On Tiling Directed Graphs with Cycles and Tournaments

by

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ABSTRACT

A tiling is a collection of vertex disjoint subgraphs called tiles. If the tiles are all isomorphic to a graph H then the tiling is an H-tiling. If a graph G has an H-tiling which covers all of the vertices of G then the H-tiling is a perfect H-tiling or an H-factor. A goal of this study is to extend theorems on sufficient minimum degree conditions for perfect tilings in graphs to directed graphs.

Corrádi and Hajnal proved that every graph G on $3k$ vertices with minimum degree $\delta(G) \geq 2k$ has a K_3-factor, where K_s is the complete graph on s vertices. The following theorem extends this result to directed graphs: If D is a directed graph on $3k$ vertices with minimum total degree $\delta(D) \geq 4k-1$ then D can be partitioned into k parts each of size 3 so that all of parts contain a transitive triangle and $k-1$ of the parts also contain a cyclic triangle. The total degree of a vertex v is the sum of $d^-(v)$ the in-degree and $d^+(v)$ the out-degree of v. Note that both orientations of C_3 are considered: the transitive triangle and the cyclic triangle. The theorem is best possible in that there are digraphs that meet the minimum degree requirement but have no cyclic triangle factor. The possibility of added a connectivity requirement to ensure a cycle triangle factor is also explored.

Hajnal and Szemerédi proved that if G is a graph on sk vertices and $\delta(G) \geq (s-1)k$ then G contains a K_s-factor. As a possible extension of this celebrated theorem to directed graphs it is proved that if D is a directed graph on sk vertices with $\delta(D) \geq 2(s-1)k-1$ then D contains k disjoint transitive tournaments on s vertices. We also discuss tiling directed graph with other tournaments.

This study also explores minimum total degree conditions for perfect directed cycle tilings and sufficient semi-degree conditions for a directed graph to contain an anti-directed Hamilton cycle. The semi-degree of a vertex v is $\min\{d^+(v), d^-(v)\}$ and an anti-directed Hamilton cycle is a spanning cycle in which no pair of consecutive edges form a directed path.

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Chapter 1

HISTORY AND SUMMARY OF RESULTS

1.1 Preliminary definitions and notation

We start with a very brief introduction to some of the fundamental definitions and concepts from graph theory.

For any $k \in \mathbb{Z}^+$, call a set of cardinality k a k-set. For any set V, let

$$\binom{V}{k} := \{U \subseteq V : |U| = k\}.$$

Call the ordered pair $G = (V, E)$ a *graph* if $E \subseteq \binom{V}{2}$ and define $V(G) := V$ to be the *vertices* and $E(G) := E$ to be the *edges* of G respectively. Let $|G| := |V(G)|$ be the *order* of the graph G and set $\|G\| := |E(G)|$. We normally denote $\{x, y\} \in E(G)$ by xy for convenience. The union of the graphs $G_1, \ldots, G_d$ is

$$(V(G_1) \cup \cdots \cup V(G_d), E(G_1) \cup \cdots \cup E(G_d)).$$

Two graph G_1 and G_2 are *vertex disjoint* or just *disjoint* if $V(G_1)$ and $V(G_2)$ are disjoint and *edge disjoint* if $E(G_1)$ and $E(G_2)$ are disjoint. The edges e, f are *disjoint* or *independent* if $e \cap f = \emptyset$.

We say that two vertices $x, y \in G$ are adjacent if $xy \in E(G)$ and we say that the edge $e \in E(G)$ is *incident* to $x \in V(G)$ if $x \in e$. Let $N_G(x)$ be the set of vertices adjacent to x in G or the *neighbors* of x. For any $v \in V(G)$, let $d(v)$ denote the number of edges incident to v or the *degree* of v, let $\delta(G) := \min\{d(v) : v \in V(G)\}$ be the *minimum degree of* G and let $\Delta(G) := \max\{d(v) : v \in V(G)\}$ by the *maximum degree of* G. Let K_s be the *complete graph on* s *vertices*, the graph on s vertices in which every pair of vertices is adjacent. The *path on* s *vertices*, P_s, is the graph on s

vertices in which the vertices can be ordered $v_1, \ldots, v_s$ so that v_iv_{i+1} for $1 \leq i \leq s-1$ are the edges of P_s. We say that v_1 and v_s are the *ends* of the path. If P is a path with ends x and y, we say that P is an x,y-path and that P *joins* x and y. A *cycle on* s *vertices*, C_s, is an x, y-path on s vertices with the additional edge xy. We normally denote paths and cycles by just listing the vertices (without commas) in their natural order. A cycle which is a spanning subgraph is called a *Hamilton* or *hamiltonian* cycle. We sometimes call $C_3 = K_3$ a *triangle*.

We say that H is a *subgraph* of G and write $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is a *spanning subgraph* if $V(H) = V(G)$. For any $U \subseteq V(G)$, $G[U] = (U, \binom{U}{2} \cap E(G))$ is the graph *induced* by U. We say that $U \subseteq V(G)$ is an *independent set* if $G[U]$ has no edges. If a graph G consists of a single vertex or there is a partition $\{A, B\}$ of $V(G)$ such that both A and B are independent sets we say that the G is a *bipartite graph* (or *bigraph*) and call G an A, B*-bipartite graph*. Define $\overline{U} := V(G) \setminus U$ to be the complement of U and define $G - U$ to be $G[\overline{U}]$. The complement $\overline{G}$ of a graph G is the graph on $V(G)$ where $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$, that is $E(\overline{G}) = \binom{V(G)}{2} \setminus E(G)$. Two graphs G and H are isomorphic if there is a bijection $f : V(G) \to V(H)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. We will say G contains H if there is a subgraph of G which is isomorphic to H.

If there is an x, y-path in G we say that x and y are *connected*. If G is a graph we can define a relation $\sim$ on $V(G)$ by $x \sim y$ if and only x and y are connected. This relation is an equivalence relation, and the components of G are the graphs induced by the equivalence classes of this relation. A *trivial component* is component of order 1. We say that a graph is connected if it has one component.

A *multigraph* is similar to a graph except that there can be multiple edges between two vertices. That is, a multigraph is an ordered pair $M = (V, E)$ where E is a multiset of $\binom{V}{2}$. For a multigraph M and edge $e \in E(M)$, let $\mu_M(e)$ be the multiplicity of e. If $e \notin E(M)$ then we say that $\mu_M(e) = 0$. Most of the notation and terminology for multigraphs translates directly to graphs, but note that the degree of $v \in V(M)$ is

$$d_M(v) := \sum_{u \in V - v} \mu_M(uv).$$

Let the *multiplicity of* M be $\mu(M) := \max_{e \in \binom{V(M)}{2}} \{\mu_M(e)\}$. Set $\|M\| := \sum_{e \in \binom{V(M)}{2}} \mu_M(e)$.

A *directed graph* or *digraph* is an order pair $D = (V, E)$ where $E \subseteq V^2$. For $(u, v), (v, u) \in E$, we write uv and vu respectively and say that uv is an edge *oriented* from u to v. A *loop* is an edge of the form (v, v). We only consider *simple* digraphs, those having no loops. Much of the notation and terminology for graphs translates directly to digraphs. For example, the definition of the *directed path on s vertices* $\overrightarrow{P}_s$ and the *directed cycle on s vertices* $\overrightarrow{C}_s$ is completely analogous to the related definitions for graphs. Note that in digraphs $\overrightarrow{C}_2$ make sense, we call such cycles 2-*cycles*. An edge is *heavy* if it is contained in a 2-cycle and we call all other edges *light*.

The *in-neighborhood* and *out-neighborhood* of $v \in V(D)$ are $N^-_D(v) := \{u \in V(G) : (v, u) \in E(D)\}$ and $N^+_D(v) := \{u \in V(G) : (u, v) \in E(D)\}$. The *in-degree* and *out-degrees* of $v \in V(D)$ are of a vertex v are $d^-_D(v) := |N^-_D(v)|$ and $d^+_D(v) := |N^+_D(v)|$; the *total degree* of v is the sum $d_D(v) := d^-_D(v) + d^+_D(v)$. The *minimum semi-degree* of G is $\delta^0(G) := \min\{\min\{d^+_D(v), d^-_D(v)\} : v \in V\}$ and the *maximum semi-degree* of G is $\Delta^0(G) := \max\{\max\{d^+_D(v), d^-_D(v)\} : v \in V\}$. The *minimum total degree* of G is $\delta(G) := \min\{d_D(v) : v \in V\}$ and the *maximum total degree* of G is $\Delta(G) := \max\{d_D(v) : v \in V\}$.

A *tournament* is a directed graph in which there is exactly one edge between every pair of vertices. A tournament is *transitive* if it contains no directed cycles. Among digraphs on p vertices, let $\overrightarrow{T}_p$ is the transitive tournament on p vertices and let $\overrightarrow{K}_p$ be the complete digraph, that is, the digraph with all possible edges in both directions.

1.2 Tilings and perfect matchings

One way to study graphs is to analyze the subgraphs they contain. We will focus on the following special type of spanning subgraph. All of the following definitions are essentially the same for digraphs and multigraphs.

Let G be a graph. A subgraph of G is a *tiling* if it is the union of vertex disjoint subgraphs called *tiles*. If a tiling is a spanning subgraph it is called a *perfect tiling* or a *factor*. We will tend to use the term factor instead of perfect tiling in this introduction. If each tile is isomorphic to a graph H, then the tiling is called an *H-tiling* or an *H-factor* when it is a spanning subgraph. We call an H-tilling of G an *ideal H-tiling* if it has $\left\lfloor \frac{|G|}{|H|} \right\rfloor$ tiles, so an ideal H-tiling is a H-factor when $|H|$ divides $|G|$.

The following two simple examples are a good place to start.

Example 1.2.1. For any $n, m \in \mathbb{Z}^+$ with $n \geq m$, let $G := G_I(n, m)$ be the graph on n vertices that contains an independent set A of size m and, subject to this, all possible edges. Note that $\delta(G) = n - m$.

Let $s \geq 2$ and let H be graph of order s. Suppose $n = ks$ and $m = k + 1$ for some $k \in \mathbb{Z}^+$. If an H-factor of G exists it contains k tiles, and one of these tiles must intersect A in at least 2 vertices. In particular, G has no K_s-factor. We also have that

$$\delta(G) = \left(\frac{s-1}{s}\right) n - 1.$$

Example 1.2.2. For any $n \in \mathbb{Z}^+$, let $G := G_S(n)$ be the disjoint union of the graphs $K_{\lfloor n/2 \rfloor}$ and $K_{\lceil n/2 \rceil}$. Note that when n is even $\delta(G) = (n-2)/2$ and when n is odd $\delta(G) = (n-3)/2$.

Let H be a connected graph and let $s := |H|$. There is an H-factor of G if and only if s divides $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. In particular, if $s \geq 2$ and s divides n then G has an H-factor if and only if $n = 0 \pmod{2s}$.

One of the simplest and most studied type of H-factor is the K_2-factor or *perfect matching*. All of the theorems that will be presented here are in some way related to the following proposition. It is a corollary to Proposition 2.1.1, the proof of which is quite short.

Proposition 1.2.3. *If G is a graph on n vertices and $\delta(G) \geq \frac{n}{2}$ then G has an ideal K_2-tiling.*

Note that Example 1.2.1 and Example 1.2.2 both show that the degree condition in Proposition 1.2.3 is tight.

For any graph H define $o(H)$ to the number of components of H that have odd order. It is not hard to see that in order for G to have a K_2-factor it must be that $o(G - S) \leq |S|$ for every $S \subseteq V(G)$. Tutte showed that this is also sufficient.

Theorem 1.2.4 (Tutte 1947 [35])**.** *Every graph G has a K_2-factor if and only if*

$$o(G - S) \leq |S| \text{ for every } S \subseteq V(G) \ .$$

1.3 Perfect tilings with tiles of order 3

When H is connected and $|H| \geq 3$ the problem becomes more complicated, and simple characterizations like Theorem 1.2.4 seem unlikely. Furthermore, in this case, deciding if a graph has an H-factor is NP-complete [27]. We therefore turn to sufficient minimum degree conditions like Proposition 1.2.3. The following two theorem deal with the cases when H is of order exactly 3.

Corollary 1.3.1 (Corrádi & Hajnal 1963 [5])**.** *If G is a graph on n vertices and $\delta(G) \geq \frac{2n}{3}$ then G has an ideal K_3-tiling.*

Theorem 1.3.2 (Enomoto, Kaneko & Tuza 1987 [14]). *If G is a connected graph on n vertices and $\delta(G) \geq \frac{n}{3}$ then G has an ideal P_3-tiling.*

Both theorems are tight. Indeed, $G_I(3k, k+1)$ does not have a K_3-factor and $G_I(3k, 2k+1)$ does not have a P_3-factor. Furthermore, when $n = 3 \pmod 6$, $G_S(n)$ does not contain a P_3 factor, so the connectively condition in Theorem 1.3.2 can only be dropped if we force $\delta(G) \geq n/2$.

We now consider this same type of problem when G and H are directed graphs. We will first explore the problem when $|H| = 3$ and $G(H) = K_3$. We call such graphs *digraph triangles.* As in the undirected case, we begin with two important classes of digraphs. The first is a directed analogue to Example 1.2.1.

Example 1.3.3. Define $D = \overrightarrow{G_I}(n, m)$ to be the directed graph formed from $G_I(n, m)$ by replacing all edges with 2-cycles. Clearly $\delta(D) \geq 2(n - m)$ and if $n = ks$ and $m = k + 1$, $\delta(D) \geq \frac{2k}{k+1} - 2$.

We say a directed graph D is *strongly d-connected* if for any $(d-1)$-set U of $V(D)$ and any two vertices $x, y \in V(D - U)$ there is a directed path from x to y and a directed path from y to x in $G - U$. If D is strongly 1-connected we say that D is *strongly connected* or just *strong.*

Example 1.3.4. For any $n \in \mathbb{Z}^+$, let $D = \overrightarrow{G_S}(n)$ be the disjoint union of the graphs $A_1 = \overrightarrow{K}_{\lfloor n/2 \rfloor}$ and $A_2 = \overrightarrow{K}_{\lceil n/2 \rceil}$ and all possible edges from A_1 to A_2. Note that when n is even $\delta(D) = (3n-4)/2$ and when n is odd $\delta(D) = (3n-5)/2$.

Let H be a strongly connected digraph and let $s := |H|$. Every copy of H must be contained in $D[A_1]$ or $D[A_2]$ since there are no edges directed from A_2 to A_1. Therefore, there is an H-factor of D if and only if s divides $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. In particular, if $s \geq 2$ and s divides n then D has an H-factor if and only if $n = 0 \pmod{2s}$.

As with graphs, we will look for sufficient minimum total degree conditions for H-factors and ideal H-tilings. The following theorem of Wang is one of the first theorems of this type and a starting point for our investigations. Note that Example 1.3.4 shows that the degree condition is tight. It is a corollary to Theorem 1.3.12, which has a short proof that is presented later in this document.

Theorem 1.3.5 (Wang 2000 [36])**.** *If G is a directed graph on n vertices and $\delta(G) \geq \frac{3n-3}{2}$ then G has an ideal $\overrightarrow{C}_3$-tiling.*

We also prove the following theorem which gives a fairly complete picture of the case when $H \in \{\overrightarrow{C}_3, \overrightarrow{T}_3\}$ and no connectivity condition is imposed.

Theorem 1.3.6 (Czygrinow, Kierstead & Molla 2012 [7])**.** *If G is a digraph on n vertices and $\delta(G) \geq \frac{4}{3}n - 1$, and $c \geq 0$ and $t \geq 1$ are integers with $3(c+t) \leq n$. Then G has a tiling in which c tiles are isomorphic to $\overrightarrow{C}_3$ and t tiles are isomorphic to $\overrightarrow{T}_3$.*

It has the obvious corollary.

Corollary 1.3.7 (Czygrinow, Kierstead & Molla 2012 [7])**.** *If G is a digraph on n vertices and $\delta(G) \geq \frac{4n}{3} - 1$ then G has an ideal $\overrightarrow{T}_3$-tiling.*

For any $k \in \mathbb{Z}^+$, $\overrightarrow{G_I}(3k, k+1)$ shows that both Theorem 1.3.6 and Theorem 1.3.7 are tight. Theorem 1.3.6 gives us a tiling that is almost a $\overrightarrow{C}_3$-factor, but Example 1.3.4 forces us to raise the minimum total degree condition significantly to ensure the digraph has $\overrightarrow{C}_3$-factor. This suggests that, in analogy with Theorem 1.3.2, we might be able to lower the degree condition by imposing a connectivity condition. The first thought might be to require the digraph to be strongly connected, the following example shows that this is not sufficient.

Example 1.3.8. For any $n \in \mathbb{Z}^+$, let $D := \overrightarrow{G_{S'}}(n)$ be the disjoint union of the graphs $A_1 = \overrightarrow{K}_{\lfloor n/2 \rfloor - 1}$ and $A_2 = \overrightarrow{K}_{\lceil n/2 \rceil}$ and another vertex v. D contains all possible edges from A_1' to A_2 all possible edges from A_2 to v and all possible edges between v and A_1'. Let $A_1 := A_1' + v$. We have that $\delta(D) = \delta(\overrightarrow{G_S}(n))$. Note that D is strongly connected and every $\overrightarrow{C}_3$ in D is either contained in $D[A_1]$ or contained in $D[A_2]$ or contains v and a vertex from A_1'. When $n = 3 \pmod 6$, $|A_1| = 1 \pmod 3$ and $|A_2| = 2 \pmod 3$, so D has no $\overrightarrow{C}_3$-factor.

Therefore, the following conjecture would be best possible.

Conjecture 1.3.9 (Czygrinow, Kierstead & Molla 2013 [6]). *If D is a strongly 2-connected digraph on n vertices such that $\delta(D) \geq \frac{4}{3}n - 1$ then D has an ideal $\overrightarrow{C}_3$-factor.*

We prove Conjecture 1.3.9 is asymptotically true, that is, we prove the following weaker theorem. This and all of the asymptotic results presented in this document use the probabilistic absorbing method of Rödl, Ruciński, and Szemerédi [34, 33, 31].

Theorem 1.3.10 (Czygrinow, Kierstead & Molla 2013). *For any $\varepsilon > 0$ there exists n_0 such that if D is a directed graph on $n \geq n_0$ vertices, D is strongly 2-connected and $\delta(D) \geq \left(\frac{4}{3} + \varepsilon\right) n$ then D has an ideal $\overrightarrow{C}_3$-factor.*

There are other digraphs triangles we could consider. Call a digraph triangle with at least one heavy edge a 4-triangle (it contains 4-edges). Similarly, if a digraph triangle has 2 or 3 heavy edges we will call it a 5-triangle or 6-triangle respectively (they contain 5 or 6 edges respectively). Note that there are three different directed triangles that have exactly one heavy edge and all of them contain $\overrightarrow{T}_3$. Furthermore, one of these three 4-triangles does not contain $\overrightarrow{C}_3$. Clearly a 5-triangle contains both $\overrightarrow{T}_3$ and $\overrightarrow{C}_3$, so a 5-triangle factor, contains both a $\overrightarrow{T}_3$-factor and a $\overrightarrow{C}_3$-factor. It is also clear that if we are looking for a 5-triangle-factor the orientation of the edges is no longer important. Therefore, the question is no longer a digraph problem, but a multigraph problem.

Removing the orientation from the edges of a directed graph D leaves a loopless multigraph M such that every edge has multiplicity at most 2. Call such a multigraph *standard*, and say that $M(D)$ is the multigraph underlying D. For a fixed standard multigraph M, let $G(M)$,$H(M)$ and $L(M)$ be the simple graphs on $V(M)$ containing the edges of M with multiplicity at most 1, exactly 2 and exactly 1 respectively. If $M = M(D)$ then the edges of $H(M)$ and $L(M)$ arise from the heavy and light edges of D, respectively; we extend this terminology to standard multigraphs. If T is a K_3 in $G(M)$ then we call T a k-triangle if $\|T\| \geq k$ and say that T is a T_k. Clearly a k-triangle in $M(D)$ corresponds to a k-triangle in $M(D)$.

The following is our main theorem on triangles in standard multigraphs. It has Theorem 1.3.6 as a corollary. By considering the operation of transforming a graph G into a standard multigraph by giving every $e \in E(G)$ multiplicity 2, it is easy to see that the following theorem implies Theorem 1.3.1. We will call such an operation *doubling the edges of* G. We will also call the similar operation of creating a digraph from a graph by replacing the edges with 2-cycles doubling the edges.

Theorem 1.3.11 (Czygrinow, Kierstead & Molla 2012 [7]). *Every standard multigraph M with $\delta(M) \geq \frac{4n}{3} - 1$ has a tiling in which one tile is a 4-triangle and $\lfloor \frac{n}{3} \rfloor - 1$ tiles are 5-triangles.*

We also prove the following strengthening of Theorem 1.3.5.

Theorem 1.3.12 (Czygrinow, Kierstead & Molla 2012 [7]). *Every standard multigraph M on n vertices with $\delta(M) \geq \frac{3n-3}{2}$ has an ideal 5-triangle tiling.*

We conjecture that the following is true and prove an asymptotic version.

Conjecture 1.3.13 (Czygrinow, Kierstead & Molla 2013 [7]). *If M is a standard multigraph on n vertices, $H(M)$ is connected and $\delta(M) \geq \frac{4}{3}n - 1$ then M has an ideal 5-triangle-tiling.*

Theorem 1.3.14 (Czygrinow, Kierstead & Molla 2013). *For any $\varepsilon > 0$ there exists n_0 such that if M is a standard multigraph on $n \geq n_0$ vertices, $H(M)$ is connected and $\delta(M) \geq \left(\frac{4}{3} + \varepsilon\right) n$ then M has an ideal 5-triangle-tiling.*

Note that Conjecture 1.3.13 implies Theorem 1.3.1 and Theorem 1.3.2 (if we double the edge of a graph G and add a light edge between every two non-adjacent vertices a 5-triangle tiling corresponds to P_3 tiling in G). It also implies Theorem 1.3.5, because $H(M)$ is always connected when $\delta(M) \geq (3n-3)/2$. Conjecture 1.3.13 and Theorem 1.3.14 are closely related to Conjecture 1.3.9 and Theorem 1.3.10

1.4 Hajnal-Szemerédi for digraphs

The following generalization of Corollary 1.3.1 was conjectured by Erdős [15] in 1963, and proved seven years later:

Theorem 1.4.1 (Hajnal & Szemerédi 1970 [19]). *If G is a graph on $n = ks$ vertices and $\delta(G) \geq \frac{s-1}{s} n$ then G has a K_s-factor.*

A k-coloring f of G is a function from $V(G)$ to $[k]$. The color classes of f are the sets $\{f^{-1}(\{1\}), \ldots, f^{-1}(\{k\})\}$. A k-coloring f is proper if $f(x) \neq f(x)$ for every $xy \in E(G)$. Equivalently, f is a proper k-coloring if its color classes are independent sets. An *equitable* k-coloring of G is a proper k-coloring whose color classes form an equitable partition of $V(G)$. That is, the color class differ in size by at most 1. If $V(G) = ks$ then the color classes are each isomorphic to $\overline{K_s}$, the graph on s vertices with no edges. Since $|G| = \Delta(G) + \delta(\overline{G}) + 1$, Theorem 1.4.1 has the following complementary form, in which Hajnal and Szemerédi stated their proof of Erdős' conjecture.

Theorem 1.4.2 (Hajnal & Szemerédi 1970 [19]). *If G is a graph on n vertices and $\Delta(G) \leq k - 1$ then G has an equitable k-coloring.*

Example 1.2.1 shows that the degree bounds in these theorems are tight. The original proof of Theorem 1.4.1 was quite involved, and only yielded an exponential time algorithm. Short proofs yielding polynomial time algorithms appear in [24, 25]; the following theorem provides a fast algorithm.

Theorem 1.4.3 (Kierstead, Kostochka, Mydlarz & Szemerédi 2010 [26])**.** *Every graph G on n vertices with $\Delta(G) \leq k-1$ can be equitably k-colored in $O(kn^2)$ steps.*

We wish to extend Theorem 1.4.1 to digraphs. For any digraph D, we define $\overline{D}$, the complement of D, to be the digraph on $V(D)$ where $xy \in E(\overline{D})$ if and only if $xy \notin E(D)$. Note that the complement of an s-tournament is another s-tournament and the complement of $\overrightarrow{T}_s$ is $\overrightarrow{T}_s$.

By doubling the edges of a graph, it is clear that the following theorem generalizes Theorem 1.4.2.

Theorem 1.4.4 (Czygrinow, Kierstead & Molla 2013 [6])**.** *Every digraph G with $|G| = n = sk$ and $\delta(G) \geq 2\frac{s-1}{s}n - 1$ has a $\overrightarrow{T}_s$-factor.*

Note that the case $s = 3$ is equivalent to Corollary 1.3.7. We prove Theorem 1.4.4 in its following stronger complementary form by extending ideas developed in [24, 23, 26, 25]. An *equitable acyclic coloring* of a digraph is a coloring whose classes induce acyclic subgraphs (subgraphs with no directed cycles, including 2-cycles), and differ in size by at most one.

Theorem 1.4.5 (Czygrinow, Kierstead & Molla 2013 [6])**.** *Every digraph G with $\Delta(G) \leq 2k-1$ has an equitable acyclic k-coloring.*

To see that Theorem 1.4.5 implies Theorem 1.4.4, consider a digraph G with $|G| = n = sk$ and $\delta(G) \geq 2\frac{s-1}{s}n - 1$. Its complement H satisfies $\Delta(H) \leq 2n - 2 - (2(1 - 1/s)n - 1) \leq 2k - 1$. By Theorem 1.4.5, H has an equitable acyclic k-coloring.

Since each color class is acyclic it can be embedded in a transitive s-tournament, whose complement is another transitive tournament contained in G. Thus the tiles in G induced by the color classes of H contain transitive s-tournaments.

In an effort to prove theorems more general than Theorem 1.4.4, we shift our attention from digraphs to standard multigraphs. As a step toward our eventual goal, we make the following conjectures which clearly implies Theorem 1.4.4 and Theorem 1.4.5. The standard multigraph M is acyclic if it contains no cycles, including 2-cycles. In other words, $G(M)$ is acyclic and $H(M)$ contains no edges. If $G(M)$ is a complete graph we call M a clique.

Conjecture 1.4.6 (Czygrinow, Kierstead & Molla 2013 [6]). *Every standard multigraph M with $\Delta(M) \leq 2k-1$ has an equitable acyclic k-coloring.*

We normally state Conjecture 1.4.6 in the following complimentary form. The *complement* $\overline{M}$ of a standard multigraph M is the standard multigraph on $V(G)$ where for any distinct $x, y \in V(G)$ the multiplicity of the edge xy in $\overline{M}$ is equal to $2-\mu_M(xy)$, The complement of an acyclic standard multigraph on s-vertices is called a *full s-clique.*

Conjecture 1.4.7 (Czygrinow, Kierstead & Molla 2013 [6]). *For every $s, k \in \mathbb{N}$, if M is a standard multigraph on sk vertices and $\delta(M) \geq 2(s-1)k-1$ then M contains k disjoint full s-cliques.*

Let K be an s-clique and let $\mathcal{D}$ be the set of all simple digraphs D such that $K = M(D)$ (equivalently the set of all simple digraphs obtained by orienting the edges of K); we say K is *universal* if for all $D \in \mathcal{D}$, D contains every tournament on s vertices. For example, the 5-triangle is universal: It contains both $\overrightarrow{C}_3$ and $\overrightarrow{T}_3$. Our goal is to factor standard multigraphs into universal tiles.

Note that K is universal if and only if for every tournament T on s vertices and every orientation D of $L(K)$ there is an embedding of D into T (after embedding D into T, every other edge of T corresponds to a heavy edge of K). The following Theorem of Havet and Thomassé and famous conjecture of Sumner, which has been proved for large values of n [30], allow us to concisely say which cliques are universal. (This definition is not important at this point, but an *anti-directed path* is a path in which no pair of consecutive edges from a directed path. We will have more to say about anti-directed paths and cycles later in this document.)

Theorem 1.4.8 (Havet & Thomassé 2000 [20])**.** *Every tournament T on n vertices contains every oriented path P on n vertices except when P is an anti-directed path and $n \in \{3, 5, 7\}$.*

Conjecture 1.4.9 (Sumner 1971)**.** *Every orientation of every tree on n vertices is a subgraph of every tournament on $2n - 2$ vertices.*

With Theorem 1.4.8, we can state Conjecture 1.4.9 in a form that is more useful for our goal.

Conjecture 1.4.10. *Let T be a tournament on n vertices and F be a forest on at most n vertices with c non-trivial components. If F has at most $n/2 + c - 1$ edges then T contains every orientation of F.*

Proposition 1.4.11. *Theorem 1.4.8 and Conjecture 1.4.9 imply Conjecture 1.4.10.*

In light of this, we make the following definition: a full s-clique K is *acceptable* if the forest $L(K)$ has c non-trivial components and at most $s/2 + c - 1$ edges.

If Sumner's conjecture is true then, with Proposition 1.4.11, acceptable s-cliques are universal s-cliques. We make the following conjectures.

Conjecture 1.4.12. *For every $s \geq 4$ and $k \in \mathbb{N}$, if M is a standard multigraph on $n = sk$ vertices with $\delta(M) \geq 2\frac{s-1}{s}n - 1$ then M can be tiled with k disjoint acceptable s-cliques.*

Note that the case where $s = 3$ is covered by Conjecture 1.3.14, because 5-triangles are acceptable.

We support Conjecture 1.4.12 with the following two related theorems. Theorem 1.4.14 proves that Conjecture 1.4.12 is asymptotically true.

Theorem 1.4.13. *For any $s \geq 4$ and any standard multigraph M on on n vertices with $\delta(M) \geq 2\frac{s-1}{s}n - 1$, there exists a disjoint collection of acceptable s-cliques that tile all but at most $s(s-1)(2s-1)/3$ vertices of M.*

Theorem 1.4.14. *For all $s \geq 4$ and $\varepsilon > 0$ there exists n_0 such that if M is a standard multigraph on $n \geq n_0$ vertices, where n is divisible by s, then the following holds. If $\delta(M) \geq 2\frac{s-1}{s}n + \varepsilon n$ then there exists a perfect tiling of M with acceptable s-cliques.*

With Proposition 1.4.11 and the fact that Conjecture 1.4.9 is true for large trees [30], we have the following corollary to Theorem 1.4.13.

Corollary 1.4.15. *There exists s_0 such that for any $s \geq s_0$ and any $\varepsilon > 0$ there exists n_0 such that if D is a directed graph on $n \geq n_0$ vertices, where n is divisible by s, the following holds. If $\delta(D) \geq 2\frac{s-1}{s}n + \varepsilon n$, then D can be partitioned into tiles of order s such that each tile contains every tournament on s vertices.*

If we combine Theorem 1.3.6, Conjectures 1.3.9, 1.4.9 and 1.4.12 with Proposition 1.4.11 we have the following conjecture.

Conjecture 1.4.16. *For any $s, k \in \mathbb{N}$, if D is a strongly 2-connected digraph on sk vertices and $\delta(D) \geq 2(s-1)k - 1$ then D contains any combination of k disjoint tournaments on s vertices.*

1.5 Tiling directed graphs with cycles

Call a graph r-*regular* if $d(v) = r$ for every vertex v and a call a digraph r-*regular* $d^+(v) = d^-(v) = r$ for every vertex v. Note that a 2-regular graph consists entirely of disjoint cycles and 1-regular digraphs consists entirely of disjoint directed cycles.

The following is another well known extension of Corollary 1.3.1.

Theorem 1.5.1 (Aigner & Brandt 1994 [2]). *If G is a graph on n vertices and $\delta(G) \geq \frac{2n-1}{3}$ then G contains any 2-regular subgraph of order at most n.*

The following conjecture of El-Zahar, which has been proved for large n by Abbasi [1], suggests that the degree condition of Theorem 1.5.1 can be relaxed depending on the type of 2-regular subgraph desired.

Conjecture 1.5.2 (El-Zahar 1984 [12]). *Let G be a graph on n vertices and $n_1, \dots, n_d$ be integers greater than 2 such that $n = \sum_{i=1}^{d} n_i$. If $\delta(G) \geq \sum_{i=1}^{d} \left\lceil \frac{n_i}{2} \right\rceil$ then G contains d vertex independent cycles $C_1, \dots, C_d$ such that $|C_i| = n_i$ for every $i \in [d]$.*

In addition to proving Theorem 1.3.5 in [36], Wang also made the following conjecture, which can be seen as a analogue of Theorem 1.5.1 for digraphs.

Conjecture 1.5.3 (Wang 2000 [36]). *If D is a digraph on n vertices and $\delta(D) \geq \frac{3n-3}{2}$ then D contains any 1-regular subdigraph of order at most n.*

Towards proving this conjecture we have proved the following theorem.

Theorem 1.5.4 (Czygrinow, Kierstead & Molla 2013 [8]). *For any odd $k \geq 5$ there exists n_0 such that the following holds. If D is a digraph on $n \geq n_0$ vertices, n is divisible by k and $\delta(D) \geq \frac{3n-3}{2}$ then D contains a $\overrightarrow{C}_k$-factor.*

To prove Theorem 1.5.4, we transform the problem into a multigraph problem and actually prove more. For any $k \geq 3$, we say that the standard multigraph C is a heavy k-cycle if $G(C)$ is a cycle on k vertices and $\|M\| \geq 2k-1$. We say C is a heavy 2-cycle if C consists of two vertices and a heavy edge between them. We actually prove the following theorem.

Theorem 1.5.5 (Czygrinow, Kierstead & Molla 2013 [8]). *For any odd $k \geq 5$ there exists n_0 such that the following holds. If M is a standard multigraph on $n \geq n_0$ vertices, n is divisible by k and $\delta(M) \geq \frac{3n-3}{2}$ then M contains a heavy k-cycle-factor.*

This proof uses the Many-Color Regularity Lemma [29] and the Blow-Up Lemma [28] with the stability approach.

Note that when $\delta(M) \geq \frac{3n-3}{2}$,

$$\delta(H(M)) \geq \delta(M) - n - 1 \geq \frac{n-1}{2},$$

so if we replace an odd $k \geq 5$ with an even $k \geq 4$ in the statement of Theorem 1.5.5, Conjecture 1.5.2 gives us that $H(M)$ has a k-cycle factor, and hence M has a heavy k-cycle factor. Therefore, since Conjecture 1.5.2 has been proved for large n, with Proposition 1.2.3, Theorem 1.3.12 and Theorem 1.5.5 we have the following theorem.

Theorem 1.5.6 (Czygrinow, Kierstead & Molla 2013 [8]). *For any $k \geq 2$ there exists n_0 such that the following holds. If M is a standard multigraph on $n \geq n_0$ vertices, n is divisible by k and $\delta(M) \geq \frac{3n-3}{2}$ then M contains a heavy k-cycle-factor.*

When we convert this back into the language of digraphs we get the following.

Corollary 1.5.7 (Czygrinow, Kierstead & Molla 2013 [8]). *For any $k \geq 2$ there exists n_0 such that the following holds. If D is a digraph on $n \geq n_0$ vertices, n is divisible by k and $\delta(D) \geq \frac{3n-3}{2}$ then D can be partitioned into tiles of order k such that each tile contains every orientation of a cycle on k vertices.*

By Example 1.3.4, these results are tight, but we think it might be interesting to explore the possibility that, as with Conjecture 1.3.9, we can lower the degree condition in Conjecture 1.5.3 by requiring the directed graph to be strongly 2-connected. We could possibly lower the minimum degree condition to $\frac{4}{3}n - 1$ and prove an extension of Theorem 1.5.1 to digraphs.

1.6 Orientations of Hamilton cycles in digraphs

Closely related to tiling problems are the Hamilton cycle problems. The following is a fundamental result in graph theory

Theorem 1.6.1 (Dirac 1952 [10]). *If G is a graph on $n \geq 3$ vertices and $\delta(G) \geq \frac{n}{2}$ then G contains a Hamilton cycle.*

Dirac's Theorem has the following analogue for directed graphs.

Theorem 1.6.2 (Ghouila-Houri 1960 [17]). *If G is a directed graph on n vertices and $\delta_0(G) \geq \frac{n}{2}$ then G has a directed Hamilton cycle, that is $\overrightarrow{C}_n \subseteq G$.*

An *anti-directed cycle* is a cycle in which no two consecutive edges form a directed path. It is not hard to see that all anti-directed cycles are on an even number of vertices. In 1983, Cai showed that for any $n \in \mathbb{Z}^+$ there exists a directed graph G on $2n$ vertices with $\delta_0(G) = n$ that does not contain an anti-directed Hamilton cycle [3]. Therefore, a proof of the following conjecture would be a tight result.

Conjecture 1.6.3 (Diwan, Frye, Plantholt & Tipnis 2011 [11]). *Let G be a directed graph on $2n$ vertices. If $\delta_0(G) \geq n + 1$ then G has an anti-directed Hamilton cycle.*

We will present a proof of this conjecture for large graphs.

Theorem 1.6.4 (DeBiasio & Molla 2013 [9]). *There exists n_0 such that if D is a directed graph on $2n \geq n_0$ vertices and $\delta_0(D) \geq n + 1$ then D has an anti-directed Hamilton cycle.*

This proof uses the stability method and the probabilistic absorbing method (as opposed to the regularity/blow-up method).

The following theorem shows that if the minimum semi-degree condition of Theorem 1.6.4 is increased slightly, we can find all possible orientations of a Hamilton cycle.

Theorem 1.6.5 (Häggkvist & Thomason 1995 [18]). *There exists n_0 such that if G is a digraph on $n \geq n_0$ vertices and $\delta_0(G) \geq \frac{n}{2} + n^{5/6}$ then G contains every orientation of a Hamilton cycle.*

The next question may be to determine if if there exist some constant C such that every digraph G on n vertices with $\delta_0(G) \geq \frac{n}{2} + C$ contains every orientation of a Hamilton cycle.

1.7 Additional Notation

For a digraph D we set

$$E_D^+(X,Y) = E_D^-(Y,X) = \{xy \in E : x \in X \wedge y \in Y\},$$

and $E_D(X,Y) = E_D^+(X,Y) \cup E_D^-(X,Y)$. Set $\|X,Y\|_D = |E_D(X,Y)|$, $\vec{e}(X,Y) = \|X,Y\|_D^+ = |E_D^+(X,Y)|$ and $\|X,Y\|_D^- = |E^-(X,Y)|_D$. Let $\|X,Y\|_D^h$ denote the number of 2-cycles contained in $E_D(X,Y)$. Then $2\|X,Y\|_D^h$ is the number of heavy edges in $E_D(X,Y)$. Let $\|X,Y\|_D^l$ denote the number of light edges in $E_D(X,Y)$. For a multigraph M the definitions are similar. For any $U, W \subseteq V(M)$ set $E_M(U,W) = \{xy \in E(M) : x \in U \wedge y \in W\}$. We let

$$\|v,U\|_M := \deg_M(v,U) := \sum_{e \in E(x,U)} \mu_M(e),$$

and $\|U,W\|_M := \sum_{u \in U} \deg(u,W)$. By viewing graphs as multigraphs with multiplicity at most 1, we use the same definitions as above for graphs. We shorten $E_D(\{x\},Y)$

to $E_D(x, Y)$ and $E_D(X, V)$ to $E_D(X)$, etc. We also let $\deg^+_D(x, Y) := |E^+_D(x, Y)|$ and $\deg^-_D(x, Y) := |E^-_D(x, Y)|$.

For a d-tuple $T := (v_1, \ldots, v_d) \in V^d$, let $\mathrm{im}(T) := \{v_1, \ldots, v_d\}$ denote the image of T. For numbers x, y and c we say that $x = y \pm c$ if $|x - y| \leq c$.

Let D be a digraph D and $A, B \subseteq V(D)$. Let $\{y' : y \in B\}$ be a set of new vertices and $D[A, B]$ be the bipartite graph on $A \cup B'$ defined by $xy' \in E(D[A, B])$ if and only if $xy \in E(D)$, $x \in A$ and $y \in B$. We normally just identify B' with B if no confusion can occur. If G is a graph and $A, B \subseteq V(G)$ we define the bigraph $G[A, B]$ similarly.

Let G be a graph. For any subsets U and W of $V(G)$ let

$$d_G(U, W) := \frac{\|U, W\|_G}{|U||W|}.$$

We will call $d_G(U, W)$ the *density* of the pair (U, W) and we will use the same definition for multigraphs. If $u \in V$, we will let

$$d_G(u, W) := d_G(\{u\}, W) = \deg(u, W)/|W|$$

For a digraph D, we define

$$d^+_D(U, W) := \frac{\|U, W\|^+_G}{|U||W|}, d^-_D(U, W) := \frac{\|U, W\|^-_G}{|U||W|} \text{ and } d_D(U, W) := \frac{\|U, W\|_G}{|U||W|}$$

For all of the preceding definitions we will often drop the subscript if the relevant graph,digraph or multigraph is clear from context.

We will often use the following simple fact, which is obvious from the definition, without explicit mention.

Proposition 1.7.1. *Let G is a graph and A and B be non empty vertex subsets. If $\{A_1, \ldots, A_p\}$ and $\{B_1, \ldots, B_q\}$ are partitions of A and B respectively then*

$$d(A, B) = \sum_{i \in [p], j \in [q]} d(A_i, B_j) \frac{|A_i||B_j|}{|A||B|}.$$

The following corollary and its contrapositive are useful.

Proposition 1.7.2. *Let G is a graph and A and B be non empty vertex subsets. If $\{A_1, \ldots, A_p\}$ and $\{B_1, \ldots, B_q\}$ are partitions of A and B respectively and*

$$d(A_i, B_j) \leq d \text{ for every } i \in [p] \text{ and } j \in [q]$$

then $d(A, B) \leq d$.

Proof. This follows by the previous proposition and the fact that

$$\sum_{i\in[p], j\in[q]} |A_i||B_j| = (|A_1| + \cdots + |A_p|)(|B_1| + \cdots + |B_q|) = |A||B|.$$

□

Chapter 2

GENERAL LEMMAS

2.1 Some simple general lemmas

Here we collect a few simple propositions that are used throughout the document. For any graph G, a matching is a collection of pairwise disjoint edges.

Proposition 2.1.1. *If G is a graph then there is a matching in G of order*

$$\min\{\delta(G), \lfloor |V(G)|/2 \rfloor\}.$$

Proof. Suppose M is a maximal matching in G. Let U be the vertices that are incident to an edge M. If $|M| < \min\{\delta(G), \lfloor |V(G)|/2 \rfloor\}$ then there exists distinct $x, y \in \overline{U}$. By the maximality of M, $\overline{U}$ is an independent set. Therefore, $\|\{x, y\}, U\| \geq 2\delta(G) > 2|M|$, and there exists $e \in M$ such that $\|\{x, y\}, e\| > 2$. So there are 2 disjoint edges in $G[\{x, y\} \cup e]$ and, hence, G contains a matching larger than M. □

Proposition 2.1.2. *If G is an X, Y-bipartite graph with $|X| \geq |Y|$ then there is a matching in G of order* $\min\{2\delta(G), |Y|\}$.

Proof. This proof is very similar to the proof of Proposition 2.1.1. Suppose M is a maximal matching in G. Let U be the vertices that are incident to an edge M. If $|M| < \min\{2\delta(G), |Y|\}$ then there exists $x \in X \cap \overline{U}$ and $y \in Y \cap \overline{U}$. By the maximality of M, $\overline{U}$ is an independent set. Therefore, $\|\{x, y\}, U\| \geq 2\delta(G) > |M|$ so there exists $e \in M$ such that $\|\{x, y\}, e\| > 1$. So, if $y' = e \cap Y$ and $x' = e \cap X$ then xy' and yx' are disjoint edges, so M contains a matching larger than M □

Proposition 2.1.3. *If G is a graph $V_1, V_2 \subseteq V(G)$ are disjoint and $d_G(V_1, V_2) \geq c$ then there exists a path on at least $c \cdot \min\{|V_1|, |V_2|\}$ vertices in $G[V_1 \cup V_2]$*

Proof. Let $s = \min\{|V_1|, |V_2|\}$. Initially set $V_i' = V_i$ for $i \in \{1,2\}$. If there exists $i \in \{1,2\}$ and $v \in V_i$ such that $\deg(v, V_{3-i}') \leq c/2|V_{3-i}|$, then reset $V_i' := V_i' - v$ and repeat this process. We must stop with a non-empty graph because we can only remove less than $c/2|V_1||V_2| + c/2|V_2||V_1| = c|V_1||V_2|$ edges. Now $\delta(G[V_1', V_2']) \geq c/2 \cdot s$ so we can greedily construct the desire path on cs vertices. □

2.2 Probabilistic lemmas

The following is a version of the Chernoff [4] bound for the binomial distributions and Hoeffding [21] bound for the hypergeometric distribution (see Section 2.1 in [22]),

Theorem 2.2.1. *If X is a random variable with binomial or hypergeometric distribution and $\mathbb{E}[X] = \gamma > 0$ then*

$$\Pr(X \geq \gamma + t) \leq \exp\left(-\frac{t^2}{2(\gamma + t/3)}\right) \quad \text{and}$$
$$\Pr(X \leq \gamma - t) \leq \exp\left(-\frac{t^2}{2\gamma}\right).$$

The following is just a convenient simplification of the preceding result:

Corollary 2.2.2. *For any $1 \geq p \geq 0$ and $1 \geq \varepsilon > 0$. and random variable X with binomial or hypergeometric distribution and $\mathbb{E}[X] = pn$*

$$\Pr(X \geq (p+\varepsilon)n), \Pr(X \leq (p-\varepsilon)n) < e^{-\varepsilon^2 n/3} \quad \text{and}$$
$$\Pr(|X - pn| \geq \varepsilon n) < 2e^{-\varepsilon^2 n/3}.$$

Proof. If $p = 0$ the statement is trivial, so assume $p \geq 0$. Since $2(p + \varepsilon/3) \leq 8/3 < 3$, Theorem 2.2.1 gives us that

$$\Pr(X \geq (p+\varepsilon)n), \Pr(X \leq (p-\varepsilon)n) \leq \exp\left(-\frac{\varepsilon^2 n^2}{2(p+\varepsilon/3)n}\right) < e^{\varepsilon^2 n/3}.$$

□

Lemma 2.2.3. *Let $1 \geq \varepsilon > 0$, V be an n-set and $\mathcal{V}$ a collection of subsets of V. If U is selected uniformly at random from all m-subset of V then and $m \leq n/2$ then the probability that*

$$|S \cap U|/m, |S \cap \overline{U}|/(n-m) = |S|/n \pm \varepsilon$$

for all $S \in \mathcal{S}$ is at most $|\mathcal{V}|2e^{-\varepsilon^2 m/3}$

Proof. Pick U uniformly at random from all m sets of V. Note that since $|U| \leq |\overline{U}|$, if the inequality holds for U it will also hold for $|\overline{U}|$. For any $S \in \mathcal{V}$, $|S \cap U|$ is a random variable with hypergeometric distribution and $E[|S \cap U|] = \frac{|S|}{n} \cdot m$. By Corollary 2.2.2,

$$\Pr\left(\left||S \cap U| - |S|\frac{m}{n}\right| \geq \varepsilon m\right) \leq 2e^{\varepsilon^2 n/3}.$$

The result then follows from an application of the union bound. □

Lemma 2.2.4. *Let $1 \geq \xi, \gamma > 0$ let G be a graph on n vertices and let $n_1 + n_2 = n$ with $\min\{n_1, n_2\} \geq \gamma n$. If $\{V_1, V_2\}$ is a partition of V such that $|V_i| = n_i$ for $i \in \{1, 2\}$. Then with high probability*

$$|N(x) \cap V_i|/n_i = |N(x)|/n \pm \xi$$

for every $x \in V(G)$.

Proof. Pick $i \in \{1, 2\}$ so that $n_i \leq n_{3-i}$. Then apply Lemma 2.2.3 with $V = V(G)$, $m = n_i$, $\varepsilon = \xi$ and $\mathcal{V} = \{N(x) : x \in V(G)\}$. □

Lemma 2.2.5. *Let $m, d \in \mathbb{N}$, $a > 0$, $b \in (0, \frac{a}{2d})$ and $c \in \left(0, 2b\left(\frac{a}{2d} - b\right)\right)$. There exists n_0 such that when V is a set of order $n \geq n_0$ and $\mathcal{V}$ is a set of order less than some polynomial of n the following holds. For every $S \in \mathcal{V}$, let $f(S)$ be a subset of V^d. Call $T \in V^d$ a* good tuple *if $T \in f(S)$ for some $S \in \mathcal{V}$. If $|f(S)| \geq an^d$ for every $S \in \mathcal{V}$ then there exists a set $\mathcal{F}$ of at most bn/d good tuples such that $|f(S) \cap \mathcal{F}| \geq cn$ for every $S \in \mathcal{V}$ and the images of distinct elements of $\mathcal{F}$ are disjoint.*

Proof. Pick $\varepsilon > 0$ so that

$$(1+a)\varepsilon < \frac{ab}{d} - 2b^2 - c.$$

Let $b' := \frac{b}{d}$, $p := b' - \varepsilon$ and $c' := c + (d^2+1)p^2$. Let $\mathcal{F}'$ be a random subset of V^d where each $T \in V^d$ is selected independently with probability pn^{1-d}. Let

$$\mathcal{O} := \left\{ \{T, T'\} \in \binom{V^d}{2} : \operatorname{im}(T) \cap \operatorname{im}(T') \neq \emptyset \right\}$$

and $\mathcal{O}_{\mathcal{F}'} := \mathcal{O} \cap \binom{\mathcal{F}'}{2}$.

We only need to show that, for sufficiently large n_0, with positive probability $|\mathcal{O}_{\mathcal{F}'}| < (d^2+1)p^2 n$, $|\mathcal{F}'| < b'n$ and $|f(S) \cap \mathcal{F}'| > c'n$ for every $S \in \mathcal{V}$. We can then remove at most $(d^2+1)p^2 n$ tuples from such a set $\mathcal{F}'$ so that the images of the remaining tuples are disjoint. After also removing every $T \in \mathcal{F}'$ for which there is no $S \in \mathcal{V}$ for which $f(S) = T$, the resulting set $\mathcal{F}$ will satisfy the conditions of the lemma.

Clearly,

$$|\mathcal{O}| \leq n \cdot d^2 \cdot n^{2d-2} = d^2 n^{2d-1},$$

and for any $\{T, T'\} \in \binom{V^d}{2}$, $\Pr(\{T, T'\} \subseteq \mathcal{F}') = p^2 n^{2-2d}$. Therefore, by the linearity of expectation, $\mathbb{E}[|\mathcal{O}_{\mathcal{F}'}|] < d^2 p^2 n$. So, by Markov inequality,

$$\Pr\left(|\mathcal{O}_{\mathcal{F}'}| \geq (d^2+1)p^2 n\right) \leq \frac{d^2}{d^2+1}.$$

Note that $\mathbb{E}[|\mathcal{F}'|] = pn$ and $pn \geq \mathbb{E}[|f(S) \cap \mathcal{F}'|] \geq apn$ for every $S \in \mathcal{V}$. Therefore, by the Chernoff inequality, $\Pr(|\mathcal{F}'| \geq b'n) \leq e^{-\varepsilon^2 n/3}$ and, since

$$ap - c' = \frac{ab}{d} - a\varepsilon - (d^2+1)\left(\frac{b}{d} - \varepsilon\right)^2 - c \geq \frac{ab}{d} - 2b^2 - c - a\varepsilon > \varepsilon,$$

$\Pr(|\mathcal{F}' \cap f(S)| \leq c'n) < e^{-\varepsilon^2 n/3}$ for every $S \in \binom{V}{m}$. Therefore, for sufficiently large n_0,

$$\Pr\left(|\mathcal{O}_{\mathcal{F}'}| \geq (d^2+1)p^2\right) + \Pr(|\mathcal{F}'| \geq b'n) + \sum_{S \in \mathcal{V}} \Pr(|\mathcal{F}' \cap f(S)| \leq c'n) < 1. \qquad \square$$

2.3 Extremal graphs

In many of the theorems we will be investigating, the minimum degree condition is tight. It is an important fact that, if the degree condition is relaxed slightly, the graphs that do not have the desired tiling must "look like" one of example graphs from the introduction that prove the degree condition is tight. This fact is used to prove theorems using what is called the stability method: The exact degree condition is used to prove a result for graphs that look like one of the examples, while the result is proved with a slightly weaker degree condition for all other graphs. These two cases are called the extremal case and the non-extremal case respectively. We use this method to prove Theorem 1.5.4 and Theorem 1.6.4.

In this section, we will provide two general lemmas that may help to prove theorems in this manner. Lemma 2.3.7 can be used to apply the non-extremal case of a proof regarding bigraphs to the non-extremal case of a related graph or digraph theorem. Although we do not use this lemma in our proof of Theorem 1.6.4, it could be used to give a short proof of the non-extremal case that relies on a previous result. Lemma 2.3.10 essentially states that if a graph in the non-extremal case is large enough its induced subgraphs are also non-extremal if they are sufficiently large. This is useful in our proof of Theorem 1.5.4. While these lemmas are written so as to be useful for us, this general approach could be applied to a wider range of problems.

To make the proofs easier, some of the definition are non-standard. We begin defining our notation of an extremal bipartite graph.

Definition 2.3.1. A V_1, V_2-bipartite graph G is (α, k)-*extremal* if

$$\delta(V_1, V_2) \geq \left(\frac{k-1}{k} - \alpha\right)|V_2|$$

and there exists $A \subseteq V_1$ and $B \subseteq V_2$ such that $|A| \geq \left(\frac{1}{k} - \alpha\right)|V_1|$, $|B| \geq \left(\frac{1}{k} - \alpha\right)|V_2|$ and $\|A, B\| \leq \alpha|V_1||V_2|$. We will refer to the ordered pair of sets (A, B) as an (α, k)-*extremal pair.*

Definition 2.3.2. For any V_1, V_2-bipartite graph G call any $x, y \in V_1$ α-*similar* in G if

$$|N(x) \triangle N(y)| \leq \alpha|V_2|.$$

Let $S_G^\alpha(x)$ be the set vertices in G that are α-similar with x. Call a vertex $x \in V_1$ an (α, k)-extremal vertex in G if $\deg(x, V_2) \leq \left(\frac{k-1}{k} + \alpha\right)|V_2|$ and $|S_G^\alpha(x)| \geq \left(\frac{1}{k} - \alpha\right)|V_1|$.

For a directed graph G, we say that G is (α, k)-extremal if $G[V(G), V(G)]$ is (α, k)-extremal. That is, we will call a directed graph G on n vertices (α, k)-*extremal* if

$$\delta^+(G) \geq \left(\frac{k-1}{k} - \alpha\right)n$$

and there exists $A, B \subseteq V(G)$ such that $|A|, |B| \geq \left(\frac{1}{k} - \alpha\right)n$ and $\|A, B\|^+ \leq \alpha n^2$. In the same way, we use the definition of extremal pairs, similar vertices and S_G^α for bigraphs to make the analogous definitions for digraphs. We can apply all of the following results to graphs by considering the digraph formed by replacing every edge of a graph with 2-cycles. Graphs will be discussed in greater detail later in this section.

Proposition 2.3.3. *If G is a V_1, V_2-bigraph such that $\delta(V_1, V_2) \geq \left(\frac{k-1}{k} - \alpha\right)|V_2|$ and $x \in V(G)$ is a (k, α)-extremal vertex then G is (k, α)-extremal with (k, α)-extremal pair $(S_G^\alpha(x), V_2 \setminus N(x))$.*

Proof. Let $A := S_G^\alpha(x)$ and $B := V_2 \setminus N(x)$ note that $|B| \geq \left(\frac{1}{k} - \alpha\right)|V_2|$. If $y \in A$ then, because y is α-similar to x and x has no out-neighbors in B, y has at most $\alpha|V_2|$ neighbors in B. So $\|A, B\| \leq |A|\alpha|V_2| \leq \alpha|V_1||V_2|$. □

Proposition 2.3.4. *Let $\alpha > 0$ and $\beta \geq 8\alpha^{1/2}$. If G is a V_1, V_2-bigraph that is (α, k)-extremal with extremal pair (A, B), there exists $A' \subseteq A$ such that*

$$|A'| \geq \left(\frac{1}{k} - 2\alpha^{1/2}\right)|V_1|,$$

and, for every $x \in A'$, x is (β, k)-extremal, $A' \subseteq S_G^\beta(x)$ and

$$\deg(x, B) \leq \alpha^{1/2}|V_2|.$$

Proof. Let $A' := \{a \in A : \|x, B\| \leq \alpha^{1/2}|V_2|\}$. We claim that A' is the desired set.

First note that $|A'| \geq |A| - \alpha^{1/2}|V_1| \geq \left(\frac{1}{k} - 2\alpha^{1/2}\right)|V_1|$. Let $x, y \in A'$ and define $X := N(x)$ and $Y := N(y)$. Note that $|V_2 \setminus B| \leq \left(\frac{k-1}{k} + \alpha\right)|V_2|$. Because $\deg(x, B), \deg(y, B) \leq \alpha^{1/2}|V_2|$, we have that $|(X \triangle Y) \cap B| \leq 2\alpha^{1/2}|V_2|$ and

$$\deg(x, V_2), \deg(y, V_2) \leq |V_2 \setminus B| + \alpha^{1/2}n \leq \left(\frac{k-1}{k} + 2\alpha^{1/2}\right)|V_2|.$$

With the minimum degree condition,

$$\deg(x, V_2 \setminus B), \deg(y, V_2 \setminus B) \geq \left(\frac{k-1}{k} - 2\alpha^{1/2}\right)|V_2|,$$

so $|\overline{X} \cap (V_2 \setminus B)|, |\overline{Y} \cap (V_2 \setminus B)| \leq 3\alpha^{1/2}|V_2|$. Hence $|(X \triangle Y) \cap (V_2 \setminus B)| \leq 6\alpha^{1/2}|V_2|$ and $|X \triangle Y| \leq 8\alpha^{1/2}|V_2|$. □

Proposition 2.3.5. *If G is a V_1, V_2-bigraph and $x \in V_1$ is (α, k)-extremal than every $y \in S_G^\alpha(x)$ is $(2\alpha, k)$-extremal.*

Proof. We have that

$$\deg(y, V_2) \leq \deg(x, V_2) + \alpha|V_2| \leq \left(\frac{k-1}{k} + 2\alpha\right)|V_2|$$

and, for any $z \in S_G^\alpha(x)$

$$|N(y) \triangle N(z)| \leq |N(y) \triangle N(x)| + |N(x) \triangle N(z)| \leq 2\alpha|V_2|.$$

□

Lemma 2.3.6. *For any $\alpha > 0$, and $\alpha' := 8\alpha^{1/2}$ $\beta \geq 16\alpha^{1/2}$ and $\xi > 0$ the following holds with high probability. Let G be a directed graph on n vertices and let n_1, n_2 be positive integers greater than ξn such that $n_1 + n_2 = n$. Let $\{V_1, V_2\}$ be a partition of $V(G)$ where $|V_i| = n_i$ for every $i \in [2]$ selected uniformly at random from all such partitions.*

Let $1 \leq i \leq j \leq 2$ and define $H := G[V_i, V_j]$. For any $x \in V_i$,

$$S_H^{\alpha'}(x) \subseteq S_G^{\beta}(x) \cap V_i \text{ and}$$
$$S_G^{\alpha'}(x) \cap V_i \subseteq S_H^{\beta}(x),$$

and, for any $\gamma \leq 2\alpha'$, if x is (γ, k)-extremal in H, x is (β, k)-extremal in G; and if x is (γ, k)-extremal in G, x is (β, k)-extremal in H.

Furthermore, If $\delta^+(G) \geq \left(\frac{k-1}{k} - \beta\right) n$ and H is (α, k)-extremal then G is (β, k)-extremal; and if G is (α, k)-extremal then H is (β, k)-extremal.

Proof. Let $\varepsilon := \beta - 2\alpha'$ Define $\mathcal{V}(G)$ to be the collection of sets $N^+(x)$, $S_G^{\alpha'}(x)$ for every $x \in V(G)$ and $N^+(x) \triangle N^+(y)$ for every pair $x, y \in V(G)$. By Lemma 2.2.3, with high probability, for any $i \in \{1, 2\}$ and $U \in \mathcal{V}(G)$,

$$|U \cap V_i|/n_i = |U|/n \pm \varepsilon. \tag{2.3.1}$$

Let $x, y \in V_i$ and $0 < \gamma < 2\alpha'$. Because $\gamma + \varepsilon \leq \beta$ and (2.3.1), if x and y are γ similar in H then x and y are β similar in G. So $S_H^{\gamma}(x) \subseteq S_G^{\beta}(x) \cap V_i$. Similarly, $S_G^{\gamma}(x) \cap V_i \subseteq S_H^{\beta}(x)$.

Since this gives us that

$$|S_G^{\beta}(x)|/n \geq |S_G^{\beta}(x) \cap V_i|/n_i - \varepsilon \geq |S_H^{\gamma}(x)|/n_i - \varepsilon$$

we have that if x is (γ, k)-extremal in H then x is (β, k)-extremal in G. Similarly, the fact that

$$|S_H^{\beta}(x)|/n_i \geq |S_G^{\gamma}(x) \cap V_i|/n_i \geq |S_G^{\gamma}(x)|/n - \varepsilon$$

gives us that if x is (γ, k)-extremal in G then x is (β, k)-extremal in H.

Assume $\delta^+(G) \geq \left(\frac{k-1}{k} - \beta\right) n$ and H is α-extremal. By Proposition 2.3.4, there exists $x \in V_i$ such that x is α'-extremal in H. Therefore x is β-extremal in G and, by Proposition 2.3.3, G is (β, k)-extremal.

If G is α-extremal then, by Proposition 2.3.4, there exists an (α', k)-extremal vertex $x \in V$. Therefore, by (2.3.1), there exists $y \in S_G^{\alpha'}(x) \cap V_i$ if n is large enough. Since Proposition 2.3.5 gives us that y is $(2\alpha', k)$-extremal in G we have that y is (β, k)-extremal in H. By Proposition 2.3.3, and because (2.3.1) also gives that $\delta(V_1, V_2) \geq \left(\frac{k-1}{k} - \beta\right) |V_2|$, we have that H is (β, k)-extremal. □

The first results follows directly from Lemma 2.3.6 and is relevant to the section on anti-directed hamilton cycles.

Lemma 2.3.7. *For any $\alpha > 0$, $\beta \geq 16\alpha^{1/2}$, and $\alpha > \varepsilon > \varepsilon' > 0$ there exists $n_0 := n_0(\alpha, \beta, \varepsilon', \varepsilon)$ such that the following holds. If G is a digraph on $2n \geq n_0$ vertices such that $\delta^+(G) \geq \left(\frac{1}{2} - \varepsilon'\right) 2n$ and G is not β-extremal then there exists $\{A, B\}$ an equitable partition of $V(G)$ such that $\delta(G[A, B]) \geq \left(\frac{1}{2} - \varepsilon\right) n$ and $G[A, B]$ is not α-extremal.*

Suppose now that G is a graph on n vertices and $A, B \subseteq V(G)$ such that $|A|, |B| \geq \left(\frac{1}{k} - \alpha\right) n$. If $\delta(G) \geq \left(\frac{k-1}{k} - \alpha\right) n$ and $|A \cup B| \geq \left(\frac{1}{k} + 2\alpha\right) n$, then every vertex in $A \cap B$ has at least αn neighbors in $A \cup B$. So $\|A, B\| \geq \alpha n^2$ whenever $\left(\frac{1}{k} - 4\alpha\right) \geq |A \cap B| \geq \alpha n$. Therefore, if G is (α, k)-extremal with (α, k)-extremal pair (A, B) we either have that A and B are either nearly disjoint or nearly identical. For simplicity, we will now focus when A and B are nearly disjoint and $k = 2$, because this relates directly to Theorem 1.5.4. All of what follows could be stated more generally.

Say that a graph G is (α, k)-*splittable* if $\delta(G) \geq (\frac{1}{k} - \alpha)n$ and there exists $A \subseteq V(G)$ such that $|A| \geq (\frac{1}{k} \pm \alpha)n$ and such that $\|A, \overline{A}\| \leq \alpha n^2$. Say that a graph G is (α, k)-

independent if $\delta(G) \geq (\frac{1}{k} - \alpha)n$ and there exists $A \subseteq V(G)$ such that $|A| \geq (\frac{k-1}{k} - \alpha)n$ and $|E(G[A])| \leq \alpha n^2$.

Proposition 2.3.8. *For any $0 < \alpha < 10^{-4}$ define $\beta := 8\alpha^{1/2}$. If G is an $(\alpha, 2)$-splittable graph on n vertices then there exists $x \in V(G)$ such that x is an $(\beta, 2)$-extremal vertex and $|S_G^\beta(x) \cap N(x)| \geq \left(\frac{1}{2} - \beta\right) n$.*

Proof. Let $A \subseteq V(G)$ be such that $|A| \geq \left(\frac{1}{2} - \alpha\right) n$ and $\|A, \overline{A}\| \leq \alpha n$. This implies that G is $(\alpha, 2)$-extremal with extremal pair $(A, \overline{A})$. Let $A' \subseteq A$ be the set guaranteed by Proposition 2.3.4. That is, $|A'| \geq (\frac{1}{2} - 2\alpha^{1/2})n$, and, for any $x \in A'$, x is $(\beta, 2)$-extremal, $\deg(x, \overline{A}) \leq 2\alpha^{1/2}n$ and $A' \subseteq S_G^\beta(x)$.

Since $\deg(x, A') \leq |A \setminus A'| + \deg(x, B) \leq 4\alpha^{1/2}$, by the minimum degree condition, we have that

$$|S_G^\beta(x)| \cap N(x)| \geq \deg(x, A') \geq \left(\frac{1}{2} - 5\alpha^{1/2}\right) n.$$

□

Proposition 2.3.9. *If $\delta(G) \geq (\frac{1}{2} - \alpha)n$ and there exists $x \in G$ such that $|N(x)| \leq \left(\frac{1}{2} + \alpha\right) n$ and $|S_G^\alpha(x) \cap N(x)| \geq \left(\frac{1}{2} - \alpha\right) n$ then G is $(3\alpha, 2)$-splittable.*

Proof. Let $A := S_G^\alpha(x) \cap N(x)$. Note that $\deg(x, \overline{A}) \leq |N(x)| - |A| \leq 2\alpha n$. So, for every $y \in B$, $\deg(y, \overline{A}) \leq 3\alpha n$ and $\|A, \overline{A}\| \leq 3\alpha n^2$. □

Lemma 2.3.10. *For any $\alpha > 0$, $\beta > 2^5 \cdot \alpha^{1/2}$, $\rho > 0$, and $\alpha > \varepsilon > \varepsilon' > 0$ there exists $n_0 := n_0(\alpha, \beta, \varepsilon', \varepsilon, \rho)$ such that for any graph G on $n \geq n_0$ vertices and $\delta(G) \geq \left(\frac{1}{2} - \varepsilon'\right) n$ the following holds. If G is not β-splittable and n_1, n_2 are positive integers greater than ρn such that $n_1 + n_2 = n$ there exists $\{V_1, V_2\}$ a partition of $V(G)$ such that for every $i \in [2]$, $|V_i| = n_i$, $G[V_i]$ is not α-splittable and* $\deg(v, V_i) \geq \left(\frac{1}{2} - \varepsilon\right) n_i$ *for every $v \in V$.*

Proof. Let $\beta' = \beta/2^2$, $\alpha' = \alpha^{1/2} \cdot 2^3$ and $\delta := \beta' - \alpha'$. With high probability the conditions of Lemma 2.2.4 (with $\xi = \varepsilon - \varepsilon'$) and Lemma 2.3.6 (with β' instead of β) hold. By Lemma 2.2.3, we can also assume

$$|S_G^{\beta'}(x) \cap N(x) \cap V_i|/n_i = |S_G^{\beta'}(x) \cap N(x)|/n \pm \delta. \tag{2.3.2}$$

for every $x \in V(G)$ and $i \in [2]$.

We will prove the contrapositive. So assume there exists $H := G[V_i]$ that is α-splittable. Therefore, by Proposition 2.3.8, there exists $x \in V_i$ such that x is $(\alpha', 2)$-extremal in H and

$$|S_H^{\alpha'}(x) \cap N_H(x)| \geq \left(\frac{1}{2} - \alpha'\right) n_i.$$

By Lemma 2.3.6, x is $(\beta', 2)$-extremal in G, so $|N_G(x)| \leq \left(\frac{1}{2} + \beta'\right) n$, and $S_H^{\alpha'}(x) \subseteq S_G^{\beta'}(x) \cap V_i$. Furthermore, by (2.3.2),

$$\begin{aligned}|S_G^{\beta'}(x) \cap N_G(x)|/n &\geq |S_G^{\beta'}(x) \cap N_G(x) \cap V_i|/n_i - \delta \\ &\geq |S_H^{\alpha'}(x) \cap N_H(x)|/n_i - \delta \geq \frac{1}{2} - \alpha' - \delta \geq \frac{1}{2} - \beta'.\end{aligned}$$

Therefore, by Proposition 2.3.9, G is β-splittable. □

2.4 The regularity and blow-up lemmas

Let G be a graph. We call a function f from $E(G)$ to $[r]$ an *r-edge-coloring* and we say that the edges of G are *r-colored*. Given an r-edge-coloring and disjoint $U, W \subseteq V(G)$, we define $d_c(U, W) = \frac{\|U,W\|_c}{|U||W|}$ where $\|U, W\|_c$ is the number of edges colored c in $E(U, W)$. We will use the following version of the Szemerédi's Regularity Lemma [29].

Lemma 2.4.1 (Many-Color Regularity Lemma)**.** *For every $\varepsilon > 0$ and $r, t_0 \in \mathbb{Z}^+$ there exists $T \in \mathbb{Z}^+$ such that if the edges of a graph G on n vertices are r-colored then the vertex set can be partitioned into sets $V_0, V_1, \ldots, V_t$ for some $t_0 \leq t \leq T$ so* *that* $|V_0| < \varepsilon n$, $m := |V_1| = |V_2| = \cdots = |V_t|$, *and all but at most* εt^2 *pairs* (V_i, V_j) *with* $1 \leq i < j \leq t$ *satisfy the following regularity condition: For every* $X \subseteq V_i$ *and* $Y \subseteq V_j$ *such that* $|X|, |Y| > \varepsilon m$ *and any color* $c \in \{1, \ldots, r\}$

$$|d_c(V_i, V_j) - d_c(X, Y)| < \varepsilon.$$

Let M be a standard multigraph and $G := M_G$ and $H := M_H$ and $L := M_L$. We will call two disjoint vertex sets A and B *ε-regular* if for every $X \subseteq A$ and $Y \subseteq B$ such that $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$

$$|d_G(A, B) - d_G(X, Y)| < \varepsilon$$

and

$$|d_H(A, B) - d_H(X, Y)| < \varepsilon.$$

Note that

$$`|d_G(V_i, V_j) - d_G(X, Y)| = |(d_L(V_i, V_j) + d_H(V_i, V_j)) - (d_L(X, Y) + d_H(X, Y))|$$
$$\leq |d_H(V_i, V_j) - d_H(X, Y)| + |d_H(V_i, V_j) - d_H(X, Y)|.$$

Also, for any multigraph M we can view the view the light edges and the heavy edges as a 2-edge-coloring of $G(M)$. Therefore, by applying Lemma 2.4.1 with $\varepsilon/2$ we have the following corollary.

Corollary 2.4.2 (Standard Multigraph Regularity Lemma)**.** *For every* $\varepsilon > 0$ *and* $t_0 \in \mathbb{Z}^+$ *there exists* $T \in \mathbb{Z}^+$ *such that if* M *is a standard multigraph on* n *vertices the vertex set can be partitioned into sets* $V_0, V_1, \ldots, V_t$ *for some* $t_0 \leq t \leq T$ *so that* $|V_0| < \varepsilon n$, $|V_1| = |V_2| = \cdots = |V_t|$, *and all but at most* εt^2 *pairs* (V_i, V_j) *with* $1 \leq i < j \leq t$ *are* ε*-regular.*

We call V_0 an *exceptional* cluster and $V_1, \ldots, V_t$ *non-exceptional* clusters.

We will also need the following two slightly modified version of standard lemmas about ε-regular pairs. The follow directly from their corresponding graph versions.

Lemma 2.4.3 (Slicing Lemma)**.** *Let M be a standard multigraph with $G := M_G$ and $H := M_G$ and let (U, W) be an ε-regular pair with $d_H(U, V) \geq d_H$ and $d_G(U, V) \geq d_G$. For some $\upsilon > \varepsilon$ let $U' \subset U$ and $W' \subset W$ with $|U'| \geq \upsilon|U|$ and $|W'| \geq \upsilon|W|$. Then (U', W') is an ε'-regular pair with $\varepsilon' = \max\{\frac{\varepsilon}{\upsilon}, 2\varepsilon\}$, $d_H(U', W') > d_H - \varepsilon$ and $d_G(U', W') > d_G - \varepsilon$*

Lemma 2.4.4. *Let (U, V) be an ε-regular pair with $d_H(U, V) \geq d_H$ and $d_G(U, V) \geq d$ and let $Y \subseteq V$ such that $|Y| \geq \varepsilon|V|$. Then all but fewer than $\varepsilon|U|$ vertices in U have less than $(d_H - \varepsilon)|Y|$ heavy-neighbors in Y and all but fewer than $\varepsilon|U|$ vertices in U have less than $(d_G - \varepsilon)|Y|$ neighbors in Y.*

If (A, B) is an ε-regular pair we say that that the pair (A, B) is (ε, δ)-super regular if $\deg(a, B) \geq \delta|B|$ and $\deg(b, B) \geq \delta|A|$ for every $a \in A$ and $b \in B$.

We will use the following lemma.

Lemma 2.4.5 (Komlós, Sárközy & Szemerédi 1995 [28])**.** *Given a graph R of order r and positive parameters, δ and Δ there exists a positive $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds. Let $n_1, \ldots, n_r$ be arbitrary positive integers and let us replace the vertices $v_1, \ldots, v_r$ with pairwise disjoint sets $V_1, \ldots, V_r$ of sizes $n_1, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex set $V = \bigcup V_i$. The first graph R is obtained by replacing each edge $\{v_i, v_j\}$ of R with the complete bipartite graph between the corresponding vertex-sets V_1 and V_j. The second graph G is constructed by replacing each edge $\{v_i, v_j\}$ arbitrarily with an (ε, δ)-super-regular pair between V_i and V_j. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into R then it is already embeddable into G.*

Chapter 3

TRIANGLE TILINGS

3.1 Three short proofs on triangle tilings

In this section we proving the standard multigraph generalizations of Theorem 1.3.5 and the case $c = 0$ of Theorem 1.3.6. For completeness, and to illustrate the origins of our methods, we begin with a short proof of Corollary 1.3.1 based on a related theorem proved by Enomoto [13]. We use the symbol $\oplus$ to indicate addition modulo k, where k should be clear from context. Let $T = xyz$ be triangle in either a graph or multigraph. We sometime represent T by ez or ze where e is the edge xy for convenience.

Corollary 1.3.1 (Corrádi & Hajnal 1963 [5]). *If G is a graph on n vertices and $\delta(G) \geq \frac{2n}{3}$ then G has an ideal K_3-tiling.*

Proof. First note that we can assume 3 divides n. If $n = 1 \pmod 3$ remove any vertex to create G', and if $n = 2 \pmod 3$ add a new vertex adjacent to every vertex in $V(G)$ to create G'. In both cases, $\delta(G') \geq 2|G'|/3$ and a K_3-factor of G' corresponds to an ideal K_3-tiling of G.

Let $G = (V, E)$ be an edge-maximal counterexample. Then $n = 3k$, $\delta(G) \geq 2k$, G does not contain a C_3-factor (so $G \neq K_{3k}$), but the graph G^+ obtained by adding a new edge a_1a_3 does have a C_3-factor. So G has a *near triangle factor* $\mathcal{T}$, i.e., a factor such that $A := a_1a_2a_3 \in \mathcal{T}$ is a path and every $H \in \mathcal{T} - A$ is a triangle.

Claim. Suppose $\mathcal{T}$ is a near triangle factor of G with path $A := a_1a_2a_3$ and triangle $B := b_1b_2b_3$. If $\|\{a_1, a_3\}, B\| \geq 5$ then $\|a_2, B\| = 0$.

Proof. Choose notation so that $\|a_1, B\| = 3$ and $\|a_3, B\| \geq 2$. Suppose $b_i \in N(a_2)$. Then either $\{a_1 b_{i\oplus 1} b_{i\oplus 2}, a_2 a_3 b_i\}$ or $\{a_1 a_2 b_i, a_3 b_{i\oplus 1} b_{i\oplus 2}\}$ is a C_3-factor of $G[A \cup B]$, depending on whether $b_i \in N(a_3)$. Regardless, this contradicts the minimality of G. □

Since $\|\{a_1, a_3\}, G\| \geq 4k$, but $\|\{a_1, a_3\}, A\| = 2 < 4$, there is a triangle $B := b_1 b_2 b_3 \in \mathcal{T}$ with $\|\{a_1, a_3\}, B\| \geq 5$. Choose notation so that $b_1, b_2, b_3 \in N(a_1)$ and $b_2, b_3 \in N(a_3)$. So $b_1 a_1 a_2 a_3$ is a path with every vertex except a_2 adjacent to both b_2 and b_3. Applying the claim to A yields $\|a_2, B\| = 0$. Thus

$$2\|\{b_1, a_2\}, A \cup B\| + \|\{a_1, a_3\}, A \cup B\| \leq 2(4+2) + 2(1+3) = 20 < 24 = 6 \cdot 2 \cdot 2.$$

Since $2\|\{b_1, a_2\}, G\| + \|\{a_1, a_3\}, G\| \geq 12k$, some triangle $C := c_1 c_2 c_3 \in \mathcal{T}$ satisfies:

$$2\|\{b_1, a_2\}, C\| + \|\{a_1, a_3\}, C\| \geq 13.$$

Then $\|a_2, C\|, \|\{a_1, a_3\}, C\| > 0$. By Claim, $\|\{a_1, a_3\}, C\| \leq 4$; so $\|\{b_1, a_2\}, C\| \geq 5$. Claim applied to

$$\mathcal{T} \cup \{b_1 a_1 a_2, a_3 b_2 b_3\} \smallsetminus \{A, B\}$$

yields $\|a_1, C\| = 0$. So $\|a_3, C\| > 0$ and either $\|\{b_1, a_2\}, C\| = 6$ or $\|a_3, C\| = 3$. Thus some $i \in [3]$ satisfies $c_i a_2, c_i a_3, c_{i\oplus 1} b_1, c_{i\oplus 2} b_1 \in E(G)$. So

$$\mathcal{T} \cup \{c_i a_2 a_3, b_1 c_{i\oplus 1} c_{i\oplus 2}, a_1 b_2 b_3\} \smallsetminus \{A, B, C\}$$

is a C_3-factor of G. Note that $a_1 b_2 b_3$ is a triangle, so it suffices to show that there is a triangle factorization of $G[\{b_1, a_2, a_3\} \cup V(C)]$. If $\|\{b_1, a_2\}, C\| = 6$ then $\|a_3, C\| \geq 1$. Let c_i be a neighbor of a_3. Then $a_2 a_3 c_i$ and $b_1 c_{i\oplus 1} c_{i\oplus 2}$ are disjoint triangles. So assume $\|a_3, C\| = 3$. Since $\|\{b_1, a_2\}, C\| \geq 5$, there exists $c_i \in V(C)$ such that $\|\{b_1, a_2\}, c_i\| \geq 1$ and $\|\{b_1, a_2\}, c_{i\oplus 1}\| = \|\{b_1, a_2\}, c_{i\oplus 2}\| = 2$. If $a_2 c_i \in E$ then $a_2 a_3 c_i$ is a triangle and $b_1 c_{i\oplus 1} c_{i\oplus 2}$ is a triangle. Otherwise, $b_1 c_i \in E$ and $b_1 c_i c_{i\oplus 1}$ and $a_2 a_3 c_{i\oplus 2}$ are disjoint triangles. □

Next we use Corollary 1.3.1 to prove Theorem 3.1.1. This gives us Corollary 1.3.7 another way.

Theorem 3.1.1. *Every standard multigraph M on n vertices with $\delta(M) \geq \frac{4n}{3} - 1$ contains $\lfloor \frac{n}{3} \rfloor$ independent 4-triangles.*

Proof. We consider three cases depending on $n \pmod 3$.

Case 0: $n \equiv 0 \pmod 3$. Since $\delta(M(G)) \geq \lceil \frac{1}{2}\delta(M) \rceil \geq \frac{2}{3}n$, Corollary 1.3.1 implies M has a triangle factor $\mathcal{T}$. Choose $\mathcal{T}$ having the maximum number of 4-triangles. We are done, unless $\|A\| = 3$ for some $A = a_1a_2a_3 \in \mathcal{T}$. Since $\|A, M\| \geq 3\left(\frac{4n-3}{3}\right)$,

$$\|A, M - A\| \geq 4n - 3 - \|A, A\| = 4n - 9 > 12\left(\frac{n-3}{3}\right).$$

Thus $\|A, B\| \geq 13$ for some $B = b_1b_2b_3 \in \mathcal{T}$. Suppose $\|a_1, B\| \geq \|a_2, B\| \geq \|a_3, B\|$. Then $5 \leq \|a_1, B\| \leq 6$ and $\|\{a_2, a_3\}, B\| \geq 7$. Hence, $\|\{a_2, a_3\}, b_i\| \geq 3$ for some $i \in [3]$; so $\mathcal{T} \cup \{a_2a_3b_i, a_1b_{i\oplus1}b_{i\oplus2}\} \smallsetminus \{A, B\}$ is a 4-triangle factor of M. a set of k independent 4-triangles.

Case 1: $n \equiv 1 \pmod 3$. Pick $v \in V$, and set $M' := M - v$. Then $|M'| \equiv 0 \pmod 3$, and

$$\delta(M') \geq \delta(M) - \mu(M) \geq \left\lceil \frac{4n-9}{3} \right\rceil = \frac{4(n-1)-3}{3} \geq \frac{4|M'|-3}{3}.$$

By Case 0, M', and also M, contains $\lfloor \frac{|M'|}{3} \rfloor = \lfloor \frac{n}{3} \rfloor$ independent 4-triangles.

Case 2: $n \equiv 2 \pmod 3$. Form $M^+ \supseteq M$ by adding a new vertex x and heavy edges xv for all $v \in V(M)$. Then $|M^+| \equiv 0 \pmod 3$ and $\delta(M^+) \geq \frac{4|M^+|-3}{3}$. By Case 0, M^+ contains $\frac{|M^+|}{3}$ independent 4-triangles. So $M = M^+ - x$ contains $\frac{|M^+|}{3} - 1 = \lfloor \frac{n}{3} \rfloor$ of them. $\square$

Now we consider 5-triangle tilings. First we prove Proposition 3.1.2, which is also needed in the next section. Then we strengthen Wang's Theorem to standard multigraphs. Now we consider 5-triangle tilings. First we prove Theorem 1.3.12 strengthening Theorem 1.3.5.

Proposition 3.1.2. *Let $T = v_1v_2v_3 \subseteq M$ be a 5-triangle, and $x \in V(M-T)$. If $3 \leq \|x,T\| \leq 4$ then xe is a $(\|x,T\|+1)$-triangle for some $e \in E(T)$.*

Proof. Suppose $v_1v_2, v_1v_3 \in E_H$. If $N(x) \subseteq \{v_2v_3\}$ then xv_2v_3 is a $(\|x,T\|+1)$-triangle. Else, $\|x, v_1v_i\| \geq \|x,T\| - 1$ for some $i \in \{2,3\}$. So xv_1v_i is a $(\|x,T\|+1)$-triangle. □

Proposition 3.1.3. *Let $T \subseteq M$ be a 5-triangle. If $x \in V(M-T)$ and $\|x,T\| \geq 3$ then there exists a 4-triangle consisting of x and 2 vertices from $V(T)$.*

Proof. If $\|x,T\| > 3$ then apply Proposition 3.1.2. If $\|x,T\| = 3$, add a parallel edge to some $e \in E(x,T)$ such that $\mu(e) = 1$ and then apply Proposition 3.1.2. □

The following is a generalization of the result of [36] for the case where the number of vertices is divisible by 3.

Theorem 1.3.12 (Czygrinow, Kierstead & Molla 2012 [7])**.** *Every standard multigraph M on n vertices with $\delta(M) \geq \frac{3n-3}{2}$ has an ideal 5-triangle tiling.*

Proof. Consider two cases depending on whether $n \equiv 2 \pmod 3$.

Case 1: $n \not\equiv 2 \pmod 3$. By Theorem 3.1.1, M has a tiling $\mathcal{T}$ consisting of $\lfloor \frac{n}{3} \rfloor$ independent 4- and 5-triangles. Over all such tilings, select $\mathcal{T}$ with the maximum number of 5-triangles. We are done, unless there exists $A = a_1a_2a_3 \in \mathcal{T}$ such that $\|A\| = 4$. Assume a_1a_2 is the heavy edge of A. By the case, $L := V \smallsetminus \bigcup \mathcal{T}$ has at most one vertex. If $L \neq \emptyset$ then let $a_3' \in L$; otherwise set $a_3' := a_3$. Also set $A' := A + a_3'$. Then $\|a_3', A\| \leq 2 + 2(|A'|-3) = 2|A'|-4$, since otherwise $G[A']$ contains a 5-triangle. So $\|A, A' \smallsetminus A\| \leq 4(|A'|-3)$.

For $B \in \mathcal{T}$, define $f(B) := \|A, B\| + \|a_3', B\|$. Then $f(A) = 8 + \|a_3', A\| \leq 4 + 2|A'|$. So

$$\sum_{B \in \mathcal{T}} f(B) = d(a_1) + d(a_2) + d(a_3) + d(a_3') - \|A, A' \smallsetminus A\| \geq 4 \cdot \frac{3n-3}{2} - \|A, A' \smallsetminus A\|$$
$$\geq 6n - 6 - 4(|A'| - 3) = 6(n - |A'|) + (4 + 2|A'|) + 2$$
$$> 18\,(|\mathcal{T}| - 1) + f(A).$$

Thus $f(B) \geq 19$ for some $B \in \mathcal{T} - A$. If B is a 4-triangle then set $B' := B + e'$, where e' is parallel to some $e \in E(B)$ with $\mu(e) = 1$, and set $M' := M + e'$. Otherwise, set $B' := B$ and $M' := M$. It suffices to prove that $M'[A' \cup B']$ contains two independent 5-triangles, since in either case another 5-triangle can be added to $\mathcal{T}$, a contradiction.

Label the vertices of B' as b_1, b_2, b_3 so that b_1b_2 and b_1b_3 are heavy edges. Since a_1a_2 is a heavy edge, if $\|\{a_1, a_2\}, b\| \geq 3$ then a_1a_2b is a 5-triangle for all $b \in V(B)$. Consider three cases based on $k := \max\{\|a_3, B\|, \|a_3', B\|\}$. Let $a \in \{a_3, a_3'\}$ satisfy $\|a, B\| = k$. Since $f(B) \geq 19$, we have $4 \leq \|a, B\| \leq 6$.

If $\|a, B\| = 4$ then $\|\{a_1, a_2\}, B\| \geq 11$. By Proposition 3.1.2, there exists $i \in [3]$ such that $ab_ib_{i\oplus 1}$ is a 5-triangle, and $a_1a_2b_{i\oplus 2}$ is another disjoint 5-triangle.

If $\|a, B\| = 5$ then $\|\{a_1, a_2\}, B\| \geq 9$. So there exists $i \in \{2, 3\}$ such that $a_1a_2b_i$ is a 5-triangle; and ab_1b_{5-i} is another 5-triangle.

Finally, if $\|a, B\| = 6$ then $\|\{a_1, a_2\}, B\| \geq 7$. So there exists $i \in [3]$ such that $a_1a_2b_i$ is a 5-triangle; and $ab_{i\oplus 1}b_{i\oplus 2}$ is another 5-triangle.

Case 2: $n \equiv 2 \pmod 3$. Form $M' \supseteq M$ by adding a vertex x and heavy edges xv for all $v \in V$. Then $|M'| \equiv 0 \pmod 3$ and $\delta(M') \geq \frac{3|M'|-3}{2}$. By Case 1, M' contains $\frac{|M^+|}{3}$ independent 5-triangles. So $M = M' - x$ contains $\frac{|M'|}{3} - 1 = \frac{n}{3}$ of them. $\square$

3.2 Main triangle result

In this section we prove our main result on triangles, Theorem 1.3.11. Let M be a standard multigraph with $\delta(M) \geq \frac{4n-3}{3}$. We start with three Propositions used in the proof.

Proposition 3.2.1. *Suppose $T = v_1v_2v_3 \subseteq M$ is a 5-triangle, and $x_1, x_2 \in V(M-T)$ are distinct vertices with $\|\{x_1, x_2\}, T\| \geq 9$. Then $M[\{x_1, x_2\} \cup V(T)]$ has a factor containing a 5-triangle and an edge e such that e is heavy if $\min_{i\in[2]}\{\|x_i, T\|\} \geq 4$.*

Proof. Label so that $v_1v_2, v_1v_3 \in E_H$ and $\|x_1, T\| \geq \|x_2, T\|$.

First suppose $\|x_2, T\| \geq 4$. If $x_2v_i \in E_H$ for some $i \in \{2, 3\}$ then $\{x_1v_1v_{5-i}, x_2v_i\}$ works. Else $V(T) \subseteq N(x_2)$. Also $x_1v_j \in E_H$ for some $j \in \{2, 3\}$. So $\{x_1v_j, x_2v_1v_{5-j}\}$ works.

Otherwise, $\|x_2, T\| = 3$ and $\|x_1, T\| = 6$. So $\{x_1v_{i\oplus1}v_{i\oplus2}, x_2v_i\}$ works for some $i \in [3]$. □

Proposition 3.2.2. *Suppose $T = v_1v_2v_3 \subseteq M$ is a 5-triangle, and $e_1, e_2 \in E(M-T)$ are independent heavy edges with $\|e_1, T\| \geq 9$ and $\|e_2, T\| \geq 7$. Then $M[e_1 \cup e_2 \cup V(T)]$ contains two independent 5-triangles.*

Proof. Choose notation so that $\|e_1, v_i\| \geq 3$ for both $i \in [2]$. There exists $j \in [3]$ so that $\|e_2, v_j\| \geq 3$. Pick $i \in [2] - j$. Then e_1v_i and e_2v_j are disjoint 5-triangles. □

Proposition 3.2.3. *Suppose $T \subseteq M$ is a 5-triangle, and xyz is a path in $H(M)-T$. If $\|xz, T\| \geq 9$ and $\|y, T\| \geq 1$ then $M[\{x, y, z\} \cup V(T)]$ has a factor containing a 5- and a 4-triangle.*

Proof. Choose notation so that $\|x, T\| \geq \|z, T\|$, and $T = v_1v_2v_3$ with $v_1 \in N(y)$. We identify a 4-triangle A and a 5-triangle B depending on several cases.

Suppose $\|x, T\| = 6$ and $\|z, T\| \geq 3$. If $zv_1 \in E$ then set $A := yzv_1$ and $B := xv_2v_3$; else set $A := zv_2v_3$ and $B := xyv_1$. Otherwise $\|x, T\| = 5$ and $\|z, T\| \geq 4$.

If $zv_1 \notin E$ then set $A := xyv_1$ and $B := zv_2v_3$. Otherwise $zv_1 \in E$.

If zv_1 is heavy then set $A := xv_2v_3$ and $B := zyv_1$; if xv_1 is light then set $A := zyv_1$ and $B := xv_2v_3$. Otherwise zv_1 is light and xv_1 is heavy. Set $A := zv_2v_3$ and $B := xyv_1$. $\square$

Theorem 1.3.11 (Czygrinow, Kierstead & Molla 2012 [7]). *Every standard multigraph M with $\delta(M) \geq \frac{4n}{3} - 1$ has a tiling in which one tile is a 4-triangle and $\lfloor \frac{n}{3} \rfloor - 1$ tiles are 5-triangles.*

Proof. We consider three cases depending on $n \pmod 3$.

Case 0: $n \equiv 0 \pmod 3$. Let $n =: 3k$, and let M be a maximal counterexample. Let $\mathcal{T}$ be a maximum T_5-tiling of M and $U = \bigcup_{T \in \mathcal{T}} V(T)$.

Claim 1. $|\mathcal{T}| = k - 1$.

Proof. Let $e \in \overline{E}$. By the maximality of M, $M + e$ has a factor $\mathcal{T}'$ consisting of 5-triangles and one 4-triangle A_1. If $e \in A_1$ then the 5-triangles are contained in M, and so we are done. Otherwise, $e \in E(A_2^+)$ for some 5-triangle $A_2^+ \in \mathcal{T}'$. Set $A_2 := A_2^+ - e$, and $A := A_1 \cup A_2$. Then A satisfies: (i) $|A| = 6$, (ii) $M[A]$ contains two independent heavy edges, and (iii) $M[V \setminus A]$ has a T_5-factor. Over all vertex sets satisfying (i–iii), select A and independent heavy edges $e_1, e_2 \in M[A]$ so that $\|z_1z_2\|$ is maximized, where $\{z_1, z_2\} := A \setminus (e_1 \cup e_2)$. Let $\mathcal{T}'$ be a T_5-factor of $M[V \setminus A]$. Set $A_i := e_i + z_i$, for $i \in [2]$.

If $M[A]$ contains a 5-triangle we are done. Otherwise $\|x, A_2\| \leq 4$ for all $x \in V(A_1)$, and so $\|A\| = \|A_1\| + \|A_2\| + \|A_1, A_2\| \leq 20$. Thus

$$\|A, V \setminus A\| \geq 6\left(\frac{4}{3}n - 1\right) - 40 > 24(k - 2).$$

So $\|A,B\| \geq 25$ for some $B = b_1b_2b_3 \in \mathcal{T}'$. It suffices to show that $M[A \cup B]$ contains two independent T_5.

Suppose $\|\{z_1, z_2\}, B\| \geq 9$. If there exists $h \in [2]$ such that $\|z_h, B\| = 6$ then choose $i \in [3]$ with $\|b_i, A\| \geq 9$. There exists $j \in [2]$ with $\|b_i e_j\| \geq 5$; also $\|z_h b_{i\oplus 1} b_{i\oplus 2}\| \geq 5$. So we are done. Otherwise, $\|z_h, B\| \geq 4$ and $\|z_{3-h}, B\| \geq 5$ for some $h \in [2]$. By Proposition 3.2.1, $M[V(B) + z_1 + z_2]$ has a factor consisting of a heavy edge and a T_5, implying, by the maximality of $\|z_1z_2\|$, that $\|z_1z_2\| = 2$. Set $e_3 := z_1z_2$. Choose distinct $i, j \in [3]$ so that $\|e_i, B\| \geq 9$ and $\|e_j, B\| \geq 7$. By Proposition 3.2.2, there are two T_5 in $M[e_i \cup e_j \cup V(B)]$. □

By Claim 1, $W := V \setminus U$ satisfies $|W| = 3$. Choose $\mathcal{T}$ with $\|W\|_H$ maximum.

Claim 2. $\|W\| \geq 4$.

Proof. Suppose not. Then $\|W, U\| \geq 3(\frac{4}{3}n - 1) - 3 > 12(k-1)$. So $\|W, T\| \geq 13$ for some $T \in \mathcal{T}$. Thus there exist $w, w' \in W$ with $\|w, T\| \geq 4$ and $\|w', T\| \geq 5$. By Proposition 3.2.1, $M[V(T) \cup \{w, w'\}]$ has a factor containing a T_5 and a heavy edge. By the choice of W this implies $\|W\|_H = 1$. Set $W =: \{x, y, z\}$ where xy is heavy. Since

$$2\|z, U\| + \|xy, U\| \geq 4\left(\frac{4n}{3} - 1\right) - 2 - 5 > 16\left(\frac{n}{3} - 1\right) = 16(k-1),$$

some $T = v_1v_2v_3 \in \mathcal{T}$ satisfies $2\|z, T\| + \|xy, T\| \geq 17$. Suppose $v_1v_2, v_1v_3 \in E_H$. To contradict the maximality of $\|W\|$, it suffices to find $i \in [3]$ so that

$$\{M[\{x, y, v_i\}], M[\{z, v_{i\oplus 1}, v_{i\oplus 2}\}]\}$$

contains a T_5, and a graph with at least four edges.

If $\|z, T\| = 3$ then $\|xy, T\| \geq 11$. Choose $i \in \{2, 3\}$ so that $\|z, v_1v_i\| \leq 1$.

If $\|z, T\| = 4$ then $\|xy, T\| \geq 9$. Choose $i \in \{2, 3\}$ so that xyv_i is a 5-triangle.

If $\|z, T\| = 5$ then $\|xy, T\| \geq 7$. Choose $i \in \{2, 3\}$ so that $\|xy, v_i\| \geq 2$.

Otherwise, $\|z,T\| = 6$ and $\|xy,T\| \geq 5$. Choose $i \in [3]$ so that $\|xy, v_i\| \geq 2$. $\square$

Since M is a counterexample and $\|W\| \geq 4$, we have $M[W] =: xyz$ is a path in M_H.

Claim 3. There exists $A \in \mathcal{T}$ and a labeling $\{a_1, a_2, a_3\}$ of $V(A)$ such that

(a) x is adjacent to a_1;

(b) one of xa_2a_3 and za_2a_3 is a 5-triangle and the other is at least a 4-triangle;

(c) if xa_1 is light then both xa_2a_3 and za_2a_3 are 5-triangles; and

(d) $\|y, A\| = 0$.

Proof. There exists $A = a_1a_2a_3 \in \mathcal{T}$ such that $\|xz, A\| \geq 9$, since

$$\|xz, U\| \geq 2\left(\frac{4n}{3} - 1\right) - \|xz, W\| \geq \frac{8n}{3} - 2 - 4 > 8\left(\frac{n}{3} - 1\right) = 8(k-1).$$

Say $\|x, A\| \geq \|z, A\|$. Since M is a counterexample, Proposition 3.2.3 implies (d) $\|y, A\| = 0$.

If $\|z, A\| = 3$ then, by Proposition 3.1.2, za_2a_3 is a 4-triangle for some $a_2, a_3 \in A$. In this case $\|x, A\| = 6$, so xa_2a_3 is a 5-triangle and xa_1 is a heavy edge. So (a–c) hold.

If $\|z, A\| \geq 4$ then, by Proposition 3.1.2, za_2a_3 is a 5-triangle for some $a_2a_3 \in A$. In this case $\|x, A\| \geq 5$, so xa_2a_3 is a 4-triangle and x is adjacent to a_1. Furthermore, if xa_1 is light then xa_2a_3 is a 5-triangle. Again (a–c) hold. $\square$

Claim 4. There exists $B \in \mathcal{T} - A$ such that $2\|a_1y, B\| + \|xz, B\| \geq 25$.

Proof. Set $U' := U \smallsetminus V(A)$. Since $xz \notin E$ and $\|y, A\| = 0$,

$$\begin{aligned} 2\|a_1y, U'\| + \|xz, U'\| &\geq 6\left(\frac{4}{3}n - 1\right) - 2\|a_1y, W \cup V(A)\| - \|xz, W \cup A\| \\ &\geq \frac{24n}{3} - 6 - 2(8+4) - (8+8) > 24\left(\frac{n}{3} - 2\right) = 24(k-2). \end{aligned}$$

So there exists $B \in \mathcal{T} - A$ with $2\|a_1y, B\| + \|xz, B\| \geq 25$. $\square$

Let $W' := W \cup \{a_1\} \cup V(B)$. For any edge $e \in \{a_1x, xy, yz\}$ define

$$Q(e) := \{u \in V(B) : \|e, u\| \geq 3\}$$

and for any vertex $v \in \{a_1, x, y, z\}$ and $k \in \{4, 5\}$ define

$$P_k(v) := \{u \in B : T_k \subseteq M[B - u + v]\}.$$

Claim 5. If $v \notin e$ and there exists $u \in P_k(v) \cap Q(e)$ then $M[(V(B) \cup e) + v]$ can be factored into a $(3 + \|e\|)$-triangle and a k-triangle. Moreover:

$$\begin{aligned} &\text{(a) } |Q(e)| \geq \frac{\|e, B\| - 6}{2} \quad \text{(b) } |P_5(v)| \geq \|v, B\| - 3 \\ &\text{(c) } |P_4(v)| = 3 \text{ if } \|v, B\| \geq 5 \text{ and } |P_4(v)| \geq (\|v, B\| - 2) \text{ otherwise.} \end{aligned} \tag{3.2.1}$$

Proof. For the first sentence apply definitions; for (3.2.1) check each argument value. $\square$

To obtain a contradiction, it suffices to find two independent triangles $C, D \subseteq M[W' - w]$ for some $w \in \{a_1, x, y\}$ so that $\{C, D, wa_2a_3\}$ is a factor of $M[W \cup V(A) \cup V(B)]$ consisting of two 5-triangles and one 4-triangle. We further refine this notation by setting $D := vb_{i\oplus 1}b_{i\oplus 2}$ and $C := b_ie$, where $v \in \{a_1, x, y, z\}$, $b_i \in B$ and $e \in E(\{a_1, x, y, z\} - v)$. Then w is defined by $w \in (\{a_1, x, y, z\} \smallsetminus e) - v$; set $W^* := wa_2a_3$.

Claim 6. None of the following statements is true:

(a) $P_4(v) \cap Q(e) \neq \emptyset$ for some $e \in \{xy, yz\}$ and $v \in \{x, z\} \smallsetminus e$.

(b) $P_5(v) \cap Q(e) \neq \emptyset$ for some $e \in \{a_1x, xy, yz\}$ and $v \in \{a_1, x, y, z\} \smallsetminus e$ such that $y \in e + v$.

(c) $P_4(a_1) \cap Q(xy) \neq \emptyset$ and $P_4(a_1) \cap Q(yz) \neq \emptyset$.

(d) There exists $b_i \in P_5(a_1)$ such that $x, y, z \in N(b_i)$.

Proof. By Claim 3, each case implies M is not a counterexample, a contradiction:

(a) Then $w = a_1$, and so $\|W^*\| \geq 5$, $\|C\| \geq 5$ and $\|D\| \geq 4$.

(b) Then $\|D\| \geq 5$. If $\|e\| = 2$ then $\|C\| \geq 5$ and $\|W^*\| \geq 4$; otherwise $e = a_1x$ and, by Claim 3 (c), $\|W^*\| \geq 5$ and $\|C\| \geq 4$.

(c) By Claim 3 (b) , $\|wa_2a_3\| \geq 5$ for some $w \in \{x, z\}$. Set $v := a_1$ and $e := \{x, y, z\} - w$. Then $\|W^*\| \geq 5$, $\|C\| \geq 5$ and $\|D\| \geq 4$.

(d) Set $v := a_1$, choose $w \in \{x, z\}$ so that $\|W^*\| \geq 5$, and set $e := \{x, y, z\} - w$. Then $\|D\| \geq 5$ and $\|C\| \geq 4$. □

Claim 7. $\|a_1, B\| < 5$.

Proof. Suppose not. Let $\{x', z'\} = \{x, z\}$, where $\|x', B\| \geq \|z', B\|$. For $k \in \{4, 5\}$, define

$$s_k(e, v) := |Q(e)| + |P_k(v)|.$$

Then $s_k(e, v) > 3$ implies $Q(e) \cap P_k(v) \neq \emptyset$. We use Claim 5 to calculate $s_k(e, v)$. Observe

$$25 - 2\|a_1, B\| \leq \|x'y, B\| + \|yz', B\|, \text{ and so } \|x'y, B\| \geq 13 - \|a_1, B\|.$$

If $\|a_1, B\| = 6$ then $s_5(x'y, a_1) \geq 1 + 3$, contradicting Claim 6 (b). Otherwise, $\|a_1, B\| = 5$. Either $\|x'y, B\| \geq 9$ or $\|z'y, B\| \geq 7$. In the first case, $s_5(x'y, a_1) \geq 2 + 2$, contradicting Claim 6 (b). In the second case, $s_4(x'y, a_1), s_4(z'y, a_1) > 1 + 3$, contradicting Claim 6 (c). □

Claim 8. $\|\{a_1, y\}, B\| < 9$.

Proof. Suppose $\|\{a_1, y\}, B\| \geq 9$. We consider several cases.

Case 1: $\|a_1, B\| = 4$ and $\|y, B\| = 6$. By Proposition 3.1.2 there are distinct $b, b', b'' \in V(B)$ with $b \in P_5(a_1)$ and $1 \leq \|a_1, b'\| \leq \|a_1, b''\| = 2$. Claim 6 (b) implies $b \notin Q(xy) \cup Q(yz)$; so $x, z \notin N(b)$, since $\|b, y\| = 2$. By Claim 5 (b), $P_5(y) = B$; so

$Q(a_1, x) = \emptyset$ by Claim 6 (b). Thus $\|x, b'\| \leq 2 - \|a_1, b'\|$ and $\|x, b''\| = 0$. By the case $\|\{x, z\}, B\| \geq 5$; thus $\|x, b'\| = 1$ and $\|z, \{b', b''\}\| = 4$; so $\|a_1, b'\| = 1 = \|a_1, b\|$. Thus $b' \in P_4(a_1) \cap Q(xy) \cap Q(yz)$, contradicting Claim 6 (c).

Case 2: $\|a_1, B\| = 3$ and $\|y, B\| = 6$. Then $\|\{x, z\}, B\| \geq 7$. For $\{u, v\} = \{x, y\}$, we have $\|u, B\| \geq 1$. So Claim 5 (a) implies $|Q(uy)| \geq 1$. Thus Claim 6 (a) implies $|P_4(v)| \leq 2$. So Claim 5 (c) implies $\|v, B\| \leq 4$. Thus $3 \leq \|x, B\|, \|z, B\| \leq 4$.

By Proposition 3.1.2, $b \in P_4(a_1)$ for some $b \in B$. So $b \notin Q(xy) \cap Q(yz)$ by Proposition 6 (b). Thus $b \notin N(x) \cap N(z)$. Also $P_5(y) = B$ by $\|y, B\| = 6$. By Claim 6 (b), $Q(a_1 x) = \emptyset$. Thus $\|x, B - b\| \leq 2$. So $xb \in E$ and $\|z, B - b\| = \|z, B\| \geq 3$. Thus $b \in P_4(z) \cap Q(xy)$, contradicting Claim 6 (a).

Case 3: $\|a_1, B\| = 4$ and $\|y, B\| = 5$. Then (i) $\|\{x, z\}, B\| \geq 7$; let $\{x', z'\} := \{x, z\}$, where $\|x', B\| \geq \|z', B\|$. Claim 5 (b) implies $|P_5(y)| \geq 2$; Proposition 3.1.2 implies $b_i \in P_5(a_1)$ for some $i \in [3]$. So by Claim 6 (b,d), (ii) $|Q(a_1 x)| \leq 1$, (iii) $b_i \notin Q(xy) \cup Q(yz)$, and (iv) $xyz \nsubseteq N(b_i)$. Thus by (iii) and Claim 5 (a), $\|x', B\| \leq 5$, and so by (i), $\|z', B\| \geq 2$. By Claim 5 (a), $|Q(x'y)| \geq 2$ and $|Q(yz')| \geq 1$. Claim 6 (a) then implies $|P_4(x')| \leq 2$ meaning (v) $4 \geq \|x', B\| \geq \|z', B\| \geq 3$ and, therefore, by Claim 5 (a), (vi) $|Q(a_1 x)| \geq 1$. By Claim 6 (b,c) $|P_5(a_1)| \leq 1$ and $|P_4(a_1)| \leq 2$. This implies (vii) $b_i \notin N(a_1)$: Otherwise, since $b_i \in P_5(a_1)$ implies $\|b_i, a_1\| \leq 1$, there exist $h, j \in [3] - i$ with $\|b_h, a_1\| = 2$ and $\|b_i, a_1\| = 1 = \|b_j, a_1\|$. If $\|b_i b_j\| = 1$ then $|P_5(a_1)| = 2$; if $\|b_i b_j\| = 2$ then $|P_4(a_1)| = 3$. Either is a contradiction.

By (ii), (vi) and (vii), $Q(a_1x) = \{b_j\}$ for some $j \in [3]-i$, and $\|a_1, b_hb_j\| = 4$, where $h = 6-i-j$. Thus $N(x) \cap B = \{b_i, b_j\}$. By (iv), $z \notin N(b_i)$, and so $N(z) \cap B = \{b_h, b_j\}$. So

$$\begin{aligned}
\|yzb_h\| &= \|z, B\| + \|y, zb_h\| - \|zb_j\| \geq 3 + 3 - 2 = 4 && \text{by (v)} \\
\|xb_ib_j\| &= \|x, B\| + \|b_ib_j\| \geq 3 + 1 = 4 && \text{by (v)} \\
\|yzb_h\| + \|xb_ib_j\| &= \|\{x, z\}, B\| + \|y, zb_h\| - \|zb_j\| + \|b_ib_j\| \\
&\geq \|\{x, z\}, B\| + 2 \geq 9 && \text{by (i)}
\end{aligned}$$

Thus $\{yzb_h, xb_ib_j, A\}$ is a factor of $M(A \cup B \cup W)$ with two T_5 and a T_4, a contradiction.

□

Claim 9. $\|\{a_1, y\}, B\| \geq 9$.

Proof. Suppose $\|\{a_1, y\}, B\| \leq 8$. Then $\|\{x, z\}, B\| \geq 9$ and $\|y, B\| \geq 1$. Proposition 3.2.3 implies there exist independent 4- and 5-triangles in $M[W \cup V(B)]$, a contradiction. □

Observing that Claim 8 contradicts Claim 9, completes the proof of Case 0.

Case 1: $n \equiv 1 \pmod 3$. Choose any vertex $v \in V$, and set $M' = M - v$. By Case 0, M', and so M, contains $\frac{n-4}{3} = \lfloor \frac{n-3}{3} \rfloor$ independent 5-triangles and a 4-triangle.

Case 2: $n \equiv 2 \pmod 3$. Add a new vertex x together with all edges of the form $xv, v \in V$ to M to get M^+. By Case 0, M^+ contains $\frac{n-2}{3} = \lfloor \frac{n-3}{3} \rfloor + 1$ independent 5-triangles and a 4-triangle, at most one of them contains x. So M contains $\lfloor \frac{n-3}{3} \rfloor$ independent 5-triangles and a 4-triangle. □

3.3 Asymptotic results

In this section we prove Theorem 1.3.14, Theorem 1.3.10. We rely on ideas from Levitt, Sárközy and Semerédi [31] throughout.

We first prove the following lemma, which is a large part of the proof of both Theorem 1.3.14 and Theorem 1.3.10.

Lemma 3.3.1. *For every $\varepsilon, \alpha > 0$ there exists $n_0 := n_0(\varepsilon, \alpha)$ such that for every standard multigraph $M = (V, E)$ on $n \geq n_0$ vertices the following holds. If $\delta(M) \geq \left(\frac{4}{3} + \varepsilon\right) n$ and $H(M)$ is not α-splittable then M has an ideal 5-triangle factor.*

Proof. Let $\varepsilon, \alpha > 0$ and let $M = (V, E)$ be standard multigraph on n vertices such that for every $u \in V$

$$\begin{aligned} &\text{(i) } d_M(u) \geq \left(\frac{4}{3} + \varepsilon\right) n \text{ which implies} \\ &\text{(ii) } d_G(u) \geq \left(\frac{2}{3} + \frac{\varepsilon}{2}\right) n \text{ and} \\ &\text{iii) } d_H(u) \geq \left(\frac{1}{3} + \varepsilon\right) n \end{aligned} \tag{3.3.1}$$

where $H := H(M)$ and $G := G(M)$. We will assume throughout that n is sufficiently large. Let $0 < \sigma < \min\{\frac{\varepsilon}{12}, \frac{\sqrt{\alpha}}{16}\}$ and $\tau := \frac{\sigma^{45}}{4}$.

For any $U \subseteq V$ and $k \geq 1$ define $Q_k(U) := \{v \in V : \|v, U\| \geq k\}$. For any $e \in E$,

$$2\left(\frac{4}{3} + \varepsilon\right) n \leq \|e, V\| \leq |Q_4(e)| + 3|Q_3(e)| + 2|\overline{Q_3(e)}| = |Q_4(e)| + |Q_3(e)| + 2n.$$

Therefore,

$$|Q_4(e)| + |Q_3(e)| \geq \left(\frac{2}{3} + 2\varepsilon\right) n, \tag{3.3.2}$$

and since $Q_4(e) \subseteq Q_3(e)$,

$$|Q_3(e)| \geq \left(\frac{1}{3} + \varepsilon\right) n. \tag{3.3.3}$$

For any $u \in V$, let $F(u) := \{e \in E_H : u \in Q_3(e)\}$. Note that

$$2|F(u)| \geq \sum_{v \in N_H(u)} |N(u) \cap N_H(v)|,$$

and for every $v \in N_H(u)$, $|N(u) \cap N_H(v)| \geq \left(\frac{2}{3} + \frac{\varepsilon}{2}\right) n + \left(\frac{1}{3} + \varepsilon\right) n - n = \frac{3\varepsilon}{2} n$. So

$$|F(u)| \geq \frac{1}{2}\left(\frac{1}{3} + \varepsilon\right) \frac{3\varepsilon}{2} n^2 > \frac{\varepsilon}{4} n^2. \tag{3.3.4}$$

Definition 3.3.2. For any disjoint $X, Y \subseteq V$, we will say that Y *absorbs* X if $M[Y]$ and $M[Y \cup X]$ both have 5-triangle factors. Let $Z := (z_1, \ldots, z_{45}) \in V^{45}$. For any $X \in \binom{V}{3}$, call Z an X-*sponge* when $|\operatorname{im}(Z)| = 45$ and $\operatorname{im}(Z)$ absorbs X, and let $f(X)$ be the set of X-sponges. Two sponges Z, Z' are *disjoint* if $\operatorname{im}(Z)$ and $\operatorname{im}(Z')$ are disjoint. For any collection of sponges $\mathcal{F}$ let $V(\mathcal{F}) := \bigcup_{Z \in \mathcal{F}} \operatorname{im}(Z)$.

Definition 3.3.3. For $k > 0$ the tuple $(z_1, \ldots, z_{3k-1}) \in V^{3k-1}$ is a k-*chain* if

(a) $z_1, \ldots, z_{3k-1}$ are distinct vertices,

(b) $z_{3i-2}z_{3i-1}$ is a heavy edge for $1 \leq i \leq k$, and

(c) $z_{3i} \in Q_3(z_{3i-2}z_{3i-1}) \cap Q_3(z_{3i+1}z_{3i+2})$ for $1 \leq i \leq k-1$.

For $u, v \in V$ if $u \in Q_3(z_1z_2)$ and $v \in Q_3(z_{3i-2}z_{3i-1})$ for some $1 \leq i \leq k$ and $u, v \notin \{z_1, \ldots, z_{3k-1}\}$ then we say that the k-chain *joins* u *and* v (see Figure 3.1). For $k > 0$, if there are at least $(\sigma n)^{3k-1}$ k-chains that join u and v we say that u *is* k-*joined with* v.

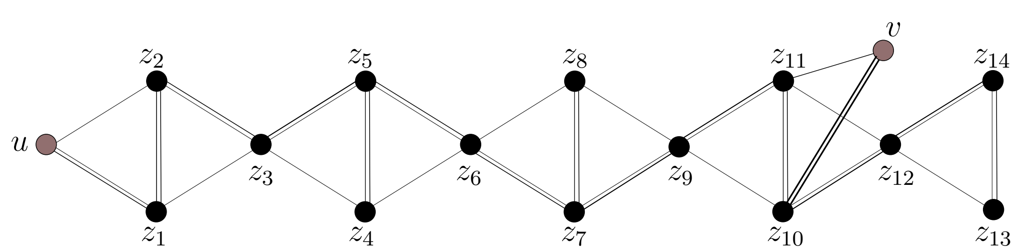

Figure 3.1: The 5-chain $(z_1, \ldots, z_{14})$ joins u and v.

Note that for $1 \leq i < k \leq 5$ if u is i-joined with v then u is k-joined with v. Indeed, using (3.3.3) and (3.3.4), we can extend any i-chain that joins u and v by iteratively picking a vertex $z_{3j} \in Q_3(z_{3j-2}, z_{3j-1})$ and then a heavy edge $z_{3j+1}z_{3j+1} \in F(z_{3j})$ that avoids the vertices $\{u, v, z_1, \ldots, z_{3j-1}\}$ for j from $i+1$ to k in at least $(\sigma n)^{3j}$ ways. For any $u \in V$ define

$$L_k(u) := \{v \in V : v \text{ is } k\text{-joined with } u \text{ for some } 1 \leq i \leq k\}.$$

Note that, by the previous comment, $L_1(u) \subseteq \cdots \subseteq L_5(u)$.

Let $\{x_1, x_2, x_3\} := X \in \binom{V}{3}$, $Y := (z_1, \ldots, z_{45}) \in V^{45}$, and define $m(i) := 15(i-1)$. It is not hard to see that $Y \in f(X)$ if Y satisfies the following for $i \in [3]$:

- the vertices $z_1, \ldots, z_{45}$ are distinct,
- $M[\{z_{m(1)+1}, z_{m(2)+1}, z_{m(3)+1}\}]$ is a 5-triangle, and
- $(z_{m(i)+2}, \ldots, z_{m(i)+15})$ is a 5-chain that joins $z_{m(i)+1}$ and x_i.

Our plan is to use Lemma 2.2.5 to show (i) there is a small set $\mathcal{F}$ of disjoint sponges such that for all 3-sets $X \in \binom{V}{3}$ there exists an X-sponge $Y \in \mathcal{F}$, and (ii) there exists a 3-set $X \subseteq V \setminus V(\mathcal{F})$ such that $M - (X \cup V(\mathcal{F}))$ has a 5-triangle factor. Since there exists an X-sponge in $\mathcal{F}$, this will imply that M has a 5- triangle factor.

To prove (i) we first show that every 3-set is absorbed by a positive fraction of all 45-tuples, and then apply Lemma 2.2.5. The following claim is our main tool.

Claim 1. $L_5(x) = V$ for every vertex $x \in V$.

Proof. We will first show that, for every $u \in V$,

$$|L_1(u)| \geq \left(\frac{1}{3} + \frac{\varepsilon}{3}\right) n \text{ and } u \in L_1(u).$$

By (3.3.4), $|F(u)| \geq (\alpha n)^2$, so $u \in L_1(u)$. Let $t := \sum_{e \in F(u)} |Q_3(e)|$. By (3.3.3), $t \geq |F(u)| \left(\frac{1}{3} + \varepsilon\right) n$. If $v \notin L_1(u)$ then there are less than $(\sigma n)^2 < \varepsilon(\alpha n)^2 \leq \varepsilon |F(u)|$ edges $e \in F(u)$ for which $v \in Q_3(e)$. Therefore,

$$\begin{aligned} |F(u)| \left(\frac{1}{3} + \varepsilon\right) n \leq t &< |F(u)||L_1(u)| + \varepsilon |F(u)||\overline{L_1(u)}| \\ &\leq \varepsilon |F(u)| n + (1 - \varepsilon) |F(u)||L_1(u)|, \end{aligned}$$

and $|L_1(u)| > \frac{n}{3} \cdot (1 - \varepsilon)^{-1} > \left(\frac{1}{3} + \frac{\varepsilon}{3}\right) n$.

Note that for any $u, v \in V$ if $|L_i(u) \cap L_j(v)| \geq 2\sigma n$ and $1 \leq i, j \leq 2$ then $v \in L_{i+j}(u)$. Indeed, we can pick $w \in L_i(u) \cap L_j(v)$ in one of $2\sigma n$ ways and we can then pick an i-chain $(u_1, \ldots, u_{3i-1})$ that joins u and w and a j-chain $(v_1, \ldots, v_{3i-1})$ that joins v and w so that $u, u_1, \ldots, u_i, w, v_j, \ldots, v_1$ and v are all distinct in $\frac{1}{2}(\sigma n)^{3(i+j)-2}$ ways. Since $(u_1, \ldots, u_i, w, v_j, \ldots, v_1)$ is a $(i+j)$-chain that joins u and v and there are $(\sigma n)^{3(i+j)-1}$ such 2-chains, $v \in L_2(u)$.

Let $x \in V$ and suppose, by way of contradiction, that there exists $y \in V$ such that $y \notin L_5(x)$. If there exists $z \notin L_2(x) \cup L_2(y)$, from the preceding argument, we have $|L_1(u) \cap L_1(v)| < 2\sigma n$ for any distinct $u, v \in \{x, y, z\}$. But this is a contradiction, because $3\left(\frac{1}{3} + \frac{\varepsilon}{3}\right) n - 3(2\sigma)n > n$. Therefore, if we let $X := L_2(x)$ and $Y := L_2(y) \setminus L_2(x)$, $\{X, Y\}$ is a partition of V. We have that $|X| \geq |L_1(x)| \geq \left(\frac{1}{3} + \frac{\varepsilon}{3}\right) n$ and, since $y \notin L_4(x)$, $|L_2(y) \cap L_2(x)| < 2\sigma n$ so $|Y| \geq |L_1(y)| - 2\sigma n \geq \left(\frac{1}{3} + \frac{\varepsilon}{6}\right) n$

Call a 4-tuple (v_1, v_2, v_3, v_4) *connecting* if $v_1 \in X$ and $v_4 \in Y$, $v_2v_3 \in E_H$ and $v_1, v_4 \in Q_3(v_2v_3)$. Since M is not α-splittable, $|E_H(X, Y)| \geq \alpha n^2$. Pick some $e := x'y' \in E_H(X, Y)$ where $x' \in X$ and $y' \in Y$. We will show that there are at least $(\sigma n)^2$ connecting 4-tuples which contain x' and y'. Since $M[v_1, v_2, v_3, v_4]$ can contain at most 4 edges from $E_H(X, Y)$, this will imply that there are at least $\frac{1}{4} \cdot \alpha n^2 \cdot (\sigma n)^2 \geq 4(\sigma n)^4$ connecting 4-tuples and this will prove that $y \in L_5(x)$, a contradiction. Indeed, select a connecting 4-tuple (v_1, v_2, v_3, v_4) in $4(\sigma n)^4$ ways. Since v_1 is 2-joined with x there are at least $\frac{1}{2}(\sigma n)^5$ 2-chains that join x and v_1 and avoid $\{v_1, v_2, v_3, v_4\}$. Similarly, there are $\frac{1}{2}(\sigma n)^5$ 2-chains that join v_4 and y and avoid all previously selected vertices. Therefore, there are at least $(\sigma n)^{14}$ 5-chains that join x and y. So, by way of contradiction, assume there are less than $(\sigma n)^2$ connecting 4-tuples containing e.

Suppose $|Q_4(e)| \geq \sigma n$ and pick $z \in Q_4(e)$ and let $T := \{x', y', z\}$. Note that $M[T]$ is a 6-triangle and that

$$3\left(\frac{4}{3} + \varepsilon\right) n \leq \|T, V\| \leq 6|Q_5(T)| + 4|\overline{Q_5(T)}| = 2|Q_5(T)| + 4n$$

so $|Q_5(T)| \geq \frac{3}{2}\varepsilon n$. Pick $w \in Q_5(T)$. Note that there are at least $\sigma n \cdot \frac{3}{2}\varepsilon n \geq (\sigma n)^2$ choices for the pair (z, w) and that if $w \in X$ then (w, x', z, y') is a connecting 4-tuple and if $w \in Y$ then (x', z, y', w) is a connecting 4-tuple. Therefore, we can assume $|Q_4(e)| < \sigma n$ which, by (3.3.2), implies that $|Q_3(e)| \geq \left(\frac{2}{3} + \varepsilon\right) n$.

For any $v_1 \in Q_3(e) \cap X$ and $v_4 \in Q_3(e) \cap Y$, (v_1, x', y', v_4) is a connecting 4-tuple. Therefore, we cannot have $|Q_3(e) \cap X| \geq \sigma n$ and $|Q_3(e) \cap Y| \geq \sigma n$. So suppose $|Q_3(e) \cap X| < \sigma n$. Then $|Y| \geq |Q_3(e) \cap Y| > \frac{2n}{3}$ which contradicts the fact that $|X| > \frac{n}{3}$. Since a similar argument holds when $|Q_3(e) \cap Y| < \sigma n$, the proof is complete. □

Claim 2. For every $X \in \binom{V}{3}$, $|f(X)| \geq 4\tau n^{45}$.

Proof. Recall that $m(i) := 15(i-1)$ and let $\{x_1, x_2, x_3\} := X$. Pick $v_{m(1)+1} v_{m(2)+1} := e$ from one of the at least $\frac{1}{3}n^2$ edges in $H - X$. By (3.3.3), we can pick $v_{m(3)+1}$ from one of the more than $\frac{1}{3}n$ vertices in $Q_3(e) \setminus X$. We have that $M[\{v_{m(1)+1}, v_{m(2)+1}, v_{m(3)+1}]$ is a 5-triangle. For $i \in [3]$, pick a 5-chain $(v_{m(i)+2}, \ldots, v_{m(i)+15})$ that joins $v_{m(i)+1}$ and x_i in one of $\frac{1}{2}(\sigma n)^{14}$ ways. Note that $(v_1, \ldots, v_{45}) \in f(X)$ and there are at least $\frac{1}{72}\sigma^{42} n^{45} \geq 4\tau n^{45}$ such tuples. □

Let $\mathcal{F}$ be the set of 45-tuples guaranteed by Lemma 2.2.5 applied with f, and, say, $a = 4\tau$, $b = \tau/25$ and $c = \tau^2/25$. Let $A := V(\mathcal{F})$ and note that $|A| \leq \tau n/25 < \varepsilon n/2$. Let $M' = M - A$. By Theorem 1.3.11, there is a 5-triangle tiling of $M' - X$ where $X \subseteq V \setminus A$ and $|X| = n - 3(\lfloor (n - |A|)/3 \rfloor - 1)$. Let $X' \subseteq X$ be a 3-set. By Lemma 2.2.5, there exists $Z \in f(X') \cap \mathcal{F}$. By the definition of an X'-sponge there is an ideal 5-triangle tiling of $M[X' \cup \mathrm{im}(Z)]$ and, since every tuple in $\mathcal{A}$ is a sponge, there is a 5-triangle factor of $M[A \setminus \mathrm{im}(Z)]$. This completes the proof. □

Theorem 1.3.14 (Czygrinow, Kierstead & Molla 2013)**.** *For any $\varepsilon > 0$ there exists n_0 such that if M is a standard multigraph on $n \geq n_0$ vertices, $H(M)$ is connected and $\delta(M) \geq \left(\frac{4}{3} + \varepsilon\right) n$ then M has an ideal 5-triangle-tiling.*

Theorem 1.3.10 (Czygrinow, Kierstead & Molla 2013)**.** *For any $\varepsilon > 0$ there exists n_0 such that if D is a directed graph on $n \geq n_0$ vertices, D is strongly 2-connected and $\delta(D) \geq \left(\frac{4}{3} + \varepsilon\right) n$ then D has an ideal $\overrightarrow{C}_3$-factor.*

Proof of Theorem 1.3.10 and Theorem 1.3.14. Set $\alpha := \frac{\varepsilon}{10}$. Let D be a digraph on n vertices such that n is sufficient large and divisible by 3; and

$$\delta(D) \geq \left(\frac{4}{3} + \varepsilon\right) n.$$

Let M be the underlying multigraph of D and let $H := H(M)$. Note that equations (3.3.1) and (3.3.3) hold for M. To prove Theorem 1.3.10, We will show that if H is connected then there is an ideal 5-triangle tiling of M. In the case when H is not connected, we will show D has an ideal cyclic triangle tiling, proving Theorem 1.3.10.

Assume $n \geq n_0(\varepsilon, \alpha^3)$, where n_0 is the function associated with Lemma 3.3.1. We can then also assume that H is α^3-splittable, as otherwise Lemma 3.3.1 implies that M has a 5-triangle factor. So partition $V := V(H)$ as $\{A_1, A_2\}$ so that the quantity $\|A_1, A_2\|_H$ is minimized subject to $|A_1|, |A_2| \geq \left(\frac{1}{3} - \alpha\right) n$. Since H is α^3 splittable, $\|A_1, A_2\|_H \leq \alpha^3 n^2$. Let $F := E_H(A_1, A_2)$. We also now have $|A_i| > n/3$, since, by (3.3.1)(iii), $|F| \geq |A_i|(\left(\frac{1}{3} + \varepsilon)n\right) - (|A_i| - 1))$. Therefore, since $\|A_1, A_2\|_H$ is minimized, for any $x \in A_i$,

$$|N_H(x) \cap A_i| \geq |N_H(x) \cap A_{3-i}|. \tag{3.3.5}$$

The proof proceeds as follows: First, we find a set $\mathcal{T}$ of up to two disjoint triangles such that their removal leaves one of $|A_1|$ or $|A_2|$ divisble by 3, that is, if $A'_i := A_i \setminus \bigcup_{T \in \mathcal{T}} V(T)$ then $|A'_i| = 0 \pmod 3$ for some $i \in \{1, 2\}$. Note that if n is divible by 3 if one of A'_1 or A'_2 is divisible by 3 the other is as well. If $F \neq \emptyset$ the triangles in $\mathcal{T}$ are 5-triangles, otherwise they are cyclic triangles. We then find a 5-triangle factor in both $M[A'_1]$ and $M[A'_2]$. Note that this will prove both theorems since we will find an ideal 5-triangle factor in all cases except when H is disconnected, and, in that case, we will find an ideal cyclic-triangle factor.

Call a triangle a *spanning* if it contains vertices in A_1 and A_2. We call a spanning triangle *type-i* if it has one vertex in A_i and two vertices in A_{3-i}. To show that the desired set $\mathcal{T}$ exists, in the case when $F \neq \emptyset$, we will show that either there are two disjoint 5-triangles or there is a type-1 spanning 5-triangle and a type-2 spanning 5-triangle that are not necessarily disjoint. The case when $F = \emptyset$ is similar except we will find spanning cyclic triangles instead of spanning 5-triangles.

For any $x \in A_i$, define

$$L(x) := \begin{cases} N^+(x) & \text{if } d^+(x) \geq d^-(x) \\ N^-(x) & \text{if } d^+(x) < d^-(x) \end{cases} \quad \text{and}$$

$$S(x) := \begin{cases} N^-(x) \cap A_{3-i} & \text{if } d^+(x) \geq d^-(x) \\ N^+(x) \cap A_{3-i} & \text{if } d^+(x) < d^-(x). \end{cases}$$

Clearly, if $\deg^0(x, A_{3-i}) \geq 1$ then $S(x) \neq \emptyset$ and that for every $y \in S(x)$ and $z \in L(x) \cap N_H(y)$ we have that xyz is a spanning cyclic triangle. Also note that xyz is a spanning 5-triangle if xy is a heavy edge. Furthermore, by (3.3.1) and the definition of $L(x)$,

$$\begin{aligned} |L(x) \cap N_H(y)| &= |L(x)| + |N_H(y)| - |L(x) \cup N_H(y)| \\ &\geq \left(\frac{2}{3} + \varepsilon/2\right)n + \left(\frac{1}{3} + \varepsilon\right)n - n > \varepsilon n. \end{aligned} \tag{3.3.6}$$

Therefore all vertices $x \in V$ for which $S(x) \neq \emptyset$ are contained in a spanning triangle. In particular, if $N_H(x) \cap A_{3-i} \neq \emptyset$ then x is contained in a spanning 5-triangle.

Claim 1. If $F \neq \emptyset$ then there are either two disjoint spanning 5-triangles; or a type-1 spanning 5-triangle and a type-2 spanning 5-triangle that are not necessarily disjoint.

Proof. If there are two independent edges xy and $x'y'$ in F then, by (3.3.3), we can pick $z \in Q_3(xy)$ and $z' \in Q_3(x'y') - z$. to form two disjoint spanning 5-triangles xyz and $x'y'z'$.

So we can assume that no two edges in F are independent, this implies that every edge in F is incident to a vertex x. Let $\{i, j\} = \{1, 2\}$ so that $x \in A_i$ and let $y = N_H(x) \cap A_j$. By (3.3.6), there exists $z \in L(x) \cap N_H(y) \cap A_j$ because $N_H(y) \cap A_i = \{x\}$. Note that xyz is a type-i spanning 5-triangle.

So assume there are no type-j spanning 5-triangles. This implies that there are no edges between $N_H(x) \cap A_i$ and $N_H(x) \cap A_j$. For both $k \in \{1, 2\}$, define $\ell_k := N_H(x) \cap A_k$ and let $y_k \in N_H(x) \cap A_k$. Both y_1 and y_2 must exist by the selection of x and the fact that $\|A, B\|_H$ is minimized. Note that if $k = i$, then y_k has no heavy neighbors in A_{3-k} and if $k = j$, then the only heavy neighbor of y_k in A_{3-k} is x. Therefore, in either case,

$$\begin{aligned}\left(\frac{4}{3} + \varepsilon\right) n &\leq \deg_M(y_k, A_{3-k}) + \deg_M(y_k, A_k) \\ &\leq (|A_{3-k}| - l_{3-k} + 1) + 2(|A_k| - 1) = n + |A_k| - l_{3-k} - 1\end{aligned}$$

so $|A_k| \geq \left(\frac{1}{3} + \varepsilon\right) n + l_{3-k}$. By (3.3.1), $l_1 + l_2 = d_H(x) \geq \left(\frac{1}{3} + \varepsilon\right) n$, so

$$|A_1| + |A_2| \geq 2\left(\frac{1}{3} + \varepsilon\right) n + l_2 + l_1 > n.$$

This contradicts the fact that A_1 and A_2 are disjoint. □

Claim 2. If $F = \emptyset$ there are either two disjoint spanning cyclic-triangles; or a type-1 spanning cyclic triangle and a type-2 spanning cyclic triangle that are not necessarily disjoint.

Proof. We will first show that for any y there are distinct $x, x' \in V - y$ such that $S(x) - y, S(x') - y \neq \emptyset$. To this end let $D' = D - y$ and, for $i \in \{1,2\}$, let $A'_i = A_i - y$ and

$$A_i^+ := \{v \in A'_i : \deg^+(v, A'_{3-i}) > 0\} \text{ and}$$
$$A_i^- := \{v \in A'_i : \deg^-(v, A'_{3-i}) > 0\}.$$

Note first that since G' is strongly connected each of these four sets is non-empty. Also note that it is sufficient to show that $|(A_1^+ \cap A_1^-) \cup (A_2^+ \cap A_2^-)| \geq 2$. So assume the contrary and fix $i \in \{1,2\}$ so that $|A'_i| \geq |A'_{3-i}|$. Further assume that $|A_i^-| \leq |A_i^+|$. The case when $|A_i^+| > |A_i^-|$ is completely analogous. Let $y \in A_i^-$ and $z \in N^-(y) \cap A'_{3-i}$. Note that $z \in A_{3-i}^+$ and, since $y' \in N^+z \cap A_i^+$ implies $y' \in A_i^+ \cap A_i^-$, that $\deg^+(z, A_i^+) \leq 1$. It is also the case that $z \in A_{3-i}^-$, because the fact that

$$|A_i^-| \leq (|A'_i| + |A_i^- \cap A_i^+|)/2 \leq |A'_i|/2 + 1/2,$$

implies

$$\begin{aligned}\deg^+(z, A_i^-) + \deg^+(z, A_i^+) + \deg_M(z, A'_{3-i}) &\leq |A_i^-| + 1 + 2(|A_{3-i}| - 1)\\ &\leq |A_i|/2 + 3/2 + 2(n - |A'_i| - 1) \leq 2n - 3|A_i|/2 - 1/2 \leq 5n/4 - 1/2.\end{aligned}$$

so that $\deg^-(z, A'_i) > 0$. Therefore, $N^-(y) \cap A_{3-i'} \subseteq A_{3-i}^- \cap A_{3-i}^+$. We are done if $y \in A_i^+ \cap A_i^-$, so assume $y \notin A_i^+$. But this implies

$$|A_{3-i}^- \cap A_{3-i}^+| \geq \deg^-(y, A_{3-i}) \geq \varepsilon n > 2,$$

because $\deg_M(y, A_i) \leq 2(|A_i| - 1) < 4n/3$.

The preceding argument clearly gives us distinct $x, x' \in V$ such that $S(x), S(x') \neq \emptyset$. Fix $\{i, j\} = \{1, 2\}$ so that $x \in A_i$. If $x' \in A_j$ then, by (3.3.6), we have a type-i spanning cyclic triangle and a type-j spanning cyclic triangle.

So assume $S(y) = \emptyset$ for every $y \in A_j$. If $S(x) = S(x') = \{y\}$, then, by the preceding argument, there exists $x'', x''' \in G - y$ such that $S(x') - y, S(x'') - y \neq \emptyset$, so we can reset x' to be an element of $\{x'', x'''\} - x$. In any case, since $S(x)$ and $S(x')$ are not empty, $x, x' \in A_i$ and there exists $y \in S(x)$ and $y' \in S(x') - y$. Therefore xy and $x'y'$ are distinct edges, so, by (3.3.6), there are two disjoint spanning cyclic triangles. $\square$

So we have $\mathcal{T}$ our desired set of disjoint triangles. That is, $|\mathcal{T}| \leq 2$ and if we let $V' := V \setminus \bigcup_{T \in \mathcal{T}} V(T)$, $M' := M[V']$, $H' := H[V']$ and $A'_i := A_i \cap V'$ then $A'_1 = A'_2 = 0 \pmod 3$. Let

$$W := \{v \in V' : |E_H(v) \cap F| \geq \alpha n\}$$

and note that $|W| \leq \alpha^2 n$. For every $w \in W$, by (3.3.5), if $w \in A_i$ then $|N_H(w) \cap A_i| \geq \alpha n$ so $|N_H(w) \cap A'_i| \geq \alpha n/2$. Therefore, we can greedily find K_i a matching in H between $A'_i \cap W$ and $A'_i \setminus W$. Let $wx \in K_i$ with $x \notin W$. In M, x has at most $3|\mathcal{T}| + \alpha n < \varepsilon n - 4$ heavy neighbors outside of A'_i. Therefore, because $|A'_i| + 4 \geq \frac{n}{3}$,

$$\begin{aligned}
\|wx, A'_i\| &\geq 2\delta(M) - \|wx, V \setminus A'_i\|_G - \deg_H(w, V \setminus A'_i) - \deg_H(x, V \setminus A'_i) \\
&\geq 2\left(\frac{4}{3} + \varepsilon\right) n - 2(n - |A'_i|) - (n - |A'_i|) - (\varepsilon n - 4) \\
&\geq 3|A'_i| + 4 - \frac{n}{3} n + \varepsilon n \geq 2|A'_i| + \varepsilon n.
\end{aligned}$$

So there are at least $\varepsilon n/2$ vertices $z \in A'_i$ for which $3 \leq \|z, wx\| \leq 4$. Therefore, for every $e \in K_i$ we can greedily select $z_e \in A'_i$ so that $\mathcal{W}_i := \bigcup_{e \in K_i} e z_e$ is a collection of disjoint 5-triangles in $M[A'_i]$.

For both $i \in \{1, 2\}$, remove the vertices from the triangles in $\mathcal{W}_1 \cup \mathcal{W}_2$ from A'_i to form A''_i and let $M'' := M[A''_1 \cup A''_2]$. Let $x \in A''_i$. Since x is not in W, x has at most $3|\mathcal{T}| + 3|W| + 2\alpha n < \varepsilon n$ heavy neighbors outside of A''_i. This with the fact that $\frac{n}{3} \geq |A_i|/2 \geq |A''_i|/2$ and the degree condition, gives us that

$$\begin{aligned}
\|x, A''_i\| &\geq \delta(M) - \deg_G(x, V \setminus A'') - \deg_H(x, V \setminus A'') \\
&\geq \left(\frac{4}{3} + \varepsilon\right) n - (n - |A''_i|) - \varepsilon n = |A''_i| + \frac{n}{3} \geq \frac{3}{2}|A''_i|
\end{aligned}$$

Hence, by Theorem 1.3.12, both $M[A_1'']$ and $M[A_2'']$ have 5-triangle factors. $\square$

Chapter 4

TOURNAMENT TILINGS

4.1 Transitive tournament tilings

In this section we prove Theorem 1.4.5. Our proof is based on the proof of Theorem 1.4.3. Although we do not go into the details, it also provides an $O(kn^2)$ algorithm. Otherwise, our proof could be slightly simplified by avoiding the use of $\mathcal{B}'$.

For simplicity, we shorten *equitable acyclic* to *good*.

Theorem 1.4.5 (Czygrinow, Kierstead & Molla 2013 [6]). *Every digraph G with $\Delta(G) \leq 2k - 1$ has an equitable acyclic k-coloring.*

Proof. We may assume $|G| = sk$, where $s \in \mathbb{N}$: If $|G| = sk - p$, where $1 \leq p < k$, then let G' be the disjoint union of G and $\overrightarrow{K}_p$. Then $|G'|$ is divisible by k, and $\Delta(G') \leq 2k - 1$, any good k-coloring of G' induces a good k-coloring of G.

Argue by induction on $\|G\|$. The base step $\|G\| = 0$ is trivial; so suppose u is a non-isolated vertex. Set $G' := G - E(u)$. By induction, G' has a good k-coloring f. We are done unless some color class U of f contains a cycle C with $u \in C$. Since $\Delta(G) \leq 2k - 1$, for some class W either $\|u, W\|^- = 0$ or $\|u, W\|^+ = 0$. Moving u from U to W yields an acyclic k-coloring of G with all classes of size s, except for one *small* class $U - u$ of size $s - 1$ and one *large* class $W + u$ of size $s + 1$. Such a coloring is called a *nearly equitable* acyclic k-coloring. We shorten this to *useful* k-coloring.

For a useful k-coloring f, let $V^- := V^-(f)$ be the small class and $V^+ := V^+(f)$ be the large class of f, and define an auxiliary digraph $\mathcal{H} := \mathcal{H}(f)$, whose vertices are the color classes, so that UW is a directed edge if and only if $U \neq W$ and $W + y$

is acyclic for some $y \in U$. Such a y is called a *witness* for UW. If $W + y$ contains a directed cycle C, then we say that y is *blocked* in W by C. If y is blocked in W, then

$$\|W, y\| \geq 2. \tag{4.1.1}$$

Let $\mathcal{A}$ be the set of classes that can reach V^- in $\mathcal{H}$, $\mathcal{B}$ be the set of classes not in $\mathcal{A}$, and $\mathcal{B}'$ be the set of classes that can be reached from V^+. Call a class $W \in \mathcal{A}$ *terminal*, if every $U \in \mathcal{A} - W$ can reach V^- in $\mathcal{H} - W$; so V^- is terminal if and only if $\mathcal{A} = \{V^-\}$. Let $\mathcal{A}'$ be the set of terminal classes. A class in $\mathcal{A}$ with maximum distance to V^- in $\mathcal{H}$ is terminal; so $\mathcal{A}' \neq \emptyset$. For any $W \in V(\mathcal{H})$ and any $x \in W$ we say x is *q-movable* if it witnesses exactly q edges in $E^+_{\mathcal{H}}(W, \mathcal{A})$. If x is q-movable for $q \geq 1$, call x *movable*. Set $a := |\mathcal{A}|$, $a' := |\mathcal{A}'|$, $b := |\mathcal{B}|$, $b' := |\mathcal{B}'|$, $A := \bigcup \mathcal{A}$, $A' := \bigcup \mathcal{A}'$, $B := \bigcup \mathcal{B}$ and $B' := \bigcup \mathcal{B}'$. An edge $e \in E(A, B)$ is called a crossing edge; denote its ends by e_A and e_B, where $e_A \in A$.

Claim 1. If $V^+ \in \mathcal{A}$, then G has a good k-coloring.

Proof. Let $\mathcal{P} = V_1 \ldots V_k$ be a V^+, V^--path in $\mathcal{H}$. Moving witnesses y_j of $V_j V_{j+1}$ to V_{j+1} for all j yields a good k-coloring of G. □

Establishing the next lemma completes the proof; notice the weaker degree condition.

Lemma 4.1.1. *A digraph G has a good k-coloring provided it has a useful k-coloring f with*

$$d(v) \leq 2k - 1 \ (= 2a + 2b - 1) \text{ for every vertex } v \in A' \cup B. \tag{4.1.2}$$

Proof. Arguing by induction on k, assume G does not have a good k-coloring.

A crossing edge e with $e_A \in W \in \mathcal{A}$ is *vital* if $G[W + e_B]$ contains a directed cycle C with $e \in E(C)$. In particular if xy is a crossing edge with $\|x, y\| = 2$, then both

xy and yx are vital. For sets $S \subseteq A$ and $T \subseteq B$ denote the number of vital edges in $E(S,T)$, $E^-(S,T)$, and $E^+(S,T)$ by $\nu(S,T)$, $\nu^-(S,T)$, and $\nu^+(S,T)$, respectively. If $S = \{x\}$ or $T = \{y\}$, we drop the braces. Every $y \in B$ is blocked in W; so $\nu^+(W,y), \nu^-(W,y) \geq 1$ and

$$\nu(W,y) \geq 2. \tag{4.1.3}$$

Claim 2. For any $x \in W \in \mathcal{A}'$, if x is q-movable, then

$$\text{(a) } \|x,B\| \leq 2(b+q)+1-\|x,W\| \text{ and (b) } \nu(x,B) \leq 2(b+q).$$

Proof. (a) There are $(a-1)-q$ classes in $\mathcal{A} \setminus W$ in which x is blocked. So (4.1.1) gives that $\|x, A \setminus W\| \geq 2a-2q-2$. With (4.1.2), this implies

$$\|x,B\| \leq 2a+2b-1-\|x,A \setminus W\| - \|x,W\| \leq 2(b+q)+1-\|x,W\|.$$

(b) By (a), $\nu(x,B) \leq 2(b+q)+1-\|x,W\|$. So the desired inequality holds if $\|x,W\| \geq 1$ or $\nu(x,B)$ is even. If $\|x,W\| = 0$, then every vital edge incident to x must be heavy. This implies that $\nu(x,B)$ is even. □

Claim 3. V^- is not terminal.

Proof. If V^- is terminal, then $\mathcal{A} = \{V^-\}$ and $a = 1$; thus there are no movable vertices. Claim 2(b) implies $\nu(u,B) \leq 2b$ for all $u \in A$ and (4.1.3) implies $\nu(A,w) \geq 2$ for all $w \in B$. This yields the contradiction

$$2(bs+1) = 2|B| \leq \nu(A,B) \leq 2b(s-1).$$

□

Using Claim 1 and Claim 3, $V^+ \in \mathcal{B}$ and $\mathcal{A} \neq \mathcal{A}'$; thus

$$|A| = as-1, |A'| = a's, |B| = bs+1, \text{ and } |B'| = b's+1. \tag{4.1.4}$$

The next claim provides a key relationship between vertices in A' and vertices in B.

Claim 4. For all $x \in W \in \mathcal{A}'$, and $y \in B$:

(a) if $G[W - x + y]$ is acyclic, then x is not movable; and

(b) there is no $y' \in B' - y$ such that $G[W - x + y + y']$ is acyclic.

Proof. By Claim 1 and Claim 3, $W \notin \{V^-, V^+\}$. Suppose there exists $y \in B$ such that $G[W - x + y]$ is acyclic. If there exists $y' \in B' - y$ such that $G[W - x + y + y']$ is acyclic, put $y_1 := y'$, $y_2 := y$ and $Y := \{y_1, y_2\}$; else put $y_1 := y$ and $Y := \{y_1\}$. Since $W \in \mathcal{A}$, it contains a movable vertex. If x is movable put $x' := x$; else let $x' \in W$ be any movable vertex; say x' witnesses WU, where $U \in \mathcal{A}$. Let $X := \{x', x\}$ and $W' := W \smallsetminus X + y_1$.

Moving x' to U and switching witnesses along a U, V^--path in $\mathcal{H} - W$ yields a good $(a-1)$-coloring f_1 of $G_1 := G[A \smallsetminus W + x']$. Also f induces a b-coloring f_2 of $G_2 := G[B - y_1]$. It is good if $y_1 \in V^+$; else it is useful. Since every $v \in B - y_1$ is blocked in every color class in $\mathcal{A}(f)$, (4.1.1) and (4.1.2) imply $\Delta(G_2) \leq 2k - 1 - 2a = 2b - 1$. By induction, there is a good b-coloring g_2 of G_2. (For algorithmic considerations, note that if $y_1 \in B'$, as when $|Y| \geq 2$, then g_2 is immediately constructible from f_2 using Claim 1, since then $V^+(f_2) \in \mathcal{A}(f_2)$.)

If $|X| = 1$, then $|W'| = s$. So g_1, W' and g_2 form a good k-coloring of G. This completes the proof of (a) (see Figure 4.1). To prove (b), suppose $|X| = |Y| = 2$. It suffices to show that $G_3 := G[(B - y_1) + (W' + x)]$ has a good $(b+1)$-coloring. By the case, x is blocked in every class of $\mathcal{A} \smallsetminus W$; so $\|x, A \smallsetminus W\| \geq 2a - 2$ by (4.1.1). Thus $Z + x$ is acyclic for some class $Z \in \mathcal{B} + W'$. So G_3 has a useful $(b+1)$-coloring f_3 with $V^-(f_3) = W'$ and $V^+(f_3) = Z + x$, or $Z = W'$ and f_3 is already good. Since $W' + y_2$ is acyclic, $W' \notin \mathcal{A}'(f_3) \cup \mathcal{B}(f_3)$. By the definitions of x and $B(f)$, every $v \in V(G_3) \smallsetminus W'$ is blocked in every color class in $\mathcal{A}(f) - W$. Thus, by (4.1.1) and (4.1.2), $\|v, V(G_3)\| \leq 2(b+1) - 1$. So, by induction, there exists a good $(b+1)$-coloring g_3 of G_3 (see Figure 4.2). □

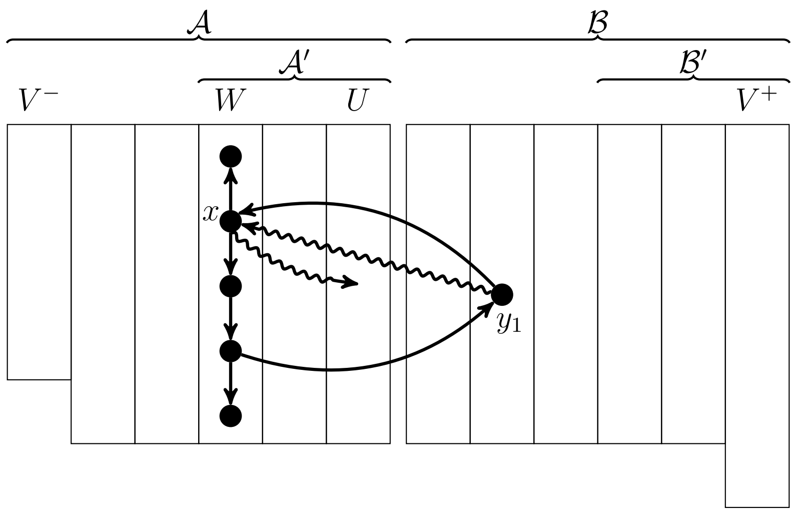

Figure 4.1: After moving x and y_1 as indicated, switching witness along a U, V^--path in $\mathcal{H} - W$ creates a good a-coloring of $G[A + y_1]$. By induction, there is a good b-coloring of $G[B - y_1]$.

A crossing edge $e \in E(W, B)$ is *lonely* if it is vital and either (i) $e \in E^-(W, B)$ and $\nu^-(W, e_B) = 1$ or (ii) $e \in E^+(W, B)$ and $\nu^+(W, e_B) = 1$. If (i), then e is *in-lonely*; if (ii), then e is *out-lonely*. If e is lonely, then $G[W - e_A + e_B]$ is acyclic. For sets $S \subseteq A$ and $T \subseteq B$ denote the number of lonely, in-lonely and out-lonely edges in $E(S, T)$ by $\lambda(S, T)$, $\lambda^-(S, T)$ and $\lambda^+(S, T)$, respectively; drop braces for singletons. If $y \in B$, then y is blocked in W. So

$$\nu(W, y) + \lambda(W, y) \geq 4. \tag{4.1.5}$$

Claim 5. $a' > b$.

Proof. Assume $a' \leq b$. Order $\mathcal{A}$ as $X_1 := V^-, X_2, \ldots, X_a$ so that for all $j > 1$ there exists $i < j$ with $X_j X_i \in E(\mathcal{H})$, and subject to this, order $\mathcal{A}$ so that l is maximum, where l is the largest index of a non-terminal class. Set $W := X_a$.

The deletion of any non-terminal class leaves some class which can no longer reach V^- in $\mathcal{H}$; thus $l < a$, i.e., W is terminal. Also $N^+_{\mathcal{H}}(W) \subseteq \mathcal{A}' + X_l$, since otherwise

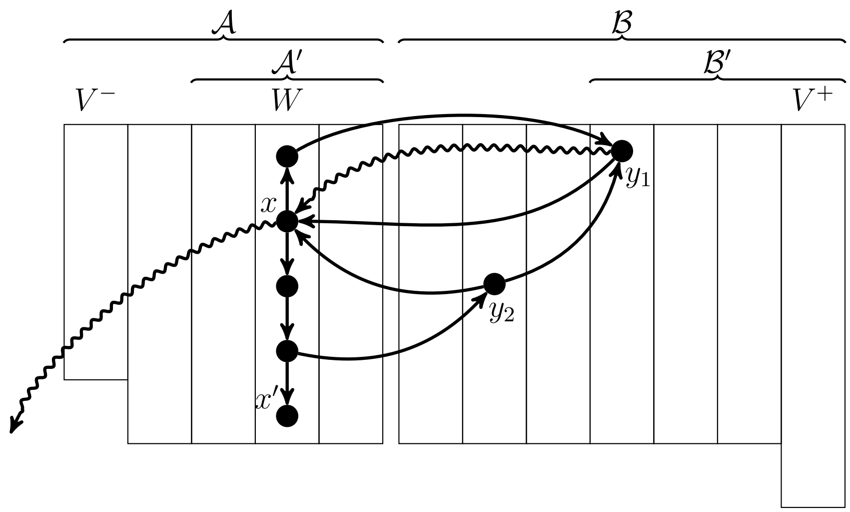

Figure 4.2: After moving x and y_1 as indicated, switching witnesses (one of which is x') creates a good $(a-1)$-coloring of $G[A - W + x']$ and a good b-coloring g_2 of $G[B - y_1]$. Placing x in a color class of g_2 gives a useful $(b+1)$-coloring of $G[B+W-x']$ with small class $W' := W - x - x' + y_1$. By induction, there is a good $(b+1)$-coloring of $G[B + W - x']$ because $G[W' + y_2]$ is acyclic.

we could increase the index l by moving W in front of X_l. So if $x \in W$ is q-movable, then

$$q \leq a'. \tag{4.1.6}$$

If $\lambda(x, B) \geq 1$, then there exists $y \in B$ such that $\nu^+(x, y) = 1$ or $\nu^-(x, y) = 1$. In either case, $W - x + y$ is acyclic. Therefore, Claim 4(a) implies $q = 0$; since every lonely edge is vital, this and Claim 2(b) imply, $\lambda(x, B) \leq \nu(x, B) \leq 2b$. If $\lambda(x, B) = 0$, then Claim 2(b) and (4.1.6) gives $\nu(x, B) \leq 2(b + a') \leq 4b$. Regardless, $\lambda(x, B) + \nu(x, B) \leq 4b$. So

$$\lambda(W, B) + \nu(W, B) = \sum_{x \in W} \lambda(x, B) + \nu(x, B) \leq 4b|W| \leq 4bs.$$

This is a contradiction, since (4.1.5) and (4.1.4) imply

$$\lambda(W, B) + \nu(W, B) = \sum_{y \in B} \lambda(W, y) + \nu(W, y) \geq 4|B| > 4bs. \qquad \square$$

A crossing edge $e \in E(W, B)$ is *solo* if either (i) $e \in E^{-}(W, B)$ and $\|W, e_B\|^{-} = 1$ or (ii) $e \in E^{+}(W, B)$ and $\|W, e_B\|^{+} = 1$. If (i), then e is *in-solo*; if (ii), then e is *out-solo*. For sets $S \subseteq A$ and $T \subseteq B$ denote the number of solo, in-solo and out-solo edges in $E(S, T)$ by $\sigma(S, T)$, $\sigma^{-}(S, T)$ and $\sigma^{+}(S, T)$, respectively; drop braces for singletons. If $y \in B$, then y is blocked in W. So

$$\|W, y\| + \sigma(W, y) \geq 4. \tag{4.1.7}$$

Every $y \in B'$ is blocked in every color class in $\mathcal{A} \cup (\mathcal{B} \smallsetminus \mathcal{B}')$. So (4.1.1) and (4.1.2) give

$$\|A', y\| \leq 2a+2b-1-\|A \smallsetminus A', y\|-\|B \smallsetminus B', y\|-\|y, B'\| \leq 2a'+2b'-1-\|y, B'\|. \tag{4.1.8}$$

Using (4.1.7) and (4.1.8) we have

$$\begin{aligned} \sigma(A', y) &\geq \sum_{W \in \mathcal{A}'} (4 - \|W, y\|) = 4a' - \|A', y\| \geq 2a' - 2b' + \|y, B'\| + 1 \\ &= 2(a' - b') + 2\|y, B'\|_H + \|y, B'\|_L + 1. \end{aligned} \tag{4.1.9}$$

Choose a maximal set I subject to $V^{+} \subseteq I \subseteq B'$ and $G[I]$ contains no 2-cycle. Let

$$J := \{y \in I : \sigma(A', y) = 2(a' - b') + 2\,\|y, B'\|^{h} + 1\}.$$

Note that, by (4.1.9), the vertices in J have the minimum possible number of solo-neighbors in A' and additionally are incident with no light edges in B'.

Claim 6. Every $x \in A'$ satisfies $\sigma(x, I) \leq 2$. Furthermore, if there are distinct $y_1, y_2 \in I$ such that $\sigma(x, y_1), \sigma(x, y_2) \geq 1$, then $\{y_1, y_2\} \subseteq I \smallsetminus J$.

Proof. Suppose $\sigma(x, I) \geq 3$ for some $x \in W \in \mathcal{A}'$. By Claim 3, $W \neq V^{-}$. There exist distinct $y_1, y_2 \in I$ such that either $\sigma^{+}(x, \{y_1, y_2\}) = 2$ or $\sigma^{-}(x, \{y_1, y_2\}) = 2$. Suppose $\sigma^{+}(x, \{y_1, y_2\}) = 2$. Then $\|y_i, W - x + y_i\|^{+} = 0$ for each $i \in [2]$. The choice of I

implies $\|y_1, y_2\| \leq 1$. So there exists $i \in [2]$ with $\|y_i, W - x + y_1 + y_2\|^+ = 0$. Thus $G[W - x + y_1 + y_2]$ is acyclic, contradicting Claim 4(b).

Now suppose there exist distinct $y_1 \in J$ and $y_2 \in I$ with $\sigma(x, y_1), \sigma(x, y_2) \geq 1$. By the definition of J and (4.1.9), $\|y_1, B'\|_L = 0$. Therefore, by the definition of I, $\|y_1, I\| = 0$ and in particular $\|y_1, y_2\| = 0$. So again $G[W - x + y_1 + y_2]$ is acyclic, contradicting Claim 4(b). $\square$

The maximality of I implies that for all $y \in B \smallsetminus I$ there exists $v \in I$ with $\|y, v\| = 2$. Therefore,

$$\sum_{y \in I}(2\|y, B'\|_H + 2) = 2\|B' \smallsetminus I, I\|_H + 2|I| \geq 2|B' \smallsetminus I| + 2|I| = 2|B'|. \tag{4.1.10}$$

Also, by (4.1.9) and the definition of J, for every $y \in I \smallsetminus J$,

$$\sigma(A', y) \geq 2(a' - b') + 2\|y, B'\|_H + 2. \tag{4.1.11}$$

Therefore, by (4.1.11), (4.1.10), (4.1.4), Claim 5, and the fact that $|I| \geq |V^+| > s$,

$$\begin{aligned}
\sigma(A', I) + |J| &= \sum_{y \in I \setminus J} \sigma(A', y) + \sum_{y \in J} (\sigma(A', y) + 1) \\
&\geq \sum_{y \in I} (2\,(a' - b') + 2\|y, B'\|_H + 2) > 2s(a' - b') + 2|B'| > 2|A'| \\
\sigma(A', I) &> 2|A'| - |J|.
\end{aligned} \tag{4.1.12}$$

Claim 6 only gives $\sigma(A', I) \leq 2|A'|$, so we have not reached a contradiction yet. However, we will be saved by the fact that every vertex in J forces at least one fewer solo edge between A' and I. Formally, let $A'_1 := \{x \in A' : \sigma(x, I) \leq 1\}$ and note that we can now write

$$\sigma(A', I) \leq 2|A'| - |A'_1|. \tag{4.1.13}$$

Claim 7. $|A'_1| \geq |J|$

Proof. For any $y \in J$, by the definition of J, $\sigma(A', y)$ is odd. This implies that there exists $x \in A'$ such that $\sigma(x, y) = 1$. By Claim 6, $\sigma(x, y') = 0$ for all $y' \in I - y$. Therefore $x \in A'_1$. □

Finally by (4.1.12), (4.1.13), and Claim 7,

$$2|A'| - |J| < \sigma(A', I) \leq 2|A'| - |A'_1| \leq 2|A'| - |J|,$$

a contradiction. This completes the proof of Lemma 4.1.1. □

Applying Lemma 4.1.1 to the useful k-coloring f completes the proof of Theorem 1.4.5. □

4.2 Universal tournament tilings

In this section we prove the following theorem.

Theorem 1.4.14. *For all $s \geq 4$ and $\varepsilon > 0$ there exists n_0 such that if M is a standard multigraph on $n \geq n_0$ vertices, where n is divisible by s, then the following holds. If $\delta(M) \geq 2\frac{s-1}{s}n + \varepsilon n$ then there exists a perfect tiling of M with acceptable s-cliques.*

We begin with a proof of Proposition 1.4.11.

Proposition 1.4.11. *Theorem 1.4.8 and Conjecture 1.4.9 imply Conjecture 1.4.10.*

Proof. Assume Conjecture 1.4.9 is true. Let D be an orientation of F. We will argue by induction on c. Let D_1 be the largest component in D, $D_2 := D - D_1$ and $m_i := \|D_i\|$ for $i \in \{1, 2\}$. We can assume that $m_1 \geq 3$. Indeed, if $m_1 \leq 2$ then F is a collection of disjoint paths each on at most 3 vertices. Since $\|F\| \leq 1$ when $n = 3$, Theorem 1.4.8 implies that there is an embedding of D into T.

Because there are $c - 1$ non-trivial components in D_2, $m_2 \geq c - 1$. Therefore, $m_1 \leq n/2$ and, since D_1 is a tree, $2|D_1| - 2 \leq 2(n/2 + 1) - 2 = n$. Conjecture 1.4.9

then implies that there is an embedding ϕ of D_1 into T. Note that this handles the case when $c = 1$.

Let $T_2 := T - \phi(V(D_1))$. Since $m_1 \geq 3$ we have that $m_1 \geq (m_1 + 1)/2 + 1$. Therefore, $\|D_2\| \leq n/2+c-1-m_1 \leq (n-m_1-1)/2+(c-1)-1$. Since $|T_2| = n-m_1-1$, there is an embedding of D_2 into T_2 by induction. □

Let K be a full clique on at most s vertices. It is *fit* if $\|K\|_L \leq \max\{0, |K| - s/2\}$. It is a *near matching* if either $\|v, K\|_L \leq 1$ for every vertex $v \in K$; or $|K| = s$, $\|v, K\|_L \leq 2$ for every vertex $v \in K$ and $\|v, K\|_L = 2$ for at most one vertex $v \in K$. It is not hard to see that both fit and near matching cliques are *acceptable*.

We now show with Theorem 1.4.13 that for fixed s we can tile all but at most a constant number of vertices of M with universal s-cliques.

The following is a key step in the proof.

Lemma 4.2.1. *Let $1 \leq t \leq s-1$ and suppose M is a standard multigraph. If X_1 and X_2 are fit t-cliques, Y is a fit s-clique, and $\|X_i, Y\| \geq 2(s-1)t+2-i$ for $i \in [2]$, then $M[X_1 \cup X_2 \cup Y]$ contains two disjoint fit cliques with orders $t+1$ and s respectively.*

Proof. Put $Y_i^c := \{y \in Y : \|X_i, y\| = 2t - c\}$ and choose $x_1 \in X_1$ with $\|x_1, Y\|_L \leq 1$.

Assume there exists $y \in Y_1^0 \cup Y_2^0$ such that $\|y, Y\|_L \geq 1$. If $y \in Y_2^0$, then $Y - y + x_1$ and $X_2 + y$ are fit. If $y \in Y_1^0 \setminus Y_2^0$, then $X_1 + y$ is fit and $\|X_2, Y - y\| \geq 2(s-1)t - (2t - 1) = 2(s-2)t + 1$ so there exists $x_2 \in X$ such that $\|x_2, Y - y\|_L \leq 1$ and $Y - y + x_2$ is fit.

So we can assume $\|Y_1^0 \cup Y_2^0, Y\|_L = 0$. Since

$$|Y_i^0| + (2t-1)s \geq 2|Y_i^0| + |Y_i^1| + (2t-2)s \geq ||X_i, Y|| \geq 2(s-1)t + 2 - i,$$

we have

$$\text{(a) } |Y_i^0| \geq s - 2t + 2 - i \quad \text{and} \quad \text{(b) } |Y_i^0| + \frac{1}{2}|Y_i^1| \geq s - t + 1 - \frac{i}{2}. \qquad (4.2.1)$$

By (4.2.1), if $t < s/2$ there exists $y_2 \in Y_2^0$ and if $t \geq s/2$ there exists $y_2 \in Y_2^0 \cup Y_2^1$. Note that in either case $X_2 + y_2$ is fit. As Y is full, $\alpha(L[Y_1^1]) \geq \frac{1}{2}|Y_1^1|$. So by (4.2.1.b) there exists $I_1 \subseteq Y_1^1$ such that $\|I_1 \cup Y_1^0\|_L = 0$ and $|I_1 \cup Y_1^0| \geq s - t + 1$. Therefore we can select $Z_1 \subseteq I_1 \cup Y_1^0 - y_2$ such that $|Z_1| = s - t$ and, by (4.2.1.a) $|Z_1 \cap Y_1^0| \geq s - 2t$. $X_1 \cup Z_1$ is full and, because $\|Z_1, X_1\|_L = |Z_1 \cap Y_1^1| \leq \min\{t, s-t\} \leq s/2$, $X_1 \cup Z_1$ is fit.

□

Theorem 1.4.13. *For any $s \geq 4$ and any standard multigraph M on on n vertices with $\delta(M) \geq 2\frac{s-1}{s}n - 1$, there exists a disjoint collection of acceptable s-cliques that tile all but at most $s(s-1)(2s-1)/3$ vertices of M.*

Proof. Let $\mathcal{M}$ be a set of disjoint fit cliques in M, each having at most s vertices. Let p_i be the number of i-cliques in $\mathcal{M}$ and pick $\mathcal{M}$ so that $(p_s, \ldots, p_1)$ is maximized lexicographically. Put $\mathcal{Y} := \{Y \in \mathcal{M} : |Y| = s\}$ and $\mathcal{X} = \mathcal{M} - \mathcal{Y}$. Set $U := \bigcup_{X \in \mathcal{X}} V(X)$, $W := \bigcup_{Y \in \mathcal{Y}} V(Y)$. Assume, for a contradiction, that $|U| > s(s-1)(2s-1)/3$. We claim that for all $X, X' \in \mathcal{X}$ with $|X| \leq |X'|$, $\|X, X'\| \leq 2\frac{s-2}{s-1}|X'||X|$.

If $X = X'$, then $\|X, X'\| \leq 2(|X|-1)|X| \leq 2\frac{s-2}{s-1}|X|^2$.

If $X \neq X'$, then the maximality of $\mathcal{M}$ implies $x + X'$ is not a fit $(|X'|+1)$-clique for any $x \in X$. Thus

$$\|X, X'\| \leq \begin{cases} (2|X'|-1)|X| = 2\frac{2|X'|-1}{2|X'|}|X'||X| \leq 2\frac{s-2}{s-1}|X'||X| & \text{if } |X'| \leq \frac{s-1}{2}; \\ (2|X'|-2)|X| = 2\frac{|X'|-1}{|X'|}|X'||X| \leq 2\frac{s-2}{s-1}|X'||X| & \text{if } \frac{s}{2} \leq |X'| \leq s-1. \end{cases}$$

Therefore by the claim,

$$\|X, U\| \leq 2\frac{s-2}{s-1}|U||X| = 2\frac{s-1}{s}|U||X| - \frac{2}{s(s-1)}|U||X| < 2\frac{s-1}{s}|U||X| - |X|.$$

By the degree condition,

$$\|X, W\| > 2\frac{s-1}{s}|W||X|. \tag{4.2.2}$$

Since

$$\sum_{t=1}^{s-1} 2t^2 = \frac{s(s-1)(2s-1)}{3} < |U| = \sum_{t=1}^{s-1} tp_t,$$

there exists $t \in [s-1]$ with $p_t \geq 2t+1$. Choose $\mathcal{X}' \subseteq \mathcal{X}$ such that $|\mathcal{X}'| = 2t+1$ and $|X| = t$ for every $X \in \mathcal{X}'$. Put $U' := \bigcup_{X \in \mathcal{X}'} V(X)$.

By (4.2.2), there exists $Y \in \mathcal{Y}$ such that $\|U', Y\| \geq 2\frac{s-1}{s}|U'||Y| + 1 = 2(s-1)t|\mathcal{X}'| + 1$. Let $X_1, \ldots, X_{2t+1}$ be an ordering of $\mathcal{X}'$ such that $\|X_i, Y\| \geq \|X_{i+1}, Y\|$ for $i \in [2t]$. Clearly, $2(s-1)t + 2t \geq \|X_1, Y\| \geq 2(s-1)t + 1$, so

$$\|U' - V(X_1), Y\| \geq 2(s-1)t(2t+1) + 1 - (2(s-1)t + 2t) = (2(s-1)t - 1)2t + 1.$$

This implies $\|X_2, Y\| \geq 2(s-1)t$. Lemma 4.2.1 applied to X_1, X_2 and Y then gives a contradiction to the maximality of $\mathcal{M}$. □

Lemma 4.2.2. *Let $s \geq 2$, $\varepsilon > 0$, and $M = (V, E)$ be a standard multigraph on n vertices. If $\delta(M) \geq 2\frac{s-1}{s}n + \varepsilon n$, then for all distinct $x_1, x_2 \in V$, there exists $A \subseteq V^{s-1}$ such that $|A| \geq (\varepsilon n)^{s-1}$ and for every $T \in A$ both $\mathrm{im}(T) + x_1$ and $\mathrm{im}(T) + x_2$ are near matching s-cliques.*

Proof. For $0 \leq t \leq s-1$, the t-tuple $T \in V^t$ is called *useful* if, for both $i \in \{1, 2\}$, $\mathrm{im}(T) + x_i$ is a near matching and $\|x_i, \mathrm{im}(T)\|_L \leq \max\{0, t - s + 3\}$. To complete the proof we will show that there exists a set $A \subseteq V^{s-1}$ such that $|A| \geq (\varepsilon n)^{s-1}$ and every $T \in A$ is useful. Suppose there is no such set.

Let $0 \leq t < s-1$ be the maximum integer for which there exists $A \subseteq V^t$ such that $|A| \geq (\varepsilon n)^t$ and every $T \in A$ is useful (note that 0 is a candidate since the empty function $f : \emptyset \to V$ is useful and $V^0 = \{f\}$) ; select A so that in addition $\sum_{T \in A} \|M[\mathrm{im}(T)]\|$ is maximized. Since t is maximized, there exists $(v_1, \ldots, v_t) \in A$ with less than εn extensions $(v_1, \ldots, v_t, v)$ to a useful $(t+1)$-tuple.

Let $m := n/s$, $Y := \{x_1, x_2, v_1, \ldots, v_t\}$, and $V_c := \{v \in V : \|v, Y\| \geq 2t + 4 - c\}$. Then $|V_0| \leq \varepsilon n$, since each $v \in V_0$ extends $(v_1, \ldots, v_t)$. Define

$$Z := \begin{cases} V_1 \cap N_H(x_1) \cap N_H(x_2) & \text{if } t \leq s - 4 \\ V_1 & \text{if } s - 3 \leq t \leq s - 2 \end{cases}.$$

We claim that $|Z| \geq (t+1)\varepsilon n$. Since $(t+2)(2s-2)m + (t+2)\varepsilon n \leq \|Y, V\| \leq |V_0| + |V_1| + (2t + 4 - 2)sm$, we have

$$|V_1| \geq 2(s - 2 - t)m + (t+2)\varepsilon n - |V_0| \geq (t+1)\varepsilon n. \tag{4.2.3}$$

So we are done unless $t \leq s - 4$. In this case, note that $|N_H(x_i)| \geq (s-2)m + \varepsilon n$ for $i \in \{1, 2\}$, which combined with (4.2.3) gives

$$|Z| \geq 2(s - 2 - t)m + (t+1)\varepsilon n - 4m \geq (t+1)\varepsilon n.$$

So there exists $z \in Z \subseteq V_1$ such that $(v_1, \ldots, v_t, z)$ is not useful. Let $\{y\} = N_L(z) \cap Y$. The definitions of useful and Z imply $y \notin \{x_1, x_2\}$ and $\|y, Y\| = 1$. But then $\|Y - y + z\| > \|Y\|$, contradicting the maximality of $\sum_{T \in A} \|M[\text{im}(T)]\|$. □

Theorem 1.4.14. *For all $s \geq 4$ and $\varepsilon > 0$ there exists n_0 such that if M is a standard multigraph on $n \geq n_0$ vertices, where n is divisible by s, then the following holds. If $\delta(M) \geq 2\frac{s-1}{s}n + \varepsilon n$ then there exists a perfect tiling of M with acceptable s-cliques.*

Proof. Assume $s \geq 2$ as otherwise the theorem is trivial. Let $d := s^2$ and $\alpha := \frac{\varepsilon^d}{2}$. For any $S \in \binom{V}{s}$ call $Z \in \binom{V-S}{d}$ an *S-sponge* if both $M[Z]$ and $M[Z \cup S]$ have a perfect acceptable s-clique tiling. Define $f : \binom{V}{s} \to 2^{V^d}$ by

$$f(S) := \{T \in V^d : \text{im}(T) \text{ is an } S\text{-sponge}\}.$$

Claim. $|f(S)| \geq \alpha n^d$ for every $S \in \binom{V}{s}$.

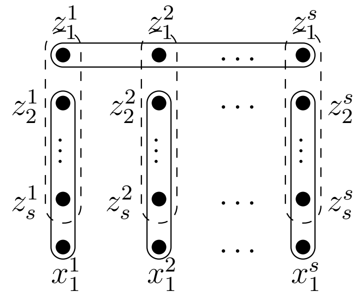

Figure 4.3: An S-sponge. Note that the tuples indicated by the dashed lines form a tiling and the tuples indicated by the solid lines form a larger tiling.

Proof. Let $S := \{x_1^1, \dots, x_1^s\} \in \binom{V}{s}$. By Lemma 4.2.2 there are (many) more than $(\varepsilon n)^s$ tuples $T_0 \in V^s$ such that $\mathrm{im}(T_0)$ is an acceptable s-clique and $\mathrm{im}(T_0) \cap S = \emptyset$. Let $(z_1^1, \dots z_1^s)$ be one such tuple. Again by Lemma 4.2.2, for every $i \in [s]$ there are at least $(\varepsilon n)^{s-1}$ tuples $T_i = (z_2^i, \dots, z_s^i) \in V^{s-1}$ such that $x_1^i + \mathrm{im}(T_i)$ and $z_1^i + \mathrm{im}(T_i)$ are both acceptable s-cliques. Therefore, when n is sufficiently large, there are at least $(\varepsilon n)^s((\varepsilon n)^s)^{s-1} \geq \alpha n^d$ tuples $T := (z_1^1, \dots, z_s^1, \dots, z_1^s, \dots, z_s^s)$ such that if we define $Z := \mathrm{im}(T)$, $Z_0 := \{z_1^1, \dots, z_1^s\}$ and $Z_i := \{z_2^i, \dots, z_s^i\}$ for every $i \in [s]$, then

- $Z \in \binom{V-S}{d}$;
- $\{z_1^i + Z_i : i \in [s]\}$ is a perfect acceptable s-clique tiling of $M[Z]$; and
- $Z_0 + \{x_1^i + Z_i : i \in [s]\}$ is a perfect acceptable s-clique tiling of $M[Z \cup S]$. □

With $a = \alpha$, let $b, c < \min\{a, \frac{\varepsilon}{2}\}$ be constants that satisfy the hypothesis of Lemma 2.2.5, and let $\mathcal{F} \subset V^d$ be a set guaranteed by the lemma. Let $Q := \bigcup_{T \in \mathcal{F}} \mathrm{im}(T)$ and note that $|Q| = d|\mathcal{F}| < \frac{\varepsilon}{2} n$. Let $M' := M - Q$.

We now can apply Theorem 1.4.13 to M' to tile all of the vertices of M' with acceptable s-cliques except a set X of order at most $s(s-1)(2s-1)/3$. Partition X into sets of size s. If n is sufficiently large, $|X| \leq scn$. Therefore, for every set S in the partition of X, we can choose a unique $T \in f(S) \cap \mathcal{F}$. This implies that there is a perfect acceptable s-clique tiling of $M[X \cup Q]$ which completes the proof. □

Chapter 5

ODD HEAVY CYCLE TILINGS

In this section we prove the following theorem.

Theorem 1.5.5 (Czygrinow, Kierstead & Molla 2013 [8])**.** *For any odd $k \geq 5$ there exists n_0 such that the following holds. If M is a standard multigraph on $n \geq n_0$ vertices, n is divisible by k and $\delta(M) \geq \frac{3n-3}{2}$ then M contains a heavy k-cycle-factor.*

This proof follows the stability approach, and we start with the extremal case. To avoid confusion, since the letter d is used for density extensively in this section we use $\deg_G(v)$ for the degree of v in G.

5.1 The extremal case

In this section we will prove the following Lemma. It is quite a bit more than is necessary for Theorem 1.5.5, and may be useful in proofing of Conjecture 1.5.3.

Lemma 5.1.1. *For any $0 < \beta < 10^{-6}$ and positive integer $n_0 := n_0(\beta)$ the following holds. Let M be a standard multigraph on $n \geq n_0$ vertices, that is β-splittable and such that $\delta(M) \geq (3n-3)/2$. If $n_1, \ldots, n_d$ are positive integers greater than 1 such that $n_1 + \cdots + n_d \leq n$ then M contains disjoint heavy cycles $C_1, \ldots, C_d$ such that $|C_i| = n_i$ for every $i \in [d]$.*

We will prove Lemma 5.1.1 by finding a subgraph of M that is very close to spanning square path in $H := M_H$. We will now describe the structure of this subgraph. After the description, we will prove that such a subgraph exists in M.

For any sequence of vertices $x_1 \ldots x_m$ call $e \in E[G(M)]$ a d-chord if $e = x_i x_{i+d}$ for some $i \in [m-d]$. A sequence is a square path if all possible 1-chords and 2-chords exist. Call a d-chord a heavy or light d-chord if it is a heavy or light edge. We will find an ordering $x_1 \dots x_n$ of the vertices of M for which all possible 1-chords and 2-chords exists and are heavy except, for some $4 \le p \le n-2$, the 1-chord $x_p x_{p+1}$ and 2-chord $x_p x_{p+2}$ may be light. The heavy 3-chord $x_{p-3}x_p$ and the light 3-chord $x_{p-1}x_{p+2}$ will also exist.

Define $y_{p-1} = x_p$, $y_p = x_{p-1}$ and $y_i = x_i$ for all $i \in [n] \setminus \{p-1, p\}$. All possible 1-chords and 2-chords exist in the sequence $y_1 \dots, y_n$. In addition, all of these chords are heavy except possibly the 2-chords $y_p y_{p+2}$ and $y_{p-1}y_{p+1}$. For any $1 \le i < j \le n$ define $x(i,j)$ to be the sequence $x_i \dots x_j$ and $y(i,j)$ to be $y_i, \dots, y_j$. When $j - i \ge 2$, it is not hard to see that the 2-chords in $x(i,j)$ and the 1-chords $x_i x_{i+1}$ and $x_{j-1}x_j$ form a cycle and an analogous statement is clearly true for $y(i,j)$. Therefore, there is a heavy cycle on the vertices of $x(i,j)$ unless $i = p$ and a heavy cycle on the vertices of $y(i,j)$ unless $i \le p-1$ and $j \ge p+2$. Recall that we consider a heavy edge to be a heavy cycle on 2-vertices, so the preceding statement is true even when $j - i = 1$.

Now suppose that $n_1, \dots, n_d$ are positive integers greater than 1 such that $n_1 + \dots + n_d \le n$ and define $s_i := \sum_{j=1}^{i-1} n_j$. If there exists $j \in [d]$ such that $p = s_j + 1$ then define $C_i := y(s_i + 1, s_i + n_i)$, otherwise let $C_i := x(s_i + 1, s_i + n_i)$. In either case and for every $i \in [d]$, $|C_i| = n_i$ and there is a heavy cycle on the vertices of C_i.

We will use the following result and corollary

Theorem 5.1.2 (Fan and Häggvist 1994 [16]). *Let G be a graph on n vertices. If $\delta(G) \ge \frac{5}{7}n$, then G contains the square of a hamiltonian cycle.*

Corollary 5.1.3. *For any $\frac{2}{63} \ge \alpha \ge 0$ and any graph G on n vertices the following holds. If $\delta(G) \ge 6\alpha n$ and all but at most αn vertices in G have degree at least $(1-\alpha)n$ then G contains the square of a hamiltonian cycle.*

Proof. Let W be the set of vertices in G with degree less than $(1-\alpha)\,n$ and $U := \overline{W}$. For every $w \in W$, $|N(w) \cap U| \geq 5\alpha n$, and $\deg(u, N(w) \cap U) \geq 4\alpha n$ for every $u \in N(w) \cap U$. Therefore, there exists a set $\{P_w : w \in W\}$ of disjoint paths on 4 vertices such that $V(P_w) \subset N(w) \cap U$ for every $w \in W$. Note that if $abcd = P_w$ then $abwcd$ is a square path. Let $W' := W \cup \bigcup_{w\in W} V(P_w)$ and G' be the graph formed by, for every $w \in W$, replacing w and $V(P_w)$ with a new vertex adjacent to

$$\left(\bigcap_{v\in V(P_w)} N(v)\right) \setminus W'.$$

Let G' be the resulting graph and note that $\delta(G') \geq n - 4\alpha n - |W'| \geq \frac{5}{7}|G'|$. Theorem 5.1.2 then implies that G', and hence G, contains the square of a hamiltonian cycle. □

Proof of Lemma 5.1.1. Let $H := M_H$ and $G := M_G$ and note that $\delta(H) \geq 3(n-1)/2 - (n-1) \geq (n-1)/2$ and $\delta(G) \geq 3(n-1)/4$. Since M is β-splittable, there exists $A \subset V(M)$ such that $|A| = (1/2 \pm \beta)n$ and $\|A, \overline{A}\|_H \leq \beta n^2$.

Let

$$W := \{v \in V(G) : |E_H(v) \cap E_H(A, \overline{A})| \geq \beta^{1/2} n\},$$

so $|W| \leq \beta^{1/2} n$. Let $A_1 := A \setminus W$, $A_2 := \overline{A} \setminus W$,

$$B_1 := A_1 \cup \{w \in W : \deg_H(w, A_1) \geq \deg_H(w, A_2)\} \text{ and}$$

$$B_2 := A_2 \cup \{w \in W : \deg_H(w, A_2) > \deg_H(w, A_1)\}.$$

Note that A_1 and A_2 are disjoint and $|A_1|, |A_2| \geq (1/2 - 2\beta^{1/2})n$.

For every $v \in A_i$, $\deg_H(v) \le \max\{|A|, |\overline{A}|\} + \beta^{1/2}n$ which implies, because $\deg_H(v) + \deg_G(v) = \deg_M(v) \ge 3(n-1)/2$, that $\deg_G(v) \ge (1 - 2\beta^{1/2})n$. Therefore, for any $i \in \{1, 2\}$, $a \in A_i$ and $b \in B_i$,

$$\begin{aligned}
\deg_H(a, A_i) &\ge \delta(H) - |W| - \deg_H(a, A_{3-i}) \ge (1/2 - 3\beta^{1/2})n, \\
\deg_G(a, A_{3-i}) &\ge |A_{3-i}| - 2\beta^{1/2}n, \\
\deg_H(b, A_i) &\ge (\delta(H) - |W|)/2 \ge (1/4 - 2\beta^{1/2})n \text{ and} \\
\deg_G(b, A_{3-i}) &\ge \delta(G) - |\overline{A_{3-i}}| \ge (1/4 - 3\beta^{1/2})n;
\end{aligned}$$

so, since $|A_1|, |A_2| \le (1/2 + 2\beta^{1/2})n$,

$$\begin{aligned}
d_H(a, A_i), d_G(a, A_{3-i}) &\ge 1 - 10\beta^{1/2} \ge 1 - 10^{-2} \text{ and} \\
d_H(b, A_i) d_G(b, A_{3-i}) &\ge 1/2 - 10\beta^{1/2} \ge 1/2 - 10^{-2}.
\end{aligned} \tag{5.1.1}$$

Pick $i \in 1, 2$ so that $|B_i| \le |B_{3-i}|$ and note that, by the degree condition,

$$\Delta(H[B_i, B_{3-i}]) \ge 1.$$

By (5.1.1), we can iteratively pick

1. $x_{p-1} \in A_i$,
2. $x_{p+1} \in N_H(x_{p-1}) \cap B_{3-i}$,
3. $x_p \in N_H(x_{p-1}) \cap N_G(x_{p+1}) \cap A_i$,
4. $x_{p+2} \in N_G(x_{p-1}) \cap N_G(x_p) \cap N_H(x_{p+1}) \cap A_{3-i}$,
5. $x_{p-2} \in N_H(x_{p-1}) \cap N_H(x_p) \cap A_i$ and
6. $x_{p-3} \in N_H(x_{p-2}) \cap N_H(x_{p-1}) \cap N_H(x_p) \cap A_i$

so that $x_{p-3}, \ldots, x_{p+2}$ are all distinct.

Replace x_{p-3}, x_{p-2}, x_{p-1} and x_p in $H[B_i]$ with a new vertex adjacent to

$$N_H(x_{p-3}) \cap N_H(x_{p-2}) \cap (B_i - x_p - x_{p-1})$$

and let H_i' be the resulting graph. Similarly, to create H_{3-i}', remove x_{p+1} and x_{p+2} from $H[B_{3-i}]$ and add a new vertex adjacent to

$$N_H(x_{p+1}) \cap N_H(x_{p+2}) \cap B_{3-i}.$$

By (5.1.1) and the fact that

$$|B_1 \setminus A_1|, |B_2 \setminus A_2| \leq |W| \leq \beta^{1/2}n,$$

Corollary 5.1.3 gives us a Hamilton square cycle in both H_i' and H_{3-i}'. Therefore, there is a Hamilton square path P_i in $H[B_i]$ that ends with $x_{p-3}x_{p-2}x_{p-1}x_p$ and a Hamilton square path P_{3-i} in $H[B_{3-i}]$ that begins with $x_{p+1}x_{p+2}$. Hence, P_iP_{3-i} gives the desired ordering of $V(M)$. □

5.2 Non-extremal case

In this section we complete the proof of Theorem 1.5.5 by proving the following lemma.

Lemma 5.2.1. *For any odd integer $k \geq 5$ and any real number $80^{-2} > \alpha > 0$ there exists n_0 and $\varepsilon > 0$ such that the following holds. Let M be a standard multigraph on $n \geq n_0$ and vertices such that k divides n and $\delta(G) \geq (\frac{3}{2} - \varepsilon/2)n$. If M_H is not α-splittable then M has a heavy C^k-factor.*

Define h so that $k = 2h+1$ and let $1/10 > \beta > 8\alpha^{1/2}$. Define constants γ, σ and δ so that $\gamma \leq \frac{\beta}{35}$, $\sigma \leq \left(\frac{\gamma}{42h}\right)^2$ and $\delta \leq \frac{\sigma}{16}$. Let $\varepsilon \leq \left(\frac{\delta}{64k}\right)^2$ and also small enough so that the conditions of the Lemma 2.4.5 are satisfied with 8ε, $d = \delta/2$ and $\Delta = 2$. By Lemma 2.3.10 applied to H with α, β, $\rho = 1/k$ and $\varepsilon' = \varepsilon/2$, if n is large enough there exists $\{W, U\}$ a partition V such that $|W| = n/k$ and $|U| = 2hn/k$, $H[U]$ is not β-splittable and $\delta(M[U]), \delta(M[W, U]) \geq (\frac{3}{2} - \varepsilon)|U|$.

Let M be a standard multigraph on n vertices that satisfies the conditions of the lemma. We will assume throughout that n is sufficiently large.

Constructing cluster triangles

Claim 1. There are partitions $U_0, U_1, \ldots, U_{2t}$ of U and $W_0, W_1, \ldots, W_t$ of W such that:

1. $2h|W_0| = |U_0| \leq \delta|U|$;
2. $t \geq 1/\varepsilon$;
3. $|W_i| = m$ and $|U_i| = hm$ for some $m \geq 1/\varepsilon$ for every $i \geq 1$; and
4. U_i and every W_i is ε-regular with all but at most δt other clusters for every $i \geq 1$.

Proof. Let $\varepsilon' < \varepsilon/2h$. Since in the proof of the Regularity Lemma and, therefore, the proof of Lemma 2.4.1, a given partition is refined to yield the desired partition we can start initially with the partition $\{U, W\}$ and obtain a partition $U_0, \ldots, U_{t_1}$ of U and a partition $W_0, \ldots, W_{t_2}$ of W, such that: $|U_0| + |W_0| \leq \varepsilon' n$, every non-exceptional cluster is of order $m' > 1/\varepsilon'$, $t' := t_1 + t_2 \geq 1/\varepsilon'$ and all but at most $\varepsilon' t'^2$ pairs of non-exceptional clusters are ε'-regular. Note that $m' \leq n/t' \leq \varepsilon' n$. The clusters U_0 and W_0 will be called exceptional and all other clusters will be called non-exceptional. We will add vertices to U_0 and W_0 but, for simplicity, we will continue to refer to the sets as U_0 and W_0. It is assumed throughout that vertices added to U_0 are elements of U and vertices added to W_0 are elements of W.

First, move the non-exceptional clusters that are ε'-regular with at most $(1-\sqrt{\varepsilon'})t'$ other non-exceptional clusters to U_0 or W_0. Note that now $|U_0| + |W_0| \leq 2\sqrt{\varepsilon'} n$.

Second, for p as small as possible, move clusters to U_0 so that there are $2h \cdot p$ clusters remaining in U and p clusters remaining in W. If $2h|W_0| > |U_0|$, this can be accomplished by moving clusters to U_0 until $2h|W_0| = |U_0|$. If $2h|W_0| < |U_0|$, we must first move at most $2h - 1$ clusters to U_0 so that the number of non-exceptional cluster in U remaining is divisible by $2h$, and then move non-exceptional clusters in W to W_0. In either case, this can be done so that $|U_0| \leq 4h\sqrt{\varepsilon'}n = 2k\sqrt{\varepsilon'}|U|$.

For the third and final step, let $m := \lfloor m'/h \rfloor$ and divide every non-exception cluster in W into h clusters of order m and move the $q < h$ vertices left over to W_0. Also, move q vertices from each non-exceptional cluster in U to U_0 so that such clusters have order hm.

Clearly (3) holds because $m > m'/2h \geq 1/\varepsilon$. To show (1) holds, note that we have only added at most qt' vertices to $W_0 \cup U_0$ in the third step so $2h|W_0| = |U_0| \leq 2k\sqrt{\varepsilon'}|U| + qt' \leq \delta|U|$. Since $\varepsilon > 2h\varepsilon'$, Lemma 2.4.3, gives us that every non-exceptional cluster is ε-regular with at most $h\sqrt{\varepsilon'}t'$ other clusters. Furthermore, if we let t be the number of non-exceptional clusters remaining in W we have that $tm \geq |W|/2 = n/2k \geq (hmt')/2k$ so $t \geq ht'/2k \geq 1/\varepsilon$ and $h\sqrt{\varepsilon'}t' \leq 2k\sqrt{\varepsilon'}t \leq \delta t$. This proves (2) and (4) which completes the proof. □

Call a triple (P, Q, Q') of clusters an (a, b)-*regular triangle* if $h|P| = |Q| = |Q'|$, the clusters P, Q and Q' are pairwise a-regular and $d_H(U, Q'), d_H(W, Q)$ and $d_G(W, Q')$ are all greater than b. Similarly, call a (a, b)-regular triangle (P, Q, Q') a (a, b)-*super-regular triangle* if the pairs (P, Q) and (Q, Q') are (a, b)-super-regular in H and the pair (P, Q') is (a, b)-super-regular in G.

Note that by our selection of ε and δ and by the Lemma 2.4.5, there exists a heavy C^k-factor of the graph induced by the vertices of any $(8\varepsilon, \delta/2)$-super-regular triangle if the clusters are sufficiently large.

Claim 2. For some $r \geq (1-15\delta)t$, there exists a reordering of the clusters $U_1, \ldots, U_{2t}$ and $W_1, \ldots, W_t$ so that $\{W_i, U_{2i-1}, U_{2i}\}$ is an (ε, δ)-regular triangle for every $i \in [r]$.

Proof. R_1 be the graph on $U_1, \ldots, U_{2t}$ in which $XY \in E(R)$ if $\{X, Y\}$ is an ε-regular pair and $d_H(X, Y) \geq \delta$. Note that $d_H(U_i, U \setminus U_0) \leq \delta + (1-\delta)d_{R_1}(U_i)/2t$, so since $|U \setminus U_0|/|U| \leq 1$, for any $U_i \in V(R_1)$

$$1/2 - \varepsilon \leq d_H(U_i, U) \leq \delta + (1-\delta)d_{R_1}(U_i)/2t + |U_0|/|U| \leq d_{R_1}(U_i)/2t + 2\delta$$

so $\delta(R_1) \geq (1/2 - 3\delta)|R_1|$. By Proposition 2.1.1, there is a matching M of size at least $(1/2 - 3\delta)2t$ in R_1. Let U^* be union of the clusters not saturated by M and note that $|U^*| \leq 3\delta|U|$.

Now construct a bipartite graph R_2 on M and $\{W_1, \ldots, W_t\}$ where W_i is adjacent to $\{U_j, U_{j'}\} \in M$ if, for some $\{l, l'\} = \{j, j'\}$, $d_H(W_i, U_l) \geq \delta$ and $d_G(W_i, U_{l'}) \geq \delta$, that is, if (W_i, U_l, U_l') form an (ε, δ)-triangle. It not hard to see to see that that if $\|W_i, U_j \cup U_{j'}\| > (2+2\delta)hm^2$ then $\{U_j, U_{j'}\}$ is adjacent to W_i. Therefore, $\|W_i, U_j \cup U_{j'}\|/2hm^2 \leq (1+\delta)$ when $\{U_j, U_{j'}\}$ is not adjacent to W_i. Hence, for any W_i

$$3/2 - \varepsilon \leq d_M(W_i, U) \leq (1-\delta)(\deg_{R_2}(W_i)/|M| + |U^*|/|U| + |U_0|/|U|) + (1+\delta)$$

so $\deg_{R_2}(W_i) > (1/2 - 6\delta)|M|$. Similarly, for any $\{U_j, U_j'\} := e \in M$,

$$3/2 - \varepsilon \leq d_M(U_j \cup U_j', W) \leq (1-\delta)(\deg_{R_2}(e)/t + |W_0|/|W|) + (1+\delta)$$

so $\deg_{R_2}(e) > (1/2 - 3\delta)t$. Therefore, by Proposition 2.1.2, we can find a matching of size r in R_2, where

$$r \geq (1 - 12\delta)|M| \geq (1 - 15\delta)t.$$

We can then reordering the clusters so that W_i is matched to $\{U_{2i-1}, U_{2i}\}$ and $d_H(W_i, U_{2i-1}) \geq \delta$ for every $i \in [r]$. □

Move the clusters $\{U_{2r+1}, \ldots, U_{2t}\}$ to U_0 and move the clusters $\{W_{r+1}, \ldots, W_t\}$ to W_0. Now $2h|W_0| = |U_0| \leq \sigma|U|$.

We will refer to a triangle by its index, that is, for $1 \leq i \leq r$, triangle i will be (W_i, U_{2i-1}, U_{2i}). Furthermore, for any $1 \leq i \leq 2r$, let

$$\bar{i} := \begin{cases} i-1 & \text{if } i \text{ is even} \\ i+1 & \text{if } i \text{ is odd} \end{cases}$$

and let $\underline{i} := (i + \bar{i} + 1)/4$. Note that if i is odd, $(W_{\underline{i}}, U_i, U_{\bar{i}})$ is triangle $\underline{i}$ and if i is even, $(W_{\underline{i}}, U_{\bar{i}}, U_i)$ is triangle $\underline{i}$.

Distribution procedure

We will first iteratively construct a set $\mathcal{C}$ of disjoint heavy C^k. We will define Z to be the set of vertices covered by the cycles in $\mathcal{C}$ at any point of the construction and define $U'_{2i-1} := U_{2i-1} \setminus Z$, $U'_{2i} := U_{2i} \setminus Z$ and $W'_i := W_i \setminus Z$.

When this process completes, $W_0 \cup U_0$ will be a subset of Z and for every $i \in [r]$, $(U'_{2i-1}, U'_{2i}, W'_i)$ will be a $(8\varepsilon, \delta/2)$ super-regular triangle. The Lemma 2.4.5 applied to each of these super-regular triangles will then complete the proof.

Call a vertex *used* if it is in Z and *unused* if it is not. We will ensure that at most $\gamma/3$ of the vertices in any non-exceptional cluster are used and this fact is assumed in the following claims.

Recall that

$$1 \gg \beta \gg \gamma \gg \sigma \gg \delta \gg \varepsilon > 0$$

and we have the following inequalities $\beta \geq 35\gamma$, $\gamma \geq 42h\sqrt{\sigma}$, $\sigma \geq 16\delta$ and $\delta \geq 15\sqrt{\varepsilon}$.

Call a vertex v γ-*good* for triangle i if, for some $j \in \{2i-1, 2i\}$,

$$\deg_H(v, U_j), \deg_G(v, U_{\bar{j}}) \geq \gamma hm.$$

Claim 3. If $v \in V(G)$ is γ-good for triangle i, then there exists a heavy C^k on unused vertices that contains v, h vertices from U_{2i-1} and h vertices from U_{2i}.

Proof. Fix $\{j,k\} = \{2i-1, 2i\}$ so that $\deg_H(v, U_j) \geq \gamma hm$ and $\deg_G(v, U_k) \geq \gamma hm$. Since U_j and U_k are ε-regular, $d_H(N_H(v) \cap U'_j, N_G(v) \cap U'_k) \geq \delta - \varepsilon$. Therefore, by Proposition 2.1.3, there exists a path P on $2h$ vertices in $H[N_H(v) \cap U'_j, N_G(v) \cap U'_k]$ and wPw is the desired cycle. □

Claim 4. If for some $v \in V$ and $j \in [2r]$, $\deg_H(v, U_{\bar{j}}) \geq (2\gamma/3)hm$ then there exists a heavy C^k on unused vertices that contains v, $h-1$ vertices from U_j, h vertices from $U_{\bar{j}}$ and one vertex from $W_{\underline{j}}$.

Proof. Assume $\bar{j}$ is odd. The case when $\bar{j}$ is even is similar. We have that

$$d_H(U_{\bar{j}}, W_{\underline{j}}) \geq \delta \text{ and } d_G(U_j, W_{\underline{j}}) \geq \delta.$$

Because $(U_{\bar{j}}, W_{\underline{j}})$ and $(W_{\underline{j}}, U_j)$ are are ε-regular pairs, we can iteratively pick $u \in N_H(v) \cap U'_{\bar{j}}$ and then $w \in N_H(u) \cap W'_{\underline{j}}$ so that

$$d_H(u, W'_{\underline{j}}), d_G(w, U'_j) \geq \delta - \varepsilon.$$

Since U_j and $U_{\bar{j}}$ are ε-regular,

$$d_H(N_H(v) \cap U'_{\bar{j}}, N_G(w) \cap U'_j) \geq \delta - \varepsilon,$$

so, by Proposition 2.1.3, there exists a path P on $2h-2$ vertices in $H[N_H(v) \cap U'_{\bar{j}}, N_G(w) \cap U'_j]$, that avoids u, and $vPwuv$ is the desired cycle. □

Claim 5. For any distinct $X, Y \in \{U_1, \ldots, U_{2r}, W_1, \ldots, W_r\}$ if $d_H(X,Y) \geq \gamma$ there exists an unused vertex $x \in X$ such that $\deg_H(x, Y) \geq (2\gamma/3)|Y|$

Proof. Let X' and X^* be the set of unused and used vertices in X respectively. Since $|X^*| \leq \gamma|X|/3$,

$$d_H(X', Y) \geq d_H(X', Y)|X'|/|X| = d_H(X,Y) - d_H(X^*, Y)|X^*|/|X| \geq 2\gamma/3,$$

and the conclusion follows. □

Recall that r is the number of triangles.

Claim 6. Every $w \in W$ is γ-good for at least $(1/2 - 2\gamma)r$ triangles.

Proof. Let x be the number of triangles for which w is γ-good. If w is not γ-good for triangle i than we have that both $\deg_H(w, U_{2i-1}) < \gamma hm$ and $\deg_H(w, U_{2i}) < \gamma hm$; or we have that either $\deg_G(w, U_{2i-1}) < \gamma hm$ or $\deg_G(w, U_{2i}) < \gamma hm$. Since neither of these two cases is possible if $\deg_M(v, U_{2i-1} \cup U_{2i}) \geq (2 + 2\gamma)hm$,

$$3/2 - \varepsilon \leq d_M(v, U) \leq (1 - \gamma)(x/r + |U_0|/|U|) + (1 + \gamma)$$

so $x \geq (1/2 - 2\gamma)r$. □

The following two lemmas are the key parts of the distribution procedure.

Lemma 5.2.2. *For every unused $u \in U$, cluster U_p and $I \subseteq [r] \setminus \{\underline{p}\}$ of order at least $(1-\gamma)r$ there exists $J \subseteq I$ with $|J| \in \{1, 2\}$ such that there is a set of $|J|+1$ disjoint heavy C^k that covers a set X of unused vertices such that X consists of:*

- *u,*
- *$h - 1$ vertices from U_p,*
- *h vertices from $U_{\bar{p}}$,*
- *1 vertex from $W_{\underline{p}}$, and*
- *for every $j \in J$, h vertices from both U_{2j-1} and U_{2j} and 1 vertex from U_j.*

Proof. Let u, U_p and I be as in the statement of the lemma and let $\hat{I} := \{i \in [2r] : \underline{i} \in I\}$. With Claims 3, 4 and 5, the proof of the lemma is complete if any of the following three conditions is satisfied:

1. there exists $i \in \hat{I}$ such that $\deg_H(u, U_{\bar{i}}) \geq \gamma hm$ and $d_H(U_i, U_{\bar{p}}) \geq \gamma$;

2. there exists $i \in I$ such that u is γ-good for i and $d_H(W_i, U_{\bar{p}}) \geq \gamma$; or

3. there exist distinct $i, j \in \hat{I}$ such that $\deg_H(u, U_{\bar{i}}) \geq \gamma hm$, $d_H(U_i, U_j) \geq \gamma$, and $d_H(U_{\bar{j}}, U_{\bar{p}}) \geq \gamma$.

Let $U' := U \setminus U_0$ and $W' := W \setminus W_0$ and note that for every $v \in V$

$$\begin{aligned} d_M(v, U'), d_M(v, W') &\geq 3/2 - \gamma, \text{ and} \\ d_H(v, U'), d_H(v, W') &\geq 1/2 - \gamma \end{aligned} \tag{5.2.1}$$

Let $S := \{i \in [2r] : d_H(U_i, U_{\bar{p}}) \geq \gamma\}$ and let $T := \{i \in [2r] : d_H(u, U_{\bar{i}}) \geq \gamma\}$. We have

$$d_H(u, U') \leq \gamma + (1-\gamma)|T|/2r, \text{ and} \tag{5.2.2}$$

$$d_H(U_{\bar{p}}, U') \leq \gamma + (1-\gamma)|S|/2r \tag{5.2.3}$$

so $|S|, |T| \geq (1/2 - 2\gamma)2r$.

If $|S \cap T| > 2\gamma r$ then $|S \cap T \cap \hat{I}| > 0$ so condition 1 is satisfied. Therefore, let us assume $|S \cap T| \leq 2\gamma r$. This, together with the lower bound for $|S|$, implies that $|T| \leq (1/2 + 3\gamma)2r$. From this upper bound for $|T|$ and (5.2.2), we have that $d_H(u, U') \leq 1/2 + 4\gamma$ and, furthermore, that

$$d_G(u, U') \geq 1 - 5\gamma, \tag{5.2.4}$$

because

$$d_G(u, U') + d_H(u, U') = d_M(u, U') \geq 3/2 - \gamma.$$

Let $T_1 := \{i \in [r] : 2i - 1 \in T \text{ or } 2i \in T\}$.

Case 1: $|T_1| \geq (1/2 + 10\gamma)r$.

In this case we will show that condition 2 is satisfied. Let $i \in T_1$ and note that then $d_H(u, U_j) \geq \gamma$ for some $j \in \{2i-1, 2i\}$. Therefore, if u is not γ-good for i then u then $d_G(u, U_{\bar{j}}) \leq \gamma$. So $d_{\overline{G}}(u, U_j \cup U_{\bar{j}}) \geq 1 - \gamma$ and if x is the number of indices in T_1 for which u is not γ-good, by (5.2.4),

$$6\gamma \geq d_{\overline{G}}(u, U') \geq (1-\gamma)x/r$$

so $x \le 7\gamma r$. Hence, there are at least $(1/2+3\gamma)r$ indices in T_1 for which u is γ-good. Since $d_H(U_{\bar{p}}, W') \ge 1/2-\gamma$, an inequality analogous to (5.2.3), gives that there are at least $(1/2-2\gamma)r$ indices i for which $d_H(U_{\bar{p}}, W_i) \ge \gamma$. Therefore, condition 2 is satisfied.

Case 2: $|T_1| < (1/2+10\gamma)r$.

In this case we will show that condition 3 is satisfied. Let

$$T_2 := \{i \in [r] : 2i-1 \in T \text{ and } 2i \in T\}.$$

Since $(1-4\gamma)r \le |T| = |T_1|+|T_2|$, we have that $|T_2| > (1/2-14\gamma)r$. Let S_1 and S_2 be defined analogously to T_1 and T_2. Because $|S_1 \cap T_2| \le |S \cap T| < 2\gamma r$ and $|S_1 \cup T_2| \le r$, $|S_1| < (1/2+16\gamma)r$. Therefore, because $|S| \ge (1-4\gamma)r$, $|S_2| > (1/2-20\gamma)r$. Let $A = \bigcup_{i \in T_2 \cap I}\{U_{2i-1}, U_{2i}\}$ and $B = \bigcup_{i \in S_2 \cap I}\{U_{2i-1}, U_{2i}\}$. Note that because $S \cap T \cap \hat{I}$ is empty $T_2 \cap S_2 \cap I$ is empty, so A and B are disjoint. Also, note that $|A|, |B| \ge (1/2-21\gamma)|U'| \ge (1/2-22\gamma)|U|$. Furthermore, If there exists $i \in T_2 \cap I$ and $j \in S_2 \cap I$ such that $d_H(U_{2i-1} \cup U_{2i}, U_{2j-1} \cup U_{2i}) \ge \gamma$ then it is not hard to see that condition 3 is satisfied. So we can assume $d_H(A, B) < \gamma$ which implies and

$$\|A, U \setminus A\|_H < |A|(\gamma|B| + |U \setminus (A \cup B)|) \le 45\gamma|U||A| < 50\gamma|U|^2.$$

This is a contradiction because $H[U]$ is not β-splittable. □

For any $X \subseteq V(G)$ and any $i \in [r]$ define $w_i(X) := |X \cap W_i|/m$. Call X *evenly distributed* if $2h|X \cap W| = |X \cap U|$ and $|X \cap U_{2i-1}| = |X \cap U_{2i}| = hm \cdot w_i(X)$ for every $i \in [r]$.

Lemma 5.2.3. *Let X, Y be disjoint evenly distributed subsets of $V(G)$ and let $Y_W = Y \cap W$ and $c = 2h\varepsilon + 6k\gamma^{-1}|Y_W|/|W|$. If $w_i(X \cup Y) \le \gamma/3 - c$ for every $i \in [r]$ then there exists an evenly distributed subset Z of $V(G)$ that is disjoint from $X \cup Y$ such that $w_i(Z) \le c$ for every $i \in [r]$ and such that there exists a heavy C^k factor of $M[Y \cup Z]$.*

Proof. Iteratively we will construct a set $\mathcal{C}$ of at most $3k|Y|$ disjoint heavy C^k. During this procedure $Z := Z(\mathcal{C})$ will be set of vertices used by the cycles of $\mathcal{C}$ that are not in the set Y. Note that we will always then have $|Z| \leq 3k|Y|$, and since Z will eventually be evenly distributed, $|Z \cap W| \leq 3k|Y_W|$. Let $J := J(Z)$ be the indices of triangles for which $w_i(Z) \geq c - 2h\varepsilon$ and $I := I(Z) := [r] \setminus J(Z)$.

During this procedure, we will select triangles and then construct heavy cycles using vertices from the clusters of the triangle. Triangles must be in I to be selected and when a triangle is selected at most $2h$ vertex of W_i will be added to Z, unless it is selected again. Since $(2h\varepsilon)m \leq 2h$, this will ensure that we maintain the condition that $w_i(Z) \leq c$ for every $i \in [r]$. We have

$$|Z \cap W|/m \geq |J|(c - 2h\varepsilon) = 6k\gamma^{-1}|J||Y_W|/|W| \geq 2\gamma^{-1}|J||Z \cap W|/|W|$$

so $|J| \leq \gamma|W|/2m < \gamma r$ and $|I| > (1-\gamma)r$.

Suppose Z is currently evenly distributed and let Y' be the vertices in Y that have not been used in a previously constructed heavy C^k. Since Y and Z are evenly distributed there exists $w \in Y' \cap W$ and $A \subseteq Y' \cap U$ such that $|A| = 2h$. We can use Claim 6 to select $i \in I$ such that w is good for triangle i. Using Claim 3, we can construct a heavy C^k that contains w and has h vertices in both U_{2j-1} and U_{2j}. Note that at this point, Z is not evenly distributed since $w_i(Z) = 0$ and $|Z \cap U_{2i-1}| = |Z \cap U_{2i}| = h$. Next, for any $u \in A$ and with $p = 2i - 1$ we can use Lemma 5.2.2 to find at most 3 cycles that avoid the triangles in J that meet the condition of the lemma. Now we have that $w_i(Z) = 1$, $|Z \cap U_{2i-1}| = 2h - 1$ and $|Z \cap U_{2i}| = 2h$ and for any $j \in [r] \setminus \{i\}$, $|Z \cap U_{2j-1}| = |Z \cap U_{2j}| = h \cdot w_j(Z)$. We then repeat this construction for $h-1$ additional vertices of A. After this step $w_i(Z) = h$, $|Z \cap U_{2i-1}| = h + h(h-1) = h^2$ and $|Z \cap U_{2i}| = h^2 + h$. For the remaining h vertices of A, we will do the same, but with $p = 2i$, so that after this step $w_i(Z) = 2h$,

$|Z \cap U_{2i-1}| = 2h^2$ and $|Z \cap U_{2i}| = h + h^2 + h(h-1) = 2h^2$. Since we have extended Z while maintaining the conditions of the lemma the proof is complete. □

We know finish the proof of Lemma 5.2.1. Let $Y = W_0 \cup U_0$ and $X = \emptyset$. By Lemma 5.2.3, there exists a cycle covering $\mathcal{C}_1$ of $Y \cup Z$ where $Z \subseteq V(G) \setminus Y$, Z is evenly distributed and for every $i \in [r]$

$$w_i(Z) \leq k\varepsilon + 6k\gamma^{-1}|W_0|/|W| \leq \gamma/6.$$

We will now rename $Y \cup Z$ as X and set $Y = \emptyset$. Note that now $w_i(X) = w_i(Z)$ for every $i \in [r]$.

Let $W_i' := W_i \setminus (X \cup Y)$, $U_{2i-1}' := U_{2i-1} \setminus (X \cup Y)$ and $U_{2i}' := U_{2i} \setminus (X \cup Y)$ for every $i \in [r]$. By Lemma 2.4.3, $\{W_i', U_{2i-1}', U_{2i}'\}$ is a $(2\varepsilon, \delta - \varepsilon)$ regular triangle. Our goal now is to add a small number of vertices to Y so that the triangle become super-regular triangle

By Lemma 2.4.4, there are at most $2\varepsilon|W_i'|$ vertices in W_i' that have less than $(\delta - 3\varepsilon)|U_{2i-1}'|$ heavy-neighbors in U_{2i-1}' and there are at most $2\varepsilon|W_i'|$ vertices in W_i' that have less than $(\delta - 3\varepsilon)|U_{2i-1}'|$ neighbors in U_{2i}'. Using similar logic for both U_{2i-1}' and U_{2i}', there exists $a_i \leq 4\varepsilon|W_i'|$ such that we can add a_i vertices from W_i' to Y and ha_i vertices from both U_{2i-1}' and U_{2i}' to Y so that, with Lemma 2.4.3, $(W_i', U_{2i-1}', U_{2i}')$ is a $(4\varepsilon, \delta - 7\varepsilon)$ super-regular triangle.

Let $Y_W := Y \cap W$ and note that $|Y_W| \leq 4\varepsilon|W|$. We can now use Lemma 5.2.3 to find a cycle covering $\mathcal{C}_2$ of $Y \cup Z$ where $Z \subseteq V(G) \setminus (X \cup Y)$ such that for every $i \in [r]$

$$w_i(Z) \leq k\varepsilon + 6k\gamma^{-1}|Y_W|/|W| \leq \sqrt{\varepsilon}$$

Let $W_i'' := W_i' \setminus Z$, $U_{2i-1}'' := U_{2i-1}' \setminus Z$ and $U_{2i}'' := U_{2i}' \setminus Z$, for every $i \in [r]$. Note that, since $7\varepsilon + \sqrt{\varepsilon} \leq \delta/2$, and by Lemma 2.4.3 $(W_i'', U_{2i-1}'', U_{2i}'')$ is a $(8\varepsilon, \delta/2)$ super-regular triangle. We can now apply the blow-up lemma to each triangle $(W_i'', U_{2i-1}'', U_{2i}'')$ to complete the desired heavy C^k-factor of M.

Chapter 6

ANTI-DIRECTED HAMILTON CYCLES

In this section we will prove the following theorem.

Theorem 1.6.4 (DeBiasio & Molla 2013 [9])**.** *There exists* n_0 *such that if* D *is a directed graph on* $2n \geq n_0$ *vertices and* $\delta_0(D) \geq n+1$ *then* D *has an anti-directed Hamilton cycle.*

6.1 Overview

To get the exact result, we use the now common stability technique where we split the proof into two cases depending on whether D is "close" to an extremal configuration or not (see Figure 6.3). If D is close to an extremal configuration, then we use some ad-hoc techniques which rely on the exact minimum semi-degree condition and if D is not close to an extremal configuration then we use the recent absorbing method of Rödl, Ruciński, and Szemerédi (as opposed to the regularity/blow-up method).

To formally say what we mean by "close" to an extremal configuration we need the following definition, which is essentially equivalent to the definition of $(\alpha, 2)$-extremal digraphs given in Section 2.3.

Definition 6.1.1. Let D be a directed graph on $2n$ vertices. We say D is α-extremal if there exists $A, B \subseteq V(D)$ such that $(1-\alpha)n \leq |A|, |B| \leq (1+\alpha)n$ and $\Delta^+(A, B) \leq \alpha n$ and $\Delta^-(B, A) \leq \alpha n$.

This definition is more restrictive than simply bounding the number of edges, thus it will help make the extremal case less messy. However, a non-extremal set still has many edges from A to B.

Observation 6.1.2. Let $0 < \alpha \ll 1$. Suppose D is not α-extremal, then for $A, B \subseteq V(D)$ with $(1-\alpha/2)n \le |A|, |B| \le (1+\alpha/2)n$, we have $\vec{e}(A,B) \ge \frac{\alpha^2}{2}n^2$.

Proof. Let $A, B \subseteq V(D)$ with $(1-\alpha/2)n \le |A|, |B| \le (1+\alpha/2)n$. Since D is not α-extremal, there is some vertex $v \in A$ with $\deg^+(v, B) \ge \alpha n$ or $v \in B$ with $\deg^-(v, A) \ge \alpha n$. Either way, we get at least αn edges. Now delete v, and apply the argument again to get another αn edges. We may repeat this until $|A|$ or $|B|$ drops below $(1-\alpha)n$, i.e. for at least $\frac{\alpha}{2}n$ steps. This gives us at least $\frac{\alpha^2}{2}n^2$ edges in total. □

Finally, we make two more observations which will be useful when working with non-extremal graphs.

Observation 6.1.3. Let $0 < \lambda \le \alpha \ll 1$ and let D be a directed graph on n vertices. If D is not α-extremal and $X \subseteq V(D)$ with $|X| \le \lambda n$, then $D' = D - X$ is not $(\alpha - \lambda)$-extremal.

Proof. Let $A', B' \subseteq V(D') \subseteq V(D)$ with $(1-\alpha+\lambda)|D'| \le |A'|, |B'| \le (1+\alpha-\lambda)|D'|$. Note that

$$(1-\alpha)n \le (1-\alpha+\lambda)(1-\lambda)n \le (1-\alpha+\lambda)|D'| \le |A'|, |B'| \le (1+\alpha-\lambda)|D'| \le (1+\alpha)n$$

thus there exists $v \in A'$ such that $\deg^+(v, B') \ge \alpha n \ge (\alpha - \lambda)|D'|$ or $v \in B'$ such that $\deg^-(v, A') \ge \alpha n \ge (\alpha - \lambda)|D'|$. □

Lemma 6.1.4. *Let $X, Y \subseteq V(D)$. If $\vec{e}(X,Y) \ge c|X||Y|$, then there exists*

(i) *$X' \subseteq X$, $Y' \subseteq Y$ such that $X' \cap Y' = \emptyset$ and $\delta^+(X', Y') \ge \frac{c}{8}|Y|, \delta^-(Y', X') \ge \frac{c}{8}|X|$ and*

(ii) *a proper anti-directed path in $D[X \cup Y]$ on at least $\frac{c}{4} \cdot \min\{|X|, |Y|\}$ vertices.*

Proof. *(i)* Let $X^* = X \backslash Y$ and $Y^* = Y \backslash X$. Delete all edges not in $\vec{E}(X, Y)$. Choose a partition $\{X'', Y''\}$ of $X \cap Y$ which maximizes $\vec{e}(X^* \cup X'', Y^* \cup Y'')$ and set $X_0 = X^* \cup X''$ and $Y_0 = Y^* \cup Y''$. Note that $\vec{e}(X_0) + \vec{e}(Y_0) + \vec{e}(X_0, Y_0) + \vec{e}(Y_0, X_0) = \vec{e}(X, Y)$. We have that

$$\vec{e}(X_0) = \sum_{v \in X_0} \deg^+(v, X_0) = \sum_{v \in X''} \deg^-(v, X_0) \leq \sum_{v \in X''} \deg^+(v, Y_0) \leq \vec{e}(X_0, Y_0)$$

where the inequality holds since if $\deg^-(v, X_0) > \deg^+(v, Y_0)$ for some $v \in X''$, then we could move v to Y'' and increase the number of edges across the partition. Similarly, $\vec{e}(X_0, Y_0) \geq \vec{e}(Y_0)$. Thus $\vec{e}(X_0, Y_0) \geq \frac{1}{4}\vec{e}(X, Y) \geq \frac{c}{4}|X||Y|$.

If there exists $v \in X_0$ such that $\deg^+(v, Y_0) < \frac{c}{8}|Y|$ or $v \in Y_0$ such that $\deg^-(v, X_0) < \frac{c}{8}|X|$, then delete v and set $X_1 = X_0 \setminus \{v\}$ and $Y_1 = Y_0 \setminus \{v\}$. Repeat this process until there no vertices left to delete. This process must end with a non-empty graph because fewer than $|X|\frac{c}{8}|Y| + |Y|\frac{c}{8}|X| = \frac{c}{4}|X||Y|$ edges are deleted in this process. Finally, let X' and Y' be the sets of vertices which remain after the process ends.

(ii) Apply Lemma 6.1.4.(i) to obtain sets $X' \subseteq X$, $Y' \subseteq Y$ such that $X' \cap Y' = \emptyset$ and $\delta^+(X', Y') \geq \frac{c}{8}|Y|$ and $\delta^-(Y', X') \geq \frac{c}{8}|Y|$. Let G be an auxiliary bipartite graph on X', Y' with $E(G) = \{\{x, y\} : (x, y) \in \vec{E}(X', Y')\}$. Note that $\delta(G) \geq \frac{c}{8}\min\{|X|, |Y|\}$ and thus G contains a path on at least $2\delta(G) \geq \frac{c}{4} \cdot \min\{|X|, |Y|\}$ vertices, which starts in X. This path contains a proper anti-directed path in D on at least $\frac{c}{4} \cdot \min\{|X|, |Y|\}$ vertices.

□

6.2 Non-extremal Case

In this section we will prove that if D satisfies the conditions of Theorem 1.6.4 and D is not α-extremal, then D has an ADHC. We actually prove a stronger statement which in some sense shows that the extremal condition is “stable," i.e. graphs which do not satisfy the extremal condition do not require the tight minimum semi-degree condition.

Theorem 6.2.1. *For any $\alpha \in (0, 1/32)$ there exists $\varepsilon > 0$ and n_0 such if $D = (V, E)$ is a directed graph on $2n \geq 2n_0$ vertices, D is not α-extremal and $\delta_0(D) \geq (1-\varepsilon)n$, then D contains an anti-directed Hamiltonian cycle.*

Lemma 6.2.2. *For all $0 < \epsilon \ll \beta \ll \lambda \ll \alpha \ll 1$ there exists n_0 such that if $n \geq n_0$, D is a directed graph on $2n$ vertices, $\delta^0(D) \geq (1-\varepsilon)n$, and D is not α-extremal, then there exists a proper anti-directed path P^* with $|P^*| \leq \lambda n$ such that for all $W \subseteq V(D) \setminus V(P^*)$ with $2w := |W| \leq \beta n$, $D[V(P^*) \cup W]$ contains a spanning proper anti-directed path with the same endpoints as P^*.*

Lemma 6.2.3. *For all $0 < \epsilon \ll \beta \ll \lambda \ll \sigma \ll \alpha \ll 1$ there exists n_0 such that if $n \geq n_0$, D is a directed graph on $2n$ vertices, $\delta^0(D) \geq (1-\varepsilon)n$, D is not α-extremal, and P^* is a proper anti-directed path with $|P^*| \leq \lambda n$, then D contains an anti-directed cycle on at least $(2-\beta)n$ vertices which contains P^* as a segment.*

First we use Lemma 6.2.2 and Lemma 6.2.3 to prove Theorem 6.2.1.

Proof. Let $\alpha \in (0, 1/32)$ and choose $0 < \epsilon \ll \beta \ll \lambda \ll \sigma \ll \alpha$. Let n_0 be large enough for Lemma 6.2.2 and Lemma 6.2.3. Let D be a directed graph on $2n$ vertices with $\delta^0(D) \geq (1-\varepsilon)n$. Apply Lemma 6.2.2 to obtain an anti-directed path P^* having the stated property. Now apply Lemma 6.2.3 to obtain an anti-directed cycle C^* which contains P^* as a segment. Let $W = D - C^*$ and note that since C^* is an anti-directed cycle, $|C^*|$ is even which implies $|W|$ is even, since $|D|$ is even. Finally apply the property of P^* to the set W to obtain an ADHC in D. □

Let $\mathcal{P} := V^2 - \{(x,x) : x \in V\}$. For any $(x,y) \in \mathcal{P}$, call $(a,b,c,d) \in V^4$ an (x,y)-*absorber* if $abcd$ is a proper anti-directed path and $axcbyd$ is a proper anti-directed path (see Figure 6.1) and call $(a,b) \in V^2$ an (x,y)-*connector* if $xaby$ is an anti-directed path where (a,b) is an edge (note that specifying one edge dictates the directions of all the other edges).

Note that if $(x',x),(y,y') \in \vec{E}(D)$ and (a,b) is an (x,y)-connector disjoint from $\{x',y'\}$ then $x'xabyy'$ is an anti-directed path.

For all $(x,y) \in \mathcal{P}$, let $f_{\text{abs}}(x,y) = \{T \in V^4 : T \text{ is an } (x,y)\text{-absorber}\}$ and $f_{\text{con}}(x,y) = \{T \in V^2 : T \text{ is an } (x,y)\text{-connector}\}$.

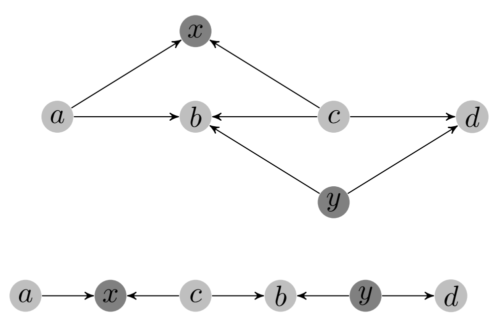

Figure 6.1: (a,b,c,d) is an (x,y)-absorber

Claim 1. Let D satisfy the conditions of Lemma 6.2.2. For all $(x,y) \in \mathcal{P}$ we have

(i) $|f_{abs}(x,y)| \geq \alpha^{12} n^4$ and

(ii) $|f_{\text{con}}(x,y)| \geq \alpha^3 n^2$.

Proof. Let $(x,y) \in \mathcal{P}$ and let $A = N^-(x)$ and $B = N^+(y)$.

(i) By Observation 6.1.2 and Lemma 6.1.4, there exists $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cap B' = \emptyset$ and $\delta^+(A',B'), \delta^-(B',A') \geq \frac{\alpha^2}{16}(1-\varepsilon)n \geq \alpha^3 n + 1$. For all

$(b,c) \in \vec{E}(A', B')$, we have $|N^+(b) \cap B'| \geq \alpha^3 n + 1$ and $|N^-(c) \cap A'| \geq \alpha^3 n + 1$. So there are more than $(\alpha^3 n)^2$ choices for (b,c), $\alpha^3 n$ choices for a and $\alpha^3 n$ choices for d, i.e. $|f_{abs}(x,y)| \geq \alpha^{12} n^4$.

(ii) Similarly, by Observation 6.1.2, we have $\vec{e}(A,B) \geq \frac{\alpha^2}{2} n^2 \geq \alpha^3 n^2$, each of which is a connector.

□

Claim 2 (Connecting-Reservoir)**.** For all $0 < \gamma \ll \alpha$ and $D' \subseteq D$ such that $|D'| \geq (2-\lambda)n$, there exists a set of ordered pairs $\mathcal{R}$ such that if $R = \cup_{(a,b)\in\mathcal{R}}\{a,b\}$, $R \subseteq V(D')$, $|R| \leq \gamma n$ and for all distinct $x, y \in V(D)$, $|f_{con}(x,y) \cap \mathcal{R}| \geq \gamma^2 n$.

Proof. For every $(x,y) \in \mathcal{P}$

$$|\{(a,b) \in f_{con}(x,y) : a, b \in V(D')\}| \geq |f_{con}(x,y)| - |D - D'|n \geq \alpha^3 n^2/2.$$

Therefore, we can apply Lemma 2.2.5 to obtain a set $\mathcal{R}$ of disjoint good ordered pairs such that $|\mathcal{R}| \leq \gamma n/2$ and $|f_{con}(x,y) \cap \mathcal{R}| \geq \gamma\alpha^3 n/4 - 2\gamma^2 n \geq \gamma^2 n$ and $\mathcal{R} \subseteq V(D')^2$. □

Now we prove Lemma 6.2.2.

Proof. Since $|f_{abs}(x,y) \cap \mathcal{P}(V')| \geq \alpha^{12} n^4$ we apply Lemma 2.2.5 to D obtain a set $\mathcal{A}$ of disjoint good 4-tuples $\{A_1, \ldots, A_\ell\}$ such that $|\mathcal{A}| \leq \lambda n/8$ and $|f_{abs}(x,y) \cap \mathcal{A}| \geq \lambda\alpha^{12} n/8 - 2(\lambda/2)^2 n \geq \lambda^2 n$. Let $A = \cup_{(a,b,c,d)\in\mathcal{A}}\{a,b,c,d\}$ and note that $|A| \leq \lambda n/2$.

Let $(a_i, b_i, c_i, d_i) := A_i$ for every $i \in [l]$, so $a_i b_i c_i d_i$ is a proper ADP. Note that there are less than $|A|n$ ordered pairs that contain a vertex from A, so since $\lambda \ll \alpha$, we can greedily choose vertex disjoint $(x_i, y_i) \in f_{con}(d_i, a_{i+1})$ for each $i \in [l-1]$ such that $x_i, y_i \notin A$. Set $P^* := A_1 x_1 y_1 A_2 x_2 y_2 A_2 \ldots A_{l-1} x_{l-1} y_{l-1} A_l$ and note that $|P^*| \leq \lambda n$ and $|P^*|$ is a proper ADP.

To see that P^* has the desired property, let $W \subseteq V \setminus V(P^*)$ such that $2w = |W| \leq \beta n$. Arbitrarily partition W into pairs and since $\beta \ll \lambda$, we can greedily match the disjoint pairs from W with 4-tuples in $\mathcal{A}$. By the way we have defined an (x, y)-absorber, $D[V(P') \cup W]$ contains a spanning proper anti-directed path starting with an out-edge from a_1 and ending with an in-edge to d_ℓ.

□

Covering

The main challenge in the proof of Lemma 6.2.3 is to show that if a maximum length anti-directed path is not long enough, then we can build a constant number of vertex disjoint anti-directed paths whose total length is sufficiently larger.

Claim 3. Let $m = \lceil \frac{1}{4} \log n \rceil$; and n be large enough so that

$$n \geq \frac{2m2^{2m}}{\varepsilon^2 \beta} \text{ and } m > 10\beta^{-4}\varepsilon^{-1}; \tag{6.2.1}$$

and P be a proper anti-directed path with beginning segment P^* such that $|P^*| \leq \lambda n$ and $|P^*|$ is even. Let $D' = D - P$. If $|P| < (2 - \beta)n$, then there exist disjoint proper anti-directed paths $Q_1, \ldots, Q_r \subseteq D[V(P) \cup V(D')]$, such that $r \leq 6$, Q_1 contains P^* as an initial segment and

$$|Q_1| + \cdots |Q_r| \geq |P| + \varepsilon m.$$

First we show how this implies Lemma 6.2.3.

Proof. Let P^* be a proper anti-directed path with $|P^*| \leq \lambda n$. Let $D' = D - P^*$. Now apply Claim 2 with $\gamma = \beta^2$ to get $\mathcal{R}$ and R such that $|f_{con}(x, y) \cap \mathcal{R}| \geq \beta^4 n$ for every $(x, y) \in \mathcal{P}$ and $|R| \leq \beta^2 n$.

Let P be a maximum length anti-directed path on an even number of vertices in $D - R$ that begins with P^*. If $|P| < (2 - \beta)n$, then we apply Claim 3. Now connect

$Q_1, \dots, Q_r$ into a longer path using at must 5 pairs from $\mathcal{R}$. Delete these vertices from R and reset $\mathcal{R}$. We may repeat this process as long as there are sufficiently many pairs remaining in $\mathcal{R}$. On each step, $|f_{con}(x, y) \cap \mathcal{R}|$ may be reduced by at most 5. However, in less than $\frac{2n}{\varepsilon m}$ steps, we will have a path of length greater than $(2-\beta)n$ in which case we would be done. By (6.2.1), $5 \cdot \frac{2n}{\varepsilon m} < \beta^4 n$, so we can repeat the process sufficiently many times. Once we have a path P with $|P| \geq (2-\beta)n$, we use one more pair from $\mathcal{R}$ to connect the endpoints of P to form an anti-directed cycle C, which is possible since $|P|$ is even. Note that C contains P^* as a segment by construction.

□

Proof of Claim 3. Let P be a maximum length proper ADP in D containing P^* as an initial segment. Let $v_1 \dots v_p := P - P^*$, $T := V \setminus V(P)$, and $P_i := v_{2m(i-1)+1} \dots v_{2mi}$ for $i \in [s]$ where $s := \left\lfloor \frac{p}{2m} \right\rfloor$. Note that $|P_i| = 2m$ for every $i \in [s]$. Let $P' := P_1 \dots P_s$. Assume $|T| > \beta n$.

Claim 4. Let $c \in (\varepsilon^2 - 1, 1)$, $d \in (\varepsilon^2, 1+c)$, and $b := \lceil (1+c-d)m \rceil$. If $\vec{e}(T, P_i) \geq (1+c)m|T|$, then there exists $X_i \subseteq V(P_i)$ and $Y_i \subseteq T$ such that $|X_i| = b$, $|Y_i| \geq 2m$ and $X_i \subseteq N^+(y)$ for every $y \in Y_i$. In particular, $D[V(P_i) \cup T]$ contains a proper anti-directed path on $2b$ vertices.

Proof. Let $T' = \{v \in T : \deg^+(v, P_i) \geq b\}$ and since

$$(1+c)m|T| \leq \vec{e}(T, P_i) \leq (|T| - |T'|)(b-1) + |T'|(2m-b+1) \leq |T|(1+c-d)m + |T'|2m$$

which implies $|T'| \geq \frac{d}{2}|T|$. Together with (6.2.1) we have

$$|T'| \geq \frac{d}{2}|T| \geq \varepsilon^2 \beta n \geq 2m 2^{2m} > 2m \binom{2m}{b},$$

which by the pigeonhole principle implies that there exists $X_i \subseteq V(P_i)$ with $|X_i| = b$ and $Y_i \subseteq T'$ such that $|Y_i| \geq 2m$ and $X_i \subseteq N_H(y)$ for every $y \in Y_i$. □

By Claim 4, if $\vec{e}(T, P_i) \geq (1+\varepsilon)|T|m$ there exists a proper anti-directed path Q_3 of length

$$2\left\lceil (1+\varepsilon-\varepsilon^2)m \right\rceil > (2+\varepsilon)m \text{ in } D[T \cup P_i].$$

Letting $Q_1 := P_A P_1 \cdots P_{i-1}$ and $Q_2 := P_{i+1} \cdots P_q$ then satisfies the condition of the lemma. Therefore, we can assume that,

$$\vec{e}(T, P_i) < (1+\varepsilon)|T|m \text{ for every } i \in [s]. \tag{6.2.2}$$

We can also assume that

$$\vec{e}(T, T) < \varepsilon|T|^2. \tag{6.2.3}$$

Otherwise by Lemma 6.1.4.(ii) there exists a proper anti-directed path Q_2 of length $(\varepsilon/4)|T| \geq \varepsilon m$ in $D[T]$. Then $Q_1 := P$ and Q_2 satisfy the condition of the lemma.

So (6.2.3) implies that

$$\vec{e}(T, P') \geq (1-\varepsilon)n|T| - (|R| + |P_A| + m)|T| - \vec{e}(T, T) \geq (1-2\lambda)n|T| \tag{6.2.4}$$

Let

$$I := \{i \in [s] : \vec{e}(T, P_i) \geq (1-\sigma)|T|m\}.$$

By (6.2.2) and (6.2.4),

$$\begin{aligned}(1-2\lambda)n|T| \leq \vec{e}(T, P') &\leq (1-\sigma)m(s-|I|)|T| + (1+\varepsilon)m|I||T| \\ &\leq (1-\sigma)n|T| + (\sigma+\varepsilon)m|I||T|\end{aligned}$$

which implies that $m|I| \geq \frac{\sigma-2\lambda}{\sigma+\varepsilon}n > (1-\alpha)n$. Also note that $n \geq |P|/2 \geq m|I|$.

For every $i \in I$, let $X_i \subseteq P_i$ and $Y_i \subseteq T$ be the sets guaranteed by Claim 4 with $c := -\sigma$, $d := \sigma$ and $b := \lceil (1-2\sigma)m \rceil$. Let $Z_i := V(P_i) \setminus X_i$ for $i \in [I]$ and let $Z := \bigcup_{i \in I} Z_i$. Note that $|Z_i| = 2m - b$ for every $i \in I$ so $|Z| = (2m-b)|I|$ and

$$(1+\alpha)n > (1+2\sigma)n \geq (2m-b)|I| \geq m|I| > (1-\alpha)n.$$

Therefore by Observation 6.1.2, $\vec{e}(Z,Z) \geq \frac{\alpha^2}{2}|Z|^2$. Because

$$\frac{\alpha^2}{2} \leq \frac{\vec{e}(Z,Z)}{|Z|^2} = \frac{1}{|I|^2}\sum_{i\in I}\sum_{j\in I}\frac{\vec{e}(Z_i,Z_j)}{(2m-b)^2},$$

there exists $i,j \in I$ such that $\vec{e}(Z_i,Z_j) \geq \alpha^2(2m-b)^2/2$. Removing P_i and P_j divides P into three disjoint anti-directed paths. Note that some of these paths may be empty. Label these paths Q_1, Q_2 and Q_3 so that $P^* \subseteq Q_1$. By Lemma 6.1.4.(ii) there exists a proper anti-directed path Q_4 of length at least $(\alpha^2/8)(2m-b) \geq (\alpha^2/8)m$ in $D[Z_i \cup Z_j]$. By Claim 4, there also exists a proper anti-directed path $Q_5 \subseteq D[X_i \cup Y_i]$ such that $|Q_5| \geq 2(1-2\sigma)m$.

If $i = j$ then $Q_4 \subseteq D[Z_i]$ and $|Q_1|+|Q_2|+|Q_3| = |P|-2m$. Therefore it is enough to observe that $|Q_4|+|Q_5| \geq 2(1-2\sigma)m + (\alpha^2/8)m \geq 2m+\varepsilon m$.

If $i \neq j$, then $Y_j' := Y_j \setminus V(Q_4)$ has order at least $2m-b \geq m$. So there exists a path $Q_6 \subseteq D[X_j \cup Y_j']$ such that $|Q_6| \geq 2(1-2\sigma)m$. Since $|Q_1|+|Q_2|+|Q_3| = |P|-4m$ and $|Q_4|+|Q_5|+|Q_6| \geq 4(1-2\sigma)m+(\alpha^2/8)m \geq 4m+\varepsilon m$, the proof is complete. □

6.3 Extremal Case

Let $1 \gg \gamma \gg \beta \gg \alpha > 0$. Let D be a directed graph on $2n$ vertices with $\delta^0(D) \geq n+1$ and suppose that D satisfies the extremal condition with parameter α. We will first partition $V(D)$ in the preprocessing section, then we will handle the main proof.

Preprocessing

The point of this section is to make the following statement precise: If D satisfies the extremal condition, then D is very similar to the digraph in Figure 6.3.

Proposition 6.3.1. *If there exists an α-extreme pair of sets $A, B \subseteq V(G)$, then there exists a partition $\{X_1', X_2', Y_1', Y_2', Z\}$ of $V(G)$ such that*

(i) $|Z'| \leq 3\alpha^{2/3}n$, $||X_1'|-|X_2'||, ||Y_1'|-|Y_2'|| \leq 3\alpha^{2/3}n$ *and*

(ii) $\delta^0(X'_{3-i}, X'_i), \delta^-(Y'_{3-i}, X'_i), \delta^+(Y'_i, X'_i) \geq |X'_i| - 2\alpha^{1/3}n$ *and* $\delta^0(Y'_i, Y'_i), \delta^-(X'_i, Y'_i), \delta^+(X'_{3-i}, Y'_i) \geq |Y'_1| - 2\alpha^{1/3}n$ *for* $i = 1, 2$.

Proof. Let $A, B \subseteq V(D)$ such that $(1-\alpha)n \leq |A|, |B| \leq (1+\alpha)n$, $\Delta^+(A, B) \leq \alpha n$, and $\Delta^-(B, A) \leq \alpha n$. We have that

$$\delta^+(A, \overline{B}) \geq (1-\alpha)n, \text{ and} \tag{6.3.1}$$

$$\delta^-(B, \overline{A}) \geq (1-\alpha)n. \tag{6.3.2}$$

Set $\widetilde{X_1} = V \setminus (A \cup B)$, $\widetilde{X_2} = A \cap B$, $\widetilde{Y_1} = A \setminus B$, $\widetilde{Y_2} = B \setminus A$. Note that $\widetilde{Y_1} \cup \widetilde{X_2} = A$, $\widetilde{Y_2} \cup \widetilde{X_2} = B$ so $||\widetilde{Y_1}| - |\widetilde{Y_2}|| \leq 2\alpha n$, and $||\widetilde{X_1}| - |\widetilde{X_2}|| \leq 2\alpha n$, because $|\widetilde{X_1}| - |\widetilde{X_2}| = |V| - |A| - |B|$.

Let

$$\hat{Y_1} = \{v \in \widetilde{Y_1} : \deg^-(v, \widetilde{X_2}) < |\widetilde{X_2}| - \alpha^{1/3}n \text{ or } \deg^-(v, \widetilde{Y_1}) < |\widetilde{Y_1}| - \alpha^{1/3}n\},$$

$$\hat{Y_2} = \{v \in \widetilde{Y_2} : \deg^+(v, \widetilde{X_2}) < |\widetilde{X_2}| - \alpha^{1/3}n \text{ or } \deg^+(v, \widetilde{Y_2}) < |\widetilde{Y_2}| - \alpha^{1/3}n\},$$

$$\hat{X_1} = \{v \in \widetilde{X_1} : \deg^-(v, \widetilde{Y_1}) < |\widetilde{Y_1}| - \alpha^{1/3}n \text{ or } \deg^+(v, \widetilde{Y_2}) < |\widetilde{Y_2}| - \alpha^{1/3}n \text{ or } \deg^0(v, \widetilde{X_2}) < |\widetilde{X_2}| - \alpha^{1/3}n\},$$

$\hat{B} = \hat{Y_1} \cup \hat{X_1}$ and $\hat{A} = \hat{Y_2} \cup \hat{X_1}$. Note that $\hat{B} \subseteq \overline{B}$ and $\hat{A} \subseteq \overline{A}$. Now we show that each of these sets are small.

Claim 1. $|\hat{Y_1}|, |\hat{Y_2}|, |\hat{X_1}| \leq 2\alpha^{2/3}n$ and $|\hat{Y_1}| + |\hat{Y_2}| + |\hat{X_1}| \leq 3\alpha^{2/3}n$

Proof. By (6.3.1) and the definition of $\hat{X_1}, \hat{Y_1}$, we have

$$|\widetilde{Y_1} \cup \widetilde{X_2}|(1-\alpha)n = |A|(1-\alpha)n \leq \vec{e}(A, \overline{B}) \leq (|\overline{B}| - |\hat{B}|)|A| + |\hat{B}|(|A| - 2\alpha^{1/3}n)$$

This implies

$$|\hat{Y_1} \cup \hat{X_1}| = |\hat{B}| \leq \frac{|A|(|\overline{B}| - (1-\alpha)n)}{2\alpha^{1/3}n} \leq \frac{(1+\alpha)n((1+\alpha)n - (1-\alpha)n)}{2\alpha^{1/3}n} = (1+\alpha)\alpha^{2/3}n$$

Now using (6.3.2), the same calculation (with the symbol A exchanged with the symbol B) gives that $|\hat{Y}_2 \cup \hat{X}_1| = |\hat{A}| \leq (1+\alpha)\alpha^{2/3}n$. Thus $|\hat{Y}_1| + |\hat{Y}_2| + |\hat{X}_1| \leq 2(1+\alpha)\alpha^{2/3}n \leq 3\alpha^{2/3}n$. □

Let $X_1' = \widetilde{X_1} \setminus \hat{X}_1$, $X_2' = \widetilde{X_2}$, $Y_i' = \widetilde{Y_i} \setminus \hat{Y}_i$ for $i = 1, 2$, and $Z = \hat{X}_1 \cup \hat{Y}_1 \cup \hat{Y}_2$. Note that $|Z| \leq 3\alpha^{2/3}n$ and $||X_1'| - |X_2'||, ||Y_1'| - |Y_2'|| \leq 2\alpha n + 2\alpha^{2/3}n < 3\alpha^{2/3}n$. The required degree conditions all follow from (6.3.1) and (6.3.2); the definitions of $\hat{X}_1$, $\hat{Y}_1$ and $\hat{Y}_2$; and Claim 1 . □

Finding the ADHC

The following facts immediately follow from the Chernoff-type bound for the hypergeometric distribution (Theorem 2.2.1).

Lemma 6.3.2. *For any $\varepsilon > 0$, there exists n_0 such that if D is a digraph on $n \geq n_0$ vertices, $S \subseteq V(D)$, $m \leq |S|$ and $c := m/|S|$ then there exists $T \subseteq S$ of order m such that for every $v \in V$*

$$||N^{\pm}(v) \cap T| - c|N^{\pm}(v) \cap S|| \leq \varepsilon n \quad \textit{and}$$
$$||N^{\pm}(v) \cap (S \setminus T)| - (1-c)|N^{\pm}(v) \cap S|| \leq \varepsilon n.$$

We will need the following theorem and corollary.

Theorem 6.3.3 (Moon & Moser [32])**.** *If G is a balanced bipartite graph on n vertices such that for every $1 \leq k \leq n/4$ there are less than k vertices v such that $\deg(v) \leq k$ then G has a Hamilton cycle.*

Corollary 6.3.4. *Let G be a U, V-bipartite graph on n vertices such that n is sufficiently large and $0 \leq |U| - |V| \leq 1$ and let $C \geq 3$ be a positive integer. If n is even, let $a \in U$ and $b \in V$ and if n is odd, let $a, b \in U$. If $\delta(G) > 2C$ and $\deg(v) > 2n/5$ for all but at most C vertices v then G has a Hamilton path with ends a and b.*

Proof. If n is even then iteratively pick $v_0 \in N(b) - a$, $v_1 \in N(v_0) - b$ and $v_2 \in N(a)-b-v_1$ and set $R = \{a, b, v_0, v_1, v_2\}$. If n is odd then iteratively pick $v_1 \in N(a)-b$ and $v_2 \in N(b) - v_1$. and set $R = \{a, b, v_1, v_2\}$. In both cases, we can select v_1, v_2 to have degree greater than $2n/5$. Applying Theorem 6.3.3 to the graph formed by removing R from the graph and adding a new vertex to V which is adjacent to $N(v_1) \cap N(v_2) \setminus R$ completes the proof. $\square$

Looking ahead (in what will be the main case), we are going to distribute vertices from Z to the sets X_1', X_2', Y_1', Y_2' to make sets X_1, X_2, Y_1, Y_2. Then we are going to partition each of the sets $X_1 = X_1^1 \cup X_1^2$, $X_2 = X_2^1 \cup X_2^2$, $Y_1 = Y_1^1 \cup Y_1^2$, and $Y_2 = Y_2^1 \cup Y_2^2$ (so that each set is approximately split in half). Then we are going to look at the bipartite graphs induced by edges from $X_2^1 \cup Y_1^1$ to $X_1^1 \cup Y_1^2$ and $X_1^2 \cup Y_2^2$ to $X_2^2 \cup Y_2^1$ respectively (see Figure 6.3). By the degree conditions for X_1', X_2', Y_1', Y_2', these bipartite graphs will be nearly complete, however we must be sure that the vertices from Z each have degree at least γn in the bipartite graph. This next claim shows that the vertices of Z can be distributed so that this condition is satisfied.

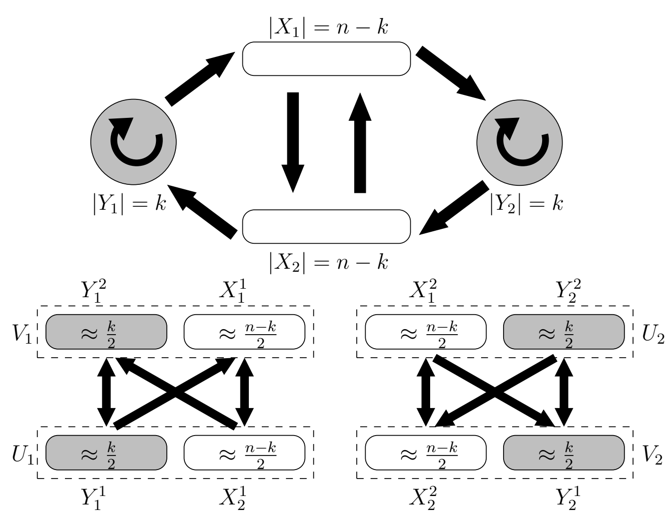

Figure 6.2: The objective partition.

Definition 6.3.5. For $z \in Z$ and $A, B \in \{X_1', X_2', Y_1', Y_2'\}$, we say $z \in Z(A, B)$ if $\deg^+(z, B) \geq 5\gamma n$ and $\deg^-(z, A) \geq 5\gamma n$.

Claim 2. Every vertex in Z belongs to at least one of the following sets:

(i) $Z(X_i', X_i')$,

(ii) $Z(Y_i', Y_i')$,

(iii) $Z(X_i', X_{3-i}')$,

(iv) $Z(Y_i', Y_{3-i}')$,

(v) $Z_1 := \bigcap_{1 \leq i,j \leq 2} Z(Y_i', X_j')$ or

(vi) $Z_2 := \bigcap_{1 \leq i,j \leq 2} Z(X_i', Y_j')$.

Proof. Let $v \in Z$ and suppose that v is in none of the sets $(i) - (iv)$. Note that v must have at least $(n - |Z|)/4$ out-neighbors in some set $A \in \{X_1', X_2', Y_1', Y_2'\}$.

Assume $A = X_i'$ for some $i = 1, 2$. Because of the degree condition and the fact that v is in none of the sets $(i) - (iv)$, we have

$$\deg^-(v, Y_1 \cup Y_2) \geq n - 10\gamma n - |Z| \geq (1 - 11\gamma)n, \text{ and}$$
$$\deg^+(v, X_1 \cup X_2) \geq n - 10\gamma n - |Z| \geq (1 - 11\gamma)n.$$

This implies, $||X_1 \cup X_2| - n|, ||Y_1 \cup Y_2| - n| \leq 11\gamma n$. With Proposition 6.3.1, we have that $(1/2 - 6\gamma)n \leq |X_1|, |X_2|, |Y_1|, |Y_2| \leq (1/2 + 6\gamma)n$ so $v \in Z_1$.

If $A = Y_i'$ for some $i = 1, 2$, the previous argument (with the symbol X exchanged with the symbol Y) gives us that $v \in Z_2$. □

Since a vertex may be in multiple sets $(i) - (vi)$, we arbitrarily pick one set for each vertex if necessary. Now we distribute vertices from Z.

Procedure 6.3.6. (Distributing the vertices from Z) For $1 \leq i \leq 2$, set

- $X_i := X_i' \cup Z(X_{3-i}', X_{3-i}') \cup Z(Y_i', Y_{3-i}')$ and
- $Y_i := Y_i' \cup Z(Y_i', Y_i') \cup Z(X_{3-i}', X_i') \cup Z_i$.

By Claim 2, $\{X_1, X_2, Y_1, Y_2\}$ is a partition of V. (We allow empty sets in our partitions). Note that the vertices from $Z_1 \cup Z_2$ have no obvious place to be distributed, thus our choice is arbitrary.

Call a partition of a set into two parts *nearly balanced* if the sizes of the two part differ by at most $2\beta n$. Call a partition $\bigcup_{1 \leq i,j \leq 2}\{X_i^j, Y_i^j\}$ of V a *splitting* of D if $\{X_i^1, X_i^2\}$ is a nearly balanced partition of X_i and $\{Y_i^1, Y_i^2\}$ is a nearly balanced partition of Y_i. Define $U_i := X_{3-i}^i \cup Y_i^i$ and $V_i := X_i^i \cup Y_i^{3-i}$ (see Figure 6.3). Note that, with Proposition 6.3.1, $||A| - n/2| \leq 3\beta n$ for any $A \in \{U_1, U_2, V_1, V_2\}$. Furthermore, if $u \in U_i \setminus Z$, by Proposition 6.3.1, $\deg^+(u, X_i' \cup Y_i') \geq |X_i' \cup Y_i'| - 4\alpha^{1/3}$, so

$$\deg^+(u, V_i) \geq |V_i| - 4\alpha^{1/3} - |Z| \geq |V_i| - 2\beta n. \tag{6.3.3}$$

Similarly, if $v \in V_i \setminus Z$, then

$$\deg^-(v, U_i) \geq |U_i| - 2\beta n. \tag{6.3.4}$$

Let G be the bipartite graph on vertex sets $U := U_1 \cup U_2, V := V_1 \cup V_2$ such that $\{u, v\} \in E(G)$ if and only if $u \in U, v \in V$, and $(u, v) \in E(D)$. Let $G_i := G[U_i, V_i]$ and $Q_i = \{v \in V(G_i) : \deg_G(v) < (1-\gamma)n/2\}$. Call a splitting *good* if $\delta(G_i) \geq \gamma n$ and $|Q_i| \leq \beta n$ for $i \in 1, 2$. If $x \in X_i$ is mapped to some X_i^j we say that x is *preassigned* to X_i^j. Similarly, if $y \in Y_i$ is mapped to some Y_i^j we say that y is *preassigned* to Y_i^j.

Claim 3. If P is a set of preassigned vertices such that $|P| \leq \beta n$ and for all $1 \leq i, j \leq 2$, x_i^j and y_i^j are non-negative integers such that:

(i) x_i^j and y_i^j are at least as large as the number of vertices preassigned to X_i^j and Y_i^j respectively;

(ii) $x_i^1 + x_i^2 = |X_i|$ and $y_i^1 + y_i^2 = |Y_i|$; and

(iii) $||X_i|/2 - x_i^j|, ||Y_i|/2 - y_i^j| \leq \beta n$

then there exists a good splitting of V such that $|X_i^j| = x_i^j$ and $|Y_i^j| = y_i^j$ and every vertex in P is in its preassigned set.

Proof. We can split $X_i \setminus P$ and $Y_i \setminus P$ so that, after adding every vertex in P to its preassigned set, $|X_i^j| = x_i^j$ and $|Y_i^j| = y_i^j$. When $|X_i| \geq 5\gamma n$, by Lemma 6.3.2, we can also ensure that for every $v \in V$,

$$\begin{aligned} |N^{\pm}(v) \cap X_i^j| &\geq |N^{\pm}(v) \cap (X_i \setminus P)| \frac{x_i^j - |P|}{|X_i \setminus P|} - \alpha n \\ &\geq \big(|N^{\pm}(v) \cap X_i| - \beta n\big)\,(1/2 - 2\beta n/|X_i|) - \alpha n \\ &\geq |N^{\pm}(v) \cap X_i|/2 - \gamma n, \end{aligned}$$

since $2\beta/5\gamma \ll \gamma$. By a similar calculation, if $|Y_i| \geq 5\gamma n$ we can partition Y_i so that $|N^{\pm}(v) \cap Y_i^j| \geq |N^{\pm}(v) \cap Y_i|/2 - \gamma n$ for every $v \in V$.

Let $v \in V(G_i)$ for some $i \in \{1, 2\}$. If $v \in Z$, by the previous calculation, Claim 2 and Procedure 6.3.6, $d_{G_i}(v) \geq \gamma n$. If $v \notin Z$, by 6.3.3 and 6.3.4, $d_{G_i}(v) \geq (1-\gamma)n/2$. Therefore, $\delta(G_i) \geq \gamma n$ and $|Q_i| \leq \beta n$. □

Claim 4. If there exists a good splitting of D and two independent edges uv and $u'v'$ such that either

(i) $u \in U_1$, $v \in V_2$, $u' \in U_2$, $v' \in V_1$ and $|U_i| = |V_{3-i}|$ for $i = 1, 2$; or

(ii) there exists $i = 1, 2$ such that $u, u' \in U_i$, $v, v' \in V_{3-i}$, $|U_i| = |V_i| + 1$ and $|V_{3-i}| = |U_{3-i}| + 1$

then D contains an ADHC.

Proof. Apply Corollary 6.3.4 to get a Hamilton path P_i in G_i so that the ends of P_1 and P_2 are the vertices $\{u, u', v, v'\}$. These paths and the edges uv and $u'v'$ correspond to an ADHC in D. □

Note that the edges uv and $u'v'$ played a special role in the previous proposition. Now we discuss what properties these edges must have and how we can find them (this will be the bottleneck of the proof in each case and is the only place where the exact degree condition will be needed).

Definition 6.3.7. Let uv be an edge in D. We call uv a *connecting edge* if for some $i = 1, 2$, $u \in X_i$ and either $v \in X_i$ or $v \in Y_i$; or $u \in Y_i$ and either $v \in Y_{3-i}$ or $v \in X_{3-i}$.

Basically, connecting edges are edges which do not behave like edges in the graph shown in Figure 6.3.

The following simple equations are used to help find connecting edges and follow directly from the degree condition. For any $A \subseteq V$ and $v \in A$

$$\deg^0(v, A) \geq n + 1 - |\overline{A}| \tag{6.3.5}$$

$$\deg^0(v, \overline{A}) \geq n + 1 - (|A| - 1) = n + 2 - |A|. \tag{6.3.6}$$

At this point, we take different routes depending on the order of the sets Y_1 and Y_2.

Case 1: $\min\{|Y_1|, |Y_2|\} > \beta n$

Claim 5. For each $i = 1, 2$, there exists a partition of X_i as $\{X_i^1, X_i^2\}$ with $||X_i^1| - |X_i^2|| \leq \alpha n$ and $W_i := Y_i \cup X_1^i \cup X_2^i$ such that either

(i) $|W_1|$, $|W_2|$ are odd and there are two independent connecting edges directed from W_j to W_{3-j} for some $j = 1, 2$; or

(ii) $|W_1|, |W_2|$ are even and there are two independent connecting edges, one directed from W_1 to W_2 and the other directed from W_2 to W_1.

Proof. Without loss of generality suppose $|X_1 \cup Y_1| \geq |X_2 \cup Y_2|$. By the case, we can choose distinct $u, u' \in Y_1$. By (6.3.6), $\deg^0(u, X_2 \cup Y_2), \deg^0(u', X_2 \cup Y_2) \geq 2$. Thus we can choose distinct $v \in N^+(u) \cap (X_2 \cup Y_2)$ and $v' \in N^+(u') \cap (X_2 \cup Y_2)$. For $i = 1, 2$, let $\{X_i^1, X_i^2\}$ be a partition of X_i such that $||X_i^1| - |X_i^2|| \leq \alpha n$ and $W_i := Y_i \cup X_1^i \cup X_2^i$ with $u, u' \in W_1$ and $v, v' \in W_2$.

If this can be done so that $|W_1|$ and $|W_2|$ are odd then we are done, so suppose not. Then it must be the case that $X_1 = \emptyset$ and $X_2 \subseteq \{v, v'\}$. Hence, $W_1 = Y_1$ and $W_2 = X_2 \cup Y_2$. Therefore $|W_1| \geq |W_2|$, so by (6.3.6) we have $\deg^0(w, W_1) \geq 2$ for all $w \in W_2$. In this case we choose $u'' \in Y_2 \setminus \{v\}$ and then $v'' \in N^+(u'') \cap (W_1 \setminus \{u\})$ giving us the desired connecting edges uv and $u''v''$. □

By Claim 5 and Proposition 6.3.1 for $i = 1, 2$ we have $||X_1^i| - |X_2^i|| \leq \alpha n + 3\alpha^{2/3} n$. So since $|Y_i| \geq \beta n$ we can assume that after we apply Proposition 3, $||U_i| - |V_i|| \leq 1$.

Let uv and $u'v'$ be the connecting edges from Claim 5. Suppose Claim 5.(i) holds and fix $i \in \{1, 2\}$ so that $u, u' \in W_i$ and $v, v' \in W_{3-i}$. Preassign u, u', v and v' so that, after splitting D with Proposition 3, $u, u' \in U_i$ and $v, v' \in V_{3-i}$. Since $|W_1|$ and $|W_2|$ are odd, we can ensure that $|U_i| = |V_i| + 1$ and $|V_{3-i}| = |U_{3-i}| + 1$. We can then apply Claim 4.(ii) to find an ADHC. Now suppose Claim 5.(ii) holds and let $u, v' \in W_1$, $v, u' \in W_2$ so that uv and $u'v'$ are the connecting edges. Preassign u, u', v and v' so that, after splitting D with Proposition 3, $u \in U_1$,$v \in V_2$, $u' \in U_2$ and $v' \in V_1$. Since $|W_1|$ and $|W_2|$ are even, we can apply Claim 4.(i) to find an ADHC.

Case 2: $\min\{|Y_1|, |Y_2|\} \leq \beta n$

Without loss of generality, suppose $|X_1| \geq |X_2|$. If $|X_1| > n$, then let $X_1'' \subseteq \{v \in X_1 : \deg^-(v, X_1) \geq 5\gamma n\}$ be as large as possible subject to $|X_1''| \leq |X_1| - n$. Reset $X_1 := X_1 \setminus X_1''$ and, because $\deg^-(v, X_1) \geq 5\gamma n$ and $\deg^+(v, X_2) \geq 5\gamma n$ for every

$v \in X_1''$, reset $Y_2 := Y_2 \cup X_1''$. By Proposition 6.3.1, $|X_1'| \leq n + 2\alpha^{2/3}$ and $|Z| \leq 3\alpha^{2/3}$, thus $|X_1''| \leq 5\alpha^{3/2} \ll \beta n$. Therefore, the conclusions of Claim 3 still hold with the redefined sets $\{X_1, X_2, Y_1, Y_2\}$.

Case 2.1: $|X_1| \leq n$.

If $|X_1 \cup Y_1| = |X_2 \cup Y_2| = n$, then choose $u \in X_1$ and $u' \in X_2$. By (6.3.5), we have $\deg^0(u, X_1 \cup Y_1), \deg^0(u', X_2 \cup Y_2) \geq 1$, so choose $v \in N^+(u) \cap (X_1 \cup Y_1)$ and $v' \in N^+(u') \cap (X_2 \cup Y_2)$. Preassign u to X_1^2, u' to X_2^1 and v, v' so that $v \in V_1$ and $v' \in V_2$. Because $|X_1 \cup \{u, v\}|, |X_2 \cup \{u', v'\}| \leq n$, we can apply Claim 3, so that $|U_1| + |V_2| = |U_2| + |V_1| = n$, $|U_1| = |V_1|$ and $|U_2| = |V_2|$. Applying Claim 4.(i) then gives the desired ADHC.

Now suppose $|X_i \cup Y_i| > |X_{3-i} \cup Y_{3-i}|$ for some $i = 1, 2$. By (6.3.5), $\deg^0(u, X_i \cup Y_i) \geq 2$ for all $u \in X_i \cup Y_i$. Let $u, u' \in X_i$ and choose distinct $v \in N^+(u) \cap (X_i \cup Y_i)$ and $v' \in N^+(u') \cap (X_i \cup Y_i)$ with a preference for choosing v and v' in X_i. Note that if $|X_i| = n$, then, by (6.3.5), $\deg^+(u, X_i) \geq 1$. So we can assume, in all cases, that $|X_i \cup \{u, u', v, v'\}| \leq n + 1$. Therefore, after preassigning u, u' to X_i^{3-i} and v, v' to X_i^i or Y_i^{3-i} as appropriate, we can apply Claim 3 to get $|U_{3-i}| + |V_i| = n + 1$, $|U_{3-i}| = |V_{3-i}| + 1$ and $|V_i| = |U_i| + 1$. Applying Claim 4.(ii) then completes this case.

Case 2.2: $|X_1| \geq n + 1$.

Definition 6.3.8. A star with k-leaves in which every edge is oriented away from the center is called a k-out star. A star with k-leaves in which every edge is oriented towards the center is called a k-in star.

Lemma 6.3.9. *Let G be a directed graph on n vertices and let $d \geq 1$. If $\delta^+(G) \geq d+1$ and $\Delta^-(G) \leq D$, then G has at least $\frac{d}{3(d+D)}n$ disjoint 2-in-stars.*

Proof. Let M be a maximum collection of m vertex disjoint 2-in stars and let $L = V(G) \setminus V(M)$. Note that

$$\sum_{v \in L} \deg^+(v, L) \leq |L| = n - 3m$$

otherwise $\sum_{v \in L} \deg^-(v, L) = \sum_{v \in L} \deg^+(v, L) > |L|$ would give a 2-in star disjoint from M. Thus

$$d(n - 3m) \leq (d + 1)(n - 3m) - \sum_{v \in L} \deg^+(v, L) \leq \vec{e}(L, M) \leq 3mD$$

which gives $m \geq \frac{d}{3(d+D)} n$. □

Set $d = |X_1| - n$ and recall that $d \ll \beta n$. By (6.3.5), $\delta^+(D[X_1]) \geq d + 1$. By the case, $X_1'' \cap X_1 = \emptyset$, so $\Delta^-(D[X_1]) < 5\gamma n$ and $\frac{d}{3d+15\gamma n} n \geq d + 1$. Applying Lemma 6.3.9 gives $\{S_1, \ldots, S_{d+1}\}$ a collection of $d + 1$ vertex disjoint 2-in stars in $D[X_1]$. Let uv and $u'v'$ be one edge in S_d and S_{d+1} respectively. Preassign the vertices in $S_1, \ldots, S_{d-1}$ and the vertices u and u' to X_1^2. Also, preassign v and v' to X_1^1. Recall that $X_1^1 \cup X_1^2 \subseteq U_2 \cup V_1$, so we can use Claim 3, to get a good splitting of D such that $|U_2| = \lceil n/2 \rceil + d$, $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil - d + 1$ and $|U_1| = \lfloor n/2 \rfloor - 1$. We then use Corollary 6.3.4, to find a Hamilton path P_1 in G_1 with ends v and v'.

We now move the roots of the stars $S_1, \ldots, S_{d-1}$ from U_2 to V_2 and then use Corollary 6.3.4 to complete the proof. Formally, we greedily find a matching M between the leaves of the stars $S_1, \ldots, S_{d-1}$ and the vertices in V_2 of degree at least $(1 - \gamma)n/2$ in G_2. For each $1 \leq i \leq d - 1$, let a_i and b_i be the vertices matched to the leaves of S_i and replace $V(S_i) \cup \{a_i, b_i\}$ in G_2 with a new vertex adjacent to $N_{G_2}(a_i) \cap N_{G_2}(b_i)$ minus the vertices of the stars. Apply Corollary 6.3.4 to get a Hamilton path P_2 in the resulting graph with ends u and u'. The stars $S_1, \ldots, S_{d-1}$; the edges in M; the paths P_1 and P_2; and the edges uv and $u'v'$ correspond to an ADHC in D.

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