On Minimal Levels of Iwasawa Towers

by

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ABSTRACT

In 1959, Iwasawa proved that the size of the p-part of the class groups of a $\mathbb{Z}_p$-extension grows as a power of p with exponent $\mu p^m + \lambda\, m + \nu$ for m sufficiently large. Broadly, I construct conditions to verify if a given m is indeed sufficiently large.

More precisely, let CG_m^i (class group) be the ϵ_i-eigenspace component of the p-Sylow subgroup of the class group of the field at the m-th level in a $\mathbb{Z}_p$-extension; and let $IACG_m^i$ (Iwasawa analytic class group) be $\mathbb{Z}_p[[T]]/((1+T)^{p^m}-1, f(T,\omega^{1-i}))$, where f is the associated Iwasawa power series. It is expected that CG_m^i and $IACG_m^i$ be isomorphic, providing us with a powerful connection between algebraic and analytic techniques; however, as of yet, this isomorphism is unestablished in general.

I consider the existence and the properties of an exact sequence

$$0 \longrightarrow \ker \longrightarrow CG_m^i \longrightarrow IACG_m^i \longrightarrow \text{coker} \longrightarrow 0.$$

In the case of a $\mathbb{Z}_p$-extension where the Main Conjecture is established, there exists a pseudo-isomorphism between the respective inverse limits of CG_m^i and $IACG_m^i$. I consider conditions for when such a pseudo-isomorphism immediately gives the existence of the desired exact sequence, and I also consider work-around methods that preserve cardinality for otherwise. However, I primarily focus on constructing conditions to verify if a given m is sufficiently large that the kernel and cokernel of the above exact sequence have become well-behaved, providing similarity of growth both in the size and in the structure of CG_m^i and $IACG_m^i$; as well as conditions to determine if any such m exists.

The primary motivating idea is that if $IACG_m^i$ is relatively easy to work with, and if the relationship between CG_m^i and $IACG_m^i$ is understood; then CG_m^i becomes easier to work with.

Moreover, while the motivating framework is stated concretely in terms of the cyclotomic $\mathbb{Z}_p$-extension of p-power roots of unity, all results are generally applicable to arbitrary $\mathbb{Z}_p$-extensions as they are developed in terms of Iwasawa-Theory-inspired, yet abstracted, algebraic results on maps between inverse limits.

To Katie Nicole Elledge, for being my wife and muse.

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I would like to thank my advisor Dr. Nancy Childress for her guidance and endurance; my committee members Dr. Andrew Bremner, Dr. Susanna Fishel, Dr. John Jones, and Dr. Julien Paupert for their willingness and patience to endure reading outside their primary areas of research; Dr. Lawrence Washington for showing me a significantly shorter and clearer proof of Lemma 4.0.2 and for doing me the honor of reviewing my dissertation; Debbie Olson for helping me navigate the maze of graduate school and for patiently answering nearly a decade of incessant questions; my colleagues past and present Scott Zinzer, Dr. Ahmed Matar, and Dr. Chase Franks for helpful conversations; Dr. Gary Lovejoy for his skillful guidance; my family and friends for their support throughout; every single teacher, in particular Dr. Glenn Hurlbert for the many lessons imparted during undergraduate research, and the math teachers of my childhood John Bohan, Royce Freitag, and Phil Gille; and everybody who has assisted me in my pursuit of knowledge.

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LIST OF IMPORTANT DIAGRAMS

LIST OF NOTATIONS AND DEFINITIONS

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NOTATION OR DEFINITION		Page
j	a positive integer index	9
$IACG^i_m$	$\Lambda/((1+T)^{p^m}-1, f(T,\omega^{1-i}))$, for odd $i \in \mathbb{Z} \cap [3, p-2]$	9
m	a nonnegative integer index	11
n	a nonnegative integer index	11
N	a general norm map	11
Res	a general restriction map	11
Lrg	a general coset enlargement map	11
ι	a general injection map	11
F_m	$\mathbb{Q}(\zeta_{p^{m+1}})$, the prime power cyclotomic field	11
F	$\bigcup_{m\geq 0} F_m$, the union of the prime power cyclotomic fields	11
F^+_m	$\mathbb{Q}(\zeta_{p^{m+1}} + \zeta^{-1}_{p^{m+1}})$, the maximal real subfield of F_m	11
F^+	$\bigcup_{m\geq 0} F^+_m$, the maximal real subfield of F	11
$\mathscr{O}_m$	$\mathbb{Z}[\zeta_{p^{m+1}}]$, the ring of algebraic integers of F_m	12
$\mathscr{O}$	$\bigcup_{m\geq 0} \mathscr{O}_m$, the ring of algebraic integers of F	12
$\mathscr{O}^+_m$	$\mathbb{Z}[\zeta_{p^{m+1}} + \zeta^{-1}_{p^{m+1}}]$, the ring of algebraic integers of F^+_m	12
$\mathscr{O}^+$	$\bigcup_{m\geq 0} \mathscr{O}^+_m$, the ring of algebraic integers of F^+.	12
H_m	the p-Hilbert class field of F_m	12
H	$\bigcup_{m\geq 0} H_m$, the p-Hilbert class field of F	12
M_m	the maximal abelian p-extension of F_m unramified outside of p	12
M	$\bigcup_{m\geq 0} M_m$	12
CG_m	the p-Sylow subgroup of the class group of F_m	13
CG^i_m	the ϵ_i-eigenspace component of CG_m	13
N^n_m	the ideal norm map $CG_n \longrightarrow CG_m$	13
CG	$\varprojlim CG_m$	13
CG^i	$\epsilon_i CG_m = \varprojlim CG^i_m$	14
ι^n_m	an injection $CG^i_n \longrightarrow CG^i_m$, for odd $i \in \mathbb{Z} \cap [3, p-2]$	14
$\overrightarrow{CG}$	$\varinjlim CG_m$	14
$\overrightarrow{CG}^i$	$\varinjlim CG^i_m$, for odd $i \in \mathbb{Z} \cap [3, p-2]$	14
HCG_m	$\mathrm{Gal}(H_m/F_m)$, the p-Hilbert class group	14
j	a nonnegative integer index	14
b_j	the coefficients of a p-adic integer	14

NOTATION OR DEFINITION		Page
τ	a Galois automorphism of H_m/F_m	14
γ	a Galois automorphism of F_m/F_0	14
τ^γ	γ acting on τ	14
$\widetilde{\gamma}$	a Galois automorphism of H_m/F_0, an extension of γ	15
HCG_m^i	$\epsilon_i HCG_m$, the ϵ_i-eigenspace component of the p-Hilbert class group	15
Res_m^n	the restriction map $HCG_n \longrightarrow HCG_m$	15
τ	a Galois automorphism of H_{m+1}/F_{m+1}	15
HCG	$\varprojlim HCG_m = \mathrm{Gal}(H/F)$	15
HCG^i	$\varprojlim HCG_m^i = \epsilon_i HCG$	15
MCG_m	$\mathrm{Gal}(M_m/F)$	15
j	a nonnegative integer index	15
b_j	the coefficients of a p-adic integer	15
τ	a Galois automorphism of M_m/F	15
γ	a Galois automorphism of F_m/F_0	15
τ^γ	γ acting on τ	15
$\widetilde{\gamma}$	a Galois automorphism of M_m/F_0, an extension of γ	15
Res_m^n	the restriction map $MCG_n \longrightarrow MCG_m$	16
MCG	$\varprojlim MCG_m = \mathrm{Gal}(M/F)$	16
δ	1 for odd characters, and 0 for even characters	16
MCG_m^i	$\epsilon_{p-i} MCG_m = \epsilon_{p-i}\mathrm{Gal}(M_m/F)$, for odd $i \in \mathbb{Z} \cap [3, p-2]$	16
MCG^i	$\varprojlim MCG_m^i = \epsilon_{p-i} MCG$	16
$\mu_{p^{m+1}}$	the p^{m+1}-th roots of unity in $\mathbb{C}_p$	16
ρ_m^n	the p-power map $\mu_{p^{n+1}} \longrightarrow \mu_{p^{m+1}}$	16
N_m^n	the norm map $N_m^n : F_n \longrightarrow F_m$	16
$T = T_p(\mathscr{O})$	$\varprojlim \mu_{p^{m+1}}$, the p-adic Tate module of $\mathscr{O}$	16
t	an element of the Tate module	16
$T^{(-1)}$	$\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$, the twisted Tate module	17
Υ	an element of the twisted Tate module	17
$MCG_m^i(-1)$	$MCG_m^i \otimes_{\mathbb{Z}_p} T^{(-1)}$, for odd $i \in \mathbb{Z} \cap [3, p-2]$	17
$MCG^i(-1)$	$MCG^i \otimes_{\mathbb{Z}_p} T^{(-1)} = \varprojlim MCG_m^i(-1)$, for odd $i \in \mathbb{Z} \cap [3, p-2]$	17
x	an element of $F^\times$	17

NOTATION OR DEFINITION		Page

NOTATION OR DEFINITION		Page
$(n_i)_{i\in I}$	a typical element of N	34
θ_i	the map $M_i \longrightarrow N_i$ of a transformation	34
θ	the map $M \longrightarrow N$ induced from a transformation	34
ϕ_i^j	the induced kernel map $\ker\theta_j \longrightarrow \ker\theta_i$	35
$\overline{\psi}_i^j$	the induced cokernel map $\operatorname{coker}\theta_j \longrightarrow \operatorname{coker}\theta_i$	35
$\overline{M}_i^j$	$\ker\phi_i^j \cap \ker\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\psi_i^j) = \ker(\ker\theta_j \longrightarrow \ker\theta_i)$	35
$\widetilde{N}_i^j$	$\operatorname{coker}(\ker\phi_i^j \longrightarrow \ker\psi_i^j)$	35
s-min	strong minimal level	37
c-min	common minimal level	37
w-min	weak minimal level	37
k	an integer index, an element of the indexing set I	38
n_0	the $\operatorname{coker}\theta_i$ have fixed size for all $i \geq n_0$	40
p	a prime	40
μ	an integer, intended to remind us of Iwasawa's μ	40
λ	an integer, intended to remind us of Iwasawa's λ	40
v	an integer, intended to remind us of Iwasawa's ν	40
w	an integer, intended to remind us of Iwasawa's ν	40
n_0	$\lvert M_i\rvert = p^{\mu p^i+\lambda i+v}$ and $\lvert N_i\rvert = p^{\mu p^i+\lambda i+w}$ for all $i \geq n_0$	40
n_0'	$\max\{n_0, \text{w-min}\,\theta\}$	40
I'	$I \cap [n_0', \infty)$, the restricted index set	40
d	$\lvert\operatorname{coker}\theta_i\rvert = p^d$ for all $i \geq \text{w-min}\,\theta$	41
e_i	$\lvert\ker\theta_i\rvert = p^{e_i}$	41
$\{Q_i, \rho_i^j\}$	an inverse system of R-modules over I	41
Q	$\varprojlim Q_i$, the inverse limit	42
τ_i	the map $N_i \longrightarrow Q_i$ of a transformation	41
τ	the map $N \longrightarrow Q$ induced from a transformation	42
$\overline{M}_i^j$	$\ker\phi_i^j \cap \ker\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\psi_i^j) = \ker(\ker\theta_j \longrightarrow \ker\theta_i)$	43
$\widetilde{N}_i^j$	$\operatorname{coker}(\ker\phi_i^j \longrightarrow \ker\psi_i^j)$	43
$\overline{N}_i^j$	$\ker\psi_i^j \cap \ker\tau_j = \ker(\ker\psi_i^j \longrightarrow \ker\rho_i^j) = \ker(\ker\tau_j \longrightarrow \ker\tau_i)$	43
$\widetilde{Q}_i^j$	$\operatorname{coker}(\ker\psi_i^j \longrightarrow \ker\rho_i^j)$	43
$\overline{M}_i^{j\,\prime}$	$\ker\phi_i^j \cap \ker\tau_j\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\rho_i^j) = \ker(\ker\tau_j\theta_j \longrightarrow \ker\tau_i\theta_i)$	43

NOTATION OR DEFINITION		Page
$\widetilde{Q}_i^{j\,\prime}$	coker $(\ker\phi_i^j \longrightarrow \ker\rho_i^j)$	43
θ'	an inducible R-module homomorphism $M \longrightarrow N$	46
θ_i'	$\psi_i^*\theta'(\phi_i^*)^{-1} : M_i \longrightarrow N_i$, the maps of the inducing transformation	46
	inducible	46
	inducing transformation	47
$\overline{M}_i^j$	$\ker\phi_i^j \cap \ker\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\psi_i^j) = \ker(\ker\theta_j \longrightarrow \ker\theta_i)$	49
$\widetilde{N}_i^j$	coker $(\ker\phi_i^j \longrightarrow \ker\psi_i^j)$	49
$\overline{M}_j^*$	$\ker\phi_j^* \cap \ker\theta = \ker(\ker\phi_j^* \longrightarrow \ker\psi_j^*) = \ker(\ker\theta \longrightarrow \ker\theta_j)$	49
$\widetilde{N}_j^*$	coker $(\ker\phi_j^* \longrightarrow \ker\psi_j^*)$	49
$\overline{M}_i^*$	$\ker\phi_i^* \cap \ker\theta = \ker(\ker\phi_i^* \longrightarrow \ker\psi_i^*) = \ker(\ker\theta \longrightarrow \ker\theta_i)$	49
$\widetilde{N}_i^*$	coker $(\ker\phi_i^* \longrightarrow \ker\psi_i^*)$	49
$\overline{M}_{*i}^{*j}$	$\ker(\ker\phi_i^* / \ker\phi_j^* \longrightarrow \ker\psi_i^* / \ker\psi_j^*)$	49
$\widetilde{N}_{*i}^{*j}$	coker $(\ker\phi_i^* / \ker\phi_j^* \longrightarrow \ker\psi_i^* / \ker\psi_j^*)$	49
x	of the xyz-plane used to orient Rectangular Prism (5.6)	50
y	of the xyz-plane used to orient Rectangular Prism (5.6)	50
z	of the xyz-plane used to orient Rectangular Prism (5.6)	50
co-w-min	co-weak minimal level	55
n_0	the $\overline{M}_i^*$ have fixed size for all $i \geq n_0$	55
R	a commutative ring (assumed unitary)	63
z	an integer, the minimum of the indexing set I	63
I	$\mathbb{Z} \cap [z, \infty)$, the indexing set	63
$\{A_i, \alpha_i^j\}$	an inverse system of R-modules over I	63
i	an integer index, an element of the indexing set I	63
j	an integer index, an element of the indexing set I	63
A	$\varprojlim A_i$, the inverse limit	63
α_j^*	the natural map $A \longrightarrow A_j$	63
a_i	a typical element of A_i	63
$(a_i)_{i\in I}$	a typical element of A	63
B_i	a non-increasing sequence of subsets of $\ker\alpha_i^*$	64
$\widehat{A}_i$	A/B_i	64
γ_i	projection $A \longrightarrow \widehat{A}_i$	64

PREFACE

I present an organization of the chapters, some notational conventions, and some fairly pervasive literature references.

This dissertation can be divided into two parts: the first part, Chapters 1, 2, and 3; and the second part, Chapters 4, 5, 6, and 7. The first part uses the language of Iwasawa Theory, while the second the language of algebra. The two parts are logically independent; however, the first part motivates the second.

Chapter 1 is a concrete introduction to the objects that we consider and why they are of interest.

In Chapter 2 we develop several background results necessary to fluidly discuss Iwasawa Theory. We also present the Kummer-Vandiver Conjecture and some of its consequences, which then serve as motivation for many of the results below, as well as heuristic confidence that these results will be of use and interest to the research community.

Chapter 3 exhibits all the maps between all the inverse systems to which I think the results of this dissertation may be applicable; that is, we do not restrict ourselves solely to the map $CG_m^i \longrightarrow IACG_m^i$.

It is also important to note that the first three chapters are specialized to the cyclotomic $\mathbb{Z}_p$-extension of p-power roots of unity; however, all of the subsequence results are applicable to arbitrary $\mathbb{Z}_p$-extensions. I have chosen not to generalize the motivating development of the first three chapters to arbitrary $\mathbb{Z}_p$-extensions to avoid unnecessary complications, and since the cyclotomic $\mathbb{Z}_p$-extensions of p-power roots of unity has a richness of interesting maps to explore. The Greenberg Conjecture is the generalization of the Kummer-Vandiver Conjecture to arbitrary $\mathbb{Z}_p$-extensions.

Chapter 4 produces lemmas of use throughout the remainder of this dissertation. Some of these lemmas present fairly simple, and perhaps standard, results in order to facilitate later results. However, one lemma in particular, Lemma 4.0.2, is of singular importance.

Chapter 5 defines the notion of minimal levels that is the central idea of this dissertation, and develops our understanding thereof. This chapter represents the primary development of the work presented in this dissertation. Also, the methods of minimal levels, as presented in this chapter, are of use only when given a transformation between two inverse systems; and we address precisely this issue in the following two chapters.

Chapter 6 develops a method by which we modify an inverse system, while attempting to preserve as much of its structure as possible. This chapter may be viewed as lemmas to facilitate the techniques of the next chapter.

Chapter 7 develops a method by which we may apply the techniques of minimal levels when given only a map between the inverse limits, whether the map is or is not induced from a transformation. Chapter 7 represents a quick continuation of the primary development of Chapter 5.

Chapter 8 is a quick summary of the results and their applicability within the $\mathbb{Z}_p$-extensions that inspired their existence.

Chapters 1, 2, and 3 are notationally cumulative; however, Chapters 4, 5, 6, and 7 are pair-wise notationally independent as well as independent from all other chapters. For example, the connecting morphisms α_i^j of the inverse system $\{A_i, \alpha_i^j\}$ are assumed surjective in Chapter 6, but not necessarily so throughout Chapter 4. Much of the notation of Chapters 5 and 7 corresponds as a memory aide; however, they are ultimately independent. Similarly, the notation of Chapters 4 and 6 have been made similar as they both serve the same functional purpose of developing lemmas to streamline the main development of Chapters 5 and 7; however again, they are ultimately independent. Chapter 8 has no notation excepting references to Chapters 1, 2, and 3; which it does using their notation.

Also, while every effort has been made to avoid reusing notation, some exceptions have been forced by tradition, the Pigeonhole Principle, and the desire to achieve comprehensibility. Particularly important examples include the letters f, i, δ, μ, and θ.

It is also worth mentioning that the indexing of the ϵ_i-eigenspaces in Chapters 2 and 3 below varies greatly among the literature. We chose to index the ϵ_i-eigenspaces consistently throughout this dissertation; however as a result, the indexing here often does not correspond to that of the literature, even in the cases where the indexing has become fairly standard.

The three primary references for Chapters 1, 2, and 3 as well as for Iwasawa Theory in general are Washington (1997); Lang (1990); and Neukirch, Schmidt, and Wingberg (2008). The algebraic terminology and notational trends used throughout, but primarily in the four algebraic chapters, originate from Rotman (2002).

Chapter 1

BERNOULLI NUMBERS AND CLASS GROUPS OF CYCLOTOMIC FIELDS

[1]Let p be an odd prime.

We define the Bernoulli numbers via the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!};$$

giving

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots.$$

Since $B_k = 0$ for all odd $k \geq 3$, this is sometimes referred to as even index notation. It is not uncommon to remove negatives or re-index giving $B_1^* = \frac{1}{6},\ B_2^* = \frac{1}{30},\ B_3^* = \frac{1}{42}, \ldots$. Generally, even index notation became popular somewhere around 1970. (It is also not uncommon to change the left-hand-side to $\frac{te^t}{e^t-1}$ rendering $B_1 = +\frac{1}{2}$, which is often more convenient in combinatorics.)

An odd prime p is called regular if it does not divide the numerator of any Bernoulli number B_k for $k = 2, 4, 6, \ldots, p-3$; and p is called irregular otherwise (3 is considered regular and 2 is ignored). As such, we say that (p, k) is an irregular pair if p divides the numerator of B_k for $k = 2, 4, 6, \ldots, p-3$. E.g., $(691, 12)$ is an irregular pair since $B_{12} = -\frac{691}{2730}$. The irregularity index $i(p)$ is the number of irregular pairs (p, k). So $i(p) \geq 1$ if and only if p is irregular. There are infinitely many irregular primes; see the original Jensen (1915), or the simple Carlitz (1954). The first few irregular primes are

$$37,\ 59,\ 67,\ 101,\ 103,\ 131,\ 149,\ 157, \ldots.$$

For a positive integer r, let ζ_r denote a primitive r-th root of unity. For consistency and ease of notation below, we assume $\zeta_r \in \mathbb{C}_p$. For $m \geq 0$, let $F_m = \mathbb{Q}(\zeta_{p^{m+1}})$, let $F = \bigcup_{m \geq 0} F_m$, let CG_m be the p-Sylow subgroup of the class group of F_m, let $CG = \varprojlim CG_m$ (with respect to the ideal norm), let h_m be the p-part of the class number, and let $e_m = \operatorname{ord}_p h_m$. So

$$|CG_m| = h_m = p^{e_m}.$$

[1]Washington (1997); Lang (1990); and Neukirch, Schmidt, and Wingberg (2008) are general references for most results, definitions, and notation.

It is known that p is regular if and only if $h_0 = 1$; see the original Kummer (1850); or Herbrand (1932) for the forward direction decomposed by eigenspaces as in §2.4 below; or Ribet (1976) for the converse decomposed by eigenspaces; or Washington (1997), pages 101-102, Theorems 6.17 and 6.18. It is also known that if $h_0 = 1$, then $h_m = 1$ for all $m \geq 0$; see the original Furtwängler (1911); or Iwasawa (1956) for an alternate proof that relates the p-divisibility of the class numbers of two number fields that form a Galois extension that is cyclic of p-power degree, fully ramified at some prime, and unramified elsewhere.[2] It is also known that there exist integers $\lambda \geq 0$ and ν independent of m such that

$$e_m = \lambda\, m + \nu, \quad \text{for all } m \geq n_0,$$

for $n_0 \geq 0$ some integer; see Iwasawa (1959a) or Iwasawa (1959b) for the general formula, and Ferrero and Washington (1979) for $\mu = 0$. So, for a regular prime $\lambda = \nu = n_0 = 0$. It has been numerically verified for all odd primes $p < 163\,577\,856$ that $\lambda = \nu = i(p)$ and $n_0 = 0$, so $e_m = (m+1)i(p)$; and moreover that CG_m is the direct product of $i(p)$ cyclic groups of order p^{m+1}. The string of references that gives this is Iwasawa and Sims (1966); Johnson (1973); Johnson (1974); Johnson (1975); Wagstaff (1978); Tanner and Wagstaff (1987); Ernvall and Metsänkylä (1991); Ernvall and Metsänkylä (1992); Buhler, Crandall, and Sompolski (1992); Buhler, Crandall, Ernvall, and Metsänkylä (1993); Buhler, Crandall, Ernvall, Metsänkylä, and Shokrollahi (2001); and Buhler and Harvey (2011).

Broadly speaking, we will be investigating the minimal n_0 where $e_m = \lambda\, m + \nu$ for all $m \geq n_0$; but more specifically, the minimal n_0 where the structure of CG_m grows in a regular fashion for all $m \geq n_0$.

[2] Iwasawa (1956) is an accessible article that contains the simple fundamental ideas that would later grow into Iwasawa Theory, and in my opinion, marks its beginning.

Chapter 2

BACKGROUND RESULTS

[1]We state several background results and establish notation.

2.1 Generalized Bernoulli Numbers and p-adic L-Functions

Let

$$\omega : \mathbb{Z}_p^\times \longrightarrow \mu_{p-1}$$

be the Teichmüller character, where μ_{p-1} is the $(p-1)$-th roots of unity in $\mathbb{C}_p$, so $\mu_{p-1} \subset \mathbb{Z}_p^\times$. Also, define

$$\langle\cdot\rangle : \mathbb{Z}_p^\times \longrightarrow 1 + p\mathbb{Z}_p \qquad \text{to be} \qquad \langle a \rangle = \frac{a}{\omega(a)}.$$

This gives a multiplicative isomorphism

$$\mathbb{Z}_p^\times \longrightarrow (1 + p\mathbb{Z}_p) \times \mu_{p-1} \qquad \text{by} \qquad a \mapsto \Big(\langle a \rangle \ , \ \omega(a)\Big)$$

or a non-canonical isomorphism

$$\mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p \times \mu_{p-1} \qquad \text{by} \qquad a \mapsto \Big(\frac{\log_p\langle a \rangle}{\log_p(1+p)} \ , \ \omega(a)\Big),$$

where here we take $\mathbb{Z}_p$ as an additive group.

Let

$$\chi : (\mathbb{Z}/f\mathbb{Z})^\times \longrightarrow \mu_{\phi(f)}$$

be a p-adic Dirichlet character of conductor f, where ϕ is the Euler totient function and $\mu_{\phi(f)}$ is the $\phi(f)$-th roots of unity in $\mathbb{C}_p$. Then we define the (p-adic) generalized Bernoulli number $B_{k,\chi}$ via the exponential generating function

$$\sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{k\geq 0} B_{k,\chi}\frac{t^k}{k!}.$$

Since $\omega(a)$ depends only on a mod $p\mathbb{Z}_p$, we may consider the Teichmüller character to be a p-adic Dirichlet character of conductor p

$$\omega : (\mathbb{Z}_p/p\mathbb{Z}_p)^\times \longrightarrow \mu_{p-1};$$

[1]Washington (1997); Lang (1990); and Neukirch, Schmidt, and Wingberg (2008) are general references for most results, definitions, and notation.

whence it has order $p-1$, and it generates all p-adic Dirichlet characters of conductor dividing p (all characters are assumed primitive).

The salient result is that the p-adic L-functions can be interpolated from the values at the non-positive integers

$$L_p(1-n,\chi) = -(1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n}, \quad n \geq 1;$$

see the original Kubota and Leopoldt (1964), or the modern Washington (1997), page 57, Theorem 5.11.

2.2 Galois Groups and Character Decompositions

Let $\alpha \in \mathbb{Z} \cap [1,p^2)$ be a primitive root modulo p^2 so that $\alpha + p^{m+1}\mathbb{Z}_p$ generates the multiplicative group $(\mathbb{Z}_p/p^{m+1}\mathbb{Z}_p)^\times$ for all $m \geq 0$, which is cyclic of order $(p-1)p^m$. Let $\beta = \alpha^{p-1} \in (1+p\mathbb{Z}) \cap [1, p^{2p-2})$ so that $\beta(1+p^{m+1}\mathbb{Z}_p)$ generates the multiplicative group $(1+p\mathbb{Z}_p)/(1+p^{m+1}\mathbb{Z}_p)$ for all $m \geq 0$, which is (canonically isomorphic to) a cyclic subgroup of order p^m inside of $(\mathbb{Z}_p/p^{m+1}\mathbb{Z}_p)^\times$. Note that it is unnecessary to restrict the size of $\alpha < p^2$ and $\beta < p^{2p-2}$ as I have done here; this restriction is solely to establish concreteness.

For

$$a = \sum_{i \geq 0} a_i p^i \in \mathbb{Z}_p^\times,$$

define $\sigma_a \in \mathrm{Gal}(F/\mathbb{Q})$ by

$$\sigma_a : \zeta_{p^{m+1}} \mapsto \zeta_{p^{m+1}}^a = \prod_{i \geq 0} \zeta_{p^{m+1}}^{a_i p^i} \qquad \text{for any } m,$$

and extending by linearity. (σ_a is well-defined independent of choice of m.) This product is finite since $\zeta_{p^{m+1}}^{a_i p^i} = 1$ whenever $i \geq m+1$. In particular, for any m, we may also consider $\sigma_a \in \mathrm{Gal}(F_m/\mathbb{Q})$ by restricting to F_m, since σ_a would be determined by a mod $p^{m+1}\mathbb{Z}_p$. Let

$$G_m = \mathrm{Gal}(F_m/\mathbb{Q}) = \{\sigma_a : a \in (\mathbb{Z}_p/p^{m+1}\mathbb{Z}_p)^\times\},$$

which is cyclic of order $(p-1)p^m$ generated by σ_α, let $Res^n_m : G_n \longrightarrow G_m$ be the restriction map, then $\{G_m, Res^n_m\}$ is an inverse system, and let

$$G = \varprojlim G_m = \mathrm{Gal}(F/\mathbb{Q}) = \{\sigma_a : a \in \mathbb{Z}_p^\times\},$$

for which σ_α is a topological generator.

Let

$$\Gamma_m = \mathrm{Gal}(F_m/F_0) = \{\sigma_a : a \in (1+p\mathbb{Z}_p)/(1+p^{m+1}\mathbb{Z}_p)\},$$

which is cyclic of order p^m generated by σ_β, let $Res^n_m : \Gamma_n \longrightarrow \Gamma_m$ be the restriction map, then $\{\Gamma_m, Res^n_m\}$ is an inverse system, and let

$$\Gamma = \varprojlim \Gamma_m = \mathrm{Gal}(F/F_0) = \{\sigma_a : a \in 1+p\mathbb{Z}_p\},$$

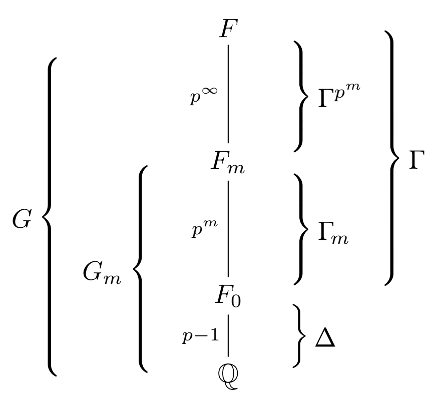
for which σ_β is a topological generator. So, $\Gamma \cong \mathbb{Z}_p$ non-canonically by $\sigma_a \mapsto \log_p a/\log_p(1+p)$. It is often convenient to instead choose a topological generator σ_β with $\beta = 1+p$, so that it would be represented by $1 \in \mathbb{Z}_p$.

If we define $\Delta = G_0 = \mathrm{Gal}(F_0/\mathbb{Q})$, then we can decompose G_m and G as

$$G_m \cong \Delta \times \Gamma_m \qquad \text{and} \qquad G \cong \Delta \times \Gamma.$$

Note that the elements of Γ that fix F_m are those in Γ^{p^m}, so that $\Gamma_m = \Gamma/\Gamma^{p^m}$. Here are the Galois groups heretofore defined.

(2.1)



For $\sigma_a \in G_m$, we may decompose

$$\sigma_a = \delta(a)\gamma_m(a), \quad \text{with } \delta(a) \in \Delta,\ \gamma_m(a) \in \Gamma_m.$$

If χ is a p-adic Dirichlet character with conductor a power of p, then we may take χ to be a Galois character

$$\chi : G_m \longrightarrow \mu_{(p-1)p^m}$$

for some $m \geq 0$; and hence may decompose

$$\chi = \theta\psi, \quad \text{with } \theta : \Delta \longrightarrow \mu_{p-1},\ \psi : \Gamma_m \longrightarrow \mu_{p^m},$$

and θ is of the first kind with conductor $p-1$, and ψ is of the second kind with conductor p^m. The character ψ must be even. (If $\mathbb{B}_m$ is the unique subfield of $\mathbb{Q}(\zeta_{p^{m+1}})$ of degree p^m over $\mathbb{Q}$, then ψ could also be considered a Galois character of $\mathrm{Gal}(\mathbb{B}_m/\mathbb{Q})$; whence ψ must be even since $\mathbb{B}_m$ is a real field.) Since $L_p(s,\chi)$ is identically zero for odd χ, we will only need to consider even χ; in which case θ must also be even.

2.3 $\mathbb{Z}_p[[\Gamma]]$-Modules

Let $\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[\Gamma_m]$ be the pro-finite completion of $\mathbb{Z}_p[\Gamma]$. Assume that $\{V_m, \kappa_m^n\}$ is an inverse systems of $\mathbb{Z}_p[\Gamma_m]$-modules such that κ_m^n commutes with the Γ_m-action; that is

$$\kappa_m^n \underbrace{\sigma_a}_{\in\Gamma_n} = \underbrace{\sigma_a}_{\in\Gamma_m} \kappa_m^n.$$

Since κ_m^n is a $\mathbb{Z}_p$-module homomorphism, it must also commute with the $\mathbb{Z}_p[\Gamma_m]$-action. First, we can form the inverse limit $V = \varprojlim V_m$ as a $\mathbb{Z}_p$-module. If $y = (y_m)_{m\in\mathbb{N}} \in \mathbb{Z}_p[[\Gamma]]$ and $v = (v_m)_{m\in\mathbb{N}} \in V$, then $yv = (y_m v_m)_{m\in\mathbb{N}} \in V$ since $\kappa_m^n(y_n v_n) = (\kappa_m^n y_n)(v_n) = (y_m \kappa_m^n)(v_n) = y_m(\kappa_m^n v_n) = y_m v_m$; whence V is a $\mathbb{Z}_p[[\Gamma]]$-module. See Washington (1997), page 199; or Lang (1990), page 125.

2.4 Eigenspace Decompositions and the Kummer-Vandiver Conjecture

We have that CG_m is a $\mathbb{Z}_p$-module via

$$\underbrace{\Big(\sum_{j\geq 0} b_j p^j\Big)}_{\in\mathbb{Z}_p} \cdot \underbrace{[\mathfrak{a}]}_{\in CG_m} = \prod_{j\geq 0}[\mathfrak{a}^{b_j p^j}],$$

which is a finite product since CG_m has p-power order. Together with the Galois action we have that CG_m is a $\mathbb{Z}_p[[\Gamma]]$-module, which is the module structure of primary interest; see §3.2.1 below. However, evidenced by Herbrand (1932) and Ribet (1976), it is well to decompose CG_m by viewing it as a $\mathbb{Z}_p[\Delta]$-module.

For $0 \leq i \leq p-2$, define

$$\epsilon_i = \frac{1}{p-1}\sum_{a=1}^{p-1} \omega^i(a)\sigma_a^{-1} \in \mathbb{Z}_p[\Delta],$$

$$\epsilon^+ = \sum_{\substack{i=0 \\ i \text{ even}}}^{p-3} \epsilon_i = \frac{1+\sigma_{-1}}{2} \in \mathbb{Z}_p[\Delta], \quad \text{and} \quad \epsilon^- = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} \epsilon_i = \frac{1-\sigma_{-1}}{2} \in \mathbb{Z}_p[\Delta].$$

The ϵ_i form a system of orthogonal idempotents for $\mathbb{Z}_p[\Delta]$, as do $\epsilon^{\pm}$. Thus we may decompose CG_m into

$$CG_m = \bigoplus_{i=0}^{p-2} CG_m^i = CG_m^+ \oplus CG_m^-,$$

where $CG_m^i = \epsilon_i CG_m$ and $CG_m^{\pm} = \epsilon^{\pm} CG_m$. Recall that $\Delta = G_0$, and note that equivalently we could have also viewed $\epsilon_i \in \mathbb{Z}_p[G_m]$ and decomposed CG_m as a $\mathbb{Z}_p[G_m]$-module, or viewed $\epsilon_i \in \mathbb{Z}_p[G]$ and decomposed CG_m as a $\mathbb{Z}_p[G]$-module. To establish notation, we take

$$\begin{array}{llll} |CG_m| = h_m = p^{e_m} & \text{with} & e_m = \lambda\, m + \nu, & \text{for all } m \text{ sufficiently large;} \\ |CG_m^i| = h_{m,i} = p^{e_{m,i}} & \text{with} & e_{m,i} = \lambda_i\, m + \nu_i, & \text{for all } m \text{ sufficiently large;} \\ |CG_m^{\pm}| = h_m^{\pm} = p^{e_m^{\pm}} & \text{with} & e_m^{\pm} = \lambda^{\pm} m + \nu^{\pm}, & \text{for all } m \text{ sufficiently large;} \end{array}$$

that is, λ and ν are the Iwasawa invariants. Note that we have the relations

$$\begin{array}{lllllllll} h_m & = & \prod_{i=0}^{p-2} h_{m,i} & = & h_m^+ h_m^-, & h_m^+ & = & \prod_{\substack{i=0 \\ i \text{ even}}}^{p-3} h_{m,i}, & h_m^+ = \prod_{\substack{i=1 \\ i \text{ odd}}}^{p-2} h_{m,i}, \\ e_m & = & \sum_{i=0}^{p-2} e_{m,i} & = & e_m^+ + e_m^-, & e_m^+ & = & \sum_{\substack{i=0 \\ i \text{ even}}}^{p-3} e_{m,i}, & e_m^+ = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} e_{m,i}, \\ \lambda & = & \sum_{i=0}^{p-2} \lambda_i & = & \lambda^+ + \lambda^-, & \lambda^+ & = & \sum_{\substack{i=0 \\ i \text{ even}}}^{p-3} \lambda_i, & \lambda^+ = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} \lambda_i, \\ \nu & = & \sum_{i=0}^{p-2} \nu_i & = & \nu^+ + \nu^-, & \nu^+ & = & \sum_{\substack{i=0 \\ i \text{ even}}}^{p-3} \nu_i, & \nu^+ = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} \nu_i. \end{array}$$

Also note that $CG_{m,0} = CG_{m,1} = 0$; see Washington (1997), page 101, Proposition 6.16.

F_m is a CM field with maximal real subfield $F_m^+ = \mathbb{Q}(\zeta_{p^{m+1}} + \zeta_{p^{m+1}}^{-1})$, also $F^+ = \bigcup_{m \geq 0} F_m^+$; and CG_m^+ is the p-Sylow subgroup of the class group of F_m^+ (σ_{-1} is complex conjugation). The Kummer-Vandiver Conjecture conjectures that $p \nmid h_0^+$. If the Kummer-Vandiver Conjecture were true, it would follow that $p \nmid h_m^+$ for all $m \geq 0$; and the CG_m^i for even i would all be trivial. The Kummer-Vandiver Conjecture has been numerically verified for all primes $p < 163\,577\,856$; see Lehmer, Lehmer, and Vandiver

(1954); Vandiver (1954); Selfridge, Nicol, and Vandiver (1955); and the string of references at the end of Chapter 1 on page 2 above.

(Greenberg has also conjectured that $\lambda = \mu = 0$ for any totally real number field and any p; see Greenberg (1976).)

2.5 The Iwasawa Power Series

Let $\mathbb{Z}_p[[T]]$ be the power series ring over $\mathbb{Z}_p$ in the indeterminant T. Then we have the Iwasawa isomorphism (which depends on the choice of topological generator)

$$\mathbb{Z}_p[[\Gamma]] \xrightarrow{\cong} \mathbb{Z}_p[[T]] \qquad \text{determined by} \qquad \sigma_\beta \mapsto 1 + T.$$

As such we take $\Lambda = \mathbb{Z}_p[[\Gamma]]$ or $\Lambda = \mathbb{Z}_p[[T]]$ to be whichever is more convenient at the time, and we refer to Λ-modules as Iwasawa modules. For even θ, define

$$\xi_m(\theta) = -\frac{1}{p^{m+1}} \sum_{\substack{a=1 \\ p \nmid a}}^{p^{m+1}-1} a\theta\omega^{-1}(a)\gamma_m(a)^{-1} \in \mathbb{Q}_p[\Gamma_m], \quad \xi(\theta) = \varprojlim \xi_m(\theta) \in \mathbb{Q}_p[[\Gamma]],$$

$$\tau_m = (1 - (1+p)\gamma_m(1+p)^{-1}) \in \mathbb{Z}_p[\Gamma_m], \quad \tau = \varprojlim \tau_m \in \mathbb{Z}_p[[\Gamma]],$$

$$\eta_m(\theta) = \tau_m\, \xi_m(\theta) \in \mathbb{Z}_p[\Gamma_m], \quad \eta(\theta) = \varprojlim \eta_m(\theta) \in \mathbb{Z}_p[[\Gamma]].$$

($\xi_m(\theta)$ is derived from the Stickelberger element and the idempotent of $\omega\theta^{-1}$.) By the Iwasawa isomorphism, define

$$\tau \mapsto h(T, \theta),$$

$$\eta(\theta) \mapsto g(T, \theta),$$

$$f(T, \theta) = \frac{g(T, \theta)}{h(T, \theta)}.$$

If $\theta \neq 1$, then $\xi_m(\theta) \in \mathbb{Z}_p[\Gamma_m]$, $\xi(\theta) \in \mathbb{Z}_p[[\Gamma]]$, and $\xi(\theta) \mapsto f(T, \theta)$.

Now we have the following result. If $\chi = \theta\psi$ is an even Dirichlet character and $\zeta_\psi = \psi(1+p)^{-1} = \chi(1+p)^{-1}$ (which is a root of unity of p-power order); then

$$L_p(s, \chi) = f(\zeta_\psi(1+p)^s - 1, \theta);$$

see Iwasawa (1969); Iwasawa (1972); and Washington (1997), page 123, Theorem 7.10.

2.6 The Structure of the p-Class Group Assuming the Kummer-Vandiver Conjecture

If $p \nmid h_0^+$, then, for $i = 3, 5, \ldots, p-2$,

$$CG_m^i \cong \mathbb{Z}_p[[T]]/((1+T)^{p^m} - 1, f(T, \omega^{1-i}))$$

and

$$CG^i \cong \mathbb{Z}_p[[T]]/(f(T, \omega^{1-i}))$$

as Λ-modules, where $f(T, \omega^{1-i})$ satisfies $L_p(s, \omega^{1-i}) = f((1+p)^s - 1, \omega^{1-i})$ as above; see Washington (1997), pages 199-201, Theorem 10.16.

With some additional assumptions, we get the decomposition into cyclic groups. Let p be an irregular prime, for $j = 1, \ldots, i(p)$ let (p, i_j) be an irregular pair, and suppose $p \nmid h_0^+$. If

$$B_{1,\omega^{i_j - 1}} \quad \not\equiv \quad 0 \quad \bmod p^2 \tag{2.2}$$

and

$$\frac{B_{i_j}}{i_j} \quad \not\equiv \quad \frac{B_{i_j+p-1}}{i_j+p-1} \quad \bmod p^2 \quad \text{for all } j = 1, \ldots, i(p); \tag{2.3}$$

then

$$CG_m = \bigoplus_{j=1}^{i(p)} CG_{m,p-i_j} \cong \bigoplus_{j=1}^{i(p)} (\mathbb{Z}/p^{m+1}\mathbb{Z}) \cong (\mathbb{Z}/p^{m+1}\mathbb{Z})^{i(p)} \quad \forall m \geq 0;$$

see Iwasawa and Sims (1966); or Washington (1997), pages 202-203, Corollary 10.17. Conditions (2.2) and (2.3) have been numerically verified for all odd primes $p < 163\,577\,856$; see the string of references at the end of Chapter 1 on page 2 above.

Note that we will not be assuming the Kummer-Vandiver Conjecture anywhere below; that is, we will not be using the results of §2.6 directly. However §2.6 does motivate the analytic class group definitions made below in §§3.2.9, 3.2.11, 3.2.12. More specifically, in §3.2.9 below we define the Iwasawa analytic class group to be

$$IACG_m^i = \Lambda/((1+T)^{p^m} - 1, f(T, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

So it is expected that CG_m^i and $IACG_m^i$ be isomorphic, providing us with a powerful connection between algebraic and analytic techniques; however, as of yet, this isomorphism is unestablished in general. As such, below we consider exact sequences of the form

$$0 \longrightarrow \ker \longrightarrow CG_m^i \longrightarrow IACG_m^i \longrightarrow \text{coker} \longrightarrow 0, \tag{2.4}$$

and we thereby hope to gain similarity between CG_m^i and $IACG_m^i$ by constraining the kernel and cokernel.

The primary motivating idea is that if $IACG_m^i$ is relatively easy to work with, and if the relationship between CG_m^i and $IACG_m^i$ is understood; then CG_m^i becomes easier to work with. Via the methods referenced above, it has been numerically verified for all odd primes $p < 163\,577\,856$ that the Weierstrass degree of f is either 0 or 1; and in the first case $IACG_m^i$ is trivial, and in the second case $IACG_m^i$ is cyclic of order p^{m+1}.

Chapter 3

THE PSEUDO-ISOMORPHISMS OF IWASAWA THEORY

[1]This is a development of the inverse system, transformation, and pseudo-isomorphism perspective of Iwasawa Theory that motivates the algebraic development of the remaining chapters. It is important to note that in the literature, most of the limits below are constructed directly; however, for our purposes, we opt to construct via inverse and direct systems. This approach is messier and requires a substantial attention to detail; and as such, we have made every effort to explicitly give all the module structures. The most complete references for the module structures are Washington (1997) and Coates (1977). Since there is a dizzying array of Λ-modules, and since notations vary so greatly, as a memory aide we opt to name everything after somebody or something, and then use (pseudo-)acronyms; e.g., we have already seen CG for the class group. I would recommend taking a glance at Diagram (3.2) on page 28 below for an orientating preview of what is about to happen in this chapter.

The inverse and direct systems and their maps will all be indexed over $\mathbb{N}$; that is, let $m, n \in \mathbb{N}$ and assume $m \leq n$. Also, we will be abusing notation severally by using N, Res, Lrg, and ι to respectively denote multiple different norm maps, restriction maps, coset enlargement maps, and injection maps.

3.1 Some Fields and Rings

We begin with some fields and rings.

3.1.1 F, the Field

Recall $F_m = \mathbb{Q}(\zeta_{p^{m+1}})$, the tower of fields that is our focus; and also $F = \bigcup_{m\geq 0} F_m$, $F_m^+ = \mathbb{Q}(\zeta_{p^{m+1}} + \zeta_{p^{m+1}}^{-1})$, and $F^+ = \bigcup_{m\geq 0} F_m^+$. (For convenience, all fields are contained in $\mathbb{C}_p$.)

[1]Washington (1997); Lang (1990); and Neukirch, Schmidt, and Wingberg (2008) are general references for most results, definitions, and notation.

3.1.2 $\mathscr{O}$, the Ring of Integers

Define $\mathscr{O}_m = \mathbb{Z}[\zeta_{p^{m+1}}]$, the ring of algebraic integers of F_m; and also define $\mathscr{O} = \bigcup_{m \geq 0} \mathscr{O}_m$, the ring of algebraic integers of F (n.b., $\mathscr{O}$ is not a Dedekind domain). We also define $\mathscr{O}_m^+ = \mathbb{Z}[\zeta_{p^{m+1}} + \zeta_{p^{m+1}}^{-1}]$, the ring of algebraic integers of F_m^+; and also define $\mathscr{O}^+ = \bigcup_{m \geq 0} \mathscr{O}_m^+$, the ring of algebraic integers of F^+.

3.1.3 H, the p-Hilbert Class Field

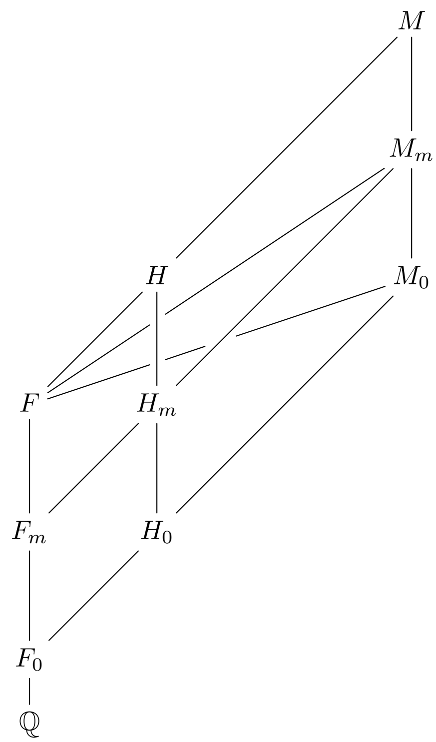
Define H_m to be the p-Hilbert class field of F_m; that is, the maximal unramified abelian p-extension (all contained in the algebraically closed $\mathbb{C}_p$); and also define $H = \bigcup_{m \geq 0} H_m$, the p-Hilbert class field of F; see the original Furtwängler (1906); or the modern Childress (2009), page 153, Proposition 4.1. Note that $H_m \cap F_{m+1} = F_m$. (If $\mathfrak{p}$ is a prime $\mathscr{O}_m$-ideal lying above $p\mathbb{Z}$, then $\mathfrak{p}$ is totally ramified in $\mathscr{O}_{m+1}$; and there are no proper, non-trivial, intermediate fields between F_{m+1}/F_m.)

3.1.4 M, the Maximal Abelian p-Extension Unramified Outside of p

Define M_m to be the maximal abelian p-extension of F_m unramified outside of p (all contained in the algebraically closed $\mathbb{C}_p$); and also define $M = \bigcup_{m \geq 0} M_m$, the maximal abelian p-extension of F unramified outside of p. Note that $F \subseteq M_m$.

3.1.5 A Field Diagram

Here are the fields heretofore defined.



(3.1)

This field diagram is not intended to be complete; that is, there may be additional subset relationships not represented, and neither does it give information about intersections or composites.

3.2 Inverse Systems

We now define the inverse systems of interest, as well as two direct systems.

3.2.1 CG, the p-Ideal Class Group

Recall that CG_m is the p-Sylow subgroup of the ideal class group of $\mathscr{O}_m$, and that we decompose CG_m into eigenspaces by defining $CG_m^i = \epsilon_i CG_m$. Then $\{CG_m, N_m^n\}$ is an inverse system of $\mathbb{Z}_p[[\Gamma]]$-modules where $N_m^n : CG_m \longrightarrow CG_m$ is the (ideal) norm map (which is surjective); whence we define $CG = \varprojlim CG_m$ and also

$CG^i = \epsilon_i CG_m = \varprojlim CG^i_m$. For the surjectivity of N, see Washington (1997), page 185, Theorem 10.1. We can recover the finite levels from the infinite since

$$CG_m = CG/CG^{\sigma_\beta^{p^m}-1};$$

see Lang (1990), page 140, Corollary.

We will also have occasion for another perspective. For $n \geq m$, the natural injective map from the ideal group of $\mathscr{O}_m$ into the ideal group of $\mathscr{O}_n$ gives a map $CG_m \longrightarrow CG_n$; see Iwasawa (1973), page 259 and pages 263-264. Then, for odd $i \in \mathbb{Z} \cap [3, p-2]$ we have an injection $\iota^n_m : CG^i_n \longrightarrow CG^i_m$; see Washington (1997), page 288, Proposition 13.26, page 294. Note that $N^{m+1}_m \iota^m_{m+1}$ is the p-power map; in particular, note that N and ι are not inverses. Thus $\{CG_m, \iota^n_m\}$ and $\{CG^i_m, \iota^n_m\}$ are direct systems, so we take

$$\overrightarrow{CG} = \varinjlim CG_m \overset{\dagger}{=} \bigcup_{m \geq 0} CG_m$$

and

$$\overrightarrow{CG}^i = \varinjlim CG^i_m \overset{\dagger}{=} \bigcup_{m \geq 0} CG^i_m \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2],$$

where † requires that we associate $CG_m \subseteq CG_{m+1}$; see Rotman (2002), page 509, Example 7.99. The rudiments of why we consider both the direct and inverse limit can be seen in Iwasawa (1973), page 259; Mazur and Wiles (1984), page 192; and Rotman (2002) page 506, Proposition 7.96.

3.2.2 HCG, the p-Hilbert Class Group

Define $HCG_m = \text{Gal}(H_m/F_m)$, the Galois group of the p-Hilbert class field. We have that HCG_m is a $\mathbb{Z}_p$-module via

$$\underbrace{\Big(\sum_{j \geq 0} b_j p^j\Big)}_{\in \mathbb{Z}_p} \cdot \underbrace{\tau}_{\in HCG_m} = \prod_{j \geq 0} \tau^{b_j p^j},$$

which is a finite product since HCG_m has p-power order. Note that H_m/F_0 is Galois and HCG_m is a normal subgroup. The Γ_m-action on HCG_m is inner-automorphism (conjugation)

$$\underbrace{\gamma}_{\in \Gamma_m} \cdot \underbrace{\tau}_{\in HCG_m} = \tau^\gamma = \widetilde{\gamma}\tau(\widetilde{\gamma})^{-1},$$

where $\widetilde{\gamma} \in \mathrm{Gal}(H_m/F_0)$ is some extension of γ, which gives a well-defined action since H_m/F_m is abelian. Thus HCG_m is a $\mathbb{Z}_p[\Gamma_m]$-module. Thus we also define $HCG_m^i = \epsilon_i HCG_m$.

We define the restriction map $Res_m^{m+1} : HCG_{m+1} \longrightarrow HCG_m$. If $\tau \in HCG_{m+1}$, then $\tau : H_{m+1} \longrightarrow H_{m+1}$ leaving F_{m+1} fixed; but if we restrict τ to H_m, then it will leave $H_m \cap F_{m+1} = F_m$ fixed, and hence $Res_m^{m+1}\tau \in HCG_m$. Composing as necessary, we may define $Res_m^n : HCG_n \longrightarrow HCG_m$ for $n > m$. Then $\{HCG_m, Res_m^n\}$ is an inverse system; whence we define $HCG = \varprojlim HCG_m = \mathrm{Gal}(H/F)$. Also define $HCG^i = \varprojlim HCG_m^i = \epsilon_i HCG$.

Note that we have $HCG_m \cong \mathrm{Gal}(H_m F/F)$. We can recover the finite levels from the infinite since

$$HCG_m = HCG/HCG^{\sigma_\beta^{p^m}-1}.$$

See Lang (1990), page 140, Corollary; and Washington (1997), page 285, Proposition 13.22.

3.2.3 MCG, the Galois Group of the Maximal Abelian p-Extension Unramified Outside of p

Consider $\mathrm{Gal}(M_m/F_m)$, the Galois group of the maximal abelian p-extension of F_m unramified outside of p. Noting that $F_m \subseteq F \subseteq M_m$, we are interested the in the subgroup that fixes F. Define $MCG_m = \mathrm{Gal}(M_m/F)$. As with the p-Hilbert class group, we have that MCG_m is a $\mathbb{Z}_p$-module via

$$\underbrace{\Big(\sum_{j\geq 0} b_j p^j\Big)}_{\in \mathbb{Z}_p} \cdot \underbrace{\tau}_{\in MCG_m} = \prod_{j\geq 0} \tau^{b_j p^j},$$

which is a finite product since MCG_m is p-power torsion. Note that M_m/F_0 is Galois and MCG_m is a normal subgroup. Similar to the p-Hilbert class group, the Γ_m-action is inner-automorphism (conjugation)

$$\underbrace{\gamma}_{\in \Gamma_m} \cdot \underbrace{\tau}_{\in MCG_m} = \tau^\gamma = \widetilde{\gamma}\tau(\widetilde{\gamma})^{-1},$$

where $\widetilde{\gamma} \in \mathrm{Gal}(M_m/F_0)$ is some extension of γ, which gives a well-defined action since $\mathrm{Gal}(M_m/F_m)$ is abelian. (This Galois action can be placed on all of $\mathrm{Gal}(M_m/F_m)$.) Thus

MCG_m is a $\mathbb{Z}_p[\Gamma_m]$-module. Then $\{MCG_m, Res_m^n\}$ is an inverse system where $Res_m^n : MCG_n \longrightarrow MCG_m$ is the restriction map; whence we define $MCG = \varprojlim MCG_m = \mathrm{Gal}(M/F)$.

We now want to examine the eigenspace decomposition,

$$MCG_m = \bigoplus_{i=0}^{p-1} \epsilon_i \mathrm{Gal}(M_m/F) = \underbrace{\Big(\bigoplus_{\substack{i=0 \\ i \text{ even}}}^{p-1} \epsilon_i \mathrm{Gal}(M_m/F) \Big)}_{\mathrm{Gal}(M_m^+/F)} \oplus \Big(\bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{p-1} \epsilon_i \mathrm{Gal}(M_m/F) \Big).$$

The $\mathbb{Z}_p$-rank of the eigenspace $\epsilon_i \mathrm{Gal}(M_m/F)$ is

$$\delta(\omega^i)\, p^m = \begin{cases} p^m & \text{for odd } i, \\ 0 & \text{for even } i; \end{cases}$$

so the odd eigenspaces are really big; however, we will be using the ω^{p-i}-eigenspace for odd i. See Washington (1997), pages 292-297; Mazur and Wiles (1984), page 194; Coates (1977), page 279, Theorem 1.8; Iwasawa (1973); Greenberg (1976). So we define

$$MCG_m^i = \epsilon_{p-i} MCG_m = \epsilon_{p-i} \mathrm{Gal}(M_m/F) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2]$$

(which is finite), and also $MCG^i = \varprojlim MCG_m^i = \epsilon_{p-i} MCG$. Note that $MCG_m = MCG/MCG^{\sigma_\beta^{p^m}-1}$, so we can recover the finite levels from the limit.

3.2.4 $MCG(-1)$, the Tate Twist of the Galois Group of the Maximal Abelian p-Extension Unramified Outside of p

Let $\mu_{p^{m+1}}$ be the p^{m+1}-th roots of unity in $\mathbb{C}_p$, which is a cyclic multiplicative group of order p^{m+1}; and also the torsion subgroup of F_m. Let $\rho_m^{m+1} : \mu_{p^{m+2}} \longrightarrow \mu_{p^{m+1}}$ be the p-power map $\eta \mapsto \eta^p$; which is equivalent to the restriction of the norm map $N_m^{m+1} : F_{m+1} \longrightarrow F_m$. Composing as necessary, define $\rho_m^n : \mu_{p^{n+1}} \longrightarrow \mu_{p^{m+1}}$ for $n > m$.

The group $\mu_{p^{m+1}}$ should be multiplicative, but we're going to write it additively. It still has the $\mathbb{Z}_p[\Gamma_n]$-module structure. So we have that $\{\mu_{p^{m+1}}, \rho_m^n\}$ is an inverse system of $\mathbb{Z}_p[\Gamma_n]$-modules; and thus we define $T = \varprojlim \mu_{p^{m+1}}$. So $T = T_p(\mathscr{O})$ is the p-adic Tate module of $\mathscr{O}$. Note that $T \cong \mathbb{Z}_p$ as additive abelian groups. We write the Galois action on T additively; that is, $\sigma_a(t) = at$ (which is really exponentiation t^a) for $t \in T$, $a \in \mathbb{Z}_p^\times$.

Then we define $T^{(-1)} = \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$, the twisted Tate module. So we have a Γ-action on $T^{(-1)}$ given by $\sigma_a \Upsilon = a^{-1}\Upsilon$ for $\Upsilon \in T^{(-1)}$, since $(\sigma_a \Upsilon)(t) = \sigma_a(\Upsilon(\sigma_a^{-1} t)) \stackrel{\dagger}{=} \Upsilon(\sigma_a^{-1} t) = \Upsilon(a^{-1} t) = a^{-1}\Upsilon(t)$, where † follows since Γ is trivial on $\mathbb{Z}_p$.

Now, for odd $i \in \mathbb{Z} \cap [3, p-2]$ we define

$$MCG_m^i(-1) = MCG_m^i \otimes_{\mathbb{Z}_p} T^{(-1)},$$

which as a $\mathbb{Z}_p$-module is the same as MCG_m^i, but the Galois action has changed

$$\sigma_a(\tau \otimes \Upsilon) = \sigma_a(\tau) \otimes a^{-1}\Upsilon = a^{-1}\sigma_a(\tau) \otimes \Upsilon.$$

(So basically, the effect that we want is that, if $\sigma_a(\tau)$ is what results when $\sigma_a \in \Gamma$ acts on $\tau \in MCG_m$, then $a^{-1}\sigma_a(\tau)$ is what would result if $\sigma_a \in \Gamma$ acted on τ were τ an element of $MCG_m(-1)$, but the tensor-product-with-the-Tate-Module method is a convenient way to not need to keep track of two different Γ-actions on the same module. From a certain perspective, this is nothing more than a different perspective.) We also define $MCG^i(-1) = MCG^i \otimes_{\mathbb{Z}_p} T^{(-1)} = \varprojlim MCG_m^i(-1)$ for odd $i \in \mathbb{Z} \cap [3, p-2]$. See Washington (1997), page 295.

3.2.5 IKCG, Iwasawa's Kummer Pairing Class Group

Consider $F^\times \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p)$, the elements of which can be written in the form $x \otimes p^{-c}$ for $x \in F^\times$ and $c \in \mathbb{N}$; see Iwasawa (1973), page 271-277; or Coates (1977), pages 283-284; or Washington (1997), pages 294-295. Then, for each $m \geq 0$, there is a subgroup

$$IKCG_m \subset F^\times \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p)$$

such that

$$M_m = F(\{x^{1/p^c} : x \otimes p^{-c} \in IKCG_m\}).$$

We have $IKCG_m \subset IKCG_n$ for $n \geq m$, giving inclusion $\iota_m^n : IKCG_m \longrightarrow IKCG_n$; giving the direct system $\{IKCG_m, \iota_m^n\}$; and also the direct limit

$$IKCG = \varinjlim IKCG_m = \bigcup_{m \geq 0} IKCG_m;$$

which also gives

$$M = F(\{x^{1/p^c} : x \otimes p^{-c} \in IKCG\}).$$

We have that Γ acts on $IKCG$ diagonally, but trivially on $\mathbb{Q}_p/\mathbb{Z}_p$; that is, for $\sigma \in \Gamma$,

$$\sigma(x \otimes p^{-c}) = \sigma x \otimes \sigma p^{-c} = \sigma x \otimes p^{-c}.$$

Now we define

$$IKCG_m^i = \epsilon_i IKCG_m \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2];$$

and then we similarly define

$$IKCG^i = \lim_{\rightarrow} IKCG_m^i = \bigcup_{m \geq 0} IKCG_m^i \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.6 DCG, the Dual Class Group

Define

$$DCG_m^i = \mathrm{Hom}_{\mathbb{Z}_p}(IKCG_m^i, \mathbb{Q}_p/\mathbb{Z}_p) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2],$$

the Pontryagin dual group. (Since $IKCG_m^i$ is p-power-torsion, a $\mathbb{Z}_p$-module homomorphism is nothing more than a $\mathbb{Z}$-module homomorphism). DCG_m^i is a $\mathbb{Z}_p$-module under the trivial action, and Γ acts via

$$(\sigma\Upsilon)(x \otimes p^{-c}) = \sigma(\Upsilon(\sigma^{-1}(x \otimes p^{-c}))),$$

for $\Upsilon \in DCG_m^i$. Then $\{DCG_m^i, Res_m^n\}$ is an inverse system where Res_m^n is restriction, and thus we define

$$DCG^i = \lim_{\leftarrow} DCG_m^i = \mathrm{Hom}_{\mathbb{Z}_p}(IKCG^i, \mathbb{Q}_p/\mathbb{Z}_p) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.7 TDCG, the Twisted Dual Class Group

Let μ_{p^∞} be the p-power roots of unity in $\mathbb{C}_p$, which is a multiplicative pro-p-group; and also the torsion subgroup of F, and as such it has a Γ-action. Define

$$TDCG_m^i = \mathrm{Hom}_{\mathbb{Z}_p}(IKCG_m^i, \mu_{p^\infty}) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Then $\{TDCG_m^i, Res_m^n\}$ is an inverse system where Res_m^n is restriction, and thus we define

$$TDCG^i = \lim_{\leftarrow} TDCG_m^i = \mathrm{Hom}_{\mathbb{Z}_p}(IKCG^i, \mu_{p^\infty}) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Note that DCG and $TDCG$ are equivalent as $\mathbb{Z}_p$-modules; that is, as groups they are just two incarnations of the Pontryagin dual group. Moreover, Γ still acts via

$$(\sigma\Upsilon)(x \otimes p^{-c}) = \sigma(\Upsilon(\sigma^{-1}(x \otimes p^{-c}))),$$

for $\Upsilon \in DCG^i_m$. Ultimately, the difference between DCG and $TDCG$ lies in that Γ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p$, but non-trivially on μ_{p^∞}, which is why we define the Pontryagin dual in two equivalent, but different ways following Coates (1977), pages 283-284; or Washington (1997), pages 294-296. (Again, from a certain perspective, this is nothing more than a different perspective.)

3.2.8 ICG, the Iwasawa Class Group

[2]Recall the definition of ξ from §2.5 above on page 8. Define

$$ICG^i_m = \mathbb{Z}_p[[\Gamma]]/(\xi(\omega^{1-i}), \sigma_\beta^{p^m} - 1) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Then $\{ICG^i_m, Lrg^n_m\}$ is an inverse system of $\mathbb{Z}_p[[\Gamma]]$-modules, where $Lrg^n_m : ICG^i_n \longrightarrow ICG^i_m$ is coset enlargement. Then we define

$$ICG^i = \lim_{\leftarrow} ICG^i_m = \mathbb{Z}_p[[\Gamma]]/(\xi(\omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.9 IACG, the Iwasawa Analytic Class Group

Recall the definition of f from §2.5 above on page 8. Define

$$IACG^i_m = \Lambda/((1+T)^{p^m} - 1, f(T, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Then $\{IACG^i_m, Lrg^n_m\}$ is an inverse system of $\mathbb{Z}_p[[T]]$-modules, where $Lrg^n_m : IACG^i_n \longrightarrow IACG^i_m$ is coset enlargement. Then we define

$$IACG^i = \lim_{\leftarrow} IACG^i_m = \Lambda/(f(T, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.10 ICG′, the Iwasawa Class Group Under Construction

Recall the definition of ξ_m from §2.5 above on page 8. Define

$$ICG'^i_m = \mathbb{Z}_p[\Gamma_m]/(\xi_m(\omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

[2]Owing to Iwasawa (1969) and Iwasawa (1972), I have decided to designate this module as the Iwasawa class group; and the associated elementary Iwasawa module as the Iwasawa analytic class group as in §3.2.9.

Under the restriction map $Res_m^n : \Gamma_n \longrightarrow \Gamma_m$, which can be thought of as coset enlargement $Lrg_m^n : \Gamma/\Gamma^{p^n} \longrightarrow \Gamma/\Gamma^{p^n}$, we have $\xi_n(\omega^{1-i}) \mapsto \xi_m(\omega^{1-i})$; see Washington (1997), page 119, Proposition 7.6c. This gives us the connecting morphism $\kappa_m^n : ICG'^{i}_n \longrightarrow ICG'^{i}_m$; and thus the inverse system $\{ICG'^{i}_m, \kappa_m^n\}$, whence also the inverse limit

$$ICG'^{i} = \lim_{\leftarrow} ICG'^{i}_m \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.11 MACG, the Analytic Class Group of the Galois Group of the Maximal Abelian p-Extension Unramified Outside of p

Define

$$MACG_m^i = \Lambda/((1+T)^{p^m} - 1, f(\tfrac{1+p}{1+T} - 1, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Then $\{MACG_m^i, Lrg_m^n\}$ is an inverse system of $\mathbb{Z}_p[[T]]$-modules, where $Lrg_m^n : MACG_n^i \longrightarrow MACG_m^i$ is coset enlargement. Then

$$MACG^i = \lim_{\leftarrow} MACG_m^i = \Lambda/(f(\tfrac{1+p}{1+T} - 1, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.12 DACG, the Dual Analytic Class Group

Define

$$DACG_m^i = \Lambda/((1+T)^{p^m} - 1, f(\tfrac{1}{1+T} - 1, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Then $\{DACG_m^i, Lrg_m^n\}$ is an inverse system of $\mathbb{Z}_p[[T]]$-modules, where $Lrg_m^n : DACG_n^i \longrightarrow DACG_m^i$ is coset enlargement. Then

$$DACG^i = \lim_{\leftarrow} DACG_m^i = \Lambda/(f(\tfrac{1}{1+T} - 1, \omega^{1-i})) \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.2.13 SCG, the Stickelberger Class Group

Let

$$\theta_m = \frac{1}{p^m} \sum_{\substack{a=1 \\ p \nmid a}}^{p^m} a\sigma_a^{-1} \in \mathbb{Q}[G_m]$$

be the Stickelberger element, and define

$$SCG_m^- = \epsilon^- \mathbb{Z}[G_m]/(\theta_m \mathbb{Z}[G_m] \cap \epsilon^- \mathbb{Z}[G_m]).$$

See Iwasawa (1962); Washington (1997), page 102, Theorem 6.19, and also pages 106-107. The restriction map $Res^n_m : G_n \longrightarrow G_m$ may induce a connecting morphism; however, this has not been established.

3.2.14 UCG, the Unit Class Group

Let $CU_m \subset \mathscr{O}_m^\times$ be the p-power-torsion cyclotomic units of F_m; that is, if $V_m \subset F_m^\times$ is the multiplicative subgroup generated by $\{\pm\zeta_{p^{m+1}}, 1-\zeta_{p^{m+1}}^k : 1 \le k < p^{m+1}\}$, then CU_m is the p-Sylow subgroup of $V_m \cap \mathscr{O}_m^\times$. Let $CU_m^+ \subset (\mathscr{O}_m^+)^\times$ be the p-power-torsion cyclotomic units of F_m^+; that is, $CU_m^+ = CU_m \cap (\mathscr{O}_m^+)^\times$. Now we define

$$UCG_m^i = \epsilon_i(\mathscr{O}_m^\times / CU_m) \qquad \text{for even } i \in \mathbb{Z} \cap [2, p-3]$$

and also $UCG_m^+ = (\mathscr{O}_m^+)^\times / CU_m^+$. See Washington (1997), page 145, Theorem 8.2, and also page 146; and also page 342, Theorem 15.7. Establishing a limit UCG^i is also problematic here as above since no connecting morphism has been established. The inclusion map $\mathscr{O}_m^\times \subset \mathscr{O}_n^\times$ would be indicative of a direct limit, however the inclusion $CU_m \subset CU_n$ would be indicative of a coset enlargement type map of an inverse limit.

3.2.15 LUCG, the Local Unit Class Group

Let

$$L1U_m = \left\{x \in \mathbb{Z}_p[\zeta_{p^{m+1}}]^\times : x \equiv 1 \mod (\zeta_{p^{m+1}} - 1)\right\}$$

be the local 1-units of $\mathbb{Q}_p(\zeta_{p^{m+1}})$, which is a $\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^{m+1}})/\mathbb{Q}_p)]$-module, and also a $\mathbb{Z}_p[\Gamma_m]$-module. Then the restriction of the norm map $N_m^{m+1} : \mathbb{Q}_p(\zeta_{p^{m+2}}) \longrightarrow \mathbb{Q}_p(\zeta_{p^{m+1}})$ gives a map $N_m^{m+1} : L1U_{m+1} \longrightarrow L1U_m$. Define $LC1U_m = \overline{CU_m \cap L1U_m}$, the closure of $CU_m \cap L1U_m$ in $L1U_m$, which is a $\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^{m+1}})/\mathbb{Q}_p)]$-module, and also a $\mathbb{Z}_p[\Gamma_m]$-module. Define

$$LUCG_m = L1U_m / LC1U_m,$$

which is a $\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^{m+1}})/\mathbb{Q}_p)]$-module, and also a $\mathbb{Z}_p[\Gamma_m]$-module. Define

$$LUCG = \lim_{\leftarrow} LUCG_m$$

with respect to the norm map. We also define

$$LUCG_m^i = \epsilon_{p-i} LUCG_m$$

and

$$LUCG^i = \lim_{\leftarrow} LUCG^i_m = \epsilon_{p-i} LUCG \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

See Washington (1997), pages 312-318.

3.3 Maps

Generally, the references that apply to the objects also apply to the maps between.

3.3.1 Artin: $CG \longrightarrow HCG$

The Artin map

$$[\mathfrak{a}] \mapsto \left(\frac{H_m/F_m}{\mathfrak{a}} \right)$$

gives an isomorphism $CG_m \cong HCG_m$, and hence $CG^i_m \cong HCG^i_m$ follows; but it also gives a transformation of inverse systems giving an isomorphism $CG \cong HCG$, and hence $CG^i \cong HCG^i$. See Iwasawa (1973), pages 259-260 for a proof that the Artin map commutes with the connecting morphism.

3.3.2 Kummer: $IKCG \longrightarrow CG$

We have a surjective map $IKCG \longrightarrow \overrightarrow{CG}$ given by $x \otimes p^{-c} \mapsto [\ \sqrt[p^c]{(x)}\ ]$ as in Iwasawa (1973), page 275 (n.b., A not A'); or Washington (1997), page 295. Decomposing on idempotents gives $IKCG^i \longrightarrow \overrightarrow{CG}^i$ for odd $i \in \mathbb{Z} \cap [3, p-2]$. The ϵ_i-eigenspace component of $\ker(IKCG \longrightarrow \overrightarrow{CG})$ is trivial, giving us an isomorphism $IKCG^i \xrightarrow{\cong} \overrightarrow{CG}^i$. (We do not necessarily have an isomorphism for the even eigenspaces.) Since this map is an isomorphism, and since $IKCG$ is a direct limit to boot, we concern ourselves only with the map between the limits.

3.3.3 Dual: $IKCG \longrightarrow DCG$, $IKCG \longrightarrow TDCG$

By Pontryagin duality, we immediately have the non-canonical abelian group isomorphisms $IKCG^i \xrightarrow{\cong} DCG^i$ and $IKCG^i_m \xrightarrow{\cong} TDCG^i_m$; but we also have Λ-module isomorphism under the given Γ actions. As above, we concern ourselves only with the maps between the limits.

3.3.4 Pairing: $MCG \longrightarrow TDCG$, $MCG(-1) \longrightarrow DCG$

We have a non-degenerate pairing

$$MCG \times IKCG \longrightarrow \mu_{p^\infty}$$

given by

$$(\tau, x \otimes p^{-c}) \mapsto \langle \tau, x \otimes p^{-c} \rangle = \frac{\tau(\sqrt[p^c]{x}\,)}{\sqrt[p^c]{x}},$$

which additionally satisfies

$$\sigma\langle \tau, x \otimes p^{-c} \rangle = \langle \sigma \cdot \tau, \sigma(x \otimes p^{-c}) \rangle = \langle \sigma\tau\sigma^{-1}, \sigma x \otimes p^{-c} \rangle \qquad \text{for all } \sigma \in \Gamma.$$

We in fact also have a non-degenerate pairing

$$MCG^i \times IKCG^i \longrightarrow \mu_{p^\infty} \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2],$$

which gives

$$MCG^i \xrightarrow{\cong} TDCG^i \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

Note that $IKCG_m^\perp = \mathrm{Gal}(M/M_m) \cong MCG/MCG_m$ is the annihilator of $IKCG_m$ in MCG with respect to the above pairing; which gives us the induced pairing

$$MCG_m \times IKCG_m \longrightarrow \mu_{p^\infty}.$$

Combining we have

$$MCG_m^i \xrightarrow{\cong} TDCG_m^i \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

The map $MCG^i(-1) \longrightarrow DCG^i$ is an alternate, but equivalent formulation. Note that these maps are non-canonical; and as such, they a priori need not commute with the connecting morphisms. This raises the question, under what conditions can such maps be constructed that commute with the connecting morphisms, and under what conditions would the induced map be equivalent to the map on the limits?

3.3.5 Iwasawa: $ICG \longrightarrow IACG$

Recall from §2.5, that the Iwasawa isomorphism given by $\sigma_\beta \mapsto 1 + T$ maps $\xi(\theta) \mapsto f(T, \theta)$. Thus under the Iwasawa isomorphism we have

$$ICG_m^i \xrightarrow{\cong} IACG_m^i \qquad \text{and} \qquad ICG^i \xrightarrow{\cong} IACG^i \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2].$$

3.3.6 $ICG \longrightarrow ICG'$

I would like to construct the "obvious" map $ICG \longrightarrow ICG'$ given by

$$(y_m)_{m\in\mathbb{N}} \mod (\xi(\omega^{1-i}), \sigma_\beta^{p^m} - 1) \mapsto \Big(y_m \mod (\xi_m(\omega^{1-i}))\Big)_{m\geq 0}$$

for $(y_m)_{m\in\mathbb{N}} \in \mathbb{Z}_p[[\Gamma]]$; however, there is no immediate reason why this should even be well-defined, let alone commute with the connecting morphisms. Regardless, should these conditions be met, it would indeed be very interesting to understand the minimal m where this map began to behave in a regular fashion and what that would mean.

3.3.7 Mazur-Wiles: $HCG \longrightarrow IACG$, $HCG \longrightarrow ICG$, $DCG \longrightarrow DACG$, $TDCG \longrightarrow DACG$, $MCG \longrightarrow MACG$, $MCG(-1) \longrightarrow MACG$

[3]Mazur and Wiles (1984) proves the Main Conjecture of Iwasawa Theory for cyclotomic $\mathbb{Z}_p$-extensions over totally real number fields establishing the existence all the pseudo-isomorphisms

$$\begin{aligned}
HCG^i &\longrightarrow IACG^i \\
HCG^i &\longrightarrow ICG^i \\
MCG^i &\longrightarrow MACG^i \\
MCG^i(-1) &\longrightarrow MACG^i \\
DCG^i &\longrightarrow DACG^i \\
TDCG^i &\longrightarrow DACG^i
\end{aligned}$$

for odd $i \in \mathbb{Z} \cap [3, p-2]$ by justifying, in simplistic terms, that the characteristic polynomials of HCG, MCG, and DCG are respectively $f(T, \omega^{1-i})$, $f(\frac{1+p}{1+T} - 1, \omega^{1-i})$, and $f(\frac{1}{1+T} - 1, \omega^{1-i})$. However, none of these maps are explicitly constructed, and it remains a fundamental question as to whether these maps can be induced from transformations on the inverse systems, and if so, for which m do these transformations become well-behaved. It is important to recall that if the Kummer-Vandiver Conjecture

[3]Owing to Wiles (1980), and Mazur and Wiles (1984), I have decided to designate the maps constituting the Main Conjecture as the Mazur-Wiles map.

holds, then such a transformation does exist, and it is an isomorphisms for all $m \geq 0$, and it induces an isomorphism. However, the gap between what we expect to be true and what we can prove remains significant. A primary motivation of the work of the subsequent chapters is to provide some understanding to the task of bridging this gap.

3.3.8 Transform: $IACG \longrightarrow DACG$

We already have a pseudo-isomorphism

$$IACG^i \longrightarrow DACG^i \qquad \text{given by} \qquad T \mapsto \frac{1}{1+T};$$

however, for all the complications it is reassuring to see it verified directly. If $f(T) = a_0 + a_1T + a_2T^2 + a_3T^3 + \cdots \in \mathbb{Z}_p[[T]]$, then $a_i \in p\mathbb{Z}_p$ for $0 \leq i < \lambda$ and $a_\lambda \in \mathbb{Z}_p^\times$ if and only if

$$f(T) \equiv a_\lambda T^\lambda \not\equiv 0 \mod (p, T^{\lambda+1}).$$

Since

$$\frac{1}{1+T} = 1 - T + T^2 - T^3 + \cdots$$

and

$$\begin{aligned} f\Big(\tfrac{1}{1+T} - 1\Big) &= a_0 + a_1\Big(\tfrac{1}{1+T} - 1\Big) + a_2\Big(\tfrac{1}{1+T} - 1\Big)^2 + a_3\Big(\tfrac{1}{1+T} - 1\Big)^3 + \cdots \\ &\equiv a_\lambda\Big(\tfrac{1}{1+T} - 1\Big)^\lambda \\ &= a_\lambda\Big(-T + T^2 - T^3 + \cdots\Big)^\lambda \\ &\equiv (-1)^\lambda a_\lambda T^\lambda \\ &\equiv (-1)^\lambda f(T) \mod (p, T^{\lambda+1}), \end{aligned}$$

we have that $f(T)$ and $f(\frac{1}{1+T} - 1)$ have the same Weierstrass degree; and thus $IACG^i$ and $DACG^i$ are pseudo-isomorphic.

3.3.9 Transform: $IACG \longrightarrow MACG$

Similar to the above, we already have a pseudo-isomorphism

$$IACG^i \longrightarrow MACG^i \qquad \text{given by} \qquad T \mapsto \frac{1+p}{1+T};$$

which we verify directly. Since

$$\frac{1+p}{1+T} = (1+p) - (1+p)T + (1+p)T^2 - (1+p)T^3 + \cdots$$

and

$$\begin{aligned}
& f\Big(\tfrac{1+p}{1+T} - 1\Big) \\
= \; & a_0 + a_1\Big(\tfrac{1+p}{1+T} - 1\Big) + a_2\Big(\tfrac{1+p}{1+T} - 1\Big)^2 + a_3\Big(\tfrac{1+p}{1+T} - 1\Big)^3 + \cdots \\
\equiv \; & a_\lambda\Big(\tfrac{1+p}{1+T} - 1\Big)^\lambda \\
= \; & a_\lambda\Big(p - (1+p)T + (1+p)T^2 - (1+p)T^3 + \cdots\Big)^\lambda \\
\equiv \; & a_\lambda\Big(p - (1+p)T\Big)^\lambda \\
= \; & a_\lambda\Big(p^\lambda - \lambda p^{\lambda-1}(1+p)T + \cdots + (-1)^{\lambda-1}\lambda p(1+p)^{\lambda-1}T^{\lambda-1} + (-1)^\lambda(1+p)^\lambda T^\lambda\Big) \\
\equiv \; & (-1)^\lambda a_\lambda (1+p)^\lambda T^\lambda \\
= \; & (-1)^\lambda a_\lambda \Big(1 + \lambda p + \cdots + \lambda p^{\lambda-1} + p^\lambda\Big) T^\lambda \\
\equiv \; & (-1)^\lambda a_\lambda T^\lambda \\
\equiv \; & (-1)^\lambda f(T) \mod (p, T^{\lambda+1}),
\end{aligned}$$

we have that $f(T)$ and $f(\frac{1+p}{1+T} - 1)$ have the same Weierstrass degree; and thus $IACG^i$ and $MACG^i$ are pseudo-isomorphic.

3.3.10 Iwasawa: $SCG \longrightarrow CG$

Iwasawa has shown that SCG_m^- and CG_m^- have the same size, and it was conjectured for a time that they might be isomorphic, but it is now known that they are not isomorphic in general. However, if there is indeed a connecting morphism that allows for the construction of a limit SCG^-, then it seems reasonable to conjecture that SCG^- and CG^- may well be pseudo-isomorphic.

3.3.11 Kummer: $UCG \longrightarrow CG$

Originally due to Kummer in 1851, it is known for nonzero even $i \in \mathbb{Z} \cap [2, p-3]$ that UCG_m^i and CG_m^i have the same order, however both are conjectured to be zero by the Kummer-Vandiver Conjecture. Although, without assuming the Kummer-Vandiver

Conjecture, if there is a limit UCG^i, then it seems reasonable to conjecture that UCG^i and CG^i may well be pseudo-isomorphic.

3.3.12 Iwasawa: $LUCG \longrightarrow IACG$

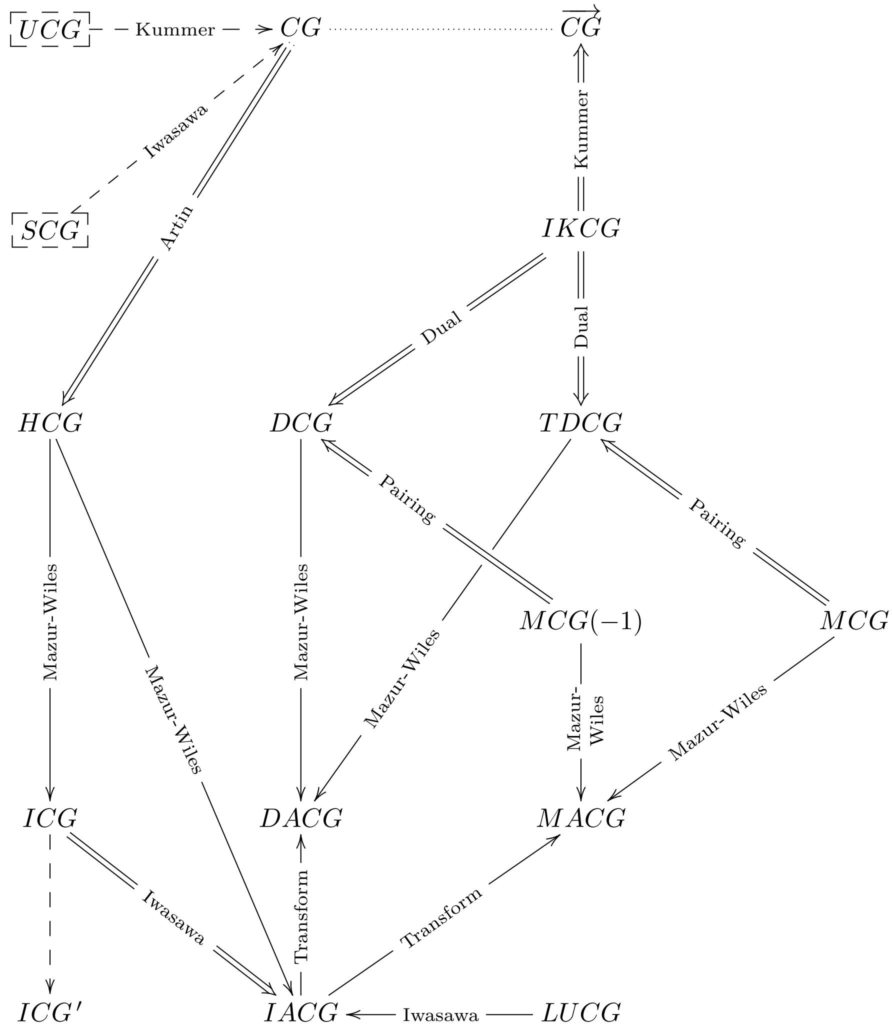
In this case we do have well-defined limits, as well as an isomorphism

$$LUCG^i \longrightarrow IACG^i \qquad \text{for odd } i \in \mathbb{Z} \cap [3, p-2]$$

established by Iwasawa in 1964; see Washington (1997), page 316, Theorem 13.56; and Neukirch, Schmidt, and Wingberg (2008), page 782, Theorem 11.6.18. However, as with the Mazur-Wiles Main Conjecture Map, it remains a fundamental question as to whether these maps can be induced from transformations on the inverse systems, and if so, for which m do these transformations become well-behaved. Fortunately, in this case the map can be explicitly constructed.

3.4 A Graph of the Maps

Here is a graph of all the maps heretofore considered.



(3.2)

I have dropped the $_i$'s. The dashed box around UCG and SCG remind us that the existence of these limits is not established. A double shafted solid arrow indicates a known isomorphism, a single shafted solid arrow indicates a known pseudo-isomorphism, a dashed arrow indicates a conjectured pseudo-isomorphism, and a dotted line indicates two objects that are direct and inverse limits of similar objects. Several of the maps are a transformation between two systems, however several maps are defined only on the limit. Most of the maps known to exist are only defined for odd $i \in \mathbb{Z} \cap [3, p-2]$; however, one is defined for all i. Of the three conjectured maps, one is only for odd $i \in \mathbb{Z} \cap [3, p-2]$, one is only for even $i \in \mathbb{Z} \cap [2, p-3]$, and one is only for the entire odd eigenspace. The primary map currently under considerations is Mazur-Wiles $\circ$ Artin : $CG^i \longrightarrow IACG^i$.

Chapter 4

LEMMATA

[1]We now initiate our Iwasawa-Theory-inspired development of inverse systems from a removed algebraic perspective.

Lemmas 4.0.1 and 4.0.3 are fairly standard results included for clarity sake−so they can be referenced−and moreover they illustrate well several issues of primary concern.

Lemma 4.0.2 expounds a phenomenon that unexpectedly is a deciding factor in later results.

Lastly, we finish the chapter with Lemma 4.0.4, which serves only to facilitate a proof in §5.5 below. It seems like it should be a standard result, but I've never seen it before, which is why we prove it; moreover, while it is simple enough as presented in Lemma 4.0.4, it is not so simple in §5.5.

The notation of Chapter 4 is independent of all other chapters.

Let R be a commutative ring (assumed unitary).

<u>Lemma 4.0.1</u>. Let $z \in \mathbb{Z}$; let $I = \mathbb{Z} \cap [z, \infty)$; let $\{A_i, \alpha_i^j\}$ be an inverse system of R-modules over I; let $A = \varprojlim A_i$ be the inverse limit; and let $\alpha_j^* : A \longrightarrow A_j$ be the natural maps $\alpha_j^* : (a_i)_{i\in I} \mapsto a_j$. If we have that the α_i^j are all surjective (for i sufficiently large), then we have that the α_j^* are all surjective (for all $j \geq i$). If we have that the α_i^j are all injective (for i sufficiently large), then we have that the α_j^* are all injective (for all $j \geq i$). If we have that the α_i^j are all bijective (for i sufficiently large), then we have that the α_j^* are all bijective (for all $j \geq i$).

<u>Lemma 4.0.2</u>. Let $z \in \mathbb{Z}$; let $I = \mathbb{Z} \cap [z, \infty)$; let $\{A_i, \alpha_i^j\}$ be an inverse system of R-modules over I, without assuming that the α_i^j are either injective or surjective; let $A = \varprojlim A_i$ be the inverse limit; and let $\alpha_j^* : A \longrightarrow A_j$ be the natural maps $\alpha_j^* : (a_i)_{i\in I} \mapsto a_j$, which are not necessarily surjective since we have not assumed the α_i^j to be surjective. If we have that the A_i are all eventually finite with bounded cardinality, then we have that α_j^* is an injection for all j sufficiently large.

[1]Rotman (2002) is a general reference for definitions and notational trends.

Proof. Let $B_i = \alpha_i^* A$. For $j \geq i$, we have $\alpha_i^j \alpha_j^* = \alpha_i^*$; and hence the restricted map $\alpha_i^j : B_j \longrightarrow B_i$ is well-defined and surjective. Note that $A = \varprojlim B_i$. Since the A_i are all eventually finite with bounded cardinality, so are the B_i. Then, the sequence $\{|B_i|\}_{i \in I}$ is a bounded non-decreasing sequence of natural numbers, and thus must stabilize (the Bounded Monotonic Sequence Convergence Theorem and discrete sets lack accumulation points). Once it stabilizes, say by n_0, we have $|B_j| = |B_i|$ for all $j \geq i \geq n_0$; whence the already surjective $\alpha_i^j : B_j \longrightarrow B_i$ must now be injective. Thus, by Lemma 4.0.1, we have that the restriction $\alpha_i^* : A \longrightarrow B_i$ is an isomorphism for all $i \geq n_0$; and thus α_i^* maps A isomorphically onto a subset of A_i, thither $\alpha_i^* : A \longrightarrow A_i$ is injective. ☐

<u>Lemma 4.0.3</u>. For $r \in \mathbb{Z} \cap [2, \infty)$, let

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_r \longrightarrow 0$$

be a finite length exact sequence of R-modules. Then

$$\prod_{\text{odd } i} |A_i| = \prod_{\text{even } i} |A_i|.$$

Proof. If $r = 2$, then

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow 0$$

gives $A_1 \cong A_2$, and thus $|A_1| = |A_2|$.

If $r = 3$, then

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$$

together with the First Isomorphism Theorem, gives us $A_2/fA_1 = A_2/\ker g \cong gA_2 = A_3$ and $A_1 \cong fA_1$. If we assume $|A_1| \cdot |A_2| \cdot |A_3| < \infty$, then

$$|A_1| \cdot |A_3| = |fA_1| \cdot |A_2/fA_1| = |fA_1| \cdot \frac{|A_2|}{|fA_1|} = |A_2|.$$

If $|A_1| = \infty$, then $|A_2| = \infty$ since f is injective; and the equality holds. If $|A_3| = \infty$, then $|A_2| = \infty$ since g is surjective; and the equality holds. Thus, we may assume that $|A_1| \cdot |A_3| < \infty$; but this gives $|A_2| < \infty$ since otherwise we would have the contradiction $|A_3| = |A_2/fA_1| = \infty$.

We now induct on r using $r = 3$ as the base case, which is done. So we assume $r > 3$ and that the result holds for all $s \in \mathbb{Z} \cap [2, r)$. Then we can split the exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_{r-2} \xrightarrow{f} A_{r-1} \xrightarrow{g} A_r \longrightarrow 0$$

into the two exact sequences

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_{r-2} \xrightarrow{f} \operatorname{im} f \longrightarrow 0$$

$$0 \longrightarrow \ker g \longrightarrow A_{r-1} \xrightarrow{g} A_r \longrightarrow 0$$

where $\ker g \longrightarrow A_{r-1}$ is inclusion; whence the induction hypothesis gives

$$\prod_{\substack{\text{odd } i \\ i \le r-2}} |A_i| = |\operatorname{im} f| \cdot \prod_{\substack{\text{even } i \\ i \le r-3}} |A_i| \qquad \text{for odd } r,$$

$$|\operatorname{im} f| \cdot \prod_{\substack{\text{odd } i \\ i \le r-3}} |A_i| = \prod_{\substack{\text{even } i \\ i \le r-2}} |A_i| \qquad \text{for even } r,$$

and

$$|\operatorname{im} f| \cdot |A_r| = |\ker g| \cdot |A_r| = |A_{r-1}|.$$

Thus

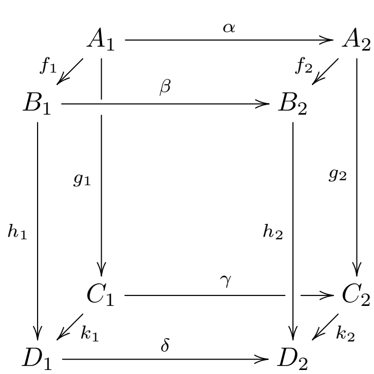
$$\prod_{\text{odd } i} |A_i| = |A_r| \cdot \prod_{\substack{\text{odd } i \\ i \le r-2}} |A_i| = |A_r| \cdot |\operatorname{im} f| \cdot \prod_{\substack{\text{even } i \\ i \le r-3}} |A_i| = |A_{r-1}| \cdot \prod_{\substack{\text{even } i \\ i \le r-3}} |A_i| = \prod_{\text{even } i} |A_i| \qquad \text{for odd } r;$$

and

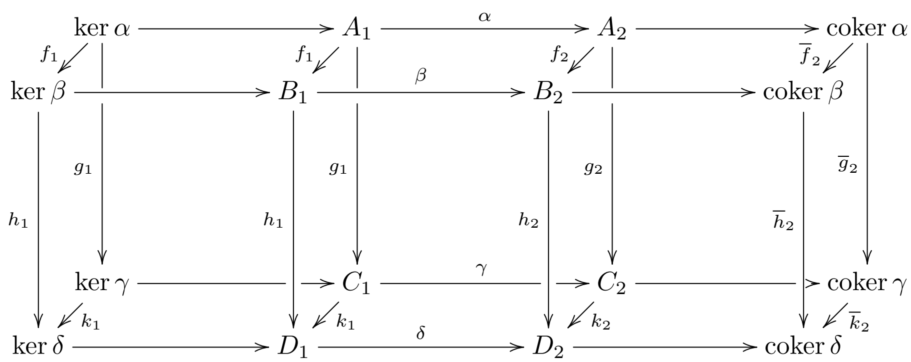
$$\prod_{\text{odd } i} |A_i| = |A_{r-1}| \cdot \prod_{\substack{\text{odd } i \\ i \le r-3}} |A_i| = |A_r| \cdot |\operatorname{im} f| \cdot \prod_{\substack{\text{odd } i \\ i \le r-3}} |A_i| = |A_r| \cdot \prod_{\substack{\text{even } i \\ i \le r-2}} |A_i| = \prod_{\text{even } i} |A_i| \qquad \text{for even } r.$$

□

Lemma 4.0.4. If $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are R-modules and the cube



commutes, then the lattice



commutes with the natural induced maps: $\ker\alpha \longrightarrow A_1$ is inclusion, $A_2 \longrightarrow \operatorname{coker}\alpha$ is projection, the induced kernel map $\ker\alpha \xrightarrow{f_1} \ker\beta$ is the restriction of f_1, the induced cokernel map $\operatorname{coker}\alpha \xrightarrow{\overline{f}_2} \operatorname{coker}\beta$ is given by $a + \operatorname{im}\alpha \mapsto f_2 a + \operatorname{im}\beta$, and similarly for all other maps.

Proof. Note that a diagram commutes if and only if each minimal polygon commutes. All maps are well-defined and the induced kernel and cokernel maps commute with their inducing maps. It remains only to verify that the kernel and cokernel squares on the ends commute. The kernel square commutes since the $A_1 - B_1 - C_1 - D_1$ square commutes, and the cokernel square commutes since the $A_2 - B_2 - C_2 - D_2$ square commutes. □

Chapter 5

THE MINIMAL LEVELS OF A TRANSFORMATION BETWEEN INVERSE SYSTEMS

[1]The notation of Chapter 5 is independent of all other chapters.

Let R be a commutative ring (assumed unitary); let $z \in \mathbb{Z}$; let $I = \mathbb{Z} \cap [z, \infty)$; let $\{M_i, \phi_i^j\}$ and $\{N_i, \psi_i^j\}$ be inverse systems of R-modules over I; assume that ϕ_i^j and ψ_i^j are both surjective for all $i \leq j$; let $M = \varprojlim M_i$ and $N = \varprojlim N_i$ be the inverse limits; let $\phi_i^* : M \longrightarrow M_i$ and $\psi_i^* : N \longrightarrow N_i$ be the natural (projection) maps $\phi_i^* : (m_j)_{j \in I} \mapsto m_i$ and $\psi_i^* : (n_j)_{j \in I} \mapsto n_i$, which are surjective by Lemma 4.0.1; let $\{\theta_i : M_i \longrightarrow N_i\} : \{M_i, \phi_i^j\} \longrightarrow \{N_i, \psi_i^j\}$ be a transformation, so that $\theta_i : M_i \longrightarrow N_i$ are R-homomorphisms such that

$$\begin{array}{ccc} M_j & \xrightarrow{\phi_i^j} & M_i \\ {\scriptstyle \theta_j}\downarrow & & \downarrow{\scriptstyle \theta_i} \\ N_j & \xrightarrow{\psi_i^j} & N_i \end{array}$$

commutes for all $i \leq j$; and let $\theta : M \longrightarrow N$ be the induced R-homomorphism, so that $\theta : (m_i)_{i \in I} \mapsto (\theta_i m_i)_{i \in I}$, where $(\theta_i m_i)_{i \in I} \in N$ since $\psi_i^j \theta_j m_j = \theta_i \phi_i^j m_j = \theta_i m_i$. For $i \leq j$, we have that

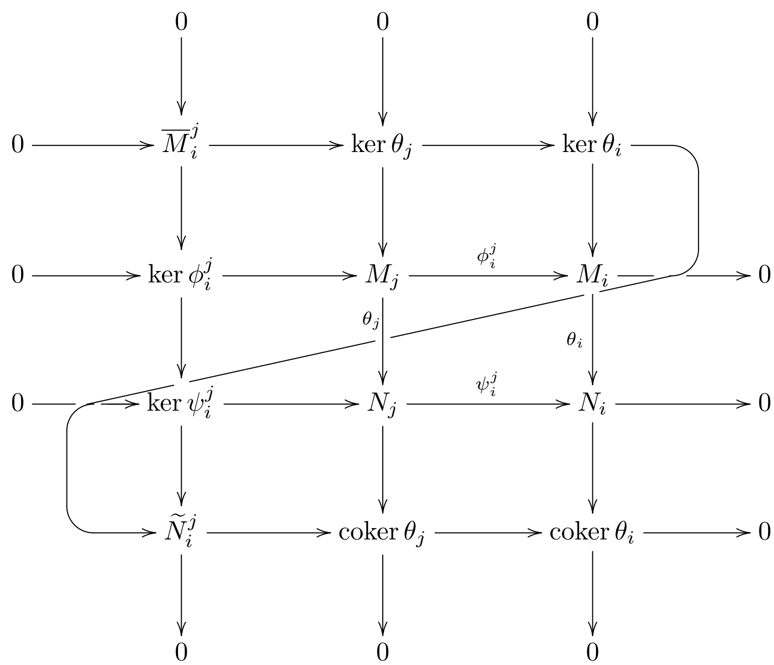
$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \ker\theta_j & \xrightarrow{\phi_i^j} & \ker\theta_i \\ \downarrow & & \downarrow \\ M_j & \xrightarrow{\phi_i^j} & M_i \\ {\scriptstyle \theta_j}\downarrow & & \downarrow{\scriptstyle \theta_i} \\ N_j & \xrightarrow{\psi_i^j} & N_i \\ \downarrow & & \downarrow \\ \operatorname{coker}\theta_j & \xrightarrow{\overline{\psi}_i^j} & \operatorname{coker}\theta_i \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

[1]Rotman (2002) is a general reference for definitions and notational trends.

commutes and is exact vertically, where $\ker\theta_j \xrightarrow{\phi_i^j} \ker\theta_i$ is the restriction of ϕ_i^j and $\operatorname{coker}\theta_j \xrightarrow{\overline{\psi}_i^j} \operatorname{coker}\theta_i$ is the induced map $\overline{\psi}_i^j : n + \operatorname{im}\theta_j \mapsto \psi_i^j n + \operatorname{im}\theta_i$. We have that $\{\ker\theta_i, \phi_i^j\}$ and $\{\operatorname{coker}\theta_i, \overline{\psi}_i^j\}$ are inverse systems.

<u>Remark 5.0.5</u>. It would be nice if the θ_i of interest were all isomorphisms, however this may be too high of an aim; so rather we seek to control the θ_i by regulating the inverse systems $\{\ker\theta_i, \phi_i^j\}$ and $\{\operatorname{coker}\theta_i, \overline{\psi}_i^j\}$, ideally by forcing the induced maps ϕ_i^j and $\overline{\psi}_i^j$ to eventually be isomorphisms, but secondarily by forcing the $\ker\theta_i$ and $\operatorname{coker}\theta_i$ to have an eventually fixed size.

For $i \le j$, the Snake Lemma gives us that



(5.1)

commutes and is exact horizontally and vertically (and snake-ishly), where

$$\begin{aligned} \overline{M}_i^j &= \ker\phi_i^j \cap \ker\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\psi_i^j) = \ker(\ker\theta_j \longrightarrow \ker\theta_i) \\ \widetilde{N}_i^j &= \operatorname{coker}(\ker\phi_i^j \longrightarrow \ker\psi_i^j). \end{aligned}$$

and

$$\ker\theta_i \longrightarrow \widetilde{N}_i^j$$

is the connecting homomorphism given by $m_i \mapsto \theta_j(\phi_i^j)^{-1}m_i + \theta_j \ker\phi_i^j$. Since the ψ_i^j are surjective, so are the $\overline{\psi}_i^j$, whence the natural (projection) maps

$\overline{\psi}_i^* : \lim\limits_{\leftarrow} \operatorname{coker}\theta_j \longrightarrow \operatorname{coker}\theta_i$ given by $\overline{\psi}_i^* : (n_j + \operatorname{im}\theta_j)_{j\in I} \mapsto n_i + \operatorname{im}\theta_i$ are also surjective. However, since the maps $\ker\theta_j \xrightarrow{\phi_i^j} \ker\theta_i$ need not be surjective, the natural maps $\phi_i^* : \lim\limits_{\leftarrow} \ker\theta_j \longrightarrow \ker\theta_i$ given by $\phi_i^* : (m_j)_{j\in I} \mapsto m_i$ need not be surjective. Since inverse limits are left exact, we have

$$\ker\theta = \lim_{\leftarrow} \ker\theta_i;$$

and since the cokernel maps are surjective, we have an R-epimorphism

$$\operatorname{coker}\theta \twoheadrightarrow \lim_{\leftarrow} \operatorname{coker}\theta_i.$$

Remark 5.0.6. So the kernels are well-behaved since the inverse limits are left exact, but ill-behaved since they are not necessarily surjective; and the cokernels are well-behaved since they are surjective, but ill-behaved since inverse limits are not necessarily right exact.

Proof. We justify the surjectivity of the cokernel map

$$\operatorname{coker}\theta \twoheadrightarrow \lim_{\leftarrow} \operatorname{coker}\theta_i$$

which we define by mapping $n + \theta M = (n_i)_{i\in I} + \theta M \in \operatorname{coker}\theta$ to $(n_i + \theta_i M_i)_{i\in I} \in \lim\limits_{\leftarrow} \operatorname{coker}\theta_i$.

Let $(n_i + \theta_i M_i)_{i\in I} \in \lim\limits_{\leftarrow} \operatorname{coker}\theta_i$. We may not have $\psi_i^j n_j \stackrel{?}{=} n_i$, that is we may not have $(n_i)_{i\in I} \stackrel{?}{\in} N$. So, we construct $n' = (n_i')_{i\in I} \in N$ such that $(n_i' + \theta_i M_i)_{i\in I} = (n_i + \theta_i M_i)_{i\in I} \in \lim\limits_{\leftarrow} \operatorname{coker}\theta_i$ and also $n' + \theta M \mapsto (n_i + \theta_i M_i)_{i\in I}$. So we must show for all $j \in I$, that $n_j' + \theta_j M_j = n_j + \theta_j M_j \in \operatorname{coker}\theta_j$ and $\psi_i^j n_j' = n_i'$ $\forall i \le j$.

We induct on j. Let $j = \min I$, and define $n_j' = n_j$. Then $n_j' + \theta_j M_j = n_j + \theta_j M_j \in \operatorname{coker}\theta_j$ and $\psi_i^j n_j' = n_i'$ $\forall i \le j$.

Let $j > \min I$ and assume true for all indices less than j. Then $\psi_{j-1}^j n_j + \theta_{j-1} M_{j-1} = \psi_{j-1}^j (n_j + \theta_j M_j) = n_{j-1} + \theta_{j-1} M_{j-1} = n_{j-1}' + \theta_{j-1} M_{j-1}$. So $\psi_{j-1}^j n_j - n_{j-1}' \in \theta_{j-1} M_{j-1}$, so there exists $m_{j-1} \in M_{j-1}$ such that $\theta_{j-1} m_{j-1} = \psi_{j-1}^j n_j - n_{j-1}'$. Since ϕ_{j-1}^j is surjective, there exists $m_j \in M_j$ such that $\phi_{j-1}^j m_j = m_{j-1}$. Then $\psi_{j-1}^j \theta_j m_j = \theta_{j-1}\phi_{j-1}^j m_j = \theta_{j-1} m_{j-1} = \psi_{j-1}^j n_j - n_{j-1}'$.

Define $n'_j = n_j - \theta_j m_j$, then

$n'_j + \theta_j M_j = n_j - \theta_j m_j + \theta_j M_j = n_j + \theta_j M_j \in \operatorname{coker}\theta_j$. We also have

$\psi^j_{j-1} n'_j = \psi^j_{j-1}(n_j - \theta_j m_j) = \psi^j_{j-1} n_j - \psi^j_{j-1}\theta_j m_j = \psi^j_{j-1} n_j - (\psi^j_{j-1} n_j - n'_{j-1}) = n'_{j-1}$.

If $i \leq j-1$, then $\psi^j_i n'_j = \psi^{j-1}_i \psi^j_{j-1} n'_j = \psi^{j-1}_i n'_{j-1} = n'_i$. Thus

$\psi^j_i n'_j = n'_i \ \forall i \leq j$. □

5.1 Definitions and Basic Facts

We define three notions of the minimal level of θ and give some basic facts.

Define the *strong minimal level* of θ to be

$$\begin{aligned}
\text{s-min}\,\theta &= \min\Big(\{i \in I : \ker\theta_j \overset{\cong}{\to} \ker\theta_i \text{ and } \operatorname{coker}\theta_j \overset{\cong}{\to} \operatorname{coker}\theta_i \ \forall j \geq i\} \cup \{\infty\}\Big) \\
&= \min\Big(\{i \in I : \ker\theta_{j+1} \overset{\cong}{\to} \ker\theta_j \text{ and } \operatorname{coker}\theta_{j+1} \overset{\cong}{\to} \operatorname{coker}\theta_j \ \forall j \geq i\} \cup \{\infty\}\Big);
\end{aligned}$$

the *common minimal level* of θ to be

$$\begin{aligned}
\text{c-min}\,\theta &= \min\Big(\{i \in I : |\ker\theta_j| = |\ker\theta_i| \text{ and } \operatorname{coker}\theta_j \overset{\cong}{\to} \operatorname{coker}\theta_i \ \forall j \geq i\} \cup \{\infty\}\Big) \\
&= \min\Big(\{i \in I : |\ker\theta_{j+1}| = |\ker\theta_j| \text{ and } \operatorname{coker}\theta_{j+1} \overset{\cong}{\to} \operatorname{coker}\theta_j \ \forall j \geq i\} \cup \{\infty\}\Big);
\end{aligned}$$

and the *weak minimal level* of θ to be

$$\begin{aligned}
\text{w-min}\,\theta &= \min\Big(\{i \in I : \operatorname{coker}\theta_j \overset{\cong}{\to} \operatorname{coker}\theta_i \ \forall j \geq i\} \cup \{\infty\}\Big) \\
&= \min\Big(\{i \in I : \operatorname{coker}\theta_{j+1} \overset{\cong}{\to} \operatorname{coker}\theta_j \ \forall j \geq i\} \cup \{\infty\}\Big).
\end{aligned}$$

So

$$\text{w-min}\,\theta \leq \text{c-min}\,\theta \leq \text{s-min}\,\theta.$$

Note that the minimal levels here are defined in terms of a transformation between inverse limits.

Fact 5.1.1. If $\text{w-min}\,\theta < \infty$, then we have

$$\varprojlim \operatorname{coker}\theta_j \cong \operatorname{coker}\theta_i \qquad \text{for all } i \geq \text{w-min}\,\theta.$$

Proof. If we assume $\text{w-min}\,\theta < \infty$, and let $i \geq \text{w-min}\,\theta$, then the natural map $\overline{\psi}^*_i : \varprojlim \operatorname{coker}\theta_j \longrightarrow \operatorname{coker}\theta_i$ given by $\overline{\psi}^*_i : (n_j + \theta_j M_j)_{j\in I} \mapsto n_i + \theta_i M_i$ is an isomorphism by Lemma 4.0.1. □

Fact 5.1.2. If s-min $\theta < \infty$, then we have

$$\ker\theta = \varprojlim \ker\theta_j \cong \ker\theta_i \quad \text{and} \quad \operatorname{coker}\theta \cong \varprojlim \operatorname{coker}\theta_j \cong \operatorname{coker}\theta_i \quad \text{for all } i \geq \text{s-min}\,\theta.$$

Proof. We assume s-min $\theta < \infty$, and let $i \geq$ s-min θ. Then the natural map $\phi_i^* : \varprojlim \ker\theta_j \longrightarrow \ker\theta_i$ given by $\phi_i^* : (m_j)_{j\in I} \mapsto m_i$ is an isomorphism by Lemma 4.0.1. We already had $\ker\theta = \varprojlim \ker\theta_j$ by left exactness; and we have $\varprojlim \operatorname{coker}\theta_j \cong \operatorname{coker}\theta_i$ by Fact 5.1.1 above or by Lemma 4.0.1. Thus it remains to show $\operatorname{coker}\theta \cong \varprojlim \operatorname{coker}\theta_j$, the which we do by showing that the well-defined and surjective cokernel map

$$\operatorname{coker}\theta \twoheadrightarrow \varprojlim \operatorname{coker}\theta_j$$

given by $(n_j)_{j\in I} + \theta M \mapsto (n_j + \theta_j M_j)_{j\in I}$ is injective. Note that we will use the surjectivity of the induced kernel maps $\ker\theta_k \xrightarrow{\phi_j^k} \ker\theta_j$ to show the injectivity of the cokernel map $\operatorname{coker}\theta \twoheadrightarrow \varprojlim \operatorname{coker}\theta_j$.

If $n + \theta M = (n_j)_{j\in I} + \theta M \in \operatorname{coker}\theta$, maps to $(n_j + \theta_j M_j)_{j\in I} = (0 + \theta_j M_j)_{j\in I} \in \varprojlim \operatorname{coker}\theta_j$, so $n_k \in \theta_k M_k$ for all k; then we want to show that $n \in \theta M$ so that $n + \theta M = 0 + \theta M$. Since $n_k \in \theta_k M_k$ for all k, there exists $m_k \in M_k$ such that $\theta_k m_k = n_k$; however we may not have $\phi_j^k m_k \stackrel{?}{=} m_j$ for all $j \leq k$, that is we may not have $(m_j)_{j\in I} \stackrel{?}{\in} M$. So we want to construct $m'_k \in M_k$ such that $\theta_k m'_k = n_k$ and $\phi_j^k m'_k = m'_j$ for all $j \leq k$ so that $m' = (m'_j)_{j\in I} \in M$ and $\theta m' = n$.

We first consider $k \leq i$. Define $m'_i = m_i$ $(= \phi_i^i m'_i)$ and $m'_k = \phi_k^i m'_i$ for all $k < i$. Then $\theta_k m'_k = \theta_k \phi_k^i m'_i = \psi_k^i \theta_i m'_i = \psi_k^i \theta_i m_i = \psi_k^i n_i = n_k$ for all $k \leq i$; and for $j \leq k \leq i$, we have $\phi_j^k m'_k = \phi_j^k \phi_k^i m'_i = \phi_j^i m'_i = m'_j$. So we're done for $k \leq i$.

For $k \geq i$, we induct on k. The base case $k = i$ is done.

Let $k > i$ and assume true for all indices less than k (but at least as large as i, however those cases are true because we just proved it so). Then $\theta_{k-1}\phi_{k-1}^k m_k = \psi_{k-1}^k \theta_k m_k = \psi_{k-1}^k n_k = n_{k-1} = \theta_{k-1} m'_{k-1}$, where the last equality is by the induction hypothesis. So $\theta_{k-1}(\phi_{k-1}^k m_k - m'_{k-1}) = \theta_{k-1}\phi_{k-1}^k m_k - \theta_{k-1} m'_{k-1} = 0$ and $\phi_{k-1}^k m_k - m'_{k-1} \in \ker\theta_{k-1}$. Thus, by the surjectivity of the induced map $\ker\theta_k \xrightarrow{\phi_{k-1}^k} \ker\theta_{k-1}$, there exists $m''_k \in \ker\theta_k$ such that $\phi_{k-1}^k m''_k = \phi_{k-1}^k m_k - m'_{k-1}$. Define $m'_k = m_k - m''_k$, so that we have

$\theta_k m'_k = \theta_k(m_k - m''_k) = \theta_k m_k - \theta_k m''_k = \theta_k m_k - 0 = \theta_k m_k = n_k$, and also $\phi^k_{k-1} m'_k = \phi^k_{k-1}(m_k - m''_k) = \phi^k_{k-1} m_k - \phi^k_{k-1} m''_k = m'_{k-1}$.

Let $j \leq k$. If $j = k$, then $\phi^k_j m'_k = m'_j$, otherwise $j < k$. If $j \geq i$, then $\phi^k_j m'_k = \phi^{k-1}_j \phi^k_{k-1} m'_k = \phi^{k-1}_j m'_{k-1} = m'_j$, where the last equality is by the induction hypothesis. If $j \leq i$, then we have $\phi^k_j m'_k = \phi^i_j \phi^k_i m'_k = \phi^i_j m'_i = m'_j$, where the penultimate equality is by the previous sentence. □

Remark 5.1.3. Note that since we used the surjectivity of the induced kernel maps $\ker\theta_k \xrightarrow{\phi^k_j} \ker\theta_j$ to show the injectivity of the cokernel map $\operatorname{coker}\theta \twoheadrightarrow \varprojlim \operatorname{coker}\theta_j$, there is no immediate version of Facts 5.1.1 or 5.1.2 for the common minimal level. In the case where the $\ker\theta_i$ are all eventually finite, if the common minimal level is finite, then we do eventually get injections

$$\ker\theta = \varprojlim \ker\theta_j \longrightarrow \ker\theta_i$$

by Lemma 4.0.2; but these injections would be surjective if and only if the strong minimal level were also finite. Moreover, this bespeaks nothing of when the surjective cokernel map

$$\operatorname{coker}\theta \twoheadrightarrow \varprojlim \operatorname{coker}\theta_j$$

might be injective. These kernel and cokernel maps will be explored in greater detail in §5.5 below.

Fact 5.1.4. If $\text{s-min}\,\theta < \infty$, and if the M_i and N_i are finite for all $i \geq \text{s-min}\,\theta$; then

$$\frac{|N_i|}{|M_i|} = \frac{|\operatorname{coker}\theta_i|}{|\ker\theta_i|} = \frac{|\operatorname{coker}\theta|}{|\ker\theta|} = \text{ constant} \qquad \text{for all } i \geq \text{s-min}\,\theta;$$

so the M_i and N_i grow at the same rate for all $i \geq \text{s-min}\,\theta$.

Proof. Since the M_i and N_i are finite; so must also $\ker\theta_i$ and $\operatorname{coker}\theta_i$ be finite; whence $|\ker\theta| = |\ker\theta_i| < \infty$ and $|\operatorname{coker}\theta| = |\operatorname{coker}\theta_i| < \infty$ by Fact 5.1.2. Since the sequences

$$0 \longrightarrow \ker\theta_i \longrightarrow M_i \xrightarrow{\theta_i} N_i \longrightarrow \operatorname{coker}\theta_i \longrightarrow 0$$

are exact, we have

$$|\ker\theta_i| \cdot |N_i| = |M_i| \cdot |\operatorname{coker}\theta_i|$$

by Lemma 4.0.3. □

Fact 5.1.5. If c-min $\theta < \infty$, and if the M_i and N_i are finite for all $i \geq$ c-min θ; then

$$\frac{|N_i|}{|M_i|} = \frac{|\operatorname{coker}\theta_i|}{|\ker\theta_i|} = \text{ constant} \qquad \text{for all } i \geq \text{c-min}\,\theta;$$

so the M_i and N_i grow at the same rate for all $i \geq$ c-min θ.

Remark 5.1.6. So basically, a finite strong minimal level, s-min $\theta < \infty$, is what we hope for; a finite common minimal level, c-min $\theta < \infty$, is good enough; and a finite weak minimal level, w-min $\theta < \infty$, is a convenience so that we can get the cokernel maps out of the way.

5.2 Finiteness Conditions

Proposition 5.2.1. If either $\operatorname{coker}\theta$ is finite, or $\varprojlim \operatorname{coker}\theta_i$ is finite; then the weak minimal level is also finite.

Proof. Since the cokernel maps $\operatorname{coker}\theta \twoheadrightarrow \varprojlim \operatorname{coker}\theta_j$ and $\overline{\psi}_i^* : \varprojlim \operatorname{coker}\theta_j \twoheadrightarrow \operatorname{coker}\theta_i$ are surjective; if $\operatorname{coker}\theta$ is finite, then so is $\varprojlim \operatorname{coker}\theta_j$; and if $\varprojlim \operatorname{coker}\theta_j$ is finite, then so are the $\operatorname{coker}\theta_i$. Then, the sequence $\{|\operatorname{coker}\theta_i|\}_{i\in I}$ is a non-decreasing sequence of natural numbers bounded above by $|\varprojlim \operatorname{coker}\theta_j|$, and thus must stabilize (the Bounded Monotonic Sequence Convergence Theorem and discrete sets lack accumulation points). Once it stabilizes, say by n_0, we have $|\operatorname{coker}\theta_i| = |\operatorname{coker}\theta_j|$ for all $j \geq i \geq n_0$; whence the already surjective $\operatorname{coker}\theta_j \longrightarrow \operatorname{coker}\theta_i$ must now be injective. Thus
w-min $\theta \leq n_0 < \infty$. □

Proposition 5.2.2. Let p be a prime; and let $\mu, \lambda, v, w \in \mathbb{Z}$. If we have that w-min $\theta < \infty$, that the M_i and N_i are both (eventually) finite for all $i \in I$, and also that

$$|M_i| = p^{\mu p^i + \lambda i + v} \qquad \text{and} \qquad |N_i| = p^{\mu p^i + \lambda i + w} \qquad \text{for } i \text{ sufficiently large;}$$

then it follows that c-min $\theta < \infty$.

Proof. Let $n_0 \in I$ be large enough that for all $i \geq n_0$ we have $|M_i| = p^{\mu p^i + \lambda i + v}$ and $|N_i| = p^{\mu p^i + \lambda i + w}$. Let $n_0' = \max\{n_0, \text{w-min}\,\theta\}$, and define $I' = I \cap [n_0', \infty)$. For the remainder of the proof, we assume $i, j \in I'$.

Since M_i is a finite p-group, so must be $\ker\theta_i$; and since N_i is a finite p-group, so must be $\operatorname{coker}\theta_i$. Since $\operatorname{coker}\theta_i \cong \operatorname{coker}\theta_j$, they must be same size, say $|\operatorname{coker}\theta_i| = |\operatorname{coker}\theta_j| = p^d$, for some $d \in \mathbb{N}$. Let $|\ker\theta_i| = p^{e_i}$, for some $e_i \in \mathbb{N}$. Then $|\ker\theta_i| \cdot |N_i| = |M_i| \cdot |\operatorname{coker}\theta_i|$ gives us, $p^{e_i} \cdot p^{\mu p^i + \lambda i + w} = p^{\mu p^i + \lambda i + v} \cdot p^d$, or $e_i + (\mu p^i + \lambda i + w) = (\mu p^i + \lambda i + v) + d$, or $e_i = v - w + d$, which is constant for all $i \in I'$.

Thus c-min $\theta \leq n_0' < \infty$. □

Remark 5.2.3. Proposition 5.2.2 above represents one of the original motivating ideas for the study of minimal levels.

Theorem 5.2.4. If $\ker\theta$ is finite, then we have s-min $\theta < \infty$ if and only if c-min $\theta < \infty$ and $|\ker\theta| \geq |\ker\theta_i|$ for some (or all) $i \geq$ c-min θ.

Remark 5.2.5. Theorem 5.2.4 above was the original motivating reason behind Lemma 4.0.2. Basically, at least from the perspective of Iwasawa-Theory-inspired algebra, what we have is that a finite strong minimal level is indeed a high aim for which to hope, and we should be content with a finite common minimal level; that is, Iwasawa theoretically we will get nothing for free from the algebra theoretical perspective beyond a clear understand of precisely which algebraic properties with which we must contend (though we have indeed shown that we *must* contend with these issues). I had hoped this to be otherwise, but what's true is true and you can't prove something that's false. Corollary 5.2.6 below serves to illustrates this point a bit more clearly.

Corollary 5.2.6. If θ is injective, then we have s-min $\theta < \infty$ if and only if c-min $\theta < \infty$ and θ_i is injective for some (or all) $i \geq$ c-min θ; which is when and only when w-min $\theta < \infty$ and θ_i is injective for all i sufficiently large.

5.3 The Minimal Levels Under Composition

We additionally let $\{Q_i, \rho_i^j\}$ be an inverse systems of R-modules over I; assume that ρ_i^j is surjective for all $i \leq j$; let $\{\tau_i : N_i \longrightarrow Q_i\} : \{N_i, \psi_i^j\} \longrightarrow \{Q_i, \rho_i^j\}$ be a transformation, so that $\tau_i : N_i \longrightarrow Q_i$ are R-homomorphisms such that

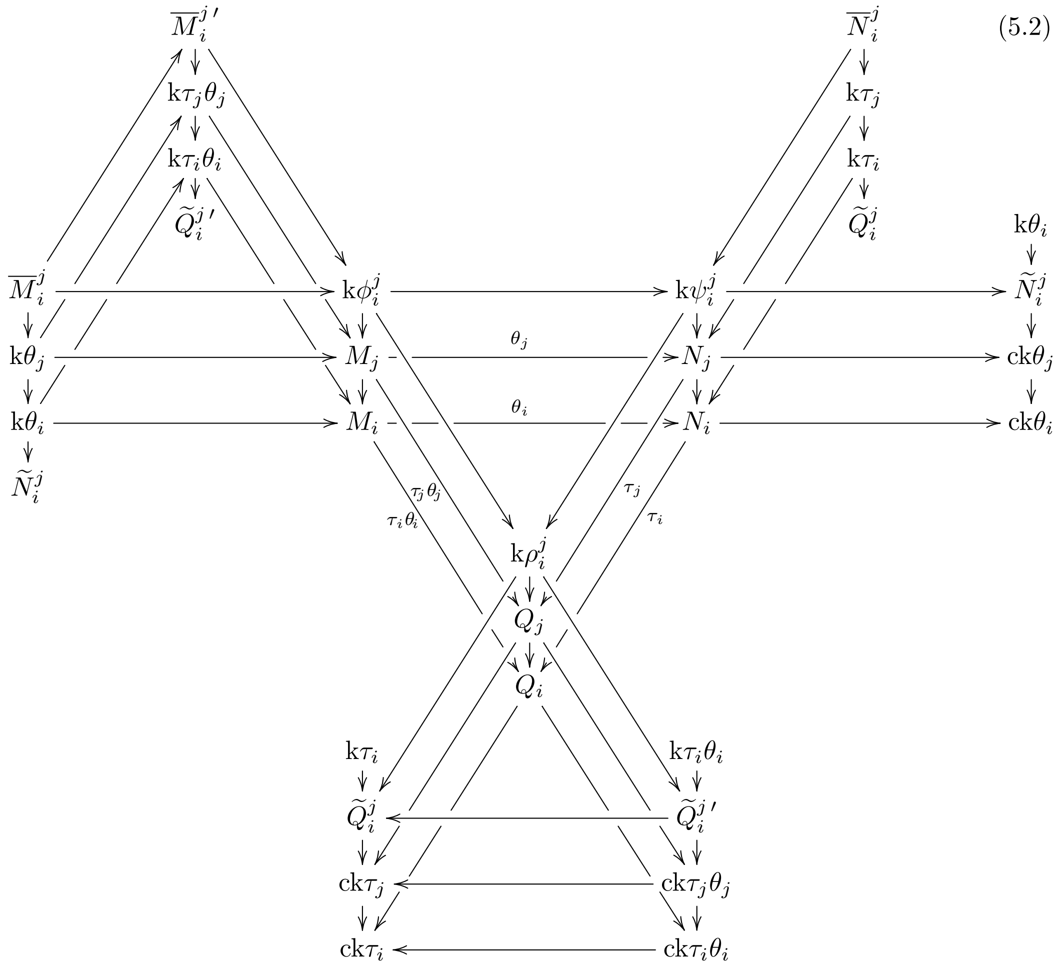
$$\begin{array}{ccc} N_j & \xrightarrow{\psi_i^j} & N_i \\ {\scriptstyle \tau_j}\downarrow & & \downarrow{\scriptstyle \tau_i} \\ Q_j & \xrightarrow{\rho_i^j} & Q_i \end{array}$$

commutes for all $i \leq j$; let $Q = \varprojlim Q_i$ be the inverse limit; and let $\tau : N \longrightarrow Q$ be the induced R-homomorphism, so we have that

$$\begin{array}{ccc} M & \xrightarrow{\theta} & N \\ & {\scriptstyle \tau\theta}\searrow \quad \swarrow{\scriptstyle \tau} & \\ & Q & \end{array}$$

commutes.

Then, applying the Snake Lemma severally gives us that the triangular prism[2]



(5.2)

commutes and is exact in straight lines, where we dropped the zero maps for visibility sake, we abbreviated ker and coker to k and ck, we set

$$\begin{aligned}
\overline{M}_i^j &= \ker\phi_i^j \cap \ker\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\psi_i^j) = \ker(\ker\theta_j \longrightarrow \ker\theta_i)\\
\widetilde{N}_i^j &= \operatorname{coker}(\ker\phi_i^j \longrightarrow \ker\psi_i^j)\\
\overline{N}_i^j &= \ker\psi_i^j \cap \ker\tau_j = \ker(\ker\psi_i^j \longrightarrow \ker\rho_i^j) = \ker(\ker\tau_j \longrightarrow \ker\tau_i)\\
\widetilde{Q}_i^j &= \operatorname{coker}(\ker\psi_i^j \longrightarrow \ker\rho_i^j)\\
\overline{M}_i^{j\prime} &= \ker\phi_i^j \cap \ker\tau_j\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\rho_i^j) = \ker(\ker\tau_j\theta_j \longrightarrow \ker\tau_i\theta_i)\\
\widetilde{Q}_i^{j\prime} &= \operatorname{coker}(\ker\phi_i^j \longrightarrow \ker\rho_i^j),
\end{aligned}$$

[2]Note that the various snake maps have been indicated by placing the map in two different places, with the intension that these places be identified. E.g., consider the map $\mathrm{k}\theta_i \longrightarrow \widetilde{N}_i^j$, which appears in two different places in the vertical plane that moves left-to-right across the page. This is meant to indicate the connecting map produced from the Snake Lemma in that plane, which is precisely Diagram (5.1) on page 35 above. If you remove the middle-most triangular prism and its connecting maps, then what remains seems best visualized as in Split Ladder (5.3) on page 44 below.

we have that

$$
\begin{aligned}
\ker\theta_i &\longrightarrow \widetilde{N}_i^j\\
\ker\tau_i &\longrightarrow \widetilde{Q}_i^j\\
\ker\tau_i\theta_i &\longrightarrow \widetilde{Q}_i^{j\,\prime}
\end{aligned}
$$

are the connecting morphisms of the Snake Lemma, and the top three rungs of the split ladder

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_i^j & \longrightarrow & \overline{M}_i^{j\,\prime} & & \overline{N}_i^j & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\theta_j & \longrightarrow & \ker\tau_j\theta_j & & \ker\tau_j & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\theta_i & \longrightarrow & \ker\tau_i\theta_i & & \ker\tau_i & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \widetilde{N}_i^j & & \widetilde{Q}_i^{j\,\prime} & \longrightarrow & \widetilde{Q}_i^j & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker}\theta_j & & \operatorname{coker}\tau_j\theta_j & \longrightarrow & \operatorname{coker}\tau_j & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker}\theta_i & & \operatorname{coker}\tau_i\theta_i & \longrightarrow & \operatorname{coker}\tau_i & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad (5.3)
$$

are inclusions and the bottom three are coset enlargement.

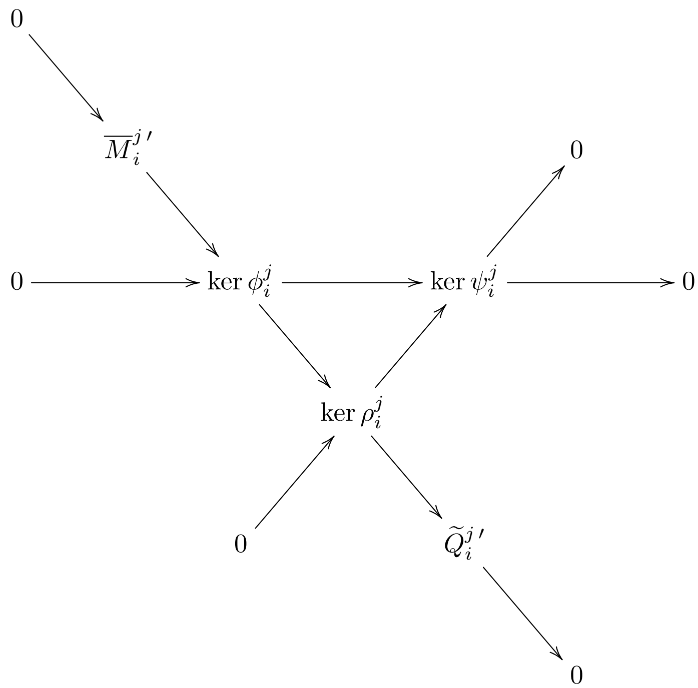
Proof. Note that a diagram commutes if and only if each minimal polygon commutes. Most everything is a direct application of the Snake Lemma in Diagram (5.1) on page 35 and similar. The middle horizontal triangles are composition; the horizontal triangles of kernels are inclusion; and the horizontal triangles of cokernels are coset enlargement. It remains only to verify that Split Ladder (5.3) commutes and is exact horizontally; however, these follow since the top three rungs are inclusion, and the bottom three rungs are coset enlargement. Well-definedness is a non-issue established by the Snake Lemma. □

From the above, we get the following

Theorem 5.3.1.

$$\text{s-min}\,\tau\theta \le \max\{\text{s-min}\,\theta, \text{s-min}\,\tau\}.$$

Proof. If either s-min $\theta = \infty$ or s-min $\tau = \infty$, then we're done. Otherwise, assume s-min $\theta \cdot$ s-min $\tau < \infty$. Let $j \geq i \geq \max\{\text{s-min}\,\theta, \text{s-min}\,\tau\}$. Then we have $\overline{M}_i^j = \widetilde{N}_i^j = \overline{N}_i^j = \widetilde{Q}_i^j = 0$, whence the top slice of Triangular Prism (5.2) becomes



giving us $\overline{M}_i^{j\,\prime} = \widetilde{Q}_i^{j\,\prime} = 0$; that is $\ker \phi_i^j \xrightarrow{\cong} \ker \rho_i^j$. ☐

<u>Remark 5.3.2</u>. There is, a priori, no immediate version of Theorem 5.3.1 for the common minimal level; however see Corollary 5.5.9 on page 58 below.

5.4 Inducing a Transformation from a Map on the Limits

For all $i \in I$, we have that $\theta(\ker \phi_i^*) \subset \ker \psi_i^*$, and that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \phi_i^* & \longrightarrow & M & \xrightarrow{\phi_i^*} & M_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_i} & & \\ 0 & \longrightarrow & \ker \psi_i^* & \longrightarrow & N & \xrightarrow{\psi_i^*} & N_i & \longrightarrow & 0 \end{array} \tag{5.4}$$

commutes and is exact horizontally, where $\ker \phi_i^* \xrightarrow{\theta} \ker \psi_i^*$ is the restriction of θ. We can also see

$$M_i \cong M / \ker \phi_i^* \qquad \text{and} \qquad N_i \cong N / \ker \psi_i^*.$$

Now let's consider this from the opposite direction. Assume that we have an R-module homomorphism $\theta' : M \longrightarrow N$ such that $\theta'(\ker \phi_i^*) \subset \ker \psi_i^*$ for all $i \in I$. Then we can define a well-defined map $\theta_i' : M_i \longrightarrow N_i$ given by $\theta_i' = \psi_i^* \theta' (\phi_i^*)^{-1}$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \phi_i^* & \longrightarrow & M & \xrightarrow{\phi_i^*} & M_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta_i'} & & \\ 0 & \longrightarrow & \ker \psi_i^* & \longrightarrow & N & \xrightarrow{\psi_i^*} & N_i & \longrightarrow & 0 \end{array}$$

commutes and is exact horizontally; $\{\theta_i' : M_i \longrightarrow N_i\} : \{M_i, \phi_i^j\} \longrightarrow \{N_i, \psi_i^j\}$ is a transformation; and the induced R-homomorphism $M \longrightarrow N$ agrees with the original $\theta' : M \longrightarrow N$.

Proof. First, θ_i' is well-defined. Let $m_i \in M_i$, and let $m = (m_j)_{j\in I} \in M$ such that $\phi_i^* m = m_i$. Assume we also have $m' = (m_j')_{j\in I} \in M$ such that $\phi_i^* m' = m_i' = m_i$. Then $\phi_i^*(m - m') = m_i - m_i' = 0$, so that $m - m' \in \ker \phi_i^*$. Thus $\theta'(m - m') \in \ker \psi_i^*$, and hence $\psi_i^*\theta' m - \psi_i^*\theta' m' = \psi_i^*\theta'(m - m') = 0$. Thus $\psi_i^*\theta' m = \psi_i^*\theta' m'$ and θ_i' is well-defined.

The left square commutes since the horizontal maps are inclusion and one of the vertical maps is the restriction of the other. The right square commutes since

$$\theta_i'\phi_i^* = \psi_i^*\theta'(\phi_i^*)^{-1}\phi_i^* = \psi_i^*\theta'.$$

Since

$$\theta_i'\phi_i^j = \psi_i^*\theta'(\phi_i^*)^{-1}\phi_i^j = \psi_i^*\theta'(\phi_j^*)^{-1} = \psi_i^j\psi_j^*\theta'(\phi_j^*)^{-1} = \psi_i^j\theta_j',$$

we have a transformation.

For the moment, let $\theta'' : M \longrightarrow N$ be the induced map $\theta''(m_i)_{i\in I} = (\theta_i' m_i)_{i\in I}$. Let $m = (m_i)_{i\in I} \in M$. Then

$$\theta' m = n = (n_i)_{i\in I} = (\psi_i^* n)_{i\in I} = (\psi_i^*\theta' m)_{i\in I} \overset{\dagger}{=} (\psi_i^*\theta'(\phi_i^*)^{-1} m_i)_{i\in I} = (\theta_i' m_i)_{i\in I} = \theta'' m,$$

where $n = (n_j)_{j\in I} \in N$ such that $\theta' m = n$, and † follows for the same reason that θ_i' is well-defined; or rather, since well-definedness is established, we may take $m = (\phi_i^*)^{-1} m_i$. □

As such, an R-homomorphism θ' such that $\theta'(\ker \phi_i^*) \subset \ker \psi_i^*$ for all $i \in I$, will be referred to as *inducible*, and the resulting transformation

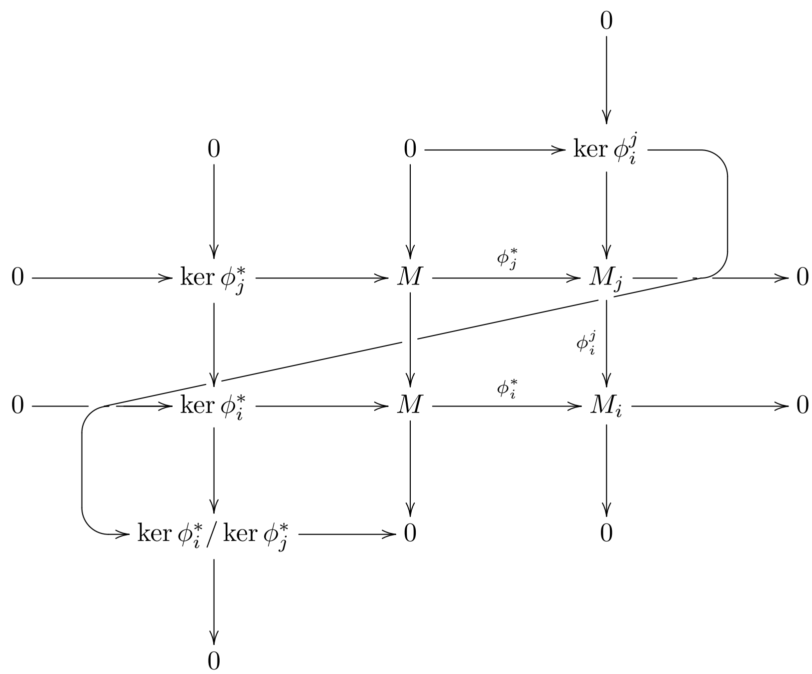
$\{\theta_i' = \psi_i^* \theta' (\phi_i^*)^{-1} : M_i \longrightarrow N_i\} : \{M_i, \phi_i^j\} \longrightarrow \{N_i, \psi_i^j\}$ will be referred to as the *inducing transformation.*

Remark 5.4.1. It would be great if all R-homomorphisms of interest were inducible; however, this a high hope. In Chapters 6 and 7 we develop techniques for the case when a non-inducible homomorphism is given.

5.5 Verifying Criteria for the Minimal Levels

Considering the diagram

(5.5)



we can see

$$\ker \phi_i^* / \ker \phi_j^* \cong \ker \phi_i^j,$$

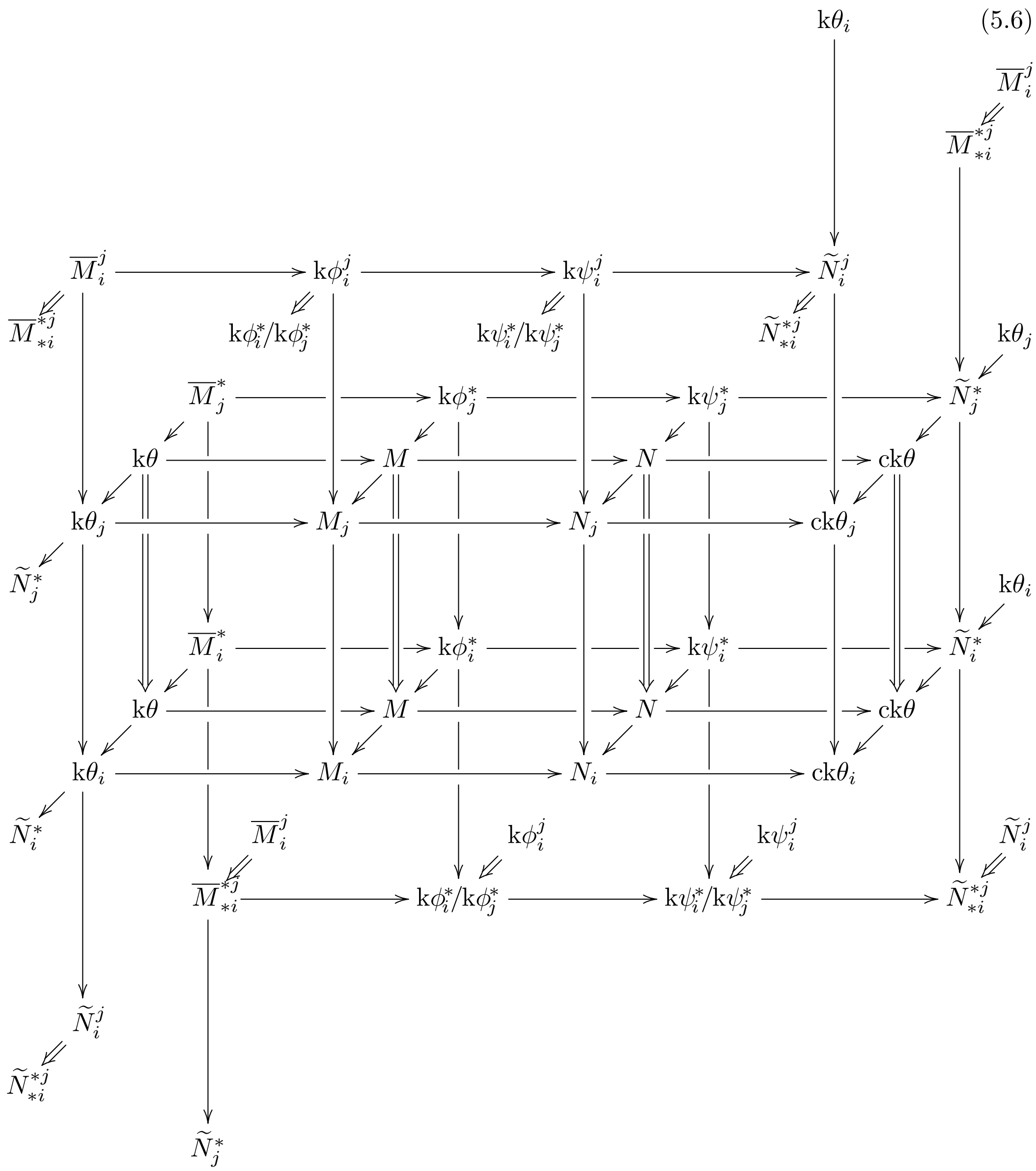
where the map is given by $m \mapsto (\phi_j^*)^{-1} m + \ker \phi_j^*$; and similarly

$$\ker \psi_i^* / \ker \psi_j^* \cong \ker \psi_i^j,$$

where the map is given by $n \mapsto (\psi_j^*)^{-1} n + \ker \psi_j^*$.

Putting our three perspectives together, we get that the rectangular prism[3]

(5.6)



commutes and is exact in straight lines, where we dropped the zero maps for visibility sake, a double shafted arrow indicates isomorphism (or equality), we abbreviated ker and

[3]Note that the various snake maps have been indicated by placing the map in at least two different places, with the intension that these places be identified. E.g., consider the map $\mathrm{k}\phi_i^j \Longrightarrow \mathrm{k}\phi_i^*/\mathrm{k}\phi_j^*$, which appears in two different places in the plane $\{y = 1\}$. This is meant to indicate the connecting map produced from the Snake Lemma in that plane, which is precisely Diagram (5.5) on page 47 above. If you consider the "outside" shell of Rectangular Prism (5.6) by removing the middle-most square and its connecting maps, so remove $\{0 < x < 2$ and $0 < y < 3$ and $2 \leq z \leq 3\}$, then what remains seems best visualized as an immersion with a double point into a subset of the 3-torus $T^3 = S^1 \times S^1 \times S^1$. If you instead remove $\{0 < y < 3\}$ entirely, then Snake Tessellation (5.7) on page 54 below seems the best visualization.

coker to k and ck, we set

$$\begin{aligned}
\overline{M}_i^j &= \ker\phi_i^j \cap \ker\theta_j = \ker(\ker\phi_i^j \longrightarrow \ker\psi_i^j) = \ker(\ker\theta_j \longrightarrow \ker\theta_i)\\
\widetilde{N}_i^j &= \operatorname{coker}(\ker\phi_i^j \longrightarrow \ker\psi_i^j)\\
\overline{M}_j^* &= \ker\phi_j^* \cap \ker\theta = \ker(\ker\phi_j^* \longrightarrow \ker\psi_j^*) = \ker(\ker\theta \longrightarrow \ker\theta_j)\\
\widetilde{N}_j^* &= \operatorname{coker}(\ker\phi_j^* \longrightarrow \ker\psi_j^*)\\
\overline{M}_i^* &= \ker\phi_i^* \cap \ker\theta = \ker(\ker\phi_i^* \longrightarrow \ker\psi_i^*) = \ker(\ker\theta \longrightarrow \ker\theta_i)\\
\widetilde{N}_i^* &= \operatorname{coker}(\ker\phi_i^* \longrightarrow \ker\psi_i^*)\\
\overline{M}_{*i}^{*j} &= \ker(\ker\phi_i^* / \ker\phi_j^* \longrightarrow \ker\psi_i^* / \ker\psi_j^*)\\
\widetilde{N}_{*i}^{*j} &= \operatorname{coker}(\ker\phi_i^* / \ker\phi_j^* \longrightarrow \ker\psi_i^* / \ker\psi_j^*),
\end{aligned}$$

we have that

$$\ker\theta_j \longrightarrow \widetilde{N}_j^*$$

$$\ker\theta_i \longrightarrow \widetilde{N}_i^*$$

$$\ker\theta_i \longrightarrow \widetilde{N}_i^j$$

$$\overline{M}_{*i}^{*j} \longrightarrow \widetilde{N}_j^*$$

$$\ker\phi_i^j \xrightarrow{\cong} \ker\phi_i^* / \ker\phi_j^*$$

$$\ker\psi_i^j \xrightarrow{\cong} \ker\psi_i^* / \ker\psi_j^*$$

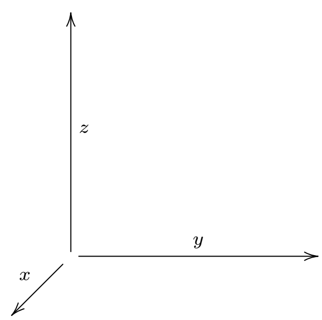
are the connecting morphisms of the Snake Lemma, and the induced maps connecting the bottom two connecting isomorphisms have

$$\overline{M}_i^j \xrightarrow{\cong} \overline{M}_{*i}^{*j}$$

$$\widetilde{N}_i^j \xrightarrow{\cong} \widetilde{N}_{*i}^{*j}$$

for induced kernel and cokernel maps.

Proof. We orient ourselves on Rectangular Prism (5.6) per



with the origin at the lowest point in the back-left-bottom corner, and the highest point is $(2, 3, 5)$. Every square lies in at least one of ten planes, as does every exact sequence, including the ones that snake around. The plane $\{y = 1\}$ is Diagram (5.5) on page 47, and the plane $\{y = 2\}$ is similar. The plane $\{x = 2\}$ is Diagram (5.1) on page 35. The plane $\{x = 1\}$ is clear with the vertical maps being equality. The plane $\{z = 2\}$ is Diagram (5.4) on page 45 together with the Snake Lemma, and the plane $\{z = 3\}$ is similar.

It remains to verify the commutativity of the five planes $\{x = 0\}$, $\{y = 0\}$, $\{y = 3\}$, $\{z = 4\}$, and $\{z = 1\}$, noting that the second and third and also the fourth and fifth are pairwise equivalent; and it also remains to verify the exactness and well-definedness of the two equivalent planes $\{z = 4\}$ and $\{z = 1\}$, and of the two lines $\{x = y = 0\}$ and $\{x = 0 \text{ and } y = 3\}$ which snake into each other.

Since the cube $\{1 \le x \le 2 \text{ and } 1 \le y \le 2 \text{ and } 2 \le z \le 3\}$ commutes, Lemma 4.0.4 on page 32 give that the cubes $\{1 \le x \le 2 \text{ and } 0 \le y \le 1 \text{ and } 2 \le z \le 3\}$, $\{1 \le x \le 2 \text{ and } 2 \le y \le 3 \text{ and } 2 \le z \le 3\}$, and $\{0 \le x \le 1 \text{ and } 1 \le y \le 2 \text{ and } 2 \le z \le 3\}$ commute. Then, similarly, the cubes $\{0 \le x \le 1 \text{ and } 0 \le y \le 1 \text{ and } 2 \le z \le 3\}$, $\{0 \le x \le 1 \text{ and } 2 \le y \le 3 \text{ and } 2 \le z \le 3\}$, and $\{0 \le x \le 1 \text{ and } 1 \le y \le 2 \text{ and } 1 \le z \le 2\}$ commute (recall the undrawn zero maps). Then, again similarly, the cubes $\{0 \le x \le 1 \text{ and } 0 \le y \le 1 \text{ and } 1 \le z \le 2\}$ and $\{0 \le x \le 1 \text{ and } 2 \le y \le 3 \text{ and } 1 \le z \le 2\}$ commute.

Now that the plane $\{x = 0\}$ commutes, and the kernel and cokernel maps are the natural induced maps, the Snake Lemma gives the exactness and well-definedness of the two lines $\{x = y = 0\}$ and $\{x = 0 \text{ and } y = 3\}$ as snaked together.

The square $\{2 \leq x \leq 3 \text{ and } 1 \leq y \leq 2 \text{ and } z = 4\}$ has the maps

$$\begin{array}{ccc} \ker \phi_i^j & \longrightarrow & \ker \phi_i^* / \ker \phi_j^* \\ \downarrow & & \downarrow \\ \ker \psi_i^j & \longrightarrow & \ker \psi_i^* / \ker \psi_j^* \end{array}$$

given by

$$\begin{array}{ccc} m_j & \longmapsto & (\phi_j^*)^{-1} m_j + \ker \phi_j^* \qquad m + \ker \phi_j^* \\ \downarrow & & \downarrow \\ \theta_j m_j & & \theta m + \ker \psi_j^* \\ n_j & \longmapsto & (\psi_j^*)^{-1} n_j + \ker \psi_j^* \end{array}$$

where $m_j \in M_j$, $n_j \in N_j$, and $m \in M$ (subsets thereof that is). The well-definedness of these maps has already been established above, as well as the bijectivity of the horizontal maps. If we let $m_j \in \ker \phi_i^j$, then down-then-right maps to $(\psi_j^*)^{-1} \theta_j m_j + \ker \psi_j^*$, and right-then-down maps to $\theta (\phi_j^*)^{-1} m_j + \ker \psi_j^*$; whence commutativity follows by Diagram (5.4) on page 45. Thus the plane $\{z = 4\}$, or $\{z = 1\}$, follows since the Snake Lemma now gives us that

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_i^j & \longrightarrow & \overline{M}_{*i}^{*j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \phi_i^j & \longrightarrow & \ker \phi_i^* / \ker \phi_j^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \psi_i^j & \longrightarrow & \ker \psi_i^* / \ker \psi_j^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \widetilde{N}_i^j & \longrightarrow & \widetilde{N}_{*i}^{*j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

commutes and is exact.

Thus it now remains only to show that

$$
\begin{array}{ccc}
\overline{M}_i^j & \longrightarrow & \overline{M}_{*i}^{*j} \\
\downarrow & & \downarrow \\
\ker \theta_j & \longrightarrow & \widetilde{N}_j^* \\
\downarrow & & \downarrow \\
\ker \theta_i & \longrightarrow & \widetilde{N}_i^* \\
\downarrow & & \downarrow \\
\widetilde{N}_i^j & \longrightarrow & \widetilde{N}_{*i}^{*j}
\end{array}
$$

commutes. The maps are all well-defined and given by

$$m_j \longmapsto (\phi_j^*)^{-1} m_j + \ker \phi_j^* \qquad\qquad m + \ker \phi_j^* \longmapsto \theta m + \theta \ker \phi_j^*$$

$$m_j \longmapsto m_j$$

$$m_j \longmapsto \theta(\phi_j^*)^{-1} m_j + \theta \ker \phi_j^* \qquad\qquad n + \theta \ker \phi_j^* \longmapsto n + \theta \ker \phi_i^*$$

$$m_j \longmapsto \phi_i^j m_j$$

$$m_i \longmapsto \theta(\phi_i^*)^{-1} m_i + \theta \ker \phi_i^* \qquad\qquad n + \theta \ker \phi_i^* \longmapsto (n + \ker \psi_j^*) + \overline{\theta}(\ker \phi_i^* / \ker \phi_j^*)$$

$$m_i \longmapsto \theta_j(\phi_i^j)^{-1} m_i + \theta_j \ker \phi_i^j$$

$$(n + \ker \psi_j^*) + \overline{\theta}(\ker \phi_i^* / \ker \phi_j^*) \longmapsto \psi_j^* n + \theta_j \ker \phi_i^j$$

$$n_j + \theta_j \ker \phi_i^j \longmapsto ((\psi_j^*)^{-1} n_j + \ker \psi_j^*) + \overline{\theta}(\ker \phi_i^* / \ker \phi_j^*)$$

where $m_j \in M_j$, $m_i \in M_i$, $n_j \in N_j$, $m \in M$, $n \in N$ (subsets thereof that is); $\ker \phi_i^* / \ker \phi_j^* \xrightarrow{\overline{\theta}} \ker \psi_i^* / \ker \psi_j^*$ is the induced cokernel map whose image we use; and noting that the bottom map is an isomorphism.

If we let $m_j \in \overline{M}_i^j$, then down-then-right maps to $\theta(\phi_j^*)^{-1} m_j + \theta \ker \phi_j^*$, as does right-then-down; whence the commutativity of the upper square follows.

If we let $m_j \in \ker \theta_j$, then down-then-right maps to $\theta(\phi_i^*)^{-1} \phi_i^j m_j + \theta \ker \phi_i^*$, and right-then-down maps to $\theta(\phi_j^*)^{-1} m_j + \theta \ker \phi_i^*$. Since well-definedness has been established, we may choose $m \in M$ such that $\phi_j^* m = m_j$ for $m = (\phi_j^*)^{-1} m_j$. By the commutativity of the square $\{1 \leq x \leq 2 \text{ and } y = 1 \text{ and } 2 \leq z \leq 3\}$, we have $\phi_i^j m_j = \phi_i^j \phi_j^* m = \phi_i^* m$. Since well-definedness has been established, we may also choose m for $m = (\phi_i^*)^{-1} \phi_i^j m_j$. This gives the equality, and thus the commutativity of the middle square.

If we let $m_i \in \ker \theta_i$, then down maps to $\theta_j(\phi_i^j)^{-1} m_i + \theta_j \ker \phi_i^j$, and right-then-down-then-left maps to $\psi_j^* \theta(\phi_i^*)^{-1} m_i + \theta_j \ker \phi_i^j$. Since well-definedness has been established, choose $m \in M$ such that $\phi_i^* m = m_i$ for $m = (\phi_i^*)^{-1} m_i$; and let $m_j = \phi_j^* m$. Again by the commutativity of the square $\{1 \leq x \leq 2 \text{ and } y = 1 \text{ and } 2 \leq z \leq 3\}$, we have $\phi_i^j m_j = \phi_i^j \phi_j^* m = \phi_i^* m = m_i$, so we may also choose $m_j = (\phi_i^j)^{-1} m_i$. If $\theta m = (n_k)_{k \in I}$, then $\theta_j m_j = n_j$, and $\psi_j^*(n_k)_{k\in I} = n_j$. This gives the equality, and thus the commutativity of the bottom square. $\square$

Putting together four of the connecting maps, and the induced kernel and cokernel maps of the other two, we get that the snake tessellation

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & & & & & 0 & \longrightarrow & \overline{M}_j^* & \longrightarrow & \cdots \\
 & & & & & & & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & & & & & 0 & \longrightarrow & \overline{M}_i^* & \longrightarrow & \cdots \\
 & & & & & & & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & & & & 0 & & 0 & \longrightarrow & \overline{M}_i^j & \longrightarrow & \overline{M}^{*j}_{*i} & \longrightarrow & 0 \\
 & & & & & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & & & 0 & \longrightarrow & \overline{M}_j^* & \longrightarrow & \mathrm{k}\theta & \longrightarrow & \mathrm{k}\theta_j & \longrightarrow & \widetilde{N}_j^* & \longrightarrow & \cdots \\
 & & & & & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & & & 0 & \longrightarrow & \overline{M}_i^* & \longrightarrow & \mathrm{k}\theta & \longrightarrow & \mathrm{k}\theta_i & \longrightarrow & \widetilde{N}_i^* & \longrightarrow & \cdots \\
 & & & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & \longrightarrow & \overline{M}_i^j & \longrightarrow & \overline{M}^{*j}_{*i} & \longrightarrow & 0 & \longrightarrow & \widetilde{N}_i^j & \longrightarrow & \widetilde{N}^{*j}_{*i} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \overline{M}_j^* & \longrightarrow & \mathrm{k}\theta & \longrightarrow & \mathrm{k}\theta_j & \longrightarrow & \widetilde{N}_j^* & \longrightarrow & \mathrm{ck}\theta & \longrightarrow & \mathrm{ck}\theta_j & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \overline{M}_i^* & \longrightarrow & \mathrm{k}\theta & \longrightarrow & \mathrm{k}\theta_i & \longrightarrow & \widetilde{N}_i^* & \longrightarrow & \mathrm{ck}\theta & \longrightarrow & \mathrm{ck}\theta_i & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \overline{M}_i^j & \longrightarrow & \overline{M}^{*j}_{*i} & \longrightarrow & 0 & \longrightarrow & \widetilde{N}_i^j & \longrightarrow & \widetilde{N}^{*j}_{*i} & \longrightarrow & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & & & & & \\
\cdots & \longrightarrow & \mathrm{k}\theta_j & \longrightarrow & \widetilde{N}_j^* & \longrightarrow & \mathrm{ck}\theta & \longrightarrow & \mathrm{ck}\theta_j & \longrightarrow & 0 & & & & & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & & & & & & & \\
\cdots & \longrightarrow & \mathrm{k}\theta_i & \longrightarrow & \widetilde{N}_i^* & \longrightarrow & \mathrm{ck}\theta & \longrightarrow & \mathrm{ck}\theta_i & \longrightarrow & 0 & & & & & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & & & & & & & \\
0 & \longrightarrow & \widetilde{N}_i^j & \longrightarrow & \widetilde{N}^{*j}_{*i} & \longrightarrow & 0 & & 0 & & & & & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & & & & & & & \\
\cdots & \longrightarrow & \mathrm{ck}\theta_j & \longrightarrow & 0 & & & & & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & & & & & \\
\cdots & \longrightarrow & \mathrm{ck}\theta_i & \longrightarrow & 0 & & & & & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & & & & & \\
 & & 0 & & & & & & & & & & & & & & & &
\end{array} \tag{5.7}
$$

commutes and is exact, where we abbreviated ker and coker to k and ck.

This motivates another minimal level definition comparable in function to the weak minimal level; that is, a convenience to get an underlying portion of the kernel maps out of the way; more specifically, to regulate the kernel of the map $\ker\theta \longrightarrow \ker\theta_i$.

Define the *co-weak minimal level* of θ to be

$$\begin{aligned}\text{co-w-min}\,\theta &= \min\Big(\{i \in I : \overline{M}^*_j = \overline{M}^*_i \ \forall j \geq i\} \cup \{\infty\}\Big) \\ &= \min\Big(\{i \in I : \overline{M}^*_{j+1} = \overline{M}^*_j \ \forall j \geq i\} \cup \{\infty\}\Big).\end{aligned}$$

Proposition 5.5.1. If $\ker\theta$ is finite, then the co-weak minimal level is also finite.

Proof. Similar to Proposition 5.2.1. Since $\overline{M}^*_i \subset \ker\theta$, we have that the sequence $\{|\overline{M}^*_i|\}_{i\in I}$ is a non-increasing sequence of natural numbers bounded below by 1, and thus must stabilize (the Bounded Monotonic Sequence Convergence Theorem and discrete sets lack accumulation points). Once it stabilizes, say by n_0, we have $|\overline{M}^*_i| = |\overline{M}^*_j|$ for all $j \geq i \geq n_0$; whence the subset $\overline{M}^*_j \subset \overline{M}^*_i$ must now be equality. Thus co-w-min $\theta \leq n_0 < \infty$. □

Proposition 5.5.2. We have

$$\text{co-w-min}\,\theta \leq \text{s-min}\,\theta;$$

and if moreover s-min $\theta < \infty$, then $\overline{M}^*_i = 0$ for all $i \geq$ co-w-min θ.

Proof. If s-min $\theta = \infty$, then we're done. Otherwise, assume s-min $\theta < \infty$, and let $i \geq$ s-min θ. Then $\ker\theta \xrightarrow{\cong} \ker\theta_i$ is an isomorphism by Fact 5.1.2; but $\overline{M}^*_i$ is the kernel of precisely that map; and hence $\overline{M}^*_i = 0$. For $j \geq i$, we have $\overline{M}^*_j \subset \overline{M}^*_i = 0$, which must then be equality. □

Proposition 5.5.3. If c-min $\theta < \infty$ with the $\ker\theta_i$ all (eventually) finite, then co-w-min $\theta < \infty$ and $\overline{M}^*_i = 0$ for all $i \geq$ co-w-min θ.

Proof. Lemma 4.0.2 gives us that $\ker\theta \longrightarrow \ker\theta_i$ is an injection for i sufficiently large, and hence $\overline{M}^*_i = 0$ for i sufficiently large since it is the kernel of precisely that map. □

Remark 5.5.4. Note that in general the co-weak minimal level may be smaller than both the weak and common minimal levels, larger than both, or in between the two.

Proposition 5.5.5. If $\ker\theta$ and coker θ are both finite, and the $\ker\theta_i$ are all (eventually) finite (e.g., if the M_i are all (eventually) finite); then everything in Snake Tessellation

(5.7) is (eventually) finite, and both the weak and the co-weak minimal levels are finite. Moreover, for all $j \geq i \geq \max\{\text{w-min}\,\theta, \text{co-w-min}\,\theta\}$, we have that

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & \longrightarrow & \overline{M}_i^j & \longrightarrow & \overline{M}_{*i}^{*j} & \longrightarrow & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_j^* & \longrightarrow & \ker\theta & \longrightarrow & \ker\theta_j & \longrightarrow & \widetilde{N}_j^* & \longrightarrow & \operatorname{coker}\theta & \rightarrow & \operatorname{coker}\theta_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_i^* & \longrightarrow & \ker\theta & \longrightarrow & \ker\theta_i & \longrightarrow & \widetilde{N}_i^* & \longrightarrow & \operatorname{coker}\theta & \rightarrow & \operatorname{coker}\theta_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & \longrightarrow & \widetilde{N}_i^j & \longrightarrow & \widetilde{N}_{*i}^{*j} & \longrightarrow & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & & & \\
 & & & & & & 0 & & 0 & & & & & &
\end{array} \tag{5.8}$$

commutes and is exact. Moreover again, if the $\ker\theta_i$ are all (eventually) bounded in size (e.g., if $\text{c-min}\,\theta < \infty$); then everything in Snake Tessellation (5.7) is (eventually) bounded in size.

Proof. Since $\operatorname{coker}\theta$ and $\ker\theta$ are both finite, we have that the weak and the co-weak minimal levels must both be finite by Propositions 5.2.1 and 5.5.1.

Since $\ker\theta$ is finite, $\overline{M}_i^*$ must be finite for all i. Since $\operatorname{coker}\theta$ is finite, $\operatorname{coker}\theta_i$ must be finite for all i. Then $\widetilde{N}_i^*$ must be finite for all i (sufficiently large) since there cannot be only one infinite thing in an exact sequence. Since $\overline{M}_i^j \subset \ker\theta_j$, it must also be finite for all j (sufficiently large); and hence also its isomorphic copy $\overline{M}_{*i}^{*j}$. Now $\widetilde{N}_i^j$ and $\widetilde{N}_{*i}^{*j}$ must be finite for all j (sufficiently large) since there cannot be only one infinite thing in an exact sequence. Thus everything in Snake Tessellation (5.7) is finite for all $j \geq i$ (with i sufficiently large).

If $j \geq i \geq \max\{\text{w-min}\,\theta, \text{co-w-min}\,\theta\}$, then $\overline{M}_j^* \xrightarrow{=} \overline{M}_i^*$ and $\operatorname{coker}\theta_j \xrightarrow{\cong} \operatorname{coker}\theta_i$ are isomorphisms; and thence $\overline{M}_i^* \xrightarrow{0} \overline{M}_{*i}^{*j}$ and $\widetilde{N}_i^j \xrightarrow{0} \operatorname{coker}\theta_j$ are zero maps, and thus Snake Tessellation (5.7) reduces to Diagram (5.8).

Since $\ker\theta$ is finite, $\overline{M}_i^*$ is bounded for all i. Since $\operatorname{coker}\theta$ is finite, $\operatorname{coker}\theta_i$ is bounded for all i. Since $\overline{M}_i^j \subset \ker\theta_j$, it must be (eventually) bounded for all j (sufficiently large); and hence also its isomorphic copy $\overline{M}_{*i}^{*j}$. Similarly, $\widetilde{N}_i^j$ and $\widetilde{N}_{*i}^{*j}$ must be (eventually) bounded for all j (sufficiently large) since the first is a quotient of something (eventually) bounded. By Diagram (5.8) and Lemma 4.0.3, we have

$$|\overline{M}_i^*| \cdot |\ker\theta_i| \cdot |\mathrm{coker}\,\theta| = |\ker\theta| \cdot |\widetilde{N}_i^*| \cdot |\mathrm{coker}\,\theta_i|,$$

and since all the terms are finite, we may rearrange to get

$$|\widetilde{N}_i^*| = \frac{|\overline{M}_i^*|}{|\ker\theta|} \cdot \frac{|\ker\theta_i| \cdot |\mathrm{coker}\,\theta|}{|\mathrm{coker}\,\theta_i|} \le 1 \cdot \frac{|\ker\theta_i| \cdot |\mathrm{coker}\,\theta|}{1} = |\ker\theta_i| \cdot |\mathrm{coker}\,\theta|,$$

which is (eventually) bounded since the $\ker\theta_i$ are. Thus everything in Snake Tessellation (5.7) is (eventually) bounded for all $j \ge i$ (with i sufficiently large). □

<u>Remark 5.5.6</u>. So for the applications that we have in mind, even though M and N are infinite, and the M_i and N_i grow unboundedly; their relative behaviors may be described well by modules that are known to be finite, bounded, and eventually fixed in size (and in many cases, expected to be trivial or cyclic).

<u>Lemma 5.5.7</u>. Assume $\ker\theta$ and $\mathrm{coker}\,\theta$ are both finite, and the M_i and N_i are all (eventually) finite, so that Proposition 5.5.5 applies. Then, for $i \in I$, the following are equivalent:

1. $|\overline{M}_i^j| = |\widetilde{N}_i^j|$ for all $j \ge i$;
2. $|\overline{M}_{*i}^{*j}| = |\widetilde{N}_{*i}^{*j}|$ for all $j \ge i$;
3. $|\ker\phi_i^j| = |\ker\psi_i^j|$ for all $j \ge i$;
4. $|\ker\phi_i^* / \ker\phi_j^*| = |\ker\psi_i^* / \ker\psi_j^*|$ for all $j \ge i$.

Proof. By Snake Tessellation (5.7) we have $\overline{M}_i^j \cong \overline{M}_{*i}^{*j}$ and $\widetilde{N}_i^j \cong \widetilde{N}_{*i}^{*j}$, whence 5.5.7.1 and 5.5.7.2 are equivalent. Since $\ker\phi_i^j \subset M_j$ and $\ker\psi_i^j \subset N_j$, we have that $\ker\phi_i^j$ and $\ker\psi_i^j$ must be finite, whence the line $\{x = 2 \text{ and } z = 4\}$ of Rectangular Prism (5.6) and Lemma 4.0.3 give

$$|\overline{M}_i^j| \cdot |\ker\psi_i^j| = |\ker\phi_i^j| \cdot |\widetilde{N}_i^j|;$$

and thus 5.5.7.1 and 5.5.7.3 are equivalent. Finally, Diagram (5.5) and similarly for ψ, or the plane $\{z = 1\}$ of Rectangular Prism (5.6), gives that 5.5.7.3 and 5.5.7.4 are equivalent. □

<u>Theorem 5.5.8</u>. Assume $\ker\theta$ and $\mathrm{coker}\,\theta$ are both finite, and the M_i and N_i are all (eventually) finite, so that Proposition 5.5.5 applies. Then, for $i \ge \text{w-min}\,\theta$, the following are equivalent:

1. $i \geq \text{c-min}\,\theta$;
2. $|\ker\theta_j| = |\ker\theta_i|$ for all $j \geq i$;
3. $|\overline{M}_i^j| = |\widetilde{N}_i^j|$ for all $j \geq i$;
4. $|\overline{M}_{*i}^{*j}| = |\widetilde{N}_{*i}^{*j}|$ for all $j \geq i$;
5. $|\ker\phi_i^j| = |\ker\psi_i^j|$ for all $j \geq i$;
6. $|\ker\phi_i^* / \ker\phi_j^*| = |\ker\psi_i^* / \ker\psi_j^*|$ for all $j \geq i$.

Proof. First, 5.5.8.1 implies 5.5.8.2 by definition, and the reverse implication follows since $i \geq \text{w-min}\,\theta$; whence 5.5.8.1 and 5.5.8.2 are equivalent. By Snake Tessellation (5.7) and Lemma 4.0.3, we have

$$|\overline{M}_i^j| \cdot |\ker\theta_i| \cdot |\text{coker}\,\theta_j| = |\ker\theta_j| \cdot |\widetilde{N}_i^j| \cdot |\text{coker}\,\theta_i|,$$

and since $i \geq \text{w-min}\,\theta$, we have $|\text{coker}\,\theta_i| = |\text{coker}\,\theta_j|$; and thus

$$|\overline{M}_i^j| \cdot |\ker\theta_i| = |\ker\theta_j| \cdot |\widetilde{N}_i^j|;$$

whence 5.5.8.2 and 5.5.8.3 are equivalent. The remaining equivalences follow from Lemma 5.5.7. □

<u>Corollary 5.5.9</u>. Recall the notation of §5.3 on page 41; specifically, the inverse system $\{Q_i, \rho_i^j\}$ and the map $\tau : N \longrightarrow Q$. Assume $\ker\theta$, $\text{coker}\,\theta$, $\ker\tau$, $\text{coker}\,\tau$, $\ker\tau\theta$, and $\text{coker}\,\tau\theta$ are all finite; and the M_i, N_i, and Q_i are all (eventually) finite; so that Proposition 5.5.5 applies to all three. Then,

$$\text{c-min}\,\tau\theta \leq \max\{\text{c-min}\,\theta, \text{c-min}\,\tau, \text{w-min}\,\tau\theta\}.$$

Proof. If either $\text{c-min}\,\theta = \infty$ or $\text{c-min}\,\tau = \infty$, then we're done. Otherwise, assume $\text{c-min}\,\theta \cdot \text{c-min}\,\tau < \infty$. Note that everything else is finite. Let $i \geq \max\{\text{c-min}\,\theta, \text{c-min}\,\tau, \text{w-min}\,\tau\theta\}$. Then, by Theorem 5.5.8, we have

$$|\ker\phi_i^j| = |\ker\psi_i^j| = |\ker\rho_i^j| \qquad \text{for all } j \geq i,$$

and thus also $i \geq \text{c-min}\,\tau\theta$. □

<u>Theorem 5.5.10</u>. Assume $\ker\theta$ and $\text{coker}\,\theta$ are both finite, and the M_i and N_i are all (eventually) finite, so that Proposition 5.5.5 applies. Then, for $i \geq \text{co-w-min}\,\theta$, the following are equivalent:

1. $|\widetilde{N}_j^*| = |\widetilde{N}_i^*|$ for all $j \geq i$;
2. $|\overline{M}_i^j| = |\widetilde{N}_i^j|$ for all $j \geq i$;
3. $|\overline{M}_{*i}^{*j}| = |\widetilde{N}_{*i}^{*j}|$ for all $j \geq i$;
4. $|\ker \phi_i^j| = |\ker \psi_i^j|$ for all $j \geq i$;
5. $|\ker \phi_i^* / \ker \phi_j^*| = |\ker \psi_i^* / \ker \psi_j^*|$ for all $j \geq i$.

Proof. By Snake Tessellation (5.7) and Lemma 4.0.3, we have

$$|\overline{M}_j^*| \cdot |\overline{M}_{*i}^{*j}| \cdot |\widetilde{N}_i^*| = |\overline{M}_i^*| \cdot |\widetilde{N}_j^*| \cdot |\widetilde{N}_{*i}^{*j}|,$$

and since $i \geq \text{co-w-min}\,\theta$, we have $|\overline{M}_i^*| = |\overline{M}_j^*|$; and thus

$$|\overline{M}_{*i}^{*j}| \cdot |\widetilde{N}_i^*| = |\widetilde{N}_j^*| \cdot |\widetilde{N}_{*i}^{*j}|,$$

whence 5.5.10.1 and 5.5.10.3 are equivalent. The remaining equivalences follow from Lemma 5.5.7. □

Corollary 5.5.11. Assume $\ker\theta$ and $\operatorname{coker}\theta$ are both finite, and the M_i and N_i are all (eventually) finite, so that Proposition 5.5.5 applies. Then, for $i \geq \max\{\text{w-min}\,\theta, \text{co-w-min}\,\theta\}$, the following are equivalent:

1. $i \geq \text{c-min}\,\theta$;
2. $i \geq \text{c-min}\,\theta$ and $\overline{M}_j^* = 0$ for all $j \geq i$;
3. $|\ker\theta_j| = |\ker\theta_i|$ for all $j \geq i$;
4. $|\widetilde{N}_j^*| = |\widetilde{N}_i^*|$ for all $j \geq i$;
5. $|\overline{M}_i^j| = |\widetilde{N}_i^j|$ for all $j \geq i$;
6. $|\overline{M}_{*i}^{*j}| = |\widetilde{N}_{*i}^{*j}|$ for all $j \geq i$;
7. $|\ker \phi_i^j| = |\ker \psi_i^j|$ for all $j \geq i$;
8. $|\ker \phi_i^* / \ker \phi_j^*| = |\ker \psi_i^* / \ker \psi_j^*|$ for all $j \geq i$.

Proof. First, 5.5.11.2 clearly implies 5.5.11.1, and the reverse implication follows from Proposition 5.5.3; and thus 5.5.11.1 and 5.5.11.2 are all equivalent. The remaining equivalences follow from Theorems 5.5.8 and 5.5.10. □

Theorem 5.5.12. Assume $\ker\theta$ and $\operatorname{coker}\theta$ are both finite, and the M_i and N_i are all (eventually) finite, so that Proposition 5.5.5 applies. Then, for $i \geq \max\{\text{w-min}\,\theta, \text{co-w-min}\,\theta\}$, the following are equivalent:

1. $i \geq \text{s-min}\,\theta$;
2. $i \geq \text{s-min}\,\theta$ and $\overline{M}_j^* = 0$ for all $j \geq i$;
3. $\ker\theta_j \xrightarrow{\cong} \ker\theta_i$ for all $j \geq i$;
4. $\widetilde{N}_j^* \xrightarrow{\cong} \widetilde{N}_i^*$ for all $j \geq i$;
5. $\overline{M}_i^j = \widetilde{N}_i^j = 0$ for all $j \geq i$;
6. $\overline{M}_{*i}^{*j} = \widetilde{N}_{*i}^{*j} = 0$ for all $j \geq i$;
7. $\ker\phi_i^j \xrightarrow{\cong} \ker\psi_i^j$ for all $j \geq i$;
8. $\ker\phi_i^* / \ker\phi_j^* \xrightarrow{\cong} \ker\psi_i^* / \ker\psi_j^*$ for all $j \geq i$;
9. the map

   $$\ker\theta_j \longrightarrow \widetilde{N}_j^*$$

   is the zero map for all $j \geq i$;
10. the sequence

    $$0 \longrightarrow \overline{M}_j^* \longrightarrow \ker\theta \longrightarrow \ker\theta_j \longrightarrow 0$$

    is exact for all $j \geq i$;
11. the sequence

    $$0 \longrightarrow \widetilde{N}_j^* \longrightarrow \operatorname{coker}\theta \longrightarrow \operatorname{coker}\theta_j \longrightarrow 0$$

    is exact for all $j \geq i$;
12. $\widetilde{N}_j^* = 0$ for all $j \geq i$;
13. $\widetilde{N}_j^* = \overline{M}_j^* = 0$ for all $j \geq i$;
14. $\ker\phi_j^* \xrightarrow{\cong} \ker\psi_j^*$ for all $j \geq i$.

Proof. First, 5.5.12.2 clearly implies 5.5.12.1, and the reverse implication follows from Proposition 5.5.3; and thus 5.5.12.1 and 5.5.12.2 are all equivalent. Next, 5.5.12.1 implies 5.5.12.3 by definition, and the reverse implication follows since $i \geq \text{w-min}\,\theta$; and thus 5.5.12.1 and 5.5.12.3 are all equivalent. By Diagram (5.8), we have that 5.5.12.3 is equivalent to 5.5.12.5 is equivalent to 5.5.12.6 is equivalent to 5.5.12.4. Next, the line $\{x = 2 \text{ and } z = 4\}$ of Rectangular Prism (5.6) gives that 5.5.12.5 and 5.5.12.7 are equivalent. Next, Diagram (5.5) and similarly for ψ, or the plane $\{z = 1\}$ of Rectangular Prism (5.6), gives that 5.5.12.7 and 5.5.12.8 are equivalent.

We next show that 5.5.12.9, 5.5.12.10, and 5.5.12.11 are equivalent. We have that

$$0 \longrightarrow \overline{M}_j^* \longrightarrow \ker\theta \longrightarrow \ker\theta_j \longrightarrow \widetilde{N}_j^* \longrightarrow \operatorname{coker}\theta \longrightarrow \operatorname{coker}\theta_j \longrightarrow 0$$

is exact. If 5.5.12.10 holds, then $\ker\theta_j = \operatorname{im}(\ker\theta \to \ker\theta_j) = \ker(\ker\theta_j \to \widetilde{N}_j^*)$ gives 5.5.12.9. If 5.5.12.11 holds, then $0 = \ker(\widetilde{N}_j^* \to \operatorname{coker}\theta) = \operatorname{im}(\ker\theta_j \to \widetilde{N}_j^*)$ gives 5.5.12.9. If 5.5.12.9 holds, then $\operatorname{im}(\ker\theta \to \ker\theta_j) = \ker(\ker\theta_j \to \widetilde{N}_j^*) = \ker\theta_j$ gives 5.5.12.10. If 5.5.12.9 holds, then $\ker(\widetilde{N}_j^* \to \operatorname{coker}\theta) = \operatorname{im}(\ker\theta_j \to \widetilde{N}_j^*) = 0$ gives 5.5.12.11.

If 5.5.12.10 holds, then Diagram (5.8) gives that

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \overline{M}_i^j & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_j^* & \longrightarrow & \ker\theta & \longrightarrow & \ker\theta_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_i^* & \longrightarrow & \ker\theta & \longrightarrow & \ker\theta_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & \widetilde{N}_i^j & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

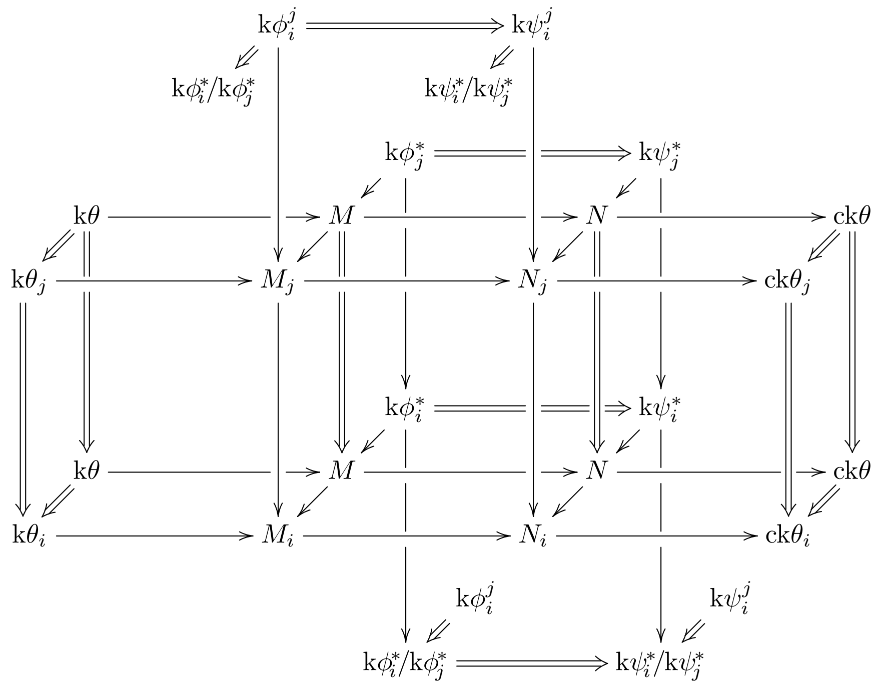
commutes and is exact; whence the Snake Lemma gives 5.5.12.5. Again considering Diagram (5.8), if 5.5.12.2 holds, then Lemma 4.0.1 gives that $\ker\theta \to \ker\theta_j$ is an isomorphism, whence $\ker(\ker\theta \to \ker\theta_j) = 0 = \overline{M}_j^*$, and thus 5.5.12.10 follows.

Next, by Fact 5.1.2, 5.5.12.2 and 5.5.12.11 imply 5.5.12.13. Then 5.5.12.13 clearly implies 5.5.12.12; and 5.5.12.12 implies 5.5.12.6 by Diagram (5.8) on page 56.

Lastly, 5.5.12.13 and 5.5.12.14 are equivalent by the line $\{x = 0 \text{ and } z = 3\}$ of Rectangular Prism (5.6). □

Remark 5.5.13. As we can see, a finite strong minimal level is indeed a rather strong condition. This can also be seen in that Rectangular Prism (5.6) for $j \geq i \geq \text{s-min}\,\theta$

becomes


$k\phi_i^j$
$k\psi_i^j$
$k\phi_i^*/k\phi_j^*$
$k\psi_i^*/k\psi_j^*$
$k\phi_j^*$
$k\psi_j^*$
$k\theta$
M
N
$ck\theta$
$k\theta_j$
M_j
N_j
$ck\theta_j$
$k\phi_i^*$
$k\psi_i^*$
$k\theta$
M
N
$ck\theta$
$k\theta_i$
M_i
N_i
$ck\theta_i$
$k\phi_i^j$
$k\psi_i^j$
$k\phi_i^*/k\phi_j^*$
$k\psi_i^*/k\psi_j^*$

(5.9)

Chapter 6

INVERSE SYSTEM EXPANSION

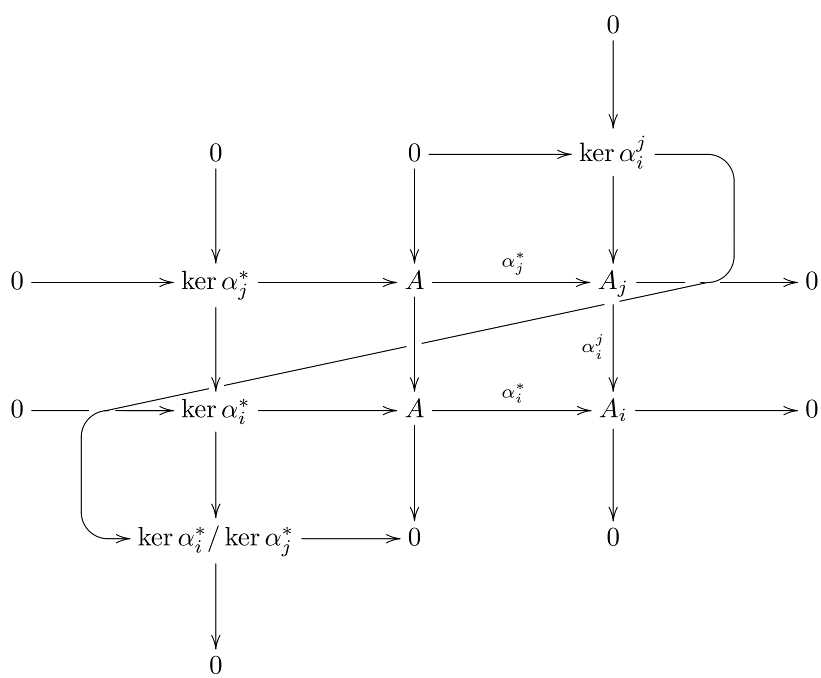
[1]The notation of Chapter 6 is independent of all other chapters.

Let R be a commutative ring (assumed unitary); let $z \in \mathbb{Z}$; let $I = \mathbb{Z} \cap [z, \infty)$; let $\{A_i, \alpha_i^j\}$ be an inverse system of R-modules over I; assume that the α_i^j are surjective for all $i \leq j$; let $A = \varprojlim A_i$ be the inverse limit; and let $\alpha_j^* : A \longrightarrow A_j$ be the natural (projection) maps $\alpha_j^* : (a_i)_{i \in I} \mapsto a_j$, which are necessarily surjective by Lemma 4.0.1.

6.1 The Sequence of the Kernels of the Projection Maps

Considering the diagram

(6.1)



we can see $A_i \cong A / \ker \alpha_i^*$ and $\ker \alpha_i^j \cong \ker \alpha_i^* / \ker \alpha_j^*$, where the second map is given by $a_j \mapsto (\alpha_j^*)^{-1} a_j + \ker \alpha_j^*$. (This is the same as Diagram (5.5) on page 47.) We have that $\{A_i\}_{i \in I}$ forms a sequence of modules of non-decreasing size, but not (necessarily) nested inside each other. However, $\{\ker \alpha_i^*\}_{i \in I}$ forms a non-increasing sequence of submodules of A such that $\bigcap_{i \in I} \ker \alpha_i^* = 0$.

[1]Rotman (2002) is a general reference for definitions and notational trends.

Proof. If $a = (a_j)_{j \in I} \in \bigcap_{i \in I} \ker \alpha_i^*$, then $\alpha_j^* a = a_j = 0$ for all $j \in I$. □

As such, these two sequences contain the same basic information, but from different perspectives. It will be convenient for us to consider an enlargement of $A_i \cong A/\ker \alpha_i^*$ by substituting something smaller for $\ker \alpha_i^*$.

In the case where the A_i are all finite, we have that the $\ker \alpha_i^*$ all have finite index in A.

6.2 Definitions and Basic Facts

Assume for each $i \in I$ that we are given $B_i \subset \ker \alpha_i^*$ such that $B_j \subset B_i$ for all $j \geq i$. Then we define

$$\widehat{A}_i = A/B_i,$$

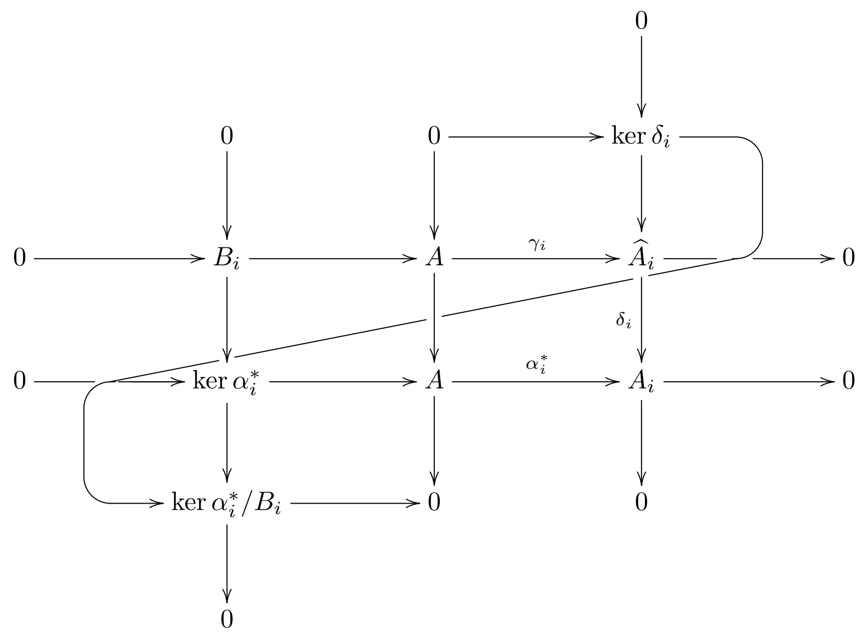
the surjective map $\gamma_i : A \longrightarrow \widehat{A}_i$ to be projection $\gamma_i : a \mapsto a + B_i$, and the surjective map $\delta_i : \widehat{A}_i \longrightarrow A_i$ given by $\delta_i = \alpha_i^*(\gamma_i)^{-1} : a + B_i \mapsto a_i$ for $a = (a_j)_{j \in I} \in A$. Moreover, $\ker \delta_i \cong \ker \alpha_i^*/B_i$ by $a + B_i \mapsto a + B_i$.

Proof. γ_i is a well-defined surjection.

We first show that δ_i is well-defined. Let $a + B_i = (a_j)_{j \in I} + B_i \in \widehat{A}_i$ and also let $a' + B_i = (a'_j)_{j \in I} + B_i \in \widehat{A}_i$ such that $a + B_i = a' + B_i$, so that $a - a' \in B_i \subset \ker \alpha_i^*$. Then $\alpha_i^* a - \alpha_i^* a' = \alpha_i^*(a - a') = 0$ and $\alpha_i^* a = \alpha_i^* a'$; that is $a_i = a'_i$; whence δ_i is well-defined.

We next show that δ_i is surjective. Let $a_i \in A_i$. Let $a = (a_j)_{j \in I} \in A$ such that $\alpha_i^* a = a_i$ by Lemma 4.0.1. Then $\delta_i(a + B_i) = a_i$.

So we have

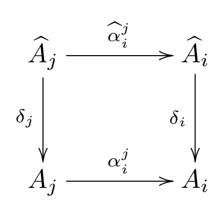


(6.2)

whence we get $\ker \delta_i \cong \ker \alpha_i^*/B_i$ by $a + B_i \mapsto a + B_i$. ☐

Now we have the surjective maps $\widehat{\alpha}_i^j : \widehat{A}_j \longrightarrow \widehat{A}_i$ given by coset enlargement; and thus $\{\widehat{A}_i, \widehat{\alpha}_i^j\}$ is an inverse system, so we define $\widehat{A} = \varprojlim \widehat{A}_i$; and we also have the natural (projection) maps $\widehat{\alpha}_i^* : \widehat{A} \longrightarrow \widehat{A}_i$, which are necessarily surjective by Lemma 4.0.1. We also have that $\{\delta_i : \widehat{A}_i \longrightarrow A_i\} : \{\widehat{A}_i, \widehat{\alpha}_i^j\} \longrightarrow \{A_i, \alpha_i^j\}$ is a transformation.

Proof. We need only verify that we have a transformation. Let $a + B_j \in \widehat{A}_j$ where $a = (a_k)_{k\in I} \in A$. Then $\delta_i \widehat{\alpha}_i^j(a + B_j) = \delta_i(a + B_i) = a_i$, and $\alpha_i^j \delta_j(a + B_j) = \alpha_i^j a_j = a_i$. Thus $\delta_i \widehat{\alpha}_i^j = \alpha_i^j \delta_j$, and hence we have that



commutes. ☐

So let $\delta : \widehat{A} \longrightarrow A$ be the induced R-homomorphism. Note that a typical element of $\widehat{A}$ is of the form $(a^{(i)} + B_i)_{i\in I} \in \widehat{A}$ where the $a^{(i)} = (a_k^{(i)})_{k\in I} \in A$ form a sequence of elements in A such that $\widehat{\alpha}_i^j(a^{(j)} + B_j) = a^{(j)} + B_i = a^{(i)} + B_i$, or $a^{(j)} - a^{(i)} \in B_i$ for all $j \geq i$. It follows that $a_i^{(j)} = a_i^{(i)}$ for all $j \geq i$; in fact, if follows that $a_i^{(k)} = a_i^{(j)}$ for all $i \leq j \leq k$. Thus

$$\delta : (a^{(i)} + B_i)_{i \in I} \mapsto (\delta_i(a^{(i)} + B_i))_{i \in I} = (a_i^{(i)})_{i \in I},$$

which gives a well-defined element of A since, as we have shown, $\alpha_i^j a_j^{(j)} = a_i^{(j)} = a_i^{(i)}$ for all $j \geq i$.

<u>Fact 6.2.1</u>. δ is surjective.

Proof. Let $a \in A$, then $(a + B_i)_{i \in I} \in \widehat{A}$ since $\widehat{\alpha}_i^j(a + B_j) = (a + B_i)$ for all $j \geq i$. Then $\delta(a + B_i)_{i \in I} = (\delta_i(a + B_i))_{i \in I} = (a_i)_{i \in I} = a$. □

<u>Proposition 6.2.2</u>. We have

$$\widehat{A}_i \cong A_i \oplus \ker \delta_i \qquad \text{and} \qquad \widehat{A} \cong A \oplus \ker \delta.$$

Proof. Given $a_i \in A_i$, map $a_i \mapsto \big((\alpha_i^*)^{-1} a_i + B_i\big)_{i \in I}$. If $(\alpha_i^*)^{-1} a_i \in B_i$, then $(\alpha_i^*)^{-1} a_i \in \ker \alpha_i^*$, and hence $\alpha_i^*(\alpha_i^*)^{-1} a_i = a_i = 0$; whence the map is injective. Next, $a_i \mapsto \big((\alpha_i^*)^{-1} a_i + B_i\big)_{i \in I} \overset{\delta_i}{\mapsto} a_i$. Thus

$$0 \longrightarrow \ker \delta_i \longrightarrow \widehat{A}_i \underset{\delta_i}{\overset{\curvearrowleft}{\longrightarrow}} A_i \longrightarrow 0$$

splits.

Given $a \in A$, map $a \mapsto (a + B_i)_{i \in I}$. If $a \in B_i$ for all $i \in I$, then $a \in \bigcap_{i \in I} B_i \subset \bigcap_{i \in I} \ker \alpha_i^* = 0$; whence the map is injective. Next, $a \mapsto (a + B_i)_{i \in I} \overset{\delta}{\mapsto} a$. Thus

$$0 \longrightarrow \ker \delta \longrightarrow \widehat{A} \underset{\delta}{\overset{\curvearrowleft}{\longrightarrow}} A \longrightarrow 0$$

splits. □

<u>Proposition 6.2.3</u>. We have that $\widehat{A}_i$ is finite if and only if A_i and $\ker \delta_i$ are both finite, which is when and only when A_i is finite and B_i has finite index in $\ker \alpha_i^*$.

Proof. This follows from Diagram (6.2) or Proposition 6.2.2 above. □

We call the inverse system $\{\widehat{A}_i, \widehat{\alpha}_i^j\}$, or its limit $\widehat{A}$, the *expansion* of $\{A_i, \alpha_i^j\}$, or of A, by $\{B_i\}_{i\in I}$; and the transformation $\{\delta_i : \widehat{A}_i \longrightarrow A_i\} : \{\widehat{A}_i, \widehat{\alpha}_i^j\} \longrightarrow \{A_i, \alpha_i^j\}$, or its induced map δ, will be referred to as the $\{B_i\}$-*expansion transformation*, and the $\ker\delta_i \cong \ker\alpha_i^*/B_i$ will be referred to as the *expansion quotients*.

Note that the expansion of A by $\{\ker\alpha_i^*\}_{i\in I}$ gives isomorphisms $\delta_i : \widehat{A}_i \xrightarrow{\cong} A_i$ with trivial expansion quotients.

6.3 The Minimal Levels of an Expansion Transformation

We now consider the minimal levels of the expansion transformation δ given above.

Proposition 6.3.1.

$$\text{w-min}\,\delta = \min I.$$

Proof. The δ_i are all surjective. □

Proposition 6.3.2.

$$\begin{aligned}\text{c-min}\,\delta &= \min\Big(\{i\in I : |\ker\delta_j| = |\ker\delta_i|\ \forall j\geq i\}\cup\{\infty\}\Big)\\ &= \min\Big(\{i\in I : [\ker\alpha_j^* : B_j] = [\ker\alpha_i^* : B_i]\ \forall j\geq i\}\cup\{\infty\}\Big).\end{aligned}$$

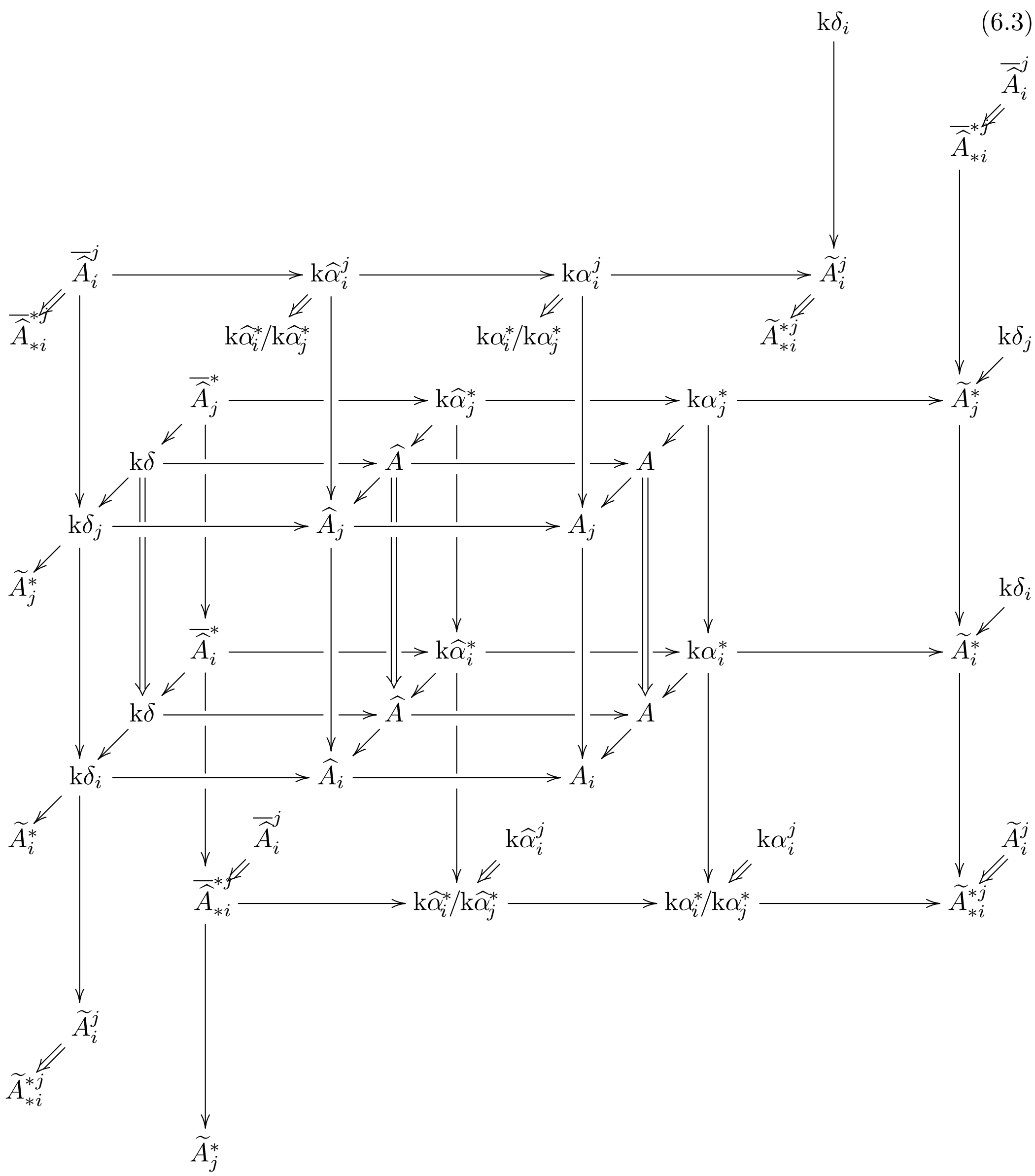
Proof. From Diagram (6.2), we have the set equality

$$\{i\in I : |\ker\delta_j| = |\ker\delta_i|\ \forall j\geq i\} = \{i\in I : [\ker\alpha_j^* : B_j] = [\ker\alpha_i^* : B_i]\ \forall j\geq i\}.$$

Then, given Proposition 6.3.1 above, the results follows. □

For ease of reference and to avoid confusion later, we record Rectangular Prism (5.6) and Snake Tessellation (5.7) applied to the expansion transformation $\delta : \widehat{A} \longrightarrow A$. Recall that δ and the δ_i are surjective so that their cokernels are trivial. This gives us that the rectangular prism

(6.3)

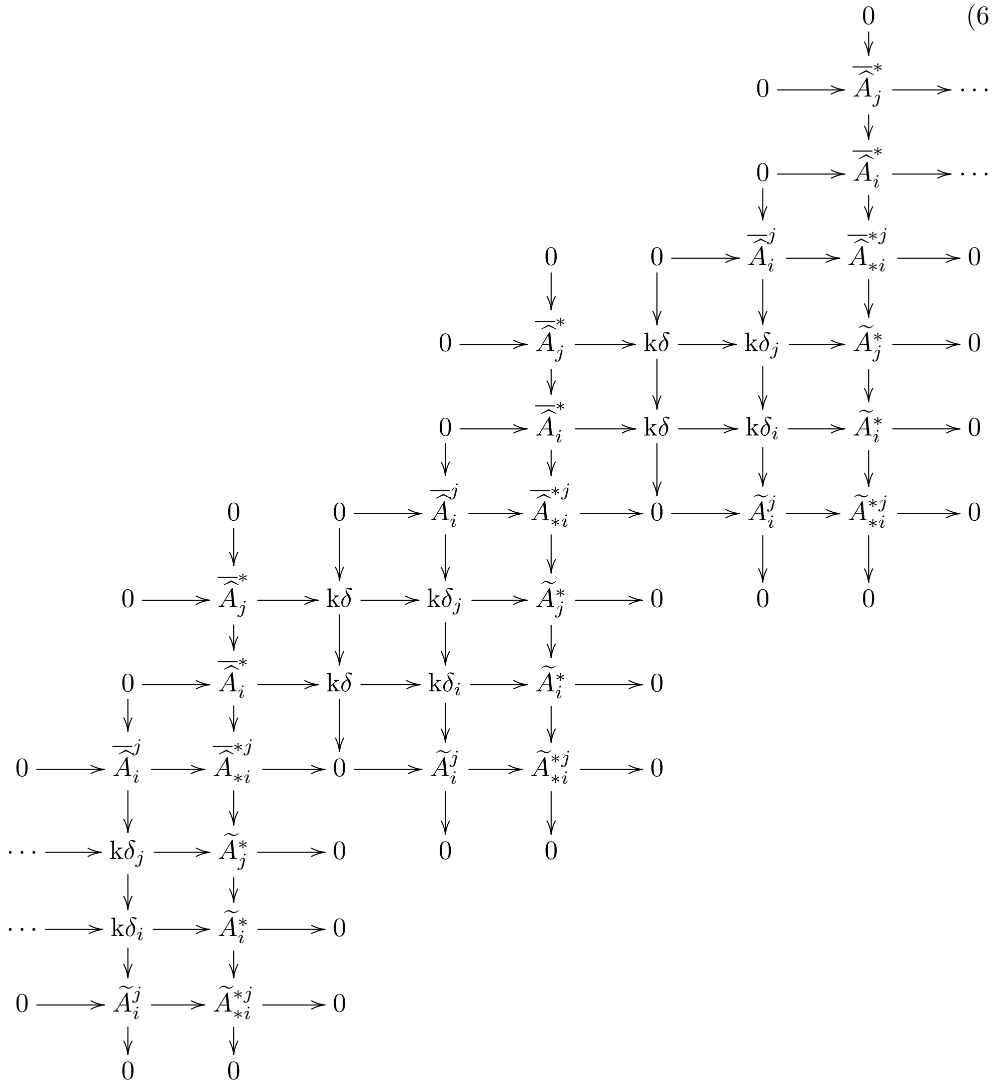


commutes and is exact in straight lines, where we dropped the zero maps for visibility sake, a double shafted arrow indicates isomorphism (or equality), we abbreviated ker and coker to k and ck, and we set

$$
\begin{aligned}
\overline{\widehat{A}}{}_i^j &= \ker \widehat{\alpha}_i^j \cap \ker \delta_j = \ker(\ker \widehat{\alpha}_i^j \longrightarrow \ker \alpha_i^j) = \ker(\ker \delta_j \longrightarrow \ker \delta_i) \\
\widetilde{A}_i^j &= \operatorname{coker}(\ker \widehat{\alpha}_i^j \longrightarrow \ker \alpha_i^j) \\
\overline{\widehat{A}}{}_j^* &= \ker \widehat{\alpha}_j^* \cap \ker \delta = \ker(\ker \widehat{\alpha}_j^* \longrightarrow \ker \alpha_j^*) = \ker(\ker \delta \longrightarrow \ker \delta_j) \\
\widetilde{A}_j^* &= \operatorname{coker}(\ker \widehat{\alpha}_j^* \longrightarrow \ker \alpha_j^*) \\
\overline{\widehat{A}}{}_i^* &= \ker \widehat{\alpha}_i^* \cap \ker \delta = \ker(\ker \widehat{\alpha}_i^* \longrightarrow \ker \alpha_i^*) = \ker(\ker \delta \longrightarrow \ker \delta_i) \\
\widetilde{A}_i^* &= \operatorname{coker}(\ker \widehat{\alpha}_i^* \longrightarrow \ker \alpha_i^*) \\
\overline{\widehat{A}}{}_{*i}^{*j} &= \ker(\ker \widehat{\alpha}_i^* / \ker \widehat{\alpha}_j^* \longrightarrow \ker \alpha_i^* / \ker \alpha_j^*) \\
\widetilde{A}_{*i}^{*j} &= \operatorname{coker}(\ker \widehat{\alpha}_i^* / \ker \widehat{\alpha}_j^* \longrightarrow \ker \alpha_i^* / \ker \alpha_j^*);
\end{aligned}
$$

and we also have that the snake tessellation

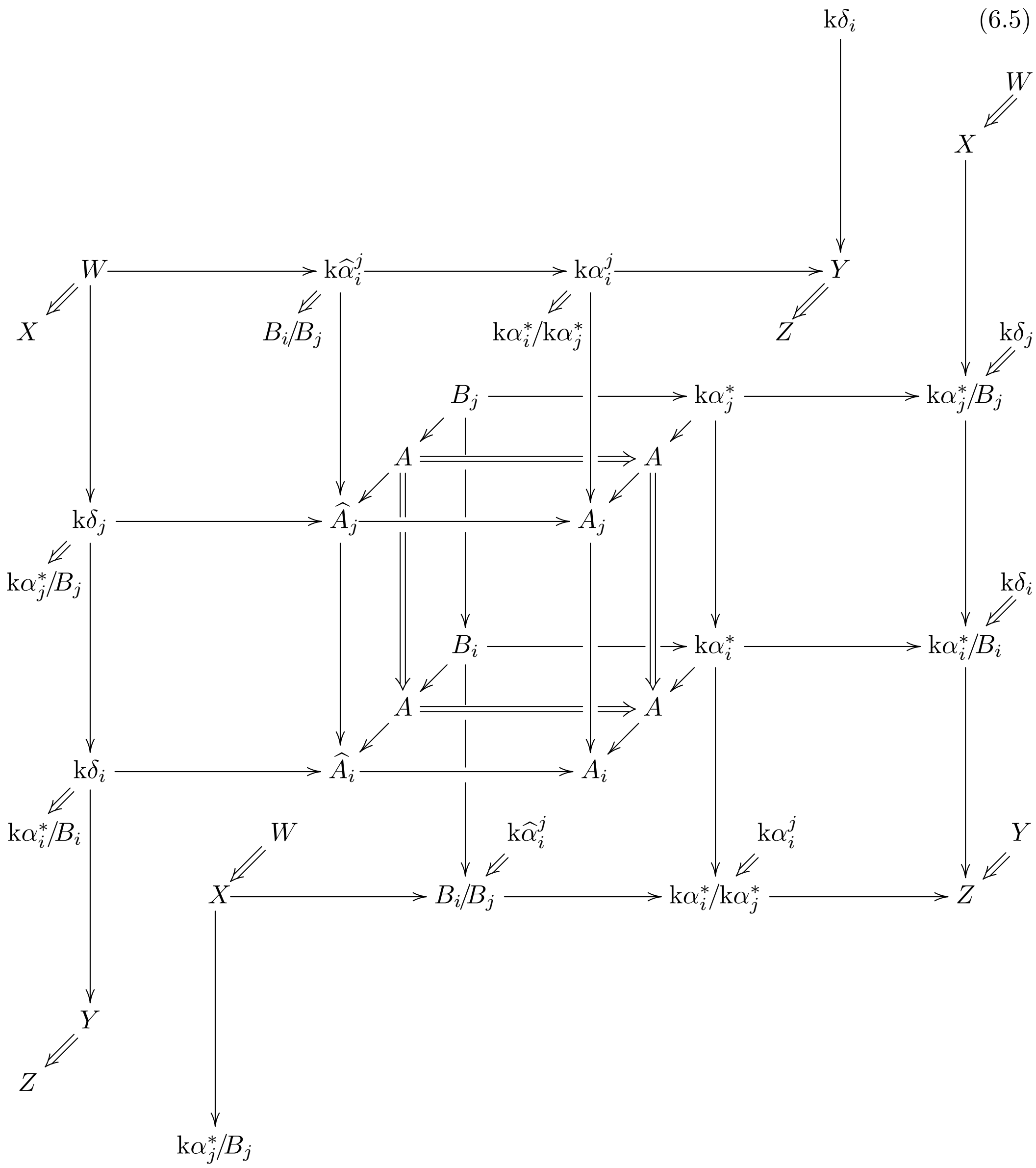
(6.4)



commutes and is exact, where we abbreviated ker and coker to k and ck.

It would be nice if we could integrate Diagram (6.2) into Rectangular Prism (6.3); however, since it can often be difficult to typeset a four-dimensional lattice, we will be satisfied with the below. That is, we have that the rectangular prism

(6.5)



commutes and is exact in straight lines, where we dropped the zero maps for visibility sake, a double shafted arrow indicates isomorphism (or equality), we abbreviated ker to k, we set

$$
\begin{aligned}
W &= \ker \widehat{\alpha}_i^j \cap \ker \delta_j = \ker(\ker \widehat{\alpha}_i^j \longrightarrow \ker \alpha_i^j) = \ker(\ker \delta_j \longrightarrow \ker \delta_i) \\
Y &= \operatorname{coker}(\ker \widehat{\alpha}_i^j \longrightarrow \ker \alpha_i^j) \\
X &= \ker(B_i/B_j \longrightarrow \ker \alpha_i^* / \ker \alpha_j^*) \\
Z &= \operatorname{coker}(B_i/B_j \longrightarrow \ker \alpha_i^* / \ker \alpha_j^*),
\end{aligned}
$$

we have that

$$\ker \delta_j \xrightarrow{\cong} \ker \alpha_j^* / B_j$$

$$\ker \delta_i \xrightarrow{\cong} \ker \alpha_i^* / B_i$$

$$\ker \delta_i \longrightarrow Y$$

$$X \longrightarrow \ker \alpha_j^* / B_j$$

$$\ker \widehat{\alpha}_i^j \xrightarrow{\cong} B_i / B_j$$

$$\ker \alpha_i^j \xrightarrow{\cong} \ker \alpha_i^* / \ker \alpha_j^*$$

are the connecting morphisms of the Snake Lemma, and the induced maps connecting the bottom two connecting isomorphisms have

$$W \xrightarrow{\cong} X$$

$$Y \xrightarrow{\cong} Z$$

for induced kernel and cokernel maps. Putting together four of the connecting maps, and the induced kernel and cokernel maps of the other two, we get that

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \delta_j & \longrightarrow & \ker \alpha_j^* / B_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \delta_i & \longrightarrow & \ker \alpha_i^* / B_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array} \qquad (6.6)$$

is exact.

Proof. The justification here is similar to that for Rectangular Prism (5.6) on page 48 above. However, note the important differences between Rectangular Prism (6.5) and Rectangular Prism (6.3); e.g., A (and 0) entirely occupies the plane $\{x = 1\}$. That being said, all the maps are the, perhaps iteratively, induced kernel, cokernel, and snake maps of the middle cube $\{1 \leq x \leq 2 \text{ and } 1 \leq y \leq 2 \text{ and } 2 \leq z \leq 3\}$; only more are known to become zero maps in this case. At this point, all that would remain would be to show that the square $\{2 \leq x \leq 3 \text{ and } 1 \leq y \leq 2 \text{ and } z = 4\}$ commutes and that Diagram (6.6) commutes; however, we need only the exactness of the latter, which we have, and so we leave commutativity unjustified. $\square$

Proposition 6.3.3. Assume $\ker\delta$ is finite, and the $\widehat{A}_i$ and A_i are all (eventually) finite, so that Proposition 5.5.5 applies. Then

$$\begin{aligned}
\text{c-min}\,\delta &= \min\Big(\{i \in I : |\ker\delta_j| = |\ker\delta_i|\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : [\ker\alpha_j^* : B_j] = [\ker\alpha_i^* : B_i]\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : |\overline{\widehat{A}}{}_i^j| = |\widetilde{A}_i^j|\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : |\overline{\widehat{A}}{}_{*i}^{*j}| = |\widetilde{A}_{*i}^{*j}|\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : |\ker\widehat{\alpha}_i^j| = |\ker\alpha_i^j|\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : |\ker\widehat{\alpha}_i^* / \ker\widehat{\alpha}_j^*| = |\ker\alpha_i^* / \ker\alpha_j^*|\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : |B_i/B_j| = |\ker\alpha_i^* / \ker\alpha_j^*|\ \forall j \geq i\} \cup \{\infty\}\Big).
\end{aligned}$$

Proof. Given Proposition 6.3.1 above, this is a combination of Proposition 6.3.2, Theorem 5.5.8, and Rectangular Prism (6.5). $\square$

Proposition 6.3.4.

$$\begin{aligned}
\text{s-min}\,\delta &= \min\Big(\{i \in I : \ker\delta_j \overset{\cong}{\to} \ker\delta_i\ \forall j \geq i\} \cup \{\infty\}\Big)\\
&= \min\Big(\{i \in I : \ker\alpha_j^*/B_j \overset{\cong}{\to} \ker\alpha_i^*/B_i\ \forall j \geq i\} \cup \{\infty\}\Big),
\end{aligned}$$

where the map $\ker\alpha_j^*/B_j \longrightarrow \ker\alpha_i^*/B_i$ is given by $a + B_j \mapsto a + B_i$, which is not coset enlargement even though it is determined by enlarging the cosets. That is, $\ker\alpha_j^*/B_j \longrightarrow \ker\alpha_i^*/B_i$ is the induced kernel map of the induced kernel map $\ker\alpha_j^* \longrightarrow \ker\alpha_i^*$ given by inclusion, which is induced from equality $A \longrightarrow A$; so while the cosets B_i are (possibly) getting larger, their containing modules $\ker\alpha_i^*$ are (possibly) getting larger too.

Proof. Given Proposition 6.3.1 above, this follows from Diagram (6.6). That is, we have $j \geq i \geq \text{s-min}\,\delta$ if and only if $\ker \delta_j \overset{\cong}{\to} \ker \delta_i$, if and only if $W = Y = 0$, if and only if $X = Z = 0$, if and only if $\ker \alpha_j^*/B_j \overset{\cong}{\to} \ker \alpha_i^*/B_i$. □

Chapter 7

FORCING A TRANSFORMATION FROM A MAP ON THE LIMITS

[1]The notation of Chapter 7 is independent of all other chapters.

Let R be a commutative ring (assumed unitary); let $z \in \mathbb{Z}$; let $I = \mathbb{Z} \cap [z, \infty)$; let $\{M_i, \phi_i^j\}$ and $\{N_i, \psi_i^j\}$ be inverse systems of R-modules over I; assume that ϕ_i^j and ψ_i^j are both surjective for all $i \leq j$; let $M = \varprojlim M_i$ and $N = \varprojlim N_i$ be the inverse limits; and let $\phi_i^* : M \longrightarrow M_i$ and $\psi_i^* : N \longrightarrow N_i$ be the natural (projection) maps $\phi_i^* : (m_j)_{j \in I} \mapsto m_i$ and $\psi_i^* : (n_j)_{j \in I} \mapsto n_i$, which are surjective by Lemma 4.0.1.

Let $\kappa : M \longrightarrow N$ be an arbitrary R-module homomorphism (not necessarily inducible). We may not have $\kappa \ker \phi_i^* \overset{?}{\subset} \ker \psi_i^*$; however, we may consider the restriction of κ

$$\ker \phi_i^* \cap \kappa^{-1} \ker \psi_i^* \xrightarrow{\kappa} \ker \psi_i^*.$$

So it may be well to consider expanding the inverse systems a la Chapter 6.

Let B_i be a subset of M such that

$$B_i \subset \ker \phi_i^* \cap \kappa^{-1} \ker \psi_i^* \qquad \text{and} \qquad B_j \subset B_i \text{ for all } j \geq i.$$

If convenient, we could take $\ker \phi_i^* \cap \kappa^{-1} \ker \psi_i^*$ for B_i.

Proof. We have $\ker \phi_j^* \cap \kappa^{-1} \ker \psi_j^* \subset \ker \phi_i^* \cap \kappa^{-1} \ker \psi_i^*$ since $\ker \phi_j^* \subset \ker \phi_i^*$ and $\kappa^{-1} \ker \psi_j^* \subset \kappa^{-1} \ker \psi_i^*$, where the latter follows since $\ker \psi_j^* \subset \ker \psi_i^*$. □

Let C_i be a submodule on N such that

$$\kappa B_i \subset C_i \subset \ker \psi_i^* \qquad \text{and} \qquad C_j \subset C_i \text{ for all } j \geq i.$$

If convenient, we could take any of the three $\kappa B_i \subset \ker \psi_i^* \cap \kappa \ker \phi_i^* \subset \ker \psi_i^*$ for C_i.

Proof. $B_j \subset B_i$ gives $\kappa B_j \subset \kappa B_i$. We already have $\ker \psi_j^* \subset \ker \psi_i^*$. We have $\ker \psi_j^* \cap \kappa \ker \phi_j^* \subset \ker \psi_i^* \cap \kappa \ker \phi_i^*$ since $\ker \psi_j^* \subset \ker \psi_i^*$ and $\kappa \ker \phi_j^* \subset \kappa \ker \phi_i^*$, where the latter follows since $\ker \phi_j^* \subset \ker \phi_i^*$. Note also that we indeed have the relation

[1]Rotman (2002) is a general reference for definitions and notational trends.

$\kappa B_i \subset \ker \psi_i^* \cap \kappa \ker \phi_i^* \subset \ker \psi_i^*$, where the latter is clear and the prior follows since $\kappa B_i \subset \ker \psi_i^*$ and since $B_i \subset \ker \phi_i^*$ gives $\kappa B_i \subset \kappa \ker \phi_i^*$. $\square$

We now consider the restriction of κ

$$B_i \xrightarrow{\kappa} C_i$$

and the respective expansions of M and N by $\{B_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$,

$$\widehat{M} = \lim_{\leftarrow} \widehat{M_i} \qquad \text{and} \qquad \widehat{N} = \lim_{\leftarrow} \widehat{N}_i,$$

where

$$\widehat{M_i} = M/B_i \qquad \text{and} \qquad \widehat{N}_i = N/C_i.$$

Let

$$\delta_i : \widehat{M_i} \longrightarrow M_i \qquad \text{and} \qquad \epsilon_i : \widehat{N}_i \longrightarrow N_i$$

given by

$$\delta_i : m + B_i \mapsto \phi_i^* m \qquad \text{and} \qquad \epsilon_i : n + C_i \mapsto \psi_i^* n$$

be the expansion transformations; and define

$$\widehat{\kappa}_i : \widehat{M_i} \longrightarrow \widehat{N}_i \qquad \text{by} \qquad \widehat{\kappa}_i : m + B_i \mapsto \kappa m + C_i;$$

which gives a transformation.

Proof. We first show that $\widehat{\kappa}_i$ is well-defined. Let $m + B_i \in \widehat{M_i}$ and assume $m' + B_i \in \widehat{M_i}$ with $m + B_i = m' + B_i$, so $m - m' \in B_i$. Thus $\kappa m - \kappa m' = \kappa(m - m') \in C_i$.

We now verify that we indeed have a transformation. Let $m + B_j \in \widehat{M_j}$ where $m = (m_k)_{k\in I} \in M$. Then $\widehat{\kappa}_i \widehat{\phi}_i^j (m + B_j) = \widehat{\kappa}_i(m + B_i) = \kappa m + C_i$, and $\widehat{\psi}_i^j \widehat{\kappa}_j (m + B_j) = \widehat{\psi}_i^j (\kappa m + C_j) = \kappa m + C_i$. Thus $\widehat{\kappa}_i \widehat{\phi}_i^j = \widehat{\psi}_i^j \widehat{\kappa}_j$, and hence we have that

$$\begin{array}{ccc}
\widehat{M_j} & \xrightarrow{\widehat{\phi}_i^j} & \widehat{M_i} \\
{\scriptstyle \widehat{\kappa}_j}\downarrow & & \downarrow{\scriptstyle \widehat{\kappa}_i} \\
\widehat{N}_j & \xrightarrow{\widehat{\psi}_i^j} & \widehat{N}_i
\end{array}$$

commutes. $\square$

Then we have the maps

$$\begin{array}{ccc} \widehat{M}_i & \xrightarrow{\delta_i} & M_i \\ {\scriptstyle \widehat{\kappa}_i}\downarrow & & \vdots\ {\scriptstyle ¿\ ?} \\ \widehat{N}_i & \xrightarrow{\epsilon_i} & N_i \end{array}$$

where $M_i \xrightarrow{\kappa_i} N_i$ exists for all i precisely when κ is inducible; in which case the diagram commutes.

Proof. Assume κ is inducible; that is, we have a well-defined map $\kappa_i : M_i \longrightarrow N_i$ given by $\kappa_i = \psi_i^* \kappa (\phi_i^*)^{-1}$, see §5.4 on page 45. Let $m + B_i \in \widehat{M_i}$ for $m = (m_k)_{k \in I} \in M$. Then

$$\epsilon_i \widehat{\kappa}_i (m + B_i) = \epsilon_i (\kappa m + C_i) = \psi_i^* \kappa m$$

and

$$\kappa_i \delta_i (m + B_i) = \kappa_i \phi_i^* m = \psi_i^* \kappa m.$$

□

Let δ, ϵ, and $\widehat{\kappa}$ be the induced maps. Then we have that

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\delta} & M \\ {\scriptstyle \widehat{\kappa}}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \widehat{N} & \xrightarrow{\epsilon} & N \end{array} \qquad \text{commutes if and only if } \widehat{\kappa} \ker \delta \subset \ker \epsilon. \tag{7.1}$$

Proof. Let $m \in M$, and consider $(m + B_i)_{i \in I} \in \widehat{M}$. If we let $n = (n_k)_{k \in I} \in N$ such that $\kappa\, m = n$, then

$$\epsilon \widehat{\kappa} (m + B_i)_{i \in I} = \epsilon \big(\widehat{\kappa}_i (m + B_i) \big)_{i \in I} = \epsilon (\kappa\, m + C_i)_{i \in I}$$

$$= \big(\epsilon_i (\kappa\, m + C_i) \big)_{i \in I} = (\psi_i^* \kappa\, m)_{i \in I} = (\psi_i^* n)_{i \in I} = (n_i)_{i \in I} = n$$

and

$$\kappa \delta (m + B_i)_{i \in I} = \kappa \big(\delta_i (m + B_i) \big)_{i \in I} = \kappa (\phi_i^* m)_{i \in I} = \kappa (m_i)_{i \in I} = \kappa\, m = n.$$

So if δ were an isomorphism, we would have commutativity.

However, by Proposition 6.2.2 on page 66 above, we have that an arbitrary element of $\widehat{M}$ may be decomposed as $(m + B_i)_{i \in I} + (m^{(i)} + B_i)_{i \in I} \in \widehat{M}$ where $m \in M$ and $(m^{(i)} + B_i)_{i \in I} \in \ker \delta$. We have seen that the first part of the decomposition commutes just fine. So it remains to consider the second part; which, when mapped right-then-down, gives zero. Thus, the diagram commutes if and only if, when the second part is mapped down-then-right, we still get zero; which happens when and only when we have $\widehat{\kappa} \ker \delta \subset \ker \epsilon$. □

We say that the transformation $\{\widehat{\kappa}_i : \widehat{M}_i \longrightarrow \widehat{N}_i\} : \{\widehat{M}_i, \widehat{\phi}_i^j\} \longrightarrow \{\widehat{N}_i, \psi_i^j\}$, or its induced map $\widehat{\kappa}$, is the $\{B_i, C_i\}$-*forced transformation*; we call the $\{B_i, C_i\}$ a *sequence of forcing submodules*; and we say that κ is *almost inducible* if there exists a forced transformation $\widehat{\kappa}$ where $\widehat{\kappa} \ker \delta \subset \ker \epsilon$.

We now define a notion of the minimal level of an arbitrary R-homomorphism. Since these techniques will be of interest primarily when we have no direct map $M_i \longrightarrow N_i$, and since the weak minimal level of an extension transformation is trivial, we expect only one of the four notions of minimal level to be of primary concern; although the other three are entirely analogous.

Define the $\{B_i, C_i\}$-*generalized common minimal level* of κ to be

$$\{B_i, C_i\}\text{-g-c-min}\,\kappa \quad = \quad \max\{\text{c-min}\,\delta, \text{c-min}\,\epsilon, \text{c-min}\,\widehat{\kappa}\}.$$

Proposition 7.0.5. If $\{B_i, C_i\}$-g-c-min $\kappa < \infty$; and if M_i, N_i, $[\ker \phi_i^* : B_i]$, and $[\ker \psi_i^* : C_i]$ are finite for all $i \geq \{B_i, C_i\}$-g-c-min κ; then

$$\frac{|N_i|}{|M_i|} = \frac{|\text{coker}\,\widehat{\kappa}_i| \cdot |\ker \delta_i|}{|\ker \widehat{\kappa}_i| \cdot |\ker \epsilon_i|} = \text{constant} \qquad \text{for all } i \geq \{B_i, C_i\}\text{-g-c-min}\,\kappa;$$

so the M_i and N_i grow at the same rate for all $i \geq \{B_i, C_i\}$-g-c-min κ.

Proof. Let $i \geq \{B_i, C_i\}$-g-c-min κ. Since M_i, N_i, $[\ker \phi_i^* : B_i]$, and $[\ker \psi_i^* : C_i]$ are finite, we have that $\widehat{M}_i$ and $\widehat{N}_i$ are also finite by Proposition 6.2.3 on page 66 above. Then, by Fact 5.1.5 on page 40 above, we have

$$\frac{|\widehat{N}_i|}{|\widehat{M}_i|} = \frac{|\text{coker}\,\widehat{\kappa}_i|}{|\ker \widehat{\kappa}_i|} = \text{constant}',$$

$$\frac{|M_i|}{|\widehat{M}_i|} = \frac{|\operatorname{coker} \delta_i|}{|\ker \delta_i|} = \frac{1}{|\ker \delta_i|} = \text{constant}'',$$

$$\frac{|N_i|}{|\widehat{N}_i|} = \frac{|\operatorname{coker} \epsilon_i|}{|\ker \epsilon_i|} = \frac{1}{|\ker \epsilon_i|} = \text{constant}''';$$

whence

$$\frac{|N_i|}{|M_i|} = \frac{|\operatorname{coker} \widehat{\kappa}_i| \cdot |\ker \delta_i|}{|\ker \widehat{\kappa}_i| \cdot |\ker \epsilon_i|} = \frac{\text{constant}''' \cdot \text{constant}'}{\text{constant}''} = \text{constant}.$$

□

However, finding well-chosen $\{B_i, C_i\}$ remains entirely to be seen.

Chapter 8

RETURN TO IWASAWA THEORY

Even after a century and a half of investigation, the Kummer-Vandiver Conjecture remains one of the largest unsolved problems in cyclotomic fields, or perhaps one of the largest impediments toward advancement in certain areas of the study of cyclotomic fields. It stands to argue that a proof would be complicated or a counterexample enormous. If the Kummer-Vandiver conjecture holds, then $CG \longrightarrow IACG$ is an inducible isomorphism with trivial strong minimal level. However, if the Kummer-Vandiver conjecture does not hold, then $CG \longrightarrow IACG$ may not be inducible, may not be an isomorphism, or may have non-trivial minimal levels. In particular, to disprove the Kummer-Vandiver Conjecture, it would suffice to show for a given prime that there are no inducible pseudo-isomorphisms $CG \longrightarrow IACG$, or that any inducible pseudo-isomorphism has a non-trivial minimal level, or to construct an inducible pseudo-isomorphism where at any finite level the kernel and cokernel had different size. These criteria may produce easier, or numerically faster, methods to verify if the Kummer-Vandiver Conjecture does or does not hold for a given prime. Also, if the Kummer-Vandiver Conjecture does hold, then understanding why the pseudo-isomorphism must be inducible, or why the minimal levels must be trivial, may aid in the understanding of why the conjecture also would hold. Therefore, if our goal is to disprove the Kummer-Vandiver Conjecture, then we have another tool available; and if our goal is to prove the Kummer-Vandiver Conjecture, then we have detailed what may well be several intermediate step towards a proof, or perhaps steps towards understanding the underlying phenomena that will later lead towards a proof. Or also interestingly, what if the Kummer-Vandiver Conjecture were false, but some or all of the minimal levels were still trivial?

I have framed the above discussion in terms of the Kummer-Vandiver Conjecture only as an example. Beyond the Kummer-Vandiver Conjecture, there is still much to be gleaned from understanding the minimal levels in other settings. The development given herein is generally applicable to arbitrary $\mathbb{Z}_p$-extension for which the Main Conjecture has been proven. I would not be surprised to find non-trivial minimal levels among arbitrary $\mathbb{Z}_p$-extensions. What would it mean were a minimal level non-trivial? Having detailed the algebraic aspects of this search, it remains only to understand how these concepts translate to Iwasawa Theory in general; that is, what are the underlying Galois-Theoretic properties of a $\mathbb{Z}_p$-extension that would produce a situation where the lower levels behaved fundamentally differently than the higher levels?

Moreover, most of the results here are stated in terms of biconditionals; that is, if we are to understand at what level the limiting Iwasawa behaviors begin, then we *must* grapple with these issues, whether sooner or later. Having developed them in an abstracted algebra theoretical setting, provides us with an understanding of the minimal properties that need to be understood or proven. Moreover, this development would have been quite cumbersome had we chosen to do so in the thick of Iwasawa Theory. Rather, this Iwasawa-Theory-inspired algebraic development seems to be a far cleaner and more readily comprehensible setting.

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