(\sigma)$. $\square$

Corollary 2.4. *Let $\tau \in \mathfrak{S}_n$, $n \geq 2$. Then $G(\tau)$ is an induced subgraph of $G(w_0)$.*

Proof. This result is an immediate consequence of the previous corollary. $\square$

We now have a way to break $G(\sigma)$ into subgraphs. Using this new method, we can now begin to ask questions about the braid edge and commutation edge degrees for vertices v in $G(\sigma)$.

Definition 2.5. Let $\sigma \in \mathfrak{S}_n$. For $G(\sigma)$, we define $d_b(v)$ to be the braid edge degree of a vertex v. That is, $d_b(v)$ counts the number of braid edges incident to v in the graph $G(\sigma)$.

Definition 2.6. Let $\sigma \in \mathfrak{S}_n$. For $G(\sigma)$, we define $d_c(v)$ to be the commutation edge degree of a vertex v. That is, $d_c(v)$ counts the number of commutation edges incident to v in the graph $G(\sigma)$.

Definition 2.7. Let $\sigma, \tau \in \mathfrak{S}_n$ be such that $\ell(\sigma) = \ell(\tau) = l$. Let $s_{i_1}s_{i_2}\dots s_{i_l}$ be a reduced decomposition of σ. We say that σ and τ are *equivalent* if there exists a reduced decomposition of τ, $s_{j_1}s_{j_2}\dots s_{j_l}$ such that $i_a - i_{a+1} = j_a - j_{a+1}$ for all $1 \leq a < l$. The existence of one such pair of matched decompositions will mean that there is a way to match both sets of reduced words with each other, and that this equivalence is on the sets of reduced words for the two permutations.

For example, $\tau = [321456]$ and $\sigma = [123654]$ are equivalent permutations because we can match 5 with 2 and 4 with 1 in order to get a matching between the reduced decompositions $s_2s_1s_2$ and $s_5s_4s_5$. This will also extend to the other elements in $R(\tau)$ and $R(\sigma)$.

Definition 2.8. Let $w_0^{(k,i)}$ be defined as follows

$$w_0^{(k,i)} = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & i+k-1 & i+k & \dots n \\ 1 & \dots & i-1 & i+k-1 & i+k-2 & \dots & i & i+k & \dots n \end{pmatrix}$$

Note that $w_0^{(k,i)} \in \mathfrak{S}_n$ is equivalent to w_0 in $\mathfrak{S}_k$, where $k \leq n$. We are allowed to say these are equivalent in $\mathfrak{S}_n$ because $\mathfrak{S}_k$ is a subgroup of $\mathfrak{S}_n$ for all $2 \leq k \leq n$. We also note that s_i will be the transposition with the smallest index in the reduced decomposition for $w_0^{(k,i)}$. This tells us that the descent set will be $\{i, i+1 \dots, i+k-2\}$, which we will sometimes refer to using the interval notation $[i, i+k-2]$.

For example, $s_1s_2s_1 \in \mathfrak{S}_6$ and $s_4s_5s_4 \in \mathfrak{S}_6$ are both equivalent to $w_0 \in \mathfrak{S}_3$, but we would write them as

$$w_0^{(3,1)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix}, \text{ and } w_0^{(3,4)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$$

For a permutation $\sigma \in \mathfrak{S}_n$, we want to look at the degree of a vertex u in $G(\sigma)$. This vertex can also be considered as part of an induced subgraph $G(\tau)$, where $\tau s_i = \sigma$, and $\ell(\tau) = \ell(\sigma) - 1$. We will denote this as $\overline{u} = ui$, where $u \in R(\tau)$ and $\overline{u} \in R(\sigma)$.

We note that the degree of $\overline{u}$ in $G(\sigma)$ could be larger than the degree of u in $G(\tau)$. We would like to find a way to relate $\sum_{u \in G(\tau)} d_b(u)$ and $\sum_{\overline{u} \in G(\sigma)} d_b(\overline{u})$.

Example 2.9. Let $\sigma, \tau \in \mathfrak{S}_n$. Recall from Theorem 1.28 that we can partition $R(\sigma)$ over the descent set. We partition into disjoint subsets $R(\sigma s_i)$ where $i \in D(\sigma)$.

Each $R(\sigma s_i)$ forms the vertex set for the induced subgraph $G(\sigma s_i)$. This also tells us that for a reduced decomposition of σ, this final s_i determines which induced subgraph the vertex is contained in. From Proposition 2.2, we also note that any edge between words of the form ws_i and vs_j where $i \neq j$ will not be properly contained in any of the induced subgraphs.

Consider $w_0 \in \mathfrak{S}_4$. The graph $G(w_0)$ has been drawn in Figure 2.1.

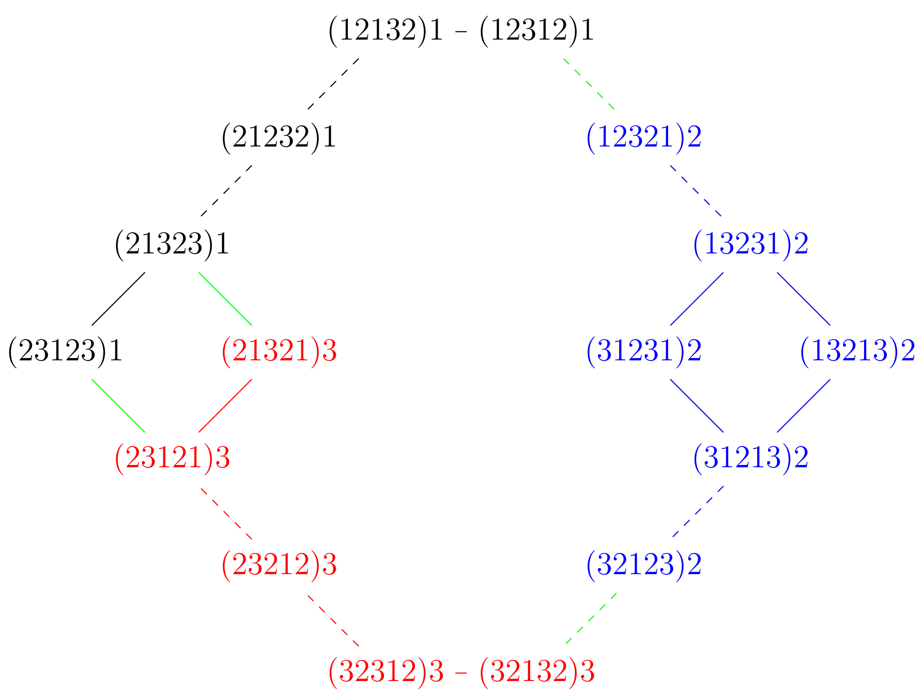

Figure 2.1: The Graph $G(w_0)$ for $w_0 \in \mathfrak{S}_4$

The induced subgraphs highlighted in different colors: $G(w_0s_1)$ is drawn black, $G(w_0s_2)$ in blue, and $G(w_0s_3)$ in red. There are also four edges in green that only appear in $G(w_0)$.

The edges between subgraphs are what we are particularly interested in. We will now focus on counting how many of these edges will exist in $G(\sigma)$.

Proposition 2.10. *Let $\sigma \in \mathfrak{S}_n$, and $i, j \in D(\sigma)$. Suppose that $ws_i, us_j \in R(\sigma)$, where $w \in R(\alpha)$, $u \in R(\beta)$ and $\alpha, \beta \lessdot \sigma$. Suppose that $(ws_i) - (us_j)$ is an edge in the graph $G(\sigma)$.*

1. *If $u \neq w$, but $i = j$, then $\alpha = \beta$ and the edge appears in the induced subgraph $G(\alpha)$.*

2. *If $i \neq j$, then the vertices appear in two disjoint subgraphs of $G(\sigma)$, and so this edge is not properly contained in any induced subgraph.*

Proof. Case 1 was already proven in Proposition 2.2. For Case 2, if $i \neq j$, then ws_i appears in $R(\sigma s_i)$, which is disjoint from $R(\sigma s_j)$ which contains us_j. This means that the two vertices appear in disjoint induced subgraphs. Thus the edge $(ws_i) - (us_j)$ connects the two subgraphs in $G(\sigma)$, but is not properly contained in either one. $\square$

Now that we have a better idea of what edges appear only in $G(\sigma)$ and not in any of the induced subgraphs, we have the following proposition:

Proposition 2.11. *Let $\sigma, \tau \in \mathfrak{S}_n$, $n \geq 2$ such that $\tau \lessdot \sigma$ in the weak order lattice. Suppose that $\tau s_i = \sigma$, where $i \in D(\sigma)$.*

1. *If $i-1,\ i,\ i+1 \in D(\sigma)$, then there are $|R(\sigma s_i s_{i+1} s_i)| + |R(\sigma s_i s_{i-1} s_i)|$ braid edges incident to the vertices of $G(\tau)$ that are not contained in the induced subgraph $G(\tau)$.*

2. *If only* $i,\ i+1 \in D(\sigma)$*, (or* $i-1, i \in D(\sigma)$*), then there are* $|R(\sigma s_i s_{i+1} s_i)|$ *braid edges incident to the vertices of* $G(\tau)$ *that are not contained in the induced subgraph* $G(\tau)$*.*

3. *If neither of these cases is true, then the only braid edges incident to the vertices of* $G(\tau)$ *are the braid edges that are incident to two vertices from* $G(\tau)$*.*

Proof. Consider $\sigma, \tau \in \mathfrak{S}_n$ such that $\tau \lessdot \sigma$, and suppose that $i, i+1 \in D(\sigma)$ and $\sigma s_i = \tau$.

Note that since $i, i+1 \in D(\sigma)$, then $\sigma = [a_1 \ldots a_n]$ will be such that $a_i > a_{i+1} > a_{i+2}$. We can consider $\sigma s_i = [\ldots a_{i+1} a_i a_{i+2} \ldots]$, and $\sigma s_{i+1} = [\ldots a_i a_{i+2} a_{i+1} \ldots]$. We note that $i \in D(\sigma s_{i+1})$ and $i+1 \in D(\sigma s_i)$. We can continue in this manner to get the portion of weak order lattice in Figure 2.2.

$$\begin{tikzcd}[column sep=small]
 & \sigma \arrow[dl, dash] \arrow[dr, dash] & \\
\sigma s_i \arrow[d, dash] & & \sigma s_{i+1} \arrow[d, dash] \\
\sigma s_i s_{i+1} \arrow[dr, dash] & & \sigma s_{i+1} s_i \arrow[dl, dash] \\
 & \sigma s_i s_{i+1} s_i &
\end{tikzcd}$$

Figure 2.2: When $i, i+1 \in D(\sigma)$, We Have This Sublattice In $W(\mathfrak{S}_n)$

Let $\alpha = \sigma s_i s_{i+1} s_i$ so that $\sigma = \alpha s_i s_{i+1} s_i$. Equivalently, we have that $\sigma = \alpha s_{i+1} s_i s_{i+1}$.

We want to be able to count the number of reduced words of σ that have a braid move in those last three positions that use the letters i and $i+1$. This means we really need to count the number of ways we can write α. This will simply be $|R(\sigma s_i s_{i+1} s_i)|$.

We note that all vertices that end in the letters $i,\ i+1,\ i$ will be contained in the subgraph $G(\sigma s_i)$, where $\tau = \sigma s_i$. All vertices that end in the letters $i+1,\ i,\ i+1$

will be contained in the subgraph $G(\sigma s_{i+1})$. Then the number of braid edges between these subgraphs will be equal to $|R(\sigma s_i s_{i+1} s_i)|$.

The same reasoning allows us to conclude that if $i-1,\ i,\ i+1 \in D(\sigma)$, there are $|R(\sigma s_i s_{i+1} s_i)| + |R(\sigma s_i s_{i-1} s_i)|$ "new" braid edges incident to vertices in $G(\tau)$.

Similarly, if we cannot write $\sigma = \alpha s_i s_{i\pm1} s_i$ for this particular i, then the only possible braid edges that can be incident to the vertices of $G(\tau)$ must be fully contained in the induced subgraph. $\square$

The following is an immediate consequence of Proposition 2.11.

Corollary 2.12. *Let $\sigma \in \mathfrak{S}_n$. Then*

$$\sum_{v \in G(\sigma)} d_b(v) = \left(\sum_{i \in D(\sigma)} \sum_{u \in G(\sigma s_i)} d_b(u) \right) + 2 \cdot \sum_{i, i+1 \in D(\sigma)} |R(\sigma s_i s_{i+1} s_i)|$$

where we consider $d_b(u)$ in $G(\sigma s_i)$ on the right hand side, and $\overline{u} = ui \in G(\sigma)$ on the left hand side.

This is a brand new way to count the number of braid edges in a graph of a set of reduced words. Now we have methods of breaking $G(\sigma)$ into subgraphs, and a way to count the braid edges in $G(\sigma)$. Since we want to understand the congruence classes in $B(\sigma)$, this is a step closer to that goal.

Because the arguments are very similar, we decided to prove a similar result that will allow us to count the number of commutation edges in $G(\sigma)$.

Proposition 2.13. *Let $\sigma, \tau \in \mathfrak{S}_n$, $n \geq 2$ such that $\tau \lessdot \sigma$ in the weak order lattice. Suppose that $\tau s_i = \sigma$, where $i \in D(\sigma)$. Let $I_i = \{j \mid j \in D(\sigma) \text{ and } |j - i| > 1\}$.*

1. *If $I_i \neq \emptyset$, then there are $\sum_{j \in I_i} |R(\sigma s_i s_j)|$ commutation edges incident to the vertices of $G(\tau)$ that are not contained in the induced subgraph.*

2. *If $I_i = \emptyset$, then the only commutation edges incident to the vertices of $G(\tau)$ are the commutation edges that are incident to two vertices from $G(\tau)$.*

Proof. Consider $\sigma, \tau \in \mathfrak{S}_n$ such that $\tau \lessdot \sigma$. Suppose that $i, j \in D(\sigma)$ are such that $|i - j| > 1$, and $\sigma s_i = \tau$.

Since $i, j \in D(\sigma)$, and letters i and j commute with each other, there exist reduced decompositions of σ such that $\sigma = \alpha s_i s_j$ and $\sigma = \alpha s_j s_i$. We want to be able to count the number of reduced words of σ that have a commutation move in those last two positions that use the letters i and j. This means we really need to count the number of ways we can write α. This will simply be $|R(\sigma s_i s_j)|$.

We note that all vertices that end in the letters j i will be contained in the subgraph $G(\sigma s_i)$, where $\tau = \sigma s_i$. All vertices that end in the letters i j will be contained in the subgraph $G(\sigma s_j)$. Then the number of commutation edges between these subgraphs will be equal to $|R(\sigma s_i s_j)|$.

The same reasoning allows us to conclude that for all $k \in I_i$, then there are $|R(\sigma s_k s_i)|$ "new" commutation edges incident to vertices in $G(\tau)$.

Similarly, if $I_i = \emptyset$ then we cannot write $\sigma = \beta s_j s_i = \beta s_i s_j$ for this particular i, then the only possible commutation edges that can be incident to $G(\tau)$ must be fully contained in the induced subgraph. $\square$

The following is an immediate consequence of Proposition 2.13.

Corollary 2.14. *Let $\sigma \in \mathfrak{S}_n$. Then if $d_c(v)$ counts the number of commutation edges incident to v in $G(\sigma)$,*

$$\sum_{v \in G(\sigma)} d_c(v) = \left(\sum_{i \in D(\sigma)} \sum_{u \in G(\sigma s_i)} d_c(u) \right) + 2 \cdot \sum_{i,j \in D(\sigma)} \sum_{i - j > 1} |R(\sigma s_j s_i)|$$

where we consider $d_b(u)$ in $G(\sigma s_i)$ on the right hand side, and $\overline{u} = ui \in G(\sigma)$ on the left hand side.

Now that we have a way to count both types of edges over induced subgraphs, we need to consider the types of induced subgraphs we could encounter.

2.2 The Weak Order Lattice

In addition to understanding the number of new braid or commutation edges we will have in $G(\sigma)$, we want information about what types of induced subgraphs we might be dealing with. What sorts of permutations might we have for $\tau \lessdot \sigma$? We already know a certain amount about $w_0 \in \mathfrak{S}_n$, so we ask if we will ever encounter a graph $G(\sigma)$ where multiple subgraphs are equivalent to a $G(w_0^{(k,i)})$ for some $k, i \in \mathbb{N}$.

If we look at Figure 1.1, we see that $w_0^{(3,1)}$ and $w_0^{(3,2)}$ are both present in the lattice. We also note that they do not cover the same permutations, and they are not covered by the same permutations. This leads us to the following proposition.

Proposition 2.15. *Let $\sigma \in \mathfrak{S}_n$ be such that σ is not equivalent to a $w_0^{(k,i)}$, for $k > 2$. Further suppose that it is not covered by a permutation equivalent to $w_0^{(k,i)}$ for any i. The permutations covered by σ are $\{\sigma s_j \mid j \in D(\sigma)\}$.*

1. *There exists k, i such that σs_j is equivalent to $w_0^{(k,i)}$ for at most one j,*
2. *There exists k, i, m and $i \leq m \leq i + k - 1$ such that σs_j is equivalent to a $w_0^{(k,i)} s_m$ for at most one j.*

Proof. This will come from the descent sets for each σs_j, and the lattice structure of $W(\mathfrak{S}_n)$. We will consider σs_x and σs_y for $x, y \in D(\sigma)$. Let $\ell(\sigma) = l$.

1. Suppose that $\sigma s_x = w_0^{(k_x,i)}$ and $\sigma s_y = w_0^{(k_y,j)}$. Since these permutations each have the same length, $l - 1$, $k_x = k_y$ while the minimum index i and j could be distinct.

Consider σs_x first. We can draw part of the weak order lattice containing σ and σs_x in order to get an idea of what the descent sets must look like.

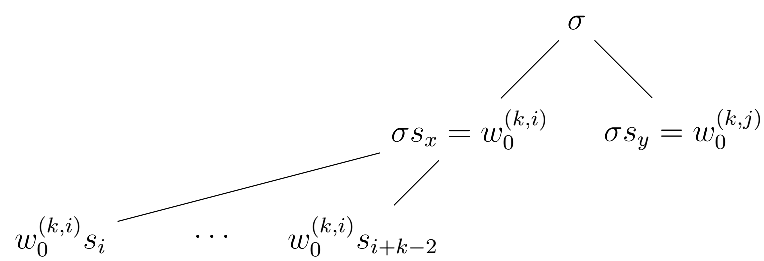

Figure 2.3: Proposition 2.15 Part 1

We write $D(\sigma s_x) = \{i, i+1, \dots, i+k-2\}$. Let $A \subset [n]$ such that $x \in A$, $A \cap [i, i+k-2] = \emptyset$ and $\{i-1, i+k-1\} \subset A$. We note that x could be $i-1$ or $i+k-1$, and we could end up with $D(\sigma) = [i, i+k-3] \cup \{x\}$, or $D(\sigma) = [i+1, i+k-2] \cup \{x\}$. However, in both of these cases, $y \in D(\sigma)$ would be such that $D(\sigma s_y) \neq D(w_0^{(k,i)})$ for any i, k.

Since $D(\sigma s_y)$ only contains consecutive elements, this tells us two things: (i) x must be either $i-1$ or $i+k-1$, and (ii) $D(\sigma) = D(\sigma s_x) \cup \{x\}$.

Similar reasoning tells us that in order for $D(\sigma s_x)$ to only contain consecutive elements then y must either equal x, or one of the following: if $x = i+k-1$, then $y = i$ and if $x = i-1$, then $y = i+k-2$. Without loss of generality, consider $D(\sigma) = \{x = i-1, i, i+1, \dots, i+(k-2) = y\}$.

Then since we claim both σs_x and σs_y are equivalent to $w_0^{(k,j)}$ for some j, we have that $\sigma s_x = w_0^{(k,i)}$ and $\sigma s_y = w_0^{(k,i-1)}$. This also means we know two different reduced decompositions for σ using decompositions for the $w_0^{(k,j)}$'s that we discussed in Chapter 1: $(s_x s_i \dots s_{i+k-3} \dots s_x s_i s_x) s_y$ and $(s_i s_{i+1} \dots s_y \dots s_i s_{i+1} s_i) s_x$.

But this means

$$\sigma = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & i+k-2 & i+k-1 & \dots n \\ 1 & \dots & i+k-1 & i+k-3 & i+k-4 & \dots & i-1 & i+k-2 & \dots n \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & i+k-2 & i+k-1 & \ldots n \\ 1 & \ldots & i & i+k-1 & i+k-2 & \ldots & i+1 & i-1 & \ldots n \end{pmatrix}$$

which are not equivalent permutations. In fact, the only way we could have both reduced decompositions for σ is if $k = 2$, and $\sigma = s_x s_y$. Since we assume $k > 2$, σ can only cover at most one permutation equivalent to $w_0^{(k,i)}$ for some i.

2. Since the meet and join of a lattice are well defined, if σs_x and σs_y are covered by the same $\tau \equiv w_0^{(k,i)}$, then $\sigma = \tau$. So they must be covered by different τ's: $w_0^{(k,i)}$ and $w_0^{(k,j)}$. We can draw a picture of the part of the weak order lattice that contains these permutations:

$$\begin{tikzcd} w_0^{(k,i)} & \sigma & w_0^{(k,j)} \\ \sigma s_x \arrow[u, dash] \arrow[ur, dash] & & \sigma s_y \arrow[u, dash] \arrow[ul, dash] \end{tikzcd}$$

Figure 2.4: Proposition 2.15 Part 2

We know that $D(w_0^{(k,i)}) = \{i, i+1, \ldots, i+k-2\}$, and that x cannot be an element in this set. Let $\sigma s_x = w_0^{(k,i)} s_a$. Then we know that $D(\sigma s_x) = D(w_0^{(k,i)}) - \{a\}$, and that $x \neq a$. Similarly, $D(\sigma s_y) = D(w_0^{(k,j)}) - \{b\}$ for $y \neq b$, where $\sigma s_y = w_0^{(k,i)} s_b$.

So we have $\sigma = (w_0^{(k,i)} s_a) s_x = (w_0^{(k,j)} s_b) s_y$. As before, without loss of generality, let $a = i$ and $b = j$, and we would then write σ as

$$\sigma = \begin{pmatrix} \ldots & i & i+1 & \ldots & i+k-1 & i+k & \ldots x & x+1 & \ldots \\ \ldots & i+k-2 & i+k-1 & \ldots & i & i+k & \ldots x+1 & x & \ldots \end{pmatrix}$$

$$\sigma = \begin{pmatrix} \ldots & j & j+1 & \ldots & j+k-1 & j+k & \ldots y & y+1 & \ldots \\ \ldots & j+k-2 & j+k-1 & \ldots & j & j+k & \ldots y+1 & y & \ldots \end{pmatrix}$$

Even allowing for any other $a \in D(w_0^{(k,i)})$ and $b \in D(w_0^{(k,j)})$, we note that the only way that we can have this equality is if $k = 2$, $a = b$, and $i = j$, a contradiction.

Therefore, at most one of the σs_j's can be covered by a $w_0^{(k,i)}$ for some i. $\square$

We want to compare $\sum_{v \in G(\sigma)} d_b(v)$ with $|R(\sigma)|$. From Theorem 1.31, we know that for $w_0 \in \mathfrak{S}_n$, these two quantities are equal. From our work with Corollary 2.12, we know how we will be adding braid edges as we go up the weak order lattice. And we have just shown that a permutation can cover at most one element equivalent to $w_0^{(k,i)}$, which could be helpful when trying to prove that w_0 "maximizes" the braid edge degrees in its related graph.

We have now finished answering our initial questions about subgraphs in $G(\sigma)$. We had looked for existing results on subgraphs in order to answer questions about the number of braid classes in $R(\sigma)$. When we were unable to find the answers to our questions, we produced the work in this chapter. After finding these new formulas and other results, we were able to answer some of those original questions on braid classes. Those new results will be discussed in detail in Chapter 4.

Chapter 3

RATIOS OF SETS OF REDUCED WORDS

As in the last chapter, we are interested in a relationship between the structures of the sets $R(\tau)$ and $R(\sigma)$ when $\tau \lessdot \sigma$ in the Weak Order Lattice. In this chapter, we are concerned with relationships between the sizes of these sets.

As we have noted before, we have a lot of information about $w_0 \in \mathfrak{S}_n$, but less information about an arbitrary permutation. We want to have some idea of how large our induced subgraphs can get. For example, will we ever have a subgraph $G(\sigma s_i)$, $i \in D(\sigma)$, that is significantly larger that all other subgraphs?

The difficulty is that there are relatively few known relationships between these sets. What is known was discussed in Chapter 1. Another difficulty is that we have to know more about σ in order to make assumptions about the structure of $G(\sigma)$, and how that might relate to the relative size of $G(\sigma s_i)$. We will discuss some of these challenges first, followed by new results about different families of permutations.

3.1 Examples and Difficulties

For this next section, we will look at constructing $R(\sigma)$ from an inductive point of view. We will then use this understanding of the set to prove results about the relationship between $|R(\sigma)|$ and $|R(\tau)|$ when $\tau \lessdot \sigma$.

Consider a permutation $\sigma \in \mathfrak{S}_n$. We know that we can look at $i \in D(\sigma)$, and consider $\sigma = \tau s_i$ for some $\tau \in \mathfrak{S}_n$. We can use this structure of σ in order to build words in $R(\sigma)$ from words $t \in R(\tau)$ by appending the letter i in the right most spot. In addition to this sort of embedding of $R(\tau) \subset R(\sigma)$, which we discussed in Chapter 2, we would like to go further, and construct all of $R(\sigma)$.

Since $ti \in R(\sigma)$, we can generate all words in $R(\sigma)$ by considering all the commutation and braid moves possible between letters of t, and i. In particular, we want to be able to keep track of i, and so we will define the following:

Definition 3.1. Let $\sigma \in \mathfrak{S}_n$ be such that $\sigma = s_{a_1} s_{a_2} \dots s_{a_l}$, where $\ell(\sigma) = l$. We say that i is in the *support of* σ if $i = a_j$ for some $1 \leq j \leq l$.

For a reduced word $w \in R(\sigma)$, we will refer to letters j in *position* x, where $1 \leq x \leq l$.

For $i \in D(\sigma)$, define $C_x^{(i)}$ as the collection of reduced words of σ where this particular letter i sits in position x.

For an arbitrary permutation, any element $i \in D(\sigma)$ will be in the support of σ, but not all elements in the support of σ will appear in $D(\sigma)$.

For any $t \in R(\tau)$, $C_{\ell(\sigma)}^{(t)}$ is well defined, as i does not need to commute anywhere for the word ti to be in $R(\sigma)$.

Example 3.2. Let $\sigma = [153264] = s_4 s_2 s_3 s_2 s_5$ where $D(\sigma) = \{2, 3, 5\}$. Since there are three descents, we will look at three different inductive constructions of $R(\sigma)$.

Case 1: We can consider $\tau_1 = s_4 s_2 s_3 s_2$, where $\sigma = \tau_1 s_5$. We note that the only braid moves for σ come from braiding elements in τ_1, and that our extra transposition s_5 is not part of any braid moves. So we build $R(\sigma)$ in a very straightforward manner.

We see that $R(\tau_1) = \{4232, 4323, 2432\}$. We can take each of these words for τ_1 and build the set $R(\sigma)$. We will use the notation in Definition 3.1 to keep track of how we formed the words from $t \in R(\tau_1)$ and 5. Let $C_t^{(5)}$ be the collection of reduced words of $R(\sigma)$ with 5 in position t. Then we have the following collection of $C_t^{(5)}$'s:

$$C_5^{(5)} = \{42325, 43235, 24325\}$$

$$C_4^{(5)} = \{42352, 43253, 24352\}$$

$$C_3^{(5)} = \{42532, 43523, 24532\}$$

$$C_2^{(5)} = \{45232, 45323\}$$

The number of reduced words of σ we get from an element in $R(\tau_1)$ relies on where 4 is sitting in the word, as it is the only element that does not commute with 5. Additionally, there are restrictions on how far right 4 can go in a word t, since there are elements in the support of τ_1 that do not commute with 4. We will return to this difficulty later on.

We also note that in each word of σ, the letters from t appear in the same order as they did in their original word for τ_1.

Case 2: Next we have $\tau_2 = s_4 s_2 s_3 s_5$ where $\sigma = \tau_2 s_2$. We see that there are no braid moves between elements in $R(\tau_2)$. However, there are braid moves between elements in $R(\sigma)$. This means that s_2 will be used in all braid moves between elements of $R(\sigma)$. This is the difficulty that we remarked on after Definition 3.1.

We want to still keep track of where 2 sits in the word, and build the sets $C_t^{(2)}$ based on that information. However, after a braid move we will try to keep track of the right most 3 that 2 has been replaced with. We will color this letter in red after a braid move. We will denote these sets of reduced words of σ as $C_t^{(2)\prime}$.

This makes our job of building up to $R(\sigma)$ more difficult. We can use the same method as before, but we will have to also take into account some special words.

Note that $R(\tau_2) = \{4235, 2435, 2453, 4253, 4523\}$. Then we have

$$C_5^{(2)} = \{42352, 24352, 24532, 42532, 45232\}$$

$$C_4^{(2)} = \{42325, 24325\}$$

$$C_5^{(2)\prime} = \{45323, 43523, 43253\}$$

$$C_4^{(2)\prime} = \{43235\}$$

We note that for any word with a highlighted 3, the letters from $t \in R(\tau_2)$ appear in a different order than they did in the original word. We also see that there is a special symmetry for words where s_2 is part of a braid move.

Case 3: Finally we have $\tau_3 = s_4 s_3 s_2 s_5$ where $\sigma = \tau_3 s_3$. Once again, we will have to use s_3 in order to get the braid moves for words in $R(\sigma)$.

Note that $R(\tau_3) = \{4325, 4352, 4532\}$. Already we expect to have more words for σ similar to the special word from Case 2. We know that there are words in $R(\sigma)$ that do not have s_4 as the left-most transposition. We have the following:

$$C_5^{(3)} = \{43253, 43523, 45323\}$$

$$C_4^{(3)} = \{43235\}$$

$$C_5^{(3)\prime} = \{45232, 42532, 24532, 42352, 24352\}$$

$$C_4^{(3)\prime} = \{42325, 24325\}$$

It is a bit easier to see what is happening in this case. In words for $R(\tau_3)$, we have letters that cannot commute with each other, but by a braid move that uses 3, we have something that acts like a commutation move on those factors of τ_3.

We can now sum up all of our cases as follows: for permutations σ and τ_i

$$|R(\tau_1)| \geq \frac{1}{4}|R(\sigma)| \quad |R(\tau_2)| \geq \frac{1}{3}|R(\sigma)| \quad |R(\tau_3)| \geq \frac{1}{4}|R(\sigma)|$$

As we move forward in the chapter, we will be looking for bounds that will approach the bounds that we found in this example. However, we will need to be careful considering the difficulties we had in keeping track of the letters in cases 2 and 3.

3.2 Fully Commutative Permutations

There are families of permutations where the number of reduced words is easily counted without requiring the aid of another combinatorial objects. One such family is fully commutative permutations. Recall Definition 1.5 on pattern avoidance.

Definition 3.3. A permutation $\sigma \in \mathfrak{S}_n$ is call *fully commutative* if it is 321 pattern avoiding. That is, there are no braid edges in $G(\sigma)$.

Example 3.4. We note that $\sigma = [23418567] = s_1s_2s_3s_7s_6s_5$ is fully commutative, while $\tau = [24318567] = s_1s_2s_3s_7s_6s_5s_2$ is not.

While looking at fully commutative permutations $\sigma \in \mathfrak{S}_n$, we began to study the types of simple transposition blocks that could be present in a reduced decomposition. Consider the following reduced decompositions:

$$s_4s_5s_3s_4, \quad s_1s_2s_3s_4s_5, \quad \text{and } s_1s_3s_5$$

All of these reduced decompositions produce fully commutative permutations. Additionally, we can combine these patterns and still produce fully commutative permutations. For example, consider the following reduced decomposition:

$$s_6s_5s_7s_6s_4s_3s_2s_1s_{10}s_{14}$$

These combinations of simple transposition blocks leads us to the following remark.

Remark 3.5. A reduced decomposition of a fully commutative permutation may contain combinations of simple transposition blocks of the following forms:

1. Let $\epsilon \in \{\pm 1\}$ be fixed. Then we consider $s_i s_{i+\epsilon} s_{i+2\epsilon} \dots s_{i+k\epsilon}$.

2. Let I be an index set where all $i, j \in I$ are such that $|i - j| > 1$. Then each pair of elements s_{i_1}, s_{i_2} with $i_1, i_2 \in I$ commute completely with each other.

3. For $2 \leq i \leq n-2$, consider $s_i s_{i+1} s_{i-1} s_i$.

Unfortunately, this leads to many possible varieties of permutations. We will focus on certain subfamilies to help us attempt to understand the structure of $R(\sigma)$ for an arbitrary fully commutative permutation σ.

Definition 3.6. Let $\sigma \in \mathfrak{S}_n$ be such that $\sigma = s_{a_1} s_{a_2} \dots s_{a_l}$. We will call σ *completely commutative* if it is of type 2 from Remark 3.5. That is, if there are no pairs of indices a_i, a_j such that $|a_i - a_j| = 1$.

Lemma 3.7. *Let $\sigma \in \mathfrak{S}_n$ be completely commutative, containing only patterns of type 2 from Remark 3.5. Then $|R(w)| = l!$ where $\ell(\sigma) = l$. Furthermore, for all $i, j \in D(w)$, $|R(ws_i)| = |R(ws_j)|$.*

Proof. Since each s_{a_i} commutes with each of the other s_{a_j}'s in the factorization of σ, each of the l positions in w, there are no restrictions on which factor can be in that spot. Therefore, there are $l!$ ways to write σ.

From Theorem 1.28, we know that $|R(\sigma)| = \sum_{i \in D(\sigma)} |R(\sigma s_i)|$. For each $i \in D(\sigma)$, $\ell(\sigma s_i) = l - 1$. Additionally, each σs_i is completely commutative as well. So, we have $|R(\sigma s_i)| = (l-1)!$ for all $i \in D(\sigma)$, as desired. □

We can also have permutations that are almost completely commutative.

Proposition 3.8. *Let $\sigma \in \mathfrak{S}_n$ be written as $\sigma = s_{a_1} s_{a_2} \dots s_{a_l}$, where $\ell(\sigma) = l$. Suppose there is a unique pair of indices a_i, a_j such that $|a_i - a_j| = 1$. Without loss of* *generality, suppose that s_{a_i} appears to the left of s_{a_j} in the factorization of σ. Then*

$$|R(\sigma)| = \sum_{k=2}^{l}(k-1)\cdot(l-2)! = \frac{l!}{2}$$

Furthermore, for all $a_x \in D(\sigma)$ such that $a_x \neq a_j, a_i$, then

$$|R(\sigma s_{a_x})| = \sum_{k=2}^{l-1}(k-1)\cdot(l-3)!$$

If $a_x = a_j, a_i$, then $|R(\sigma s_{a_x})| = (l-1)!$.

Proof. For the first claim, we note that s_{a_i} and s_{a_j} are the only factors that do not commute with each other. Let s_{a_j} sit in position k, where $2 \leq k \leq l$. Then s_{a_i} has $k-1$ positions to the left of s_{a_j} that it can sit in. Additionally, all other factors of w have complete freedom around s_{a_i} and s_{a_j}. Thus there are $(k-1)\cdot(l-2)!$ ways to write w when s_{a_j} appears in position k. Vary over all possible k and we get the desired size of $|R(\sigma)|$.

Using this result, we get the size of $|R(\sigma s_{a_x})| = \sum_{k=2}^{l-1}(k-1)\cdot(l-3)!$. Using the previous proposition, we know that if we do not have any restrictions on what the factors can commute with, then $|R(\sigma)| = l!$. So, if $a_x = a_j$ or a_i, (depending on which is the larger integer), then ws_{a_x} will not have any restrictions on commutation. Thus, we have $|R(\sigma s_{a_x})| = (l-1)!$ as desired. $\square$

At this point, we have run through all of the families of fully commutative permutations where the size of $R(\sigma)$ is easy to calculate. We will move onto more complex families of permutations now.

Definition 3.9. A permutation is called *Grassmannian* if it has a unique descent.

These types of permutations must have the following structure:

$$\sigma = [a_1\ a_2\ \ldots a_i\ b_1\ b_2\ \ldots b_j]$$

where $a_x < a_{x+1}$ for all $1 \leq x \leq i$, $a_i > b_1$, and $b_y < b_{y+1}$ for all $1 \leq y \leq j$.

Permutations of type 1 and type 3 in Remark 3.5 are Grassmannian. The following propositions tells us more about the sets of reduced words for Grassmannian permutations.

Proposition 3.10. *Let $\sigma \in \mathfrak{S}_n$ be written as $\sigma = \tau s_b$. If $\epsilon \in \{\pm 1\}$, and τ is of the form $s_i s_{i+\epsilon} s_{i+2\epsilon} \ldots s_{i+k\epsilon}$, and $b = i + (k+1)\epsilon$, then*

$$|R(\tau)| = |R(\sigma)| = 1$$

For permutations with larger sets of reduced words, we have the following.

Proposition 3.11. *Suppose that $\sigma \in \mathfrak{S}_n$ is Grassmannian, with descent i. Then $|R(\sigma)| = |R(\sigma s_i)|$.*

Proof. Since $|R(\sigma)| = \sum_{i \in D(\sigma)} |R(\sigma s_i)|$, and $|D(\sigma)| = 1$, we have that $|R(\sigma)| = |R(\sigma s_i)|$ as desired. $\square$

Now that we have a better understanding of the structure of fully commutative permutations, we are prepared to prove the following proposition.

Proposition 3.12. *Let $\sigma \in \mathfrak{S}_n$ be fully commutative. Then for $i \in D(\sigma)$ and $\ell(\sigma) = l$,*

$$\frac{|R(\sigma s_i)|}{|R(\sigma)|} \geq \frac{1}{l}$$

Proof. Consider $\sigma = \tau s_i$ where $i \in D(\sigma)$.

First we note that since σ is fully commutative, the letter i will not be a part of any braid moves. Thus, it either commutes with a letter in $t \in R(\tau)$, or it does not.

Consider all possible sets $C_x^{(i)}$, where i sits in position x in words of σ. We know that $\sigma = \tau s_i$, so we have at least one way to place i: ti for any $t \in R(\tau)$. Thus $C_l^{(i)}$ is non-empty, and $|R(\tau)| = |C_l^{(i)}|$.

Note that for any x, because i can only commute with other letters, $|C_x^{(i)}| \leq |R(\tau)|$. Then since

$$R(\sigma) = \bigcup_{1 \leq x \leq l} C_x^{(i)}$$

we have that

$$|R(\sigma)| \geq l \cdot |R(\tau)|$$

If there is a letter in τ that will not commute with i, the inequality would be strict.

Therefore, for any $i \in D(\sigma)$,

$$\frac{|R(\sigma s_i)|}{|R(\sigma)|} \geq \frac{1}{l}$$

as desired. □

This bound for fully commutative permutations will be sharp, because we have equality in Lemma 3.7.

3.3 Vexillary Permutations

Consider $\sigma, \tau \in \mathfrak{S}_n$ such that $\tau \lessdot \sigma$, and $\ell(\sigma) = l$. Then we note that from Stanley [14], we have

$$|R(\tau)| = \sum_{\lambda \vdash l-1} a_\lambda f^\lambda \qquad \text{and} \qquad |R(\sigma)| = \sum_{\mu \vdash l} a_\mu f^\mu$$

where $a_\lambda, a_\mu \in \mathbb{Z}_{\geq 0}$.

In the previous section, we were able to find a lower bound for ratios of fully commutative elements. Using tableaux and the results from Stanley, we can arrive at the following lower bound for sets of reduced words of vexillary permutations σ.

Proposition 3.13. *Let $\sigma, \tau \in \mathfrak{S}_n$ such that $\tau \lessdot \sigma$, and let $\ell(\sigma) = l$. If both σ and τ are vexillary, then*

$$\frac{|R(\tau)|}{|R(\sigma)|} \geq \frac{1}{l}$$

Proof. If both permutations are vexillary, then there exist tableaux λ and μ such that

$$|R(\tau)| = f^{\lambda} = \frac{(l-1)!}{\prod_{(i,j)\in\lambda} h_{i,j}} \text{ and } |R(\sigma)| = f^{\mu} = \frac{l!}{\prod_{(i,j)\in\mu} h_{i,j}}$$

where the $h_{(i,j)}$'s are the *hook length numbers* that count the number of cells below, to the right of, and including the cell in row i and column j.

Note that additionally, $\lambda \lessdot \mu$, so $\prod_{(i,j)\in\lambda} h_{i,j} \leq \prod_{(i,j)\in\mu} h_{i,j}$. Then,

$$\begin{aligned}
\frac{|R(\tau)|}{|R(\sigma)|} &= \frac{f^{\lambda}}{f^{\mu}} \\
&= \frac{(l-1)!}{\prod_{(i,j)\in\lambda} h_{i,j}} \cdot \frac{\prod_{(i,j)\in\mu} h_{i,j}}{l!} \\
&= \frac{1}{l} \cdot \frac{\prod_{(i,j)\in\mu} h_{i,j}}{\prod_{(i,j)\in\lambda} h_{i,j}} \\
&\geq \frac{1}{l}
\end{aligned}$$

□

We have found the same lower bound for fully commutative and vexillary permutations.

Because of our interest in $w_0 \in \mathfrak{S}_n$, we are very interested in understanding vexillary permutations. Thus, we will work towards a better bound for some vexillary permutations.

Suppose that both τ and σ are vexillary permutations, with $\tau \lessdot \sigma$. For a removable border box in the diagram for σ, we note that we have the following ratio:

$$\frac{|R(\tau)|}{|R(\sigma)|} = \frac{a_1 \cdot a_2 \cdot \cdots \cdot a_j \cdot b_1 \cdot b_2 \cdot \cdots \cdot b_k}{(a_1-1)(a_2-1)\cdots(a_j-1)(b_1-1)(b_2-1)\cdots(b_k-1)}$$

where the a_i's are the hook lengths from the column we removed the box from, and the b_i's are the hook lengths from the row we removed the box from. This leads us to the following remark.

Remark 3.14. While the bound of $\frac{1}{l}$ is achieved by certain types of fully commutative permutations, including completely commutative permutations, we do not expect that it will be attained by most vexillary permutations. In particular, $i \in D(\sigma)$ would need to commute with everything else in the support of σ in order to achieve this bound.

From the above remark, we will consider a better bound for w_0 and the elements it covers in the weak order lattice.

We know the hook length numbers for the staircase tableau, so we should be able to 1) find calculations for any of the ratios between $|R(w_0)|$ and $|R(w_0 s_i)|$, and 2) find which one of these ratios will be the smallest. Removal of the box from row 1 and row $n-1$ will give the smallest ratios.

Proposition 3.15. *For $w_0 \in \mathfrak{S}_n$, $n \geq 3$, $j \in D(w_0)$, we know that $\ell(w_0) = \dfrac{n(n-1)}{2}$, so*

$$\frac{|R(w_0 s_j)|}{|R(w_0)|} \geq \frac{\prod_{i=1}^{n-1} 2i-1}{\prod_{i=1}^{n-2} 2i} \cdot \frac{2}{n(n-1)}$$

With equality for $j = 1$ and $j = n-1$.

Proof. The formula above is exactly the calculation for hook length ratios when $j = 1$ and $j = n-1$. We claim that this is also the minimum of all such ratios.

Consider removing the border box from row 2. Here is where the calculation will differ:

$$\frac{|R(w_0 s_j)|}{|R(w_0)|} \geq \frac{3 \cdot \prod_{i=1}^{n-2} 2i-1}{2 \cdot \prod_{i=1}^{n-3} 2i} \cdot \frac{2}{n(n-1)}$$

It has the same number of factors, but

$$\frac{3}{2} > \frac{2n-3}{2n-2}$$

Similar replacements happen in every row except for rows 1 and $n-1$. $\square$

3.4 Other Permutations

Fully commutative and vexillary permutations are well-studied families, which is why we considered them first. Now we move into arbitrary permutations that contain braid moves, but are not vexillary.

First we note that one of our results, Lemma 3.7 can be weakened and generalized for permutations containing braid moves.

Corollary 3.16. *Let $\sigma \in \mathfrak{S}_n$ be written as $\sigma = \tau\ s_b$ and let $\ell(\sigma) = l$. If $|b - i| > 1$ for all s_i in the support of τ, then*

$$|R(\tau)| = \frac{1}{l}|R(\sigma)|$$

Note that when involving a braid move, we maximize $|R(\sigma)|$ if $i(i+1)i$ commutes with everything else in the support of σ. Consider for example $\sigma = s_2s_3s_1s_2s_1$ with five reduced words versus $\omega = s_4s_5s_1s_2s_1$ with twenty.

Proposition 3.17. *Let $\sigma \in \mathfrak{S}_n$ be written as $\sigma = \tau s_i s_{i+1} s_i$ where $\ell(\tau) = \ell(\sigma) - 3$. Let $\ell(\sigma) = l$. If s_i and s_{i+1} commute with every s_j in the support of τ, then*

$$\frac{|R(\sigma s_i)|}{|R(\sigma)|} \geq \frac{1}{l}$$

Proof. Fix a decomposition of τ, and consider specifically $i \in D(\sigma)$. We want to compare $|R(\sigma)|$ and $|R(\sigma s_i)|$.

Suppose that for every s_j in the decomposition of τ, $|i - j| > 1$ and $|i + 1 - j| > 1$. Then we have

$$\binom{l}{3} \cdot 2$$

ways to place $s_i s_{i+1} s_i$. Therefore, we have

$$|R(\sigma)| = |R(\tau)|\binom{l}{3} \cdot 2$$

For σs_i, we construct a similar argument. For a fixed decomposition of τ, we have

$$\binom{l-1}{2}$$

ways to place $s_i s_{i+1}$. Therefore,

$$|R(\sigma s_i)| = |R(\tau)|\binom{l-1}{2}$$

and thus

$$\begin{aligned}
\frac{|R(\sigma s_i)|}{|R(\sigma)|} &= \frac{|R(\tau)|\binom{l-1}{2}}{|R(\tau)|\binom{l}{3}\cdot 2} \\
&= \frac{(l-1)!}{2!(l-3)!}\frac{3!(l-3)!}{2\cdot l!} \\
&= \frac{3}{2\cdot l} \\
&\geq \frac{1}{l}
\end{aligned}$$

□

For an arbitrary permutation with braid moves, we can use a similar argument to what we used for fully commutative permutations in Proposition 3.12, but the bound is worse.

Proposition 3.18. *Let $\sigma \in \mathfrak{S}_n$ be written as $\sigma = \tau s_i s_{i+1} s_i$ where $\ell(\tau) = \ell(\sigma) - 3$. Let $\ell(\sigma) = l$. Then,*

$$\frac{|R(\sigma s_i)|}{|R(\sigma)|} > \frac{1}{2(l-2)}$$

Proof. Fix a reduced decomposition for σ that ends in s_i on the right. We know that s_i can be used in a braid move, because $\sigma = \tau s_i s_{i+1} s_i = \tau s_{i+1} s_i s_{i+1}$.

Recall cases 2 and 3 from Example 3.2. For any decomposition where i can commute to the left and be in an additional braid move, we could have $i+1$ commute back out to the right, producing reduced words for σ that are not simply letters of reduced words $w \in R(\omega)$ with i in position x. In fact, so long as i did not have to commute past $i+2$, this has the potential to double the number of reduced words of σ we could get from a single reduced word of ω.

The maximum number we could get from one decomposition would be if we commute i all the way to the left, minus two spots for $i\ i+1$, perform a braid move and commute out again. This gives $2(l-2)$ new decompositions.

Therefore,

$$|R(\sigma)| < 2(l-2)|R(\sigma s_i)|$$

which in turn gives us

$$\frac{|R(\sigma s_i)|}{|R(\sigma)|} > \frac{1}{2(l-2)}$$

$\square$

Unlike the others, this bound is not tight. If $s_i s_{i+1}$ cannot move out of the left most spots, it is because s_{i+1} cannot commute to the right. Thus after the braid move, our s_{i+1} cannot commute back out, and we only have $(l-2)+1$ new words.

At the moment, this is the best bound we can prove for a non-vexillary permutation with braid moves. We did attempt to use Stanley's theorems with tableau to go further, and we will discuss the problems with that approach below.

Example 3.19. For $w_0 \in \mathfrak{S}_4$, we have the following staircase tableau:

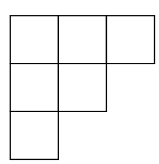

Figure 3.1: The Young Diagram for $\lambda = (3, 2, 1)$

There are three border boxes that can be removed, and each of those three tableau are used to calculate the size of the set of reduced words of one of the permutations covered by w_0 in the weak order lattice of $\mathfrak{S}_4$:

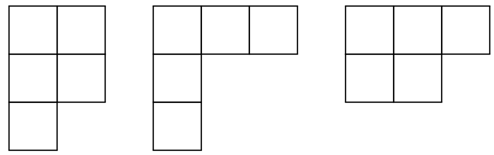

Figure 3.2: The Young Diagrams for $\lambda_1 = (2,2,1)$, $\lambda_2 = (3,1,1)$, and $\lambda_3 = (3,2)$

For vexillary permutations σ, that cover only vexillary permutations, we can use the fact that $|R(\sigma)| = f^{\mu}$ to find each $|R(\sigma s_i)|$. In particular, we use the one-to-one relationship between the elements in the sum $f^{\mu} = \sum_{\mu-} f^{\mu-}$ and the elements in the sum $|R(\sigma)| = \sum_{i \in D(\sigma)} |R(\sigma s_i)|$ to calculate each $|R(\sigma s_i)| = f^{\mu-}$ for some $\mu- \vdash \ell(\sigma) - 1$. However, it is not always this easy.

If a permutation is not vexillary, we do not have that particular information about the rows and columns. But we will have at least as many distinct shapes as we have descents.

Example 3.20. For the completely commutative non-vexillary permutation $\sigma = s_1 s_3 s_5$, we have the following calculation from Theorem 1.30:

$$|R(\sigma)| = 2 \cdot f^{(2,1)} + f^{(1,1,1)} + f^{(3)} = 2 \cdot (f^{(1,1)} + f^{(2)}) + f^{(1,1)} + f^{(2)}$$

There are three elements in the first sum, and three descents in $D(\sigma)$. Additionally,

$$|R(\tau)| = f^{(1,1)} + f^{(2)}$$

but here we lose track of which elements from the sum for $|R(\sigma)|$ will correspond with the calculations for $|R(\tau)|$. For example, we are not guaranteed that the calculation for $|R(\sigma s_1)|$ only uses elements from $f^{(2,1)}$, or only $f^{(1,1,1)} + f^{(3)}$. It could use a mix of those numbers to arrive at the calculation we have.

Remark 3.21. For fully commutative permutations τ and σ, $\tau \lessdot \sigma$, we know from Theorem 1.30 that

$$|R(\tau)| = \sum_{\lambda \vdash l-1} a_\lambda f^\lambda \qquad \text{and} \qquad |R(\sigma)| = \sum_{\mu \vdash l} a_\mu f^\mu$$

However, we have not been able to make any general statements about a relationship between a_λ and a_μ.

Example 3.22. Consider $\sigma = s_1 s_3 s_4 s_3$, and $\tau_1 = s_1 s_3 s_4$, $\tau_2 = s_1 s_3$. Note that both τ_1 and τ_2 are fully commutative. We find that

$$|R(\sigma)| = f^{(2,1,1)} + f^{(3,1)} + f^{(2,2)}$$

$$|R(\tau_1)| = f^{(1,1,1)} + f^{(2,1)}$$

$$|R(\tau_2)| = f^{(1,1)} + f^{(2)}$$

Examples 3.20 and 3.22 have led us to the following conjecture:

Conjecture 3.23. The fully commutative permutations ω where $|i - j| > 1$ for all $i, j \in D(\omega)$ will produce non-zero a_λ's for every $\lambda \vdash l(\omega)$. As soon as we introduce an extra s_i that does not commute with every s_j present in ω, we start to have some $a_\lambda = 0$.

Remark 3.24. If $\tau \lessdot \sigma$ in the weak order, then for every λ such that $a_\lambda \neq 0$ for $|R(\tau)|$, there is at least one $a_\mu \neq 0$ for $|R(\sigma)|$ such that $\lambda \lessdot \mu$ in the Young lattice. Unfortunately, this does not help us with the ratios of the sums, as there is still too much that is unknown about the relationship between these pairs $a_\lambda f^\lambda$ and $a_\mu f^\mu$

As with Chapter 2, we started this work because we were unable to find any existing papers to answer our questions about the relationship between $|R(\sigma)|$ and $|R(\sigma s_i)|$. At the beginning, we were interested in these new results as a better way to understand Theorem 1.28, and how we could use it to answer questions about braid classes. However, after the work done in this chapter, we had a much better understanding not only of the permutations $\tau \lessdot \sigma$, but also of the complications that come with braid relations.

While these complications did prevent us from proving our desired lower bound for all permutations, it lead us in new directions. From here, we were able to come up with additional methods of breaking $G(\sigma)$ into subgraphs which we will discuss in detail in the next chapter.

At some point in the future we intend to return to working on the lower bound for $|R(\sigma s_i)|/|R(\sigma)|$, but we will not continue to study that problem in this document.

Chapter 4

BRAID CLASSES AND SETS OF REDUCED WORDS

The question that motivated all of this research was as follows: if $\sigma \in \mathfrak{S}_n$ is allowed to become arbitrarily long, will there be any identifiable relationship between the number of reduced words for σ and the number of braid classes in $R(\sigma)$?

After extensive work with examples for $n = 4, 5, 6, 7, 8$, using Sage to help find the sizes of the sets of reduced words, we arrived at the following conjectures.

Conjecture 4.1. For all $\sigma \in \mathfrak{S}_n$,

$$\frac{1}{2}|R(\sigma)| \leq |B(\sigma)| \leq |R(\sigma)|.$$

Conjecture 4.2. For all $\sigma \in \mathfrak{S}_n$,

$$0 \leq |C(\sigma)| \leq \frac{1}{2}|R(\sigma)| + 1.$$

These conjectures have proven to be far more involved than they appear. We have determined that Conjecture 4.1 can be used as the key to moving forward. For one thing, the recursion for the braid classes is significantly easier to work with than the recursion for the commutation classes. Another point in favor of directing our efforts towards the braid classes is that we are expecting there to be far fewer braid edges in an arbitrary $G(w)$. This is because the conditions for a commutation move are easier to meet than the conditions for a braid move. Consider i and j in two consecutive spots in a word when $|i - j| > 1$, versus $i(i + 1)i$ in three consecutive spots.

We also determined that if we could prove Conjecture 4.1, we could use Theorem 1.26 in order to prove Conjecture 4.2 as a corollary, since it relates $|R(\sigma)|$, $|B(\sigma)|$, and $|C(\sigma)|$ with each other. We will discuss this idea more in Section 4.

4.1 General Work

First, we tried a simple application of the braid class recursion from Theorem 1.29 and induction on the length of $\sigma \in \mathfrak{S}_n$.

Base Case: Suppose that $l(\sigma) = 2$. Then the permutation either has one reduced word, $i(i \pm 1)$, or two, ij where $|i - j| > 1$. In either case, there is no way to perform a braid move in a permutation of length two, we have that $|B(\sigma)| = |R(\sigma)|$, which does not contradict our conjecture.

Induction Step: Suppose that $l(\sigma) = k + 1$ and for all permutations $\omega \in \mathfrak{S}_n$ such that $l(\omega) \leq k$, that $|B(\omega)| \geq \frac{1}{2}|R(\omega)|$. Then using the recursion from Theorem 1.29, and Theorem 1.28 we have

$$\begin{aligned}
|B(\sigma)| &= \left(\sum_{i \in D(\sigma)} |B(\sigma s_i)|\right) - \sum_{i,i+1 \in D(\sigma)} |B(\sigma s_i s_{i+1} s_i)| && (4.1)\\
&\geq \left(\sum_{i \in D(\sigma)} \frac{1}{2}|R(\sigma s_i)|\right) - \sum_{i,i+1 \in D(\sigma)} |B(\sigma s_i s_{i+1} s_i)| && (4.2)\\
&= \left(\frac{1}{2}|R(\sigma)|\right) - \sum_{i,i+1 \in D(\sigma)} |B(\sigma s_i s_{i+1} s_i)| && (4.3)\\
&\geq \left(\frac{1}{2}|R(\sigma)|\right) - \sum_{i,i+1 \in D(\sigma)} |R(\sigma s_i s_{i+1} s_i)| && (4.4)
\end{aligned}$$

which will not be greater than $\frac{1}{2}|R(\sigma)|$ so long as at least one of the $|R(\sigma s_i s_{i+1} s_i)|$'s is non-zero.

Not only were we unable to prove our conjecture, but we also note that the lower bound we found in inequality 4.4 is significantly smaller than we believe to be true. Consider for example, $\sigma = [4321]$. We find that $|B(\sigma)| = 8 = \frac{1}{2}|R(\sigma)|$. Now consider the bound calculated below:

$$\left(\frac{1}{2}|R(\sigma)|\right) - \sum_{i,i+1 \in D(\sigma)} |R(\sigma s_i s_{i+1} s_i)| = 8 - 2 = 6$$

We can also attempt this by induction on the descent set rather than the length, with similar results. Thus we determined that we cannot use induction with well known existing results to prove our conjectures.

These attempts led us to look into what is known about $|R(\sigma)|$ and $|R(\tau)|$ for $\tau \leq \sigma$ in the weak order lattice. We discovered that the relationship between $|R(\sigma)|$ and $|R(\tau)|$ had not been studied, so we began to rephrase the problem. We translated our conjectures from braid classes into braid edges. At this point, we still needed to understand how the graphs $G(\tau)$ and $G(\sigma)$ were related, and if understanding the relationship between $|R(\sigma)|$ and $|R(\tau)|$ would help us prove the Conjectures. These questions led to the new research in Chapters 2 and 3.

In the next section, we will consider a particular family of permutations, and find a smaller lower bound for $|B(\sigma)|$ than the conjectured $\frac{1}{2}|R(\sigma)|$.

4.2 A Special Case

We will restrict our attention to the permutations $w_0^{(k,i)} \in \mathfrak{S}_n$.

Theorem 4.3. *For $w_0 \in \mathfrak{S}_n$, $|B(w_0)| \geq \frac{1}{2}|R(w_0)| - 1$.*

Proof. We will consider the graph $G_b'(w_0)$. Recall that the connected components of this graph are the braid classes of $G(w_0)$. We would like to show that there are at least $\frac{1}{2}|R(w_0)| - 1$ connected components.

Suppose to the contrary that there are less than $\frac{1}{2}|R(w_0)| - 1$ connected components in $G_b'(w_0)$. Then if $|R(w_0)|$ is even, we have at most $\frac{1}{2}|R(w_0)| - 2$ components, and if $|R(w_0)|$ is odd, then there are at most $\frac{1}{2}|R(w_0)| - \frac{3}{2}$ components.

We can partition our vertex set over the connected components. Suppose there are m connected components, and let V_i be the vertex set for one of these components, for $1 \leq i \leq m < \frac{1}{2}|R(w_0)| - 1$. Then,

$$V(G'_b(w_0)) = \bigcup_{1 \leq i \leq m} V_i$$

We note that a graph with V vertices and k components must have at least $V - k$ edges, since a component with V_i vertices must have a spanning tree with $V_i - 1$ edges. We assume there are m components, where $m \leq \frac{1}{2}|R(w_0)| - i$, for $i \in \left\{\frac{3}{2}, 2\right\}$. Thus, we can subtract off a maximum of $\frac{1}{2}|R(w_0)| - i$ edges.

We also see that each of the connected components in $G'_b(w_0)$ will at minimum be trees, which means we have at least

$$\sum_{v \in V(G(w_0))} d_{G'_b(w_0)}(v) \geq 2|R(w_0)| - 2$$

Thus, for $i \in \left\{\frac{3}{2}, 2\right\}$, we have

$$\begin{aligned}
\sum_{v \in V(G(w_0))} d_{G'_b(w_0)}(v) &= 2E(G'_b(\sigma)) \\
&\geq 2|R(w_0)| - 2 - 2\left[\frac{1}{2}|R(w_0)| - i\right] \\
&> |R(w_0)|
\end{aligned}$$

for either choice of i.

This contradicts Reiner's result in Theorem 1.31 that states that the sum of the braid degrees exactly equals the size of the set of reduced words. □

How does this allow us to get closer to our conjecture? Consider the following definition.

Definition 4.4. Let $\sigma \in \mathfrak{S}_n$, and let $S \subset R(\sigma)$. We will consider $S \subset V(G(\sigma))$, and define

$$A(S) = |S| - \sum_{v \in S \subset G(\sigma)} d_b(v) \tag{4.5}$$

We will define a special case $A(\sigma)$ as

$$A(\sigma) = |R(\sigma)| - \left(\sum_{v \in G(\sigma)} d_b(v) \right) \tag{4.6}$$

If we are able to prove that for an arbitrary permutation $\sigma \in \mathfrak{S}_n$ that we have $A(\sigma) > 0$, then we would know that there are at most $\frac{1}{2}|R(\sigma)|$ braid edges in our graph $G(\sigma)$. We could use Theorem 4.3 to write a similar proof, and find that $|B(\sigma)| \geq \frac{1}{2}|R(\sigma)| - 1$. While these bounds on the congruence classes are not exactly what we conjectured, they are close. If we could prove this lower bound, we would be able to use Theorem 1.26 to prove a similar weaker result for the number of commutation classes.

4.3 Permutation Shuffling

Before we can discuss equation (4.6) in further detail, we will describe generalizations of our work in Chapter 2 for new families of subgraphs. We will also generalize our work from Chapter 3 to describe how we can generate $R(\sigma)$ by breaking the permutation into pieces. First, an example.

Example 4.5. Consider the permutation $\sigma = [2431] = s_1 s_2 s_3 s_2$ with descent set $D(\sigma) = \{2, 3\}$.

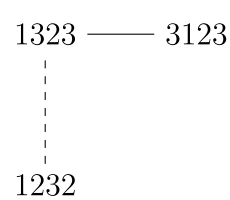

Figure 4.1: The Graph $G([2431])$

We note that the braid edge itself is a copy of $G([1432])$, which is $w_0^{(3,2)}$, and has the same descent set as σ. Starting with the words 1323 and 1232, we attempt to move the letter 1 to the right. We note that after moving the letter 1, we do not have a second full copy of $G([1432])$. We would need both the vertices 2132 and 3123, as well as a braid edge between them. There are two problems with this: there is no braid move between those reduced words, and 2132 is not a reduced word of σ.

Next we consider $\sigma = [32154] = s_4 s_1 s_2 s_1$ with descent set $D(\sigma) = \{1, 2, 4\}$. We have

$$\begin{tikzcd}
4212 \arrow[r, dash] \arrow[d, dash, dashed] & 2412 \arrow[r, dash] & 2142 \arrow[r, dash] & 2124 \arrow[d, dash, dashed] \\
4121 \arrow[r, dash] & 1421 \arrow[r, dash] & 1241 \arrow[r, dash] & 1214
\end{tikzcd}$$

Figure 4.2: The Graph $G([32154])$

In this picture, we have four copies of the vertex set of $G([321])$, with 4 sitting in different spots through the reduced words for 121. We focus on this permutation because the longest string of consecutive elements in $D(\sigma)$ is the descent set for $w_0^{(3,1)}$. As the letter 4 moves through the word from left to right, we have effectively blocked braid edges from appearing in the middle portion of the graph.

Using ideas from the graphs in this example, we will now discuss how to consider subgraphs in $G(\sigma)$ that may not be of the form $G(\sigma s_i)$, $i \in D(\sigma)$.

Lemma 4.6. *Let $\sigma \in \mathfrak{S}_n$. Suppose that $D(\sigma)$ contains a string of m consecutive elements, with smallest element in the string i: $\{i, i+1, \ldots, i+m-1\}$. Then there is a reduced word in $R(\sigma)$ of the form uv where $v \in R(w_0^{(k,i)})$, for $k = m+1$.*

Proof. Let $\sigma = [a_1 a_2 \ldots a_n] \in \mathfrak{S}_n$. We recall that $i \in D(\sigma)$ if and only if $a_i > a_{i+1}$. We also recall that for any j, $\sigma s_j = [a_1 a_2 \ldots a_{j+1} a_j \ldots a_n]$.

Suppose that $\{i, i+1, \ldots i+m-1\} \subset D(\sigma)$ is a string of consecutive elements. This means that

$$a_i > a_{i+1} > \ldots > a_{i+m-1}$$

Let $\tau_\sigma^{(j,k)}$ be defined as follows:

$$\tau_\sigma^{(j,k)} := \sigma s_j s_{j+1} \ldots s_{j+k-1} = [a_1 \ldots a_{j-1} a_{j+1} \ldots a_{j+k} a_j \ldots a_n]$$

where $\tau_\sigma^{(0,0)} = \sigma$, and $\tau_\sigma^{(j,1)} = \sigma s_j$.

We note that for the set $\{i, i+1, \ldots i+m-1\} \subset D(\sigma)$, if $j = i$ and $k = m$, we have

$$\tau_\sigma^{(i,m)} = \sigma s_i s_{i+1} \ldots s_{i+m-1} = [a_1 \ldots a_{i-1} a_{i+1} a_{i+2} \ldots a_{i+m} a_i \ldots a_n]$$

We see that $\{i, i+1, \ldots i+m-2\} \subset D(\tau_\sigma^{(i,m)})$, while $i+m-1 \notin D(\tau_\sigma^{(i,m)})$.

Furthermore, $\ell(\tau_\sigma^{(i,m)}) = \ell(\sigma) - m$. Since $i \in D(\sigma)$, $\ell(\tau_\sigma^{(i,1)}) = \ell(\sigma) - 1$. We will still have $i+1 \in D(\tau_\sigma^{(i,1)})$, and we can see that $\ell(\tau_\sigma^{(i,2)}) = \ell(\sigma) - 2$. Inductively, this process continues until we have $\ell(\tau_\sigma^{(i,m)}) = \ell(\sigma) - m$.

Let $\sigma^{(1)} = \tau_\sigma^{(i,m)}$. We will now consider

$$\tau_{\sigma^{(1)}}^{(i,m-1)} = \sigma^{(1)} s_i s_{i+1} \ldots s_{i+m-2} = [a_1 \ldots a_{i-1} a_{i+2} \ldots a_{i+m} a_{i+1} a_i \ldots a_n]$$

We note that

$$i+m-1, i+m-2 \notin D(\tau_{\sigma^{(1)}}^{(i,m-1)}) \text{ and } \{i, i+1, \ldots i+m-3\} \subset D(\tau_{\sigma^{(1)}}^{(i,m-1)})$$

We also have $\ell(\tau_{\sigma^{(1)}}^{(i,m-1)}) = \ell(\sigma^{(1)}) - (m-1) = \ell(\sigma) - m - (m-1)$ using the same argument on descents as before. Let $\sigma^{(2)} = \tau_{\sigma^{(1)}}^{(i,m-1)}$.

Inductively, for $1 \leq x \leq m-3$, we have that the permutation $\sigma^{(x+1)} = \tau_{\sigma^{(x)}}^{(i,m-x)}$ is such that

$$\sigma^{(x+1)} = \sigma^{(x)} s_i s_{i+1} \ldots s_{i+m-(x+1)} = [a_1 \ldots a_{i-1} a_{i+x} \ldots a_{i+m} a_{i+x-1} \ldots a_{i+1} a_i \ldots a_n]$$

where $\{i, \ldots, i+m-(x+2)\} \subset D(\sigma^{(x+1)})$, $\{i+m-(x+1), \ldots, i+m-1\} \cap D(\sigma) = \emptyset$, and $\ell(\sigma^{(x+1)}) = \ell(\sigma) - \sum_{b=0}^{m-x} m - b$.

Now consider $\sigma^{(m-2)} = \tau_{\sigma^{(m-3)}}^{(i,m-(m-3))}$:

$$\sigma^{(m-2)} = [a_1 \ldots a_{i-1} a_{i+m-2} a_{i+m-1} a_{i+m} a_{i+m-3} \ldots a_{i+1} a_i \ldots a_n]$$

From the above one line notation, we see that $\sigma^{(m)} := \sigma^{(m-2)} s_i s_{i+1} s_i$ will be such that $\{i, i+1, \ldots i+m-2\} \cap D(\sigma^{(m)}) = \emptyset$. Additionally,

$$\ell(\sigma^{(m)}) = \left(\ell(\sigma) - \sum_{b=0}^{m-3} m - b\right) - 3 = \ell(\sigma) - \binom{m+1}{2}$$

Let $u \in R(\sigma^{(m)})$. Tracing our products from $\sigma^{(m)}$ back to $\sigma^{(1)}$, let v be the word

$$i(i+1)i \ (i+2)(i+1)i \ldots (i+k-1)(i+k-2) \ldots (i+1)i$$

We see that $\ell(v) = \binom{m+1}{2}$, and that we can verify that $v \in R(w_0^{(m+1,i)})$.

By construction, we have $uv \in R(\sigma)$ as desired. $\square$

Definition 4.7. Let $\sigma \in \mathfrak{S}_n$ be such that $[i, i+k-2] \subset D(\sigma)$ is a set of consecutive elements. From Lemma 4.6, we note that there are permutations $\alpha, \beta \in \mathfrak{S}_n$ such that $\beta = w_0^{(k,i)}$, and $u = u_1 u_2 \ldots u_{l(\alpha)} \in R(\alpha)$ and $v = v_1 v_2 \ldots v_{l(\beta)} \in R(w_0^{(k,i)})$ can be concatenated to produce $uv \in R(\sigma)$. In order to distinguish the letters, we will color the letters of u blue, and the letters of v red.

A *shuffle* of the letters of uv will be defined in the following way:

1. A commutation or braid move using only the blue letters of u, or using only the red letters of v.

2. A commutation move using one blue letter u_i of u and one red letter v_j of v.

3. A braid move that uses two blue letters $u_i u_{i+1}$ of u and one red letter v_j of v, or vice versa.

None of the above shuffle types will change the color of the letters. After any finite sequence of shuffles of any type, we will no longer have the word uv. We will still refer to each new shuffle as a shuffle of the letters of uv.

Example 4.8. Consider $\sigma = [165324] = s_4s_5s_2s_3s_4s_2s_3s_2$. We can use Lemma 4.6 to write $45 \in R(\alpha)$, $234232 \in R(\beta)$, and $45234232 \in R(\sigma)$.

1. Using a shuffles of type 1, we have $45434234 \in R(\sigma)$.

2. Using a shuffle of type 2, we have $42534232 \in R(\sigma)$.

3. Using a shuffle of type 3, we have $54534234 \in R(\sigma)$

We see that for the first type of shuffle, we start with $u \in R(\alpha)$ and shuffle letters to get $a \in R(\alpha)$. For a fixed pair $a \in R(\alpha)$ and $b \in R(\beta)$, the second type of shuffle will commute where the letters of a and b sit. The third shuffle type will be a mix of blue letters forming a word a', and red letters form a word b'. However, $a' \notin R(\alpha)$ and $b' \in R(\beta)$, which will make the words formed after a shuffle of type 3 more difficult to discuss.

We know that we can start with any word in $R(\sigma)$, and generate the full set by performing all possible commutation and braid moves among the letters. So we can use the letters of uv to fully generate $R(\sigma)$ in the standard way.

The question is whether we can keep track of the letters of $u \in R(\alpha)$ and $v \in R(\beta)$ as we perform the shuffle process?

Lemma 4.9. *Let $\alpha, \beta, \sigma \in \mathfrak{S}_n$, with $u \in R(\alpha)$, $v \in R(\beta)$ and $uv \in R(\sigma)$ as described in Lemma 4.6 and Definition 4.7. We can fully construct $R(\sigma)$ by looking at all possible ways that the letters of uv can shuffle through each other.*

Proof. Consider $uv \in R(\sigma)$, where $u = u_1 \dots u_{l(\alpha)} \in R(\alpha)$ and $v = v_1 \dots v_{l(\beta)} \in R(\beta)$.

We will color all the letters descended from u blue, and all the letters descended from v red.

Let $w \in R(\sigma)$ be an arbitrary word that is distinct from uv. We know that there is a finite sequence of commutation and braid moves that will transform uv into w. Since $G(\sigma)$ is connected, let us consider this as a path of m vertices in $G(\sigma)$.

$$(uv) - (a_2) - \cdots - (a_{m-1}) - (w)$$

To travel from uv to a_2, we either perform a commutation move or a braid move of the letters of uv. This will be a shuffle of the letters of uv.

Inductively, each a_j in this path will have all the letters colored blue and red, since no shuffle type will change the colors of the letters. We will always have the same number of blue letters and red letters, since none of the shuffles will recolor the letters. Thus, the letters of w will be a mix of $\ell(\alpha)$ blue letters and $\ell(\beta)$ red letters.

Because w was an arbitrary element of $R(\sigma)$, every element of the set will be formed from a series of shuffles of the letters of uv. $\square$

Our work on splitting σ into pieces α and β, and shuffling the letters of their respective reduced words, is similar to permutation inflations. Permutation inflations are related to grid drawings of permutations, and use patterns in the one line notation in consecutive spots to write reduced words. For more information on inflations, we recommend [1] or [4].

We pursued this new shuffle method rather than the inflations because we could use our α and β construction for any arbitrary permutation, rather than being restricted to a particular family.

Example 4.10. With the knowledge from Lemma 4.9, we will look at a slightly more complex example of $G(\sigma)$, though we will not draw the full graph: $\sigma = [3215476] =$

$s_4 s_6 s_1 s_2 s_1$ with descent set $D(\sigma) = \{1, 2, 4, 6\}$. The longest string of consecutive descents is $\{1, 2\}$.

From Lemma 4.6 $\alpha = [1235476]$ and $\beta = [3214567]$. Note that as we shuffle $a \in R(\alpha)$ and $b \in R(\beta)$, we will only perform shuffle moves of the first and second type. We will consider $46121 \in R(\sigma)$ as our starting point.

For any reduced word of σ, we can select where 121 or 212 will sit, and then the remaining two spots can have either 4 or 6. Thus we will have $|R(\sigma)| = 2\binom{5}{3} \cdot 2 = 40$.

We can begin to construct the graph as follows:

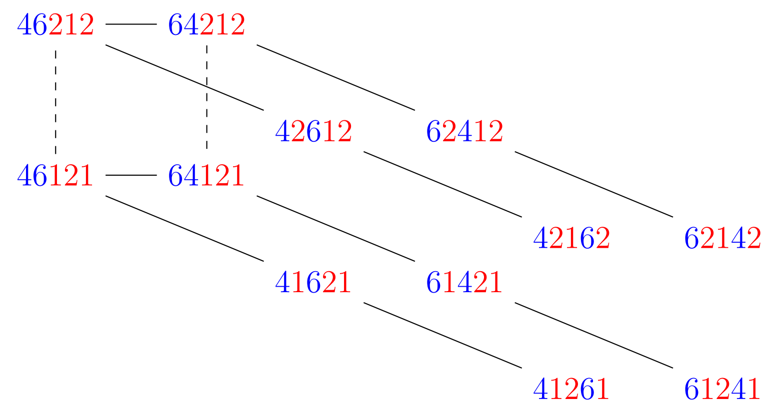

Figure 4.3: Part of the Graph $G([3215476])$

The square of vertices to the far left can be viewed as a copy of of the standard product graph $G([1235476]) \times G([3214567])$. The commutation edges in that square are copies of the single edge from $G([1235476])$, while the braids are copies of the single braid edge in $G([3214567])$. As we shuffle the elements 4 and 6 to the right, notice that we do not have edges inherited from $G([1235476])$ or $G([3214567])$ anymore. We have the vertex set of $G([1235476]) \times G([3214567])$, but no internal edges.

We would continue moving forward in this manner, sometimes with those internal edges present, but most of the time they will not be.

Definition 4.11. Let $\alpha, \beta, \sigma \in \mathfrak{S}_n$, with $a \in R(\alpha)$, $b \in R(\beta)$ and $ab \in R(\sigma)$ as described in Lemma 4.6 and Definition 4.7. Let C_I denote all the words in $R(\sigma)$ where the letters from $a \in R(\alpha)$ sit in the positions contained in I, where $I \subset [\ell(\sigma)]$ and $|I| = \ell(\alpha)$. We will call this a *configuration*, C_I.

After a shuffle of type 3, we would consider sets labeled C_I'. We would still be looking at where the red letters descended from a would sit, but we would want to be able to differentiate this set from C_I. This is similar to how we labeled the sets in Example 3.2.

Definition 4.12. Let $\alpha, \beta, \sigma \in \mathfrak{S}_n$, with $u \in R(\alpha)$, $v \in R(\beta)$ and $uv \in R(\sigma)$ as described in Lemma 4.6 and Definition 4.7. We define H_I to be the induced subgraph of G with the vertex set C_I.

If there is a commutation move or braid move between letters of $a_1, a_2 \in R(\alpha)$, then there is an edge between all vertices of the form a_1b and a_2b, for any $b \in R(\beta)$. Similarly, in a collection or reduced words C_I, any time there is a commutation or braid move between $a_1, a_2 \in R(\alpha)$, and the letters used in these particular moves are sitting in consecutive spots in the words contained in C_I, there will be an edge between those vertices for all $b \in R(\beta)$.

Note that not all configurations will result in a subgraph with $|R(\alpha)||R(\beta)|$ vertices. This is because not all letters of a reduced word of α need to commute with all letters of a word of β. See Example 4.5.

For the moment let us consider permutations $\sigma \in \mathfrak{S}_n$ where $uv \in R(\sigma)$ is such that we only have shuffles of type 1 and 2.

Lemma 4.13. *Let $\alpha, \beta, \sigma \in \mathfrak{S}_n$, with $u \in R(\alpha)$, $v \in R(\beta)$ and $uv \in R(\sigma)$ as described in Lemma 4.6 and Definition 4.7. Further suppose that for all s_i in the support of α*

and all s_j in the support of β, $|i-j| \geq 1$. That is, there will not be any shuffles of the type 3 from Definition 4.7. Then

$$|R(\sigma)| \leq |R(\alpha)||R(\beta)|\binom{\ell(\sigma)}{\ell(\alpha)}$$

Proof. We assume that there will not be any shuffles of type 3 in $R(\sigma)$. So we only need to consider whether letters from $u \in R(\alpha)$ commute with letters in $v \in R(\beta)$, or not.

Let C_I be defined as in as in Definition 4.11, where we have chosen the set I such that $|I| = \ell(\alpha)$ and $I \subset \ell(\sigma)$. There are $\binom{\ell(\sigma)}{\ell(\alpha)}$ choices for the set I. Furthermore, because there are no shuffles of type 3, we have

$$R(\sigma) = \bigcup_{I \subset [\ell(\sigma)],\ |I| = \ell(\alpha)} C_I$$

This is a disjoint union, and some sets C_I could be empty.

Let us define new sets A_I as follows:

$$A_I = \{w \mid a \in R(\alpha) \text{ in positions in } I, b \in R(\beta) \text{ in the remaining positions}\}$$

Each of the sets A_I will have size $|A_I| = |R(\alpha)||R(\beta)|$.

These sets may contain words which are not in $R(\sigma)$. Let $a = a_1 \ldots a_{\ell(\alpha)} \in R(\alpha)$, and $b = b_1 \ldots b_{\ell(\beta)} \in R(\beta)$. Suppose that there are letters in these words a_i and b_j such that $|a_i - b_j| = 1$. Then any index set I that places the letter a_i to the right of letter b_j will mean that A_I contains words that are not in $R(\sigma)$.

For any index set I, $C_I \subset A_I$ so that $|C_I| \leq |A_I|$. There is no way for C_I to be larger than A_I, since A_I already contains all possible words formed from $a \in R(\alpha)$ in positions in I.

If $C_I = A_I$ for all index sets I, then

$$|R(\sigma)| = |R(\alpha)||R(\beta)|\binom{\ell(\sigma)}{\ell(\alpha)}$$

If there are letters in a word $a \in R(\alpha)$ that do not commute with letters in $b \in R(\beta)$, then there is some index set J such that $C_J \neq A_J$. In which case,

$$|R(\sigma)| < |R(\alpha)||R(\beta)|\binom{\ell(\sigma)}{\ell(\alpha)}$$

Therefore, we have the desired inequality. $\square$

The shuffling process and configuration subgraphs that are described in the lemmas and definition above gives us a way to divide our graphs up into copies of $G(\alpha)$ and $G(\beta)$.

Lemma 4.14. *Let $\alpha, \beta, \sigma \in \mathfrak{S}_n$, with $u \in R(\alpha)$, $v \in R(\beta)$ and $uv \in R(\sigma)$ as described in Lemma 4.6 and Definition 4.7.*

If we arrive at C_I using only shuffles of type 1 and 2, then H_I will be a subgraph of $G(\alpha) \times G(\beta)$. H_I does not need to be an induced subgraph. If a shuffle of type 1 from Definition 4.7 exists in this configuration, then each of those edges from $G(\alpha)$ is replaced with $|R(\beta)|$ copies of the same edge.

If we arrive at C_I using all three types of shuffles, H_I will not necessarily be a subgraph of $G(\alpha) \times G(\beta)$, because a type 3 shuffle changes α and β.

The reason we want to look at this family of subgraphs is because we know a lot about $G(w_0^{(k,i)})$, and we know exactly what $A(\sigma)$ is when $\sigma = w_0^{(k,i)}$. However, for an arbitrary σ, we have less information about $G(\sigma s_j)$ for $j \in D(\sigma)$.

Lemma 4.15. *Let $\alpha, \beta \in \mathfrak{S}_n$. Then*

$$\sum_{v \in G(\alpha) \times G(\beta)} d_b(v) = |R(\alpha)| \cdot \sum_{u \in G(\beta)} d_b(u) + |R(\beta)| \cdot \sum_{t \in G(\alpha)} d_b(t)$$

Proof. We will consider $G(\alpha) \times G(\beta)$ as follows: let $v \in G(\alpha) \times G(\beta)$ be labeled as (a, b) for $a \in R(\alpha)$ and $b \in R(\beta)$.

There is a braid edge between (a, b) and (a', b') if and only if there is a braid move between a and a' and $b = b'$, or if there is a braid move between b and b' and $a = a'$.

Therefore, the braid degree of $v \in G(\alpha) \times G(\beta)$ depends on the braid degrees of $a \in G(\alpha)$ and $b \in G(\beta)$. That is, $v = (a, b)$ is such that $d_b(v) = d_b^{G(\alpha)}(a) + d_b^{G(\beta)}(b)$. In order to calculate $\sum_{v \in G(\alpha)\times G(\beta)} d_b(v)$, we have to be able to vary over $a \in R(\alpha)$ and $b \in R(\beta)$.

Therefore, our calculation will be

$$\begin{aligned}
\sum_{v\in G(\alpha)\times G(\beta)} d_b(v) &= \sum_{t\in G(\alpha)} \sum_{u\in G(\beta)} d_b(t) + d_b(u) \\
&= \sum_{t\in G(\alpha)} \left(\sum_{u\in G(\beta)} d_b(t) + \sum_{u\in G(\beta)} d_b(u) \right) \\
&= \sum_{t\in G(\alpha)} \left(|R(\beta)| d_b(t) + \sum_{u\in G(\beta)} d_b(u) \right) \\
&= \sum_{t\in G(\alpha)} |R(\beta)| d_b(t) + \sum_{t\in G(\alpha)} \sum_{u\in G(\beta)} d_b(u) \\
&= |R(\beta)| \cdot \sum_{t\in G(\alpha)} d_b(t) + |R(\alpha)| \cdot \sum_{u\in G(\beta)} d_b(u)
\end{aligned}$$

as desired □

Before moving on, we need to discuss a few features of these $w_0^{(k,i)}$ permutations.

Lemma 4.16. *The permutations $w_0^{(k,i)}$ are invariant under conjugation. If we want to know how many braid moves we will have in position j, where $1 \le j \le \ell(w_0^{(k,i)}) - 2$, we need only consider how many braid edges we will have when $j = \ell(w_0^{(k,i)}) - 2$.*

While this was not explicitly proven by Reiner in [9], it is a consequence of the work done in that paper.

Proof. From Chapter 2, we note that there are going to be

$$\sum_{j,j+1\in D(\sigma)} |R(\sigma s_j s_{j+1} s_j)|$$

braid edges that use the last three positions in decompositions of σ. These are the braid edges that connect the induced subgraphs of $G(\sigma)$.

We also note that because of the structure of w_0, any of these moves can be cycled into any position inside w_0 and be performed there. The number $|R(\sigma s_j s_{j+1} s_j)|$ represents all the possible decompositions of w_0 with a particular type of braid move at the end, and will allow us to count the number of decompositions of σ with that braid move shifted in as well. We sum over all possibilities of $i, i+1 \in D(w_0^{(k,i)})$ to account for all braid moves. $\square$

Between the work we have done on subgraphs in this section, and in Chapter 2, we now have enough information to move forward with our question about a lower bound for $|B(\sigma)|$.

4.4 Cases for the Lower Bound for $|B(\sigma)|$

Lemma 4.17. *Suppose that $D(\sigma) = [i, i+k-1] \cup J$, where k is the length of the longest string of consecutive elements in $D(\sigma)$. Then*

1. *If $k = 1$,*
$$A(\sigma) = |R(\sigma)| - \left(\sum_{v \in G(\sigma)} d_b(v) \right) > 0$$
2. *If $J = \{j\}$ and $\sigma = s_j w_0^{(k,i)}$, then $A(\sigma) > 0$*
3. *If $J = \emptyset$ and $\sigma = w_0^{(k,i)}$, then $A(\sigma) = 0$*

Proof. First, we note that point three above is exactly Theorem 1.31. Thus we will focus on proving the other two parts of our lemma.

Let $\sigma \in \mathfrak{S}_n$ where $l(\sigma) = l$ and $D(\sigma) = [i, i+k-1] \cup J$ for some $k, i \in \mathbb{Z}$ such that no string of consecutive elements in J has length greater than k.

Suppose that for all $\tau \in \mathfrak{S}_n$ where $l(\tau) < l$, exactly one of the following is true:

1. $\tau = w_0^{(k,i)}$, and $A(\tau) = 0$,

2. $\tau \neq w_0^{(k,i)}$, and $A(\tau) > 0$.

We want to show that $A(\sigma) > 0$ as well.

Case 1: Let $k = 1$. That is, suppose that $D(\sigma)$ does not contain any consecutive elements. Then we select any $j \in D(\sigma)$, and consider the word $aj \in R(\sigma)$ where $a \in R(\sigma s_j)$.

If $D(\sigma) = \{j\}$, Proposition 3.11 notes that $|R(\sigma)| = |R(\sigma s_j)|$. Then $A(\sigma) = A(\sigma s_j) > 0$ by our induction hypothesis.

If $|D(\sigma)| > 1$, Proposition 2.15 notes that at most one $j \in D(\sigma)$ will produce $\sigma s_j = w_0^{(k,i)}$. Using Corollary 2.12, we have

$$\begin{aligned} \sum_{v \in G(\sigma)} d_b(v) &= \left(\sum_{x \in D(\sigma)} \sum_{u \in G(\sigma s_x)} d_b(u) \right) + 2 \cdot \sum_{x, x+1 \in D(\sigma)} |R(\sigma s_x s_{x+1} s_x)| \\ &= \sum_{x \in D(\sigma)} \sum_{u \in G(\sigma s_x)} d_b(u) \end{aligned}$$

because there are no consecutive elements in the set $D(\sigma)$.

We assume that $A(\sigma s_x) = 0$ for at most one descent x, and $A(\sigma s_y) > 0$ for all other descents y. There are at least two elements in $D(\sigma)$, and therefore,

$$\begin{aligned} \sum_{v \in G(\sigma)} d_b(v) &= \sum_{x \in D(\sigma)} \sum_{u \in G(\sigma s_x)} d_b(u) \\ &< \sum_{x \in D(\sigma)} (|R(\sigma s_x)|) \\ &= |R(\sigma)| \end{aligned}$$

and therefore, $A(\sigma) > 0$.

Case 2: Let $\ell(\alpha) = 1$. Suppose that $\alpha = s_j$, $\beta = w_0^{(k,i)}$, so that $jv \in R(\sigma)$ for $v \in R(w_0^{(k,i)})$. Because $D(\sigma) = [i, i+k-1] \cup \{j\}$, we will know that $j \notin \{i-1, i+k\}$. Then

j commutes with every letter in $u \in R(\beta)$, so we can split up $G(\sigma)$ into subgraphs H_t as defined in Definition 4.12, where $1 \leq t \leq \ell(\beta) + 1$.

Since j commutes with everything in $u \in R(\beta)$, we know that j cannot be used in a braid move, and that $|V(H_t)| = |R(\beta)|$. Thus each of the H_t's is joined to another H_{t-1} and H_{t+1} by commutation edges.

In any three consecutive positions in β, there are $\sum_{m,m+1 \in D(\beta)} |R(\beta s_m s_{m-1} s_m)|$ braid moves, corresponding to braid edges. Since β is a $w_0^{(k,i)}$, we also know that

$$|R(\beta)| = 2 \cdot |E_b(\beta)| = \sum_{v \in G(\beta)} d_b(v)$$

which means that

$$|R(\beta)| = 2 \cdot \left((l(\beta) - 2) \sum_{m,m+1 \in D(\beta)} |R(\beta s_m s_{m-1} s_m)| \right) = \sum_{v \in G(\beta)} d_b(v)$$

For $t = 1, l(\beta)+1$, the subgraph H_t has exactly $|R(\beta)|$ vertices, and $\sum_{v \in H_t} d_b(v) = |V(H_t)|$.

For $t = 2, l(\beta)$, the subgraph H_t has exactly $|R(\beta)|$ vertices, but

$$\sum_{v \in H_t} d_b(v) = 2 \cdot \left((l(\beta) - 3) \sum_{m,m+1 \in D(\beta)} |R(\beta s_m s_{m-1} s_m)| \right) < |R(\beta)|$$

For all other $3 \leq t \leq l(\beta) - 1$, the subgraph H_t has exactly $|R(\beta)|$ vertices, but

$$\sum_{v \in H_t} d_b(v) = 2 \cdot \left((l(\beta) - 4) \sum_{m,m+1 \in D(\beta)} |R(\beta s_m s_{m-1} s_m)| \right) < |R(\beta)|$$

Thus

$$\sum_{v \in G(\sigma)} d_b(v) < |R(\sigma)|$$

so that $A(\sigma) > 0$ as desired. $\square$

Note that the cases we have proven in Lemma 4.17 relied heavily on the work we did in Chapter 2, and the work we did constructing the shuffles and the subgraphs they create in Section 4.3. This Lemma also covers all of the cases we have been able to prove.

Remark 4.18. Consider $\alpha = s_j$ where $j \in \{i-1, i+k\}$, and $\sigma = s_j w_0^{(k,i)}$. Then s_j will not commute with every letter in a $w \in R(w_0^{(k,i)})$, so the argument above will not work.

Suppose that $j = i-1$. Let $\mathfrak{S}_n^{(i)} = \langle s_i, s_{i+1}, \ldots, s_{i+n-1} \rangle$. Let H_t be the subgraph with vertices formed from the reduced words with j sitting in position t. Each H_t will be joined to H_{t-1} and H_{t+1}, if they exist, by commutation edges. We also know that

$$H_t = \bigcup_{\tau \in \mathfrak{S}_{k-1}^{(i+1)},\ l(\tau)=t-1} G(\tau) \times G(\tau^{-1} w_0^{(k,i)})$$

So we know that $1 \leq t \leq \binom{k-1}{2} + 1$.

Working through examples by hand and using Maple, we found that

$$0 < A(s_1 w_0^{(3,2)}) < A(s_1 w_0^{(4,2)}) < A(s_1 w_0^{(5,2)}) < A(s_1 w_0^{(6,2)})$$

However, for $k > 4$, when we look at $H_{\binom{k-1}{2}}$ and use Definition 4.4, $A(H_{\binom{k-1}{2}}) < 0$. In each of our examples, $A(H_2)$ was considerably larger than $A(H_{\binom{k-1}{2}})$. Similarly, $A(H_{\binom{k-1}{2}-1}) < 0$, but $A(H_3) - A(H_{\binom{k-1}{2}-1}) > 0$, and so on. We considered pairing off the subgraphs in this manner, but we were unable to use this to prove that $A(s_1 w_0^{(k,2)}) > 0$ for all k. We have also been unable to find the first m such that $A(H_m) < 0$ for each $s_1 w_0^{(k,2)}$.

We have also been unable to prove that we will always have $A(\sigma) > 0$. It must be true, because 1 cannot contribute to any braid moves. But we have not been able to write a rigorous argument.

Remark 4.19. In order to use induction to prove in general that $A(\sigma) > 0$, we would need to be able to understand the family of σ's in Remark 4.18. We would also need to be able to understand the structure of $G(\alpha w_0^{(k,i)})$ where $l(\alpha) > 1$ that have descent sets $D(\alpha) \cap \{i-1, i+k\} \neq \emptyset$.

We would also need to understand what happens for $a \in R(\alpha)$ that share letters with $b \in R(w_0^{(k,i)})$.

Theorem 4.20. *Let $\sigma \in \mathfrak{S}_n$ be one of the permutations covered by Lemma 4.17. Then*

$$|B(\sigma)| \geq \frac{1}{2}|R(\sigma)| - 1 \quad \text{and} \quad |C(\sigma)| \leq \frac{1}{2}|R(\sigma)| + 2$$

Proof. We can follow all of the steps of the proof of Lemma 4.3, replacing w_0 with σ. The proof is dependent on arriving at

$$\sum_{v \in G(w_0)} d_b(v) > |R(w_0)|$$

which is a contradiction for w_0.

If σ is one of the permutations in Lemma 4.17, then $A(\sigma) > 0$, and this is also a contradiction for σ.

Thus $|B(\sigma)| \geq \frac{1}{2}|R(\sigma)| - 1$. By Theorem 1.26,

$$|B(\sigma)| + |C(\sigma)| - 1 \leq |R(\sigma)|$$

We can now rewrite this as

$$\frac{1}{2}|R(\sigma)| - 1 + |C(\sigma)| - 1 \leq |R(\sigma)|$$

which means that

$$|C(\sigma)| \leq \frac{1}{2}|R(\sigma)| + 2$$

as desired. □

While not complete for arbitrary permutations, this theorem represents all current knowledge of the relationship between $|B(\sigma)|$ and $|R(\sigma)|$. We used all of our new results on subgraphs to arrive at this result, and believe that additional work on our new families of subgraphs will be required to prove any more cases.

We also believe that $A(\sigma) > 0$ for all $\sigma \in \mathfrak{S}_n$, for $n > 2$. If we can make additional progress on this part of the problem, we would be able to prove more cases. We intend to return to this in the future, as well as considering the problem from other angles.

Chapter 5

CONCLUSION

When we began working on this problem, we hoped to be able to use existing recursions in our proofs along with induction. Because of the weaknesses inherent in this type of argument, we had to go a different direction and were able to find brand new information about the structure of $R(\sigma)$.

Working on the graphs $G(\sigma)$, we discovered new facts about induced subgraphs $G(\sigma s_i)$ for $i \in D(\sigma)$. Using this new information about how to break $G(\sigma)$ into a union of subgraphs, we were also able to find new recursions for the number of braid and commutation edges in $G(\sigma)$. Progressing from there, we were able to find the maximum number of subgraphs that will have the same structure as $w_0^{(k,i)}$. We were also able to go from σ into σs_i and s_i, to considering reduced decompositions of σ as a product α and β. This gave us yet another new method of splitting $G(\sigma)$ into induced subgraphs.

At the same time as we were investigating subgraph structures, we also found lower bounds for $R(\sigma s_i)/|R(\sigma)|$ for $i \in D(\sigma)$. We worked on this problem, because it was one of the road blocks we encountered when trying to use existing recursions. Proving these bounds gave us a greater understanding of how we can move from one induced subgraph in $G(\sigma)$ to another, and allowed us to define our permutation shuffles.

While neither of these topics were part of our original problem, they were both important as they led to a greater understanding of $R(\sigma)$, and the structure of $G(\sigma)$, which is what we wished to do over the course of this dissertation research.

Finally, we returned to the main question that got us started on this work. Using the work we did on the subgraphs of $G(\sigma)$, we were able to prove lower bounds for $|B(\sigma)|$ in certain cases.

At this point, we still have a few open questions. We wish to return to the work done in Chapter 3, in order to improve our lower bounds for $|R(\tau)/|R(\sigma)|$ when $\tau \lessdot \sigma$. We would like to consider other families of permutations ω that are counted by combinatorial objects, and use those objects to calculate the size of $R(\omega)$. It was through the construction of these lower bounds that we were able to construct our permutation shuffles, so further work in this area should lead to greater understanding of general permutations as well.

We also wish to continue to work on proving our lower bound for $|B(\sigma)|$ for arbitrary permutations $\sigma \in \mathfrak{S}_n$. We believe that we can continue to make progress using Lemma 4.6 to split up σ into α and β, but we need to continue to work on smaller cases and our inductive hypothesis. Because our proof Lemma 4.17 relied so heavily on the framework of subgraphs and permutation shuffles that we created in this document, we may need to continue generalizing our study of subgraphs, and their average braid degrees, in order to be able to prove Theorem 4.20 for arbitrary permutations. We hope to be able to answer this question in the future, as well as consider other tools that may allow us to prove the conjectured bounds for $|B(\sigma)|$ and $|C(\sigma)|$.

REFERENCES

[1] Albert, M.H., N. Ruškuc and V. Vatter, “Inflations of geometric grid classes of permutations”, Isr. J. Math. **205**, 73–108, (2015).

[2] Björner, A. and F. Brenti, *Combinatorics of Coxeter groups*, (Springer, 2005).

[3] Bergeron, N., C. Ceballos and J.P. Labbé, “Fan Realizations of type A subword complexes and multi-associahedra of rank 3”, Discrete Comput. Geom., **54**, no. 1, 195–231, (2015).

[4] Dahlberg, S. and Y. Kim, “Diameters of Graphs on Reduced Words of 12 and 21-Inflations”, arXiv:2010.15758, (2020).

[5] Edelman, P. and C. Greene, “Balanced tableaux”, Adv. in Math. **63**, no. 1, 42–99, (1987).

[6] Elnitsky, S., “Rhombic tilings of polygons and classes of reduced words in Coxeter groups”, J. Combin. Theory Ser. A **77**, no. 2, 193–221, (1997).

[7] Fishel, S., E. Milićević, R. Patrias and B.E. Tenner, “Enumerations relating braid and commutation classes”, European J. Combin., **74**, 11-26, (2018).

[8] Fulton, W. *Young Tableaux: With Applications to Representation Theory and Geometry*, (Cambridge University Press, Cambridge, 1997).

[9] Reiner, V., “Note on the expected number of Yang-Baxter moves applicable to reduced decompositions”, European J. Combin., **26**, no. 6, 1019-1021, (2005).

[10] Reiner, V. and Y. Roichman, “Diameter of Graphs of Reduced Words and Galleries”, Trans. Amer. Math. Soc. **365**, no. 5, 2779-2802, (2013).

[11] Sagan, B. E., *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, (2nd Ed.), (Springer, 2013).

[12] Schensted, C., “Longest Increasing and Decreasing Subsequences”, Canadian J. Math, **13**, 179-191, (1961).

[13] Schilling, A., N.M. Thiéry, G. White and N. Williams, “Braid moves in commutation classes of the symmetric group”, European J. Combin., **62**, 15–34, (2017).

[14] Stanley, R. P., “On the reduced decompositions of elements of Coxeter groups”, European J. Combin., **5**, 359-372, (1984).

[15] Stanley, R.P., *Enumerative Combinatorics, Vol. 1* (2nd Ed.), (Cambridge University Press, Cambridge, 2012).

[16] Tenner, B.,“Reduced decompositions and permutation patterns” J. Algebraic Combin. **24**, no. 3, 263–284 (2006).

[17] Tenner, B., “On the expected number of commutations in reduced words”, Australas. J. Combin. **62**, 147–154 (2015).