Some Questions on Uniqueness and the Preservation of Structure for the Ricci Flow

by

Mary Cook

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved June 2021 by the
Graduate Supervisory Committee:

Brett Kotschwar, Chair
Susanna Fishel
Steven Kaliszewski
Matthias Kawski
Julien Paupert

ARIZONA STATE UNIVERSITY

August 2021

ABSTRACT

This thesis explores several questions concerning the preservation of geometric structure under the Ricci flow, an evolution equation for Riemannian metrics. Within the class of complete solutions with bounded curvature, short-time existence and uniqueness of solutions guarantee that symmetries and many other geometric features are preserved along the flow. However, much less is known about the analytic and geometric properties of solutions of potentially unbounded curvature. The first part of this thesis contains a proof that the full holonomy group is preserved, up to isomorphism, forward and backward in time. The argument reduces the problem to the preservation of reduced holonomy via an analysis of the equation satisfied by parallel translation around a loop with respect to the evolving metric. The subsequent chapter examines solutions satisfying a certain instantaneous, but nonuniform, curvature bound, and shows that when such solutions split as a product initially, they will continue to split for all time. This problem is encoded as one of uniqueness for an auxiliary system constructed from a family of time-dependent, orthogonal distributions of the tangent bundle. The final section presents some details of an ongoing project concerning the uniqueness of asymptotically product gradient shrinking Ricci solitons, including the construction of a certain system of mixed differential inequalities which measures the extent to which such a soliton fails to split.

ACKNOWLEDGMENTS

First, I would like to thank my advisor, Professor Brett Kotschwar, for his help and guidance during my Ph.D. study. His kindness, knowledge, and advice have been invaluable to me, and I would not be where I am today without his mentorship.

I would also like to thank the rest of my supervisory committee, Professors Susanna Fishel, Matthias Kawski, Steven Kaliszewski, and Julien Paupert, for their support and encouragement throughout my time at Arizona State.

Finally, I would like to thank my family: in particular, my parents Anne and Gary, my sister Ellen, and my husband Danny. I am forever grateful for your love and support.

TABLE OF CONTENTS

CHAPTER Page

LIST OF FIGURES

Figure Page

Chapter 1

INTRODUCTION

The primary subject of this thesis is the Ricci flow, the equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Rc}(g(t)) \tag{1.1}$$

for families of Riemannian manifolds $(M, g(t))$, which was introduced by Richard Hamilton [32] in 1982. The study of the Ricci flow belongs to what might be called the "heat flow" method in geometry, a program which has its origins in the work of Eells-Sampson [23] on harmonic maps. The idea is to take a reasonably generic initial metric and deform it into something which is more uniform or canonical–in the best case, perhaps an Einstein or constant curvature manifold.

The Ricci flow has proven to be a powerful tool for solving problems in geometry and topology. In his original paper, Hamilton showed that on any compact three-manifold with positive Ricci curvature, the volume-normalized Ricci flow exists for all time and converges to a metric of constant positive curvature. Thus the manifold itself is a quotient of S^3. Subsequent work of Hamilton and others laid the groundwork for a program of Ricci flow with surgery in dimension three (e.g., [37, 38, 36, 35]). A version of this program was completed by Perelman [63, 64, 65], culminating in his landmark proof of Thurston's Geometrization Conjecture (which subsumes the Poincaré conjecture). Since Perelman's work, the Ricci flow has featured in the proofs of a variety of other notable results, including Böhm and Wilking's proof that compact manifolds with 2-positive curvature operator are space forms [9], Brendle and Schoen's proof of the Differentiable Sphere Theorem [10], and Bamler and Kleiner's proof of the Generalized Smale Conjecture [5].

The results in this thesis belong to two categories of problems having to do with fundamental analytic properties of the Ricci flow. The first concerns questions of uniqueness and the preservation of structures under the Ricci flow, while the second pertains to the classification of complete noncompact shrinking Ricci solitons according to their asymptotic behavior.

1.1 The Preservation of Structure Under the Ricci Flow

Much of the utility of the Ricci flow rests on the fact that it is fundamentally a geometric equation–that it preserves (in some generality) any symmetry or otherwise special structure the initial metric might have. It is important to remember that this preservation of structure is not an automatic feature of the equation itself, but a consequence of various statements of uniqueness within classes of solutions whose behavior is not overly pathological. For example, in the class of complete solutions with bounded curvature (the traditional setting for the study of the Ricci flow), one has both the short-time existence and uniqueness of solutions ([15],[22],[32], [68], see Theorem 3.2.1). The uniqueness of solutions immediately implies that symmetries are preserved in the class of complete solutions with bounded curvature, since if ϕ is an isometry of g_0, then $\phi^*g(t)$ and $g(t)$ are both solutions to (1.1) with the same initial condition (see Section 3.2.1).

Similarly, short-time existence and uniqueness together imply the preservation of other structural features under the flow. Consider, for example, a solution $g(t)$ on a product manifold $M = \hat{M} \times \check{M}$, which splits initially as $g(0) = \hat{g}_0 \oplus \check{g}_0$. Then, the short-time existence of solutions implies we can construct solutions $\hat{g}(t)$ and $\check{g}(t)$ on the factors $\hat{M}$ and $\check{M}$ respectively. Their product $\hat{g}(t) \oplus \check{g}(t)$ is a solution on all of M which is complete and of bounded curvature. By uniqueness, $\hat{g}(t) \oplus \check{g}(t)$ and $g(t)$ must coincide for all time, so the solution $g(t)$ continues to split as a product (see

Section 3.3).

However, outside of the class of complete solutions with bounded curvature, the basic analytic properties of the Ricci flow are still not yet well-understood. It remains an important question how much these assumptions on the class of solutions can be weakened while still ensuring uniqueness of solutions or the preservation of certain geometric structures under the flow. While completeness is easily seen to be necessary to ensure the preservation of symmetry and other structural features, it is less clear what, if any, other global restrictions are needed. For example, if $(M, g(t))$ is a smooth complete solution to (1.1) for $t \in [0, T)$, the following questions are still unknown in general:

- If $g(0)$ splits as a product, does $g(t)$ continue to split as a product for all $t \in [0, T)$?
- If $g(0)$ is Kähler, is $g(t)$ Kähler for all $t \in [0, T)$?
- Does the isometry group of $g(t)$ remain fixed for all $t \in [0, T)$?

A common thread tying these types of questions together is their relationship to the uniqueness of solutions to the Ricci flow. Beyond these applications, a stronger theory of well-posedness for the Ricci flow on general complete manifolds would have applications to Ricci flow with surgery and the analysis of singularity models. There are by now many natural constructions of solutions to (4.2) (see, e.g., [11], [39], [70]) which have unbounded curvature on each time-slice or whose curvature is instantaneously, but not uniformly, bounded. In this setting, solutions can exhibit behavior which is unexpected or counter-intuitive, such as curvature which becomes unbounded and then bounded again [29]. A characterization of the conditions under which solutions to the Ricci flow must be unique would help to clarify the above questions of preservation of structure.

There are some results in this direction, notably in the work of Giesen and Topping ([28], [27], [73]), which has established an essentially complete theory of well-posedness for Ricci flow in dimension two. Their work shows that *any* surface, even one which is incomplete or has unbounded curvature, admits a unique instantaneously complete short-time solution to the Ricci flow. However, the proof is crucially dependent on the conformal nature of the equation in dimension two. In [55], Lee proves that solutions are unique in the class of metrics bounded above by ϵ/t where ϵ is a constant depending only on the dimension (see Section 3.2.2). In general, it is unknown to what extent one can weaken the assumption that curvature is uniformly bounded and still guarantee uniqueness.

1.2 Preservation of Holonomy

The Riemannian holonomy is an algebraic invariant of a manifold (M, g) which measures, roughly, the extent to which parallel translation relative to the Levi-Civita connection of g deviates from that of Euclidean space. It is an old observation of Hamilton ([33], [37]) that, in the class of complete solutions with bounded curvature, the reduced holonomy group $\mathrm{Hol}^0(g(t))$ cannot expand. This can be proven using the existing theory in a variety of ways. The simplest, perhaps, is to use Berger's classification and apply an argument similar to the one sketched above for product structures. It was later proven in [45] that the reduced holonomy also cannot contract. Here, however, the same type of argument does not apply to every case: since the terminal value problem for the Ricci flow is ill-posed, one cannot construct the "competitor" product or Kähler solutions one needs in order to appeal to the backward uniqueness of the Ricci flow. Instead, the problem is framed as one of backward uniqueness of the solutions to a related prolonged system.

Left open by the previous work (at least in full generality) is the question of whether the full holonomy is preserved under the flow. In Theorem 4.1.1 of Chapter 4, we show that the matter of the preservation of the full holonomy can be reduced to the preservation of the reduced holonomy. In particular, this shows that the full holonomy is preserved for complete solutions of bounded curvature. As a consequence, one sees that if a complete solution to (4.2) with bounded curvature is Kähler or splits locally as a product on any time-slice, it does so on any time-slice. The proof we give is symmetric in time and is based on an explicit analysis of the equation satisfied by the parallel transport of a vector around a fixed loop with respect to the evolving metric.

1.2.1 The Preservation of Holonomy for Solutions of Potentially Unbounded Curvature

Theorem 4.1.1 raises the question of what global hypotheses on the class of solutions are needed to ensure preservation of holonomy. When the initial metric is not complete, it is easy to cook up examples where the reduced holonomy is not preserved (see, e.g., "flat-sided sphere" in Section 3.2.2, a two-dimensional, simply-connected solution which has trivial holonomy initially and holonomy SO(2) for all positive time). It is less clear, however, the role that bounded curvature plays, and whether that condition can be relaxed. One important test case is the question of the preservation of product structures under the flow when the solution no longer is assumed to have a uniform curvature bound.

Given the variety of recent constructions of solutions (e.g., [39], [54], [70]) with instantaneous, but not uniform curvature bounds, it is natural to ask whether a complete solution to (4.2) satisfying $|\operatorname{Rm}| \leq \frac{C}{t}$ and such that $g(0)$ splits as a product continues to split for $t > 0$. In Theorem 5.1.1 of Chapter 5, we obtain a partial result in this direction. We show that there is a constant $\epsilon = \epsilon(n)$ depending only on the dimension such that, if the initial metric $g(0)$ for a solution $(M, g(t))$ to (1.1) splits and the curvature satisfies $|\operatorname{Rm}| \leq \frac{\epsilon}{t}$, then the solution continues to split for all time such that it exists.

This result should be compared to that in the recent paper of Lee [55]. There, the author has established (by somewhat different methods) the uniqueness of general solutions satisfying a curvature bound of the form ϵ/t for some ϵ sufficiently small. Whereas we have seen that uniqueness of classical solutions to Ricci flow enables of the preservation of product structures in that class, our result in Chapter 5 cannot be derived as a simple consequence of Lee's result in the same way. Since we make no assumption on g_0 other than that it is complete, there is no statement of short-time existence to which we can appeal in order to solve the flow on the individual factors of M to produce a competitor solution to which we might apply the theorem in [55]. Instead, our proof frames the problem as one of uniqueness for a certain auxillary system of differential inequalities. From there, we proceed analytically as in the work of Huang-Tam [40] and Liu-Székelyhidi [56], making use of an adapted version of the maximum principle from these papers. The details of this work appear in the preprint [18]

1.3 Future Directions: Asymptotically Product Shrinking Ricci Solitons

To understand the singular behavior of the Ricci flow, it is important to study the classification of *shrinking Ricci solitons*, Riemannian manifolds which correspond to homothetically contracting self-similar solutions to the Ricci flow. Via this correspondence, determining whether or not two solitons which are asymptotic to each other along an end much be isometric can be framed as a (possibly singular) question of backward uniqueness to the Ricci flow or a related system.

This point of view has been adopted, e.g., in the work of [50], [51], [75], [74]. In the manuscript [19], currently in preparation, we combine this perspective together with the framework of Chapter 5 to study the problem of uniqueness of shrinking Ricci solitons which agree to infinite order at infinity along an end with a product shrinker of the form $\mathbb{R}^{n-k} \times \Sigma^k$, where Σ is a compact shrinker. In Chapter 6, we detail the reduction of the problem of uniqueness to one of unique continuation for a system of mixed differential inequalities.

Chapter 2

GEOMETRIC PRELIMINARIES

Before we begin the presentation of our primary results, we will review some of fundamental geometric concepts on which they rely, fixing some notation as we go.

2.1 Connections and Parallel Translation

Let M be a smooth manifold and TM its tangent bundle. For any vector bundle E over M, we define $C^\infty(E)$ to be the set of smooth sections of E. Suppose $\pi : E \to M$ is a vector bundle over M. A *connection* is a map $\nabla : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$, denoted by $\nabla(X, Y) = \nabla_X Y$, which satisfies the following properties:

1. For any smooth functions $f_1, f_2 : M \to \mathbb{R}$, $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$,
2. For any $c_1, c_2 \in \mathbb{R}$, $\nabla_X (c_1 Y_1 + c_2 Y_2) = c_1 \nabla_X Y_1 + c_2 \nabla_X Y_2$,
3. ∇ satisfies the product rule $\nabla_X (fY) = f \nabla_X Y + (Xf)Y$ for any smooth function $f : M \to \mathbb{R}$.

Suppose ∇ is a connection on E and $p \in E$. Let $\{E_i\}$ and $\{F_\alpha\}$ be smooth local frames for TM and E defined on a neighborhood $\mathcal{U}$ of $\pi(p) \in M$. The functions $\Gamma^\beta_{i\alpha} : \mathcal{U} \to \mathbb{R}$ defined by

$$\nabla_{E_i} F_\alpha = \Gamma^\beta_{i\alpha} F_\beta$$

are called the *connection coefficients* for ∇.

A connection ∇ on the vector bundle E defines a connection on the dual bundle E^* via the correspondence

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

for $\omega \in C^\infty(E^*)$. Furthermore, the connection can also act on sections A of $E^{(k,l)} := (E^*)^{\otimes^k} \otimes E^{\otimes^l}$ via

$$\begin{aligned}(\nabla_X A)(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l) = X(A(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l)) \\ - \sum_{i=1}^{k} A(X_1, \ldots, \nabla_X X_i, \ldots, X_k, \omega^1, \ldots, \omega^l) \\ - \sum_{i=1}^{l} A(X_1, \ldots, X_k, \omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^l).\end{aligned}$$

In terms of the local frames $\{E_i\}$, $\{F_\alpha\}$, and the dual coframe $\{F^\gamma\}$ for E^*, this can be written as

$$\nabla_m A^{\gamma_1,\ldots,\gamma_l}_{\alpha_1,\ldots,\alpha_k} = E_m(A^{\gamma_1,\ldots,\gamma_l}_{\alpha_1,\ldots,\alpha_k}) + \sum_{n=1}^{l} A^{\gamma_1,\ldots,\beta,\ldots,\gamma_l}_{\alpha_1,\ldots,\alpha_k} \Gamma^{\gamma_n}_{m\beta} - \sum_{n=1}^{k} A^{\gamma_1,\ldots,\gamma_l}_{\gamma_1,\ldots,\beta,\ldots,\gamma_k} \Gamma^{\beta}_{m\alpha_n}.$$

Now consider a smooth curve $\gamma : [0,1] \to M$. The connection ∇ on E induces a connection on the pullback bundle $\gamma^*(E)$ over $[0,1]$ called the *pullback connection.* A section $Y \in C^\infty(\gamma^*(E))$ is said to be *parallel along* γ if its derivative in the direction $\dot\gamma(t)$ with respect to the pullback connection is zero. As before, the pullback connection can be extended to bundles of tensors of any rank, and sections of these bundles along γ are also said to be parallel along γ if their covariant derivative is zero.

Let $p = \gamma(0)$ and $q = \gamma(1)$. Then, for any nonzero $X_p \in E_p$, there is a unique vector field X along γ such that $X|_0 = X_p$ and X is parallel along γ with respect to ∇. This defines a map

$$\begin{aligned} P_{p,\gamma(t)} :& E_p \to E_{\gamma(t)} \\ & X_p \mapsto X_{\gamma(t)}. \end{aligned}$$

for each $t \in [0,1]$, called the *parallel translation along* γ. Again, this can be extended to tensors of any rank. For example, if $A \in T^2(T_p^*M)$, then we define $PA = P_{p,\gamma(t)}A \in T^2(T_p^*M)$ by

$$PA(X,Y) = A(P^{-1}X, P^{-1}Y),$$

for any $X, Y \in T_{\gamma(t)}M$.

2.2 The Levi-Civita Connection

A connection $\nabla : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ is said to be *compatible with the metric* g if it satisfies

$$\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

We say that it is *torsion-free* if it satisfies

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

It is a fundamental result of Riemannian geometry that there exists a unique connection on any Riemannian manifold which is both compatible with the metric and torsion-free. This connection is called the *Levi-Civita connection.* Note that there are several different ways we can characterize compatibility with the metric. For example, it is equivalent to the condition $\nabla g \equiv 0$ or to the property that the inner product of any two parallel vector fields along a curve is constant along that curve.

2.3 Curvature

Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection of g. The *Riemann curvature endomorphism* is defined as the map $R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is easy to check that this map is linear over $C^\infty(M)$ in every argument, and thus defines a $(3,1)$ tensor field over M, which we call the *(3,1)-Riemann curvature tensor.* We will also often use the *(4,0)-Riemann curvature tensor*, given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

In local coordinates, we write this as

$$R_{ijkl} = g_{lm} R_{ijk}^{m},$$

i.e., our convention is to lower the index to the fourth position. The curvature tensor has the following algebraic symmetries:

- $R(X, Y, Z, W) = -R(Y, X, Z, W)$,
- $R(X, Y, Z, W) = -R(X, Y, W, Z)$,
- $R(X, Y, Z, W) = R(Z, W, X, Y)$,
- $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.

The last of these is known as the *first Bianchi identity.* The curvature tensor also satisfies

$$\nabla_V R(X, Y, Z, W) + \nabla_Z R(X, Y, W, V) + \nabla_W R(X, Y, V, Z) = 0.$$

This equation, known as the *second Bianchi identity*, has the form

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0,$$

and taking two successive traces of both sides of this equation yields

$$\nabla_k R_{ij} - \nabla_j R_{ik} + \nabla_m R_{ijk}^{m} = 0,$$

and

$$\nabla^k R_{kj} = \frac{1}{2} \nabla_j R.$$

This final equality is called the *contracted second Bianchi identity.*

2.4 The Holonomy Group of a Connection

We will now review holonomy, a fundamental algebraic invariant of a connection. For this discussion, let M be a connected manifold, E a vector bundle over M, and ∇ a connection on E.

Consider a piecewise smooth loop $\gamma : [0,1] \to M$ such that $\gamma(0) = \gamma(1) = p$. Parallel translation of elements of the fiber E_p around γ induces a linear transformation of E_p. The *holonomy group* based at p is defined by

$$\mathrm{Hol}_p(\nabla) := \{P_\gamma = P_{\gamma(0),\gamma(1)} : \gamma \text{ is a piecewise smooth loop based at } p\}.$$

This set has a natural group structure, where the group operation corresponds to the composition of linear maps. The uniqueness of parallel transport implies that the elements of $\mathrm{Hol}_p(\nabla)$ are invertible, and that the inverse of each element $P_{\gamma(0),\gamma(1)}$ is given by $P_{\overline{\gamma}(0),\overline{\gamma}(1)}$, where $\overline{\gamma}(t) = \gamma(1-t)$ is the reverse parameterization of γ. In light of this, we may view $\mathrm{Hol}_p(\nabla)$ as a subgroup of $\mathrm{GL}(E_p)$.

It is not hard to see that the holonomy group is independent of basepoint, up to conjugation. So, we will usually omit the basepoint and simply write $\mathrm{Hol}(\nabla)$. This gives a natural representation of $\mathrm{Hol}(\nabla)$ as a subgroup of $\mathrm{GL}(E_p)$, which is called the *holonomy representation*.

The *reduced holonomy* $\mathrm{Hol}^0_p(\nabla)$ of ∇ at p is defined as

$$\mathrm{Hol}^0_p(\nabla) := \{P_\gamma : \gamma \text{ is a nullhomotopic loop based at } p\}.$$

The group $\mathrm{Hol}^0_p(\nabla)$ is a connected Lie subgroup of $\mathrm{SO}(n)$, and coincides with the identity component of $\mathrm{Hol}_p(\nabla)$. As with the full holonomy group, this is independent of basepoint, up to conjugation. Clearly, if M is simply connected, $\mathrm{Hol}(\nabla) = \mathrm{Hol}^0(\nabla)$. Because $\mathrm{Hol}^0(\nabla)$ is the identity component of $\mathrm{Hol}(\nabla)$, the two have isomorphic Lie algebras, which we denote by $\mathfrak{hol}(\nabla)$.

When $E = TM$ and ∇ is the Levi-Civita connection of a Riemannian metric on M, we call $\mathrm{Hol}(\nabla)$ the *Riemannian holonomy group*, and will often denote it by $\mathrm{Hol}(g)$.

Since parallel translation along a curve γ with respect to ∇ preserves orthonormal frames for TM, it defines an isometry from $(T_{\gamma(0)}M, g)$ to $(T_{\gamma(t)}M, g)$. Thus for Riemannian holonomy we have $\mathrm{Hol}_p(g) \leq \mathrm{O}(T_pM)$ (respectively $\mathrm{SO}(T_pM)$ if M is orientable), and $\mathrm{Hol}^0(g) \leq \mathrm{SO}(T_pM)$.

If $\widetilde{M}$ is the universal cover of M, and $\widetilde{\nabla}$ is the Levi-Civita connection of the lift $\tilde{g}$ of the metric on M to $\widetilde{M}$, then $\mathrm{Hol}^0(\nabla) \cong \mathrm{Hol}(\widetilde{\nabla})$.

2.4.1 Berger's Classification

When the holonomy $\mathrm{Hol}(g)$ of a manifold (M, g) is isomorphic to a proper subgroup of $\mathrm{SO}(n)$, we say that the manifold has *restricted holonomy.* According to a fundamental theorem of Berger [7], the possibilities for a Riemannian manifold to have restricted holonomy are fairly limited. Before stating this theorem, we will make a few simplifying observations.

First, in light of the discussion above, in order to determine which groups may arise as the reduced holonomy of a Riemannian manifold, it is enough to consider the holonomy of the universal cover. Thus, we may restrict our attention to simply connected manifolds.

Next, recall that a manifold (M, g) is called *reducible* if it is isometric to a Riemannian product manifold, and *locally reducible* if each point of M has a neighborhood which is reducible. For a product manifold, we have

$$\mathrm{Hol}(g_1 \oplus g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2).$$

So, the holonomy representation of a reducible manifold is itself reducible. De Rham's

Theorem offers a partial converse: if the holonomy is reducible, then the underlying manifold is at least locally reducible [8]. Thus, in order to consider only irreducible holonomy groups, it suffices to consider only irreducible manifolds.

Finally, we say that a manifold (M, g) is a *(Riemannian) symmetric space* if, for every $p \in M$, there exists an involutive isometry $i_p : M \to M$ such that p is a fixed point of i_p. We say that (M, g) is *locally symmetric* if it is locally isometric to a simply connected symmetric Riemannian manifold. These spaces and their holonomy groups have been classified by Cartan, and thus are excluded from this classification.

The following theorem effectively classifies the possibilities for restricted reduced holonomy.

Theorem 2.4.1. *[[7], see also [42]] Suppose M is a simply-connected manifold of dimension n, and g is an irreducible, nonsymmetric Riemannian metric on M. Then exactly one of the following seven cases holds:*

1. $\mathrm{Hol}(\nabla) = \mathrm{SO}(n)$,
2. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(\nabla) \cong \mathrm{U}(m)$ *in* $\mathrm{SO}(n)$,
3. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(\nabla) \cong \mathrm{SU}(m)$ *in* $\mathrm{SO}(n)$,
4. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(\nabla) \cong \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,
5. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(\nabla) \cong \mathrm{Sp}(m)\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,
6. $n = 7$ *and* $\mathrm{Hol}(\nabla) \cong G_2$ *in* $\mathrm{SO}(7)$, *or*
7. $n = 8$ *and* $\mathrm{Hol}(\nabla) \cong \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Remark 2.4.1. *In cases 3-7 (and also when g is locally symmetric), g is necessarily Einstein. In fact, in cases 3,4,6, and 7, g is necessarily Ricci-flat.*

2.4.2 Parallel Tensors and Fixed Points of the Holonomy Group

The action of the holonomy group on TM can naturally be extended to an action on tensors of any rank. For example, for $P \in \mathrm{Hol}_p(\nabla)$, $A \in T^2(T_p^*M)$, we can say

$$PA_p(X, Y) = A_p(PX, PY)$$

for $X, Y \in T_pM$. A basic fact about the Riemannian holonomy group is that, on a connected manifold, fixed points of this action are in one-to-one correspondence with parallel tensors, i.e., tensors whose covariant derivative vanishes on all of M. More precisely, we have the following proposition.

Proposition 2.4.2. *Let (M, g) be a connected Riemannian manifold, and A a parallel tensor on M. Then, for $p \in M$, A_p is fixed by the action of $\mathrm{Hol}_p(\nabla)$. Conversely, if $A_p \in T_pM$ is fixed by the action of $\mathrm{Hol}_p(\nabla)$, then there exists some parallel tensor A on M such that $A|_p = A_p$.*

Chapter 3

THE RICCI FLOW

3.1 The equation

We say that a one-parameter family of Riemannian metrics $(M, g(t)), t \in [0, T)$ is a solution to the Ricci flow if it satisfies

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))$$

for all $t \in [0, T)$.

The Ricci flow is often described as a heat equation for Riemannian metrics, given its formal and qualitative resemblance to the equation

$$\frac{\partial}{\partial t} u = \Delta u.$$

for scalar functions. For example, in *harmonic coordinates*, i.e., local coordinates $\{x^i\}$ satisfying $\Delta_g x^i = 0$, the right hand side of the Ricci flow equation has the form

$$-2R_{ij} = \Delta_g(g_{ij}) + Q_{ij}(g^{-1}, \partial g),$$

where Q is quadratic in g^{-1} and first derivatives of g. Thus, $-2\,\mathrm{Rc}$ can be regarded as a type of Laplacian for metrics. See [16], Chapter 3 for additional details.

Heuristically, the heat equation smooths and averages out an initial temperature distribution u_0, lowering the temperature in areas with high heat and raising it in areas with low heat to approach something more uniform. This is also what we expect from the Ricci flow, at least in the short term–the flow will tend to make the geometry of the space more homogeneous, at least on a small time scale. Note, too, that the

Ricci flow tends to shrink distances in regions with positive curvature and expand them in regions with negative curvature. This phenomenon completely describes the dynamics in the following simple example.

Recall that an *Einstein manifold* (M, g) is a manifold on which $\operatorname{Rc}(g) = \lambda g$ for some $\lambda \in \mathbb{R}$. These include among them the constant curvature spaces, in particular Euclidean space $\mathbb{R}^n$ and the round sphere S^n.

Example 3.1.1. *Suppose (M, g_0) is Einstein. Consider the one-parameter family of metrics*

$$g(t) = (1 - 2\lambda t) g_0.$$

Then $g(0) = g_0$, and because the Ricci tensor is invariant under scaling of the metric,

$$\frac{\partial}{\partial t} g = -2\lambda g_0 = -2\operatorname{Rc}(g_0) = -2\operatorname{Rc}(g(t)).$$

Thus, $g(t)$ is a solution to Ricci flow with initial condition (M, g_0). The interval for which the flow exists depends on λ. When $\lambda > 0$, the flow exists for $t \in (-\infty, \frac{1}{2\lambda})$. When $\lambda < 0$, the flow exists for $t \in (\frac{1}{2\lambda}, \infty)$. Finally, when $\lambda = 0$, (i.e., (M, g_0) is Ricci flat) the solution is static and exists for all time.

3.2 Well-posedness for the Ricci flow

Perhaps the most fundamental question one can ask of any PDE is: when does it admit solutions, and when are they unique? These (and related questions) are often discussed under the heading of "well-posedness", and for evolution equations such as the Ricci flow, generally in reference to the restriction of the equation to some fixed class of solutions and initial data. Here, we discuss the theory of well-posedness for the Ricci flow.

3.2.1 Ricci flow with bounded curvature

The Ricci flow has traditionally been considered within the class of complete solutions with uniformly bounded curvature, that is, the class of solutions on $M \times [0, T)$ such that $(M, g(t))$ is complete for each t and

$$\sup_{M\times[0,a]} |\operatorname{Rm}| < \infty$$

for all $a < T$. We will refer to members of this traditional class as *classical solutions.* The problems of existence and uniqueness for classical solutions have been completely resolved.

Theorem 3.2.1. *[Hamilton [32], DeTurck [22], Shi [68], Chen-Zhu [15]] Let* (M, g_0) *be a smooth, complete Riemannian manifold satisfying*

$$\sup_{M} |\operatorname{Rm}| \leq K_0$$

for some $K_0 > 0$*. Then there exist a unique* $T = T(K_0, n) > 0$ *and a smooth solution to the Ricci flow* $g(t)$ *for* $t \in [0, T)$ *satisfying* $g(0) = g_0$*. Moreover,* $(M, g(t))$ *is complete for each* t *and*

$$\sup_{M\times[0,a]} |\operatorname{Rm}| < \infty \tag{3.1}$$

for any $a \in [0, T)$*, and* T *is maximal in the sense that if* $T < \infty$*,*

$$\limsup_{t\nearrow T} \big(\sup_{M} |\operatorname{Rm}|\big) = \infty.$$

Finally, if $\tilde{g}(t)$ *is any other complete solution for* $t \in [0, \tilde{T})$ *with* $\tilde{g}(0) = g_0$ *which satisfies (3.1), then* $\tilde{g}(t) = g(t)$ *for all* $t \in [0, \min(T, \tilde{T}))$*.*

This statement was proven by Hamilton in [32] in the case that the initial manifold (M, g_0) is compact. His proof, which made use of a variation on the Nash-Moser

Inverse Function Theorem, was shortly thereafter simplified by DeTurck [22]. Short-time existence for the case of a complete, noncompact initial metric with bounded curvature was proven by W.-X. Shi in [68]. The full statement of uniqueness for classical solutions was proven by Chen and Zhu in [15].

By now, the short-time analytic behavior of classical solutions is fairly well-understood. For example, it is known that classical solutions to the Ricci flow are also unique going backward in time [52], in the sense that if $g(t), \tilde{g}(t)$ are two solutions to Ricci flow on a manifold M and $g(t_0) = \tilde{g}(t_0)$ for some $t_0 > 0$, then $g(t) = \tilde{g}(t)$ for $0 \leq t \leq t_0$. An interesting consequence of forward and backward uniqueness is that the isometry group is preserved under the Ricci flow. One one hand, the forward uniqueness of solutions implies that the isometry group cannot shrink. Indeed, if $(M, g(t))$ is a solution to Ricci flow and $\varphi \in \mathrm{Isom}(g_0)$, then $\tilde{g}(t) = \varphi^* g(t)$ is another solution to Ricci flow with the same initial conditions. This is a consequence of the diffeomorphism invariance of the Ricci tensor:

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}(t) &= \varphi^*\big(\frac{\partial}{\partial t} g(t)\big) \\ &= -2\varphi^* \operatorname{Rc}(g(t)) \\ &= -2 \operatorname{Rc}(\varphi^* g(t)).\end{aligned}$$

Thus, by forward uniqueness, $\tilde{g}(t) = g(t)$, or in other words, φ is an isometry of $(M, g(t))$ for all $t \in [0, T]$. Similarly, backward uniqueness implies that the group of isometries cannot expand, so that the group of isometries is exactly preserved by the flow.

In fact, solutions to Ricci flow with uniformly bounded curvature are real analytic in both space and time for $t > 0$. Instantaneous analyticity in space was proven in the compact case by Bando in [6].

Theorem 3.2.2 ([6])**.** *Let $(M, g(t))$ be a solution to Ricci flow. Then at any $t \in (0, T]$,*

$g(t)$ is analytic with respect to normal coordinates.

Analyticity in time and local space-time analyticity was proven by Kotschwar in [53], [46] (see also [67]).

3.2.2 Ricci Flow of Spaces of Potentially Unbounded Curvature

The existence and uniqueness results discussed above imply that many geometric properties are preserved along the flow for classical solutions (we have already discussed how uniqueness implies that the isometry group is preserved, and we expand this to the preservation of other structures in Section 3.3). It is an important problem to determine the extent to which the same is true for general smooth solutions to the Ricci flow.

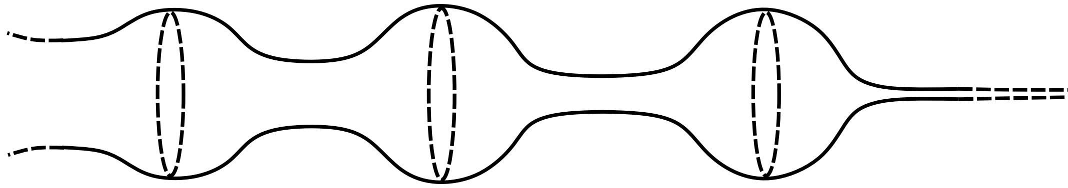
Question 3.2.1. *Under what conditions on a Riemannian manifold (M, g_0) is there guaranteed to exist a smooth short-time solution $(M, g(t))$ to (1.1) with $g(0) = g_0$?*

Question 3.2.2. *Under what conditions on solutions $(M, \hat{g}(t))$ and $(M, \check{g}(t))$ to (1.1) satisfying $\hat{g}(0) = \check{g}(0)$ is it guaranteed that $\hat{g}(t) = \check{g}(t)$ for all t such that both solutions exist?*

The following two examples show that one cannot expect *unconditional* statements of existence and uniqueness.

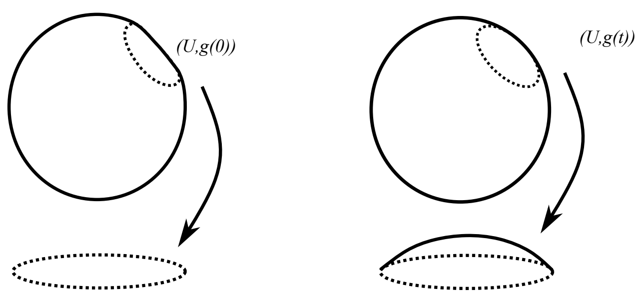
Example 3.2.3. *Suppose $n \geq 3$ and let M be given by $S^{n-1} \times \mathbb{R}$ with infinitely many necks, the radii of which approach zero as the manifold approaches spatial infinity, and which are separated by spheres of a fixed radius. Since a cylindrical solution forms a singularity in time proportional to the square of radius of the cylinder, it is expected that a complete Ricci flow with this initial condition could not exist on any interval $[0, \epsilon)$ (see [72]).*

Figure 3.1: $S^{n-1} \times \mathbb{R}$ with Infinitely Many Necks



Example 3.2.4. *[Flat-sided sphere] Let $U \subset \mathcal{S}^2$ be an open neighborhood with $\overline{U} \neq \mathcal{S}^2$. Let g_0 be a metric that is flat on U, but has nonnegative Gaussian curvature everywhere else, and strictly positive curvature somewhere (such a metric exists by a theorem of Kazdan and Warner, [43]). Consider the Ricci flow $g(t)$ on U given by the restriction of the Ricci flow with initial condition $(\mathcal{S}^2, g_0)$. At any positive time, $(U, g(t))$ will have uniformly positive curvature (this follows from an application of the strong maximum principle to the flow on all of S^2). However, the restriction of g_0 to U is flat, so the constant metric $\tilde{g}(t) = g_0$ is another solution to the Ricci flow on U with the same initial condition.*

Figure 3.2: Flat Sided Sphere



Example 3.2.4 indicates that completeness (or some appropriate substitute) is needed to ensure uniqueness of solutions in general. One might ask if completeness alone is sufficient. In dimension two, the recent work of Giesen and Topping shows that this is indeed true (see Section 3.2.3 below). However, in higher dimensions it

is less clear what to expect, as Example 3.2.3 shows that short time existence likely does not hold for general initial metrics. The possibility remains that there is some other condition on the curvature which ensures uniqueness and short-time existence of solutions to Ricci flow. For example, in the work of [11], [39], and [54], there are constructions which produce solutions emanating from initial metrics satisfying only (one-sided) lower bounds on curvature. It is unknown whether the solutions produced by these constructions are unique, however, under certain additional non-collapsing conditions, these solutions satisfy an instantaneous curvature bound of the form

$$|\operatorname{Rm}| \leq \frac{c}{t},$$

for some constant $c > 0$. This type of curvature bound also arises in a number of other recent constructions of non-classical solutions, for example, [66] and [69]. This suggests that solutions satisfying such instantaneous curvature bounds are a natural class to test the extent to which the uniqueness result of Chen-Zhu can be generalized. Lee [55] has proven a recent result in this direction.

Theorem 3.2.5 ([55])**.** *For any $m \in \mathbb{N}$, $\exists \epsilon = \epsilon(m) > 0$ such that the following holds: Suppose (M, g_0) is a complete noncompact manifold satisfying*

$$|\operatorname{Rm}(g_0)| \leq C_0(d(x,p)+1)^m$$

for some $C_0 > 0$ and a fixed point $p \in M$. If $g(t)$ and $\tilde{g}(t)$ are two smooth solutions to Ricci flow on $M \times [0,T]$ with $g(0) = \tilde{g}(0) = g_0$ for which

$$|\operatorname{Rm} \tilde{g}(t)|_{\tilde{g}(t)} + |\operatorname{Rm}(g(t))|_{g(t)} \leq \frac{\epsilon}{t}$$

on $M \times (0,T]$, then $g(t) = \tilde{g}(t)$ for all $t \in [0,T]$.

3.2.3 Instantaneously Complete Ricci Flow on Surfaces

There is one situation in which the assumptions that guarantee existence and uniqueness of solutions have been essentially completely determined. The program of Topping and Giesen-Topping [27, 28, 30, 31, 71, 73] has extablished existence of a Ricci flow with arbitrary initial metric (M, g_0) when M has dimension two. Moreover, this solution will be complete for any $t > 0$, and unique among solutions with this property. These instantaneously complete solutions are a useful source of insight into the nature of solutions to Ricci flow with potentially unbounded curvature.

Theorem 3.2.3 ([27, 28, 73])**.** *Let (M^2, g_0) be any smooth Riemannian surface, possibly incomplete and/or with unbounded curvature. Depending on the conformal type, we define $T \in (0, \infty]$ by*

$$T := \begin{cases} \frac{1}{4\pi\chi(M)} \mathrm{Vol}_{g_0} M & \text{if } (M, g_0) \cong S^2, \mathbb{C}, \text{ or } \mathbb{R}P^2, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a unique smooth Ricci flow $g(t)$ on M^2, defined for $t \in [0, T)$ such that

1. *$g(0) = g_0$;*

2. *$g(t)$ is instantaneously complete.*

The existence of such a solution with an arbitrary initial metric was proven by Giesen and Topping in [28]. Uniqueness for solutions whose initial metric has uniformly negative curvature was shown by the same authors in [27], and the full result was proven in [73] by Topping. The unique flow guaranteed to exist by the theorem is called the *instantaneously complete Ricci flow*.

As the method of Giesen and Topping relies on the conformal nature of the flow in dimension two, it is unclear how much of this theory might reasonably be hoped to extend to higher dimensions. In particular, in view of Example 3.2.3, there is not much hope for a similar statement of existence in dimensions three and above. Still, the work of Giesen and Topping shows that in dimension two, completeness is enough to imply the uniqueness of solutions on its own. This suggests the following question.

Question 3.2.6. *If $(M, \hat{g}(t))$ and $(M, \check{g}(t))$ are complete solutions to (1.1) with $\hat{g}(0) = \check{g}(0)$, must $\hat{g}(t) = \check{g}(t)$ for as long as both solutions exist?*

3.3 Well-posedness and the Preservation of Geometric Structures

As discussed in Section 3.2.1, uniqueness of solutions to the Ricci flow implies that the isometry group of the initial metric is preserved. It is easy to see that well-posedness implies that certain other structures are preserved as well. The following result, in particular, is standard: we give a detailed proof here in order to provide context for the result of Chapter 5.

Proposition 3.3.1. *Let $(M, g_0) = (\hat{M} \times \check{M}, \hat{g}_0 \oplus \check{g}_0)$ be a complete manifold which splits as a product, and let $(M, g(t))$ be a solution to (1.1) for $t \in [0, T]$ with $g(0) = g_0$ and with uniformly bounded curvature. Then there exist metrics $\hat{g}(t)$, $\check{g}(t)$ on $\hat{M}$ and $\check{M}$, respectively, with $\hat{g}(0) = \hat{g}_0$ and $\check{g}(0) = \check{g}_0$ such that $g(t) = \hat{g}(t) \oplus \check{g}(t)$ for all $t \in [0, T]$.*

Proof. Using the existence component of Theorem 3.2.1, we know that there exist solutions $\hat{g}(t), \check{g}(t)$ to (1.1) with $\hat{g}(0) = \hat{g}_0$ and $\check{g}(0) = \check{g}_0$, both with uniformly bounded curvature. Let $\hat{T}$ be the maximal existence time for $\hat{g}$, and $\check{T}$ the maximal existence time for $\check{g}$. Their product $(\hat{g} \oplus \check{g})(t)$ is then a solution on M with the required initial condition, and also has bounded curvature. Thus, by the uniqueness component of Theorem 3.2.1, we must have $g(t) = (\hat{g} \oplus \check{g})(t)$ for $t \in [0, \min(\hat{T}, \check{T}, T))$. Suppose without loss of generality that $\hat{T} < T$. Then, again by Theorem 3.2.1, we must have

$$\limsup_{t \nearrow T} \big(\sup_{M} | \operatorname{Rm} |_{\hat{g}(t)} | \big) = \infty$$

as $t \nearrow \hat{T}$. But, this would imply that

$$\limsup_{t \nearrow T} \big(\sup_{M} | \operatorname{Rm}_{\hat{g}(t) \oplus \check{g}(t)} | \big) \to \infty$$

as $t \nearrow \hat{T}$, contradicting the assumption that $g(t)$ must satisfy

$$\sup_{M \times [0, \hat{T}]} | \operatorname{Rm}_{g(t)} | \leq K$$

for some $K < \infty$. Therefore, we must have $\hat{T}, \check{T} \geq T$, and $g(t) = \hat{g}(t) \oplus \check{g}(t)$ for all $t \in [0, T)$. □

In Chapter 5, we extend this statement to the class of solutions satisfying a curvature bound of the form $| \operatorname{Rm} | \leq \epsilon / t$, where $\epsilon = \epsilon(n)$ depends only on the dimension. Our proof, however, relies on a substantially different approach. Note that we cannot simply follow the template of the theorem above, substituting the theorem of Lee for that of Chen-Zhu, as the argument for Proposition 3.3.1 relies on our ability to construct the 'competitor' solutions $\hat{g}(t)$ and $\check{g}(t)$ on the factors $\hat{M}$ and $\check{M}$. As we assume nothing other than the completeness of g_0, there is no short-time existence result to which we could appeal. Instead, we reformulate the problem as one of uniqueness of solutions to a certain auxiliary system. The components of this system are constructed from an evolving family of complementary orthogonal projections onto subspaces of the tangent bundle which, in a sense, mimic the natural projections induced by the product structure at time $t = 0$. The evolution equations of the components of this system can be organized into a system of mixed differential inequalities. Adapting a version of the maximum principle from [40] (see also [56]) for systems of this type, we are able to show that the components of this system must vanish, and thus that the time-dependent projections remain parallel with respect to $g(t)$. This implies that the solution continues to split for all time.

Chapter 4

PRESERVATION OF THE HOLONOMY GROUP UNDER THE RICCI FLOW

This chapter is joint work with Brett Kotschwar and an article based on its contents has been accepted for publication in the Proceedings of the American Mathematical Society ([20]). With his permission, it has been faithfully reproduced here with only minor modifications. The question addressed here was brought to the authors' attention by Thomas Leinster and Miles Simon. The authors are grateful to them for their interest and for subsequent related discussions.

4.1 Introduction

In this chapter, we consider the holonomy of a family of manifolds $(M, g(t))$ evolving by the Ricci flow. It is an old observation of Hamilton [33, 37] that, under mild hypotheses on the solution $g(t)$, the reduced holonomy $\mathrm{Hol}^0(g(t))$ *cannot expand*: if $\mathrm{Hol}^0(g(t))$ is initially restricted to some subgroup $G \subset \mathrm{SO}(n)$, then it remains so, provided that the solution is complete and of bounded curvature or otherwise belongs to some class in which the equation is well-posed. As we have discussed previously, this can be proven with a short argument using only general ingredients. First, one passes to the universal cover and applies Berger's classification (see Theorem 2.4.1). It follows from this classification that one need only verify that Einstein, product, and Kähler structures are preserved by the flow. Using the short-time existence component of Theorem 3.2.1 where needed, one can construct complete Einstein, product, and Kähler solutions to the flow starting from given initial data with those characteristics. The uniqueness results of Hamilton and Chen-Zhu [32, 15] (Theorem 3.2.1) then imply that these special solutions are the only solutions within the class with the given initial data.

In [45], Kotschwar later showed that the reduced holonomy $\mathrm{Hol}^0(g(t))$ of a complete solution of uniformly bounded curvature also cannot contract and, consequently, that $\mathrm{Hol}^0(g(t)) \cong \mathrm{Hol}^0(g(0))$ for all time t. However, even with Berger's classification, the problem of non-contraction does not reduce in the same way to one of backward uniqueness of solutions to the Ricci flow. While it is still only necessary to verify that the above three special structures are preserved under the flow, it is not in general possible to solve the parabolic terminal-value problems needed to obtain "competitor" solutions with these special structures to compare against the original solution. (The one exception is the Einstein case, in which suitable competitors can be obtained by scaling the initial metric homothetically.) Instead, in [45], the problem is framed as one of backward uniqueness of the solutions to a related prolonged system which may in turn be treated by the general methods of [52, 47]. This formulation also leads to an alternative proof of the non-expansion of $\mathrm{Hol}^0(g(t))$ (which is closer in spirit to that suggested in [37] than the argument sketched above).

The above results leave open the question of whether the full holonomy $\mathrm{Hol}(g(t))$ is preserved by the Ricci flow. Both $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$ are fundamentally global invariants of the manifold, however, at each p, the reduced holonomy $\mathrm{Hol}^0_p(g)$ is the connected component of the identity in $\mathrm{Hol}_p(g)$, and is determined by the holonomy Lie algebra $\mathfrak{hol}_p(g)$. Since the latter contains the image of the curvature operator Rm at each point p, it is possible to test for the preservation of the reduced holonomy along the flow by studying the kernel of the curvature operator of the solution.

For the full holonomy, which carries information about the global topology of the manifold, there is no such convenient infinitesimal characterization. Assuming the invariance of the reduced holonomy along the flow, the essence of the problem one faces is this: if the lift $(\tilde{M}, \tilde{g}(t))$ to the universal cover $\tilde{M}$ of a solution $(M, g(t))$ to the Ricci flow admits a parallel family of tensors $\tilde{A}(t)$, and if, at some time t_0, the tensor $\tilde{A}(t_0)$ descends to a parallel tensor $A(t_0)$ on M, then does $\tilde{A}(t)$ descend to a smooth parallel family of tensors $A(t)$ on M at all times t? One natural strategy to attack this problem is to extend $A(t_0)$ to a family of tensors defined for $t > t_0$ as the solution of an appropriate heat-type equation coupled with the Ricci flow and to argue from the maximum principle that $\nabla_{g(t)} A(t) \equiv 0$; however, this approach does not apply to times $t < t_0$.

Here, we show that the invariance (up to isomorphism) of $\mathrm{Hol}(g(t))$ along a solution $g(t)$ to the Ricci flow can nevertheless be obtained from that of $\mathrm{Hol}^0(g(t))$ by a direct argument which applies equally well forward and backward in time. The precise statement of the main theorem of this chapter is the following.

Theorem 4.1.1. *Let $g(t)$ be a solution to the Ricci flow on $M \times [0, T]$ such that $(M, g(t))$ is complete for each $t \in [0, T]$, and $\sup_{M\times[0,T]} |\mathrm{Rm}|(x, t) < \infty$. Then, for all $q \in M$ and $t \in [0, T]$,*

$$\mathrm{Hol}_q(g(t)) = \psi_t \circ \mathrm{Hol}_q(g(0)) \circ \psi_t^{-1}, \tag{4.1}$$

where $\psi_t \in \mathrm{O}(T_qM, g_q(t))$ satisfies

$$\frac{d\psi_t}{dt} = \mathrm{Rc} \circ \psi_t, \quad \psi_0 = \mathrm{Id},$$

for $t \in [0, T]$. Here $\mathrm{Rc} = \mathrm{Rc}(g_q(t)) : T_qM \to T_qM$.

In particular, Theorem 4.1.1 implies that if a complete Ricci flow with bounded curvature is Kähler or splits as a product on any time-slice, it must be Kähler or split as a product on all time-slices. In Section 4.4 we show that the complex and product structures will in these cases be independent of time. The preservation of such structures *forward* in time is of course a well-known property of the Ricci flow. We have already sketched one proof of this fact above; one feature of the argument below (when used in conjunction with [45]) is that it does not make use of the short-time existence and uniqueness of solutions to the equation.

In [49], Kotschwar has also previously shown that the preservation of global Kählerity under the flow follows from the preservation of local Kählerity. Our proof of Theorem 4.1.1 is in some sense a generalization of the argument given there. Further results concerning the preservation of the Kähler property for Ricci flows with instantaneously bounded curvature can be found in [40, 56].

4.2 The Preservation of Reduced Holonomy and a Reformulation

Let $g(t)$ be a smooth solution to the Ricci flow

$$\frac{\partial}{\partial t} g = -2\operatorname{Rc}(g), \tag{4.2}$$

on $M \times [0, T]$. The holonomy groups $\operatorname{Hol}^0_p(g(t))$ and $\operatorname{Hol}_p(g(t))$ based at a point $p \in M$ are naturally represented as subgroups of the orthogonal group $\operatorname{O}(T_pM, g_p(t))$ relative to the time-varying inner product $g_p(t)$ at p. Using Uhlenbeck's trick, we can transform Theorem 4.1.1 into an equivalent statement for a family of connections whose holonomy groups are instead realized as subgroups of some *fixed* representation of the orthogonal group.

4.2.1 Uhlenbeck's Trick

Fix $t_0 \in [0, T]$ and let E be a vector bundle isomorphic to TM by some fixed isomorphism $\imath_{t_0} : E \to TM$. Using $\imath_{t_0}$, we equip E with the bundle metric h defined by

$$h_q(V, W) = g_q(t_0)(\imath_{t_0} V, \imath_{t_0} W)$$

for $q \in M$ and $V, W \in E_q$. Then, we extend $\imath_{t_0}$ forward and backward in time as the solution to the fiber-wise ODE

$$\frac{\partial \imath_t}{\partial t} = \mathrm{Rc} \circ \imath_t, \tag{4.3}$$

where $\mathrm{Rc} = \mathrm{Rc}(g_q(t))$, to obtain a family $\imath_t : E \to TM$ of bundle isomorphisms for $t \in [0, T]$. With this extension, $\imath_t : (E, h) \to (TM, g(t))$ is in fact a bundle isometry for each $t \in [0, T]$.

Let $\nabla = \nabla^t$ denote the Levi-Civita connection of $g(t)$. We will study the holonomy of ∇ via that of the family of pull-back connections $\overline{\nabla} = \overline{\nabla}^t$ on E defined by

$$\overline{\nabla}^t_X V = \imath_t^{-1}(\nabla^t_X(\imath_t V)) \tag{4.4}$$

for $X \in TM$ and $V \in \Gamma(E)$, where $\Gamma(E)$ denotes the space of smooth sections of E. The metric h is compatible with the connection $\overline{\nabla}^t$, and the holonomy groups of $\overline{\nabla}^t$ and ∇^t are related by

$$\mathrm{Hol}_q(\overline{\nabla}^t) = \imath_t^{-1} \circ \mathrm{Hol}_q(\nabla^t) \circ \imath_t. \tag{4.5}$$

4.2.2 Preservation of Reduced Holonomy

As discussed in the introduction, the results in [37], [45] imply that when $(M, g(t))$ is complete and of uniformly bounded curvature, the reduced holonomy group of ∇^t is isomorphic to the reduced holonomy of ∇^{t_0} for any t, $t_0 \in [0, T]$. While the analytic arguments used in the verification of this fact for $t < t_0$ are fundamentally different from those used for the case $t_0 < t$, the backward-time and forward-time problems can still be formulated in a unified way in terms of the image of the curvature operator $\mathrm{Rm} = \mathrm{Rm}(g(t))$ of the solution in the bundle of two-forms.

Let $\mathfrak{hol}(\nabla^t)$ be the subbundle of $\mathrm{End}(TM)$ whose fiber $\mathfrak{hol}_q(\nabla^t) \subset \mathfrak{so}(T_qM, g_q(t))$ at q is the Lie algebra of $\mathrm{Hol}^0_q(\nabla^t) \subset \mathrm{O}(T_qM, g_q(t))$. Let $\mathcal{H}(\nabla^t) \subset \wedge^2 T^*M$ be the bundle of two-forms isomorphic to $\mathfrak{hol}(\nabla^t)$ via the correspondence

$$A \in \mathfrak{hol}_q(\nabla^t) \mapsto g_q(t)(A\cdot,\cdot) \in \mathcal{H}_q(\nabla^t),$$

and let $\mathfrak{hol}(\overline{\nabla}^t) \subset \operatorname{End}(E)$ and $\mathcal{H}(\overline{\nabla}^t) \subset \wedge^2 E^*$ denote the analogous families of bundles relative to the connection $\overline{\nabla}^t$.

In Theorem 1.4 and Appendix A of [45] (compare Theorem 4.1 of [37]), it is shown that $\mathcal{H}(\nabla^t)$ is time-invariant by first showing that the family of subbundles

$$H(t) = (\imath_t)_* \mathcal{H}(\overline{\nabla}^{t_0}) \subset \wedge^2(T^*M)$$

is a ∇^t-parallel subalgebra which contains the image of $\operatorname{Rm}(g(t))$. From this, we deduce that $H(t)$ must coincide with $\mathcal{H}(\nabla^t)$. Then, using the definition of $\imath_t$ and the fact that $H(t)$ contains the image of $\operatorname{Rm}(g(t))$, one verifies by a short calculation that $H(t)$ must actually be independent of time. Thus,

$$\mathcal{H}(\nabla^t) = H(t) = H(t_0) = \imath_{t_0}^* \mathcal{H}(\overline{\nabla}^{t_0}) = \mathcal{H}(\nabla^{t_0}).$$

But, $\mathcal{H}(\overline{\nabla}^t) = \imath_t^* \mathcal{H}(\nabla^t)$, so

$$\mathcal{H}(\overline{\nabla}^t) = \imath_t^* H(t) = \mathcal{H}(\overline{\nabla}^{t_0}),$$

and $\mathcal{H}(\overline{\nabla}^t)$ is also independent of time.

Whereas the fibers of $\mathfrak{hol}(\nabla^t)$ are related to the fibers of $\mathcal{H}(\nabla^t)$ via the time-dependent isomorphisms $A \mapsto g(t)(A\cdot,\cdot)$, the fibers of $\mathfrak{hol}(\overline{\nabla}^t)$ and $\mathcal{H}(\overline{\nabla}^t)$ are related by the *time-independent* isomorphism $A \mapsto h(A\cdot,\cdot)$. Thus, the fibers $\mathfrak{hol}_q(\overline{\nabla}^t) \subset \mathfrak{so}(E_q, h)$ are also independent of time, and it follows that the same is true of $\operatorname{Hol}^0_q(\overline{\nabla}^t) \subset \operatorname{O}(E_q, h)$.

In terms of the framework we have established, the preservation of reduced holonomy can be restated precisely as follows

Theorem 4.2.1 ([37], [45]). *Let $g(t)$, ∇^t, and $\overline{\nabla}^t$ be as above, and assume that $(M, g(t))$ is complete and of uniformly bounded curvature for $t \in [0, T]$. Then $\mathfrak{hol}_q(\overline{\nabla}^t) \subset \mathfrak{so}(E_q, h)$ is independent of t for all $q \in M$. Hence,*

$$\mathrm{Hol}^0_q(\overline{\nabla}^t) = \mathrm{Hol}^0_q(\overline{\nabla}^{t_0}), \quad \mathrm{Hol}^0_q(\nabla^t) = \psi_t \circ \mathrm{Hol}^0_q(\nabla^{t_0}) \circ \psi_t^{-1},$$

for all $q \in M$, $t \in [0, T]$, where $\psi_t = \imath_t \circ \imath_{t_0}^{-1}$.

Theorem 4.1.1 can now also be restated in terms of the family of connections $\overline{\nabla}^t$.

Theorem 4.2.2. *Provided $(M, g(t))$ is complete and of uniformly bounded curvature,*

$$\mathrm{Hol}_q(\overline{\nabla}^t) = \mathrm{Hol}_q(\overline{\nabla}^{t_0})$$

for all $t \in [0, T]$ and $q \in M$. Consequently,

$$\mathrm{Hol}_q(\nabla^t) = \psi_t \circ \mathrm{Hol}_q(\nabla^{t_0}) \circ \psi_t^{-1},$$

where $\psi_t = \imath_t \circ \imath_{t_0}^{-1}$.

4.3 Invariance of the Full Holonomy Group

Given a piecewise smooth curve $\gamma : [a, b] \to M$, we will use $D_s = D_s^t$ to denote the covariant derivative along γ induced by $\overline{\nabla} = \overline{\nabla}^t$. We will temporarily suppress the subscript t on the maps $\imath = \imath_t$.

The key to the proof of Theorem 4.2.2 is the following identity.

Proposition 4.3.1. *Let $\gamma : [a, b] \to M$ denote a smooth curve and $V = V(s, t)$ be a smooth family of smooth sections of E along $\gamma = \gamma(s)$ which is parallel along γ with respect to $D_s = D_s^t$ for all $t \in [0, T]$. Then V satisfies*

$$D_s \frac{\partial}{\partial t} V = -\imath^{-1} \mathrm{div}\, \mathrm{Rm}(\dot{\gamma}) \imath V.$$

Here, $\operatorname{div}\mathrm{Rm}$ is the section of $T^*M \otimes \mathrm{End}(TM)$ defined as follows: for any $p \in M$ and $X \in T_pM$, $\operatorname{div}\mathrm{Rm}_p(X) \in \mathrm{End}(T_pM)$ acts on $Y \in T_pM$ by

$$\operatorname{div}\mathrm{Rm}_p(X)Y = \sum_{l=1}^{n} \nabla_{e_l} R_p(e_l, X)Y,$$

where $\{e_l\}_{l=1}^n$ is a $g(t)$-orthonormal basis of T_pM. Note that $\operatorname{div}\mathrm{Rm}_p(X) \in \mathfrak{hol}_p(\nabla^t)$ for any $X \in T_pM$. Indeed, the curvature endomorphisms $R_p \in T^{(3,1)}(T_pM)$ belong to $\mathcal{H}_p(t) \otimes \mathfrak{hol}_p(\nabla^t)$, and so do $\nabla_{X_1}\nabla_{X_2}\cdots\nabla_{X_k}R_p$ for any $X_1, X_2, \ldots, X_k \in T_pM$.

Proof of Proposition 4.3.1. Let $s_0 \in [a,b]$ be fixed. Choose local frames $(E_\alpha)_{\alpha=1}^n$ for E and $(e_i)_{i=1}^n$ for TM on a neighborhood of $\gamma(s_0)$, and let $\overline{\Gamma}_{i\alpha}^\beta$ be the coefficients of $\overline{\nabla}$ in terms of these frames, i.e., $\overline{\nabla}_{e_i}E_\alpha = \overline{\Gamma}_{i\alpha}^\beta E_\beta$.

First, since $V(\cdot,t)$ is parallel for all t,

$$0 = \frac{\partial}{\partial t} D_s V = \frac{\partial^2 V^\alpha}{\partial s \partial t} E_\alpha + \frac{\partial V^\alpha}{\partial t} \overline{\nabla}_{\dot\gamma} E_\alpha + \dot\gamma^i V^\alpha \frac{\partial \overline{\Gamma}_{i\alpha}^\beta}{\partial t} E_\beta$$

at any (s_0, t). On the other hand, for s near s_0,

$$\frac{\partial V}{\partial t}(s,t) = \frac{\partial V^\alpha}{\partial t}(s,t) E_\alpha(\gamma(s)),$$

and so

$$\begin{aligned} D_s \frac{\partial V}{\partial t} &= \frac{\partial^2 V^\alpha}{\partial s \partial t} E_\alpha + \frac{\partial V^\alpha}{\partial t} \overline{\nabla}_{\dot\gamma} E_\alpha \\ &= -\dot\gamma^i V^\alpha \frac{\partial \overline{\Gamma}_{i\alpha}^\beta}{\partial t} E_\beta \end{aligned} \tag{4.6}$$

at (s_0, t).

Now, fix α and temporarily write $X = \imath E_\alpha$. Then we have $\frac{\partial}{\partial t} X = \mathrm{Rc}(X)$, and, using that

$$\frac{\partial \Gamma_{ij}^k}{\partial t} = \nabla^k R_{ij} - \nabla_i R_j^k - \nabla_j R_i^k, \quad \nabla^k R_{ji} - \nabla_j R_i^k = g^{lm} \nabla_l R_{mij}^k,$$

where Γ_{ij}^k denotes the components of ∇ in terms of $\{e_i\}_{i=1}^n$, we see that

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla_{e_i} X) &= \nabla_{e_i}\left(\frac{\partial}{\partial t} X\right) + \left(\frac{\partial \Gamma_{ij}^k}{\partial t}\right) X^j e_k \\ &= \nabla_{e_i}(\mathrm{Rc}(X)) - (\nabla_{e_i} \mathrm{Rc})(X) - (\nabla_X \mathrm{Rc})(e_i) + (\nabla^k \mathrm{Rc})(e_i, X) e_k \\ &= \mathrm{Rc}(\nabla_{e_i} X) + \operatorname{div} \mathrm{Rm}(e_i) X.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial \overline{\Gamma}_{i\alpha}^\beta}{\partial t} E_\beta &= \frac{\partial}{\partial t}\left(\overline{\nabla}_{e_i} E_\alpha\right) \\ &= \frac{\partial}{\partial t}\left(\imath^{-1} \nabla_{e_i} \imath(E_\alpha)\right) \\ &= -\imath^{-1} \circ \mathrm{Rc}(\nabla_{e_i} \imath E_\alpha) + \imath^{-1}\left(\frac{\partial}{\partial t}(\nabla_{e_i} \imath E_\alpha)\right) \\ &= \imath^{-1} \operatorname{div} \mathrm{Rm}(e_i) \imath E_\alpha.\end{aligned}$$

Inserting this expression into (4.6) for $D_s \frac{\partial V}{\partial t}$ completes the proof. $\square$

Next we use Proposition 4.3.1 to determine the evolution of parallel transport along a fixed loop.

Proposition 4.3.2. *Let $q \in M$ and let $\gamma : [0,1] \to M$ be a piecewise smooth loop with $\gamma(0) = \gamma(1) = q$. Let $P_{s,t} : E_q \to E_{\gamma(s)}$ be parallel translation along γ with respect to $D_s = D_s^t$. Then*

$$\frac{\partial}{\partial t} P_{1,t} = P_{1,t} B$$

for some $B = B(t) \in \mathfrak{hol}_q(\overline{\nabla})$.

Proof. It suffices to show that

$$P_{1,t}^{-1} \frac{\partial}{\partial t} P_{1,t} \in \mathfrak{hol}_q(\overline{\nabla}).$$

Let $0 = a_0 < a_1 < \ldots < a_k = 1$ be such that $\gamma|_{[a_{i-1}, a_i]}$ is smooth and fix an arbitrary $W \in E_q$. Applying the previous lemma to $V = P_{s,t} W$ on any subinterval $[a_{i-1}, a_i]$, we find that

$$\frac{d}{ds}\left(P_{s,t}^{-1}\frac{\partial}{\partial t}P_{s,t}W\right) = P_{s,t}^{-1}\left(D_s\frac{\partial}{\partial t}P_{s,t}W\right)$$
$$= -P_{s,t}^{-1}\left(\imath^{-1}\operatorname{div}\operatorname{Rm}_{\gamma(s)}(\dot\gamma)\imath(P_{s,t}W)\right).$$

In other words,

$$\frac{d}{ds}\left(P_{s,t}^{-1}\frac{\partial}{\partial t}P_{s,t}\right) = -P_{s,t}^{-1}\circ\imath^{-1}\circ\operatorname{div}\operatorname{Rm}_{\gamma(s)}(\dot\gamma)\circ\imath\circ P_{s,t} \doteqdot A(s,t).$$

But $\operatorname{div}\operatorname{Rm}_{\gamma(s)}(\dot\gamma)\in\mathfrak{hol}_{\gamma(s)}(\nabla)$ for each s, so $\imath^{-1}\circ\operatorname{div}\operatorname{Rm}_{\gamma(s)}(\dot\gamma)\circ\imath\in\mathfrak{hol}_{\gamma(s)}(\overline{\nabla})$ for each s. Since $\mathfrak{hol}(\overline{\nabla})$ is invariant under parallel translation, it follows that $A(s,t)\in\mathfrak{hol}_q(\overline{\nabla})$ for all $s\in(a_{i-1},a_i)$ and $t\in[0,T]$.

Now let $\mathfrak{hol}_q^{\perp}(\overline{\nabla})$ denote the orthogonal complement of $\mathfrak{hol}_q(\overline{\nabla})$ in $\operatorname{End}(E)$ and let $L\in\mathfrak{hol}_q^{\perp}(\overline{\nabla})$ be arbitrary. Then

$$F(s) = \left\langle L, P_{s,t}^{-1}\frac{\partial}{\partial t}P_{s,t}\right\rangle_{h_q} = \left\langle P_{s,t}\circ L\circ P_{s,t}^{-1}, \frac{\partial}{\partial t}P_{s,t}\circ P_{s,t}^{-1}\right\rangle_{h_{\gamma(s)}}$$

is continuous on $[0,1]$ and smooth on each interval (a_{i-1},a_i). For s in any such interval,

$$F'(s) = \langle L, A(s,t)\rangle_{h_q} = 0.$$

Thus $F|_{[a_{i-1},a_i]}$ is constant for each i.

But $P_{0,t}=P_{0,t}^{-1}=\operatorname{Id}$ for all t, so $P_{0,t}^{-1}\frac{\partial}{\partial t}P_{0,t}=0$ and $F(0)=0$. Thus $F(s)=0$ for all $s\in[0,1]$. Since $L\in\mathfrak{hol}_q^{\perp}(\overline{\nabla})$ was arbitrary, it follows that

$$B(t)\doteqdot P_{1,t}^{-1}\frac{\partial}{\partial t}P_{1,t}\in\mathfrak{hol}_q(\overline{\nabla}),$$

completing the proof. □

We will use Proposition 4.3.2 in conjunction with the following simple fact.

Lemma 4.3.3. *Suppose H is a Lie subgroup of the Lie group G, $B(t)$ is a smooth family of tangent vectors in $T_eH \subset T_eG$ for $t \in [0,T]$, and $X = X(g,t)$ is the left-invariant extension of $B(t)$ to G for each t. If $\alpha : [0,T] \to G$ is an integral curve of X passing through $a \in H$ at $t = t_0$ then $\alpha(t) \in H$ for all $t \in [0,T]$.*

Proof. Since $B(t) \in T_eH$, we may separately form the left-invariant extension $\overline{X}$ of $B(t)$ on H and obtain $\overline{\alpha} : [0,T] \to H$ solving $\overline{\alpha}'(t) = \overline{X}(\overline{\alpha}(t),t)$ with $\overline{\alpha}(t_0) = a$. Then, the inclusion $\iota \circ \overline{\alpha}$ of $\overline{\alpha}$ into G will be an integral curve of X passing through a at $t = t_0$ whose image lies in $H \subset G$. By uniqueness, it must coincide with α. □

Now we put the above pieces together to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. Fix $q \in M$ and $t_0 \in [0,T]$. We will show that $\mathrm{Hol}_q(\overline{\nabla}^t) \subset \mathrm{Hol}_q(\overline{\nabla}^{t_0})$ for all $t \in [0,T]$. Let $\gamma : [0,1] \to M$ be an arbitrary piecewise-smooth loop based at q and let $P(t) = P_{1,t} : E_q \to E_q$ be parallel translation along γ with respect to the covariant derivative $D_s = D_s^t$ relative to $\overline{\nabla}^t$. By Proposition 4.3.2, $\frac{\partial P}{\partial t} = PB$ for some $B = B(t)$ in the time-invariant subalgebra $\mathfrak{hol}_q(\overline{\nabla}^t) = \mathfrak{hol}_q(\overline{\nabla}^{t_0}) \subset \mathfrak{so}(E_q,h)$. For each $t \in [0,T]$, let $X(\cdot,t)$ be the left-invariant extension of $B(t)$ to all of $O(E_q,h)$, given by $X(A,t) = AB(t)$.

Then $P(t)$ is an integral curve of the left-invariant vector field $X(\cdot,t)$, and applying Lemma 4.3.3 with $H = \mathrm{Hol}_q(\overline{\nabla}^{t_0})$, $G = \mathrm{O}(E_q,h)$, $\alpha(t) = P(t)$, and $a = P(t_0)$, we obtain that $P(t) \in \mathrm{Hol}_q(\overline{\nabla}^{t_0})$ for all t. But $P(t)$ represents $\overline{\nabla}^t$-parallel translation along an arbitrary piecewise smooth loop γ based at q, so $\mathrm{Hol}_q(\overline{\nabla}^t) \subset \mathrm{Hol}_q(\overline{\nabla}^{t_0})$ for all t as claimed. □

4.4 Preservation of Parallel Tensors

One consequence of Theorem 4.1.1 is that if $g(t)$ is a complete solution to the Ricci flow of uniformly bounded curvature on $M \times [0,T]$ and A_0 is a smooth ∇^{t_0}-parallel tensor for some $t_0 \in [0,T]$, then there is a smooth family $A(t)$ of ∇^t-parallel tensors on $M \times [0,T]$ with $A(t_0) = A_0$.

Corollary 4.4.1. *If the tensor field $A_0 \in \Gamma(T^{k,l}(M))$ is ∇^{t_0}-parallel for some $t_0 \in [0,T]$, then $A(t) = (\imath_t)_* \imath_{t_0}^* A_0$ is ∇^t-parallel for all t.*

Indeed, the section $B_0 = \imath_{t_0}^* A_0$ of the corresponding tensor product of E is $\overline{\nabla}^{t_0}$-parallel, and Theorem 4.2.2 shows that $\mathrm{Hol}_q(\overline{\nabla}^t)$ is independent of time for each q. So B_0 is $\overline{\nabla}^t$-parallel, and $A(t) = (\imath_t)_* B_0$ therefore ∇^t-parallel, for each $t \in [0,T]$. The family $A(t)$ in Corollary 4.4.1 can be explicitly described as the solution of the fiberwise linear system

$$\frac{\partial}{\partial t} A^{a_1 \ldots a_l}_{b_1 \ldots b_k} = R^c_{b_1} A^{a_1 \ldots a_l}_{c b_2 \ldots b_k} + \cdots + R^c_{b_k} A^{a_1 \ldots a_l}_{b_1 b_2 \ldots c} - R^{a_1}_c A^{c a_2 \ldots a_l}_{b_1 \ldots b_k} - \cdots - R^{a_l}_c A^{a_1 \ldots c}_{b_1 \ldots b_k},$$
$$A(t_0) = A_0,$$

of ordinary differential equations.

In some cases, the extended family of parallel tensors $A(t)$ will be independent of time. This is true, for example, when the time-slice $(M, g(t_0)) = (\tilde{M} \times \hat{M}, \tilde{g} \oplus \hat{g})$ is a Riemannian product, and A_0 is one of the associated complementary orthogonal projections $\tilde{P}$, $\hat{P} \in \mathrm{End}(TM)$. It is also true when the time-slice $(M, g(t_0))$ is Kähler and $A_0 = J$ is its complex structure. We give the argument for these two special cases below.

Product structures

Suppose $M = \tilde{M} \times \hat{M}$ and $g(t_0) = \tilde{g} \oplus \hat{g}$ is a Riemannian product. Let $\tilde{P}_0$ and $\hat{P}_0$ denote the orthogonal projections onto the subbundles of TM isomorphic to $T\tilde{M}$ and $T\hat{M}$, respectively. These sections of $\mathrm{End}(TM)$ are parallel at $t = t_0$, and therefore, by Corollary 4.4.1, their extensions defined by

$$\begin{aligned}\frac{\partial \tilde{P}}{\partial t} &= \tilde{P} \circ \mathrm{Rc} - \mathrm{Rc} \circ \tilde{P}, \quad \tilde{P}(t_0) = \tilde{P}_0, \\ \frac{\partial \hat{P}}{\partial t} &= \hat{P} \circ \mathrm{Rc} - \mathrm{Rc} \circ \hat{P}, \quad \hat{P}(t_0) = \hat{P}_0,\end{aligned} \tag{4.7}$$

are ∇^t-parallel for each t. It also follows directly from (4.7) that $\tilde{P}(t)$ and $\hat{P}(t)$ will remain complementary $g(t)$-orthogonal projections. But these properties imply that $\tilde{P} \circ \mathrm{Rc} = \mathrm{Rc} \circ \tilde{P}$ and $\hat{P} \circ \mathrm{Rc} = \mathrm{Rc} \circ \hat{P}$ identically on $M \times [0, T]$. So $\frac{\partial \tilde{P}}{\partial t} = \frac{\partial \hat{P}}{\partial t} = 0$, and the product structure these projections define is constant in time.

Complex Structures

Similarly, suppose $(M, g(t_0))$ is Kähler with complex structure J_0. As above, the family $J = J(t)$ defined by

$$\frac{\partial J}{\partial t} = J \circ \mathrm{Rc} - \mathrm{Rc} \circ J, \quad J(t_0) = J_0,$$

will be ∇^t-parallel and will satisfy $J^2 = -\mathrm{Id}$ and $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ for all t. However, these conditions likewise imply that $\mathrm{Rc} \circ J = J \circ \mathrm{Rc}$ for each t, and hence that $\frac{\partial J}{\partial t} = 0$, so $(M, g(t))$ is Kähler relative to the *fixed* complex structure J_0 for all t. (Compare Section 3 of [49].)

Chapter 5

THE PRESERVATION OF PRODUCT STRUCTURES UNDER THE RICCI FLOW

5.1 Introduction

In this chapter, we consider the problem of whether a solution to the Ricci flow which splits as a product at $t = 0$ continues to do so for all time. As we have discussed in Section 3.3, this problem is closely related, but not strictly equivalent, to the question of uniqueness of solutions to the Ricci flow. For example (see Proposition 3.3.1), when $(\hat{M} \times \check{M}, \hat{g}_0 \oplus \check{g}_0)$, Shi's existence theorem [68] implies that there exist complete, bounded curvature solutions $(\hat{M}, \hat{g}(t))$ and $(\check{M}, \check{g}(t))$ with initial conditions $\hat{g}_0$ and $\check{g}_0$, respectively, which exist on some common time interval $[0, T]$. Then, $\hat{g}(t) \oplus \check{g}(t)$ solves (1.1) on $\hat{M} \times \check{M}$ for $t \in [0, T]$ and is also complete and of bounded curvature. But, according to the uniqueness results of Hamilton [32] and Chen-Zhu [15], such a solution is unique among those which are complete and have bounded curvature. Thus, any solution in that class starting at $\hat{g}_0 \oplus \check{g}_0$ continues to split as a product.

Outside of this class, less is known. While there are elementary examples which show that without completeness, a solution may instantaneously cease to be a product (see Example 3.2.4), the extent to which the uniform curvature bound can be relaxed is less well-understood. (One exception is in dimension two, where the work of Giesen and Topping [28, 27] has established an essentially complete theory of existence and uniqueness for (1.1). In particular, in [73], Topping shows that any two complete solutions with the same initial data must agree. See Section 3.2.3.)

One class of particular interest is that of solutions satisfying a curvature bound of the form c/t for some constant c, which arise naturally as limits of exhaustions (see, e.g., [11], [39], [70]). The purpose of this note is to prove the following.

Theorem 5.1.1. *Let* $(\hat{M}, \hat{g}_0)$ *and* $(\check{M}, \check{g}_0)$ *be two Riemannian manifolds and let* $M = \hat{M} \times \check{M}$ *and* $g_0 = \hat{g}_0 \oplus \check{g}_0$*. Then there exists a constant* $\epsilon = \epsilon(n) > 0$*, where* $n = \dim(M)$*, such that if* $g(t)$ *is a complete solution to (1.1) on* $M \times [0, T]$ *with* $g(0) = g_0$ *satisfying*

$$|\operatorname{Rm}|(x,t) \leq \frac{\epsilon}{t}, \tag{5.1}$$

then $g(t)$ *splits as a product for all* $t \in [0, T]$*, i.e.,* $g(t) = \hat{g}(t) \oplus \check{g}(t)$*, where* $\hat{g}(t)$ *and* $\check{g}(t)$ *are solutions to (1.1) on* $\hat{M}$ *and* $\check{M}$*, respectively, for* $t \in [0, T]$*.*

Lee [55] has already established the uniqueness of complete solutions satisfying the bound (5.1). However, his result does not directly imply Theorem 5.1.1: without any restrictions on the curvatures of $\hat{g}_0$ and $\check{g}_0$, we lack the short-time existence theory to guarantee that there are *any* solutions on $\hat{M}$ and $\check{M}$, respectively, with the given initial data, let alone solutions satisfying a bound of the form (5.1) for sufficiently small ϵ. Thus we are unable to construct a competing product solution on $\hat{M} \times \check{M}$ to which we might apply Lee's theorem.

Instead, we frame the problem as one of uniqueness for a related system, using a perspective similar to that of [56] and [45]. The key ingredient is a maximum principle closely based on one due to Huang-Tam [40] and modified by Liu-Székelyhidi [56]. These references establish, among other things, related results concerning the preservation of Kähler structures under instantaneous curvature bounds.

5.2 Tracking the Product Structure

Our first step toward proving Theorem 5.1.1 is to construct a system associated to a solution to Ricci flow which measures the degree to which a solution which initially splits as a product fails to remain a product. Consider a Riemannian product $(M, g_0) = (\hat{M} \times \check{M}, \hat{g}_0 \oplus \check{g}_0)$, and let $g(t)$ be a smooth solution to the Ricci flow on $M \times [0, T]$ with $g(0) = g_0$. For the time being, we make no assumptions on the completeness of $g(t)$ or bounds on its curvature.

5.2.1 Extending the Projections

Let $\hat{\pi} : M \to \hat{M}$ and $\check{\pi} : M \to \check{M}$ be the projections on each factor, and let $\hat{H} = \ker(d\check{\pi})$ and $\check{H} = \ker(d\hat{\pi})$. We define $\hat{P}_0, \check{P}_0 \in \mathrm{End}(TM)$ to be the orthogonal projections onto $\hat{H}$ and $\check{H}$ determined by g_0.

Following [45], we extend each of them to a time-dependent family of projections for $t \in [0, T]$ by solving the fiber-wise ODEs

$$\begin{cases} \partial_t \hat{P}(t) = \mathrm{Rc} \circ \hat{P} - \hat{P} \circ \mathrm{Rc} \\ \hat{P}(0) = \hat{P}_0 \end{cases}, \qquad \begin{cases} \partial_t \check{P}(t) = \mathrm{Rc} \circ \check{P} - \check{P} \circ \mathrm{Rc} \\ \check{P}(0) = \check{P}_0 \end{cases}. \tag{5.2}$$

From $\hat{P}$ and $\check{P}$, we construct time-dependent endomorphisms $\mathcal{P}, \overline{\mathcal{P}} \in \mathrm{End}(\Lambda^2 T^* M)$ by

$$\mathcal{P}\omega(X, Y) = \omega(\hat{P}X, \check{P}Y) + \omega(\check{P}X, \hat{P}Y),$$

$$\overline{\mathcal{P}}\omega(X, Y) = \omega(\hat{P}X, \hat{P}Y) + \omega(\check{P}X, \check{P}Y).$$

Let $\mathrm{Rm} : \Lambda^2 T^* M \to \Lambda^2 T^* M$ be the curvature operator, and define the following:

$$\mathcal{R} = \mathrm{Rm} \circ \mathcal{P}, \qquad \overline{\mathcal{R}} = \mathrm{Rm} \circ \overline{\mathcal{P}},$$

$$\mathcal{S} = (\nabla \mathrm{Rm}) \circ (\mathrm{Id} \times \mathcal{P}), \qquad \mathcal{T} = (\nabla\nabla \mathrm{Rm}) \circ (\mathrm{Id} \times \mathrm{Id} \times \mathcal{P}).$$

In order to study the evolution of $\mathcal{R}$, it will be convenient to introduce an operator Λ_b^a which acts algebraically on tensors via

$$\Lambda_b^a A_{i_1\dots i_l}^{j_1\dots j_k} = \delta_{i_1}^a A_{bi_2\dots i_l}^{j_1\dots j_k} + \dots + \delta_{i_l}^a A_{i_1\dots}^{j_1\dots j_k} - \delta_b^{j_1} A_{i_1\dots i_l}^{a\dots j_k} - \dots - \delta_b^{j_k} A_{i_1\dots i_l}^{j_1\dots a}.$$

We will also consider the operator

$$D_t := \partial_t + R_{ab} g^{bc} \Lambda_c^a.$$

This operator has the property that $D_t g = 0$, and for any time-dependent tensor fields A and B,

$$D_t \langle A, B \rangle = \langle D_t A, B \rangle + \langle A, D_t B \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the metric induced by $g(t)$. Note that by construction the projections satisfy

$$D_t \hat{P} \equiv 0, \quad D_t \check{P} \equiv 0, \quad D_t \mathcal{P} \equiv 0, \quad D_t \overline{\mathcal{P}} \equiv 0.$$

5.2.2 Evolution Equations

In order to determine how the components of $\mathbf{X}$ and $\mathbf{Y}$ evolve, we will make use of the following commutation formulas (see [45], Lemma 4.3):

$$[D_t, \nabla_a] = \nabla_p R_{pacb} \Lambda_c^b + R_{ac} \nabla_c, \tag{5.3}$$

$$[D_t - \Delta, \nabla_a] = 2 R_{abdc} \Lambda_d^c \nabla_b + 2 R_{ab} \nabla_b. \tag{5.4}$$

Additionally, we will need to examine the sharp operator on endomorphisms of two forms. For any $A, B \in \mathrm{End}(\Lambda^2 T^* M)$,

$$\langle A \# B(\varphi), \psi \rangle = \frac{1}{2} \sum_{\alpha,\beta} \langle [A(\omega_\alpha), B(\omega_\beta)], \varphi \rangle \cdot \langle [\omega_\alpha, \omega_\beta], \psi \rangle, \tag{5.5}$$

where $\varphi, \psi \in \Lambda^2 T^* M$ and $\{\omega_\alpha\}$ is an orthonormal basis for $\Lambda^2 T^* M$. Recall that the curvature operator evolves according to

$$(D_t - \Delta) \,\mathrm{Rm} = \mathcal{Q}(\mathrm{Rm}, \mathrm{Rm}),$$

under the Ricci flow, where $\mathcal{Q}(A, B) = \frac{1}{2}(AB + BA) + A\#B$.

Proposition 5.2.1. *We have the following evolution equations for the projection* $\mathcal{P}$*:*

$$D_t \nabla \mathcal{P} = \operatorname{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S},$$

$$D_t \nabla^2 \mathcal{P} = \operatorname{Rm} * \nabla^2 \mathcal{P} + \nabla \operatorname{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{T} + \nabla \operatorname{Rm} * \mathcal{P} * \nabla \mathcal{P}.$$

In particular, there exists a constant $C = C(n)$ *such that*

$$\begin{aligned} |D_t \nabla \mathcal{P}| &\leq C(|\operatorname{Rm}||\nabla \mathcal{P}| + |\mathcal{S}|), \\ |D_t \nabla^2 \mathcal{P}| &\leq C(|\operatorname{Rm}||\nabla^2 \mathcal{P}| + |\nabla \operatorname{Rm}||\nabla \mathcal{P}| + |\mathcal{T}|). \end{aligned} \tag{5.6}$$

Here, for tensors A, B, the notation $A * B$ refers to some finite linear combination of contractions of $A \otimes B$.

Proof. Using equation (5.3) and the fact that $D_t \mathcal{P} = 0$, we can see that $D_t \nabla \mathcal{P} = [D_t, \nabla]\mathcal{P}$. With some additional computation, we can then see (as in Propositions 4.5 and 4.6 from [45]) that

$$D_t \nabla \mathcal{P} = \operatorname{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S}.$$

Similarly, using this equation together with (5.3) and the fact that $\nabla \mathcal{S} = \mathcal{T} + \nabla \operatorname{Rm} * \nabla \mathcal{P}$, we have

$$\begin{aligned} D_t \nabla^2 \mathcal{P} &= [D_t, \nabla]\nabla \mathcal{P} + \nabla(D_t \nabla \mathcal{P}) \\ &= \operatorname{Rm} * \nabla^2 \mathcal{P} + \nabla \operatorname{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{T} + \nabla \operatorname{Rm} * \mathcal{P} * \nabla \mathcal{P}, \end{aligned}$$

as claimed. □

In order compute similar evolution equations for $\mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, we will need the following lemma.

Lemma 5.2.2. *Let $A, B \in \mathrm{End}(\Lambda^2 T^*M)$ be self-adjoint operators. There exists $C = C(n) > 0$ such that*

$$|\mathcal{Q}(A,B)\circ\mathcal{P}| \leq C\left(|A\circ\mathcal{P}||B| + |A||B\circ\mathcal{P}|\right).$$

Proof. Clearly,

$$|(A\circ B + B\circ A)\circ\mathcal{P}| \leq |A\circ\mathcal{P}||B| + |A||B\circ\mathcal{P}|.$$

Furthermore, for $\eta \in \Lambda^2 T^*M$,

$$\begin{aligned}
\big((A\#B)\circ\mathcal{P}\big)(\eta) &= \frac{1}{2}\sum_{\alpha,\beta}\langle[A\omega_\alpha, B\omega_\beta], \mathcal{P}\eta\rangle\cdot[\omega_\alpha,\omega_\beta] \\
&= \frac{1}{2}\sum_{\alpha,\beta}\langle[\mathcal{P}\circ A\omega_\alpha, \mathcal{P}\circ B\omega_\beta], \mathcal{P}\eta\rangle\cdot[\omega_\alpha,\omega_\beta] \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta}\langle[\overline{\mathcal{P}}\circ A\omega_\alpha, \mathcal{P}\circ B\omega_\beta], \mathcal{P}\eta\rangle\cdot[\omega_\alpha,\omega_\beta] \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta}\langle[\mathcal{P}\circ A\omega_\alpha, \overline{\mathcal{P}}\circ B\omega_\beta], \mathcal{P}\eta\rangle\cdot[\omega_\alpha,\omega_\beta] \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta}\langle[\overline{\mathcal{P}}\circ A\omega_\alpha, \overline{\mathcal{P}}\circ B\omega_\beta], \mathcal{P}\eta\rangle\cdot[\omega_\alpha,\omega_\beta],
\end{aligned}$$

where $\{\omega_\alpha\}$ is an orthonormal basis for $\Lambda^2 T^*M$. The final term on the right hand side is zero (see [45], Lemma 3.5); the point is that the image of $\overline{\mathcal{P}}$ is closed under the bracket and is perpendicular to the image of $\mathcal{P}$. Moreover, $\mathcal{P}\circ A = (A\circ\mathcal{P})^*$ and $\mathcal{P}\circ B = (B\circ\mathcal{P})^*$, so it follows that

$$\begin{aligned}
|(A\#B)\circ\mathcal{P}| &\leq C\big(|A\circ\mathcal{P}||B\circ\mathcal{P}| + |A\circ\overline{\mathcal{P}}||B\circ\mathcal{P}| + |A\circ\mathcal{P}||B\circ\overline{\mathcal{P}}|\big) \\
&\leq C\big(|A\circ\mathcal{P}||B| + |A||B\circ\mathcal{P}|\big),
\end{aligned}$$

completing the proof. ∎

Proposition 5.2.3. *As defined above, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ satisfy the inequalities*

$$\begin{aligned}
|(D_t - \Delta)\mathcal{R}| &\le C(|\operatorname{Rm}||\mathcal{R}| + |\nabla \operatorname{Rm}||\nabla \mathcal{P}| + |\operatorname{Rm}||\nabla^2 \mathcal{P}|), \\
|(D_t - \Delta)\mathcal{S}| &\le C(|\nabla \operatorname{Rm}||\mathcal{R}| + |\operatorname{Rm}||\mathcal{S}| + |\nabla^2 \operatorname{Rm}||\nabla \mathcal{P}| + |\nabla \operatorname{Rm}||\nabla^2 \mathcal{P}|), \\
|(D_t - \Delta)\mathcal{T}| &\le C(|\nabla^2 \operatorname{Rm}||\mathcal{R}| + |\nabla \operatorname{Rm}||\mathcal{S}| + |\operatorname{Rm}||\mathcal{T}| \\
&\quad + (|\nabla \operatorname{Rm}||\operatorname{Rm}| + |\nabla^3 \operatorname{Rm}|)|\nabla \mathcal{P}| + |\nabla^2 \mathcal{R}||\nabla^2 \mathcal{P}|),
\end{aligned} \tag{5.7}$$

where $C = C(n) > 0$*.*

Proof. Using the evolution equation for Rm, we have

$$(D_t - \Delta)\mathcal{R} = \mathcal{Q}(\operatorname{Rm}, \operatorname{Rm}) \circ \mathcal{P} + \operatorname{Rm} \circ \Delta \mathcal{P} + 2\nabla \operatorname{Rm} * \nabla \mathcal{P}.$$

The first inequality then follows immediately from Lemma 5.2.2.

We now compute the evolution equation for $\mathcal{S}$. First, note that

$$\begin{aligned}
(D_t - \Delta)\mathcal{S} &= ([D_t - \Delta, \nabla]\operatorname{Rm}) \circ \mathcal{P} + \nabla((D_t - \Delta)\operatorname{Rm}) \circ \mathcal{P} \\
&\quad + \nabla^2 \operatorname{Rm} * \nabla \mathcal{P} + \nabla \operatorname{Rm} * \nabla^2 \mathcal{P}.
\end{aligned} \tag{5.8}$$

For the first term, using the commutator (5.4), we have

$$[(D_t - \Delta), \nabla_a] R_{ijkl} = 2 R_{abdc} \Lambda_d^c \nabla_b R_{ijkl} + 2 R_{ab} \nabla_b R_{ijkl}.$$

As in the computation in Proposition 4.13 from [45], we have

$$R_{abdc} \Lambda_d^c \nabla_b R_{mnkl} \mathcal{P}_{ijmn} = \operatorname{Rm} * \mathcal{S} + \nabla \operatorname{Rm} * \mathcal{R} * \mathcal{P},$$

which gives us

$$([D_t - \Delta, \nabla]\operatorname{Rm}) \circ \mathcal{P} = \operatorname{Rm} * \mathcal{S} + \nabla \operatorname{Rm} * \mathcal{R} * \mathcal{P}. \tag{5.9}$$

We then compute

$$\begin{aligned}
\nabla((D_t - \Delta)\operatorname{Rm}) &= \nabla \mathcal{Q}(\operatorname{Rm}, \operatorname{Rm}) \\
&= \nabla \operatorname{Rm} \circ \operatorname{Rm} + \operatorname{Rm} \circ \nabla \operatorname{Rm} + \nabla \operatorname{Rm} \# \operatorname{Rm} + \operatorname{Rm} \# \nabla \operatorname{Rm} \\
&= 2\mathcal{Q}(\nabla \operatorname{Rm}, \operatorname{Rm}),
\end{aligned}$$

where we regard $\nabla \operatorname{Rm}$ as a one form with values in $\operatorname{Sym}(\Lambda^2 T^*M)$. Then, applying Lemma 5.2.2 and combining the result in (5.9) with (5.8), we obtain the second inequality.

For the third inequality, we begin with the identity

$$(D_t - \Delta)\mathcal{T} = \big((D_t - \Delta)\nabla^2 \operatorname{Rm}\big) \circ \mathcal{P} + \nabla^2 \operatorname{Rm} * \nabla^2 \mathcal{P} + \nabla^3 \operatorname{Rm} * \nabla \mathcal{P}.$$

The first term can be rewritten as

$$\begin{aligned}((D_t - \Delta)\nabla^2 \operatorname{Rm}) \circ \mathcal{P} &= ([D_t - \Delta, \nabla]\nabla \operatorname{Rm}) \circ \mathcal{P} + (\nabla[D_t - \Delta, \nabla] \operatorname{Rm}) \circ \mathcal{P} \\ &\quad + (\nabla\nabla(D_t - \Delta) \operatorname{Rm}) \circ \mathcal{P}.\end{aligned}$$

Applying equation (5.4) once again gives us

$$((D_t - \Delta)\nabla_a \nabla \operatorname{Rm}) \circ \mathcal{P} - (\nabla_a (D_t - \Delta)\nabla \operatorname{Rm}) \circ \mathcal{P} = (2R_{abdc}\Lambda^c_d \nabla_b \nabla \operatorname{Rm} + 2R_{ab}\nabla_b \nabla \operatorname{Rm}) \circ \mathcal{P},$$

and we have

$$R_{abdc}\Lambda^c_d \nabla_b \nabla \operatorname{Rm} \circ \mathcal{P} = \operatorname{Rm} * \mathcal{T} + \nabla^2 \operatorname{Rm} * \mathcal{R} * \mathcal{P}$$

(again see [45], Proposition 4.13, also [56]). We can see that

$$\begin{aligned}(\nabla[D_t - \Delta, \nabla] \operatorname{Rm}) \circ \mathcal{P} &= \nabla(([D_t - \Delta, \nabla] \operatorname{Rm}) \circ \mathcal{P}) + ([D_t - \Delta, \nabla] \operatorname{Rm}) * \nabla \mathcal{P} \\ &= \nabla \operatorname{Rm} * \mathcal{S} + \operatorname{Rm} * \mathcal{T} + \operatorname{Rm} * \nabla \operatorname{Rm} * \nabla \mathcal{P} + \nabla^2 \operatorname{Rm} * \mathcal{R} * \mathcal{P} + \nabla \operatorname{Rm} * \mathcal{S} * \mathcal{P} \\ &\quad + \nabla \operatorname{Rm} * \operatorname{Rm} * \nabla \mathcal{P} * \mathcal{P} + \nabla \operatorname{Rm} * \mathcal{R} * \nabla \mathcal{P} + \operatorname{Rm} * \nabla \operatorname{Rm} * \nabla \mathcal{P}\end{aligned}$$

where we again use the facts that $\nabla \mathcal{R} = \mathcal{S} + \operatorname{Rm} * \nabla \mathcal{P}$ and $\nabla \mathcal{S} = \mathcal{T} + \nabla \operatorname{Rm} * \nabla \mathcal{P}$. Additionally,

$$(\nabla\nabla(D_t - \Delta) \operatorname{Rm}) \circ \mathcal{P} = 2\mathcal{Q}(\nabla^2 \operatorname{Rm}, \operatorname{Rm}) \circ \mathcal{P} + 2\mathcal{Q}(\nabla \operatorname{Rm}, \nabla \operatorname{Rm}) \circ \mathcal{P}.$$

Combining the above identities and again applying Lemma 5.2.2 to the last term, we obtain the third inequality. □

5.2.3 Constructing a PDE-ODE System

With an eye toward Theorem 5.1.1, we now organize the tensors $\nabla\mathcal{P}$, $\nabla^2\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ into groupings which satisfy a closed system of differential inequalities. Let

$$\mathcal{X} = \mathcal{T}^4(T^*M) \oplus \mathcal{T}^5(T^*M) \oplus \mathcal{T}^6(T^*M), \quad \mathcal{Y} = \mathcal{T}^5(T^*M) \oplus \mathcal{T}^6(T^*M),$$

and define families of sections $\mathbf{X} = \mathbf{X}(t)$ of $\mathcal{X}$ and $\mathbf{Y} = \mathbf{Y}(t)$ of $\mathcal{Y}$ for $t \in (0, T]$ by

$$\mathbf{X} = \left(\frac{\mathcal{R}}{t}, \frac{\mathcal{S}}{t^{1/2}}, \mathcal{T}\right), \qquad \mathbf{Y} = \left(\frac{\nabla\mathcal{P}}{t^{1/2}}, \nabla^2\mathcal{P}\right). \tag{5.10}$$

Proposition 5.2.4. *If $g(t)$ is a smooth solution to Ricci flow on $M \times [0, T]$ with $|\operatorname{Rm}|(x,t) < a/t$ for some $a > 0$, then there exists a constant $C = C(a, n) > 0$ such that $\mathbf{X}$ and $\mathbf{Y}$ satisfy*

$$|(D_t - \Delta)\mathbf{X}| \leq C\left(\frac{1}{t}|\mathbf{X}| + \frac{1}{t^2}|\mathbf{Y}|\right), \quad |D_t\mathbf{Y}| \leq C\left(|\mathbf{X}| + \frac{1}{t}|\mathbf{Y}|\right), \tag{5.11}$$

on $M \times (0, T]$.

Remark 5.2.1. *Inspection of the proof reveals that the constant C in fact has the form $C = a\tilde{C}$, where $\tilde{C}$ depends only on n and $\max\{a, 1\}$.*

This follows directly from Propositions 5.2.1 and 5.2.3 with the help of the following curvature bounds, which can be obtained from the classical estimates of Shi [68] with a simple rescaling argument.

Proposition 5.2.5. *Suppose $(M, g(t))$ is a complete solution to Ricci flow for $t \in [0, T]$ which satisfies*

$$|\operatorname{Rm}|(x,t) \leq \frac{a}{t}$$

for some constant $a > 0$. Then for each $m > 0$, there exists a constant $C = C(m, n)$ such that

$$|\nabla^{(m)}\operatorname{Rm}|(x,t) \leq \frac{aC}{t^{m/2+1}}(1 + a^{m/2}).$$

Proof of Proposition 5.2.4. Throughout this proof, C will denote a constant which may change from line to line but depends only on n and a. Using (5.6) in combination with the curvature estimates, we obtain

$$\begin{aligned} |D_t\mathbf{Y}| &\leq \frac{1}{2}t^{-3/2}|\nabla\mathcal{P}| + t^{-1/2}|D_t\nabla\mathcal{P}| + |D_t\nabla^2\mathcal{P}| \\ &\leq Ct^{-1/2}|\mathcal{S}| + C|\mathcal{T}| + Ct^{-3/2}|\nabla\mathcal{P}| + Ct^{-1}|\nabla^2\mathcal{P}| \\ &\leq C|\mathbf{X}| + \frac{C}{t}|\mathbf{Y}|. \end{aligned}$$

Applying the curvature estimates to the inequalities (5.7) for $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$, we get

$$|(D_t - \Delta)\mathcal{R}| \leq Ct^{-1}|\mathcal{R}| + Ct^{-3/2}|\nabla\mathcal{P}| + Ct^{-1}|\nabla^2\mathcal{P}|,$$

$$|(D_t - \Delta)\mathcal{S}| \leq Ct^{-3/2}|\mathcal{R}| + Ct^{-1}|\mathcal{S}| + Ct^{-2}|\nabla\mathcal{P}| + Ct^{-3/2}|\nabla^2\mathcal{P}|,$$

and

$$|(D_t - \Delta)\mathcal{T}| \leq Ct^{-2}|\mathcal{R}| + Ct^{-3/2}|\mathcal{S}| + Ct^{-1}|\mathcal{T}| + Ct^{-5/2}|\nabla\mathcal{P}| + t^{-2}|\nabla^2\mathcal{P}|.$$

Combining these equations, we have

$$\begin{aligned} |(D_t - \Delta)\mathbf{X}| \leq\ & t^{-1}|(D_t - \Delta)\mathcal{R}| + t^{-2}|\mathcal{R}| + t^{-1/2}|(D_t - \Delta)\mathcal{S}| + \frac{1}{2}t^{-3/2}|\mathcal{S}| \\ &+ |(D_t - \Delta)\mathcal{T}| \\ \leq\ & Ct^{-2}|\mathcal{R}| + Ct^{-3/2}|\mathcal{S}| + Ct^{-1}|\mathcal{T}| + Ct^{-5/2}|\nabla\mathcal{P}| + Ct^{-2}|\nabla^2\mathcal{P}| \\ \leq\ & Ct^{-1}|\mathbf{X}| + Ct^{-2}|\mathbf{Y}|, \end{aligned}$$

as desired. □

5.3 A General Uniqueness Theorem for PDE-ODE Systems

We now aim to show that $\mathbf{X}$ and $\mathbf{Y}$ vanish using a maximum principle from [40] by adapting it to apply to a general PDE-ODE system. The following theorem is essentially a reformulation of Lemma 2.3 in [40] and Lemma 2.1 in [56].

Theorem 5.3.1. *Let $M = M^n$ and $\mathcal{X}$ and $\mathcal{Y}$ be finite direct sums of $T^k_l(M)$. There exists an $\epsilon = \epsilon(n) > 0$ with the following property: Whenever $g(t)$ is a smooth, complete solution to the Ricci flow on M satisfying*

$$|\operatorname{Rm}|(x,t) \leq \frac{\epsilon}{t}$$

on $M \times (0, T]$, and $\mathbf{X} = \mathbf{X}(t)$ and $\mathbf{Y} = \mathbf{Y}(t)$ are families of smooth sections of $\mathcal{X}$ and $\mathcal{Y}$ satisfying

$$|(D_t - \Delta)\mathbf{X}| \leq \frac{C}{t}|\mathbf{X}| + \frac{C}{t^2}|\mathbf{Y}|, \quad |D_t\mathbf{Y}| \leq C|\mathbf{X}| + \frac{C}{t}|\mathbf{Y}|,$$

$$D_t^k\mathbf{Y} = 0, \quad D_t^k\mathbf{X} = 0 \text{ for } k \geq 0 \text{ at } t = 0,$$

and

$$|\mathbf{X}| \leq Ct^{-l},$$

for some $C > 0$, $l > 0$, then $\mathbf{X} \equiv 0$ and $\mathbf{Y} \equiv 0$ on $M \times [0, T]$.

The key ingredient in the proof of Theorem 5.3.1 is an the following scalar maximum principle due to Huang-Tam [40] (and its variant in [56]). Though the statement has been slightly changed from its appearance in [40], the proof is nearly identical. We detail here the modifications we make for completeness.

Proposition 5.3.2 (c.f. [40], Lemma 2.3 and [56], Lemma 2.1)**.** *Let M be a smooth n-dimensional manifold. There exists an $\epsilon = \epsilon(n) > 0$ such that the following holds: Whenever $g(t)$ is a smooth complete solution to the Ricci flow on $M \times [0, T]$ such that the curvature satisfies $|\operatorname{Rm}|(x,t) \leq \epsilon/t$ for some and $f \geq 0$ is a smooth function on $M \times [0, T]$ satisfying*

1. $(\partial_t - \Delta) f(x,t) \leq at^{-1} \max_{0 \leq s \leq t} f(x,s)$,
2. $\partial_t^k\big|_{t=0} f = 0$ *for all* $k \geq 0$,

3. $\sup_{x\in M} f(x,t) \le Ct^{-l}$ *for some positive integer* l *for some constant* C*,*

then $f \equiv 0$ *on* $M \times [0,T]$*.*

Proof. For the time-being, we will assume $\epsilon > 0$ is fixed and that $g(t)$ is a smooth, complete solution to Ricci flow on $M \times [0,T]$ satisfying $|\operatorname{Rm}| \le \epsilon/t$. We will then specify ϵ over the course of the proof.

As in [40] we may assume $T \le 1$. We will first show that for any $k > a$, there exists a constant B_k such that

$$\sup_{x\in M} f(x,t) \le B_k t^k.$$

Let ϕ be a cutoff function as in [40], i.e., choose $\phi \in C^\infty([0,\infty))$ such that $0 \le \phi \le 1$ and

$$\phi(s) = \begin{cases} 1 & 0 \le s \le 1, \\ 0 & 2 \le s, \end{cases} \qquad -C_0 \le \phi' \le 0, \quad |\phi''| \le C_0,$$

for some constant $C_0 > 0$. Then let $\Phi = \phi^m$ for $m > 2$ to be chosen later and define $q = 1 - \frac{2}{m}$. Then

$$0 \ge \Phi' \ge -C(m)\Phi^q, \quad |\Phi''| \le C(m)\Phi^q.$$

where $C(m) > 0$ is a constant depending only on m (and on C_0).

Fix a point $y \in M$. As in Lemma 2.2 of [40], there exists some $\rho \in C^\infty(M)$ such that

$$d_{g(T)}(x,y) + 1 \le \rho(x) \le C'(d_{g(T)}(x,y)+1), \quad |\nabla_{g(T)}\rho|_{g(T)} + |\nabla^2_{g(T)}\rho|_{g(T)} \le C',$$

where C' is a positive constant depending only on n and $\frac{\epsilon}{T}$. This function then also satisfies

$$|\nabla\rho| \le C_1 t^{-c\epsilon}, \quad |\Delta\rho| \le C_2 t^{-1/2-c\epsilon},$$

where C_1, C_2 are constants depending only on n, T and ϵ, and $c > 0$ depends only on the dimension n. We may assume ϵ is small enough so that $c\epsilon < 1/4$. Let $\Psi(x) = \Psi_r(x) = \Phi(\rho(x)/r)$ for $r \gg 1$. Define also $\theta = \exp(-\alpha t^{1-\beta})$, where $\alpha > 0$ and $0 < \beta < 1$. By the estimates on the derivatives of ρ, we have

$$|\nabla\Psi| = r^{-1}|\Phi'(\rho/r)||\nabla\rho| \leq r^{-1}C(m)C_1\Phi^q(\rho/r)t^{-c\epsilon} \leq C(m)\Psi^q t^{-1/4}$$

and

$$\begin{aligned}|\Delta\Psi| &= |r^{-2}\Phi''(\rho/r)|\nabla\rho|^2 + r^{-1}\Phi'(\rho/r)\Delta\rho| \\ &\leq r^{-2}C(m)\Phi^q(\rho/r)t^{-2c\epsilon} + r^{-1}C(m)\Phi^q(\rho)t^{-1/2-c\epsilon} \\ &\leq C(m)\Psi^q t^{-3/4}.\end{aligned}$$

For $k > a$, let $F = t^{-k}f$. Then F satisfies

$$\begin{aligned}(\partial_t - \Delta)F &= -kt^{-k-1}f + t^{-k}(\partial_t - \Delta)f \\ &\leq -kt^{-k-1}f(x,t) + at^{-k-1}\max_{0\leq s\leq t} f(x,s)\end{aligned}$$

and $F \leq Ct^{-l-k}$.

Let $H = \theta\Psi F$ and suppose that H attains a positive maximum at the point (x_0, t_0). Then, at this point, we have $\Psi > 0$ and both $(\partial_t - \Delta)H \geq 0$ and $\nabla H = 0$. Since $\nabla H = 0$, we have

$$\nabla\Psi \cdot \nabla F = -\frac{F|\nabla\Psi|^2}{\Psi}.$$

Additionally, since Ψ is independent of time,

$$\theta(s)F(x_0, s) \leq \theta(t_0)F(x_0, t_0)$$

for all $s \leq t_0$. Because θ is decreasing, we have

$$s^{-k}f(x_0, s) = F(x_0, s) \leq F(x_0, t_0) = t_0^{-k}f(x_0, t_0)$$

for $s \le t_0$, which in turn implies

$$\max_{0\le s\le t_0} f(x,s) = f(x,t_0).$$

Thus, at (x_0,t_0) we have

$$(\partial_t - \Delta)F \le (-k+a)t_0^{-1}F \le 0.$$

Thus at (x_0,t_0) we have

$$\begin{aligned}
\Delta H &= \theta F\Delta\Psi + \theta\Psi\Delta F + 2\theta\nabla F\cdot\nabla\Psi \\
&= \theta F\Delta\Psi + \theta\Psi\Delta F - 2\theta\frac{F|\nabla\Psi|^2}{\Psi} \\
&\ge -C(m)\theta F\Psi^q t_0^{-3/4} - C(m)\theta F\Psi^{2q-1}t_0^{-3/4} + \theta\Psi\Delta F
\end{aligned}$$

and

$$\partial_t H = -\alpha(1-\beta)t_0^{-\beta}\theta\Psi F + \theta\Psi\partial_t F.$$

We can then compute

$$\begin{aligned}
0 &\le (\partial_t - \Delta)H \\
&\le \theta\Psi(\partial_t-\Delta)F - \alpha(1-\beta)t_0^{-\beta}\theta\Psi F + C(m)\theta\Psi^q F t_0^{-3/4} + C(m)\theta\Psi^{2q-1}Ft_0^{-3/4} \\
&\le -\alpha(1-\beta)t_0^{-\beta}\theta\Psi F + C(m)\theta(\Psi F)^q t_0^{-3/4-(1-q)(l+k)} \\
&\quad + C(m)\theta(\Psi F)^{2q-1}t_0^{-3/4-(2-2q)(l+k)}.
\end{aligned}$$

We now choose m and β so that the powers of t_0 in the denominators of the last two terms are less than β. We take β to be $7/8$ (any $\beta \in (3/4,1)$ will do). Recalling that $q = 1-2/m$, we choose m large enough so that $7/8 > 3/4 + (1-q)(l+k)$ and $7/8 > 3/4 + (2-2q)(l+k)$. Then

$$\frac{\alpha}{8}\Psi F = \alpha(1-\beta)\Psi F \le C(m)\left((\Psi F)^q + (\Psi F)^{2q-1}\right).$$

Finally, we choose α large enough so that $\alpha > 16C(m)$. Then

$$2\Psi F \le (\Psi F)^q + (\Psi F)^{2q-2},$$

implying that $(\Psi F)(x_0, t_0) \leq 1$, and hence $H \leq 1$ everywhere. In particular, for any $x \in \{\rho \leq r\}$, $f(x,t) = t^k F(x,t) \leq e^{\alpha} t^k := B_k t^k$. Sending r to infinity then proves that $f(x,t) \leq B_k t^k$.

Next, again as in [40], we define the function $\eta(x,t) = \rho(x) \exp\left(\frac{2C_2}{1-b} t^{1-b}\right)$ for $b > 1$. Since $|\Delta \rho| \leq C_2 t^{-b}$, we have

$$(\partial_t - \Delta)\, \eta > 0, \quad \partial_t \eta > 0.$$

Let $F = t^{-a} f$. Fix $\delta > 0$ and consider the function $F - \delta\eta - \delta t$. Note that by our previous argument, $F \leq Ct^2$, and in particular is bounded. For some $t_1 > 0$ depending on δ and c, $F - \delta t < 0$ for $t \leq t_1$ and for $t \geq t_1$, $F - \delta\eta < 0$ outside some compact set. So, if $F - \delta\eta - \delta t$ is ever positive, there must exist some $(x_0, t_0) \in M \times (0, T]$ at which it attains a positive maximum. Because $-\delta\eta - \delta t$ is decreasing in time, for any $s < t_0$ from the inequality

$$F(x_0, s) - \delta\eta(x_0, s) - \delta s \leq F(x_0, t_0) - \delta\eta(x_0, t_0) - \delta t_0,$$

we conclude

$$F(x_0, s) \leq F(x_0, t_0).$$

As in our previous argument, this implies that $f(x_0, t_0) = \max_{0 \leq s \leq t_0} f(x_0, s)$, so that at (x_0, t_0)

$$(\partial_t - \Delta)(F - \delta\eta - \delta t) < 0,$$

a contradiction. Thus, for any $\delta > 0$, $F - \delta\eta - \delta t \leq 0$. Taking $\delta \to 0$ then implies that $F = 0$. □

We can now prove Theorem 5.3.1.

Proof of Theorem 5.3.1. For $k > 0$ to be determined later, define the functions F and G on $M \times [0, T]$ by $F = t^{-k}|\mathbf{X}|^2$, $G = t^{-(k+1)}|\mathbf{Y}|^2$ for $t \in (0, T]$ and $F(x, 0) =$

$G(x, 0) = 0$. From the assumption that $D_t^l \mathbf{X} = D_t^l \mathbf{Y} = 0$ for all $l \geq 0$, it follows that both F and G are smooth on $M \times [0, T]$ and that $\partial_t^l F = \partial_t^l G = 0$ for all $l \geq 0$.

We have

$$\begin{aligned}
(\partial_t - \Delta) F &= -kt^{-(k+1)} |\mathbf{X}|^2 + 2t^{-k} \langle (D_t - \Delta)\mathbf{X}, \mathbf{X} \rangle - 2t^{-k} |\nabla \mathbf{X}|^2 \\
&\leq -kt^{-(k+1)} |\mathbf{X}|^2 + 2t^{-k} |(D_t - \Delta)\mathbf{X}||\mathbf{X}| \\
&\leq t^{-(k+1)}(2C - k)|\mathbf{X}|^2 + 2Ct^{-(k+2)}|\mathbf{X}||\mathbf{Y}| \\
&\leq t^{-1}(3C - k)F + Ct^{-2}G
\end{aligned}$$

and

$$\begin{aligned}
\partial_t G &= -(k+1)t^{-(k+2)}|\mathbf{Y}|^2 + 2t^{-(k+1)} \langle D_t \mathbf{Y}, \mathbf{Y} \rangle \\
&\leq (2C - k - 1)t^{-(k+2)}|\mathbf{Y}|^2 + 2Ct^{-(k+1)}|\mathbf{X}||\mathbf{Y}| \\
&\leq CF + t^{-1}(3C - k - 1)G.
\end{aligned}$$

Choosing $k > 3C$, this becomes

$$(\partial_t - \Delta) F \leq t^{-2} C G, \qquad \partial_t G \leq CF.$$

In particular this implies that

$$G(x, t) \leq Ct \max_{0 \leq s \leq t} F(x, s),$$

and therefore

$$(\partial_t - \Delta) F \leq t^{-1} C^2 \max_{0 \leq s \leq t} F(x, s).$$

By our assumption on $\mathbf{X}$, $F \leq Ct^{-2l-k}$. Thus F satisfies the hypotheses of Proposition 5.3.2, and must vanish identically. We then conclude that G, hence $\mathbf{Y}$, vanishes as well. □

5.4 Proof of Theorem 5.1.1

We are now almost ready to prove Theorem 5.1.1. We just need to first verify that $\mathbf{X}$ and $\mathbf{Y}$ satisfy the last major remaining hypothesis of Theorem 5.3.1, that is, that all time derivatives of $\mathbf{X}$ and $\mathbf{Y}$ vanish at $t = 0$.

5.4.1 Vanishing of Time Derivatives

We begin by recording a standard commutator formula, which is in fact valid (with obvious modifications) for any family of smooth metrics.

Proposition 5.4.1. *Let $(M, g(t))$ be a smooth solution to the Ricci flow for $t \in [0, T]$. Then, for any $l \geq 1$, the formula*

$$[D_t, \nabla^{(l)}]\mathcal{A} = \sum_{k=1}^{l} \nabla^{(k-1)}[D_t, \nabla]\nabla^{(l-k)}\mathcal{A} \tag{5.12}$$

is valid for any smooth family of tensor fields $\mathcal{A}$ on $M \times [0, T]$.

Proof. We proceed by induction on l. The base case, $l = 1$, is trivial. Now, suppose that (5.12) holds for $l \leq m$ for some $m \geq 1$. Then,

$$\begin{aligned}
[D_t, \nabla^{(m+1)}]\mathcal{A} &= D_t\nabla^{(m+1)}\mathcal{A} - \nabla^{(m+1)}D_t\mathcal{A} \\
&= [D_t, \nabla^{(m)}]\nabla\mathcal{A} + \nabla^m D_t\nabla\mathcal{A} - \nabla^{(m+1)}D_t\mathcal{A} \\
&= [D_t, \nabla^{(m)}]\nabla\mathcal{A} + \nabla^{(m)}[D_t, \nabla]\mathcal{A} \\
&= \sum_{k=1}^{m} \nabla^{(k-1)}[D_t, \nabla]\nabla^{(m-k)}(\nabla\mathcal{A}) + \nabla^{(m)}[D_t, \nabla]\mathcal{A} \\
&= \sum_{k=1}^{m+1} \nabla^{(k-1)}[D_t, \nabla]\nabla^{(m+1-k)}\mathcal{A},
\end{aligned}$$

as desired. □

Now we argue inductively that $D_t^k\mathbf{X} = 0$ and $D_t^k\mathbf{Y} = 0$ at $t = 0$.

Proposition 5.4.2. *Let $M = \hat{M} \times \check{M}$ be a smooth manifold and $g(t)$ be a smooth, complete solution to the Ricci flow such that $g(0)$ splits as a product. Define $\mathcal{P}$ and $\mathcal{R}$ as in Section 2. The following equations hold at $t = 0$ for all $k, l \geq 0$:*

$$D_t^k\nabla^{(l)}\mathcal{R} = 0, \quad D_t^k\nabla^{(l+1)}\mathcal{P} = 0.$$

Proof. We proceed by induction on k, beginning with the base case $k = 0$. Because the metric splits as a product initially, at $t = 0$ we have $\nabla^{(l)}\hat{P} \equiv \nabla^{(l)}\check{P} \equiv 0$ for all $l \geq 0$ and

$$R(\hat{P}(\cdot), \check{P}(\cdot), \cdot, \cdot) \equiv 0.$$

From this we get that, for any $X, Y, Z, W \in TM$,

$$\mathcal{R}(X^* \wedge Y^*)(Z, W) = 2R(\hat{P}X, \check{P}Y, W, Z) + 2R(\check{P}X, \hat{P}Y, W, Z) = 0.$$

Combining these facts, we conclude

$$\nabla^{(l+1)}\mathcal{P} \equiv 0, \quad \nabla^{(l)}\mathcal{R} \equiv 0, \quad \nabla^{(l)}\mathcal{R}^* \equiv 0,$$

at $t = 0$, where $\mathcal{R}^* = \mathcal{P} \circ \mathrm{Rm}$ denotes the adjoint of $\mathcal{R}$ with respect to g.

Now starting the induction step, suppose that for some $k \geq 0$, for all $l \geq 0$ and any $m \leq k$,

$$D_t^m \nabla^{(l+1)}\mathcal{P} = 0, \quad D_t^m \nabla^{(l)}\mathcal{R} = 0,$$

hence also $D_t^m \nabla^{(l)}\mathcal{R}^* = 0$. Recall that

$$(D_t - \Delta)\,\mathrm{Rm} = \mathcal{Q}(\mathrm{Rm}, \mathrm{Rm}).$$

As in [45], Lemma 4.9, $\mathcal{Q}(\mathrm{Rm}, \mathrm{Rm}) \circ \mathcal{P} = \mathcal{R} * \mathcal{U}_1 + \mathcal{R}^* * \mathcal{U}_2$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are smooth families of tensors on M. Thus we can compute

$$\begin{aligned} D_t\mathcal{R} &= (D_t \,\mathrm{Rm}) \circ \mathcal{P} + \mathrm{Rm} \circ (D_t\mathcal{P}) \\ &= (\Delta\,\mathrm{Rm}) \circ \mathcal{P} + \mathcal{Q}(\mathrm{Rm}, \mathrm{Rm}) \circ \mathcal{P}, \end{aligned}$$

and thus

$$D_t^{k+1}\mathcal{R} = D_t^k\left((\Delta\,\mathrm{Rm}) \circ \mathcal{P}\right) + D_t^k\left(\mathcal{Q}(\mathrm{Rm}, \mathrm{Rm}) \circ \mathcal{P}\right). \tag{5.13}$$

Because

$$\Delta\mathcal{R} = (\Delta\,\mathrm{Rm}) \circ \mathcal{P} + \mathrm{Rm} \circ \Delta\mathcal{P} + 2\nabla_i \,\mathrm{Rm} \circ \nabla_i\mathcal{P},$$

by the induction hypothesis $D_t^k((\Delta \operatorname{Rm}) \circ \mathcal{P}) \equiv 0$ at $t = 0$. Similarly,

$$D_t^k\big(\mathcal{Q}(\operatorname{Rm}, \operatorname{Rm}) \circ \mathcal{P}\big) = D_t^k(\mathcal{R} * \mathcal{U}_1) + D_t^k(\mathcal{R}^* * \mathcal{U}_2) = 0.$$

We conclude that $D_t^{k+1}\mathcal{R} \equiv 0$, and thus $D_t^{k+1}\mathcal{R}^* \equiv 0$.

Now, using the commutator from equation (5.3) and Proposition 5.4.1, for any $l > 0$ we have

$$\begin{aligned} D_t \nabla^{(l)} \mathcal{R} &= \sum_{m=1}^{l} \nabla^{(m-1)} [D_t, \nabla] \nabla^{(l-m)} \mathcal{R} + \nabla^{(l)} D_t \mathcal{R} \\ &= \sum_{m=1}^{l} \nabla^{(m-1)} \left(\nabla \operatorname{Rm} * \nabla^{(l-m)} \mathcal{R} + \operatorname{Rm} * \nabla^{(l-m+1)} \mathcal{R}\right) + \nabla^{(l)} D_t \mathcal{R}, \end{aligned}$$

and thus

$$D_t^{k+1} \nabla^{(l)} \mathcal{R} = \sum_{m=1}^{l} D_t^k \nabla^{(m-1)} \left(\nabla \operatorname{Rm} * \nabla^{(l-m)} \mathcal{R} + \operatorname{Rm} * \nabla^{(l-m+1)} \mathcal{R}\right) + D_t^k \nabla^{(l)} D_t \mathcal{R}.$$

Expanding using the product rule and applying the induction hypothesis, all terms in the first sum vanish at $t = 0$. For the remaining term, we again use the evolution equation for $\mathcal{R}$. We have

$$D_t^k \nabla^{(l)} D_t \mathcal{R} = D_t^k \nabla^{(l)} \left((\Delta \operatorname{Rm}) \circ \mathcal{P} + \mathcal{Q}(\operatorname{Rm}, \operatorname{Rm}) \circ \mathcal{P}\right).$$

As before, rewriting $\mathcal{Q}(\operatorname{Rm}, \operatorname{Rm}) \circ \mathcal{P}$ in terms of $\mathcal{R}$ and $\mathcal{R}^*$ and expanding using the product rule, it follows that $D_t^k \nabla^{(l)} D_t \mathcal{R} \equiv 0$ at $t = 0$.

We now move on to the derivatives of $\mathcal{P}$. Recall that

$$D_t \nabla \mathcal{P} = [D_t, \nabla]\mathcal{P} = \operatorname{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S}.$$

Applying this in combination with Proposition 5.4.1, we get, for any $l \geq 1$,

$$\begin{aligned} D_t^{k+1} \nabla^{(l)} \mathcal{P} &= \sum_{m=1}^{l} D_t^k \nabla^{(m-1)} [D_t, \nabla] \nabla^{(l-m)} \mathcal{P} + D_t^k \nabla^{(l)} D_t \mathcal{P} \\ &= \sum_{m=1}^{l-1} D_t^k \nabla^{(m-1)} \big(\nabla \operatorname{Rm} * \nabla^{(l-m)} \mathcal{P} + \operatorname{Rm} * \nabla^{(l-m+1)} \mathcal{P}\big) \\ &\quad + D_t^k \nabla^{(l-1)} [D_t, \nabla] \mathcal{P} + D_t^k \nabla^{(l)} D_t \mathcal{P}. \end{aligned}$$

As before, every term in the first sum vanishes by the induction hypothesis, while the final term vanishes because $D_t\mathcal{P} \equiv 0$. Finally we can see that

$$D_t^k \nabla^{(l-1)}[D_t, \nabla]\mathcal{P} = D_t^k \nabla^{(l-1)}(\mathrm{Rm} * \nabla\mathcal{P} + \mathcal{P} * \mathcal{S}),$$

and because $\mathcal{S} = (\nabla \mathrm{Rm}) \circ \mathcal{P} = \nabla\mathcal{R} + \mathrm{Rm} * \nabla\mathcal{P}$, $D_t^k \nabla^{(l-1)}[D_t, \nabla]\mathcal{P} \equiv 0$ at $t = 0$. This completes the proof. □

5.4.2 Preservation of Product Structures

In the proof of Theorem 5.1.1, we will use the operator $\mathcal{F} : \Lambda^2 T^*M \to \Lambda^2 T^*M$ defined by

$$\mathcal{F}\omega(X, Y) = \omega(\hat{P}X, \check{P}Y) - \omega(\check{P}X, \hat{P}Y).$$

(See, for example, Section 2.2 of [49].) Observe that

$$\begin{aligned}
\mathcal{P} \circ \mathcal{F}\omega(X, Y) &= \mathcal{F}\omega(\hat{P}X, \check{P}Y) + \mathcal{F}\omega(\check{P}X, \hat{P}Y) \\
&= \omega(\hat{P}^2X, \check{P}^2Y) - \omega(\check{P}\hat{P}X, \hat{P}\check{P}Y) + \omega(\hat{P}\check{P}X, \check{P}\hat{P}Y) - \omega(\check{P}^2X, \hat{P}^2Y) \\
&= \omega(\hat{P}X, \check{P}Y) - \omega(\check{P}X, \hat{P}Y).
\end{aligned}$$

Therefore $\mathcal{P} \circ \mathcal{F} \equiv \mathcal{F}$.

Proof of Theorem 5.1.1. We have shown in Propositions 5.2.4 and 5.4.2 that the system $\mathbf{X}, \mathbf{Y}$ satisfies the first two hypotheses of Theorem 5.3.1. Additionally, the curvature bounds from Proposition 5.2.5 imply that $|\mathbf{X}| \leq Ct^{-2}$. Thus, $\mathbf{X} \equiv 0$ and $\mathbf{Y} \equiv 0$ on $M \times [0, T]$. In particular, we know that $\mathcal{R} \equiv 0$ and $\nabla\mathcal{P} \equiv 0$ on $M \times [0, T]$.

We claim that $\nabla\hat{P} \equiv \nabla\check{P} \equiv 0$ and $\partial_t\hat{P} \equiv \partial_t\check{P} \equiv 0$. Similar to the proof of Lemma 7 in [49], if we define $W = \nabla\hat{P}$, then

$$D_t W_{ai}^j = [D_t, \nabla_a]\hat{P}_i^j = \nabla_p R_{pai}^c \hat{P}_c^j - \nabla_p R_{pab}^j \hat{P}_i^b + R_a^c W_{ci}^j.$$

Note that the first two terms combine to give

$$\begin{aligned}
&\langle \nabla_{e_p} R(e_p, e_a)\check{P}e_i, \hat{P}e_j\rangle - \langle \nabla_{e_p} R(e_p, e_a)\hat{P}e_i, \check{P}e_j\rangle \\
&\quad = \frac{1}{2}\left(\nabla_{e_p} \operatorname{Rm}(\check{P}e_i^* \wedge \hat{P}e_j^*)(e_p, e_a) - \nabla_{e_p} \operatorname{Rm}(\hat{P}e_i^* \wedge \check{P}e_j^*)(e_p, e_a)\right) \\
&\quad = -\frac{1}{2}\nabla_{e_p} \operatorname{Rm} \circ \mathcal{F}(e_i^* \wedge e_j^*)(e_p, e_a).
\end{aligned}$$

But, since $\mathcal{P} \circ \mathcal{F} = \mathcal{F}$,

$$\nabla \operatorname{Rm} \circ \mathcal{F} = \nabla \operatorname{Rm} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{S} \circ \mathcal{F} = 0,$$

so $D_t W_{ai}^j = R_a^c W_{ci}^j$. Thus, for any point $x \in M$, the function $f(t) = |\nabla \hat{P}|^2(x,t)$ satisfies

$$f'(t) \leq Cf$$

for some C depending on x. Since $f(0) = 0$, f is identically zero. Thus $\hat{P}$ (and similarly $\check{P}$) remain parallel.

Hence, we have

$$R(\cdot, \cdot, \hat{P}(\cdot), \check{P}(\cdot)) = 0,$$

which implies that $\operatorname{Rc} \circ \hat{P} = \hat{P} \circ \operatorname{Rc}$ and $\operatorname{Rc} \circ \check{P} = \check{P} \circ \operatorname{Rc}$, and thus, from (5.2), $\partial_t \hat{P} = \partial_t \check{P} = 0$ on $[0, T]$. Theorem 5.1.1 follows. □

5.5 Further Questions: Preservation of Holonomy Under a Non-uniform Curvature Bound

It would be interesting to know if the statement of Theorem 5.1.1 holds for solutions satisfying the more general curvature bound

$$\sup_{x \in M} |\operatorname{Rm}|(x,t) \leq \frac{c}{t}, \tag{5.14}$$

where $c > 0$ is an arbitrary constant. It isn't clear whether the restriction to $c \leq \epsilon(n)$ is essential or simply an artifact of our particular proof of the maximum principle (Theorem 5.3.1).

Question 5.5.1. *Let* $(M, g(t))$ *be a complete solution to (1.1) satisfying*

$$\sup_M |\operatorname{Rm}(g(t))| \leq \frac{c}{t}$$

for some $c > 0$*. Then, if* $g(0)$ *splits as a product, must* $g(t)$ *split for all time such that the solution exists?*

The analogous statement in the context of Kähler structures has been considered by Huang and Tam [40].

Notice that the geometric assumptions of Theorem 5.1.1 were really only used in the derivation of the PDE-ODE system (5.11). After that, the argument only made use of the structure of (5.11), and not of the internal structure of our specific choices of $\mathbf{X}$ and $\mathbf{Y}$. In [44], [48], [55] the problem of uniqueness for the Ricci flow is reduced to that of solutions to a PDE-ODE system of a similar, but not identical, form. It is an interesting question whether a version of the maximum principle (Theorem 5.3.1) might be proven for systems of this form. Such a maximum principle could potentially be of use in the study of the uniqueness of solutions to the Ricci flow satisfying a general instantaneous curvature bound.

Chapter 6

ASYMPTOTICALLY PRODUCT SHRINKING RICCI SOLITONS

6.1 Asymptotic Behavior of Shrinking Ricci Solitons

Recall that a *gradient shrinking Ricci soliton* is a Riemannian manifold (M, g) satisfying

$$\mathrm{Rc}(g) + \nabla\nabla f = \frac{g}{2} \tag{6.1}$$

for some $f \in C^\infty(M)$.

Ricci solitons correspond to generalized fixed points of the flow: self-similar solutions which change only by scaling and diffeomorphisms. Shrinking Ricci solitons are of particular importance to the study of the singular behavior of the flow. The work of [37], [61], [63], culminating in [24], has shown, e.g., that from a blow-up sequence taken about a Type-I singular point, one can extract a sequence converging to a nontrivial shrinking gradient Ricci soliton. Recent work of Bamler [2, 3, 4] has now shown that essentially any singularity can be modeled by a shrinking gradient Ricci soliton (except maybe on a singular set of lower dimension). For potential future topological applications of the flow, it is desirable to have as complete a classification of gradient shrinkers as possible.

Complete shrinking Ricci solitons are completely classified in dimensions two and three. It is a result of Hamilton [34] that the only nontrivial complete shrinking Ricci solitons in dimension two are the round metrics on S^2 and $\mathbb{R}P^2$. In dimension three, the combined results of Hamilton [37], Ivey [41], Perelman [63], Ni-Wallach [62], and Cao-Chen-Zhu [12] show that the only nontrivial complete examples are quotients of S^3 or $\mathbb{R} \times S^2$, both with their standard metrics. While a complete classification in dimension four and above is not expected, the class of 4-D complete noncompact shrinking Ricci solitons may be rigid enough to support a classification, perhaps in terms of their asymptotic geometries. Broadly, we are interested in exploring the following question:

- What are the possibilities for the asymptotic geometry of a gradient shrinking Ricci soliton?

Cao-Zhou [13] have proven sharp bounds on the potential function for a gradient shrinking Ricci soliton of any dimension, as well as sharp upper and lower bounds on its volume growth. In 4-D, the picture is becoming more refined. The work of Munteanu and Wang [57, 58, 59, 60] has established a near dichotomy: If $R \geq c$, then each end of the soliton is either asymptotic to a quotient of $\mathbb{R} \times S^3$ or converges to a quotient of $\mathbb{R}^2 \times S^2$ along integral curves of ∇f. If $R \to 0$ along an end, then the soliton is asymptotic to a cone (recall that by [14], $R \geq 0$ on any complete shrinker). At present, all complete noncompact examples in any dimension either split locally or are asymptotic to a cone, and in the latter case, there are still relatively few known examples (to the author's knowledge, the examples found in [1], [21], [26], and [76] form a complete list). In dimension four, there is currently only one nontrivial example known–the construction due to Feldman, Ilmanen, and Knopf [26]. According to [17], it is unique among Kähler examples.

These observations suggest the following questions.

- If a gradient shrinker is asymptotic to a product along some end, does it split globally as a product?

- If a gradient shrinker is, in some sense, sufficiently close to a known shrinker along some end, are the two isometric?

In this direction, Kotschwar and Wang have proven the following.

Theorem 6.1.1 ([50], [51]). *Let (M, g, f) be a shrinking gradient Ricci soliton.*

(1) *If two shrinkers are asymptotic to a regular cone $((0,\infty)\times\Sigma, dr^2+r^2 g_\Sigma)$ along some end, then the two are isometric on a neighborhood of infinity.*

(2) *If a gradient shrinking soliton (M, g) is strongly asymptotic to an end of the standard cylinder $\mathbb{R}^{n-k}\times S^k, k\geq 2$ along some end $E\subset M$, then it is isometric to the cylinder along that end.*

Here, strongly asymptotic means that g converges to the cylindrical metric to "infinite order at spatial infinity", that is,

$$\sup_{(z,\theta)\in\mathbb{R}^{n-k}\times S^k}\left\{|z|^l|\nabla^{(m)}_{g_{cyl}}(\Phi^* g - g_{cyl})|(z,\theta)\right\} < \infty$$

for all $l, m\geq 0$, where $\Phi : \tilde{E}\to E$ is some diffeomorphism from an end $\tilde{E}$ of the cylinder to E. The proof in [51] converts the problem into problem of backward uniqueness for a PDE-ODE system, which is then analyzed using Carleman inequalities, similar to the method from [25].

Although the assumption that the soliton is actually *strongly* asymptotic to the cylinder is apparently very restrictive, it is likely that, in the generality of the statement, the condition cannot be significantly relaxed. Wang [75] has shown, for example, that there are (incomplete) self-shrinkers to the mean curvature flow which are not rotationally symmetric but agree to arbitrary finite order with the round cylinder $\mathbb{R}\times S^{n-1}$ at infinity. Similar examples likely also exist for the Ricci flow.

The work above suggests several immediate questions.

- If a gradient shrinking Ricci soliton is strongly asymptotic to a general product along some end, does it locally split as a product?

- If a complete gradient shrinking Ricci soliton is asymptotic to a cone, cylinder, or other product, must it approach the model at some fixed minimum rate?

In a current project [19], motivated by the first question, we consider the uniqueness of shrinkers strongly asymptotic to a general product of the form $\mathbb{R}^{n-k} \times \Sigma$ where Σ is an arbitrary compact shrinker. While the large-scale strategy is much the same as in [51], our approach differs fundamentally in a few key places. After some initial normalizations, we connect the problem to one of parabolic unique continuation for a PDE-ODE system akin to the one from [18] discussed above. Below, we discuss this reduction process in greater detail.

6.2 Constructing a PDE-ODE System

A shrinking Ricci soliton corresponds (at least locally) to a solution $(M, g(\tau))$ to the *backward* Ricci flow, i.e., the equation

$$\frac{\partial}{\partial \tau} g = 2\,\mathrm{Rc}\,. \tag{6.2}$$

Indeed, if (M, g_1, f) solves (6.1), then, solving the equation

$$\frac{\partial}{\partial \tau}\Phi = -\frac{1}{\tau}\nabla_{g_1} f \circ \Phi, \qquad \Phi_1 = \mathrm{Id},$$

we obtain a locally defined solution $g(\tau)$ to (6.2) with $g(1) = g_1$ by setting $g(\tau) = \tau \Phi_\tau^* g_1$. When (M, g_1) is complete, it is a theorem of Zhang [77] that $\nabla_{g_1} f$ is a complete vector field, and in this case the solution $g(\tau)$ will be defined on $M \times (0, \infty)$.

In [19], after some initial reductions, the problem boils down to the consideration of the following situation: we have an unknown shrinking soliton (M, g, f) which is strongly asymptotic to $(M, \tilde{g}, \tilde{f})$, where

$$M = M_a = \begin{cases} \left(\mathbb{R}^{n-k} \setminus \overline{B_a(0)}\right) \times \Sigma, & 2 \leq k < n-1, \\ (a, \infty) \times \Sigma, & k = n-1, \end{cases}$$

$(\Sigma, g_\Sigma, f_\Sigma)$ is a compact k-dimensional shrinking soliton, $\tilde{g} = \overline{g} \oplus g_\Sigma$ is the product of the Euclidean metric $\overline{g}$ with g_Σ, and $\tilde{f}(z,\sigma) = \frac{|z|^2}{4} + f_\Sigma(\sigma)$. As in [51], we may assume further that

$$\nabla f = \tilde{\nabla}\tilde{f} := X = \sum_{i=1}^{n-k} \frac{z^i}{2}\frac{\partial}{\partial z^i} + \nabla_\Sigma f_\Sigma.$$

In this case, the solutions to (6.2) associated to the unknown and model solitons flow along the *same* vector field and are of the form $g(\tau) = \tau\Phi_\tau^* g$ and $\tilde{g}(\tau) = \tau\Phi_\tau^*\tilde{g}$, where Φ_τ has the form $\Phi_\tau(z,\sigma) = (\frac{z}{\sqrt{\tau}}, \varphi_\tau(\sigma))$. These (incomplete) solutions are defined on $M \times (0,1]$. Note that $\tilde{g}(\tau) = \overline{g} \oplus \tau\varphi_\tau^* g_\Sigma$.

The assumption that g is strongly asymptotic to $\tilde{g}$ on M implies that we have the space-time decay bounds

$$\sup_{M\times(0,1]} \left\{ \frac{|z|^{2l}}{\tau^l} |\tilde{\nabla}^{(m)}(\tilde{g} - g)|_{\tilde{g}}(z,\sigma,\tau) \right\} < \infty. \tag{6.3}$$

Thus, our original problem of unique continuation at infinity has been converted to a problem of parabolic unique continuation for the backward Ricci flow at the singular time.

Rather than to try to estimate the difference of $\tilde{g}(\tau)$ and $g(\tau)$ directly as in [51], we instead analyze the solution $g(\tau)$ directly and show that the fact that it is asymptotic to a product implies that it is itself a product. We measure the failure of $g(\tau)$ to be a product as follows.

From the bounds (6.3), we obtain (from a limiting argument similar to that in Appendix 4 of [51]) families of smooth sections $\hat{P}(\tau), \check{P}(\tau)$ of $\mathrm{End}(TM)$ for $\tau \in (0,1]$ which satisfy

1. $\hat{P}$ and $\check{P}$ are projections onto complementarary $g(\tau)$-orthogonal families of subbundles

$$H = H_\tau = \hat{P}_\tau(TM), \quad K = K_\tau = \check{P}_\tau(TM).$$

2. $D_\tau \hat{P} \equiv D_\tau \check{P} \equiv 0$.

3. $\hat{P}$ and $\check{P}$ agree to infinite order as $|z| \to \infty$ and $\tau \to 0$ with the $\tilde{g}$-orthogonal projections $\hat{P}_{\tilde{g}}$ and $\check{P}_{\tilde{g}}$, and in fact,

$$\sup_{M\times(0,1]} \left\{ \frac{|z|^{2l}}{\tau^l} |\nabla^{(m)}_{g(\tau)} \hat{P}_\tau|_{g(\tau)} \right\} < \infty$$

for all l and $m \geq 1$ (and similarly for $\check{P}_\tau$).

Next, in the spirit of Chapter 5, from $\hat{P}$ and $\check{P}$ we define smooth families of projections $\mathcal{P} = \mathcal{P}(\tau), \overline{\mathcal{P}} = \overline{\mathcal{P}}(\tau) \in C^\infty(\mathcal{E})$, where $\mathcal{E} = \mathrm{End}(\Lambda^2(T^*M))$, by

$$\mathcal{P}\omega(X,Y) = \omega(\hat{P}X, \check{P}Y) + \omega(\check{P}X, \hat{P}Y),$$
$$\overline{\mathcal{P}}\omega(X,Y) = \omega(\hat{P}X, \hat{P}Y) + \omega(\check{P}X, \check{P}Y).$$

Then we let

$$\mathcal{R} = \mathrm{Rm} - \mathcal{P} \circ R \circ \mathcal{P}, \qquad E = \mathcal{P} \circ R \circ \mathcal{P},$$
$$\mathcal{S} = \nabla R - \mathcal{P} \circ \nabla \,\mathrm{Rm} \circ \mathcal{P}, \qquad F = \mathcal{P} \circ \nabla R \circ \mathcal{P},$$

for $\mathrm{Rm} \in C^\infty(\mathcal{E}_s)$, where $\mathcal{E}_s$ denotes self-adjoint elements of $\mathcal{E}$ and Rm again denotes the Riemann curvature operator. Under the assumption (6.3), both $\mathcal{R}$ and $\mathcal{S}$ will vanish to infinite order as $\tau \to 0$ and $|z| \to \infty$.

As in Chapter 5, to show that $(M, g(\tau))$ splits, it is enough to show that $\nabla \hat{P}(\tau) \equiv 0$, however, the evolution of $\nabla \hat{P}(\tau)$ depends on the curvature tensor of $g(\tau)$ and its derivatives. The reason we instead consider a system consisting of $\nabla\mathcal{P}$, $\nabla^2\mathcal{P}$ and $\mathcal{S}$ (as in [45] and [18]) is that these objects together satisfy a *closed* system of differential inequalities. The approach in [19] is to argue from this closed system that $\nabla\mathcal{P}$, $\nabla^2\mathcal{P}$, and $\mathcal{S}$ vanish identically. This implies that $\nabla\hat{P}$ vanishes identically, and is in fact independent of time. Hence the splitting for g is independent of time. Below, we discuss the derivation of this system in detail.

6.3 The PDE-ODE System

First, we introduce two operators which will appear in the evolution equation for $\mathcal{S}$. We define $\mathcal{L}, \mathcal{Q} : (T^*M \otimes \mathcal{E}) \times \mathcal{E} \to T^*M \otimes \mathcal{E}$ by

$$\begin{aligned}\mathcal{L}(A,B)_{eabcd} &= A_{pqbcd}B_{aqep} - A_{pqacd}B_{bqep} + A_{pqdab}B_{cqep} - A_{pqcab}B_{dqep} \\ &= -(B_{peqm}(\delta_{mn}\Lambda^n_q - \delta_{qn}\Lambda^n_m)A_{pabcd}),\end{aligned}$$

where Λ^n_q is defined as in Chapter 5, and

$$\mathcal{Q}(A,B) = AB + BA + 2A\#B.$$

Here, $A\#B(V,\omega,\eta) := (A(V,\cdot)\#B)(\omega,\eta)$, and $\#$ is defined as in Chapter 5. There is a natural identification of the space $\mathcal{E}$ with $\Lambda^2(T^*M)\otimes\Lambda^2(T^*M)$, where an element $A \in \Lambda^2(T^*M)\otimes\Lambda^2(T^*M)$ is identified with the operator $A \in \mathcal{E}$ given by

$$A(\eta)_{ab} = A(\eta)(e_a,e_b) = -A_{abcd}\eta_{cd},$$

where $\{e_i\}$ is a local orthonormal frame for TM. (The minus sign on the right hand side is chosen so that, by our convention for the $(4,0)$-curvature tensor, the curvature operator Rm has positive eigenvalues on the sphere.) In particular $\omega_{ij} = e_i^* \wedge e_j^*$, then

$$\langle A(\omega_{ab}), \omega_{cd}\rangle = -4A_{abcd}.$$

Finally, we define $\mathcal{B} : (T^*M\otimes\mathcal{E}_s)\times\mathcal{E}_s \to T^*M\otimes\mathcal{E}_s$ by $\mathcal{B}(\cdot) = \mathcal{L}(\cdot,E) + \mathcal{Q}(\cdot,E)$. It is not hard to see that $\mathcal{B}$ is self-adjoint.

Proposition 6.3.1. *The sections $\mathbf{X} = \mathcal{S}, \mathbf{Y} = (\nabla\mathcal{P}, \nabla^2\mathcal{P})$ satisfy the system*

$$|(D_\tau + \Delta + \mathcal{B})\mathbf{X}| \le C(|\mathcal{R}||\mathbf{X}| + (|\nabla \operatorname{Rm}| + |\nabla^2 \operatorname{Rm}| + |\nabla\mathcal{P}|)|\mathbf{Y}|),$$

$$|D_\tau \mathbf{Y}| \le C(|\mathbf{X}| + |\nabla\mathbf{X}| + (|\operatorname{Rm}| + |\nabla \operatorname{Rm}|)|\mathbf{Y}|),$$

on $M\times(0,1]$ for some constant $C > 0$.

Here, $D_\tau = \frac{\partial}{\partial \tau} - R_q^p \Lambda_p^q$. This system should be compared to Proposition 4.1 in [51]. The advantage of this formulation (based on the elliptic operator $\Delta + \mathcal{B}$) is the that the coefficient of $|\mathbf{X}|$ on the right hand side of the first equation vanishes to infinite order in space and time.

To prove Proposition 6.3.1, we first compute evolution equations for $\mathcal{S}, \nabla\mathcal{P}$, and $\nabla^2\mathcal{P}$. As in Chapter 5, recall that for tensors A, B, the notation $A * B$ refers to some finite linear combination of contractions of $A \otimes B$.

Proposition 6.3.2.

$$(D_\tau + \Delta + \mathcal{B})\mathcal{S} = \mathcal{S} * \mathcal{R} + \mathcal{R} * F + \Delta\mathcal{P} * \nabla \operatorname{Rm} * \mathcal{P} + (\nabla\mathcal{P})^2 * \nabla \operatorname{Rm} + \mathcal{P} * \nabla\mathcal{P} * \nabla^2 \operatorname{Rm}.$$

Proof. We can immediately see that

$$\begin{aligned}(D_\tau + \Delta)\mathcal{S} &= (D_\tau + \Delta)(\nabla \operatorname{Rm} - \mathcal{P} \circ \nabla \operatorname{Rm} \circ \mathcal{P}) \\ &= [D_\tau + \Delta, \nabla] \operatorname{Rm} - \mathcal{P} \circ \big([D_\tau + \Delta, \nabla] \operatorname{Rm}\big) \circ \mathcal{P} + l(\mathcal{P}, \nabla\mathcal{P}, \nabla^2\mathcal{P}, \nabla \operatorname{Rm}) \\ &\quad + \nabla(D_\tau + \Delta) \operatorname{Rm} - \mathcal{P} \circ \big(\nabla(D_\tau + \Delta) \operatorname{Rm}\big) \circ \mathcal{P},\end{aligned}$$

where l denotes lower order terms involving covariant derivatives of $\mathcal{P}$ (see (6.5) below). We split this equation into three terms:

$$T^{(1)} = [D_\tau + \Delta, \nabla] \operatorname{Rm} - \mathcal{P} \circ \big([D_\tau + \Delta, \nabla] \operatorname{Rm}\big) \circ \mathcal{P},$$

$$T^{(2)} = \nabla(D_\tau + \Delta) \operatorname{Rm} - \mathcal{P} \circ \big(\nabla(D_\tau + \Delta) \operatorname{Rm}\big) \circ \mathcal{P},$$

$$T^{(3)} = l(\mathcal{P}, \nabla\mathcal{P}, \nabla^2\mathcal{P}, \nabla \operatorname{Rm}).$$

We will further analyze the commutators using the formula (see [45], Chapter 4)

$$[D_\tau + \Delta, \nabla_a] = -R_{abdc}\nabla_b(\delta_{cb}\Lambda_d^b - \delta_{db}\Lambda_c^b).$$

Then,

$$
\begin{aligned}
([(D_\tau + \Delta), \nabla_e] \operatorname{Rm})_{abcd} &= -\frac{1}{2} \left(R_{peqm} \nabla_p (\delta_{mn} \Lambda^n_q - \delta_{qn} \Lambda^n_m) R_{abcd} \right) \\
&= -\frac{1}{2} (2R_{aqep} \nabla_p R_{qbcd} - 2R_{bqcp} \nabla_p R_{qacd} + 2R_{cqep} \nabla_p R_{qdab} \\
&\quad - 2R_{dqep} \nabla_p R_{qcab}) \\
&= -\mathcal{L}(\nabla \operatorname{Rm}, \operatorname{Rm})_{eabcd},
\end{aligned}
$$

and thus

$$
T^{(1)} = \mathcal{P} \circ \mathcal{L}(\nabla \operatorname{Rm}, \operatorname{Rm}) \circ \mathcal{P} - \mathcal{L}(\nabla \operatorname{Rm}, \operatorname{Rm}).
$$

For a local orthonormal frame $\{e_i\}$, let $\omega_{ij} = e_i^* \wedge e_j^*$ for $i < j$. Note that, for $A \in T^*M \otimes \mathcal{E}_s$, $B \in \mathcal{E}_s$, respectively, we have

$$
\begin{aligned}
\mathcal{L}(A, B)_{eabcd} &= \frac{1}{4} \left(\langle [B(\omega_{ep}), A_p(\omega_{cd})], \omega_{ab} \rangle + \langle [B(\omega_{ep}), A_p(\omega_{ab})], \omega_{cd} \rangle \right) \\
&= -\frac{1}{4} \langle \mathcal{L}_e(A, B)(\omega_{cd}), \omega_{ab} \rangle,
\end{aligned}
\tag{6.4}
$$

where $\mathcal{L}_e(\cdot, \cdot) = \mathcal{L}(e_e, \cdot, \cdot)$ and $A_p(\cdot) = A(e_p, \cdot)$.

Now, we can rewrite $T^{(1)}$ as

$$
\begin{aligned}
T^{(1)} &= \mathcal{P} \circ \mathcal{L}(\mathcal{S}, \mathcal{R}) \circ \mathcal{P} - \mathcal{L}(\mathcal{S}, \mathcal{R}) + \mathcal{P} \circ \mathcal{L}(\mathcal{S}, E) \circ \mathcal{P} - \mathcal{L}(\mathcal{S}, E) + \mathcal{P} \circ \mathcal{L}(F, \mathcal{R}) \circ \mathcal{P} \\
&\quad - \mathcal{L}(F, \mathcal{R}) + \mathcal{P} \circ \mathcal{L}(F, E) \circ \mathcal{P} - \mathcal{L}(F, E).
\end{aligned}
$$

Note that (see Lemma 3.5, [45])

$$
\langle [\mathcal{P}(\cdot), \mathcal{P}(\cdot)], \overline{\mathcal{P}}(\cdot) \rangle = -\langle [\mathcal{P}(\cdot), \overline{\mathcal{P}}(\cdot)], \mathcal{P}(\cdot) \rangle = 0.
$$

Using this we can see that

$$
\begin{aligned}
\langle \mathcal{P} \circ \mathcal{L}_e(F, E) \circ \mathcal{P}(\phi), \eta \rangle &= -\langle [E(\omega_{ep}), F_p(\mathcal{P}\phi)], \mathcal{P}(\eta) \rangle - \langle [E(\omega_{ep}), F_p(\mathcal{P}\eta)], \mathcal{P}(\phi) \rangle \\
&= \langle \mathcal{L}_e(F, E)(\phi), \eta \rangle,
\end{aligned}
$$

and, similarly,

$$\begin{aligned}\langle \mathcal{P}\circ\mathcal{L}_e(\mathcal{S},E)\circ\mathcal{P}(\phi),\eta\rangle &= -\langle[E(\omega_{ep}),\mathcal{S}_p\circ\mathcal{P}(\phi)],\mathcal{P}(\eta)\rangle\\ &\quad -\langle[E(\omega_{ep}),\mathcal{S}_p\circ\mathcal{P}(\eta)],\mathcal{P}(\phi)\rangle\\ &= -\langle[E(\omega_{ep}),\overline{\mathcal{P}}\circ\nabla_p\operatorname{Rm}\circ\mathcal{P}(\phi)],\mathcal{P}(\eta)\rangle\\ &\quad -\langle[E(\omega_{ep}),\overline{\mathcal{P}}\circ\nabla_p\operatorname{Rm}\circ\mathcal{P}(\eta)],\mathcal{P}(\phi)\rangle\\ &= 0.\end{aligned}$$

Thus,

$$T^{(1)} = \mathcal{P}\circ\mathcal{L}(\mathcal{S},\mathcal{R})\circ\mathcal{P} - \mathcal{L}(\mathcal{S},\mathcal{R}) - \mathcal{L}(\mathcal{S},E) + \mathcal{P}\circ\mathcal{L}(F,\mathcal{R})\circ\mathcal{P} - \mathcal{L}(F,\mathcal{R}).$$

We now analyze the term $T^{(2)}$.

$$T^{(2)} = \nabla(D_\tau+\Delta)\operatorname{Rm} - \mathcal{P}\circ(\nabla(D_\tau+\Delta)\operatorname{Rm})\circ\mathcal{P}.$$

Here, we use the fact that

$$(D_t-\Delta)\operatorname{Rm} = \operatorname{Rm}^2 + \operatorname{Rm}^{\#},$$

which implies

$$(D_\tau+\Delta)\operatorname{Rm} = -\operatorname{Rm}\circ\operatorname{Rm} - \operatorname{Rm}\#\operatorname{Rm},$$

and finally

$$\begin{aligned}T^{(2)} &= -\nabla\operatorname{Rm}\circ\operatorname{Rm} - \operatorname{Rm}\circ\nabla\operatorname{Rm} - 2(\nabla\operatorname{Rm}\#\operatorname{Rm})\\ &\quad -\mathcal{P}\circ(-\nabla\operatorname{Rm}\circ\operatorname{Rm} - \operatorname{Rm}\circ\nabla\operatorname{Rm} - 2(\nabla\operatorname{Rm}\#\operatorname{Rm}))\circ\mathcal{P}.\end{aligned}$$

So, again we need to examine how these operations interact with the projections. We can rewrite a term of the form $\nabla\operatorname{Rm}\#\operatorname{Rm}$ as

$$\begin{aligned}\nabla\operatorname{Rm}\#\operatorname{Rm} &= (\mathcal{S}+F)\#(\mathcal{R}+E)\\ &= \mathcal{S}\#\mathcal{R} + \mathcal{S}\#E + F\#\mathcal{R} + F\#E.\end{aligned}$$

Now, we can see that

$$\begin{aligned}
\langle \mathcal{P} \circ (F \# E) \circ \mathcal{P}(\phi), \eta\rangle &= \frac{1}{2}\sum_{\alpha,\beta} \langle [F(\omega_\alpha), E(\omega_\beta)], \mathcal{P}(\phi)\rangle \cdot \langle [\omega_\alpha, \omega_\beta], \mathcal{P}(\eta)\rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta} \langle [F(\omega_\alpha), E(\omega_\beta)], \phi\rangle \cdot \langle [\mathcal{P}(\omega_\alpha), \mathcal{P}(\omega_\beta)], \eta\rangle \\
&= \langle (F\#E)(\phi), \eta\rangle, \\
\langle \mathcal{P} \circ (\mathcal{S} \# E) \circ \mathcal{P}(\phi), \eta\rangle &= \frac{1}{2}\sum_{\alpha,\beta} \langle [\mathcal{S}(\omega_\alpha), E(\omega_\beta)], \mathcal{P}(\phi)\rangle \cdot \langle [\omega_\alpha, \omega_\beta], \mathcal{P}(\eta)\rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta} \langle [\mathcal{S}(\omega_\alpha), E(\omega_\beta)], \mathcal{P}(\phi)\rangle \cdot \langle [\omega_\alpha, \mathcal{P}(\omega_\beta)], \mathcal{P}(\eta)\rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta} \langle [(\overline{\mathcal{P}} \circ \nabla R \circ \mathcal{P})(\omega_\alpha), E(\omega_\beta)], \mathcal{P}(\phi)\rangle \\
&\quad \cdot \langle [\mathcal{P}(\omega_\alpha), \mathcal{P}(\omega_\beta)], \mathcal{P}(\eta)\rangle \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathcal{P} \circ (F \# \mathcal{R}) \circ \mathcal{P}(\phi), \eta\rangle &= \frac{1}{2}\sum_{\alpha,\beta} \langle [F(\omega_\alpha), \mathcal{R}(\omega_\beta)], \mathcal{P}(\phi)\rangle \cdot \langle [\omega_\alpha, \omega_\beta], \mathcal{P}(\eta)\rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta} \langle [F(\omega_\alpha), (\overline{\mathcal{P}} \circ R \circ \mathcal{P})(\omega_\beta)], \mathcal{P}(\phi)\rangle \\
&\quad \cdot \langle [\mathcal{P}(\omega_\alpha), \mathcal{P}(\omega_\beta)], \mathcal{P}(\eta)\rangle \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\mathcal{P} \circ (\nabla \operatorname{Rm} \# \operatorname{Rm}) \circ \mathcal{P} - \nabla \operatorname{Rm} \# \operatorname{Rm} = \mathcal{P} \circ (\mathcal{S} \# \mathcal{R}) \circ \mathcal{P} - \mathcal{S}\#\mathcal{R} - \mathcal{S}\#E - F\#\mathcal{R}.$$

As for the other parts of $T^{(2)}$, we have

$$\mathcal{P} \circ (\nabla \operatorname{Rm} \circ \operatorname{Rm}) \circ \mathcal{P} - \nabla \operatorname{Rm} \circ \operatorname{Rm} = \mathcal{P}(\mathcal{S} \circ \mathcal{R})\mathcal{P} - \mathcal{S} \circ \mathcal{R} - F \circ \mathcal{R} - \mathcal{S} \circ E,$$

and similarly for $\mathcal{P} \circ (\operatorname{Rm} \circ \nabla \operatorname{Rm}) \circ \mathcal{P} - \operatorname{Rm} \circ \nabla \operatorname{Rm}$. So,

$$T^{(2)} = \mathcal{P} \circ (\mathcal{Q}(\mathcal{S}, \mathcal{R})) \circ \mathcal{P} - \mathcal{Q}(\mathcal{S}, \mathcal{R}) - \mathcal{Q}(F, \mathcal{R}) - \mathcal{Q}(\mathcal{S}, E).$$

The term $T^{(3)}$ consists only of the lower order terms:

$$T^{(3)} = -\Delta\mathcal{P} \circ \nabla \operatorname{Rm} \circ \mathcal{P} - 2\nabla_i \mathcal{P} \circ \nabla_i \nabla \operatorname{Rm} \circ \mathcal{P} - 2\nabla_i \mathcal{P} \circ \nabla \operatorname{Rm} \circ \nabla_i \mathcal{P} \\ - 2\mathcal{P} \circ \nabla_i \nabla \operatorname{Rm} \circ \nabla_i \mathcal{P} - \mathcal{P} \circ \nabla \operatorname{Rm} \circ \Delta\mathcal{P}. \tag{6.5}$$

We have finally determined that

$$(D_\tau + \Delta)\mathcal{S} = \mathcal{P} \circ \big(\mathcal{L}(\mathcal{S}, \mathcal{R}) + \mathcal{Q}(\mathcal{S}, \mathcal{R})\big) \circ \mathcal{P} - L(\mathcal{S}, \mathcal{R}) - \mathcal{Q}(\mathcal{S}, \mathcal{R}) - \mathcal{L}(\mathcal{S}, E) - \mathcal{Q}(\mathcal{S}, E) \\ - \mathcal{L}(F, \mathcal{R}) - \mathcal{Q}(F, \mathcal{R}) + \mathcal{P} \circ \mathcal{L}(F, \mathcal{R}) \circ \mathcal{P}\Delta\mathcal{P} \circ \nabla \operatorname{Rm} \circ \mathcal{P} \\ - 2\nabla_i \mathcal{P} \circ \nabla_i \nabla \operatorname{Rm} \circ \mathcal{P} - 2\nabla_i \mathcal{P} \circ \nabla \operatorname{Rm} \circ \nabla_i \mathcal{P} - 2\mathcal{P} \circ \nabla_i \nabla \operatorname{Rm} \circ \nabla_i \mathcal{P} \\ - \mathcal{P} \circ \nabla \operatorname{Rm} \circ \Delta\mathcal{P}.$$

Rearranging this equation yields

$$(D_\tau + \Delta + \mathcal{B})\mathcal{S} = \mathcal{S} * \mathcal{R} + \mathcal{R} * F + \Delta\mathcal{P} * \nabla \operatorname{Rm} * \mathcal{P} + (\nabla\mathcal{P})^2 * \nabla \operatorname{Rm} + \mathcal{P} * \nabla\mathcal{P} * \nabla^2 \operatorname{Rm}.$$

□

Proposition 6.3.3.

$$D_\tau \nabla\mathcal{P} = \mathcal{S} * \mathcal{P} + \operatorname{Rm} * \nabla\mathcal{P},$$

$$D_\tau \nabla^2\mathcal{P} = \operatorname{Rm} * \nabla^2\mathcal{P} + \nabla \operatorname{Rm} * \nabla\mathcal{P} + \nabla\mathcal{S} * \mathcal{P} + \mathcal{S} * \mathcal{P}.$$

Proof. We follow Section 4 from [45], starting with the commutator formula

$$[D_\tau, \nabla_a] = -\nabla_p R_{pacb} \Lambda_c^b - R_{ac} \nabla_c. \tag{6.6}$$

Note that, following Proposition 4.5 from [45],

$$\nabla_p R_{pacb} \Lambda_c^b \mathcal{P}_{ijkl} = (\nabla \operatorname{Rm} \circ \mathcal{P})_{ppacb} \Lambda_c^b \mathcal{P}_{ijkl} \\ = ((\mathcal{S} + F) \circ \mathcal{P})_{ppacb}) \Lambda_c^b \mathcal{P}_{ijkl}.$$

Let $\{e_i\}$ be a local orthonormal frame for TM and define $\omega_{ij} = e_i^* \wedge e_j^*$. As in Lemma 4.4 from [45] and the fact that $\mathrm{Im}(\mathcal{P})$ is closed under the bracket, we have

$$\begin{aligned} -4(\mathcal{P} \circ \nabla \mathrm{Rm} \circ \mathcal{P})_{ppacb} \Lambda_c^b \mathcal{P}_{ijkl} &= \langle [\mathcal{P}(\omega_{kl}), \mathcal{P} \circ \nabla_p \mathrm{Rm} \circ \mathcal{P}(\omega_{pa})], \omega_{ij} \rangle \\ &\quad + \langle [\mathcal{P}(\omega_{ij}), \mathcal{P} \circ \nabla_p \mathrm{Rm} \circ \mathcal{P}(\omega_{pa})], \omega_{kl} \rangle \\ &= \langle [\mathcal{P}(\omega_{kl}), \mathcal{P} \circ \nabla_p \mathrm{Rm} \circ \mathcal{P}(\omega_{pa})], \mathcal{P}(\omega_{ij}) \rangle \\ &\quad + \langle [\mathcal{P}(\omega_{ij}), \mathcal{P} \circ \nabla_p \mathrm{Rm} \circ \mathcal{P}(\omega_{pa})], \mathcal{P}(\omega_{kl}) \rangle, \end{aligned}$$

which vanishes by the full antisymmetry of the map $(\omega, \eta, \phi) \mapsto \langle [\omega, \eta], \phi \rangle$. Now, using the fact that $D_\tau \nabla \mathcal{P} = [D_\tau, \nabla]\mathcal{P}$, we obtain

$$D_\tau \nabla \mathcal{P} = \mathcal{S} * \mathcal{P} + \mathrm{Rm} * \nabla \mathcal{P}.$$

For the second identity, we proceed as in Chapter 5, using again the fact that

$$D_\tau \nabla^2 \mathcal{P} = [D_\tau, \nabla] \nabla \mathcal{P} + \nabla D_\tau \nabla \mathcal{P}.$$

Applying (6.6) again, we obtain

$$[D_\tau, \nabla] \nabla \mathcal{P} = \mathrm{Rm} * \nabla^2 \mathcal{P} + \nabla \mathrm{Rm} * \nabla \mathcal{P} + \nabla \mathcal{S} * \mathcal{P},$$

from which the conclusion follows.

□

We are now ready to prove Proposition 6.3.1.

Proof of Proposition 6.3.1. The second inequality follows immediately from Proposition 6.3.3. As for the first inequality, commuting two covariant derivatives of $\hat{P}$ gives

$$\begin{aligned} \nabla_k \nabla_l \hat{P}_{ij} - \nabla_l \nabla_k \hat{P}_{ij} &= -R(e_k, e_l, e_i, \hat{P}(e_j)) - R(e_k, e_l, e_j, \hat{P}(e_i)) \\ &= -R(e_k, e_l, \check{P}(e_i), \hat{P}(e_j)) - R(e_k, e_l, \hat{P}(e_i), \check{P}(e_j)) \\ &= \frac{1}{4} \langle \mathrm{Rm} \circ \overline{\mathcal{P}}(e_i^* \wedge e_j^*), e_k^* \wedge e_l^* \rangle. \end{aligned}$$

Therefore,

$$|\mathcal{P} \circ \mathrm{Rm} \circ \overline{\mathcal{P}}| \leq |\mathrm{Rm} \circ \overline{\mathcal{P}}| \leq C|\nabla^2 \hat{P}|,$$

and similarly

$$|\overline{\mathcal{P}} \circ \mathrm{Rm} \circ \overline{\mathcal{P}}| \leq C|\nabla^2 \hat{P}|.$$

Finally, $\overline{\mathcal{P}} \circ \mathrm{Rm} \circ \mathcal{P}$ is the adjoint of $\mathcal{P} \circ \mathrm{Rm} \circ \overline{\mathcal{P}}$, so the two have the same norm. Thus,

$$|\mathcal{R}| \leq C|\nabla^2 \hat{P}| \leq C|\nabla^2 \mathcal{P}|$$

and finally

$$|\mathcal{R}||F| \leq C|\nabla^2 \mathcal{P}||\nabla \mathrm{Rm}|.$$

This, combined with Proposition 6.3.2, implies the first inequality.

□

6.4 Future Work

From here, the approach we pursue in [19] roughly follows that in [51], [75], using the method of Carleman estimates to conclude that $\mathbf{X}$ and $\mathbf{Y}$ vanish identically. For this we derive two sets of Carleman estimates for the system from Proposition 6.3.1: one to show that $\mathbf{X}$ and $\mathbf{Y}$ vanish at least at an exponential rate, and the second, which makes use of this rapid decay, to show that they must be zero (see [25], [51], and [75] for applications of this type of estimates to similar systems). Our particular formulation of the PDE-ODE system, and in particular our consideration of the operator $\Delta + \mathcal{B}$, is designed to enable a more efficient proof of the exponential decay portion than that in [51].

Finally, though here we consider the case that an unknown soliton is asymptotic to a model soliton of the form $(\mathbb{R}^{n-k} \times \Sigma^k, \overline{g} \oplus g_\Sigma)$, we expect that our approach carries over to more general products of the form $N^{n-k} \times \Sigma^k$ where N^{n-k} is asymptotically conical. We are interested more generally in the extent to which the decay assumption can be weakened when (M, g) is assumed to be complete.

REFERENCES

[1] Angenent, S. B. and D. Knopf, "Ricci solitons, conical singularities, and nonuniqueness", arXiv:1909.08087 (2021).

[2] Bamler, R. H., "Compactness theory of the space of super Ricci flows", arXiv:2008.09298 (2020).

[3] Bamler, R. H., "Entropy and heat kernel bounds on a Ricci flow background", arXiv:2008.07093 (2020).

[4] Bamler, R. H., "Structure theory of non-collapsed limits of Ricci flows", arXiv:2009.03243 (2020).

[5] Bamler, R. H. and B. Kleiner, "Ricci flow and diffeomorphism groups of 3-manifolds", arXiv:1712.06197 (2017).

[6] Bando, S., "Real analyticity of solutions of Hamilton's equation", Math. Z. **195**, 1, 93–97 (1987).

[7] Berger, M., "Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes", Bull. Soc. Math. France **83**, 279–330 (1955).

[8] Besse, A. L., *Einstein manifolds*, Classics in Mathematics (Springer-Verlag, Berlin, 2008), reprint of the 1987 edition.

[9] Böhm, C. and B. Wilking, "Manifolds with positive curvature operators are space forms", Ann. of Math. (2) **167**, 3, 1079–1097 (2008).

[10] Brendle, S. and R. M. Schoen, "Classification of manifolds with weakly 1/4-pinched curvatures", Acta Math. **200**, 1, 1–13 (2008).

[11] Cabezas-Rivas, E. and B. Wilking, "How to produce a Ricci flow via Cheeger-Gromoll exhaustion", J. Eur. Math. Soc. (JEMS) **17**, 12, 3153–3194 (2015).

[12] Cao, H.-D., B.-L. Chen and X.-P. Zhu, "Recent developments on Hamilton's Ricci flow", in "Surveys in differential geometry. Vol. XII. Geometric flows", vol. 12 of *Surv. Differ. Geom.*, pp. 47–112 (Int. Press, Somerville, MA, 2008).

[13] Cao, H.-D. and D. Zhou, "On complete gradient shrinking Ricci solitons", J. Differential Geom. **85**, 2, 175–185 (2010).

[14] Chen, B.-L., "Strong uniqueness of the Ricci flow", J. Differential Geom. **82**, 2, 363–382 (2009).

[15] Chen, B.-L. and X.-P. Zhu, "Uniqueness of the Ricci flow on complete noncompact manifolds", J. Differential Geom. **74**, 1, 119–154 (2006).

[16] Chow, B. and D. Knopf, *The Ricci flow: an introduction*, vol. 110 of *Mathematical Surveys and Monographs* (American Mathematical Society, Providence, RI, 2004).

[17] Conlon, R. J., A. Deruelle and S. Sun, "Classification results for expanding and shrinking gradient Kähler-Ricci solitons", arXiv:1904.00147 (2019).

[18] Cook, M., "The preservation of product structures for Ricci flows with instantaneously bounded curvature", Preprint (2021).

[19] Cook, M., "Shrinking Ricci solitons asymptotic to products", In preparation (2021).

[20] Cook, M. and B. Kotschwar, "The full holonomy group under the Ricci flow", Proc. Amer. Math. Soc. **148**, 12, 5313–5321 (2020).

[21] Dancer, A. S. and M. Y. Wang, "On Ricci solitons of cohomogeneity one", Ann. Global Anal. Geom. **39**, 3, 259–292 (2011).

[22] DeTurck, D. M., "Deforming metrics in the direction of their Ricci tensors", J. Differential Geom. **18**, 1, 157–162 (1983).

[23] Eells, J., Jr. and J. H. Sampson, "Harmonic mappings of Riemannian manifolds", Amer. J. Math. **86**, 109–160 (1964).

[24] Enders, J., R. Müller and P. M. Topping, "On type-I singularities in Ricci flow", Comm. Anal. Geom. **19**, 5, 905–922 (2011).

[25] Escauriaza, L., G. Seregin and V. Šverák, "Backward uniqueness for the heat operator in half-space", Algebra i Analiz **15**, 1, 201–214 (2003).

[26] Feldman, M., T. Ilmanen and D. Knopf, "Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons", J. Differential Geom. **65**, 2, 169–209 (2003).

[27] Giesen, G. and P. M. Topping, "Ricci flow of negatively curved incomplete surfaces", Calc. Var. Partial Differential Equations **38**, 3-4, 357–367 (2010).

[28] Giesen, G. and P. M. Topping, "Existence of Ricci flows of incomplete surfaces", Comm. Partial Differential Equations **36**, 10, 1860–1880 (2011).

[29] Giesen, G. and P. M. Topping, "Ricci flows with unbounded curvature", Math. Z. **273**, 1-2, 449–460 (2013).

[30] Giesen, G. and P. M. Topping, "Ricci flows with unbounded curvature", Math. Z. **273**, 1-2, 449–460 (2013).

[31] Giesen, G. and P. M. Topping, "Ricci flows with bursts of unbounded curvature", Comm. Partial Differential Equations **41**, 5, 854–876 (2016).

[32] Hamilton, R. S., "Three-manifolds with positive Ricci curvature", J. Differential Geometry **17**, 2, 255–306 (1982).

[33] Hamilton, R. S., "Four-manifolds with positive curvature operator", J. Differential Geom. **24**, 2, 153–179 (1986).

[34] Hamilton, R. S., "The Ricci flow on surfaces", in "Mathematics and general relativity (Santa Cruz, CA, 1986)", vol. 71 of *Contemp. Math.*, pp. 237–262 (Amer. Math. Soc., Providence, RI, 1988).

[35] Hamilton, R. S., "The Harnack estimate for the Ricci flow", J. Differential Geom. **37**, 1, 225–243 (1993).

[36] Hamilton, R. S., "A compactness property for solutions of the Ricci flow", Amer. J. Math. **117**, 3, 545–572 (1995).

[37] Hamilton, R. S., "The formation of singularities in the Ricci flow", in "Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)", pp. 7–136 (Int. Press, Cambridge, MA, 1995).

[38] Hamilton, R. S., "Four-manifolds with positive isotropic curvature", Comm. Anal. Geom. **5**, 1, 1–92 (1997).

[39] Hochard, R., "Short-time existence of the Ricci flow on complete, non-collapsed 3-manifolds with Ricci curvature bounded from below", arXiv:1603.08726 (2016).

[40] Huang, S. and L.-F. Tam, "Kähler-Ricci flow with unbounded curvature", Amer. J. Math. **140**, 1, 189–220 (2018).

[41] Ivey, T., "Ricci solitons on compact three-manifolds", Differential Geom. Appl. **3**, 4, 301–307 (1993).

[42] Joyce, D. D., *Riemannian holonomy groups and calibrated geometry*, vol. 12 of *Oxford Graduate Texts in Mathematics* (Oxford University Press, Oxford, 2007).

[43] Kazdan, J. L. and F. W. Warner, "Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures", Ann. of Math. (2) **101**, 317–331 (1975).

[44] Kotschwar, B., "An energy approach to the problem of uniqueness for the Ricci flow", Comm. Anal. Geom. **22**, 1, 149–176 (2014).

[45] Kotschwar, B., "Ricci flow and the holonomy group", J. Reine Angew. Math. **690**, 133–161 (2014).

[46] Kotschwar, B., "Time-analyticity of solutions to the Ricci flow", Amer. J. Math. **137**, 2, 535–576 (2015).

[47] Kotschwar, B., "A short proof of backward uniqueness for some geometric evolution equations", Internat. J. Math. **27**, 12, 1650102, 17 (2016).

[48] Kotschwar, B., "Short-time persistence of bounded curvature under the Ricci flow", Math. Res. Lett. **24**, 2, 427–447 (2017).

[49] Kotschwar, B., "Kählerity of shrinking gradient Ricci solitons asymptotic to Kähler cones", J. Geom. Anal. **28**, 3, 2609–2623 (2018).

[50] Kotschwar, B. and L. Wang, “Rigidity of asymptotically conical shrinking gradient Ricci solitons”, J. Differential Geom. **100**, 1, 55–108 (2015).

[51] Kotschwar, B. and L. Wang, “A uniqueness theorem for asymptotically cylindrical shrinking Ricci solitons”, arXiv:1712.03185 (2020).

[52] Kotschwar, B. L., “Backwards uniqueness for the Ricci flow”, Int. Math. Res. Not. IMRN , 21, 4064–4097 (2010).

[53] Kotschwar, B. L., “A local version of Bando’s theorem on the real-analyticity of solutions to the Ricci flow”, Bull. Lond. Math. Soc. **45**, 1, 153–158 (2013).

[54] Lai, Y., “Producing 3d Ricci flows with non-negative Ricci curvature via singular Ricci flows”, arXiv:2004.05291 (2020).

[55] Lee, M.-C., “On the uniqueness of Ricci flow”, J. Geom. Anal. **29**, 4, 3098–3112 (2019).

[56] Liu, G. and G. Székelyhidi, “Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature bounded below”, arXiv:1804.08567 (2020).

[57] Munteanu, O. and J. Wang, “Geometry of shrinking Ricci solitons”, Compos. Math. **151**, 12, 2273–2300 (2015).

[58] Munteanu, O. and J. Wang, “Topology of Kähler Ricci solitons”, J. Differential Geom. **100**, 1, 109–128 (2015).

[59] Munteanu, O. and J. Wang, “Conical structure for shrinking Ricci solitons”, J. Eur. Math. Soc. (JEMS) **19**, 11, 3377–3390 (2017).

[60] Munteanu, O. and J. Wang, “Structure at infinity for shrinking Ricci solitons”, Ann. Sci. Éc. Norm. Supér. (4) **52**, 4, 891–925 (2019).

[61] Naber, A., “Noncompact shrinking four solitons with nonnegative curvature”, J. Reine Angew. Math. **645**, 125–153 (2010).

[62] Ni, L. and N. Wallach, “On a classification of gradient shrinking solitons”, Math. Res. Lett. **15**, 5, 941–955 (2008).

[63] Perelman, G., “The entropy formula for the Ricci flow and its geometric applications”, (2002).

[64] Perelman, G., “Finite extinction time for the solutions to the Ricci flow on certain three-manifolds”, arXiv:math/0307245 (2003).

[65] Perelman, G., “Ricci flow with surgery on 3-manifolds”, arXiv:math/0303109 (2003).

[66] Schulze, F. and M. Simon, “Expanding solitons with non-negative curvature operator coming out of cones”, Math. Z. **275**, 1-2, 625–639 (2013).

[67] Shao, Y., “A family of parameter-dependent diffeomorphisms acting on function spaces over a Riemannian manifold and applications to geometric flows”, NoDEA Nonlinear Differential Equations Appl. **22**, 1, 45–85 (2015).

[68] Shi, W.-X., “Deforming the metric on complete Riemannian manifolds”, J. Differential Geom. **30**, 1, 223–301 (1989).

[69] Simon, M., “Ricci flow of almost non-negatively curved three manifolds”, J. Reine Angew. Math. **630**, 177–217 (2009).

[70] Simon, M. and P. Topping, “Local control on the geometry in 3d limit flow”, arXiv:1611.06137 (2017).

[71] Topping, P. M., “Uniqueness and nonuniqueness for Ricci flow on surfaces: reverse cusp singularities”, Int. Math. Res. Not. IMRN , 10, 2356–2376 (2012).

[72] Topping, P. M., “Ricci flows with unbounded curvature”, in “Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II”, pp. 1033–1048 (Kyung Moon Sa, Seoul, 2014).

[73] Topping, P. M., “Uniqueness of instantaneously complete Ricci flows”, Geom. Topol. **19**, 3, 1477–1492 (2015).

[74] Wang, L., “Uniqueness of self-similar shrinkers with asymptotically conical ends”, J. Amer. Math. Soc. **27**, 3, 613–638 (2014).

[75] Wang, L., “Uniqueness of self-similar shrinkers with asymptotically cylindrical ends”, J. Reine Angew. Math. **715**, 207–230 (2016).

[76] Yang, B., “A characterization of noncompact Koiso-type solitons”, Internat. J. Math. **23**, 5, 1250054, 13 (2012).

[77] Zhang, Z.-H., “On the completeness of gradient Ricci solitons”, Proc. Amer. Math. Soc. **137**, 8, 2755–2759 (2009).