A Pairwise Comparison Matrix Framework

for Large-Scale Decision Making

by

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ABSTRACT

A Pairwise Comparison Matrix (PCM) is used to compute for relative priorities of criteria or alternatives and are integral components of widely applied decision making tools: the Analytic Hierarchy Process (AHP) and its generalized form, the Analytic Network Process (ANP). However, a PCM suffers from several issues limiting its application to large-scale decision problems, specifically: (1) to the curse of dimensionality, that is, a large number of pairwise comparisons need to be elicited from a decision maker (DM), (2) inconsistent and (3) imprecise preferences maybe obtained due to the limited cognitive power of DMs. This dissertation proposes a PCM Framework for Large-Scale Decisions to address these limitations in three phases as follows.

The first phase proposes a binary integer program (BIP) to intelligently decompose a PCM into several mutually exclusive subsets using interdependence scores. As a result, the number of pairwise comparisons is reduced and the consistency of the PCM is improved. Since the subsets are disjoint, the most independent pivot element is identified to connect all subsets. This is done to derive the global weights of the elements from the original PCM. The proposed BIP is applied to both AHP and ANP methodologies. However, it is noted that the optimal number of subsets is provided subjectively by the DM and hence is subject to biases and judgement errors.

The second phase proposes a trade-off PCM decomposition methodology to decompose a PCM into a number of optimally identified subsets. A BIP is proposed to balance the: (1) time savings by reducing pairwise comparisons, the level of PCM inconsistency, and (2) the accuracy of the weights. The proposed methodology is applied to the AHP to demonstrate its advantages and is compared to established methodologies.

In the third phase, a beta distribution is proposed to generalize a wide variety of imprecise pairwise comparison distributions via a method of moments methodology. A Non-Linear Programming model is then developed that calculates PCM element weights which maximizes the preferences of the DM as well as minimizes the inconsistency simultaneously.

Comparison experiments are conducted using datasets collected from literature to validate the proposed methodology.

DEDICATION

There are only 11_2 people in this world I would dedicate this dissertation to:

- to both my parents who taught me the value of education over material things,

- and to my one and only who gave me level 3 luck in life.

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INTRODUCTION

1.1   Background and Rationale

Making a decision is one of the integral functions of management. The fast
changing business environment nowadays warrants managers to make swift decisions
without sacrificing quality. These managers deal with numerous types of decision making
problems: whether it would be strategic decisions on significant infrastructure
investments, tactical decisions like selecting the best supplier, and operational ones like
scheduling daily personnel. A majority of the decision problems have multiple decision
criteria. A simple example of a multiple criteria decision making (MCDM) tool is the
weighted sum method (WSM). However, according to Triantaphyllou (2000), the
estimation of criteria weights is highly subjective and thus would vary across different
decision makers (DM). One tool that addresses this weakness is called a Pairwise
Comparison Matrix (PCM). An $m \times m$ PCM denoted by A is a reciprocal matrix that is
composed of pairwise comparisons $a_{ij} \in [1/9, 9]$ which represent the scaled relative
importance scores of element i as compared to element j. Typically, a PCM is generated
from repetitive pairwise comparisons elicited from a DM to estimate criteria or alternative
priorities. A widely applied MCDM methodology that uses PCMs is the Analytic
Hierarchy Process (AHP) developed by Saaty (1977). Since its development, the AHP has
been successfully implemented in a wide-range of MCDM problems specifically: supplier
selection (Gencer & Gurpinar, 2007; Ghodsypour & O'Brien, 1998), resource allocation
(Ramanathan & Ganesh, 1995), defense R&D selection (Greiner et al., 2003), information
systems outsourcing (Schniederjans & Wilson, 1991), marketing (Lu et al., 1994; Kwak et
al., 2005), facility location planning (Badri, 1999), product design (H. Wang et al., 1998),
the environment (Masozera et al., 2006) and education (Saaty et al., 1991) just to name a
few. Conversely, the AHP fails to account for the interdependencies of the criteria and
alternatives, and hence it assumes that all criteria and alternatives are independent. If left
unchecked, any DM using the AHP would then provide inaccurate decisions. To address

this, the Analytic Network Process (ANP) has been developed by Saaty & Takizawa (1986) as a generalization of the AHP. The ANP requires additional pairwise comparisons to estimate the inner and outer dependencies of the criteria and alternatives. Although it addresses the limitations of the AHP, the ANP still use PCMs which are faced with the following issues: (1) numerous pairwise comparisons are required for decision making, (2) inconsistent pairwise comparisons are obtained when numerous pairwise comparisons are elicited from the DM and (3) pairwise comparisons may be imprecise due to the limited cognitive powers of the DM. These limitations are explained in detail as follows.

To illustrate the issue of numerous pairwise comparisons, consider a PCM with m elements. A total of $m(m-1)/2$ pairwise comparisons are needed to obtain the weights of the elements. Additionally, for the ANP, m^2 comparisons are needed to estimate the inner dependencies of the criteria. This could be a time consuming ordeal for any DM when m is large. Saaty (1977) argues that the redundancy of the questioning process provides weights which are much less sensitive to biases and judgement errors. As an example, in a case study by Lin et al. (2008), it took two and a half hours on average to complete a three-level AHP decision problem per DM and a total of 380 man-hours to complete all pairwise comparisons. This could be greater for the case of the ANP. There are generally three reasons why a decision maker (DM) is reluctant to complete the required comparisons specifically: (1) there is insufficient time to complete all comparisons; (2) the DM is unwilling to make direct comparisons of two alternatives and (3) the DM is unsure of some of the comparisons (Harker, 1987a). In a Monte-Carlo simulation study (Carmone et al., 1997), comparisons are deleted from large matrices and results show that at most 50% of the comparisons can be deleted without significantly affecting the weights of the criteria. Furthermore, to obtain a reasonable and consistent PCM, Saaty (1977) recommends that the number of criteria or alternatives within a PCM should only be at most seven. Hence, any PCM with eight or more elements is considered large. Unfortunately, lots of decision problems far exceed this maximum threshold. There exist several articles that attempt to address the issue of numerous pairwise comparisons, one

of which are called decomposition methodologies (Shen et al., 1992; Islam et al., 1997a; Triantaphyllou, 1995; Islam & Abdullah, 2006; Ishizaka, 2008, 2012). It is known that to overcome the limited cognitive processing powers of a DM, a complex problem is decomposed into smaller pieces to bring it within a DM's cognitive ability (Wright, 1985). When a PCM A of n alternatives is decomposed into k subsets, pairwise comparisons are elicited only in those subsets. Since the dimensions of the decomposed matrices are smaller than A, a reduction in the number of pairwise comparisons is realized. Unfortunately, these methods are not without any disadvantages. Firstly, these methodologies focus on decomposing PCMs for alternatives. Lots of pairwise comparisons can be saved when these methodologies are extended to decompose the criteria PCMs since both the AHP and ANP can have multiple criteria PCMs within the hierarchy or network respectively. Secondly, when a PCM is decomposed into subsets, the obtained relative weights of the elements are valid only within those subsets and the problem arises when aggregating the results back to its global counterpart. As a result, a pivot element is selected arbitrarily and assigned to all subsets and is used as a basis for comparing the criteria across all disjoint subsets. The global weights can then be estimated as is done in (Shen et al., 1992; Triantaphyllou, 2000; Ishizaka, 2012). Please note that pivot element selection is a challenging issue as the decisions should consider reducing the number and inconsistency of the pairwise comparisons. Thirdly, these methodologies lack guidelines to assign PCM elements to respective subsets since they are done arbitrarily. This does not guarantee that the elements within subsets are independent. Lastly, to the best of our knowledge, the number of subsets is known as a prior and are subject to DM biases and judgement errors.

The second weakness of using PCMs is attributed to the consistency of the pairwise comparisons elicited from the DM when the number of elements within the PCM is large. As the number of pairwise comparisons elicited from the DM increases, the consistency of these comparisons is expected to be less reliable and results to inconsistent decisions (Weiss & Rao, 1987). A performance metric called consistency index (CI) is

generally used to estimate the inconsistency of a PCM A (Saaty, 1977). The CI is computed by obtaining the eigenvalue of the PCM using Eq. 1.1:

$$CI(A) = \frac{\lambda_{max} - m}{m - 1} \tag{1.1}$$

where m is the dimension of the PCM A and λ_{max} is the maximal eigenvalue of matrix A. The consistency ratio (CR) is the ratio of CI and RI and is computed using Eq. 1.2:

$$CR(A) = \frac{CI(A)}{RI(m)} \tag{1.2}$$

where $RI(m)$ is the random index obtained from the average CI of 500 randomly filled matrices and is dependent on the value of the m. If a PCM A has $CR < 10\%$, then A is considered to have an acceptable level of consistency (Saaty, 1977). On the other hand, the process of eliciting pairwise comparisons from the DM shall be repeated until a consistent matrix is obtained. This limitation is evident in the same case study by Lin et al. (2008) since it took 380 man-hours to complete all pairwise comparisons which included repeats. There are extensive methodologies that focus on improving the consistency of a PCM by changing pairwise comparison values individually (Zeshui & Cuiping, 1999; Cao et al., 2008; Saaty, 2003; Benítez, Delgado-Galván, Gutiérrez, & Izquierdo, 2011; Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2011). Upon executing these methodologies, a more consistent PCM is obtained. Yet, these methods require a significant amount of time to complete since a large PCM is required as input. Additionally, these methodologies may alter the individual pairwise comparisons that greatly differ from the original preferences of the DM. It is therefore evident that reducing the number of pairwise comparisons and bringing it within the bounds of a DM's cognitive ability would be the only recourse to improve the overall consistency of the pairwise comparisons.

The third limitation of PCMs is attributed to the existence of uncertainty in the elicitation process of the pairwise comparisons. Conventionally, crisp pairwise comparisons are elicited from the DM to calculate the weights of the elements. Yet, these crisp values may not be sufficient to model the presence of ambiguity (A. H. I. Lee et al., 2008) due to the limited cognitive powers of DMs. To address this limitation, the concept of interval

judgements or interval pairwise comparisons is proposed in (Saaty & Vargas, 1987; Arbel, 1989). Upper and lower limit pairwise comparison values (LL,UL) are used to model the imprecise pairwise comparisons. Essentially, these pairwise comparisons are treated as random variables that follow a uniform probability distribution. Other stochastic distributions have also been explored including the the triangular distribution (Banuelas & Antony, 2006), binomial (Hahn, 2003) and the Cauchy distribution (Lipovetsky & Tishler, 1999). Fuzzy sets are also proposed in studies done by (Mikhailov, 2000, 2004; Dagdeviren & Yüksel, 2008; T. C. Wang & Chen, 2008). However, these methodologies are not without any disadvantages. Firstly, in order to complete the decision making process, weights need to be estimated from interval distributions. The conventional AHP eigenvector method for calculating weights is no longer applicable when pairwise comparisons are imprecise (Arbel, 1989). These stochastic AHP models need to explore the use of methodologies like linear programming and simulation to calculate the weights. Note, most stochastic AHP methodologies focus on either maximizing the preferences of the DM only even if they may be inconsistent (Y. M. Wang & Elhag, 2007; Banuelas & Antony, 2006), or minimizing the inconsistency only which may result the decision deviated from the DM's true preference (Y. M. Wang et al., 2005a; Mikhailov, 2000, 2004). A balanced consideration of both objectives is of necessity, especially, the use of stochastic distributions may lead to increased inconsistency if the weights are not properly derived from the pairwise comparisons. Another notable issue of existing literature is most stochastic AHP methodologies employ one single type of pairwise comparison distribution. Knowing DMs that use the AHP are faced with the issues on bounded rationality (Simon, 1955, 1972) that is, limited cognitive power to precisely depict the preferences over large number of comparisons; it is desirable to provide DMs the flexibility to choose different distributions for different pairwise comparisons during the elicitation process.

1.2   Research Contributions

To address these issues, this dissertation proposes a PCM framework which is composed of three phases. The contributions of each phase is explained as follows:

The first phase proposes a PCM decomposition methodology which addresses the first two limitations of large-scale PCMs. To reduce the number and the inconsistency of the pairwise comparisons of a large scale PCM, a binary integer programming (BIP) model is developed to segment PCM elements into k mutually exclusive subsets. Since the subsets are disjoint, a pivot element is selected that better connects all the disjoint subsets. By assigning the pivot element to all subsets, all elements can then be compared across all subsets. Then the global weights of the PCM elements can be estimated. The methodology is applied to both AHP and ANP methodologies to demonstrate its effectiveness.

However, the PCM decomposition methodology is limited in providing an assignment of PCM elements into k subsets where k is elicited from the DM. Selecting a specific value of k is subject to human bias and judgement error. Furthermore, the proposed PCM decomposition methodology selects the pivot element greedily after decomposition. This setup does not guarantee that the amount of dependence among elements within a subset is minimized and in effect the number of pairwise comparisons needed after decomposition. The second phase proposes a trade-off PCM decomposition methodology by proposing a BIP model that is capable of: (1) solving for the optimal number of subsets to segregate the PCM elements using the weighted preferences of the DM, (2) optimally assigning all elements to all the PCM subsets and (3) selecting the optimal pivot element for the calculation of the global element weights. The BIP methodology is applied to AHP dataset and is compared to existing decomposition methodologies.

The third phase addresses the third PCM limitation by proposing a stochastic based decision making methodology that mainly addresses issue of imprecise pairwise comparisons elicited from the DM. First, a beta distribution is proposed to model the varying types of probability distributions for the different types of stochastic pairwise comparisons elicited from the DM. The beta distribution has interesting properties, one of which is its ability to model other probability distributions, and it is differentiable over its

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domain making it ideal for optimization algorithms. The method of moments methodology is applied to fit any input pairwise comparison distribution into beta distributed pairwise comparisons. Next, a Non-Linear programming (NLP) model is developed to calculate crisp element weights that maximize the probability likelihood of varying types of imprecise pairwise comparisons or in a sense the preferences of the DM and at the same time minimizing the inconsistency of the pairwise comparisons.

1.3   Dissertation Organization

This dissertation is organized into six interrelated chapters that address the aforementioned issues of the PCMs. The reader may encounter some level of redundancy in the writing of this dissertation since chapters 3, 4 and 5 are written as standalone papers for scholarly journals. Chapter 2 provides a generalized literature review on the existing methodologies on large-scale PCMs. Chapter 3 presents the proposed PCM decomposition methodology for large scale PCMs. The proposed methodology is applied to both the AHP and the ANP methodologies. Furthermore, Chapter 4 presents the trade-off BIP that addresses the limitations of the proposed PCM decomposition methodology. Comparison experiments between the proposed methodology and existing methodologies are conducted using datasets collected from literature. Chapter 5 presents a stochastic decision making methodology that addresses the imprecise pairwise comparison issue. Lastly, Chapter 6 presents the conclusions of this dissertation as well as avenues for future research. The appendices summarize the proofs of propositions and theorems and the Matlab code used to validate all methodologies.

Chapter 2

REVIEW OF RELATED LITERATURE

This chapter provides a literature review on each of the PCM limitations. Specifically, section 2.1 reviews existing literature that addresses the numerous number of elicited pairwise comparisons. Section 2.2 reviews methodologies that address the inconsistency component of PCMs while Section 2.3 summarizes existing literature on imprecise pairwise comparisons. Additionally, Section 2.4 provides an analysis of the gaps in literature.

2.1   The Curse of Dimensionality of PCMs

There exist methodologies that address the curse of dimensionality and can be mainly classified as: (1) optimization methods and (2) heuristic methods. Each category is reviewed as follows.

Optimization Methods

Optimization models start with a handful of pairwise comparisons only. The remaining pairwise comparisons are estimated using optimization algorithms by taking advantage of the matrix properties of A. Starting with a of minimum $m-1$ comparisons, a gradient descent method is proposed to select the next pairwise comparison that would have the biggest information gain (Harker, 1987b). Additionally, stopping rules are provided for terminating the pairwise comparison elicitation process. The methodology by Bozoki et al. (2009) uses nonlinear optimization with exponential scaling to estimate the missing pairwise comparisons from available ones. However, all possible combinations of connecting paths must be considered. The number of combinations exponentially grows as the number of missing comparisons increases and thus would be inefficient to solve. A linear programming formulation by Triantaphyllou (1995) is used to estimate the missing pairwise comparisons of A by considering two arbitrary subsets s_1 and s_2 of the criteria PCM where $s_1 \cup s_2 \neq \emptyset$. By solving the linear programming problem, the global weights of

the m criteria of the PCM can be estimated. Nevertheless, the algorithm only focuses on dividing the PCM A into two subsets. If m is large, then the two subsets are still large. Moreover, the error rates of estimating the missing comparisons are dependent on the number of common elements of subsets s_1 and s_2. The smaller the $s_1 \cup s_2$, the estimation of the missing comparisons is expected to be less accurate and deviation error increases significantly. In general, the challenge of optimization based approaches lies to the scalability of the problem. That is, as most decision problems tend to have large number of alternatives and criteria, these analytical approaches may suffer.

Heuristic Methods

There are several types of heuristic methodologies that propose to reduce the number of pairwise comparisons elicited from the DM. Saaty (1990) proposes the idea to group alternatives into subsets according to a common decision criterion. Islam et al. (1997a) propose to assign the alternatives into k subsets based on a subjective absolute scale in which alternatives that have close "magnitudes" are grouped together. By using several pivot alternatives that is common to at most two subsets, the global priorities of the alternatives are then obtained. Note, the definition of close "magnitudes" is not well defined and is highly subjective. Furthermore, no guidelines are provided to determine which alternatives are assigned to which subset. Shen et al. (1992) propose an arbitrary decomposition of the alternative PCM into k subsets such that these k subsets have one common pivot alternative. Pairwise comparisons are first performed on each subset and local priorities are calculated. The global priority is then derived by using common pivot alternative and local priorities of each subset. Ishizaka (2012) applies the same decomposition algorithm on supplier selection. There exist models that use the concept of a balanced incomplete block designs in which subsets of the PCM A are assigned to different DMs treated as replicates in contrast to having all DMs focus on the large PCM (Weiss & Rao, 1987; Takahashi, 1990). The computation of the global alternative weights is done by using the geometric mean. To the best of our knowledge, no methodology has tried to optimally assign PCM elements to subsets that minimize the number of pairwise

comparisons, the amount of dependence among criteria and the consistency of the pairwise comparisons. Any methodology that reduces the required number of pairwise comparisons for a PCM is fruitful for wider adoption of the AHP and ANP methodologies (Ishizaka & Labib, 2011). Triantaphyllou (2000) develops a method to reduce the number of pairwise comparisons via the duality approach when the number of alternatives is greater than the number of criteria plus one. The question asked to quantify a pairwise comparison is "What is the relative importance of criterion C when it is compared with criterion C in terms of alternative A?" Instead of pairwise comparisons on the alternatives, the criteria are the subject of the questioning. This approach works due to the assumption that in a given MCDM problem the criteria influence the perception of the alternatives and vice-versa. On the other hand, it is only applicable for problems when the number of alternatives is much greater than the number of criteria; otherwise the reduction on the number of pairwise comparisons is not significant enough to warrant a change on the classical method of questioning. Islam & Abdullah (2006) consider reducing the number decision criteria by the nominal group technique. The decision criteria that have insignificant weights are eliminated from future pairwise comparison elicitation process.

2.2   Inconsistencies of Large PCMs

There are several articles (Cao et al., 2008; Saaty, 2003; Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2011) that focus on reducing the inconsistency of a given PCM without reducing the number of pairwise comparisons. Benítez, Delgado-Galván, Izquierdo, & Pérez-García (2011) propose a linearization heuristic that provides the closest consistent PCM which is later extended to balance the consistency and the preferences of the DM (Benítez, Delgado-Galván, Gutiérrez, & Izquierdo, 2011). Still, upon applying these methodologies, the original pairwise comparisons significantly deviate from the resulting consistent pairwise comparisons and thus would not reflect the actual preferences of the DM. Ishizaka & Lusti (2004) implement an expert module that helps the DM make a pairwise comparison one at a time within control limits. Decker et al. (2008) suggest a method to identify erroneous pairwise comparisons using the geometric

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mean method and proposed changes to improve the consistency. Wu et al. (2010) present to include scale and judgment errors into the PCM. By treating pairwise comparisons as random variables, an estimator to calculate weights is proposed. However, these methods focus on improving a given PCM which could be large. If the number of decision criteria or alternatives is large, these heuristics would still take a lot of time to complete and are more subject to human error when providing the initial pairwise comparisons. Furthermore, due to their limited cognitive processing powers, the DMs are not expected to provide consistent pairwise comparisons all throughout the pairwise comparison elicitation process especially when the number of pairwise comparisons is large. Therefore, a methodology that reduces the pairwise comparisons elicited from the DM would lead to improved consistency levels since only a handful of pairwise comparisons are elicited and would not be cognitively taxing to the DM.

2.3   Imprecise Pairwise Comparisons

The concept of interval pairwise comparisons or interval judgements is originally proposed by Saaty & Vargas (1987). In this setup, instead of eliciting crisp pairwise comparisons, the DM provides the minimum and maximum values that the unknown pairwise comparison value can have. This elicitation process handles the ambiguity issue of the pairwise comparison whenever the DM is unsure of its true value. In Saaty and Vargas's methodology, the calculation of global weights is done using a Monte Carlo simulation methodology by sampling feasible crisp weights that satisfy all interval judgements. However, this setup would not provide an optimal weight solution and is time consuming. As such, Arbel (1989) proposes a linear programming model to estimate the weights from interval judgements. However, Kress (1991) argues that the solution from (Arbel, 1989) exists only in completely consistent interval judgements. To date, there exist goal programming models to estimate global weights from interval pairwise comparisons (Bryson, 1995; Xu, 2004; Y. M. Wang & Elhag, 2007; Z. J. Wang & Li, 2012). These methodologies seek an optimal set of satisficing weights which is calculated by minimizing deviations of the optimal weights from all feasible interval judgements or in

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a sense maximizing preferences of the DM. Other similar goal programming variants include min-max goal programming (Despotis & Derpanis, 2008), logarithmic goal programming (Y. M. Wang et al., 2005b) and lexicographic programming (Islam et al., 1997b). Unfortunately, the lexicographic programming model provides unreliable priority estimates as shown in (Y. M. Wang, 2006). Hence, Y. M. Wang et al. (2005a) propose to minimize the consistency ratio (CR) using a NLP approach. Salo & Hämäläinen (1995) propose a preference programming approach with interactive decision support from the DM. Guo & Tanaka (2010) suggest the use of subjective pairwise comparisons of the likelihood of events for all possible alternative ranking outcomes. Quadratic programming is applied to estimate the final weights. Guo & Wang (2011) extend the model of Guo & Tanaka (2010) by using dual interval probabilities and linear programming. Please note only one type of distribution is studied in these mathematical models which may limit its application to large-scale decision problems where the DM tends to have different preference knowledge over different pairwise comparisons.

In modeling uncertainty, the concept of fuzzy sets has also been applied. Mikhailov (2000, 2004) applies fuzzy sets to model uncertainty and fuzzy preference programming method to estimate crisp weights. A similar methodology is proposed in (A. H. I. Lee et al., 2008, 2009). According to Y. M. Wang & Chin (2011), these methodologies may produce conflicting priority vectors that lead to inaccurate decisions as such they propose to use a logarithmic fuzzy preference programming methodology for priority derivations. In 2006, Y. M. Wang & Chin (2006) propose a combination of the eigenvector method and linear programming to estimate crisp priorities from fuzzy comparison matrices. Additionally, Yang et al. (2012) propose a cloud Delphi hierarchical analysis with fuzzy interval weights to model the uncertainty of the AHP. These models also use a single type of fuzzy distribution (e.q. triangular fuzzy numbers) which may not be applicable to model the varying preferences of the DM.

There are methodologies that handle the uncertainty of comparisons and weights calculation by applying statistical modeling techniques. Moreno-Jimenez & Vargas (1993)

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develop a methodology to estimate the probability of all possible alternative preference rankings from uniformly distributed interval judgements. These probabilities are analytically calculated whenever perfect consistent judgements are obtained. On the other hand, simulation is used for inconsistent cases. In (Haines, 1998), Haines proposes a statistical based algorithm to study the effect of using uniform and convex distributions on interval judgements. The mean of the distributions is used to rank the alternatives. Lipovetsky & Tishler (1999) propose to model interval judgements in terms of a Cauchy distribution and a non-linear approximation methodology to calculate priorities. Lipovetsky & Tishler (1997) also propose several other distributions like the triangular, normal, Laplace or Cauchy however they used it individually. In (Sugihara et al., 2004), Sugihara et al. suggest an interval regression model to estimate interval priorities from interval pairwise comparison judgements. Using uniform interval judgements, Stam & Silva (1997) recommend multivariate statistical techniques to estimate points and confidence intervals for rank reversal probabilities. When the rank reversal probability is low, then the interval judgements are accepted. Hahn (2003) proposes a Bayesian approach specifically, a weighted hierarchical multinomial logit model to obtain final weights. Furthermore, inference on these weights is done using a Markov chain Monte Carlo sampling method. Recently, Liu et al. (2011) suggest a probability distribution aggregation and mathematical programming to combine pairwise comparisons modelled as probability distributions.

2.4   Literature Gap Analysis

In summary, the following five gaps in literature can be gleaned from the review as follows:

- Existing decomposition methodologies focus on decomposing the PCMs that contain decision alternatives. Additional pairwise comparisons can be saved when decomposing algorithms are extended to criteria PCMs.

- These methodologies lack guidelines to assign PCM elements that minimize the number of pairwise comparisons elicited, as well as the interdependence of elements within each subset. There exist no guidelines to decompose a PCM into an appropriate number of subsets.

- These methodologies select the pivot element arbitrarily and no rules are provided in literature.

- Existing methodologies focus on modeling the imprecise pairwise comparisons using a single distribution type. However, it is expected that the variability of the pairwise comparisons would not be constant due to bounded rationality issues.

- Existing methodologies focus on either maximizing the preferences of the DM or minimizing the consistency ratio for pairwise comparisons that follow a single type of distribution. To the best of our knowledge, there exists no model that addresses these two objectives simultaneously and efficiently.

This dissertation proposes a PCM framework for large-scale decision making that address these gaps in succeeding sections.

Chapter 3

INTELLIGENT DECOMPOSITION OF PAIRWISE COMPARISON MATRICES FOR

LARGE-SCALE DECISIONS

A Pairwise Comparison Matrix (PCM) has been used to compute for relative priorities of elements and are integral components in widely applied decision making tools: the Analytic Hierarchy Process (AHP) and its generalized form, the Analytic Network Process (ANP). However, PCMs suffer from several issues limiting their applications to large-scale decision problems. These can be attributed to the curse of dimensionality, that is, a large number of pairwise comparisons need to be elicited from a decision maker. Due to the limited cognitive power of decision makers, inconsistent preferences may be obtained. To address these limitations, this research proposes a PCM decomposition methodology. A binary integer program is proposed to intelligently decompose a PCM into several smaller subsets using interdependence scores among elements. Since the subsets are disjoint, the most independent pivot element is identified to connect all subsets to derive the global weights of the elements from the original PCM. As a result, the number of pairwise comparison is reduced. In addition, the consistency is improved. The proposed decomposition methodology is applied to both AHP and ANP to demonstrate its advantages.

3.1   Introduction

An $m \times m$ pairwise comparison matrix (PCM) denoted by A is a reciprocal matrix which is composed of pairwise comparisons $a_{ij} \in [1/9, 9]$ that represent the scaled relative importance scores of element i as compared to element j. Typically, a PCM is generated from pairwise comparisons elicited from a decision maker (DM) to estimate criteria or alternative priorities for any decision problem. One of the most widely used multiple criteria decision making (MCDM) methodology that use PCMs is the Analytic Hierarchy Process (AHP) developed by Saaty (1977). Currently, there are several successful applications of the AHP in a wide-range of MCDM problems (Ishizaka & Labib, 2011).

Conversely, the AHP fails to account for the interdependencies of the criteria and alternatives, and hence it assumes that all criteria and alternatives are independent. If left unchecked, any DM using the AHP would then provide inaccurate decisions. To address this, the Analytic Network Process (ANP) has been developed by Saaty & Takizawa (1986) as a generalization of the AHP. The ANP requires additional pairwise comparisons to estimate the inner and outer dependencies of the criteria and alternatives. Although it addresses the limitations of the AHP, the ANP still use PCMs which are faced with the following issues: (1) numerous pairwise comparisons are required for decision making and (2) inconsistent pairwise comparisons are obtained when numerous pairwise comparisons are elicited from the DM are large.

The first limitation is attributed to the fact that both methodologies suffer from the curse of dimensionality. Consider a PCM of m criteria. A total of $m(m-1)/2$ pairwise comparisons are needed to obtain the priorities. Additionally, for the ANP, m^2 comparisons are needed to estimate the inner dependencies of the criteria. This would be impractical when m is large. Saaty (1977) argues that the redundancy of the questioning process provides weights which are much less sensitive to biases and judgement errors. In a case study by Lin et al. (2008), it took two and a half hours on average to complete a three-level AHP decision problem per DM and a total of 380 man-hours to complete all pairwise comparisons. This could be greater for the case of the ANP. There are generally three reasons why a DM is reluctant to complete the required comparisons specifically: (1) there is insufficient time to complete all comparisons; (2) the DM is unwilling to make direct comparisons of two alternatives and (3) the DM is unsure of some of the comparisons (Harker, 1987a). In a Monte-Carlo simulation study (Carmone et al., 1997), comparisons are deleted from large matrices and results show that at most 50% of the comparisons can be deleted without significantly affecting the weights of the criteria. Furthermore, to obtain a reasonable and consistent PCM, Saaty (1977) recommends that the number of criteria or alternatives within a PCM should only be at most seven. Hence, any PCM with eight or more elements is considered large. Unfortunately, lots of decision

problems far exceed this maximum threshold. There exist several articles that attempt to address the issue of numerous pairwise comparisons, one of which are called decomposition methodologies (Shen et al., 1992; Islam et al., 1997a; Triantaphyllou, 1995; Ishizaka, 2008, 2012). It is known that to overcome the limited cognitive processing powers of a DM, a complex problem is decomposed into smaller pieces to bring it within a DM's cognitive ability (Wright, 1985). When a PCM A of n alternatives is decomposed into k subsets, pairwise comparisons are elicited only in those subsets. Since the dimensions of the decomposed matrices are smaller than A, a reduction in the number of pairwise comparisons is realized. Unfortunately, these methods are not without any disadvantages. Firstly, these methodologies focus on decomposing PCMs for alternatives. Lots of pairwise comparisons can be saved when these methodologies are extended to decompose the criteria PCMs since both the AHP and ANP can have multiple criteria PCMs within the hierarchy or network respectively. Secondly, when a PCM is decomposed into subsets, the obtained relative weights of the elements are valid only within those subsets and the problem arises when aggregating the results back to its global counterpart. As a result, a pivot element is selected arbitrarily and assigned to all subsets and is used as a basis for comparing the elements across all disjoint subsets. The global weights can then be estimated (Shen et al., 1992; Islam et al., 1997a; Ishizaka, 2008). Please note that pivot element selection is a challenging issue as the decisions should consider reducing the number and inconsistency of the pairwise comparisons, as well as reducing the amount of dependence present among elements within each subset. Thirdly, these methodologies lack guidelines to assign criteria or alternatives to respective subsets since they are done arbitrarily. This does not guarantee that the number of pairwise comparisons is reduced optimally.

The second limitation related to the curse of dimensionality is attributed to the consistency of the pairwise comparisons elicited from the DM when the number of alternatives or criteria is large. As the number of pairwise comparisons increases, the consistency of these comparisons is expected to be less reliable and results to inconsistent

decisions (Weiss & Rao, 1987). A performance metric called consistency index (CI) is generally used to estimate the inconsistency of a PCM A (Saaty, 1977). The CI is computed by obtaining the eigenvalue of the PCM using Eq. 3.1:

$$CI(A) = \frac{\lambda_{max} - m}{m - 1} \tag{3.1}$$

where m is the dimension of the PCM A and λ_{max} is the maximal eigenvalue of matrix A. The consistency ratio (CR) is the ratio of CI and RI and is computed using Eq. 3.2:

$$CR(A) = \frac{CI(A)}{RI(m)} \tag{3.2}$$

where $RI(m)$ is the random index obtained from the average CI of 500 randomly filled matrices and is dependent on the value of the m. According to Saaty (1977), if a PCM A has CR < 10%, then A is considered to have an acceptable level of consistency. Nevertheless, DMs that use the PCMs are faced with the issues on bounded rationality (Simon, 1955, 1972). With this, due to their limited cognitive processing powers, the DMs are not expected to provide consistent pairwise comparisons all throughout the pairwise comparison elicitation process especially when the number of pairwise comparisons is large. Therefore, a methodology that reduces the pairwise comparisons elicited from the DM would lead to improved consistency levels since only a handful of pairwise comparisons are elicited and would not be cognitively taxing to the DM.

This research proposes the PCM Decomposition Methodology (PDM) to address the limitations of PCMs when used in either the AHP or the ANP methodology. The contributions of the PDM are twofold: (1) The PDM seeks to reduce the number of pairwise comparisons elicited from the DM thereby increasing its consistency. A binary integer programming (BIP) model is proposed to accomplish this by intelligently decomposing PCMs into smaller and manageable subsets. Only pairwise comparisons within those subsets are elicited from the DM. The BIP uses the inner dependence comparisons of the elements to assign these elements into mutually exclusive subsets. Hence, interdependent elements are separated as much as possible thereby reducing the

amount of interdependencies among subsets. (2) Since the subsets are disjoint, a pivot element is optimally selected and is used to connect all pairwise comparison matrices within each PCM. The pivot is selected that minimizes the interdependencies of the elements. Using the pivot element and the local weights, the global weights of the elements of the PCM are then calculated.

The rest of this paper is organized as follows. Section 3.2 reviews existing literature that tries to solve the aforementioned problems. Section 3.3 illustrates the steps of the proposed PDM, while Section 3.4 describes the application of the PDM in reducing the number of pairwise comparisons in an AHP problem. On the other hand, section 3.5 presents the application of the PDM in reducing the number of pairwise comparisons in an ANP problem. Finally, Section 3.6 concludes the paper and proposes further research areas.

3.2   Review of Related Literature

There exist methodologies that address the limitations for the PCMs and can be mainly classified as: (1) optimization methods and (2) heuristic methods. Each category is reviewed as follows.

Optimization Methods

Optimization models start with a handful of pairwise comparisons only. The remaining pairwise comparisons are estimated using optimization algorithms by taking advantage of the matrix properties of A. Starting with a of minimum $m - 1$ comparisons, a gradient descent method is proposed to select the next pairwise comparison that would have the biggest information gain (Harker, 1987b). Additionally, stopping rules are provided for terminating the pairwise comparison elicitation process. The methodology by Bozoki et al. (2009) uses nonlinear optimization with exponential scaling to estimate the missing pairwise comparisons from available ones. On the other hand, all possible combinations of connecting paths must be considered. The number of combinations exponentially grows as the number of missing comparisons increases and thus would be

inefficient to solve. A linear programming formulation by Triantaphyllou (1995) is used to estimate the missing pairwise comparisons of A by considering two arbitrary subsets s_1 and s_2 of the criteria PCM where $s_1 \cup s_2 \neq \emptyset$. By solving the linear programming problem, the global weights of the m criteria of the PCM can be estimated. Nevertheless, the algorithm only focuses on dividing the PCM A into two subsets. If m is large, then the two subsets are still large. Moreover, the error rates of estimating the missing comparisons are dependent on the number of common elements of subsets s_1 and s_2. The smaller the $s_1 \cup s_2$, the estimation of the missing comparisons is expected to be less accurate and deviation error increases significantly. In general, the challenge of optimization based approach lies to the scalability of the problem. That is, as most decision problems tend to have large number of alternatives and criteria, these analytical approaches may suffer.

Heuristic Methods

There are several types of heuristic methodologies that propose to reduce the number of pairwise comparisons elicited from the DM. Saaty (1990) proposes the idea to group alternatives into subsets according to a common decision criterion. Islam et al. (1997a) propose to assign the alternatives into k subsets based on a subjective absolute scale in which alternatives that have close "magnitudes" are grouped together. By using several pivot alternatives that is common to at most two subsets, the global priorities of the alternatives are then obtained. Note, the definition of close "magnitudes" is not well defined and is highly subjective. Furthermore, no guidelines are provided to determine which alternatives are assigned to which subset. Shen et al. (1992) propose an arbitrary decomposition of the alternative PCM into k subsets such that these k subsets have one common pivot alternative. Pairwise comparisons are first performed on each subset and local priorities are calculated. The global priority is then derived by using common pivot alternative and local priorities of each subset. Ishizaka (2012) applies the same decomposition algorithm on supplier selection. There exist models that use the concept of a balanced incomplete block designs in which subsets of the PCM A are assigned to different DMs treated as replicates in contrast to having all DMs focus on the large

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hierarchy (Weiss & Rao, 1987; Takahashi, 1990). The computation of the global alternative weights is done by using the geometric mean. To the best of our knowledge, no methodology has tried to optimally assign PCM elements to subsets that minimize the number of pairwise comparisons, the amount of dependence among criteria and the consistency of the pairwise comparisons. Any methodology that reduces the required number of pairwise comparisons for a PCM is fruitful for wider adoption of the AHP and ANP methodologies (Ishizaka & Labib, 2011). Triantaphyllou (2000) develops a method to reduce the number of pairwise comparisons via the duality approach when the number of alternatives is greater than the number of criteria plus one. The question asked to quantify a pairwise comparison is "What is the relative importance of criterion C when it is compared with criterion C in terms of alternative A?" Instead of pairwise comparisons on the alternatives, the criteria are the subject of the questioning. This approach works due to the assumption that in a given MCDM problem the criteria influence the perception of the alternatives and vice-versa. On the other hand, it is only applicable for problems when the number of alternatives is much greater than the number of criteria; otherwise the reduction on the number of pairwise comparisons is not significant enough to warrant a change on the classical method of questioning. Islam & Abdullah (2006) consider reducing the number decision criteria by the nominal group technique. The decision criteria that have insignificant weights are eliminated from future pairwise comparison elicitation process.

There are several articles (Cao et al., 2008; Saaty, 2003; Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2011) that focus on reducing the inconsistency of a given PCM without reducing the number of pairwise comparisons. Benítez, Delgado-Galván, Izquierdo, & Pérez-García (2011) propose a linearization heuristic that provides the closest consistent PCM which is later extended to balance the consistency and the preferences of the DM (Benítez, Delgado-Galván, Gutiérrez, & Izquierdo, 2011). Still, upon applying these methodologies, the original pairwise comparisons significantly deviate from the resulting consistent pairwise comparisons and thus would not reflect the actual

preferences of the DM. Ishizaka & Lusti (2004) implement an expert module that helps the DM make a pairwise comparison one at a time within control limits. Decker et al. (2008) suggest a method to identify erroneous pairwise comparisons using the geometric mean method and proposed changes to improve the consistency. Wu et al. (2010) propose to include scale and judgment errors into the PCM. By treating pairwise comparisons as random variables, an estimator to calculate weights is proposed. However, these methods focus on improving a given PCM which could be large. If the number of decision criteria or alternatives is large, these heuristics would still take a lot of time to complete and are more subject to human error when providing the initial pairwise comparisons.

In summary, the following four gaps in literature can be gleaned from the review as follows:(1) Existing decomposition methodologies focus on decomposing the PCMs with alternative elements. Additional pairwise comparisons can be saved when decomposing algorithms are extended to criteria PCMs. (2) These methodologies lack guidelines to assign elements to subsets of the PCMs that minimize the number of pairwise comparisons elicited, as well as the independence of elements within each subset. (3) These methodologies select the pivot element arbitrarily and no rules are provided in literature. (4) To the best of our knowledge, there is a lack of methodologies that try to reduce the number of pairwise comparisons of a PCM for the AHP but not for the ANP. Any methodology that can simplify the ANP would be beneficial for any decision with interdependent criteria and alternatives.

3.3   Proposed PCM Decomposition Methodology

This section outlines the proposed PCM Decomposition Methodology (PDM). Figure 3.1 presents a high level overview of the proposed PDM.

We illustrate the decomposition of a PCM A with m elements (criteria or alternatives). In step 1, the m elements are collected and a value of the number of subsets $k \in [2, m-1]$ is elicited from the DM. The pairwise comparisons that measure the inner dependencies of the elements are qualitatively elicited or quantitatively gathered.
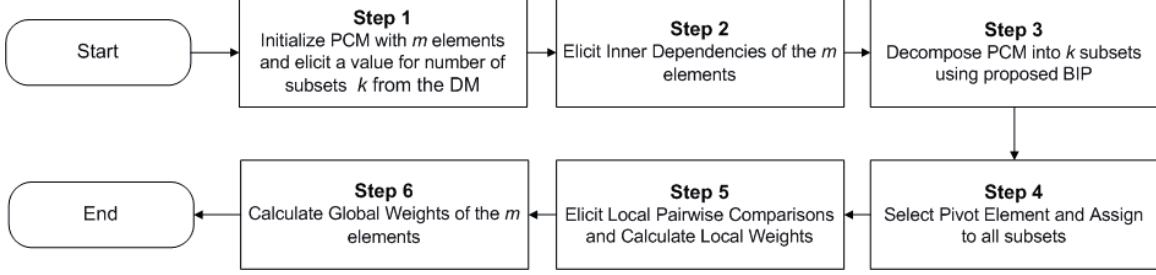
Figure 3.1. The Proposed PCM Decomposition Methodology

Quantitative pairwise comparisons are direct observations from the attributes of alternatives, while qualitative comparisons are elicited from the DM to quantify the degree of preference between any two elements. This is completed in step 2 (see section 3.3.1). Let these comparisons be $R = \{r_{ii'}|i, i' = 1, 2, .., m\}$. A symmetric interdependence matrix for the elements is derived from the R scores. Using the obtained interdependence matrices, the m elements are decomposed into k mutually exclusive subsets $s_l \in S$ using the proposed BIP decomposition methodology (see section 3.3.1). Additionally, in step 4, the pivot element is selected by choosing the most independent one and assigned to all subsets (see section 3.3.2). In step 5, local pairwise comparisons a_{ij} are then elicited for all subsets of S and local weights are calculated (see section 3.3.3). In step 6, global priorities are estimated from the local pairwise comparisons for all elements (see section 3.3.4).

Decompose PCM into Subsets

We start with the elicited m^2 inner dependence pairwise comparisons denoted by matrix R as follows:

$$
R = \begin{bmatrix}
r_{11} & r_{21} & \cdots & r_{m1} \\
r_{12} & r_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
r_{1m} & \cdots & \cdots & r_{mm}
\end{bmatrix}
\tag{3.3}
$$

According to Saaty & Takizawa (1986) if all elements are independent, then $R = I_m$ where I_m is an identity matrix of size m. Otherwise, a score r_{ij} is used to denote

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the dependence of element i to element j. Since it is unintuitive to partition the elements using a directed graph, we transform the directed graph into a symmetric undirected graph as follows:

$$\tilde{R} = \frac{R + R^T}{2} \tag{3.4}$$

We then obtain a symmetric matrix $\tilde{R}$ with scores $\tilde{r}_{ij} \in \tilde{R}$ where $\tilde{r}_{ij} = \tilde{r}_{ji}$. Given this, independent elements are intelligently assigned into the k subsets using the proposed BIP formulation as follows:

Let the decision variables be:

$\tilde{r}_{ij} :=$ dependence of element i to i'

$$x_{il} = \begin{cases} 1 & \text{if element } i \text{ is assigned to subset with index } l \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ii'l} := x_{il}x_{i'l} \begin{cases} 1 & \text{if both elements } i \text{ and } i' \text{ is assigned to } l \\ 0 & \text{otherwise} \end{cases}$$

Objective Function:

$$\max \quad \sum_{l=1}^{k}\sum_{i=1}^{m}\sum_{i'<i} \tilde{r}_{ii'}y_{ii'l} \tag{3.5}$$

Constraints:

$$x_{il} - y_{ii'l} \geq 0, \qquad\qquad i \in [1,m], l \in [1,k] \tag{3.6}$$

$$x_{i'l} - y_{ii'l} \geq 0, \qquad\qquad i' \in [1,m], l \in [1,k] \tag{3.7}$$

$$x_{i'l} + x_{il} - y_{ii'l} \leq 1, \qquad\qquad i, i' \in [1,m], l \in [1,k] \tag{3.8}$$

$$\sum_{l=1}^{k} x_{il} = 1 \qquad\qquad i \in [1,m] \tag{3.9}$$

$$\sum_{i=1}^{m} x_{il} \geq 1 \qquad\qquad l \in [1,k] \tag{3.10}$$

$$y_{ii'l} \in \mathbb{B}^{mC2 \times k}, \ x_{i'l} \in \mathbb{B}^{m \times k} \qquad\qquad i, i' \in [1,m], l \in [1,k] \tag{3.11}$$

The output of the BIP is a mutually exclusive assignment of the m elements to subsets $S = \{s_l | l = 1, 2..k\}$. Equation 3.5 describes the objective function of minimizing the inner dependencies of the elements to be assigned in each subset s_l. The BIP formulation would work hard to assign two elements to two different subsets if $\tilde{r}_{ii'} > 0$. Constraint sets 3.6 to

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3.8 are constraints that linearize the quadratic constraint $y_{ii'l} := x_{il}x_{i'l}$. Constraint set 3.9 forces each element to be a member of a subset while constraint set 3.10 forces all subsets to have at least one element. Eq. 3.11 defines x_{il} and $y_{ii'l}$ as binary integer variables.

Given the BIP formulation, the following properties can be realized: (1) the minimum number of elements assigned to a given subset is one and (2) the maximum number of elements assigned to a given subset is $m - k$. The minimum number of elements assigned follows from the BIP formulation; specifically constraint set 3.10 forces the number of elements assigned to subsets to be at least 1. In terms of the maximum, since the $k - 1$ subsets would have at least 1 element, then the kth subset has $m - k$ elements assigned.

Select Pivot Element

When the PCM elements are decomposed into k subsets, the elements are disjoint since there are no pairwise comparisons across subsets. In order to determine the relative priorities of the elements across subsets, a pivot element is selected and assigned to all subsets and is used as a basis for the global weights. To select the best pivot element, we select the element that minimizes the following function:

$$Pivot \; Element \; i = argmin_i \left(\sum_{i'} \tilde{r}_{ii'} \right) \tag{3.12}$$

Eq. 3.12 selects the least interdependent element as compared to all other elements.

Elicit Local Pairwise Comparisons and Calculate Local Weights

After decomposition, local pairwise comparisons are elicited from the DM for all subsets after the elements are assigned to subsets. The local pairwise comparisons for elements subset s_l are illustrated in matrix form A_l as shown in Eq. 3.13 as follows:

$$A_l = \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,m_l} \\ 1/a_{,2} & 1 & \cdots & a_{2,m_l} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1,m_l} & 1/a_{2,m_l} & \cdots & 1 \end{bmatrix}, \forall s_l \in S \tag{3.13}$$

A new performance measure is proposed to keep track of the consistency of the pairwise comparisons. The original definition of the CR of matrix A is no longer applicable since the m elements are assigned into k subsets. With this, a new definition of consistency is proposed as follows:

Definition 3.1. *The Average Consistency Ratio (ACR) performance measure of an AHP problem decomposed into k subsets is defined as:*

$$ACR = \frac{1}{m + k - 1} \sum_{l=1}^{k} m_l CR(A_l) \tag{3.14}$$

where $CR(A_l)$ is the consistency ratio of the PCM A_l of subset l.

In simple terms, the ACR is the weighted average of the CR of each of the local pairwise comparison matrices. The ACR is used to estimate the overall CR of the pairwise comparisons across all subsets.

Given the assignment of m elements into k subsets and the addition of the pivot element in all subsets, we define a quantity to determine the reduction in the number of pairwise comparisons elicited from the DM as follows:

Definition 3.2. *Given the set of the assigned elements into the k subsets with one pivot element, the total number of required pairwise comparisons needed to obtain the local priorities of the elements denoted by $d(A^k)$ is computed as:*

$$d(A^k) = \sum_{l=1}^{k} \binom{m_l}{2} = \sum_{l=1}^{k} \frac{m_l(m_l - 1)}{2} \tag{3.15}$$

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Remark: The maximum number of elements assigned to k subsets after including the pivot elements is:

$$\sum_{l=1}^{k} m_l \leq m + k - 1 \tag{3.16}$$

The total number of pairwise comparisons after decomposition is dependent on the distribution of the assignment of elements to the k subsets and as such is shown using Eq. 3.16. If $k = 1$, no decomposition is performed, then all the required $m(m-1)/2$ pairwise comparisons are elicited. On the other hand when $k = m - 1$, then the pivot element is compared to all other elements with a total of only $m - 1$ pairwise comparisons. The following propositions can be drawn with regards to the number of pairwise comparisons.

Proposition 3.1. *Given values of k, function $d(A^k)$ is maximized when $m_l = m + (k-1)$ for $l \in S$ and $m_{l'} = 2, \forall l \in S \setminus \{l\}$*

Proof. See Appendix. □

Proposition 3.2. *Given values of k, function $d(A^k)$ is minimized when:*

- $m_l = \frac{m+(k-1)}{k}$ *for $l \in S$ if $\frac{m+(k-1)}{k} \in \mathbb{Z}$ or*

- $m_l = \left\lfloor \frac{m+(k-1)}{k} \right\rfloor$ *for some $l \in S$ and $m_{l'} = \left\lceil \frac{m+(k-1)}{k} \right\rceil$ for some l' if $\frac{m+(k-1)}{k} \notin \mathbb{Z}$*

Proof. See Appendix. □

Definition 3.3 illustrates the difference between the original total number of pairwise comparisons $m(m-1)/2$ and the amount of pairwise comparisons needed, $D(A^k)$ when the elements are assigned to subsets.

Definition 3.3. *The difference between the original number of pairwise comparisons of matrix A and number of pairwise comparisons after decomposition into k subsets denoted by $D(A^k)$ is given by:*

$$D(A^k) = \binom{m}{2} - d(A^k) = \frac{m(m-1)}{2} - \sum_{l=1}^{k} \binom{m_l}{2} \tag{3.17}$$

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Given Proposition 3.1 and Proposition 3.2, the amount of time saved by the DM in terms of the reduction of the number of pairwise comparisons can be generalized in terms of Theorem 3.1.

Theorem 3.1. *Given m elements grouped into k subsets, the total number of required local pairwise comparison saved is bounded by:*

$$\left\lceil (k-1)\left(m - \frac{k+2}{2}\right) \right\rceil \leq D(A^k) \leq \left\lfloor \frac{(k-1)(m-1)^2}{2k} \right\rfloor \tag{3.18}$$

Proof. See Appendix. □

Theorem 3.1 provides the pessimistic and the optimistic estimates of the reduction of the required $m(m-1)/2$ pairwise comparisons of A. This performance metric would be a good yardstick to determine the amount time saved by the DM when making a complex decision using the proposed PDM.

After eliciting local pairwise comparisons for all subsets A_l we now define the local element weights computed for each subset. Let $w(A_l)$ be the vector of local weights from A_l where $w(i,l) \in w(A_l)$ is the local weight of element i. The original eigenvector methodology is used to calculate the local element weights as follows:

$$w(A_l) = \begin{bmatrix} \frac{\sum_{j=1}^{m_l} \tilde{a}_{1j}}{m_l} \\ \vdots \\ \frac{\sum_{j=1}^{m_l} \tilde{a}_{m_l j}}{m_l} \end{bmatrix} = \begin{bmatrix} w(1,l) \\ w(2,l) \\ \vdots \\ w(m_l, l) \end{bmatrix}, l = 1, 2, \cdots, k \tag{3.19}$$

Calculate Global Weights of the PCM

This subsection describes the methodology to compute global weights of the elements of the decomposed PCM from the local subsets using a pivot element. Given values of k there will be $m + k - 1$ instances of $w(i,l)$. The local element weight in subset l is divided by the weight of pivot element in that subset and is repeated for all subsets.

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To illustrate this, let $\tilde{w}(A_l)$ be the vector of normalized weights where each $\tilde{w}(i,l) \in \tilde{w}(A_l)$ is computed using Eq. 3.20.

$$\tilde{w}(i,l) = \frac{1}{w(i = c_p, l)}[w(i,l)], \quad \forall i \in s_l, \forall s_l \in S \tag{3.20}$$

Given this, the normalized pivot element weight in each subset has a value equal to one. Since all normalized pivot element weights have a value equal to one, all the other elements in the other subsets can be compared to the pivot elements. For the computation of the global weights, let $w'(A)$ be the vector of global weights where $w'(i) \in w'(A)$ is computed using Eq. 3.21.

$$w'(i) = \frac{1}{\sum_{l=1}^{k} \sum_{i \in s_l} \tilde{w}(i,l) - k + 1}\tilde{w}(i,l), \quad \forall i \in [1,m] \tag{3.21}$$

3.4 Decomposing PCMs for the AHP Methodology

This section illustrates the application of the PDM on an AHP problem. The AHP assumes that the criteria and alternatives are independent and hence, we seek an alternative way to measure the interdependence of the elements. Specifically, the correlation of the alternative scores r_{ij} is used to estimate the inner dependencies of the criteria. The correlation of the alternative scores could act as an alternative to estimate inner dependencies of the criteria as is done by Yurdakul & Tansel (2009). The alternative scores r_{ij} represent the raw rating score of alternative j on criterion i. We formally define the correlation denoted by $R(c_i, c_{i'})$ as follows:

Definition 3.4. *Let $R(c_i, c_{i'})$ be the correlation between criterion c_i and criterion $c_{i'}$ which is calculated using (4.22) as follows*

$$R(c_i, c_{i'}) = \left| \frac{\sum_{j=1}^{n}(r_{ij} - \bar{r}_{ij})(r_{i'j} - \bar{r}_{i'j})}{\sqrt{\sum_{j=1}^{n}(r_{ij} - \bar{r}_{ij})^2 \sum_{j=1}^{n}(r_{i'j} - \bar{r}_{i'j})^2}} \right| \tag{3.22}$$

These correlation coefficients will be used in the proposed BIP to group uncorrelated criteria. A value $R(c_i, c_{i'}) = 1$ means that the criterion c_i is positively or

negatively correlated to criterion $c_{i'}$. Hence, the PDM can be applied to decompose the lowest level criteria PCM, directly above the alternatives within an AHP decision problem. Otherwise, we decompose all other AHP PCMs arbitrarily as is done in existing literature. Figure 3.2 illustrates the proposed PDM as applied to a 3-level AHP problem.
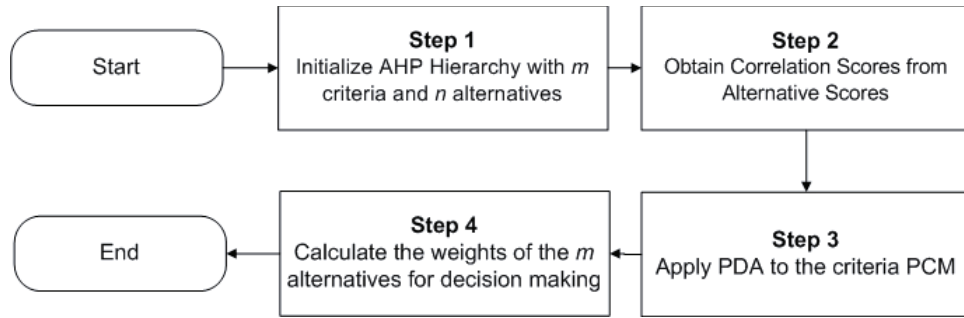
Figure 3.2. Application of the PDM for a 3-Level AHP Problem

In step 1, the DM decides on the m criteria and n alternatives and the three-level decision hierarchy (see section 3.4.1). Step 2 is the alternative scores pairwise comparison elicitation stage and the corresponding correlation matrix is computed (see section 3.4.2). The PDM is applied to decompose the criteria PCM and the global weights of the m criteria are calculated in step 3 (see section 3.4.3). Lastly, step 4 computes the weighted scores of the n alternatives for decision making (see section 3.4.4). A peer reviewed AHP dataset from existing literature is used to illustrate the application of the PDM for AHP. Önüt et al. (2010) use a fuzzy AHP model for shopping center site selection and is illustrated in this subsection.

Initialize AHP Hierarchy

The m criteria and n alternatives are identified and arranged into a three-level decision hierarchy. In terms of the dataset from Önüt et al. (2010), the goal, eight criteria and six alternatives are setup as a hierarchy as is done in the traditional AHP methodology. Figure 3.3 illustrates the proposed three-level AHP hierarchy structure.

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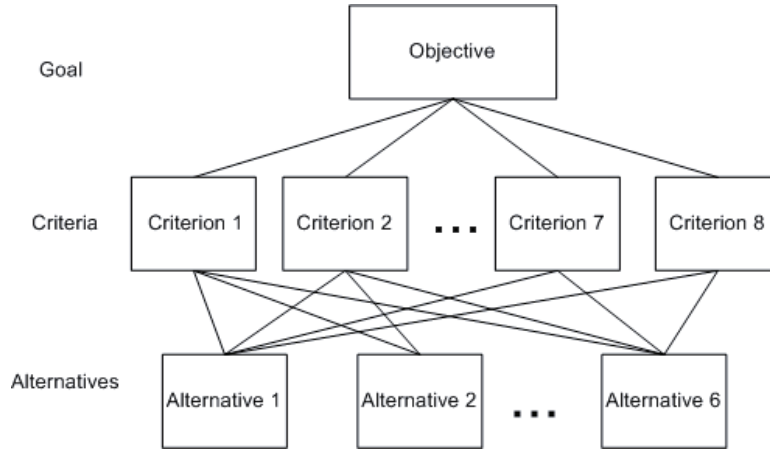
Figure 3.3. AHP Hierarchy Structure for the Dataset of Önüt et al. (2010)

Compute Correlations of Criteria

Table 3.1 presents the most likely scores of the six site alternatives over the eight selection criteria. Additionally, the corresponding 8×8 correlation matrix is computed and is presented in table 3.2.

Table 3.1. Raw Data of the Scores of Each Alternative on Each Criterion

Alternative	Criterion							
	1	2	3	4	5	6	7	8
A	5	7	5	9	5	5	3	3
B	7	7	5	7	7	7	5	5
C	5	5	7	5	5	5	9	7
D	5	5	8	3	5	5	9	5
E	7	9	5	5	7	7	3	3
F	5	5	7	3	5	5	9	5

Apply PCM to the Criteria PCM

This subsection illustrates the decomposition of the criteria PCM and the calculation of the global weights of the eight decision criteria. Specifically, the criteria PCM is decomposed into two subsets ($k = 2$), then a pivot criteria is selected.

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Table 3.2. Correlation Matrix Computed From Table 3.1

Criterion	1	2	3	4	5	6	7	8
1	1.000	0.791	0.680	0.221	1.000	1.000	0.600	0.343
2	0.791	1.000	0.860	0.489	0.791	0.791	0.922	0.759
3	0.680	0.860	1.000	0.794	0.680	0.680	0.933	0.633
4	0.221	0.489	0.794	1.000	0.221	0.221	0.758	0.417
5	1.000	0.791	0.680	0.221	1.000	1.000	0.600	0.343
6	1.000	0.791	0.680	0.221	1.000	1.000	0.600	0.343
7	0.600	0.922	0.933	0.758	0.600	0.600	1.000	0.824
8	0.343	0.759	0.633	0.417	0.343	0.343	0.824	1.000

Furthermore, the local pairwise comparisons are collected for each subset and the corresponding global weights are calculated.

Decompose Criteria PCM

The proposed BIP methodology from section 3.3.1 is applied on the PCM using the correlation scores from table 3.2. After executing the proposed BIP methodology, criteria 1, 4, 6 and 7 are assigned to subset s_1 while criteria 2, 3, 5 and 8 are assigned to subset s_2.

Select Pivot Element

The optimal pivot criterion is selected by applying equation 3.12 and table 3.3 presents the results of the sum of the individual correlations of criterion i to all other criteria. It is evident from table 3.3 that criterion 4 is the least independent criterion.

Table 3.3. Sum of Correlation

	Criterion							
	1	2	3	4	5	6	7	8
Correlation	5.6346	6.402	6.2592	4.1196	5.6346	5.6346	6.2365	4.6615

And thus criterion 4 is assigned to all subsets. Hence, subsets s_1 and s_2 now have 1, 4, 6 & 7 and 2, 3, 4, 5 & 8 criteria respectively.

Elicit Local Pairwise Comparisons

To illustrate the eliciting of local pairwise comparisons, the original most likely fuzzy values are used from the original 8×8 fuzzy AHP criteria PCM. Table 3.4 and table 3.5 summarize the local pairwise comparison matrices for the two subsets respectively.

Table 3.4. PCM for Subset 1

	1	4	6	7	Local Priority
1	1.00	0.33	0.14	1.50	10.31%
4	3.00	1.00	1.00	3.00	33.65%
6	7.00	1.00	1.00	5.00	46.99%
7	0.67	0.33	0.20	1.00	9.05%

Table 3.5. PCM for Subset 2

	2	3	4	5	8	Local Priority
2	1.00	0.33	0.20	0.33	1.00	7.43%
3	3.00	1.00	3.00	0.33	3.00	26.39%
4	5.00	0.33	1.00	1.00	3.00	24.24%
5	3.00	3.00	1.00	1.00	5.00	34.94%
8	1.00	0.33	0.33	0.20	1.00	7.00%

A total of $4C2 + 5C2 = 16$ pairwise comparisons are elicited in this setup, which is a reduction of $D(A^k) = 12$, as compared to the original 28 required pairwise comparisons when the original AHP methodology is used. The average consistency of these two priority matrices is computed using Eq. 3.14 as follows:

$$ACR = \frac{1}{m+k-1} \sum_{l=1}^{k} m_l CR(A_l) = \frac{4}{9}0.0547 + \frac{5}{9}0.119 = 0.0904 \qquad (3.23)$$

The original 8×8 AHP matrix has a CR of 12.96% which is highly inconsistent while the decomposed matrix has an ACR of only 9.04%. Hence, in this setup, several inconsistent pairwise comparisons are excluded from the decision making. Furthermore the subset PCMs have a smaller dimensions ($dim(s_1) = 4$, $dim(s_2) = 5$) thus making the elicitation of pairwise comparisons less taxing for the DM.

Calculate Global Weights of the Criteria PCM

After calculating the priorities of the local criteria, the corresponding global priorities of the criteria need to be calculated. Using the local weights from subset 1 and subset 2 and equation 3.20, all local weights are divided by the local weight of pivot criterion 4. And as such, we obtain normalized weights for subset 1: $\tilde{w}(A_1) = [0.31, 1.00, 1.40, 0.27]^T$ and subset 2: $\tilde{w}(A_2) = [0.31, 1.00, 1.00, 1.44, 0.29]^T$. Using the normalized weights and using equation 3.21 we sum the normalized weights and we obtain the following global weights as summarized in table 3.6.

Table 3.6. Results of the PDM as Compared to the Original AHP Criteria PCM

Criterion	Original Criteria PCM Weights	PDM Weights
5	26.09%	23.64%
6	21.95%	22.90%
4	15.69%	17.86%
3	15.36%	16.40%
1	7.52%	5.027%
2	5.24%	5.025%
8	4.46%	4.736%
7	3.69%	4.409%
$D(A^2)$	None	12
ACR	12.96%	9.63%

Based on the results, the weights obtained from the proposed PDM is relatively similar to the value of the original weights with a lower average consistency ratio. We observe from table 3.6 that we have saved 12 pairwise comparisons for the criteria PCM while having similar weights. This savings is attributed to the decomposition of the PCM into two subsets in which the pairwise comparisons of the criteria across subsets are not elicited. Furthermore, a reduction of the CR is observed from 12.96% to 9.63%. In this setup, lots of inconsistent pairwise comparisons are omitted. Furthermore, the PCMs for the subsets are smaller well below Saaty's threshold of seven elements.

However, the PDM can only be applied to the special case of a three-level AHP problem. Furthermore, by definition of the correlation, we only measure the linear

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interdependencies of the criteria from the alternatives. Hence, the application of the PDM for AHP problems is limited. Therefore, we illustrate the full potential of the PDM in terms of an ANP network in section 3.5.

Calculate Weights of Alternatives

After computing for the global weights of the eight criteria, we now compute the weights of the six alternatives. Table 3.7 presents the results of the weighted scores by multiplying the scores from table 3.1 with the obtained global weights from table 3.6. Hence, alternative B is selected since it has the highest weighted score followed by alternative E, then A, C, D and E.

Table 3.7. Weighted Scores of the Six Alternatives

Alternative	Weighted Score
A	5.6319
B	6.4889
C	5.5989
D	5.3110
E	6.0493
F	5.1470

3.5   Decomposing PCMs for the ANP Methodology

This section illustrates the application of the PDM on an ANP problem. Since all PCMs that form clusters within the ANP network require inner dependencies to measure the interdependence of elements, these inner dependencies are then used to decompose all PCMs within the network. Hence, all PCMs in the network can be decomposed using the PDM which leads to larger pairwise comparison savings. We illustrate the decomposition of the eight decision criteria into three subsets using figure 3.4 as follows.

In step 1, the ANP network structure is developed with the n alternatives and m criteria (see section 3.5.1). In Step 2, the inner dependencies of all the elements of each PCM are elicited (see section 3.5.2). Furthermore, we apply the PDM to all PCMs of the network in step 3 (see section 3.5.3). Finally, step 4, we calculate the limiting weights of the alternatives for decision making (see section 3.5.4). A dataset from existing literature
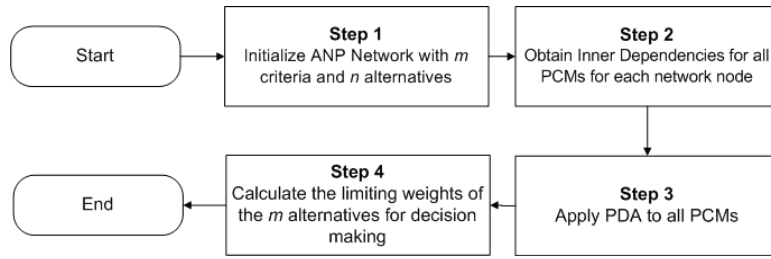
Figure 3.4. Application of the PDM for an ANP Problem

is used to demonstrate the application of the PDM for the ANP. Cheng & Li (2004) propose an ANP methodology for contractor selection. This dataset is selected since the eight decision criteria are considered simultaneously in a single cluster. Furthermore, the decision focuses on three alternatives.

Initialize ANP Network

The goal, eight criteria and four alternatives are setup as a network as is done in the traditional ANP methodology. Figure 3.5 illustrates the proposed three-cluster ANP network structure.
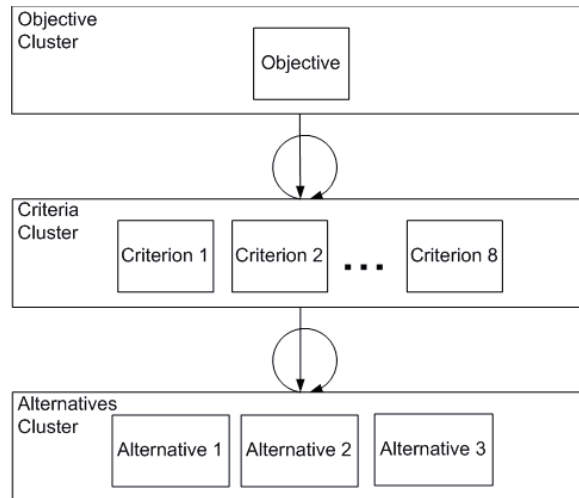
Figure 3.5. The ANP Network Structure for the Dataset of Cheng & Li (2004)

Elicit Inner Dependence

Inner dependencies are elicited from the DM to estimate the interdependencies of the criteria. Table 3.8 summarizes the elicited inner dependency scores of all eight decision criteria which are elicited from the DM from Cheng & Li (2004). We then compute the corresponding symmetric matrix using equation 3.4. Table 3.9 presents the symmetric matrix.

Table 3.8. Inner Dependency Scores for the Eight Criteria

Criterion	1	2	3	4	5	6	7	8
1	0.00	0.56	0.33	0.31	0.21	0.20	0.17	0.18
2	0.14	0.00	0.37	0.32	0.24	0.26	0.17	0.14
3	0.20	0.06	0.00	0.20	0.21	0.18	0.17	0.17
4	0.11	0.06	0.06	0.00	0.18	0.18	0.09	0.16
5	0.20	0.06	0.06	0.04	0.00	0.07	0.17	0.23
6	0.11	0.06	0.06	0.04	0.09	0.00	0.17	0.03
7	0.11	0.06	0.06	0.06	0.04	0.03	0.00	0.10
8	0.11	0.12	0.06	0.03	0.04	0.07	0.04	0.00

Table 3.9. Symmetric Inner Dependency Scores for the Eight Criteria

Criterion	1	2	3	4	5	6	7	8
1	0.00	0.35	0.27	0.21	0.21	0.16	0.14	0.15
2	0.35	0.00	0.22	0.19	0.15	0.16	0.12	0.13
3	0.27	0.22	0.00	0.13	0.14	0.12	0.12	0.12
4	0.21	0.19	0.13	0.00	0.11	0.11	0.08	0.10
5	0.21	0.15	0.14	0.11	0.00	0.08	0.11	0.14
6	0.16	0.16	0.12	0.11	0.08	0.00	0.10	0.05
7	0.14	0.12	0.12	0.08	0.11	0.10	0.00	0.07
8	0.15	0.13	0.12	0.10	0.14	0.05	0.07	0.00

Apply PDM to all PCMs

This subsection illustrates the application of the PDM to decompose all PCMs. In our example, we illustrate the decomposition of the criteria cluster into three subsets. Furthermore, the optimal pivot criterion is selected and is used to link all three subsets. The local pairwise comparisons are then elicited and local weights are calculated. The

corresponding global criteria PCM weights are computed from the local weights illustrated as follows.

Decompose Criteria PCM

The proposed BIP methodology is applied from section 4.1 on the symmetric inner dependency scores from table 3.9 . Hence, criteria 2, 5, and 7 are assigned to subset s_1 while criteria 1, 6 and 8 are assigned to subset s_2 and lastly criteria 3 and 4 are assigned to subset s_3.

Select Pivot Element

The optimal pivot criterion is selected by applying equation 3.12 and table 3.10 presents the results of the sum of the individual inner dependencies of criterion i to all other criteria.

Table 3.10. Sum of Inner Dependencies

	Criterion							
	1	2	3	4	5	6	7	8
Sum of Inner Dependence	1.47	1.31	1.095	0.92	0.92	0.775	0.72	0.74

It is evident from table 3.10 that criterion 7 is the least independent criterion. And thus criterion 7 is assigned to all subsets. Hence, subset s_1 has criteria 2, 5 and 7, subset s_2 has criteria 1, 6, 7 and 8 and subset s_3 has 3, 4 and 7.

Elicit Local Pairwise Comparisons

Local pairwise comparisons are elicited for all subsets of the criteria cluster. Table 3.11 to table 3.13 show the local priorities of the decomposed AHP obtained from the dataset of Cheng & Li (2004).

A total of $3C2 + 4C2 + 3C2 = 12$ pairwise comparisons are elicited in this setup. A reduction of $D(A^k) = 16$ pairwise comparisons is realized from the original 28 required pairwise comparisons. The average consistency of these three PCMs is computed using

Table 3.11. PCM for Subset 1

	2	5	7	Local Priority
2	1	6	7	76.38%
5	1/6	1	1	12.11%
7	1/7	1	1	11.51%

Table 3.12. PCM for Subset 2

	1	6	7	8	Local Priority
1	1	9	9	9	73.28%
6	1/9	1	1/2	1/2	6.01%
7	1/9	2	1	1/2	8.79%
8	1/9	2	2	1	11.93%

Table 3.13. PCM for Subset 3

	3	4	7	Local Priority
3	1	3	3	58.89%
4	1/3	1	2	25.19%
7	1/3	1/2	1	15.93%

Eq. 3.14 as follows:

$$ACR = \frac{1}{m+k-1}\sum_{l=1}^{k} m_l CR(A_l) = \frac{3}{10}0.0042 + \frac{4}{10}0.0847 + \frac{3}{10}0.0607 = 0.05335 \quad (3.24)$$

The original 8×8 ANP cluster has a CR of 9.44% which is around the threshold of 10% while the decomposed matrix has an ACR of only 5.335%. Again we observe a reduction of the inconsistency of the PCMs. Several inconsistent pairwise comparisons are not elicited in this setup and the dimensions of the three subsets are considerably less than the original criteria PCM.

Calculate Global Weights of the Criteria PCM

Again using the local weights from subset 1, 2 and 3 and equation 3.20, all local weights are divided by the local weight of pivot criterion 7. And as such we obtain normalized weights for subset 1: $\tilde{w}(A_1) = [6.64, 1.00, 1.00]^T$, subset 2: $\tilde{w}(A_2) = [8.64, 0.68, 1.00, 1.36]^T$ and subset 3: $\tilde{w}(A_3) = [3.70, 1.58, 1.00]^T$ Using the

normalized weights and using equation 3.21 we sum the normalized weights and we obtain the following global weights as summarized in table 3.14.

Table 3.14. Results of the PDM as compared to the original ANP Criteria PCM

Criterion	Original Criteria PCM Weights	PDM Weights
1	33.9%	33.3%
2	27.3%	27.3%
3	14.8%	15.2%
4	9.1%	6.5%
8	3.4%	5.6%
5	3.2%	4.3%
7	3.8%	4.1%
6	4.5%	2.8%
$D(A^3)$	None	16
ACR	9.44%	5.34%

The weights obtained from the proposed PDM is relatively similar to the value of the original weights with a lower average consistency ratio. Additionally, we observe from table 3.14 that a reduction of 18 pairwise comparisons is realized for the criteria PCM while having similar weights. This savings is again attributed to the decomposition of the criteria PCM into three subsets in which the pairwise comparisons of the criteria across subsets are not elicited. Furthermore, a reduction of the CR is observed from 9.44% to 5.34%. In this setup, lots of inconsistent pairwise comparisons are omitted. Furthermore, the three subset PCMs are smaller which are within Saaty's threshold of seven elements.

We further test the PDM by changing the number of subsets. The same dataset from Cheng & Li (2004) is used in terms of the decomposition of the 8 criteria into $k \in [2, 7]$ subsets. The traditional ANP is used and the results are presented in table 3.15.

Table 3.15. Performance of the PDM

Metric	Methodology						
	No Decomposition	Proposed PDM					
		$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$D(A^k)$	0	12	16	18	19	20	21
ACR	9.44%	8.99%	5.34%	5.24%	5.17%	4.82%	0.00%

It is observed from table 3.15 that as the value of k increases, we generally observe an increase in the number of pairwise comparisons saved. This is attributed to smaller subset PCMs when we increase the number of subsets. Furthermore, we observe a gradual decrease on the average CR of the PCMs. However, lots of pairwise comparisons are not elicited for values of $k \geq 4$ hence, the redundancy advantage of PCMs is reduced and thus the preferences of the DM would be subject to more biases and judgement errors. A trade-off mechanism must be developed to address this which is the subject of our future research.

Calculate Weights of Alternatives

After computing the weights of the criteria cluster for the eight decision criteria, we now calculate the weights of the three alternatives using the traditional ANP methodology. Table 3.16 presents the results of the three alternatives. Hence, alternative A is selected since it has the highest limiting score followed by alternative B then alternative C.

Table 3.16. Weighted Scores of the Three Alternatives

Alternative	Weighted Score
A	0.47
B	0.27
C	0.26

3.6   Conclusions and Future Work

A Pairwise Comparison Matrix (PCM) is an integral component of decision making methodologies: Analytic Hierarchy Process (AHP) and Analytic Network Process (ANP). These are used to determine relative weights of criteria and alternatives. However, a PCM suffers from the curse of dimensionality and hence the issue of inconsistent pairwise comparisons when elicited from a decision maker (DM). The proposed PCM Decomposition Methodology (PDM) addresses these disadvantages. The PDM decomposes all PCMs into smaller manageable subsets using binary integer programming

with inner dependency scores. As a result, the number and the inconsistency of pairwise comparisons elicited are reduced. Since the subsets are disjoint, the most independent pivot element is selected to connect all disjoint subsets. Hence the inner dependencies of the elements are minimized within each subset. Using local priorities and the pivot element, global priorities are then estimated for the elements of the PCM.

The PDM is applied to a three-level AHP problem to decompose the criteria PCM. Correlation of the criteria from alternative scores is used as an alternative to estimate the interdependencies of the criteria. The proposed methodology does indeed reduce the number of pairwise comparisons and the consistency ratio. Nevertheless, more pairwise comparisons is saved when the PDM is applied to the ANP methodology. The PDM can be applied to all cluster PCMs within the network since inner dependencies of the elements are elicited for each PCM.

The authors plan to extend the framework by determining the optimal number of subsets k for each ANP cluster by balancing the (1) time savings by reducing pairwise comparisons, (2) the amount of inner dependency among criteria and alternatives (3) the level of consistency, and (4) the accuracy of the global weights. Furthermore, multiple pivot elements are to be studied to further improve the estimation of the global weights. Although improvements in the average consistency of the local pairwise comparisons are observed, the authors plan to implement the methodology proposed by Benítez, Delgado-Galván, Gutiérrez, & Izquierdo (2011) to further improve consistency.

Chapter 4

OPTIMAL DECOMPOSITION OF AHP PAIRWISE COMPARISON MATRICES

A Pairwise Comparison Matrix (PCM) is a widely used tool to compute relative priorities of elements and is an integral component of the Analytic Hierarchy Process (AHP), a well-accepted multi-criteria decision making tool. However, a PCM suffers from several issues limiting its application to large scale decision problems. The issues can be attributed to the curse of dimensionality due to the large number of pairwise comparisons. This leads to inconsistencies and inaccurate decisions. It is noted that the extensive research that address these limitations decompose PCMs into smaller sized subsets arbitrarily and no guidelines are provided to segment the elements. Additionally, the optimal number of subsets is not identified. This research explores the interdependencies among elements and thus proposes a trade-off decomposition methodology to decompose the PCM elements into an optimal number of mutually exclusive subsets. As a result, the number and the inconsistencies of the pairwise comparisons are reduced. Furthermore, the model identifies a pivot element which is used to obtain the appropriate global weights for all elements across all subsets. The proposed methodology is applied to the AHP to demonstrate its advantages.

4.1 Introduction

An $m \times m$ pairwise comparison matrix (PCM) denoted by A is a reciprocal matrix that is composed of pairwise comparisons $a_{ij} \in [1/9, 9]$ that represent the scaled relative importance scores of element i as compared to element j. Typically, a PCM is generated from pairwise comparisons elicited from a decision maker (DM) to estimate criteria or alternative priorities. A widely applied multiple criteria decision making (MCDM) methodology that uses PCMs is the Analytic Hierarchy Process (AHP) developed by Saaty (1977). Currently, there are several successful applications of the AHP in a wide-range of MCDM problems (Ishizaka & Labib, 2011). On the other hand, the AHP that uses PCMs has two interrelated limitations, specifically: (1) a large number of

pairwise comparisons need to be elicited from the DM and (2) the inconsistency obtained during the elicitation of these pairwise comparisons.

To illustrate the issue of the large number of required pairwise comparisons, let us consider a PCM with m elements. A total of $m(m-1)/2$ pairwise comparisons are needed to obtain the weights of the elements. In a case study by Lin et al. (2008), it took two and a half hours on average to complete a three-level AHP decision problem per decision maker (DM) and a total of 380 man-hours to complete all pairwise comparisons. Due to the limited cognitive powers of DMs, Saaty (1990) recommends that the number of elements within a PCM should be at most seven. Any decision that has exceeded this threshold is considered large. Methodologies are proposed to decompose large AHP PCMs into smaller subsets to minimize the number of pairwise comparisons at the alternative level (Shen et al., 1992; Islam et al., 1997a; Triantaphyllou, 1995; Islam & Abdullah, 2006; Ishizaka, 2008, 2012). For example, the alternatives are subjectively assigned into $k \in [2, m-1]$ mutually exclusive subsets. Pairwise comparisons are elicited from the DM only in those local PCMs. In effect, a reduction in the number of pairwise comparisons is realized since the dimensions of the decomposed matrices are much smaller than the original PCM. On the other hand, when the PCM is decomposed into subsets, the obtained relative weights are valid only within those subsets and the problem arises when aggregating the local weights back to its global counterpart. To address this issue, these methodologies propose the use of a pivot element. The pivot element is a common AHP element assigned to all subsets and is used as a basis for comparing the elements across all the disjoint subsets. The global weights of the elements are then calculated for decision making. However, the decomposition methodologies reviewed above are not without any disadvantages. Firstly, to the best of our knowledge, the number of subsets is known as a prior and the assignments of the elements to each subset are done arbitrarily which are subject to DM biases and judgement errors. Secondly, most methods lack a methodology to identify an appropriate pivot element to be used as shown in (Shen et al., 1992; Triantaphyllou, 2000; Ishizaka, 2012). Please note that pivot selection is a challenging

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issue as the decisions should consider reducing the number and inconsistency of the pairwise comparisons simultaneously.

The second limitation of the PCMs, which is related to the curse of dimensionality, is attributed to the consistency of the pairwise comparisons. When the number of required pairwise comparisons increases, the consistency of these comparisons is expected to plummet and would result to inconsistent decisions (Weiss & Rao, 1987). To analytically measure the amount of inconsistency of a PCM A, Saaty (1980) proposed the consistency index (CI). The CI is computed by obtaining the eigenvalue of the PCM using Eq. (4.1) as follows:

$$CI(A) = \frac{\lambda_{max} - m}{m - 1} \tag{4.1}$$

where m is the dimension of the PCM A and λ_{max} is its maximum eigenvalue. The consistency ratio (CR) is the ratio of CI and RI and is computed using Eq. (4.2) as follows:

$$CR(A) = \frac{CI(A)}{RI(m)} \tag{4.2}$$

where $RI(m)$ is the random index obtained from the average CI of 500 randomly filled matrices and is a function of m. If matrix A has $CR < 10\%$, then A is considered to have an acceptable level of consistency. The DM can then use the matrix A for decision making. On the other hand if $CR \geq 10\%$, the process of eliciting pairwise comparisons from the DM is repeated until a consistent matrix is obtained. Extensive research focus on improving the consistency of a PCM by changing pairwise comparison values individually (Zeshui & Cuiping, 1999; Cao et al., 2008; Saaty, 2003; Benítez, Delgado-Galván, Gutiérrez, & Izquierdo, 2011; Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2011). Upon executing these methodologies, a more consistent PCM is obtained. Yet, these methods require a significant amount of time to complete since a large PCM is required as input. With this, due to their limited cognitive processing powers, the DMs are not expected to provide consistent pairwise comparisons all throughout the pairwise comparison elicitation process especially when the number of pairwise comparisons is large. Therefore, a methodology that reduces the pairwise comparisons elicited from the

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DM would lead to improved consistency levels since only a handful of pairwise comparisons are elicited and would not be cognitively taxing to the DM.

Noting the limitations are interrelated, however, most research attempts to address these limitations individually. In this paper, a novel trade-off PCM decomposition methodology is proposed that aims to tackle the aforementioned limitations and thus improve the applicability of the PCMs for large scale MCDM problems. A binary integer program (BIP) is developed to (1) identify the optimal number of subsets (k), (2) to assign each element to the appropriate subset, and (3) to identify the appropriate element to be the pivot element. The BIP balances the number of pairwise comparisons saved, the inconsistency of the PCM as well as the accuracy of the global PCM element weights. A number of comparison experiments are conducted to validate the proposed model using datasets from existing literature.

The rest of this paper is organized as follows. Section 4.2 provides a literature review on related articles that address the limitations of the AHP. Section 4.3 illustrates the steps of the proposed decomposition methodology. Section 4.4 describes the computational experiments done to validate the proposed methodology. Finally, section 4.5 concludes the paper and proposes further research areas.

4.2   Review of Related Literature

Saaty (1990) proposes the idea to group alternatives into subsets according to the attributes of the alternatives to reduce the number of pairwise comparisons. Using this concept, Shen et al. (1992) propose an arbitrary assignment of alternatives into k subsets such that these k subsets have one common pivot alternative. Pairwise comparisons are first elicited on each subset and local priorities are calculated using the standard AHP methodology. The global priority is then derived by using a common pivot criterion and local priorities of each subset. Additional guidelines to segment alternatives into k subsets based on a subjective scale are provided in (Islam et al., 1997a). According to the model, alternatives that have close magnitudes are grouped together. To link all the subsets

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together, a single linkage method in which the largest alternative within a given subset is assigned to the next closest subset. Then the largest alternative from that subset is assigned to the next and so on. On the other hand, take note that the definition of magnitudes is highly subjective since it is provided by the DM. Ishizaka (2012) applies the same methodology on supplier selection by extending the close magnitude concept on alternatives. However, these methods do not consider grouping the decision criteria together to further minimize the number of pairwise comparisons. Islam & Abdullah (2006) consider reducing the number of decision criteria by the nominal group technique. The decision criteria that have insignificant weights are eliminated from future pairwise comparison elicitation process. A similar methodology is proposed in (Malakooti, 2000) where the final weights are obtained using partial information on the preferences elicited from the DM. Benítez et al. (2012) on the other hand proposed a methodology that uses partial and/or incomplete DM preferences at multiple points within the decision making process. Weiss & Rao (1987) and Takahashi (1990) develop models that use balanced incomplete block designs where arbitrary subsets of a PCM A are assigned to different DMs for elicitation. The multiple assignments are treated as replicates in contrast when all DMs focus on the large PCM. The geometric mean is used to consolidate the weights. However, the total number of pairwise comparisons has increased considerably and the decision problem would take too long to solve. Rating scales methodologies also address the issue of numerous pairwise comparisons (Liberatore, 1987; Singh et al., 2007; Chamodrakas et al., 2010; Önüt et al., 2010). These methodologies focus on the alternative level in which a five-point scale (outstanding, good, average, fair and poor) scores are elicited for each alternative for each criterion. Instead of providing $mn(n-1)/2$ comparisons, only nm are provided at the alternative level. Triantaphyllou (1995) and Triantaphyllou (2000) develop a linear programming model to estimate the missing pairwise comparisons of A by selecting two arbitrary subsets of the criteria s_1 and s_2, where $s_1 \cup s_2 = C$ and $s_1 \cap s_2$. The proposed linear programming problem, calculates the missing comparisons and the global weights can then be estimated. However, the algorithm only focused on dividing the PCM A into two subsets. If m is large, then the

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subsets are still large. Moreover, the deviation error is dependent on the number of common elements of subsets s_1 and s_2. The smaller the $s_1 \cap s_2$, the estimation of the missing comparisons is expected to be less accurate and deviation error rates increase significantly.

Methodologies are developed to individually change pairwise comparison values that have the greatest effect in improving consistency (Zeshui & Cuiping, 1999; Cao et al., 2008; Saaty, 2003; Benítez, Delgado-Galván, Gutiérrez, & Izquierdo, 2011; Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2011). These methodologies identify the most inconsistent entry in the PCM and seek to change its value towards a more consistent matrix. This is done in an iterative fashion until a certain level of consistency is reached. There are methods that start with the minimum $m - 1$ pairwise comparisons and seek to estimate the missing comparisons to preserve consistency. Zhang & Chen (2009) use a combination of nonlinear programming and genetic simulated annealing while Bozoki et al. (2009) use nonlinear optimization with exponential scaling to estimate the missing pairwise comparisons from available ones. However, these methods are found to be ineffective for problems with large number of criteria. This is due to the fact that all possible combinations of connecting paths must be considered (Fedrizzi & Giove, 2007, 2009). The number of connecting paths exponentially grows as the number of missing comparisons increases and thus would be inefficient to solve.

In summary, the existing methodologies focus on reducing the number of pairwise comparisons. We conclude most methods suffer from being subjective, considering alternative level decomposition and tend to have large deviation errors. We also find research that concentrates on improving the consistency of a PCM by changing individual pairwise comparisons. However, these methodologies provide final pairwise comparisons may not reflect the original preference of a DM since individual pairwise comparisons are changed. It is also noted most research attempts to address the limitations of the AHP individually. Knowing the issues are interrelated, this research proposes a methodology for PCM decomposition considering element interdependencies to reduce the number of

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pairwise comparisons as well as the inconsistency of the PCM. The next section presents the proposed PCM decomposition methodology.

4.3   Proposed Decomposition Methodology

We illustrate a decomposition of a PCM A with m elements. Figure 4.1 illustrates an overview of the proposed decomposition methodology.
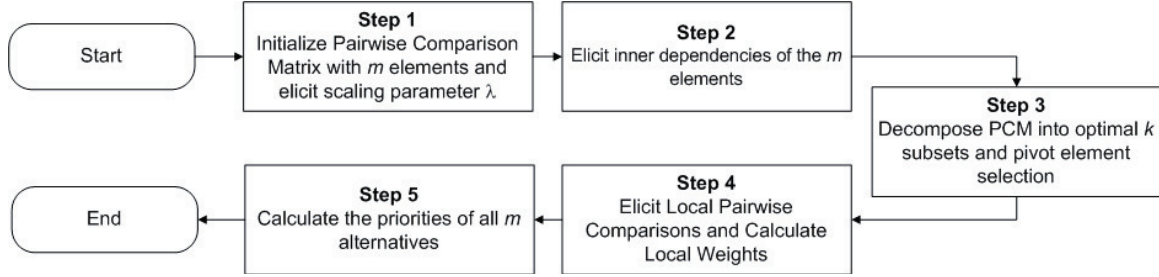
Figure 4.1. High Level Overview of Proposed Decomposition Methodology for PCMs

In step 1, the m elements of the PCM are collected as well as the $\lambda \in \mathbb{R}^+$ trade-off parameter is elicited from the DM. In step 2, inner dependency scores r_{ij} are elicited from the DM to measure the interdependence of between element i and j. The $r_{ii'}$ rating scores are elicited from the DM qualitatively or gathered by observing the actual alternative quantitative attributes in step 2. Using the $r_{ii'}$ scores, step 3 decomposes the m elements into k mutually exclusive subsets $S = \{s_l | l = 1, 2, \cdots, k\}$ in which each subset $s_l = \{s_{li} | i = 1, 2, \cdots, m_l\}$, where $\sum_{l=1}^{k} m_l = m + k - 1$ using a BIP (see section 4.3.1). Additionally, the pivot element is identified. Local pairwise comparisons for all elements assigned to their respective subsets are elicited and local global weights are calculated using the original AHP methodology in step 5 (see section 4.3.2). Finally in step 6, with the use of a pivot element, the global weight of each element is calculated (see section 4.3.3).

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Binary Integer Programming Model

Applying the concept of correlation, the linear interdependencies of the elements can be analytically estimated as is done in (Yurdakul & Tansel, 2009). To compute for the dependencies of the elements we define the following value:

Definition 4.1. *Let $r_{ii'}$ be the correlation between element i and element i' which is calculated using (4.3) as follows*

$$r_{ii'} = \left| \frac{cov(i,i')}{\sigma_i \sigma_{i'}} \right| \tag{4.3}$$

The correlation coefficients are tabulated in an $m \times m$ correlation matrix where $r_{ii} = 1$ and $r_{ii'} = r_{i'i}$. These correlation coefficients will be used in a BIP decomposition algorithm to group uncorrelated elements. A value $r_{ii'} = 1$ indicates strong positive or negative correlation between elements i and i'.

The proposed PCM decomposition model is based on the amount of interdependence among the elements with the objective being to group the most dissimilar elements together into smaller subsets. In this research, we propose a BIP model with the objective function being: (1) based on the independency in which the model aims to minimize the amount of dependency among elements in a subset and (2) the deviation error when the weights of the decomposed matrix is compared to the weights of original complete PCM.

Interdependencies Component of the Objective Function

Based on the obtained correlation matrix in Eq. (4.3) we seek to minimize the amount of dependence or correlation among elements assigned to subsets and as such Eq. (4.4) is proposed as an objective function to satisfy this requirement.

$$min \ Z = \sum_{l=1}^{k} \sum_{i=1}^{m} \sum_{i'<i} r_{ii'} y_{ii'l} \tag{4.4}$$

Where variable $y_{ii'l} = 1$ if both element i and i' are assigned to subset l and $y_{ii'l} = 0$ otherwise. The objective is to assign dependent elements to different subsets appropriately. Additionally, we seek to determine the optimal value of the number of subsets k with feasible values $k \in [2, m-1]$. However, treating k as a variable leads to another problem in which an erroneous optimal solution is solved. It is intuitive from the objective function in Eq. (4.4) that the optimal solution will always have large values of k (e.g. $k = m-1$). Since the elements are assigned to more subsets, the less pairwise comparisons will be needed. In this setup, significant deviation errors are obtained which may result in inaccurate decisions. This is discussed by Saaty (1980) in which the provision of repetitive pairwise comparisons would result to weights that are less sensitive to biases and judgement errors and a higher risk of erroneous global weights can be realized when only $m-1$ pairwise comparisons are used to calculate the global weights. To address this issue, a second component is introduced in the next subsection to reduce the deviation error by penalizing large values of k.

Deviation Error Component of the Objective Function

To analytically measure Saaty's claim on the deviation error of the global weights, we define the following performance metric:

Definition 4.2. *The accuracy of the decomposition of A into k subsets is estimated using the weight difference error (WDE) performance metric which is calculated using Eq. (4.5) as follows:*

$$WDE(A^k) = \sum_{i=1}^{m} \left(w(i) - w'(i) \right)^2 \tag{4.5}$$

where $w(i)$ are the weights of the undecomposed PCM and while $w'(i)$ are the priorities of the elements obtained using the proposed decomposition methodology.

Yet, the WDE is not measured beforehand since the true global values of the elements are unknown and only the local pairwise comparisons for the decomposed matrices are elicited in the decomposition methodology. In this regard, we investigate

other measurable parameters as a surrogate to be included in the objective function to mirror the behavior of the WDE for different values of k.

Consider the case when m elements are decomposed into k subsets. Let $d(A^k) = \sum_{l=1}^{k} C(m_l, 2)$ be the number of pairwise comparisons needed after decomposition. It is observed when the elements are assigned to $k = m - 1$ subsets, the number of pairwise comparisons needed after decomposing would only be $m - 1$ and large values of the WDE are obtained. On the other hand, when $k = 2$, then lots of pairwise comparisons are needed since the m elements are distributed evenly into two subsets. This would result into smaller WDE values since more redundant pairwise comparisons are used. As such the number of pairwise comparisons after decomposition is proposed to estimate the value of the WDE given the number of subsets k. We define the number of pairwise comparisons after decomposition as follows:

Definition 4.3. *The difference between the original number of pairwise comparisons of matrix A and number of pairwise comparisons after decomposition into k subsets denoted by $D(A^k)$ is given by:*

$$D(A^k) = C(m, 2) - d(A^k) = \frac{m(m-1)}{m} - \sum_{l=1}^{k} C(m_l, 2) \tag{4.6}$$

It is easy to show that function $D(A^k)$ is minimized when: $m_l = \frac{m-1}{k} + 1, \forall l \in S$ if $\frac{m-1}{k} + 1 \in \mathbb{Z}$ or $\lfloor m_l = \frac{m-1}{k} + 1 \rfloor$ for some $l \in S$ and $\lfloor m_l = \frac{m-1}{k} + 1 \rfloor + 1$ for some $l' \in S \backslash \{l\}$ if $\frac{m-1}{k} + 1 \notin \mathbb{Z}$.

To quantify and illustrate the relationship between $D(A^k)$ and the WDE, the following propositions are presented to support our claim that the number of pairwise comparisons after decomposition affects the WDE. Consider an arbitrary decomposition of an $m \times m$ PCM A into k subsets with corresponding decomposed matrices $A^k = \{A_l^k = \{a_{ij}\} | l = 1, 2, \cdots, k, \forall (i, j) \in s_l\}$. It is easy to show that if A is perfectly consistent $(CR(A) = 0)$ then the global weights of the decomposed matrices A^k are equal to the original weights of the PCM A and thus $WDE(A^k) = 0$. Now we consider the case when A is inconsistent.

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Proposition 4.1. *If A is inconsistent $(CR(A) > 0)$, then $WDE(A^k) > 0$ for any*

$k \in [2, m-1]$.

Proof. See Appendix. □

Proposition 4.1 states that if A is inconsistent $(CR(A) > 0)$, then there exist at least one inconsistent pairwise comparison in A. We can conclude that the global weights of the decomposed matrices A^k are not equal to the original weights of the PCM A and thus $WDE(A^k) > 0$. Now we consider the relationship of the WDE for different k values.

Proposition 4.2. *If A is inconsistent then $WDE(A^{k+1}) \geq WDE(A^k)$*

Proof. See Appendix. □

Proposition 4.2 states that by comparing the decompositions of A into A^{k+1} or A^k, one can expect that $WDE(A^{k+1}) \geq WDE(A^k)$ if A is inconsistent. We generalize this relationship for any value of $k \in [2, m-1]$ in theorems 4.1 and 4.2 as follows:

Theorem 4.1. *If A is inconsistent then a decomposition of A into A^{k+1} or A^k will result in $D(A^{k+1}) > D(A^k)$ and $WDE(A^{k+1}) \geq WDE(A^k)$*

Proof. Consider an inconsistent matrix A with decompositions A^k and A^{k+1}. Calculating the number of pairwise comparisons after decomposition, by definition, we have:

$$D(A^k) = C(m, 2) - k\left(\frac{m-1}{k} + 1\right)$$

$$D(A^k) = C(m, 2) - m - k + 1 \tag{4.7}$$

Computing for A^{k+1} we have:

$$D(A^{k+1}) = C(m, 2) - (k+1)\left(\frac{m-1}{k+1} + 1\right)$$

$$D(A^{k+1}) = C(m, 2) - m - k \tag{4.8}$$

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Therefore, for any $k \in [1, m-1]$, we have

$$D(A^{k+1}) \geq D(A^k) \qquad (4.9)$$

Since $WDE(A^{k+1}) \geq WDE(A^k)$ by proposition 4.2, therefore larger pairwise comparison losses result to larger WDE. For any $D(A^k)$ and $D(A^{k'})$ where $k > k'$ we have

$WDE(A^k) \geq WDE(A^{k'})$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Theorem 4.2. *$WDE(A^k)$ is monotone increasing as the values of k is increased.*

Proof. Based on theorem 4.1 since, $D(A^{k+1}) > D(A^k)$ then $WDE(A^{k+1}) \geq WDE(A^k)$. Consider $D(A^{k+2})$. Then we have $D(A^{k+2}) > D(A^{k+1})$ and $D(A^{k+2}) > D(A^k)$, by theorem 1, then $WDE(A^{k+2}) \geq WDE(A^{k+1}) \geq WDE(A^k)$ $\qquad\qquad\qquad\qquad$ □

Since the proofs are based on a single erroneous pairwise comparison, a Monte Carlo simulation is performed on a peer reviewed AHP dataset (Önüt et al., 2010) from existing literature to further illustrate the relationship between $D(A^k)$ and the $WDE(A^k)$. In (Önüt et al., 2010), Önüt et al. propose a fuzzy AHP model for shopping center site selection. The row scores of the alternatives over elements are presented in Table 4.1. These scores represent the most likely scores of the fuzzy pairwise comparison using a 10-point rating scale. The 8×8 correlation matrix is then obtained and is presented in Table 4.2.

Table 4.1. Raw data of the Scores of Each Alternative on Each Element

Alternative	1	2	3	4	5	6	7	8
A	5	7	5	9	5	5	3	3
B	7	7	5	7	7	7	5	5
C	5	5	7	5	5	5	9	7
D	5	5	8	3	5	5	9	5
E	7	9	5	5	7	7	3	3
F	5	5	7	3	5	5	9	5

By varying the raw scores and the most likely local pairwise comparisons by $\pm 10\%$, the response of the $D(A^k)$ and the WDE performance metrics are tracked to

Table 4.2. Correlation Matrix Computed from Table 4.1

Element	1	2	3	4	5	6	7	8
1	1.00	0.79	0.68	0.22	1.00	1.00	0.60	0.34
2	0.79	1.00	0.86	0.49	0.79	0.79	0.92	0.76
3	0.68	0.86	1.00	0.79	0.68	0.68	0.93	0.63
4	0.22	0.49	0.79	1.00	0.22	0.22	0.76	0.42
5	1.00	0.79	0.68	0.22	1.00	1.00	0.60	0.34
6	1.00	0.79	0.68	0.22	1.00	1.00	0.60	0.34
7	0.60	0.92	0.93	0.76	0.60	0.60	1.00	0.82
8	0.34	0.76	0.63	0.42	0.34	0.34	0.82	1.00

determine their relationship. The decomposition is done on values of $k \in [2, m-1]$ using a MATLAB 2011b programming environment. The simulation is run with 10,000 replications and a 95% confidence interval plot on the mean of the $WDE(A^k)$ plotted over values of $k \in [1, 7]$ with $D(A^k)$ is presented in Figure 4.2.
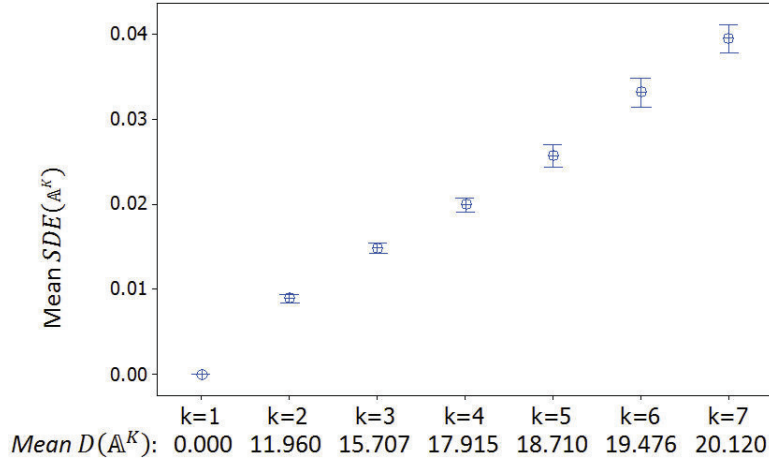
Figure 4.2. The Simulated Effect of different k values with mean of $D(A^k)$ on the mean of the WDE

The effect of increasing the values of $D(A^k)$ on the WDE is clearly presented. A monotone increase of the WDE is observed as the values of $D(A^k)$ is increased which supports the findings of theorem 4.2. Based on theorem 4.2 and the results presented in figure 4.2, we seek to minimize $D(A^k)$ or equivalently by definition of $D(A^k)$ to maximize the number of pairwise comparisons $\sum_{l=1}^{k} C(m_l, 2)$. This is done to obtain small values of the WDE which is consistent with the recommendation of Saaty (1980).

BIP Formulation

The proposed trade-off objective function is presented in Eq. (4.10).

$$min \ Z = \sum_{l=1}^{k} \sum_{i=1}^{m} \sum_{i'<i} r_{ii'} y_{ii'l} + \lambda D(A^k) \qquad (4.10)$$

The trade-off parameter λ is included as a scaling factor that control the optimal solution of the BIP which is elicited from the DM. Small values of λ provide an optimal solution that minimizes the interdependencies among elements and seeks to assign all m elements to at most $m-1$ subsets. On the other hand, large values of λ provide an optimal solution that prioritizes the WDE and assigns the m elements into at least 2 subsets. Furthermore, since $D(A^k) = C(m,2) - \sum_{l=1}^{k} C(m_l, 2)$ and $C(m,2)$ is a constant, then only the $\sum_{l=1}^{k} C(m_l, 2)$ term is included in the revised objective function presented in equation 4.11.

The proposed BIP formulation is a modification of the quadratic clustering formulation of Song and Hitomi (Song & Hitomi, 1992). A linearization technique is applied to linearize the quadratic elements to obtain a linear binary integer formulation with additional decision variables. The definition of all the decision variables as well as the formulation is presented as follows:

Let:

$r_{ii'} :=$ correlation(dependence) bet. element i and i'

$$x_{il} = \begin{cases} 1 & \text{if element } i \text{ is assigned to subset with index } l \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ii'l} := x_{il}x_{i'l} \begin{cases} 1 & \text{if both elements } i \text{ and } i' \text{ are assigned to } l \\ 0 & \text{otherwise} \end{cases}$$

$$z_i := \begin{cases} 1 & \text{if element } i \text{ is a pivot element} \\ 0 & \text{otherwise} \end{cases}$$

$$s_l^f := \begin{cases} 1 & \text{if subset with index } l \text{ can be assigned elements} \\ 0 & \text{otherwise} \end{cases}$$

$$k_L := \begin{cases} 1 & \text{if there are } L = 2, 3, \cdots, m - 1 \text{ subsets} \\ 0 & \text{otherwise} \end{cases}$$

$$m_{il} := \begin{cases} 1 & \text{if there are } m_l \text{ elements in subset } l \\ 0 & \text{otherwise} \end{cases}$$

$$w_{iL} := z_i k_L := \begin{cases} 1 & \text{if } k = L \text{ and } z_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Objective Function:

$$min\ Z = \sum_{l=1}^{k}\sum_{i=1}^{m}\sum_{i'<i} r_{ii'} y_{ii'l} + \lambda \sum_{i=1}^{k} \sum_{i\in[0,2,\cdots,m-1]} C(i,2)m_{il} \tag{4.11}$$

s.t.

$$x_{il} - y_{ii'l} \geq 0, \qquad\qquad i \in [1, m], l \in [1, m-1] \tag{4.12}$$

$$x_{i'l} - y_{ii'l} \geq 0, \qquad\qquad i \in [1, m], l \in [1, m-1] \tag{4.13}$$

$$x_{i'l} + x_{il} - y_{ii'l} \leq 1, \qquad\qquad i \in [1, m], l \in [1, m-1] \tag{4.14}$$

$$\sum_{i=1}^{m} x_{il} \geq 2s_l^f, \qquad\qquad l \in [1, m-1] \tag{4.15}$$

$$\sum_{i=1}^{m} x_{il} \leq ms_l^f, \qquad\qquad l \in [1, m-1] \tag{4.16}$$

$$\sum_{l=1}^{k} x_{il} \geq 1, \qquad\qquad i \in [1, m], l \in [1, m-1] \tag{4.17}$$

$$k_l - w_{iL} \geq 0, \qquad\qquad L \in [2, m-1], i \in [1, m] \tag{4.18}$$

$$z_i - w_{iL} \geq 0, \qquad\qquad L \in [2, m-1], i \in [1, m] \tag{4.19}$$

$$k_L + z_i - w_{iL} \leq 1, \qquad\qquad L \in [2, m-1], i \in [1, m] \tag{4.20}$$

$$\sum_{i=1}^{m} x_{il} \leq \sum_{L=2}^{m-1} \left\lceil \frac{m-1}{L} - 1 \right\rceil k_L, \qquad\qquad L \in [2, m-1], i \in [1, m] \tag{4.21}$$

$$\sum_{l=1}^{k} x_{il} \leq \sum_{l=1}^{k} l w_{iL} - z_i + 1, \qquad\qquad L \in [2, m-1], i \in [1, m] \tag{4.22}$$

$$\sum_{i=1}^{k} z_i = 1, \tag{4.23}$$

$$\sum_{l=1}^{k} s_l^f = Lk_L, \qquad\qquad L \in [2, m-1] \tag{4.24}$$

$$\sum_{l=1}^{k} k_l = 1 \tag{4.25}$$

$$\sum_{i=1}^{m} x_{il} = \sum_{I \in [0,2,\cdots,m-1]} Im_{Il} \qquad\qquad l \in [1, m-1] \tag{4.26}$$

$$\sum_{i=1}^{m} m_{il} = 1, \qquad\qquad l \in [1, m-1] \tag{4.27}$$

$$x_{il}, y_{ii'l} \in \mathbb{B}^{m \times (m-1)}, z_i \in \mathbb{B}^m, s_L^f \in \mathbb{B}^{m-1}$$
$$k_L \in \mathbb{B}^{m-2}, wiL \in \mathbb{B}^{m \times (m-2)}, m_{Il} \in \mathbb{B}^{(m-1) \times (m-1)} \tag{4.28}$$

Equation 4.11 shows the revised objective function. Equations sets 4.12 to 4.14 linearize the quadratic relationship ($y_{ii'l} := x_{il}x_{i'l}$). Equation set 4.15 forces each subset with index l, ($s_l^f = 1$) to have at least 2 elements and at least zero assignments otherwise. Equation set 4.16 provides an upper bound for the number of elements assigned to subset with index l if l is feasible ($s_l^f = 1$) and zero assignments otherwise. Equation set 4.17 forces each element i to be assigned to at least a single subset. Equations 4.18 to 4.20 linearize the quadratic relationship $w_{iL} := z_i k_L$. Equation set 4.21 serves as an upper bound (($m + (l-1)/l$)) for the number of elements assigned to subset l based on the number of subsets k_l. Equation set 4.22 assigns element i to all feasible subsets if element i is a pivot element ($z_i = 1$) and subset l is feasible ($s_l^f = 1$). Equation 4.23 sets the number of pivot elements to 1. Equation set 4.24 calculates the number of feasible subsets s_l^f to the number of subsets k_L. Equation 4.25 ensures that there is at least 1 $k_l = 1$ option. Equation set 4.26 counts the number of elements assigned to a subset and relating it to m_{il}. Equation set 4.27 ensures that there is at least one $m_{il} = 1$. Equation set 4.28 declares all variables as binary variables.

The optimal output solution of the proposed BIP is an assignment of the m elements into optimally selected k subsets denoted by $x_{il} = 1$ if element i is assigned to

subset s_l. Additionally, the optimal pivot element p is selected and assigned to all subsets. Furthermore, the BIP also provides a listing of the feasible subsets s_l^f that have elements assigned to it where $\sum_{l=1}^{m-1} s_l^f = k$.

Elicit Local Pairwise Comparisons and Calculate Local Weights

After decomposition, local pairwise comparisons are elicited from the DM for all subsets after the elements are assigned to subsets. The local pairwise comparisons for elements subset s_l are illustrated in matrix form A_l as shown in Eq. 4.29:

$$A_l = \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,m_l} \\ 1/a_{,2} & 1 & \cdots & a_{2,m_l} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1,m_l} & 1/a_{2,m_l} & \cdots & 1 \end{bmatrix}, \forall s_l \in S \tag{4.29}$$

Let $w(A_l)$ be the vector of local weights from A_l where $w(i,l) \in w(A_l)$ is the local weight of elements i. The original eigenvector methodology is used to calculate the local element weights as follows:

$$w(A_l) = \begin{bmatrix} \frac{\sum_{j=1}^{m_l} \tilde{a}_{1j}}{m_l} \\ \vdots \\ \frac{\sum_{j=1}^{m_l} \tilde{a}_{m_l j}}{m_l} \end{bmatrix} = \begin{bmatrix} w(1,l) \\ w(2,l) \\ \vdots \\ w(m_l,l) \end{bmatrix}, l = 1, 2, \cdots, k \tag{4.30}$$

A new performance measure is needed to keep track of the consistency of the pairwise comparisons. The original definition of the CR of matrix A is no longer applicable since the m elements are assigned into k subsets. With this, a new definition of consistency is proposed as follows:

Definition 4.4. *The Average Consistency Ratio (ACR) performance measure of an PCM decomposed into k subsets is defined by:*

$$ACR = \frac{1}{m+k-1} \sum_{l=1}^{k} m_l CR(A_l) \tag{4.31}$$

where $CR(A_l)$ is the consistency ratio of the PCM subset A_l.

In simple terms, the ACR is the weighted average of the CR of each of the local PCM. The ACR is used to estimate the overall CR of the pairwise comparisons across all subsets.

Calculate Global Weights

Given values of k there will be $m + k - 1$ instances of $w(i, l)$. The local elements weight in subset l is divided by the weight of pivot element p in that subset and is repeated for all subsets. To illustrate this, let $\tilde{w}(A_l)$ be the vector of normalized weights where each $\tilde{w}(i, l) \in \tilde{w}(A_l)$ is computed using Eq. 4.32.

$$\tilde{w}(i, l) = \frac{1}{w(i = p, l)}[w(i, l)], \quad \forall i \in s_l, \forall s_l \in S \tag{4.32}$$

Given this, the normalized pivot element weight in each subset has a value equal to one. Since all normalized pivot element weight has a value equal to one, all the other elements in the other subsets can be compared to the pivot elements. For the computation of the global weights, let $w'(A)$ be the vector of global weights where $w'(i) \in w'(A)$ is computed using Eq. 4.33.

$$w'(i) = \frac{1}{\sum_{l=1}^{k} \sum_{i \in s_l} \tilde{w}(i, l) - K + 1} \tilde{w}(i, l), \quad \forall i \in C \tag{4.33}$$

Illustrative Examples

This section presents an example that illustrate the capability of the proposed methodology in decomposing a criteria PCM. Consider the same dataset specifically the

fuzzy AHP site selection data from Önüt et al. (2010). Consider a scenario where the DM chooses to have a solution that have equal priorities for the dependence and the amount of deviation error $\left(\sum_{i=1}^{m} \sum_{i'<i} r_{ii'} y_{ii'l} = \lambda D(A^k) \right)$. Equal weights are given to the independence and deviation error component of the objective function. Four performance metrics are considered, specifically: the ACR, $D(A^k), WDE$ and the amount of dependence after decomposition: TDS= $\sum_{i=1}^{m} \sum_{i'<i} r_{ii'} y_{ii'l}$. Table 4.3 summarizes the results of the proposed model for a balanced objective function.

Table 4.3. Proposed Model Optimal Results for Balanced Dependence and WDE

Subset Index	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	1	0	1	1	0	0	0	1
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	1	0	1	0	1	0	0
6	0	0	0	1	1	0	1	0
7	0	0	0	0	0	0	0	0

Criterion 4 is chosen as a pivot criterion and is assigned to three feasible subsets. Criteria 1, 3, 4 and 8 are assigned to subset with index 2, criteria 2, 4 and 6 are assigned to subset with index 5, and criteria 4, 5 and 7 are assigned to subset index with 6. The total amount of dependence after decomposition is $\sum_{l=1}^{k} \sum_{i=1}^{m} \sum_{i'<i} r_{ii'} y_{ii'l} = 6.1663$, while the total number of pairwise comparisons after decomposition is $28 - 12 = 16$. Table 4.4 to Table 4.6 show the local priorities of the decomposed AHP obtained from the original 8×8 PCM.

Table 4.4. Subset 1 Local Priorities

Criteria	1	3	4	8	Priority
1	1	0.33	0.33	1	12.20%
3	3	1	3	3	47.32%
4	3	0.33	1	3	28.27%
8	1	0.33	0.33	1	12.20%

The average consistency ratio (ACR) is computed by obtaining the weighted average of the individual priorities of the 3 subsets as:

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Table 4.5. Subset 2 Local Priorities

Criteria	4	5	7	Priority
4	1	1	3	40.55%
5	1	1	5	47.96%
7	0.33	0.2	1	11.50%

Table 4.6. Subset 3 Local Priorities

Criteria	2	4	6	Priority
2	1	0.2	0.2	9.09%
4	5	1	1	45.45%
6	5	1	1	45.45%

$$ACR = \frac{1}{m+k-1}\sum_{l=1}^{k} m_l CR(A_l) \tag{4.34}$$

$$= \frac{4}{10}8.08\% + \frac{3}{10}3.09\% + \frac{3}{10}0.00\% = 4.16\%$$

Using the pivot criterion 4, the global weights are as follows:

$$w' = [0.07, 0.12, 0.28, 0.16, 0.19, 0.04, 0.05, 0.07]^T \tag{4.35}$$

Two other scenarios are considered in which the DM chooses to have a solution that minimizes the amount of dependence among criteria ($\lambda = 0$) and the DM chooses to have a solution that minimizes the WDE, ($\lambda \gg 0$). A summary of the results for all three scenarios including the results of the performance measures for the original AHP methodology is presented in Table 4.7.

Table 4.7. Summary of Results on Each Performance Metric

Performance Metric	Original	Balanced Priority	Independence Priority	WDE Priority
ACR	12.96%	4.16%	0.00%	7.11%
$D(A^k)$	0	16	21	12
WDE	0	5.599E-02	9.24E-02	8.03E-03
TDS	18.29	6.166	3.120	9.4814

It is evident from Table 4.7 that the proposed decomposition methodology illustrate a trade-off relationship between different λ values. For the undecomposed PCM, we can observe an inconsistent PCM (12.96%), with no reductions on the number of pairwise comparisons (time consuming), zero WDE due to complete comparisons and lots of interdependence (18.29). On the other hand, a balanced priority objective function provides a trade-off solution with acceptable inconsistency (4.16%), 16 pairwise comparison reductions, some deviation error (5.99E-02) and a 67% reduction in the amount of dependence (6.166). Extreme solutions are obtained for both independence and WDE priority settings. If the priority is independence, we have consistent matrices (0.00%) and lowest amount of dependence (3.120). Conversely, if the priority is the WDE, we obtain some error (8.30E-03) at a cost of 12 pairwise comparison reductions and more consistent local pairwise comparisons (7.11%).

4.4   Validation of the Proposed Methodology

Although preliminary results performed on established AHP data are promising, further validation of the proposed decomposition methodology is done using Monte Carlo simulation on a MATLAB 2011b environment. By varying the similarity coefficients and the local pairwise comparisons by $\pm 10\%$, the sensitivity of the four performance metrics are measured. The continuity assumption of the pairwise comparisons is believed to be reasonable since the variation can be viewed as a degree of belonging to a fuzzy set (Emblemsvg & Tonning, 2003; Triantaphyllou, 1995). This section presents the results of the two validation phases. The first phase tests the validity of the proposed decomposition methodology as compared an undecomposed PCM. Likewise, the second phase compares the model to three other decomposition models, specifically the methodologies of Ishizaka (2012), Shen et al. (1992) and Triantaphyllou (1995). Table 4.8 summarizes the methodologies and their corresponding settings. The same dataset from (A) Önüt et al. (2010) is used to validate the methodology including two other datasets from peer reviewed papers specifically the ones of (B) Al-Harbi (2001) and (C) Y. Lee & Kozar (2006).

Table 4.8. Validation Parameters

Parameters	Methodology				
	Proposed Model	Undecomposed	Ishizaka (2012)	Shen et al. (1992)	Triantaphyllou (1995)
Number of Subsets	BIP	1	Unknown Brute Force	Unknown Brute Force	2
Assignment of Criteria	BIP	N/A	Attribute Magnitudes	Random	Random
Selection of Pivot Criterion	BIP	N/A	Random	Random	Random
Calculation of Global Weights	Pivot Criterion	N/A	Single Linkage	Pivot Criterion	LP

Comparison with the Original AHP Methodology

This section presents the results of the first phase of the validation in which the main performance metric is the weighted objective function value sum computed by Eq. 4.11. In this phase, the optimal Z value of the BIP is compared to the solution when the original AHP PCM is used where $k = 1$. A paired t-test on the mean is used to determine if there is significant difference between the optimal solution provided by the proposed model(μ_P) and the optimal solution provided by the original AHP methodology (μ_A) on the objective function value obtained. The null hypothesis tested is: $\mu_A = \mu_P$ and alternative hypothesis: $\mu_A \neq \mu_P$ at a 95% confidence level. Each simulation is run with 10,000 iterations. Table 4.9 shows the simulation results when the optimal Z value of the proposed model is compared to Z value when the original AHP is used.

It is observed in Table 4.9, that the proposed methodology outperforms the original AHP PCM in terms of the weighted objective function value Z when the WDE and dependence are equally prioritized and when dependence is solely prioritized. On the other hand, the original methodology outperforms the proposed methodology when the WDE is prioritized. This can be attributed to the fact that the undecomposed PCM would lead to zero deviations since all original pairwise comparisons are used.

Table 4.9. Comparison of Z Value Results From the Original AHP and the Proposed Model

Settings	DataSet	AHP		Proposed Model		t-test	p-Val
		Mean	Std.Dev.	Mean	Std.Dev.		
	A	6.82	2.10	3.99	1.21	$\mu_A > \mu_P$	0.00
Balanced	B	-0.13	0.38	-2.77	0.51	$\mu_A > \mu_P$	0.00
	C	7.48	2.25	-0.64	2.21	$\mu_A > \mu_P$	0.00
	A	146.17	10.30	24.59	3.18	$\mu_A > \mu_P$	0.00
Dependence	B	34.75	1.116	1.69	0.90	$\mu_A > \mu_P$	0.00
	C	168.62	13.28	14.01	3.74	$\mu_A > \mu_P$	0.00
	A	-10.78	0.92	-3.51	0.58	$\mu_A < \mu_P$	0.00
WDE	B	-10.04	0.11	-4.89	0.10	$\mu_A < \mu_P$	0.00
	C	-8.40	1.29	-2.91	0.63	$\mu_A < \mu_P$	0.00

Comparison with other Decomposition Models

The second validation stage extends the first validation stage by considering $k \in [2, m-1]$ subsets. Therefore, an extension of Ishizaka (2012)'s model for PCM decomposition is used as a comparison. The purpose of Ishizaka's model is similar to the purpose of the proposed decomposition model and as such we compare the two models in terms of four performance metrics. A brute force approach is done where the optimal Z value of the proposed model is compared to the Z values when Ishizaka's methodology is applied for all possible $k \in [2, m-1]$ values.

Furthermore, this section compares the four performance metrics of the model, when the value of k is solved optimally, to that of the original decomposition methodology by Shen et al. (1992) with the same k value. Whatever k value is provided by the BIP, the same k is run through the methodology proposed by Shen for decomposition at the criteria level.

Additionally, a comparison of the proposed decomposition methodology on the one proposed by Triantaphyllou (1995) on all four performance metrics is also done. Since Triantaphyllou's model is limited to only two subsets, we compare the performance of the proposed model with the $D(A^k)$ prioritized. Since larger weights are placed on $D(A^k)$, solutions of $k = 2$ subsets are obtained. The proposed model is now comparable to the

methodology of Triantaphyllou when we chose the number of criteria for subset 1 to be $n_1 = m/2 + 1$ and subset 2 to be $n_2 = m/2$.

A paired t-test on the mean is used to determine if there is significant difference on each performance metric $D(A^k)$, WDE, ACR and TDS. The null hypothesis tested is $\mu_* = \mu_P$ and alternative hypothesis: $\mu_* \neq \mu_{BIP}$ at a 95% confidence level. Each simulation is run with 10,000 iterations. Table 4.10 summarizes the results of the simulation for all methodologies in terms of wins, ties and loses. A win (W) is defined as statistical significant improvement of a performance metric, a tie (T) denotes not that there is not enough evidence to reject $\mu_* = \mu_P$ and a lose(L) denotes a statistical significant degradation of a performance metric when the proposed model is compared to each of the decomposition methodologies.

Table 4.10. Comparison of other Decomposition Methodologies and the Proposed Model

Settings	DataSet	Ishizaka et al.			Shen et al.			Triantaphyllou		
		W	T	L	W	T	L	W	T	L
Balanced	A	5	1	0	2	2	0	N/A	N/A	N/A
	B	4	0	0	3	0	1	N/A	N/A	N/A
	C	6	0	0	3	1	0	N/A	N/A	N/A
Dependence	A	6	0	0	2	1	1	N/A	N/A	N/A
	B	4	0	0	3	0	1	N/A	N/A	N/A
	C	6	0	0	2	0	2	N/A	N/A	N/A
WDE	A	6	0	0	3	1	0	3	1	0
	B	4	0	0	4	0	0	3	1	0
	C	6	0	0	2	2	0	3	1	0
Totals		47	1	0	24	7	5	9	3	0

Based on Table 4.10 it is observed that the optimal trade-off objective function value of the proposed model dominates all possible objective function values for $k \in [2, m-1]$ provided by the methodology of Ishizaka. The weighted optimal solution provided by the model generally provides a better solution on all performance metrics since the proposed model selects the best number of subsets, provides optimal assignment of criteria to all subsets and selects the least independent pivot criterion as compared to

Ishizaka's methodology that does not have any capability of doing it. Therefore, given the preferences of the DM, the best number of subsets k can be determined thus eliminating the DM's judgement error and bias.

Furthermore, the proposed decomposition methodology improves on the methodology proposed by Shen specifically on 24 out of 36 tests. The wins are mainly attributed to the $D(A^k)$ and TDS performance metrics. This means that the proposed model has lesser required pairwise comparisons after decomposition as compared the methodology proposed by Shen. This translates to faster decision making and lesser time for the DM to setup the AHP model. Furthermore, the proposed model clearly segregates uncorrelated criteria into the same subsets. A reduction in the amount of dependence in criteria means that the criteria assigned to subsets are much more independent and thus the provision of pairwise comparisons would not be subject to biases and error. Comparable performance is generally obtained when comparing the proposed model and Shen's methodology on the ACR and WDE.

It is also observed in Table 4.10 that the proposed model improves on the methodology proposed by Triantaphyllou on 9 out of 12 tests while the three ties can be attributed to the no significant difference result on the $D(A^k)$ performance metric. The proposed model improves on the ACR, which means that the proposed model has generally more consistent comparisons. Furthermore, statistically lesser deviations are observed when the global weights calculation methodology using a pivot criterion is used as compared to the linear programming estimation methodology proposed by Triantaphyllou. In terms of the TDS, the proposed model has statistically lesser amount of dependence among criteria as compared to the model of Triantaphyllou on all three datasets.

4.5   Conclusions and Future Work

Due to the inherent limitations of large PCMs and the existing methodologies on PCM decomposition, this paper proposes a novel decomposition methodology to

decompose the elements of a PCM into smaller subsets. The decomposition methodology which consists of a Binary Integer Program (BIP) which uses a trade-off objective function that balances the deviation error between the original undecomposed PCM weights and the decomposed PCM weights (WDE) and the amount of dependence among elements (TDS). The proposed model is capable of (1) solving for the optimal number of PCM subsets, (2) assigning the PCM elements to all subsets, and (3) selecting the best pivot element to be assigned to all subsets. These three contributions address the limitations of the existing methodologies. Based on the experimental results, the following specific conclusions can be drawn:

The proposed methodology improves on the AHP methodology when the issue of independence is given priority over the deviations of the global weights. Additionally, the proposed model surpasses the methodology proposed by Ishizaka (2012) in a majority of the statistical t-tests performed on the optimal objective function Z values. This implies that the proposed model determines the best the number of PCM subsets (k) thus avoiding judgement and bias error from the DM. The proposed decomposition methodology can be used as a decision support tool for DMs to determine the optimal k value given weighting preferences on the amount of dependence present among elements in a subset and the deviations of the global weights as well. Furthermore, it can be used if the DM's preference is to balance the two performance metrics as well as prioritizing only the dependence among elements. On the other hand, DM can use the original AHP methodology if his preference is solely the accuracy of the weights and the time to provide the comparisons is not an issue.

Additionally, the proposed model outperforms the decomposition model proposed by Shen et al. (1992) on the TDS and number of pairwise comparisons after decomposition ($D(A^k)$) performance metrics. Additionally, the BIP improves on the methodology proposed by Triantaphyllou (1995) on the average consistency ratio (ACR), WDE and TDS performance metrics. This shows that a reduction on the amount of dependencies among elements is obtained which can lead to more accurate decisions since

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the elements in the local subsets are more independent. Likewise, a statistically significant decrease on the required number of pairwise comparisons is obtained thus saving the DM time in making a large scale decision. Consequently, we can infer from this that the optimal assignment of criteria is an effective means to balance the amount of dependence among elements and number of pairwise comparisons after decomposition.

The authors plan to extend the model by considering more AHP levels and applying it in the ANP methodology. An extension of using multiple pivot criteria in the BIP formulation is also being considered to further reduce the deviation errors as well as inclusion of interval and fuzzy pairwise comparisons that handle imprecise pairwise comparisons.

Chapter 5

A GENERALIZED STOCHASTIC AHP DECISION MAKING METHODOLOGY

Existing methodologies address imprecise Analytic Hierarchy Process (AHP) pairwise comparisons by modeling crisp pairwise comparisons as fuzzy sets or a single type of probability distribution (e.g., uniform, triangular). However, one common issue faced by decision makers (DM) is bounded rationality, that is, DMs have limited cognitive powers in specifying their preferences over multiple pairwise comparisons. Hence, there is a need of presenting the imprecise comparisons in a generalized form which can be used to represent different types of distributions. In addition, given the ultimate goal of imprecise AHP is to make the decision, computing appropriate weights for the alternatives and criteria from the imprecise comparisons is a must. Therefore, a second advantage in using a generalized distribution function is it can represent the varied distributions making the computation of the weights simplified. In this research, a beta distribution is proposed which can represent a wide variety of distributions based on the information available via method-of-moments methodology. A Non-Linear Programming model is then developed that calculates weights which maximizes the preferences of the DM as well as minimizes the inconsistency simultaneously. Comparison experiments are conducted using datasets collected from literature to validate the proposed methodology.

5.1   Introduction

The Analytic Hierarchy Process (AHP), developed by Saaty (1977) is a practical and useful multiple criteria decision making (MCDM) tool. It has been widely accepted by industries and applied in various MCDM problems (Ishizaka & Labib, 2011). The AHP methodology uses pairwise comparisons a_{ij} between criteria or alternative i and j to calculate their respective weights for decision making. These pairwise comparisons usually are quantitative or qualitative in nature. Quantitative pairwise comparisons are direct observations from the attributes of alternatives while qualitative comparisons are elicited from the decision maker (DM) to quantify the degree of preference. Conventionally,

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pairwise comparisons are modelled as crisp values $a_{ij} \in [1/9, 9]$. Yet, these crisp values may not be sufficient to model the presence of ambiguity (A. H. I. Lee et al., 2008). To address this limitation, the concept of interval judgements or interval pairwise comparisons is proposed in (Saaty & Vargas, 1987; Arbel, 1989). Upper and lower limit pairwise comparison values (LL,UL) are used to model the imprecise pairwise comparisons. Essentially, these pairwise comparisons are treated as random variables that follow a uniform probability distribution. Other stochastic distributions have also been explored including the triangular distribution (Banuelas & Antony, 2006), binomial (Hahn, 2003) and the Cauchy distribution (Lipovetsky & Tishler, 1999). Fuzzy sets are also proposed in various studies (Mikhailov, 2000, 2004; Dagdeviren & Yüksel, 2008; T. C. Wang & Chen, 2008).

However, these methodologies are not without any disadvantages. Firstly, in order to complete the decision making process, weights need to be estimated from interval distributions. The conventional AHP eigenvector method for calculating weights is no longer applicable when pairwise comparisons are imprecise (Arbel, 1989). Existing stochastic AHP models need to explore the use of methodologies like linear programming and simulation to calculate the weights. Note, most stochastic AHP methodologies focus on either maximizing the preferences of the DM only even if they may be inconsistent (Y. M. Wang & Elhag, 2007; Banuelas & Antony, 2006), or minimizing the inconsistency only which may result the decision deviated from the DM's true preference (Y. M. Wang et al., 2005a; Mikhailov, 2000, 2004). A balanced consideration of both objectives is of necessity, especially, the use of stochastic distributions may lead to increased inconsistency if the weights are not properly derived from the pairwise comparisons.

Another notable issue of existing literature is most stochastic AHP methodologies employ one single type of pairwise comparison distribution. Knowing DMs that use the AHP are faced with the issues on bounded rationality (Simon, 1955, 1972) that is, limited cognitive power to precisely depict the preferences over large number of comparisons; it is desirable to provide DMs the flexibility to choose different distributions for different

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pairwise comparisons during the elicitation process. For example, a crisp comparison may be used in the cases the DM are sure of the comparisons; a uniform distribution may apply in the cases the DM is totally unsure; and a triangular distribution may be used if it is the case in between. The underline challenge imposed by this idea though is it is difficult if not impossible for optimization methodologies such as LP to solve for crisp pairwise comparisons from varying pairwise comparison distributions. The only alternative maybe simulation as is done in (Saaty & Vargas, 1987; Banuelas & Antony, 2006; Lipovetsky & Tishler, 1997), which may not locate the global optimum and suffers from expensive computational costs. As such, a methodology that efficiently computes for weights that maximizes the preferences of the DM and at the same time computes for consistent crisp pairwise comparisons from varying types of imprecise pairwise comparison preferences is of urgent need.

In this paper, a Generalized Stochastic AHP decision making methodology is proposed. First, a beta distribution is developed to model the varying types of probability distributions for the different pairwise comparisons elicited from the DM. The beta distribution has interesting properties, one of which is its ability to model other probability distributions, and it is differentiable over its domain making it ideal for optimization algorithms. The method of moments methodology is applied to fit any input pairwise comparison distribution into beta distributed pairwise comparisons. Next, a Non-Linear programming (NLP) model is developed to calculate crisp criteria or alternative weights that maximize the probability likelihood of varying types of imprecise pairwise comparisons or in a sense the preferences of the DM and at the same time minimizing the inconsistency of the pairwise comparisons.

The rest of this paper is organized as follows. Section 5.2 reviews existing literature that attempts to solve the aforementioned problems. Section 5.3 illustrates the steps of the proposed methodology, while Section 5.4 describes the computational experiments done to validate and address the research questions. Finally, Section 5.5 concludes the paper and proposes further research areas.

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5.2   Review of Related Literature

The concept of interval pairwise comparisons or interval judgements is originally proposed by Saaty & Vargas (1987). In this setup, instead of eliciting crisp pairwise comparisons, the DM provides the minimum and maximum values that the unknown pairwise comparison value can have. This elicitation process handles the ambiguity issue of the pairwise comparison whenever the DM is unsure of its true value. In Saaty and Vargas's methodology, the calculation of global weights is done using a Monte Carlo simulation methodology by sampling feasible crisp weights that satisfy all interval judgements. However, this setup would not provide an optimal weight solution and is time consuming. As such, Arbel (1989) proposes a linear programming model to estimate the weights from interval judgements. However, Kress (1991) argues that the solution from (Arbel, 1989) exists only in completely consistent interval judgements. To date, there exist goal programming models to estimate global weights from interval pairwise comparisons (Bryson, 1995; Xu, 2004; Y. M. Wang & Elhag, 2007; Z. J. Wang & Li, 2012). These methodologies seek an optimal set of satisficing weights which is calculated by minimizing deviations of the optimal weights from all feasible interval judgements or in a sense maximizing preferences of the DM. Other similar goal programming variants include min-max goal programming (Despotis & Derpanis, 2008), logarithmic goal programming (Y. M. Wang et al., 2005b) and lexicographic programming (Islam et al., 1997b). Unfortunately, the lexicographic programming model provides unreliable priority estimates as shown in (Y. M. Wang, 2006). Hence, Y. M. Wang et al. (2005a) propose to minimize the consistency ratio (CR) using a NLP approach. Salo & Hämäläinen (1995) propose a preference programming approach with interactive decision support from the DM. Guo & Tanaka (2010) suggest the use of subjective pairwise comparisons of the likelihood of events for all possible alternative ranking outcomes. Quadratic programming is applied to estimate the final weights. Guo & Wang (2011) extend the model of Guo & Tanaka (2010) by using dual interval probabilities and linear programming. Please note

only one type of distribution is studied in these mathematical models which may limit its application to large-scale decision problems where the DM tends to have different preference knowledge over different pairwise comparisons.

In modeling uncertainty, the concept of fuzzy sets has also been applied. Mikhailov (2000, 2004) applies fuzzy sets to model uncertainty and fuzzy preference programming method to estimate crisp weights. A similar methodology is proposed in (A. H. I. Lee et al., 2008, 2009). According to Y. M. Wang & Chin (2011), these methodologies may produce conflicting priority vectors that lead to inaccurate decisions as such they propose to use a logarithmic fuzzy preference programming methodology for priority derivations. In 2006, Y. M. Wang & Chin (2006) propose a combination of the eigenvector method and linear programming to estimate crisp priorities from fuzzy comparison matrices. Additionally, Yang et al. (2012) propose a cloud Delphi hierarchical analysis with fuzzy interval weights to model the uncertainty of the AHP. These models also use a single type of fuzzy distribution (e.q. triangular fuzzy numbers) which may not be applicable to model the varying preferences of the DM.

There are methodologies that handle the uncertainty of comparisons and weights calculation by applying statistical modeling techniques. Moreno-Jimenez & Vargas (1993) develop a methodology to estimate the probability of all possible alternative preference rankings from uniformly distributed interval judgements. These probabilities are analytically calculated whenever perfect consistent judgements are obtained. On the other hand, simulation is used for inconsistent cases. In (Haines, 1998), Haines proposes a statistical based algorithm to study the effect of using uniform and convex distributions on interval judgements. The mean of the distributions is used to rank the alternatives. Lipovetsky & Tishler (1999) propose to model interval judgements in terms of a Cauchy distribution and a non-linear approximation methodology to calculate priorities. Lipovetsky & Tishler (1997) also propose several other distributions like the triangular, normal, Laplace or Cauchy however they used it individually. In (Sugihara et al., 2004), Sugihara et al. suggest an interval regression model to estimate interval priorities from

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interval pairwise comparison judgements. Using uniform interval judgements, Stam & Silva (1997) recommend multivariate statistical techniques to estimate points and confidence intervals for rank reversal probabilities. When the rank reversal probability is low, then the interval judgements are accepted. Hahn (2003) proposes a Bayesian approach specifically, a weighted hierarchical multinomial logit model to obtain final weights. Furthermore, inference on these weights is done using a Markov chain Monte Carlo sampling method. Recently, Liu et al. (2011) suggest a probability distribution aggregation and mathematical programming to combine pairwise comparisons modelled as probability distributions.

Though extensive research has been proposed, it is observed that firstly: existing methodologies focus on either maximizing the preferences of the DM or minimizing the consistency ratio for pairwise comparisons that follow a single type of distribution. To the best of our knowledge, there exists no model that addresses these two objectives simultaneously and efficiently in terms of varying pairwise comparisons. Secondly, existing methodologies focus on modeling imprecise pairwise comparisons using a single distribution type. However, it is expected that the amount of imprecision of the pairwise comparisons would not be constant due to bounded rationality issues. This research proposes a generalized stochastic AHP to address these gaps which is explained in the following section.

5.3   Proposed Model Framework

Figure 5.1 presents an overview of the proposed Generalized Stochastic AHP methodology for decision making. In step 1, the DM decides on the n criteria and m alternatives for the decision making problem. In step 2, the criteria are decomposed to appropriate sublevels and a hierarchy is proposed as is done in the traditional AHP methodology. Step 3 is the elicitation of the stochastic pairwise comparisons for all n criteria and m alternatives while Step 4 transforms these random variables into beta distributed pairwise comparisons (see section 5.3.1). Using the beta distributed pairwise comparisons; the priority weights for all PCMs are estimated using the proposed NLP

methodology in Step 5 (see section 5.3.2). Finally in step 6, the global priority of each alternative is calculated for decision making.
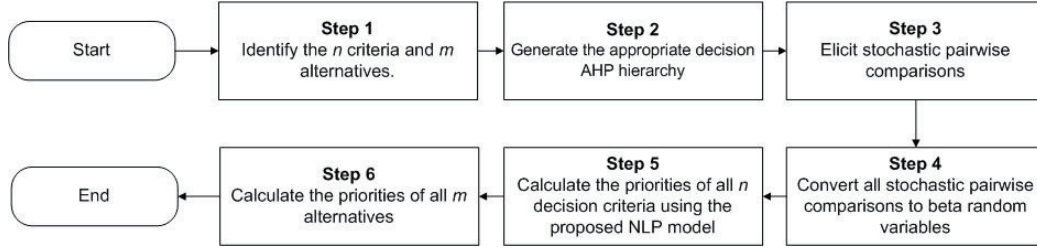
Figure 5.1. Proposed Generalized Stochastic AHP Methodology

Stochastic Pairwise Comparison Elicitation and Transformation

After identifying n criteria and m alternatives as well as the AHP decision hierarchy, this subsection presents the proposed modeling of the stochastic pairwise comparisons as beta distributions. We first formally define a stochastic pairwise comparison as follows:

Definition 5.1. *A stochastic pairwise comparison a_{ij} is a random variable that can take a set of possible crisp values that occur according to probability density function (PDF) $f_{ij}(a_{ij}|\theta_{ij})$ with parameters θ_{ij}*

These stochastic pairwise comparisons are observed from quantitative attribute data or qualitatively elicited from the DM. Furthermore, as shown in Eq. 5.1 these distributions are tabulated in a stochastic reciprocal pairwise comparison matrix (PCM) A as follows.

$$A = \begin{bmatrix} 1 & f_{12}(a_{12}|\theta_{12}) & \cdots & f_{ij}(a_{1n}|\theta_{1n}) \\ \frac{1}{f_{12}(a_{12}|\theta_{12})} & 1 & \cdots & f_{2n}(a_{2n}|\theta_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{f_{1n}(a_{1n}|\theta_{1n})} & \frac{1}{f_{2n}(a_{2n}|\theta_{2n})} & \cdots & 1 \end{bmatrix} \tag{5.1}$$

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Given any stochastic pairwise comparison a_{ij} a transformation algorithm which converts the probability distribution to a generalized Beta distribution is developed. Let us define a Beta distributed pairwise comparison $\tilde{a}_{ij}$ as follows:

Definition 5.2. *A stochastic pairwise comparison $\tilde{a}_{ij}$ is said to follow a beta distribution if its PDF (Probability Density Function) is defined as follows (Gupta & Nadarajah, 2004):*

$$B(\tilde{a}|\alpha,\beta,LL,UL) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(\tilde{a}-LL)^{\alpha-1}(UL-\tilde{a})^{\beta-1}}{(UL-LL)^{\alpha+\beta}} \tag{5.2}$$

where $LL \leq \tilde{a} \leq UL, \alpha, \beta \geq 1$, and $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ is a gamma function.

The beta distribution can generally be used to model other probability distributions due to the flexibility of the shape (α, β) and location (LL, UL) parameters. For example:

- in the case when $a_{ij} \sim U(LL_{ij}, UL_{ij})$, then a_{ij} can be modelled using the beta distribution with parameters $a_{ij} \sim B_{ij}(\tilde{a}_{ij}|1, 1, LL_{ij}, UL_{ij})$

- the triangular distribution and beta distribution can also be substituted for each other as seen in (Johnson, 2002) when $\alpha > 1, \beta > 1$, and $\alpha > \beta$ or $\alpha < \beta$ for skewed triangular distributions or $\alpha = \beta$ for symmetric triangular distributions.

Furthermore, the PDF of a beta distribution is differentiable over $[LL, UL]$ which would be beneficial for optimization (as compared to a triangular distribution) in succeeding sections. Additionally, the beta distribution has advantages in terms of Bayesian statistics which has closed form solutions. This benefit will come in handy when group decision making is considered in future research extensions of the methodology.

To explicitly model all a_{ij} as beta random variables $\tilde{a}_{ij}$, the shape $(\alpha_{ij}, \beta_{ij})$ and location (LL_{ij}, UL_{ij}) parameters need to be estimated. One of the commonly used methods to estimate distribution parameters is by the use of the maximum likelihood estimation methodology. However, for the beta distribution, there exists no closed form solution and as such the estimation of these parameters is difficult (Beckman & Jen,

1978). On the other hand, the method of moments (MOM) estimation methodology as is done in (AbouRizk et al., 1994; Owen, 2008) equates the first and second moments of the beta distribution to the sample mean and variance of the data sample. To illustrate this, given that if $\tilde{a}_{ij}$ is beta distributed with PDF and parameters $B(\tilde{a}_{ij}|\alpha, \beta, LL, UL)$ then the first moment of $\tilde{a}_{ij}$ is calculated as:

$$E[\tilde{a}_{ij}] = LL + \frac{\alpha}{\alpha + \beta}(UL - LL) \tag{5.3}$$

and the variance of $\tilde{a}_{ij}$ is calculated as:

$$Var(\tilde{a}_{ij}) = (UL - LL)\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \tag{5.4}$$

Equating equations 5.3 and 5.4 to the sample mean a $\bar{a}_{ij}$ and sample variance S_{ij}^2 respectively, and solving for α and β, we obtain the following closed form estimates of the shape parameters:

$$\hat{\alpha}_{ij} = \left(\frac{\bar{a}_{ij} - LL}{UL - LL}\right)\left(\frac{\left(\frac{\bar{a}_{ij} - LL}{UL - LL}\right)\left(1 - \left(\frac{\bar{a}_{ij} - LL}{UL - LL}\right)\right)}{\frac{S_{ij}^2}{(UL - LL)^2}} - 1\right) \tag{5.5}$$

$$\hat{\beta}_{ij} = \left(1 - \frac{\bar{a}_{ij} - LL}{UL - LL}\right)\left(\frac{\left(\frac{\bar{a}_{ij} - LL}{UL - LL}\right)\left(1 - \left(\frac{\bar{a}_{ij} - LL}{UL - LL}\right)\right)}{\frac{S_{ij}^2}{(UL - LL)^2}} - 1\right) \tag{5.6}$$

Algorithm 1 summarizes the MOM methodology when applied to transforming stochastic pairwise comparisons in an algorithm. The outputs of algorithm 1 are beta distributed pairwise comparisons.

Data:

  (i) $a_{ij} \sim f_{ij}(a_{ij}|\theta_{ij})$

Result: $\tilde{a}_{ij} \sim B(\hat{\alpha}_{ij}, \hat{\beta}_{ij}, LL_{ij}, UL_{ij})$
for *all* $a_{ij} \sim f_{ij}(a_{ij}|\theta_{ij})$ **do**

    if a_{ij} *is crisp* **then**
      | **return** $\tilde{a}_{ij} = a_{ij}$;
    else if $a_{ij} \sim U(LL_{ij}, UL_{ij})$ **then**
      | **return** $\tilde{a}_{ij} \sim B(\hat{\alpha}_{ij} = 1, \hat{\beta}_{ij} = 1, LL_{ij}, UL_{ij})$;
    else if $a_{ij} \sim T(LL_{ij}, ML_{ij}, UL_{ij})$ **then**
        $\bar{a}_{ij} = (LL_{ij} + ML_{ij} + UL_{ij})/3$;
        $\bar{S}_{ij}^2 = (LL_{ij}^2 + ML_{ij}^2 + UL_{ij}^2 - LL_{ij}ML_{ij} - LL_{ij}UL_{ij} - ML_{ij}UL_{ij})/18$;
        Compute $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ using equations 5.5 and 5.6 ;
        return $\tilde{a}_{ij} \sim B(\hat{\alpha}_{ij}, \hat{\beta}_{ij}, LL_{ij}, UL_{ij})$;
    else
        for $l = 1, 2, \cdots, 10,000$ **do**
        | Generate a random variate $x_l \sim f_{ij}(a_{ij}|\theta_{ij})$;
        end
        Compute for $\bar{a}_{ij} = \sum_{l=1}^{10,000} x_l/10,000$;
        Compute for $\bar{S}_{ij}^2 = \sum_{l=1}^{10,000} (x_l - \bar{a}_{ij})^2/9,999$;
        Compute $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ using equations 5.5 and 5.6 ;
        return $\tilde{a}_{ij} \sim B(\hat{\alpha}_{ij}, \hat{\beta}_{ij}, LL_{ij}, UL_{ij})$;
    end
end

Algorithm 1: Transformation of Stochastic Pairwise Comparisons to Beta Distributed Pairwise Comparisons Methodology

Computation of Weights for all Pairwise Comparison Matrices

      To estimate the crisp weights for each beta distributed PCM with n criteria or alternatives, we seek to find crisp pairwise comparison values $\tilde{x}_{ij} = \frac{w_i}{w_j}$ that maximize each beta likelihood probabilities $B(\tilde{a}_{ij} = \tilde{x}_{ij}|\alpha, \beta, LL, UL)$ and at the same time minimizes the inconsistency of the optimal crisp pairwise reciprocal comparison $n \times n$ matrix $\tilde{A}$. Conventionally, to analytically measure whether a crisp PCM $\tilde{A}$ is consistent, Saaty (1980) proposes the consistency index (CI). The CI is computed by obtaining the eigenvalue of $\tilde{A}$ using Eq. 5.7 as follows:

$$CI(\tilde{A}) = \frac{\lambda_{max} - n}{n - 1} \tag{5.7}$$

where n is the dimension of the PCM $\tilde{A}$ and λ_{max} is its maximum eigenvalue. The consistency ratio (CR) is the ratio of CI and RI and is computed using Eq. 5.8 as follows:

$$CR(\tilde{A}) = \frac{CI(\tilde{A})}{RI(n)} \tag{5.8}$$

where $RI(n)$ is the random index obtained from the average CI of 500 randomly filled matrices and is a function of n.

In this research, the CR is used as the measure of inconsistency. A quantifiable relationship is developed to relate the weights w_i, the crisp pairwise comparisons $\tilde{x}_{ij}$, and the CR of $\tilde{A}$. Following the logic of Y. M. Wang et al. (2005a) and by using the conventional AHP methodology for computing weights, the weights of the crisp PCM $\tilde{A}$ is computed as follows:

$$\tilde{A}w = \lambda_{max}w \tag{5.9}$$

where the vector of weights w corresponds to the eigenvector corresponding to the largest eigenvalue of $\tilde{A}$. Rewriting equations 5.7 and 5.8 we have:

$$\lambda_{max} = n + (n-1)RI(n)CR(\tilde{A}) \tag{5.10}$$

and substituting 5.9 and 5.10 we have:

$$\tilde{A}w = \Big[n + (n-1)RI(n)CR(\tilde{A})\Big]w$$
$$\tilde{A}w = \Big[n + kCR(\tilde{A})\Big]w \tag{5.11}$$

Where $k = (n-1)RI(n)$. Given this, we obtain a relationship between the crisp pairwise comparisons $\tilde{x}_{ij}$, the weights w_i and the $CR(\tilde{A})$ of PCM $\tilde{A}$. Thus an NLP formulation is proposed as follows:

Let the decision variables be:

$\tilde{x}_{ij} \qquad (i,j) \in \tilde{A}$ be the observed PC value from distribution $\tilde{a}_{ij}$

$w_i \qquad i \in \tilde{A}$ weight of element i

$CR(\tilde{A})$ be the CR of $\tilde{A}$

Objective Function:

$$\max \quad \sum_{i=1}^{n}\sum_{j>1}^{n} f(\tilde{x}_{ij}|\alpha_{ij},\beta_{ij},LL_{ij},UL_{ij}) - \lambda CR(\tilde{A}) \tag{5.12}$$

Constraints:

$$\sum_{j=1}^{i-1}\frac{w_i}{\tilde{x}_{ij}} - \left[n + kCR(\tilde{A})\right]w_i + \sum_{j=i+1}^{n}\tilde{x}_{ij}w_j = 0 \qquad \forall i \in \tilde{A} \tag{5.13}$$

$$\sum_{i=1}^{n} w_i = 1 \tag{5.14}$$

$$LL_{ij} \leq \tilde{x}_{ij} \qquad \forall(i,j),(i<j) \in \tilde{A} \tag{5.15}$$

$$UL_{ij} \geq \tilde{x}_{ij} \qquad \forall(i,j),(i<j) \in \tilde{A} \tag{5.16}$$

$$CR(\tilde{A}) \geq 0 \tag{5.17}$$

$$w_i \geq 0 \qquad \forall i \in \tilde{A} \tag{5.18}$$

where Eq. 5.12 seeks to find crisp $\tilde{x}_{ij}$ pairwise comparison values that maximize the probability of the distribution of the ij^{th} beta distributed pairwise comparison $\tilde{a}_{ij}$ and minimizing the corresponding $CR(\tilde{A})$ of PCM $\tilde{A}$. The λ parameter serves as a weighting parameter to obtain a consistent crisp pairwise comparison matrix. Equations sets 5.13 are a direct extension of equation 5.11 and relates all $\tilde{x}_{ij}$ to the weights of w_i and w_j and $CR(\tilde{A})$. Equation 5.14 ensures that all weights sum to unity while equation sets 5.15 and 5.16 forces all crisp pairwise comparisons to be within the limits of the distributions. Equation 5.17 forces the inconsistency of $\tilde{A}$ to be ≥ 0 and equation sets 5.18 sets all weights to be non-negative.

The outputs of the NLP methodology are crisp pairwise comparisons and weights from the stochastic PCM. The NLP methodology is repeated for all beta distributed pairwise comparison matrices of a given decision hierarchy. The final weights of all decision alternatives are then obtained by calculating for the weighted scores of all alternatives as is done in the traditional AHP methodology.

Illustrative Example

To illustrate the proposed methodology, consider a three-level MCDM problem adapted from Islam et al. (1997b). The decision problem is composed of a single goal G, four decision criteria (1, 2, 3 and 4) and four alternatives (A, B, C and D). A total of 30 interval pairwise comparisons are elicited and as such the hierarchy is summarized in Figure 5.2.
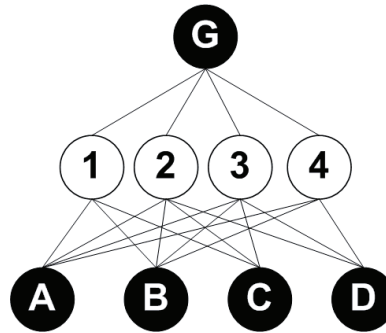
Figure 5.2. Example Hierarchy Adapted from Islam et al. (1997b)

Since all pairwise comparisons are interval based (uniformly distributed), we convert some of the pairwise comparisons to triangular distributions with the same lower limits and upper limits and some into crisp comparisons. These modified comparisons are summarized in Eqs. 5.19 to 5.23. Specifically, stochastic pairwise comparison matrices of all four alternatives for each criterion is summarized in Eqs. 5.19 to 5.22 while the stochastic PCM for all four criteria is presented in Eq. 5.23.

$$
A_1 = \begin{pmatrix} 1 & U(1/4, 1/3) & U(3, 4) & U(1/6, 1/5) \\ & 1 & T(6, 6.75, 7) & U(1/5, 1/4) \\ & & 1 & U(1/7, 1/6) \\ & & & 1 \end{pmatrix} \tag{5.19}
$$

$$A_2 = \begin{pmatrix} 1 & T(3,3.5,4) & U(4,5) & U(6,7) \\ & 1 & T(3,3.5,4) & U(5,6) \\ & & 1 & U(4,5) \\ & & & 1 \end{pmatrix} \tag{5.20}$$

$$A_3 = \begin{pmatrix} 1 & 1 & T(1/6,1/5.5,1/5) & U(1/4,1/3) \\ & 1 & U(1/6,1/5) & U(1/4,1/3) \\ & & 1 & U(4,5) \\ & & & 1 \end{pmatrix} \tag{5.21}$$

$$A_4 = \begin{pmatrix} 1 & U(3,4) & 6 & U(6,7) \\ & 1 & T(3,3.75,4) & U(3,4) \\ & & 1 & U(3,4) \\ & & & 1 \end{pmatrix} \tag{5.22}$$

$$A_C = \begin{pmatrix} 1 & T(3,3.5,4) & U(5,6) & U(6,7) \\ & 1 & T(4,4.5,5) & U(5,6) \\ & & 1 & 3.5 \\ & & & 1 \end{pmatrix} \tag{5.23}$$

Transformation procedure from section 5.3.1, specifically algorithm 1 is applied to transform all stochastic pairwise comparisons that are summarized in Eqs. 5.19 to 5.23 to beta distributed pairwise comparisons. The converted distributions are summarized in Eqs. 5.24 to 5.28. The transformed beta distributed pairwise comparison matrices of all four alternatives for each criterion is summarized in Eqs. 5.24 to 5.27 while the transformed beta PCM for all four criteria is presented in Eq. 5.28.

$$\tilde{A}_1 = \begin{pmatrix} 1 & B(1,1,1/4,1/3) & B(1,1,3,4) & B(1,1,1/6,1/5) \\ & 1 & B(2.56,1.83,6,7) & B(1,1,1/5,1/4) \\ & & 1 & B(1,1,1/7,1/6) \\ & & & 1 \end{pmatrix} \tag{5.24}$$

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$$\tilde{A}_2 = \begin{pmatrix} 1 & B(2.5, 2.5, 3, 4) & B(1, 1, 4, 5) & B(1, 1, 6, 7) \\ & 1 & B(2.5, 2.5, 3, 4) & B(1, 1, 5, 6) \\ & & 1 & B(1, 1, 4, 5) \\ & & & 1 \end{pmatrix} \tag{5.25}$$

$$\tilde{A}_3 = \begin{pmatrix} 1 & 1 & B(2.5, 2.5, 1/6, 1/5) & B(1, 1, 1/4, 1/3) \\ & 1 & B(1, 1, 1/6, 1/5)1.1 & B(1, 1, 1/4, 1/3) \\ & & 1 & B(1, 1, 4, 5) \\ & & & 1 \end{pmatrix} \tag{5.26}$$

$$\tilde{A}_4 = \begin{pmatrix} 1 & B(1, 1, 3, 4) & 6 & B(1, 1, 6, 7) \\ & 1 & B(2.56, 1.87, 3, 4) & B(1, 1, 3, 4) \\ & & 1 & B(1, 1, 3, 4) \\ & & & 1 \end{pmatrix} \tag{5.27}$$

$$\tilde{A}_C = \begin{pmatrix} 1 & B(2.5, 2.5, 3, 4) & B(1, 1, 5, 6) & B(1, 1, 6, 7) \\ & 1 & B(2.5, 2.5, 4, 5) & B(1, 1, 5, 6) \\ & & 1 & 3.5 \\ & & & 1 \end{pmatrix} \tag{5.28}$$

Next, we illustrate the application of the NLP methodology from section 5.3.2. The appropriate λ value is first determined for each stochastic pairwise comparison matrix. Here, we start the illustration of the NLP on matrix $\tilde{A}_C$ with $\lambda = 0$. This implies that only the preferences of the DM is maximized. However, upon solving the NLP, we obtain an inconsistent crisp pairwise comparison matrix with $CR(\tilde{A}) = 20.38\%$ which is way above the threshold of 10 %. Hence, we increase the value of λ incrementally and figure 5.3 presents the results of the $CR(\tilde{A})$ for different values of λ.

Hence, it is observed from figure 5.3 that for a value of $\lambda = 1e - 2$, we obtain a consistent matrix with $CR(\tilde{A}_C) = 9.49\%$. Therefore, we can keep on increasing the value
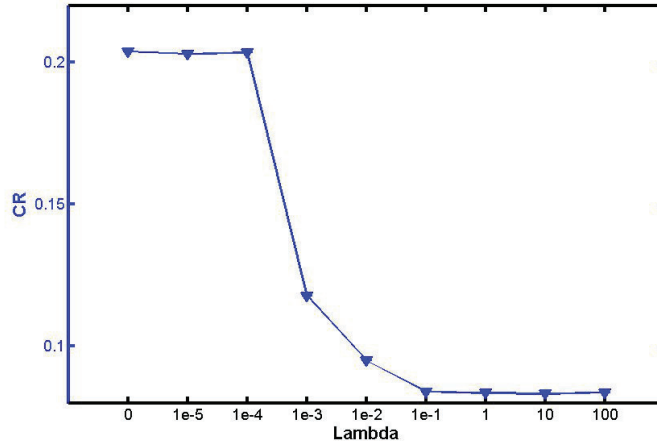
Figure 5.3. Matrix $\tilde{A}_C$ $CR(\tilde{A})$ values on Different λ Values

of λ until a target CR for the crisp pairwise comparison matrix $\tilde{A}$ is obtained. By applying the NLP methodology on all stochastic pairwise comparison matrices with their corresponding λ values, we obtain the optimal weights for each stochastic pairwise comparison matrix, and the corresponding weighted average scores. The results are presented in table 5.1.

Table 5.1. Crisp Weights Obtained Using the Proposed NLP Methodology

Criteria	$\tilde{A}_C$	Alternative				CR
		A	B	C	D	
$\tilde{A}_1$	55.10%	10.63%	25.13%	4.86%	59.37%	7.92%
$\tilde{A}_2$	30.01%	53.56%	29.32%	12.42%	4.70%	8.73%
$\tilde{A}_3$	10.05%	9.28%	8.78%	58.54%	23.40%	1.92%
$\tilde{A}_4$	4.85%	57.20%	26.49%	10.55%	5.75%	6.75%
Weighted Average:		25.64%	24.81%	12.80%	36.75%	

In this regard, the proposed methodology solves for consistent weights for each PCM since all $CR < 10\%$. Furthermore, alternative D is chosen to be the best alternative, followed by alternative A, B and lastly C.

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5.4   Comparison Experiments

In this section we compare the proposed methodology to established stochastic AHP methodologies. The first phase compares the proposed methodology to the simulation methodology of Banuelas & Antony (2004). This is done to determine the capability of modeling varying types of imprecise pairwise comparisons as beta distributions. The second phase compares the proposed NLP methodology to the goal programming methodology which maximizes the DM's preferences as proposed by Y. M. Wang & Elhag (2007).

Comparison Experiment on the Use of Beta Distributions

This phase is done to determine if there is significant difference when the untransformed stochastic pairwise comparison distribution is used as is done in (Banuelas & Antony, 2004) to the proposed transformed beta distributions. The validation is done by sampling 10,000 crisp pairwise comparison sets from the dataset adapted from the complete AHP problem hierarchy of Islam et al. (1997b), specifically from Eqs. 5.19 to 5.23. These comparisons are compared to the corresponding 10,000 sampled crisp pairwise comparison sets obtained from the transformed beta distributions, specifically from Eqs. 5.24 to 5.27. The corresponding alternative weights for each are calculated using the traditional AHP eigenvector method. The Monte Carlo simulation is done in a Matlab 2012a environment. Table 5.2 summarizes the results of this validation phase.

Table 5.2. Simulation Results on Alternatives Weights Using the Raw and Beta Distributions

Alternative	Raw Distribution		Beta Distribution		Difference
	Mean	Std. Dev.	Mean	Std. Dev.	
A	25.44%	5.58E-04	25.46%	4.52E-04	-4.01E-4
B	24.21%	5.02E-04	24.11%	4.72E-04	9.63E-4
C	13.86%	2.51E-04	13.87%	2.43E-04	-4.41E-4
D	36.44%	5.97E-04	36.46%	5.61E-04	-2.47E-4

Based on Table 5.2, the weights obtained from the original stochastic pairwise

comparisons are similar to the weights obtained from beta distributed pairwise comparisons. Additionally, the ranks of the alternatives have not changed across methodologies. We conclude that the beta distribution generalizes the raw stochastic pairwise comparisons well and can henceforth be used as a generalizing distribution.

Comparison with Goal Programming Methodology of Y. M. Wang & Elhag (2007)

This subsection presents the results when the proposed methodology is compared to the goal programming methodology proposed by Y. M. Wang & Elhag (2007). The purpose of this phase is to determine the effectiveness of the proposed NLP methodology in estimating crisp weights. To do this, we define a new performance metric that measures accuracy of the weights.

Definition 5.3. *The accuracy of the weights is measured by the Weight Deviation Error (WDE):*

$$WDE = \|w(i) - w'(i)\|_2^2 \tag{5.29}$$

where $w'(i)$ is the mean weight vector obtained from the proposed stochastic algorithm and $w(i)$ is the weight vector obtained by using the traditional AHP eigenvector methodology on the crisp mean values of the stochastic pairwise comparisons.

We can interpret the WDE as the distance of the weights obtained from the mean of the optimally computed distribution to weights computed using the mean of the original preferences of the DM. A small WDE is desired which implies that the optimally computed weights are similar to the most likely weights of the stochastic PCM. The three datasets that are used in this subsection are: (1) a consistent uniformly distributed interval matrix from Sugihara et al. (2004), (2) an inconsistent interval matrix from Islam et al. (1997b), and (3) the triangular dataset from Banuelas & Antony (2004). Since the dataset from Banuelas & Antony (2004) has triangular distributions, only the upper limit and lower limit parameters are used as input to the methodology proposed by Y. M. Wang & Elhag (2007) which take only uniformly distributed comparisons. The

resulting weights are then compared to the proposed methodology weights which converts the triangular distributions to beta distributions.

The parameters of the raw stochastic pairwise comparisons from the datasets are varied $\pm 10\%$ of their original value. The WDE is computed for all replications using the proposed methodology of Y. M. Wang & Elhag (2007) and the proposed methodology. Ten thousand replicates are simulated in a Matlab 2012a environment. Furthermore, we define μ_w as the mean WDE when the goal programming methodology is used while μ_g is the mean when the NLP methodology is used. A paired t-test is applied to determine if there is significant difference between the methodologies with null hypothesis $H_0 : \mu_w = \mu_g$ and alternative hypothesis $H_1 : \mu_w \neq \mu_g$ at $\alpha = 0.05$. Table 5.3 summarizes the WDE and t-test results of this validation phase.

Table 5.3. Simulation Results of the Proposed Methodology and Y. M. Wang & Elhag (2007)

Dataset	Wang et al.		Proposed		t-Test	
	Mean	Std. Dev.	Mean	Std. Dev.	Result	p-Value
Sugihara et al. (2004)	8.24E-4	2.56E-4	1.50E-5	2.00E-5	$\mu_w \geq \mu_g$	0
Islam et al. (1997b)	3.92E-3	3.93E-4	3.16E-3	5.72E-4	$\mu_w \geq \mu_g$	0
Banuelas & Antony (2004)	1.09E-2	3.29E-4	8.43E-3	5.08E-4	$\mu_w \geq \mu_g$	0

It is observed from table 5.3 that the proposed methodology has lower WDE for all datasets. This implies that the proposed methodology provides weights that are more accurate and adhere to the most likely preferences of the DM. This can be attributed to the proposed objective function in Eq. 5.12 in which the probability likelihood of each realized pairwise comparison is maximized and at the same time minimizing the CR.

5.5   Conclusions and Future Work

One of the inherent limitations of the AHP when applied to multi criteria decision making problems is the issue of bounded rationality present with the decision maker (DM). Given this, the DM might provide varying types of pairwise comparison preferences modelled as different types of probability distributions. Existing methodologies focus on modeling this uncertainty with a single type of probability distribution which is not

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enough to model the varying preferences of the DM. Furthermore, existing methodologies are inefficient in calculating weights. These methodologies focus on maximizing the preferences of the DM or minimizing inconsistency but not simultaneously from a single type of pairwise comparison distribution. In this regard, this paper proposes the Generalized Stochastic AHP methodology to address these limitations by proposing to model the different stochastic pairwise comparisons as beta distributed pairwise comparisons. Moreover, a Non-Linear Programming (NLP) methodology is proposed to maximize the likelihood probability of pairwise comparisons and at the same time minimizing the inconsistency.

It is observed that the proposed methodology which uses beta distributions is comparable to the methodology proposed by Banuelas & Antony (2004). Moreover, the proposed methodology provides better weight estimates using the proposed NLP methodology that maximizes the preferences of the DM as compared to the methodology of Y. M. Wang & Elhag (2007) which uses a goal programming approach.

While promising, there are several future directions to further improve the model. The first one is to extend the model for group DMs. The beta distribution has interesting properties especially for Bayesian analysis in which additional information from multiple DMs to update a prior beta distribution to a posterior beta distribution. Secondly, a known issue of AHP is order effect, that is, when comparing two elements A and B, and there are conflicting preferences when the order A then B is used as compared to when the order of B then A (Hogarth & Einhorn, 1992). In the context of the pairwise comparison elicitation process, variability could be observed when the DM is asked to provide pairwise comparison values. It is our intention to explore the application of the Generalized Stochastic AHP methodology to address this effect.

Chapter 6

CONCLUSIONS AND FUTURE RESEARCH

A Pairwise Comparison Matrix (PCM) is an integral component of decision making methodologies: the Analytic Hierarchy Process (AHP) and the Analytic Network Process (ANP). These are used to determine relative weights of criteria and alternatives. However, PCMs suffer from several issues limiting their applications to large-scale decision problems, specifically: (1) the curse of dimensionality due to a large number of pairwise comparisons elicited from the decision maker (DM), (2) the issue of inconsistent pairwise comparisons and (3) the existence of uncertainty and limited cognition of DMs results to imprecise pairwise comparisons. This dissertation proposes a PCM framework for large-scale decision making to address these limitations in three phases as follows.

The PCM Decomposition Methodology (PDM) proposed in the first phase addresses the first two limitations. The PDM decomposes a PCM into smaller manageable subsets using binary integer programming (BIP) with inner dependency scores of elements. As a result, the number and the inconsistency of pairwise comparisons elicited are reduced. Since the subsets are disjoint, the most independent pivot element is selected to connect all disjoint subsets. Hence the inner dependencies of the elements are minimized within each subset. Using local priorities and the pivot element, global priorities are then estimated for the elements of the PCM. The PDM is applied to a three-level AHP problem to decompose the criteria PCM. Correlation of the criteria from alternative scores is used as an alternative to estimate the interdependencies of the criteria. The proposed methodology does indeed reduce the number of pairwise comparisons and the consistency ratio. Nevertheless, more pairwise comparisons is saved when the PDM is applied to the ANP methodology. The PDM can be applied to all cluster PCMs within the network since inner dependencies of the elements are elicited for each PCM.

The PDM suffers from a major limitation in which the number of subsets elicited from the DM is subjectively elicited. Hence, the second phase of this dissertation proposes

a methodology to address this limitation. A BIP is proposed which consists of a trade-off objective function balances the deviation error between the original PCM weights and the decomposed weights and the amount of dependence among elements. The proposed methodology is capable of (1) solving for the optimal number of PCM subsets, (2) assigning the PCM elements to all subsets, and (3) selecting the best pivot element to be assigned to all subsets. The BIP is applied to an AHP decision to demonstrate its effectiveness.

The third phase of this dissertation addresses imprecise pairwise comparisons. Existing methodologies address this by modeling crisp pairwise comparisons as fuzzy sets or a single type of probability distribution (e.g., uniform, triangular). However, one common issue faced by DMs is bounded rationality, that is, DMs have limited cognitive powers in specifying their preferences over multiple pairwise comparisons. Hence, the DM might be sure of some comparisons and unsure with others. In addition, computing appropriate weights for the alternatives and criteria from the imprecise comparisons is a must since the original PCM eigenvector methodology is no longer applicable. The third phase proposes a Generalized Stochastic AHP Decision Making methodology that uses beta distributions to represent a wide variety of distributions via a method-of-moments methodology. Furthermore, a Non-Linear Programming model is then developed that calculates weights which maximizes the preferences of the DM as well as minimizes the inconsistency simultaneously. Comparison experiments with established stochastic AHP methodologies are conducted using datasets collected from literature and results shows that the proposed stochastic methodology outperforms existing methodologies.

While promising, there are several future research directions that arise from this dissertation. One could explore an extension of using multiple pivot elements. This is done to further reduce the deviation errors when computing for global weights. Secondly, these three phases can be extended to model the dynamics of group decision making or multiple DMs. The beta distribution has interesting properties especially for Bayesian analysis in which additional information from multiple DMs to update a prior beta

distribution to a posterior beta distribution. Thirdly, another known issue of PCM elicitation is order effect, that is, when comparing two elements A and B, and there are conflicting preferences when the order A then B is used as compared to when the order of B then A (Hogarth & Einhorn, 1992). In the context of the pairwise comparison elicitation process, variability could be observed when the DM is asked to provide pairwise comparison values. An extension of the third phase could be of worthwile importance to address this limitation.

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APPENDIX A

Proofs of Propositions and Theorems

Proof of Proposition 3.1

Maximizing the value of $d(A^k)$ is determined by solving the following IP problem:

$$\max d(A^k) = \sum_{l=1}^{k} \binom{m_l}{2} = \sum_{l=1}^{k} \frac{m_l(m_l - 1)}{2} \tag{A.1}$$

Constraints:

$$\sum_{l=1}^{k} m_l \leq m + (k-1), \qquad\qquad l \in [1, k] \tag{A.2}$$

$$m_l \geq 2, \qquad\qquad l \in [1, k] \tag{A.3}$$

$$m_l \in \mathbb{Z}^k, \qquad\qquad l \in [1, k] \tag{A.4}$$

It is obvious that the integer program tends to assign all elements to a single subset while minimizing the assignment to other subsets. Therefore, Eq. A.3 would be a binding lower bound. And thus, the solution is $m_l = m + (k-1)$ for $l \in S$ and $m_{l'} = 2, \forall l' \in S \setminus \{l\}$ maximizes $d(A^k)$ where:

$$d(A^k) = \frac{(m - k + 1)(m - k)}{2} + k - 1 \tag{A.5}$$

Proof of Proposition 3.2

The proof for proposition 3.2 is similar to the proof of proposition 3.1 but the IP problem is set to minimize and the other extreme point is obtained. However, by solving the IP problem, if $\frac{m+(k-1)}{k} \in \mathbb{Z}$, then the solution of the IP problem would be equal to the LP relaxation problem where $m_l = \frac{m+(k-1)}{k}$ in which all elements are equally distributed among the k subsets. However, when $\frac{m+(k-1)}{k} \notin \mathbb{Z}$, then the solution of the LP relation of the IP problem is different and thus the solution would require round off values of $m_l = \left\lfloor \frac{m+(k-1)}{k} \right\rfloor$ for some $l \in S$ and $m_{l'} = \left\lceil \frac{m+(k-1)}{k} \right\rceil$ for all $l' \in S \setminus \{l\}$

Proof of Theorem 3.1

Part 1 (Lower Limit): Given that there are $m(m-1)/2$ required pairwise comparisons for an original PCM, the minimum number of pairwise comparisons reduced is bounded by:

$$D(A^k) \geq \frac{m(m-1)}{2} - \max d(A^k) \tag{A.6}$$

Based on proposition 3.1, by substituting the values $m_l = m + (k-1)$ for $l \in S$ and $m_{l'} = 2, \forall l' \in S \setminus \{l\}$ that maximizes $d(A^k)$, we have:

$$D(A^k) \geq \frac{m(m-1)}{2} - \left[\frac{(m-k+1)(m-k)}{2} + k - 1 \right] \tag{A.7}$$

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Simplifying Eq A.7 we have:

$$D(A^k) \geq \left\lceil (k-1)\left(m - \frac{k+2}{2}\right) \right\rceil \tag{A.8}$$

Part 2 (Upper Limit): Given that there are $m(m-1)/2$ required pairwise comparisons for the original PCM, the maximum number of pairwise comparisons reduced is bounded by:

$$D(A^k) \geq \frac{m(m-1)}{2} - \min d(A^k) \tag{A.9}$$

Based on proposition 3.1, by substituting the values $m_l = \frac{m+(k-1)}{k}$ if $\frac{m+(k-1)}{k} \in \mathbb{Z}$, we have:

$$D(A^k) \leq \frac{m(m-1)}{2} - \frac{k}{2}\left[\left(\frac{m+(k-1)}{k}\right)\left(\frac{m+(k-1)}{k} - 1\right)\right] \tag{A.10}$$

Simplifying Eq A.10 we have:

$$D(A^k) \leq \left\lfloor \frac{(k-1)(m-1)^2}{2k} \right\rfloor \tag{A.11}$$

Proof of Proposition 4.1

$Proof$: Since A is inconsistent, then there exists at least one inconsistent pairwise comparison $\tilde{a}_{ij} \in A$ where, $\tilde{a}_{ij} = a_{ij} + \epsilon$ and $\epsilon > 0$. Without loss of generality, consider c_p as pivot, where $p \neq i'$. The weights of A computed using the standard AHP methodology would be as follows:

$$w(A) = \begin{bmatrix} w(1)' & w(2)' & \cdots & w(m)' \end{bmatrix}^T \tag{A.12}$$

Since only criteria $c_{i'}$ and $c_{j'}$ are affected by the inconsistent comparison, the relative comparisons of the other criteria remain the same and by using criteria c_p as pivot, dividing all weights by $w(i = c_p)$ we have:

$$w(A) = \begin{bmatrix} \frac{w(1)}{w(i=c_p)} & \cdots & \frac{w(i')}{w(i=c_p)} + \epsilon_{ic_p} & \cdots & \frac{w(m)}{w(i=c_p)} \end{bmatrix}^T \tag{A.13}$$

Where ϵ_{ic_p} is the resulting deviation due to the inconsistency To compute for the global weights of the decomposed matrices using pivot criterion c_p we consider two possible cases:

(i.) Consider the case where $\tilde{a}_{ij} \notin A^k$. Since $\tilde{a}_{ij} \notin A^k$, then all A_l^k are perfectly consistent, we have:

$$w(A) = \begin{bmatrix} \frac{w(1)}{w(i=c_p)} & \cdots & \frac{w(i')}{w(i=c_p)} & \cdots & \frac{w(m')}{w(i=c_p)} \end{bmatrix}^T$$

$$\therefore WDE(A^k) = \sum_{i=1}^{m}(w(i) - w'(i))^2 = (\epsilon_{ic_p} + \epsilon_{jc_p})^2 \tag{A.14}$$

(ii.) Consider the case where $\tilde{a}_{ij} \in A^k$. Since $\tilde{a}_{ij} \in A^k$, then all A_l^k except one local matrix is perfectly consistent. Suppose $\tilde{a}_{ij} \in A_l^k$ then the local weights are as follows:

$$w(A_1^k) = \begin{bmatrix} \frac{w(1,1)}{w(i=c_p)} & \frac{w(2,1)}{w(i=c_p)} & \cdots & 1 & \cdots & \frac{w(m_1,1)}{w(i=c_p)} \end{bmatrix}^T$$

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$$w(A_l^k) = \left[\frac{w(i',l)}{w(i=c_p)} + \epsilon_{i'p_h} \quad \cdots \quad 1 \quad \cdots \quad \frac{w(m_1,1)}{w(i=c_p)} \right]^T$$

$$w(A_k^k) = \left[\frac{w(1,k)}{w(i=c_p)} \quad \frac{w(2,k)}{w(i=c_p)} \quad \cdots \quad 1 \quad \cdots \quad \frac{w(m_1,k)}{w(i=c_p)} \right]^T \qquad (A.15)$$

Using criterion c_p as pivot, we have:

$$w'(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \frac{w(2)}{w(i=c_p)} + \epsilon_{i'p_h} \quad \cdots \quad 1 \quad \cdots \quad \frac{w(m_1)}{w(i=c_p)} \right]^T \qquad (A.16)$$

Since the $dim(A_l^k) < dim(A)$, $\forall l \in C$, then $\epsilon_{i'p_h} \neq \epsilon_{i'p_h}$, $\forall i, j \in C$ then:

$$\therefore WDE(A^k) = \sum_{i=1}^{m} (w(i) - w'(i))^2$$

$$= (\epsilon_{ic_p} + \epsilon_{jc_p} - \epsilon_{i'p_h} - \epsilon_{j'p_h})^2 > 0 \blacksquare \qquad (A.17)$$

Proof of Proposition 4.2

$Proof$: Based on proposition 4.1, we have:

$$w(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} + \epsilon_{ic_p} \quad \cdots \quad \frac{w(m)}{w(i=c_p)} \right]^T \qquad (A.18)$$

(i.) Consider the case where $\tilde{a}_{ij} \notin A^k$ and $\tilde{a}_{ij} \notin A^{k+1}$. Since $\tilde{a}_{ij} \notin A^k$ and $\tilde{a}_{ij} \notin A^{k+1}$ then all A_l^k and A_l^{k+1} are perfectly consistent. For A^k we have:

$$w(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} \quad \cdots \quad \frac{w(m)}{w(i=c_p)} \right]^T$$

$$\therefore WDE(A^k) = \sum_{i=1}^{m} (w(i) - w'(i))^2 = (\epsilon_{ic_p} + \epsilon_{jc_p})^2 \qquad (A.19)$$

Additionally the weights for A^{k+1} are:

$$w(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} \quad \cdots \quad \frac{w(m)}{w(i=c_p)} \right]^T$$

$$\therefore WDE(A^{k+1}) = \sum_{i=1}^{m} (w(i) - w'(i))^2 = (\epsilon_{ic_p} + \epsilon_{jc_p})^2$$

$$\therefore WDE(A^k) = WDE(A^{k+1}) \qquad (A.20)$$

(ii.) Consider the case where $\tilde{a}_{ij} \in A^k$ and $\tilde{a}_{ij} \notin A^{k+1}$. Since $\tilde{a}_{ij} \in A^k$ then all A^k except one is perfectly consistent and all A_l^{k+1} are perfectly consistent. Suppose $\tilde{a}_{ij} \in A_l^k$, we have:

$$w(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} \epsilon_{i'c_p} \quad \cdots \quad \frac{w(m)}{w(i=c_p)} \right]^T$$

$$\therefore WDE(A^k) = \sum_{i=1}^{m} (w(i) - w'(i))^2$$

$$= (\epsilon_{ic_p} + \epsilon_{jc_p} - \epsilon_{i'c_p} - \epsilon_{j'c_p})^2 \qquad (A.21)$$

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Additionally the weights for A^{k+1} are:

$$w(A) = \left[\frac{w(1)'}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} \quad \cdots \quad \frac{w(m')}{w(i=c_p)} \right]^T$$

$$\therefore WDE(A^{k+1}) = \sum_{i=1}^{m}(w(i) - w'(i))^2 = (\epsilon_{ic_p} + \epsilon_{jc_p})^2$$

$$\therefore WDE(A^k) = WDE(A^{k+1}) \tag{A.22}$$

(iii.) Consider the case where $\tilde{a}_{ij} \in A^k$ and $\tilde{a}_{ij} \in A^{k+1}$. Since $\tilde{a}_{ij} \in A^k$ and $\tilde{a}_{ij} \in A^{k+1}$ then all except one from A^k and one from A^{k+1} are perfectly consistent. Suppose $\tilde{a}_{ij} \in A_l^k$ and $\tilde{a}_{ij} \in A_l^{k+1}$, then from proposition 1 we have:

$$w(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} + \epsilon_{i'c_p} \quad \cdots \quad \frac{w(m)}{w(i=c_p)} \right]^T$$

$$\therefore WDE(A^k) = \sum_{i=1}^{m}(w(i) - w'(i))^2$$

$$= (\epsilon_{ic_p} + \epsilon_{jc_p} - \epsilon_{i'c_p} - \epsilon_{j'c_p})^2 \tag{A.23}$$

Additionally the weights for A^{k+1} are:

$$w(A) = \left[\frac{w(1)}{w(i=c_p)} \quad \cdots \quad \frac{w(i')}{w(i=c_p)} + \epsilon'_{i'c_p} \quad \cdots \quad \frac{w(m)}{w(i=c_p)} \right]^T$$

$$\therefore WDE(A^k) = \sum_{i=1}^{m}(w(i) - w'(i))^2$$

$$= (\epsilon_{ic_p} + \epsilon_{jc_p} - \epsilon'_{i'c_p} - \epsilon'_{j'c_p})^2 \tag{A.24}$$

To compare $\epsilon_{i'c_p} - \epsilon_{j'c_p}$ and $\epsilon'_{i'c_p} - \epsilon'_{j'c_p}$ we study the dimensions of all the matrices of A_l^k and A_l^{k+1}. Since $dim(A_l^k) > dim(A_l^{k+1}), \forall l \in S$ and since the weights are computed as $(\sum_{i=1}^{m_l} a_{ij})/dim(A_l^k)$, then $\frac{\epsilon}{dim(A_l^k)} \leq \frac{\epsilon}{dim(A_l^{k+1})}$ then $\epsilon_{i'c_p} - \epsilon_{j'c_p} \leq \epsilon'_{i'c_p} - \epsilon'_{j'c_p}$. Then:

$$\therefore WDE(A^k) \leq WDE(A^{k+1}) \blacksquare \tag{A.25}$$

APPENDIX B

Matlab Code

```matlab
1  function [S ] = Set( ...
        A,R,k,q,d,numiter,alpha,beta,gamma,pivotassign,lpdeviation, opttype,c)
2  S.A = A;
3  S.R = R;
4  S.k = k;
5  S.q = q;
6  S.d = d;
7  S.numiter = numiter;
8  S.alpha = alpha;
9  S.beta = beta;
10 S.gamma = gamma;
11 S.pivotassign = pivotassign;
12 S.lpdeviation = lpdeviation;
13 S.opttype = opttype;
14 S.c = c
15 end
16
17 function [ R] = IterateCMACD3(S)
18 numiter=S.numiter;
19 i=1;
20 R.SDEP=numiter:1;
21 R.ACRP=numiter:1;
22 R.PSP=numiter:1;
23 R.TDSP=numiter:1;
24 R.SDEO=numiter:1;
25 R.ACRO=numiter:1;
26 R.PSO=numiter:1;
27 R.TDSO=numiter:1;
28
29 while i ≤ numiter
30         [  R.ACRP(i,1),R.ACRO(i,1), R.PSP(i,1), R.PSO(i,1),R.SDEP(i,1), ...
              R.SDEO(i,1),R.TDSP(i,1), R.TDSO(i,1)] = CMACD3(S);
31         disp(i);
32         i=i+1;
33 end
34 end
35
36 function [ACRP,ACRO, PSP, PSO,SEP, SEO,TDSP, TDSO] = CMACD3(S)
37 p=0.1;
38 A = GenerateRMatrix(S.A,p,1);
39 R = GenerateRMatrix(S.R,0.1,2);
40 pivotassign = S.pivotassign;
41 I = abs(corrcoef(R'))−eye(size(corrcoef(R'),1));
42 q=S.q;
43
44  %compute AHP Weights
45 [WOI,¬]=ComputeWeights(A,[],1,1);
46 %WOI = I*WO;
47
48 %Solve the BIP Formulation and Select Pivot
49 %Solve Optimal BIp Formulation
50 [SolRev,PivotSetRev ] = SolveBIPRev7(I,S.q,S.alpha,S.beta,S.gamma);
51 % SolRev = [1,0,0,1,1,0,1,1;0,0,1,1,0,1,0,0;0,1,0,1,0,0,0,0];
52 % PivotSetRev = 4;
53 k = size(SolRev,1);
54 %Solve Original BIP Formulation
55 [SolOrig ] = SolveBIP( I, k );
```

```
56  [SolOrig,PivotSetOrig] = SelectPivot( I, k,S.q,pivotassign,SolOrig );
57
58  [ TDSP ] = ComputeTDS( I, SolRev );
59  [ TDSO ] = ComputeTDS( I, SolOrig );
60
61  i=1;
62  WP=zeros(k,size(A,1));
63  CRP=zeros(k,1);
64  SRev=SolRev
65  while i≤k
66      Temp=GenerateMatrix(A,SolRev,i);
67      [WP(i,:),CRP(i,1)]=ComputeWeights(Temp,SolRev,i,2);
68      i=i+1;
69  end
70
71  i=1;
72  WO=zeros(k,size(A,1));
73  SOrigp=SolOrig
74  CRO=zeros(k,1);
75  while i≤k
76      Temp=GenerateMatrix(A,SolOrig,i);
77      [WO(i,:),CRO(i,1)]=ComputeWeights(Temp,SolOrig,i,2);
78      i=i+1;
79  end
80
81  WP=CombinePType3Clusters(WP,PivotSetRev');
82  WO=CombinePType3Clusters(WO,PivotSetOrig);
83  WP = WP';
84  WO =WO';
85
86  %Compute SSE
87  SEP=sum((WOI−WP).^2);
88  SEO=sum((WOI−WO).^2);
89  if isnan(SEP)==1
90      Test=1;
91  end
92  %Compute ACR
93  ACRP=sum(CRP.*sum(SolRev,2)./sum(sum(SolRev,2)));
94  ACRO=sum(CRP.*sum(SolOrig,2)./sum(sum(SolOrig,2)));
95
96  %Compute P(S)
97  ps=size(A,1)*(size(A,1)−1)/2;
98  PSP=ps−((sum(SolRev,2)−ones(k,1))')*sum(SolRev,2)/2 + q*(q−1)*(k−1)/2;
99  PSO=ps−((sum(SolOrig,2)−ones(k,1))')*sum(SolOrig,2)/2 + q*(q−1)*(k−1)/2;
100 end
101
102 function [ RN ] = GenerateRMatrix( A, p, type )
103 %Upper Random Triangular Matrix
104 if type == 1
105     RN= triu(rand(size(A,1),size(A,2)),1)+ eye(size(A,1)) ;
106     %Compute New Values
107     j=2;
108     i=1;
109     while j≤size(A,1)
110         while i<j
111             RN(i,j)=A(i,j)*(1−p)+(A(i,j)*(1+p)−A(i,j)*(1−p))*RN(i,j);
112             i=i+1;
113         end
```

```matlab
114         j=j+1;
115         i=1;
116     end;
117 %Compute Reciprocals
118     i=2;
119     j=1;
120     while i≤size(A,1)
121         while j<i
122             RN(i,j)=1/RN(j,i);
123             j=j+1;
124         end
125     i=i+1;
126     j=1;
127     end;
128 else
129     RN= rand(size(A,1),size(A,2));
130     j=1;
131     i=1;
132     while i≤size(A,1)
133         while j≤size(A,2)
134             RN(i,j)=A(i,j)*(1−p)+(A(i,j)*(1+p)−A(i,j)*(1−p))*RN(i,j);
135             j=j+1;
136         end
137         i=i+1;
138         j=1;
139     end;
140 end
141 end
142
143 function [W,CR] = ComputeWeights(A,S,r,type)
144 %Type=1 Compute weights of generic matrix
145 if type==1
146     W=mean((A.*(repmat(sum(A),size(A,1),1).^−1)),2);
147     RI=[0,0,0.58,0.9,1.12,1.24,1.32,1.41,1.45,1.49,1.51,1.53,1.55,1.57,1.58];
148     if size(A,1)==2
149         CR=0;
150     else
151         CR=(((dot(W,sum(A))−size(A,1)))/(size(A,1)−1))/RI(size(A,1));
152     end
153 else
154 %Type = 2 Compute Weights
155     W1=mean((A.*(repmat(sum(A),size(A,1),1).^−1)),2);
156     RI=[0,0,0.58,0.9,1.12,1.24,1.32,1.41,1.45,1.49,1.51,1.53,1.55,1.57,1.58];
157     if size(A,1)≤2
158         CR=0;
159     else
160         CR=(((dot(W1,sum(A))−size(A,1)))/(size(A,1)−1))/RI(size(A,1));
161     end
162     W=zeros(1,size(S,2));
163     i=1;
164     j=1;
165     while i ≤ size(S,2)
166         if S(r,i)==1
167             W(1,i)=W1(j,1);
168             j=j+1;
169         end
170         i=i+1;
171     end
```

```
172  end
173  end
174
175  function [  Sol,PivotSet ] = SolveBIPRev7(I,q,alpha,beta,gamma)
176  m = size(I,1);
177  k=m-1;
178  mtaken2 = m*(m-1)/2;
179  %──────────Generate F
180
181  H=I;
182  H=(H+H')/2;
183  H=H-tril(H,0);
184  %H=H*-1;
185  i=1;
186  %Xcoefficients
187  F = zeros(1,m*k);
188
189  %Y Coefficients Distribute Upper Traig B into 1-d F
190  Ftemp=H(i,i+1:size(H,1));
191  i=i+1;
192  while i<m
193      Ftemp=[Ftemp H(i,i+1:size(H,1))];
194      i=i+1;
195  end;
196  Ycoef = Ftemp*alpha;
197  Ftemp = repmat(Ftemp*alpha,1,k);
198  F=[F Ftemp];
199
200  %S^F and Z Coefficients
201  F=[F zeros(1,k+m)];
202
203  %k^l Coefficients
204  ktemp = zeros(1,m-2);
205  i=2;
206  while i ≤ k
207      ktemp(1,i-1) = (i-1)*((m-q)^2)/(2*i);
208      i=i+1;
209  end
210
211  %Coeff 0.7= onut, 0.9 = lee, 0.6 alharbi
212  ktemp = ktemp*beta*0.7;
213  F=[F ktemp];
214
215  %Wil coefficients
216  F=[F zeros(1,(k-1)*m)];
217
218  %mil coefficients
219  ftempgamma=0;
220  i=2;
221  while i ≤ k
222      ftempgamma = [ftempgamma i*(i-1)/2];
223      i=i+1;
224  end
225  ftempgamma = ftempgamma*gamma*-.75;
226  ftemp = ftempgamma;
227
228  F=[F repmat(ftemp,1,k)];
229
```

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```matlab
230  %Transpose row vector to columns vector
231  F=F';
232
233  %———————————Generate A
234  %yii'l constraints xi(1)
235  A = GenerateXCoefficients1(m);
236  A = [A eye(mtaken2*k)];
237  A = [A zeros(mtaken2*k,k+m+k-1+(k-1)*m+k*(m-1))];
238
239  %yii'l constraints xi' (2)
240  Atemp = GenerateXCoefficients2(m);
241  Atemp = [Atemp eye(mtaken2*k)];
242  Atemp = [Atemp zeros(mtaken2*k,k+m+k-1+(k-1)*m+(k)*(m-1))];
243  A=[A;Atemp];
244
245  %yii'l constraints xi and xi' (3)
246  Atemp = GenerateXCoefficients3(m);
247  Atemp = [Atemp eye(mtaken2*k)*-1];
248  Atemp = [Atemp zeros(mtaken2*k,k+m+k-1+(k-1)*m+(k)*(m-1))];
249  A=[A;Atemp];
250
251
252  %constraint (4) Xsum accros i for all l ≥ q*kopen
253  Atemp=[];
254  Atempdiag=ones(1,m);
255  i=2;
256  while i≤k
257      Atempdiag =blkdiag(Atempdiag, ones(1,m));
258      i=i+1;
259  end
260  Atemp=[Atemp Atempdiag*-1];
261  Atemp=[Atemp zeros(k,mtaken2*k)];
262  Atemp=[Atemp eye(k)*(1+q)];
263  Atemp=[Atemp zeros(k,m+k-1+(k-1)*m+(k)*(m-1))];
264  A=[A;Atemp];
265
266  %constraint (5) Xsum accros i for all l ≤ m*kopen
267  Atemp=[];
268  Atemp=[Atemp Atempdiag];
269  Atemp=[Atemp zeros(k,mtaken2*k)];
270  Atemp=[Atemp eye(k)*-m];
271  Atemp=[Atemp zeros(k,m+k-1+(k-1)*m+(k)*(m-1))];
272  A=[A;Atemp];
273
274  %constraint (6) Xsum accros l for all i ≥1
275  Atemp=[];
276  Atemp=[Atemp repmat(eye(m),1,k)*-1];
277  Atemp=[Atemp zeros(m,mtaken2*k+k+m+k-1+(k-1)*m+(k)*(m-1))];
278  A=[A;Atemp];
279
280  % wil constraints, (8)
281  Atemp = zeros(m*(k-1),m*k+mtaken2*k+k);
282  Atemp = [Atemp repmat(eye(m),k-1,1)*-1];
283  Atemp = [Atemp zeros(m*(k-1),k-1)];
284  Atemp = [Atemp eye((k-1)*m)];
285  Atemp = [Atemp zeros(m*(k-1),(k)*(m-1))];
286  A=[A;Atemp];
287
```

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```matlab
288  % wil constraints, (7)
289  Atemp = zeros(m*(k-1),m*k+mtaken2*k+k+m);
290  Atemp1=ones(m,1)*-1;
291  i=1;
292  while i < k-1
293      Atemp1 = blkdiag(Atemp1, ones(m,1)*-1);
294      i=i+1;
295  end
296  Atemp = [Atemp Atemp1];
297  Atemp = [Atemp eye(m*(k-1))];
298  Atemp = [Atemp zeros(m*(k-1),(k)*(m-1))];
299  A=[A;Atemp];
300
301  % wil constraints, (9)
302  Atemp = zeros(m*(k-1),m*k+mtaken2*k+k);
303  Atemp = [Atemp repmat(eye(m),k-1,1)];
304  Atemp = [Atemp Atemp1*-1];
305  Atemp = [Atemp eye(m*(k-1))*-1];
306  Atemp = [Atemp zeros(m*(k-1),(k)*(m-1))];
307  A=[A;Atemp];
308
309  Atemp = Atempdiag;
310  Atemp = [Atemp zeros(k,mtaken2*k+k+m)];
311  i=1;
312  while i <= k-1
313      Atemp10(1,i)=(m+q*i)/(i+1);
314      i=i+1;
315  end
316  Atemp10 = repmat((ceil( Atemp10))*-1,k,1);
317  Atemp = [Atemp  Atemp10];
318  Atemp = [Atemp zeros(k,(k-1)*m+k*k)];
319  A=[A;Atemp];
320
321  %——————————Generate Aeq
322
323  %Constaint (11) - relate X and Z and wil
324  Aeq=[];
325  Aeqtemp=[];
326  Aeqtemp=[Aeqtemp repmat(eye(m),1,k)];
327  Aeqtemp=[Aeqtemp zeros(m,mtaken2*k+k)];
328  Aeqtemp=[Aeqtemp eye(m)];
329  Aeqtemp=[Aeqtemp zeros(m,k-1)];
330  %——————Generate coefficients of wil
331  i=3;
332  Aeqtemp1=eye(m)*-2;
333  while i <=k
334      Aeqtemp1 =[Aeqtemp1 eye(m)*-i];
335      i=i+1;
336  end
337  Aeqtemp=[Aeqtemp Aeqtemp1];
338  Aeqtemp=[Aeqtemp zeros(m,(k)*(m-1))];
339  Aeq = [Aeq; Aeqtemp];
340
341  %Constaint (12) sum of zi = q
342  Aeqtemp=zeros(1,m*k+mtaken2*k+k);
343  Aeqtemp=[Aeqtemp ones(1,m)];
344  Aeqtemp=[Aeqtemp zeros(1,k-1+(k-1)*m +(k)*(m-1))];
345  Aeq = [Aeq; Aeqtemp];
```

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```matlab
346
347   %Constaint (13) sum of kopen = l*kl
348   Aeqtemp=zeros(1,m*k+mtaken2*k);
349   Aeqtemp=[Aeqtemp ones(1,k)];
350   Aeqtemp=[Aeqtemp zeros(1,m)];
351   Aeqtemp=[Aeqtemp (2:k)*-1];
352   Aeqtemp=[Aeqtemp zeros(1,m*(k-1)+(k)*(m-1))];
353   Aeq = [Aeq; Aeqtemp];
354
355   %Constaint (14) sum of k= 1
356   Aeqtemp=zeros(1,m*k+mtaken2*k+k+m);
357   Aeqtemp=[Aeqtemp ones(1,k-1)];
358   Aeqtemp=[Aeqtemp zeros(1,m*(k-1)+(k)*(m-1))];
359   Aeq = [Aeq; Aeqtemp];
360
361   %Constaint (15) sum of xil = imil
362   Aeqtemp=Atempdiag;
363   Aeqtemp=[Aeqtemp zeros(k,mtaken2*k+k+m+k-1+m*(k-1))];
364   Aeqtemp1=[];
365   i=1;
366   Aeqtemp2 = [0 (2:m-1)*-1];
367   while i <= k
368       Aeqtemp1 = blkdiag(Aeqtemp1, Aeqtemp2);
369       i=i+1;
370   end
371   Aeqtemp=[Aeqtemp Aeqtemp1];
372   Aeq = [Aeq; Aeqtemp];
373
374   %Constaint (16) sum of xil  = 1
375   Aeqtemp=[];
376   Aeqtemp=[Aeqtemp zeros(k,m*k+mtaken2*k+k+m+k-1+m*(k-1))];
377   Aeqtemp1=[];
378   i=1;
379   while i <= k
380       Aeqtemp1 = blkdiag(Aeqtemp1, ones(1,k));
381       i=i+1;
382   end
383   Aeqtemp=[Aeqtemp Aeqtemp1];
384   Aeq = [Aeq; Aeqtemp];
385   %---------------generate Beq
386   Beq=[];
387   Beq = [Beq; ones(m,1)];
388   Beq = [Beq; q];
389   Beq = [Beq; 0];
390   Beq = [Beq; 1];
391   Beq = [Beq; zeros(k,1)];
392   Beq = [Beq; ones(k,1)];
393   %Beq = [Beq; 1];
394   %-----------------Generate B
395   B=zeros(mtaken2*k*2,1);
396   B=[B;ones(mtaken2*k,1)];
397   B=[B;zeros(k*2,1)];
398   B=[B;ones(m,1)*-1];
399   B=[B;zeros((k-1)*m*2,1)];
400   B=[B;ones((k-1)*m,1)];
401   B=[B;zeros(k,1)];
402
403   [x,fval] = cplexbilp(F,A,B,Aeq,Beq);
```

```matlab
404  Soltemp =reshape(x(1:m*k),m,k)';
405
406  %Clean Up Solution
407  i=1;
408  j=1;
409  while i ≤ size(Soltemp,1)
410      while j ≤ size(Soltemp,2)
411          if Soltemp(i,j) < 0.5
412              Soltemp(i,j)=0;
413          else
414              Soltemp(i,j)=1;
415          end
416          j=j+1;
417      end
418      j=1;
419      i=i+1;
420  end
421
422  Z = x(m*k+mtaken2*k+k+1:m*k+mtaken2*k+k+m,1)';
423  Sol = Soltemp;
424  Sol(¬any(Soltemp,2),:)=[];
425
426  %Clean Z
427  i=1;
428  while i ≤ size(Z,2)
429      if Z(1,i) <0.5
430          Z(1,i)=0;
431      else
432          Z(1,i)=1;
433      end
434      i=i+1;
435  end
436  i=1;
437
438  PivotSet = find(Z);
439
440  if size(PivotSet, 2) >1
441      test=1;
442  end
443
444  yiil = x(m*k+1:m*k+mtaken2*k,1);
445  yiil = reshape(x(m*k+1:m*k+mtaken2*k,1),k,mtaken2);
446  Ycoef = repmat(Ycoef,k,1);
447  fvalyiil = sum(sum(yiil.*Ycoef));
448
449  kopen = x(m*k+mtaken2*k+1:m*k+mtaken2*k+k,1)';
450
451  kval = x(m*k+mtaken2*k+k+m+1:m*k+mtaken2*k+k+m+k−1,1)';
452  kfval = kval*ktemp';
453  mil = reshape(x(m*k+mtaken2*k+k+m+k−1+(k−1)*
454  m+1:m*k+mtaken2*k+k+m+k−1+(k−1)*m+k*(m−1),1)',k,k)';
455  cmilcoeff = repmat(ftemp, k,1);
456  fvalmil = sum(sum(mil.*cmilcoeff));
457  end
458
459  function [A] = GenerateXCoefficients1(m)
460  mcurr = m−1;
461  Atemp =[];
```

```
462  while mcurr ≥ 0
463      Atemp = blkdiag(Atemp,ones(mcurr,1)*−1);
464      mcurr = mcurr−1;
465  end
466  A=Atemp;
467  i=2;
468  while i < m
469      A = blkdiag(A,Atemp);
470      i=i+1;
471  end
472  end
473
474  function [A] = GenerateXCoefficients2(m)
475  mcurr = m−1;
476  A=[];
477  Atemp =[];
478  while mcurr ≥ 0
479      Atemp = zeros(mcurr,m−mcurr);
480      Atemp = [Atemp eye(mcurr)*−1];
481      A = [A; Atemp];
482      mcurr = mcurr − 1;
483  end
484  Atemp=A;
485  i=2;
486  while i < m
487      A = blkdiag(A,Atemp);
488      i=i+1;
489  end
490  end
491
492  function [ A ] = GenerateXCoefficients3(m)
493  mcurr = m−1;
494  A =[];
495  while mcurr > 0
496
497      Atemp = zeros(mcurr,m−1−mcurr);
498      Atemp = [Atemp ones(mcurr,1)];
499      Atemp = [Atemp eye(mcurr)];
500      A = [A;Atemp];
501      mcurr = mcurr − 1;
502  end
503  Atemp = A;
504  i=2;
505  while i < m
506      A = blkdiag(A,Atemp);
507      i=i+1;
508  end
509  end
510
511  function [ Sol ] = SolveBIP(I,k)
512  %Preprare Interdependence Metrix
513  I = I + I';
514  I = I − diag(diag(I,0));
515
516  %Generate Hessian
517  H=I;
518  i=1;
519  while i<k
```

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```matlab
520      H=blkdiag(H,I);
521      i=i+1;
522  end
523  H=(H+H')/2;
524
525  %Generate F
526  F=zeros(size(I,1)*k,1);
527
528  %Generate First Set Constraint (Assign Criteria to only 1 Cluster)
529  Aeq  = eye(size(I,1));
530  i=1;
531  while i<k
532      Aeq=[Aeq eye(size(I,1))];
533      i=i+1;
534  end
535  Beq = ones(size(I,1),1);
536  lb = zeros(size(I,1)*k,1);
537  ub = ones(size(I,1)*k,1);
538
539  %Solve BIP Formulation
540  opts = optimset('Algorithm','active-set','Display','off');
541  X =quadprog(H,F,[],[],Aeq,Beq,lb,ub,[], opts);
542  m=size(I,1);
543
544  %Generate Matrix Solution Sol
545  Sol=zeros(k,m);
546  i=1;
547  while i≤k
548      Sol(i,1:m)=(X(1+m*(i-1):m+m*(i-1),1))';
549      i=i+1;
550  end
551
552  %Clean Soluton for 0.0000
553  i=1;
554  j=1;
555  while i ≤ size(Sol, 1)
556      while j ≤ size(Sol,2)
557          if Sol(i,j) >0.5
558              Sol(i,j) = 1;
559          else
560              Sol(i,j) = 0;
561          end
562          j=j+1;
563      end
564      j=1;
565      i=i+1;
566  end
567  end
568
569  function [NewSol,PivotSet ] = SelectPivot( I, k,q,pivotassign,Sol )
570  %Include Pivot Element in Sol
571  %Type 1 = CMACD, Type 2 = CMACD2
572  if pivotassign == 1
573      i=1;
574      j=1;
575      l=1;
576      m=1;
577      n=1;
```

```matlab
578        NewSol = Sol;
579        DS=zeros(1,size(I,1));
580        %Generate Criteria Set
581        CSet = 1:size(I,1);
582        PivotSet = zeros(q,1);
583        while l ≤q
584            while i ≤ k
585                while j≤size(I,1)
586                    if Sol(i,j)==1
587                        DS(1,j)= CalculateDS(i,j,Sol,I);
588                    end
589                    j=j+1;
590                end
591                j=1;
592                i=i+1;
593            end
594            %Generate Temporary Variables
595            TempDS = 1:size(I,1);
596            TempDS(2,:) = DS;
597            TempDS2 = TempDS;
598            %Remove Pivot Criteria Option
599            while n ≤ q
600                while m ≤size(TempDS2,2)
601                    if PivotSet(n,1)==TempDS2(1,m)
602                        TempDS2(:,m)=[];
603                    end
604                    m=m+1;
605                end
606                m=1;
607                n=n+1;
608            end
609            n=1;
610            m=1;
611            %Get Index of Minimum
612            [¬,Index] = min(TempDS2(2,:));
613            %Insert Minimum Criterion Using Index to Pivot Set
614            PivotSet(l,1)= TempDS2(1,Index);
615            %Assign to all Clusters
616            while m ≤k
617                Sol(m,TempDS2(1,Index))=1;
618                m=m+1;
619            end
620            m=1;
621            i=1;
622            l=l+1;
623        end
624        NewSol = Sol;
625    elseif pivotassign == 2
626        %Determine First Cluster to Select Pivot
627        NewSol = Sol;
628        MaxSolution  = DetermineBiggestMat(NewSol);
629        PivotSet = zeros(q,1);
630        j=1;
631        while j≤q
632            %Initialize
633            Dist = zeros(size(MaxSolution,1),1);
634            MinimumCriterion = zeros(size(MaxSolution,1),1);
635
```

```matlab
636            %Compute Distances and Find Minimum Distance of Similar Sized ...
                   Clusters
637            i=1;
638            while i≤size(MaxSolution,1)
639                [ Dist(i,1), MinimumCriterion(i,1)] = FindMinimumCriteria( ...
                       Sol,i,MaxSolution,I,PivotSet);
640                i=i+1;
641            end
642            [¬,temp] = min(Dist);
643            PivotSet(j,1) = MinimumCriterion(temp,1);
644
645            %Assign Pivot Criterion to Clusters
646            i=1;
647            while i≤k
648                NewSol(i,PivotSet(j,1)) = 1;
649                i=i+1;
650            end
651            %Select jth Cluster to SelectPivot
652            MaxSolution  = DetermineBiggestMat(NewSol);
653        j=j+1;
654        end
655  end
656  end
657
658  function [DS ] = CalculateDS( x,y,Sol,I )
659  DS=0;
660  i=1;
661  j=1;
662  m = size(Sol, 2);
663  k = size(Sol, 1);
664  while i≤k
665      if x ≠ i
666          while j≤ m
667              if Sol(i,j) == 1
668                  DS =DS+ I(y,j);
669              end
670              j=j+1;
671          end
672
673      end
674
675      j=1;
676      i=i+1;
677  end
678  end
679
680  function [ TDS ] = ComputeTDS( B, Sol )
681  TDS=0;
682  k=size(Sol,1);
683      K=1;
684          m=1;
685          while K≤k
686          while m≤ size(B,1)
687              if Sol(K,m)≠0
688                  i=m;
689                  j=m+1;
690                  while j≤ size(B,1)
691                      if Sol(K,j)≠0
```

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```matlab
692                          TDS= TDS+ B(i,j);
693                      end
694                      j=j+1;
695                  end
696              end
697
698          m=m+1;
699      end
700      m=1;
701      K=K+1;
702      end
703 end
704
705 function [ Acomp ] = GenerateMatrix(A,CiC,r)
706 %Generate Matrix from solution CiC query from A with row number r)
707 %Return all Non Zero Elements of CiC on Row r
708 indeces=find(CiC(r,1:size(A,1)));
709  %Generate Blank
710 Ainc=ones(size(indeces,2),size(indeces,2));
711 i=1;
712 j=1;
713
714 %Copy Values from A to Ci
715     while i≤size(indeces,2)
716         while j≤size(indeces,2)
717             Ainc(i,j)=A(indeces(1,i),indeces(1,j));
718             j=j+1;
719         end
720         j=1;
721         i=i+1;
722     end
723 Acomp = Ainc;
724 end
725
726 function [ W ] = CombinePType3Clusters( WP,P2 )
727 i=1;
728 while i ≤size(P2,1);
729     GMean(i,:)=CombineClusters3(WP,P2(i,1));
730
731     i=i+1;
732 end
733 W = geomean(GMean,1);
734 End
735
736 function [ W] = CombineClusters3( WP,Pivot )
737 i=1;
738 wtemp = WP;
739 while i ≤ size(WP, 1)
740     wtemp(i,:)= WP(i,:)./WP(i,Pivot);
741     i=i+1;
742 end
743
744 i=1;
745 j=1;
746 ws=0;
747 W = zeros(1,size(WP,2));
748 while i ≤size(WP,2)
749     if nnz(WP(:,i))== 1
```

```matlab
750          while j ≤size(WP,1);
751              if WP(j,i)≠0
752                  ws=ws+wtemp(j,i);
753                  W(1,i)=wtemp(j,i);
754              end
755              j=j+1;
756          end
757          j=1;
758      else
759          W(1,i)=geomean(wtemp(:,i));
760          ws=ws+W(1,i);
761      end
762      i=i+1;
763  end
764  W = W ./ ws;
765  W = W ./ sum(W);
766  end
767
768  function [ Results ] = CompleteCompare( R1,R2 )
769  Results.ACR = Compare( R1.ACR,R2.ACR);
770  Results.PS = Compare( R1.PS,R2.PS);
771  Results.SDE = Compare( R1.SDE,R2.SDE);
772  Results.TDS = Compare( R1.TDS,R2.TDS);
773  end
774
775  function [ Results] = Compare( X1,X2)
776  Results = zeros(size(X1,2),6);
777  i=1;
778  while i ≤ size(X1,2)
779      %Perform T-Test
780      Results(i,1) = mean(X1(:,i));
781      Results(i,2) = std(X1(:,i));
782      Results(i,3) = mean(X2(:,i));
783      Results(i,4) = std(X2(:,i));
784      [h,p,ci] = ttest(X1(:,i),X2(:,i));
785      Results(i,5) = h;
786      Results(i,6) = p;
787      i=i+1;
788  end
789  end
```