Robust Interval Observer Design for Uncertain Nonlinear and Hybrid Dynamical Systems

by

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ABSTRACT

The objective of this thesis is to propose two novel interval observer designs for different classes of linear and hybrid systems with nonlinear observations. The first part of the thesis presents a novel interval observer design for uncertain locally Lipschitz continuous-time (CT) and discrete-time (DT) systems with noisy nonlinear observations. The observer is constructed using mixed-monotone decompositions, which ensures correctness and positivity without additional constraints/assumptions. The proposed design also involves additional degrees of freedom that may improve the performance of the observer design. The proposed observer is input-to-state stable (ISS) and minimizes the L_1-gain of the observer error system with respect to the uncertainties. The observer gains are computed using mixed-integer (linear) programs.

The second part of the thesis addresses the problem of designing a novel asymptotically stable interval estimator design for hybrid systems with nonlinear dynamics and observations under the assumption of known jump times. The proposed architecture leverages mixed-monotone decompositions to construct a hybrid interval observer that is guaranteed to frame the true states. Moreover, using common Lyapunov analysis and the positive/cooperative property of the error dynamics, two approaches were proposed for constructing the observer gains to achieve uniform asymptotic stability of the error system based on mixed-integer semidefinite and linear programs, and additional degrees of freedom are incorporated to provide potential advantages similar to coordinate transformations. The effectiveness of both observer designs is demonstrated through simulation examples.

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INTRODUCTION

1.1 Background and Motivation

State estimation is a fundamental problem in real-world engineering applications such as autonomous vehicles and robots (cf. Figure 1.1), where the objective is to estimate the unmeasured states of a system using the available measurements for decision-making and control. The accuracy and robustness of the state estimator are crucial for achieving stability and optimal performance in control systems. However, in many practical applications, the dynamics of the system are uncertain and nonlinear, and the measurements may be corrupted by noise or other disturbances, e.g., in Tahir and Açıkmeşe (2021). To address these challenges, various techniques have been developed, including Kalman filters, Bayesian filters, etc. To handle bounded uncertainties without knowledge of their statistics such as means and covariances as well as nonlinearities, interval observers are designed to provide a guaranteed bound on the state estimate even in the presence of unknown bounded disturbances. In contrast, Kalman filters linearize the system and the noise is assumed to be Gaussian and white, which makes the solution less accurate for highly nonlinear systems and when the noise is non-Gaussian.

1.2 Literature Review

Numerous studies have been conducted on the development of set-valued/interval observers for a wide range of systems. These include linear, nonlinear, cooperative/monotone, mixed-monotone, and Metzler dynamics (Wang *et al.* (2015); Cheb-

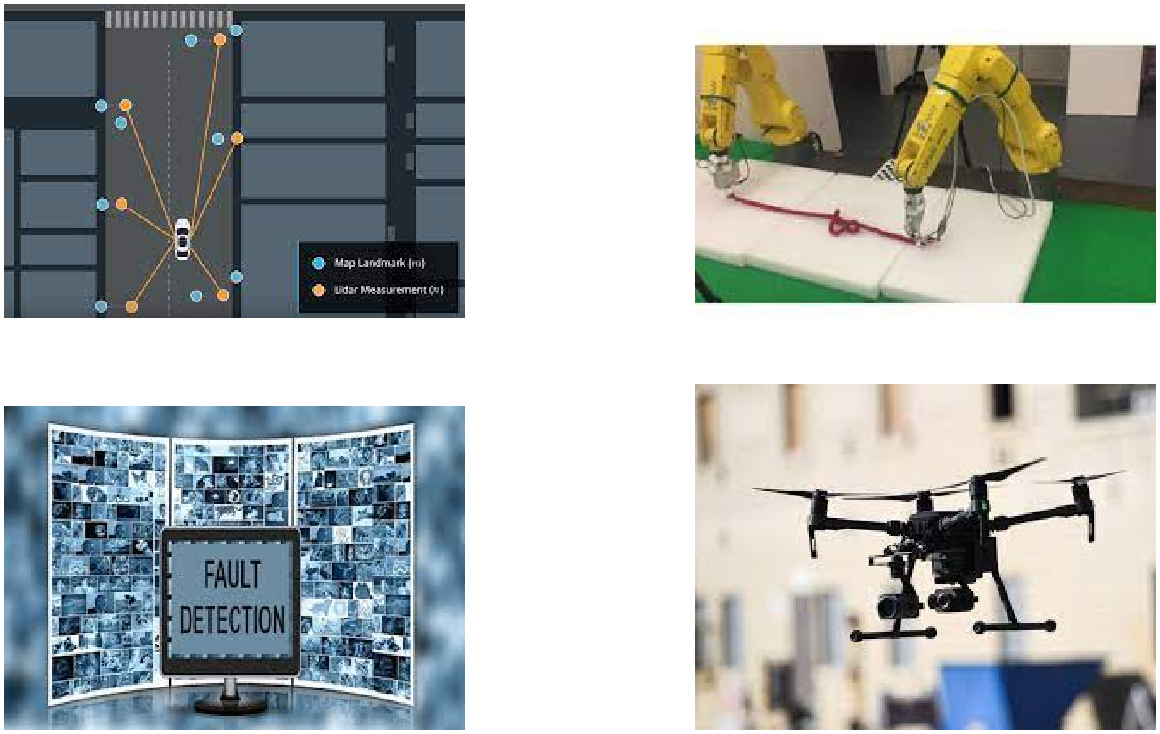

Figure 1.1: Various Real-World Applications Where State Estimation is Important.

otarev *et al.* (2015); Tahir and Açıkmeşe (2021); Khajenejad *et al.* (2021); Khajenejad and Yong (2020); Farina and Rinaldi (2000)). The main idea behind the literature is to design observer gains that ensure the observer error dynamics are Schur/Hurwitz stable and positive/cooperative. Incorporating all of these constraints can result in theoretical and computational complexities. For specific classes of systems, to overcome these difficulties, some studies have proposed leveraging interval arithmetic-based approaches, transforming the system into a positive form, and/or using (time-varying/invariant) state transformations, such as the ones discussed in prior studies, before designing the interval observer.

Additionally, for more general classes of nonlinear systems, various types of bounding mappings/decomposition functions (Khajenejad and Yong (2021)) have been leveraged to cast the observer design problem into semi-definite programs (SDPs) with LMI constraints (Wang *et al.* (2012, 2015); Efimov and Raïssi (2016); Briat and Khammash (2016); Tahir and Açıkmeşe (2021); Khajenejad *et al.* (2022)). However, the obtained LMIs might still be restrictive and solutions may not exist for some systems, i.e., the LMIs might be infeasible for some systems, due to several imposed conditions and upper bounding. To tackle this problem, coordinate transformations have been proposed to relax the design conditions and to facilitate obtaining feasible observer gains (Dinh *et al.* (2014); Mazenc and Bernard (2011)). However, unfortunately, the coordinate transformation and observer gains cannot be simultaneously synthesized/designed. Recently, robustness against noise/uncertainty has been considered for designing uncertain systems. This involves solving SDPs that satisfy framer, stabilizability, and noise attenuation/mitigation constraints. This approach may lead to conservatism and computational complexity (Khajenejad and Yong (2022)). In the case of continuous-time LPV systems, L_1/L_2 performance is often considered.

On the other hand, the design of observers for hybrid systems is more challenging because hybrid systems combine both continuous/flow dynamics and discrete/jump dynamics. An observer design framework was introduced for hybrid systems with linear flows/jump dynamics in Bernard and Sanfelice (2018). Moreover, the especially challenging task of designing hybrid interval observers was tackled for specific classes of hybrid systems like linear impulsive systems (Degue *et al.* (2021, 2018); Briat and Khammash (2017)), switched linear systems (Wang *et al.* (2018)) and switched nonlinear systems (He and Xie (2016)). However, to our best knowledge, there are no existing interval observer designs for general hybrid systems with nonlinear dynamics and nonlinear observations, which is the focus of the second part of this thesis.

Chapter 2

PRELIMINARIES

2.1 Notations

$\mathbb{R}^n$, $\mathbb{R}^n_{>0}$, $\mathbb{R}^{n\times p}$, $\mathbb{N}_n$ and $\mathbb{N}$ denote the n-dimensional Euclidean space, positive vectors of size n, matrices of size n by p, natural numbers up to n and natural numbers, respectively. For a vector $v \in \mathbb{R}^n$, its vector p-norm is given by $\|v\|_p \triangleq (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$, while for a matrix $M \in \mathbb{R}^{n\times p}$, M_{ij} represents its j-th column and i-th row entry, $\text{sgn}(M)$ represents its element-wise signum function, $M^{\oplus} \triangleq \max(M, \mathbf{0}_{n\times p})$, $M^{\ominus} \triangleq M^{\oplus} - M$, and $|M| \triangleq M^{\oplus} + M^{\ominus}$ is its element-wise absolute value. Moreover, M^{d} is a diagonal matrix with the diagonal elements of a square matrix $M \in \mathbb{R}^{n\times n}$, $M^{\text{nd}} \triangleq M - M^{\text{d}}$ is the matrix with only its off-diagonal elements, and $M^{\text{m}} \triangleq M^{\text{d}} + |M^{\text{nd}}|$ is the "Metzlerized" matrix. A Metzler matrix is a square matrix in which all the off-diagonal components are nonnegative (equal to or greater than zero). Further, all vector and matrix inequalities are element-wise inequalities, and the matrices of zeros and ones of dimension $n \times p$ are denoted as $\mathbf{0}_{n\times p}$ and $\mathbf{1}_{n\times p}$, respectively. Finally, a function $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, positive definite (i.e., $\alpha(x) = 0$ for $x = 0$; $\alpha(x) > 0$ otherwise), and strictly increasing, and is of class $\mathcal{K}_\infty$ if it is also unbounded, while $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\lambda(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\lambda(s, t)$ decreases to zero as $t \to \infty$.

2.2 Definitions

Definition 1 (Interval)**.** *An (n-dimensional) interval $\mathcal{I} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{R}^n$ is the set of vectors $z \in \mathbb{R}^{n_z}$ such that $\underline{z} \leq z \leq \overline{z}$. A similar definition applies to matrix intervals.*

Definition 2 (Jacobian Sign-Stability). *A vector-valued function $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ is Jacobian sign-stable (JSS), if in its domain $\mathcal{Z}$, the entries of its Jacobian matrix do not change signs, i.e., if one of the following hold:*

$$J^f_{ij}(z) \geq 0 \quad or \; J^f_{ij}(z) \leq 0$$

for all $z \in \mathcal{Z}, \forall i \in \mathbb{N}_p, \forall j \in \mathbb{N}_{n_z}$, where $J^f(z)$ represents the Jacobian matrix of the mapping f evaluated at $z \in \mathcal{Z}$.

Definition 3 (Mixed-Monotonicity and Decomposition Functions). *Consider the uncertain dynamical system with initial state $x_0 \in \mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}^n$ and process noise $w_t \in \mathcal{W} \triangleq [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$:*

$$x_t^+ = g(z_t) \triangleq g(x_t, w_t), z_t \triangleq [x_t^\top \; w_t]^\top, \tag{2.1}$$

where $x_t^+ \triangleq x_{t+1}$ if (2.1) is a DT system and $x_t^+ \triangleq \dot{x}_t$ if (2.1) is a CT system. Moreover, $g : \mathcal{Z} \subset \mathbb{R}^n_z \to \mathbb{R}^n$ is the vector field with augmented state $z_t \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{W} \subset \mathbb{R}^{n_z}$ as its domain, where $\mathcal{X}$ is the entire state space and $n_z = n + n_w$.

Suppose (2.1) is a DT system. Then, a mapping $g_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^p$ is a DT mixed-monotone decomposition function with respect to g, if i) $g_d(z, z) = g(z)$, ii) g_d is monotone increasing in its first argument, i.e., $\hat{z} \geq z \Rightarrow g_d(\hat{z}, z') \geq g_d(z, z')$, and iii) g_d is monotone decreasing in its second argument, i.e., $\hat{z} \geq z \Rightarrow g_d(z', \hat{z}) \leq g_d(z', z)$.

On the other hand, if (2.1) is a CT system, a mapping $g_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^p$ is a CT mixed-monotone decomposition function with respect to g, if i) and iii) for the DT system hold, while ii) is modified to the following: ii') g_d is monotone increasing in its first argument with respect to "off-diagonal" arguments, i.e., $\forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_{n_z} \wedge (i \neq j), \hat{z}_j \geq z_j, \hat{z}_i = z_i \Rightarrow g_{d,i}(\hat{z}, z') \geq g_{d,i}(z, z')$.

Definition 4 (Embedding System). *For an n-dimensional system (2.1) with any decomposition function $g_d(\cdot)$, its* embedding system is defined as the following $2n$-

dimensional dynamical system with initial condition $\begin{bmatrix}\overline{x}_0^\top & \underline{x}_0^\top\end{bmatrix}^\top$:

$$\begin{bmatrix}\underline{x}_t^+ \\ \overline{x}_t^+\end{bmatrix} = \begin{bmatrix}\underline{g}_d(\begin{bmatrix}(\underline{x}_t)^\top & \underline{w}^\top\end{bmatrix}^\top, \begin{bmatrix}(\overline{x}_t)^\top & \overline{w}^\top\end{bmatrix}^\top) \\ \overline{g}_d(\begin{bmatrix}(\overline{x}_t)^\top & \overline{w}^\top\end{bmatrix}^\top, \begin{bmatrix}(\underline{x}_t)^\top & \underline{w}^\top\end{bmatrix}^\top)\end{bmatrix}. \tag{2.2}$$

Definition 5 (Correct Interval Framers and Framer Errors)**.** *Let Assumptions 2–1 hold. Given the nonlinear plant* $\mathcal{G}$ *in* (3.1)*,* $\overline{x}, \underline{x} : \mathbb{T} \to \mathbb{R}^n$ *are called upper and lower framers for the system states, if* $\underline{x}_t \leq x_t \leq \overline{x}_t,\ \forall t \in \mathbb{T}, \forall w_t \in \mathcal{W}, \forall v_t \in \mathcal{V}$*. Further,* $\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$ *is called the* framer error *at time* t*. Any dynamical system whose states are correct framers for the system states of the plant* $\mathcal{G}$*, i.e., with* $\varepsilon_t \geq 0, \forall t \in \mathbb{T}$*, is called a* correct *interval framer for system* (3.1)*.*

Definition 6 (L_1-Robust Interval Observer)**.** *An interval framer* $\hat{\mathcal{G}}$ *is* L_1*-robust and optimal, if the* L_1*-gain of the framer error system* $\tilde{\mathcal{G}}$*, defined below, is minimized:*

$$\|\tilde{\mathcal{G}}\|_{L_1} \triangleq \sup_{\|\Delta\|_{L_1}=1} \|\varepsilon\|_{L_1}, \tag{2.3}$$

where $\|v\|_{L_1} \triangleq \int_0^\infty \|v_t\|_1 dt$ *is the* L_1 *signal norm for* $v \in \{\varepsilon, \Delta\}$*, and* ε_t *and* $\Delta_t = \Delta \triangleq \begin{bmatrix}\Delta w^\top & \Delta v^\top\end{bmatrix}^\top$ *are the framer error and combined noise signals, respectively, with* $\Delta w \triangleq \overline{w} - \underline{w}$ *and* $\Delta v \triangleq \overline{v} - \underline{v}$*.*

2.3 Propositions

Proposition 1 (Jacobian Sign-Stable Decomposition)**.** *For a mapping* $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$*, if* $J^f(z) \in [\underline{J}^f, \overline{J}^f]$ *for all* $z \in \mathcal{Z}$*,, where* $\underline{J}^f, \overline{J}^f \in \mathbb{R}^{p \times n_z}$ *are known matrices, then the function* f *can be decomposed to a JSS mapping* $\mu(\cdot)$ *and a (remainder) affine mapping* Hz *(that is also JSS), in an additive remainder-form:*

$$\forall z \in \mathcal{Z}, f(z) = Hz + \mu(z), \tag{2.4}$$

where the matrix $H \in \mathbb{R}^{p \times n_z}$, *satisfies*

$$\forall (i,j) \in \mathbb{N}_p \times \mathbb{N}_{n_z}, H_{ij} = \underline{J}^f_{ij} \ \vee H_{ij} = \overline{J}^f_{i,j}. \tag{2.5}$$

Proposition 2 (Tight and Tractable Decomposition Functions for JSS Mappings)**.** *Let* $\mu : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ *be a JSS mapping on its domain. Then, it admits a tight decomposition function for each* $\mu_i(\cdot)$, $i \in \mathbb{N}_p$ *as follows:*

$$\mu_{d,i}(z_1, z_2) = \mu_i(D^i z_1 + (I_{n_z} - D^i) z_2), \tag{2.6}$$

for any ordered $z_1, z_2 \in \mathcal{Z}$ *where* D^i *is a binary diagonal matrix determined by which vertex of the interval* $[z_2, z_1]$ *or* $[z_1, z_2]$ *that maximizes (if* $z_2 \leq z_1$*) or minimizes (if* $z_2 > z_1$*) the function* $\mu_i(\cdot)$ *that can be found in closed-form as:*

$$D^i = \mathrm{diag}(\max(\mathrm{sgn}(\overline{J}^\mu_i), \mathbf{0}_{1,n_z})). \tag{2.7}$$

Consequently, a tight and tractable remainder-form decomposition function for the system in (2.1) can be found by applying Proposition 2 to the Jacobian sign-stable decomposition from Proposition 1.

2.4 Assumptions

Assumption 1. *The disturbance/noise bounds* $\underline{w}$, $\overline{w}$, $\underline{v}$, *and* $\overline{v}]$ *as well as the signals* y_t *(output) and* u_t *(input, if any) are known at all times. Moreover, the initial state* x_0 *is such that* $x_0 \in \mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ *with known bounds* $\underline{x}_0$ *and* $\overline{x}_0$.

Assumption 2. *The mappings/functions* $f(\cdot)$ *and* $h(\cdot)$ *are known, locally Lipschitz, differentiable and mixed-monotone in their domain* [1] *. Moreover, the lower and upper*

[1] Both assumptions of locally Lipschitz continuity and differentiability are primarily for ease of exposition and can be relaxed to a much weaker continuity assumption (cf. Khajenejad and Yong (2021) for more details).

bounds of their Jacobian matrices, $\underline{J}^f, \overline{J}^f \in \mathbb{R}^{n \times n_z}$ *and* $\underline{J}^h, \overline{J}^h \in \mathbb{R}^{l \times n_\zeta}$ *are known, where* $n_z = n + n_w$ *and* $n_\zeta = n + n_v$*.*

Assumption 3. *The mappings* f_c, f_d, h_c, h_d *are known, differentiable, and locally Lipschitz in their domain with a priori known lower and upper bounds for their Jacobian matrices,* $\underline{J}^{f_c}, \overline{J}^{f_c}, \underline{J}^{f_d}, \overline{J}^{f_d} \in \mathbb{R}^{n \times n}$ *,* $\underline{J}^{h_c}, \overline{J}^{h_c} \in \mathbb{R}^{m_1 \times n}$ *and* $\underline{J}^{h_d}, \overline{J}^{h_d} \in \mathbb{R}^{m_2 \times n}$*, respectively.*

Assumption 4. *The plant jump times, as well as the outputs* y_c *during flows and/or* y_d *at the jumps, are known.*

Assumption 5. *Some information about the flow/dwell time between successive jumps is known, namely a closed subset* $\mathcal{I}$ *of* $\mathbb{R}_{\geq 0}$ *such that any hybrid arc* x *in the maximal solution set* $\mathcal{S}_\mathcal{H}(\mathcal{X}_0)$ *with the initial state in* $\mathcal{X}_0$ *satisfies:*

- $t_{j+1}(x) - t_j(x) \in \mathcal{I},\ \forall j \in \{1, ..., J(x) - 1\}$ *if* $J(x) < +\infty$ *or* $\forall j \in \mathbb{N}_{\geq 0}$ *if* $J(x) = +\infty$*,*
- $0 \leq t - t_j(x) \leq \sup \mathcal{I},\ \forall (t, j) \in dom(x)$*,*

where $t_j(x)$ *is the time stamp corresponding to jump* j *and* $J(x) \triangleq \sup dom_j(x)$*.*

Chapter 3

L_1-ROBUST INTERVAL OBSERVER

This chapter considers the design of L_1-robust interval observers, which is published in Pati *et al.* (2022).

3.1 Problem Formulation

System Dynamics: The uncertain/noisy discrete-time (DT) or continuous-time (CT) nonlinear systems considered in this work is given as:

$$\mathcal{G}: \begin{cases} x_t^+ = \hat{f}(x_t, u_t) + W w_t \triangleq f(x_t) + W w_t, \\ y_t = \hat{h}(x_t, u_t) + V v_t \triangleq h(x_t) + V v_t, \end{cases} \tag{3.1}$$

for all $t \in \mathbb{T}$, where $x_t^+ = x_{t+1}, \mathbb{T} = \{0\} \cup \mathbb{N}$ if $\mathcal{G}$ is a DT system and $x_t^+ = \dot{x}_t, \mathbb{T} = \mathbb{R}_{\geq 0}$, if $\mathcal{G}$ is a CT system. Moreover, $x_t \in \mathcal{X} \subset \mathbb{R}^n$, $w_t \in \mathcal{W} \triangleq [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}, v_t \in \mathcal{V} \triangleq [\underline{v}, \overline{v}] \subset \mathbb{R}^{n_v}$, $u_t \in \mathbb{R}^s$ and $y_t \in \mathbb{R}^l$ are state, process noise, measurement noise, known control input and output measurement signals, respectively. $\hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the nonlinear state vector field and $\hat{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ denotes observation/constraint functions, from which the mappings/functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ are well-defined, since the input signal $u_t \in \mathbb{R}^m$ is known. Moreover, the noise matrices W and V are known.

The goal is to estimate the state trajectories of the plant $\mathcal{G}$ in (3.1), when the initial state satisfies $x_0 \in \mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathcal{X}$.

Problem 1. *Given the nonlinear system in* (3.1)*, as well as Assumptions 1–2, design a correct and L_1-robust interval observer (cf. Definitions 6) whose framer error (cf.*

Definition 5) is input-to-state stable (ISS), i.e.,

$$\|\varepsilon_t\|_2 \leq \beta(\|\varepsilon_0\|_2, t) + \rho(\|\Delta\|_{L_\infty}), \forall t \in \mathbb{T}, \tag{3.2}$$

where β and ρ are functions of classes $\mathcal{KL}$ and $\mathcal{K}_\infty$, respectively, with the L_∞ signal norm $\|\Delta\|_{L_\infty} = \sup_{s\in[0,\infty)} \|\Delta_s\|_2 = \|\Delta\|_2$ and Δ_s defined as in (2.3)*.*

3.2 Interval Observer Design

In order to the propose the interval observer design, we first provide an equivalent representation of the system dynamics for the plant $\mathcal{G}$ in (3.1).

Lemma 1. *Consider plant $\mathcal{G}$ in* (3.1) *and suppose that Assumptions 1–2 hold. Let $L, N \in \mathbb{R}^{n\times l}$ and $T \in \mathbb{R}^{n\times n}$ be arbitrary matrices that satisfy $T + NC = I_n$. Then, the system dynamics* (3.1) *can be equivalently written as*

$$\begin{aligned} \xi_t^+ &= (TA - LC - NA_2)x_t + T\phi(x_t) - N\rho(x_t, w_t) \\ &\quad + (TW - NB_2)w_t - L\psi(x_t) + L(y_t - Vv_t), \\ x_t &= \xi_t + Ny_t - NVv_t, \end{aligned} \tag{3.3}$$

where $A \in \mathbb{R}^{n\times n}$, $C, A_2 \in \mathbb{R}^{l\times n}$, and $B_2 \in \mathbb{R}^{l\times n_w}$ are chosen such that the following decompositions hold $\forall x \in \mathcal{X}$, $w \in \mathcal{W}$ (cf. Definition 2 and Proposition 1):

$$\begin{gathered} f(x) = Ax + \phi(x), \;\; h(x) = Cx + \psi(x), \\ \psi^+(x, w) = A_2 x + B_2 w + \rho(x, w), \end{gathered} \tag{3.4}$$

such that ϕ, ψ, ρ are JSS, with $\psi^+(x, w) = \dot{\psi}(x, w) = \frac{\partial \psi}{\partial x}(f(x) + Ww)$ if $\mathcal{G}$ is a CT system and $\psi^+(x, w) = \psi(x^+) = \psi(f(x) + Ww)$ if $\mathcal{G}$ is a DT system.

Proof. This can be proved by defining an auxiliary state $\xi_t \triangleq x_t - Ny_t + NVv_k$. Then, from (3.1) and (3.4), we have $\xi_t = x_t - N(y_t - Vv_k) = x_t - N(Cx_t + \psi(x_t))$, and moreover, by choosing N to satisfy $T+NC=I_n$, we obtain $\xi_t=Tx_t-N\psi(x_t)$ that has the following dynamics:

$$\begin{aligned}\xi_t^+ &= Tx_t^+ - NC\psi^+(x_t,w_t)\\ &= T(Ax_t+\phi(x_t)+Ww_t)-N(A_2x_t+B_2w_t+\rho(x_t,w_t)),\end{aligned}$$

with x_t^+ from (3.1) and $f(x_t)$ and $\psi^+(x_t,w_t)$ from (3.4). Finally, adding a 'zero term' $L(y_t-Cx_t-\psi(x_t)-Vv_t)=0$ (cf. (3.1) and (3.4)) to the above yields (3.3), where x_t can be recovered from the definition of ξ_t. □

Then, using the equivalent system in (3.3), we propose a unified interval observer $\hat{\mathcal{G}}$ based on the construction of an embedding system

$$\begin{aligned}\underline{\xi}_t^+ &= M^{\uparrow}\underline{x}_t - M^{\downarrow}\,\overline{x}_t + Ly_t + T^{\oplus}\phi_d(\underline{x}_t,\overline{x}_t) - T^{\ominus}\phi_d(\overline{x}_t,\underline{x}_t)\\ &\quad +(LV)^{\oplus}\overline{v}+(LV)^{\ominus}\underline{v}+(TW-NB_2)^{\oplus}\underline{w}\\ &\quad -(TW-NB_2)^{\ominus}\overline{w}-L^{\oplus}\psi_d(\overline{x}_t,\underline{x}_t)+L^{\ominus}\psi_d(\underline{x}_t,\overline{x}_t)\\ &\quad -N^{\oplus}\rho_d(\overline{x}_t,\overline{w},\underline{x}_t,\underline{w})+N^{\ominus}\rho_d(\underline{x}_t,\underline{w},\overline{x}_t,\overline{w})\\ &\quad +MNy-(MNV)^{\oplus}\underline{v}+(MNV)^{\ominus}\overline{v},\\ \overline{\xi}_t^+ &= M^{\uparrow}\overline{x}_t - M^{\downarrow}\,\overline{x}_t + Ly_t + T^{\oplus}\phi_d(\overline{x}_t,\underline{x}_t) - T^{\ominus}\phi_d(\underline{x}_t,\overline{x}_t)\\ &\quad +(LV)^{\oplus}\underline{v}+(LV)^{\ominus}\overline{v}+(TW-NB_2)^{\oplus}\overline{w}\\ &\quad -(TW-NB_2)^{\ominus}\underline{w}-L^{\oplus}\psi_d(\underline{x}_t,\overline{x}_t)+L^{\ominus}\psi_d(\overline{x}_t,\underline{x}_t)\\ &\quad -N^{\oplus}\rho_d(\underline{x}_t,\underline{w},\overline{x}_t,\overline{w})+N^{\ominus}\rho_d(\overline{x}_t,\overline{w},\underline{x}_t,\underline{w})\\ &\quad +MNy-(MNV)^{\oplus}\overline{v}+(MNV)^{\ominus}\underline{v},\end{aligned}$$

$$\begin{aligned}\underline{x}_t &= \underline{\xi}_t + Ny_t - (NV)^{\oplus}\overline{v} + (NV)^{\ominus}\underline{v},\\ \overline{x}_t &= \overline{\xi}_t + Ny_t - (NV)^{\oplus}\underline{v} + (NV)^{\ominus}\overline{v},\end{aligned} \tag{3.5}$$

where $\overline{\xi}_t,\underline{\xi}_t,\overline{\xi}_t^+,\underline{\xi}_t^+\in\mathbb{R}^n$ are auxiliary variables, $M\triangleq TA-LC-NA_2$ and if $\mathcal{G}$ is a CT system, then

$$\overline{x}_t^+\triangleq\dot{\overline{x}}_t,\;\; \underline{x}_t^+\triangleq\dot{\underline{x}}_t,\;\; M^{\uparrow}\triangleq M^{\mathrm{d}}+M^{\mathrm{nd},\oplus},\;\; M^{\downarrow}\triangleq M^{\mathrm{nd},\ominus}, \tag{3.6}$$

and if $\mathcal{G}$ is a DT system, then

$$\overline{x}_t^+ \triangleq \overline{x}_{t+1},\ \underline{x}_t^+ \triangleq \underline{x}_{t+1},\ M^{\uparrow} \triangleq M^{\oplus},\ M^{\downarrow} \triangleq M^{\ominus}. \tag{3.7}$$

Further, $\phi_d, \psi_d : \mathbb{R}^{2n} \to \mathbb{R}^n$ and $\rho_d : \mathbb{R}^{2l} \to \mathbb{R}^l$ are tight mixed-monotone decomposition functions of ϕ, ψ, and ρ, respectively (cf. (3.4), Definition 3), which are JSS and thus, can be computed using (2.6) and (2.7). Finally, $N, L \in \mathbb{R}^{n\times l}$ and $T \in \mathbb{R}^{n\times n}$ are the observer gain matrices to be designed with T and N satisfying $T + NC = I_n$. The detailed derivation of the observer design $\hat{\mathcal{G}}$ in (3.5) and its desired properties are given in the next subsections.

Remark 1. *Note that the above observer structure is inspired by Wang* et al. *(2012) to introduce additional degrees of freedom, where we now have three to-be-designed observer gains N, T, L, in contrast to only one observer gain L in our previous work (Khajenejad and Yong (2022)). While this is very helpful from a performance perspective, it is found in Section 5.1.1 to be not an exact substitute for coordinate transformations that may help to make the observer gain design problem in Theorem 2 feasible, presumably because the latter are nonlinear operations and the proposed structure only introduces linear terms. In this case, a coordinate transformation can be applied in a straightforward manner, similar to (Tahir and Açıkmeşe, 2021, Section V) (omitted for brevity; cf. Mazenc and Bernard (2021) and references therein for more discussions).*

3.3 Observer Correctness (Framer Property)

In this subsection, we show that by construction, the proposed interval observer $\hat{\mathcal{G}}$ in (3.5) is a correct framer for the system $\mathcal{G}$ (and equivalently, for (3.3) in Lemma 3.2) in the sense of Definition 5 for both the CT (3.6) and DT (3.7) cases.

Theorem 1 (Correctness)**.** *Consider the nonlinear plant $\mathcal{G}$ in* (3.1) *and suppose Assumptions 1 and 2 hold. Then,* $\underline{x}_t \leq x_t \leq \overline{x}_t, \forall t \in \mathbb{T}, \forall w_t \in \mathcal{W}, \forall v_t \in \mathcal{V}$*, where* x_t *and* $[\underline{x}_t^\top \overline{x}_t^\top]^\top$ *are the state vectors in* $\mathcal{G}$ *and* $\hat{\mathcal{G}}$ *at time* $t \in \mathbb{T}$*, respectively. In other words, the dynamical system* (3.5) *constructs a correct interval framer for the nonlinear plant* $\mathcal{G}$ *in* (3.1)*.*

Proof. Here by framing all the constituent items in the right hand side of system (3.3) (which is equivalent to the original system (3.1)), using Proposition 1, we obtain (3.5). Hence, (3.5) is a framer system for the original plant $\mathcal{G}$, since it is straightforward to see that the summation of the embedding systems/framers of constituent systems constructs an embedding system for the summation of constituent systems. In order to compute framers for the constituent systems in the right hand side of (3.3), we split them into three groups, as follows. i) Known/certain terms that are independent of state and noise and so, their upper and lower bounds are equal to their original known value; ii) Linear terms with respect to the state and noise, which can be upper and lower framed by applying (Efimov *et al.*, 2013, Lemma 1). Note that there is a subtle difference in computing the upper and lower framing/embedding systems for the CT and DT cases, which is reflected in the definition of $M^\uparrow, M^\downarrow$ in (3.6) and (3.7) (cf. Definition 4); iii) Nonlinear terms in the state and noise that can be upper and lower framed by leveraging Propositions 1 and 2

Summing up the constituent embedding systems/framers in i)-iii) yields the embedding system (3.5). Finally, the correctness property follows from the framer property of embedding systems. $\square$

3.4 L_1-Robust Observer Design

In addition to the correctness property, it is important to guarantee the stability of the proposed framer i.e., we aim to design the observer gains T, N, and L to obtain input-to-state stability (ISS) and L_1-robustness.

Theorem 2 (L_1-Robust and L_1-ISS Observer Design). *If Assumptions 1–2 hold for the nonlinear system $\mathcal{G}$ in (3.1), then the correct interval framer $\hat{\mathcal{G}}$ proposed in (3.5) is L_1-robust (cf. Definition 6), if there exist $Q, \tilde{T} \in \mathbb{R}^{n\times n}$, $\tilde{N}, N, \tilde{L} \in \mathbb{R}^{n\times l}$, $p \in \mathbb{R}^n_{>0}$, and $\gamma > 0$ that solve the following mixed-integer program (MIP):*

$$
\begin{aligned}
&(\gamma^*, p^*, Q^*, \tilde{T}^*, \tilde{L}^*, \tilde{N}^*, N^*) \in \\
&\quad {}_{\{\gamma,p,Q,\tilde{T},\tilde{L},\tilde{N},N\}}\ \gamma \\
&\quad s.t.\ \ \mathbf{1}_{1\times n}\begin{bmatrix}\Omega & \Lambda & \Upsilon\end{bmatrix} < \begin{bmatrix}\sigma & \gamma\mathbf{1}_{1\times n_w} & \gamma\mathbf{1}_{1\times n_v}\end{bmatrix}, \\
&\qquad\quad \tilde{T} + \tilde{N}C = Q,\ \tilde{N} = \Gamma,\ p > 0,\ \gamma > 0,
\end{aligned}
\tag{3.8}
$$

where $Q \triangleq diag(p)$ denotes a matrix whose diagonal entries are the elements of p, $\Lambda \triangleq |\tilde{T}W - \tilde{N}B_2| + |\tilde{N}|\overline{F}^{\rho}_w$, and

1. *for a DT system $\mathcal{G}$: $\sigma \triangleq p^\top - \mathbf{1}_{1\times n}$, $\Omega \triangleq |M| + |\tilde{T}|\overline{F}^{\phi}_x + |\tilde{N}|\overline{F}^{\rho}_x + |\tilde{L}|\overline{F}^{\psi}_x$, $\Upsilon \triangleq |\tilde{L}V| + |\tilde{N}V|$, and $\Gamma \triangleq \tilde{N}$;*

2. *for a CT system $\mathcal{G}$: $\sigma \triangleq -\mathbf{1}_{1\times n}$, $\Omega \triangleq M^m + |\tilde{T}|\overline{F}^{\phi}_x + |\tilde{N}|\overline{F}^{\rho}_x + |\tilde{L}|\overline{F}^{\psi}_x$, $\Upsilon \triangleq |\tilde{L}V| + Z$, and $\Gamma \triangleq QN$ if $V \neq 0$ and $\Gamma \triangleq \tilde{N}$ otherwise,*

with $M \triangleq TA - LC - NA_2$ and $Z \triangleq (|M| - M^m)|NV|$. Furthermore, in both cases, $\overline{F}^{\phi}_x, \overline{F}^{\psi}_x, \overline{F}^{\rho}_x, \overline{F}^{\rho}_w$ are computed from the JSS functions ϕ, ψ and ρ as follows:

$$
\overline{F}^{\mu} \triangleq [\overline{F}^{\mu}_x\ \overline{F}^{\mu}_w] \triangleq (\overline{J}^{\mu})^{\oplus} + (\underline{J}^{\mu})^{\ominus},
\tag{3.9}
$$

with $\overline{J}^{\mu} = \overline{J}^{f} - H$, $\underline{J}^{\mu} = \underline{J}^{f} - H$, $\overline{F}^{\mu}_x \in \mathbb{R}^{n\times n}$ and $\overline{F}^{\mu}_w \in \mathbb{R}^{n\times n_w}$ (cf. Proposition 1 and (Khajenejad and Yong, 2022, Lemma 3)).

Then, the corresponding L_1-robust stabilizing observer gains T^*, L^*, and N^* can be obtained as $T^* = (Q^*)^{-1}\tilde{T}^*, L^* = (Q^*)^{-1}\tilde{L}^*$ and $N^* = (Q^*)^{-1}\tilde{N}^*$. Moreover, the interval observer is ISS, i.e., it satisfies (3.2).

Proof. We start by deriving the framer error ($\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$) from (3.5), before proving that the DT and CT error systems satisfy the condition in (2.3) and (3.2), i.e., $\hat{\mathcal{G}}$ is L_1-robust and ISS. To do this, we define $\Delta s \triangleq \overline{s} - \underline{s}, \Delta_d^\mu \triangleq \mu_d(\overline{x}, \overline{s}, \underline{x}, \underline{s}) - \mu_d(\overline{x}, \overline{s}, \underline{x}, \underline{s})$ for all $s \in \{w, v\}$ and $\mu \in \{\phi, \psi, \rho\}$.

First, from (3.5), the framer error is $\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t = \overline{\xi}_t - \underline{\xi}_t + |NV|\Delta v$. Then, the DT observer error dynamics $\tilde{\mathcal{G}}$ obtained from (3.5) and (3.7) can be written as:

$$\begin{aligned}
\varepsilon_t^+ &= |TA - LC - NA_2|\varepsilon_t + |T|\Delta_d^\phi + |N|\Delta_d^\rho + |L|\Delta_d^\psi \\
&\quad + |TW - NB_2|\Delta w + (|LV| + |NV|)\Delta v + |MNV|\Delta v \\
&\leq (|TA - LC - NA_2| + |T|\overline{F}_x^\phi + |N|\overline{F}_x^\rho + |L|\overline{F}_x^\psi)\varepsilon_t \\
&\quad + (|TW - NB_2| + |N|\overline{F}_w^\rho)\Delta w + (|LV| + |NV|)\Delta v,
\end{aligned} \tag{3.10}$$

where the inequality holds since Δ_d^μ, $\mu \in \{\phi, \psi, \rho\}$ satisfy $\Delta_d^\mu \leq \overline{F}_x^\mu \varepsilon_t + \overline{F}_w^\mu \Delta w$, with $\overline{F}_x^\mu, \overline{F}_w^\mu$ defined in (3.9) by (Khajenejad and Yong, 2022, Lemma 3) and their pre-multiplier matrices $|\cdot|$ are non-negative. Further, by the *Comparison Lemma* (Khalil, 2002, Lemma 3.4), the actual framer error system is stable if the comparison system on the right hand side in (3.10) is stable.

Defining $\tilde{A} \triangleq |TA - LC - NA_2| + |T|\overline{F}_x^\phi + |N|\overline{F}_x^\rho + |L|\overline{F}_x^\psi$, $\tilde{B} \triangleq \begin{bmatrix} (|TW - NB_2| + |N|\overline{F}_w^\rho) & (|LV| + |NV|) \end{bmatrix} \geq 0$, $\tilde{C} \triangleq I_{n\times n} \geq 0$, and $\tilde{D} \triangleq \mathbf{0}_{n\times n} \geq 0$, the DT comparison system with $z_t = \varepsilon_t$ can be written as:

$$\varepsilon_t^+ \leq \tilde{A}\varepsilon_t + \tilde{B}[(\Delta w)^\top (\Delta v)^\top]^\top, z_t = \tilde{C}\varepsilon_t + \tilde{D}[(\Delta w)^\top (\Delta v)^\top]^\top. \tag{3.11}$$

Since $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are non-negative, the error system in (3.11) is positive. Then, by (Chen *et al.*, 2013, Theorem 1), the comparison system (3.11) is asymptotically stable with $\gamma > 0$ as the L_1-gain (cf. Definition 2.3), if there exists $p \in \mathbb{R}^n_{>0}$ such that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^\top \begin{bmatrix} p \\ \mathbf{1}_{n\times 1} \end{bmatrix} < \begin{bmatrix} p \\ \gamma \mathbf{1}_{(n_w+n_v)\times 1} \end{bmatrix}. \tag{3.12}$$

Next, by defining $Q = Q^\top \triangleq diag(p) > 0$, $\tilde{T} = QT$, $\tilde{N} = QN$, and $\tilde{L} = QL$, we have $p = Q\mathbf{1}_{n\times 1}$ and $\tilde{C}^\top \mathbf{1}_{n\times 1} = \mathbf{1}_{n\times 1}$. Further defining $\Omega \triangleq Q\tilde{A}$ and $\begin{bmatrix} \Lambda & \Upsilon \end{bmatrix} \triangleq Q\tilde{B}$, we obtain that (3.12) is equivalent to the inequality constraint in (3.8). Similarly, we can pre-multiply $T + NC = I_n$ by the invertible Q matrix to obtain the equality constraint in (3.8). Hence, by solving the MILP in (3.8) results in observer gains $T^* = (Q^*)^{-1}\tilde{T}^*, N^* = (Q^*)^{-1}\tilde{N}^*$ and $L^* = (Q^*)^{-1}\tilde{L}^*$ that result in a L_1-robust comparison system (3.11) i.e., it satisfies (2.3) with γ^*. Moreover, since the comparison system is linear, asymptotically stability also implies that it is ISS (Sontag (2008)). Consequently, by the Comparison Lemma (Khalil, 2002, Lemma 3.4), the actual DT framer error system on the left hand side of (3.10) is also L_1-robust and ISS.

The CT case is similar to the DT case, where from (3.5) and (3.6), we obtain CT observer error dynamics $\tilde{\mathcal{G}}$:

$$\begin{aligned} \dot{\varepsilon}_t &= (TA - LC - NA_2)^m \varepsilon_t + |T|\Delta^\phi_d + |N|\Delta^\rho_d + |L|\Delta^\psi_d \\ &\quad + |TW - NB_2|\Delta w + |LV|\Delta v + |MNV|\Delta v \\ &\leq ((TA - LC - NA_2)^m + |T|\overline{F}^\phi_x + |N|\overline{F}^\rho_x + |L|\overline{F}^\psi_x)\varepsilon_t \\ &\quad + (|TW - NB_2| + |N|\overline{F}^\rho_w)\Delta w + (|LV| + Z)\Delta v, \end{aligned}$$

with $Z \triangleq (|M| - M^m)|NV| \geq 0$. Further, defining $\tilde{A} \triangleq (TA - LC - NA_2)^m + |T|\overline{F}^\phi_x + |N|\overline{F}^\rho_x + |L|\overline{F}^\psi_x$, $\tilde{B} \triangleq \begin{bmatrix} (|TW - NB_2| + |N|\overline{F}^\rho_w) & (|LV| + Z) \end{bmatrix} \geq 0$, $\tilde{C} \triangleq I_{n\times n} \geq 0$, and $\tilde{D} \triangleq \mathbf{0}_{n\times n} \geq 0$, the CT comparison system with $z_t = \varepsilon_t$ is given by (3.11). Since $\tilde{A}$ is Metzler and $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are non-negative, the error comparison system is positive. Then, by (Briat, 2013, Lemma 1), the CT comparison system is asymptotically stable with L_1-gain, $\gamma > 0$, if there exists $p \in \mathbb{R}^n_{>0}$ such that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^\top \begin{bmatrix} p \\ \mathbf{1}_{n\times 1} \end{bmatrix} < \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \gamma \mathbf{1}_{(n_w+n_v)\times 1} \end{bmatrix}. \tag{3.13}$$

By defining Q, $\tilde{T}$, $\tilde{N}$, and $\tilde{L}$ similar to the DT case , $\tilde{M} \triangleq QM$, and $\tilde{Z} \triangleq QZ$, the inequality in 3.13 is equivalent to the inequality condition (3.8). The equality constraint can also be obtained as in the DT case. Finally, a similar argument to the DT case implies that the CT interval observer is L_1-robust and is also ISS. □

Note that in some cases, a coordinate transformation may help to make the MIP in (3.8) feasible (cf. Remark 4). Further, we found that it is often helpful to multiply Υ by a factor representing the ratio of the magnitudes of the measurement to process noise signals to penalize their effects equally.

Remark 2. *In the CT case (only), the presence of the term* $\tilde{Z} = (|\tilde{M}| - \tilde{M}^m)|NV|$ *leads to bilinear constraints, but the MIP remains solvable with off-the-shelf solvers, e.g., Gurobi (Gurobi Optimization, Inc. (2015)). Nonetheless, in the absence of measurement noise, i.e.,* $V = 0$*, or by choosing* $N = 0$ *(at the cost of losing the extra degrees of freedom with* T *and* N*), the MIP reduces to a mixed-integer linear program (MILP) similar to DT case.*

Remark 3. *We have a mixed-integer problem in* (3.8) *due to the presence of terms (Note that absolute values are internally converted into a mixed-integer formulation in off-the-shelf tools, e.g., YALMIP (Löfberg (2004)), where a binary variable is introduced to indicate if* $|x| = x$ or $|x| = -x$*.) involving absolute values* $|M|$ *and "Metzlerization"* $M^m = M^d + |M^{nd}|$*. If desired, extra positivity constraints can be imposed (i.e., by setting* $M \geq 0$, $M^{nd} \geq 0$ *and replacing* $|M|$, $|M^{nd}|$ *with* M, M^{nd}*), similar to the literature on SDP/LMI-based interval observer designs, to obtain a linear program. This addition is found to sometimes not incur any conservatism (e.g.,* *in the DT example in Section 5.1.2) but the problem becomes infeasible in others (e.g., in the CT example in Section 5.1.1.*

Chapter 4

HYBRID INTERVAL OBSERVER

This chapter considers the design of a hybrid interval observers, which was submitted to the 2023 IEEE Conference on Decision and Control.

4.1 Problem Formulation

Consider the hybrid plant with flow and jump dynamics characterized by mappings $f_c : \mathbb{R}^n \to \mathbb{R}^n$ and $f_d : \mathbb{R}^n \to \mathbb{R}^n$, respectively and corresponding output mappings $h_c : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $h_d : \mathbb{R}^n \to \mathbb{R}^{m_2}$, and an equivalent hybrid system composed of JSS decompositions (Proposition 1) of f_c, f_d, h_c, and, h_d given as:

$$\mathcal{H}\begin{cases} \dot{x} &= f_c(x) = A_c x + \phi_c(x), \quad x \in \mathcal{C}, \\ x^+ &= f_d(x) = A_d x + \phi_d(x), \quad x \in \mathcal{D}, \\ \\ y_c &= h_c(x) = H_c x + \psi_c(x), \quad x \in \mathcal{C}, \\ y_d &= h_d(x) = H_d x + \psi_d(x), \quad x \in \mathcal{D}, \end{cases} \tag{4.1}$$

with an *uncertain* initial state $x_{0,0}$ satisfying $x_{0,0} \in \mathcal{X}_0 \triangleq [\underline{x}_{0,0}, \overline{x}_{0,0}] \subset \mathcal{X}$, where $\underline{x}_{0,0}$ and $\overline{x}_{0,0}$ are known and the JSS mappings $\phi_c, \phi_d, \psi_c, \psi_d$ and matrices A_c, A_d, H_c, H_d are obtained via Proposition 1, while $x \in \mathbb{R}^n$ is the state and $y = (y_c, y_d)$ is the output with $y_c \in \mathbb{R}^{m_1}$ and $y_d \in \mathbb{R}^{m_2}$. The sets $\mathcal{C} \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n$ represent the flow set and the jump/guard set of the hybrid plant $\mathcal{H}$, respectively.

Additionally, a hybrid arc is the solution x of the hybrid plant $\mathcal{H}$ defined on a hybrid time domain denoted by $dom(x) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for any $(T, J) \in dom(x)$,

$\exists\ t_0 \leq t_1 \leq \cdots \leq t_J$ that satisfy:

$$dom(x) \cap ([0,T] \times \{0,1,...,J\}) = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j).$$

Further, $dom_t(x)$ and $_j(x)$ represent the projection of $dom(x)$ in its first and second dimension, respectively.

Additionally in this section, we describe the construction of the proposed hybrid interval observer as well as analyze its correctness and asymptotic stability properties. Note that in the rest of the paper, for brevity we drop the explicit dependence on the hybrid time (t,j) unless explicitly necessary.

4.2 Interval Observer Design

Inspired by our work on interval observers for discrete- and continuous-time systems in Pati *et al.* (2022), we propose the following hybrid interval observer $\hat{\mathcal{H}}$ for the hybrid system $\mathcal{H}$ in (4.1), which can be obtained by first finding its equivalent representation with a transformation that satisfies $T_c + N_c H_c = I_n$ and $T_d + N_d H_d = I_n$ similar to (Pati *et al.*, 2022, Lemma 1) and then constructing the corresponding embedding systems for the flow and jump dynamics (cf. Definition 4):

$$\begin{aligned}
\dot{\underline{\xi}} &= (M_c^d + M_c^{nd,\oplus})\underline{x} - M_c^{nd,\ominus}\ \overline{x} + L_c y_c + T_c^{\oplus}\phi_{c,\delta}(\underline{x},\overline{x}) - T_c^{\ominus}\phi_{c,\delta}(\overline{x},\underline{x}) \\
&\quad - L_c^{\oplus}\psi_{c,\delta}(\overline{x},\underline{x}) + L_c^{\ominus}\psi_{c,\delta}(\underline{x},\overline{x}) - N_c^{\oplus}\rho_{c,\delta}(\overline{x},\underline{x}) + N_c^{\ominus}\rho_{c,\delta}(\underline{x},\overline{x}) + M_c N_c y, \\
\dot{\overline{\xi}} &= (M_c^d + M_c^{nd,\oplus})\overline{x} - M_c^{nd,\ominus}\ \underline{x} + L_c y_c + T_c^{\oplus}\phi_{c,\delta}(\overline{x},\underline{x}) - T_c^{\ominus}\phi_{c,\delta}(\underline{x},\overline{x}) \\
&\quad - L_c^{\oplus}\psi_{c,\delta}(\underline{x},\overline{x}) + L_c^{\ominus}\psi_{c,\delta}(\overline{x},\underline{x}) - N_c^{\oplus}\rho_{c,\delta}(\underline{x},\overline{x}) + N_c^{\ominus}\rho_{c,\delta}(\overline{x},\underline{x}) + M_c N_c y, \\
\underline{x} &= \underline{\xi} + N_c y_c, \\
\overline{x} &= \overline{\xi} + N_c y_c,
\end{aligned}$$

$$
\begin{aligned}
\underline{\zeta}^+ &= M_d^{\oplus}\underline{x} - M_d^{\ominus}\,\overline{x} + L_d y_d + T_d^{\oplus}\phi_{d,\delta}(\underline{x},\overline{x}) - T_d^{\ominus}\phi_{d,\delta}(\overline{x},\underline{x}) \\
&\quad - L_d^{\oplus}\psi_{d,\delta}(\overline{x},\underline{x}) + L_d^{\ominus}\psi_{d,\delta}(\underline{x},\overline{x}) - N_d^{\oplus}\rho_{d,\delta}(\overline{x},\underline{x}) + N_d^{\ominus}\rho_{d,\delta}(\underline{x},\overline{x}) + M_d N_d y, \\
\overline{\zeta}^+ &= M_d^{\oplus}\overline{x} - M_d^{\ominus}\,\underline{x} + L_d y_d + T_d^{\oplus}\phi_{d,\delta}(\overline{x},\underline{x}) - T_d^{\ominus}\phi_{d,\delta}(\underline{x},\overline{x}) - L_d^{\oplus}\psi_{d,\delta}(\underline{x},\overline{x}) + L_d^{\ominus}\psi_{d,\delta}(\overline{x},\underline{x}) \\
&\quad - N_d^{\oplus}\rho_{d,\delta}(\underline{x},\overline{x}) + N_d^{\ominus}\rho_{d,\delta}(\overline{x},\underline{x}) + M_d N_d y, \\
\underline{\xi}^+ &= \underline{\zeta}^+ + N_d y_d - N_c y_c, \\
\overline{\xi}^+ &= \overline{\zeta}^+ + N_d y_d - N_c y_c,
\end{aligned}
$$

where $\overline{x}, \underline{x} \in \mathbb{R}^n$ are upper and lower framers of the state x, respectively, $\overline{\xi}, \underline{\xi} \in \mathbb{R}^n$ are continuous-time auxiliary framers and $\overline{\zeta}, \underline{\zeta} \in \mathbb{R}^n$ are discrete-time auxiliary framers, with $M_c \triangleq T_c A_c - L_c H_c - N_c A_{2c}$ and $M_d \triangleq T_d A_d - L_d H_d - N_d A_{2d}$. Additionally, the JSS mappings $\rho_c : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $\rho_d : \mathbb{R}^n \to \mathbb{R}^{m_2}$ are obtained from JSS decomposition (via Proposition 1 under the assumption of known Jacobian bounds) of $\dot{\psi}_c(x) = \frac{\partial \psi_c}{\partial x} f_c(x) = A_{2c}x + \rho_c(x)$ and $\psi_d(x^+) = \psi_d(f_d(x)) = A_{2d}x + \rho_d(x)$, with linear JSS mappings $A_{2c} \in \mathbb{R}^{m_1 \times n}$ and $A_{2d} \in \mathbb{R}^{m_2 \times n}$.

Further, $\phi_{c,\delta}, \phi_{d,\delta} : \mathbb{R}^{2n} \to \mathbb{R}^n$, $\psi_{c,\delta}, \rho_{c,\delta} : \mathbb{R}^{2n} \to \mathbb{R}^{m_1}$ and $\psi_{d,\delta}, \rho_{d,\delta} : \mathbb{R}^{2n} \to \mathbb{R}^{m_2}$ are tight mixed-monotone decomposition functions of ϕ_c, ϕ_d, ψ_c, ρ_c, ψ_d, and ρ_d, respectively (cf. (4.1), Definition 3), which are JSS and thus, can be computed using (2.6). Finally, $N_c, L_c \in \mathbb{R}^{n \times m_1}$, $N_d, L_d \in \mathbb{R}^{n \times m_2}$, and $T_c, T_d \in \mathbb{R}^{n \times n}$ are the observer gain matrices to be designed that satisfies $T_c + N_c H_c = T_d + N_d H_d = I_n$. The detailed properties of correctness and asymptotic stability of the observer design $\hat{\mathcal{H}}$ are proven in the next subsections.

Remark 4. *Note that the observer design presented in this section is inspired by Wang* et al. *(2012) and Degue* et al. *(2021) and involves six observer gains, N_c, T_c,*

L_c, N_d, T_d and L_d, which provide additional degrees of freedom when compared to existing hybrid observer designs, e.g., in Bernard and Sanfelice (2018), which often only have two degrees of freedom with gains L_c and L_d.

4.3 Hybrid Observer Correctness

In this section, we demonstrate that by construction, the hybrid interval estimator $\hat{\mathcal{H}}$ proposed in (4.2) for the hybrid plant $\mathcal{H}$ is *correct*, i.e., its framers bound the true states.

Theorem 3. *[Correctness] Suppose Assumptions 3-2.4 hold for the hybrid plant $\mathcal{H}$ considered in* (4.1) *and let $\hat{\mathcal{H}}$ be its corresponding hybrid interval observer built according to* (4.2). *Then, their respective solutions $x_{t,j}$ and $[\underline{x}_{t,j}^\top, \overline{x}_{t,j}^\top]^\top$ satisfy $\underline{x}_{t,j} \leq x_{t,j} \leq \overline{x}_{t,j}, \forall (t,j) \in dom(x)$, i.e., the hybrid interval observer $\hat{\mathcal{H}}$ functions as a correct interval framer for the hybrid plant $\mathcal{H}$.*

Proof. We start by considering the base case, $\underline{x}_{0,0} \leq x_{0,0} \leq \overline{x}_{0,0}$, which is trivially true because of the assumption on the initial condition $x_{0,0} \in \mathcal{X}_0 \triangleq [\underline{x}_{0,0}, \overline{x}_{0,0}] \subset \mathcal{X}$. Next, assuming that $\underline{x}_{t,j} \leq x_{t,j} \leq \overline{x}_{t,j}$ holds for some $(t,j) \in dom(x)$ with $t_j \leq t \leq t_{j+1}$, we will show that $\underline{x}_{t,j+1} \leq x_{t,j+1} \leq \overline{x}_{t,j+1}$ holds for $(t, j+1) \in dom(x)$ with $t_{j+1} \leq t \leq t_{j+2}$. By construction, the continuous-time embedding system during flow in (4.2) guarantees the framer properties by (Khajenejad and Yong, 2021, Proposition 3), i.e., $\underline{x}_{t',j} \leq x_{t',j} \leq \overline{x}_{t',j}$, $\forall t \leq t' \leq t_{j+1}$. Then, by the construction of the discrete-time embedding system during jumps in (4.2) and (Khajenejad and Yong, 2021, Proposition 3), we have $\underline{x}_{t_{j+1},j+1} \leq x_{t_{j+1},j+1} \leq \overline{x}_{t_{j+1},j+1}$, $(t_{j+1}, j+1) \in dom(x)$. Finally, by (Khajenejad and Yong, 2021, Proposition 3) again for the flow, we obtain $\underline{x}_{t,j+1} \leq x_{t,j+1} \leq \overline{x}_{t,j+1}$ for $(t, j+1) \in dom(x)$ with $t_{j+1} \leq t \leq t_{j+2}$. Thus, by the principle of mathematical induction, the theorem holds. □

In addition to proving correctness, it is essential to ensure the stability of the proposed hybrid framer. Thus, we propose two variants for designing the observer gains T_c, T_d, N_c, N_d, L_c, and L_d to asymptotically stabilize (cf. (3.2)) the error dynamics of the hybrid interval observer.

Q-Hybrid Interval Observer

First, we outline the first variant that is based on the use of a *quadratic* common Lyapunov function to prove asymptotic stability.

Theorem 4. *[Q-Hybrid Interval Observer] If Assumptions 3-2.4 hold for the hybrid plant* $\mathcal{H}$ *in* (4.1)*, then the hybrid interval observer* $\hat{\mathcal{H}}$ *in* (4.2) *is asymptotically stable if there exist* $a_c, a_d \in \mathbb{R}$*,* $\tilde{T}_c, \tilde{T}_d \in \mathbb{R}^{n \times n}$*,* $\tilde{N}_c, \tilde{L}_c \in \mathbb{R}^{n \times m_1}$*,*$\tilde{N}_d, \tilde{L}_d \in \mathbb{R}^{n \times m_2}$ *and a diagonal matrix* $P \succ 0$ *such that:*

$$\Gamma^T + \Gamma \preceq a_c P, \tag{4.2a}$$

$$\begin{bmatrix} P & \Omega \\ \Omega^T & e^{a_d} P \end{bmatrix} \succeq 0, \tag{4.2b}$$

$$a_c \tau + a_d < 0, \quad \forall \tau \in \mathcal{I}, \tag{4.2c}$$

$$\tilde{T}_c + \tilde{N}_c H_c = P, \tag{4.2d}$$

$$\tilde{T}_d + \tilde{N}_d H_d = P, \tag{4.2e}$$

where $\Gamma \triangleq (\tilde{T}_c A_c - \tilde{L}_c H_c - \tilde{N}_c A_{2c})^m + |\tilde{T}_c| \overline{F}_{\phi_c} + |\tilde{L}_c| \overline{F}_{\psi_c} + |\tilde{N}_c| \overline{F}_{\rho_c}$*,* $\Omega \triangleq |\tilde{T}_d A_d - \tilde{L}_d H_d - \tilde{N}_d A_{2d}| + |\tilde{T}_d| \overline{F}_{\phi_d} + |\tilde{L}_d| \overline{F}_{\psi_d} + |\tilde{N}_d| \overline{F}_{\rho_d}$*, and* $\mathcal{I}$ *satisfies Assumption 5. Further,* $\overline{F}_\xi$*,* $\forall \xi \in \Xi \triangleq \{\phi_c, \psi_c, \rho_c, \phi_d, \psi_d, \rho_d\}$ *are computed from the JSS functions* $\xi \in \Xi$ *with Jacobian matrices* $J^\xi \in [\underline{J}^\xi, \overline{J}^\xi]$ *as follows:* $\overline{F}_\xi \triangleq (\overline{J}^\xi)^\oplus + (\underline{J}^\xi)^\ominus$*.*

Then, the observer gains in (4.2) *can be obtained as* $T_c \triangleq P^{-1}\tilde{T}_c$*,* $N_c \triangleq P^{-1}\tilde{N}_c$*,* $L_c \triangleq P^{-1}\tilde{L}_c$*,* $T_d \triangleq P^{-1}\tilde{T}_d$*,* $N_d \triangleq P^{-1}\tilde{N}_d$ *and* $L_d \triangleq P^{-1}\tilde{L}_d$*.*

Proof. We begin by constructing the hybrid framer error dynamics from (4.2), before proving that the feasibility of (4.2a)-(4.2e) implies that the hybrid error dynamics are asymptotically stable. From (4.2), the framer error is $\varepsilon \triangleq \overline{x} - \underline{x} = \overline{\xi} - \underline{\xi} = \overline{\zeta} - \underline{\zeta}$ and the hybrid framer error dynamics of $\tilde{\mathcal{H}}$ can be expressed as

$$\tilde{H}\begin{cases} \dot{\varepsilon} \ = (T_cA_c - L_cH_c - N_cA_{2c})^m\varepsilon_t \\ \quad +|T_c|\Delta_d^{\phi_c} + |N_c|\Delta_d^{\rho_c} + |L_c|\Delta_d^{\psi_c} \ \ x \in C, \\ \quad \le E_c\varepsilon \\ \varepsilon^+ \ = |T_dA_d - L_dH_d - N_dA_{2d}|\varepsilon \\ \quad +|T_d|\Delta_d^{\phi_d} + |N_d|\Delta_d^{\rho_d} + |L_d|\Delta_d^{\psi_d} \ \ x \in D, \\ \quad \le E_d\varepsilon \end{cases} \tag{4.3}$$

where we define $\Delta_\delta^\xi \triangleq \xi_\delta(\overline{x}, \underline{x}) - \xi_\delta(\underline{x}, \overline{x})$ for all $\xi \in \Xi$ that satisfy $\Delta_\delta^\xi \le \overline{F}_\xi$ by (Khajenejad *et al.*, 2022, Lemma 3), as well as $E_c \triangleq (T_cA_c - L_cH_c - N_cA_{2c})^m + |T_c|\overline{F}_{\phi_c} + |L_c|\overline{F}_{\psi_c} + |N_c|\overline{F}_{\rho_c}$ and $E_d \triangleq |T_dA_d - L_dH_d - N_dA_{2d}| + |T_d|\overline{F}_{\phi_d} + |L_d|\overline{F}_{\psi_d} + |N_d|\overline{F}_{\rho_d}$.

Consequently, by applying (Bernard and Sanfelice, 2018, Theorem 3.1) (that uses a *quadratic* common Lyapunov function) to the linear comparison hybrid system in (4.3), the error dynamics is global asymptotically stable if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and scalars a_d and a_c such that:

$$E_c^T P + PE_c \preceq a_c P, \tag{4.4a}$$

$$E_d^T PE_d \preceq e^{a_d} P, \tag{4.4b}$$

$$a_c\tau + a_d < 0, \quad \forall\tau \in \mathcal{I}. \tag{4.4c}$$

Moreover, since P is diagonal (by assumption), then it can be trivially shown that $PM^m = (PM)^m$ and $P|M| = |PM|$ for any matrix M. Given this, as well as defining

Γ and Ω as in Theorem 4, (4.4a) can be written as in (4.2a), while (4.4b) can be represented as

$$\Omega^T P^{-1} \Omega \leq e^{a_d} P. \tag{4.5}$$

Then, by applying the Schur complement, (4.5) is equivalent to (4.2b). The conditions in (4.2d) and (4.2e) can be recovered by left multiplying the two linear transformations $T_c + N_c H_c = T_d + N_d H_d = I_n$ introduced in the hybrid observer design in (4.2) with the matrix P and defining $\tilde{M} \triangleq PM$, $\forall M \in \{T_c, T_c, N_c, N_d\}$. Hence, the feasibility of the constraints (4.2a)-(4.2e) prove the asymptotic stability of the error comparison linear hybrid system in (4.3). Consequently, the error hybrid dynamics in (4.3) and the hybrid interval observer in (4.2) are asymptotically stable according to Comparison Lemma (Khalil, 2002, Lemma 3.4). □

L-Hybrid Interval Observer

Next, we introduce the second variant that is based on the use of a *linear* common Lyapunov function by leveraging the fact that our hybrid framer error dynamics are cooperative/positive by design.

Theorem 5. *[L-Hybrid Interval Observer] If Assumptions 3-2.4 hold for the hybrid plant $\mathcal{H}$ in* (4.1)*, then the hybrid interval observer $\hat{\mathcal{H}}$ in* (4.2) *is asymptotically stable if there exist $a_c, a_d \in \mathbb{R}$, $\tilde{T}_c, \tilde{T}_d \in \mathbb{R}^{n \times n}$, $\tilde{N}_c, \tilde{L}_c \in \mathbb{R}^{n \times m_1}$, $\tilde{N}_d, \tilde{L}_d \in \mathbb{R}^{n \times m_2}$ and a* *positive vector* $z \in \mathbb{R}^n_{>0}$ *such that:*

$$\Gamma^\top \mathbf{1}_{n\times 1} \le a_c z, \tag{4.6a}$$

$$\Omega^\top \mathbf{1}_{n\times 1} \le e^{a_d} z, \tag{4.6b}$$

$$a_c\tau + a_d < 0, \quad \forall \tau \in \mathcal{I} \tag{4.6c}$$

$$\tilde{T}_c + \tilde{N}_c H_c = P, \tag{4.6d}$$

$$\tilde{T}_d + \tilde{N}_d H_d = P, \tag{4.6e}$$

where $P \triangleq diag(z)$ *(a diagonal matrix with* z *as its diagonal elements),* $\Gamma \triangleq (\tilde{T}_c A_c - \tilde{L}_c H_c - \tilde{N}_c A_{2c})^m + |\tilde{T}_c|\overline{F}_{\phi_c} + |\tilde{L}_c|\overline{F}_{\psi_c} + |\tilde{N}_c|\overline{F}_{\rho_c}$, $\Omega \triangleq |\tilde{T}_d A_d - \tilde{L}_d H_d - \tilde{N}_d A_{2d}| + |\tilde{T}_d|\overline{F}_{\phi_d} + |\tilde{L}_d|\overline{F}_{\psi_d} + |\tilde{N}_d|\overline{F}_{\rho_d}$, $\mathcal{I}$ *satisfies Assumption 5 and* $\overline{F}_\xi$, $\forall \xi \in \Xi \triangleq \{\phi_c, \psi_c, \rho_c, \phi_d, \psi_d, \rho_d\}$ *are computed as described in Theorem 4.*

Then, the observer gains in (4.2) *can be obtained as* $T_c \triangleq P^{-1}\tilde{T}_c$, $N_c \triangleq P^{-1}\tilde{N}_c$, $L_c \triangleq P^{-1}\tilde{L}_c$, $T_d \triangleq P^{-1}\tilde{T}_d$, $N_d \triangleq P^{-1}\tilde{N}_d$ *and* $L_d \triangleq P^{-1}\tilde{L}_d$.

Proof. The proof follows similar steps as the proof of Theorem 4. Since the flow dynamics and jump dynamics of our proposed hybrid interval observer in (4.2) are correct by construction according to Theorem 3, the framer error hybrid dynamics in (4.3) are also cooperative/positive by construction. Hence, by Propositions (Rantzer, 2011, Proposition 1 and 2), a *linear* common Lyapunov function $V(\varepsilon) = z^\top \varepsilon$ can be considered. Consequently, by applying (Goebel *et al.*, 2012, Proposition 3.29) to the linear comparison hybrid system in (4.3), the error dynamics is globally asymptotically stable if there exists a vector $z > 0$, and scalars a_d and a_c such that:

$$z^\top E_c \le a_c z^\top, \tag{4.7a}$$

$$z^\top E_d \le e^{a_d} z^\top, \tag{4.7b}$$

$$a_c\tau + a_d < 0, \quad \forall \tau \in \mathcal{I}. \tag{4.7c}$$

Moreover, defining $P \triangleq diag(z)$ and therefore, $z = P\mathbf{1}_{n\times 1}$, it can be trivially shown that for any matrix M, we have $M^m P = (MP)^m$ and $|M|P = |MP|$, and defining Γ and Ω as in Theorem 5, inequalities (4.6a) and (4.6b) can be obtained from (4.7a) and (4.7b), respectively. Further, similar to the proof of Theorem 4, the constraints (4.6d) and (4.6e) are consequences of left multiplying the linear transformations $T_c + N_c H_c = T_d + N_d H_d = I_n$ with the matrix P and defining $\tilde{M} \triangleq PM$, $\forall M \in \{T_c, T_c, N_c, N_d\}$. Hence, the linear hybrid system in (4.3) and by Comparison Lemma (Khalil, 2002, Lemma 3.4), the hybrid error dynamics in (4.3) and the hybrid interval observer in (4.2) are asymptotically stable. $\square$

Remark 5. *Note that the presence of absolute value terms* $|M|$ *and "Metzlerization"* $M^m = M^d + |M^{nd}|$ *results in mixed-integer optimization problems in Theorems 4 and 5. Additionally, due to the presence of the term* e^{a_d} *in inequalities* (4.6b) *and* (4.2b), *the optimizations in Theorems 4 and 5 are non-trivial. However, by (line) searching over* a_c *and* a_d *that satisfy* $a_c\tau + a_d < 0, \forall \tau \in \mathcal{I}$, *the optimization problems in Theorems 4 and 5 can be simplified to mixed-integer semidefinite programs (MISDP) and mixed-integer linear programs (MILP), respectively, which can be solved using off-the-shelf tools. If desired, extra positivity constraints can be imposed (i.e., by setting* $M \geq 0$, $M^{nd} \geq 0$ *and replacing* $|M|$, $|M^{nd}|$ *with* M, M^{nd}*), then the MISDP in Theorem* (4) *and the MILP in Theorem* (5) *can be further simplified to semidefinite programs (SDP) and linear programs (LP), respectively, that are often more computationally amenable.*

Chapter 5

SIMULATIONS

5.1 Examples for L_1-Robust Interval Observer Design

In this section, we consider both CT and DT examples to demonstrate the effectiveness of our approaches. The MILPs in (3.8) are solved using YALMIP (Löfberg (2004)) and Gurobi (Gurobi Optimization, Inc. (2015)).

5.1.1 CT System Example

Let us consider the CT system in (Dinh *et al.*, 2014, Eq. (30)):

$$\dot{x}_1 = x_2 + w_1, \quad \dot{x}_2 = b_1 x_3 - a_1 \sin(x_1) - a_2 x_2 + w_2,$$
$$\dot{x}_3 = -a_3(a_2 x_1 + x_2) + \frac{a_1}{b_1}(a_4 \sin(x_1) + \cos(x_1) x_2) - a_4 x_3 + w_3,$$

with output $y = x_1$ (i.e., $V = 0$ and we have an MILP formulation (cf. Remark 3)) and the following parameters: $a_1 = 35.63, b_1 = 15, a_2 = 0.25, a_3 = 36, a_4 = 200, \mathcal{X}_0 = [19.5, 9] \times [9, 11] \times [0.5, 1.5], \mathcal{W} = [-0.1, 0.1]^3$.

The problem in (3.8), even with the additional degrees of freedom, as well as the observer designs in Khajenejad and Yong (2022); Dinh *et al.* (2014) are infeasible without a state transformation (also cf. Remark 4). Hence, similar to (Khajenejad and Yong, 2022, Section V-A), we consider a similarity transformation $z = Sx$ with $S = \begin{bmatrix} 20 & 0.1 & 0.1 \\ 0 & 0.01 & 0.06 \\ 0 & -10 & -0.4 \end{bmatrix}$, and added and subtracted $5y$ from the dynamics of $\dot{x}_1$. Further, adding positivity constraints to cast (3.8) as an LP led to infeasibility; thus, the MILP formulation is less conservative (cf. Remark 3).

By solving (3.8), we obtain the following observer gains:

$T = \begin{bmatrix} -1.569 & 4.138 & -0.021 \\ 0.016 & 0.973 & 0.0001 \\ 50.200 & -80.860 & 1.421 \end{bmatrix}$, $N = \begin{bmatrix} 51.386 \\ -0.330 \\ -1004.014 \end{bmatrix}$, $L = \begin{bmatrix} 7869.338 \\ 0.186 \\ 0 \end{bmatrix}$,

$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & 0.25 & 15 \\ 0 & 36 & 200 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A_2 = \emptyset$, and $B_2 = \emptyset$. From Figure 5.1 (only results for x_3 is depicted for brevity; other states show the same trends), we observe that the state framers obtained by our proposed approach, $\underline{x}, \overline{x}$, are tighter than those obtained by both the interval observers in Khajenejad and Yong (2022), $\underline{x}^{\mathcal{H}_\infty}, \overline{x}^{\mathcal{H}_\infty}$, and in Dinh *et al.* (2014), $\underline{x}^{DMN}, \overline{x}^{DMN}$. Moreover, the framer error $\varepsilon_t = \overline{x}_t - \underline{x}_t$ is observed to converge exponentially to a steady-state value.

As shown in Figure 5.1, state, x_3, and its upper and lower framers and error of our proposed observer, $\overline{x}_3, \underline{x}_3, \varepsilon_3$, and by the observers in Khajenejad and Yong (2022), $\overline{x}_3^{\mathcal{H}_{infty}}, \underline{x}_3^{\mathcal{H}_{infty}}, \varepsilon_3^{\mathcal{H}_{infty}}$, and Dinh *et al.* (2014), $\overline{x}_3^{DMN}, \underline{x}_3^{DMN}, \varepsilon_3^{DMN}$.

5.1.2 DT System Example

Next, we consider the noisy DT Hénon chaos system (Khajenejad and Yong, 2022, Section V-B): $x_{t+1} = Ax_t + r[1 - x_{t,1}^2] + Bw_t,\ y_t = x_{t,1} + v_t$, where $A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}$, $B = I$, $r = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}$, $\mathcal{X}_0 = [-2, 2] \times [-1, 1]$, $\mathcal{W} = [-0.01, 0.01]^2$ and $\mathcal{V} = [-0.025, 0.025]$. Note that in this example, the problem in (3.8) is feasible without any coordinate transformation and further, additional positivity constraints can be added to cast (3.8) as an LP without any performance loss (cf. Remark 3). By solving (3.8), we obtain the following observer gains: $T = \begin{bmatrix} 0.5448 & 0 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 0.4552 \\ 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

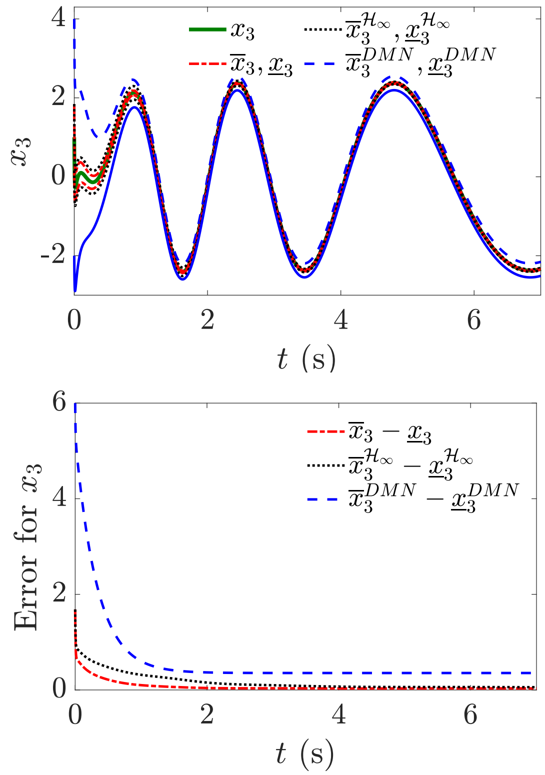

Figure 5.1: State, x_3, as well as its upper and lower framers and error returned by our proposed observer, $\overline{x}_3, \underline{x}_3, \varepsilon_3$, and by the observer in Dinh *et al.* (2014), $\overline{x}_3^{DMN}, \underline{x}_3^{DMN}, \varepsilon_3^{DMN}$ for the CT System example.

$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $A_2 = \emptyset$, and $B_2 = \emptyset$. From Figure 5.4 (x_2 omitted for brevity), the estimates from our proposed approach are tighter than the methods in Tahir and Açıkmeşe (2021) and Khajenejad and Yong (2022) that instead minimizes the $\mathcal{H}_\infty$-gain.

As shown in the Figure 5.2, state, x_1, and its upper and lower framers and error of our proposed observer, $\overline{x}_1, \underline{x}_1, \varepsilon_1$, and by the observers in Khajenejad and Yong (2022), $\overline{x}_1^{\mathcal{H}_\infty}, \underline{x}_1^{\mathcal{H}_\infty}, \varepsilon_1^{\mathcal{H}_\infty}$, and Tahir and Açıkmeşe (2021), $\overline{x}_1^{TA}, \underline{x}_1^{TA}, \varepsilon_1^{TA}$.

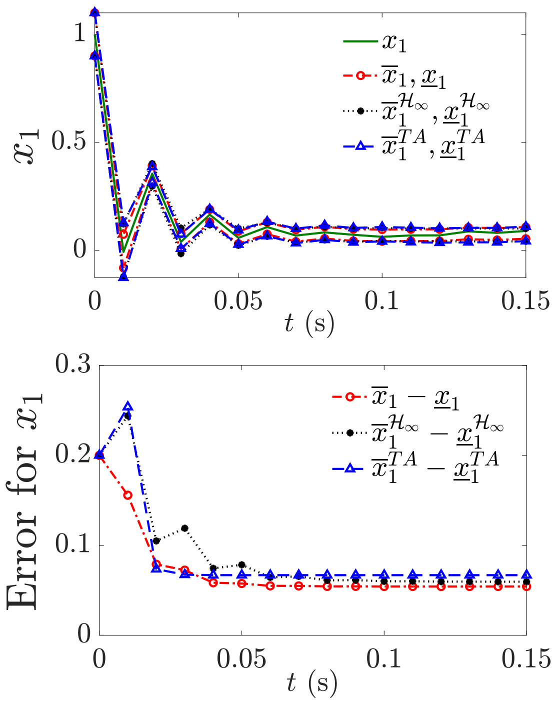

Figure 5.2: State, x_2, and its Upper and Lower Framers, Returned by our Proposed Observer, $\overline{x}_2, \underline{x}_2$, and by the Observer in Tahir and Açıkmeşe (2021), $\overline{x}_2^{TA}, \underline{x}_2^{TA}$ (left) and Norm of Framer Error (right) for the DT System Example.

5.2 Examples for Hybrid Interval Observer Designs

5.2.1 Bouncing Ball

Consider a bouncing ball with gravity coefficient $g > 0$, restitution coefficient $\lambda > 0$ and (nonlinear) drag coefficient $\beta > 0$, modeled as system (4.1) with

$$f_c(x) = \begin{bmatrix} x_2 \\ -g - \beta x_2 |x_2| \end{bmatrix}, f_d(x) = \begin{bmatrix} x_1 \\ -\lambda x_2 \end{bmatrix},$$

$$\mathcal{C} = \mathbb{R}_{\geq 0} \times \mathbb{R}, \mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\},$$

with output $h_c(x) = h_d = x_1$, where state x_1 represents the position (above ground) and x_2 represents the velocity. Next, we consider two cases where the minimum dwell time τ_m is zero ($\lambda < 1$) or non-zero (sufficiently large λ).

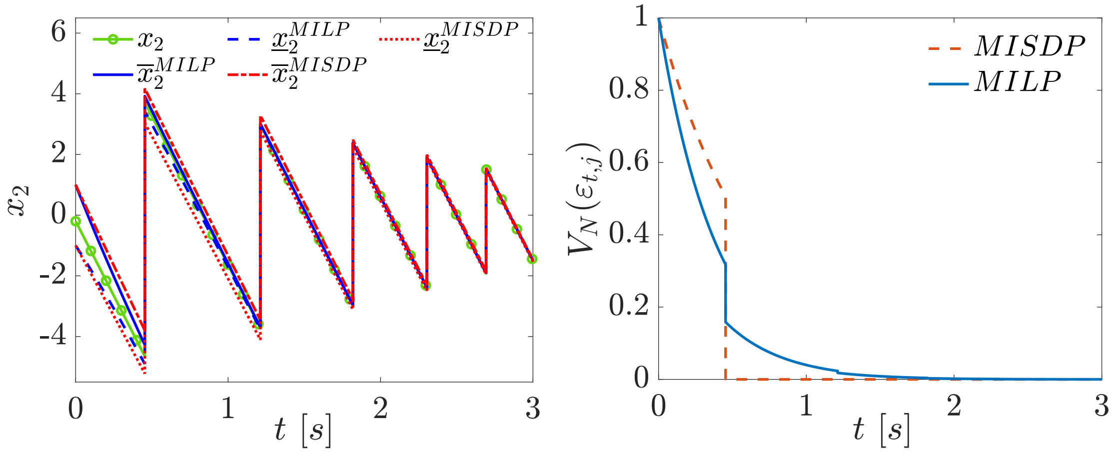

Figure 5.3: Linear Bouncing Ball with $\tau_m = 0$: State x_2, its Lower ($\underline{x}_2$) and Upper ($\overline{x}_2$) Framers, and the Normalized Lyapunov Function ($V_N(\varepsilon_{t,j}) = V(\varepsilon_{t,j})/V(\varepsilon_{0,0})$) of Both Hybrid Observer Variants.

Zero minimum dwell time

First, we consider the case of the linear ($\beta = 0$) bouncing ball problem with $\lambda = 0.8$, i.e., the overall the system losses energy and exhibits Zeno behavior. Hence, for this case, we have minimum dwell time $\tau_m = 0$. In this scenario, from Fig. 5.3, it can be observed that the $MILP$ approach (using Theorem 5) estimates the unmeasured velocity x_2 faster than the $MISDP$ approach (using Theorem 4). Moreover, from the analysis of normalized Lyapunov function values (normalized by the initial value) from Fig. 5.3, both the observers have comparable performance in asymptotically stabilizing the framer errors.

From the Figure 5.3, for the Linear Bouncing Ball (with $\tau_m = 0$): the state x_2, its lower ($\underline{x}_2$) and upper ($\overline{x}_2$) framers, and the normalized Lyapunov function ($V_N(\varepsilon_{t,j}) = V(\varepsilon_{t,j})/V(\varepsilon_{0,0})$) of both hybrid observer variants.

Non-zero minimum dwell time

Next, we consider the case of a nonlinear bouncing ball problem with $\beta = 0.02$, i.e., the system losses energy during flow due to nonlinear drag forces, but due to a

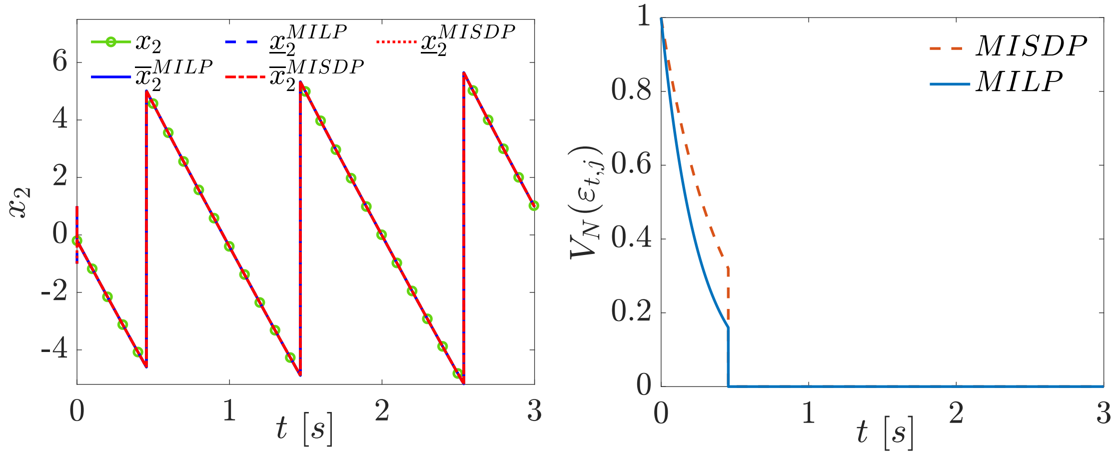

Figure 5.4: Nonlinear Bouncing Ball with $\tau_m \neq 0$: State x_2, its Lower ($\underline{x}_2$) and Upper ($\overline{x}_2$) Framers and the Normalized Lyapunov Function ($V_N(\varepsilon_{t,j}) = V(\varepsilon_{t,j})/V(\varepsilon_{0,0})$) of Both Hybrid Observer Variants.

sufficiently large coefficient of restitution $\lambda = 1.09$ (such a system can be realized in practice by considering an actuated table), the system gains more energy than it loses during flows. Hence, the system has a non-zero minimum dwell time τ_m (no Zeno behavior). From Fig. 5.4, the velocity x_2 framers of both the MILP and MISDP approaches (Using Theorems 4 and 5, respectively) converge very fast to the true values. Moreover, from Fig. 5.4, it is evident that both variants have comparable performances in the sense of their normalized Lyapunov function values.

From the Figure 5.4, for the Nonlinear Bouncing Ball (with $\tau_m \neq 0$): State x_2, its lower ($\underline{x}_2$) and upper ($\overline{x}_2$) framers and the normalized Lyapunov function ($V_N(\varepsilon_{t,j}) = V(\varepsilon_{t,j})/V(\varepsilon_{0,0})$) of both hybrid observer variants.

Chapter 6

CONCLUSION

This thesis proposed two interval observer designs, the first was developed uncertain locally Lipschitz CT and DT systems with nonlinear noisy observations and the second is designed for hybrid systems with known jump times and nonlinear dynamics and observations. In the former case, the proposed L_1 observer is input-to-state stable (ISS) and minimizes the L_1-gain of the observer error system. Unlike most existing interval observers, the design involves mixed-integer (linear) programs instead of semi-definite programs with linear matrix inequalities and offers additional degrees of freedom that can be simultaneously designed. On the other hand, the proposed observer for hybrid systems has cooperative flow dynamics and positive jump dynamics, and it is correct by construction without any additional positivity constraints. The proposed observer designs involve solving mixed-integer semidefinite programs or mixed-integer linear programs to compute the observer gains, including the additional degrees of freedom from a system transformation. Both designs were demonstrated to be effective in several simulation examples.

6.1 Future Work

In our future work, we will extend our proposed framework to consider noisy or uncertain hybrid systems with unobserved discrete modes and unknown jump times. Additionally, it would be interesting to investigate the performance of the proposed observer design on more complex hybrid systems and to compare it with other observer designs.

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