From Stochastic N Particle Systems to Deterministic Differential Equations - with Applications to Econophysics and Averaging Dynamics

by

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ABSTRACT

The mechanisms behind the emergence of collective behaviors arising from physics, biology, economics and many other related fields have drawn a lot of attention among the applied math community in the last few decades. Broadly speaking, collective behaviors in natural, life and social sciences are all modelled by interacting particle systems, in which a bulk of N particles are engaging in some simple binary pair-wise interactions. In this dissertation, some prototypical interacting particle systems having applications in econophysics and statistical averaging dynamics are investigated. It is also emphasized that there is an increasing tendency among the applied math community to apply tools or concepts for studying many particle systems to the (rigorous) investigation of artificial (deep) neural networks.

DEDICATION

I devote this dissertation to my parents, who always give me the best possible support. I also want to express my gratitude to Xiaoqian Gong, a former Ph.D who graduated from SoMSS a few years ago and helped me a lot during my stay at Arizona. Finally, I thank my Ph.D advisor Sebastien Motsch as well as one of his coauthors Pierre-Emmanuel Jabin for their tremendous academic support.

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Chapter 1

INTRODUCTION

How does the group of certain animals exhibits collective behaviors? What are the mechanisms behind several pattern formation phenomena? Why the opinions among a large community/society can reach a consensus or develop polarization? How does the echo chamber effect on social media emerge? Why the wealth distributions in many developed countries have a similar shape? How on earth one can replace training infinitely-width artificial deep neural networks by kernel regression methods? Mathematically, all of the aforementioned questions can be reformulated in the framework of *interacting particle systems*, in which the word *particle* can be understood as animal, agent, molecular, and neuron, just to name a few. In general, one of the main goals for building an interacting particle system is to *reproduce* certain phenomenon observed in reality (or at the macroscopic level) based on simple agent-based interaction rules at the microscopic level. It is definitely possible that different agent-based interaction rules at the microscopic level may lead to the same macroscopic observation, whence (as a rule thumb) one must not convince himself/herself that the observed macroscopic phenomenon originates from a specific interaction rule that he/she designs manually. For instance, one can not view the process of generating a 'Cayley tree' as the actual mechanism behind the growth/development of a real tree! Despite of the preceding fact, the possibility of reconstructing certain macroscopic phenomena by building individual-based interaction rules still shed some light on the better understanding of the formation of certain complex patterns in reality.

Another main motivation for investigating agent-based interacting particle systems is to link agent-based behaviors at the microscopic level to partial differential equations at the macroscopic level. The quest for such a linkage resides in the underlying philosophical debate as to whether our real world is 'continuous' or 'discrete'. One might argue that the continuous space variable in partial differential equations such as heat equations or porous medium equations is not realistic because the porous media in reality are 'discrete'. However, it turns out in many cases, once can design an interacting particle system such that the empirical measure of the particles converges (in some appropriate sense) to the solution of our target partial differential equation as the number of particles approaches infinity. This means that many important partial differential equations of theoretical and practical interests may admit ad-hoc and clever approximations by agent-based models, thereby enhancing the quest for further examination of these particle systems.

1.1 Interacting Particle Systems

Interacting particle systems emerge in a vast number of problems in applied mathematics, ranging from biological sciences, physical sciences, social and life sciences Agueh *et al.* [2], Aldana *et al.* [3], Aldana and Huepe [4], Aoki [9], Barbaro and Degond [19], Belmonte *et al.* [22], Bertin *et al.* [25], Bolley *et al.* [31], Carrillo *et al.* [44], Cucker and Smale [64]. A generic model of interacting particle systems consists of N indistinguishable random variables $\{X_i\}_{1\leq i\leq N} \in \mathbb{R}^d$ ($d \in \mathbb{N}_+$), among which interactions based on certain rules take place as time advances. Depending on the areas of application in one's mind, the set of random variables $\{X_i\}_{1\leq i\leq N}$ can be identified as (i) the locations of particles/animals, or (ii) the opinions of a group of agents/individuals, or (iii) the amount of dollars each agent possesses, just to name a few. The identification of $\{X_i\}_{1\leq i\leq N}$ as the locations of particles/animals is universal in literatures on statistical physics and flocking dynamics, while the identifications (ii) and (iii) are often used to study opinion dynamics (typically on certain graphs) and models arising from econophysics, respectively. We refer to figure 1.1 for a illustration.

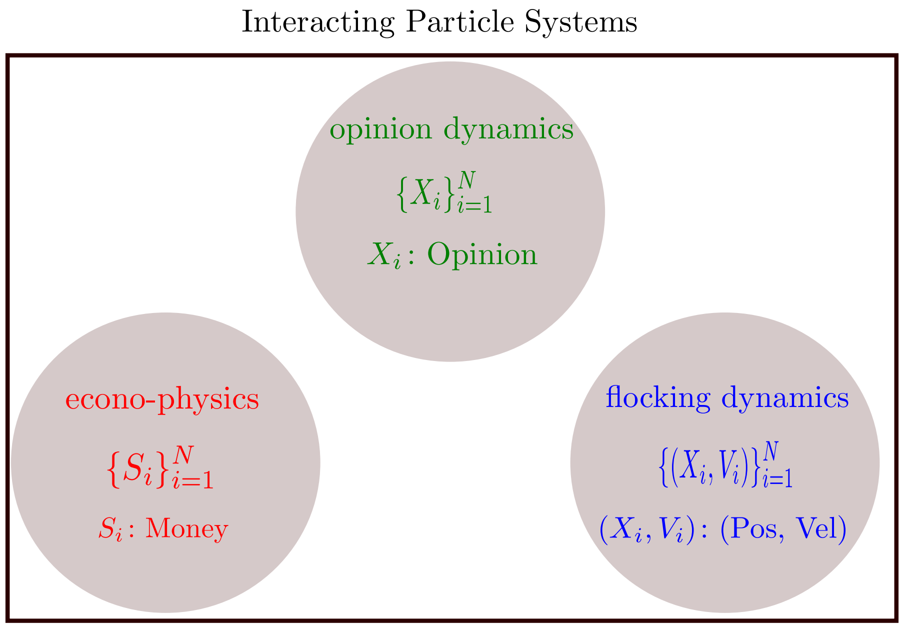

Figure 1.1: Overview of Some Typical Types of Interacting Particle Systems and Their Interpretations in Different Contexts.

We emphasize here that other interpretations of the aforementioned random variables $\{X_i\}_{1\leq i\leq N}$ are also possible, for instance once might think of $\{X_i\}_{1\leq i\leq N}$ as neurons in real or artificial neural networks Mei *et al.* [117], Rotskoff and Vanden-Eijnden [138]. We will however restrict our attentions to the three scenarios mentioned at the very beginning of this report, in order to present some basic and concrete models that will be analyzed in detail.

1.1.1 Econophysics

Econophysics is an emerging branch of statistical physics that apply concepts and techniques of traditional physics to economics and finance Savoiu [140], Chatterjee *et al.* [57], Dragulescu and Yakovenko [79]. It has attracted considerable attention in recent years raising challenges on how various economical phenomena could be explained by universal laws in statistical physics, and we refer to Chakraborti *et al.* [53, 54], Pereira *et al.* [125], Kutner *et al.* [103] for a general review.

The primary motivation for study models arising from econophysics is at least two-fold: from the perspective of a policy maker, it is important to deal with the raise of income inequality Dabla-Norris *et al.* [66], De Haan and Sturm [70] in order to establish a more egalitarian society. From a mathematical point of view, we have to understand the fundamental mechanisms, such as money exchange resulting from individuals, which are usually agent-based models. Given an agent-based model, one is expected to identify the limit dynamics as the number of individuals tends to infinity and then its corresponding equilibrium when run the model for a sufficiently long time (if there is one), and this guiding approach is carried out in numerous works across different fields among literatures of applied mathematics, see for instance Naldi *et al.* [122], Barbaro and Degond [19], Carlen *et al.* [42].

1.1.2 Self-Organized Dynamics

The collective behavior of various particle systems is a subject of intensive research that has potential applications in biology, physics, economics, and engineering Naldi *et al.* [122], Belmonte *et al.* [22], Chuang *et al.* [59]. Different models are proposed to study the emergence of flocking of birds, formation of consensus in opinion dynamics, and phase transitions in network models Motsch and Tadmor [121], Porfiri and Ariel [131], Chaté *et al.* [55], Barbaro and Degond [19]. Broadly speaking, all of the aforementioned models are instances of interacting particle systems, under various interaction rules among the particles. We refer the readers to Liggett [112] for a general introduction into this branch of applied mathematics.

1.1.3 Opinion Dynamics

Opinion dynamics such as the voter model is of interest to both social scientists and applied mathematicians Das *et al.* [68], Proskurnikov *et al.* [134]. In general, agents in such dynamics have their opinions which vary in time due to the interaction of agents with their neighbors. Depending on the model, the evolution of individual's option can be deterministic or stochastic Weber *et al.* [150]. It is also possible to take into account the spatial structure of the interaction, leading to opinion dynamics on (random) graphs Lanchier [107], Sood and Redner [142]. The literatures in this direction is also enormous and hence we refer interested readers to Castellano *et al.* [49] and references therein for a detailed review.

1.2 Derivation of Macroscopic Models

1.2.1 Motivation

In some physical relevant situations, gas dynamics for instance, the number of particles N under consideration is overwhelmingly large (and is in the order of 10^{23}). Even in the context of sociological and economical models, the number of agents N is usually in the order of millions. It is therefore almost impossible to keep track of each individual's behavior over time, even numerically. In the pioneering work of Ludwig Boltzmann Boltzmann [32], Boltzmann suggested that we should take advantage of the fact that N is very large and seek an statistical description instead. That being said, it is expected that when N is very large, some averaging effect may take place and a (typically nonlinear) partial differential equation for the one-particle marginal probability density function can be hoped for. See figure 1.2 for a sketch of the reasoning along this line.

Under the famous "molecular chaos assumption" proposed by Boltzmann Boltz-

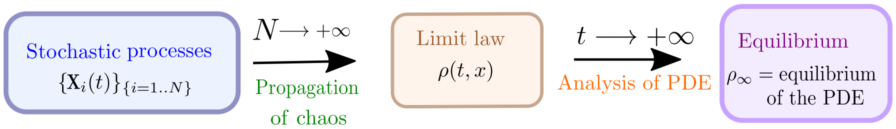

Figure 1.2: Sketch of a General Approaching for Analyzing Interacting Particle Systems. Under the Limit $N \to \infty$, We Expect to Have a (Partial) Differential Equation Governing the Evolution of the (Marginal) Law of X_1, This Deterministic Formulation of the Problem Can Then Be Investigated Using Standard PDE Approach in Order to Understand the Long Time Behavior of the Dynamics.

mann [32], he managed to derive such a partial differential equation (for motion of (ideal) gases) which bears his name - the Boltzmann equation. The so-called "molecular chaos assumption" has been made rigorous thanks to Mark Kac's foundation work Kac [99] and is now known as Kac's propagation of chaos, this is the central concept of our the next subsection to which we now turn.

1.2.2 Propagation of Chaos

Definition

We propose to review several methods used to prove the so-called propagation of chaos. But first we need to carefully define what propagation of chaos means. Roughly speaking, if $\{X_i\}_{1\leq i\leq N}$ are independent identically distributed (i.i.d) initially, this independency will be lost at later times due to the underlying interaction mechanism. But as $N \to \infty$, we can often hope for a recovery of such independency property. We now turn to a rigorous definition of propagation of chaos. With this aim, we consider a (stochastic) $N-$particle system denoted $(S_1,\ldots,S_N)$ where particles are indistinguishable. In other words, the particle system is invariant by permutation, i.e. for any test function φ and permutation $\sigma \in \mathcal{S}_N$:

$$\mathbb{E}[\varphi(S_1,\ldots,S_N)] = \mathbb{E}[\varphi(S_{\sigma(1)},\ldots,S_{\sigma(N)})].$$

In particular, all the single processes S_i for $i = 1,\ldots,N$ have the same law (but they are in general not independent). Denote by $\mathbf{p}^{(N)}(s_1,\ldots,s_N)$ the density distribution of the $N-$process and let $\mathbf{p}_k^{(N)}$ be the marginal density, i.e. the law of the process $(S_1, ..., S_k)$ (for $1 \leq k \leq N$):

$$\mathbf{p}_k^{(N)}(s_1, \dots, s_k) = \int_{s_{k+1},\dots,s_N} \mathbf{p}^{(N)}(s_1, \dots, s_N)\, \mathrm{d}s_{k+1} \dots \mathrm{d}s_N.$$

Consider now a *limit* stochastic process $(\overline{S}_1, \dots, \overline{S}_k)$ where $\{\overline{S}_i\}_{i=1,\dots,k}$ are independent and identically distributed. Denote by $\mathbf{p}_1$ the law of a single process, thus by independence assumption the law of *all* the processes is given by:

$$\mathbf{p}_k(s_1, \dots, s_k) = \prod_{i=1}^{k} \mathbf{p}_1(s_i).$$

Definition 1 *We say that the stochastic process* $(S_1, \dots, S_N)$ *satisfies the propagation of chaos if for any fixed* k*:*

$$\mathbf{p}_k^{(N)} \quad \overset{N\to+\infty}{\rightharpoonup} \quad \mathbf{p}_k \tag{1.1}$$

which is equivalent to have for any test function φ*:*

$$\mathbb{E}[\varphi(S_1, \dots, S_k)] \overset{N\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(\overline{S}_1, \dots, \overline{S}_k)]. \tag{1.2}$$

Remark. For binary collision models Carlen *et al.* [43, 42], proving propagation of chaos is equivalent to show that $\mathbf{p}_2^{(N)}(s_1, s_2) \approx \mathbf{p}_1^{(N)}(s_1)\, \mathbf{p}_1^{(N)}(s_2)$, i.e. collisions come from two independent particles.

Coupling Method

The *coupling* method Sznitman [145] consists in generating the two processes $(S_1, \dots, S_N)$ and $(\overline{S}_1, \dots, \overline{S}_k)$ *simultaneously* in such a way that:

i) $(S_1, \dots, S_k)$ and $(\overline{S}_1, \dots, \overline{S}_k)$ satisfy their respective law,

ii) S_i and $\overline{S}_i$ are *closed* for all $1 \leq i \leq k$.

The main difficulty is that $\{\overline{S}_i\}_{i=1,\dots,k}$ are independent but $\{S_i\}_{i=1,\dots,N}$ are <u>not</u>, thus the two processes cannot be *too* closed. In practice, we expect to find a bound of the form:

$$\mathbb{E}[|S_i - \overline{S}_i|] \leq \frac{C}{\sqrt{N}} \overset{N\to+\infty}{\longrightarrow} 0 \quad , \quad \text{for all } 1 \leq i \leq k. \tag{1.3}$$

Such result is sufficient[1] to prove (2.2) and therefore one deduces propagation of chaos.

In a more abstract point of view, the inequality (2.3) gives an upper bound for the Wasserstein distance between $\mathbf{p}_k^{(N)}$ and the limit density $\mathbf{p}_k$. Since convergence in Wasserstein distance is equivalent to weak-* convergence for measures, we can conclude about the propagation of chaos (2.1).

Empirical Distribution - Tightness of Measure

Another approach to prove propagation of chaos is to study the so-called empirical measure:

$$\mathbf{p}_{emp}^{(N)}(s) = \frac{1}{N}\sum_{i=1}^{N} \delta_{S_i}(s) \tag{1.4}$$

where δ is the Delta distribution, i.e. for a smooth test function $\varphi(s)$ the duality bracket is defined as:

$$\langle \mathbf{p}_{emp}^{(N)}, \varphi \rangle = \frac{1}{N}\sum_{i=1}^{N} \varphi(S_i). \tag{1.5}$$

Notice that $\mathbf{p}_{emp}^{(N)}$ is a distribution of a single variable, thus the domain of $\mathbf{p}_{emp}^{(N)}$ remains the same as N increases which simplifies its study. However, $\mathbf{p}_{emp}^{(N)}$ is also a *stochastic* measure, i.e. $\mathbf{p}_{emp}^{(N)}$ is a random variable on the space of measures Billingsley [26]. The link between propagation of chaos and empirical distribution relies on the following lemma.

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[1] using as a test function $\varphi(s_1,\dots,s_k) = \varphi_1(s_1)\dots\varphi_k(s_k)$

Lemma 1.2.1 *The stochastic process $(S_1, \ldots, S_N)$ satisfies the propagation of chaos (2.1) if and only if:*

$$\mathbf{p}_{emp}^{(N)} \stackrel{N \to +\infty}{\rightharpoonup} \mathbf{p}_1, \tag{1.6}$$

i.e. for any test function φ the random variable $\langle \mathbf{p}_{emp}^{(N)}, \varphi \rangle$ converges in law to the constant value $\mathbb{E}[\varphi(\overline{S}_1)]$.

The proof can be found in Sznitman [145] and is hence skipped.

A Toy Interacting Particle System

In literature, the notion of propagation of chaos is sometimes referred to as the mean-field limit, the latter terminology is perhaps more illuminating due to the following informal reasoning: Fix $i \in \{1, \ldots, N\}$ and suppose there are $\mathcal{O}(N)$ particles in the neighborhood of X_i $(1 \le i \le N)$, assume further that each of the neighbors of X_i can only influence the behavior of X_i by a factor of $\mathcal{O}(1/N)$, then under the limit $N \to \infty$, any finitely many particles (say $\{X_i\}_{1 \le i \le k}$ with k being independent of N) will never interact with each almost surely over a finite time span, and loosely speaking each of them will interact (independently of each other) with a single 'ghost' particle which represents the average behavior of the particle system. The phrase 'mean-field' emphasizes this averaging effect when the number of particles N tends to infinity.

We finish this subsection with a toy example taken from and analyzed in Lacker [104], with the aim of giving the readers a favour of such type of argument. Consider the toy model

$$\mathrm{d}X_t^i = a(\overline{X}_t - X_t^i)\mathrm{d}t + \sigma \mathrm{d}W_t^i,$$

where $a, \sigma > 0$ and $\overline{X}_t = \frac{1}{N} \sum_{k=1}^N X_t^k$ is the empirical mean. Each particle experiences an independent white noise, and the drift term pushes each particle toward the empirical mean. Notice that the system $(X^1, \ldots, X^n)$ is *exchangeable.* To get a first vague sense of how a mean-field limit works in a model of this form, simply average the N particles to find the dynamics for the empirical average:

$$\mathrm{d}\overline{X}_t = \frac{\sigma}{n} \sum_{k=1}^{n} \mathrm{d}W_t^k.$$

In integrated form, we also have

$$\overline{X}_t = \frac{1}{N} \sum_{k=1}^{N} X_0^k + \frac{\sigma}{N} \sum_{k=1}^{N} W_t^k.$$

If (X_0^k) are i.i.d. with mean m, then the law of large numbers tells us that $\overline{X}_t \to m$ almost surely as $N \to \infty$, since of course Brownian motion has mean zero. If we focus now on a fixed particle i in the N-particle system, we find that as $N \to \infty$ the behavior of particle i should look like

$$\mathrm{d}X_t^i = a(m - X_t^i)\mathrm{d}t + \sigma \mathrm{d}W_t^i.$$

Since m is constant, the 'limiting' evolution consists of i.i.d. particles. In summary, as $N \to \infty$, the particles become *asymptotically i.i.d.*, and the behavior of each one is governed by an Ornstein-Uhlenbeck process. As a concluding remark, the $N \to \infty$ limit in this toy model can be studied quite easily by taking advantage of the special form of the model, but for more sophisticated models such simple computations will not be possible and one has to resort to more advanced tools.

1.3 Analysis of the Limit Dynamics

After the passage from a stochastic N particle system to a deterministic (partial) differential equation of the form $\partial_t \rho = Q[\rho]$, where $Q \colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, the next natural question comes to the understanding of the large time (i.e., $t \to \infty$) behavior of the partial differential equation for the one-particle marginal density function

$\rho(t,x)$. Typically, we are interested to show that $\rho \to \rho_\infty$ (in some appropriate sense) as $t \to \infty$, in which ρ_∞ is the unique probability density function satisfying $Q[\rho_\infty] = 0$. See figure 1.3 for a illustration.

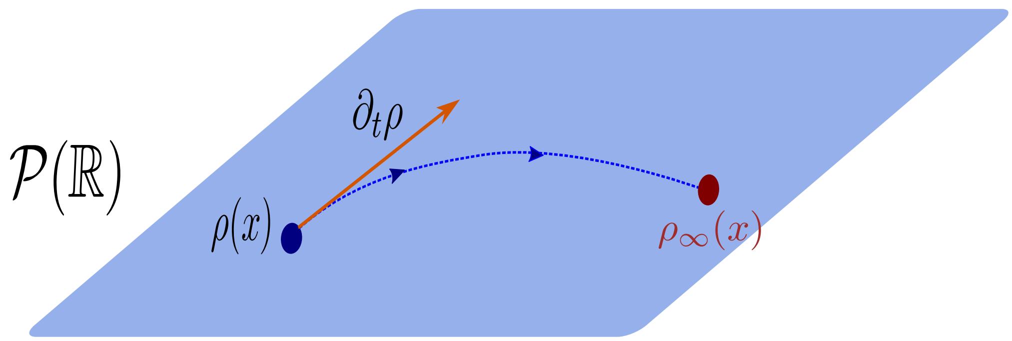

Figure 1.3: Sketch of the Dynamics Described by the PDE for $\rho(t,x)$, The Goal is to Show That $\rho \to \rho_\infty$ As $t \to \infty$, Where ρ_∞ is the Unique Equilibrium Probability Density.

In this report, we will mainly focus on the convergence of probability density functions in the sense of relative entropy. We shall review the basic principles in section 1.3.1, then we end this section with an introduction to a specific technique used in the context of entropy methods, the so-called Bakry-Emery approach.

1.3.1 Entropy Methods

We intend to briefly discuss about the spirit behind the so-called entropy methods Jüngel [97], Matthes [114] in the study of many kinetic equations (Boltzmann type equations and Fokker-Planck type equations, just to name a few). For this purpose, let us define the relative entropy from ρ to ρ_∞ as

$$\mathrm{H}[\rho] := \int \rho(x) \log \frac{\rho(x)}{\rho_\infty(x)} \,\mathrm{d}x,$$

where we require that $\rho_\infty > 0$ in the domain of our interest. It is a standard fact that (see for instance Cover [62]) $\mathrm{H}[\rho]$ is always non-negative and $\mathrm{H}[\rho] =$ if and only of $\rho = \rho_\infty$, hence the quantity $\mathrm{H}[\rho]$ may serve as a measure of closeness between

ρ and ρ_∞. Note that relative entropy $\mathrm{H}[\rho]$ is also known as the Kullback-Leibler distance and is often denoted by $\mathrm{D_{KL}}(\rho \,||\, \rho_\infty)$. However, in general we do not have $\mathrm{D_{KL}}(\rho \,||\, \rho_\infty) = \mathrm{D_{KL}}(\rho_\infty \,||\, \rho)$, thus the Kullback-Leibler distance is not a true distance between two (smooth) probability densities.

In many applications, it can be shown that the entropy production, defined by

$$\mathrm{D}[\rho] := -\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(x)\,\log\frac{\rho(x)}{\rho_\infty(x)}\,\mathrm{d}x,$$

is non-negative. In the context of the gas dynamics, this fact amounts to the celebrated second law of thermodynamics, i.e., the entropy of a isolated (ideal) gas system can not decrease as time moves forward. If we can show a **entropy-entropy production inequality** of the form

$$\mathrm{D}[\rho] \geq \kappa\,\mathrm{H}[\rho], \tag{1.1}$$

where $\kappa > 0$ is some universal constant independent of $\rho \in \mathcal{P}(\mathbb{R})$, then the classical Gronwall's inequality will lead us to $\mathrm{H}[\rho(t)] \leq \mathrm{H}[\rho(0)]\,\mathrm{e}^{-\kappa t}$. That is, the relative entropy will decay exponentially fast in time! As a rule of thumb, proving (1.3.1) for Boltzmann-type evolution equations is in general very challenging (see discussions in Desvillettes and Villani [76], Rezakhanlou *et al.* [136], Villani [147]). However, for Fokker-Planck type equations, one can often establish by employing the so-called Bakry-Emery approach. We will present (from a PDE point of view) the essential skeleton of this approach in the upcoming section, together with a baby example demonstrating one application of this technique.

1.3.2 Bakry-Emery Approach

The original method of Bakry and Emery Bakry and Émery [15] is of probabilistic nature and is very abstract. In its original form, this method is closely related to the so-called Γ-calculus and we refer interested readers to Bakry *et al.* [14], Matthes [114] for a comprehensive discussion. Our goal in this section is to present the PDE perspective of the Bakry-Emery approach, introduced in Arnold *et al.* [11], along with a elementary example in order to convince the readers of the usefulness of this technique.

Let us introduce

$$\mathrm{R}[\rho] := \frac{\mathrm{d}^2}{\mathrm{d}^2 t} \int \rho(x) \log \frac{\rho(x)}{\rho_\infty(x)} \, \mathrm{d}x,$$

which is just the second time derivative of the (relative) entropy $\mathrm{H}[\rho]$. If we can show

$$\mathrm{R}[\rho] \geq \kappa \, \mathrm{D}[\rho] \tag{1.2}$$

for some universal constant $\kappa > 0$, then upon integration over (t, ∞) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho(t)] - \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho(t)] \leq -\kappa \left(\mathrm{H}[\rho(t)] - \lim_{t\to\infty} \mathrm{H}[\rho(t)] \right).$$

Thanks to Gronwall's inequality and (1.2), we have $\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho(t)] = 0$. Assume one can manage to show that $\lim_{t\to\infty} \mathrm{H}[\rho(t)] = 0$, then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho(t)] \leq -\kappa \, \mathrm{H}[\rho(t)],$$

leading to the exponentially fast in time decay of $\mathrm{H}[\rho]$.

Remark. Depending on the context, the crucial inequality (1.2) is referred to as a curvature dimension condition, or a logarithmic Sobolev inequality.

Finally, we present a simplest example (taken from Matthes [114]) to illustrate in detail how this technique can be applied in certain initial boundary value problems. Let us consider the following heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t \rho = \partial_{xx} \rho, & x \in [0,1], t \geq 0, \\ \partial_x \rho(0) = \partial_x \rho(1) = 0, & t \geq 0, \\ \rho(x,0) = \rho_0(x), & x \in [0,1], \end{cases} \tag{1.3}$$

in which the initial data ρ_0 satisfies $\rho(x) > 0$ for all $x \in [0,1]$ and $\int_0^1 \rho_0(x)\,\mathrm{d}x = 1$. It is not hard to show that (1.3) admits a unique (smooth) solution $\rho(t,x)$ with $\rho(t,x) > 0$ for all $x \in [0,1]$ and $t > 0$. Moreover, $\int_0^1 \rho(t,x)\,\mathrm{d}x = 1$ for all $t \geq 0$ and $\rho_\infty(x) \equiv 1$ is the unique equilibrium of the problem (1.3). Thus, our (relative) entropy boils down to $\mathrm{H}[\rho] = \int_0^1 \rho(x)\,\log\rho(x)\,\mathrm{d}x$ and we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho] = \int_0^1 (\partial_t\rho)\,\log\rho = \int_0^1 \Delta\rho\,\log\rho = -\int_0^1 \frac{|\partial_x\rho|^2}{\rho} \leq 0.$$

Thus, the entropy production $\mathrm{D}[\rho]$ is given by

$$\mathrm{D}[\rho] = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho] = \int_0^1 \frac{|\partial_x\rho|^2}{\rho},$$

which is just the Fisher information of the probability density ρ. To lower bound the second time derivative of the entropy in terms of the entropy production $\mathrm{D}[\rho]$, a straightforward computation together with a Poincaré inequality yield

$$\mathrm{R}[\rho] \geq 2\,\pi^2\,\mathrm{D}[\rho]. \tag{1.4}$$

By standard PDE theory, the solution ρ to (1.3) converges to the homogeneous steady state ρ_∞ in C^∞, implying that $\lim_{t\to\infty}\mathrm{D}[\rho(t)] = 0$ and that $\lim_{t\to\infty}\mathrm{H}[\rho(t)] = 0$. Consequently, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\rho] \leq -2\,\pi^2\,\mathrm{H}[\rho],$$

whence $\mathrm{H}[\rho(t)] \leq \mathrm{H}[\rho(0)]\,\mathrm{e}^{-2\,\pi^2\,t}$ for all $t \geq 0$.

Remark. Interestingly, the problem (1.3) admits other entropies as well. For instance, if we define

$$\mathrm{H}_0[\rho] := -\int_0^1 \log\rho(x)\,\mathrm{d}x,$$

we can still show that

$$\mathrm{H}_0[\rho(t)] \leq \mathrm{H}_0[\rho(0)]\,\mathrm{e}^{-2\,\pi^2\,t}$$

via a pretty similar argument. We refer interested readers to Day [69] for details.

Chapter 2

DERIVATION OF WEALTH DISTRIBUTIONS FROM BIASED EXCHANGE OF MONEY

Chapter 2 is the pre-print [40] submitted to *Annals of Applied Probability*, in collaboration with Sebastien Motsch.

2.1 Abstract

In the manuscript, tools from kinetic theory are employed to understand the time evolution of wealth distribution and their large scale behavior such as the evolution of inequality (e.g. Gini index). Three types of dynamics denoted unbiased, poor-biased and rich-biased exchange models are investigated. At the individual level, one agent is picked randomly based on its wealth and one of its dollar is redistributed among the population. Proving the so-called propagation of chaos, it is possible to identify the limit of each dynamics as the number of individual approaches infinity using both coupling techniques Sznitman [145] and martingale-based approach Merle and Salez [118]. Equipped with the limit equation, it is possible to identify and prove the convergence to specific equilibrium for both the unbiased and poor-biased dynamics. In the rich-biased dynamics however, a more complex behavior where a dispersive wave emerges is observed. Although the dispersive wave is vanishing in time, its also accumulates all the wealth leading to a Gini approaching 1 (its maximum value). Numerical behavior of dispersive wave can be characterized but further analytic investigation is needed to derive such dispersive wave directly from the dynamics.

2.2 Introduction

Econophysics is an emerging branch of statistical physics that apply concepts and techniques of traditional physics to economics and finance Savoiu [140], Chatterjee *et al.* [57], Dragulescu and Yakovenko [79]. It has attracted considerable attention in recent years raising challenges on how various economical phenomena could be explained by universal laws in statistical physics, and we refer to Chakraborti *et al.* [53, 54], Pereira *et al.* [125], Kutner *et al.* [103] for a general review.

The primary motivation for study models arising from econophysics is at least two-fold: from the perspective of a policy maker, it is important to deal with the raise of income inequality Dabla-Norris *et al.* [66], De Haan and Sturm [70] in order to establish a more egalitarian society. From a mathematical point of view, we have to understand the fundamental mechanisms, such as money exchange resulting from individuals, which are usually agent-based models. Given an agent-based model, one is expected to identify the limit dynamics as the number of individuals tends to infinity and then its corresponding equilibrium when run the model for a sufficiently long time (if there is one), and this guiding approach is carried out in numerous works across different fields among literatures of applied mathematics, see for instance Naldi *et al.* [122], Barbaro and Degond [19], Carlen *et al.* [42].

Although we will only consider three distinct binary exchange models in the present work, other exchange rules can also be imposed and studied, leading to different models. To name a few, the so-called immediate exchange model introduced in Heinsalu and Patriarca [90] assumes that pairs of agents are randomly and uniformly picked at each random time, and each of the agents transfer a random fraction of its money to the other agents, where these fractions are independent and uniformly distributed in $[0, 1]$. The so-called uniform reshuffling model investigated in Dragulescu and Yakovenko [79] and Lanchier and Reed [109] suggests that the total amount of money of two randomly and uniformly picked agents possess before interaction is uniformly redistributed among the two agents after interaction. For models with saving propensity and with debts, we refer the readers to Chakraborti and Chakrabarti [52], Chatterjee *et al.* [56] and Lanchier and Reed [110].

2.2.1 Unbiased/Poor-Biased/Rich-Biased Dynamics

In this work, we consider several dynamics for money exchange in a closed economical system, meaning that there are a fixed number of agents, denoted by N, with an (fixed) average number of dollar m. We denote by $S_i(t)$ the amount of dollars the agent i has at time t. Since it is a closed economical system, we have:

$$S_1(t) + \cdots + S_N(t) = \text{Constant} \qquad \text{for all } t \geq 0. \tag{2.1}$$

As a first example of money exchange, we review the model proposed in Dragulescu and Yakovenko [79]: at random time (exponential law), an agent i is picked at random (uniformly) and if it has one dollar (i.e. $S_i \geq 1$) it will give it to another agent j picked at random (uniformly). If i does not have one dollar (i.e. $S_i = 0$), then nothing happens. From now on we will call this model as **unbiased exchange model** as all the agents are being picked with equal probability. We refer to this dynamics as follow:

$$\textbf{unbiased:} \qquad (S_i, S_j) \overset{\lambda}{\rightsquigarrow} (S_i - 1, S_j + 1) \quad (\text{if } S_i \geq 1). \tag{2.2}$$

In other words, any agents with at least one dollar gives to all of the others agents at a fixed rate. Later on, we will adjust the rate λ (more exactly $\lambda \mathbb{1}_{[1,+\infty)}(S_i)$) by normalizing by N in order to have the correct asymptotic as $N \to +\infty$ (the rate of one agent giving a dollar per unit time is of order N otherwise).

Another possible dynamics is to pick the giver agent, i.e. agent i, with higher probability if the agent is rich, i.e. S_i large. Thus, *poor* agent will have a lower frequency of being picked. From now on we will call this model as **poor-biased model** and it illustrates as follow:

$$\textbf{poor-biased:} \qquad (S_i, S_j) \overset{\lambda S_i}{\rightsquigarrow} (S_i - 1, S_j + 1). \tag{2.3}$$

Notice that since the rate of giving is S_i, an agent with no money, i.e. $S_i = 0$, will never have to give. As for the unbiased dynamics (2.2), we will also adjust the rate, normalizing by N.

Our third dynamics that we would like to explore is the **rich-biased model**: we reverse the bias compared to the previous dynamics, rich agents are *less* likely to give:

$$\textbf{rich-biased:} \qquad (S_i, S_j) \overset{\lambda/S_i}{\rightsquigarrow} (S_i - 1, S_j + 1) \quad (\text{if } S_i \geq 1). \tag{2.4}$$

As a consequence of this dynamics, rich agents will tend to become even richer compared to poor agents creating a feedback that could lead to singular behavior. The adjustment of the rate for this dynamics is more delicate since the sum of the rates λ/S_i is no longer constant. In particular, we will see that a normalization of the rates to have a constant rate of giving a dollar per agent will lead to finite time blow-up of the dynamics in the limit $N \to +\infty$.

We illustrate the dynamics in figure 2.1-left. The key question of interest is the exploration of the limiting money distribution among the agents as the total number of agents and the number of time steps become large. We illustrate numerically (see figure 2.2) the three previous dynamics using $N = 500$ agents. In the unbiased dynamics (pink), the wealth distribution is (approximately) exponential with the proportion of agent decaying as wealth increases. On the contrary, the poor-biased dynamics (blue) has the bulk of its distribution around \$10 (the average capital per agent). For the rich-biased dynamics (green), most of the agents are left with no money and few with large amounts (more than \$30). To visualize the temporal evolution of the three dynamics, we estimate the Gini index G after each iteration in figure 2.1-right:

$$G = \frac{1}{2\mu} \sum_{1 \leq i,j \leq N} |S_i - S_j|, \tag{2.5}$$

where μ is the average wealth ($\mu = \frac{1}{N}\sum_{i=1}^{N} S_i$). The widely used inequality indicator Gini index G measures the inequality in the wealth distribution and ranges from 0 (no inequality) to 1 (extreme inequality). Since all agents have the same amount of dollar initially ($S_i(t=0) = \mu$), the Gini index starts at zero (i.e. $G(t=0) = 0$). In the unbiased dynamics, the Gini index stabilizes around .5 (which corresponds to the Gini index of an exponential distribution). The Gini index is strongly reduced in the poor-biased dynamics ($G \approx .19$). On the contrary, the Gini index keeps increasing in the rich-biased dynamics and seems to approach 1 (its maximum). We study in more details this phenoma in section 2.6.3. We emphasize that the "rich-get-richer" phenomenon, numerically observed in the rich-biased dynamics in the present work, has also been reported in other models from econophysics, and we refer interested readers to Boghosian *et al.* [28, 29] and references therein.

2.2.2 Asymptotic Dynamics: $N \to +\infty$ and $t \to +\infty$

One of the main difficulty in any rigorous mathematical treatment lies in the general fact that models in econophysics typically consist of a large number of interacting (coupled) economic agents. Fortunately the framework of kinetic theories allows simplification of the mathematical analysis of certain such models under some appropriate limit processes. For the unbiased model (2.2) and the poor-biased model (2.3), instead of taking the large time limit and then the large population limit as in Lanchier [106], we first take the large population limit to achieve a transition from the large stochastic system of interacting agents to a deterministic system of ordi-

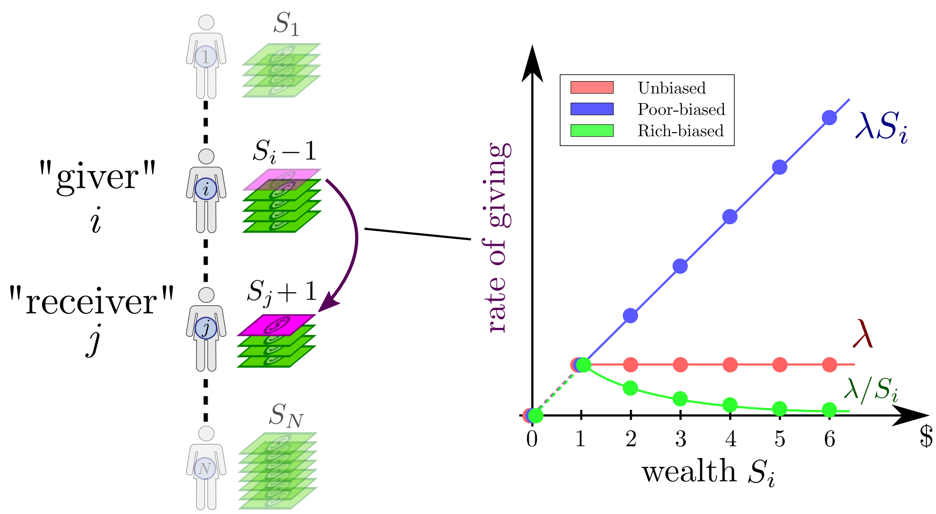

Figure 2.1: Left: Illustration of the 3 Dynamics: at Random Time, One Dollar is Passed From a "Giver" i to a "Receiver" j. **Right:** The Rate of Picking the "Giver" i Depends on the Wealth S_i.

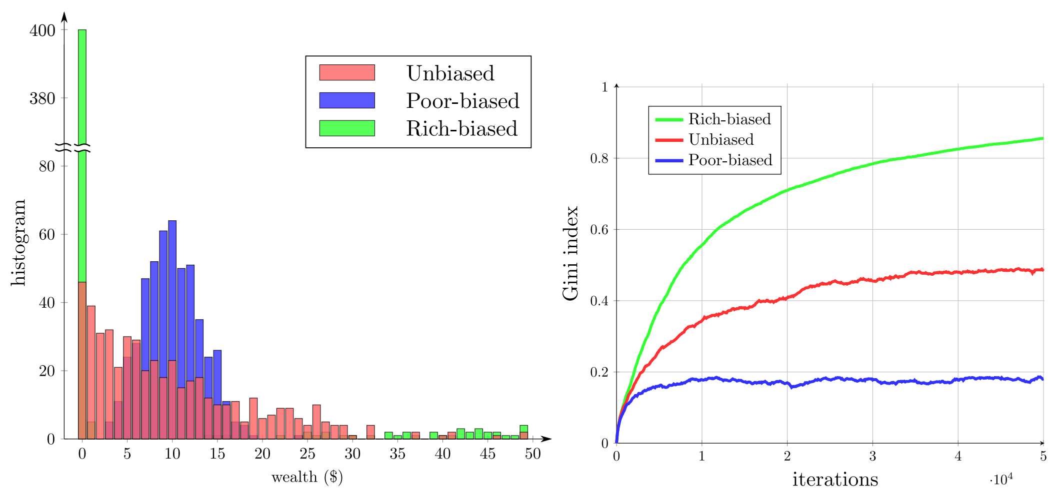

Figure 2.2: Left: Distribution of Wealth for the Three Dynamics After $50,000$ Steps. The Distribution Decays for the Unbiased Dynamics (Pink) i.e. Poor Agents are More Frequent Than Rich Agents, Whereas in the Poor-Biased Dynamics, the Distribution (Blue) is Centered at the Average \$10. For the Rich-Biased Dynamics, Almost All Agents Have Zero Dollars Except a Few with a Large Amount (More Than \$30). **Right:** Evolution of the Gini Index (2.5) for the Three Dynamics. The Gini Index is Lower for the Poor-Biased Dynamics (Less Inequality) Whereas it is Approaching 1 for the Rich-Biased Dynamics.

nary differential equations by proving the so-called propagation of chaos Sznitman [145], Merle and Salez [118], Méléard and Roelly-Coppoletta [119], Oelschlager [123] through a well-designed coupling technique, see figure 2.3 for a illustration of these strategies. After that, analysis of the deterministic description is then built on its (discrete) Fokker-Planck formulation and we investigate the convergence toward an equilibrium distribution by employing entropy methods. Arnold *et al.* [11], Matthes [114], Jüngel [97]. For the rich-biased model, we prove the propagation of chaos by virtue of a novel martingale-based technique introduced in Merle and Salez [118], and we report some interesting numerical behavior of the associated ODE system. We illustrate the various (limiting) ODE systems obtained in the present work in figure 2.4.

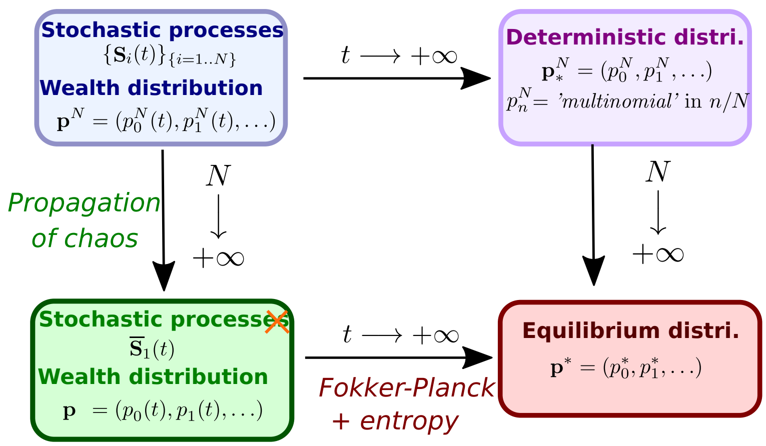

Figure 2.3: Schematic Illustration of the Strategy of Proof: The Approach of Sending $t \to \infty$ First and then Taking $N \to \infty$ is Carried Out in Lanchier [106] (See Also Lanchier and Reed [109, 110] for Usage of this Approach Applied for a Variety of Models in Econophysics). Our Strategy is to Perform the Limit $N \to \infty$ Before Investigating the Time Asymptotic $t \to \infty$.

For the poor-biased model, we present an explicit rate of convergence of its associated system of ordinary differential equations toward its equilibrium via the Bakry-Emery approach Bakry and Émery [15]. Then, we resort to numerical simulation in the determination of the sharp rate of convergence and a heuristic argument is used

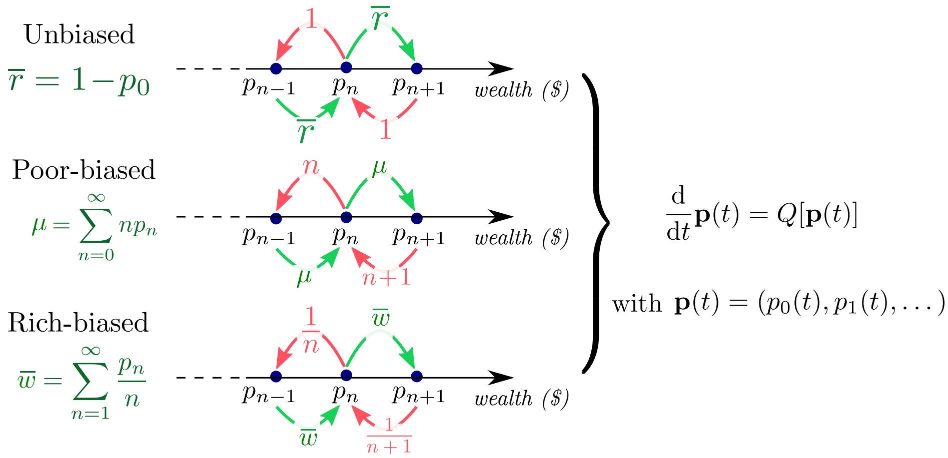

Figure 2.4: Summary of the Limit ODE Systems Obtained in this Manuscript. The Exact Form of the Operator Q will be Model-Dependent.

in support of our numerical observation.

This paper is organized as follows: in section 2.3, we briefly review different approaches to tackle the propagation of chaos. Section 2.4 is devoted to the investigation of the unbiased exchange model, where the rigorous large population limit $N \to \infty$ is carried out via a coupling argument and the limiting system of ODEs is studied in detail. We perform the analysis, for the poor-biased model in section 2.5 and for the rich-biased model in section 2.6, in a parallel fashion that resembles section 2.4. A subsection is dedicated in 2.6.3 to the emergence of a dispersive traveling wave in the rich-biased dynamics. Finally, a conclusion is drawn in section 2.7.

2.3 Review Propagation of Chaos

2.3.1 Definition

We propose to review the method used to prove the so-called propagation of chaos. But first we need to carefully define what propagation of chaos means. With this aim, we consider a (stochastic) $N-$particle system denoted $(S_1, \dots, S_N)$ where particles are indistinguishable. In other words, the particle system is invariant by permutation, i.e. for any test function φ and permutation $\sigma \in \mathcal{S}_N$:

$$\mathbb{E}[\varphi(S_1, \ldots, S_N)] = \mathbb{E}[\varphi(S_{\sigma(1)}, \ldots, S_{\sigma(N)})].$$

In particular, all the single processes S_i for $i = 1, \ldots, N$ have the same law (but they are in general not independent). Denote by $\mathbf{p}^{(N)}(s_1, \ldots, s_N)$ the density distribution of the $N-$process and let $\mathbf{p}_k^{(N)}$ be the marginal density, i.e. the law of the process $(S_1, ..., S_k)$ (for $1 \leq k \leq N$):

$$\mathbf{p}_k^{(N)}(s_1, \ldots, s_k) = \int_{s_{k+1}, \ldots, s_N} \mathbf{p}^{(N)}(s_1, \ldots, s_N)\, \mathrm{d}s_{k+1} \ldots \mathrm{d}s_N.$$

Consider now a *limit* stochastic process $(\overline{S}_1, \ldots, \overline{S}_k)$ where $\{\overline{S}_i\}_{i=1,\ldots,k}$ are independent and identically distributed. Denote by $\mathbf{p}_1$ the law of a single process, thus by independence assumption the law of *all* the processes is given by:

$$\mathbf{p}_k(s_1, \ldots, s_k) = \prod_{i=1}^{k} \mathbf{p}_1(s_i).$$

Definition 2 *We say that the stochastic process $(S_1, \ldots, S_N)$ satisfies the propagation of chaos if for any fixed k:*

$$\mathbf{p}_k^{(N)} \quad \overset{N \to +\infty}{\rightharpoonup} \quad \mathbf{p}_k \tag{2.1}$$

which is equivalent to have for any test function φ:

$$\mathbb{E}[\varphi(S_1, \ldots, S_k)] \overset{N \to +\infty}{\longrightarrow} \mathbb{E}[\varphi(\overline{S}_1, \ldots, \overline{S}_k)]. \tag{2.2}$$

Remark. For binary collision models Carlen *et al.* [43, 42], proving propagation of chaos is equivalent to show that $\mathbf{p}_2^{(N)}(s_1, s_2) \approx \mathbf{p}_1^{(N)}(s_1)\, \mathbf{p}_1^{(N)}(s_2)$, i.e. collisions come from two independent particles.

2.3.2 Coupling Method

The *coupling* method Sznitman [145] consists in generating the two processes $(S_1, \ldots, S_N)$ and $(\overline{S}_1, \ldots, \overline{S}_k)$ *simultaneously* in such a way that:

i) $(S_1, \ldots, S_k)$ and $(\overline{S}_1, \ldots, \overline{S}_k)$ satisfy their respective law,

ii) S_i and $\overline{S}_i$ are *closed* for all $1 \leq i \leq k$.

The main difficulty is that $\{\overline{S}_i\}_{i=1,\ldots,k}$ are independent but $\{S_i\}_{i=1,\ldots,N}$ are not, thus the two processes cannot be *too* closed. In practice, we expect to find a bound of the form:

$$\mathbb{E}[|S_i - \overline{S}_i|] \leq \frac{C}{\sqrt{N}} \overset{N \to +\infty}{\longrightarrow} 0 \quad , \quad \text{for all } 1 \leq i \leq k. \tag{2.3}$$

Such result is sufficient[1] to prove (2.2) and therefore one deduces propagation of chaos.

In a more abstract point of view, the inequality (2.3) gives an upper bound for the Wasserstein distance between $\mathbf{p}_k^{(N)}$ and the limit density $\mathbf{p}_k$. Since convergence in Wasserstein distance is equivalent to weak-* convergence for measures, we can conclude about the propagation of chaos (2.1).

2.3.3 Empirical Distribution - Tightness of Measure

Another approach to prove propagation of chaos is to study the so-called empirical measure:

$$\mathbf{p}_{emp}^{(N)}(s) = \frac{1}{N} \sum_{i=1}^{N} \delta_{S_i}(s) \tag{2.4}$$

where δ is the Delta distribution, i.e. for a smooth test function $\varphi(s)$ the duality bracket is defined as:

$$\langle \mathbf{p}_{emp}^{(N)}, \varphi \rangle = \frac{1}{N} \sum_{i=1}^{N} \varphi(S_i). \tag{2.5}$$

Notice that $\mathbf{p}_{emp}^{(N)}$ is a distribution of a single variable, thus the domain of $\mathbf{p}_{emp}^{(N)}$ remains the same as N increases which simplifies its study. However, $\mathbf{p}_{emp}^{(N)}$ is also a *stochastic* measure, i.e. $\mathbf{p}_{emp}^{(N)}$ is a random variable on the space of measures Billingsley [26]. The

[1] using as a test function $\varphi(s_1, \ldots, s_k) = \varphi_1(s_1) \ldots \varphi_k(s_k)$

link between propagation of chaos and empirical distribution relies on the following lemma.

Lemma 2.3.1 *The stochastic process* $(S_1, \ldots, S_N)$ *satisfies the propagation of chaos* (2.1) *if and only if:*

$$\mathbf{p}^{(N)}_{emp} \overset{N \to +\infty}{\rightharpoonup} \mathbf{p}_1, \tag{2.6}$$

i.e. for any test function φ *the random variable* $\langle \mathbf{p}^{(N)}_{emp}, \varphi \rangle$ *converges in law to the constant value* $\mathbb{E}[\varphi(\overline{S}_1)]$.

The proof can be found in Sznitman [145] but for completeness we write our own in Appendix A.

2.4 Unbiased Exchange Model

2.4.1 Definition and Limit Equation

We consider first the unbiased model that is briefly mentioned in the introduction above. For the three models investigated in this work, we consider a (closed) economic market consisting of N agents with μ dollars per agents for some (fixed) $\mu \in \mathbb{N}_+$, i.e. there are a total of μN dollars. We denote by $S_i(t)$ the amount of dollars that agent i has (i.e. $S_i(t) \in \{0, \ldots, \mu N\}$ and $\sum_{i=1}^N S_i(t) = \mu N$ for any $t \geq 0$).

Definition 3 (Unbiased Exchange Model) *The dynamics consist in choosing with uniform probability a "giver"* i *and a "receiver"* j*. If the receiver* i *has at least one dollar (i.e.* $S_i \geq 1$*), then it gives one dollar to the receiver* j*. This exchange occurs according to a Poisson process with frequency* $\lambda/N > 0$*.*

The unbiased exchange model can be written as a stochastic differential equation Privault [133], Shreve [141]. Introducing $\{\mathrm{N}_t^{(i,j)}\}_{1 \leq i,j \leq N}$ independent Poisson processes with constant intensity $\frac{\lambda}{N}$, the evolution of each S_i is given by:

$$\mathrm{d}S_i(t) = -\sum_{j=1}^{N} \underbrace{\mathbb{1}_{[1,\infty)}\Big(S_i(t-)\Big)\mathrm{dN}_t^{(i,j)}}_{\text{“}i\text{ gives to }j\text{”}} + \sum_{j=1}^{N} \underbrace{\mathbb{1}_{[1,\infty)}\Big(S_j(t-)\Big)\mathrm{dN}_t^{(j,i)}}_{\text{“}j\text{ gives to }i\text{”}}. \tag{2.1}$$

To gain some insight of the dynamics, we focus on $i = 1$ and introduce some notations:

$$\mathbf{N}_t^1 = \sum_{j=1}^{N} \mathrm{N}_t^{(1,j)}, \quad \mathbf{M}_t^1 = \sum_{j=1}^{N} \mathrm{N}_t^{(j,1)}.$$

The two Poisson processes $\mathbf{N}_t^1$ and $\mathbf{M}_t^1$ are of intensity λ. The evolution of $S_1(t)$ can be written as:

$$\mathrm{d}S_1(t) = -\mathbb{1}_{[1,\infty)}\Big(S_1(t-)\Big)\mathrm{d}\mathbf{N}_t^1 + Y(t-)\mathrm{d}\mathbf{M}_t^1, \tag{2.2}$$

with $Y(t)$ Bernoulli distribution with parameter $r(t)$ (i.e. $Y(t) \sim \mathcal{B}(r(t))$) representing the proportion of “rich” people:

$$r(t) = \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{[1,\infty)}\Big(S_j(t)\Big). \tag{2.3}$$

Thus, the dynamics of S_1 can be seen as a compound Poisson process.

Motivated by (2.2), we give the following definition of the limiting dynamics of $S_1(t)$ as $N \to \infty$ from the process point of view.

Definition 4 (Asymptotic Unbiased Exchange Model) *We define $\overline{S}_1(t)$ to be the (nonlinear) compound Poisson process satisfying the following SDE:*

$$\mathrm{d}\overline{S}_1(t) = -\mathbb{1}_{[1,\infty)}\Big(\overline{S}_1(t-)\Big)\mathrm{d}\overline{\mathbf{N}}_t^1 + \overline{Y}(t-)\mathrm{d}\overline{\mathbf{M}}_t^1, \tag{2.4}$$

in which $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ are independent Poisson processes with intensity λ, and $\overline{Y}(t) \sim \mathcal{B}(\bar{r}(t))$ independent Bernoulli variable with parameter

$$\bar{r}(t) \;:=\; \mathbb{P}\Big(\overline{S}_1(t) > 0\Big) \;=\; 1 - \mathbb{P}\Big(\overline{S}_1(t) = 0\Big). \tag{2.5}$$

We denote by $\mathbf{p}(t) = \Big(p_0(t), p_1(t), \ldots\Big)$ the law of the process $\overline{S}_1(t)$, i.e. $p_n(t) = \mathbb{P}\Big(\overline{S}_1(t) = n\Big)$. Its time evolution is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, Q_{unbias}[\mathbf{p}(t)] \tag{2.6}$$

with:

$$Q_{unbias}[\mathbf{p}]_n := \begin{cases} p_1 - \bar{r}\, p_0 & \text{if } n = 0 \\ p_{n+1} + \bar{r}\, p_{n-1} - (1+\bar{r})p_n & \text{for } n \geq 1 \end{cases} \tag{2.7}$$

and $\bar{r} = 1 - p_0$.

2.4.2 *Coupling for the Unbiased Exchange Model*

We now provide the coupling strategy to link the $N-$particle system $(S_1, \dots, S_N)$ with the limit dynamics $(\overline{S}_1, \dots, \overline{S}_k)$. In Sznitman [145], the core of the method is to use the same "noise" in both the $N-$particle system and the limit system. Unfortunately, it is not possible in our settings: the clocks $\mathrm{N}_t^{(i,j)}$ cannot be used "as it" since they would correlate the jump of $\overline{S}_i$ with the jump of $\overline{S}_j$ which is not acceptable. Indeed, if $\overline{S}_i(t)$ and $\overline{S}_j(t)$ are independent, they cannot jump at (exactly) the same time.

For this reason, we have to introduce an intermediate dynamics, denoted by $\{\widehat{S}_i\}_{i\geq 1}$, which employs exactly the same "clocks" as our original dynamics (2.1), but the property of being rich or poor is decoupled.

Definition 5 (Intermediate model) *We define for $\{\widehat{S}_i\}_{1\leq i\leq N}$ to be a collection of identically distributed (nonlinear) compound Poisson processes satisfying the following SDEs for each $1 \leq i \leq N$:*

$$\begin{aligned} \mathrm{d}\widehat{S}_i(t) \;&=\; -\sum_{j=1, j\neq i}^{N} \mathbb{1}_{[1,\infty)}\Big(\widehat{S}_i(t-)\Big)\mathrm{dN}_t^{(i,j)} + \sum_{j=1, j\neq i}^{N} \overline{Y}(t-)\mathrm{dN}_t^{(j,i)} && (2.8) \\ & \quad -\mathbb{1}_{[1,\infty)}\Big(\widehat{S}_i(t-)\Big)\mathrm{d}\overline{\mathrm{N}}_t^{(i,i)} + \overline{Y}(t-)\mathrm{d}\overline{\mathrm{M}}_t^{(i,i)} && (2.9) \end{aligned}$$

in which $\overline{Y}(t) \sim \mathcal{B}(\bar{r}(t))$, the Poisson clocks $\mathrm{N}_t^{(i,j)}$ $(1 \leq i \neq j \leq N)$ are the same as those used in (2.1)*, the two extra clocks $\overline{\mathrm{N}}_t^{(i,i)}$ and $\overline{\mathrm{M}}_t^{(i,i)}$ are independent with rate λ/N.*

We do not use the "self-giving" clocks $\mathrm{N}_t^{(i,i)}$ since we want to decouple the receiving and giving dynamics.

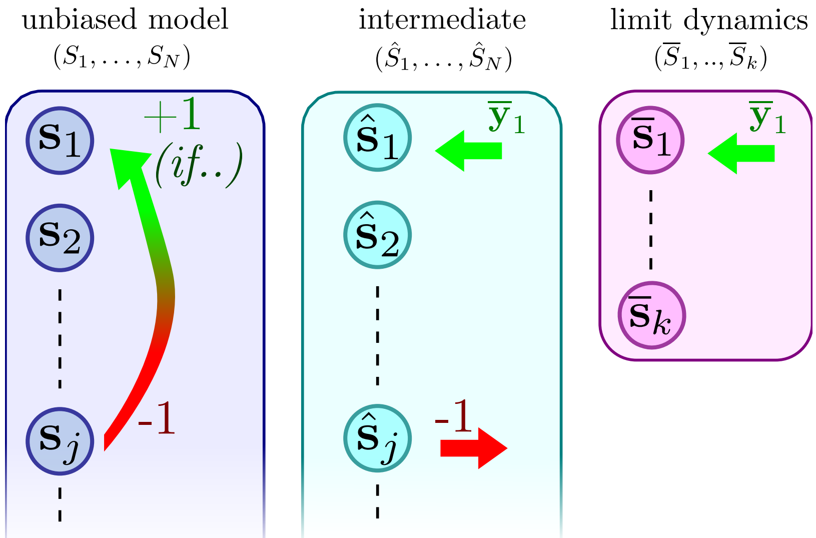

Figure 2.5: Schematic Illustration of the Coupling Strategy. We Use an Intermediate Process $(\widehat{S}_1, \ldots, \widehat{S}_N)$ to *Decouple* the "Give" and "Receive" Parts of the Dynamics.

An schematic illustration of the above coupling technique is shown in Fig 2.5 below. We first have to control the difference between the process $(S_1, \ldots, S_N)$ and the intermediate dynamics $(\widehat{S}_1, \ldots, \widehat{S}_N)$. The key idea is based on the following simple yet effective lemma that allows to create optimal coupling between two flipping coins Den Hollander [75].

Lemma 2.4.1 *For any $p, q \in (0, 1)$, there exist $X \sim \mathcal{B}(p)$ and $Y \sim \mathcal{B}(q)$ such that* $\mathbb{P}(X \neq Y) = |p - q|$.

Proof. Let $U \sim \mathcal{U}[0, 1]$ a uniform random variable. Define the Bernoulli random variables as $X := \mathbb{1}_{[0,p)}(U)$ and $Y := \mathbb{1}_{[0,q)}(U)$. It is straightforward to show that $X \sim \mathcal{B}(p)$, $Y \sim \mathcal{B}(q)$ and $\mathbb{P}(X \neq Y) = |p - q|$. $\square$

More generally, if N_t and M_t are two inhomogeneous Poisson processes with rate $\lambda(t)$

and $\mu(t)$, respectively, then there exists a coupling such that

$$\mathrm{d}\mathbb{E}[|N_t - M_t|] \leq |\lambda(t) - \mu(t)|\mathrm{d}t.$$

This leads to the following proposition.

Proposition 2.4.2 *Let* $\left(S_1, \ldots, S_N\right)$ *and* $\left(\widehat{S}_1, \ldots, \widehat{S}_N\right)$ *be solution to* (2.1) *and* (2.8) *respectively, with the same initial condition. Then for any* $1 \leq i \leq N$*, we have*

$$\mathrm{d}\mathbb{E}[|S_i(t) - \widehat{S}_i(t)|] \ \leq\ \lambda\, \mathbb{E}[|r(t) - \bar{r}(t)|]\, \mathrm{d}t \ +\ \lambda \frac{2}{N}\, \mathrm{d}t, \tag{2.10}$$

where $r(t) = \frac{1}{N}\sum_{j=1}^{N} \mathbb{1}_{[1,\infty)}\left(S_j(t)\right)$ *and* $\bar{r}(t)$ *given by* (2.5)*.*

Proof. The processes $\widehat{S}_i(t)$ and $S_i(t)$ "share" the same clocks $\mathrm{N}_t^{(i,j)}$ and $\mathrm{N}_t^{(j,i)}$ for $j \neq i$. Denote the 'rich or not' random Bernoulli random variables:

$$R_i(t) = \mathbb{1}_{[1,\infty)}\left(S_i(t)\right) \quad \text{and} \quad \widehat{R}_i(t) = \mathbb{1}_{[1,\infty)}\left(\widehat{S}_i(t)\right). \tag{2.11}$$

Once a clock $\mathrm{N}_t^{(i,j)}$ rings, the processes become:

$$\begin{aligned} (S_i, S_j) &\rightsquigarrow (S_i - R_i, S_j + R_i), \\ (\widehat{S}_i, \widehat{S}_j) &\rightsquigarrow (\widehat{S}_i - \widehat{R}_i, \widehat{S}_j + \overline{Y}). \end{aligned} \tag{2.12}$$

Notice that the difference $|S_i - \widehat{S}_i|$ can only decay after the jump from the clock $\mathrm{N}_t^{(i,j)}$ (the 'give' dynamics reduce the difference). However, the 'receive' dynamics from the clock $\mathrm{N}_t^{(j,i)}$ could increase the difference $|S_j - \widehat{S}_j|$ if $\widehat{R}_i \neq \overline{Y}$. More precisely, we find:

$$\mathrm{d}\mathbb{E}[|S_i(t) - \widehat{S}_i(t)|] \leq 0 + \sum_{j=1, j\neq i}^{N} \mathbb{E}[|R_j(t-) - \overline{Y}(t-)|] \frac{\lambda}{N}\, \mathrm{d}t \ +\ \frac{2\lambda}{N}\, \mathrm{d}t \tag{2.13}$$

where the extra $\frac{2\lambda}{N}\,\mathrm{d}t$ is due to the extra clocks $\overline{\mathrm{N}}_t^{(i,i)}$ and $\overline{\mathrm{M}}_t^{(i,i)}$ in (2.9).

Now we have to couple the Bernoulli process $\overline{Y}(t-)$ with $R_j(t-)$ in a convenient way to make the difference as small as possible. Here is the strategy:

- Step 1: generate a master Poisson clock $\mathbf{N}_t$ with intensity λN which gives a collection of jumping times.

- Step 2: to select which clock $\mathrm{N}_t^{(i,j)}$ rings, calculate the proportions of "rich people" for the $N-$particle system and for the limit dynamics:

$$r(t-) = \frac{1}{N}\sum_{j=1}^{N} \mathbb{1}_{[1,\infty)}\Big(S_j(t-)\Big) \quad , \quad \bar{r}(t-) = 1 - p_0(t-). \tag{2.14}$$

- Step 3: let $U \sim \mathcal{U}([0,1])$ a uniform random variable.

  - if $U < r(t-)$, pick an index i uniformly among the rich people (i.e. i such that $S_i(t-) > 0$), otherwise we pick i uniformly among the poor people (i.e. i such that $S_i(t-) = 0$). Pick index j uniformly among $\{1, 2, \ldots, N\}$.

  - if $U < \bar{r}(t-)$, let $\overline{Y}(t-) = 1$, otherwise $\overline{Y}(t-) = 0$ (i.e. $\overline{Y}(t-) = \mathbb{1}_{[0,\bar{r}(t-)]}(U)$).

- Step 4: if $i \neq j$, update using (2.12)

Thanks to our coupling, the 'receiving' dynamics of S_i and $\widehat{S}_i$ will differ with probability $|r - \bar{r}|$:

$$\mathbb{E}[|R_j(t-) - \overline{Y}(t-)|] = \mathbb{P}\Big(R_j(t-) \neq \overline{Y}(t-)\Big) = \mathbb{E}\left[|r - \bar{r}|\right]. \tag{2.15}$$

Plug in the expression in (2.13) concludes the proof.

□

Remark. The update formula (2.12) for $(\widehat{S}_i, \widehat{S}_j)$ highlights that the 'give' and 'receive' dynamics are now independent in the auxiliary dynamics (i.e. $\widehat{R}_i$ and $\overline{Y}$ are independent). In contrast, we use the same process R_i to update S_i and S_j.

Now we study the coupling between the auxiliary dynamics $(\widehat{S}_1, \ldots, \widehat{S}_N)$ and the limit dynamics $(\overline{S}_1, \ldots, \overline{S}_k)$ for a **fixed** k (while $N \to \infty$). The idea is to remove the clocks $\mathrm{N}_t^{(i,j)}$ for $1 \le i,j \le k$ to decouple the time of the jump in $\overline{S}_i$ and $\overline{S}_j$ as described in the figure 2.6.

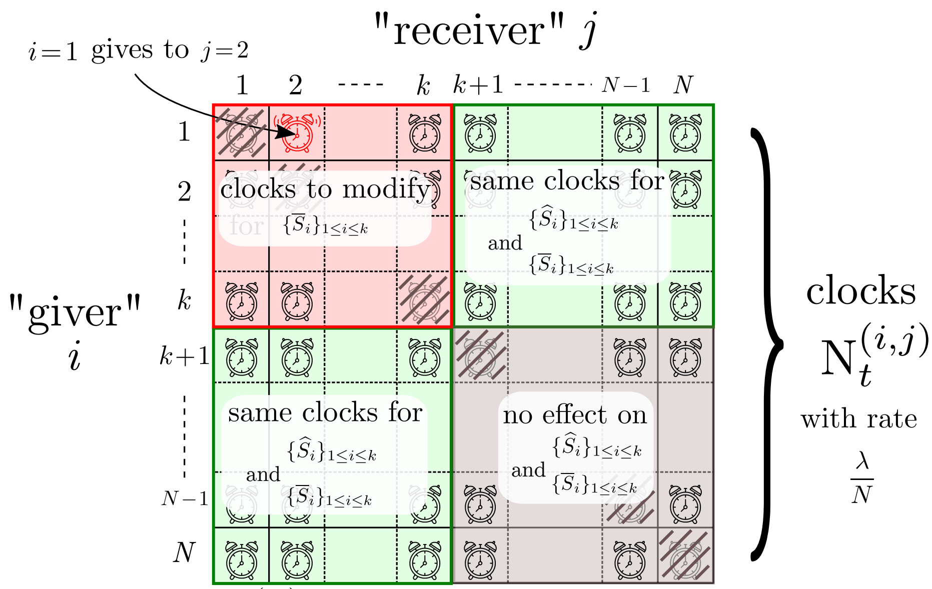

Figure 2.6: The Clocks $\mathrm{N}_t^{(i,j)}$ Used to Generate the Unbiased Dynamics (2.1) Have to Be Modified to Generate the Limit Dynamics $(\overline{S}_1(t),\ldots,\overline{S}_k(t))$ (2.4). The Processes $\overline{S}_i(t)$ and $\overline{S}_j(t)$ Have to Be Independent, Thus the Clocks $\mathrm{N}_t^{(i,j)}$ for $1 \le i,j \le k$ Cannot Be Used.

Proposition 2.4.3 *Let* $\left(\widehat{S}_1,\ldots,\widehat{S}_N\right)$ *solution to* (2.8) *and* $\{\overline{S}_i\}_{1\le i\le k}$ *independent processes solution to* (2.4). *Then for any fixed* $k \in \mathbb{N}_+$, *there exists a coupling such that for all* $t \ge 0$:

$$\mathrm{d}\mathbb{E}[|\widehat{S}_i(t) - \overline{S}_i(t)|] \le \lambda \frac{4(k-1)}{N}\,\mathrm{d}t \quad , \qquad \textit{for } 1 \le i \le k. \tag{2.16}$$

Proof. We assume $i=1$ to simplify the writing. To couple the two processes $\widehat{S}_1$ and $\overline{S}_1$, we use the same Bernoulli variable $\overline{Y}(t-)$ to generate both 'receive' dynamics:

$$\begin{cases} \mathrm{d}\widehat{S}_1(t) &= -\mathbb{1}_{[1,\infty)}\left(\widehat{S}_1(t-)\right)\mathrm{d}\widehat{\mathbf{N}}_t^1 + \overline{Y}(t-)\mathrm{d}\widehat{\mathbf{M}}_t^1, \\ \mathrm{d}\overline{S}_1(t) &= -\mathbb{1}_{[1,\infty)}\left(\overline{S}_1(t-)\right)\mathrm{d}\overline{\mathbf{N}}_t^1 + \overline{Y}(t-)\mathrm{d}\overline{\mathbf{M}}_t^1. \end{cases}$$

Meanwhile, the Poisson clocks $\widehat{\mathbf{N}}_t^1$, $\widehat{\mathbf{M}}_t^1$ are already determined in (2.8):

$$\widehat{\mathbf{N}}_t^1 = \overline{\mathrm{N}}_t^{(1,1)} + \sum_{j=2}^{N} \mathrm{N}_t^{(1,j)} \quad \text{and} \quad \widehat{\mathbf{M}}_t^1 = \overline{\mathrm{M}}_t^{(1,1)} + \sum_{j=2}^{N} \mathrm{N}_t^{(j,1)}. \tag{2.17}$$

Unfortunately, we cannot use the same definition for the clocks $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ as the clocks $\widehat{\mathbf{N}}_t^i$ and $\widehat{\mathbf{M}}_t^j$ are *not* independent (they both contain the clock $\mathrm{N}_t^{(i,j)}$). Thus, we need to remove those coupling clocks when defining $\overline{\mathbf{N}}^1$ and $\overline{\mathbf{M}}^1$. Fortunately, we only have to generate the dynamics for k process, thus we only have to replace the clocks $\mathrm{N}^{(1,i)}$ and $\mathrm{N}^{(i,1)}$ for $i = 1..k$ (see figure 2.6):

$$\overline{\mathbf{N}}_t^1 = \sum_{j=1}^{k} \overline{\mathrm{N}}_t^{(1,j)} + \sum_{j=k+1}^{N} \mathrm{N}_t^{(1,j)} \quad \text{and} \quad \widehat{\mathbf{M}}_t^1 = \sum_{j=1}^{k} \overline{\mathrm{M}}_t^{(1,j)} + \sum_{j=k+1}^{N} \mathrm{N}_t^{(j,1)} \tag{2.18}$$

where $\overline{\mathrm{N}}_t^{(1,j)}$ and $\overline{\mathrm{M}}_t^{(1,j)}$ are independent Poisson clocks with rate $\frac{\lambda}{N}$.

Using this coupling strategy, the difference $|\widehat{S}_1 - \overline{S}_1|$ could only increase (by 1) if the clocks $\overline{\mathrm{N}}_t^{(1,j)}$, $\overline{\mathrm{M}}_t^{(1,j)}$, $\mathrm{N}_t^{(1,j)}$ or $\mathrm{N}_t^{(j,1)}$ ring for $2 \leq j \leq k$ leading to (2.16).

$\square$

Finally, combining propositions 2.4.2 and 2.4.3 gives rise to the following theorem.

Theorem 1 *Let* $\left(S_1, \ldots, S_N\right)$ *to be a solution to* (2.1). *Then for any fixed* $k \in \mathbb{N}_+$ *and* $t \geq 0$, *there exists a coupling between* $\left(S_1, \ldots, S_k\right)$ *and* $\left(\overline{S}_1, \ldots, \overline{S}_k\right)$ *(with the same initial conditions) such that:*

$$\mathbb{E}[|S_i(t) - \overline{S}_i(t)|] \leq \frac{C(t)}{\sqrt{N}} \frac{(\mathrm{e}^{\lambda t} - 1)}{\lambda} + \lambda \frac{4(k-1)t}{N} \tag{2.19}$$

with $C(t) = \left(\frac{1}{4} + \lambda 4t\right)^{1/2} + \lambda \frac{2}{\sqrt{N}}$ *holding for each* $1 \leq i \leq k$.

Proof. We assume without loss of generality that $i = 1$. First, we show that the processes S_1 and $\widehat{S}_1$ remain closed. We denote:

$$R_i = \mathbb{1}_{[1,\infty)}\left(S_i\right) \quad , \quad \widehat{R}_i = \mathbb{1}_{[1,\infty)}\left(\widehat{S}_i\right) \quad , \quad \overline{R}_i = \mathbb{1}_{[1,\infty)}\left(\overline{S}_i\right).$$

We have:

$$\begin{aligned}
\mathbb{E}[|r-\bar{r}|] &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} R_i - \bar{r}\right|\right] = \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}(R_i - \widehat{R}_i) \;+\; \frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right|\right] \\
&\le \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|R_i - \widehat{R}_i|] + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right|\right] \\
&\le \mathbb{E}[|S_1 - \widehat{S}_1|] + \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right)^2\right]^{1/2},
\end{aligned}$$

where we use $|R_i - \widehat{R}_i| \le |S_i - \widehat{S}_i|$. To control the variance, we expand:

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - \bar{r})\right)^2\right] &= \frac{1}{N}\mathrm{Var}[\widehat{R}_1] + \frac{N(N-1)}{N^2}\mathrm{Cov}(\widehat{R}_1, \widehat{R}_2) \\
&\le \frac{1}{4N} + \mathrm{Cov}(\widehat{R}_1, \widehat{R}_2),
\end{aligned}$$

since $\widehat{R}_1$ is a Bernoulli variable its variance is bounded by $1/4$. Controlling the covariance of $\widehat{R}_1$ and $\widehat{R}_2$ is more delicate since the two processes are not independent due to the clocks $\mathrm{N}_t^{(1,2)}$ and $\mathrm{N}_t^{(2,1)}$. Fortunately, these clocks have a rate of only λ/N and thus the covariance has to remain small for a given time interval. To prove it, let's use the independent processes $\overline{R}_1$ and $\overline{R}_2$:

$$\mathrm{Cov}(\widehat{R}_1, \widehat{R}_2) = \mathrm{Cov}(\widehat{R}_1 - \overline{R}_1, \widehat{R}_2 - \overline{R}_2) \le \left(\mathbb{E}[|\widehat{R}_1 - \overline{R}_1|^2] \cdot \mathbb{E}[|\widehat{R}_2 - \overline{R}_2|^2]\right)^{1/2}$$

using Cauchy-Schwarz. Since the two processes $\widehat{S}_i$ and $\overline{S}_i$ remain close, we deduce:

$$\mathbb{E}[|\widehat{R}_1(t) - \overline{R}_1(t)|^2] = \mathbb{E}[|\widehat{R}_1(t) - \overline{R}_1(t)|] \le \mathbb{E}[|\widehat{S}_1(t) - \overline{S}_1(t)|] \le \lambda\frac{4t}{N},$$

using proposition 2.4.3 (with $k=2$). We conclude that:

$$\mathbb{E}[|r(t) - \bar{r}(t)|] \le \mathbb{E}[|S_1(t) - \widehat{S}_1(t)|] + \left(\frac{1}{4N} + \lambda\frac{4t}{N}\right)^{1/2}.$$

Going back to proposition 2.4.2, we find:

$$\begin{aligned}
\mathrm{d}\mathbb{E}[|S_i(t) - \widehat{S}_i(t)|] &\le \lambda\,\mathbb{E}[|S_1(t) - \widehat{S}_1(t)|]\,\mathrm{d}t + \left(\frac{1}{4N} + \lambda\frac{4t}{N}\right)^{1/2}\mathrm{d}t \;+\; \lambda\frac{2}{N}\,\mathrm{d}t \\
&\le \lambda\,\mathbb{E}[|S_1(t) - \widehat{S}_1(t)|]\,\mathrm{d}t + \frac{C(t)}{\sqrt{N}}\,\mathrm{d}t
\end{aligned}$$

with $C(t) = \left(\frac{1}{4} + \lambda 4t\right)^{1/2} + \lambda \frac{2}{\sqrt{N}} = \mathcal{O}(1)$. Using Gronwall's lemma, since $|S_i(0) - \widehat{S}_i(0)| = 0$, we obtain:

$$\mathbb{E}[|S_i(t) - \widehat{S}_i(t)|] \leq \frac{C(t)}{\sqrt{N}} \frac{(\mathrm{e}^{\lambda t} - 1)}{\lambda}. \tag{2.20}$$

We finally conclude by using proposition 2.4.3 and triangular inequality. □

Remark. In the community of Markov chains, the process $(\mathbf{S}(t) \colon t \geq 0)$ with $\mathbf{S}(t) := (S_1(t), \ldots, S_N(t))$ can serve as an example a zero-range process Spitzer [143], and it is also observed in Merle and Salez [118] that the unbiased exchange model exhibits a *cutoff* phenomenon (see for instance Diaconis [77], Aldous [5], Aldous and Diaconis [6]), which is now ubiquitous among literatures on interacting Markov chains.

2.4.3 Convergence to Equilibrium

After we achieved the transition from the interacting system of SDEs (3) to the deterministic system of nonlinear ODEs (2.6), in this section we will analyze (2.6) with the intention of proving convergence of solution of (2.6) to its (unique) equilibrium solution. The main ingredient underlying our proof lies in the reformulation of (2.6) into a (discrete) Fokker-Planck type equation, combined with the standard entropy method Arnold *et al.* [11], Matthes [114], Jüngel [97]. We emphasize here that the convergence of the solution of (2.6) has already been established in Graham [83], Merle and Salez [118], but we include a sketch of our analysis here for the sake of completeness of the present manuscript.

To study the ODE system (2.6), we introduce some properties of the nonlinear binary collision operator Q_{unbias}, whose proof is merely a straightforward calculations and will be omitted.

Lemma 2.4.4 *If* $\mathbf{p}(t) = \{p_n(t)\}_{n\geq 0}$ *is a solution of* (2.6)*, then*

$$\sum_{n=0}^{\infty} Q_{unbias}[\mathbf{p}]_n = 0 \quad , \quad \sum_{n=0}^{\infty} n\, Q_{unbias}[\mathbf{p}]_n = 0. \tag{2.21}$$

In particular, the total mass and the mean value is conserved.

Thanks to these conservations, we have $\mathbf{p}(t) \in V_\mu$ for all $t \geq 0$, where

$$V_\mu := \{\mathbf{p} \mid \sum_{n=0}^{\infty} p_n = 1,\ p_n \geq 0,\ \sum_{n=0}^{\infty} n\, p_n = \mu\}$$

is the space of probability mass functions with the prescribed mean value μ. Next, the equilibrium distribution of the limiting dynamics (2.6) is explicitly calculated.

Proposition 2.4.5 *The (unique) equilibrium distribution* $\mathbf{p}^* = \{p_n^*\}_n$ *in* V_μ *associated with the limiting dynamics* (2.6) *is given by:*

$$p_n^* = p_0^*(1 - p_0^*)^n, \quad n \geq 0, \tag{2.22}$$

where $p_0^* = \frac{1}{1+\mu}$ *if we put initially that* $\sum_{n=0}^{\infty} n\, p_n(0) = \mu$ *for some* $\mu \in \mathbb{N}_+$.

This elementary observation can be verified through straightforward computations, which we will omit here.

Next, we recall the definition of entropy Cover [62], which will play a major role in the analysis of the large time behavior of the system (2.6).

Definition 6 *(**Entropy**) For a given probability mass function* $\mathbf{p} \in V_\mu$*, the entropy of* $\mathbf{p}$ *is defined via*

$$\mathrm{H}[\mathbf{p}] = \sum_n p_n \log p_n.$$

Remark. It can be readily seen through the method of Lagrange multipliers that the geometric distribution (2.22) has the least amount of entropy among probability mass functions from V_μ.

To prove (strong) convergence of the solution of (2.6) to its equilibrium solution (2.22), a major step is first to realize that the original ODE dynamics (2.6) can be reformulated as a variant of a Fokker-Planck equation Risken [137]. Indeed, let us introduce $\mathbf{s}(t) := \{s_n(t)\}_{n\geq 0}$ with $s_n(t) := p_0(t)[\bar{r}(t)]^n$. Notice that $\frac{s_{n+1}}{s_n} = \bar{r}$. Thus, for $n \geq 1$ we can deduce that

$$p_n' = \frac{s_{n+1}}{s_{n+1}} p_{n+1} + \frac{s_n}{s_{n-1}} p_{n-1} - \frac{s_n}{s_n} p_n - \frac{s_{n+1}}{s_n} p_n.$$

Setting $q_n(t) = \frac{p_n(t)}{s_n(t)}$, we obtain

$$p_n' = s_{n+1}[q_{n+1} - q_n] - s_n[q_n - q_{n-1}], \tag{2.23}$$

with the convention that $q_{-1} \equiv 1$. This formulation leads to the following:

Proposition 2.4.6 *Let* $\{p_n(t)\}_{n\geq 0}$ *be the solution to* (2.6) *and* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *to be a continuous function, then*

$$\sum_{n=0}^{\infty} p_n'\, \varphi(n) = \sum_{n=0}^{\infty} (\bar{r} p_n - p_{n+1}) \Big(\varphi(n+1) - \varphi(n)\Big). \tag{2.24}$$

Corollary 2.4.7 *Taking* $\varphi(n) \equiv 1$ *and* $\varphi(n) = n$ *for* $n \geq 0$ *in* (2.24)*, we recover the facts that* $\sum_{n=0}^{\infty} p_n$ *and* $\sum_{n=0}^{\infty} n\, p_n$ *are preserved over time.*

Inserting $\varphi(n) = \log p_n$, we can deduce the following important result.

Proposition 2.4.8 (Entropy dissipation) *Let* $\mathbf{p}(t) = \{p_n(t)\}_{n\geq 0}$ *be the solution to* (2.6) *and* $\mathrm{H}[\mathbf{p}]$ *be the associated entropy, then for all* $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{H}[\mathbf{p}] = -\mathrm{D}_{\mathrm{KL}}\left(\mathbf{p} || \tilde{\mathbf{p}}\right) - \mathrm{D}_{\mathrm{KL}}\left(\tilde{\mathbf{p}} || \mathbf{p}\right) \leq 0,$$

where $\tilde{\mathbf{p}} := \{\tilde{p}_n\}_{n\geq 0}$ *is defined by* $\tilde{p}_0 = p_0$ *and* $\tilde{p}_n = \bar{r} p_{n-1}$ *for* $n \geq 1$.

Proof. It is worth noting that

$$\sum_{n=0}^{\infty} \tilde{p}_n = p_0 + \sum_{n=0}^{\infty} \tilde{p}_{n+1} = p_0 + \bar{r} \sum_{n=0}^{\infty} p_n = p_0 + \bar{r} = 1,$$

so that $\{\tilde{p}_n\}_n$ indeed defines a probability distribution (for all $t \geq 0$). Then we deduce from (2.24) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\mathbf{p}] &= \sum_{n=0}^{\infty} p_n' \log p_n = \sum_{n=0}^{\infty} (\bar{r} p_n - p_{n+1}) \log \frac{p_{n+1}}{p_n} \\
&= \sum_{n=0}^{\infty} (p_{n+1} - \tilde{p}_{n+1}) \Big(\log \frac{\bar{r} p_n}{p_{n+1}} - \log \bar{r} \Big) \\
&= \sum_{n=0}^{\infty} (p_{n+1} - \tilde{p}_{n+1}) \log \frac{\tilde{p}_{n+1}}{p_{n+1}} \\
&= \sum_{n=0}^{\infty} (p_n - \tilde{p}_n) \log \frac{\tilde{p}_n}{p_n} = -\sum_{n=0}^{\infty} p_n \log \frac{p_n}{\tilde{p}_n} - \sum_{n=0}^{\infty} \tilde{p}_n \log \frac{\tilde{p}_n}{p_n} \\
&= -\mathrm{D}_{\mathrm{KL}}\left(\mathbf{p} || \tilde{\mathbf{p}}\right) - \mathrm{D}_{\mathrm{KL}}\left(\tilde{\mathbf{p}} || \mathbf{p}\right) \leq 0,
\end{aligned}
$$

in which $\mathrm{D}_{\mathrm{KL}}\left(\mathbf{p} || \mathbf{q}\right) := \sum_{n=0}^{\infty} p_n \log \frac{p_n}{q_n}$ (≥ 0) is the Kullback-Leiber divergence from the probability distribution $\mathbf{q}$ to $\mathbf{p}$. $\square$

Remark. By a property of the Kullback-Leiber divergence Csiszár and Shields [63], $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}[\mathbf{p}] = 0$ if and only if $\mathbf{p} = \tilde{\mathbf{p}}$, but it can be readily shown that $\mathbf{p} = \tilde{\mathbf{p}}$ if and only if $\mathbf{p}$ coincides with the equilibrium distribution $\mathbf{p}^*$.

Our next focus is on the demonstration of the strong convergence of solutions $\mathbf{p}(t) = \{p_n(t)\}_{n\geq 0}$ of (2.6) to its unique equilibrium solution given by (2.22). First of all, we notice that V_μ is clearly closed and bounded in ℓ^p for each $1 \leq p \leq \infty$, whence there exists some $\widehat{\mathbf{p}} = \{\widehat{p}_n\}_{n\geq 0} \in V_\mu$ and a diverging sequence $\{t_k\}_k$ such that $\mathbf{p}^{(k)} := \mathbf{p}(t_k) \rightharpoonup \widehat{\mathbf{p}}$ weakly in ℓ^p $(1 < p < \infty)$ as $k \to \infty$. In particular, we have the point-wise convergence

$$
p_n^{(k)} \to \widehat{p}_n \quad \text{for each } n \geq 0.
$$

Our ultimate goal is to show that $\widehat{\mathbf{p}} = \mathbf{p}^*$, for which we first establish the following proposition.

Proposition 2.4.9 *Suppose that* $\{\mathbf{p}^{(k)}\}_k$ *is a sequence of probability distributions in* V_m *such that*

$$\mathbf{p}^{(k)} \rightharpoonup \widehat{p}$$

weakly in ℓ^p *for some* $1 < p < \infty$. *If the family* $\{\mathbf{p}^{(k)}\}_k$ *satisfies the following* uniform integrability *condition Oksendal [124]*

$$\sum_{n=0}^{\infty} n^\gamma p_n^{(k)} < \infty \quad \textit{uniformly for all } k \tag{2.25}$$

for some $\gamma > 1$, *then*

$$\sum_{n=0}^{\infty} p_n^{(k)} \log p_n^{(k)} \to \sum_{n=0}^{\infty} \widehat{p}_n \log \widehat{p}_n \quad \textit{as } k \to \infty. \tag{2.26}$$

Proof. It suffices to show that for any given $\varepsilon > 0$, there exists some universal constant $N = N(\varepsilon)$ such that

$$\sum_{n=N}^{\infty} -p_n^{(k)} \log p_n^{(k)} < \varepsilon \quad \forall k \geq 0. \tag{2.27}$$

Assume that $\sum_{n=0}^{\infty} n^\gamma p_n^{(k)} \leq C$ holds uniformly in k for some constant $\gamma > 1$, where $C > 0$ is fixed. Then $p_n^{(k)} \leq \frac{C}{n^\gamma}$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Since $g(x) := -x \log x$ is an increasing function for small $x > 0$, we have for some fixed sufficiently large N that

$$\sum_{n=N}^{\infty} -p_n^{(k)} \log p_n^{(k)} \leq \sum_{n=N}^{\infty} -\frac{C}{n^\gamma} \log \frac{C}{n^\gamma} < \varepsilon,$$

and the proof is completed.

$\square$

The next lemma ensures that the solution $\{p_n(t)\}_{n\geq 0}$ of our limiting ODE system (2.6) is uniformly integrable (in time), whose proof is elementary and is thus skipped.

Lemma 2.4.10 *Let* $\{p_n(t)\}_{n\geq 0}$ *to be the solution of* (2.6). *Assume that* $\sum_{n=0}^{\infty} p_n(0) a^n < \infty$ *for some* $a > 1$, *then for each fixed* $\gamma > 1$,

$$\sum_{n=0}^{\infty} n^\gamma p_n(t) < \infty$$

holds uniformly in time.

We are now in a position to prove the desired convergence result.

Proposition 2.4.11 *The solution* $\mathbf{p}(t) = \{p_n(t)\}_{n\geq 0}$ *of* (2.6) *converges strongly in* ℓ^p *for* $1 < p < \infty$ *as* $t \to \infty$ *to its unique equilibrium solution* $\mathbf{p}^* = \{p_n^*\}_n$ *given by* (2.22).

Proof. Our proof follows closely to the general strategy presented in Perko [126] and is in essence a continuity argument. We will denote the flow of the ODE system (2.6) with initial data $\mathbf{p}^0 \in V_\mu$ by $\phi_t(\mathbf{p}^0)$. It is recalled that we only need to show that $\widehat{\mathbf{p}} = \mathbf{p}^*$. We argue by contradiction and suppose that $\widehat{\mathbf{p}} \neq \mathbf{p}^*$. Since $\mathrm{H}[\mathbf{p}(t)]$ is strictly decreasing along trajectories of (2.6) and since $\mathbf{p}(t_k) \rightharpoonup \widehat{\mathbf{p}}$ weakly in ℓ^p $(1 < p < \infty)$ as $k \to \infty$, we deduce that $\mathrm{H}[\phi_{t_k}(\mathbf{p}^0)] \to \mathrm{H}[\widehat{\mathbf{p}}]$ by combining proposition 2.4.9 and proposition 2.4.10, whence

$$\mathrm{H}[\phi_{t_k}(\mathbf{p}^0)] > \mathrm{H}[\widehat{\mathbf{p}}]$$

for all $t > 0$. But if $\widehat{\mathbf{p}} \neq \mathbf{p}^*$, then for all $s > 0$ we must have $\mathrm{H}[\phi_s(\widehat{\mathbf{p}})] < \mathrm{H}[\widehat{\mathbf{p}}]$, and by continuity, it follows that for all $\mathbf{p} \in V_\mu$ sufficiently close to $\widehat{\mathbf{p}}$ in the ℓ^p norm $(1 < p < \infty)$ we have $\mathrm{H}[\phi_s[\mathbf{p}]] < \mathrm{H}[\widehat{\mathbf{p}}]$ for all $s > 0$. But then for $\mathbf{p} := \phi_{t_k}(\mathbf{p}^0)$ and sufficiently large k, we have

$$\mathrm{H}[\phi_{s+t_k}(\mathbf{p}^0)] < \mathrm{H}[\widehat{\mathbf{p}}],$$

which contradicts the above inequality. Therefore we must have that $\widehat{\mathbf{p}} = \mathbf{p}^*$ and hence $\mathbf{p}(t) \rightharpoonup \mathbf{p}^*$ weakly in ℓ^p $(1 < p < \infty)$ as $t \to \infty$. In particular, we have the pointwise convergence

$$p_n(t) \to p_n^* \quad \text{as } t \to \infty \text{ for each } n \geq 0.$$

Now since

$$\begin{aligned}\|\mathbf{p}(t)-\mathbf{p}^*\|_1 :&= \sum_{n=0}^{\infty}|p_n(t)-p_n^*| = \sum_{n=0}^{N}|p_n(t)-p_n^*| + \sum_{n=N+1}^{\infty}|p_n(t)-p_n^*| \\ &\leq \sum_{n=0}^{N}|p_n(t)-p_n^*| + \sum_{n=N+1}^{\infty}(p_n(t)+p_n^*),\end{aligned}$$

by taking N to be sufficiently large and independent of t, the desired strong convergence in ℓ^p for $1 < p < \infty$ follows immediately. □

2.5 Poor-Biased Exchange Model

We now investigate our second model where the 'given' dynamics is biased toward richer agent: the wealthier an agent becomes, the more likely it will give a dollar. As for the previous model, we first investigate the limit dynamics as the number of agents N goes to infinity, then we study the large time behavior and show rigorously the convergence of the wealth distribution to a Poisson distribution.

2.5.1 Definition and Limit Equation

We use the same setting as the unbiased model: there are N agents with initially the same amount of money $S_i(0) = \mu$ with $\mu \in \mathbb{N}_+$.

Definition 7 (Poor-biased exchanged model) *The dynamics consists in choosing a "giver" i with a probability proportional to its wealth (the wealthier an agent, the more likely it will be a "giver"). Then it gives one dollar to a "receiver" j chosen uniformly.*

From another point of view, the dynamics consist in taking one dollar from the common pot (tax system) and re-distribute the dollar uniformly among the individuals. Thus instead of 'taxing the agents' in the unbiased exchange model, the poor-biased model is 'taxing the dollar'.

The poor-biased model can be written in term of stochastic differential equation, the wealth S_i of agent i evolves according to:

$$\mathrm{d}S_i(t) = -\sum_{j=1}^{N} \mathrm{dN}_t^{(i,j)} + \sum_{j=1}^{N} \mathrm{dN}_t^{(j,i)}, \tag{2.1}$$

with $\mathrm{N}_t^{(i,j)}$ Poisson process with intensity $\lambda_{i,j}(t) = \frac{\lambda S_i(t)}{N}$.

Since the clocks $\{\mathrm{N}_t^{i,j}\}_{1\le i,j\le N}$ are now time dependent (in contrast to the unbiased model), the dynamics might appear more difficult to analyze. But it turns out to be simpler, since the rate of receiving a dollar is constant:

$$\sum_{j=1}^{N} \lambda_{j,i}(t) = \sum_{j=1}^{N} \frac{\lambda S_j(t)}{N} = \lambda\mu,$$

where μ is the (conserved) initial mean. In contrast, in the unbias dynamics, the rate of receiving a dollar is equal to the proportion of rich people $r(t)$ which fluctuates in time. Let's focus on $i = 1$ and sum up the clocks introducing:

$$\mathbf{N}_t^1 = \sum_{j=1}^{N} \mathrm{N}_t^{(1,j)}, \quad \mathbf{M}_t^1 = \sum_{j=1}^{N} \mathrm{N}_t^{(j,1)}, \tag{2.2}$$

where the two Poisson processes $\mathbf{N}_t^1$ and $\mathbf{M}_t^1$ have intensity λS_1 and $\lambda\mu$ (respectively). Thus, the poor-biased model leads to the equation:

$$\mathrm{d}S_1(t) = -\mathrm{d}\mathbf{N}_t^1 + \mathrm{d}\mathbf{M}_t^1. \tag{2.3}$$

Notice that $S_1(t)$ is not independent of $S_j(t)$ as both processes can jump at the same time due to the two clocks $\mathrm{N}_t^{(1,j)}$ and $\mathrm{N}_t^{(j,1)}$.

Motivated by the equation above, we give the following definition of the limiting dynamics as $N \to \infty$.

Definition 8 ***(Asymptotic Poor-biased model)*** *We define $\overline{S}_1$ to be the compound Poisson process satisfying the following SDE:*

$$\mathrm{d}\overline{S}_1(t) = -\mathrm{d}\overline{\mathbf{N}}_t^1 + \mathrm{d}\overline{\mathbf{M}}_t^1, \tag{2.4}$$

in which $\overline{\mathbf{N}}_t^1$ *and* $\overline{\mathbf{M}}_t^1$ *are independent Poisson processes with intensity* $\lambda \overline{S}_1(t)$ *and* $\lambda\mu$ *(respectively) where* μ *is the mean of* $\overline{S}_1(0)$ *(i.e.* $\mu = \mathbb{E}[\overline{S}_1(0)]$*).*

If we denote by $\mathbf{p}(t) = \Big(p_0(t), p_1(t), \ldots\Big)$ the law of the process $\overline{S}_1(t)$, its time evolution is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, Q_{poor}[\mathbf{p}(t)] \tag{2.5}$$

with:

$$Q_{poor}[\mathbf{p}]_n := \begin{cases} p_1 - \mu\, p_0 & \text{if } n = 0 \\ (n+1)p_{n+1} + \mu\, p_{n-1} - (n+\mu)\, p_n & \text{for } n \geq 1 \end{cases} \tag{2.6}$$

and $\mu = \sum_{n=0}^{+\infty} n\, p_n(t) = \sum_{n=0}^{+\infty} n\, p_n(0)$.

2.5.2 Proof of Propagation of Chaos

The aim of this subsection is to prove the propagation of chaos, i.e. that the process $\Big(S_1, \ldots, S_k\Big)$ converges to $\Big(\overline{S}_1, \ldots, \overline{S}_k\Big)$ as N goes to infinity. As for the unbiased exchange model, the key is to define the Poisson clocks for the limit dynamics $\overline{\mathbf{N}}_t^i$ and $\overline{\mathbf{M}}_t^i$ *close to* the clocks of the $N-$particle system $\mathbf{N}_t^i$ and $\mathbf{M}_t^i$ for $1 \leq i \leq k$, but at the same time making the clocks independent. With this aim, we have to 'remove' the clocks $\mathbf{N}_t^{(i,j)}$ and $\mathbf{M}_t^{(i,j)}$ for $1 \leq i, j \leq k$.

Theorem 2 *Let* $\Big(S_1, \ldots, S_N\Big)$ *to be a solution to* (2.1) *and* $\Big(\overline{S}_1, \ldots, \overline{S}_k\Big)$ *a solution to* (2.4)*. Then for any fixed* $k \in \mathbb{N}_+$*, there exists a coupling between* $\Big(S_1, \ldots, S_k\Big)$ *and* $\Big(\overline{S}_1, \ldots, \overline{S}_k\Big)$ *(with the same initial conditions) such that:*

$$\mathbb{E}[|\overline{S}_i(t) - S_i(t)|] \leq \frac{4k\lambda\mu}{N}(\mathrm{e}^{\lambda t} - 1), \tag{2.7}$$

holding for each $1 \leq i \leq k$*.*

Proof. To simplify the writing, we suppose $i = 1$. We define for $1 \leq i \leq k$ the clocks for the limit dynamics as follow:

$$\overline{\mathbf{N}}_t^1 = \overline{G}_1 \cdot \left(\sum_{j=k+1}^{N} \mathrm{N}_t^{(1,j)} \right) + \widehat{\mathrm{N}}_t^1 \quad , \quad \overline{\mathbf{M}}_t^1 = \left(\sum_{j=k+1}^{N} \mathrm{N}_t^{(j,1)} \right) + \widehat{\mathrm{M}}_t^1. \tag{2.8}$$

Here, $\overline{G}_1$ is a Bernoulli random variable that prevents the clocks to ring for $\overline{S}_1$ if the rates of the clocks $\mathrm{N}_t^{(1,j)}$ from $k+1 \le j \le N$ are too large compare to $\overline{S}_1$. The parameter of this Bernoulli random variable is given by:

$$\overline{G}_1(t) \sim \mathcal{B}\Big(1 \wedge \frac{N\overline{S}_1(t)}{(N-k)S_1(t)}\Big), \tag{2.9}$$

with $a \wedge b = \min\{a, b\}$ for any $a, b \in \mathbb{R}$. On the contrary, the two processes $\widehat{\mathrm{N}}_t^1$ and $\widehat{\mathrm{M}}_t^1$ are used to compensate if the rates of the clocks $\mathrm{N}_t^{(1,j)}$ and $\mathrm{N}_t^{(j,1)}$ from $k+1 \le j \le N$ are not large enough. Both processes $\widehat{\mathrm{N}}_t^1$ and $\widehat{\mathrm{M}}_t^1$ are independent (inhomogeneous) Poisson processes with rates respectively:

$$\widehat{\mu}(t) = \lambda \left(\overline{S}_1(t) - \frac{(N-k)S_1(t)}{N} \right)_+ \quad \text{and} \quad \widehat{\nu}(t) = \lambda \left(\mu - \sum_{j=k+1}^{N} \frac{S_j(t)}{N} \right) \tag{2.10}$$

where $a_+ = \max\{a, 0\}$ for any $a \in \mathbb{R}$. One can check that under the aforementioned setup (coupling of Poisson clocks), $\overline{\mathbf{N}}_t^1$ and $\overline{\mathbf{M}}_t^1$ are indeed independent counting processes with intensity $\lambda\, \overline{S}_i(t)$ and $\lambda\, \mu$, respectively.

The difference $|\overline{S}_1(t) - S_1(t)|$ could increase due to 3 types of events:

i) $\mathrm{N}_t^{(1,j)}$ and $\mathrm{N}_t^{(j,1)}$ ring for $1 \le j \le k$,

ii) $\widehat{\mathrm{N}}_t^1$ and $\widehat{\mathrm{M}}_t^1$ ring

iii) $\mathrm{N}_t^{(1,j)}$ ring for $j \ge k+1$ and $\overline{G}_1 = 0$.

Notice that the third type of event leads to:

$$S_1(t) = S_1(t-) - 1 \quad , \quad \overline{S}_1(t) = \overline{S}_1(t-) \tag{2.11}$$

i.e. only S_1 gives. However, the event $\{\overline{G}_1 = 0\}$ only occurs if $S_1(t-) > \overline{S}_1(t-)$. Therefore, the event iii) could only make $|\overline{S}_1(t) - S_1(t)|$ to decay.

Therefore, we deduce:

$$\begin{aligned}
\mathrm{d}\mathbb{E}[|\bar{S}_1(t) - S_1(t)|] &\le \sum_{j=1}^{k} \frac{\lambda}{N}\mathbb{E}[S_1(t)]\mathrm{d}t \;+\; \sum_{j=1}^{k} \frac{\lambda}{N}\mathbb{E}[S_j(t)]\mathrm{d}t \\
&\quad +\mathbb{E}[\widehat{\mu}(t)]\mathrm{d}t \;+\; \mathbb{E}[\widehat{\nu}(t)]\mathrm{d}t \\
&\le \frac{2k\lambda\mu}{N}\mathrm{d}t \;+\; \mathbb{E}[\widehat{\mu}(t)]\mathrm{d}t \;+\; \mathbb{E}[\widehat{\nu}(t)]\mathrm{d}t
\end{aligned} \tag{2.12}$$

using $\mathbb{E}\left[S_j(t)\right] = \mu$ for any j. Let's bound the rates $\widehat{\mu}$ and $\widehat{\nu}$:

$$\begin{aligned}
\mathbb{E}[\widehat{\mu}] &= \mathbb{E}\left[\lambda\Big(\bar{S}_1 - \frac{(N-k)S_1}{N}\Big)_+\right] \;\le\; \lambda\mathbb{E}\left[\Big(\bar{S}_i - S_i\Big)_+ + \frac{kS_1}{N}\right] \\
&\le \lambda\mathbb{E}\left[|\bar{S}_1 - S_1|\right] + \frac{\lambda k\mu}{N} \\
\mathbb{E}[\widehat{\nu}] &= \mathbb{E}\left[\lambda\Big(\mu - \sum_{j=k+1}^{N} \frac{S_j}{N}\Big)\right] = \frac{\lambda k\mu}{N}.
\end{aligned}$$

We deduce from (2.12):

$$\mathrm{d}\mathbb{E}[|\bar{S}_1(t) - S_1(t)|] \le \lambda\mathbb{E}\left[|\bar{S}_1(t) - S_1(t)|\right]\mathrm{d}t + \frac{4k\lambda\mu}{N}\mathrm{d}t. \tag{2.13}$$

Applying the Gronwall's lemma to (2.13) yields the result.

□

2.5.3 Large Time Behavior

After we achieved the transition from the interacting system of SDEs (2.1) to the deterministic system of linear ODEs (2.5), we now analyze the long time behavior of the distribution $\mathbf{p}(t)$ and its convergence to an equilibrium. The main tool behind proof relies again on the reformulation of (2.5) into a (discrete) Fokker-Planck type equation, in conjunction with the standard entropy method Arnold *et al.* [11], Matthes [114], Jüngel [97].

Let's introduce a function space to study $\mathbf{p}(t)$:

$$V_\mu := \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid \sum_{n=0}^{\infty} p_n = 1,\ p_n \ge 0,\ \sum_{n=0}^{\infty} n\, p_n = \mu\}, \tag{2.14}$$

$$\mathcal{D}(Q_{poor}) := \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid Q_{poor}[\mathbf{p}] \in \ell^2(\mathbb{N})\}, \tag{2.15}$$

where ℓ^2 denote the vector space of square-summable sequences. In contrast to the unbias model with the dynamics (2.5), the operator Q_{poor} is an unbounded operator (i.e. $\mathcal{D}(Q_{poor}) \not\subset \ell^2(\mathbb{N})$). For any $\mathbf{p} \in V_\mu \cap \mathcal{D}(Q_{poor})$, it is straightforward to show that:

$$\sum_{n=0}^{\infty} Q_{poor}[\mathbf{p}]_n = 0 \quad , \quad \sum_{n=0}^{\infty} n\, Q_{poor}[\mathbf{p}]_n = 0, \tag{2.16}$$

which express that the total mass and the mean value is conserved. Moreover, there exists a unique equilibrium $\mathbf{p}^*$ for Q_{poor} in V_μ given by a Poisson distribution:

$$p_n^* = \frac{\mu^n}{n!}\mathrm{e}^{-\mu}, \quad n \geq 0. \tag{2.17}$$

To investigate the convergence of $\mathbf{p}(t)$ solution to (2.5) to the equilibrium $\mathbf{p}_*$ (2.17), we introduce two function spaces.

Definition 9 *We define the sub-vector spaces of ℓ^2:*

$$\begin{aligned}
\mathcal{H}^0 &= \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid \sum_{n=0}^{\infty} \frac{p_n^2}{p_n^*} < +\infty\}, && (2.18)\\
\mathcal{H}^1 &= \{\mathbf{p} \in \ell^2(\mathbb{N}) \mid \sum_{n=0}^{\infty} p_n^* \left(\frac{p_{n+1}}{p_{n+1}^*} - \frac{p_n}{p_n^*}\right)^2 < +\infty\}, && (2.19)
\end{aligned}$$

and define corresponding scalar products:

$$\langle \mathbf{p}, \mathbf{q}\rangle_{\mathcal{H}^0} := \sum_{n=0}^{\infty} \frac{p_n q_n}{p_n^*} \quad , \quad \langle \mathbf{p}, \mathbf{q}\rangle_{\mathcal{H}^1} := \sum_{n=0}^{\infty} p_n^* \left(\frac{p_{n+1}}{p_{n+1}^*} - \frac{p_n}{p_n^*}\right)\left(\frac{q_{n+1}}{p_{n+1}^*} - \frac{q_n}{p_n^*}\right). \tag{2.20}$$

The advantage of using the scalar product $\langle .,.\rangle_{\mathcal{H}^0}$ is that the operator Q_{poor} becomes symmetric. To prove it, we rewrite the operator *a la* Fokker-Planck.

Lemma 2.5.1 *For any $\mathbf{p} \in \mathcal{H}^0$, we have:*

$$Q_{poor}[\mathbf{p}]_n = \mu\, D^- \left(p_n^* D^+\left(\frac{p_n}{p_n^*}\right)\right) \tag{2.21}$$

with $D^+(p_n) = p_{n+1} - p_n$, $D^-(p_n) = p_n - p_{n-1}$ and the convention $p_{-1} = p_{-1}^* = 0$.

Proof. Since $p_n^*/p_{n+1}^* = (n+1)/\mu$, we find

$$\begin{aligned}\frac{1}{\mu} Q_{poor}[\mathbf{p}]_n &= \frac{p_n^*}{p_{n+1}^*} p_{n+1} - \frac{p_{n-1}^*}{p_n^*} p_n - \left(\frac{p_n^*}{p_n^*} p_n - \frac{p_{n-1}^*}{p_{n-1}^*} p_{n-1} \right) \\ &= \mu p_n^* u_{n+1} - p_{n-1}^* u_n \;-\; \left(p_n^* u_n - p_{n-1}^* u_{n-1} \right)\end{aligned}$$

with $u_n = p_n/p_n^*$. Using the notation D^+ and D^-, we write:

$$\frac{1}{\mu} Q_{poor}[\mathbf{p}]_n \;=\; p_n^* D^+ u_n - p_{n-1}^* D^+ u_{n-1} = D^-(p_n^* D^+ u_n).$$

□

Remark. Equation (2.21) has a flavor of a Fokker-Planck equation of the form

$$\partial_t \rho = \nabla \cdot \left(\rho_\infty \nabla \left(\frac{\rho}{\rho_\infty} \right) \right), \tag{2.22}$$

where ρ_∞ is an equilibrium distribution to which ρ converges (and ρ_∞ may also depend on ρ, making the equation nonlinear).

As a consequence, we deduce that the operator Q_{poor} is symmetric on $\mathcal{H}^0$.

Proposition 2.5.2 *For any* $\mathbf{p}, \mathbf{q} \in \mathcal{H}^0$, *the operator* Q_{poor} (2.6) *satisfies:*

$$\langle Q_{poor}[\mathbf{p}], \mathbf{q} \rangle_{\mathcal{H}^0} = \langle \mathbf{p}, Q_{poor}[\mathbf{q}] \rangle_{\mathcal{H}^0} \qquad \textit{for any } \mathbf{p}, \mathbf{q} \in \mathcal{H}^0. \tag{2.23}$$

Moreover,

$$\langle Q_{poor}[\mathbf{p}], \mathbf{p} \rangle_{\mathcal{H}^0} = -\mu \sum_{n=0}^{\infty} p_n^* \left(D^+ \left(\frac{p_n}{p_n^*} \right) \right)^2 = -\mu \|\mathbf{p}\|_{\mathcal{H}^1}^2. \tag{2.24}$$

Proof. We simply use integration by parts:

$$\begin{aligned}\frac{1}{\mu} \langle Q_{poor}[\mathbf{p}], \mathbf{q} \rangle_{\mathcal{H}^0} &= \sum_{n=0}^{\infty} D^- \left(p_n^* D^+ \frac{p_n}{p_n^*} \right) \frac{q_n}{p_n^*} = -\sum_{n=0}^{\infty} p_n^* \left(D^+ \frac{p_n}{p_n^*} \right) \left(D^+ \frac{q_n}{p_n^*} \right) \\ &= \sum_{n=0}^{\infty} \frac{p_n}{p_n^*} D^- \left(p_n^* D^+ \frac{q_n}{p_n^*} \right) = \frac{1}{\mu} \langle \mathbf{p}, Q_{poor}[\mathbf{q}] \rangle_{\mathcal{H}^0}.\end{aligned}$$

□

Furthermore, the operator $-Q_{poor}$ would have a so-called spectral gap if one can show that the norm $\|.\|_{\mathcal{H}^1}$ controls the norm $\|.\|_{\mathcal{H}^0}$. To prove it, we establish a Poincaré inequality.

Lemma 2.5.3 *There exists a constant $C_p > 0$ such that for any $\mathbf{p} \in \mathcal{H}^1$ satisfying $\sum_n p_n = 1$*

$$\|\mathbf{p} - \mathbf{p}^*\|^2_{\mathcal{H}^0} \leq C_p \|\mathbf{p}\|^2_{\mathcal{H}^1} \tag{2.25}$$

where $\|.\|_{\mathcal{H}^0}$ and $\|.\|_{\mathcal{H}^1}$ are defined in (2.20) *and $\mathbf{p}^*$ is the equilibrium* (2.17)*.*

Proof. Similar to the standard proof of a classical Poincaré inequality we proceed by contradiction. Assume that no such C_p exists, then there exists a sequence (of sequence) $\mathbf{p}^{(k)}$ such that $\sum_{n=0}^{\infty} p_n^{(k)} = 1$ and

$$\|\mathbf{p}^{(k)} - \mathbf{p}^*\|_{\mathcal{H}^0} \geq k \|\mathbf{p}^{(k)}\|_{\mathcal{H}^1} \tag{2.26}$$

for all $k \in \mathbb{N}$. Denote $\mathbf{s}^{(k)} = \mathbf{p}^{(k)} - \mathbf{p}_n^*$. Then we have $\sum_{n=0}^{\infty} s_n^{(k)} = 0$ and (2.26) reads

$$\|\mathbf{s}^{(k)}\|_{\mathcal{H}^0} \geq k \|\mathbf{s}^{(k)}\|_{\mathcal{H}^1}. \tag{2.27}$$

Without loss of generality, we can assume the normalization condition $\|\mathbf{s}^{(k)}\|_{\mathcal{H}^0} = 1$ for all k and thus $\|\mathbf{s}^{(k)}\|_{\mathcal{H}^1} \leq \frac{1}{k}$. By weak compactness, there exists $\mathbf{s}^{\infty} \in \mathcal{H}^0$ such that $\mathbf{s}^{(k)} \rightharpoonup \mathbf{s}^{\infty}$ in $\mathcal{H}^0$ and in particular $s_n^{(k)} \xrightarrow{k\to\infty} s_n^{\infty}$ for all n.
Since $\|\mathbf{s}^{(k)}\|_{\mathcal{H}^1} \leq \frac{1}{k}$, we also have $\frac{s_{n+1}^{(k)}}{p_{n+1}^*} - \frac{s_n^{(k)}}{p_n^*} \xrightarrow{k\to\infty} 0$ for all k, or equivalently, $(n+1)s_{n+1}^N - \mu s_n^N \xrightarrow{k\to\infty} 0$. Thus, $(n+1)s_{n+1}^{\infty} = \mu s_n^{\infty}$ and therefore $s_n^{\infty} = \frac{\mu^n}{n!} s_0^{\infty}$ for all n. As $\sum_{n=0}^{\infty} s_n^{\infty} = 0$, we must have $s_0^{\infty} = 0$ and therefore $\mathbf{s}^{\infty} = 0$. Contradiction, $\|\mathbf{s}^{\infty}\|_{\mathcal{H}^0} = 1$ since $\|\mathbf{s}^{(k)}\|_{\mathcal{H}^0} = 1$ for all k.

□

As a result of the lemma, the operator $-Q_{poor}$ has a spectral gap of at least $1/C_p$ since:

$$\langle -Q_{poor}[\mathbf{p} - \mathbf{p}_{\infty}] \,,\, \mathbf{p} - \mathbf{p}_{\infty} \rangle_{\mathcal{H}^0} = \langle -Q_{poor}[\mathbf{p}] \,,\, \mathbf{p} \rangle_{\mathcal{H}^0} = \|\mathbf{p}\|^2_{\mathcal{H}^1} \geq \frac{1}{C_p} \|\mathbf{p} - \mathbf{p}^*\|^2_{\mathcal{H}^0}. \tag{2.28}$$

We shall establish the existence of a unique global solution to the linear ODE system (2.5). The key ingredient in our proof relies heavily on standard theory of maximal monotone operators (see for instance Chapter 7 of Brezis [34]).

Proposition 2.5.4 *Given any* $\mathbf{p}_0 \in \mathcal{D}(Q_{poor})$*, there exists a unique function*

$$\mathbf{p}(t) \in C^1\Big([0,\infty);\mathcal{H}^0\Big) \cap C\Big([0,\infty);\mathcal{D}(Q_{poor})\Big)$$

satisfying (2.5)*.*

Proof. We use the Hille-Yosida theorem and show that the (unbounded) linear operator $-Q_{poor}$ on $\mathcal{H}^0$ is a maximal monotone operator. The monotonicity of $-Q_{poor}$ follows from its symmetric property on $\mathcal{H}^0$:

$$\langle -Q_{poor}[\mathbf{v}], \mathbf{v}\rangle_{\mathcal{H}^0} = \mu \sum_{n=0}^{\infty} p_n^* \left(D^+\left(\frac{v_n}{p_n^*}\right)\right)^2 \geq 0 \quad \text{for all } \mathbf{v} \in \mathcal{D}(Q_{poor}).$$

To show the maximality of $-Q_{poor}$, it suffices to show $R(I - Q_{poor}) = \mathcal{H}^0$, i.e., for each $\mathbf{f} \in \mathcal{H}^0$, the equation $\mathbf{p} - Q_{poor}[\mathbf{p}] = \mathbf{f}$ admits at least one solution $\mathbf{p} \in \mathcal{D}(-Q_{poor})$. To this end, the weak formulation of $\mathbf{p} - Q_{poor}[\mathbf{p}] = \mathbf{f}$ reads

$$\langle \mathbf{p}, \mathbf{q}\rangle_{\mathcal{H}^0} + \langle -Q_{poor}[\mathbf{p}], \mathbf{q}\rangle_{\mathcal{H}^0} = \langle \mathbf{f}, \mathbf{q}\rangle_{\mathcal{H}^0} \quad \text{for all } \ \mathbf{q} \in \mathcal{H}^0, \tag{2.29}$$

whence the Lax-Milgram theorem yields a unique $\mathbf{p} \in \mathcal{H}^1$.

$\square$

We can now prove the convergence of $\mathbf{p}(t)$ solution of (2.5) to its equilibrium solution (2.17).

Theorem 3 *Let* $\mathbf{p}(t)$ *be the solution of* (2.5) *and* $\mathbf{p}^*$ *the corresponding equilibrium. Then:*

$$\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0} \leq \|\mathbf{p}_0 - \mathbf{p}^*\|_{\mathcal{H}^0} \mathrm{e}^{-\frac{\lambda}{C_p} t} \tag{2.30}$$

where $\mathbf{p}_0$ *is the initial condition, i.e.* $\mathbf{p}(t=0) = \mathbf{p}_0$*.*

Proof. Taking the derivative of the square norm gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{p}(t)-\mathbf{p}^*\|^2_{\mathcal{H}^0} &= \langle \mathbf{p}'(t)\,,\,\mathbf{p}(t)-\mathbf{p}^*\rangle_{\mathcal{H}^0} = \lambda\langle Q_{poor}[\mathbf{p}(t)]\,,\,\mathbf{p}(t)-\mathbf{p}^*\rangle_{\mathcal{H}^0} \\ &= \lambda\langle \mathbf{p}(t)\,,\,Q_{poor}[\mathbf{p}(t)]\rangle_{\mathcal{H}^0} = -\lambda\|\mathbf{p}(t)\|^2_{\mathcal{H}^1}, \qquad (2.31)\end{aligned}$$

using the symmetry of Q_{poor} and the relation (2.24). Using the Poincaré constant from lemma (2.5.3), we deduce:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{p}(t)-\mathbf{p}^*\|^2_{\mathcal{H}^0} \leq -\frac{\lambda}{C_p}\|\mathbf{p}(t)-\mathbf{p}^*\|^2_{\mathcal{H}^0}.$$

Applying the Gronwall's lemma leads to the result.

□

To finish our investigation of the poor-biased dynamics, we would like to find an explicit rate for the decay of the solution $\mathbf{p}(t)$ toward the equilibrium $\mathbf{p}^*$, i.e. find an explicit value for the Poincaré constant C_p in lemma (2.5.3). The key idea, due to Bakry and Emery Bakry and Émery [15], is to compute the second time derivative of $\|\mathbf{p}(t)-\mathbf{p}^*\|_{\mathcal{H}^0}$.

Lemma 2.5.5 *For any $\mathbf{p}\in V_\mu \cap \mathcal{D}(Q_{poor})$, we have:*

$$\langle Q_{poor}[Q_{poor}[\mathbf{p}]]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0} \geq -\mu\langle Q_{poor}[\mathbf{p}]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0}. \qquad (2.32)$$

Proof. Using the symmetry of Q_{poor}, we have:

$$\langle Q_{poor}[Q_{poor}[\mathbf{p}]]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0} = \langle Q_{poor}[\mathbf{p}]\,,\,Q_{poor}[\mathbf{p}]\rangle_{\mathcal{H}^0} = \mu^2\sum_{n=0}^{\infty} D^-\left(p_n^*\,z_n\right)D^-\left(p_n^*\,z_n\right)\frac{1}{p_n^*}$$

with $z_n = D^+\Big(\frac{p_n}{p_n^*}\Big)$. Since $D^-\left(p_n^*\,z_n\right)\frac{1}{p_n^*} = \mu z_n - n z_{n-1}$, integration by parts gives:

$$\begin{aligned}\frac{1}{\mu^2}\langle Q_{poor}[Q_{poor}[\mathbf{p}]]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0} &= \sum_{n=0}^{\infty} D^-\left(p_n^*\,z_n\right)(z_n - \frac{n}{\mu}z_{n-1}) \\ &= \sum_{n=0}^{\infty} D^-\left(p_n^*\,z_n\right)z_n + \sum_{n=0}^{\infty} p_n^*\,z_n D^+\left(\frac{n}{\mu}z_{n-1}\right).\end{aligned}$$

Using $D^+(\frac{n}{\mu}z_{n-1}) = \frac{n}{\mu}D^-(z_n) + \frac{z_n}{\mu}$, we deduce:

$$\begin{aligned}
\frac{1}{\mu^2}\langle Q_{poor}[Q_{poor}[\mathbf{p}]]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0} &= \sum_{n=0}^{\infty} D^-\left(p_n^*\, z_n\right) z_n \;+\; \sum_{n=0}^{\infty} p_n^*\, z_n \frac{n}{\mu} D^-(z_n) + \sum_{n=0}^{\infty} p_n^* \frac{z_n^2}{\mu} \\
&= \sum_{n=0}^{\infty}\Big(D^-\left(p_n^* z_n\right) z_n + p_n^* z_n \frac{n}{\mu} D^-(z_n)\Big) + \frac{1}{\mu}\langle Q_{poor}[\mathbf{p}],\mathbf{p}\rangle \\
&=: A \;+\; \frac{1}{\mu}\langle Q_{poor}[\mathbf{p}]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0}
\end{aligned}$$

To conclude we have to show that $A \geq 0$. Notice that $p_n^* \frac{n}{\mu} = p_{n-1}^*$, thus:

$$\begin{aligned}
A &= \sum_{n=0}^{\infty}\Big(D^-\left(p_n^*\, z_n\right) z_n + p_{n-1}^*\, z_n D^-(z_n)\Big) \\
&= -\sum_{n=0}^{\infty} p_n^*\, z_n D^+(z_n) + \sum_{n=0}^{\infty} p_n^*\, z_{n+1} D^-(z_{n+1}) \\
&= \sum_{n=0}^{\infty} p_n^*\Big(-z_n D^+(z_n) + z_{n+1} D^-(z_{n+1})\Big) \\
&= \sum_{n=0}^{\infty} p_n^*\Big(z_{n+1} - z_n\Big)^2 \geq 0.
\end{aligned}$$

$\square$

Remark. In general, the computation of the second time derivative of the energy (in our case, $\|\mathbf{p}(t)-\mathbf{p}^*\|_{\mathcal{H}^0}$) requires a number of smartly-chosen integration by parts. However, these computations can actually be made more tractable and organized to some extent. We refer interested readers to Matthes *et al.* [115], Jüngel and Matthes [98], Bukal *et al.* [35] for ample illustration of the technique known as systematic integration by parts.

Proposition 2.5.6 *The exponential decay rate in theorem* (3) *is at least* λ, *i.e.* $\|\mathbf{p}(t)-\mathbf{p}^*\|_{\mathcal{H}^0} \leq C\mathrm{e}^{-\lambda t}$.

Proof. Taking the second derivative and using the symmetry of Q_{poor} give:

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\|\mathbf{p}(t)-\mathbf{p}^*\|_{\mathcal{H}^0}^2 &= \frac{d}{dt}\lambda\langle Q_{poor}[\mathbf{p}(t)]\,,\,\mathbf{p}(t)\rangle_{\mathcal{H}^0} = 2\lambda^2\langle Q_{poor}[\mathbf{p}(t)]\,,\,Q_{poor}[\mathbf{p}(t)]\rangle_{\mathcal{H}^0} \\
&\geq -2\lambda^2\mu\langle Q_{poor}[\mathbf{p}]\,,\,\mathbf{p}\rangle_{\mathcal{H}^0} = -\lambda\frac{d}{dt}\|\mathbf{p}(t)-\mathbf{p}^*\|_{\mathcal{H}^0}^2
\end{aligned}$$

thanks to (2.32) and (2.31). Denoting $\phi(t) = \|\mathbf{p}(t) - \mathbf{p}^*\|^2_{\mathcal{H}^0}$, we have: $\phi'' \geq -2\lambda\phi'$. Integrating over the interval $(t, +\infty)$ yields:

$$0 - \phi'(t) \geq -2\lambda(0 - \phi(t)) \quad \Rightarrow \quad \phi'(t) \leq -2\lambda\phi(t)$$

and the Gronwall's lemma allows to obtain our result.

It remains to justify that $\lim_{t\to+\infty} \phi'(t) = \lim_{t\to+\infty} \phi(t) = 0$. Theorem (3) already shows that $\lim_{t\to+\infty} \phi(t) = 0$. Moreover, denoting $g(t) = -\phi'(t) \geq 0$, we have $g' \leq -2\lambda g$. Thus, by Gronwall's lemma, $g(t) \stackrel{t\to+\infty}{\longrightarrow} 0$.

$\square$

2.5.4 Numerical Illustration of Poor-Biased Model

We investigate numerically the convergence of $\mathbf{p}(t)$ solution to the poor-biased model (2.5) to the equilibrium distribution $\mathbf{p}^*$ (2.17). We use $\mu = 5$ (average money) and $\lambda = 1$ (rate of jumps) for the model. To discretize the model, we use $1,001$ components to describe the distribution $\mathbf{p}(t)$ (i.e. $(p_0(t), \ldots, p_{1000}(t))$). As initial condition, we use $p_\mu(0) = 1$ and $p_i(0) = 0$ for $i \neq \mu$. The standard Runge-Kutta fourth-order method (e.g. RK4) is used to discretize the ODE system (2.5) with the time step $\Delta t = 0.01$.

We plot in figure (2.7)-left the numerical solution $\mathbf{p}$ at $t = 12$ unit time and compare it to the equilibrium distribution $\mathbf{p}^*$. The two distributions are indistinguishable. Indeed, plotting the evolution of the difference $\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0}$ (figure (2.7)-right) shows that the difference is already below 10^{-10}. Moreover, the decay is clearly exponential as we use semi-logarithmic scale.

Notice that the numerical simulation suggests that the optimal decay rate of $\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0}$ is 2λ, which is twice the analytical decay rate λ proved in proposition 2.5.6. The reason for this discrepancy is that the solution of $\mathbf{p}(t)$ remains in the

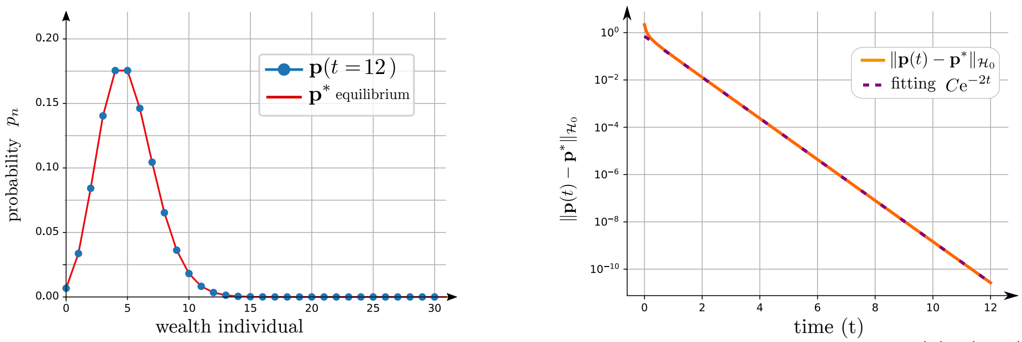

Figure 2.7: Left: Comparison Between the Numerical Solution $\mathbf{p}(t)$ (2.5) of the Poor-Bias Model and the Equilibrium $\mathbf{p}^*$ (2.17). The Two Distributions are Indistinguishable. **Right**: Decay of the Difference $\|\mathbf{p}(t) - \mathbf{p}^*\|_{\mathcal{H}^0}$ in Semilog Scale. The Decay is Exponential as Predicted by the Theorem 3.

subspace $V_\mu \cap \mathcal{D}(Q_{poor})$, i.e. the mean of $\mathbf{p}(t)$ is preserved. The analysis of the spectral gap of Q_{poor} in the proposition 2.5.6 does not take account this constraint.

We numerically investigate the spectrum of $-Q_{poor}$ denoted $\{\alpha_n\}_{n=1}^{\infty}$. The first eigenvalue satisfies $\alpha_1 = 0$ due to the equilibrium $\mathbf{p}^*$ (i.e. $Q_{poor}[\mathbf{p}^*] = 0$). The other eigenvalues are $\alpha_n = n-1$ and in particular the spectral gap is $\alpha_2 = 1$. One can find explicitly a corresponding eigenfunction given by:

$$\mathbf{p}^{(2)} = D^-(\mathbf{p}^*) = (p_0^*, p_1^* - p_0^*, \ldots, p_n^* - p_{n-1}^*, \ldots). \tag{2.33}$$

Thus, for any $\mathbf{p} \in V_\mu \cap \mathcal{D}(Q_{poor})$, we find:

$$\langle \mathbf{p}\,,\, \mathbf{p}^{(2)} \rangle_{\mathcal{H}^0} = \sum_{n=0}^{\infty} p_n (p_n^* - p_{n-1}^*) \frac{1}{p_n^*} = \sum_{n=0}^{\infty} p_n (1 - n/\mu) = 1 - \mu/\mu = 0.$$

This explains why the effective spectral gap for the dynamics is given by α_3 and not α_2: the solution $\mathbf{p}(t)$ (2.5) lives in $V_\mu \cap \mathcal{D}(Q_{poor})$ and therefore it is orthogonal to $\mathbf{p}^{(2)}$.

Remark. We can find explicitly the exact formulation of the eigenfunction $\mathbf{p}^{(k)}$ of $-Q_{poor}$ for all $k \in \mathbb{N}_+$. We find by induction:

$$\mathbf{p}^{(k)} = \left(p_0^*, p_1^* - (k-1)p_0^*, \cdots, p_n^* + \sum_{j=0}^{n-1} (-1)^{n-j} \frac{\prod_{\ell=1}^{n-j}(k-\ell)}{(n-j)!} p_j^*, \cdots \right) \tag{2.34}$$

leading to:

$$p_n^{(k)} = \sum_{j=0}^{n} \binom{k-1}{j} (-1)^j \frac{\mu^{n-j}}{(n-j)!} \mathrm{e}^{-\mu}, \quad n \geq 0, \tag{2.35}$$

with $\binom{k}{j}$ binomial coefficient (i.e. $\binom{k}{j} = \frac{k!}{(k-j)!\,j!}$). Moreover, through an induction argument and some combinatorial identities, we can verify that $\langle \mathbf{p}^{(m)}, \mathbf{p}^{(k)} \rangle_{\mathcal{H}^0} = 0$ for $m \neq k$. We speculate that $\{\mathbf{p}^{(k)}\}_{k=1}^{\infty}$ spans the entire space $\mathcal{H}^0$, but we do not have a proof for this conjecture.

2.6 Rich-Biased Exchange Model

In our third model, the selection of the 'giver' is biased toward the poor instead of the rich, i.e. the more money an individual has the less likely it will be chosen.

2.6.1 Definition and Limit Equation

As before, the definition of the model is given first.

Definition 10 ***(Rich-biased exchange model)*** *A "giver" i is chosen with inverse proportionality of its wealth. The "receiver" j is chosen uniformly.*

The rich-biased model leads to the following stochastic differential equation:

$$\mathrm{d}S_i(t) = -\sum_{j=1}^{N} \mathrm{dN}_t^{(i,j)} + \sum_{j=1}^{N} \mathrm{dN}_t^{(j,i)}, \tag{2.1}$$

with $\mathrm{N}_t^{(i,j)}$ Poisson process with intensity λ_{ij} given by:

$$\lambda_{ij} = \begin{cases} 0 & \text{if } S_i = 0 \\ \frac{\lambda}{N} \cdot \frac{1}{S_i} & \text{if } S_i > 0 \end{cases} \tag{2.2}$$

An agent i receives a dollar at rate λw where w is the inverse of the harmonic mean:

$$w = \frac{1}{N} \sum_{S_k > 0} \frac{1}{S_k}. \tag{2.3}$$

Definition 11 ***(Asymptotic Rich-biased model)***

$$\mathrm{d}\overline{S}_1(t) = -\mathrm{d}\overline{\mathbf{N}}_t^1 + \mathrm{d}\overline{\mathbf{M}}_t^1, \tag{2.4}$$

in which $\overline{\mathbf{N}}^1_t$ *and* $\overline{\mathbf{M}}^1_t$ *are independent Poisson processes with intensity* $\lambda/\overline{S}_1(t)$ *(if* $\overline{S}_1(t) > 0$*) and* $\lambda\overline{w}(t)$ *respectively. The inverse mean* $\overline{w}(t)$ *is given by:*

$$\overline{w}[\mathbf{p}(t)] := \sum_{n=1}^{\infty} \frac{p_n(t)}{n} \tag{2.5}$$

where $\mathbf{p}(t) = \Big(p_0(t), p_1(t), \dots\Big)$ *the law of the process* $\overline{S}_1(t)$*. The time evolution of* $\mathbf{p}(t)$ *is given by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \lambda\, Q_{rich}[\mathbf{p}(t)] \tag{2.6}$$

with:

$$Q_{rich}[\mathbf{p}]_n := \begin{cases} p_1 - \overline{w}\, p_0 & \text{if } n = 0 \\ \frac{p_{n+1}}{n+1} + \overline{w} p_{n-1} - \Big(\frac{1}{n} + \overline{w}\Big)\, p_n & \text{for } n \geq 1 \end{cases} \tag{2.7}$$

We will also need the weak form of the operator: for any test function φ:

$$\langle Q_{rich}[\mathbf{p}]\,,\,\varphi\rangle = \sum_{n\geq 0} p_n \left(\overline{w}\varphi(n+1) + \frac{\mathbb{1}_{\{n\geq 1\}}}{n}\varphi(n-1) - \left(\overline{w} + \frac{\mathbb{1}_{\{n\geq 1\}}}{n}\right)\varphi(n)\right) \tag{2.8}$$

2.6.2 Propagation of Chaos Using Empirical Measure

We investigate the propagation of chaos for the rich-biased dynamics using the empirical measure (see subsection 2.3.3). We consider $\{S_i(t)\}_{1\leq i\leq N}$ the solution to (2.1) and introduce the empirical measure:

$$\mathbf{p}_{emp}(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{S_i(t)}(s). \tag{2.9}$$

The goal is to show that the stochastic measure $\mathbf{p}_{emp}(t)$ converges to the deterministic density $\mathbf{p}(t)$ solution of (2.6). The main difficulty is that the empirical measure is a stochastic process on a Banach space $\ell^1(\mathbb{N})$ and thus of infinite dimension. Fortunately, the space is a discrete (i.e. $\mathbb{N}$) and therefore we do not have to consider stochastic partial differential equations which are famously difficult. Moreover, we only have to consider a finite number of possible jumps.

When agent i gives a dollar to j (i.e. $(S_i, S_j) \rightsquigarrow (S_i - 1, S_j + 1)$), the empirical measure is transformed as

$$\mathbf{p}_{emp} \ \rightsquigarrow \ \mathbf{p}_{emp} + \frac{1}{N}\Big(\delta_{S_i-1} + \delta_{S_j+1} - \delta_{S_i} - \delta_{S_j}\Big). \tag{2.10}$$

To write down the evolution equation satisfied by $\mathbf{p}_{emp}$, we regroup the agents with the same number of dollars (i.e. we project the dynamics on a subspace).

Proposition 2.6.1 *The empirical measure $\mathbf{p}_{emp}(t)$ (6.1) satisfies:*

$$\mathrm{d}\mathbf{p}_{emp}(t) = \frac{1}{N}\sum_{k=1,l=0}^{+\infty}\Big(\delta_{k-1} + \delta_{l+1} - \delta_k - \delta_l\Big)\mathrm{dN}_t^{(k,l)} \tag{2.11}$$

where $\mathrm{N}_t^{(k,l)}$ independent Poisson clock with intensity:

$$\lambda_{k,l} = N \cdot p_{emp,k} \cdot (N \cdot p_{emp,l} - \mathbb{1}_{\{k=l\}}) \cdot \frac{\lambda}{k \cdot N} \tag{2.12}$$

where $p_{emp,k}$ is the k−th coordinate of $\mathbf{p}_{emp}$.

Proof. Following the jump process given in (2.10), the empirical measure satisfies:

$$\mathrm{d}\mathbf{p}_{emp}(t) = \frac{1}{N}\sum_{i,j=1,i\neq j}^{N}\Big(\delta_{S_i-1} + \delta_{S_j+1} - \delta_{S_i} - \delta_{S_j}\Big)\mathrm{dN}_t^{(i,j)} \tag{2.13}$$

Introducing $\mathrm{N}_t^{(k,l)}$ the Poisson process regrouping all the clocks corresponding to a giver with k dollars giving to a receiver with l dollars:

$$\mathrm{N}_t^{(k,l)} = \sum_{\{i\neq j \,|\, S_i=k, S_j=l\}} \mathrm{N}_t^{(i,j)}, \tag{2.14}$$

In this sum, each clock $\mathrm{N}_t^{(i,j)}$ has the same intensity $\lambda/(S_i \cdot N) = \lambda/(k \cdot N)$. Moreover, counting the number of clocks involved in the sum (2.14) leads to (2.12). The indicator $\mathbb{1}_{\{k=l\}}$ is here to remove the self-giving clocks $\mathrm{N}_t^{(i,i)}$: when an agent gives to itself, nothing happens. $\square$

Corollary 2.6.2 *For any test function φ, the empirical measure $\mathbf{p}_{emp}(t)$ (6.1) satisfies:*

$$\mathrm{d}\mathbb{E}[\langle \mathbf{p}_{emp}(t), \varphi\rangle] = \lambda \mathbb{E}[\langle Q_{rich}[\mathbf{p}_{emp}(t)], \varphi\rangle]\mathrm{d}t \ \ - \ \ \frac{\lambda}{N}\mathbb{E}[\langle R[\mathbf{p}_{emp}(t)], \varphi\rangle]\mathrm{d}t \qquad (2.15)$$

where Q_{rich} is the operator defined in (2.7) *and R defined by:*

$$R[\mathbf{p}]_n := \frac{p_{n+1}}{n+1} + \frac{p_{n-1}}{n-1}\mathbb{1}_{\{n\geq 2\}} - \frac{2}{n}p_n\mathbb{1}_{\{n\geq 1\}}. \qquad (2.16)$$

Proof. From the proposition 2.6.1, we find:

$$\begin{aligned}
\mathrm{d}\mathbb{E}[\langle \mathbf{p}_{emp}(t), \varphi\rangle] &= \mathbb{E}\left[\sum_{k=1,l=0}^{+\infty}\Big(\varphi(k-1)+\varphi(l+1)-\varphi(k)-\varphi(l)\Big)p_{emp,k}\cdot p_{emp,l}\frac{\lambda}{k}\right] \\
&\quad -\frac{1}{N}\mathbb{E}\left[\sum_{k=1}^{+\infty}\Big(\varphi(k-1)+\varphi(k+1)-2\varphi(k)\Big)p_{emp,k}\cdot\frac{\lambda}{k}\right]\mathrm{d}t \\
&= \lambda\mathbb{E}\left[\sum_{k=1}^{+\infty}\Big(\varphi(k-1)-\varphi(k)\Big)\frac{p_{emp,k}}{k}\right]\mathrm{d}t \\
&\quad +\lambda\mathbb{E}\left[\sum_{l=0}^{+\infty}\Big(\varphi(l+1)-\varphi(l)\Big)\overline{w}[\mathbf{p}_{emp}]\cdot p_{emp,l}\right]\mathrm{d}t \\
&\quad -\frac{\lambda}{N}\mathbb{E}\left[\sum_{k=1}^{+\infty}\Big(\varphi(k-1)+\varphi(k+1)-2\varphi(k)\Big)p_{emp,k}\cdot\frac{1}{k}\right]\mathrm{d}t
\end{aligned}$$

where $\overline{w}[\mathbf{p}_{emp}]$ is defined in (2.5). We recognize the weak formulation of Q_{rich} (2.8) leading to (2.15). □

The operator R (2.16) corresponds to the *bias* in the evolution of the empirical measure $\mathbf{p}_{emp}(t)$ compared to the evolution of $\mathbf{p}(t)$ solution to the limit equation (2.6). This bias vanishes as λ/N goes to zero when the number of agents N becomes large. The other source of discrepancy between $\mathbf{p}_{emp}(t)$ and $\mathbf{p}(t)$ is the *variance* of $\mathbf{p}_{emp}(t)$ (as it is a stochastic measure). Let's review an elementary result on compensated Poisson process.

Remark. Denote $Z(t)$ a compound jump process and $M(t)$ its compensated version:

$$\mathrm{d}Z(t) = Y(t)\,\mathrm{dN}_t \quad , \quad M(t) = Z(t) - \int_0^t \mu(s)\lambda(s)\,\mathrm{d}s \qquad (2.17)$$

where $Y(t)$ denotes the (independent) jumps and N_t Poisson process with intensity $\lambda(t)$ and $\mu(t) = \mathbb{E}[Y(t)]$. The Ito's formula is given by:

$$\mathrm{d}\mathbb{E}[\varphi(M(t))] = \mathbb{E}\Big[\varphi\Big(M(t-) + Y(t-)\Big) - \varphi(M(t-)\Big]\lambda(t)\mathrm{d}t \; - \; \mathbb{E}[\varphi'(M(t))\mu(t)\lambda(t)]\,\mathrm{d}t.$$

In particular, for $\varphi(x) = x^2$, we obtain:

$$\begin{aligned} \mathrm{d}\mathbb{E}[M^2(t)] &= \mathbb{E}[2M(t-)Y(t-) + Y^2(t-)]\,\lambda(t)\mathrm{d}t - \mathbb{E}[2M(t)\mu(t)\lambda(t)]\,\mathrm{d}t \\ &= \mathbb{E}[Y^2(t)]\,\lambda(t)\mathrm{d}t. \end{aligned} \tag{2.18}$$

Here, we assume that the jump $Y(t)$ is independent of the value $Z(t)$. To generalize the formula, one has to replace $\mu(t) = \mathbb{E}[Y(t)]$ by $\mathbb{E}[Y(t)|Z(t)]$.

Motivated by this remark, we obtain the following result.

Proposition 2.6.3 *Denote $M(t)$ the compensated process of the empirical measure $\mathbf{p}_{emp}(t)$:*

$$M(t) = \mathbf{p}_{emp}(t) - \left(\mathbf{p}_{emp}(0) + \lambda \int_0^t \left(Q_{rich}[\mathbf{p}_{emp}(s)] + \frac{1}{N}R[\mathbf{p}_{emp}(s)]\right)\mathrm{d}s\right) \tag{2.19}$$

then $M(t)$ is a ℓ^1-value martingale and satisfies:

$$\mathbb{E}[\|M(t)\|_{\ell^1}] \leq \sqrt{\frac{4\lambda}{N}\,t}. \tag{2.20}$$

Proof. The key observation is that the jump (2.10) for the empirical measure are of order $\mathcal{O}(1/N)$. Indeed:

$$E\left[\left\|\frac{1}{N}(\delta_{k-1} + \delta_{l+1} - \delta_k - \delta_l)\right\|_{\ell^1}^2\right] \leq \frac{4}{N^2}. \tag{2.21}$$

Applying the formula (2.18) we obtain::

$$\mathrm{d}\mathbb{E}[\|M(t)\|_{\ell^1}^2] \leq \sum_{k=1,l=0}^{+\infty} \mathbb{E}\left[\frac{4}{N^2}\,\cdot\, Np_{emp,k} \cdot Np_{emp,l}\right]\frac{\lambda}{k\cdot N}\mathrm{d}t \;\leq\; \frac{4\lambda}{N}\mathrm{d}t. \tag{2.22}$$

Integrating in time gives (2.20). □

We are now ready to prove the propagation of chaos for the rich-biased dynamics by showing that the empirical measure $\mathbf{p}_{emp}(t)$ converges to $\mathbf{p}(t)$ as $N \to +\infty$. The key

Lemma 2.6.4 *The operator Q_{rich} (2.7) is globally Lipschitz on $\ell^1(\mathbb{N}) \cap \mathcal{P}(\mathbb{N})$ and R is an bounded on $\ell^1(\mathbb{N})$.*

$$\begin{aligned} \|Q_{rich}[\mathbf{p}] - Q_{rich}[\mathbf{q}]\|_{\ell^1} &\leq 4\|\mathbf{p}-\mathbf{q}\|_{\ell^1} \qquad \textit{for any } \mathbf{p},\mathbf{q} \in \ell^1(\mathbb{N}) \cap \mathcal{P}(\mathbb{N}) \quad (2.23) \\ \|R[\mathbf{p}]\|_{\ell^1} &\leq 4\|\mathbf{p}\|_{\ell^1} \qquad \textit{for any } \mathbf{p} \in \ell^1(\mathbb{N}) \quad (2.24) \end{aligned}$$

Proof. Since $\mathbf{p} \in \ell^1(\mathbb{N}) \cap \mathcal{P}(\mathbb{N})$, the rate of receiving $w[\mathbf{p}]$ (2.3) satisfies $0 \leq w[\mathbf{p}] \leq 1$. Thus,

$$|Q_{rich}[\mathbf{p}]_n - Q_{rich}[\mathbf{q}]_n| \leq |p_{n+1} - q_{n+1}| + |p_{n-1} - q_{n-1}| + 2|p_n - q_n|.$$

Summing in n gives the result. We proceed similarly for the operator R. $\square$

Theorem 4 *Consider $\mathbf{p}(t)$ solution to the limit equation (2.6) and $\mathbf{p}_{emp}(t)$ empirical measure (6.1). Then:*

$$\mathbb{E}[\|\mathbf{p}_{emp}(t) - \mathbf{p}(t)\|_{\ell^1}] \leq \mathcal{O}\left(\frac{te^{4\lambda t}}{\sqrt{N}}\right), \tag{2.25}$$

in particular $\mathbf{p}_{emp}(t) \overset{N\to+\infty}{\rightharpoonup} \mathbf{p}(t)$ for any $t \geq 0$.

Proof. First we write down the integral form of the equation satisfied by both $\mathbf{p}(t)$ and $\mathbf{p}_{emp}(t)$:

$$\begin{aligned} \mathbf{p}(t) &= \mathbf{p}_0 + \int_0^t Q_{rich}[\mathbf{p}(s)]\,\mathrm{d}s \\ \mathbf{p}_{emp}(t) &= \mathbf{p}_0 + \int_0^t Q_{rich}[\mathbf{p}_{emp}(s)]\,\mathrm{d}s + \frac{1}{N}\int_0^t R[\mathbf{p}_{emp}(s)]\,\mathrm{d}s + M(t) \end{aligned}$$

Combining the two equations give:

$$\begin{aligned} \|\mathbf{p}_{emp}(t) - \mathbf{p}(t)\|_{\ell^1} &\leq \lambda \int_0^t \|Q_{rich}[\mathbf{p}_{emp}(s)] - Q_{rich}[\mathbf{p}(s)]\|_{\ell^1}\,\mathrm{d}s \\ &\quad + \frac{\lambda}{N}\int_0^t \|R[\mathbf{p}_{emp}(s)]\|_{\ell^1}\,\mathrm{d}s + \|M(t)\|_{\ell^1} \\ &\leq 4\lambda \int_0^t \|\mathbf{p}_{emp}(s) - \mathbf{p}(s)\|_{\ell^1}\,\mathrm{d}s + \frac{\lambda 4t}{N} + \|M(t)\|_{\ell^1} \end{aligned}$$

using lemma 2.6.4. Denoting $\phi(t) = \mathbb{E}[\|\mathbf{p}_{emp}(t) - \mathbf{p}(t)\|_{\ell^1}]$, we deduce from the bound (2.20) of $M(t)$:

$$\phi(t) \leq 4\lambda \int_0^t \phi(s)\,\mathrm{d}s + \frac{\lambda 4t}{N} + \sqrt{\frac{4\lambda}{N}t}.$$

Applying Gronwall's lemma leads to:

$$\phi(t) \leq \left(\frac{\lambda 4t}{N} + \sqrt{\frac{4\lambda}{N}t}\right) \mathrm{e}^{4\lambda t}$$

leading to the result. □

Remark. The martingale-based technique, developed in Merle and Salez [118] and employed here for justifying the propagation of chaos, is remarkable since it does not require us to study the N-particle process $(S_1, \ldots, S_N)$ but solely its generator. One drawback is that this method might not work if the generator Q of the limit process is unbounded, which is the case for the generator Q_{poor} of the (limit) poor-biased dynamics (2.5).

2.6.3 Dispersive Wave Leading to Vanishing Wealth

As illustrated in the introduction (figure 2.2), the rich-biased dynamics tend to accentuate inequality, i.e. the Gini index $G(t)$ was approaching 1 (its maximum value) for the agent-based model (2.4) (2.1). We would like to investigate numerically the behavior of the solution to the rich-biased dynamics using the limit equation (2.6) and the distribution $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)$.

In figure 2.8, we plot the evolution of the distribution $\mathbf{p}(t)$ starting from a Dirac distribution with mean $\mu = 5$ (i.e. $p_5 = 1$ and $p_i = 0$ for $i \neq 5$). We observe that the distribution spreads in two parts: the bulk of the distribution moves toward zero whereas a smaller proportion is moving to the right. One can identify the two pieces as the "poor" and the "rich". Thus, the dynamics could be interpreted as the

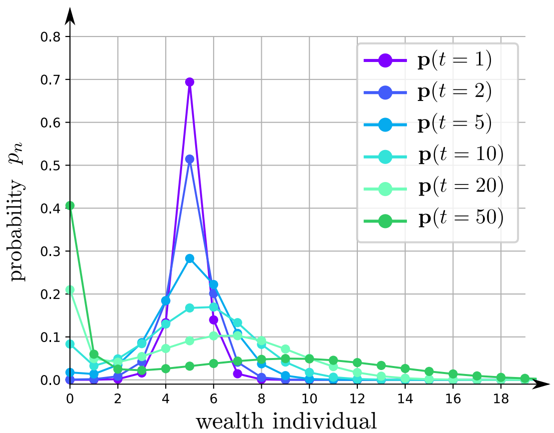

Figure 2.8: Evolution of the Wealth Distribution $\mathbf{p}(t)$ for the Rich-Biased Dynamics (2.6). The Distribution Spreads in Two Parts: A Large Proportion Starts to Concentrate at Zero ("Poor Distribution") and While the Other Part Form a Dispersive Traveling Wave. Parameters: $\Delta t = 5 \cdot 10^{-3}$, $\mathbf{p}(t) \approx (p_0(t), p_1(t), \ldots, p_{1{,}000}(t))$. A Standard Runge-Kutta of Order 4 Has Been Used to Discretize the System.

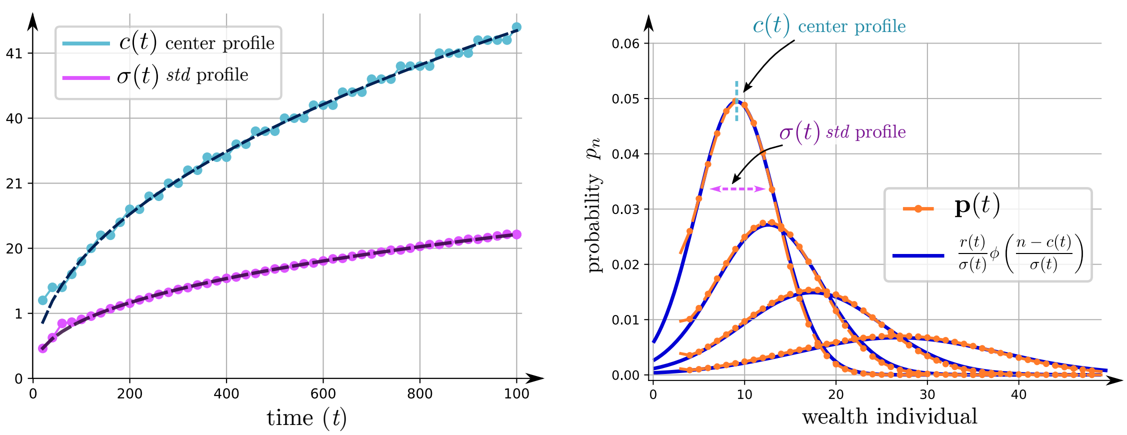

Figure 2.9: Left: Estimation of the Center $c(t)$ and Standard Deviation $\sigma(t)$ of the Dispersive Wave along with Their Parametric (Power-Law) Estimation (2.27). **Right**: Comparison of the Distribution $\mathbf{p}(t)$ (See Figure 2.8) with the Dispersive Wave Using ϕ the Standard Normal Distribution.

poor getting poorer and the rich getting richer. Notice that the proportion of poor is increasing (e.g. $p_0(t)$ is increasing) whereas the "rich" distribution resembles a dispersive traveling wave. Since both the total mass and the total amount of dollar are preserved (i.e. $\sum_n n \cdot p_n(t) = \mu$ for any t), the dispersive traveling wave contains the bulk of the money but it is also vanishing in time.

To investigate more carefully the dispersive wave, we try to fit numerically its profile. After numerically examination, we choose to approximate by a Gaussian distribution. Meanwhile we approximate the "poor" distribution by a Dirac centered at zero δ_0. Thus, we approximate the distribution $\mathbf{p}(t)$ by the following Ansatz:

$$p_n(t) \;\approx\; (1-r(t))\cdot\delta_0 \quad + \quad r(t)\cdot\frac{1}{\sigma(t)}\phi\left(\frac{n-c(t)}{\sigma(t)}\right), \tag{2.26}$$

where ϕ is the standard normal distribution (i.e. $\phi(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$), $c(t)$ is the center of the profile, $\sigma(t)$ its standard deviation and $r(t)$ the proportion of rich. The speed of the wave $c(t)$ and its standard deviation $\sigma(t)$ are estimated numerically and plotted in figure 2.9. Their growth is well-approximated by a power-law of the form:

$$c(t) = 1.4748\cdot t^{.466} \quad , \quad \sigma(t) = 0.9261\cdot t^{.399}. \tag{2.27}$$

Since the total amount of money is preserved, the proportion of rich $r(t)$ can be easily deduced from $c(t)$ since we must have $\mu = r(t)\cdot c(t)$. Such approximation leads to the fitting in figure 2.8-right (dotted-black curves). We notice that the proportion of rich in our Ansatz is vanishing:

$$r(t) = \frac{\mu}{c(t)} \xrightarrow{t\to+\infty} 0. \tag{2.28}$$

Thus, we make the conjecture that $\mathbf{p}(t)$ converges weakly toward δ_0, i.e. all the money will asymptotically disappear.

To further assess our conjecture, we measure the evolution of the Gini index for the distribution $\mathbf{p}(t)$:

$$G[\mathbf{p}] = \frac{1}{2\mu}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}|i-j|p_i p_j \tag{2.29}$$

with μ the standard mean. Using the Ansatz (2.26), we can approximate the value of the Gini index given (see Appendix B):

$$G(t) \approx 1 - \frac{\mu}{c(t)} + \frac{\mu\cdot\sigma(t)}{\sqrt{\pi}\,c^2(t)}. \tag{2.30}$$

We plot in figure 2.10-left the evolution of the Gini index $G(t)$ along with its approximation (2.30). We observe a good agreement between the two curves. To examine closely the long time behavior of the curves, we plot the evolution of $1 - G(t)$ in log-scales (figure 2.10-right) over a longer time interval (up to $t = 10^5$). Both curves seem to converges similarly toward 0 (indicating that $G(t) \xrightarrow{t\to+\infty} 1$) with a slight overshoot for the Ansatz. This overshoot might be due to our approximation that the "poor distribution" of $\mathbf{p}(t)$ is concentrated exactly at zero (i.e. $(1-r(t))\delta_0$). This approximation amplifies the inequality between the "poor" and "rich" parts of the distribution and hence increases slightly the Gini index. But overall the asymptotic behavior of the Gini index for $\mathbf{p}(t)$ matches with the formula (2.30) and thus strengthens our assumption that $\mathbf{p}(t)$ will converge (weakly) to a Dirac δ_0. However, further analytically studies are needed to derive the asymptotic behavior of $\mathbf{p}(t)$ directly from the rich-biased evolution equation (2.6).

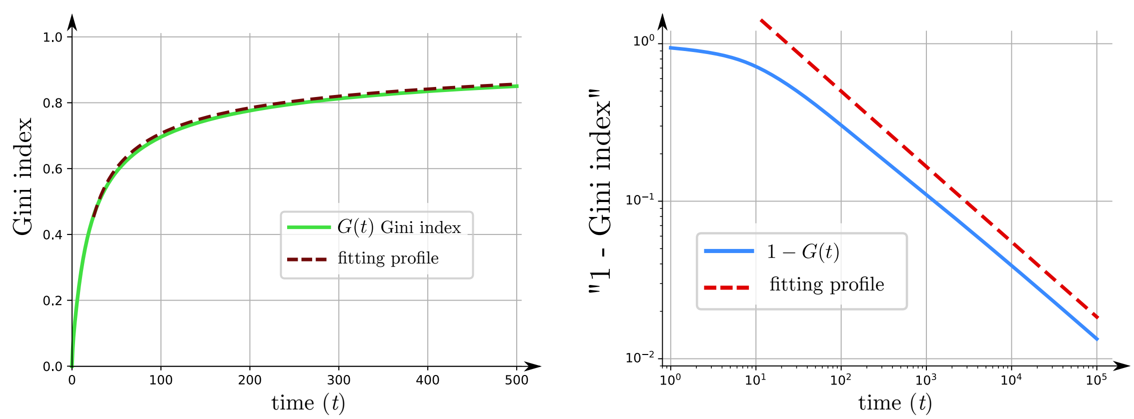

Figure 2.10: Left: Evolution of the Corresponding Gini Index (2.29) along with the Analytical Approximation Using the Dispersive Wave Profile (2.30). **Right** The Gini Index Converges to 1 Due to the Vanishing Dispersive Wave Transporting All the Wealth to Infinity.

2.7 Conclusion

In this manuscript, we have investigated three related models for money exchange originated from econophysics. For the unbiased and poor biased dynamics, we rigorously proved the so-called *propagation of chaos* by virtue of a coupling technique, and we found an explicit rate of convergence of the limit dynamics for the poor biased model thanks to the Bakry-Emery approach. We have also introduced a more challenging dynamics referred to as the rich biased model, and a propagation of chaos result was established via a powerful martingale-based argument presented in Merle and Salez [118]. In contrast to the two other dynamics, the rich-biased dynamics do not converge (strongly) to an equilibrium. Instead, we have found numerically evidence of the emergence of a (vanishing) dispersive wave. Such wave of extreme wealthy individual increases the inequality in the wealth distribution making the corresponding Gini index converging to its maximum 1.

Although we have shown numerically strong evidence of a dispersive wave, it is desirable to derive such emerging behavior directly from the evolution equation. One direction of future work would be to derive space continuous dynamics of evolution equations in order to investigate analytically the profile of traveling waves. However, space continuous description such as the uniform reshuffling model could lead to additional challenges. For instance, proving propagation of chaos using the martingale technique for the uniform reshuffling model was more involved Cao *et al.* [39].

From a modeling perspective, one should explore how selecting the ”receiver” as well as the ”giver” could impact the dynamics. Indeed, in the three dynamics studied in the manuscript, the re-distribution process (how the one-dollar is redistributed) is uniform among all the agent. It would be reasonable to have the redistribution of the dollar based on the individual wealth (e.g. poor individual being more likely to receive a dollar). The interplay between receiver and giver selection could lead to novel emerging behaviors.

Chapter 3

ENTROPY DISSIPATION AND PROPAGATION OF CHAOS FOR THE UNIFORM RESHUFFLING MODEL

Chapter 3 is the pre-print Cao *et al.* [39] submitted to *Mathematical Models and Methods in Applied Sciences*, which is a joint work with Pierre-Emmanuel Jabin and Sebastien Motsch.

3.1 Abstract

The uniform reshuffling model for money exchanges is investigated, in which two agents picked uniformly at random redistribute their dollars between them. This stochastic dynamics is of mean-field type and eventually leads to a exponential distribution of wealth. To better understand this dynamics, it is possible to investigate its limit as the number of agents goes to infinity. By proving rigorously the so-called propagation of chaos, the stochastic dynamics can be linked to a (limiting) nonlinear partial differential equation (PDE). This deterministic description, which is well-known in the literature, has a flavor of the classical Boltzmann equation arising from statistical mechanics of dilute gases. It is possible to prove its convergence toward its exponential equilibrium distribution in the sense of relative entropy.

3.2 Introduction

Econophysics is an emerging branch of statistical physics that incorporate notions and techniques of traditional physics to economics and finance Savoiu [140], Chatterjee *et al.* [57], Dragulescu and Yakovenko [79]. It has attracted considerable attention in recent years raising challenges on how various economical phenomena could be explained by universal laws in statistical physics, and we refer to Chakraborti *et al.* [53, 54], Pereira *et al.* [125], Kutner *et al.* [103] for a general review.

The primary motivation for study models arising from econophysics is at least two-fold: From the perspective of a policy maker, it is important to deal with the raise of income inequality Dabla-Norris *et al.* [66], De Haan and Sturm [70] in order to establish a more egalitarian society. From a mathematical point of view, we have to understand the fundamental mechanisms, such as money exchange resulting from individuals, which are usually agent-based models. Given an agent-based model, one is expected to identify the limit dynamics as the number of individuals tends to infinity and then its corresponding equilibrium when run the model for a sufficiently long time (if there is one), and this guiding approach is carried out in numerous works across different fields among literature of applied mathematics, see for instance Naldi *et al.* [122], Barbaro and Degond [19], Carlen *et al.* [42].

In this work, we consider the so-called uniform reshuffling model for money exchange in a closed economic system with N agents. The dynamics consists in choosing at random time two individuals and to redistribute their money between them. To write this dynamics mathematically, we denote by $X_i(t)$ the amount of dollar agent i has at time t for $1 \leq i \leq N$. At a random time generated by a Poisson clock with rate N, two agents (say i and j) update their purse according to the following rule:

$$\Big(X_i, X_j\Big) \rightsquigarrow \Big(U(X_i+X_j)\,,\,(1-U)(X_i+X_j)\Big), \tag{3.1}$$

where U is a uniform random variable over the interval $[0,1]$ (i.e. $U \sim \text{Uniform}[0,1]$). The *uniform reshuffling model* is first studied in Dragulescu and Yakovenko [79] via simulation. The agent-based numerical simulation suggests that, as the number of agents and time go to infinity, the limiting distribution of money approaches the exponential distribution as shown in Figure 3.1.

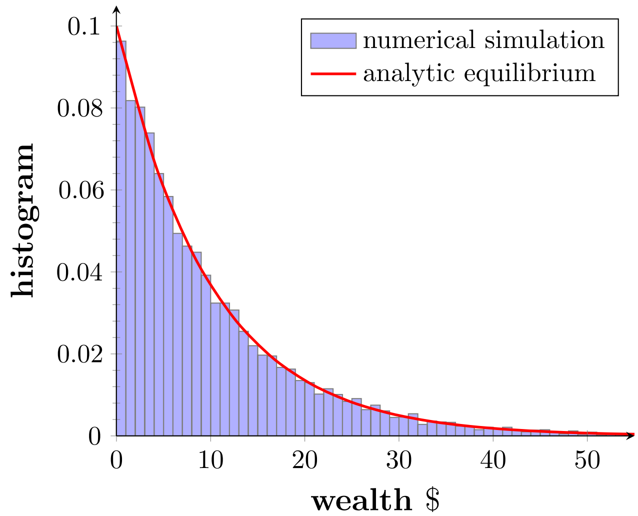

Figure 3.1: Simulation Results for the Uniform Reshuffling Model. The Blue Histogram Shows the Distribution of Money after $T = 1000$ Time Unit. The Red Solid Curve Is the Limiting Exponential Distribution Proved in Lanchier and Reed [109]. We Used $N = 10,000$ Agents, Each Starting with \$10.

It is well-known (see for instance Matthes and Toscani [116], Bassetti and Toscani [20], Düring *et al.* [80], Apenko [10]) that under the large population $N \to \infty$ limit, We can formally show that the law of the wealth of a typical agent (say X_1) satisfies the following limit PDE in a weak sense:

$$\partial_t q(t,x) = \int_0^\infty \int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(t,k) q(t,\ell)\,\mathrm{d}\ell\,\mathrm{d}k - q(t,x). \tag{3.2}$$

To our best knowledge, the rigorous derivation of the limit equation (3.2) from the particle system description is absent in most of the literature on econophysics (just like many other PDEs arising from models in econophysics Katriel [101], Heinsalu and Patriarca [90], Chakrabarti *et al.* [51]), because the propagation of chaos effect is implicitly assumed in the large N limit in most derivations. The remarkable exception is the paper Cortez [60], where the author showed a uniform-in-time propagation of chaos by virtue of a delicate coupling argument. In section 3.6 of this manuscript, we will provide an alternative rigorous justification of the equation (3.2) under the limit $N \to \infty$.

Once the limit PDE is identified from the interacting particle system, the natural next step is to study the problem of convergence to equilibrium of the PDE at hand, it has been shown in Düring *et al.* [80], Matthes and Toscani [116] that the unique (smooth) solution of (3.2) converges to its exponential equilibrium distribution exponentially fast in Wasserstein and Fourier metrics. In the present work, we demonstrate a polynomial in time convergence in relative entropy, by establishing a entropy-entropy dissipation inequality (see Theorem 9 below) which is not available among the literature. An illustration of the general strategy used in this work (and implicitly in many of the works cited above) is shown in Figure 4.1.

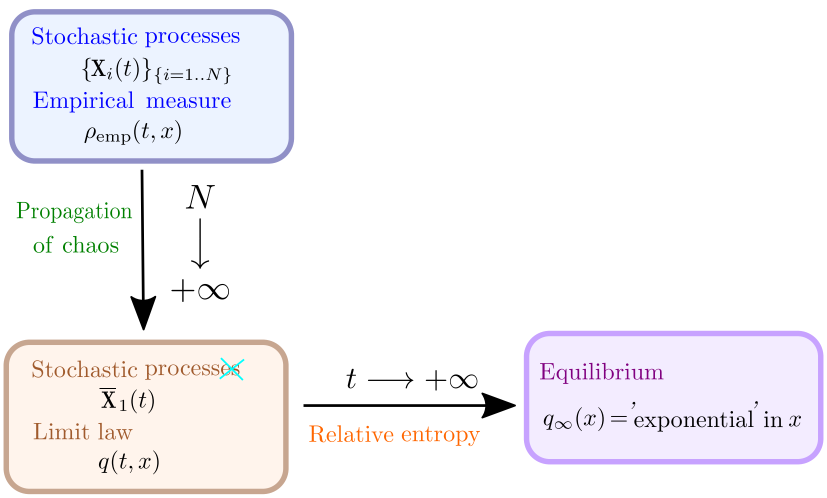

Figure 3.2: Schematic Illustration of the General Strategy of Our Treatment of the Uniform Reshuffling Dynamics.

Although only a very specific binary exchange model is explored in the present paper, other exchange rules can also be imposed and studied, leading to different models. To name a few, the so-called immediate exchange model introduced in Heinsalu and Patriarca [90] assumes that pairs of agents are randomly and uniformly picked at each random time, and each of the agents transfer a random fraction of its money to the other agents, where these fractions are independent and uniformly distributed in $[0,1]$. The uniform reshuffling model with saving propensity investigated in Chakraborti and Chakrabarti [52], Lanchier and Reed [109] suggests that the two interacting agents keep a fixed fraction λ of their fortune and only the combined remaining fortune is uniformly reshuffled between the two agents, which makes the uniform reshuffling model the particular case $\lambda = 0$. For more variants of exchange models with (random) saving propensity and with debts, we refer the readers to Chatterjee *et al.* [56] and Lanchier and Reed [110].

This manuscript is organized as follows: in section 4.3, we briefly discuss the properties of the limit equation (3.2). We show in section 3.4 convergence results for the solution of (3.2) in Wasserstein distance and in the linearized region. We take on the most delicate analysis of the entropy-entropy dissipation relation in section 3.5. Finally, we present a rigorous treatment of the propagation of chaos phenomenon in section 3.6.

3.3 The Limit Equation and Its Properties

We present a heuristic argument behind the derivation of the limit PDE (4.6) arising from the uniform reshuffling dynamics in section 5.3.1. Several elementary properties of the solution of (4.6) are recorded in section 5.3.2. Section 5.3.3 is devoted to another formulation of the uniform reshuffling model, which can be viewed as a *lifting* of the reshuffling mechanics (3.1) and is implicitly exploited in Apenko [10]. In section 3.3.4, we highlight a key ingredient known as the *micro-reversibility*, of the collision operator determined by the right side of (4.6), which allows us to construct certain Lyapunov functions associated with (4.6) (such as entropy).

3.3.1 Formal Derivation of the Limit Equation

Introducing $\mathrm{N}_t^{(i,j)}$ independent Poisson processes with intensity $1/N$, the dynamics can be written as:

$$\mathrm{d}X_i(t) = \sum_{j=1..N, j\neq i} \Big(U(t-)(X_i(t-)+X_j(t-)) - X_i(t-)\Big)\mathrm{d}\mathrm{N}_t^{(i,j)} \tag{3.1}$$

with $U(t) \sim \text{Uniform}[0,1]$ independent of $\{X_i(t)\}_{1\le i\le N}$. As the number of players N goes to infinity, one could expect that the processes $X_i(t)$ become independent and of same law. Therefore, the limit dynamics would be of the form:

$$\mathrm{d}\overline{X}(t) = \Big(U(t-)(\overline{X}(t-)+\overline{Y}(t-)) - \overline{X}(t-)\Big)\mathrm{d}\overline{\mathrm{N}}_t \tag{3.2}$$

where $\overline{Y}(t)$ is an independent copy of $\overline{X}(t)$ and $\overline{\mathrm{N}}_t$ a Poisson process with intensity 1. Taking a test function φ, the weak formulation of the dynamics is given by:

$$\mathrm{d}\mathbb{E}[\varphi(\overline{X}(t))] = \mathbb{E}[\varphi\Big(U(t)(\overline{X}(t)+\overline{Y}(t))\Big) - \varphi(\overline{X}(t))]\,\mathrm{d}t \tag{3.3}$$

In short, the limit dynamics correspond to the jump process:

$$\overline{X} \rightsquigarrow U(\overline{X}+\overline{Y}). \tag{3.4}$$

Let's denote $q(t,x)$ the law of the process $\overline{X}(t)$. To derive the evolution equation for $q(t,x)$, we need to translate the effect of the jump of $\overline{X}(t)$ via (3.4) onto $q(t,x)$.

Lemma 3.3.1 *(Hierarchy of probability distributions) Suppose X and Y two independent random variables with probability density $q(x)$ supported on $[0,\infty)$. Let $Z = U(X+Y)$ with $U \sim \text{Uniform}([0,1])$ independent of X and Y. Then the law of Z is given by $Q_+[q]$ with:*

$$\begin{aligned} Q_+[q](x) &= \int_{m=0}^{\infty} \frac{\mathbb{1}_{[0,m]}(x)}{m} \left(\int_{z=0}^{m} q(z)q(m-z)\mathrm{d}z\right)\mathrm{d}m & (3.5)\\ &= \int_{\mathbb{R}_+\times\mathbb{R}_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q(k)q(\ell)\mathrm{d}\ell\mathrm{d}k & (3.6)\end{aligned}$$

Proof Let's introduce a test function φ.

$$
\begin{aligned}
\mathbb{E}[\varphi(U(X+Y))] &= \int_{x\geq 0}\int_{y\geq 0}\int_{u=0}^{1} \varphi(u(x+y))q(x)q(y)\,\mathrm{d}u\mathrm{d}x\mathrm{d}y \\
&= \int_{m\geq 0}\int_{z=0}^{m}\int_{u=0}^{1} \varphi(um)q(z)q(m-z)\,\mathrm{d}u\mathrm{d}z\mathrm{d}m \\
&= \int_{m\geq 0}\int_{z=0}^{m}\frac{1}{m}\int_{s=0}^{m} \varphi(s)q(z)q(m-z)\,\mathrm{d}s\mathrm{d}z\mathrm{d}m
\end{aligned}
$$

using the change of variables $z = x$ and $m = x + y$ followed by $s = um$. We conclude using Fubini that:

$$
\begin{aligned}
\mathbb{E}[\varphi(U(X+Y))] &= \int_{s\geq 0}\varphi(s)\left(\int_{m\geq 0}\mathbb{1}_{[0,m]}(s)\frac{1}{m}\int_{z=0}^{m} q(z)q(m-z)\,\mathrm{d}z\mathrm{d}m\right)\mathrm{d}s \\
&= \int_{s\geq 0}\varphi(s)Q_+[q](s)\,\mathrm{d}s \qquad (3.7)
\end{aligned}
$$

with $Q_+[q]$ defined by (4.5). □

We can now write the evolution equation for the law of $\overline{X}(t)$ (3.2), the density $q(t,x)$ satisfies weakly:

$$
\partial_t q(t,x) = G[q](t,x) \qquad \text{for } t \geq 0 \text{ and } x \geq 0 \tag{3.8}
$$

with

$$
G[q](x) := Q_+[q](x) - q(x) = \int_0^\infty\int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}q(k)q(\ell)\mathrm{d}\ell\mathrm{d}k - q(x). \tag{3.9}
$$

3.3.2 Evolution of Moments

Now we will establish several elementary properties of the solution of (4.6):

Proposition 3.3.2 *Assume that $q(t,x)$ is a classical (and global in time) solution of* (4.6) *and define by $m_k(t)$ the k-th moment of q:*

$$
m_k(t) := \int_0^\infty x^k q(t,x)\mathrm{d}x. \tag{3.10}
$$

Then:

$$
m_k'(t) = \frac{1}{k+1}\sum_{j=0}^{k} C_k^j m_j(t)m_{k-j}(t) \; - \; m_k(t), \tag{3.11}
$$

where $C_k^j = \begin{pmatrix} k \\ j \end{pmatrix} = \frac{k!}{j!(k-j)!}$ *represents the binomial coefficient.*

Proof Notice that the moment can be written as: $m_k(t) = \mathbb{E}[\overline{X}^k(t)]$. Thus, we use the weak formulation of the evolution equation of $q(t,x)$ (3.3) with $\varphi(x) = x^k$ and deduce that:

$$m_k' = \mathbb{E}[\Big(U(\overline{X}+\overline{Y})\Big)^k - \overline{X}^k] \;=\; \mathbb{E}[U^k]\mathbb{E}[(\overline{X}+\overline{Y}))^k] - m_k,$$

since U is independent of $\overline{X}$ and $\overline{Y}$. Moreover, $\mathbb{E}[U^k] = \int_{u=0}^1 u^k\,\mathrm{d}u = \frac{1}{k+1}$. Using the independence of $\overline{X}$ and $\overline{Y}$ and expanding lead to (3.11). $\square$

Corollary 3.3.3 *Let $q(t,x)$ solution of* (4.6) *and $m_k(t)$ its $k-th$ moment* (3.10)*. The total mass and the mean are preserved, i.e. $m_0'(t) = m_1'(t) = 0$ and all the moments $m_k(t)$ converges in time exponentially fast.*

Proof Writing (3.11) for $k=2$ leads to:

$$m_2' = -\frac{1}{3}m_2 + \frac{2}{3}m_1^2 \tag{3.12}$$

and thus $m_2(t) = 2\,m_1^2 + \Big(m_2(0) - 2m_1^2\Big)\,\mathrm{e}^{-\frac{1}{3}t}$. More generally, we proceed by induction to show that $m_k(t)$ converges exponentially, more precisely $m_k(t)$ is of the form:

$$m_k(t) = m_k^* + \mathcal{O}(\mathrm{e}^{-\frac{k-1}{k+1}t}) \tag{3.13}$$

with m_k^* the limit value of $m_k(t)$. We first re-write the evolution equation of $m_k(t)$:

$$m_k'(t) = -\frac{k-1}{k+1}m_k(t) + P_{k-1}(t) \tag{3.14}$$

with $P_{k-1}(t) = \frac{1}{k+1}\sum_{j=1}^{k-1} C_k^j m_j(t) m_{k-j}(t)$. By induction, $P_{k-1}(t)$ has to converge in time. Using variation of constant in (3.14) gives:

$$m_k(t) = m_k(0)\mathrm{e}^{-\frac{k-1}{k+1}t} + \mathrm{e}^{-\frac{k-1}{k+1}t}\int_{s=0}^t \mathrm{e}^{\frac{k-1}{k+1}s} P_{k-1}(s)\,\mathrm{d}s, \tag{3.15}$$

which leads to (3.13). □

From the proposition, we observe that the second moment $m_2(t)$ converges exponentially toward the constant $2\,m_1^2$. This behavior could be expected as the equilibrium of the dynamics (4.6) is given by:

$$q_\infty(x) := \frac{1}{m_1}\mathrm{e}^{-\frac{x}{m_1}}\mathbb{1}_{[0,\infty)}(x) \tag{3.16}$$

for which the second moment is equal $2\,m_1^2$.

Remark. Moment calculations can be useful in the study of classical spatially homogeneous Boltzmann equation, and we refer the readers to Alonso *et al.* [8] for more information on this regard.

3.3.3 Pairwise Distribution

Before studying the evolution of the entropy of the solution $q(t,x)$, we make a detour with another formulation of the reshuffling model using a two-particles distribution. Indeed, the jump process $\overline{X}(t)$ (3.4) is a "truncated version" of the following dynamics:

$$(\overline{X},\overline{Y}) \rightsquigarrow \Big(U(\overline{X}+\overline{Y})\,,\,(1-U)(\overline{X}+\overline{Y})\Big) \tag{3.17}$$

where $U \sim \text{Uniform}([0,1])$. Introducing a test function $\varphi(x,y)$, this dynamics lead to:

$$\mathrm{d}\mathbb{E}[\varphi(\overline{X},\overline{Y})] = \mathbb{E}[\varphi\Big(U(\overline{X}+\overline{Y})\,,\,(1-U)(\overline{X}+\overline{Y})\Big) - \varphi(\overline{X},\overline{Y})]\,\mathrm{d}t. \tag{3.18}$$

We now translate this evolution equation into a PDE.

Proposition 3.3.4 *Let $f(t,x,y)$ the density distribution of the process $(\overline{X}(t),\overline{Y}(t))$. It satisfies (weakly) the linear evolution equation:*

$$\partial_t f = L_+[f] - f \tag{3.19}$$

with

$$L_+[f](x,y) = \frac{1}{x+y}\int_{z=0}^{x+y} f(z, x+y-z)\,\mathrm{d}z. \tag{3.20}$$

Proof The evolution equation (3.17) gives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x,y\geq 0} f(t,x,y)\varphi(x,y)\,\mathrm{d}x\mathrm{d}y \;&=\; \int_{u=0}^{1}\int_{x,y\geq 0} f(t,x,y)\varphi\Big(u(x{+}y),(1{-}u)(x{+}y)\Big)\,\mathrm{d}x\mathrm{d}y\mathrm{d}u \\ &\quad - \int_{x,y\geq 0} f(t,x,y)\varphi(x,y)\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{3.21}$$

To identify the operator associated with the equation, let's rewrite the "gain term" (dropping the dependency in time for simplicity) using two changes of variables:

$$\begin{aligned}&\int_{u=0}^{1}\int_{x,y\geq 0} f(x,y)\varphi\Big(u(x{+}y),(1{-}u)(x{+}y)\Big)\,\mathrm{d}x\mathrm{d}y\mathrm{d}u \\ &\qquad = \int_{u=0}^{1}\int_{m\geq 0}\int_{z=0}^{m} f(z,m-z)\varphi\Big(um,(1{-}u)m\Big)\,\mathrm{d}z\mathrm{d}m\mathrm{d}u \\ &\qquad = \int_{x',y'\geq 0}\int_{z=0}^{x'+y'} f(z,x'+y'-z)\varphi(x',y')\,\frac{1}{x'+y'}\mathrm{d}z\mathrm{d}x'\mathrm{d}y'\end{aligned}$$

with $(x' = um, y' = (1-u)m)$ leading to $\mathrm{d}x'\mathrm{d}y' = m\mathrm{d}u\mathrm{d}m$. □

Remark. Notice that the operator L (4.7) "flattens" the distribution f over the diagonals $x+y =$ Constant and thus minimizing its entropy over each diagonal (see Figure 3.3). In particular, the equilibrium for the dynamics are the distribution of the form: $f_*(x,y) = \phi(x+y)$.

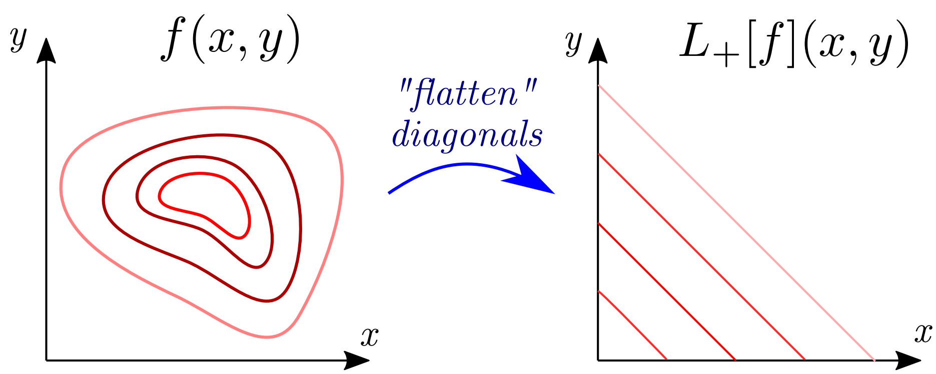

Figure 3.3: The Operator L_+ (4.7) Flattens the Distribution $f(x,y)$ Over the Diagonal Lines $x+y = \mathit{Constant}$.

The linear operator L_+ (4.7) is linked to the non-linear operator Q_+ (4.5). Indeed, assuming $\overline{X}$ and $\overline{Y}$ are independent, i.e. $f(x,y) = q(x)q(y)$, integrating $L_+[f]$ over the 'extra' variable y gives:

$$\begin{aligned}\int_{y\geq 0} L_+[f](x,y)\,\mathrm{d}y &= \int_{y\geq 0} \frac{1}{x+y}\int_{z=0}^{x+y} q(z)q(x+y-z)\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_{m=x}^{+\infty} \frac{1}{m}\int_{z=0}^{m} q(z)q(m-z)\,\mathrm{d}z\,\mathrm{d}y \quad = Q_+[q](x).\end{aligned}$$

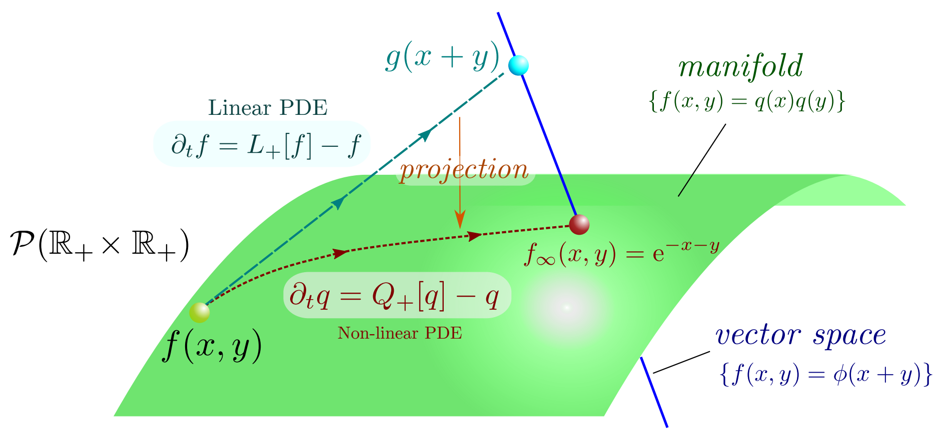

Figure 3.4: Schematic Representation of the Evolution of $f(t,x,y)$ and $q(t,x)$. If f Belongs to the Manifold of Independent Functions, i.e. $f(t,x,y) = q(t,x)q(t,y)$, Then the Evolution of Its Marginal q Satisfies *locally* the Non-Linear Equation (4.6). Notice That the Manifold of Independent Function Is Not Invariant By the Flow of the Linear PDE. Notice That We Have Assumed $m_1 = 1$ So That $f_\infty(x,y) := q_\infty(x)q_\infty(y) = \mathrm{e}^{-x-y}$. Also, the Definition of g Appears in (3.6).

3.3.4 Micro-Reversibility

The evolution equation for f (3.19) corresponds to a collisional operator with the kernel:

$$K\Big(x,y;x',y'\Big) = \frac{1}{x+y}\delta_{x+y}(x'+y') \tag{3.22}$$

where δ denote the Dirac distribution. Indeed, writing $\mathbf{z} = (x,y)$, the equation (3.19) could be written:

$$\partial_t f(\mathbf{z},t) = \int_{\tilde{\mathbf{z}}\geq 0} K\Big(\tilde{\mathbf{z}};\mathbf{z}\Big) f(\tilde{\mathbf{z}},t)\,\mathrm{d}\tilde{\mathbf{z}} - \int_{\mathbf{z}'\geq 0} K\Big(\mathbf{z};\mathbf{z}'\Big) f(\mathbf{z},t)\,\mathrm{d}\mathbf{z}' \tag{3.23}$$

where $\mathbf{z}' = (x', y')$ denotes the post-collisional position and $\tilde{\mathbf{z}} = (\tilde{x}, \tilde{y})$ the pre-collision position.

Remark. A more rigorous way to define the kernel K is through a weak formulation using a test function $\varphi(x, y)$:

$$\text{"}\int_{x',y'\geq 0} K\Big(x, y; x', y'\Big)\varphi(x', y')\,\mathrm{d}x'\mathrm{d}y'\text{"} = \frac{1}{x+y}\int_{z=0}^{x+y} \varphi(z, x+y-z)\,\mathrm{d}z. \qquad (3.24)$$

The collisional kernel K satisfies a micro-reversibility condition, namely:

$$K\Big(\mathbf{z}; \mathbf{z}'\Big) = K\Big(\mathbf{z}'; \mathbf{z}\Big) \quad \text{for any } \mathbf{z} \text{ and } \mathbf{z}' \in \mathbb{R}_+ \times \mathbb{R}_+. \qquad (3.25)$$

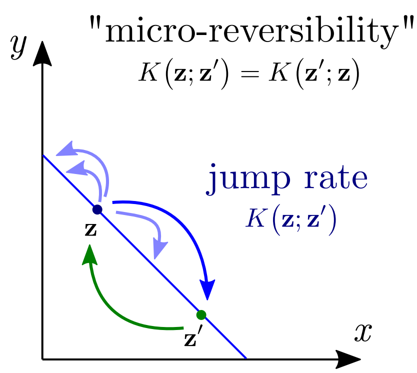

Figure 3.5: The Collision Kernel K (3.22) Satisfies a Micro-Reversibility Condition.

One has to integrate against a test function φ to make this statement rigorous. As a consequence, we deduce the lemma.

Lemma 3.3.5 *Let $\varphi(x, y)$ a (smooth) test function and $f(t, x, y)$ the solution of* (3.19). *Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{z}} f(\mathbf{z}, t)\varphi(\mathbf{z})\mathrm{d}\mathbf{z} = -\frac{1}{2}\int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z}; \mathbf{z}')\Big(f(\mathbf{z}', t) - f(\mathbf{z}, t)\Big)\Big(\varphi(\mathbf{z}') - \varphi(\mathbf{z})\Big)\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{z}'. \quad (3.26)$$

Proof We drop the dependency in time to ease the reading:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{z}} f(\mathbf{z})\varphi(\mathbf{z})\mathrm{d}\mathbf{z} &= \int_{\tilde{\mathbf{z}},\mathbf{z}} K(\tilde{\mathbf{z}};\mathbf{z})f(\tilde{\mathbf{z}})\varphi(\mathbf{z})\mathrm{d}\tilde{\mathbf{z}}\mathrm{d}\mathbf{z} - \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}')f(\mathbf{z})\varphi(\mathbf{z})\mathrm{d}\tilde{\mathbf{z}}\mathrm{d}\mathbf{z} \\
&= \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}')f(\mathbf{z})\Big(\varphi(\mathbf{z}') - \varphi(\mathbf{z})\Big)\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{z}' \\
&= \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z}';\mathbf{z})f(\mathbf{z}')\Big(\varphi(\mathbf{z}) - \varphi(\mathbf{z}')\Big)\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{z}' \\
&= \int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}')f(\mathbf{z}')\Big(\varphi(\mathbf{z}) - \varphi(\mathbf{z}')\Big)\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{z}' \\
&= \frac{1}{2}\int_{\mathbf{z},\mathbf{z}'} K(\mathbf{z};\mathbf{z}')\Big(f(\mathbf{z}) - f(\mathbf{z}')\Big)\Big(\varphi(\mathbf{z}') - \varphi(\mathbf{z})\Big)\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{z}'.
\end{aligned}$$

We deduce that both the L^2 norm and the entropy of $f(t,x,y)$ decay in time. □

3.4 Convergence to Equilibrium: Wasserstein and Linearization

We carry out an linearization analysis around the exponential equilibrium distribution of the solution of (4.6) and demonstrate an explicit rate of convergence under the linearized (weighted L^2) setting in section 3.4.1. These arguments are reinforced in section 3.4.2 into a local convergence result for the full nonlinear equation. A coupling approach is encapsulated in section 3.4.3 in order to show that solution $q(t,x)$ of (4.6) relaxes to its equilibrium q_∞ exponentially fast in the Wasserstein distance.

3.4.1 Linearization Around Equilibrium

Now we perform a linearization analysis near the global exponential equilibrium q_∞, in a fashion that is similar to Baranger and Mouhot [18]. For this purpose, we define the linear operator $\mathcal{L}$ to be

$$\mathcal{L}[h](x) := \int_0^\infty \int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k+\ell-x)\Big(h(k)+h(\ell)-h(x)-h(k+\ell-x)\Big)\,\mathrm{d}k\,\mathrm{d}\ell.$$

Setting $q = q_\infty(1+\varepsilon h)$ for $0 < \varepsilon << 1$, we deduce from (3.3) that

$$\partial_t h(x) = \mathcal{L}[h](x), \tag{3.1}$$

where $h \in L^2(q_\infty)$ is orthogonal to $\mathcal{N}(\mathcal{L}) := \mathrm{Span}\{1, x\}$ in $L^2(q_\infty)$. For the linearized equation (3.1), the natural entropy is the $L^2(q_\infty)$ norm of h, and the entropy dissipation is given by

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E} :&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} \\
&= \int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) \Big(h(k) + h(\ell) - h(k+\ell-x) - h(x)\Big) h(x)\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}x \\
&= -\frac{1}{4}\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) \Big(h(k+\ell-x) + h(x) - h(k) - h(\ell)\Big)^2\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}x.
\end{aligned}
$$

In particular, it implies that the spectrum of $\mathcal{L}$ in $L^2(q_\infty)$ is non-positive.

Remark. It is not hard to show that the linear operator $-\mathcal{L}$ enjoys a self-adjoint property on the space $L^2(q_\infty)$. Thus the existence of a spectral gap η is equivalent to

$$
\forall h \perp \mathcal{N}(\mathcal{L}), \quad -\langle \mathcal{L}[h],\ h\rangle_{L^2(q_\infty)} := -\int_0^\infty \mathcal{L}[h](x)\, h(x)\, q_\infty(x)\,\mathrm{d}x \ge \eta\, \|h\|^2_{L^2(q_\infty)}.
$$

Remark. Following Grünbaum [84], we give some comments on the space $L^2(q_\infty)$. If q is the unique solution of (4.6) and we set $q = q_\infty(1+\varepsilon h)$ as before for $h \perp \mathcal{N}(\mathcal{L})$, then

$$
\begin{aligned}
\int_0^\infty q \log q\,\mathrm{d}x &= \int_0^\infty q_\infty(1+\varepsilon h)\,\log(q_\infty(1+\varepsilon h))\,\mathrm{d}x \\
&= \int_0^\infty q_\infty\,\log q_\infty\,\mathrm{d}x + \varepsilon\int_0^\infty \left(\log\frac{1}{M} - \frac{1}{M}x\right) h\, q_\infty\,\mathrm{d}x \\
&\quad + \int_0^\infty q_\infty(1+\varepsilon h)\left(\varepsilon h - \frac{(\varepsilon h)^2}{2} \pm \cdots\right)\mathrm{d}x \\
&= \int_0^\infty q_\infty\,\log q_\infty\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_0^\infty h^2\, q_\infty\,\mathrm{d}x + O(\varepsilon^3),
\end{aligned}
$$

where we used the fact that $h \perp \mathcal{N}(\mathcal{L})$. Therefore, we can see that $\|h\|^2_{L^2(q_\infty)} = \int_0^\infty h^2\, q_\infty\,\mathrm{d}x$ gives the first-order correction to the expansion of the entropy of q around q_∞.

We will prove that the linearized entropy $\mathrm{E} = \frac{1}{2}\|h\|^2_{L^2(q_\infty)}$ decays exponentially fast in time with an explicit sharp decay rate, the essence of which lies in the following lemma.

Lemma 3.4.1 *Let* $m_1 = 1$ *and* $\mathcal{A} := \{h \in L^2(q_\infty) \mid h \in \mathcal{N}(\mathcal{L})\}$*. Then*

$$\inf_{h\in\mathcal{A}} \frac{\int_0^\infty h^2(x)\, q_\infty(x)\, \mathrm{d}x}{\int_0^\infty \frac{\mathrm{e}^{-z}}{z} \left(\int_0^z h(x)\, \mathrm{d}x\right)^2 \mathrm{d}z} = 3, \tag{3.2}$$

and the infimum in (3.2) *is attained (up to a non-zero multiplication constant) at* $h(x) = \frac{1}{2}\,(x^2 - 4x + 2)$*.*

Proof The key ingredient in the proof is the fact that the so-called Laguerre polynomials, defined by

$$L_n(x) = \frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n) = \sum_{k=0}^n \binom{n}{k}\frac{(-1)^k}{k!}x^k, \quad n \geq 0,$$

form an orthonormal basis for the weighted L^2 space $L^2(q_\infty)$ Abramowitz and Stegun [1]. Thus, for any $h \in L^2(q_\infty)$ which is not identically zero, we can write $h = \sum_{n=0}^\infty \alpha_n L_n$, in which $\alpha_n \in \mathbb{R}$ for all n. Next, notice that the condition $h \in \mathcal{A}$ implies that $\alpha_0 = \alpha_1 = 0$. Moreover, we have $\int_0^\infty h^2(x)\, q_\infty(x)\, \mathrm{d}x = \sum_{n=2}^\infty \alpha_n^2$ thanks to the orthonormality of the Laguerre polynomials $\{L_n\}_{n\geq 0}$. To proceed further, we recall that Poularikas [132] $L_n(z) - L_{n+1}(z) = \int_0^z L_n(x)\mathrm{d}x$ and $zL_n'(z) = nL_n(z) - nL_{n-1}(z)$

for all $n \geq 1$, whence

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{e}^{-z}}{z}\left(\int_0^z h(x)\mathrm{d}x\right)^2 \mathrm{d}z &= \int_0^\infty \frac{\mathrm{e}^{-z}}{z}\left(\sum_{n=2}^\infty \alpha_n (L_n(z) - L_{n+1}(z))\right)^2 \mathrm{d}z \\
&= \int_0^\infty \mathrm{e}^{-z}\left(\sum_{n,m=2}^\infty \alpha_n\alpha_m\left(\frac{L_n - L_{n+1}}{z}\right)(L_m - L_{m+1})\right)^2 \\
&= -\sum_{n,m=2}^\infty \frac{\alpha_n\alpha_m}{n+1}\int_0^\infty \mathrm{e}^{-z}(L_m(z) - L_{m+1}(z))\,\mathrm{d}L_{n+1}(z) \\
&= \sum_{n,m=2}^\infty \frac{\alpha_n\alpha_m}{n+1}\int_0^\infty L_{n+1}(z)\,\mathrm{d}\Big(\mathrm{e}^{-z}(L_m(z) - L_{m+1}(z))\Big) \\
&= \sum_{n,m=2}^\infty \frac{\alpha_n\alpha_m}{n+1}\int_0^\infty L_{n+1}(z)\,L_{m+1}(z)\,\mathrm{e}^{-z}\,\mathrm{d}z \\
&= \sum_{n=2}^\infty \frac{\alpha_n^2}{n+1} \leq \frac{1}{3}\sum_{n=2}^\infty \alpha_n^2.
\end{aligned}$$

Finally, notice that the inequality above will become an equality if and only if $\alpha_n = 0$ for all $n \geq 3$, or in other words, if and only if $h(x) = L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$ up to a non-zero multiplication constant. $\square$

We are now in a position to prove the following result.

Theorem 5 *Assume that $h \in L^2(q_\infty)$ solves the linearized equation* (3.1)*, then we have*

$$\|h(t)\|_{L^2(q_\infty)} \leq \|h(0)\|_{L^2(q_\infty)}\mathrm{e}^{-\frac{1}{3}t}. \tag{3.3}$$

Proof We will only prove the result for $m_1 = 1$, and the general case follows readily from a change of variable argument. From the discussion above, we already have that

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} = \int_{\mathbb{R}^3_+} &\frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k)q_\infty(\ell)\cdot \\
&(h(k+\ell-x) + h(x) - h(k) - h(\ell))\,h(x)\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

Thanks to $h \in \mathcal{A}$, it is not hard to see through a change of variable that

$$\int_{\mathbb{R}^3_+} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k)q_\infty(\ell)h(k+\ell-x)h(x)\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}x = 0.$$

Also, a simple calculation yields that

$$\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) h^2(x) \,\mathrm{d}k \,\mathrm{d}\ell \,\mathrm{d}x = \int_0^\infty h^2(x)\,\mathrm{e}^{-x}\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell} q_\infty(k) q_\infty(\ell) h(k) h(x) \,\mathrm{d}k \,\mathrm{d}\ell \,\mathrm{d}x = \int_0^\infty \frac{\mathrm{e}^{-z}}{z} \Big(\int_0^z h(x)\,\mathrm{d}x\Big)^2 \mathrm{d}z.$$

Consequently, (3.4) reads

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|h\|^2_{L^2(q_\infty)} &= \int_0^\infty h^2(x)\,\mathrm{e}^{-x}\,\mathrm{d}x - 2\int_0^\infty \frac{\mathrm{e}^{-z}}{z}\Big(\int_0^z h(x)\,\mathrm{d}x\Big)^2 \mathrm{d}z \\ &\geq \frac{1}{3}\int_0^\infty h^2(x)\,\mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{3}\|h\|^2_{L^2(q_\infty)}, \end{aligned}$$

in which the inequality follows directly from the previous lemma. Thus we can conclude by Gronwall's inequality. □

3.4.2 Local Convergence in L^2

We now extend the linearization argument from the previous subsection into a local convergence result for the full non-linear equation.

Theorem 6 *There exists some $\varepsilon > 0$ such that if at some time $t \geq 0$,*

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x \leq \varepsilon,$$

then q converges to q_∞ and for any $\lambda < \frac{1}{3}$, there exists C_λ such that

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x \leq C_\lambda\,\mathrm{e}^{-\lambda t}.$$

Proof For a solution q, we denote $h(t,x) = (q - q_\infty)/q_\infty$ and calculate

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\,\|h\|^2_{L^2(q_\infty)} &= -\int h\,\partial_t q = -\int h\,(Q_+[q] - q) \\ &= -\int h\,q_\infty\,\mathcal{L}[h] - \int h(x)\,q_\infty(x)\,\frac{\mathbb{1}_{x\leq k+\ell}}{k+\ell}\,q_\infty(k+\ell-x)\,h(k)\,h(\ell)\,\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}\ell. \end{aligned}$$

Denote

$$R(x) = \int \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} \, q_\infty(k+\ell-x) \, h(k) \, h(\ell) \, \mathrm{d}k \, \mathrm{d}\ell,$$

and calculate

$$\left| \int h(x) \, q_\infty(x) \, R(x) \, \mathrm{d}x \right| \le \left(\int q_\infty(x) \, \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} \, q_\infty(k+\ell-x) \, h^2(k) \, h^2(\ell) \, \mathrm{d}x \, \mathrm{d}k \, \mathrm{d}\ell \right)^{1/2}$$
$$\cdot \left(\int h^2(x) \, q_\infty(x) \, \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} \, q_\infty(k+\ell-x) \, \mathrm{d}x \, \mathrm{d}k \, \mathrm{d}\ell \right)^{1/2}.$$

So first of all,

$$\int q_\infty(x) \, \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} \, q_\infty(k+\ell-x) \, h^2(k) \, h^2(\ell) \, \mathrm{d}x \, \mathrm{d}k \, \mathrm{d}\ell$$
$$= \int \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} \, q_\infty(k) \, q_\infty(\ell) \, h^2(k) \, h^2(\ell) \, \mathrm{d}x \, \mathrm{d}k \, \mathrm{d}\ell = \|h\|^4_{L^2(q_\infty)}.$$

On the other hand,

$$\int h^2(x) \, q_\infty(x) \, \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell} \, q_\infty(k+\ell-x) \, \mathrm{d}x \, \mathrm{d}k \, \mathrm{d}\ell = \int h^2(x) \, q_\infty(x) \, \mathrm{d}x = \|h\|^2_{L^2(q_\infty)}.$$

Hence,

$$\left| \int h(x) \, q_\infty(x) \, R(x) \, \mathrm{d}x \right| \le \|h\|^3_{L^2(q_\infty)}.$$

Coming back to the equation, we have that

$$-\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|h\|^2_{L^2(q_\infty)} \ge -\int h(x) \, q_\infty(x) \, \mathcal{L}[h] \, \mathrm{d}x - \|h\|^3_{L^2(q_\infty)}.$$

Using the previous calculations on the spectral gap of $\mathcal{L}$, we can conclude that

$$-\frac{d}{dt} \frac{1}{2} \|h\|^2_{L^2(q_\infty)} \ge \frac{1}{3} \|h\|^2_{L^2(q_\infty)} - \|h\|^3_{L^2(q_\infty)},$$

which finishes the proof with a Gronwall bound. $\square$

We can couple this with an interpolation argument to modify the smallness assumption in weighted L^2 by using the relative entropy, which leads us to Theorem 7 below, whose proof will be deferred to Appendix C (as the proof of Theorem 7 relies on several a priori estimates established in section 3.5).

Theorem 7 *Assume that for some* $\lambda_0 > \frac{1}{2}$, $\sup_x \mathrm{e}^{\lambda_0 x}\, q(0,x) < \infty$. *Then there exists some* $\delta > 0$ *such that if at some time* $t \geq 0$,

$$\int q(t,x)\, \log \frac{q(t,x)}{q_\infty(x)}\, \mathrm{d}x \leq \delta,$$

we have that q *converges to* q_∞ *and for any* $\lambda < \frac{1}{3}$, *there exists* C_λ *such that*

$$\int \frac{|q(t,x) - q_\infty(x)|^2}{q_\infty(x)}\, \mathrm{d}x \leq C_\lambda\, \mathrm{e}^{-\lambda\, t}.$$

3.4.3 Coupling and Convergence in Wasserstein Distance

In this section we shall employ a coupling argument to demonstrate the convergence of the solution of (4.6) to the exponential probability density function given by (5.12). Before we state the main result of this section, we first collect several relevant definitions.

Definition 12 *For random variables* X *and* Y *taking values in* $\mathbb{R}_+$, *we write* $X \perp Y$ *to mean that* X *and* Y *are mutually independent. Also, the Wasserstein distance with exponent 2 between two probability density functions (say* f *and* g*) is defined by*

$$W_2(f,g) = \inf\left\{\sqrt{\mathbb{E}[|X-Y|^2]};\ \mathrm{Law}(X) = f,\ \mathrm{Law}(Y) = g\right\},$$

where the infimum is taken over all pairs of random variables defined on some probability space $(\Omega, \mathbb{P})$ *and distributed according to* f *and* g, *respectively.*

Next, we present a stochastic representation of the evolution equation (4.6), which is interesting in its own right.

Proposition 3.4.2 *Assume that* $q_t(x) := q(t,x)$ *is a solution of* (4.6) *with initial condition* $q_0(x)$ *being a probability density function supported on* $\mathbb{R}_+$ *with mean* m_1.

Defining $(X_t)_{t\geq 0}$ *to be a* $\mathbb{R}_+$*-valued continuous-time pure jump process with jumps of the form*

$$X_t \quad \overset{\text{rate 1}}{\rightsquigarrow} \quad U(X_t + Y_t), \tag{3.5}$$

where Y_t *is a i.i.d. copy of* X_t, $U \sim \mathrm{Uniform}[0,1]$ *is independent of* (X_t) *and* (Y_t), *and the jump occurs according to a Poisson clock running at the unit rate. If* $\mathrm{Law}(X_0) = q_0$, *then* $\mathrm{Law}(X_t) = q_t$ *for all* $t \geq 0$.

Proof Taking φ to be an arbitrary but fixed test function, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\varphi(X_t)] = \mathbb{E}[\varphi(U(X_t+Y_t))] - \mathbb{E}[\varphi(X_t)]. \tag{3.6}$$

Denoting $q(t,x)$ as the probability density function of X_t, (5.9) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} q(t,x)\varphi(x)\mathrm{d}x = \int_{\mathbb{R}_+^2}\int_0^1 \varphi(u(k+\ell))q(k,t)q(\ell,t)\mathrm{d}u\mathrm{d}k\mathrm{d}\ell - \int_{\mathbb{R}_+} q(t,x)\varphi(x)\mathrm{d}x.$$

After a simple change of variables, one arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} q(t,x)\varphi(x)\mathrm{d}x = \int_{\mathbb{R}_+} \left(Q_+[q](x,t) - q(t,x)\right)\varphi(x)\mathrm{d}x. \tag{3.7}$$

Thus, q must satisfy $\partial_t q = G[q]$ and the proof is completed. $\square$

Remark. Using a similar reasoning, we can show that if $(\overline{X}_t)_{t\geq 0}$ is a $\mathbb{R}_+$-valued continuous-time pure jump process with jumps of the form

$$\overline{X}_t \quad \overset{\text{rate 1}}{\rightsquigarrow} \quad U(\overline{X}_t + \overline{Y}_t), \tag{3.8}$$

where $\overline{Y}_t$ is a i.i.d. copy of $\overline{X}_t$, $U \sim \mathrm{Uniform}[0,1]$ is independent of $(\overline{X}_t)$ and $(\overline{Y}_t)$, and the jump occurs according to a Poisson clock running at the unit rate. Then $\mathrm{Law}(\overline{X}_0) = q_\infty$ implies $\mathrm{Law}(\overline{X}_t) = q_\infty$ for all $t \geq 0$.

The main result of this section is recorded in the following theorem:

Theorem 8 *Under the setting of Proposition 4.3.2, we have*

$$W_2(q_t, q_\infty) \leq \mathrm{e}^{-\frac{1}{6}t} W_2(q_0, q_\infty), \quad \forall t \geq 0. \tag{3.9}$$

Proof Fixing $t \in \mathbb{R}_+$, we need to couple the two densities q_t and q_∞. Suppose that $(X_t)_{t\geq 0}$ and $(\overline{X}_t)_{t\geq 0}$ are $\mathbb{R}_+$-valued continuous-time pure jump processes with jumps of the form (3.5) and (3.8), respectively. We can take (X_t, Y_t) and $(\overline{X}_t, \overline{Y}_t)$ as in the statement of Proposition 4.3.2 and Remark 3.4.3, respectively. Meanwhile, we require that $X_t \perp \overline{Y}_t$, $\overline{X}_t \perp Y_t$ and $(X_t, \overline{X}_t) \perp (Y_t, \overline{Y}_t)$, i.e., several independence assumptions can be imposed along the way when we introduce the coupling. We insist that the same uniform random variable U is used in both (3.5) and (3.8). Moreover, we impose that $\mathrm{Law}(X_0) = q_0$ and $\mathrm{Law}(\overline{X}_0) = q_\infty$. As a consequence of the previous proposition and remark, $q_t = \mathrm{Law}(X_t)$ and $\mathrm{Law}(\overline{X}_t) = q_\infty$ for all $t \geq 0$, whence $\mathbb{E}[\overline{X}_t] = \mathbb{E}[\overline{Y}_t] = m_1$ and $\mathbb{E}(\overline{X}_t^2) = \mathbb{E}(\overline{Y}_t^2) = 2\,m_1^2$, $\forall t \geq 0$. Also, we have that $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = m_1$ for all $t \geq 0$. Thanks to the aforementioned coupling, we then have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[(X_t - \overline{X}_t)^2] &= \mathbb{E}[\Big(U(X_t + Y_t - \overline{X}_t - \overline{Y}_t)\Big)^2 - (X_t - \overline{X}_t)^2] \\
&= \frac{1}{3}\Big(\mathbb{E}[(X_t - \overline{X}_t)^2] + \mathbb{E}[(Y_t - \overline{Y}_t)^2] + 2\mathbb{E}[(X_t - \overline{X}_t)(Y_t - \overline{Y}_t)]\Big) \\
&\qquad - \mathbb{E}[(X_t - \overline{X}_t)^2] \\
&= \frac{2}{3}\mathbb{E}[(X_t - \overline{X}_t)^2] + \frac{2}{3}\mathbb{E}[X_t - \overline{X}_t]\cdot\mathbb{E}[Y_t - \overline{Y}_t] - \mathbb{E}[(X_t - \overline{X}_t)^2] \\
&= -\frac{1}{3}\mathbb{E}[(X_t - \overline{X}_t)^2].
\end{aligned}
$$

Now we pick $\overline{X_0}$ with law q_∞ so that $W_2^2(q, q_\infty) = \mathbb{E}[(X_0 - \overline{X}_0)^2]$, and a routine application of Gronwall's inequality yields (3.9). $\square$

3.5 Entropy Dissipation

We state our main result, Theorem 9, in section 5.5.1 so that readers know exactly what is at stake. We will present various expressions of the entropy and entropy dissipation associated to the solution $q(t, x)$ of (4.6), along with a discussion of the strategy of the proof of Theorem 9 in section 5.5.2. A sequence of auxiliary lemmas and corollaries are recorded in section 5.5.3 and 3.5.4. Finally, a full proof of Theorem 9, built upon all of the preparatory work from 5.5.1 to 3.5.4, is shown in 3.5.5.

3.5.1 Main Result

For the integro-differential equation (4.6), a common strategy Bassetti and Toscani [20], Düring *et al.* [80], Matthes and Toscani [116] is to use the Laplace transform or Fourier transform of (4.6) to prove the exponential decay of solution of (4.6) to $q_\infty(x)$ in some Fourier metric. However, little analysis of (4.6) has been carried out without resorting to Laplace or Fourier transform. In particular, we would like to show the dissipation of relative entropy, i.e., $\mathrm{D}_{\mathrm{KL}}(q(\cdot,t) \mid\mid q_\infty)$, along solution trajectories:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty q\,\log\frac{q}{q_\infty}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty q\,\log q\,\mathrm{d}x \leq 0. \tag{3.1}$$

It is reasonable to expect the validity of (3.1) as the exponential probability density q_∞ maximizes the negative entropy $-\int_0^\infty p\,\log p\,\mathrm{d}x$ among all continuous probability density functions supported on $[0,\infty)$ with prescribed mean.

The following proposition together with its proof should be a reminiscent of the calculations carried out for a standard Boltzmann equation arising from the kinetic theory of (dilute) gases Villani [147].

Proposition 3.5.1 *Let $\varphi(x)$ be a (continuous) test function on $\mathbb{R}_+$ and assume that q is a smooth solution of* (4.6)*, then we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty q(t,x)\varphi(x)\mathrm{d}x = -\frac{1}{4}\int_{\mathbb{R}_+^3}\frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\Big(q(k+\ell-x)q(x)-q(k)q(\ell)\Big)\cdot\\ \Big(\varphi(k+\ell-x)+\varphi(x)-\varphi(k)-\varphi(\ell)\Big)\mathrm{d}k\mathrm{d}\ell\mathrm{d}x.\end{aligned}$$

Moreover, inserting $\varphi = \log q$ and employing the fact that total mass is conserved (i.e., $m_0'(t) = 0$ for all $t \geq 0$), we obtain the dissipation of relative entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty q(t,x)\log q(t,x)\mathrm{d}x = -\frac{1}{4}D[q],$$

where

$$D[q] := \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\Big(q(k+\ell-x)q(x)-q(k)q(\ell)\Big)\,\log\frac{q(k+\ell-x)q(x)}{q(k)q(\ell)}\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}x \geq 0. \tag{3.2}$$

Proof We notice that the PDE (4.6) can be rewritten as

$$\partial_t q(x) = \int_0^\infty\int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\Big(q(k)q(\ell)-q(x)q(k+\ell-x)\Big)\mathrm{d}k\mathrm{d}\ell \tag{3.3}$$

(thanks to Proposition 3.3.2). Omitting the time variable for simplicity, we deduce that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty q(x)\varphi(x)\mathrm{d}x &= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\Big(q(k)q(\ell)-q(x)q(k+\ell-x)\Big)\varphi(x)\mathrm{d}k\mathrm{d}\ell\mathrm{d}x \\
&= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}q(k)q(\ell)\Big(\varphi(x)-\varphi(\ell)\Big)\mathrm{d}k\mathrm{d}\ell\mathrm{d}x \\
&= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}q(k)q(\ell)\Big(\varphi(k+\ell-x)-\varphi(k)\Big)\mathrm{d}k\mathrm{d}\ell\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}q(k)q(\ell) \\
&\qquad \cdot\Big(\varphi(k+\ell-x)+\varphi(x)-\varphi(k)-\varphi(\ell)\Big)\mathrm{d}k\mathrm{d}\ell\mathrm{d}x \\
&= -\frac{1}{4}\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\Big(q(k+\ell-x)q(x)-q(k)q(\ell)\Big) \\
&\qquad \cdot\Big(\varphi(k+\ell-x)+\varphi(x)-\varphi(k)-\varphi(\ell)\Big)\mathrm{d}k\mathrm{d}\ell\mathrm{d}x.
\end{aligned}$$

□

Remark. The dissipation of the relative entropy can also be seen via an alternative perspective. Indeed, we fix $t \geq 0$ and assume that $X_1(t)$ and $X_2(t)$ are i.i.d $\mathbb{R}_+$-valued random variable with its probability density function given by $q(t,x)$, and we define $(Z_1,Z_2) = (U(X_1+X_2),(1-U)(X_1+X_2))$ with $U \sim \text{Uniform}[0,1]$ being independent of X_1 and X_2. Then we deduce from the PDE (4.6) and Lemma 4.3.1 that

$$\begin{aligned}
2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_{\mathrm{KL}}(q \,||\, q_\infty) &= \mathrm{H}((Z_1,Z_2),(X_1,X_2)) - \mathrm{H}((X_1,X_2)) \\
&\leq \mathrm{H}((Z_1,Z_2)) - \mathrm{H}((X_1,X_2)),
\end{aligned} \tag{3.4}$$

where $\mathrm{H}(X,Y) := \int_{\mathbb{R}} \rho_X(x) \log \rho_Y(x) \mathrm{d}x$ represents the cross entropy from Y to X, if the laws of X and Y are given by ρ_X and ρ_Y. It can be shown Apenko [10] that the joint entropy of (Z_1, Z_2) is always no more than the joint entropy of (X_1, X_2), whence the rightmost side of (3.4) is non-positive.

Corollary 3.5.2 *The exponential distribution q_∞ defined in* (5.12) *is the only (smooth) equilibrium solution of the PDE* (4.6).

Proof By Proposition 3.5.1, we see that

$$q_\infty(x) q_\infty(k+\ell-x) = q_\infty(k) q_\infty(\ell) \quad \text{for all } k,\ell,x \geq 0 \text{ such that } k+\ell \geq x.$$

Since $\int_0^\infty q_\infty(x)\,\mathrm{d}x = 1$ and $\int_0^\infty x\,q_\infty(x)\,\mathrm{d}x = m_1$, q_∞ must be the exponential probability density provided by (5.12). $\square$

We will prove that $\int q \,\log \frac{q}{q_\infty}\,\mathrm{d}x \xrightarrow{t\to\infty} 0$ occurs polynomially fast in time. Without of loss generality, throughout the argument to be presented below we will set $m_1 = 1$, i.e., $q_\infty(x) = \mathrm{e}^{-x}$ for $x \geq 0$. Our main result is stated as follows:

Theorem 9 *Under the assumptions of Lemma 3.5.7 below, we have for some constant C, $\theta > 0$ and for any $t \geq C\,\log(1/D)$ that*

$$\int_{x=0}^{+\infty} q(x,t)\,\log \frac{q(x,t)}{\mathrm{e}^{-x}}\,\mathrm{d}x \leq C\,D^\theta. \tag{3.5}$$

To our best knowledge, Theorem 9 is the first entropy-entropy dissipation inequality established for the uniform reshuffling dynamics.

3.5.2 Basic Expressions of the Entropy-Entropy Dissipation

Let us start by looking at the strong convergence of the pairwise distribution, which is essentially trivial. Indeed, we recall the linear PDE (3.19), which reads

$$\partial_t f = L_+[f] - f,$$

where

$$L_+[f](x,y) = \frac{1}{x+y} \int_{z=0}^{x+y} f(z, x+y-z)\,\mathrm{d}z.$$

Then denoting

$$g(t,\lambda) = \frac{1}{\lambda} \int_0^\lambda f(t,z,\lambda-z)\mathrm{d}z, \tag{3.6}$$

we can rewrite (3.19) as $\partial_t f(t,x,y) = g(t,x+y) - f(t,x,y)$, whence

$$\begin{aligned} \partial_t g(t,\lambda) &= \frac{1}{\lambda} \int_{z=0}^{\lambda} \partial_t f(t,z,\lambda-z)\,\mathrm{d}z \\ &= \frac{1}{\lambda} \int_0^\lambda \left(g(t,\lambda) - f(t,z,\lambda-z)\right)\,\mathrm{d}z = 0. \end{aligned}$$

Hence $g(t,\lambda) = g(0,\lambda)$ and trivially

$$|f(t,x,y) - g(0,x+y)| \le \mathrm{e}^{-t}. \tag{3.7}$$

Unfortunately this cannot be used to show the convergence on the actual equation for $q(t,x)$ because the two models are not equivalent: If $q(t,x)$ solves (4.6), which is nonlinear, then in general $f(t,x,y) = q(t,x)\,q(t,y)$ does not solve (3.19). The one exception is when $q(t,x)$ is some exponential.

This can also be seen from the fact that in the argument above f does not necessarily converge to an exponential but to whatever $g(t=0)$ was. The rate of convergence is also too fast as the second moment of q converges much slower for example.

We will still find some of the structure above in the entropy dissipation for q but that is one reason why the entropy dissipation is not easy to handle. In particular, the entropy dissipation will vanish whenever $f(x,y) = g(x+y)$ which seems to create some degeneracy.

Next, we can rewrite the dissipation term in a manner that will make the connection with the exponential more apparent. We define for simplicity $f(x,y) = q(x)\,q(y)$, and as before

$$g(\lambda) = \frac{1}{\lambda} \int_0^\lambda f(z,\lambda-z)\,\mathrm{d}z = \frac{1}{\lambda} \int_0^\lambda q(z)\,q(\lambda-z)\,\mathrm{d}z.$$

Finally, we also define

$$h(x) = \int_{\mathbb{R}_+} g(x+y)\,\mathrm{d}y.$$

We remark here that h coincides with the collision gain operator $Q_+[q]$ defined via (4.5). With these definitions, we have

Lemma 3.5.3 *One has that*

$$D = 2\int_{\mathbb{R}_+^2} q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{\mathbb{R}_+^2} g(x+y)\,\log\frac{g(x+y)}{q(x)\,q(y)}\,\mathrm{d}x\,\mathrm{d}y,$$

or as well that

$$D = 2\int_{\mathbb{R}_+^2} q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{\mathbb{R}_+^2} g(x+y)\,\log\frac{g(x+y)}{h(x)\,h(y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$+\,4\int_{\mathbb{R}_+} h(x)\,\log\frac{h(x)}{q(x)}\,\mathrm{d}x.$$

Formally this forces $g(x+y)$ to be close to $f(x,y)$ so this is a very similar term to the one that we had found when looking at equation (3.19). It is some sort of degeneracy because it does not directly force f to be close to e^{-x-y} so we will have to resolve it. Of course since $f(x,y) = q(x)\,q(y)$, $f(x,y) = g(x+y)$ forces q to be some exponential and therefore this should be possible.

Proof We can first simply rewrite

$$D = \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}\,(f(y+z-x,x) - f(y,z))\,\log\frac{f(y+z-x,x)}{f(y,z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Observe that by swapping x and z

$$\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}\,(f(y+z-x,x) - f(y,z))\,\log f(y+z-x,x)$$
$$= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+x\geq z}}{y+x}\,(f(y+x-z,z) - f(y,x))\,\log f(y+x-z,z).$$

Changing variable $y \to y' = y+x-z$, we get that

$$\int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z}\,(f(y+z-x,x) - f(y,z))\,\log f(y+z-x,x)$$
$$= \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y'+z\geq x}}{y'+z}\,(f(y',z) - f(y'+z-x,x))\,\log f(y',z).$$

Hence

$$D = 2 \int_{\mathbb{R}_+^3} \frac{\mathbb{1}_{y+z\geq x}}{y+z} \left(f(y,z) - f(y+z-x,x)\right) \log f(y,z).$$

In other words,

$$D = 2\int_{\mathbb{R}_+^2} f(y,z)\, \log f(y,z)\, dy\, dz - 2 \int_{\mathbb{R}_+^2} g(y+z)\, \log f(y,z)\, \mathrm{d}y\, \mathrm{d}z.$$

Now, we observe that

$$\int_{\mathbb{R}_+^2} f(y,z)\, \log g(y+z)\, \mathrm{d}y\, \mathrm{d}z = \int_{\mathbb{R}_+^2} g(y+z) \log g(y+z)\, \mathrm{d}y\, \mathrm{d}z.$$

Indeed, a change of variable $y = x - w$ and $z = w$ yields

$$\int_{\mathbb{R}_+^2} g(y+z) \log g(y+z)\, \mathrm{d}y\, \mathrm{d}z = \int_{\mathbb{R}_+} x\, g(x)\, \log g(x)\, \mathrm{d}x.$$

By the same change of variables, we also have

$$\int_{\mathbb{R}_+^2} f(y,z)\, \log g(y+z) = \int_{\mathbb{R}_+} \log g(x)\, \int_0^x f(x-w,w) = \int_{\mathbb{R}_+} x\, g\, \log g.$$

Hence

$$\frac{D}{2} = \int_{\mathbb{R}_+^2} f(y,z)\, \log \frac{f(y,z)}{g(y+z)}\, \mathrm{d}y\, \mathrm{d}z + \int_{\mathbb{R}_+^2} g(y+z)\, \log \frac{g(y+z)}{f(y,z)}\, \mathrm{d}y\, \mathrm{d}z.$$

Finally as $f(y,z) = q(y)\, q(z)$, we may also notice that

$$\begin{aligned}\int_{\mathbb{R}_+^2} g(y+z)\, \log \frac{g(y+z)}{f(y,z)}\, \mathrm{d}y\, \mathrm{d}z &= \int_{\mathbb{R}_+^2} g(y+z)\, \log g(y+z)\, \mathrm{d}y\, \mathrm{d}z \\ &\quad - 2 \int_{\mathbb{R}_+^2} g(y+z)\, \log q(y)\, \mathrm{d}y\, \mathrm{d}z \\ &= \int_{\mathbb{R}_+^2} g(y+z) \log g(y+z) - 2 \int_{\mathbb{R}_+} h(y) \log q(y).\end{aligned}$$

So we also have that

$$\begin{aligned}\int_{\mathbb{R}_+^2} g(y+z)\, \log \frac{g(y+z)}{f(y,z)}\, \mathrm{d}y\, \mathrm{d}z = \int_{\mathbb{R}_+^2} g(y+z)\, \log \frac{g(y+z)}{h(y)\, h(z)}\, \mathrm{d}y\, \mathrm{d}z \\ + 2 \int_{\mathbb{R}_+} h(y)\, \log \frac{h(y)}{q(y)}\, \mathrm{d}y,\end{aligned}$$

concluding the estimate. $\square$

Next, we intend to collect here some various bounds stemming from the dissipation term, the essence of those bounds lies in the following lemma.

Lemma 3.5.4 *We have that*

$$\int q(x)\,\log\frac{q(x)}{H(x)}\,dx \le \int \varphi(y)\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)}\,\mathrm{d}x\,\mathrm{d}y,$$

in which

$$H(x) = \int g(x+y)\,\varphi(y)\,\mathrm{d}y,$$

for any $\varphi \ge 0$ *such that* $\int \varphi\,q\,\mathrm{d}x = 1$*.*

Proof Indeed, as log is concave,

$$\begin{aligned}\int q(x)\,\varphi(y)\,q(y)\,\log\frac{g(x+y)}{q(x)\,q(y)}\,\mathrm{d}x\,\mathrm{d}y &\le \int q(x)\log\left(\int \frac{g(x+y)}{q(x)}\,\varphi(y)\,dy\right)\,\mathrm{d}x\\ &= \int q(x)\,\log\frac{H(x)}{q(x)}\,\mathrm{d}x,\end{aligned}$$

and the proof is completed. $\square$

As a consequence of this lemma, inserting $\phi(x) = 1$ and then $\phi(x) = x$, we then deduce that

$$\begin{aligned}\int q(x)\,\log\frac{q(x)}{h(x)}\,\mathrm{d}x &\le \int q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)}\,\mathrm{d}x\,\mathrm{d}y,\\ \int q(x)\,\log\frac{q(x)}{m(x)}\,\mathrm{d}x &\le \int x\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where

$$m(x) = \int g(x+y)\,y\,\mathrm{d}y = \int_x^\infty g(z)\,(z-x)\,\mathrm{d}z = \int_x^\infty \int_y^\infty g(z)\,\mathrm{d}z\,\mathrm{d}y = \int_x^\infty h(y)\,\mathrm{d}y.$$

Remark. We also note that $m(0) = 1$ (since $\int h\,\mathrm{d}x = \int q\,\mathrm{d}x = 1$) and so

$$\int h\,\log m\,\mathrm{d}x = -\int m'\,\log m\,\mathrm{d}x = -\int h\,\mathrm{d}x = -\int x\,h(x)\,\mathrm{d}x = -1,$$

by virtue of the fact that $\int x\, h(x)\, \mathrm{d}x = \int x\, q(x)\, \mathrm{d}x = 1$. Thus,

$$\int h \, \log \frac{h}{m} \, \mathrm{d}x = \int h \, \log \frac{h}{\mathrm{e}^{-x}} \, \mathrm{d}x.$$

This leads to a possible strategy: Control $\int h \log \frac{h}{m}$ in terms of $\int q \log \frac{q}{h}$, $\int h \log \frac{h}{q}$ and $\int q \log \frac{q}{m}$. Then control $\int q \log \frac{q}{\mathrm{e}^{-x}}$ by the previous quantities and $\int h \log \frac{h}{m}$. We can then estimate $\int x\, q(x)\, q(y) \log \frac{q(x)\, q(y)}{g(x+y)} \, \mathrm{d}x \, \mathrm{d}y$ via $\int q(x)\, q(y) \log \frac{q(x)\, q(y)}{g(x+y)} \, \mathrm{d}x \, \mathrm{d}y$ and some control on the decay of q at infinity. So in the end this would lead to some kind of bounds on $\int q \log \frac{q}{\mathrm{e}^{-x}}$ in terms of the dissipation term. We illustrate the strategy in Figure 3.6.

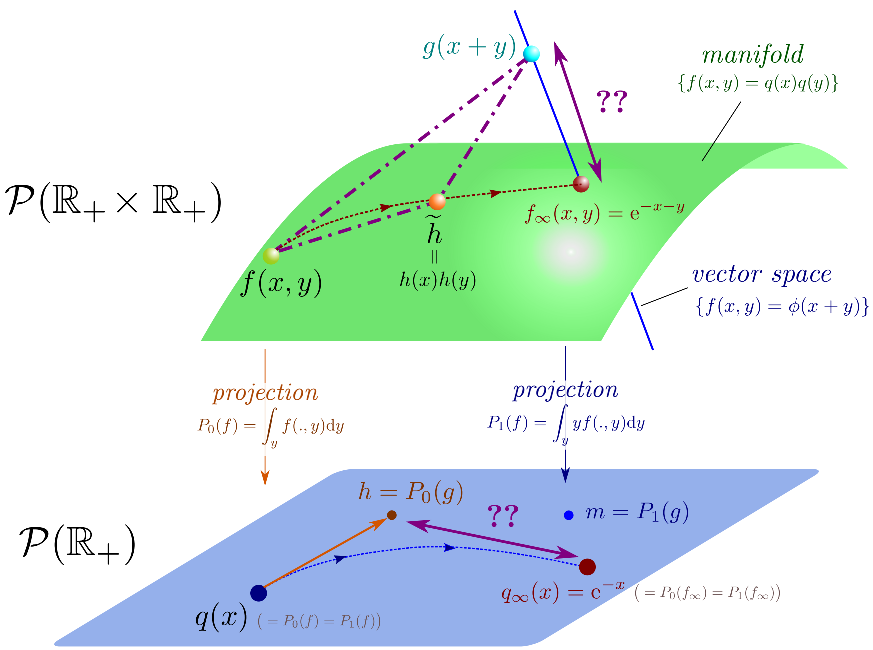

Figure 3.6: To Measure the Decay of the Relative Entropy $\int q \log \frac{q}{\mathrm{e}^{-x}}$, We Have to Control the Term $\int h \log \frac{h}{\mathrm{e}^{-x}}$ or Similarly the Term $\int g \log \frac{g}{\mathrm{e}^{-x-y}}$ (Represented In Purple). Indeed, the Dissipation Term D Already Provides a Control Over the 'Triangle' of Relative Entropies $\int f \log \frac{f}{g}$, $\int g \log \frac{g}{\widetilde{h}}$ and $\int \widetilde{h} \log \frac{\widetilde{h}}{g}$ with $\widetilde{h}(x,y) = h(x)h(y)$.

However normally it is not possible to switch relative entropy estimates. Indeed, it is not so hard to find examples of non-negative functions φ, ϕ, ψ with total mass 1 such that

$$\int \varphi \, \log \frac{\varphi}{\psi} = \infty,$$

while

$$\int \phi \log \frac{\phi}{\psi} + \int \phi \log \frac{\phi}{\varphi} + \int \varphi \log \frac{\varphi}{\phi} < \infty.$$

Therefore this strategy is not obvious to implement. It should work nicely if we had a control like $\mathrm{e}^{-x}/C \le q(x) \le C\,\mathrm{e}^{-x}$ but the general case is certainly trickier. What saves us is the key observation that here h and m are actually very nice functions in all cases. For example, m and h are monotone decreasing so bounded from above and bounded from below on any finite interval (from the propagation of moments on q). This gives us some hope when implementing the aforementioned machinery. We emphasize here that our entropy-entropy dissipation argument draws inspiration from earlier works on Becker-Döring equations and coagulation models Jabin and Niethammer [95], Cañizo *et al.* [36].

3.5.3 Switching Relative Entropies

We note that the relative entropy behaves in the following manner

Lemma 3.5.5 *For any two $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$ and for any $C \ge 2$, then*

$$\begin{aligned} &\frac{1}{2\,C} \int_{\nu/C \le \mu \le C\,\nu} \frac{(\mu-\nu)^2}{\nu} + \frac{1}{8} \int_{\mu \le \nu/C} \nu + \frac{1}{4} \int_{\mu \ge C\,\nu} \mu \log \frac{\mu}{\nu} \\ &\le \int \mu \log \frac{\mu}{\nu} \\ &\le \frac{C}{2} \int_{\nu/C \le \mu \le C\,\nu} \frac{(\mu-\nu)^2}{\nu} + \int_{\mu \le \nu/C} \nu + \int_{\mu \ge C\,\nu} \mu \log \frac{\mu}{\nu}. \end{aligned} \tag{3.8}$$

Proof We observe that

$$\int \mu \log \frac{\mu}{\nu} = \int \nu \left(\frac{\mu}{\nu} \log \frac{\mu}{\nu} + 1 - \frac{\mu}{\nu} \right).$$

On the other hand, around 1, the function $\phi(x) = x \log x + 1 - x$ satisfies that $\phi(x) \le (x-1)^2/2$ for $x \ge 1$ and $\phi(x) \le \frac{C}{2}\,(x-1)^2$ for $1/C \le x \le 1$. On the other hand $\phi(x) \ge (x-1)^2/2C$ for $1 \le x \le C$ and $\phi(x) \ge (x-1)^2/2$ for $1/C \le x \le 1$.

Furthermore ϕ lies between 1/8 and 1 when $x \leq 1/2$ and larger than $\frac{x}{4}\,\log x$ for $x \geq 2$.

$\square$

Remark. One can also rewrite a little bit the statement of Lemma 3.5.5 so that we do not need to impose that μ and ν are probability measures.

This allows us to "switch" relative entropies between two measures that are comparable.

Corollary 3.5.6 *There exists a constant $C > 0$ such that if μ_1, μ_2, $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\lambda^{-1}\,\mu_1 \leq \mu_2 \leq \lambda\,\mu_1$ and $\lambda \geq \mathrm{e}$, then*

$$\int \mu_1\,\log\frac{\mu_1}{\nu} \leq C\,\lambda^3 \int \mu_2\,\log\frac{\mu_2}{\nu} + \lambda^3 \int \mu_2\,\log\frac{\mu_2}{\mu_1}.$$

Proof Apply Lemma 3.5.5 with $C = 2\,\lambda$ first on μ_1 and ν to find

$$\int \mu_1\,\log\frac{\mu_1}{\nu} \leq \lambda \int_{\frac{\nu}{2\lambda}\leq\mu_1\leq 2\lambda\,\nu} \frac{(\mu_1-\nu)^2}{\nu} + \int_{\mu_1\leq\frac{\nu}{2\lambda}} \nu + \int_{\mu_1\geq 2\,\lambda\,\nu} \mu_1\,\log\frac{\mu_1}{\nu}.$$

Thanks to Lemma 3.5.5 again, we have

$$\int_{\mu_1\leq\frac{\nu}{2\lambda}} \nu \leq 8 \int \mu_2\,\log\frac{\mu_2}{\nu}.$$

Now if $\mu_1 \leq \frac{\nu}{2\lambda}$ then $\mu_2 \leq \frac{\nu}{2}$. Similarly if $\mu_1 \geq 2\,\lambda\,\nu$ then $\mu_2 \geq 2\,\nu$ and moreover

$$\mu_1\,\log\frac{\mu_1}{\nu} \leq \lambda\,\mu_2\,\log\frac{\lambda\,\mu_2}{\nu} \leq 3\lambda\,\log\lambda\,\mu_2\,\log\frac{\mu_2}{\nu}.$$

Conversely if $\frac{\nu}{2\lambda} \leq \mu_1 \leq 2\lambda\,\nu$ then $\frac{\nu}{2\lambda^2} \leq \mu_2 \leq 2\lambda^2\,\nu$, and

$$\frac{(\mu_1-\nu)^2}{\nu} \leq 2\left(\frac{(\mu_2-\nu)^2}{\nu} + \frac{(\mu_1-\mu_2)^2}{\nu}\right) \leq 2\,\frac{(\mu_2-\nu)^2}{\nu} + 4\,\lambda\,\frac{(\mu_1-\mu_2)^2}{\mu_1}.$$

Hence

$$\int_{\frac{\nu}{2\lambda}\leq\mu_1\leq 2\lambda\,\nu} \frac{(\mu_1-\nu)^2}{\nu} \leq 2 \int_{\frac{\nu}{2\lambda^2}\leq\mu_2\leq 2\lambda^2\,\nu} \frac{(\mu_2-\nu)^2}{\nu}$$
$$+ 4\lambda \int_{\frac{\mu_1}{\lambda}\leq\mu_2\leq\lambda\,\mu_1} \frac{(\mu_1-\mu_2)^2}{\mu_1}.$$

Note that by Lemma 3.5.5 applied with $C=\lambda$, we have that

$$\int_{\frac{\mu_1}{\lambda}\leq\mu_2\leq\lambda\,\mu_1}\frac{(\mu_1-\mu_2)^2}{\mu_1}\leq 2\,\lambda\int\mu_2\,\log\frac{\mu_2}{\mu_1}.$$

Also, Lemma 3.5.5 applied with $C=2\,\lambda^2$ gives rise to

$$\int_{\frac{\nu}{2\lambda^2}\leq\mu_2\leq 2\lambda^2\,\nu}\frac{(\mu_2-\nu)^2}{\nu}\leq 4\,\lambda^2\int\mu_2\,\log\frac{\mu_2}{\nu}.$$

Assembling these estimates, the proof is completed. □

3.5.4 Additional Estimates

This leads us to try to compare q and h. We first observe that we can get easy upper bounds.

Lemma 3.5.7 *Assume that for some* $0<\lambda_0<1$, $\int e^{\lambda_0 x}\,q(t=0,x)\,dx<\infty$. *Then we have that*

$$\sup_t\int e^{\lambda_0\,x}\,q(t,x)\,dx<\infty.$$

Proof We use a Laplace transform by defining

$$F(t,\lambda)=\int e^{\lambda\,x}\,q(t,x)\,dx,$$

and note that

$$\partial_t F=\int_{\mathbb{R}_+^2}\frac{e^{\lambda\,(y+z)}-1}{\lambda\,(y+z)}\,q(y)\,q(z)\,dy\,dz-F=\frac{1}{\lambda}\int_0^\lambda(F(\mu))^2\,d\mu-F.$$

It is useful to remark right away that the stationary solution to this equation satisfies that $F^2=\partial_\lambda(\lambda\,F)$ which has solutions of the form $\frac{1}{1-C\,\lambda}$. Those do blow-up but only for λ large enough. As a matter of fact since $\partial_\lambda F|_{\lambda=0}=1$, we can see that we should even have $C=1$. For this reason, denote now $G=(1-C\,\lambda)\,F$ with some $C<\frac{1}{\lambda}$ such that $G(t=0,\lambda)\leq 1$ on $[0,\ \lambda_0]$. We first show that $\sup_{\lambda\in[0,\lambda_0]}G(t,\lambda)\leq 1$ for all

$t \geq 0$. Indeed, let $\lambda(t)$ be such that $\sup\limits_{\lambda\in[0,\lambda_0]} G(t,\lambda) = G(t,\lambda(t))$, then

$$\partial_t \sup_{\lambda\in[0,\lambda_0]} G(t,\lambda) \leq \partial_t G(t,\lambda(t)),$$

this is because $\partial_\lambda G(t,\lambda(t)) = 0$ if $\lambda(t) < \lambda_0$, while if $\lambda(t) = \lambda_0$ then $\partial_\lambda G(t,\lambda(t)) \leq 0$ and $\lambda'(t) \leq 0$, leading to the same inequality. Now since

$$\partial_t G = (\lambda^{-1} - C) \int_0^\lambda \frac{(G(\mu))^2}{(1 - C\,\mu)^2}\,\mathrm{d}\mu - G, \tag{3.9}$$

together with $\int_0^\lambda \frac{d\mu}{(1-C\,\mu)^2} = \frac{\lambda}{1-C\,\lambda}$, we deduce that

$$\partial_t \sup_{\lambda\in[0,\lambda_0]} G(t,\lambda) \leq \left(\sup_{\lambda\in[0,\lambda_0]} G(t,\lambda)\right)^2 - \sup_{\lambda\in[0,\lambda_0]} G(t,\lambda),$$

which yields via the maximum principle that $\sup\limits_{\lambda\in[0,\lambda_0]} G(t,\lambda) \leq 1$. Now thanks to (3.9) again and the elementary observation that $\partial_t \sup\limits_{\lambda\in[0,\lambda_0]} G(t,\lambda) \leq \sup\limits_{\lambda\in[0,\lambda_0]} \partial_t G(t,\lambda)$, we arrive at

$$\partial_t \sup_{\lambda\in[0,\ \lambda_0]} G(\lambda) \leq 0,$$

which immediately proves the desired upper bound. $\square$

Remark. We believe it is possible to prove the exponential convergence of the Laplace transform $F(t,\lambda)$ to $1/(1-\lambda)$ over $\lambda \in [0,\lambda_0)$. However, this is not strictly better though than having the exponential convergence in some weak Wasserstein norm plus the control of the exponential moments that is given above, so we did not try too much in this direction.

Out of Lemma 3.5.7, we may deduce pointwise bounds on q and h, for this purpose, we need the following preparatory result.

Lemma 3.5.8 *We have that*

$$\sup_{t\geq 0} h(t,0) < \infty,$$

i.e., $h(t,0)$ is uniformly bounded in time.

Proof To show $h(t,0)$ is uniformly bounded in time, we write

$$\begin{aligned}
h(t,0) &= \int_{\mathbb{R}_+^2} \frac{q(y)\,q(z)}{y+z}\,\mathrm{d}y\,\mathrm{d}z = 2\int\int_{y\le z} \frac{q(y)\,q(z)}{y+z}\,\mathrm{d}y\,\mathrm{d}z \\
&\le 2\int\int_{y\le z} \frac{q(y)\,q(z)}{z}\,\mathrm{d}y\,\mathrm{d}z \\
&\le 2\sup_{y\le r}\int_{z\ge r} \frac{r\,q(z)}{z}\,\mathrm{d}z + 2\sup_{y\le r}\int\int_{y\le z,\,z\le r} \frac{q(z)}{z}\,\mathrm{d}y\,\mathrm{d}z + \frac{2}{r}.
\end{aligned}$$

We know that there exists some r uniformly in time such that

$$\int\int_{y\le z,\,z\le r} \frac{q(z)}{z}\,\mathrm{d}y\,\mathrm{d}z = \int_{z\le r} q(z)\,\mathrm{d}z \le \frac{1}{8}.$$

Moreover, for this r we also have $\int_{z\ge r} \frac{r\,q(z)}{z}\,\mathrm{d}z \le \frac{1}{8}$. Thus,

$$h(t,0) \le \frac{1}{2}\sup_{x\le r} q(x) + \frac{2}{r}.$$

Now we recall the equation for q to find that for any $x \le r$,

$$\partial_t q(t,x) \le h(t,0) - q(t,x) \le \frac{1}{2}\sup_{x\le r} q(t,x) + \frac{2}{r} - q(t,x),$$

so if x_* is such that $q(t,x_*) = \sup\limits_{x\le r} q(t,x)$, then

$$\partial_t q(t,x_*) \le \frac{2}{r} - \frac{1}{2}\,q(t,x_*).$$

By Gronwall's inequality, we deduce that $\sup\limits_{x\le r} q(t,x) \le \frac{4}{r}$, which allows us to finish the proof. $\square$

Corollary 3.5.9 *Assume that for some* $0 < \lambda_0 < 1$, $\int \mathrm{e}^{\lambda_0 x}\, q(0,x)\,\mathrm{d}x < \infty$, *then we have that*

$$\sup_t \int \mathrm{e}^{\lambda_0 x}\, h(t,x)\,\mathrm{d}x < \infty,\ \sup_{t,x} \mathrm{e}^{\lambda_0 x}\, h(t,x) < \infty,$$

$$q(t,x) \le C\,\mathrm{e}^{-\lambda_0 x} + q(0,x)\,\mathrm{e}^{-t} \text{ for some } C > 0.$$

Proof The first bound follows from the definition of h. Indeed, as $h = Q_+[q]$, we have

$$\begin{aligned}\int \mathrm{e}^{\lambda_0\, x}\, h(t,x)\,\mathrm{d}x &= \int \frac{\mathrm{e}^{\lambda_0\,(y+z)}-1}{\lambda_0\,(y+z)}\, q(y)\, q(z)\,\mathrm{d}y\,\mathrm{d}z \\ &\le \int \mathrm{e}^{\lambda_0\,(y+z)}\, q(y)\, q(z)\,\mathrm{d}y\,\mathrm{d}z < \infty.\end{aligned}$$

Next we observe that h is decreasing in x, so for any $x \ge 0$

$$\begin{aligned}\int_0^\infty \mathrm{e}^{\lambda_0\, y}\, h(t,y)\,\mathrm{d}y \ge \int_0^x \mathrm{e}^{\lambda_0\, y}\, h(t,y)\,\mathrm{d}y &\ge h(t,x) \int_0^x \mathrm{e}^{\lambda_0\, y}\,\mathrm{d}y \\ &= h(t,x)\,\frac{\mathrm{e}^{\lambda_0\, x}-1}{\lambda_0}.\end{aligned}$$

Since $h(t,x) \le h(t,0)$ is uniformly bounded in time, this shows the second point. Finally we recall the equation for q, which reads $\partial_t q = h - q$, so we may rewrite (4.6) as

$$q(t,x) = q(0,x)\,\mathrm{e}^{-t} + \int_0^t h(s,x)\,\mathrm{e}^{-(t-s)}\,\mathrm{d}s. \tag{3.10}$$

Moreover, notice that

$$\begin{aligned}\mathrm{e}^{\lambda_0\, x} \int_0^t h(s,x)\,\mathrm{e}^{-(t-s)}\,\mathrm{d}s &\le \sup_s \left(\mathrm{e}^{\lambda_0\, x}\, h(s,x)\right) \int_0^t \mathrm{e}^{-(t-s)}\,\mathrm{d}s \\ &\le \sup_s \left(\mathrm{e}^{\lambda_0\, x}\, h(s,x)\right).\end{aligned}$$

Combining these estimates with (3.10) ends the proof. □

We now turn to lower bounds on q and hence h. We start with a lower bound on q in terms of h.

Lemma 3.5.10 *There exists C such that for any $t \ge 1$,*

$$q(t,x) \ge \frac{1}{C}\, h(t-1,x). \tag{3.11}$$

Proof We note from the equation (4.6) that

$$\partial_t h(t,x) = 2\int_x^\infty \frac{1}{\lambda} \int_0^\lambda h(t,z)\, q(t,\lambda-z)\,\mathrm{d}z\,\mathrm{d}\lambda - 2\,h(t,x).$$

Therefore

$$\partial_t h(t,x) \geq -2\, h(t,x),$$

and we have that for any $s \leq t$ that

$$\partial_t q(t,x) \geq \mathrm{e}^{-(t-s)}\, h(s,x) - q(t,x),$$

leading for example to the claimed result

$$q(t,x) \geq \frac{h(t-1,x)}{C}$$

with $C = \frac{\mathrm{e}^2}{\mathrm{e}-1}$, thereby completing the proof. $\square$

Unfortunately, this is not enough to give us a bound between q and h which would solve everything. Instead, we can first deduce a bound near the origin.

Lemma 3.5.11 *There exists a constant C such that*

$$\inf_{t\geq 1} \inf_{x\in[0,\ 2]} h(t,x) \geq \frac{1}{C}, \qquad \inf_{t\geq 2} \inf_{x\in[0,\ 2]} q(t,x) \geq \frac{1}{C}. \tag{3.12}$$

Proof For any $x \leq 2$, we have that

$$\begin{aligned} h(t,x) &= \int \frac{\mathbb{1}_{x\leq y+z}}{y+z}\, q(t,y)\, q(t,z)\, \mathrm{d}y\, \mathrm{d}z \\ &\geq \int_{y,z\geq 1} \frac{1}{(y+1)\,(z+1)}\, q(y)\, q(z)\, \mathrm{d}y\, \mathrm{d}z = \left(\int_1^\infty \frac{q(y)}{1+y}\, \mathrm{d}y\right)^2. \end{aligned}$$

By Cauchy-Schwartz, we have that

$$\begin{aligned} \int_1^\infty q(y)\, \mathrm{d}y &\leq \left(\int_1^\infty \frac{q(y)}{1+y}\, \mathrm{d}y\right)^{1/2} \left(\int_1^\infty (1+y)\, q(y)\, \mathrm{d}y\right)^{1/2} \\ &\leq \left(\int_1^\infty \frac{q(y)}{1+y}\, \mathrm{d}y\right)^{1/2} \left(\int_0^\infty (1+y)\, q(y)\, \mathrm{d}y\right)^{1/2} \\ &= \sqrt{2} \left(\int_1^\infty \frac{q(y)}{1+y}\, \mathrm{d}y\right)^{1/2}. \end{aligned}$$

On the other hand the convergence of all moments of q shows that there exists C such that for all $t \geq 1$,

$$\int_1^\infty q(y)\, dy \geq \frac{1}{C}.$$

Therefore there exists C such that $h(t,x) \geq \frac{1}{C}$ whenever $x \leq 2$ and $t \geq 1$. Finally, we deduce the second result from Lemma 3.5.10. $\square$

We combine this with the following doubling type of argument.

Lemma 3.5.12 *There exists a constant C such that for any x and $t \geq 1$, there holds*

$$q(t,x) \geq \frac{x}{C} \left(\inf_{s\in[t-1,t]} \inf_{y\in[x/2,\,3x/4]} q(s,y) \right)^2 .$$

Proof This is a simple consequence of a lower bound on h. Indeed, we have

$$\begin{aligned} h(t,x) &= \int \frac{\mathbb{1}_{x\leq y+z}}{y+z}\, q(t,y)\, q(t,z)\, \mathrm{d}y\, \mathrm{d}z \\ &\geq \frac{2}{3\,x} \int_{y,\,z\in[x/2,\,3x/4]} q(y)\, q(z)\, \mathrm{d}y\, \mathrm{d}z. \end{aligned}$$

Therefore,

$$h(t,x) \geq \frac{x}{24} \left(\inf_{y\in[x/2,\,3x/4]} q(t,y) \right)^2 .$$

We can again conclude by virtue of Lemma 3.5.10. $\square$

Lemma 3.5.13 *There exists a constant C such that for any $t \geq 2$ and $x \geq 2$, we have*

$$q(t,x) \geq \int_{y\geq x} \frac{q(t-1,y)}{C\,y}\, \mathrm{d}y.$$

Proof This is again a consequence of a lower bound on h. Indeed,

$$\begin{aligned} h(t,x) &= \int_{\mathbb{R}_+^2} \frac{\mathbb{1}_{x\leq y+z}}{y+z}\, q(t,y)\, q(t,z)\, \mathrm{d}y\, \mathrm{d}z \\ &\geq \int_{y\leq x} \int_{z\geq x} q(y)\, \frac{q(z)}{2\,z}\, \mathrm{d}y\, \mathrm{d}z. \end{aligned}$$

Thus, by the lower bound on q on $[0,2]$ (thanks to Lemma 3.5.11), we arrive at

$$h(t,x) \geq \int_{y\geq x} \frac{q(t,y)}{C\,y}\, \mathrm{d}y.$$

Using Lemma 3.5.10, we can again conclude. $\square$

Owing to Lemma 3.5.13, we immediately deduce that

Corollary 3.5.14 *There exists some $C > 0$ such that for any $x \geq 2$ and any $t \geq$* $\max(C\,x, 1)$

$$h(t,x) \geq \frac{\mathrm{e}^{-C\,x}}{C}, \quad q(t,x) \geq \frac{\mathrm{e}^{-C\,x}}{C}.$$

Proof Define $\phi(y) = \frac{(y/x-1)_+}{y}$ for $y \leq 2\,x$ and $\phi = 1/y$ if $y \geq 2\,x$ (see Figure 3.7).

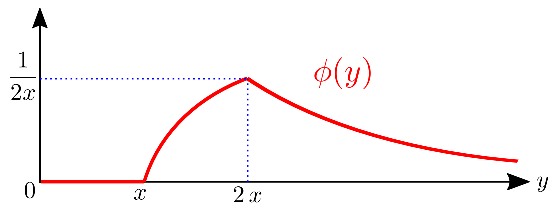

Figure 3.7: The Function ϕ Used In the Proof of Corollary 3.5.14. Notice That $\phi(y) \leq \frac{1}{y}$ for All $y > 0$.

Note that ϕ is Lipschitz with

$$\|\nabla\phi\|_{L^\infty} \leq \frac{1}{x^2}.$$

Hence

$$x^2 \int \phi(y)\,q(y)\,\mathrm{d}y \geq x^2 \int \phi(y)\,\mathrm{e}^{-y}\,\mathrm{d}y - W_1(q, \mathrm{e}^{-x}),$$

in which $W_1(q, \mathrm{e}^{-x})$ represents the Wasserstein distance (with exponent 1) between q and e^{-x}. Thanks to the exponentially fast in time of the convergence $W_1(q, \mathrm{e}^{-x}) \to 0$, which is a simple consequence of Theorem 8, we deduce that

$$\int_{y \geq x} \frac{q(y)}{y}\,\mathrm{d}y \geq \int \phi(y)\,\mathrm{e}^{-y}\,\mathrm{d}y - \frac{C}{x^2}\,\mathrm{e}^{-t/6}.$$

Note that

$$\int \phi(y)\,\mathrm{e}^{-y}\,\mathrm{d}y \geq \int_{y \geq 2x} \frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y \geq \frac{\mathrm{e}^{-3x}}{3x} \int_{2x \leq y \leq 3x} \mathrm{d}y = \frac{\mathrm{e}^{-3x}}{3}.$$

Therefore from Lemma 3.5.13, we can conclude provided that $\frac{C}{x^2}\,\mathrm{e}^{-t/6} \leq \frac{\mathrm{e}^{-3x}}{6}$. $\square$

3.5.5 Proof of the Main Result

Armed with all the previous estimates, we can finally present the proof of Theorem 9.

Proof of theorem 9. We note from our earlier estimates that

$$\int q\,\log\frac{q}{m}\,\mathrm{d}x \le \int x\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int \left(x\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)} + x\,g(x+y) - x\,q(x)\,q(y)\right)\mathrm{d}x\,\mathrm{d}y.$$

For some $K > 0$, we can separate the integral into those $x \le K$, for which

$$\int_{x\le K} \left(x\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)} + x\,g(x+y) - x\,q(x)\,q(y)\right)\mathrm{d}x\,\mathrm{d}y \le K\,D.$$

On the other hand, denoting $\phi(x) = x\,\log x + 1 - x$, which is a non-negative convex function on $\mathbb{R}_+$ and satisfies $\phi(x) \le C\,x$ for some constant C if x is bounded, we deduce for any $\lambda \in (0, \lambda_0)$ that

$$\int_{x\ge K} \left(x\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)} + x\,g(x+y) - x\,q(x)\,q(y)\right)\mathrm{d}x\,\mathrm{d}y$$
$$\le \frac{1}{\lambda}\int_{x\ge K} g(x+y)\,\phi\circ\phi\left(\frac{q(x)\,q(y)}{g(x+y)}\right)\mathrm{d}x\,\mathrm{d}y + \frac{1}{\lambda}\int_{x\ge K} \mathrm{e}^{\lambda\,x}\,g(x+y)\,\mathrm{d}x\,\mathrm{d}y,$$

where the inequality follows from the Fenchel's inequality $x\,y \le \phi(x)+\phi^*(y)$, in which ϕ^* denotes the Legendre convex conjugate of ϕ (and one can check that $\phi^*(y) = \mathrm{e}^y - 1 \le \mathrm{e}^y$ and also $\left(\frac{\phi}{\lambda}\right)^*(y) \le \frac{\mathrm{e}^{\lambda\,y}}{\lambda}$).

We can immediately note that $\phi\circ\phi \le x\log x$ for large x. Thus from Corollary 3.5.9, we have that

$$\int_{x\ge K} \left(x\,q(x)\,q(y)\,\log\frac{q(x)\,q(y)}{g(x+y)} + x\,g(x+y) - x\,q(x)\,q(y)\right)\mathrm{d}x\,\mathrm{d}y$$
$$\le \frac{D}{\lambda} + \frac{C}{\lambda}\,\mathrm{e}^{-(\lambda_0-\lambda)\,K}.$$

Combining both estimates gives rise to

$$\int q\,\log\frac{q}{m} \le (K+1)\,\frac{D}{\lambda} + \frac{C}{\lambda}\,\mathrm{e}^{-(\lambda_0-\lambda)\,K},$$

and optimizing in K leads to

$$\int q\,\log\frac{q}{m} \le C\,D\,\log\frac{1}{D}. \tag{3.13}$$

The next step is to change this to $\int h\,\log\frac{h}{m}$. We decompose again

$$\int h\,\log\frac{h}{m} = \int_{x\le K}(h\,\log\frac{h}{m}+m-h) + \int_{x\ge K}(h\,\log\frac{h}{m}+m-h).$$

We note that since $h = -\partial_x m$,

$$\int_{x\ge K} h\,\log m = -\int_{x\ge K}\partial_x m\,\log m = m(K)\,\log m(K) - m(K).$$

Applying Corollary 3.5.9 again, this shows that for some constant C, we have that

$$\int_{x\ge K}(h\,\log\frac{h}{m}+m-h) \le C\,\mathrm{e}^{-K/C}. \tag{3.14}$$

From Corollary 3.5.9 and Corollary 3.5.14, we note that on $x \le K$ there holds $\mathrm{e}^{-C\,K} \le \frac{q}{h} \le \mathrm{e}^{C\,K}$, at least provided that $t \ge C\,x$. As we will see soon, we will choose K logarithmic in $1/D$ which gives the assumption appearing in the statement of Theorem 9.

Now in the region $x \le K$, we can use Lemma 3.5.5 in exactly the same manner as what we did in Corollary 3.5.6, which yields that

$$\begin{aligned}\int(h\,\log\frac{h}{m}+m-h) &\le C\,\mathrm{e}^{C\,K}\int q\,\log\frac{q}{m} + C\,\mathrm{e}^{-K/C}\\ &\le C\,\mathrm{e}^{C\,K}\,D\,\log\frac{1}{D} + C\,\mathrm{e}^{-K/C}.\end{aligned}$$

Optimizing in K, we find that for some $\theta > 0$ (but $\theta < 1$ unfortunately),

$$\int h\,\log\frac{h}{m} \le C\,\left(D\,\log\frac{1}{D}\right)^{\theta}. \tag{3.15}$$

Now we recall that, as a simple consequence of Lemma 3.5.4, we have

$$\int h\,\log\frac{h}{m} = \int h\,\log\frac{h}{\mathrm{e}^{-x}}. \tag{3.16}$$

Therefore we now want to change back from h to q. This is the same process and leads to

$$\int q\,\log\frac{q}{\mathrm{e}^{-x}} \le C\,\left(\int h\,\log\frac{h}{\mathrm{e}^{-x}}\right)^{\theta}. \tag{3.17}$$

To finish the proof, we just need to combine (3.17) with (3.16) and (3.15). $\square$

We end this section with a numerical experiment demonstrating the entropic convergence of q to q_∞, see Figure 5.5.

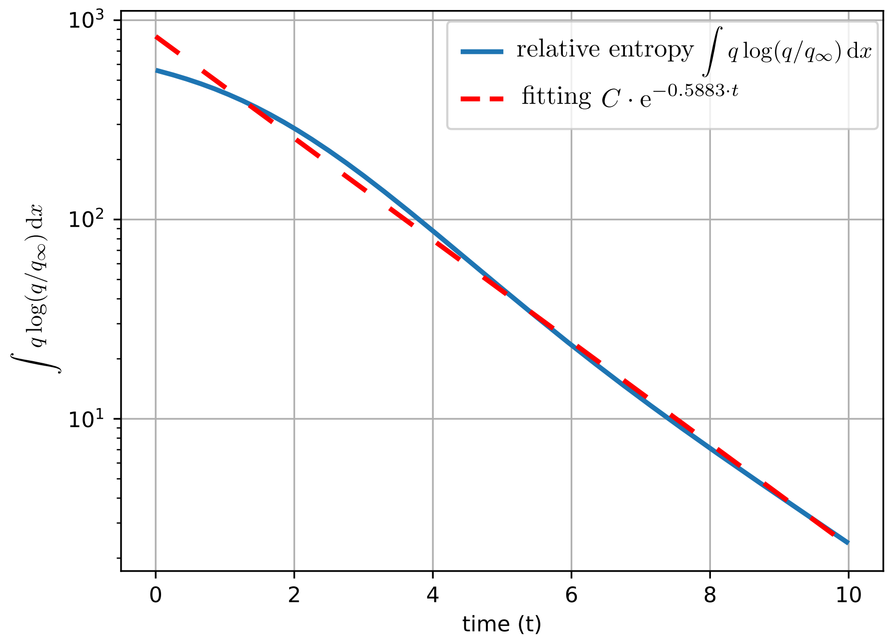

Figure 3.8: Simulation of the Relative Entropy from q to q_∞ after $t = 10$ in the Semilogy Scale. We Employed Forward Euler Method with Time Step-Size $\Delta t = 0.05$, Space Step-Size $\Delta x = 0.01$, and a "Random" Initial Condition $q(t = 0, x)$ Having Mean Value $m_1 = 5$ for the Numerical Simulation of (4.6). This Experiment Suggests That the Relaxation of $\int q \log(q/q_\infty)\,\mathrm{d}x$ Might Be Exponentially Fast in Time, Instead of Polynomially Fast in Time as Guaranteed By Theorem 9.

3.6 Propagation of Chaos

We give the statement of the propagation of chaos, Theorem 19 in 3.6.1. A technical lemma that will be employed in the proof of Theorem 19 is displayed in 3.6.2. We reveal the full proof of Theorem 19 in 3.6.3.

3.6.1 Statement of Propagation of Chaos

In this section, we try to adapt the martingale-based techniques developed in Merle and Salez [118], Hermon and Salez [91] to justify the propagation of chaos

Sznitman [145]. For this purpose, we equip the space $\mathcal{P}(\mathbb{R}_+)$ with the Wasserstein distance with exponent 1, which is defined via

$$W_1(\mu, \nu) = \sup_{\|\nabla \varphi\|_\infty \leq 1} \langle \mu - \nu, \varphi \rangle$$

for $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$. We will also need the following version of Itô's formula.

Lemma 3.6.1 *Consider an inhomogeneous Poisson process* N_t *with intensity* $\lambda(t)$*, and a random variable* $Y(t)$ *left-continuous and adapted to the filtration* $\mathcal{F}_t$ *generated by* N_t*. We define the compound jump process* $Z(t)$ *and* $M(t)$ *its associated compensated martingale by:*

$$\mathrm{d}Z(t) = Y(t)\,\mathrm{dN}_t, \quad M(t) = Z(t) - Z(0) - \int_0^t \tilde{Y}(s)\,\lambda(s)\,\mathrm{d}s, \tag{3.1}$$

where $\tilde{Y}$ *is any other left-continuous and adapted process. Itô's lemma then implies that for any* C^1 *function* Φ*,*

$$\mathrm{d}\mathbb{E}[\Phi(M(t))] = \mathbb{E}\left[\Phi\left(M(t-) + Y(t)\right) - \Phi(M(t-))\right]\lambda(t)\,\mathrm{d}t - \mathbb{E}[\nabla\Phi'(M(t))\cdot\tilde{Y}(t)\,\lambda(t)]\mathrm{d}t. \tag{3.2}$$

Our main result in this section is stated as follows.

Theorem 10 *Denote the empirical distribution of the uniform reshuffling stochastic system* (3.1) *at time* t *as*

$$\rho_{\mathrm{emp}}(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)},$$

and let $q(t)$ *be the solution of* (4.6) *with initial condition* $q(0)$*. If*

$$\mathbb{E}[W_1(\rho_{\mathrm{emp}}(0), q(0))] \to 0 \text{ as } N \to \infty, \tag{3.3}$$

then we have that

$$\mathbb{E}[W_1(\rho_{\mathrm{emp}}(t), q(t))] \to 0 \text{ as } N \to \infty,$$

holding for all $0 \leq t \leq T$ *with any prefixed* $T > 0$*.*

3.6.2 Switching Supremum and Expectation

We will also make use of the following result, which allows us to interchange the operation of supremum and of expectation.

Lemma 3.6.2 *Consider a random Radon measure Z on $\mathbb{R}$ with $\int Z(\mathrm{d}x) = 0$ and with uniformly bounded second moment $\int (1+|x|^2)\,|Z|(\mathrm{d}x) \leq m_2$ almost surely for some constant m_2. Then there exists $\theta > 0$ such that*

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \leq 1} \int \varphi\,\mathrm{d}Z\right] \leq C\,m_2 \left(\sup_{\|\nabla\varphi\|_\infty \leq 1} \mathbb{E}\left[\int \varphi\,\mathrm{d}Z\right]^2\right)^\theta.$$

Proof This is essentially an interpolation argument. First of all, we can always assume that $\varphi(0) = 0$ by subtracting a constant. Introduce a classical convolution kernel K_ε. We have that $\|K_\varepsilon \star \varphi - \varphi\|_{L^\infty} \leq C\,\varepsilon$ which implies that

$$\int \varphi\,Z(dx) \leq \int K_\varepsilon \star \varphi\,Z(\mathrm{d}x) + C\,\varepsilon.$$

Then we reduce ourselves to a compact support: since $\|\nabla\varphi\|_\infty \leq 1$ then $|\varphi(x)| \leq |x|$ and

$$\begin{aligned}\int K_\varepsilon \star \varphi\,Z(\mathrm{d}x) &\leq \int_{|x|\leq R} K_\varepsilon \star \varphi\,Z(\mathrm{d}x) + 2\int_{|x|\geq R} |x|\,|Z|(\mathrm{d}x)\\ &\leq \int_{|x|\leq R} K_\varepsilon \star \varphi\,Z(\mathrm{d}x) + 2\,\frac{m_2}{R}.\end{aligned}$$

On $[-R, R]$, we have on the other hand that $\|K_\varepsilon \star \varphi\|_{H^2} \leq \frac{C}{\varepsilon}\,\|\varphi\|_{W^{1,\infty}} \leq C\,\frac{R}{\varepsilon}$. Hence

$$\sup_{\|\nabla\varphi\|_\infty \leq 1} \int \varphi\,Z(\mathrm{d}x) \leq C\,\frac{R}{\varepsilon} \sup_{\|\varphi\|_{H^2} \leq 1} \int_{|x|\leq R} \varphi\,Z(\mathrm{d}x) + C\,m_2\left(\varepsilon + \frac{1}{R}\right).$$

Of course

$$\sup_{\|\varphi\|_{H^2} \leq 1} \int_{|x|\leq R} \varphi\,Z(\mathrm{d}x) = \|Z\|_{H^{-2}([-R,R])},$$

and by using Fourier series

$$\|Z\|^2_{H^{-2}([-R,\ R])} = \sum_k \frac{R^2}{1+k^4}\left(\int_{-R}^{R} \mathrm{e}^{-i\,k\,\pi\,x/R}\,\mathrm{d}Z\right)^2.$$

Hence by Cauchy-Schwartz,

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int \varphi\, Z(\mathrm{d}x)\right] \le C\, m_2\,\left(\varepsilon+\frac{1}{R}\right)$$
$$+\,C\,\frac{R^2}{\varepsilon}\,\left(\sum_k \frac{1}{1+k^4}\,\mathbb{E}\left[\left(\int_{-R}^{R} \mathrm{e}^{-i\,k\,\pi\,x/R}\,\mathrm{d}Z\right)^2\right]\right)^{1/2}.$$

Finally we have that

$$\|\nabla \mathrm{e}^{-i\,k\,\pi\,x/R}\|_\infty \le C\,k,$$

so that

$$\mathbb{E}\left[\left(\int_{-R}^{R} \mathrm{e}^{-i\,k\,\pi\,x/R}\,\mathrm{d}Z\right)^2\right] \le C\,k^2 \sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[\left(\int \varphi\,\mathrm{d}Z\right)^2\right].$$

This allows us to conclude that

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int \varphi\, Z(\mathrm{d}x)\right] \le C\, m_2\,\left(\varepsilon+\frac{1}{R}\right)$$
$$+\,C\,\frac{R^2}{\varepsilon}\,\left(\sum_k \frac{k^2}{1+k^4}\,\sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[\left(\int \varphi\,\mathrm{d}Z\right)^2\right]\right)^{1/2},$$

or

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int \varphi\, Z(\mathrm{d}x)\right] \le C\, m_2\,\left(\varepsilon+\frac{1}{R}\right)$$
$$+\,C\,\frac{R^2}{\varepsilon}\,\left(\sup_{\|\nabla\varphi\|_\infty\le 1}\mathbb{E}\left[\left(\int \varphi\,\mathrm{d}Z\right)^2\right]\right)^{1/2},$$

which finishes the proof by optimizing in R and ε. □

3.6.3 Proof of Propagation of Chaos

The proof of Theorem 19 occupies the rest of the section.

Proof We recall that the map $Q_+[\cdot]\colon \mathcal{P}(\mathbb{R}_+) \to \mathcal{P}(\mathbb{R}_+)$ is defined via

$$Q_+[q](x) = \int_0^\infty \int_0^\infty \frac{\mathbb{1}_{[0,k+\ell]}(x)}{k+\ell}\, q(k)q(\ell)\,\mathrm{d}k\,\mathrm{d}\ell,$$

and that a classical solution $q(t,x)$ of

$$q(t,x) = q(0,x) + \int_0^t G[q](s,x)\,\mathrm{d}s \tag{3.4}$$

exists for $0 \le t < \infty$, where $G = Q_+ - \mathrm{Id}$ and $q(0,x)$ is an continuous probability density function with mean m_1 whose support is contained in $\mathbb{R}_+$. The map Q_+ is Lipschitz continuous in the sense that

$$W_1(Q_+[f], Q_+[g]) \le W_1(f,g) \tag{3.5}$$

for any $f, g \in \mathcal{P}(\mathbb{R}_+)$. Indeed, we have

$$W_1(Q_+[f], Q_+[g]) = \sup_{\|\nabla\varphi\|_\infty \le 1} \mathbb{E}\left[\varphi(U(X_1+Y_1)) - \varphi(U(X_2+Y_2))\right],$$

where X_1, Y_1 are i.i.d with law f, X_2, Y_2 are i.i.d with law g, and $U \sim \mathrm{Uniform}[0,1]$ is independent of X_i and Y_i for $i = 1, 2$. By Lipschitz continuity of the test function φ, we obtain

$$W_1(Q_+[f], Q_+[g]) \le \mathbb{E}\left[2\,U|X_1 - X_2|\right] = \mathbb{E}[|X_1 - X_2|].$$

We now recall an alternative formulation of $W_1(f,g)$, given by

$$W_1(f,g) = \inf\left\{\mathbb{E}[|X-Y|];\ \mathrm{Law}(X) = f,\ \mathrm{Law}(Y) = g\right\},$$

so in particular, we may take a coupling of X_1 and X_2 so that $W_1(f,g) = \mathbb{E}[|X_1 - X_2|]$. Assembling these pieces together, we arrive at (D.3).

We are going to prove a more precise control than (D.3), by working directly on $Q_+[f]$. Consider now two random probability measures f and g with bounded second moment and a deterministic test function φ. We have that

$$\begin{aligned}\int \varphi(x)\,(Q_+[f] - Q_+[g])\,\mathrm{d}x &= \int \frac{\mathbb{1}_{x \le k+\ell}}{k+\ell}\varphi(x)\,(f(\mathrm{d}k) - g(\mathrm{d}k))\,(f(\mathrm{d}\ell) + g(\mathrm{d}\ell))\,\mathrm{d}x \\ &= \int (f(\mathrm{d}\ell) + g(\mathrm{d}\ell)) \int \Phi_\ell(k)\,(f(\mathrm{d}k) - g(\mathrm{d}k)),\end{aligned}$$

where we denote

$$\Phi_\ell(k) = \frac{1}{k+\ell}\int_0^{k+\ell} \varphi(x)\,\mathrm{d}x.$$

Since $\int Q_+[f] = \int Q_+[g]$, we can always assume without loss of generality that $\varphi(0) = 0$, whence $|\varphi(x)| \leq \|\nabla\varphi\|_\infty\,|x| \leq |x|$. Now we observe that Φ_ℓ is deterministic with

$$|\partial_k \Phi_\ell(k)| \leq \frac{|\varphi(k+\ell)|}{k+\ell} + \frac{1}{(k+\ell)^2}\int_0^{k+\ell} |\varphi(x)|\,\mathrm{d}x \leq 1 + \frac{1}{(k+\ell)^2}\int_0^{k+\ell} x\,\mathrm{d}x \leq \frac{3}{2}. \quad (3.6)$$

By (3.6) and recalling that again Φ_ℓ is deterministic and obtained from φ, we obtain:

$$\mathbb{E}\left[\int \Phi_\ell(k)\,(f(\mathrm{d}k) - g(\mathrm{d}k))\right] \leq \frac{3}{2}\,\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \leq 1} \int \varphi(x)\,(f(\mathrm{d}x) - g(\mathrm{d}x))\right].$$

Therefore we conclude that

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \leq 1} \int \varphi(x)\,(Q_+[f] - Q_+[g])\right] \leq 3\,\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \leq 1} \int \varphi(x)\,(f(\mathrm{d}x) - g(\mathrm{d}x))\right]. \quad (3.7)$$

We now observe that the empirical measure is a compound jump process: Define N_t a homogeneous Poisson process with constant intensity $\lambda = (N-1)/2$. Given $\tau_1, \ldots, \tau_k$ the times when N_t jumps, we take the Y_{τ_k} independent: At each τ_k, with uniform probability $\frac{2}{N\,(N-1)}$ we choose a pair $i < j$ and take

$$\begin{aligned} Y_{\tau_k} =& \frac{1}{N}\Big(\delta(x - U_k\,(X_i(\tau_k-) + X_j(\tau_k-)) + \delta(x - (1-U_k)\,(X_i(\tau_k-) + X_j(\tau_k-)) \\ &- \delta(x - X_i(\tau_k-)) - \delta(x - X_j(\tau_k-))\Big), \end{aligned}$$

where the U_k are i.i.d. in $[0,1]$.

We immediately note that

$$\begin{aligned} \lambda\,\mathbb{E}[Y_t] = \frac{1}{N^2} \sum_{i<j} \mathbb{E}\Big[&\delta(x - U\,(X_i(t-) + X_j(t-)) \\ &+ \delta(x - (1-U)\,(X_i(t-) + X_j(t-)) - \delta(x - X_i(t-)) - \delta(x - X_j(t-))\Big], \end{aligned} \quad (3.8)$$

where U is uniformly distributed in $[0,1]$ and independent of all $X_i(t-)$.

We also remark that by the standard control of moments, we immediately have that

$$\int x^2\,\rho_{\mathrm{emp}}(t, \mathrm{d}x) \leq m_2 = 2 + \int x^2\,\rho_{\mathrm{emp}}(0, \mathrm{d}x). \quad (3.9)$$

We now show that the empirical measure of the stochastic system satisfies an approximate version of (D.2). Fix a deterministic test function φ with $\|\nabla\varphi\|_\infty \le 1$, and consider the time evolution of $\langle \rho_{\mathrm{emp}}, \varphi\rangle$ where for some probability measure ν, we denote by the duality bracket $\langle \nu, \varphi\rangle = \int \varphi \,\mathrm{d}\nu$. We emphasize here that φ can also be random and will indeed be chosen according to ρ_{emp} to estimate Wasserstein distances involving ρ_{emp}. Then

$$\mathrm{d}\mathbb{E}[\langle \rho_{\mathrm{emp}}, \varphi\rangle] = \mathrm{d}\mathbb{E}\left[\langle Y_t \,\mathrm{d}N_t, \varphi\rangle\right] = \lambda \,\langle \mathbb{E}[Y_t], \varphi\rangle \,\mathrm{d}t.$$

Hence by (D.5),

$$\begin{aligned}
\mathrm{d}\mathbb{E}[\langle \rho_{\mathrm{emp}}, \varphi\rangle] &= \frac{1}{N^2} \sum_{i<j} \mathbb{E}\left[\varphi\Big(U(X_i + X_j)\Big) + \varphi\Big((1-U)(X_i + X_j)\Big) - \varphi(X_i) - \varphi(X_j)\right] \mathrm{d}t \\
&= \frac{1}{N^2} \sum_{i,j=1\ldots N, i\neq j} \mathbb{E}\left[\varphi\Big(U(X_i + X_j)\Big) - \varphi(X_i)\right] \mathrm{d}t \\
&= \frac{1}{N^2} \sum_{i,j=1}^{N} \mathbb{E}\left[\varphi\Big(U(X_i + X_j)\Big) - \varphi(X_i)\right] \mathrm{d}t + R\,\mathrm{d}t,
\end{aligned}$$

where all X_i, X_j are taken at time $t-$ and where $R = -\frac{1}{N^2} \sum_i \mathbb{E}\left[\varphi\Big(2\,U\,X_i\Big) - \varphi(X_i)\right]$. Hence $|R| \le \mathcal{O}\left(\frac{1}{N}\right)$ uniformly over φ and $t \ge 0$. On the other hand, we may calculate

$$\langle Q_+[\rho_{\mathrm{emp}}], \varphi\rangle = \frac{1}{N^2} \sum_{i,j} \int \varphi(x)\, \frac{\mathbb{1}_{x \le X_i + X_j}}{X_i + X_j} \,\mathrm{d}x = \frac{1}{N^2} \sum_{i,j} \int_0^1 \varphi(u\,(X_i + X_j))\,\mathrm{d}u,$$

by the change of variables $x = u\,(X_i + X_j)$. Therefore

$$\mathrm{d}\mathbb{E}[\langle \rho_{\mathrm{emp}}, \varphi\rangle] = \mathbb{E}\left[\langle G[\rho_{\mathrm{emp}}], \varphi\rangle\right] \,\mathrm{d}t + R\,\mathrm{d}t. \tag{3.10}$$

By Dynkin's formula, the compensated process

$$M_\varphi(t) := \langle \rho_{\mathrm{emp}}(t), \varphi\rangle - \langle \rho_{\mathrm{emp}}(0), \varphi\rangle - \int_0^t \left(\mathbb{E}[\langle G[\rho_{\mathrm{emp}}(s)], \varphi\rangle] + R(s)\right)\,\mathrm{d}s \tag{3.11}$$

is a martingale. Furthermore, comparing with (D.2), we easily obtain that

$$\begin{aligned}
\langle \rho_{\mathrm{emp}}(t) - q(t), \varphi\rangle &= M_\varphi(t) + \langle \rho_{\mathrm{emp}}(0) - q(0), \varphi\rangle \\
&\quad + \mathbb{E}\int_0^t \langle G[\rho_{\mathrm{emp}}(s)] - G[q(s)], \varphi\rangle \,\mathrm{d}s + \mathcal{O}\left(\frac{t}{N}\right).
\end{aligned}$$

Taking the supremum over φ, we therefore have that

$$\begin{aligned}\mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} \langle \rho_{\mathrm{emp}}(t) - q(t), \varphi \rangle &\le \mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} \left(|M_\varphi(t)| + \langle \rho_{\mathrm{emp}}(0) - q(0), \varphi \rangle \right) \\ &+ \int_0^t \mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} \langle G[\rho_{\mathrm{emp}}(s)] - G[q(s)], \varphi \rangle \, \mathrm{d}s + \mathcal{O}\left(\frac{t}{N} \right).\end{aligned}$$

By the definition of the W_1 distance, we deduce from (D.4) that

$$\mathbb{E}\, W_1(\rho_{\mathrm{emp}}(t), q(t)) \le \eta(t) + C \int_0^t \mathbb{E}\, W_1(\rho_{\mathrm{emp}}(t), q(t)) \, \mathrm{d}s + \frac{C\, t}{N},$$

in which we have set

$$\eta(t) := \mathbb{E} \sup_{\|\nabla\varphi\|_\infty \le 1} |M_\varphi(t)| + \mathbb{E}\, W_1(\rho_{\mathrm{emp}}(0), q(0)). \tag{3.12}$$

Thus, Gronwall's inequality gives rise to

$$\mathbb{E}\, W_1(\rho_{\mathrm{emp}}(t), q(t)) \le \left(\sup_{t \in [0,T]} \eta(t) + \frac{C\, T}{N} \right) \mathrm{e}^{C\, T}. \tag{3.13}$$

In order to establish propagation of chaos for $t \le T$, it therefore suffices to show that

$$\sup_{t \in [0,T]} \eta(t) \xrightarrow{N \to \infty} 0. \tag{3.14}$$

To prove (D.10), we treat each term appearing in the definition of $\eta(t)$ separately. The second term in (D.8) approaches to 0 as $N \to \infty$ by our assumption.

To handle the first term, let us write $Z(t) = \langle \rho_{\mathrm{emp}}(t), \varphi \rangle$ and $M(t) = M_\varphi(t)$ for notation simplicity. Of course $Z(t)$ is a compound jump process itself and by combining (D.6) and (D.7)

$$M_\varphi(t) = Z(t) - Z(0) - \int_0^t \tilde{Y}(s)\, ds, \quad \tilde{Y}(t) = \langle G[\rho_{\mathrm{emp}}(t)], \varphi \rangle + R.$$

We may hence use Itô's lemma as stated in Lemma 3.6.1, which yields

$$\mathrm{d}\mathbb{E}[M^2(t)] = \sum_{i<j} \mathbb{E}\left[M_{ij}^2(t) - M^2(t) \right] \frac{\mathrm{d}t}{N} - \mathbb{E}\left[2\, M(t) \, \langle G[\rho_{\mathrm{emp}}(t)], \varphi \rangle \right] \mathrm{d}t + \mathcal{O}\left(\frac{1}{N} \right) \mathrm{d}t,$$

where $M_{ij} = M + Y_{ij}$ and we define

$$Y_{ij} := \Big\langle \frac{1}{N}(\delta_{U_k(X_i+X_j)} + \delta_{(1-U_k)\,(X_i+X_j)} - \delta_{X_i} - \delta_{X_j}), \varphi \Big\rangle.$$

Therefore, we have

$$\begin{aligned} \mathrm{d}\mathbb{E}[M^2(t)] = &\sum_{i<j} \mathbb{E}\left[2\, M(t)\, Y_{ij} + Y_{ij}^2\right] \frac{\mathrm{d}t}{N} - \mathbb{E}\left[2\, M(t)\, \langle G[\rho_{\mathrm{emp}}(t)], \varphi\rangle\right] \mathrm{d}t \\ &+ \mathcal{O}\left(\frac{1}{N}\right) \mathrm{d}t. \end{aligned}$$

By our previous calculations

$$\begin{aligned} &\frac{1}{N} \sum_{i<j} \mathbb{E}[M(t)\, Y_{ij}] \\ &\quad = \frac{1}{N^2} \sum_{i<j} \mathbb{E}[M(t)\, (\varphi(U\,(X_i+X_j) + \varphi((1-U)\,(X_i+X_j) - \varphi(X_i) - \varphi(X_j))] \\ &\quad = \frac{1}{N^2} \sum_{i\neq j} \mathbb{E}[M(t)\, (\varphi(U\,(X_i+X_j)) - \varphi(X_i))] \\ &\quad = \frac{1}{N^2} \sum_{i,j} \mathbb{E}[M(t)\, (\varphi(U\,(X_i+X_j)) - \varphi(X_i))] + O\left(\frac{1}{N}\right), \end{aligned}$$

as U is random variable independent of $M(t)$ and $\rho_{emp}(t)$.

Therefore

$$\frac{1}{N} \sum_{i<j} \mathbb{E}[M(t)\, Y_{ij}] = \mathbb{E}\left[M(t)\, \langle G[\rho_{\mathrm{emp}}(t)], \varphi\rangle\right] + O\left(\frac{1}{N}\right),$$

and consequently

$$\mathrm{d}\mathbb{E}[M^2(t)] = \sum_{i<j} \mathbb{E}\left[Y_{ij}^2\right] \frac{\mathrm{d}t}{N} + \mathcal{O}\left(\frac{1}{N}\right) \mathrm{d}t \leq \frac{C}{N} \mathrm{d}t,$$

for a constant C that depends only on $\|\nabla\varphi\|_\infty$. This lets us deduce that

$$\sup_{\|\nabla\varphi\|_\infty \leq 1} \mathbb{E}\left[M_\varphi(t)\right] \leq \frac{C\,t}{N}.$$

Recalling the definition of $M_\varphi(t)$, we have that

$$M_\varphi(t) = \int \varphi(x)\, \mu(t, \mathrm{d}x)$$

for some random Radon measure μ with uniformly bounded second moment. Furthermore $\int \mu(t, \mathrm{d}x) = 0$ since $\int \rho_{\mathrm{emp}}(t, \mathrm{d}x) = 1 = \int \rho_{\mathrm{emp}}(0, \mathrm{d}x)$ and $\int G[\rho_{\mathrm{emp}}(t)]\, \mathrm{d}x = 0$. We may hence apply Lemma 3.6.2 to obtain that

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty \leq 1} M_\varphi(t)\right] \leq C\, \frac{t^\theta}{N^\theta},$$

which allows to conclude that $\sup_{t\in[0,T]} \eta(t) \xrightarrow{N\to\infty} 0.$ $\square$

Remark. One can readily check that

$$\|Q_+[f] - Q_+[g]\|_{L^1(\mathbb{R}_+)} \leq 2\, \|f - g\|_{L^1(\mathbb{R}_+)}$$

for all probability densities f, g whose support are contained in $\mathbb{R}_+$, but as we are working on $\mathcal{P}(\mathbb{R}_+)$, we can not use any strong distances. Hence, equipping $\mathcal{P}(\mathbb{R}_+)$ with an appropriate distance so that the operator Q_+ has enjoys a Lipschitz continuity with respect to the chosen distance is an indispensable step to make the argument above work.

Chapter 4

EXPLICIT DECAY RATE FOR THE GINI INDEX IN THE REPEATED AVERAGING MODEL

Chapter 4 is the pre-print Cao [37] submitted to *Mathematical Methods in the Applied Sciences*.

4.1 Abstract

The repeated averaging model for money exchanges is investigated, in which two agents picked uniformly at random share half of their wealth to each other. It is intuitively convincing that a Dirac distribution of wealth (centered at the initial average wealth) will be the long time equilibrium for this dynamics. In other words, the Gini index should converge to zero. To better understand this dynamics, it is possible to investigate its limit as the number of agents goes to infinity by proving the so-called propagation of chaos, which links the stochastic agent-based dynamics to a (limiting) nonlinear partial differential equation (PDE). This deterministic description has a flavor of the classical Boltzmann equation arising from statistical mechanics of dilute gases. Its convergence toward a Dirac equilibrium distribution can be proved by showing that the associated Gini index of the wealth distribution converges to zero with an explicit rate.

4.2 Introduction

Econophysics is an emerging branch of statistical physics that apply concepts and techniques of traditional physics to economics and finance Savoiu [140], Chatterjee *et al.* [57], Dragulescu and Yakovenko [79]. It has attracted considerable attention in recent years raising challenges on how various economical phenomena could be explained by universal laws in statistical physics, and we refer to Chakraborti *et al.* [53, 54], Pereira *et al.* [125], Kutner *et al.* [103] for a general review.

The primary motivation for study models arising from econophysics is at least two-fold: from the perspective of a policy maker, it is important to deal with the raise of income inequality Dabla-Norris *et al.* [66], De Haan and Sturm [70] in order to establish a more egalitarian society. From a mathematical point of view, we have to understand the fundamental mechanisms, such as money exchange resulting from individuals, which are usually agent-based models. Given an agent-based model, one is expected to identify the limit dynamics as the number of individuals tends to infinity and then its corresponding equilibrium when run the model for a sufficiently long time (if there is one), and this guiding approach is carried out in numerous works across different fields among literatures of applied mathematics, see for instance Naldi *et al.* [122], Barbaro and Degond [19], Carlen *et al.* [42].

In this work, we consider the so-called repeated averaging model for money exchange in a closed economic system with N agents. The dynamics consists in choosing at random time two individuals and to redistribute equally their combined wealth. To write this dynamics mathematically, we denote by $X_i(t)$ the amount of dollar agent i has at time t for $1 \leq i \leq N$. At a random time generated by a Poisson clock with rate N, two agents (say i and j) update their wealth according to the following rule:

$$\left(X_i, X_j\right) \rightsquigarrow \left(\frac{X_i+X_j}{2}\,, \frac{X_i+X_j}{2}\right), \tag{4.1}$$

Despite of the simplicity of the model, there are actually quite a few manuscripts in the literature which are solely dedicated to it. To the best of our knowledge, the first systematic treatment of this model is carried out by David Aldous Aldous and Lanoue [7], which is followed up by a very recent study presented in Chatterjee *et al.*

[58]. In both works, the focus is related to the estimation of the so-called mixing times and hence the targeted audiences are mathematicians from the Markov chain mixing time community. In this manuscript, we intend to give a kinetic theory perspective of the model. Indeed, under the large population $N \to \infty$ limit, We can rigorously show that the law of the wealth of a typical agent (say X_1) satisfies the following limit PDE in a weak sense:

$$\partial_t \rho(t,x) = 2(\rho * \rho)(t,2x) - \rho(t,x). \tag{4.2}$$

Once the limit PDE is identified from the interacting particle system, the natural next step is to study the problem of convergence to equilibrium of the PDE at hand. In the present work, we demonstrate that the Gini index of $\rho(t)$ converges to 0 (its minimum value), whence showing that a Dirac distribution centered at the initial average wealth is the equilibrium distribution. Moreover, this model can be served as the first example for which quantitative estimates on the convergence of Gini index can be obtained, which is our primary motivation for writing this paper. An illustration of the general strategy used in this work is shown in Figure 4.1.

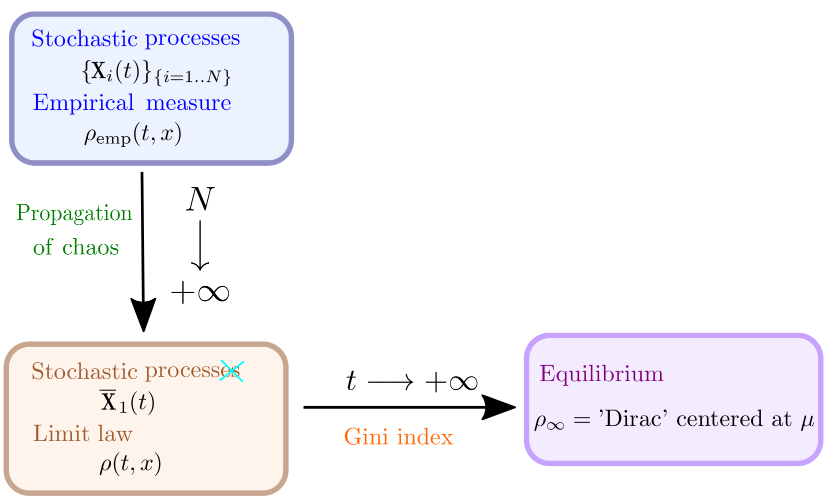

Figure 4.1: Schematic Illustration of the General Strategy of Our Treatment of the Repeated Averaging Dynamics, Where μ Represents the Initial Average Wealth.

Although only a very specific binary exchange model is explored in the present paper, other exchange rules can also be imposed and studied, leading to different models. To name a few, the so-called immediate exchange model introduced in Heinsalu and Patriarca [90] assumes that pairs of agents are randomly and uniformly picked at each random time, and each of the agents transfer a random fraction of its money to the other agents, where these fractions are independent and uniformly distributed in $[0, 1]$. The so-called uniform reshuffling model investigated in Dragulescu and Yakovenko [79], Lanchier and Reed [109], Cao *et al.* [39] suggests that the total amount of money of two randomly and uniformly picked agents possess before interaction is uniformly redistributed among the two agents after interaction. For models with saving propensity and with debts, we refer the readers to Chakraborti and Chakrabarti [52], Chatterjee *et al.* [56] and Lanchier and Reed [110]. Also, one can also modulate the rule of picking agents, leading to biased models of money exchange, see for instance Cao and Motsch [40].

This manuscript is organized as follows: in section 4.3, we briefly discuss the heuristic derivation of the limit equation (4.2) and give convergence results for the solution of (4.2) in terms of variance of the distribution as well as the Gini index. Finally, we draw a conclusion in section 4.4 and present a rigorous treatment of the propagation of chaos phenomenon in Appendix D, by applying the martingale-based technique employed in Cao *et al.* [39].

4.3 Convergence to Dirac Distribution

We present a heuristic argument behind the derivation of the limit PDE (4.2) arising from the repeated averaging dynamics in section 5.3.1, we also record a useful stochastic representation behind the PDE (4.2) on which we will heavily rely. Section 5.3.2 is devoted to the exponential decay of the variance of the solution $\rho(t)$ of (4.2). In section 5.3.3, we establish a quantitative convergence result on the Gini index of the probability distribution $\rho(t)$, by enforcing the log-concavity property of the initial datum.

4.3.1 Formal Derivation of the Limit Equation

Introducing $\mathrm{N}_t^{(i,j)}$ independent Poisson processes with intensity $1/N$, the dynamics can be written as:

$$\mathrm{d}X_i(t) = \sum_{j=1..N, j\neq i} \left(\frac{X_i(t-)+X_j(t-)}{2} - X_i(t-) \right) \mathrm{dN}_t^{(i,j)}. \tag{4.1}$$

As the number of players N goes to infinity, one could expect that the processes $X_i(t)$ become independent and of same law. Therefore, the limit dynamics would be of the form:

$$\mathrm{d}\overline{X}(t) = \left(\frac{\overline{X}(t-)+\overline{Y}(t-)}{2} - \overline{X}(t-) \right) \mathrm{d}\overline{\mathrm{N}}_t, \tag{4.2}$$

where $\overline{Y}(t)$ is an independent copy of $\overline{X}(t)$ and $\overline{\mathrm{N}}_t$ a Poisson process with intensity 1. Taking a test function φ, the weak formulation of the dynamics is given by:

$$\mathrm{d}\mathbb{E}[\varphi(\overline{X}(t))] = \mathbb{E}\left[\varphi\left(\frac{\overline{X}(t)+\overline{Y}(t)}{2} \right) - \varphi(\overline{X}(t)) \right] \mathrm{d}t. \tag{4.3}$$

In short, the limit dynamics correspond to the jump process:

$$\overline{X} \rightsquigarrow \frac{\overline{X}+\overline{Y}}{2}. \tag{4.4}$$

Let us denote $\rho(t,x)$ the law of the process $\overline{X}(t)$. To derive the evolution equation for $\rho(t,x)$, we need to translate the effect of the jump of $\overline{X}(t)$ via (4.4) onto $\rho(t,x)$.

Lemma 4.3.1 *Suppose X and Y two independent random variables with probability density $\rho(x)$ supported on $[0,\infty)$. Let $Z = (X+Y)/2$, then the law of Z is given by $Q_+[\rho]$ with:*

$$Q_+[\rho](x) = 2(\rho * \rho)(2x), \quad \forall x \geq 0. \tag{4.5}$$

The proof of this lemma is quite elementary and will be omitted. We can now write the evolution equation for the law of $\overline{X}(t)$, the density $\rho(t,x)$ satisfies weakly:

$$\partial_t \rho(t,x) = \mathcal{L}[\rho](t,x) \qquad \text{for } t \geq 0 \ \text{ and } \ x \geq 0 \tag{4.6}$$

with

$$\mathcal{L}[\rho](x) := Q_+[\rho](x) - \rho(x) = 2(\rho * \rho)(2x) - \rho(x). \tag{4.7}$$

Remark. Suppose that $\rho(0,x)$ is a probability density on $[0,\infty)$ with mean $\mu > 0$. It is readily checked that the dynamics (4.6) preserves the total mass and the mean value. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \rho(t,x)\,\mathrm{d}x = 0 \ \text{ and } \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} x\,\rho(t,x)\,\mathrm{d}x = 0.$$

For each test function φ, one can show that $\int_{\mathbb{R}_+} \varphi(x)\,G[\delta_\mu](dx) = 0$, implying that the Dirac distribution centered at μ is a equilibrium solution of (4.6).

We now present a stochastic representation of the evolution equation (4.6), which is interesting in its own right.

Proposition 4.3.2 *Assume that $\rho_t(x) := \rho(t,x)$ is a solution of (4.6) with initial condition $\rho_0(x)$ being a probability density function supported on $\mathbb{R}_+$ with mean μ. Defining $(X_t)_{t\geq 0}$ to be a $\mathbb{R}_+$-valued continuous-time pure jump process with jumps of the form*

$$X_t \ \overset{\text{rate 1}}{\rightsquigarrow} \ \frac{X_t + Y_t}{2}, \tag{4.8}$$

where Y_t is a i.i.d. copy of X_t, and the jump occurs according to a Poisson clock running at the unit rate. If $\mathrm{Law}(X_0) = \rho_0$, then $\mathrm{Law}(X_t) = \rho_t$ for all $t \geq 0$.

Proof Taking φ to be an arbitrary but fixed test function, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\varphi(X_t)] = \mathbb{E}[\varphi((X_t+Y_t)/2)] - \mathbb{E}[\varphi(X_t)]. \tag{4.9}$$

Denoting $\rho(t,x)$ as the probability density function of X_t, (4.9) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\rho(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}_+^2}\varphi((k+\ell)/2)\rho(k,t)\rho(\ell,t)\,\mathrm{d}k\,\mathrm{d}\ell - \int_{\mathbb{R}_+}\rho(t,x)\varphi(x)\,\mathrm{d}x.$$

After a simple change of variables, one arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\rho(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}_+}\left(Q_+[\rho](x,t)-\rho(t,x)\right)\varphi(x)\,\mathrm{d}x.$$

Thus, ρ has to satisfy $\partial_t\rho = \mathcal{L}[\rho]$ and the proof is completed. □

4.3.2 Exponential Decay of the Variance

Our main goal in this subsection is the proof of the following

Theorem 11 *Assume that $\rho(t,x)$ is a classical solution of (4.6) for each $t>0$, with the initial condition $\rho(0,x)$ being a probability density on $[0,\infty)$ with mean $\mu>0$ and finite variance. Then the variance of ρ at time t, denoted by $\mathrm{V}(t)$, decays exponentially in time. More specifically, we have $\mathrm{V}(t)=\mathrm{V}(0)\,\mathrm{e}^{-\frac{1}{2}t}$.*

Proof Thanks to the conservation of the mean value, we have

$$\mathrm{V}(t) = \int_{\mathbb{R}_+} x^2\,\rho(t,x)\,dx - \mu^2.$$

Thus, we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{V}(t) &= 2\int_{\mathbb{R}_+} x^2\,(\rho*\rho)(2x)\,\mathrm{d}x - \int_{\mathbb{R}_+} x^2\,\rho(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}_+} 2\,x^2\left(\int_0^{2x}\rho(y)\,\rho(2x-y)\,\mathrm{d}y\right)\,\mathrm{d}x - \int_{\mathbb{R}_+} x^2\,\rho(x)\,\mathrm{d}x \\
&= \int_{y\geq 0}\rho(y)\left(\int_{x\geq y/2} 2\,x^2\,\rho(2x-y)\,\mathrm{d}x\right)\,\mathrm{d}y - \int_{\mathbb{R}_+} x^2\,\rho(x)\,\mathrm{d}x \\
&= \int_{y\geq 0}\rho(y)\left(\int_{z\geq 0}((y+z)/2)^2\,\rho(z)\,\mathrm{d}z\right)\,\mathrm{d}y - \int_{\mathbb{R}_+} x^2\,\rho(x)\,\mathrm{d}x \\
&= -\frac{1}{2}\left(\int_{\mathbb{R}_+} x^2\,\rho(t,x)\,dx - \mu^2\right) = -\frac{1}{2}\mathrm{V}(t).
\end{aligned}$$

A simple integration yields the advertised conclusion.

Remark. The proof of Theorem 11 can also be carried out from a purely stochastic point of view, by leveraging the stochastic representation of the PDE (4.6). Indeed, suppose that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are defined as in the statement of Proposition 4.3.2. Then we can calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{V}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}[X_t] = \mathrm{Var}[(X_t + Y_t)/2] - \mathrm{Var}[X_t] = -\frac{1}{2}\mathrm{Var}[X_t] = -\frac{1}{2}\mathrm{V}(t),$$

which leads us to the same result.

4.3.3 Exponential Decay of the Gini Index

The widely used inequality indicator Gini index G measures the inequality in the wealth distribution and ranges from 0 (no inequality) to 1 (extreme inequality). We recall the definition of G here for the reader's convenience.

Definition 13 *Given a probability density function ρ supported on $\mathbb{R}_+$ with mean value $\mu > 0$. The Gini index of ρ is given by*

$$G[\rho] = \frac{1}{2\mu} \iint_{\mathbb{R}_+^2} \rho(x)\,\rho(y)\,|x-y|\,\mathrm{d}x\,\mathrm{d}y.$$

Alternatively, we can also rewrite

$$G[\rho] = \frac{1}{2\mu}\mathbb{E}[|X-Y|],$$

in which X and Y are i.i.d. random variables with law ρ.

In econophysics literature, analytical results on Gini index are comparatively rare. In certain models, the Gini index can be shown to converge to 1, which implies the emergence of the ”rich-get-richer” phenomenon and the accentuation of the wealth inequality, see for instance Boghosian *et al.* [28, 29] and references therein. There is also a recently proposed model known as the rich-biased model Cao and Motsch [40], in which the authors observe a numerical evidence for the convergence of Gini index to its maximum possible value but analytical justification is still absent. As have been indicated earlier, the limit PDE (4.6) associated with the repeated averaging model can be served as the first example for which quantitative estimates on the behavior of Gini index can be hoped. We start with the following preliminary observation.

Proposition 4.3.3 *Assume that $\rho(t,x)$ is a classical solution of (4.6) for each $t>0$, with the initial condition $\rho(0,x)$ being a probability density on $[0,\infty)$ with mean $\mu>0$. Then the Gini index $G[\rho]$ is non-increasing in time. Moreover, we have*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}G[\rho] &= -\frac{1}{\mu}\iiint_{\mathbb{R}_+^3} \rho(v)\,\rho(w)\,\rho(y)\left(\frac{|v-y|+|w-y|}{2}-\left|\frac{v+w}{2}-y\right|\right)\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}y \\
&\leq 0.
\end{aligned}
\tag{4.10}
$$

Proof By symmetry, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}G[\rho] &= \frac{1}{\mu}\iint_{\mathbb{R}_+^2} \partial_t\rho(x)\,\rho(y)\,|x-y|\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\mu}\iint_{\mathbb{R}_+^2} 2\,(\rho*\rho)(2x)\,\rho(y)\,|x-y|\,\mathrm{d}x\,\mathrm{d}y - 2\,G[\rho] \\
&= \frac{1}{\mu}\iint_{\mathbb{R}_+^2} 2\left(\int_0^{2x}\rho(z)\,\rho(2x-z)\,\mathrm{d}z\right)\rho(y)\,|x-y|\,\mathrm{d}x\,\mathrm{d}y - 2\,G[\rho] \\
&= \frac{1}{\mu}\iiint_{\mathbb{R}_+^3} \rho(v)\,\rho(w)\,\rho(y)\,\left|\frac{v+w}{2}-y\right|\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}y - 2\,G[\rho] \\
&= -\frac{1}{\mu}\iiint_{\mathbb{R}_+^3} \rho(v)\,\rho(w)\,\rho(y)\left(\frac{|v-y|+|w-y|}{2}-\left|\frac{v+w}{2}-y\right|\right)\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}y,
\end{aligned}
$$

whence the proof is finished. $\square$

Remark. In light of the previous remark and stochastic representation of the PDE (4.6). We can also provide an alternative proof of Proposition 4.3.3. Indeed, suppose that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are defined as in the statement of Proposition 4.3.2. Then we can compute

$$\frac{\mathrm{d}}{\mathrm{d}t}G[\rho] = \frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|X_t - Y_t|] = \frac{1}{\mu}\mathbb{E}[|(X_t + Z_t)/2 - Y_t|] - \frac{1}{\mu}\mathbb{E}[|X_t - Y_t|]$$
$$= -\frac{1}{\mu}\left(\mathbb{E}[|X_t - Y_t|] - \mathbb{E}[|(X_t + Z_t)/2 - Y_t|]\right) \leq 0,$$

in which Z_t is a fresh i.i.d. copy of X_t (independent of Y_t as well). This coincides with (4.10)

At this point, we may expect to bound $G[\rho]$ in terms of $-\frac{\mathrm{d}}{\mathrm{d}t}G[\rho]$ in order to extract some information on the rate of decay of G. But unfortunately, inequalities of the form $-\frac{\mathrm{d}}{\mathrm{d}t}G[\rho] \geq c \cdot G[\rho]$ can not be always fulfilled. For example, if we take $\rho = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{2\mu}$, then one can check that $\frac{\mathrm{d}}{\mathrm{d}t}G[\rho] = 0$, whereas $G[\rho] = \frac{1}{2} > 0$. However, not all hope is lost. Indeed, if we restrict the initial data $\rho(0,x)$ to be log-concave, we can prove the following

Theorem 12 *Assume that $\rho(t,x)$ is a classical solution of* (4.6) *for each $t > 0$, with the initial condition $\rho(0,x)$ being a log-concave probability density on $[0,\infty)$ with mean $\mu > 0$. Then the Gini index $G[\rho]$ converges to 0 exponentially fast in time. Moreover, we have*

$$G[\rho(t)] \leq G[\rho(0)]\mathrm{e}^{-\frac{t}{14434}}. \tag{4.11}$$

To facilitate the proof of Theorem 12, we need the following

Lemma 4.3.4 *Assume that $\rho(t,x)$ is a classical solution of* (4.6) *for each $t > 0$, with the initial condition $\rho(0)$ being a log-concave probability density on $[0,\infty)$ with mean $\mu > 0$. Then $\rho(t)$ is again log-concave for each $t > 0$.*

Proof The proof is an immediate consequence of the stochastic representation of (4.6), together with the elementary fact that log-concavity is preserved by convolution. □

Remark. Preservation of log-concavity can also be established for other PDEs, although the proofs are usually quite involved. For instance, it is well-known that evolution under the one-dimensional heat equation preserves the log-concavity of the initial datum Brascamp and Lieb [33].

Proof of Theorem 12 For notational simplicity, we write

$$G := G[\rho] \quad \text{and} \quad H := -\frac{\mathrm{d}}{\mathrm{d}t} G[\rho]. \tag{4.12}$$

In fact, we will not need the restriction that the support of the distribution ρ is $[0, \infty)$.

By approximation, without loss of generality, we may assume that $\rho(x) > 0$ for all real x. For example, one may approximate ρ by its convolution $p * \phi$ with the density ϕ of a centered normal distribution with an arbitrarily small variance. Then $\rho * g > 0$ on $\mathbb{R}$ and $\rho * \phi$ is arbitrarily close to ρ and log-concave, thanks to the preservation of log-concavity by convolution.

As ρ is a log-concave density, ρ is continuous and attains its maximum value, say $\rho_*(> 0)$, at some point $c \in \mathbb{R}$, so that $\rho_* = \rho(c) \geq \rho(x)$ for all real x. Moreover, again because ρ is log-concave, there exist (unique) real a and b such that

$$a < c < b \quad \text{and} \quad \rho(a) = \rho(b) = \rho_*/\mathrm{e}.$$

We define

$$q(x) := \begin{cases} q_1(x) := \rho_* \exp\left\{-\dfrac{x-c}{a-c}\right\} & \text{if } x < a, \\[2ex] \rho_* & \text{if } a \leq x < b, \\[2ex] q_2(x) := \rho_* \exp\left\{-\dfrac{x-c}{b-c}\right\} & \text{if } x \geq b. \end{cases}$$

Thanks to the log-concavity of ρ again, we have $\rho(x) \leq q(x)$. We refer to Figure 4.2 for an illustration.

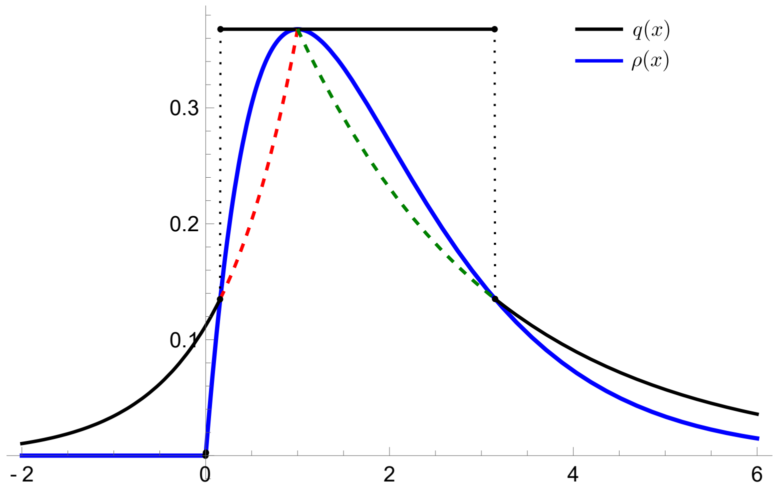

Figure 4.2: For $\rho(x) = x\,\mathrm{e}^{-x}\,\mathbb{1}_{\{x>0\}}$, Here Are the Graphs $\{(x, \rho(x)) \mid -2 \leq x \leq 6\}$ (blue), $\{(x, q(x)) \mid -2 \leq x \leq 6\}$ (Black), $\{(x, q_1(x)) \mid a \leq x \leq c\}$ (Dashed Red), and $\{(x, q_2(x)) \mid c \leq x \leq b\}$ (Dashed Green). For This Particular ρ, We Have $c = 1$, $a = -W_0\left(-1/\mathrm{e}^2\right) \approx 0.1586$, and $b = -W_{-1}\left(-1/\mathrm{e}^2\right) \approx 3.1461$, Where W_j Is the jth Branch of the Lambert W Function Lambert [105].

By shifting, we may assume with of loss of generality that $a = 0$. Thus,

$$\begin{aligned} G &\leq \iint\limits_{\mathbb{R}^2} q(x)\,q(y)\,|x-y|\,\mathrm{d}x\,\mathrm{d}y \\ &= \rho_*^2 \frac{\mathrm{e}^2 b^3 + 9\mathrm{e}b^3 + 3b^3 + 3b^2c - 12\mathrm{e}b^2c - 3bc^2 + 12\mathrm{e}bc^2}{3\mathrm{e}^2} \\ &\leq \rho_*^2 \frac{(1 + 3\mathrm{e} + \mathrm{e}^2/3)\,b^3}{\mathrm{e}^2}, \end{aligned}$$

since $0 < c < b$. Moreover, again by the log-concavity of p, we have $\rho \geq \rho_*/\mathrm{e}$ on the interval $[a, b] = [0, b]$, so that $1 = \int_{\mathbb{R}} \rho \geq \int_0^b \rho_*/\mathrm{e} = b\,\rho_*/\mathrm{e}$, whence $\rho_* \leq \mathrm{e}/b$ and

$$G \leq (1 + 3\mathrm{e} + \mathrm{e}^2/3)\,b. \tag{4.13}$$

On the other hand, because $\rho \geq \rho_*/\mathrm{e}$ on the interval $[a, b] = [0, b]$ and the integrand in the definition of H is non-negative, we have

$$\begin{aligned} H &\geq \left(\frac{p_*}{\mathrm{e}}\right)^3 \iiint\limits_{[0,b]^3} \left(\frac{|x-z| + |y-z|}{2} - \left| \frac{x-z+y-z}{2} \right| \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \left(\frac{\rho_*}{\mathrm{e}}\right)^3 \frac{b^4}{24}. \end{aligned}$$

Also, $1 = \int_{\mathbb{R}} \rho \leq \int_{\mathbb{R}} q = \rho_* b(1+1/\mathrm{e})$, so that $\rho_* \geq 1/(b(1+1/\mathrm{e}))$ and hence

$$H \geq \left(\frac{1}{(\mathrm{e}+1)b}\right)^3 \frac{b^4}{24} = \frac{b}{24(\mathrm{e}+1)^3}. \tag{4.14}$$

Comparing (4.13) and (4.14), we deduce

$$H \geq \frac{G}{24(\mathrm{e}+1)^3(1+3\mathrm{e}+\mathrm{e}^2/3)} \geq \frac{G}{14334},$$

as claimed. $\square$

Finally, we provide a numerical experiment in order to corroborate the relaxation of the Gini index guaranteed by Theorem 12, see Figure 4.3. For the initial condition, we use a gamma probability density with shape parameter $\mu = 5$ and rate parameter equal to unity, i.e., $\rho(0,x) = \mathbb{1}_{[0,\infty)}(x) \cdot x^{\mu-1}\,\mathrm{e}^{-x}/\Gamma(\mu)$. The standard forward Euler scheme (with the time step-size $\Delta t = 0.05$ and the space step-size $\Delta x = 0.01$) is enforced for the numerical solution of (4.6). Note that the Gini index of our choice of $\rho(0,x)$ has a nice closed expression $G[\rho(0)] = \frac{2^{1-2\mu}\,\Gamma(2\mu)}{\mu\,(\Gamma(\mu))^2}$, which reduces (approximately) to 0.2461 for $\mu = 5$.

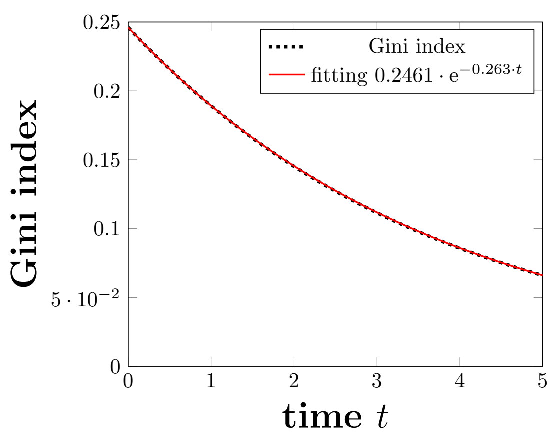

Figure 4.3: Evolution of the Gini Index of $\rho(t)$ (the Solution of (4.6)) for $0 \leq t \leq 5$, with the Initial Datum Being a Gamma Probability Density with Shape Parameter $\mu = 5$ and Rate Parameter Equal to Unity, i.e., $\rho(0,x) = \mathbb{1}_{[0,\infty)}(x) \cdot x^{\mu-1}\,\mathrm{e}^{-x}/\Gamma(\mu)$. The Black Dotted Line and the Red Smooth Curve Represent the Gini Index and Its Fitting Curve, Respectively. We Also Remark That in This Experiment These Two Curves Are Almost Indistinguishable.

4.4 Conclusion

In this manuscript, we have investigated the repeated averaging dynamics for money exchange originated from econophysics. Because of the model simplicity in its appearance, there is a comparative lack of mathematical literature that is purely dedicated to this model, although this model is a special case of the general dynamics studied in Matthes and Toscani [116]. We presented a propagation of chaos result, which links the stochastic N particle system to a deterministic nonlinear evolution equation. Although certain convergence results of the Gini index are obtained for other econophysics models, we emphasize that no quantitative estimates on the long time behavior of Gini index are available in the current literature (at least to our best knowledge). Thus, this toy model may serve as a starting point for more systematic, quantitative investigation of the large-time asymptotic of Gini index arising from other models. As a open conjecture, we speculate that the constant $1/14434$ appearing in the statement of Theorem 12 might be tremendously improved.

It would also be interesting to investigate the behavior of the Gini index for the stochastic agent-based model where the number of agents N is arbitrary but fixed. We believe that it would be relatively simple (in this setting) to demonstrate the convergence of the Gini index towards zero, but the difficulty arises when we want to obtain an explicit rate of the aforementioned convergence.

ACKNOWLEDGMENTS

It is a great pleasure to thank Iosif Pinelis for his answer to a question of myself on MathOverflow Pinelis [128], where the essential piece needed for a complete proof of Theorem 12 is presented. I would also want to express my gratitude to my Ph.D advisor Sebastien Motsch for his careful proofreading of the manuscript.

Chapter 5

K-AVERAGING AGENT-BASED MODEL: PROPAGATION OF CHAOS AND CONVERGENCE TO EQUILIBRIUM

Chapter 5 is the paper Cao [38] published on *Journal of Statistical Physics*.

5.1 Abstract

The paper treats an agent-based model with averaging dynamics which is referred to as the K-averaging model. Broadly speaking, this model can be added to the growing list of dynamics exhibiting self-organization such as the well-known Vicsek-type models Aldana and Huepe [4], Aldana *et al.* [3], Pimentel *et al.* [127]. In the K-averaging model, each of the N particles updates their position by averaging over K randomly selected particles with additional noise. To make the K-averaging dynamics more tractable, it is possible to establish a propagation of chaos type result in the limit of infinite particle number (i.e. $N \to \infty$) using a martingale technique. Then, it is possible to prove the convergence of the limit equation toward a suitable Gaussian distribution in the sense of Wasserstein distance as well as relative entropy. Additional numerical simulations are provided to illustrate both results.

5.2 Introduction

The collective behavior of various particle systems is a subject of intensive research that has potential applications in biology, physics, economics, and engineering Naldi *et al.* [122], Belmonte *et al.* [22], Chuang *et al.* [59]. Different models are proposed to study the emergence of flocking of birds, formation of consensus in opinion dynamics, and phase transitions in network models Motsch and Tadmor [121], Porfiri and Ariel [131], Chaté *et al.* [55], Barbaro and Degond [19]. Broadly speaking, all of the aforementioned models are instances of *interacting particle systems*, under various interaction rules among the particles. We refer the readers to Liggett [112] for a general introduction into this branch of applied mathematics.

In this work, we investigate a simple model to describe the collective alignment of a group of particles. The model we examine here can be classified in general as an *averaging dynamics* and will be referred to as the *K-averaging model*. The readers are encouraged to consult Carlen *et al.* [42], Bertin *et al.* [24, 25], Boissard *et al.* [30] for a variety of models in biology and physics in which averaging plays a key role in the model definition. One important inspiration for the present work is a paper of Maurizio Porfiri and Gil Ariel Porfiri and Ariel [131], which can be thought as a K-averaging model on the unit circle. In the K-averaging model considered in this manuscript, at each time step, we update the position of each particle (viewed as an element of $\mathbb{R}^d$) according to the average position of its K randomly chosen neighbors while being simultaneously subjected to additive noise (see equation (5.1)). Thus, we give the following definition.

Definition 14 *(K**-averaging model**) Consider a collection of stochastic processes $\{X_i^n\}_{1\leq i\leq N}$ evolving on $\mathbb{R}^d$, where n is the index for time. At each time step, each particle updates its value to the average of K randomly selected neighbors, subject to an independent noise term:*

$$X_i^{n+1} := \frac{1}{K}\sum_{j=1}^{K} X^n_{S_i^n(j)} + W_i^n, \quad 1 \leq i \leq N, \tag{5.1}$$

where $S_i^n(j)$ are indices taken randomly from the set $\{1,2,\ldots,N\}$ (i.e., $S_i^n(j) \sim \mathrm{Uniform}(\{1,2,\ldots,N\})$ and is independent of i,j and n), and $W_i^n \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbb{1}_d)$ is independent of i and n, in which $\mathbf{0}$ and $\mathbb{1}_d$ stands for the zero vector and the identity matrix in dimension d, respectively (see Figure 5.1 for a illustration).

We illustrate the dynamics in Figure 5.1. The key question of interest is the exploration of the limiting particle distribution as the total number of particles and the number of time steps become large. We illustrate numerically (see Figure 5.2) the evolution of the dynamics in dimension $d = 1$ using $N = 5,000$ particles after $n = 1000$ time steps. One of the main difficulty in the rigorous mathematical treatment

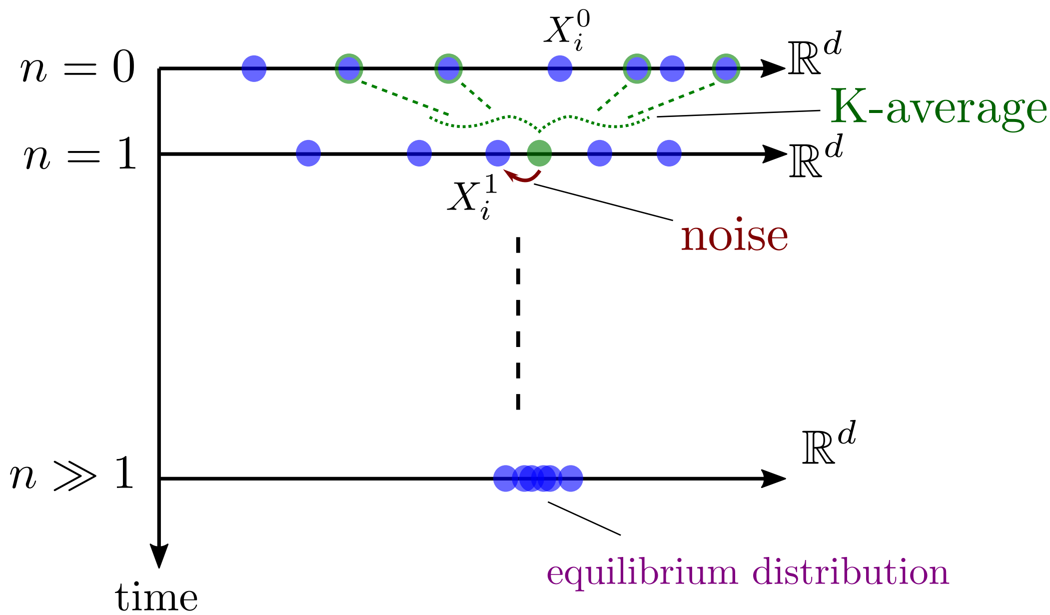

Figure 5.1: Sketch Illustration of the K-Averaging Dynamics (5.1). At Each Time Step, a Particle Updates Its Position By Taking the Averaging of K Randomly Selected Particles and Adding Some (Gaussian) Noise.

of models involving large number of interacting particles or agents lies in the general fact interaction will build up correlation over time. Fortunately the framework of kinetic theories allows possible simplification of the analysis of certain such models via suitable asymptotic analysis, see for instance Oelschlager [123], Méléard and Roelly-Coppoletta [119], Hauray and Jabin [89], Jabin and Wang [96], Merle and Salez [118], Sznitman [145]. For the model at hand, our main contribution is two fold: we first prove a result of propagation-of-chaos type under the large N limit (see Theorem 13 for a precise statement), in which interactions among particles are eliminated in finite time and a mean-field dynamics emerges. After the large population limit is carried

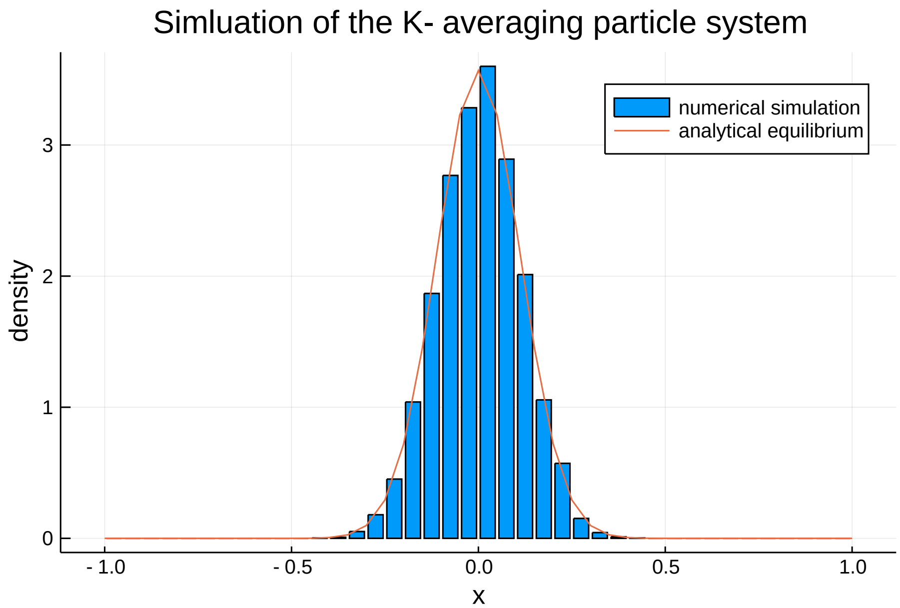

Figure 5.2: Simulation of the K-Averaging Dynamics in Dimension $d = 1$ with $K = 5$ and $N = 5000$ Particles after 1000 Time Steps, in Which We Used $\sigma = 0.1$ and Initially Each $X_i \sim \text{Uniform}(-1, 1)$. As To Be Shown Later, the Distribution of Particles Will Be Asymptotically Gaussian Under the Large N and Large Time Limits.

out and the simplified dynamics (see equation (5.5)) is obtained, we then show that the law of the limiting dynamics defined by (5.5) is asymptotically Gaussian under the large time limit, and such convergence of distribution occurs both in the Waasserstein distance (see Theorem 14) and in the sense of relative entropy (see Theorem 15). A schematic illustration of the strategy used in this manuscript is presented in Figure 5.3. We briefly explain the possible motivation of studying such a model (at least in dimension $d = 1$). In the context of a opinion dynamics model (see for instance Baumann *et al.* [21]), X_i^n may represent an evolving opinion of agent i at time step n. For a given event, agent i has a opinion X_i (which can be positive or negative) with strength $|X_i|$, and agents update their opinions based on the equation (5.1).

There remain some open questions related to our current work. First, our analysis of the model is restricted to $K \geq 2$, under which we are able to identify the equilibrium distribution and prove various results, we speculate that the propagation of chaos

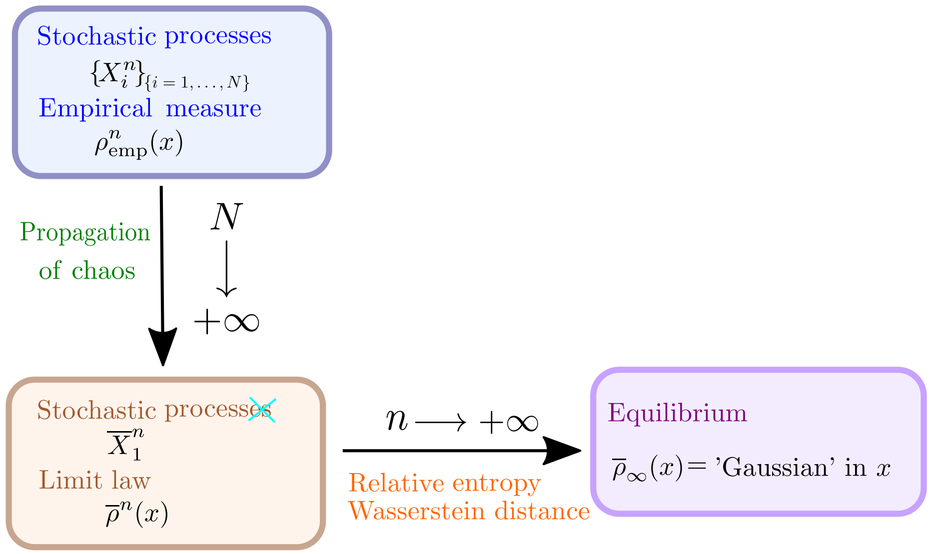

Figure 5.3: Schematic Illustration of the Limiting Procedure Carried Out for the Study of the K-Averaging Dynamics (5.1). The Empirical Measure $\rho^n_{\text{emp}}(\boldsymbol{x})$ of the System (See Equation (5.1)) Will Be Shown to Converge as $N \to \infty$ to Its Limit Law $\bar{\rho}^n$ Described By the Evolution Equation (5.10), and Then the Relaxation of $\bar{\rho}^n$ to Its Gaussian Equilibrium Will Be Established.

property will be lost if $K = 1$, yet we have not been able to find a perfect analytical justification. We also remark here that the case of $K = 1$ can be seen as a variant of the "Choose the Leader" (CL) dynamics introduced in Carlen *et al.* [42], in which each of the N particles decides to jump to the location of the other particle chosen independently and uniformly at random at every time step, though noise is injected in such a jump. Second, we think similar results can be obtained if the noise is no longer Gaussian, except that the equilibrium will not be explicit in general.

The remainder of the paper is organized as follows: In section 5.3.1, we present several preliminaries related to random probability measures and the concept of propagation of chaos. Sections 5.3.2 and 5.3.3 are concerned with the intuitive derivation of the simplified model (5.5) and related properties. We give a full proof of the propagation of chaos result in section 5.4 and the large time asymptotic of (5.5) is investigated in 5.5. We devote section 5.6 to the continuous-time counterpart of the K-averaging model studied in previous sections, and finish the paper with a conclusion in section 5.7.

5.3 K-Averaging Model

In section 5.3.1, we perform a brief review on convergence of random probability measures and the notion of propagation of chaos. Section 2.2 encapsulated a heuristic argument for the large N limit, and we prove a Lipschitz continuity property of the key operator T arising naturally from the K-averaging dynamics in section 5.3.3, which will be leveraged in the proof of Theorem 13.

5.3.1 Review Propagation of Chaos and Convergence of Random Measures

We devote this section to a quick review on propagation of chaos and convergence of random probability measures. First, we intend to briefly discuss about propagation of chaos, but we need to carefully define what propagation of chaos means. With this aim, we consider a (stochastic) N-particle system denoted by $(X_1, \ldots, X_N)$ in which particles are indistinguishable. In other words, the particle system enjoys a property known as permutation invariance, i.e. for any test function φ and permutation $\eta \in \mathcal{S}_N$:

$$\mathbb{E}[\varphi\Big(X_1, \ldots, X_N\Big)] = \mathbb{E}[\varphi\Big(X_{\eta(1)}, \ldots, X_{\eta(N)}\Big)].$$

In particular, all the single processes X_i have the same law for $1 \leq i \leq N$ (although they are in general correlated). Let $\rho^{(N)}(x_1, \ldots, x_N)$ to be the density distribution of the N-particle process and denote $\rho_k^{(N)}$ its k-particle marginal density, i.e., the law of the process $(X_1, \ldots, X_k)$:

$$\rho_k^{(N)}(x_1, \ldots, x_k) := \int_{x_{k+1}, \ldots, x_N} \rho^{(N)}(x_1, \ldots, x_N) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N.$$

Consider now a (potential) limit stochastic process $(\overline{X}_1, \ldots, \overline{X}_k)$ where $\{\overline{X}_i\}_{1 \leq i \leq k}$ are i.i.d. Denote by $\bar{\rho}_1$ the law of a single process, thus by independence assumption the law of all the process is given by:

$$\bar{\rho}_k(x_1, \ldots, x_k) = \prod_{i=1}^{k} \bar{\rho}_1(x_i), \quad \text{i.e., } \bar{\rho}_k = \bigotimes_{i=1}^{k} \bar{\rho}_1.$$

The following definition is classical and can be found for instance in Sznitman [145], Carlen *et al.* [42].

Definition 15 *We say that the stochastic process* $(X_1, \ldots, X_N)$ *satisfies the propagation of chaos if for any fixed* k*:*

$$\rho_k^{(N)} \xrightharpoonup{N\to\infty} \bar{\rho}_k, \tag{5.1}$$

which is equivalent to the validity of the following relation for any test function φ*:*

$$\mathbb{E}[\varphi\Big(X_1, \ldots, X_k\Big)] \xrightarrow{N\to\infty} \mathbb{E}[\varphi\Big(X_1, \ldots, X_k\Big)]. \tag{5.2}$$

Next, we shift to a review on convergence of random probability measures. Such topic can be found for instance in a classical book by Billingsley Billingsley [26]. However, we prefer to give a more practical treatment on convergence of random probability measures, based on Berti *et al.* [23]. Consider a sequence of random probability measures $\mu_n(\omega)$, i.e., for a given $\omega \in \Omega$, $\mu_n(\omega) \in \mathcal{P}(\mathbb{R}^d)$. We shall define the mode of convergence as follows:

Definition 16 *We say that* μ_n *converges to* $\mu \in \mathcal{P}(\mathbb{R}^d)$ *in probability, denoted by* $\mu_n \xrightarrow{\mathbb{P}} \mu$*, if*

$$\langle \mu_n(\omega), \varphi\rangle \xrightarrow{\mathbb{P}} \langle \mu(\omega), \varphi\rangle \quad \textit{for any } \varphi \in C_b(\mathbb{R}^d). \tag{5.3}$$

We record here a simple criteria to test the convergence in probability of random measures.

Lemma 5.3.1 *Suppose that the sequence of random measures* $\{\mu_n(\omega)\}_n$ *satisfies*

$$\mathbb{E}_\omega[|\langle \mu_n(\omega) - \mu(\omega), \varphi\rangle|] \xrightarrow{n\to\infty} 0 \quad \textit{for all } \varphi \in C_b(\mathbb{R}^d). \tag{5.4}$$

Then $\mu_n \xrightarrow{\mathbb{P}} \mu$.

Proof It is a direct application of the Markov's inequality. Fixing $\varphi \in C_b(\mathbb{R}^d)$ and let $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}[|\langle \mu_n(\omega), \varphi\rangle - \langle \mu(\omega), \varphi\rangle| > \varepsilon] &= \mathbb{P}[|\langle \mu_n(\omega) - \mu(\omega), \varphi\rangle| > \varepsilon] \\ &\leq \frac{\mathbb{E}_\omega[|\langle \mu_n(\omega) - \mu(\omega), \varphi\rangle|]}{\varepsilon} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Therefore, the random variables $X_n(\omega) := \langle \mu_n(\omega), \varphi\rangle$ converges in probability to $X(\omega) := \langle \mu(\omega), \varphi\rangle$. Since it is true for any $\varphi \in C_b(\mathbb{R}^d)$, we deduce that $\mu_n \xrightarrow{\mathbb{P}} \mu$. $\square$

5.3.2 Formal Limit as $N \to \infty$

We would like to investigate formally the limit as $N \to \infty$ of the dynamics, and we will provide the rigorous derivation in the next section. Motivated by the famous molecular chaos assumption (also known as propagation of chaos), which suggests that we have the statistical independence among the particle systems defined by (5.1) under the large $N \to \infty$ limit, we henceforth give the following definition of the limiting dynamics of $\overline{X}_1$ as $N \to \infty$ from the process point of view.

Definition 17 *(**Asymptotic K-averaging model**) We define a collection of random variables $\{\overline{X}^n\}_{n\geq 0}$ by setting $\overline{X}^0 = X_1^0$ and*

$$\overline{X}^{n+1} := \frac{1}{K}\sum_{j=1}^{K} \overline{Y}_j^n + W^n, \tag{5.5}$$

where $\{\overline{Y}_j^n\}_{1\leq j\leq K}$ are K i.i.d copies of $\overline{X}^n$ and $W^n \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbb{1}_d)$ is independent of n.

If we denote $\bar{\rho}$ to be the law of $\overline{X}$, then it is possible to determine the evolution of $\bar{\rho}$ with respect to time n. For this purpose, We will first collect some definitions to be used throughout the manuscript.

Definition 18 *We use $\mathcal{P}(\mathbb{R}^d)$ to represent the space of probability measures on $\mathbb{R}^d$. We will denote by ϕ the probability density of a d-dimensional Gaussian random variable $\mathcal{E} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbb{1}_d)$. For $\rho \in \mathcal{P}(\mathbb{R}^d)$, we define $T\colon \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ through*

$$T[\rho] = \phi * S_K[C_K[\rho]], \tag{5.6}$$

in which the C_K is the K-fold repeated self-convolution defined via

$$C_K[\rho] := \underbrace{\rho * \rho * \cdots * \rho}_{K \ times}, \tag{5.7}$$

and S_K is the scaling (renormalization) operator given by

$$S_K[\rho](\boldsymbol{x}) := K^d \cdot \rho(K\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^d. \tag{5.8}$$

Remark. We emphasize here that the operator T given in Definition 18 fully encodes the update rule (5.5) for the asymptotic K-averaging model. Indeed, for each valid test function φ, we have

$$\langle \bar{\rho}^{n+1}, \varphi \rangle = \mathbb{E}[\varphi(\overline{X}^{n+1})] = \mathbb{E}\left[\varphi\left(\frac{1}{K}\sum_{j=1}^{K} \overline{Y}_j^n + W^n\right)\right] = \langle T[\bar{\rho}^n], \varphi \rangle, \tag{5.9}$$

where the last equality follows because the random variable $\frac{1}{K}\sum\limits_{j=1}^{K} \overline{Y}_j^n + W^n$ has law $T[\bar{\rho}^n]$. Thus, from the density point of view, as $\bar{\rho}^n$ is the law of $\overline{X}$ at time n, then $T[\bar{\rho}^n]$ represents exactly the law of $\overline{X}$ at time $n+1$.

Equipped with Definition 18, we can write **the evolution of the limit equation** as

$$\bar{\rho}^{n+1} = T[\bar{\rho}^n], \quad n \geq 0. \tag{5.10}$$

Notice that the mean value is preserved by the dynamics (5.5), we will make a harmless assumption throughout this paper that

$$\int_{\boldsymbol{x} \in \mathbb{R}^d} \boldsymbol{x} \bar{\rho}^n(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \mathbf{0} \quad \forall n \geq 0. \tag{5.11}$$

Remark. In dimension 1, we derive from (5.5) that

$$\mathrm{Var}(\overline{X}^{n+1}) = \mathrm{Var}\left(\frac{1}{K}\sum_{j=1}^{K}\overline{Y}_j^n + W^n\right) = \frac{\mathrm{Var}(\overline{X}^n)}{K} + \sigma^2,$$

leading to $\mathrm{Var}(\overline{X}^n) \xrightarrow{n\to\infty} \frac{K\sigma^2}{K-1}$. A similar consideration demonstrates that the covariance matrix associated with $\overline{X}^n \in \mathbb{R}^d$ converges to $\frac{K\sigma^2}{K-1} \cdot \mathbb{1}_d$.

Now we can verify that a suitable Gaussian profile is a fixed point of the iteration process described by (5.10) as long as $K \geq 2$.

Lemma 5.3.2 *Fixing $K \geq 2$. Let*

$$\bar{\rho}_\infty(\boldsymbol{x}) := \frac{1}{(2\pi\sigma_\infty^2)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma_\infty^2}} \tag{5.12}$$

with $\sigma_\infty^2 := \frac{K}{K-1}\sigma^2$, then $\bar{\rho}_\infty$ is a fixed point of T. i.e, $\bar{\rho}_\infty$ satisfies $\bar{\rho}_\infty = T[\bar{\rho}_\infty]$.

Proof It is readily seen that the operator T maps a Gaussian density to another (possibly different) Gaussian density. We investigate the effect of each operator appearing in the definition of T on $\bar{\rho}_\infty$. Indeed, since $Z_1 + \cdots + Z_K \sim \mathcal{N}(\mathbf{0}, K\sigma_\infty^2\mathbb{1}_d)$ when $(Z_i)_{1\leq i\leq K}$ are i.i.d. with law $\mathcal{N}(\mathbf{0}, \sigma_\infty^2\mathbb{1}_d)$, we have

$$C_K[\bar{\rho}_\infty](\boldsymbol{x}) = \frac{1}{(2\pi K\sigma_\infty^2)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2K\sigma_\infty^2}}.$$

Next, notice that $\frac{Z}{K} \sim \mathcal{N}(\mathbf{0}, \frac{\sigma_\infty^2}{K}\mathbb{1}_d)$ if $Z \sim \mathcal{N}(\mathbf{0}, K\sigma_\infty^2\mathbb{1}_d)$, from which we deduce that

$$S_K[C_K[\bar{\rho}_\infty]](\boldsymbol{x}) = \frac{1}{(2\pi\sigma_\infty^2/K)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma_\infty^2/K}}.$$

Finally, we conclude that

$$T[\bar{\rho}_\infty](\boldsymbol{x}) = \phi * S_K[C_K[\bar{\rho}_\infty]](\boldsymbol{x}) = \frac{1}{(2\pi(\sigma_\infty^2/K+\sigma^2))^{\frac{d}{2}}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2(\sigma_\infty^2/K+\sigma^2)}} = \bar{\rho}_\infty(\boldsymbol{x}),$$

which completes the proof. □

We end this subsection with a numerical experiment demonstrating the relaxation of the solution of (5.10) to its Gaussian equilibrium $\bar{\rho}_\infty$, as is shown in Figure 5.4.

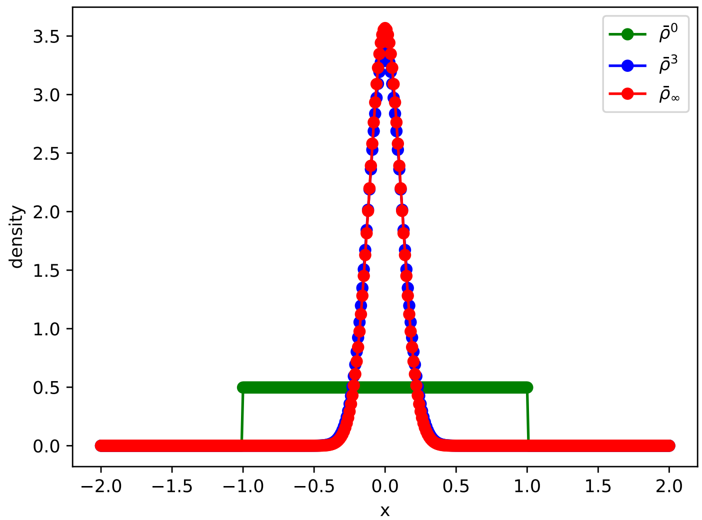

Figure 5.4: Simulation of the Discrete Evolution Equation (5.10) in Dimension $d = 1$ with $K = 5$ after 3 Time Steps, in Which We Used $\sigma = 0.1$ and a Uniform Distribution Over $[-1, 1]$ Initially $\bar{\rho}^0(x) := \frac{1}{2}\mathbb{1}_{[-1,1]}(x)$ (the Green Curve). The Blue and Red Curve Represent $\bar{\rho}^3$ and $\bar{\rho}_\infty$, Respectively. We Also Remark That in This Example $\bar{\rho}^5$ and $\bar{\rho}_\infty$ Are Almost Indistinguishable.

5.3.3 Lipschitz Continuity of the Operator T

To conclude section 2, we demonstrate a useful property of the operator T introduced in (5.6). First, we start with the following definition.

Definition 19 *For each $\mu \in \mathcal{P}(\mathbb{R}^d)$, we define the (strong) norm of μ, denoted by $|||\mu|||$, via*

$$|||\mu||| = \sup_{\|\varphi\|_\infty \leq 1} |\langle \mu, \varphi \rangle|.$$

The main result in this section lies in the Lipschitz continuity of T, to which we now turn.

Proposition 5.3.3 *For each $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, we have*

$$|||T[\mu] - T[\nu]||| \leq K|||\mu - \nu|||. \tag{5.13}$$

Proof We recall that for each $g \in \mathcal{P}(\mathbb{R}^d)$ we have

$$T[g] = \phi * S_K[C_K[g]].$$

Moreover, we have

$$\langle S_K[g], h\rangle = K^d \langle g, S_{\frac{1}{K}}[h]\rangle, \quad \forall g, h \in \mathcal{P}(\mathbb{R}^d).$$

Also, for $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and $\varphi \in C_b(\mathbb{R}^d)$, there holds

$$\langle \mu * \nu, \varphi\rangle = \langle \nu, \widehat{\mu} * \varphi\rangle,$$

where $\widehat{\mu}$ is defined via $\widehat{\mu}(\boldsymbol{x}) := \mu(-\boldsymbol{x})$. Fixing φ with $\|\varphi\|_\infty \le 1$, for each pair of probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\langle T[\mu] - T[\nu], \varphi\rangle &= \langle S_K[C_K[\mu]] - S_K[C_K[\nu]], \phi * \varphi\rangle \\
&= K^d \langle C_K[\mu] - C_K[\nu], S_{\frac{1}{K}}[\phi * \varphi]\rangle \\
&= K^d \sum_{j=0}^{K-1} \langle \underbrace{\mu * \cdots * \mu}_{j \text{ times}} * \underbrace{\nu * \cdots * \nu}_{K-1-j \text{ times}} * (\mu - \nu), S_{\frac{1}{K}}[\phi * \varphi]\rangle \\
&:= K^d \sum_{j=0}^{K-1} \langle \kappa_j * (\mu - \nu), S_{\frac{1}{K}}[\phi * \varphi]\rangle \\
&= K^d \langle \mu - \nu, \sum_{j=0}^{K-1} \widehat{\kappa_j} * S_{\frac{1}{K}}[\phi * \varphi]\rangle.
\end{aligned} \tag{5.14}$$

Setting $\psi_j = \widehat{\kappa_j} * S_{\frac{1}{K}}[\phi * \varphi]$ for each $1 \le j \le K-1$, then we have

$$\|\psi_j\|_\infty \le \|S_{\frac{1}{K}}[\phi * \varphi]\|_\infty \le \frac{\|\varphi\|_\infty}{K^d} \le \frac{1}{K^d}.$$

Thus, if we define $\varphi^{(1)} = K^d \sum_{j=0}^{K-1} \psi_j$, then $\|\varphi^{(1)}\|_\infty \le \frac{K^{d+1}}{K^d} = K$. Now taking the supremum over all φ with $\|\varphi\|_\infty \le 1$, we deduce from (5.14) that

$$|||T[\mu] - T[\nu]||| \le K|||\mu - \nu|||$$

and the proof is completed. $\square$

5.4 Propagation of Chaos

This section is devoted to the rigorous proof of propagation of chaos for the K-averaging dynamics, by employing a martingale-based technique introduced recently in Merle and Salez [118]. We will need the following definition.

Definition 20 *Let $\{X_i^n\}_{1\leq i\leq N}$ be as in Definition 14, we define*

$$\rho_{\mathrm{emp}}^n(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^n}(\boldsymbol{x}) \tag{5.1}$$

to be the empirical distribution of the system at time n. In particular, ρ_{emp}^n is s stochastic measure.

Thanks to a classical result (see for instance **Proposition 1** in Dai Pra [67] or **Proposition 2.2** in Sznitman [145]), to justify the propagation of chaos, it suffices to show that

$$\rho_{\mathrm{emp}}^n \xrightarrow{\mathcal{L}} \bar{\rho}^n \text{ as } N \to \infty.$$

i.e.,

$$\langle \rho_{\mathrm{emp}}^n, \varphi\rangle \xrightarrow{\mathcal{L}} \langle \bar{\rho}^n, \varphi\rangle \quad \text{for any } \varphi \in C_b(\mathbb{R}^d).$$

In fact, one can prove our first theorem.

Theorem 13 *Under the settings of the K-averaging model with $K \geq 2$, if*

$$\rho_{\mathrm{emp}}^0 \xrightarrow{\mathbb{P}} \bar{\rho}^0 \textit{ as } N \to \infty, \tag{5.2}$$

then for each fixed $n \in \mathbb{N}$ we have

$$\rho_{\mathrm{emp}}^n \xrightarrow{\mathbb{P}} \bar{\rho}^n \textit{ as } N \to \infty,$$

where ρ_{emp}^n and $\bar{\rho}^n$ are defined in (5.1) and (5.10), respectively.

Proof We adopt a martingale-based technique developed recently in Merle and Salez [118]. We have for each test function φ that

$$\mathbb{E}\left[\langle\rho_{\text{emp}}^{n+1},\varphi\rangle\right]=\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\varphi\left(\frac{1}{K}\sum_{j=1}^{K}Y_{i,j}^{n}+W_{i}^{n}\right)\right] \tag{5.3}$$

where $\{Y_{i,j}^n\}$ are i.i.d. with law ρ_{emp}^n. Denoting

$$Z_i^n=\frac{1}{K}\sum_{j=1}^{K}Y_{i,j}^{n}+W_i^n$$

for each $1\leq i\leq N$, since the law of Z_i^n is $T[\rho_{\text{emp}}^n]$ for each $1\leq i\leq n$ and $\{Z_i^n\}_{1\leq i\leq N}$ are i.i.d., following the reasoning behind (5.9) we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\varphi\left(Z_i^n\right)\,\middle|\,\rho_{\text{emp}}^n\right]=\langle T[\rho_{\text{emp}}^n],\varphi\rangle. \tag{5.4}$$

Now if we set

$$\begin{aligned} M^n:&=\frac{1}{N}\sum_{i=1}^{N}\varphi\left(Z_i^n\right)-\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\varphi\left(Z_i^n\right)\,\middle|\,\rho_{\text{emp}}^n\right]\\ &=\langle\rho_{\text{emp}}^{n+1},\varphi\rangle-\langle T[\rho_{\text{emp}}^n],\varphi\rangle \end{aligned} \tag{5.5}$$

for each $n\geq 0$, then $(M^n)_{n\geq 0}$ defines a martingale. Moreover, thanks to the fact that $\{\varphi(Z_i^n)\}$ are i.i.d. bounded random variables, and using the convention that the variance operation $\mathrm{Var}(X)$ is interpreted as $\mathrm{Var}(X):=\sum_{k=1}^{d}\mathrm{Var}(X_k)$ when X is a d-dimensional vector-valued random variable, we have

$$\begin{aligned} (\mathbb{E}[|M^n|])^2\leq\mathbb{E}[|M^n|^2]&=\mathbb{E}\left[\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}\varphi\left(Z_i^n\right)\,\middle|\,\rho_{\text{emp}}^n\right)\right]\\ &\leq\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}\varphi\left(Z_i^n\right)\right)\leq\frac{\|\varphi\|_\infty^2 d}{N}, \end{aligned}$$

where we have employed Popoviciu's inequality (see for instance Popoviciu [129]) for upper bounding the variance of a bounded random variable. Comparing (5.5) with

(5.10) yields

$$\mathbb{E}\Big[|\langle \rho_{\text{emp}}^{n+1} - \bar{\rho}^{n+1}, \varphi\rangle|\Big] \leq \mathbb{E}\Big[|\langle T[\rho_{\text{emp}}^{n}] - T[\bar{\rho}^{n}], \varphi\rangle|\Big] + \frac{\|\varphi\|_\infty^2 d}{\sqrt{N}}. \tag{5.6}$$

Now if $\|\varphi\|_\infty \leq 1$, we can recall the computations carried out in (5.14), which ensures the existence of some $\varphi^{(1)}$ with $\|\varphi^{(1)}\|_\infty \leq K$ such that

$$\langle T[\rho_{\text{emp}}^{n}] - T[\bar{\rho}^{n}], \varphi\rangle = \langle \rho_{\text{emp}}^{n} - \bar{\rho}^{n}, \varphi^{(1)}\rangle. \tag{5.7}$$

Then we can deduce from (5.6) and (5.7) that

$$\mathbb{E}\Big[|\langle \rho_{\text{emp}}^{n+1} - \bar{\rho}^{n+1}, \varphi\rangle|\Big] \leq \mathbb{E}\Big[|\langle \rho_{\text{emp}}^{n}) - \bar{\rho}^{n}, \varphi^{(1)}\rangle|\Big] + \frac{d}{\sqrt{N}}, \tag{5.8}$$

in which $\varphi^{(1)}$ satisfies $\|\varphi^{(1)}\|_\infty \leq K$. We can iterate (5.8) to arrive at

$$\mathbb{E}\Big[|\langle \rho_{\text{emp}}^{n} - \bar{\rho}^{n}, \varphi\rangle|\Big] \leq \mathbb{E}\Big[|\langle \rho_{\text{emp}}^{0} - \bar{\rho}^{0}, \varphi^{(n)}\rangle|\Big] + \frac{dn}{\sqrt{N}}, \tag{5.9}$$

in which $\varphi^{(n)}$ satisfies $\|\varphi^{(n)}\|_\infty \leq K^n$. Finally, combining (5.2) with (5.9) allows us to complete the proof of Theorem 13. $\square$

Remark. As we do not have a uniform-in-time propagation of chaos, we would like to know whether the convergence declared in Theorem 13 still holds if we do not fix n (i.e., if $n \to \infty$). We speculate such an uniform in time convergence can no longer be hoped for by looking at the evolution of the *center of mass* of the particle systems. Indeed, define

$$\mathcal{C}_{n+1} := \frac{1}{N}\sum_{i=1}^{N} X_i^n$$

to be the location of the center of mass, and denote by $\mathcal{F}_n$ the natural filtration generated by $(X_1^n, \cdots, X_N^n)$, then in dimension $d = 1$ we have

$$\begin{aligned}\mathbb{E}[\mathcal{C}_{n+1} \mid \mathcal{F}_n] &= \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[X_i^{n+1} \mid \mathcal{F}_n] = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{K}\sum_{j=1}^{K}\mathbb{E}[X_{S_i^n(j)}^n \mid \mathcal{F}_n] \\ &= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{K}\sum_{j=1}^{K}\frac{1}{N}\sum_{\ell=1}^{N} X_\ell^n = \mathcal{C}_n,\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[(\mathcal{C}_{n+1} - \mathcal{C}_n)^2 \mid \mathcal{F}_n] &= \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N X_i^{n+1} - \mathcal{C}_n\right)^2 \middle| \mathcal{F}_n\right] \\
&= \mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^N X_i^{n+1} \middle| \mathcal{F}_n\right] = \frac{1}{N}\mathrm{Var}[X_1^{n+1} \mid \mathcal{F}_n] \\
&\geq \frac{\sigma^2}{N},
\end{aligned}
$$

where the last equality comes from the fact that X_i^{n+1} and X_j^{n+1} are i.i.d. *given* $\mathcal{F}_n$. Thus, loosely speaking, at least in dimension $d = 1$, the center of mass of the particle systems behaves like a discrete time Brownian motion with intensity of order at least $\mathcal{O}(1/\sqrt{N})$, such an variation can accumulate in time which will eventually ruin the chaos propagation property in the long run.

5.5 Large Time Behavior

The long time behavior of the limit equation, resulted from the simplified mean-field dynamics, is treated in this section. In section 5.5.1, by employing a coupling technique and equipping the space of probability measures on $\mathbb{R}^d$ with the Wasserstein distance, we will justify the asymptotic Gaussianity of the distribution of each particle. Then we will strengthen the convergence result shown in the previous section in section 5.5.2, and numerical simulations are also performed in support of our theoretical discoveries in section 5.5.3. We emphasize here that coupling techniques will be at the core of our proof in section 5.5.1, and the technique used in section 5.5.2 depends heavily on several classical results in information theory.

5.5.1 Convergence in Wasserstein Distance

After we have achieved the transition from the interacting particle system (5.1) to the simplified de-coupled dynamics (5.5) under the limit $N \to \infty$, in this section we will analyze (5.5) and its associated evolution of its law (governed by (5.10)), with the intention of proving the convergence of $\bar{\rho}^n$ to a suitable Gaussian density. The main ingredient underlying our proof lies in a coupling technique. First, we recall the following classical definition.

Definition 21 *The Wasserstein distance (of order 2) is defined via*

$$\mathcal{W}_2^2(\mu,\nu) := \inf_{\substack{X\sim\mu \\ Y\sim\nu}} \mathbb{E}[|X-Y|^2],$$

where both μ and ν are probability measures on $\mathbb{R}^d$.

We can now state and prove our main result in this section.

Theorem 14 *Assume that the innocent-looking normalization* (5.11) *holds and $K \geq 2$, then for the dynamics* (5.10), *we have*

$$\mathcal{W}_2^2(\bar{\rho}^{n+1},\bar{\rho}_\infty) \leq \frac{1}{K}\mathcal{W}_2^2(\bar{\rho}^n,\bar{\rho}_\infty),\ \forall n \geq 0 \tag{5.1}$$

In particular, if $\bar{\rho}^0 \in \mathcal{P}(\mathbb{R}^d)$ is chosen such that $\mathcal{W}_2^2(\bar{\rho}^0,\bar{\rho}_\infty) < \infty$, then

$$\lim_{n\to\infty} \mathcal{W}_2^2(\bar{\rho}^n,\bar{\rho}_\infty) = 0.$$

Proof We first show that

$$\mathcal{W}_2^2(T(\mu),T(\nu)) \leq \frac{1}{K}\mathcal{W}_2^2(\mu,\nu) \tag{5.2}$$

for each $\mu,\nu \in \mathcal{P}(\mathbb{R}^d)$. In other words, if we equip the space $\mathcal{P}(\mathbb{R}^d)$ with the Wasserstein distance of order 2, T is a strict contraction as long as $K \geq 2$. Now we fix $\mu,\nu \in \mathcal{P}(\mathbb{R}^d)$. It is recalled that $T(\mu)$ is the law of the random variable

$$X := \frac{X_1+\cdots+X_K}{K} + \mathcal{E},$$

where $\{X_i\}_{1\leq i\leq K}$ are i.i.d. with law μ and $\mathcal{E} \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbb{1}_d)$. Thus, if we also introduce

$$Y := \frac{Y_1+\cdots+Y_K}{K} + \widetilde{\mathcal{E}},$$

in which $\{Y_i\}_{1\leq i\leq K}$ are i.i.d. with law ν and $\widetilde{\mathcal{E}} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbb{1}_d)$, then we can write

$$\begin{aligned}
\mathcal{W}_2^2(T(\mu), T(\nu)) &= \inf_{X\sim T(\mu), Y\sim T(\nu)} \mathbb{E}[|X-Y|^2] \\
&= \inf_{X_i\sim\mu, Y_i\sim\nu} \mathbb{E}\left[\left|\frac{X_1+\cdots+X_K}{K} + \mathcal{E} - \frac{Y_1+\cdots+Y_K}{K} - \widetilde{\mathcal{E}}\right|^2\right].
\end{aligned}$$

We can couple $(X_1, \cdots, X_K, \mathcal{E})$ and $(Y_1, \cdots, Y_k, \tilde{\mathcal{E}})$ as we want. First, we take $\mathcal{E} = \widetilde{\mathcal{E}}$, meaning we have a common source of noise. Second, fixing $\eta > 0$, we take (X_1, Y_1) such that

$$\mathbb{E}[|X_1 - Y_1|^2] \leq \mathcal{W}_2^2(\mu, \nu) + \eta,$$

(i.e. almost best coupling). Finally, we perform similarly for the other (X_i, Y_i) with (X_i, Y_i) independent of (X_j, Y_j) if $i \neq j$. These procedures lead us to

$$\begin{aligned}
\mathcal{W}_2^2(T(\mu), T(\nu)) &\leq \mathbb{E}\left[\left|\frac{X_1 - Y_1 + \cdots + X_K - Y_K}{K}\right|^2\right] \\
&\leq \frac{1}{K^2}\left(\mathbb{E}[|X_1-Y_1|^2] + \cdots + \mathbb{E}[|X_K - Y_K|^2]\right) \\
&\leq \frac{1}{K}\mathcal{W}_2^2(\mu,\nu) + \frac{\eta}{K}.
\end{aligned}$$

Since this is true for any $\eta > 0$, (5.2) is verified. Now we can deduce from (5.2) that

$$\mathcal{W}_2^2(\bar{\rho}^{n+1}, \bar{\rho}_\infty) = \mathcal{W}_2^2(T(\bar{\rho}^n), T(\bar{\rho}_\infty)) \leq \frac{1}{K}\mathcal{W}_2^2(\bar{\rho}^n, \bar{\rho}_\infty),$$

whence (5.1) is proved. □

5.5.2 Convergence in Relative Entropy

In this subsection we will show that the evolution of the discrete equation (5.10) relaxes to its Gaussian equilibrium $\bar{\rho}_\infty$ in the sense of relative entropy, as long as $K \geq 2$. Before stating our result, we first clarify some definitions. We refer the reader to Cover [62] for a comprehensive account of modern information theory.

Definition 22 *We use*

$$\mathrm{H}(X) := \mathrm{H}(g) = \int_{\mathbb{R}^d} g(\boldsymbol{x}) \log g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

to represent the differential entropy of a $\mathbb{R}^d$*-valued random variable* X *with law* g*. Moreover,*

$$\mathrm{D}_{\mathrm{KL}}(g||h) := \mathrm{H}(g) - \mathrm{H}(g,h) = \int_{\mathbb{R}^d} g(\boldsymbol{x}) \log g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^d} g(\boldsymbol{x}) \log h(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

denotes the relative entropy from $h \in \mathcal{P}(\mathbb{R}^d)$ *to* $g \in \mathcal{P}(\mathbb{R}^d)$*, in which*

$$\mathrm{H}(g,h) := \mathrm{H}(X,Y) = \int_{\mathbb{R}^d} g(\boldsymbol{x}) \log h(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

is the cross-entropy from g *to* h *(or equivalently, from* X *to* Y *where the laws of* X *and* Y *are* g *and* h*, respectively).*

For the reader's convenience, we explicitly state two fundamental results from information theory that we shall reply on.

Lemma 5.5.1 (Shannon-Stam) *Under the set-up of Definition 22, we have*

$$\mathrm{H}(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \leq \lambda \mathrm{H}(X) + (1-\lambda)\mathrm{H}(Y)$$

for each $\lambda \in [0,1]$.

Lemma 5.5.1 is one of the three equivalent formulations of the well-known Shannon-Stam inequality, see for instance section 1.3.2 of Rezakhanlou *et al.* [136]. The next lemma (see for instance **Theorem 1** in Artstein *et al.* [12] or equation (7) in Madiman and Barron [113]) demonstrates the monotonicity of the differential entropy along re-scaled sum of i.i.d. square-integrable random variables.

Lemma 5.5.2 *Let* $X_1, X_2, \ldots$ *be i.i.d. square-integrable random variables. Then*

$$\mathrm{H}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right) \leq \mathrm{H}\left(\frac{X_1 + \cdots + X_{n-1}}{\sqrt{n-1}}\right)$$

for each $n \geq 2$.

Theorem 15 *Assume that $\bar{\rho}$ is a solution to* (5.10)*, then for each fixed $K \geq 2$ we have*

$$\mathrm{D}_{\mathrm{KL}}(\bar{\rho}^{n+1}||\bar{\rho}_\infty) \leq \frac{1}{K}\mathrm{D}_{\mathrm{KL}}(\bar{\rho}^{n}||\bar{\rho}_\infty). \tag{5.3}$$

In particular, for each $K \geq 2$ we have $\mathrm{D}_{\mathrm{KL}}(\bar{\rho}^{n}||\bar{\rho}_\infty) \to 0$ as $n \to \infty$.

Proof Let $\{\overline{X}^n\}_{n\geq 0}$ be as in Definition 17. If we introduce a random variable $\overline{X}_\infty$ with law $\bar{\rho}_\infty$, i.e.,$\overline{X}_\infty \sim \mathcal{N}(\mathbf{0}, \sigma_\infty^2 \mathbb{1}_d)$, then for each $n \in \mathbb{N}$, we can rewrite (5.5) as

$$\overline{X}^{n+1} = \frac{1}{\sqrt{K}} \cdot \frac{1}{\sqrt{K}} \sum_{j=1}^{K} \overline{Y}_j^n + \sqrt{\frac{K-1}{K}}\overline{X}_\infty,$$

since $\sqrt{\frac{K-1}{K}}\overline{X}_\infty = W^n$ in law. Setting $\gamma = \frac{1}{\sqrt{K}}$, we obtain

$$\overline{X}^{n+1} = \sqrt{\gamma} \cdot \frac{1}{\sqrt{K}} \sum_{j=1}^{K} \overline{Y}_j^n + \sqrt{1-\gamma} \cdot \overline{X}_\infty.$$

Consequently, the Shannon-Stam inequality (see Lemma 5.5.1) together with the monotonicity of differential entropy along normalized sum of i.i.d. random variables (see Lemma 5.5.2) yields

$$\mathrm{H}(\overline{X}^{n+1}) \leq \gamma \mathrm{H}\left(\frac{1}{\sqrt{K}} \sum_{j=1}^{K} \overline{Y}_j^n\right) + (1-\gamma)\mathrm{H}(\overline{X}_\infty) \leq \gamma \mathrm{H}(\overline{X}^n) + (1-\gamma)\mathrm{H}(\overline{X}_\infty). \tag{5.4}$$

Next, we observe that the cross-entropy from each $f \in \mathcal{P}(\mathbb{R}^d)$ with mean $\mathbf{0}$ to the equilibrium distribution $\bar{\rho}_\infty$ is essentially the variance of f, meaning that

$$\mathrm{H}(f, \bar{\rho}_\infty) = -\frac{d}{2}\log(2\pi\sigma_\infty^2) - \frac{\int_{\mathbb{R}^d} |\boldsymbol{x}|^2 f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}}{2\sigma_\infty^2}.$$

In particular, if X and Y are independent random variables with mean $\mathbf{0}$ and $a^2+b^2 = 1$, then

$$\mathrm{H}(aX + bY, \overline{X}_\infty) = a^2\mathrm{H}(X, \overline{X}_\infty) + b^2\mathrm{H}(Y, \overline{X}_\infty).$$

Thus, using this formulation with $\sqrt{\gamma}$ and $\sqrt{1-\gamma}$, we find

$$\begin{aligned}\mathrm{H}(\overline{X}^{n+1}, \overline{X}_\infty) &= \gamma \mathrm{H}\left(\frac{1}{\sqrt{K}} \sum_{j=1}^{K} \overline{Y}_j^n, \overline{X}_\infty\right) + (1-\gamma)\mathrm{H}(\overline{X}_\infty, \overline{X}_\infty) \\ &= \gamma \mathrm{H}(\overline{X}^n, \overline{X}_\infty) + (1-\gamma)\mathrm{H}(\overline{X}_\infty).\end{aligned}$$

Combining this with (5.4) leads to

$$\begin{aligned}
\mathrm{D_{KL}}(\bar\rho^{n+1}||\bar\rho_\infty) &= \mathrm{H}(\overline{X}^{n+1}) - \mathrm{H}(\overline{X}^{n+1}, \overline{X}_\infty) \\
&\le \gamma \mathrm{H}(\overline{X}^{n}) + (1-\gamma)\mathrm{H}(\overline{X}_\infty) - \gamma \mathrm{H}(\overline{X}^{n}, \overline{X}_\infty) - (1-\gamma)\mathrm{H}(\overline{X}_\infty) \\
&= \gamma \mathrm{D_{KL}}(\bar\rho^{n}||\bar\rho_\infty),
\end{aligned}$$

and the proof is completed. □

Remark. By Talagrand's inequality (see for instance **Theorem 9.2.1** in Bakry *et al.* [14]), the convergence $\mathrm{D_{KL}}(\bar\rho^n||\bar\rho_\infty) \to 0$ implies the convergence $\mathcal{W}_2^2(\bar\rho^n, \bar\rho_\infty) \to 0$.

5.5.3 Numerical Illustration of Decay in Relative Entropy

We investigate numerically the convergence of the solution $\bar\rho^n$ of (5.10) to its equilibrium $\bar\rho$ in support of our Theorem 15, see Figure 5.5. We use $d = 1$ (dimension), $K = 5$ (number of neighbors to be averaged over), $\sigma = 0.1$ (the intensity of a centered Gaussian noise) in the simulation of the evolution equation (5.10). To discretize (5.10), we employ the step-size $\Delta x = 0.001$ and a cutoff threshold $M = 100{,}000$ so that the support of $\bar\rho^n$ is contained in $\{j\Delta x\}_{-M\le j\le M}$ for all n, and the total number of simulation steps is set to 15. As initial condition, we use the Laplace distribution $\bar\rho^0(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. Moreover, the simulation result is displayed in the semi-logarithmic scale, which clearly indicates a geometrically fast convergence.

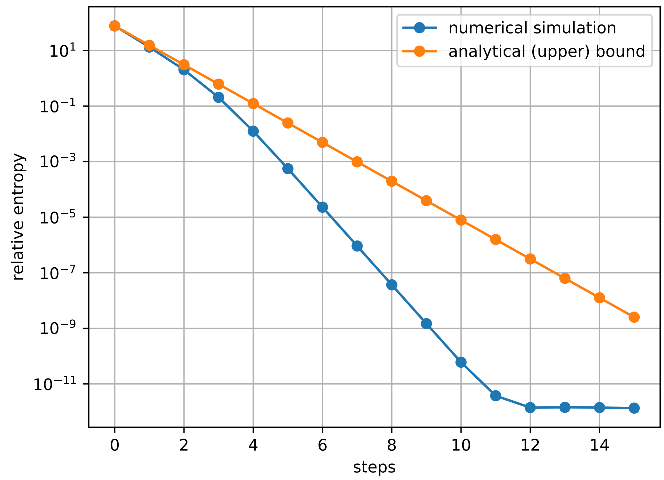

Figure 5.5: Simulation of the Relative Entropy from $\bar{\rho}$ to $\bar{\rho}_\infty$ in Dimension $d = 1$ with $K = 5$ after 15 Time Steps, in Which We Used $\sigma = 0.1$ and a Laplace Distribution $\bar{\rho}^0(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ Initially. The Blue and Orange Curve Represent the Numerical Error and the Analytical Upper Bound on the Error, Respectively. We Also Noticed That the Numerical Error Can Not Really Go Below 10^{-12}, but This Is Presumably Due to the Floating-Point Precision Error.

5.6 Continuous-Time K-Averaging Dynamics

With suitable modifications the argument used in the discrete-time applies in continuous-time as well, so in this section we briefly consider the continuous version of the K-averaging model studied in previous sections, i.e., the K-averaging occurs according to a Poisson process. First, we give a formal definition of the model.

Definition 23 *(**Continuous-time K-averaging model**) Consider a collection of stochastic processes $\{X_i(t)\}_{1\leq i\leq N}$ evolving on $\mathbb{R}^d$. At each time a Poisson clock with rate λ rings, we pick a particle $i \in \{1,\ldots,N\}$ uniformly at random and update the position of X_i according to the average position of K randomly selected neighbors, subject to an independent noise term, i.e., for each test function φ the process must* *satisfy*

$$\mathrm{d}\mathbb{E}[\varphi\Big(X_1(t),\ldots,X_N(t)\Big)] = \lambda\sum_{i=1}^{N}\mathbb{E}[\varphi\Big(X_1(t),\ldots,Z_i(t),\ldots,X_N(t)\Big) \\ -\varphi\Big(X_1(t),\ldots,X_N(t)\Big)]\mathrm{d}t, \tag{5.1}$$

where $Z_i(t) := \frac{1}{K}\sum_{j=1}^{K} X_{S_i(j)}(t) + W_i(t)$*,* $S_i(j)$ *are indices taken randomly from the set* $\{1,2,\ldots,N\}$ *(i.e.,* $S_i(j) \sim \mathrm{Uniform}(\{1,2,\ldots,N\})$ *and is independent of* i, j *and* t*), and* $W_i(t) \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbb{1}_d)$ *is independent of* i *and* t*.*

In the large N limit, we expect an emergence of a simplified dynamics, which motivates the following definition.

Definition 24 *(**Asymptotic continuous-time** K**-averaging model**) Consider a* $\mathbb{R}^d$*-valued stochastic process* $\overline{X}(t)$ *which satisfies the following relation for each test function* φ*:*

$$\mathrm{d}\mathbb{E}[\varphi\Big(\overline{X}(t)\Big)] = \lambda\mathbb{E}[\varphi\Big(\overline{Z}(t)\Big)-\varphi\Big(\overline{X}(t)\Big)]\mathrm{d}t, \tag{5.2}$$

in which $\overline{Z}(t) := \frac{1}{K}\sum_{j=1}^{K}\overline{Y}_j(t) + W(t)$*, where* $\{\overline{Y}_j(t)\}_{1\leq j\leq K}$ *are* K *i.i.d. copies of* $\overline{X}(t)$ *and* $W(t) \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbb{1}_d)$ *is independent of* t*.*

If we define $\bar{\rho}(\boldsymbol{x},t)$ to be the law of $\overline{X}$ at time t, one can readily see that the evolution of $\bar{\rho}$ is governed by

$$\partial_t\bar{\rho} = \lambda(T[\bar{\rho}]-\bar{\rho}), \quad t\geq 0. \tag{5.3}$$

Moreover, one can show the continuous-time analog of Theorem 13 and Theorem 15.

Theorem 16 *Let* $\rho_{\mathrm{emp}}(t) := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i(t)}$ *to be the empirical distribution of the system determined by* (5.1) *at time* t *and* $\bar{\rho}$ *the solution of* (5.3) *with the Gaussian equilibrium* $\bar{\rho}_\infty$ *defined in* (5.12)*, then* *(i) under the set-up of the continuous-time K-averaging model with $K \geq 2$, if*

$$\rho_{\text{emp}}(0) \xrightarrow{\mathbb{P}} \bar{\rho}(0) \text{ as } N \to \infty, \tag{5.4}$$

then we have

$$\rho_{\text{emp}}(t) \xrightarrow{\mathbb{P}} \bar{\rho}(t) \text{ as } N \to \infty,$$

holding for all $0 \leq t \leq T$ with any prefixed $T > 0$.

(ii) for each fixed $K \geq 2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_{\mathrm{KL}}(\bar{\rho}||\bar{\rho}_\infty) \leq -\lambda(1-\gamma)\mathrm{D}_{\mathrm{KL}}(\bar{\rho}||\bar{\rho}_\infty), \tag{5.5}$$

where $\gamma = \frac{1}{K}$ as before. In particular, we have

$$\mathrm{D}_{\mathrm{KL}}(\bar{\rho}(t)||\bar{\rho}_\infty) \leq \mathrm{D}_{\mathrm{KL}}(\bar{\rho}(0)||\bar{\rho}_\infty) \cdot \mathrm{e}^{-\lambda(1-\gamma)t}. \tag{5.6}$$

Proof We assume without loss of generality that $\lambda = 1$. For (i), mimic the argument in the discrete-time setting we obtain for each test function φ that

$$\begin{aligned}\mathrm{d}\mathbb{E}\left[\langle \rho_{\text{emp}}(t), \varphi \rangle\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \varphi\Big(Z_i(t)\Big) - \frac{1}{N}\sum_{i=1}^N \varphi\Big(X_i(t)\Big)\right]\mathrm{d}t \\ &= \mathbb{E}\left[\langle T[\rho_{\text{emp}}(t)] - \rho_{\text{emp}}(t), \varphi\rangle\right]\mathrm{d}t,\end{aligned}$$

in which $Z_i(t) = \frac{1}{K}\sum_{j=1}^K Y_{i,j}(t) + W_i(t)$ and $\{Y_{i,j}(t)\}$ are i.i.d. with law $\rho_{\text{emp}}(t)$. Then by Dynkin's formula, the compensated process

$$M_\varphi(t) := \langle \rho_{\text{emp}}(t), \varphi\rangle - \langle \rho_{\text{emp}}(0), \varphi\rangle - \int_0^t \langle T[\rho_{\text{emp}}(s)] - \rho_{\text{emp}}(s), \varphi\rangle \mathrm{d}s$$

defines a martingale. Comparing with (5.3) yields

$$\begin{aligned}\langle \rho_{\text{emp}}(t) - \bar{\rho}(t), \varphi\rangle| &\leq |M_\varphi(t)| + |\langle \rho_{\text{emp}}(0) - \bar{\rho}(0), \varphi\rangle| \\ &\quad + \int_0^t |\langle T[\bar{\rho}(s)] - T[\rho_{\text{emp}}(s)] - \Big(\bar{\rho}(s) - \rho_{\text{emp}}(s)\Big), \varphi\rangle|\mathrm{d}s.\end{aligned} \tag{5.7}$$

We then take the supremum over all φ with $\|\varphi\|_\infty \leq 1$ to deduce from Proposition 5.3.3 and (5.7) that

$$\|\!|\rho_{\text{emp}}(t) - \bar{\rho}(t)|\!\| \leq \eta(t) + (K+1)\int_0^t \|\!|\rho_{\text{emp}}(s) - \bar{\rho}(s)|\!\| \mathrm{d}s,$$

where we have set

$$\eta(t) := \sup_{\|\varphi\|_\infty \leq 1} |M_\varphi(t)| + \|\!|\rho_{\text{emp}}(0) - \bar{\rho}(0)|\!\|.$$

By Gronwall's inequality, we obtain

$$\sup_{t\in[0,T]} \|\!|\rho_{\text{emp}}(t) - \bar{\rho}(t)|\!\| \leq \left(\sup_{t\in[0,T]} \eta(t)\right) \mathrm{e}^{(K+1)T}.$$

In order to justify our claim (i) for $t \leq T$, it therefore suffices to show that

$$\sup_{t\in[0,T]} \eta(t) \xrightarrow[N\to\infty]{\mathbb{P}} 0. \tag{5.8}$$

To show (5.8), we address each term appearing in the definition of $\eta(t)$ separately. The second one vanishes due to our assumption (D.1). For the first one, i.e., the martingale term, we note that the i-th coordinate of M_φ is a continuous time martingale with jumps of size $\frac{1}{N}\varphi(Z_i) - \frac{1}{N}\varphi(X_i)$ whose rates of occurrence are $\lambda \cdot \mathrm{d}t = \mathrm{d}t$. Therefore,

$$\mathbb{E}[|M_\varphi(T)|^2] \leq \int_0^T \mathbb{E}\left[\sum_{i=1}^N \left|\frac{1}{N}\varphi(Z_i) - \frac{1}{N}\varphi(X_i)\right|^2\right] \mathrm{d}t \leq \frac{4\|\varphi\|_\infty^2}{N} T \leq \frac{4T}{N},$$

whence the convergence

$$\sup_{t\in[0,T]} \left(\sup_{\|\varphi\|_\infty \leq 1} |M_\varphi(t)|\right) \xrightarrow[N\to\infty]{\mathbb{P}} 0$$

follows readily from Doob's martingale inequality. For (ii), we recall that in the discrete-time case (with $\gamma = \frac{1}{K}$), (5.3) can be rewritten as

$$\mathrm{D}_{\mathrm{KL}}(\bar{\rho}^{n+1}||\bar{\rho}_\infty) - \mathrm{D}_{\mathrm{KL}}(\bar{\rho}^n||\bar{\rho}_\infty) \leq -(1-\gamma)\mathrm{D}_{\mathrm{KL}}(\bar{\rho}^n||\bar{\rho}_\infty).$$

This can be translated immediately to its continuous-time analog (5.5), whence the proof is completed. $\square$

5.7 Conclusion

In this manuscript, we have investigated a model (which we call the K-averaging model) for a system of self-propelled particles on $\mathbb{R}^d$, in both discrete-time and continuous-time settings. We also provided an rigorous proof on the convergence of the distribution of a typical particle towards a suitable Gaussian equilibrium under the large particle size $N \to \infty$ and large time $n \to \infty$ limit. Even though the majority of the work is done in discrete-time, the relevant results carry over easily to continuous-time. It would also be interesting to examine variants of this model. For instance, the K-averaging dynamics on $\mathbb{S}^1$ is closely related to several models in the literature Porfiri and Ariel [131], Aldana and Huepe [4], Aldana *et al.* [3], Pimentel *et al.* [127], Porfiri [130], and it is reasonable to expect a rigorous proof of the corresponding mean-field limit. Unfortunately, the situation on $\mathbb{S}^1$ is inevitably much more complicated since we are lacking the vector-space structure. More generally, averaging is not a straightforward operation over a manifold Degond *et al.* [72]. Other extensions of the model in the present manuscript are also possible. As of now, every agent communicate with each other. Thus, what would happen if only agents are only interacting through a pre-defined graph of neighboring few chosen neighbors? We would lose the invariance by permutation, thus the notion of limit is more challenging. This would also link the model to certain ”consensus models” Hardin and Lanchier [87], Lanchier and Li [108]. One can also explore different laws of communication between the particles (especially of the non-symmetric and non-all-to-all variety), and investigate the role of noise introduced into the system.

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Chapter 6

ASYMPTOTIC FLOCKING FOR THE THREE-ZONE MODEL

Chapter 6 is the paper Cao *et al.* [41] published on *Mathematical Biosciences and Engineering*, in collaboration with Sebastien Motsch, Alexander Reamy, and Ryan Theisen.

6.1 Abstract

In this manuscript, the asymptotic flocking behavior of a general model of swarming dynamics is proved. The model describes interacting particles encompassing three types of behavior: repulsion, alignment and attraction. This dynamics is called the three-zone model. The result expands the analysis of the so-called Cucker-Smale model where only alignment rule is taken into account. Whereas in the Cucker-Smale model, the alignment should be strong enough at long distance to ensure flocking behavior, here it is only required that the attraction is described by a confinement potential. The key for the proof is to use that the dynamics is dissipative thanks to the alignment term which plays the role of a friction term. Several numerical examples illustrate the result and it is also possible extend the proof for the kinetic equation associated with the three-zone dynamics.

6.2 Introduction

Flocking behavior is an intriguing phenomenon observed frequently in nature. However, it remains an open question how birds or fish are able to organize efficiently to form a coherent motion. Modeling has proved to be crucial in highlighting how such complex behaviors can be described by simple interaction rules. Among the different models proposed, the three-zone model has been particularly popular in biology Aoki [9], Reynolds [135], Huth and Wissel [92], Couzin *et al.* [61], Li *et al.* [111]. In the three-zone model, agents representing birds or fish engage in three types of interactions: repulsion, alignment, and attraction, depending on the relative location of their neighboring agents. The goal of this manuscript is to provide sufficient conditions for the convergence of such a system towards a *flock*, which we define to occur when all agents approach a common velocity.

Analytical studies of flocking dynamics have mainly been inspired by the seminal work of Cucker and Smale Cucker and Smale [64, 65]. In their research, they studied a simplified version of the so-called Vicsek model Vicsek *et al.* [146], where only the alignment force between agents is considered. They proved rigorously the convergence of the dynamics to a flock, given the condition that the alignment force is sufficiently *strong*. This work has been followed by many generalizations Motsch and Tadmor [120, 121], Agueh *et al.* [2] and improvements Ha and Tadmor [86], Ha and G. [85], Carrillo *et al.* [46].

One key element of proving the convergence of the Cucker-Smale model to a flock is the decay of the *kinetic energy* of the system due to the alignment rule, which acts as a source of friction. In the present manuscript, the dynamics combine the alignment rule with the additional forces of attraction and repulsion. Therefore, we have to define the energy of the system as the sum of its *kinetic energy* and *potential energy*. The attraction-repulsion term does not modify the total energy since it is Hamiltonian dynamics. However, the alignment term causes the kinetic energy, and consequently the total energy of the system, to decay with respect to time. As a result, if the attraction-repulsion force is such that the particle configuration remains spatially bounded, the influence of the alignment term will guarantee the system converges to a flock. Notice that we do not draw any conclusions about the spatial organization of the flock, though many analytic and numerical studies have studied this problem Von Brecht *et al.* [149], Carrillo *et al.* [48, 45].

Since our proof is based mainly on an energy functional, it is possible to extend the method to the kinetic equation associated with the three-zone model. However, in this case, we have to define a weaker notion of flocking for the kinetic equation. There is additional difficulty in dealing with a kinetic equation: since we are working with a continuum distribution, there is no longer a maximum distance between two agents. Despite these obstacles, we manage to prove a L^1-type convergence result by using stochastic process theory.

So far, the sufficient condition for flocking requires that the attraction term is given by a confinement potential. However, this condition can be weakened by incorporating the effect of the alignment in the non-spatial dispersion of the agents. Here, the effects of attraction and alignment are treated separately. It might be possible to improve the sufficient condition for flocking and/or to find a joint condition on the strength of attraction and alignment using commutator techniques Villani [148], Bakey *et al.* [13]. Commutator may also help to prove that the dynamics converges exponentially fast toward equilibrium. Other perspectives will be to extend the proof to other types of interactions, such as non-symmetric interactions Motsch and Tadmor [120], Karper *et al.* [100], Motsch and Tadmor [121], Jabin and Motsch [94], or those using the so-called topological distance Ballerini *et al.* [17], Haskovec [88], Blanchet and Degond [27].

The paper is organized as followed: in section 6.3, we introduce the three-zone model for agent-based dynamics. We prove the main result by finding a sufficient condition for the emergence of a flock and give several numerical illustrations. In section 6.4, we extend this result to the kinetic equation associated with the agent-based model. Finally, we draw a conclusion in section 6.5.

6.3 Agent-Based Models

6.3.1 Three-Zone Model

We consider the three-zone model, which describes agents moving according to three rules of interaction: repulsion (at short distance), alignment and attraction (long distance). A schematic representation of the model is given in figure 6.1. Each agent i is represented by a vector position $\mathbf{x}_i$ and a velocity $\mathbf{v}_i$ both belonging to $\mathbb{R}^d$ (with $d = 2$ or 3). The evolution of the N agents is governed by the following system:

$$\dot{\mathbf{x}}_i = \mathbf{v}_i \tag{6.1}$$

$$\dot{\mathbf{v}}_i = \frac{1}{N}\sum_{j=1}^{N} \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i) - \frac{1}{N}\sum_{j \neq i} \nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|). \tag{6.2}$$

Here, $\phi_{ij} = \phi(|\mathbf{x}_j - \mathbf{x}_i|)$ represents the strength of the alignment between agents i and j. We suppose that the function ϕ is strictly positive. Similarly, $\nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|)$ represents the attraction or repulsion of agent j on i. Indeed, developing the gradient gives:

$$-\nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|) = V'(|\mathbf{x}_j - \mathbf{x}_i|)\frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|}.$$

Thus, agent i is attracted to agent j if $V' > 0$ and repulsed if $V' < 0$. In fig. 6.1, we illustrate two possible choices for ϕ and V.

6.3.2 Flocking: Rigorous Results

The goal of this section is to prove conditions guaranteeing that the three-zone model (6.1)(6.2) converges to a *flock*.

Definition 25 *We say that a configuration $\{\mathbf{x}_i, \mathbf{v}_i\}_i$ converges to a flock if the following are satisfied:*

1) There exists $\mathbf{v}_\infty$ such that $\mathbf{v}_i \xrightarrow{t\to\infty} \mathbf{v}_\infty$ for all $i = 1, \ldots, N$.

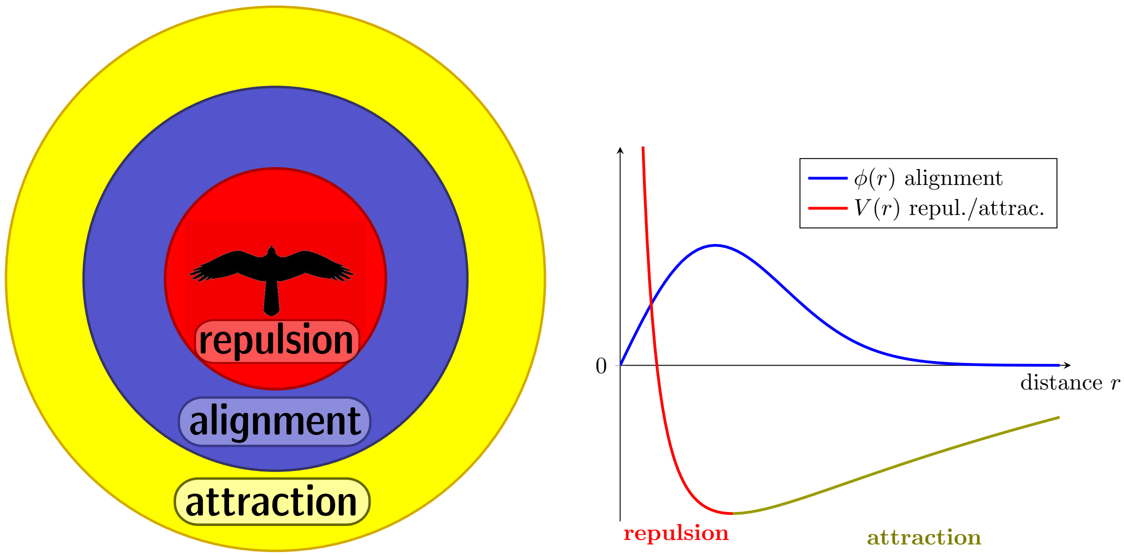

Figure 6.1: Left: Illustration of the Three-Zone Model. The Model Includes Three Types of Behavior: attraction/alignment/repulsion. **Right:** Attraction and Repulsion Are Represented Through the Function V, Alignment Is Described Via ϕ.

2) There exists M such that $|\mathbf{x}_j - \mathbf{x}_i| \leq M$ for all $i, j = 1, \ldots, N$ and for all $t \geq 0$.

In other words, in order to achieve a flock, agents should converge to a common velocity $\mathbf{v}_\infty$ and the distance between agents should remain (uniformly) bounded in time.

The key quantity for studying the emergence of a flock is the energy function, defined below:

$$\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) = \frac{1}{2N} \sum_{i=1}^{N} |\mathbf{v}_i|^2 + \frac{1}{2N^2} \sum_{i,j,i \neq j}^{N} V(|\mathbf{x}_j - \mathbf{x}_i|). \tag{6.3}$$

We can interpret this value as a sum of the kinetic and potential energy of the system.

By itself, the attraction-repulsion term (6.1)-(6.2) describes a Hamiltonian system and therefore preserves the total energy $\mathcal{E}$. However, the alignment term causes the total energy to decay with respect to time. That is, it plays the role of a 'friction term', making the system dissipative. More precisely, we can estimate the decay rate of the energy $\mathcal{E}$.

Lemma 6.3.1 *Let $\{\mathbf{x}_i, \mathbf{v}_i\}_i$ be the solution of the N-bird system* (6.1)(6.2)*. Then the energy $\mathcal{E}$* (6.3) *satisfies:*

$$\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) = -\frac{1}{2N^2}\sum_{i,j=1}^{N} \phi_{ij}|\mathbf{v}_j - \mathbf{v}_i|^2. \tag{6.4}$$

Since ϕ is a positive function, the energy $\mathcal{E}$ is decaying along the solution trajectory.

Proof. Taking the derivative in time of the energy leads to:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) &= \frac{1}{N}\sum_{i=1}^{N} \dot{\mathbf{v}}_i \cdot \mathbf{v}_i + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N} \nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|) \cdot (\mathbf{v}_j - \mathbf{v}_i) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j,j\neq i}^{N} \Big(\nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|) \cdot \mathbf{v}_i + \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\cdot \mathbf{v}_i\Big) \\
&\quad + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N} \nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|) \cdot (\mathbf{v}_j - \mathbf{v}_i).
\end{aligned}$$

By an argument of symmetry, we find:

$$\begin{aligned}
\sum_{i,j,i\neq j}^{N} \nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|) \cdot \mathbf{v}_i &= \sum_{i,j,i\neq j}^{N} V'(|\mathbf{x}_j - \mathbf{x}_i|)\frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \cdot \mathbf{v}_i \\
&= -\sum_{i,j,i\neq j}^{N} V'(|\mathbf{x}_j - \mathbf{x}_i|)\frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \cdot \mathbf{v}_j \\
&= \frac{1}{2}\sum_{i,j,i\neq j}^{N} V'(|\mathbf{x}_j - \mathbf{x}_i|)\frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \cdot (\mathbf{v}_i - \mathbf{v}_j).
\end{aligned}$$

Therefore, we can simplify:

$$\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) = \frac{1}{N^2}\sum_{i\neq j} \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\cdot \mathbf{v}_i.$$

Using now the symmetry $\phi_{ij} = \phi_{ji}$, we conclude

$$\frac{d}{dt}\mathcal{E} = \frac{1}{2N^2}\sum_{i,j=1}^{N} \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\cdot(\mathbf{v}_i - \mathbf{v}_j) = -\frac{1}{2N^2}\sum_{i,j=1}^{N} \phi_{ij}|\mathbf{v}_j - \mathbf{v}_i|^2.$$

□

Since the energy $\mathcal{E}$ is decaying, we deduce that the *potential energy* is bounded uniformly.

Lemma 6.3.2 *Take $\{\mathbf{x}_i, \mathbf{v}_i\}_i$ to be a solution of the three-zone model* (6.1)(6.2)*. There exists C such that for any time $t \geq 0$ and i, j:*

$$\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|) \quad \leq \quad C. \tag{6.5}$$

Proof. Take $C_0 = \mathcal{E}(\{\mathbf{x}_i(0), \mathbf{v}_i(0)\}_i)$. Since $\mathcal{E}$ is decaying along the solution trajectory, we deduce that for all t:

$$\frac{1}{2N}\sum_{i=1}^{N} |\mathbf{v}_i(t)|^2 + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|) \quad \leq \quad C_0,$$

Since the kinetic energy $\frac{1}{2}\sum_{i=1}^{N} |\mathbf{v}_i|^2$ is always positive, we deduce:

$$\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|) \quad \leq \quad 2C_0N^2.$$

Taking $C = 2C_0N^2$ yields the result. □

To take advantage of the lemma 6.3.2, we suppose that V is a *confinement* potential Karper *et al.* [100]:

$$V(r) \overset{r\to+\infty}{\longrightarrow} +\infty. \tag{6.6}$$

Roughly speaking, agents still experience an attraction force at long distances.

Under this assumption, we can prove the second part of flocking behavior, namely that the distances between agents are bounded.

Lemma 6.3.3 *Suppose V satisfies* (6.6)*. Then there exists r_M such that:*

$$|\mathbf{x}_j(t) - \mathbf{x}_i(t)| \leq r_M \qquad \textit{for any } i, j, \textit{ and } t \geq 0. \tag{6.7}$$

Proof. From lemma 6.3.2, we know that the potential energy is bounded, in particular:

$$V(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|) \quad \leq \quad C,$$

Since V satisfies (6.6), we deduce that there exists r_M such that:

$$V(r) > C \qquad \text{if} \quad r > r_M.$$

Since $V(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|)$ is bounded by C, we conclude that $|\mathbf{x}_j(t) - \mathbf{x}_i(t)|$ is bounded by r_M. $\square$

Remark. Similarly, if we suppose that V diverges at $r = 0$, then there exists a minimal distance r_m between agents:

$$|\mathbf{x}_j(t) - \mathbf{x}_i(t)| \geq r_m.$$

We can now conclude by deriving sufficient conditions to guarantee flocking behavior for the three-zone model.

Theorem 17 *Suppose V satisfies* (6.6) *and is bounded from below, ϕ is strictly positive and bounded. Moreover, assume that both ϕ and V have bounded first-order derivative. Then the three-zone model converges to a flock.*

Proof. Using lemma 6.3.3, we know that the distance between agents remains bounded: $|\mathbf{x}_j(t) - \mathbf{x}_i(t)| \leq r_M$. Since r_M is finite, we can take the minimum of ϕ on this interval:

$$m = \min_{s \in [0, r_M]} \phi(s).$$

Since ϕ is strictly positive, we deduce that $m > 0$. Therefore,

$$\phi_{ij} = \phi(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|) \geq m > 0.$$

Since the energy $\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i)$ is decaying and bounded from below we deduce that $\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) \stackrel{t\to\infty}{\longrightarrow} 0$ (if $\frac{d}{dt}\mathcal{E}$ is uniformly continuous). Therefore, using lemma 6.3.1,

$$\phi_{ij}|\mathbf{v}_j - \mathbf{v}_i|^2 \stackrel{t\to+\infty}{\longrightarrow} 0.$$

Since $\phi_{ij} \geq m > 0$, we conclude that $|\mathbf{v}_j - \mathbf{v}_i| \stackrel{t\to\infty}{\longrightarrow} 0$.

Moreover, the mean velocity, $\bar{\mathbf{v}} = \frac{1}{N}\sum_i \mathbf{v}_i$, is preserved by the dynamics (by symmetry), thus

$$\mathbf{v}_i(t) \stackrel{t\to\infty}{\longrightarrow} \bar{\mathbf{v}} \qquad \text{for all } i$$

which concludes the proof. To finish the proof, it suffices to establish the uniform continuity of $\frac{d}{dt}\mathcal{E}$, which is guaranteed if uniformly in time bounds on the second time derivative of $\mathcal{E}$ can be obtained. Indeed, we calculate

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}(\{\mathbf{x}_i,\mathbf{v}_i\}_i) &= -\frac{1}{2N^2}\sum_{i,j=1}^N \phi'(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|)|\mathbf{v}_j-\mathbf{v}_i|^2\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\cdot(\mathbf{v}_j-\mathbf{v}_i)\\ &\quad -\frac{1}{N^2}\sum_{i,j=1}^N \phi(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|)(\mathbf{v}_j-\mathbf{v}_i)\cdot(\dot{\mathbf{v}}_j-\dot{\mathbf{v}}_i) := \mathrm{I}+\mathrm{II},\end{aligned}$$

where II is given by

$$\begin{aligned}\mathrm{II} = &-\frac{1}{N^2}\sum_{i,j=1}^N \phi_{ij}(\mathbf{v}_j-\mathbf{v}_i)\cdot\Big(\frac{1}{N}\sum_{k=1}^N[\phi_{kj}(\mathbf{v}_k-\mathbf{v}_j)-\phi_{ki}(\mathbf{v}_k-\mathbf{v}_i)]\\ &+\frac{1}{N}\sum_{k\neq j}V'(|\mathbf{x}_k-\mathbf{x}_j|)\frac{\mathbf{x}_k-\mathbf{x}_j}{|\mathbf{x}_k-\mathbf{x}_j|}-\frac{1}{N}\sum_{k\neq i}V'(|\mathbf{x}_k-\mathbf{x}_i|)\frac{\mathbf{x}_k-\mathbf{x}_i}{|\mathbf{x}_k-\mathbf{x}_i|}\Big).\end{aligned}$$

In the sequel, we shall denote by C a time-independent constant whose value may vary from line to line (which may depend on N). Combining the boundedness of ϕ and ϕ' with the fact that $\frac{1}{2N}\sum_{i=1}^N|\mathbf{v}_i(t)|^2\le C$ holds uniformly in time, one can easily achieve $|\mathrm{II}|\le C$, where this bound on $|\mathrm{II}|$ is independent of both N and t. For the tricker term I, we have

$$|\mathrm{I}|\le\frac{C}{N^2}\sum_{i,j=1}^N|\mathbf{v}_i(t)-\mathbf{v}_j(t)|^3\le\frac{C}{N}\sum_{i=1}^N|\mathbf{v}_i(t)|^3.$$

Thus, the proof will be completed once we show that $\frac{1}{N}\sum\limits_{i=1}^N|\mathbf{v}_i(t)|^3\le C$. To this end, we calculate

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\sum_{i=1}^N|\mathbf{v}_i|^3 &= \frac{1}{N}\sum_{i,j=1}^N\phi_{ij}(\mathbf{v}_j-\mathbf{v}_i)\cdot\mathbf{v}_i|\mathbf{v}_i|+\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}V'(|\mathbf{x}_j-\mathbf{x}_i|)\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\cdot\mathbf{v}_i|\mathbf{v}_i|\\ &:= A+B.\end{aligned}$$

We have

$$A\le\frac{C}{N}\sum_{j=1}^N|\mathbf{v}_j|\cdot\sum_{i=1}^N|\mathbf{v}_i|^2-C\sum_{i=1}^N|\mathbf{v}_i|^3\le CN-C\sum_{i=1}^N|\mathbf{v}_i|^3$$

and

$$B \leq \frac{C}{N}\sum_{i,j=1}^{N}|\mathbf{v}_i|^2 \leq CN.$$

Therefore, we deduce that

$$\frac{1}{N}\frac{d}{dt}\sum_{i=1}^{N}|\mathbf{v}_i|^3 \leq C - \frac{1}{N}\sum_{i=1}^{N}|\mathbf{v}_i|^3, \quad \forall t \geq 0,\ \forall N,$$

from which we obtain the uniform bound $\frac{1}{N}\sum_{i=1}^{N}|\mathbf{v}_i(t)|^3 \leq C$ as desired. This ends the proof. $\square$

Remark. The reason that we can not achieve a bound like $|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{E}| \leq C$ with the constant C being independent of both N and t is due to the N-dependence of the constant C appearing in the lemma 6.3.2 and lemma 6.3.3.

6.3.3 Numerical Investigation

To illustrate the theorem 17, we perform numerical experiments for various choices of attraction-repulsion potentials. The theorem 17 guarantees that the velocities of the agents will converge to a single value, but there is no information about their position and in particular their relative distance. It is an open question to predict what *shape* the flock will have. All we know is that the distance between agents is bounded uniformly in time, which leaves the door open for many possible scenarios. Several studies have examined the stability of particular equilibrium solutions Fetecau *et al.* [81], Kolokolnikov *et al.* [102], Carrillo *et al.* [48], Carrillo and Huang [47], Carrillo *et al.* [45]. Models featuring self-propelled particles with an attraction-repulsion influence function have also been extensively studied D'Orsogna *et al.* [78], Chuang *et al.* [59], and several patterns observed (e.g. flock, mill formation). In our settings, however, mill formation cannot occur as the agents' velocities will converge to same value. Of particular interest is the shape of the swarm as the number of individuals N increases.

To first illustrate the theorem 17, we choose the following functions for the three-zone model (see Fig. 6.2-left):

$$V(r) = r\Big(\ln r - 1\Big) \quad , \quad \phi(r) = \frac{r}{2+r^3}. \tag{6.8}$$

The potential $V(r)$ diverges at $+\infty$ (i.e. satisfies (6.6)) and the alignment function is strictly positive (i.e. $\phi(r) > 0$), thus we can apply the theorem 17 and deduces that the agents will always converge to a flock. Notice that the alignment function $\phi(r)$ is integrable, thus without the attraction/repulsion term $V(r)$ there is no guarantee that a flock will occur. In others words, since the three-zone model (6.1)-(6.2) reduces to the Cucker-Smale model when $V' = 0$, having ϕ integrable is not a sufficient condition to guarantee flocking.

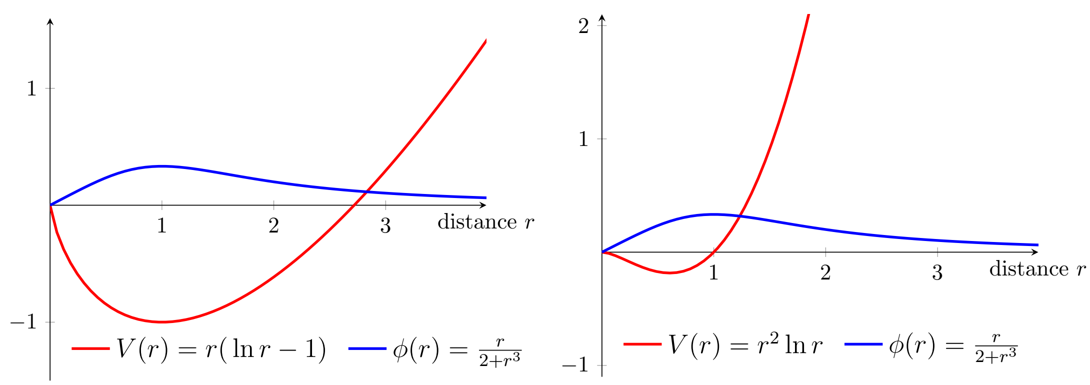

Figure 6.2: Attraction-Repulsion V and Alignment ϕ Used for the Simulations. In Both Cases, V Diverges at Infinity (i.e. Satisfies (6.6)).

We use as initial condition a uniform distribution of agents on a square of size $\sqrt{N}$. Their velocity is taken from a normal distribution. In the figure 6.3, we plot the distribution of agents after $t = 200$ time units for four different group size: $N = 20$, 50, 100 and 1000. For each group size, the agents regroup on a disc of radius close to 2 space units and the distribution is uniform inside the disc. As the number of agents N

increases, the radius remains constant and therefore the average distance between particles decreases. This type of pattern has been called *catastrophic* D'Orsogna *et al.* [78], Ruelle [139] since the density will eventually become singular as $N \longrightarrow +\infty$ and thus there is no *thermodynamic* limit. In other words, the repulsion is not strong enough to push back nearby agents.

In our second illustration, we reduce even further the repulsion force among agents using the following potential $V(r)$ (see Fig. 6.2-right):

$$V(r) = r^2 \ln r \quad , \quad \phi(r) = \frac{r}{2 + r^3}. \tag{6.9}$$

There are now two possible equilibrium points for attraction/repulsion at the distances $r = 1$ and $r = 0$. In the figure 6.4, we plot the distribution of agents after $t = 200$ time units for two group sizes ($N = 50$ and $N = 200$). We observe that the agents are now aggregating on a circle. This indicates that the equilibrium distribution might be a domain of dimension one Balagué *et al.* [16].

The evolution of the total energy $\mathcal{E}$ (6.3) for all cases is given in figure 6.5. As predicted by lemma 6.3.1, the energy is always strictly decaying. Moreover, we observe oscillatory behavior between fast and slow decay. The reason for this behavior is the repreated *contraction-expansion* of the spatial configuration as the agents approach equilibrium. The energy is decaying slower when agents are far away due to the decay of ϕ at distances greater than one space unit. These types of oscillation are also observed for the convergence of Boltzmann equation toward global equilibrium Filbet [82], Desvillettes and Villani [76].

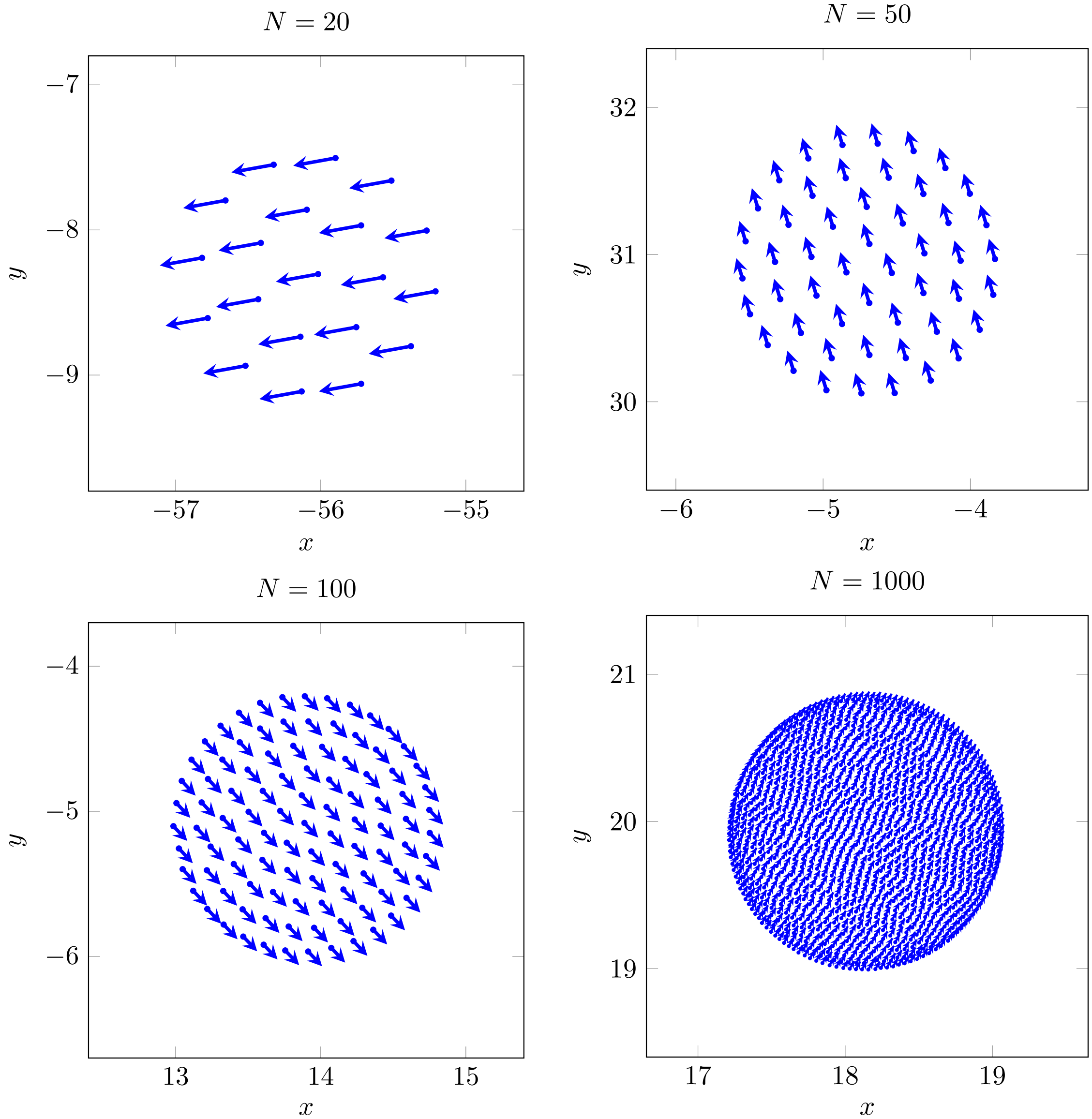

Figure 6.3: Simulation of the Three-Zone Model (6.1)-(6.2) with Potential V and Alignment Function ϕ Given By (6.8). Agents Regroup on a Disc of Size $R \approx 1.8$ for Any Group Sizes. Parameters: $\Delta t = .05$, Total Time $t = 200$ Unit Time.

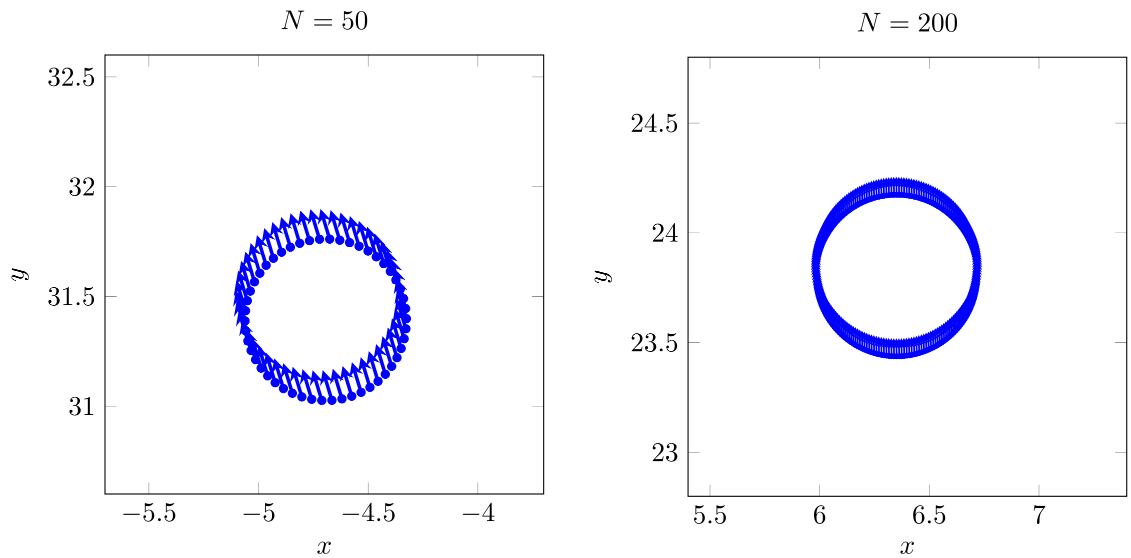

Figure 6.4: Simulation of the Three-Zone Model (6.1)-(6.2) with Potential V and Alignment Function ϕ Given By (6.9). Agents Regroup on a Circle of Size $R \approx .5$. Parameters: $\Delta t = .05$, Total Time $t = 200$ Unit Time.

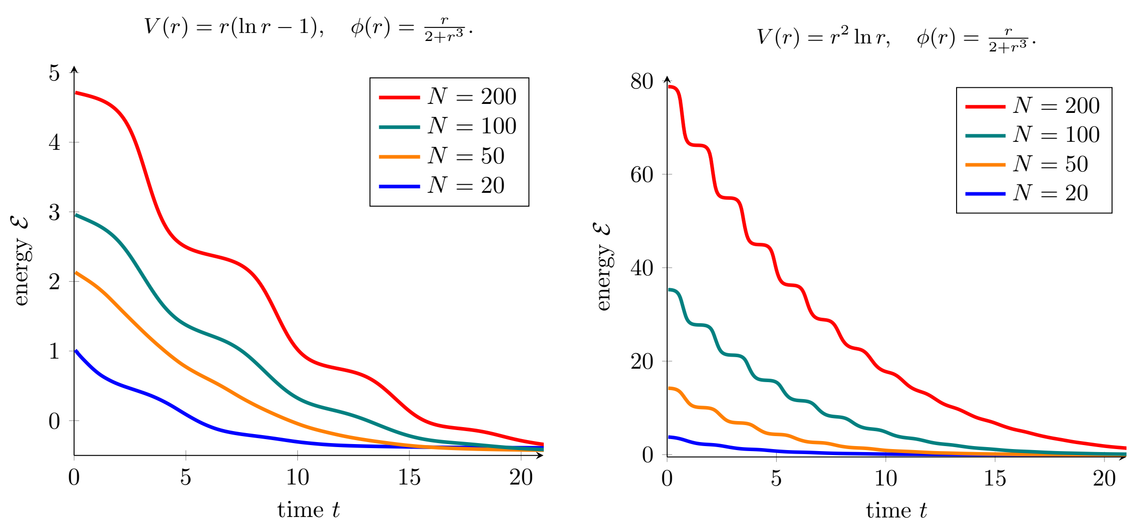

Figure 6.5: Evolution of the Energy $\mathcal{E}$ for the Solutions Depicted in Figures 6.3 and 6.4 (Left and Right Figure Respectively). The Energy Is Always Decaying but Also Oscillates Between Fast and Slow Decays. These Oscillations Can Be Explained by the Successive *Contraction-Expansion* of the Spatial Configuration. The Decay of the Energy Is Faster When Agents Are Closer to Each Other.

6.4 Kinetic Equation

6.4.1 Formal Derivation

We would like to investigate the flocking behavior of dynamics in the limit of infinitely many agents, i.e. $N \longrightarrow \infty$. With this aim, we introduce the so-called *kinetic equation* associated to *the particle dynamics* (6.1)-(6.2). To derive the kinetic equation, one can introduce the empirical distribution Bolley *et al.* [31], Carrillo *et al.* [44], Degond and Motsch [74], Degond *et al.* [73, 71], Jabin [93]:

$$f_N(\mathbf{x},\mathbf{v},t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{x}_i(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_i(t)}(\mathbf{v}), \tag{6.1}$$

where $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}$ is the solution of the system (6.1)-(6.2). By integrating the empirical distribution f_N against a test function, we can show that f_N satisfies in a weak-sense the following kinetic equation:

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot\Big(F[f]f\Big) = 0, \tag{6.2}$$

with

$$\begin{aligned} F[f](\mathbf{x},\mathbf{v}) \;\; = \;\; & -\int_{\mathbf{y}\in\mathbb{R}^n} \nabla_{\mathbf{x}} V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ & + \int_{(\mathbf{y},\mathbf{w})\in\mathbb{R}^n\times\mathbb{R}^n} \phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v}) f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}, \end{aligned} \tag{6.3}$$

and

$$\rho(\mathbf{x}) = \int_{\mathbf{v}\in\mathbb{R}^n} f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \tag{6.4}$$

the spatial distribution of particles.

The rigorous convergence of the particle dynamics (6.1)-(6.2) toward the kinetic equation (6.2) is out of the scope of the present paper. But following the methods developed in Spohn [144], Bolley *et al.* [31], Carrillo *et al.* [44], one would expect to have an error estimation between the empirical distribution f_N (6.1) and a 'classic' solution f to the kinetic equation (6.2)-(6.3). More precisely, for any time T, there exists a constant c such that the Wasserstein distance between the two distributions satisfies:

$$\mathcal{W}(f(T), f_N(T)) \leq \mathcal{W}(f(0), f_N(0))\mathrm{e}^{c \cdot T}.$$

Notice that this result cannot be used to study the long time behavior of the solution of the kinetic equation $f(t)$ since the error-bound is not uniform in time.

6.4.2 Flocking Behavior

To analyze the long-time behavior of the solution to the kinetic equation (6.2), we introduce the following energy function:

$$\mathcal{E} = \frac{1}{2}\int_{(\mathbf{x},\mathbf{v})\in\mathbb{R}^n\times\mathbb{R}^n} |\mathbf{v}|^2 f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} + \frac{1}{2}\int_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^n\times\mathbb{R}^n} V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \quad (6.5)$$

Symmetry argument shows that the energy $\mathcal{E}$ is decaying (i.e. the system is dissipative).

Lemma 6.4.1 *The functional $\mathcal{E}$ satisfies:*

$$\frac{d}{dt}\mathcal{E} = -\frac{1}{2}\int_{(\mathbf{x},\mathbf{v}),(\mathbf{y},\mathbf{w})} \phi(|\mathbf{y}-\mathbf{x}|)|\mathbf{w}-\mathbf{v}|^2 f(\mathbf{x},\mathbf{v})f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}. \quad (6.6)$$

Proof. Taking the derivative of the energy with respect to time leads to:

$$\begin{aligned}\frac{d}{dt}\mathcal{E} &= \frac{1}{2}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^2 \partial_t f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\ &\qquad + \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} V(|\mathbf{y}-\mathbf{x}|)\Big(\partial_t\rho(\mathbf{x})\rho(\mathbf{y}) + \rho(\mathbf{x})\partial_t\rho(\mathbf{y})\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &=: A + B.\end{aligned}$$

Then,

$$
\begin{aligned}
A &= -\frac{1}{2}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^2 \nabla_{\mathbf{x}} \cdot \Big(\mathbf{v} f\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} - \frac{1}{2}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^2 \nabla_{\mathbf{v}} \cdot \Big(F[f] f\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&= 0 + \int_{\mathbf{x},\mathbf{v}} \mathbf{v}\cdot F[f] f\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&= \int_{\mathbf{x},\mathbf{v}}\int_{\mathbf{y},\mathbf{w}} \mathbf{v}\cdot \phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v}) f(\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \\
&\qquad - \int_{\mathbf{x},\mathbf{y}}\int_{\mathbf{v}} \mathbf{v}\cdot \nabla_{\mathbf{x}} V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{y}) f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v} \\
&= \int_{\mathbf{x},\mathbf{v}}\int_{\mathbf{y},\mathbf{w}} \mathbf{v}\cdot \phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v}) f(\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \\
&\qquad - \int_{\mathbf{x},\mathbf{y}} \nabla_{\mathbf{x}} V(|\mathbf{y}-\mathbf{x}|)\cdot \rho(\mathbf{x})\rho(\mathbf{y}) u(\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.
\end{aligned}
$$

To compute the term B, we notice that the density distribution ρ satisfies the continuity equation:

$$
\partial_t \rho + \nabla_{\mathbf{x}}\cdot(\rho u) = 0 \quad \text{with} \;\; \rho(\mathbf{x})u(\mathbf{x}) = \int_{\mathbf{v}} \mathbf{v} f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}. \tag{6.7}
$$

We deduce:

$$
\begin{aligned}
B &= \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} V(|\mathbf{y}-\mathbf{x}|)\Big(-\nabla_{\mathbf{x}}\cdot\Big[\rho(\mathbf{x})u(\mathbf{x})\Big]\,\rho(\mathbf{y}) - \rho(\mathbf{x})\,\nabla_{\mathbf{y}}\cdot\Big[\rho(\mathbf{y})u(\mathbf{y})\Big]\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \nabla_{\mathbf{x}}\Big[V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{y})\Big]\cdot\rho(\mathbf{x})u(\mathbf{x}) + \nabla_{\mathbf{y}}\Big[V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{x})\Big]\cdot\rho(\mathbf{y})u(\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \int_{\mathbf{x},\mathbf{y}} \nabla_{\mathbf{x}} V(|\mathbf{y}-\mathbf{x}|)\cdot\rho(\mathbf{x})\rho(\mathbf{y})u(\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y},
\end{aligned}
$$

using the change of variable in the second term $\mathbf{x}\leftrightarrow\mathbf{y}$. Therefore,

$$
\begin{aligned}
A + B &= \int_{\mathbf{x},\mathbf{v}}\int_{\mathbf{y},\mathbf{w}} \mathbf{v}\cdot\phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v}) f(\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \\
&= -\frac{1}{2}\int_{\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}} \phi(|\mathbf{y}-\mathbf{x}|)|\mathbf{w}-\mathbf{v}|^2 f(\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}.
\end{aligned}
$$

□

The decay of the energy $\mathcal{E}$ is the cornerstone to prove the flocking of the dynamics. However, in the context of the kinetic equation (6.2), we deal with a continuum of agents and therefore it is more delicate to prove that the velocity of *all* agents converge to a common value. We prove a L^1 type estimate for the decay of the velocity toward its average value. The method relies mainly on stochastic process theory.

We denote by $(\mathbf{X}_t, \mathbf{V}_t)$ and $(\mathbf{Y}_t, \mathbf{W}_t)$ two independent stochastic processes with probability density function $f(\cdot, t)$ solution to (6.2). The energy $\mathcal{E}$ (6.5) can be written as:

$$\mathcal{E} = \frac{1}{2}\mathbb{E}[|\mathbf{V}|^2] + \frac{1}{2}\mathbb{E}[V(|\mathbf{X}-\mathbf{Y}|)], \tag{6.8}$$

and its decay as:

$$\frac{d}{dt}\mathcal{E} = -\frac{1}{2}\mathbb{E}[\phi(|\mathbf{X}-\mathbf{Y}|)|\mathbf{V}-\mathbf{W}|^2]. \tag{6.9}$$

We first recall an elementary lemma in stochastic process theory that will be useful later. For the sake of completeness of the manuscript, we also give the proof.

Lemma 6.4.2 *Suppose $\mathbf{X}_t$ is bounded uniformly in L^2 and that $\mathbf{X}_t$ converges in probability to 0 (i.e. $\mathbf{X}_t \xrightarrow{P} 0$). Then: $\mathbf{X}_t \xrightarrow{t\to+\infty} 0$ in L^1.*

Proof. First, we show that $\mathbf{X}_t$ bounded in L^2 implies that $\mathbf{X}_t$ is uniformly integrable. Denote C the constant such that $\mathbb{E}[|\mathbf{X}_t|^2] \leq C$ for all t. We have:

$$\begin{aligned}
\mathbb{E}[|\mathbf{X}_t|\,\mathbb{1}_{\{|\mathbf{X}_t|\geq k\}}] &\leq \left(\mathbb{E}[\mathbf{X}_t^2]\,\mathbb{E}[\mathbb{1}^2_{\{|\mathbf{X}_t|\geq k\}}]\right)^{1/2} \leq C^{1/2}\mathbb{P}(|\mathbf{X}_t| \geq k)^{1/2} \\
&\leq C^{1/2}\left(\frac{1}{k^2}\mathbb{E}[\mathbf{X}_t^2]\right)^{1/2} \leq \frac{C}{k} \quad \xrightarrow{k\to+\infty} 0,
\end{aligned}$$

using (resp.) Cauchy-Schwarz and Markov inequalities.

We now use that $\mathbf{X}_t \xrightarrow{P} 0$ to conclude. Fix $\varepsilon > 0$:

$$\mathbb{E}[|\mathbf{X}_t|] = \mathbb{E}[|\mathbf{X}_t|\,\mathbb{1}_{\{|\mathbf{X}_t|\leq\varepsilon/2\}}] + \mathbb{E}[|\mathbf{X}_t|\mathbb{1}_{\{|\mathbf{X}_t|>\varepsilon/2\}}] \leq \frac{\varepsilon}{2} + \mathbb{E}[|\mathbf{X}_t|\mathbb{1}_{\{|\mathbf{X}_t|>\varepsilon/2\}}].$$

By uniform integrability, there exists $\delta > 0$ such that:

$$\mathbb{E}[|\mathbf{X}_t|\mathbb{1}_A] \leq \varepsilon/2 \qquad \text{if} \quad \mathbb{P}(A) \leq \delta. \tag{6.10}$$

Since $\mathbf{X}_t \xrightarrow{P} 0$, there exists t_* such that $\mathbb{P}(|\mathbf{X}_t| > \varepsilon/2) \leq \delta$ for $t \geq t_*$. Combined with (6.10), we deduce $\mathbb{E}[|\mathbf{X}_t| \mathbb{1}_{\{|\mathbf{X}_t|>\varepsilon/2\}}] \leq \varepsilon/2$. Therefore, $\mathbb{E}[|\mathbf{X}_t|] \leq \varepsilon$ for $t \geq t_*$. $\square$

We now can prove our main theorem.

Theorem 18 *Under the same assumptions of theorem 17, with the additional assumption that* $\inf_{s\geq 0} \phi(s) \geq c > 0$ *for some c, the solution* f *of* (6.2) *satisfies:*

$$\int_{\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}} |\mathbf{v}-\mathbf{w}| f(\mathbf{x},\mathbf{v},t) f(\mathbf{y},\mathbf{w},t) \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \xrightarrow{t\to+\infty} 0 \tag{6.11}$$

Proof. Denote $(\mathbf{X}_t, \mathbf{V}_t)$ and $(\mathbf{Y}_t, \mathbf{W}_t)$ two independent stochastic processes with density distribution f (6.2). We first show that $\mathbf{V}_t - \mathbf{W}_t \xrightarrow{P} 0$. Fix $\delta > 0$ and $\varepsilon > 0$. We have to show that: $\mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta) < \varepsilon$ for a sufficiently large value of t.

Since the energy $\mathcal{E}$ is uniformly bounded, there exists C such that $\mathbb{E}[V(|\mathbf{X}_t - \mathbf{Y}_t|)] \leq C$ for all time t. Since the potential V satisfies (6.6), there exists L such that $V(r) \geq 2C/\varepsilon$ for $r > L$. Now we split our estimation:

$$\begin{aligned}
\mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta) &= \mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta, \, |\mathbf{X}_t - \mathbf{Y}_t| \leq L) \\
&\qquad + \mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta, \, |\mathbf{X}_t - \mathbf{Y}_t| > L) \\
&=: A + B.
\end{aligned}$$

First, we find an upper-bound for B:

$$\begin{aligned}
B &\leq \mathbb{P}(|\mathbf{X}_t - \mathbf{Y}_t| > L) \;\leq\; \mathbb{P}(V(|\mathbf{X}_t - \mathbf{Y}_t|) > 2C/\varepsilon) \\
&\leq \frac{\varepsilon}{2C} \mathbb{E}[V(|\mathbf{X}_t - \mathbf{Y}_t|)] \;\leq\; \frac{\varepsilon}{2}.
\end{aligned}$$

Then, we investigate A:

$$\begin{aligned}
A &\leq \mathbb{P}\Big(|\mathbf{V}_t - \mathbf{W}_t| > \delta \,\Big|\, |\mathbf{X}_t - \mathbf{Y}_t| \leq L\Big) \\
&\leq \frac{1}{\delta^2} \mathbb{E}\Big[|\mathbf{V}_t - \mathbf{W}_t|^2 \,\Big|\, |\mathbf{X}_t - \mathbf{Y}_t| \leq L\Big]
\end{aligned}$$

Consider $m = \inf_{r\le L} \phi(r) > 0$. We have:

$$\begin{aligned} A &\le \frac{1}{m\delta^2}\mathbb{E}\Big[\phi(|\mathbf{X}_t - \mathbf{Y}_t|)|\mathbf{V}_t - \mathbf{W}_t|^2 \,\Big|\, |\mathbf{X}_t - \mathbf{Y}_t| \le L\Big] \\ &\le -\frac{2}{m\delta^2}\frac{d\mathcal{E}}{dt}. \end{aligned}$$

If $\frac{d\mathcal{E}}{dt}$ is uniformly continuous, then we have $\frac{d\mathcal{E}}{dt} \xrightarrow{t\to+\infty} 0$, hence there exists t_* such that $A \le \varepsilon/2$ for $t \ge t_*$. Therefore, we conclude:

$$\mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta) = A + B \le \varepsilon$$

for $t \ge t_*$. Hence, $\mathbf{V}_t - \mathbf{W}_t \xrightarrow{P} 0$.

Now, since the energy $\mathcal{E}$ remains uniformly bounded, we deduce that $\mathbf{V}_t - \mathbf{W}_t$ is uniformly bounded in L^2:

$$\mathbb{E}[|\mathbf{V}_t - \mathbf{W}_t|^2] \le C.$$

Using lemma 6.4.2, we conclude that: $\mathbf{V}_t - \mathbf{W}_t \xrightarrow{t\to+\infty} 0$ in L^1 leading to the result (6.11). To complete the proof, we have to demonstrate the uniform continuity of $\frac{d}{dt}\mathcal{E}$, which is ensured if uniformly in time bounds on the second time derivative of $\mathcal{E}$ can be obtained. Actually, this fact can be shown by mimicking the computations performed in the corresponding part of the proof of theorem 17 and the computations are essentially the same. This ends the proof. $\square$

Remark. The reason that we need the somehow strong assumption that $\inf\limits_{s\ge 0} \phi(s) \ge c > 0$ is as follows: Similar to the computations performed in theorem 17, we will have $|\frac{d^2}{dt^2}\mathcal{E}| \le C$ for a time-independent constant C if we can establish a uniform-in-time bound on $\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^3 f(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$. Differentiating in time gives us

$$\frac{d}{dt}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^3 f(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \le C - \int_{\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{w}} \phi(|\mathbf{y}-\mathbf{x}|)|\mathbf{v}|^3 f(\mathbf{x},\mathbf{v})f(\mathbf{y},\mathbf{w})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w},$$

in which $C > 0$ is independent of time. With $\inf_{s\ge 0}\phi(s) \ge c > 0$ and Gronwall's inequality we obtain the desired uniform bound on $\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^3 f(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$.

6.5 Conclusion

In this study, we have derived sufficient conditions for the emergence of flock in a system of particles which includes attraction-repulsion and alignment interactions. The result sides with previous work on the Cucker-Smale model, but there is additional difficulty in the dynamics considered in this study, since energy estimates are insufficient to prove convergence. In particular, we do not have exponential decay towards equilibrium. However, it might be possible to obtain a stronger result, for instance by using commutator techniques Villani [148], Bakey *et al.* [13] to compensate the lack of Gronwall-type inequality.

It would also be interesting to adapt this model to other types of collective behavior, such as milling. Of course, this would require several adjustments to our starting assumptions: one could suppose that alignment only occurs at close distances (i.e. the function ϕ has a compact support). Another extension would be to consider non-metric interactions such as topological distance or non-symmetric interactions (e.g. presence of leaders).

Chapter 7

CONCLUSION AND FUTURE WORK

7.1 Conclusion

In this report, we introduced the notion of *interacting particle systems* which appears in enormous branches of applied mathematics, including but not limited to flocking dynamics, opinion models, econophysics, statistical physics, and artificial neural networks. One important tool in the analysis of these N particle systems at the microscopic level is the so-called *propagation of chaos*, which (occasionally) allows us to derive a partial differential equation governing the evolution of the one-particle marginal probability density function as $N \to \infty$. This transition from a microscopic, individual-based description to a macroscopic description is crucial for understanding of the large-scale asymptotic behavior of the underlying particle systems. Once a deterministic PDE is obtained, one can apply standard PDE techniques to investigate the long time behavior of its solution. In this respect, We are especially interested in the so-called *entropy methods* and convergence of probability densities in the sense of relative entropy. Entropy methods have been intertwined with information theory, logarithmic Sobolev inequalities, Bakry-Emery approach, stochastic analysis and have become increasingly popular in many interdisciplinary research projects.

7.2 Future Work

Currently we are working on other interacting particles systems. One of my current project with my Ph.D advisor is to identity the proper "limit object" in a *moderately* interacting N-particle system arising in a tumor growth model, which is quite different from the usual mean-field setting. Nevertheless, we expect that a propagation-of-chaos type result will still hold in the large N limit. Another project I am working on with my advisor consists of an attempt to perform density estimations based on geometric tools known as the mean curvature flow, with the intent to "beat" the traditional well-known kernel method (which is extremely popular in literature). Research in areas closely related to interacting particle systems is also possible, for instance we have great interest in the mathematics of the theoretic machine/deep learning.

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APPENDIX A

PROOF OF LEMMA 2.2.1

Proof. Suppose first that the stochastic process $(S_1, \ldots, S_N)$ satisfies the propagation of chaos. Let φ be a test function, $Z^{(N)} = \langle \rho^{(N)}_{emp}, \varphi \rangle$ a random variable and $\overline{Z} = \mathbb{E}[\varphi(\overline{S}_1)]$ a constant. For notation convenience, we write $[N] := \{1, 2, \ldots, N\}$. To prove that $Z^{(N)}$ converges in law to $\overline{Z}$, it is sufficient to prove the convergence in L^2:

$$
\begin{aligned}
\mathbb{E}[|Z^{(N)} - \overline{Z}]|^2] \quad &= \quad \mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^{N} \varphi(S_i) - \overline{Z} \right|^2 \right] \\
&= \quad \frac{1}{N^2} \sum_{i,j,i \neq j} \mathbb{E}[\varphi(S_i)\varphi(S_j)] + \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E}[\varphi^2(S_i)] \\
&\qquad - \frac{2}{N} \sum_{i=1}^{N} \mathbb{E}[\varphi(S_i)]\overline{Z} \quad + \quad \overline{Z}^2 \\
&\overset{N \to +\infty}{\longrightarrow} \quad \overline{Z}^2 + 0 - 2\overline{Z} \cdot \overline{Z} + \overline{Z}^2 = 0
\end{aligned}
\tag{A.1}
$$

using (2.2) with $k = 2$ and $k = 1$.

Proving the converse is more challenging. Let's take as test function $\varphi(s_1, \ldots, s_k) = \varphi_1(s_1) \ldots \varphi_k(s_k)$ and denote the random variable $Z_i = \langle \rho^{(N)}_{emp}, \varphi_i \rangle$ for all i. By assumption, Z_i converges in law to the constant $\langle \bar{\rho}_1, \varphi_i \rangle = \mathbb{E}[\varphi_i(\overline{S}_1)]$. We deduce:

$$
\begin{aligned}
\left| \mathbb{E}[\varphi_1(S_1) \cdots \varphi_k(S_k)] - \mathbb{E}[\varphi_1(\overline{S}_1) \cdots \varphi_k(\overline{S}_k)] \right| \quad &\leq \quad \mathbb{E}[\varphi_1(S_1) \cdots \varphi_k(S_k)] - \mathbb{E}[Z_1 \cdots Z_k] \\
&\qquad + \mathbb{E}[Z_1 \cdots Z_k] - \mathbb{E}[\varphi_1(\overline{S}_1) \cdots \varphi_k(\overline{S}_k)] \\
&=: \quad |A| \; + \; |B|.
\end{aligned}
$$

Since each Z_i converges to the constant $\mathbb{E}[\varphi_i(\overline{S}_i)]$, all the product in B convergence to zero using Slutsky's theorem. For A, we use the invariance by permutations:

$$
\begin{aligned}
A \quad &= \quad \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \mathbb{E}[\varphi_1(S_{\sigma(1)}) \cdots \varphi_k(S_{\sigma(k)})] - \frac{1}{N^k} \sum_{(i_1, \ldots, i_k) \in [N]^k} \mathbb{E}[\varphi_1(S_{i_1}) \cdots \varphi_k(S_{i_k})] \\
&= \quad \frac{(N-k)!}{N!} \sum_{(i_1, \ldots, i_k) \in \mathcal{P}_{N,k}} \mathbb{E}[\varphi_1(S_{i_1}) \cdots \varphi_k(S_{i_k})] - \frac{1}{N^k} \sum_{(i_1, \ldots, i_k) \in [N]^k} \mathbb{E}[\varphi_1(S_{i_1}) \cdots \varphi_k(S_{i_k})],
\end{aligned}
\tag{A.2}
$$

where $\mathcal{P}_{N,k} \subset [N]^k$ is the set of all the permutations of k elements in $[N]$ and in particular $|\mathcal{P}_{N,k}| = N!/(N-k)!$. To conclude, we split the set $[N]^k$ in two parts:

$$
\begin{aligned}
A \quad = \quad & \left(1 - \frac{1}{N^k} \cdot \frac{N!}{(N-k)!} \right) \cdot \frac{(N-k)!}{N!} \sum_{(i_1, \ldots, i_k) \in \mathcal{P}_{N,k}} \mathbb{E}[\varphi_{i_1}(S_{i_1}) \cdots \varphi_{i_k}(S_{i_k})] \\
& - \frac{1}{N^k} \sum_{(i_1, \ldots, i_k) \in [N]^k \setminus \mathcal{P}_{N,k}} \mathbb{E}[\varphi_1(S_{i_1}) \cdots \varphi_k(S_{i_k})]
\end{aligned}
$$

Thus, denoting C an upper-bound for any $\mathbb{E}[\varphi_1(S_{i_1})\cdots\varphi_k(S_{i_k})]$:

$$\begin{aligned} |A| &\leq \left(1 - \frac{N!}{N^k(N-k)!}\right) C \quad + \quad \frac{1}{N^k}(N^k - |\mathcal{P}_{N,k}|)C \qquad \text{(A.3)} \\ &= 2\left(1 - \frac{N!}{N^k(N-k)!}\right) C = 2\left(1 - \frac{(N-k+1)}{N}\cdots\frac{N-1}{N}\cdot\frac{N}{N}\right) C \quad \overset{N\to+\infty}{\longrightarrow} 0. \end{aligned}$$

$\square$

APPENDIX B

GINI INDEX DISPERSIVE WAVE

We estimate the Gini coefficient for a (continuous) distribution of the form:

$$\rho(x) = (1-r)\cdot\delta_0(x) + r\cdot\frac{1}{\sigma}\phi\left(\frac{x-c}{\sigma}\right) \tag{B.1}$$

where ϕ is the standard normal distribution, r, c, σ some positive constant with $r \in [0,1]$. The law ρ can be represented by a random variable:

$$X = (1-Y)\cdot 0 + Y\cdot(c+\sigma Z) \tag{B.2}$$

with Y random Bernoulli variable with probability r (i.e. $Y \sim B(r)$), Z a random variable with normal law (i.e. $Z \sim \mathcal{N}(0,1)$), Y and Z being independent. To estimate the Gini index of ρ, we take two independent random variables X_1 and X_2 with such law and estimate the expectation of their difference:

$$\begin{aligned} G &= \frac{1}{2\mu}\mathbb{E}[|X_1 - X_2|] = \frac{1}{2\mu}\mathbb{E}[|Y_1\cdot(c+\sigma Z_1) - Y_2\cdot(c+\sigma Z_2)|] \\ &= \frac{1}{2\mu}\mathbb{E}[|c(Y_1 - Y_2) + \sigma(Y_1 Z_1 - Y_2 Z_2)|] \end{aligned} \tag{B.3}$$

We then take the conditional expectation with respect to Y_1 and Y_2:

$$\begin{aligned} 2\mu G &= 0 + \mathbb{E}[|c+\sigma Z_1|]\mathbb{P}[Y_1 = 1, Y_2 = 0] \\ &\quad + \mathbb{E}[|-c-\sigma Z_2|]\mathbb{P}[Y_1 = 0, Y_2 = 1] \\ &\quad + \mathbb{E}[|\sigma(Z_1 - Z_2)|]\mathbb{P}[Y_1 = 1, Y_2 = 1] \\ &= 2\cdot\mathbb{E}[|c+\sigma Z_1|]r(1-r) + \mathbb{E}[|\sigma(Z_1 - Z_2)|]r^2 \end{aligned} \tag{B.4}$$

For large c, we made the approximation $\mathbb{E}[|c+\sigma Z_1|] \approx \mathbb{E}[c+\sigma Z_1] = c$. Moreover, the expectation of the difference between two standard Gaussian random variables is known explicitly: $\mathbb{E}[|Z_1 - Z_2|] = 2/\sqrt{\pi}$. We deduce:

$$2\mu G \approx 2c\cdot r(1-r) + \sigma\frac{2}{\sqrt{\pi}}r^2. \tag{B.5}$$

Furthermore, if $r = \mu/c$, we obtain:

$$G \approx 1 - \frac{\mu}{c} + \frac{\sigma\mu}{\sqrt{\pi}c^2}. \tag{B.6}$$

APPENDIX C

PROOF OF THEOREM 7

Proof The whole strategy is of course to find some δ such that if

$$\int q(t,x)\,\log\frac{q(t,x)}{q_\infty(x)}\,\mathrm{d}x\le\delta, \tag{C.1}$$

then we have for the ε of Theorem 6

$$\int\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,dx\le\varepsilon. \tag{C.2}$$

We start with using Lemma 3.5.5 for $C=2$ and note that

$$\begin{aligned}\frac{1}{4}\int_{q_\infty/2\le q\le 2\,q_\infty}&\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x+\frac{1}{8}\int_{q\le q_\infty/2}q_\infty(x)\,\mathrm{d}x+\frac{\log 2}{4}\int_{q\ge 2\,q_\infty}q(t,x)\,\mathrm{d}x\\ &\le\int q(t,x)\,\log\frac{q(t,x)}{q_\infty(x)}\,\mathrm{d}x.\end{aligned} \tag{C.3}$$

Observe that if $q\le q_\infty/2$ then

$$\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\le q_\infty(x),$$

so the first two terms already provides the straightforward bound

$$\int_{q\le 2\,q_\infty}\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x\le 8\int q(t,x)\,\log\frac{q(t,x)}{q_\infty(x)}\,\mathrm{d}x. \tag{C.4}$$

Now if $q\ge q_\infty$ then

$$\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\le\frac{(q(t,x))^2}{q_\infty(x)}.$$

Therefore for any $p>1$,

$$\begin{aligned}\int_{q\ge 2\,q_\infty}\frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x&\le\int_{q\ge 2\,q_\infty}\frac{|q(t,x)|^2}{q_\infty(x)}\,\mathrm{d}x\\ &\le\left(\int_{q\ge 2\,q_\infty}q(t,x)\,\mathrm{d}x\right)^{1-1/p}\left(\int_{q\ge 2\,q_\infty}\frac{|q(t,x)|^{p+1}}{(q_\infty(x))^p}\,\mathrm{d}x\right)^{1/p}.\end{aligned}$$

We now use Corollary 3.5.9 to find that

$$\begin{aligned}\int_{q\ge 2\,q_\infty}\frac{|q(t,x)|^{p+1}}{(q_\infty(x))^p}\,\mathrm{d}x&\le C_p\int\left(\mathrm{e}^{(-(p+1)\,\lambda_0+p)\,x}+\mathrm{e}^{p\,x}\,(q(0,x))^{p+1}\right)\mathrm{d}x\\ &\le C_p'\int\mathrm{e}^{(-(p+1)\,\lambda+p)\,x}\,\mathrm{d}x,\end{aligned}$$

in which $\lambda \in (\frac{1}{2}, \lambda_0)$. Now we take p close enough to 1 such that $p-(p+1)\lambda<0$ which is always possible if $\lambda_0 > \frac{1}{2}$. For this choice of p, we hence obtain that

$$\int_{q\geq 2q_\infty} \frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x \leq C_p \left(\int_{q\geq 2q_\infty} q(x)\,\mathrm{d}x\right)^{1-1/p}.$$

Going back to (C.3), we can conclude that

$$\int_{q\geq 2q_\infty} \frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x \leq C_p \left(\int q(t,x) \log \frac{q(t,x)}{q_\infty(x)}\,\mathrm{d}x\right)^{1-1/p},$$

and combining this with (C.4), we deduce that for some C and $\theta \in (0,1)$

$$\int \frac{|q(t,x)-q_\infty(x)|^2}{q_\infty(x)}\,\mathrm{d}x \leq C \left(\int q(t,x) \log \frac{q(t,x)}{q_\infty(x)}\,\mathrm{d}x\right)^{\theta} \leq C\,\delta^\theta.$$

It is enough to choose δ being small enough to conclude the proof. □

APPENDIX D

PROPAGATION OF CHAOS FOR REPEATED AVERAGING

In this appendix, we sketch the proof of the so-called propagation of chaos [145, 50], relying on a martingale-based technique developed in [118]. We emphasize that the proof presented here only a slight modification of the Theorem 6 in [39].

We equip the space $\mathcal{P}(\mathbb{R}_+)$ with the Wasserstein distance with exponent 1, which is defined via

$$W_1(\mu,\nu)=\sup_{\|\nabla\varphi\|_\infty\leq 1}\langle \mu-\nu,\varphi\rangle$$

for $\mu,\nu\in\mathcal{P}(\mathbb{R}_+)$. The propagation of chaos result is summarized in the following

Theorem 19 *Denote the empirical distribution of the repeated averaging N particle system* (4.1) *at time t as*

$$\rho_{\mathrm{emp}}(t):=\frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)},$$

and let $\rho(t)$ be the solution of (4.6) *with initial data $\rho(0)$. If*

$$\mathbb{E}[W_1(\rho_{\mathrm{emp}}(0),\rho(0))]\to 0 \text{ as } N\to\infty, \tag{D.1}$$

then we have that

$$\mathbb{E}[W_1(\rho_{\mathrm{emp}}(t),q(t))]\to 0 \text{ as } N\to\infty,$$

holding for all $0\leq t\leq T$ with any prefixed $T>0$.

Proof We recall that the map $Q_+[\cdot]\colon \mathcal{P}(\mathbb{R}_+)\to\mathcal{P}(\mathbb{R}_+)$ is defined via

$$Q_+[\rho](x)=2(\rho*\rho)(2x),\quad \forall x\geq 0.$$

Assume that a classical solution $\rho(t,x)$ of

$$\rho(t,x)=\rho(0,x)+\int_0^t \mathcal{L}[\rho](s,x)\,\mathrm{d}s \tag{D.2}$$

exists for $0\leq t<\infty$, where $\mathcal{L}=Q_+-\mathrm{Id}$ and $\rho(0,x)$ is a probability density function whose support is contained in $\mathbb{R}_+$. The map Q_+ is Lipschitz continuous in the sense that

$$W_1(Q_+[f],Q_+[g])\leq W_1(f,g) \tag{D.3}$$

for any $f,g\in\mathcal{P}(\mathbb{R}_+)$. Indeed, we have

$$W_1(Q_+[f],Q_+[g])=\sup_{\|\nabla\varphi\|_\infty\leq 1}\mathbb{E}\left[\varphi((X_1+Y_1)/2)-\varphi((X_2+Y_2)/2)\right],$$

where X_1,Y_1 are i.i.d with law f, X_2,Y_2 are i.i.d with law g. By Lipschitz continuity of the test function φ, we obtain

$$W_1(Q_+[f],Q_+[g])\leq \mathbb{E}[|X_1-X_2|].$$

We now recall an alternative formulation of $W_1(f,g)$, given by

$$W_1(f,g)=\inf\left\{\mathbb{E}[|X-Y|];\ \mathrm{Law}(X)=f,\ \mathrm{Law}(Y)=g\right\},$$

so in particular, we may take a coupling of X_1 and X_2 so that $W_1(f,g)=\mathbb{E}[|X_1-X_2|]$. Assembling these pieces together, we arrive at (D.3). More generally, suppose we have two random probability measures f and g with bounded second moment, taking expectation on both sides of (D.3) gives rise to

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int\varphi(x)\,(Q_+[f]-Q_+[g])\right]\le\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\le 1}\int\varphi(x)\,(f(\mathrm{d}x)-g(\mathrm{d}x))\right].\tag{D.4}$$

We now observe that the empirical measure is a compound jump process: Define N_t a homogeneous Poisson process with constant intensity $\lambda=(N-1)/2$. Given $\tau_1,\ldots,\tau_k$ the times when N_t jumps, we take the Y_{τ_k} independent: At each τ_k, with uniform probability $\frac{2}{N\,(N-1)}$ we choose a pair $i<j$ and take

$$\begin{aligned}Y_{\tau_k}=\frac{1}{N}\Big(&2\delta(x-(X_i(\tau_k-)+X_j(\tau_k-)/2))\\&-\delta(x-X_i(\tau_k-))-\delta(x-X_j(\tau_k-))\Big).\end{aligned}$$

We immediately note that

$$\begin{aligned}\lambda\,\mathbb{E}[Y_t]=\frac{1}{N^2}\sum_{i<j}\mathbb{E}\Big[&2\delta(x-(X_i(t-)+X_j(t-)/2))\\&-\delta(x-X_i(t-))-\delta(x-X_j(t-))\Big].\end{aligned}\tag{D.5}$$

We now show that the empirical measure of the stochastic system satisfies an approximate version of (D.2). Fix a deterministic test function φ with $\|\nabla\varphi\|_\infty\le 1$, and consider the time evolution of $\langle\rho_{\mathrm{emp}},\varphi\rangle$ where for some probability measure ν, we denote by the duality bracket $\langle\nu,\varphi\rangle=\int\varphi\,\mathrm{d}\nu$. Then

$$\mathrm{d}\mathbb{E}[\langle\rho_{\mathrm{emp}},\varphi\rangle]=\mathrm{d}\mathbb{E}\,[\langle Y_t\,\mathrm{d}N_t,\varphi\rangle]=\lambda\,\langle\mathbb{E}[Y_t],\varphi\rangle\,\mathrm{d}t.$$

Therefore, thanks to (D.5),

$$\begin{aligned}\mathrm{d}\mathbb{E}[\langle\rho_{\mathrm{emp}},\varphi\rangle]&=\frac{1}{N^2}\sum_{i<j}\mathbb{E}\,[2\varphi\,((X_i+X_j)/2)+-\varphi(X_i)-\varphi(X_j)]\,\mathrm{d}t\\&=\frac{1}{N^2}\sum_{i,j=1\ldots N,i\ne j}\mathbb{E}\,[\varphi\,((X_i+X_j)/2)-\varphi(X_i)]\,\mathrm{d}t\\&=\frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\,[\varphi\,((X_i+X_j)/2)-\varphi(X_i)]\,\mathrm{d}t,\end{aligned}$$

where all X_i, X_j are taken at time $t-$. On the other hand, we may calculate

$$\langle Q_+[\rho_{\mathrm{emp}}],\varphi\rangle=\int\varphi(x)\,2\,\frac{1}{N^2}\sum_{i,j=1}^{N}\delta_{X_i+X_j}(2x)\,\mathrm{d}x=\frac{1}{N^2}\sum_{i,j=1}^{N}\varphi\,((X_i+X_j)/2)\,.$$

Therefore

$$\mathrm{d}\mathbb{E}[\langle\rho_{\mathrm{emp}},\varphi\rangle]=\mathbb{E}\left[\langle\mathcal{L}[\rho_{\mathrm{emp}}],\varphi\rangle\right]\mathrm{d}t. \tag{D.6}$$

By Dynkin's formula, the compensated process

$$M_\varphi(t):=\langle\rho_{\mathrm{emp}}(t),\varphi\rangle-\langle\rho_{\mathrm{emp}}(0),\varphi\rangle-\int_0^t\mathbb{E}[\langle\mathcal{L}[\rho_{\mathrm{emp}}(s)],\varphi\rangle]\,\mathrm{d}s \tag{D.7}$$

is a martingale. Furthermore, comparing with (D.2), we easily obtain that

$$\begin{aligned}\langle\rho_{\mathrm{emp}}(t)-\rho(t),\varphi\rangle&=M_\varphi(t)+\langle\rho_{\mathrm{emp}}(0)-\rho(0),\varphi\rangle\\&\quad+\mathbb{E}\int_0^t\langle\mathcal{L}[\rho_{\mathrm{emp}}(s)]-\mathcal{L}[\rho(s)],\varphi\rangle\,\mathrm{d}s.\end{aligned}$$

Taking the supremum over φ, we therefore have that

$$\begin{aligned}\mathbb{E}\sup_{\|\nabla\varphi\|_\infty\leq 1}\langle\rho_{\mathrm{emp}}(t)-\rho(t),\varphi\rangle&\leq\mathbb{E}\sup_{\|\nabla\varphi\|_\infty\leq 1}(|M_\varphi(t)|+\langle\rho_{\mathrm{emp}}(0)-\rho(0),\varphi\rangle)\\&\quad+\int_0^t\mathbb{E}\sup_{\|\nabla\varphi\|_\infty\leq 1}\langle\mathcal{L}[\rho_{\mathrm{emp}}(s)]-\mathcal{L}[\rho(s)],\varphi\rangle\,\mathrm{d}s.\end{aligned}$$

By the definition of the W_1 distance, we deduce from (D.4) that

$$\mathbb{E}\,W_1(\rho_{\mathrm{emp}}(t),q(t))\leq\eta(t)+2\int_0^t\mathbb{E}\,W_1(\rho_{\mathrm{emp}}(t),q(t))\,\mathrm{d}s,$$

in which we have set

$$\eta(t):=\mathbb{E}\sup_{\|\nabla\varphi\|_\infty\leq 1}|M_\varphi(t)|+\mathbb{E}\,W_1(\rho_{\mathrm{emp}}(0),q(0)). \tag{D.8}$$

Thus, Gronwall's inequality gives rise to

$$\mathbb{E}\,W_1(\rho_{\mathrm{emp}}(t),\rho(t))\leq\left(\sup_{t\in[0,T]}\eta(t)\right)\mathrm{e}^{2T}. \tag{D.9}$$

In order to establish propagation of chaos for $t\leq T$, it therefore suffices to show that

$$\sup_{t\in[0,T]}\eta(t)\xrightarrow[N\to\infty]{\mathbb{P}}0. \tag{D.10}$$

To prove (D.10), we treat each term appearing in the definition of $\eta(t)$ separately. The second term in (D.8) approaches to 0 as $N\to\infty$ by our assumption. The treatment of the first term is more delicate, but can be carried out in a similar fashion as the proof of Theorem 6 in [39]. In the end, we obtain estimates of the form

$$\mathbb{E}\left[\sup_{\|\nabla\varphi\|_\infty\leq 1}|M_\varphi(t)|\right]\leq C\,\frac{t^\theta}{N^\theta}$$

for some $\theta>0$, which allows to finish the proof of (D.10). □