Solution Methods for Certain Evolution Equations

by

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ABSTRACT

Solution methods for certain linear and nonlinear evolution equations are presented in this dissertation. Emphasis is placed mainly on the analytical treatment of nonautonomous differential equations, which are challenging to solve despite the existent numerical and symbolic computational software programs available. Ideas from the transformation theory are adopted allowing one to solve the problems under consideration from a non-traditional perspective.

First, the Cauchy initial value problem is considered for a class of nonautonomous and inhomogeneous linear diffusion-type equation on the entire real line. Explicit transformations are used to reduce the equations under study to their corresponding standard forms emphasizing on natural relations with certain Riccati(and/or Ermakov)-type systems. These relations give solvability results for the Cauchy problem of the parabolic equation considered. The superposition principle allows to solve formally this problem from an unconventional point of view. An eigenfunction expansion approach is also considered for this general evolution equation. Examples considered to corroborate the efficacy of the proposed solution methods include the Fokker-Planck equation, the Black-Scholes model and the one-factor Gaussian Hull-White model.

The results obtained in the first part are used to solve the Cauchy initial value problem for certain inhomogeneous Burgers-type equation. The connection between linear (the Diffusion-type) and nonlinear (Burgers-type) parabolic equations is stress in order to establish a strong commutative relation. Traveling wave solutions of a nonautonomous Burgers equation are also investigated.

Finally, it is constructed explicitly the minimum-uncertainty squeezed states for quantum harmonic oscillators. They are derived by the action of corresponding maximal kinematical invariance group on the standard ground state solution. It is shown that the product of the variances attains the required minimum value only at the instances that one variance is a minimum and the other is a maximum, when the squeezing of one of the variances occurs. Such explicit construction is possible due to the relation between the diffusion-type equation studied in the first part and the time-dependent Schrödinger equation. A modication of the radiation field operators for squeezed photons in a perfect cavity is also suggested with the help of a nonstandard solution of Heisenberg's equation of motion.

This dissertation is dedicated to *ALL* my family.

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INTRODUCTION

1.1 Background and Motivation

At the second half of the 16th century a remarkable letter from G.W. Lebniz to G. de l'Hospital using special symbols to describe some partial processes could have originated the emergence of the branch of mathematics named later *Partial Differential Equations*(PDEs) [25]. During the preceding centuries, specially through the 18th century, outstanding pioneers such as Isaac Newton, James Bernouli, Leonhard Euler, d'Alambert, J. L. Lagrange, J. Fourier, J. Hadamard and P. Laplace, to mention some, made of PDEs the principal mode of analytical study of mechanics of continua in the physical sciences [22], [25], [27], [74]. Throughout the years these differential equations have become successful as models of physical phenomena. To date the study and analysis of PDEs, more precisely the partial differential evolution equations, e.g. PDEs with time as one of the independent variables, play a central role in the understanding of several phenomena arising not only in physics and mathematics but also in many other ramifications of science.

The study of evolution equations such as wave, Laplace, heat, and Schrödinger equations, to mention some, makes up a significant portion of the current frontier in the development of PDE theory. In such a study, developing analytic solution methods have attracted much attention both for their broad range of applicability and for the techniques developed [4], [5], [6], [22], [38], [49], [204]. The complexity and challenges in their theoretical study have attracted much interest from many mathematicians and scientists [236]. Besides scientific curiosity, the main source of motivation behind this work arise from the challenges and complexities on the topic encountered by the author of this dissertation throughout extensive readings and discussions on different scientific meetings. The work presented herein focused on the analytical treatment of the following fundamental evolution equations: Heat equation, Burgers equation and the Schrödinger equation.

1.2 The Heat equation

The one-dimensional heat equation coupled with arbitrary initial profile,

$$\frac{\partial u}{\partial t} = r\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \varphi(x) \tag{1.1}$$

was investigated first by Joseph Fourier at the beginning of the 19th century in his celebrated volume *Théorie analytique de la chaleur*(Analytic theory of heat), and has become a starting point for the extensive study of parabolic equations [27], [77], [154], [178]. The heat equation is also known as the *diffusion equation* and describes, in usual applications, the evolution in time of the density u of some quantity such as chemical concentration and temperature. In the scenario in which u stands for the temperature of certain entity, r plays the role of thermal diffusivity parameter. The heat equation and its extensions has served, for many years, as a bridge between central mathematical issues and practical applications. In the study of the heat equation and its extensions, the fundamental solution has had a great theoretical and practical importance. The significance of fundamental solutions and their importance for the solution to the Cauchy problem in general have constantly been emphasized in the literature [27], [154], [194], [195].

A general approach to solve the Cauchy initial value problem (1.1) on $\mathbb{R}$ for nonnegative time is based on the idea of the fundamental solution, idea originated from the Green's function method for solving boundary value problems [165]. For the heat equation, the fundamental solution measures the effect of concentrated heat source. Formally, because of the linearity of the heat equation, the superposition allows one to solve this initial value problem on the entire real line in the integral form

$$u(x,t) = \int_{-\infty}^{\infty} K(x,y,t)\varphi(y)dy \tag{1.2}$$

where y usually stands for a fixed shifting in the density profile. In the case of temperature dispersion the heat will diffuse away from its initial concentration and the resulting fundamental solution, or heat kernel, is denoted by $K(x,y,t)$. For convenience it is assumed also that at $t = 0$ the fundamental solution has a delta spike profile, e.g., that satisfies the delta function $K(x,y,0) = \delta(x-y)$ deviated for a fixed y. Thus, for each fixed y we have that the heat kernel satisfies

$$\frac{\partial K}{\partial t} = r\frac{\partial^2 K}{\partial x^2}. \tag{1.3}$$

In order to specify the solution in a uniquely fashion, the density shall be required to be square integrable, e.g.

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx < \infty \tag{1.4}$$

for all nonnegative t. It is widely known that on $\mathbb{R}$ the Fourier series solution to the heat equation becomes a Fourier integral. Without loss of generality the initial condition can be written as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \widehat{\varphi(k)} dk \tag{1.5}$$

where $\widehat{\varphi}$ represents the corresponding Fourier transform of φ. Furthermore, the method of separation of variables permits one to find a separable solution to the heat equation of the form

$$u(x,t) = e^{-rk^2 t} e^{ikx} \qquad \text{with} \qquad k \in \mathbb{R}. \tag{1.6}$$

Thus, the superposition principle allows one to presents the fundamental solution for the heat equation as a the Fourier integral

$$K(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-rk^2 t} e^{ikx} \widehat{\varphi(k)} dk. \tag{1.7}$$

Aditionally the Fourier transform of the Dirac delta function is given by

$$\widehat{\delta(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-y) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{iky} \tag{1.8}$$

when it is concentrated at $x = y$. Thus, substituting (1.8) into (1.7) yields

$$K(x,y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-rk^2 t} e^{ik(x-y)} dk \tag{1.9}$$

and with the assistance of the following extended Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ay^2+by} dy = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \tag{1.10}$$

the heat kernel (1.9) becomes

$$K(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4rt} \tag{1.11}$$

as expected. Consequently the solution to the Cauchy initial value problem (1.1) can be found formally by substituting (1.11) into (1.2).

1.3 The Burgers equation

The linear heat equation (1.1) is clearly related to the nonlinear Burgers equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = r \frac{\partial^2 v}{\partial x^2} \tag{1.12}$$

through the well known Cole-Hopf transformation

$$v = -2r\frac{u_x}{u} \qquad (r \textit{ represents the viscosity parameter}) \tag{1.13}$$

This is the simplest nonlinear diffusion equation and it is obtained by appending a linear diffusion term to the nonlinear transport equation. Moreover, equation (1.13) linearize and transform equation (1.12) into the heat equation (1.1). This reduction allows one to use the well known tools to treat analytically the heat equation to deal with the Burgers equation.

In fact if the corresponding time and space derivatives of the Cole-Hopf transformation (1.13) are performed we have that

$$v_t = -2r\left(\frac{u_{xt}}{u} - \frac{u_x u_t}{u^2}\right), \tag{1.14}$$

$$v_x = -2r\frac{u_x x}{u} + 2\left(\frac{u_x}{u}\right)^2, \tag{1.15}$$

$$vu_x = 2r^2\left[\left(\frac{u_x}{u}\right)^2\right]_x, \tag{1.16}$$

$$v_{xx} = -2r\left(\frac{u_{xxx}u - u_{xx}u_x}{u^2}\right) + 2r\left[\left(\frac{u_x}{u}\right)^2\right]_x. \tag{1.17}$$

After the substitution of (1.14)–(1.17) equation (1.12) reduces into

$$\frac{\partial u}{\partial t} = r\frac{\partial^2 u}{\partial x^2}, \tag{1.18}$$

with corresponding fundamental solution given by

$$K(x,y,t) = \frac{1}{\sqrt{4\pi t}}e^{-(x-y)^2/4rt} \tag{1.19}$$

as stated in the previous section. From the Cole-Hopf substitution it can be found also that

$$u(y,0) = e^{-\frac{1}{2r}\int_{-\infty}^{y} v(z,0)dz}. \tag{1.20}$$

Thus the solution of the initial value problem for (1.18) coupled with (1.20) is formally given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{\sqrt{4rt}} - \frac{1}{2r}\int_{-\infty}^{y} v(z,0)dz\right)dy. \tag{1.21}$$

Consequently the solution to the initial value problem for the Burgers equation (1.12) is given by

$$v(x,t) = -\frac{r}{\sqrt{\pi rt}}\frac{\partial}{\partial x}\ln\left[\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4rt} - \frac{1}{2r}\int_{-\infty}^{y} v(z,0)\,dz\right)dy\right] \tag{1.22}$$

for $t > 0$ and suitable initial profile $u(x,0)$ on $\mathbb{R}$ as desired. It is worth to pointing out that, as with the heat equation, the viscosity parameter r must be positive in order for the initial value problem to be well-posed in forwards time.

There are many applications in which the Burgers equation and its extensions play a crucial role in the understanding of the phenomena under consideration [45], [46], [79], [104], [185], [188], [219]. Burgers equation was first introduced by Burgers to describe the one-dimensional turbulence, and it also arise in many physical problems including sound waves in viscous medium, waves in fluid-filled viscous elastic tubes, and magnetohydrodynamic waves in a medium with finite electrical conductivity [45]. Additionally, in fluids and gases, one can interpret the right hand side as modeling the effect of viscosity, thus Burgers equation represents a simplified version of the equations of viscous fluid mechanics.

1.4 The Schrödinger equation

Classical mechanics explains matter and energy only at the macroscopic level. However, the properties that govern the macroscopic systems fail to provide a consistent description of matter on the atomic scale. The discovery of Planck's constant was probably the first indication of the invalidity of mechanically applying large-scale laws to small-scale objects [48]. In the 1920's further experimental facts were discovered revealing that the behavior of the microscopic particles differs fundamentally from that of the macroscopic world. The principles of classical mechanics were vague to examine the motion of objects at atomic levels, thus forcing eventually the abandonment of this approach. Motivated by theoretical and experimental investigations corroborating the inability of classical mechanics to describe certain microscopic phenomena, an exceptional team of physicists and mathematicians such as Planck, Bohr, Schrödinger, Heisenberg, Born, Dirac, Pauli, Hilbert and von Neumann among many others [38], [49], [87], [125], [153], [154], [204], started to developed one of the greatest intellectual endeavors the 20th century, the field of quantum mechanics. This theory, which deals effectively with both macroscopic and microscopic systems, was born during the first quarter of the 20th century and resulted in a series of outstanding articles published by Schrödinger in 1926 [194], [195], which made of wave mechanics a prominent theory. One of his key contributions was the formulation of what is known as the Schrödinger equation, which

governs the motion of a system placed in a potential. This is one of the fundamental equations of nonrelativistic quantum mechanics and can be written as

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{1.23}$$

where H, which correspond to the Hamiltonian operator, determines the evolution of the wave function(the complex solution of (1.23)) that represents the state of the system. This equation plays the same role as Hamilton's laws of motion in non-relativistic classical mechanics, and can be used to describe the quantum dynamics of a single particle or of an ensemble of particles under the influence of a variety of forces [93]. The square integrable wave function ψ contains the maximum information that nature allows concerning the state of the physical system under study at time t. Square integrable wave functions, e.g., wave functions ψ with the property

$$\int |\psi|^2 dx < \infty, \tag{1.24}$$

are normalizable. This means that they become wave functions of norm unity,

$$\int |\psi|^2 dx = 1. \tag{1.25}$$

The non-negative function $|\psi|^2$ is proportional to the probability that upon measurement of its position the particle will be detected in a given domain. If at time t, a physical state is described by the wave function ψ, the integral

$$\int_D |\psi|^2 dx, \tag{1.26}$$

of the full space of values of the variable x gives the probability that the measurement of this variable at time t will yield values within the domain D under consideration. It was Max Born who, at the end of 1926, found the correct interpretation of ψ as a probability amplitude, by analyzing experiments on the scattering of electrons on nuclei [93]. Instead of predicting what a particle actually does, the equation (1.23) can only predict the possible results of a process that a particle may undergo. There exist two variants of the Schrödinger's equation that govern such predictions, the time dependent and the time independent Schrödinger equations. The study of all of these variants is a vast and diverse field in mathematical physics.

The most general form of the equation (1.23) is the time-dependent Schrödinger equation, which determines the time evolution of the quantum system under consideration. For this case the Hamiltonian, which is the sum of kinetic $\frac{\hbar^2}{2m}\nabla^2$ and potential $U(x,t)$ energy, determines the evolution in time of the wave function. In one dimension it takes the form

$$H = \frac{\hbar^2}{2m}\nabla^2 + U(x,t) \tag{1.27}$$

with $\hbar$, m, ∇ and $U(x,t)$ representing the Planck's constant, the system's mass, the gradient operator and the potential respectively. In coordinate representation this variant of the Schrödinger equation can be written as

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(x,t)\right]\psi. \tag{1.28}$$

Among all the many solutions of the time dependent Schrödinger equation, one of the most useful are the stationary states or states of definite energy, and for these states the time dependent equation (1.28) reduces to its time independent version. It is only used when the Hamiltonian itself is not dependent on time. Hence the energy operator $i\partial\psi/\partial t$ can then be replaced by the energy eigenvalue E forming the eigenvalue equation $E\psi_n = H\psi_n$, $n \in \mathbb{N}$, and its solution ψ_n is called energy eigenstate with energy E. The explicit representation of the time-independent version of Schrödinger equation in one dimension can be given by

$$E\psi_n = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right]\psi_n \tag{1.29}$$

which is clearly an eigenvalue equation. Approximate solutions to the time-independent Schrödinger equation are commonly used to calculate the energy levels and other properties of atoms and molecules. The eigenvalues E are discrete, that is only certain energy values are allowed, all other energies are forbidden. The energy eigenvalues are also eigenstate energies. The lowest eigenstate energy is the ground state energy($n = 0$), all higher energies($n \geq 1$) are called excited state energies.

The structure of the Schrödinger equation depends on the physical situation at hand. For the case in which the potential takes the form of a classical spring the equation (1.23) takes the name of Quantum Harmonic Oscillator. The harmonic oscillator is of importance for general theory, because it forms a cornerstone in the theory of radiation [48]. The quantum harmonic oscillator in

one dimension is usually written as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi. \tag{1.30}$$

Using the separation of variables $\psi(x,t) = f(t)\Psi(x)$, one have

$$f(t) = \exp\left(\frac{-iEt}{\hbar}\right) \tag{1.31}$$

where E is a constant, and that

$$\frac{d^2\Psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2x^2}{\hbar^2}\right)\Psi = 0. \tag{1.32}$$

Notice that this last equation is very similar to the Hermit differential equation

$$y''(x) - 2xy'x + 2ny(x) = 0 \tag{1.33}$$

which solutions are given by the so called Hermit polynomials $H_n(x)$. This polynomials are extensively used in the theory of special functions [9], [11], [163], [164] and form an orthonormal set with the weight function e^{-x^2}, e.g.

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = 0 \tag{1.34}$$

where $H_m(x)$ and $H_n(x)$ are Hermit polynomials with $m \neq n$. Similar to square integrable functions, the weight function makes bounded the Hermit polynomials, which is a crucial property of the wave function. Then, the next step is to make as much similar as possible equation (1.32) to (1.33). Defining the following function:

$$Y_n(x) = e^{-x^2}H_n(x), \tag{1.35}$$

one can found that

$$H_n(x) = Y_n(x)e^{x^2/2} \tag{1.36}$$

$$H_n''(x) = \left(Y_n''(x) + Y_n(x) + xY_n'(x)\right)e^{x^2/2} + xe^{x^2/2}\left(Y_n'(x) + xY_n(x)\right). \tag{1.37}$$

By definition $H_n(x) = y(x)$, thus substituting (1.36)–(1.37) into (1.33) yields

$$Y_n''(x) + \left[(2n+1) - x^2\right]Y_n(x) = 0. \tag{1.38}$$

which is very similar to (1.33). Next step is to perform the following change of variables

$$\psi(z(x)) = Y(x) \tag{1.39}$$

with $z = \alpha x$ (α an arbitrary constant) so that

$$\frac{d^2}{dx^2}Y(x) = \alpha^2 \frac{d^2}{dz^2}\psi(z). \tag{1.40}$$

Substitution of (1.39)–(1.40) reduces equation (1.32) into

$$\psi''(z) + \left[\frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2} - z^2\right]\psi(z) = 0 \tag{1.41}$$

after making

$$\frac{mk}{\hbar\alpha^4} = 1. \tag{1.42}$$

Consequently, the solution of (1.41) can be written as

$$\psi_n(z) = e^{-x^2/2}H_n(z) \tag{1.43}$$

with $z = \alpha x$ and α can be taken from (1.42). From equations (1.38) and (1.41) one can deduce that $E = (n+1/2)\hbar$ where $n \in \mathbb{N}$. After a normalization process [9] it can be conclude that

$$\psi_n(z) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-z^2/2}H_n(z) \tag{1.44}$$

as desired.

1.5 Organization of the rest of the Dissertation

Throughout the the rest of this dissertation the three main chapters containing the core ideas are presented. All the three chapters deal with methods to construct solutions to the Cauchy initial value problem for some specific evolution equations. The solution methods to solve the main equations presented in each of the chapters are strongly related even when usual applications come from different fields of study.

In the **Chapter 2** the Cauchy initial value problem for a class of nonautonomous and inhomogeneous diffusion-type equations on $\mathbb{R}$ is considered. More specifically, explicit transformations are used to reduce the equations under study to their corresponding standard forms and emphasize natural relations with certain Riccati(and/or Ermakov)-type systems. Similar methods have been

applied to the corresponding Schrödinger equation [38], [39], [41], [42], [43], [124], [126], [133], [149], [193], [208], [209]. Alternatively, a group theoretical approach to a similar class of partial differential equations is discussed in Refs. [77], [154] and [178].

In the **Chapter 3** preliminary results from chapter 2 are used to solve the Cauchy problem for a certain inhomogeneous Burgers-type equation. The connection between linear (Diffusion-type) and nonlinear (Burgers-type) parabolic equations is explored in order to establish a commutative relation. Traveling wave solutions of a nonautonomous Burgers equation are also explored.

The **Chapter 4** is utilized to describe a six-parameter family of the minimum-uncertainty squeezed states for the harmonic oscillator in nonrelativistic quantum mechanics. They are derived by the action of corresponding maximal kinematical invariance group on the standard ground state solution. It is shown that the product of the variances attains the required minimum value $1/4$ only at the instances that one variance is a minimum and the other is a maximum, when the squeezing of one of the variances occurs. The generalized coherent states are explicitly constructed and their Wigner function is studied. The overlap coefficients between the squeezed, or generalized harmonic, and the Fock states are explicitly evaluated in terms of hypergeometric functions. The corresponding oscillating photons statistics are discussed and an application to quantum optics and cavity quantum electrodynamics is mentioned. Explicit solutions to the Heisenberg equations for radiation field operators with squeezing are also presented.

Chapter 2

CAUCHY PROBLEM FOR DIFFUSION-TYPE EQUATIONS

2.1 Introduction

It is well known that the diffusion-type equations have numerous applications in different areas of Science. Among these numerous applications, the role of fundamental solutions for these parabolic systems is essential in probability theory [44], [112]. Thus it is natural, for example, to consider an Itô diffusion process $X = \{X_t : t \geq 0\}$ which satisfies the stochastic differential equation

$$dX_t = b(X_t,t)\ dt + \sigma(X_t,t)\ dW_t, \qquad X_0 = x, \tag{2.1}$$

in which $W = \{W_t : t \geq 0\}$ is a standard Wiener process. The existence and uniqueness of solutions to (2.1) depends on the coefficients b and σ. (See Ref. [112] for conditions of a unique strong solution to (2.1).) If the equation (2.1) has a unique solution, then the expectations

$$u(x,t) = E_x[\phi(X_t)] = E[\phi(X_t)|X_0 = x] \tag{2.2}$$

are solutions of the Cauchy problem

$$u_t = \frac{1}{2}\sigma^2(x,t)\,u_{xx} + b(x,t)\,u_x, \qquad u(x,0) = \phi(x). \tag{2.3}$$

This last evolution equation is known as the Kolmogorov forward equation [44], [112]. Thus if $p(x,y,t)$ is the appropriate fundamental solution of (2.3), then one can compute the given expectations in (2.2) according to

$$E_x[\phi(X_t)] = \int_\Omega p(x,y,t)\,\phi(y)\ dy. \tag{2.4}$$

with Ω denoting the probability space where these expectations live. In this context, the fundamental solution is known as the probability transition density for the process and

$$\int_\Omega p(x,y,t)\ dy = 1. \tag{2.5}$$

See also Refs. [3] and [110] for applications to stochastic differential equations related to Fokker–Planck and Burgers equations. The Black-Scholes model of financial markets is discussed in Refs. [17], [96], [150], [151], [152], [213] (see also [192] for the one-factor Gaussian Hull-White model).

The main result of this chapter is presented in the next section with a sketch of the corresponding proof. This section contains a solution method for certain nonautonomous and inhomogeneous diffusion-type equations. The third section is devoted to explore, from a novel point of view, the symmetries of the evolution equation introduced in the second section. A second solution method, in terms of eigenfunction expansion, is introduced in the fourth section. Then several examples are presented in order to corroborate the two proposed solution methods discussed. The chapter is completed with key concluding remarks in the fifth section. Additional details in some of the crucial results for this chapter are given on the last four sections in order to establish the rigorous arguments that govern such results.

2.2 Transformation Method

The general theory of transformations is considered as a branch of analysis in the sense that it can be developed by purely analytic methods. Some of the most powerful tools for solving problems in physics and mathematics involve transform methods. A considerable amount of these analytic techniques for solving a partial differential equation require reducing it down to a set of ordinary differential equations that are hopefully easier to solve than the original partial differential equation. In this section ideas from the theory of transformations are adopted in order to construct a method to solve the Cauchy problem for a generalized diffusion-type equation.

Transformation to the Standard Form

The following resumes one of the most important results of this chapter.

Lemma 1. *The nonautonomous and inhomogeneous diffusion-type equation*

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2} - (g(t) - c(t)x)\frac{\partial u}{\partial x} + \left(d(t) + f(t)x - b(t)x^2\right)u, \tag{2.6}$$

where a, b, c, d, f, g are suitable functions of time t only, can be reduced to the standard autonomous form

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} \tag{2.7}$$

with the help of the following substitution:

$$u(x,t) = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2+\delta(t)x+\kappa(t)} v(\xi,\tau), \tag{2.8}$$
$$\xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t).$$

Here, $\mu, \alpha, \beta, \gamma, \delta, \varepsilon, \kappa$ *are time dependent functions that satisfy*

$$\frac{\mu'}{2\mu} + 2a\alpha + d = 0 \tag{2.9}$$

and

$$\frac{d\alpha}{dt} + b - 2c\alpha - 4a\alpha^2 = 0, \tag{2.10}$$
$$\frac{d\beta}{dt} - (c + 4a\alpha)\beta = 0, \tag{2.11}$$
$$\frac{d\gamma}{dt} - a\beta^2 = 0, \tag{2.12}$$
$$\frac{d\delta}{dt} - (c + 4a\alpha)\delta = f - 2\alpha g, \tag{2.13}$$
$$\frac{d\varepsilon}{dt} + (g - 2a\delta)\beta = 0, \tag{2.14}$$
$$\frac{d\kappa}{dt} + g\delta - a\delta^2 = 0. \tag{2.15}$$

Proof. The sketch of the proof is as follows. Let $S = \alpha(t)x^2 + \delta(t)x + \kappa(t)$ and $v = v(\xi,\tau)$ with ξ and τ given as in (2.8). To reduce (2.6) into the standard nonautonomous form (2.7) we need the following derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{e^S}{\sqrt{\mu(t)}}\left[-\frac{\mu'(t)}{2\mu(t)}v + (\alpha'(t)x^2 + \delta'(t)x + \kappa'(t))v\right] \\
&\quad + \frac{e^S}{\sqrt{\mu(t)}}\left[(\beta'(t)x + \varepsilon'(t))v_\xi + \gamma'(t)v_\tau\right] \qquad (2.16)\\
\frac{\partial u}{\partial x} &= \frac{e^S}{\sqrt{\mu(t)}}\left[(2\alpha(t)x + \delta(t))v + \beta(t)v_\xi\right] \qquad (2.17)\\
\frac{\partial^2 u}{\partial x^2} &= \frac{e^S}{\sqrt{\mu(t)}}\left[(2\alpha(t)x + \delta(t))^2 v + 2\beta(t)(2\alpha(t)x + \delta(t))v_\xi\right] \\
&\quad + \frac{e^S}{\sqrt{\mu(t)}}\left[2\alpha(t)v + \beta^2(t)v_{\xi\xi}\right]. \qquad (2.18)
\end{aligned}$$

Direct substitution of (2.16)–(2.18) into (2.6) leads to the desired reduced form (2.7) subject to equations (2.9)–(2.15). □

The transformation (2.8) allows one to replace the study of the original equation (2.6) by the study of the well know standard autonomous form (2.7). Equation (2.10) is called the *Riccati nonlinear differential equation* [171], [228], [230] and for terminology we shall refer to the system (2.9)–(2.15) as a *Riccati-type system*.

The substitution (2.9) reduces the nonlinear Riccati equation (2.10) to the second order linear equation

$$\mu'' - \tau(t)\,\mu' - 4\sigma(t)\,\mu = 0, \tag{2.19}$$

where

$$\tau(t) = \frac{a'}{a} + 2c - 4d, \quad \sigma(t) = ab + cd - d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{2.20}$$

Equation (2.19) shall be referred to as the *characteristic equation* [206].

Furthermore, it is also known [206] that the diffusion-type equation (2.6) admits a particular solution of the form

$$u = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)}, \tag{2.21}$$

provided that the time dependent functions $\mu, \alpha, \beta, \gamma, \delta, \varepsilon, \kappa$ satisfy the Riccati-type system (2.9)–(2.15) (the original interpretation of this system). Alternatively, a group theoretical approach to a similar class of partial differential equations is discussed in Refs. [77], [154] and [178].

Fundamental Solution

By the *superposition principle* one can formally solve the Cauchy initial value problem for the diffusion-type equation (2.6) subject to suitable initial data $u(x,0) = \varphi(x)$ on the entire real line $-\infty < x < \infty$ in the integral form:

$$u(x,t) = \int_{-\infty}^{\infty} K_0(x,y,t)\ \varphi(y)\,dy \tag{2.22}$$

with the *fundamental solution* (heat kernel) [206]:

$$K_0(x,y,t) = \frac{1}{\sqrt{2\pi\mu_0(t)}}\ e^{\alpha_0(t)x^2 + \beta_0(t)xy + \gamma_0(t)y^2 + \delta_0(t)x + \varepsilon_0(t)y + \kappa_0(t)}, \tag{2.23}$$

where a particular solution of the Riccati-type system (2.10)–(2.15) is given by:

$$\alpha_0(t) = -\frac{1}{4a(t)}\frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \tag{2.24}$$

$$\beta_0(t)=\frac{h(t)}{\mu_0(t)},\quad h(t)=\exp\left(\int_0^t (c(s)-2d(s))\,ds\right), \tag{2.25}$$

$$\gamma_0(t)=\frac{d(0)}{2a(0)}-\frac{a(t)h^2(t)}{\mu_0(t)\mu_0'(t)}-4\int_0^t\frac{a(s)\sigma(s)h(s)}{\left(\mu_0'(s)\right)^2}\,ds \tag{2.26}$$

$$=\frac{d(0)}{2a(0)}-\frac{1}{2\mu_1(0)}\frac{\mu_1(t)}{\mu_0(t)}, \tag{2.27}$$

$$\delta_0(t)=\frac{h(t)}{\mu_0(t)}\int_0^t\left[\left(f(s)+\frac{d(s)}{a(s)}g(s)\right)\mu_0(s)+\frac{g(s)}{2a(s)}\mu_0'(s)\right]\frac{ds}{h(s)}, \tag{2.28}$$

$$\varepsilon_0(t)=-\frac{2a(t)h(t)}{\mu_0'(t)}\delta_0(t)-8\int_0^t\frac{a(s)\sigma(s)h(s)}{\left(\mu_0'(s)\right)^2}\left(\mu_0(s)\delta_0(s)\right)ds \tag{2.29}$$
$$+2\int_0^t\frac{a(s)h(s)}{\mu_0'(s)}\left[f(s)+\frac{d(s)}{a(s)}g(s)\right]ds,$$

$$\kappa_0(t)=-\frac{a(t)\mu_0(t)}{\mu_0'(t)}\delta_0^2(t)-4\int_0^t\frac{a(s)\sigma(s)}{\left(\mu_0'(s)\right)^2}\left(\mu_0(s)\delta_0(s)\right)^2\,ds \tag{2.30}$$
$$+2\int_0^t\frac{a(s)}{\mu_0'(s)}\left(\mu_0(s)\delta_0(s)\right)\left[f(s)+\frac{d(s)}{a(s)}g(s)\right]ds$$

with $\delta(0)=g(0)/(2a(0))$, $\varepsilon(0)=-\delta(0)$, $\kappa(0)=0$. Here, μ_0 and μ_1 are the so-called *standard solutions* to the characteristic equation (2.19) subject to the following initial data

$$\mu_0(0)=0,\quad \mu_0'(0)=2a(0)\neq 0\qquad \mu_1(0)\neq 0,\quad \mu_1'(0)=0. \tag{2.31}$$

Solution (2.24)–(2.30) shall be referred to as a *fundamental solution* to the Riccati-type system (2.9)–(2.15); see (2.48)–(2.52) and (2.53) for the corresponding asymptotics.

It is known that the system (2.9)–(2.15) is solvable [38], [204]. Its general solution from a non-traditional point of view is discused in the following lemma.

Lemma 2. *The Riccati-type system (2.9)–(2.15) has the following general solution:*

$$\mu(t)=-2\mu(0)\mu_0(t)\left(\alpha(0)+\gamma_0(t)\right), \tag{2.32}$$

$$\alpha(t)=\alpha_0(t)-\frac{\beta_0^2(t)}{4\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.33}$$

$$\beta(t)=-\frac{\beta(0)\beta_0(t)}{2\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.34}$$

$$\gamma(t)=\gamma(0)-\frac{\beta^2(0)}{4\left(\alpha(0)+\gamma_0(t)\right)} \tag{2.35}$$

and

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)\left(\delta(0)+\varepsilon_0(t)\right)}{2\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.36}$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)\left(\delta(0)+\varepsilon_0(t)\right)}{2\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.37}$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{\left(\delta(0)+\varepsilon_0(t)\right)^2}{4\left(\alpha(0)+\gamma_0(t)\right)} \tag{2.38}$$

in terms of the fundamental solution to the Riccati-system (2.24)–(2.30) subject to arbitrary initial data $\mu(0), \alpha(0), \beta(0), \gamma(0), \delta(0), \varepsilon(0), \kappa(0)$.

Proof. Relations (2.21)–(2.23) are used with the uniqueness property of the solution and the elementary integral:

$$\int_{-\infty}^{\infty} e^{-ay^2+2by}\, dy = \sqrt{\frac{\pi}{a}}\, e^{b^2/a}, \quad a>0. \tag{2.39}$$

Then system (2.32)–(2.38) follows and this complete the proof (see Appendix *A* for a detailed proof). □

Remark 1. *It is worth noting that the transformation (2.8), combined with the standard heat kernel [162]:*

$$K_0(\xi,\eta,\tau) = \frac{1}{\sqrt{4\pi(\tau-\tau_0)}} \exp\left[-\frac{(\xi-\eta)^2}{4(\tau-\tau_0)}\right] \tag{2.40}$$

for the diffusion equation (2.7) and (2.32)–(2.38), allows one to derive the fundamental solution (2.23) of the diffusion-type equation (2.6) from a new perspective.

A detailed verification of *Remark 1* is provided in *Appendix B*. From Lemma 2 the following result, which is needed for the construction of the fundamental solution, is established.

Lemma 3. *Solution (2.32)–(2.38) implies:*

$$\mu_0 = \frac{2\mu}{\mu(0)\beta^2(0)}\left(\gamma-\gamma(0)\right), \tag{2.41}$$

$$\alpha_0 = \alpha - \frac{\beta^2}{4\left(\gamma-\gamma(0)\right)}, \tag{2.42}$$

$$\beta_0 = \frac{\beta(0)\beta}{2\left(\gamma-\gamma(0)\right)}, \tag{2.43}$$

$$\gamma_0 = -\alpha(0) - \frac{\beta^2(0)}{4\left(\gamma-\gamma(0)\right)} \tag{2.44}$$

and

$$\delta_0 = \delta - \frac{\beta(\varepsilon - \varepsilon(0))}{2(\gamma - \gamma(0))}, \tag{2.45}$$

$$\varepsilon_0 = -\delta(0) + \frac{\beta(0)(\varepsilon - \varepsilon(0))}{2(\gamma - \gamma(0))}, \tag{2.46}$$

$$\kappa_0 = \kappa - \kappa(0) - \frac{(\varepsilon - \varepsilon(0))^2}{4(\gamma - \gamma(0))}, \tag{2.47}$$

which gives the following asymptotics

$$\alpha_0(t) = -\frac{1}{4a(0)t} - \frac{c(0)}{4a(0)} + \frac{a'(0)}{8a^2(0)} + \mathscr{O}(t), \tag{2.48}$$

$$\beta_0(t) = \frac{1}{2a(0)t} - \frac{a'(0)}{4a^2(0)} + \mathscr{O}(t), \tag{2.49}$$

$$\gamma_0(t) = -\frac{1}{4a(0)t} + \frac{c(0)}{4a(0)} + \frac{a'(0)}{8a^2(0)} + \mathscr{O}(t), \tag{2.50}$$

$$\delta_0(t) = \frac{g(0)}{2a(0)} + \mathscr{O}(t), \qquad \varepsilon_0(t) = -\frac{g(0)}{2a(0)} + \mathscr{O}(t), \tag{2.51}$$

$$\kappa_0(t) = \mathscr{O}(t) \tag{2.52}$$

as $t \to 0^+$.

Equations (2.41)–(2.52) are inversions of (2.32)–(2.38) and the deriviation of the asymptotics (2.48)–(2.52) is given in the Appendix *C*. These formulas allows to establish the required asymptotic of the fundamental solution (2.23):

$$\begin{aligned} K_0(x,y,t) \sim{}& \frac{1}{\sqrt{4\pi a(0)t}} \exp\left[-\frac{(x-y)^2}{4a(0)t}\right] \\ &\times \exp\left[\frac{a'(0)}{8a^2(0)}(x-y)^2 - \frac{c(0)}{4a(0)}\left(x^2-y^2\right)\right] \exp\left[\frac{g(0)}{2a(0)}(x-y)\right]. \end{aligned} \tag{2.53}$$

(Here, $f \sim g$ as $t \to 0^+$, if $\lim_{t\to 0^+}(f/g) = 1$.)

By direct substitution one can verify that the right hand sides of (2.32)–(2.38) satisfy the Riccati-type system (2.9)–(2.15) and that the asymptotics (2.48)–(2.52) result in the continuity with respect to initial data:

$$\lim_{t\to 0^+} \mu(t) = \mu(0), \quad \lim_{t\to 0^+} \alpha(t) = \alpha(0), \quad \text{etc.} \tag{2.54}$$

The transformation property (2.32)–(2.38) permits to find the solution to the Cauchy initial value problem in terms of the fundamental solution (2.24)–(2.30), and it is referred to as a *nonlinear superposition principle* for the Riccati-type system.

2.3 Symmetries of the Nonutonomous Diffusion Equation

It is also constructive to discuss the symmetries that enjoy the nonautonomous diffusion-type equation (2.6) from the point of view of the general transformation presented in Lemma 1. In the simplest case $a=1$, $b=c=d=f=g=0$, when $u_t=u_{xx}$, the Lemma 1 provides the following general transformation

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\mu(0)(1-4\alpha(0)t)}}\exp\left(\frac{\alpha(0)x^2+\delta(0)x+\delta^2(0)t}{1-4\alpha(0)t}+\kappa(0)\right) \\ &\quad\times v\left(\frac{\beta(0)x+2\beta(0)\delta(0)t}{1-4\alpha(0)t}+\varepsilon(0),\frac{\beta^2(0)t}{1-4\alpha(0)t}+\gamma(0)\right) \end{aligned} \tag{2.55}$$

of the diffusion equation into itself [154], [178]. Here it was used $\mu'(0)=-4\alpha(0)\mu(0)$. Among the existent space-time transformations it includes the familiar *Galilei transformations*:

$$u(x,t)=\exp\left(\frac{V}{2}x+\frac{V^2}{4}t\right)v(x+Vt+x_0,t+t_0), \tag{2.56}$$

when $\alpha(0)=0$, $\beta(0)=\mu(0)=1$, $\kappa(0)=0$ and $\delta(0)=V/2$; supplemented by *dilatations*:

$$u(x,t)=v\left(lx,l^2t\right) \tag{2.57}$$

with $\alpha(0)=\gamma(0)=\delta(0)=\varepsilon(0)=\kappa(0)=0$, $\mu(0)=1$ and $\beta(0)=l$; and *expansions*:

$$u(x,t)=\frac{1}{\sqrt{1+mt}}\exp\left(-\frac{mx^2}{1+mt}\right)v\left(\frac{x}{1+mt},\frac{t}{1+mt}\right) \tag{2.58}$$

with $\beta(0)=1$, $\delta(0)=\varepsilon(0)=\kappa(0)=0$ and $\mu(0)=1$, $\mu'(0)=m$. The symmetry group of the corresponding Schrödinger equations is discussed in [77], [154], [160], [161] and [218], thus the symmetries of Schrödinger and heat equations are closely related.

2.4 Eigenfunction Expansion and Ermakov-type System

The solution of the Cauchy initial value problem for (2.6) can be found also in terms of an eigenfunction expansion similar to the case of the corresponding Schrödinger in Refs. [126] and [208]. This method is also known as the method of *Separation of Variables*, and as the *Fourier method*. With the assistance of the transformation (2.8), one can corroborate that equation (2.6) is equivalent to

$$\begin{aligned} \frac{\partial u}{\partial t} &= a(t)\frac{\partial^2 u}{\partial x^2}-(g(t)-c(t)x)\frac{\partial u}{\partial x}+\left(d(t)-c_0a(t)\beta^2\xi^2+c_0a(t)\beta^2\varepsilon^2\right) \\ &\quad+\left(\left(f(t)+2c_0a(t)\beta^3\varepsilon\right)x-\left(b(t)-c_0a(t)\beta^4\right)x^2\right)u. \end{aligned} \tag{2.59}$$

Indeed, direct substitution of (2.16)–(2.18) into (2.59) allows one to reduce the nonautonomous and inhomogeneous diffusion-type equation (2.6) into the convenient form

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} - c_0 \xi^2 v \qquad (c_0 = 0, 1) \tag{2.60}$$

whenever the system

$$\frac{d\alpha}{dt} + b - 2c\alpha - 4a\alpha^2 = c_0 a\beta^4, \tag{2.61}$$

$$\frac{d\beta}{dt} - (c + 4a\alpha)\,\beta = 0, \tag{2.62}$$

$$\frac{d\gamma}{dt} - a\beta^2 = 0, \tag{2.63}$$

$$\frac{d\delta}{dt} - (c + 4a\alpha)\,\delta = f - 2\alpha g + 2c_0 a\beta^3 \varepsilon, \tag{2.64}$$

$$\frac{d\varepsilon}{dt} + (g - 2a\delta)\,\beta = 0, \tag{2.65}$$

$$\frac{d\kappa}{dt} + g\delta - a\delta^2 = c_0 a\beta^2 \varepsilon^2 \tag{2.66}$$

$$\frac{\mu'}{2\mu} + 2a\alpha + d = 0 \tag{2.67}$$

holds. Notice also that substitution of (2.67) into (2.61) leads to the second order ordinary differential equation

$$\mu'' - \tau(t)\,\mu' - 4\sigma(t)\,\mu = -c_0 \mu \left(2a\beta^2\right)^2, \tag{2.68}$$

where $\tau(t)$ and $\sigma(t)$ are given by (2.20). The inhomogeneous characteristic equation (2.68) is an *Ermakov nonlinear differential equation*. Correspondingly, it is natural to name the extension of the Riccati-type system (2.9)–(2.15) given by (2.61)–(2.67) as the *Ermakov-type system* [126], which is integrable in quadratures in terms of solutions of the inhomogeneous characteristic equation (2.68). The corresponding integral can be found in Refs. [64], [65]. The solution of the Ermakov nonlinear differential equation (2.68) can be found in [126] and references therein.

When $c_0 = 0$ in equation (2.60) the system (2.61)–(2.67) reduces to the Riccati System (2.10)–(2.15) with corresponding solution discussed in Lemma 2. For the case of $c_0 = 1$, the solution of the Ermakov system (2.61)–(2.67) is presented in the following lemma.

Lemma 4. *The Ermakov-type system with $c_0 = 1$ has the following solution:*

$$\mu(t) = \mu_0 \mu(0) \sqrt{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)}, \tag{2.69}$$

$$\alpha(t) = \alpha_0 - \frac{\beta_0^2(\gamma_0 + \alpha(0))}{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)}, \tag{2.70}$$

$$\beta(t) = \frac{\beta(0)\beta_0}{\sqrt{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)}}, \tag{2.71}$$

$$\gamma(t) = \gamma(0) - \frac{1}{4}\ln\left[\frac{(\gamma_0 + \alpha(0)) + \frac{1}{2}\beta^2(0)}{(\gamma_0 + \alpha(0)) - \frac{1}{2}\beta^2(0)}\right] \tag{2.72}$$

and

$$\delta(t) = \delta_0 + \beta_0 \frac{\varepsilon(0)\beta^3(0) - 2(\gamma_0 + \alpha(0))(\varepsilon_0 + \delta(0))}{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)}, \tag{2.73}$$

$$\varepsilon(t) = \frac{\beta(0)(\delta(0) + \varepsilon_0) - 2\varepsilon(0)(\gamma_0 + \alpha(0))}{\sqrt{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)}}, \tag{2.74}$$

$$\begin{aligned}\kappa(t) = {} & \kappa_0 + \kappa(0) + \frac{\beta^3(0)\varepsilon(0)(\varepsilon_0 + \delta(0))}{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)} \\ & - \frac{(\gamma_0 + \alpha(0))\left[\beta^2(0)\varepsilon^2(0) + (\varepsilon_0 + \delta(0))^2\right]}{4(\gamma_0 + \alpha(0))^2 - \beta^4(0)}\end{aligned} \tag{2.75}$$

in terms of the fundamental solution of the Riccati-system (2.24)–(2.30) subject to arbitrary initial data $\mu(0), \alpha(0), \beta(0), \gamma(0), \delta(0), \varepsilon(0), \kappa(0)$. A sketch of the proof for this lemma can be found in Appendix D.

Then, a particular solution of the diffusion-type equation (2.6) has the form

$$u_n(x,t) = \frac{e^{(\alpha x^2 + \delta x + \kappa) - (2n+1)(\gamma - \gamma(0)) - (\beta x + \varepsilon)^2/2}}{\sqrt{2^n n! \mu \sqrt{\pi}}} H_n(\beta x + \varepsilon) \tag{2.76}$$

where $H_n(x)$ are the Hermite polynomials [164], provided that the solution for the Ermakov-type system (2.61)–(2.67) is given by (2.69)–(2.75). With the assistance of the superposition principle, the equation (2.76) allows one to find the solution to the Cauchy initial value problem in terms of eigenfunction expansion. In the corresponding eigenfunction expansion:

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) \tag{2.77}$$

with

$$u_n(x,0) = \frac{e^{\alpha(0)x^2 + \delta(0)x + \kappa(0)}}{\sqrt{2^n n! \mu(0) \sqrt{\pi}}} e^{-(\beta(0)x + \varepsilon(0))^2/2} H_n(\beta(0)x + \varepsilon(0)) \tag{2.78}$$

one can choose $\delta(0) = \varepsilon(0) = \kappa(0) = 0$, $\alpha(0) = 0$ and $\beta(0) = \mu(0) = 1$, when

$$\int_{-\infty}^{\infty} u_m(x,0)\, u_n(x,0)\ \ dx = \delta_{mn} \tag{2.79}$$

in view of the orthogonality property of Hermite polynomials [164]. The expansion coefficients are given by

$$c_n = \int_{-\infty}^{\infty} u_n(x,0)\, u(x,0)\ \ dx. \tag{2.80}$$

Thus, equations (2.77)–(2.80) provide the solution of the Cauchy initial value problem for equation (2.6) jointly with suitable initial data $u(x,0)$. It is worth to mention that δ_{mn} represents the familiar *Kronecker delta* which is defined as:

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases}$$

2.5 Examples

Some diffusion-type equations that are important in applications are considered from a united point of view in this section. The methods discussed previously on sections (2.2) and (2.4) are used here in order to present the corresponding solutions.

Example 1 For the standard *diffusion equation* on $\mathbb{R}$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \qquad a = \text{constant} > 0 \tag{2.81}$$

the heat kernel is given by

$$K_0(x,y,t) = \frac{1}{\sqrt{4\pi a t}} \exp\left[-\frac{(x-y)^2}{4at}\right], \qquad t > 0. \tag{2.82}$$

See [27], [162] and references therein for a detailed investigation of the classical one-dimensional heat equation.

Example 2 In the mathematical description of the nerve cell a dendritic branch is typically modeled by using cylindrical *cable equation* [101]:

$$\tau\frac{\partial u}{\partial t} = \lambda^2\frac{\partial^2 u}{\partial x^2} + u, \quad \tau = \text{constant} > 0. \tag{2.83}$$

The fundamental solution on $\mathbb{R}$ is given by

$$K_0(x,y,t) = \frac{\sqrt{\tau}e^{t/\tau}}{\sqrt{4\pi\lambda^2 t}} \exp\left[-\frac{\tau(x-y)^2}{4\lambda^2 t}\right], \quad t>0. \tag{2.84}$$

(See also [97] and references therein.)

Example 3 The fundamental solution of the *Fokker-Planck equation* [175], [233]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u \tag{2.85}$$

on $\mathbb{R}$ is given by [206]:

$$K_0(x,y,t) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp\left[-\frac{(x-e^{-t}y)^2}{2(1-e^{-2t})}\right], \quad t>0. \tag{2.86}$$

Here,

$$\lim_{t\to\infty} K_0(x,y,t) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \qquad y = \text{constant}. \tag{2.87}$$

The solution of the Cauchy initial value problem for the *Fokker-Planck equation* on $\mathbb{R}$ can also be given in terms of eigenfunction expansion with the aid of the superposition principle. In the corresponding eigenfunction expansion:

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) \tag{2.88}$$

and after choosing $\delta(0) = \varepsilon(0) = \kappa(0) = 0$, $\alpha(0) = -3/8$, $\beta(0) = 1/2$ and $\mu(0) = 1$, the corresponding eigenfunction is given by

$$u_n(x,t) = \frac{e^{-\frac{1}{2}x^2 + \frac{1}{4}(2n+1)\left(\ln\frac{1+e^{-2t}}{2}\right)}}{\sqrt{2^n n!\sqrt{\pi}\sqrt{\frac{1}{2}(1+e^{-2t})}}} H_n\left(\frac{xe^{-t}}{\sqrt{2(1+e^{-2t})}}\right) \tag{2.89}$$

$$\tag{2.90}$$

and expansion coefficients

$$c_n = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2^n n!\sqrt{\pi}}} H_n\left(\frac{x}{2}\right) u(x,0)\ dx. \tag{2.91}$$

Example 4 Equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + (g - kx)\frac{\partial u}{\partial x}, \qquad a,k > 0, \quad g \geq 0 \tag{2.92}$$

corresponds to the heat equation with linear drift when $g = 0$ [154]. In stochastic differential equations this equation corresponds the Kolmogorov forward equation for the regular Ornstein–Uhlenbech process [44]. The fundamental solution is given by

$$\begin{aligned} K_0(x,y,t) = &\frac{\sqrt{k}e^{kt/2}}{\sqrt{4\pi a \sinh(kt)}} \\ &\times \exp\left[-\frac{\left(k\left(xe^{-kt/2} - ye^{kt/2}\right) + 2g\sinh(kt/2)\right)^2}{4ak\sinh(kt)}\right], \quad t > 0. \end{aligned} \tag{2.93}$$

(See Refs. [44] and [206] for more details.)

Example 5 The Black-Scholes model provides a mathematical description of financial markets and derivative investment instruments [17], [150]. If S is the price of the stock, $V(S,t)$ is the price of a derivative as a function of time and stock price, r is the annualized risk-free interest rate, continuously compound, σ is the volatility of stock's returnes; this is the square root of the quadratic variation of the stock's log price process, the celebrated *Black-Scholes equation* is given by [17], [96], [150], [152], [151], [213]:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.94}$$

The substitution $V(S,t) = v(x,\tau)$, where $S = e^x$ (due to Euler) and $\tau = T - t$ (the time to maturity), results in the diffusion-type equation

$$v_t = \frac{1}{2}\sigma^2 v_{xx} + \left(r - \frac{1}{2}\sigma^2\right)v_x - rv, \tag{2.95}$$

which can be transformed into the standard heat equation for variable r and σ with the help of Lemma 1. The corresponding characteristic equation,

$$\mu'' - \left(4r + 2\frac{\sigma'}{\sigma}\right)\mu' + 4r\left(r - \frac{r'}{2r} + \frac{\sigma'}{\sigma}\right)\mu = 0, \tag{2.96}$$

can be solved explicitly when σ and r are constants. The standard solutions are given by

$$\mu_0 = \sigma^2 \tau e^{2r\tau}, \qquad \mu_1 = (1 - 2r\tau)e^{2r\tau} \tag{2.97}$$

and the corresponding fundamental solution can be obtain in a closed form [17]:

$$K_0(x,y,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left[-\frac{\left(x-y+\left(r-\sigma^2/2\right)\tau\right)^2}{2\sigma^2\tau}\right], \qquad \tau > 0 \tag{2.98}$$

from our equations (2.24)–(2.30). Then, by using initial conditions, V can be computed explicitly in terms of the error function, leading to Black-Scholes formula [17].

It is worth adding, concluding this example, that by Lemma 1 of the current chapter the following transformation:

$$\begin{aligned} v(x,\tau) &= \frac{1}{\sqrt{\mu(\tau)}} e^{\alpha(\tau)x^2+\delta(\tau)x+\kappa(\tau)} u(\xi,\tau_0), \\ \xi &= \beta(\tau)x+\varepsilon(\tau), \quad \tau_0 = \gamma(\tau), \end{aligned} \tag{2.99}$$

results in $u_{\tau_0} = u_{\xi\xi}$, where $\mu = \mu(0)\left(1-2\alpha(0)\sigma^2\tau\right)e^{2r\tau}$ and

$$\alpha = \frac{\alpha(0)}{1-2\alpha(0)\sigma^2\tau}, \qquad \beta = \frac{\beta(0)}{1-2\alpha(0)\sigma^2\tau}, \tag{2.100}$$

$$\gamma = \gamma(0) + \frac{\beta^2(0)\sigma^2\tau}{2\left(1-2\alpha(0)\sigma^2\tau\right)}, \tag{2.101}$$

$$\delta = \frac{\delta(0)-2\alpha(0)\delta_0\sigma^2\tau}{1-2\alpha(0)\sigma^2\tau}, \tag{2.102}$$

$$\varepsilon = \varepsilon(0) + \frac{\beta(0)\left(\delta(0)-\delta_0\right)\sigma^2\tau}{1-2\alpha(0)\sigma^2\tau}, \tag{2.103}$$

$$\kappa = \kappa(0) + \sigma^2\tau\frac{\delta^2(0)-2\delta(0)\delta_0+2\alpha(0)\delta_0^2\sigma^2\tau}{2\left(1-2\alpha(0)\sigma^2\tau\right)} \tag{2.104}$$

with $\delta_0 = \left(\sigma^2/2-r\right)/\sigma^2$. The classical subsitution [17],

$$v = u\left(x+\left(r-\sigma^2/2\right)\tau, \left(\sigma^2/2\right)\tau\right)e^{-r\tau}, \tag{2.105}$$

occurs when $\alpha(0) = \gamma(0) = \delta(0) = \varepsilon(0) = \kappa(0) = 0$ and $\beta(0) = \gamma(0) = 1$.

Example 6 In the one-factor Gaussian *Hull-White model* [192], the state of the market, at any instant time, is determined by one factor x. The interest rate $r(t)$, at time t, is given by

$$r(t) = r_0(t) + x(t), \tag{2.106}$$

where r_0 is a deterministic function, and x is a stochastically varying factor, which evolution is described by the stochastic differential equation

$$dx(t) = -\alpha x(t)\ dt + \sigma\ dB(t) \tag{2.107}$$

(α and σ are real positive constants) with respect to the pricing measure Q_0 [192]. The expectation value

$$f(x,t) = E_{Q_0}\left[e^{-\int_t^T r(s)\ ds} F(x(T))\middle| x(t) = x\right] \tag{2.108}$$

satisfies the partial differential equation

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} + \alpha x \frac{\partial f}{\partial x} + (r_0(t) + x) f \tag{2.109}$$

and the terminal condition is

$$f(x,T) = F(x) \qquad \text{for all } x \in \mathbb{R}. \tag{2.110}$$

The following substitution

$$f(x,t) = e^{-\int_t^T r_0(s)\ ds}\, g(x,\tau), \qquad \tau = T - t \tag{2.111}$$

reduces (2.109) to an autonomous form

$$g_\tau = \frac{1}{2}\sigma^2 g_{xx} - \alpha x g_x - x g. \tag{2.112}$$

The characteristic equation, $\mu'' + 2\alpha\mu' = 0$, has two standard solutions:

$$\mu_0 = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha\tau}\right), \qquad \mu_1 = 1 \tag{2.113}$$

and the corresponding Green function:

$$\begin{aligned} K_0(x,y,\tau) \quad = \quad & \frac{\sqrt{\alpha}}{\sigma\sqrt{\pi\left(e^{2\alpha\tau}-1\right)}} \exp\left[\left(\alpha + \frac{\sigma^2}{2\alpha^2}\right)\tau\right] \\ & \times \exp\left[-\frac{\alpha\left(x - y e^{\alpha\tau}\right)^2}{\sigma^2\left(e^{2\alpha\tau}-1\right)} - \frac{e^{\alpha\tau}-1}{\alpha\left(e^{\alpha\tau}+1\right)}\left(x + y + \frac{\sigma^2}{\alpha^2}\right)\right] \end{aligned} \tag{2.114}$$

for $\tau > 0$ can be found by the methods discussed in Section 1.2 of this chapter.

Example 7 Assuming (formally) $r = r_0 + r_1 V$ in the Black-Scholes equation (2.94), one gets a nonlinear equation of the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_0 \left(S\frac{\partial V}{\partial S} - V \right) + r_1 V \left(S\frac{\partial V}{\partial S} - V \right) = 0. \tag{2.115}$$

This modification of the Black-Scholes equation can be used for a mathematical description of market collapse. The substitution $V(S,t) = v(x,\tau)$, where $S = e^x$ and $\tau = T - t$, transforms (2.115) into the generalized Burgers-Huxley equation [119], [158].

Example 8 According to Ref. [181], the propagation of nonlinear magnetosonic waves is governed by a *modified Burgers equation*,

$$\frac{\partial \phi}{\partial \eta} + A(\eta)\,\phi \frac{\partial \phi}{\partial \xi} - B(\eta)\frac{\partial^2 \phi}{\partial \xi^2} + C(\eta)\,\phi = 0, \tag{2.116}$$

where $\phi(\xi,\eta)$ is the amplitude of the wave, $\xi = \int k_x\, dx + k_y y - \omega t$ and $\eta = \varepsilon x$ is the coordinate streched by a smallness parameter ε.

If $B = Ae^{-\int_0^\eta C(s)\, ds}$, the following substitution

$$\phi = e^{-\int_0^\eta C(s)\, ds}\, \psi(z,t) \tag{2.117}$$

with

$$z = \xi, \qquad t = \int_0^\eta B(\tau)\, d\tau \tag{2.118}$$

transforms the nonautonomous equation (2.116) into the Burgers equation

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial z} = \frac{\partial^2 \psi}{\partial z^2} \tag{2.119}$$

that is completely integrable. The Burgers equation and extensions of it are studied in the next chapter.

Further examples can be found in Refs. [44], [120], [133], [154] and [206].

2.6 Concluding Remarks

This chapter has been concerned with the Cauchy inital value problem for certain diffusion-type equations on $\mathbb{R}$. Two different methods were discussed to construct the explicit solution of the Cauchy problem for a generalized diffusion-type equation. In the first method ideas from the transformation theory were adopted to reduce down the inhomogeneous and nonautonomous master

equation to the standard heat equation based on the fact that the emerging Riccati-type system is completely integrable. The symmetries of the nonautonomous heat equation (2.6) were also evaluated from the point of view of the transformation presented in *Lemma* 1. In the second method discussed the Cauchy problem was solved in terms of eigenfunction expansion following similar published work on the case of the corresponding Schrödinger equation. The two methods discussed allow to connect certain nonautonomous and inhomogeneous diffusion-type equation with solutions of the Riccati-type system.

Finally, key examples were presented in order to corroborate the proposed solution methods. Examples discussed include the cable equation which appears in different fields of Science, the Fokker-Planck equation from Physics, the Kolmogorov forward equation from stochastic differential equations, the Black-Scholes equation and the Hull-White model from finance, among others. The last example presented was a modified Burgers equation that is habitually used in physics to describe the propagation of magnetosonic waves. This variety of examples confirm the utility of the results from this chapter and the wide range of positive impact it could have is unquestionable.

2.7 Appendix A: Proof of the *Lemma 2*

Proof. Consider equations (2.21), (2.22) and (2.23). By (2.22) the fundamental solution of the Cauchy initial value problem for the diffusion-type equation (2.6) subject to suitable initial data on the entire real line $-\infty < x < \infty$ can be rewritten as

$$K(x,y,t) = \int_{-\infty}^{\infty} K_0(x,z,t)\, K(z,y,0)\, dz \tag{2.120}$$

with

$$K(x,y,t) = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\delta(t)x+\varepsilon(t)y+\kappa(t)} \tag{2.121}$$

$$K_0(x,z,t) = \frac{1}{\sqrt{2\pi\mu_0(t)}}\, e^{\alpha_0(t)x^2+\beta_0(t)xz+\gamma_0(t)z^2+\delta_0(t)x+\varepsilon_0(t)z+\kappa_0(t)} \tag{2.122}$$

$$K(z,y,0) = \frac{1}{\sqrt{\mu(0)}} e^{\alpha(0)z^2+\beta(0)zy+\gamma(0)y^2+\delta(0)z+\varepsilon(0)y+\kappa(0)} \tag{2.123}$$

The right hand side (RHS) of equation (2.120) is given by

$$\int_{-\infty}^{\infty} K_0(x,z,t)\, K(z,y,0)\, dz = \frac{e^{\gamma(0)y^2+\varepsilon(0)y+\kappa(0)}}{\sqrt{2\pi\mu_0\mu(0)}} e^{\alpha_0 x^2+\delta_0 x+\kappa_0} \int_{-\infty}^{\infty} e^{f(x,y,z)} dz$$

where $f(x,y,z) = (\alpha(0)+\gamma_0)z^2 + (\beta_0 x + \beta(0)y + \varepsilon_0 + \delta(0))z$. With the aid of equality (2.39) using $a = -(\alpha(0)+\gamma_0)$ and $2b = (\beta_0 x + \beta(0)y + \varepsilon_0 + \delta(0))$ we solve the integral

$$\int_{-\infty}^{\infty} e^{f(x,y,z)}dz = \sqrt{\frac{\pi}{-(\alpha(0)+\gamma_0)}} e^{\frac{(\beta_0 x+\beta(0)y+\varepsilon_0+\delta(0))^2}{-4(\alpha(0)+\gamma_0)}}. \tag{2.124}$$

Consider in the last equality

$$F_1 = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)$$

$$F_2 = \alpha_0(t)x^2 + \gamma(0)y^2 + \delta_0(t)x + \varepsilon(0)y + (\kappa(0) + \kappa_0(t))$$

$$F_3 = \frac{(\beta_0(t)x + \beta(0)y)^2 + 2(\beta_0(t)x + \beta(0)y)(\delta(0) + \varepsilon_0(t)) + (\delta(0) + \varepsilon_0(t))^2}{-4(\alpha(0) + \gamma_0(t))}.$$

Therefore, equation (2.120) becomes

$$\frac{1}{\sqrt{\mu\,(t)}} e^{F_1} = \frac{1}{\sqrt{-2\mu(0)\mu_0(\alpha(0)+\gamma_0(t))}} e^{F_2} e^{F_3} \tag{2.125}$$

and this equality holds whenever (2.32)–(2.38) is true. Therefore, the general solution of the Riccati system (2.9)–(2.15) is given by expressions (2.32)–(2.38) as desired.

□

2.8 Appendix B: Verification of *Remark 1*

Proof. In *Lemma*1 the nonautonomous diffusion-type equation (2.6) is reduced to the standard heat equation by means of

$$u(x,t) = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2 + \delta(t)x + \kappa(t)} v(\xi, \tau) \tag{2.126}$$

where $\xi = \beta(t)x + \varepsilon(t)$ and $\tau = \gamma(t)$. Considering such transformation, let $\eta = \beta(0)y + \varepsilon(0)$ and $\tau_0 = \gamma(0)$. Then the original initial data can be rewritten as

$$u(y,0) = \frac{1}{\sqrt{\mu(0)}} e^{\alpha(0)y^2 + \delta(0)y + \kappa(0)} v(\eta, \tau_0). \tag{2.127}$$

Thus the solution in terms of ξ and τ variables has the standard form

$$v(\xi,\tau) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(\xi-\eta)^2}{4(\tau-\tau_0)}}}{\sqrt{4\pi(\tau-\tau_0)}} v(\eta,\tau_0)d\eta. \tag{2.128}$$

Notice that from (2.126)

$$v(\xi,\tau) = \sqrt{\mu(t)}u(x,t)e^{-(\alpha(t)x^2 + \delta(t)x + \kappa(t))}, \tag{2.129}$$

and similarly from (2.127)

$$v(\eta,\tau_0)=\sqrt{\mu(0)}u(y,0)e^{-(\alpha(0)y^2+\delta(0)y+\kappa(0))}. \tag{2.130}$$

For convenience we use:

$$\begin{aligned}
A_0 &= \alpha_0(t)x^2+\beta_0(t)xy+\gamma_0(t)y^2+\delta_0(t)y+\varepsilon_0(t)y+\kappa_0(t)\\
A_1 &= \alpha(t)x^2+\delta(t)x+\kappa(t)\\
A_2 &= \alpha(0)y^2+\delta(0)y+\kappa(0)\\
A_3 &= \frac{\beta^2x^2-2\beta\beta(0)xy-\beta^2(0)y^2+2(\varepsilon-\varepsilon(0))(\beta x-\beta(0)y)+(\varepsilon-\varepsilon(0))^2}{4(\gamma-\gamma(0))}.
\end{aligned}$$

Then, using (2.128) it is straightforward to see that the solution can be written as

$$u(x,t)=\beta(0)\sqrt{\frac{\mu(0)}{4\pi\mu(t)(\gamma-\gamma(0))}}\int_{-\infty}^{\infty}e^{A_1-A_2-A_3}u(y,0)dy \tag{2.131}$$

and using equation (2.23) the solution can be rewritten as

$$\frac{1}{\sqrt{2\pi\mu_0(t)}}\int_{-\infty}^{\infty}e^{A_0}u(y,0)dy=\beta(0)\sqrt{\frac{\mu(0)}{4\pi\mu(t)(\gamma-\gamma(0))}}\int_{-\infty}^{\infty}e^{A_1-A_2-A_3}u(y,0)dy$$

which holds whenever the Riccati system is true. Thus, subject to suitable initial data $u(y,0)$ the fundamental solution of (2.6) is given by (2.23) with coefficients resulting from the last equality due to solution of the Riccati system (2.10)–(2.15). □

2.9 Appendix C: Proof of *Lemma 3*

Proof. First notice that $(\alpha(0)+\gamma_0(t))$ is repeated throughout the *Riccati system*. From equation (2.44) we have that

$$(\alpha(0)+\gamma_0(t))=-\frac{\beta^2(0)}{4(\gamma(t)-\gamma(0))} \tag{2.132}$$

$$\gamma_0(t)=-\alpha(0)-\frac{\beta^2(0)}{4(\gamma(t)-\gamma(0))}. \tag{2.133}$$

Substitution of (2.132) into (2.41), (2.43) and (2.46) yields respectively to:

$$\mu_0(t)=\frac{2\mu(t)}{\mu(0)\beta^2(0)}(\gamma(t)-\gamma(0)) \tag{2.134}$$

$$\beta_0(t)=\frac{\beta(t)\beta(0)}{2(\gamma(t)-\gamma(0))} \tag{2.135}$$

$$\varepsilon_0(t)=-\delta(0)+\frac{\beta(0)(\varepsilon(t)-\varepsilon(0))}{2(\gamma(t)-\gamma(0))}. \tag{2.136}$$

Now, substituting (2.132) and (2.135) into (2.42) results in

$$\alpha_0(t) = \alpha(t) - \frac{\beta^2(t)}{4(\gamma(t)-\gamma(0))}. \tag{2.137}$$

Similarly, using (2.132) and (2.135)-(2.136) into (2.45) and (2.47) results respectively in

$$\delta_0(t) = \delta(t) - \frac{\beta(t)(\varepsilon(t)-\varepsilon(0))}{2(\gamma(t)-\gamma(0))}, \tag{2.138}$$

$$\kappa_0(t) = \kappa(t) - \kappa(0) - \frac{(\varepsilon(t)-\varepsilon(0))}{2(\gamma(t)-\gamma(0))} \tag{2.139}$$

as desired.

For the derivations of the corresponding asymptotics consider first the *Taylor expansion* of α, β and γ centered at 0 as follow:

$$\begin{aligned}
\alpha_0(t) &= \alpha(t) - \frac{\beta^2(t)}{4(\gamma(t)-\gamma(0))} \\
&\sim \alpha(0) + \alpha'(0)t - \left(\frac{\beta^2(0)+2\beta(0)\beta'(0)t}{4\gamma'(0)t}\right)\left(1 - \frac{\gamma''(0)t}{2\gamma'(0)}\right) + \mathscr{O}(t^2) \\
&\sim \alpha(0) + \alpha'(0)t - \frac{1+2(c(0)+4a(0)\alpha(0))}{4a(0)t} \\
&\quad + \frac{a'(0)+2a(0)(c(0)+4a(0)\alpha(0))}{8a^2(0)} + \mathscr{O}(t) \\
&\sim -\frac{1}{4a(0)t} - \frac{c(0)}{4a(0)t} + \frac{a'(0)}{8a^2(0)} + \mathscr{O}(t)
\end{aligned} \tag{2.140}$$

and similarly

$$\begin{aligned}
\beta_0(t) &= \frac{\beta(0)\beta(t)}{2(\gamma(t)-\gamma(0))} \\
&\sim \beta(0)\left(\frac{\beta(0)+\beta'(0)t}{2\gamma'(0)t}\right)\left(1 - \frac{\gamma''(0)t}{2\gamma'(0)}\right) + \mathscr{O}(t^2) \\
&\sim \frac{1+(c(0)+4a(0)\alpha(0))}{2a(0)t} - \frac{a'(0)+2a(0)(c(0)+4a(0)\alpha(0))}{4a^2(0)} + \mathscr{O}(t) \\
&\sim \frac{1}{2a(0)t} - \frac{a'(0)}{4a^2(0)} + \mathscr{O}(t)
\end{aligned} \tag{2.141}$$

as $t \to 0^+$. In order to obtain the corresponding asypmtotic for $\gamma_0(t)$ we can follow similar procedure as for (2.140) resulting in

$$\gamma_0(t) \sim -\frac{1}{4a(0)t} + \frac{c(0)}{4a(0)t} + \frac{a'(0)}{8a^2(0)} + \mathscr{O}(t). \tag{2.142}$$

Consider now equation (2.45). Then, for the asymptotic of $\delta_0(t)$ we have that:

$$\begin{aligned}\delta_0(t) &= \delta(t) - \frac{\beta(t)(\varepsilon(t)-\varepsilon(0))}{2(\gamma(t)-\gamma(0))} \\ &\sim \delta(0) - \frac{\beta(0)\varepsilon'(0)}{4\gamma'(0)} + \mathscr{O}(t^2) \\ &\sim \frac{g(0)}{2a(0)} + \mathscr{O}(t)\end{aligned} \tag{2.143}$$

and similarly

$$\varepsilon_0(t) \sim -\frac{g(0)}{2a(0)} + \mathscr{O}(t) \tag{2.144}$$

as $t \to 0^+$. Finally, considering the *Taylor expansion* of $\kappa(t)$ and $\varepsilon(t)$ centered at the zero, the asymptotic for $\kappa_0(t)$ results in

$$\begin{aligned}\kappa_0(t) &= \kappa(t) - \kappa(0) - \frac{(\varepsilon(t)-\varepsilon(0))}{2(\gamma(t)-\gamma(0))} \\ &\sim -\frac{\varepsilon'(0)t}{4\gamma'(0)}\left(1 - \frac{\gamma''(0)t}{2\gamma'(0)}\right) + \mathscr{O}(t^2) \\ &\sim \frac{g(0)}{2a(0)} + \mathscr{O}(t)\end{aligned} \tag{2.145}$$

as $t \to 0^+$. In the case of $\mu_0(t)$ the corresponding expansion is given by

$$\begin{aligned}\mu_0(t) &= \frac{(2\mu(0)+\mu'(0)t)(\gamma'(0)t)}{\mu(0)\beta^2(0)} + \mathscr{O}(t^3) \\ &\sim \frac{2\gamma'(0)}{\beta^2(0)}t + \mathscr{O}(t^2) \\ &\sim 2a(0)t + \mathscr{O}(t^2)\end{aligned} \tag{2.146}$$

with the aid of (2.132). Direct substitution of (2.140)-(2.146) into (2.23) results in the desired expresion (2.53). □

2.10 Appendix D: Proof of *Lemma 4*

Proof. The standard oscillatory wave functions for equation (2.60) can be given by

$$\psi_n = \frac{e^{(\alpha x^2+\delta x+\kappa)-(2n+1)\gamma-(\beta x+\varepsilon)^2/2}}{\sqrt{2^n n!\mu\sqrt{\pi}}} H_n(\beta x+\varepsilon) \tag{2.147}$$

where $H_n(x)$ are the Hermite polynomials, provided that solution to the *Ermakov-type system* is available. Considering the heat kernel given by (2.23) with corresponding coefficients given by

(2.24)–(2.30), we have that the corresponding Cauchy initial value problem can be solve formally once again by the superposition principle

$$\psi(x,t) = \int_{-\infty}^{\infty} K_0(x,y,t)\ \ \psi(y,0)\,dy \tag{2.148}$$

for certain initial data $\psi(y,0)$. Particularly, using the eigenfunction (2.147) we get

$$\psi_n(x,t) = \int_{-\infty}^{\infty} K_0(x,y,t)\ \ \psi_n(y,0)\,dy. \tag{2.149}$$

Uniqueness of the Cauchy initial value problem allows one to find the desired solution. Thus, the solution of the Ermakov-type system can be obtain by evaluating (2.149) with the help of

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{Z-\lambda^2(X-Y)^2} H_n(\upsilon Y)dy\\ &= e^Z \frac{\sqrt{\pi}}{\lambda^{n+1}} \left(\lambda^2-\upsilon^2\right)^{\frac{n}{2}} H_n\left(\frac{\lambda\upsilon X}{(\lambda^2-\upsilon^2)^{\frac{1}{2}}}\right), \qquad \lambda^2>0,\end{aligned} \tag{2.150}$$

with

$$\upsilon = a(0) \tag{2.151}$$

$$\lambda = \frac{1}{2}\beta^2(0) - (\gamma_0+\alpha(0)) \tag{2.152}$$

$$X = \frac{\beta_0 x+\varepsilon_0+\delta(0)-2\frac{\varepsilon(0)}{\beta(0)}(\gamma_0+\alpha(0))}{2\lambda^2} \tag{2.153}$$

$$Y = y+\frac{\varepsilon(0)}{\beta(0)} \tag{2.154}$$

$$Z = \lambda^2X^2+\frac{\varepsilon^2(0)}{\beta^2(0)}(\gamma_0+\alpha(0)) - \frac{\varepsilon(0)}{\beta(0)}(\beta_0 x+\varepsilon_0+\delta(0)) \tag{2.155}$$

The expression for Z arise when completing the square. □

Chapter 3

NONAUTONOMOUS BURGERS-TYPE EQUATIONS

3.1 Introduction

The study of nonlinear PDEs has been of great interest for many researchers since these models help them to understand and describe natural phenomena quite well arising in different fields of science. In Physics, it arose from the interest of researchers, like Isaac Newton, for the understanding of fluid dynamics. Newton was probably the most prominent forerunner in the study of fluids. There were many other important scientists such as Lord Kelvin, Daniel Bernoulli, Jean le Rond d'Alambert, and Leonard Euler who have added enormously to the understanding of fluid dynamics, however, Euler was the most instrumental in conceptualizing the mathematical description of a fluid flow. During the eighteen century a modification of Euler's work by Claude-Louis Navier and George Stokes lead to the Navier-Stokes equations. Early in the nineteen century (1904), Ludwig Prandtl revolutionized the understanding and analysis of fluid dynamics when introduced the concept of boundary layer in a fluid flow over a surface. Few years later, Harry Bateman (1915, [12]) first proposed the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = r\frac{\partial^2 v}{\partial x^2} \tag{3.1}$$

to illustrate the possibility of the solution of a viscous fluid becoming discontinuous when the viscosity term r approaches zero. This may be considered as the simplest equation combining both nonlinear convection $\left(v\frac{\partial v}{\partial x}\right)$ and diffusive effects $\left(\frac{\partial^2 v}{\partial x^2}\right)$. The same equation arose from the theoretical study of turbulence performed by J.M. Burgers (1948, [26]). In the context of gas dynamics, during an attempt to make equation (3.1) more tractable, Eberhard Hopf (1950, [98]) and Julian D. Cole (1951, [36]), found independently that the expression

$$v = -2r\frac{u_x}{u} \tag{3.2}$$

transforms equation (3.1) into the linear heat equation $u_t = u_{xx}$. Similar linearization procedures were used previously by A.R. Forsyth (1906, [74, p. 102]), and have been also used to solve certain generalized Riccati equations [180]. Later the equation (3.1) was named *Burgers equation* and transformation (3.2), which is of the B*ä*cklund type [180], [189], was named *Cole-Hopf* transformation. The equation (3.1) is nowdays one of the most fundamental nonlinear equations in the

study of PDE's and moreover, one of the few nonlinear solvable equations thanks to the remarkable Cole-Hopf transformation.

The Burgers equation is well known for its theoretical and applicative interest. This nonlinear PDE has a significant influence as a fluid dynamics model both for the understanding of a class of physical flows and for testing various numerical methods (2008, [237]). A vast amount of literature of numerical and analytical work regarding Burgers equation can be found in Chemistry [185], Biology [79], [104], [185], [188], [219], Engineering [45], [46], [185], [212], and in several branches of Physics [12], [26], [36], [98], [108], [181], [185] and [229].

Recently there has been considerable attention to study the solution of extensions of viscous Burgers equation, *e.g.* inhomogeneous Burgers equation, because of their applicability in diverse areas not only in fluid dynamics [4], [5], [6], [185], but also in other fields [79], [104], [185], [188], [219]. Solutions to the *viscous Burgers equation* (3.1) have been extensively investigated, see for example [12], [26], [108], [120], and [229]. However, there haven't been sufficient work focusing on the study and analysis of extensions of equation (3.1) despite the enormous importance they have in applications.

The nonautonomous and inhomogeneous extensions of equation (3.1) given by

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - r\frac{\partial^2 v}{\partial x^2} = F(x, v, v_x, t), \tag{3.3}$$

appears in several physical frameworks [111], [132], [180], [185]. Equations of such type are of great interest because they share, at some degree, a similar mathematical structure to Navier-Stokes system when the driving force term $F(x, v, v_x, t)$ is nonzero [180], [232].

The inhomogeneous part is usually considered as a deterministic or stochastic force driven by external entities [4], [5], [6], [132]. There exist a variety of physical situations in which external forcing can be realized and is of great interest [132], [185]. In fact, the forcing term may play the role of pressure gradient [232], could describe the dynamics of a physical system immersed in a system with energy pumping [168], or can be considered as a damping force [143], among many other applications.

Analytical works besides the classical *well-posed* problems on inhomogeneous Burgers equation (3.3) and extensions of it have been concentrated mostly in searching their corresponding exact

solutions. Most of the analytical works for Burgers-type equations have been performed considering the autonomous case and a few have considered the inhomogeneous case [23], [67], [170], [181], [185], [187], [188], [231]. Analytical results for inhomogeneous and nonautonomous Burgers-type equation are very scarce. This is indeed the great motivation to tackle these problems.

Throughout the rest of this chapter the focus will be set on the study of non-autonomous and inhomogeneous Burgers-type equations and its relation with diffusion equations presented in the previous chapter. The connections between linear (Diffusion-type) and nonlinear (Burgers-type) equations will be explored in order to establish a commutative relation. Traveling wave solutions of a nonautonomous Burgers equation are also studied.

3.2 Inhomogeneous Burgers Equation

A goal of this section is to solve the Cauchy initial value problem (IVP) for the inhomogeneous Burgers-type equation

$$\frac{\partial v}{\partial t}+a(t)\left(v\frac{\partial v}{\partial x}-\frac{\partial^2 v}{\partial x^2}\right)=c(t)\left(x\frac{\partial v}{\partial x}+v\right)-g(t)\frac{\partial v}{\partial x}+2(2b(t)x-f(t)) \tag{3.4}$$

coupled with certain arbitrary initial data. We shall refer to the evolution equation (3.4) as a nonautonomous *Burgers-type equation*; see also [178] and [181]. The connection of equation (3.4) with the master equation (2.6) of previous chapter is established in the following Lemma.

Lemma 5. *The following identity holds:*

$$\begin{aligned} &v_t+a\left(vv_x-v_{xx}\right)+\left(g-cx\right)v_x-cv+2\left(f-2bx\right)\\ &\qquad=-2\left(\frac{u_t-Qu}{u}\right)_x, \end{aligned} \tag{3.5}$$

if

$$v=-2\frac{u_x}{u}\qquad(\textit{The Cole–Hopf transformation}) \tag{3.6}$$

and

$$Qu=au_{xx}-\left(g-cx\right)u_x+\left(d+fx-bx^2\right)u \tag{3.7}$$

(a, b, c, d, f, g are functions of t only).

Proof. From the Cole–Hopf transformation (3.6) we have that:

$$v_t = -2\frac{u_{xt}}{u} + 2\frac{u_x u_t}{u^2}, \tag{3.8}$$

$$v_x = -2\frac{u_{xx}}{u} + \frac{1}{2}v^2, \tag{3.9}$$

$$v_{xx} = -2\frac{u_{xxx}}{u} + \frac{3}{2}vv_x - \frac{1}{4}v^3 \tag{3.10}$$

and by virtue of (3.7) we know that

$$Qu = au_{xx} - (g - cx)u_x + (d + fx - bx^2)u, \tag{3.11}$$

$$Q_x u = cu_x + (f - 2bx)u, \tag{3.12}$$

$$Qu_x = au_{xxx} - (g - cx)u_{xx} + (d + fx - bx^2)u_x. \tag{3.13}$$

In view of (3.8)–(3.13), the RHS of (3.5) becomes

$$-2\left(\frac{u_t - Qu}{u}\right)_x = v_t + \frac{2}{u}(Q_x u + Qu_x) + \frac{v}{u}Qu \tag{3.14}$$

$$\begin{aligned} &= v_t + c\left(2\frac{u_x}{u}\right) + 2(f - 2bx) + a\left(2\frac{u_{xxx}}{u}\right) - \left(2\frac{u_{xx}}{u}\right)(g - cx) \\ &+ \left(2\frac{u_x}{u}\right)(d + fx - bx^2) + v\left[a\left(\frac{u_{xx}}{u}\right) - \frac{u_x}{u}(g - cx) + (d + fx - bx^2)\right] \\ &= v_t + a(vv_x - v_{xx}) + (g - cx)v_x - cv + 2(f - 2bx) \end{aligned} \tag{3.15}$$

as desired. Backward procedure complete the proof. □

The substitution (3.6) turns the non-autonomous and inhomogeneous Burgers-type equation (3.4) into the diffusion-type equation (2.6). Thus, using the methods presented in the section (2.2) of the second chapter of this dissertation, the solution of the corresponding Cauchy initial value problem for equation (3.4) can be represented as

$$v(x,t) = -2\frac{\partial}{\partial x}\ln\left[\int_{-\infty}^{\infty} K_0(x,y,t)\exp\left(-\frac{1}{2}\int_0^y v(z,0)\ dz\right)\ dy\right], \tag{3.16}$$

where the heat kernel $K_0(x,y,t)$ is given by (2.23), for suitable initial data $v(z,0)$ on $\mathbb{R}$. Similarly, the diffusion-type equation (2.6) can be associated to another non-autonomous Burgers-type equation. This relation is contained in the following result.

Lemma 6. *The nonautonomous diffusion-type equation (2.6) can be transformed to the nonautonomous and inhomogeneous Burgers-type equation*

$$U_t + a\,(UU_x - U_{xx}) = \frac{\beta'}{\beta}\,(U + xU_x) + \frac{\varepsilon'}{\beta}U_x \tag{3.17}$$

with the aid of the extended Cole-Hopf transformation

$$U = -2\frac{u_x}{u} + 2(2\alpha x + \delta). \tag{3.18}$$

Proof. Let $\widetilde{U} = -2\frac{u_x}{u}$. Then we have from (3.18) that

$$\widetilde{U} = U - 2(2\alpha x + \delta). \tag{3.19}$$

From Lemma 5 we know that

$$\widetilde{U}_t + a\left(\widetilde{U}\widetilde{U}_x - \widetilde{U}_{xx}\right) + (g - cx)\,\widetilde{U}_x - c\tilde{U} + 2\,(f - 2bx) = -2\left(\frac{u_t - Qu}{u}\right)_x. \tag{3.20}$$

Substitution of space and time derivatives of (3.19) into (3.20) leads to

$$U_t + a\,(UU_x - U_{xx}) - \frac{\beta'}{\beta}\,(U + xU_x) - \frac{\varepsilon'}{\beta}U_x = -2\left(\frac{u_t - Qu}{u}\right)_x, \tag{3.21}$$

from which the desired commutative relation can be established. □

Following the same strategy, if we consider the transformation

$$U = \beta(t)V(\xi, \tau) \tag{3.22}$$

with $\xi = \beta(t)x + \varepsilon(t)$ and $\tau = \gamma(t)$, the corresponding space and time derivatives will be given by

$$U_t = \beta' V + \beta(\beta' x + \varepsilon')V_\xi + \beta\gamma' V_\tau \tag{3.23}$$

$$U_x = \beta^2 V_\xi \tag{3.24}$$

$$U_{xx} = \beta^3 V_{\xi\xi}. \tag{3.25}$$

Susbtitution of (3.22)–(3.25) into (3.17) yields

$$\beta' V + \beta(\beta' x + \varepsilon')V_\xi + \beta\gamma' V_\tau + a\beta^3\left(VV_\xi - V_{\xi\xi}\right) = \beta'(V + \beta x V_\xi) + \varepsilon'\beta V_\xi \tag{3.26}$$

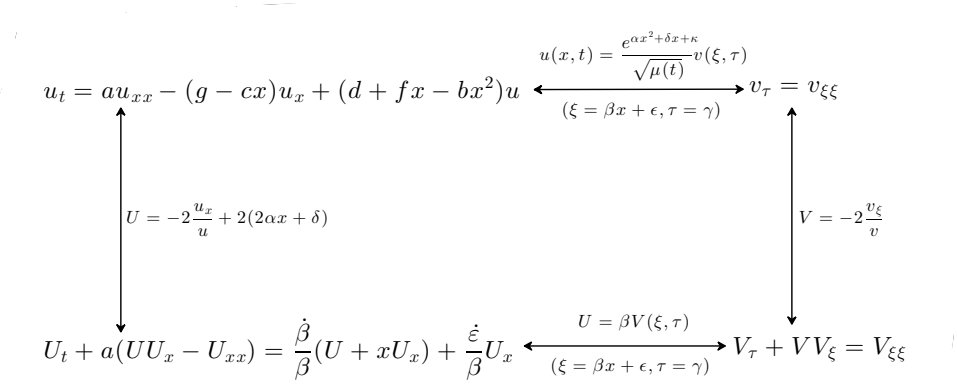

Figure 3.1: Commutative diagram summarizing the relations between the non-autonomous and inhomogeneous diffusion-type equation (2.6), the linear heat equation (2.7) and the Burgers-type equations (3.17) and (3.28). It is worth to mention that all the coefficients are time dependent.

simplifying to

$$\beta\gamma' V_\tau + a\beta^3 \left(VV_\xi - V_{\xi\xi}\right) = 0. \tag{3.27}$$

After dividing by $a\beta^3$ and using (2.12), equation (3.27) reduces to the Burgers equation

$$V_t + VV_x = V_{xx} \tag{3.28}$$

The connections between the inhomogeneous diffusion equation (2.6), the linear heat equation (2.7), the Burgers equation (3.28) and the non-autonomous and inhomogeneous Burgers equation (3.17) is portrayed in Figure 1. Equations (3.4) and (3.17) may be useful generalizations in a wide range of physical contexts, and could be used to test certain numerical schemes.

3.3 Traveling Wave Solutions

Traveling wave solutions describe a wide class of phenomena in different areas of science. These solutions often determine the behavior of the solutions of Cauchy-type problems [221]. The pioneers in studying the existence of such solutions for parabolic systems were A.N. Kolmogorov, I.G. Petrovskii, and N.S. Piskunov. Their mathematical results in existence of traveling wave solutions arose in connections with the 1937 Fisher's model for a propagation of dominant genes [122, 221]. After this prominent result the study of the propagation of waves, described specifically by parabolic equations, has become a very important subject in the understanding of many events occurring in chemistry, biology and physics.

Burgers equation is tremendously linked with situations involving wave phenomena. Traveling wave argument has been extensively used by many authors to solve the Burgers equation. Consider now the viscous Burgers equation (3.1). Following the original Bateman paper [12] with slightly different nomenclature, equation (3.1) possesses a solution of the form:

$$v = F(x+Vt), \qquad V = \text{constant} \tag{3.29}$$

if

$$VF' + FF' = aF'', \tag{3.30}$$

or

$$(F+V)^2 \pm A^2 = 2aF', \tag{3.31}$$

where A is a positive constant. The solution is thus either

$$v + V = A\tan\left[\frac{A(x+Vt-c)}{2a}\right] \tag{3.32}$$

or

$$\frac{A-v-V}{A+v+V} = \exp\left[\frac{A}{a}(x+Vt-c)\right], \tag{3.33}$$

according as the $+$ or $-$ sign is taken. In the first case there is no definite value of v when a tends to zero, while in the second case the limiting value of v is either $A-V$ or $A+V$ according as $x+Vt$ is less or greater than c. The limiting form of the solution is thus discontinuous [12].

Generalized Traveling Wave Solution.

It is well known that traveling wave solutions of partial differential equations are solutions of specific shape that usually don't change in time. The study of traveling wave transformation with non-autonomous coefficients seems to be poorly studied in the available literature. Then, looking for solutions to equation (3.4) in the general form

$$v = \beta(t)F(\beta(t)x+\gamma(t)) = \beta F(z), \quad z = \beta x+\gamma \tag{3.34}$$

(β and γ are functions of t only), one gets

$$F'' = (c_0+c_1)F' + FF' + 2c_2z + c_3 \tag{3.35}$$

provided that

$$\beta' = c\beta, \qquad \gamma' = c_0 a\beta^2, \tag{3.36}$$

$$g = c_1 a\beta, \qquad b = -\frac{1}{2}c_2 a\beta^4, \tag{3.37}$$

$$f = \frac{1}{2}a\beta^3\left(2c_2\gamma + c_3\right) \tag{3.38}$$

(c_0, c_1, c_2, c_3 are constants). Integration of (3.35) leads to:

$$F' = (c_0 + c_1)F + \frac{1}{2}F^2 + c_2 z^2 + c_3 z + c_4, \tag{3.39}$$

where c_4 is a constant of integration. The substitution

$$F = -2\frac{\mu'}{\mu} \tag{3.40}$$

transforms the Riccati equation (3.39) into a special case of generalized equation of hypergeometric type:

$$\mu'' - (c_0 + c_1)\mu' + \frac{1}{2}\left(c_2 z^2 + c_3 z + c_4\right)\mu = 0, \tag{3.41}$$

which can be solved in general by methods of Ref. [163]. Elementary solutions are discussed, for example, in [119] and [120].

3.4 Concluding Remarks

In this chapter, the emphasis was devoted to the study of Burgers-type equations and its relations with the master diffusion-type (2.6) equation presented in the second chapter of this disertation. The results from the second chapter were the key tool to establish such relations. Traveling wave solutions of the Burgers-type equations were also discussed in terms of the Riccati system. The results presented in this chapter are another tool to justify the utility and efficacy of the solution methods for the proposed generalized diffusion-type equations presented in the second chapter. It is believed that the explicit results of this chapter could be used to corroborate the efficacy of numerical algorithms to solve familiar systems.

Chapter 4

THE MINIMUM-UNCERTAINTY SQUEEZED STATES FOR QUANTUM HARMONIC OSCILLATORS

4.1 Introduction

The two previous chapters were devoted to study diffusion-type equations of the form,

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2} - (g(t) - c(t)x)\frac{\partial u}{\partial x} + \left(d(t) + f(t)x - b(t)x^2\right)u \tag{4.1}$$

and its relation with certain Burgers-type equations. Following W. Miller in [154], the substitution $t \to -it$, $a \to -a$, $b \to -b$, $c \to -ic$, $d \to -id$, $f \to -if$ and $g \to -ig$ tranform the master equation (4.1) into the one dimensional time–dependent Schrödinger equation given by,

$$i\frac{\partial u}{\partial t} = -a(t)\frac{\partial^2 u}{\partial x^2} + b(t)x^2 u - i\left(c(t)x - d(t)\frac{\partial u}{\partial x}\right) - f(t)xu + ig(t)\frac{\partial u}{\partial x}. \tag{4.2}$$

This equation has the most general variable quadratic Hamiltonian and has been extensively studied [38], [39], [41], [42], [43], [124], [126], [133], [149], [193], [208], [209] due to a wide range of applications in different areas of physics, particularly in quantum optics where it has a close relation with the process of dynamic amplification. Another application can be found in the Casimir effect, where coherent and squeezed states play an important role.

From the very beginning, nonclassical states of the linear Planck oscillator, in particular the coherent and squeezed states, have been a subject of considerable interest in quantum physics [49], [56], [80], [113], [115], [194], [195] and the references therein. They occur naturally on an atomic scale [24], [107] and, possibly, can be observed among vibrational modes of crystals and molecules [59], [70]. A single monochromatic mode of light also represents a harmonic oscillator system for which nonclassical states can be generated very efficiently by using the interaction of laser light with nonlinear optical media [21], [130], [139], [140], [142], [174], [191], [223]. Generation of squeezed light with a single atom has been experimentally demonstrated [166]. On a macroscopic scale, the squeezed states are utilized for detection of gravitational waves [94] below the so-called vacuum noise level and without violation of the uncertainty relation [1], [61], [169], [216].

The past decades progress in generation of pure quantum states of motion of trapped particles provides not only a clear illustration of basic principles of quantum mechanics, but it also manifests

the ultimate control of particle motion. These states are of interest from the standpoint of quantum measurement concepts and facilitate other applications including quantum computation [19], [30], [33], [35], [78], [85], [86], [107], [129], [146], [155], [156], [167], [177], [183].

It is well known that the harmonic quantum states can be analyzed through the dynamics of a single, two-level atom which radiatively couples to the single mode radiation field in the Jaynes–Cummings(–Paul) model [29], [32], [106], [129], [186], [198], [220] extensively studied in the cavity QED [60], [85], [172], [173]. Creation and detection of thermal, Fock, coherent, and squeezed states of motion of a single $^{9}Be^{+}$ ion confined in a rf Paul trap was reported in [146], where the state of atomic motion had been observed through the evolution of the atom's internal levels (e.g., collapse and revival) under the influence of a Jaynes–Cummings interaction realized with the application of external (classical) fields. The distribution over the Fock states is deduced from an analysis of Rabi oscillations.

Moreover, Fock, coherent, and squeezed states of motion of a harmonically bound cold cesium atoms were experimentally observed in a $1D$ optical lattice [19], [156]. This method gives a direct access to the momentum distribution through the square of the modulus of the wave function in velocity space (see also [31], [32], [34], [37], [47], [86], [105], [107], [129], [167], [217] and the references therein regarding cold trapped ions and their nonclassical states; progress in atomic physics and quantum optics using superconducting circuits is reviewed in [76], [234]).

Recent reports on observations of the dynamical Casimir effect [123], [227] strengthen the interest to the nonclassical states of generalized harmonic oscillators [49], [50], [54], [55], [57], [84], [144], [145], [159], [209] and [222]. The amplification of quantum fluctuations by modulating parameters of an oscillator is closely related to the process of particle production in quantum fields [50], [103], [145], and [159]. Other dynamical amplification mechanisms include the Unruh effect [215] and Hawking radiation [16], [89], [90].

The purpose of this chapter is to construct the minimum-uncertainty squeezed states for quantum harmonic oscillators, which are important in these applications, in the most simple closed form. The approach adopted here reveals the "hidden" quantum numbers/integrals of motion of the squeezed states in terms of solution of certain Ermakov-type system [134], [135]. The corresponding generalizations of Fock states, which were originally found in [147] and recently rediscovered

in [135], are discussed in detail. As a result, the probability amplitudes of these nonclassical states of motion are explicitly evaluated in terms of hypergeometric functions. Their experimental observations in cavity QED and quantum optics are briefly reviewed. Moreover, the radiation field operators of squeezed photons, which can be created from the QED vacuum, are introduced by second quantization with the aid of hidden symmetry of harmonic oscillator problem in the Heisenberg picture.

In summary, experimental recognitions of the nonclassical harmonic states of motion have been achieved through reconstruction of the Wigner function in optical quantum-state tomography [21], [142], from a Fourier analysis of Rabi oscillations of a trapped atom [146], and/or by a direct observation of the square of the modulus of the wave function for a large sample of cold cesium atoms in a $1D$ optical lattice [19], [156]. The theoretical consideration presented herein complements all of these advanced experimental techniques by identifying the state quantum numbers from first principles. This approach may provide a guidance for engineering more advanced nonclassical states.

The rest of the chapter is organized as follows. In sections 4.2 and 4.3, the minimum-uncertainty squeezed states for the linear harmonic oscillator in the coordinate representation is described. The generalized coherent, or TCS states, are constructed in section 4.4. In sections 4.4 and 4.5, the Wigner and Moyal functions of the squeezed states are evaluated directly from the corresponding wave functions and their classical time evolution is verified with the help of a computer algebra system. The eigenfunction expansions of the squeezed (or generalized harmonic) states in terms of the standard Fock ones are derived in section 4.6 (see also [53], [58] and the references therein for important special cases). Some experiments on engineering of nonclassical states of motion are analyzed in section 4.7. Here, the experimentally observed probability distributions are derived from the explicit expression for the probability amplitudes obtained in the previous section. In section 4.8, the radiation field quantization in a perfect cavity, which is important for applications to quantum optics is revisited. Nonstandard solutions of the Heisenberg equations of motion for the electromagnetic field operators, that naturally describe squeezing in the Heisenberg picture, are found. The variance of the number operator, which together with the eigenfunction expansion allows to compare the obtained results with experimentally observed squeezed photon statistics [21], [191], is evaluated from first principles in section 4.9. A brief summary is provided in section 4.10.

A convenient complex parametrization of the Schrödinger group is provided in section 4.11.

4.2 The Minimum-Uncertainty Squeezed States

The Heisenberg Uncertainty Principle is one of the fundamental laws of nature and the coherent states that minimize this uncertainty relation are well known. But, equally important in recent developments, minimum-uncertainty squeezed states are not so familiar outside a relatively narrow group of experts. Here these states are constructed as explicitly as possible and some of their remarkable features are discussed.

The time-dependent Schrödinger equation for the simple harmonic oscillator in one dimension,

$$2i\psi_t + \psi_{xx} - x^2\psi = 0, \tag{4.3}$$

has the following square integrable solution (Gaussian wave packet)

$$\psi_0(x,t) = \frac{e^{i\left(\alpha(t)x^2+\delta(t)x+\kappa(t)+\gamma(t)\right)}}{\sqrt{\mu(t)\sqrt{\pi}}}\, e^{-(\beta(t)x+\varepsilon(t))^2/2}, \tag{4.4}$$

where

$$\mu(t) = \mu_0\sqrt{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2}, \tag{4.5}$$

$$\alpha(t) = \frac{\alpha_0\cos 2t + \sin 2t\,\left(\beta_0^4+4\alpha_0^2-1\right)/4}{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2}, \tag{4.6}$$

$$\beta(t) = \frac{\beta_0}{\sqrt{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2}}, \tag{4.7}$$

$$\gamma(t) = \gamma_0 - \frac{1}{2}\arctan\frac{\beta_0^2\tan t}{1+2\alpha_0\tan t}, \tag{4.8}$$

$$\delta(t) = \frac{\delta_0(2\alpha_0\sin t+\cos t)+\varepsilon_0\beta_0^3\sin t}{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2}, \tag{4.9}$$

$$\varepsilon(t) = \frac{\varepsilon_0(2\alpha_0\sin t+\cos t)-\beta_0\delta_0\sin t}{\sqrt{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2}}, \tag{4.10}$$

$$\kappa(t) = \kappa_0 + \sin^2 t\,\frac{\varepsilon_0\beta_0^2(\alpha_0\varepsilon_0-\beta_0\delta_0)-\alpha_0\delta_0^2}{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2} + \frac{1}{4}\sin 2t\,\frac{\varepsilon_0^2\beta_0^2-\delta_0^2}{\beta_0^4\sin^2 t + (2\alpha_0\sin t+\cos t)^2} \tag{4.11}$$

($\mu_0 > 0$, α_0, $\beta_0 \neq 0$, γ_0, δ_0, ε_0, κ_0 are real initial data of the corresponding Ermakov-type system; a complex form of equations (4.5)–(4.11) is provided in section 4.11 and the invariants are given by

(4.62)–(4.63)). It is worth to mention that this solution is invariant under the time reversal $t \to -t$, $\psi \to \psi^*$ with $\alpha_0 \to -\alpha_0$, $\gamma_0 \to -\gamma_0$, $\delta_0 \to -\delta_0$, and $\kappa_0 \to -\kappa_0$. This quantum state is the special case $n = 0$ of a 'nonclassical' oscillator solution found in [147], which has also been recently derived in a unified approach to generalized harmonic oscillators (see, for example, [38], [41], [126], [135] and the references therein). These solutions are verified by a direct substitution with the aid of `Mathematica` computer algebra system [116], [117], [135], and [137]. The simplest special case $\mu_0 = \pm\beta_0 = 1$ and $\alpha_0 = \gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$ reproduces the ground oscillator state [73], [81], [125], [153]; see also the original Schrödinger papers [194], [195]. For the coherent states $\alpha_0 = 0$ and $\beta_0 = \pm 1$, see [195] and a more general case when $\alpha_0 = 0$ is discussed in [92], [95]. More details on the derivation of these formulas can be found in Refs. [134], [135], and [147]. An analog of Berry's phase is evaluated in Refs. [210], [211].

The "dynamic harmonic oscillator ground state" (4.4)–(4.11) is the eigenfunction,

$$E(t)\,\psi_0(x,t) = \frac{1}{2}\,\psi_0(x,t)\,, \tag{4.12}$$

of the time-dependent dynamic invariant,

$$\begin{aligned} E(t) &= \frac{1}{2}\left[\frac{(p-2\alpha x-\delta)^2}{\beta^2} + (\beta x+\varepsilon)^2\right] \\ &= \frac{1}{2}\left[\widehat{a}(t)\,\widehat{a}^{\dagger}(t) + \widehat{a}^{\dagger}(t)\,\widehat{a}(t)\right], \quad \frac{d}{dt}\langle E\rangle = 0, \end{aligned} \tag{4.13}$$

with the required operator identity [55], [57], [182]:

$$\frac{\partial E}{\partial t} + i^{-1}[E,H] = 0, \qquad H = \frac{1}{2}\left(p^2+x^2\right). \tag{4.14}$$

The time-dependent annihilation $\widehat{a}(t)$ and creation $\widehat{a}^{\dagger}(t)$ operators are given by

$$\begin{aligned} \widehat{a}(t) &= \frac{1}{\sqrt{2}}\left(\beta x+\varepsilon+i\frac{p-2\alpha x-\delta}{\beta}\right), \\ \widehat{a}^{\dagger}(t) &= \frac{1}{\sqrt{2}}\left(\beta x+\varepsilon-i\frac{p-2\alpha x-\delta}{\beta}\right) \end{aligned} \tag{4.15}$$

with $p = i^{-1}\partial/\partial x$ in terms of solutions (4.6)–(4.11) [135]. These operators satisfy the canonical commutation relation,

$$\widehat{a}(t)\,\widehat{a}^{\dagger}(t) - \widehat{a}^{\dagger}(t)\,\widehat{a}(t) = 1, \tag{4.16}$$

and the oscillator-type spectrum of the dynamic invariant E can be obtained in a standard way by using the Heisenberg–Weyl algebra of the rasing and lowering operators (a "second quantization", the Fock states [135]). In particular,

$$\widehat{a}(t)\,\Psi_0(x,t)=0,\quad \psi_0(x,t)=e^{i\gamma(t)}\,\Psi_0(x,t)\,, \tag{4.17}$$

with $\varphi_0(t)=-\gamma(t)$ being the nontrivial Lewis phase [131], [182].

This form of quadratic dynamic invariant and the corresponding creation and annihilation operators for the generalized harmonic oscillators have been introduced recently in Ref. [182] (see also [41], [208] and the references therein for important special cases). An application to the electromagnetic-field quantization and a generalization of the coherent states are discussed in Refs. [118] (see also section 4.8) and [127], respectively.

The key ingredients, the maximum kinematical invariance groups of the free particle and harmonic oscillator, were introduced in [7], [8], [83], [102], [160], and [161] (see also [20], [109], [154], [178], [206], [207] and the references therein). The connection with the Ermakov-type system allows to bypass a complexity of the traditional Lie algebra approach [134], [135] (see [66], [128] and the references therein regarding the Ermakov equation). (A general procedure of obtaining new solutions by acting on any set of given ones by enveloping algebra of generators of the Heisenberg–Weyl group is described in [57]; see also [10], [13], [55], and [147] regarding the corresponding wavefunctions.) Finally, it is worth noting that the maximal invariance group of the generalized driven harmonic oscillators is isomorphic to the Schrödinger group of the free particle [134], [135], [160], and [161].

4.3 The Uncertainty Relation and Squeezing

A quantum state is said to be "squeezed" if its oscillating variances $\langle(\Delta p)^2\rangle$ and $\langle(\Delta x)^2\rangle$ become smaller than the variances of the "static" vacuum state $\langle(\Delta p)^2\rangle=\langle(\Delta x)^2\rangle=1/2$ (with $\hbar=1$). For the harmonic oscillator, the product of the variances attains a minimum value only at the instances when one variance is a minimum and the other is a maximum. If the minimum value of the product is equal to $1/4$, then the state is called a minimum-uncertainty squeezed state (see, for example, [60], [88], [197], [202], [203], [224], and [235]). This property can be easily verified for solution (4.4).

According to (4.15), the corresponding expectation values oscillate sinusoidally in time

$$\langle x\rangle = -\frac{1}{\beta_0}\left[(2\alpha_0\varepsilon_0 - \beta_0\delta_0)\sin t + \varepsilon_0\cos t\right], \quad \frac{d}{dt}\langle x\rangle = \langle p\rangle, \tag{4.18}$$

$$\langle p\rangle = -\frac{1}{\beta_0}\left[(2\alpha_0\varepsilon_0 - \beta_0\delta_0)\cos t - \varepsilon_0\sin t\right], \quad \frac{d}{dt}\langle p\rangle = -\langle x\rangle \tag{4.19}$$

with the initial data $\langle x\rangle|_{t=0} = -\varepsilon_0/\beta_0$ and $\langle p\rangle|_{t=0} = -\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)/\beta_0$. This provides a classical interpretation of the "hidden" parameters.

The expectation values $\langle x\rangle$ and $\langle p\rangle$ satisfy the classical equation for harmonic motion, $y'' + y = 0$, with the total "classical mechanical energy" given by

$$\frac{1}{2}\left[\langle p\rangle^2 + \langle x\rangle^2\right] = \frac{\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 + \varepsilon_0^2}{2\beta_0^2} = \frac{1}{2}\left[\langle p\rangle^2 + \langle x\rangle^2\right]\Big|_{t=0}. \tag{4.20}$$

For the standard deviations on solution (4.4)–(4.11), one gets

$$\begin{aligned}\left\langle(\Delta p)^2\right\rangle &= \langle p^2\rangle - \langle p\rangle^2 \qquad (4.21)\\ &= \frac{1 + 4\alpha_0^2 + \beta_0^4 + \left(4\alpha_0^2 + \beta_0^4 - 1\right)\cos 2t - 4\alpha_0\sin 2t}{4\beta_0^2},\\ \left\langle(\Delta x)^2\right\rangle &= \langle x^2\rangle - \langle x\rangle^2 \qquad (4.22)\\ &= \frac{1 + 4\alpha_0^2 + \beta_0^4 - \left(4\alpha_0^2 + \beta_0^4 - 1\right)\cos 2t + 4\alpha_0\sin 2t}{4\beta_0^2},\end{aligned}$$

and

$$\begin{aligned}\left\langle(\Delta p)^2\right\rangle\left\langle(\Delta x)^2\right\rangle &= \frac{1}{16\beta_0^4} \qquad (4.23)\\ &\times\left[\left(1 + 4\alpha_0^2 + \beta_0^4\right)^2 - \left(\left(4\alpha_0^2 + \beta_0^4 - 1\right)\cos 2t - 4\alpha_0\sin 2t\right)^2\right].\end{aligned}$$

Here,

$$\sigma_p = \left\langle(\Delta p)^2\right\rangle = \frac{4\alpha^2 + \beta^4}{2\beta^2}, \quad \sigma_x = \left\langle(\Delta x)^2\right\rangle = \frac{1}{2\beta^2}, \tag{4.24}$$

$$\sigma_{px} = \frac{1}{2}\langle\Delta p\Delta x + \Delta x\Delta p\rangle = \frac{\alpha}{\beta^2} \tag{4.25}$$

with two invariants:

$$\sigma_p + \sigma_x = \frac{4\alpha^2 + \beta^4 + 1}{2\beta^2} = \frac{4\alpha_0^2 + \beta_0^4 + 1}{2\beta_0^2}, \tag{4.26}$$

$$\text{and} \quad \begin{vmatrix} \sigma_p & \sigma_{px} \\ \sigma_{px} & \sigma_x \end{vmatrix} = \sigma_p\sigma_x - \sigma_{px}^2 = \frac{1}{4}$$

(More invariants are given by in (4.62)–(4.63).) A family of minimum-uncertainty states, when $\langle(\Delta p)^2\rangle = \langle(\Delta x)^2\rangle = 1/2$, is defined by taking $\alpha_0 = 0$ and $\beta_0^2 = 1$.

By adding (4.20)–(4.22), follows that

$$\begin{aligned}\langle H\rangle &= \frac{1}{2}\left[\langle p^2\rangle + \langle x^2\rangle\right] \\ &= \frac{1+4\alpha_0^2+\beta_0^4}{4\beta_0^2} + \frac{\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 + \varepsilon_0^2}{2\beta_0^2} \geq \frac{1}{2}\end{aligned} \tag{4.27}$$

for the total "quantum mechanical energy" in terms of the "hidden" parameters or integrals of motion presented herein (the vacuum value $1/2$ occurs when $\beta_0 = \pm 1$ and $\alpha_0 = \delta_0 = \varepsilon_0 = 0$). See also [21] and [58].

Therefore, the upper and lower bound in the Heisenberg uncertainty relation are given by

$$\max\left[\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle\right] = \frac{\left(1+4\alpha_0^2+\beta_0^4\right)^2}{16\beta_0^4}, \quad \text{if} \quad \cot 2t = \frac{4\alpha_0}{4\alpha_0^2+\beta_0^4-1} \tag{4.28}$$

and

$$\min\left[\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle\right] = \frac{1}{4}, \qquad \text{when} \quad \tan 2t = -\frac{4\alpha_0}{4\alpha_0^2+\beta_0^4-1}. \tag{4.29}$$

The explicit formulas (4.21)–(4.22) show that the product of the variances attains the minimum value $1/4$ only at the instances that one variance is a minimum and the other is a maximum as stated in [88]. The corresponding squeezing of one of the variances is also explicitly described by the formulas presented above. Indeed, one gets

$$\left(4\alpha_0^2+\beta_0^4-1\right)\cos 2t - 4\alpha_0 \sin 2t = \pm\left(4\alpha_0^2+\left(\beta_0^2+1\right)^2\right)^{\frac{1}{2}}\left(4\alpha_0^2+\left(\beta_0^2-1\right)^2\right)^{\frac{1}{2}},$$

under the minimization condition (4.29) and at the minimum

$$\begin{aligned}\langle(\Delta p)^2\rangle &= \frac{1}{4\beta_0^2}\left[1+4\alpha_0^2+\beta_0^4 \pm \left(4\alpha_0^2+\left(\beta_0^2+1\right)^2\right)^{\frac{1}{2}}\left(4\alpha_0^2+\left(\beta_0^2-1\right)^2\right)^{\frac{1}{2}}\right], \\ \langle(\Delta x)^2\rangle &= \frac{1}{4\beta_0^2}\left[1+4\alpha_0^2+\beta_0^4 \mp \left(4\alpha_0^2+\left(\beta_0^2+1\right)^2\right)^{\frac{1}{2}}\left(4\alpha_0^2+\left(\beta_0^2-1\right)^2\right)^{\frac{1}{2}}\right]\end{aligned}$$

for all real values of our parameters. At this instant the squeezing occur:

$$\langle(\Delta p)^2\rangle > \frac{1}{2}\left(< \frac{1}{2}\right), \qquad \langle(\Delta x)^2\rangle < \frac{1}{2}\left(> \frac{1}{2}\right)$$

(for upper and lower signs, respectively). As a result, the minimum-uncertainty squeezed states for the simple harmonic oscillator is presented in the closed form (4.6)–(4.11) (see also [88] for

numerical simulations). These states form a six-parameter family (a natural generalization will be discussed in the next section). The corresponding wave function in the momentum representation is derived by the (inverse) Fourier transform of (4.4) and (4.5)–(4.11) [135]. Experimentally observed time-oscillations of the velocity variance [156] reveal certain damping, which can be explain in models of quantum damped oscillators discussed in [40], [41], and [55].

Example. In a special case, one simplifies

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{1-2\alpha_0 \sin 2t}{2\beta_0^2}, \tag{4.30}$$

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{1+2\alpha_0 \sin 2t}{2\beta_0^2}, \tag{4.31}$$

provided that $4\alpha_0^2 + \beta_0^4 = 1$. In the case of the Schrödinger ground state "static" solution [195], when $\alpha_0 = \delta_0 = \varepsilon_0 = 0$ and $\beta_0 = \pm 1$, we arrive at $\langle x\rangle = \langle p\rangle \equiv 0$ and

$$\langle(\Delta p)^2\rangle = \langle(\Delta x)^2\rangle = \frac{1}{2} \tag{4.32}$$

as presented in the textbooks [73], [81], [82], [88], [125], [153]. In general, dependence on the quantum number n, which disappears from the Ehrenfest theorem [62], [87], is coming back at the level of the higher moments of distribution [135].

According to (4.30)–(4.31),

$$\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle = \frac{1-4\alpha_0^2 \sin^2 2t}{4\beta_0^4}, \qquad 4\alpha_0^2 + \beta_0^4 = 1 \tag{4.33}$$

and the product is equal to $1/4$, if $\sin^2 2t = 1$. (For the coherent states $\alpha_0 = 0$ and $\beta_0 = \pm 1$, which describes a two-parameter family with the initial data $\langle x\rangle|_{t=0} = \mp\varepsilon_0$ and $\langle p\rangle|_{t=0} = \pm\delta_0$.)

The formulas (4.30)–(4.31) show that once again the product of the variances attains the minimum value $1/4$ only at the instances when one variance is a minimum and the other is a maximum [88], [235]. The corresponding squeezing of one of the variances is also explicitly described. For example, if $\sin 2t = 1$,

$$\langle(\Delta p)^2\rangle = \frac{1-2\alpha_0}{2\beta_0^2} < \frac{1}{2}, \qquad \langle(\Delta x)^2\rangle = \frac{1+2\alpha_0}{2\beta_0^2} > \frac{1}{2} \tag{4.34}$$

provided that $0 < \alpha_0 < 1/2$ and $4\alpha_0^2 + \beta_0^4 = 1$.

4.4 An Extension: the TCS States

An analog of the coherent states (generalized coherent, or the TCS states in the terminology of Ref. [235]) is constructed in the following standard manner

$$\begin{aligned}\psi(x,t) &= e^{-|\zeta|^2/2}\sum_{n=0}^{\infty}\psi_n(x,t)\ \frac{\zeta^n}{\sqrt{n!}} \\ &= e^{-|\eta|^2/2}e^{i\gamma}\sum_{n=0}^{\infty}\Psi_n(x,t)\ \frac{\eta^n}{\sqrt{n!}}, \quad \eta=\zeta e^{2i\gamma},\end{aligned} \tag{4.35}$$

where ζ is an arbitrary complex parameter and the "dynamic" wave functions are given by equations (1.2) and (1.16) of Ref. [135]:

$$\psi_n(x,t) = \frac{e^{i\left(\alpha x^2+\delta x+\kappa\right)+i(2n+1)\gamma}}{\sqrt{2^n n!\mu\sqrt{\pi}}}\ e^{-\xi^2/2}\,H_n(\xi), \qquad \xi=\beta x+\varepsilon \tag{4.36}$$

(see also [55] and [147]), where $H_n(x)$ are the Hermite polynomials [164]. In the explicit form [195],

$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{\mu\sqrt{\pi}}}e^{-\left(\xi^2+|\eta|^2\right)/2}e^{i\left(\alpha x^2+\delta x+\kappa+\gamma\right)}\sum_{n=0}^{\infty}\left(\frac{\eta}{\sqrt{2}}\right)^n\frac{H_n(\xi)}{n!} \\ &= \frac{1}{\sqrt{\mu\sqrt{\pi}}}e^{\left(\eta^2-|\eta|^2\right)/2}e^{i\left(\alpha x^2+\delta x+\kappa+\gamma\right)}e^{-\left(\xi-\sqrt{2}\eta\right)^2/2},\end{aligned} \tag{4.37}$$

and the eigenvalue problem is given by [235]:

$$\widehat{a}(t)\,\psi(x,t)=\eta\,\psi(x,t). \tag{4.38}$$

An elementary calculation shows that on these "dynamic coherent states",

$$\begin{aligned}\langle x\rangle &= \frac{1}{\beta\sqrt{2}}(\eta+\eta^*)-\frac{\varepsilon}{\beta}, \\ \langle x\rangle|_{t=0} &= \frac{\sqrt{2}}{\beta_0}|\zeta|\cos\left(2\left(\gamma_0+\phi\right)\right)-\frac{\varepsilon_0}{\beta_0},\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}\langle p\rangle &= \frac{\beta}{i\sqrt{2}}(\eta-\eta^*)+\frac{\alpha\sqrt{2}}{\beta}(\eta+\eta^*)+\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right), \\ \langle p\rangle|_{t=0} &= \beta_0\sqrt{2}\,|\zeta|\sin\left(2\left(\gamma_0+\phi\right)\right)+2^{3/2}\frac{\alpha_0}{\beta_0}|\zeta|\cos\left(2\left(\gamma_0+\phi\right)\right) \\ &\quad +\left(\delta_0-\frac{2\alpha_0\varepsilon_0}{\beta_0}\right),\end{aligned} \tag{4.40}$$

if $\zeta = |\zeta| e^{2i\phi}$. Moreover, a direct `Mathematica` verification shows that these expectation values satisfy the required classical equation for simple harmonic motion.

A similar calculation reveals that the corresponding oscillating variances $\langle (\Delta p)^2 \rangle$ and $\langle (\Delta x)^2 \rangle$ coincide with those for the "dynamic vacuum states" given by (4.21)–(4.22). The "dynamic coherent states" (4.37) are also the minimum-uncertainty squeezed states but they are not eigenfunctions of the time-dependent dynamic invariant (4.13) when $\eta \neq 0$.

The Wigner function [91], [130], [186], [226],

$$W(x,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^*(x+y/2)\, \psi(x-y/2)\, e^{ipy}\, dy, \tag{4.41}$$

for the TCS states (4.37) is given by

$$W(x,p) = \frac{1}{\pi \mu_0 \beta_0} \exp\left[-\left(P + i\frac{\eta - \eta^*}{\sqrt{2}} \right)^2 - \left(Q - \frac{\eta + \eta^*}{\sqrt{2}} \right)^2 \right], \tag{4.42}$$

where

$$P = \frac{p - 2\alpha x - \delta}{\beta}, \qquad Q = \beta x + \varepsilon. \tag{4.43}$$

The formulas (4.39)–(4.40), leads to the following expression of the Wigner function:

$$W(x,p) = \frac{1}{\pi} \exp\left[-\frac{(p - \langle p \rangle)^2}{\beta^2} + \frac{4\alpha}{\beta^2} (p - \langle p \rangle)(x - \langle x \rangle) - \frac{4\alpha^2 + \beta^4}{\beta^2} (x - \langle x \rangle)^2 \right], \tag{4.44}$$

in terms of the classical trajectories $\langle x \rangle$ and $\langle p \rangle$ and solutions of the Ermakov-type system (4.6)–(4.7) provided $\mu_0 \beta_0 = 1$. Taking into account the time-dependent variances (4.24), one gets [53], [58], [201]:

$$W(x,p) = \frac{1}{\pi} \exp\left[-2\left(\sigma_x (p - \langle p \rangle)^2 - 2\sigma_{px} (p - \langle p \rangle)(x - \langle x \rangle) + \sigma_p (x - \langle x \rangle)^2 \right) \right], \tag{4.45}$$

where σ_p, σ_x, and σ_{px} are given by (4.24). Then

$$W(x,p;t) = W(x\cos t - p\sin t, x\sin t + p\cos t; t = 0) \tag{4.46}$$

by a direct calculation — the graph of Wigner function rotates in the phase plane without changing its shape [201]. In a traditional approach, the quantum Liouville equation of motion for Wigner function of the corresponding quadratic system is used in order to determine this time evolution [186]. The same result have been obtained herein directly from the wave functions. Some `Mathematica` animations for the Wigner function can be found in Ref. [121]. From these animations, few snapshots are presented in Figures 4.1 – 4.3.

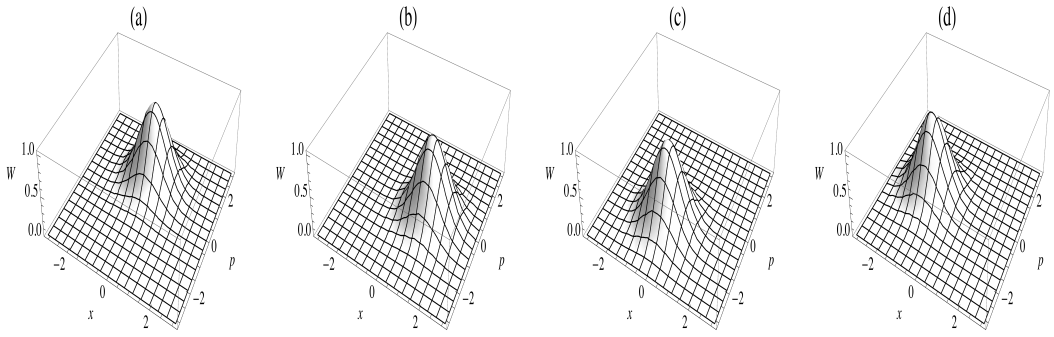

Figure 4.1: The set of images (a)-(d) represent a few snapshots taken from the Mathematica movie animation provided in [121]. From left to wright, starting with (a) they denote coherent state Wigner function (4.44) for parameters $\alpha_0 = \gamma_0 = \varepsilon_0 = \kappa_0 = 0$, $\beta_0 = 1$ and $\delta_0 = 1$. From (a) to (d) the density is rotating(in phase space) clockwise in a circular manner around the origin.

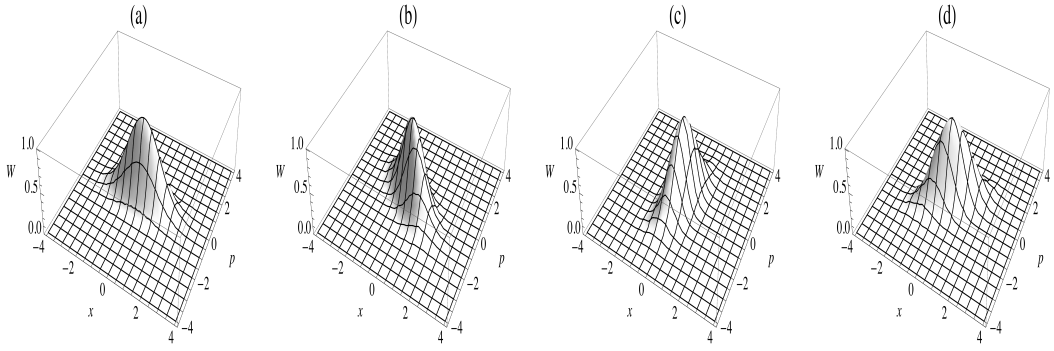

Figure 4.2: The subfigures (a)-(d) represent a few snapshots taken from the Mathematica movie animation provided in [121]. From left to wright, starting with (a) they denote squeezed coherent state Wigner function (4.44) for parameters $\alpha_0 = \gamma_0 = \varepsilon_0 = \kappa_0 = 0$, $\beta_0 = 1$ and $\delta_0 = 1$. From (a) to (d) the density isis rotating(in phase space) clockwise in a circular manner exactly in the origin.

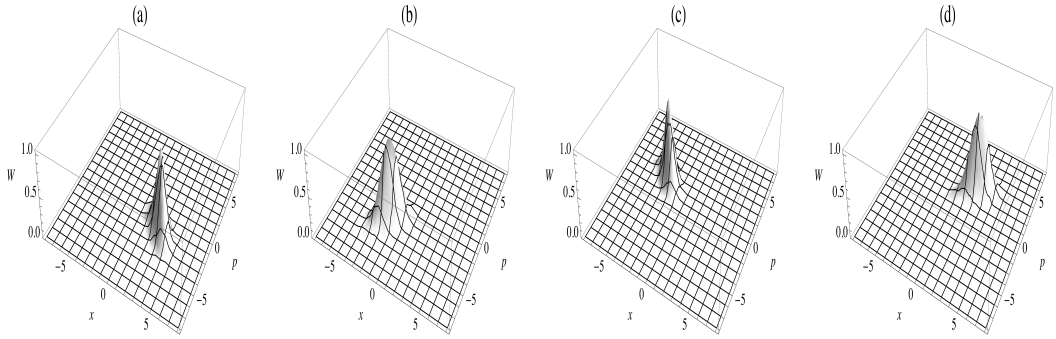

Figure 4.3: The set of subfigures (a)-(d) represent a few stills taken from the Mathematica movie animation provided in [121]. From left to wright, starting with (a) they denote the TCS squeezed state Wigner function (4.44) for parameters $\alpha_0 = \gamma_0 = \varepsilon_0 = \kappa_0 = 0$, $\beta_0 = 2/3$ *and* $\delta_0 = 1$. *From (a) to (d) the density is is rotating(in phase space) clockwise in a circular manner around the origin. One of the corner stay at the origin and the other moves clockwise around the origin.*

4.5 The Moyal Functions

The total energy of a "dynamic harmonic state" (4.36) can be presented as

$$\langle H\rangle = \frac{1}{2}\left[\langle p^2\rangle + \langle x^2\rangle\right] = \left(n+\frac{1}{2}\right)\frac{1+4\alpha_0^2+\beta_0^4}{2\beta_0^2} + \frac{\left(2\alpha_0\varepsilon_0-\beta_0\delta_0\right)^2+\varepsilon_0^2}{2\beta_0^2} \tag{4.47}$$

by (A.3)–(A.5) of Ref. [135].

The Moyal functions [157] for the "dynamic harmonic states" (4.36):

$$W_{mn}(x,p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_m^*(x+y/2,t)\,\psi_n(x-y/2,t)\,e^{ipy}\,dy \tag{4.48}$$

can be evaluated in terms of Laguerre and Charlier polynomials in a standard way [164], [186]:

$$\begin{aligned} W_{mn}(x,p,t) &= \frac{(-1)^m e^{2i(n-m)\gamma}}{\pi} e^{-Q^2-P^2/\beta^2} 2^{(m-n)/2}\sqrt{\frac{m!}{n!}} \\ &\quad\times\left(Q-\frac{iP}{\beta}\right)^{n-m} L_m^{n-m}\left(2\left(Q^2+\frac{P^2}{\beta^2}\right)\right). \end{aligned} \tag{4.49}$$

Once again, the time evolution of the corresponding Wigner function $W_{nn}(x,p,t)$ is defined by equation (4.46).

In the case of an arbitrary linear combination,

$$\psi(x,t) = \sum_m c_m \psi_m(x,t), \tag{4.50}$$

the Wigner function can be obtain as a double sum of the Moyal functions:

$$W(x,p,t) = \sum_{m,n} c_m^* c_n W_{mn}(x,p,t). \tag{4.51}$$

A coherent superposition of two states with $n=0$ and $n=1$ was experimentally realized in Ref. [156]. Moreover, the state of the electromagnetic field can be chosen anywhere between the single-photon and squeezed state in [100].

4.6 Eigenfunction Expansions

Experimentally observed statistics for various squeezed states of photons and ions in a box [21], [85], [129], [142], [146], [191] can be naturally explained in terms of explicit developments with respect to the Fock states. For a linear harmonic oscillator in the coordinate representation, a general case is discussed by using the wave functions and known expansions in Hermite polynomials [124], [133], [164]. Group-theoretical properties are discussed elsewhere.

Familiar Expansions

For the stationary harmonic oscillator wave functions,

$$\Psi_n(x) = \frac{e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}} H_n(x), \tag{4.52}$$

there are two well known expansions:

$$e^{i(\Gamma+Bx)}\Psi_n(x+A) = \sum_{m=0}^{\infty} T_{mn}(A,B,\Gamma)\ \Psi_m(x), \tag{4.53}$$

where

$$\begin{aligned} T_{mn}(A,B,\Gamma) &= \int_{-\infty}^{\infty} \Psi_m^*(x)\, e^{i(\Gamma+Bx)}\Psi_n(x+A)\ dx \\ &= \frac{i^{m-n}}{\sqrt{m!n!}}\, e^{i(\Gamma-AB/2)}\ e^{-\nu/2}\ \left(\frac{iA+B}{\sqrt{2}}\right)^m \left(\frac{iA-B}{\sqrt{2}}\right)^n \\ &\quad \times {}_2F_0\left(-n,\ -m;\ -\frac{1}{\nu}\right) \end{aligned} \tag{4.54}$$

with $\nu = \left(A^2+B^2\right)/2$ (see, for example, [133], [164] for relations with the Heisenberg–Weyl group, Charlier polynomials, and Poisson distribution) and

$$e^{i\alpha x^2}\Psi_n(\beta x) = \sum_{m=0}^{\infty} M_{mn}(\alpha,\beta)\ \Psi_m(x). \tag{4.55}$$

By the orthogonality,

$$M_{mn}(\alpha,\beta)=\int_{-\infty}^{\infty}\Psi_m^*(x)\,e^{i\alpha x^2}\Psi_n(\beta x)\;dx, \tag{4.56}$$

and one can use the integral evaluated by Bailey:

$$\begin{aligned}&\int_{-\infty}^{\infty}e^{-\lambda^2x^2}H_m(ax)\,H_n(bx)\;dx \\ &\quad=\frac{2^{m+n}}{\lambda^{m+n+1}}\Gamma\left(\frac{m+n+1}{2}\right)\left(a^2-\lambda^2\right)^{m/2}\left(b^2-\lambda^2\right)^{n/2} \\ &\qquad\times{}_2F_1\left(\begin{array}{c}-m,\quad -n\\ \frac{1}{2}(1-m-n)\end{array};\frac{1}{2}\left(1-\frac{ab}{\sqrt{(a^2-\lambda^2)(b^2-\lambda^2)}}\right)\right),\end{aligned} \tag{4.57}$$

$Re\lambda^2>0$, if $m+n$ is even; the integral vanishes by symmetry if $m+n$ is odd; see Refs. [11], [138] and the references therein for earlier works on these integrals, some of their special cases and extensions. As a result,

$$\begin{aligned}M_{mn}(\alpha,\beta)&=i^n\sqrt{\frac{2^{m+n}}{m!n!\pi}}\,\Gamma\left(\frac{m+n+1}{2}\right) \\ &\quad\times\frac{\left(\frac{1-\beta^2}{2}+i\alpha\right)^{m/2}\left(\frac{1-\beta^2}{2}-i\alpha\right)^{n/2}}{\left(\frac{1+\beta^2}{2}-i\alpha\right)^{(m+n+1)/2}} \\ &\quad\times{}_2F_1\left(\begin{array}{c}-m,\quad -n\\ \frac{1}{2}(1-m-n)\end{array};\frac{1}{2}\left(1\pm\frac{2i\beta}{\sqrt{4\alpha^2+(\beta^2-1)^2}}\right)\right).\end{aligned} \tag{4.58}$$

The terminating hypergeometric function can be transformed as follows

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}-k,\quad -n\\ \frac{1}{2}(1-k-n)\end{array};\quad\frac{1}{2}(1+i\zeta)\right) \\ &=\begin{cases}\dfrac{(1/2)_r(1/2)_s}{(1/2)_{r+s}}\;{}_2F_1\left(\begin{array}{c}-r,\quad -s\\ 1/2\end{array};\quad-\zeta^2\right), & \text{if } k=2r,\ n=2s,\\[2ex] -\dfrac{(3/2)_r(3/2)_s}{(3/2)_{r+s}}\;i\zeta\;{}_2F_1\left(\begin{array}{c}-r,\quad -s\\ 3/2\end{array};\quad-\zeta^2\right), & \text{if } k=2r+1,\ n=2s+1.\end{cases}\end{aligned} \tag{4.59}$$

It is valid in the entire complex plane; the details are given in Appendix B of [124]. The transformation (4.59) completes evaluation of the Bailey integral (4.57) and the matrix elements (4.58)

in terms of the hypergeometric functions. (Relations with the group $SU(1,1)$, Meixner polynomials [164], and with two special cases of the negative binomial, or Pascal, distribution [124] are discussed elsewhere.)

Probability Amplitudes

Expansions (4.53) and (4.55) results in

$$\psi_n(x,t) = \frac{e^{i(2n+1)(\gamma-\gamma_0)}}{\sqrt{\mu}} \sum_{m=0}^{\infty} C_{mn}(t)\ \Psi_m(x), \tag{4.60}$$

where

$$\begin{aligned} C_{mn}(t) &= \sum_{k=0}^{\infty} M_{mk}(\alpha,\beta)\ T_{kn}\left(\varepsilon, \frac{\delta}{\beta}, \kappa\right) \\ &= \sum_{k=0}^{\infty} T_{mk}\left(\frac{\varepsilon}{\beta}, \delta - \frac{2\alpha\varepsilon}{\beta}, \kappa - \frac{\alpha\varepsilon^2}{\beta^2}\right) M_{kn}(\alpha,\beta). \end{aligned} \tag{4.61}$$

The invariants are

$$\frac{4\alpha^2+\beta^4+1}{2\beta^2} = \frac{4\alpha_0^2+\beta_0^4+1}{2\beta_0^2}, \qquad \kappa - \frac{\delta\varepsilon}{2\beta} = \kappa_0 - \frac{\delta_0\varepsilon_0}{2\beta_0}, \tag{4.62}$$

$$\varepsilon^2 + \frac{\delta^2}{\beta^2} = \varepsilon_0^2 + \frac{\delta_0^2}{\beta_0^2}, \quad \frac{\varepsilon^2}{\beta^2} + \left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)^2 = \frac{\varepsilon_0^2}{\beta_0^2} + \left(\delta_0 - \frac{2\alpha_0\varepsilon_0}{\beta_0}\right)^2 \tag{4.63}$$

by the direct calculation. Another useful identity is given by

$$\frac{4\alpha^2+\beta^4+1}{2\beta^2} \pm 1 = \frac{4\alpha^2+\left(\beta^2\pm 1\right)^2}{2\beta^2} = \sigma_p + \sigma_x \pm 1. \tag{4.64}$$

Thus all arguments of the hypergeometric functions in (4.61) are constants. Moreover, the time-dependencies of the matrix elements are given only by complex phase factors:

$$T_{mn}\left(\varepsilon, \frac{\delta}{\beta}, \kappa\right) = e^{2i(m-n)(\gamma-\gamma_0)}\ T_{mn}\left(\varepsilon_0, \frac{\delta_0}{\beta_0}, \kappa_0\right), \tag{4.65}$$

$$T_{mn}\left(\frac{\varepsilon}{\beta}, \delta - \frac{2\alpha\varepsilon}{\beta}, \kappa - \frac{\alpha\varepsilon^2}{\beta^2}\right) = e^{i(n-m)t}\ T_{mn}\left(\frac{\varepsilon_0}{\beta_0}, \delta_0 - \frac{2\alpha_0\varepsilon_0}{\beta_0}, \kappa_0 - \frac{\alpha_0\varepsilon_0^2}{\beta_0^2}\right)$$

and

$$M_{mn}(\alpha,\beta) = e^{-i(2m+1)(\gamma-\gamma_0)} e^{-i(n+1/2)t} \sqrt{\frac{\mu}{\mu_0}} M_{mn}(\alpha_0,\beta_0) \tag{4.66}$$

in view of the following identities

$$\frac{\delta}{\beta}+i\varepsilon = \left(\frac{\delta_0}{\beta_0}+i\varepsilon_0\right)e^{2i(\gamma-\gamma_0)} \tag{4.67}$$

$$\delta-\frac{2\alpha\varepsilon}{\beta}+i\frac{\varepsilon}{\beta} = \left(\delta_0-\frac{2\alpha_0\varepsilon_0}{\beta_0}+i\frac{\varepsilon_0}{\beta_0}\right)e^{-it} \tag{4.68}$$

$$\frac{1-\beta^2}{2}+i\alpha = e^{-it}\left(\frac{1-\beta_0^2}{2}+i\alpha_0\right)/\left(2\alpha_0\sin t+\cos t+i\beta_0^2\sin t\right) \tag{4.69}$$

$$\frac{1+\beta^2}{2}-i\alpha = e^{it}\left(\frac{1+\beta_0^2}{2}-i\alpha_0\right)/\left(2\alpha_0\sin t+\cos t+i\beta_0^2\sin t\right) \tag{4.70}$$

and some of their complex conjugates (see also section 4.11 for a complex parametrization of the Schrödinger group).

Finally, the eigenfunction expansion takes the form

$$\psi_n(x,t)=\frac{1}{\sqrt{\mu_0}}\sum_{m=0}^{\infty}c_{mn}\,e^{-i(m+1/2)t}\,\Psi_m(x), \tag{4.71}$$

where the time-independent coefficients are explicitly given by

$$\begin{aligned} c_{mn} &= \sum_{k=0}^{\infty}M_{mk}(\alpha_0,\beta_0)\;T_{kn}\left(\varepsilon_0,\frac{\delta_0}{\beta_0},\kappa_0\right) \\ &= \sum_{k=0}^{\infty}T_{mk}\left(\frac{\varepsilon_0}{\beta_0},\delta_0-\frac{2\alpha_0\varepsilon_0}{\beta_0},\kappa_0-\frac{\alpha_0\varepsilon_0^2}{\beta_0^2}\right)\;M_{kn}(\alpha_0,\beta_0) \end{aligned} \tag{4.72}$$

in terms of the initial data/integrals of motion (of the corresponding Ermakov-type system). The total probability amplitude is a product of two infinite matrices related to the Poisson and Pascal distributions.

Moreover, a combination of (4.35) and (4.71) gives the eigenfunction expansion of the TCS states. It is worth noting also that expansion (4.71) gives an independent verification of the fact that the "missing" solutions (4.36) do satisfy the time-dependent Schrödinger equation (4.3). Indeed, they are written as an explicit superposition of the standard solutions.

4.7 Nonclassical Harmonic States of Motion and Photon Statistics

A fundamental manifestation of the interaction between an atom and a field mode at resonance in an ideal cavity is the Rabi oscillations [85]. The first observation of the nonclassical radiation field of a micromaser is reported in [172] (the statistical and discrete nature of the photon field leads to collapse and revivals in the Rabi nutation [173]). Implementation of light for purposes of quantum information relies on the ability to synthesize, manipulate, and characterize various quantum

states of the electromagnetic field. A review [142] covers the latest developments in quantum-state tomography of optical fields and photons.

Various classes of motional states in ion traps are discussed, for example, in [129]. The expansion formula (4.72) is consistent with statistics for the coherent, squeezed, and Fock states observed in Refs. [21] and [146] for ions and photons in a box (see also [58] and [129]). A method to measure the quantum state of a harmonic oscillator through instantaneous probe-system interaction, preventing decoherence from disturbing the measurement, is proposed in [183].

Coherent States

In breakthrough experiments of the NIST group on engineering ionic states of motion, the coherent states of a single $^9Be^+$ ion confined in a Paul trap were produced from the ground state by a spatially uniform classical driving field and by "moving standing wave" (see [129], [146] and the references therein for details). For the data presented in [146], the authors used the first method. The Poissonian distribution with the fitted mean quantum number $\overline{n} = 3.1 \pm 0.1$ was identified from Fourier analysis of Rabi oscillations. In our notation, $\alpha_0 = 0$, $\beta_0 = 1$, and $\overline{n} = \left(\delta_0^2 + \varepsilon_0^2\right)/2$.

Time evolution of the coherent state of cold Cs atoms was measured in [156]. For experimentally observed coherent photon states [80], see, for example, [21] and [139].

Squeezed Vacuum and Fock States

The minimum-uncertainty squeezed state with $\gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$ is called the squeezed vacuum (see [58], [115], and [129] when $\alpha_0 = 0$). Expansion (4.71) simplifies to

$$\begin{aligned} \psi_0(x,t) &= \frac{e^{i\left(\alpha(t)x^2+\gamma(t)\right)}}{\sqrt{\mu(t)\sqrt{\pi}}}\, e^{-\beta^2(t)x^2/2} \\ &= \frac{1}{\sqrt{\mu_0}} \sum_{p=0}^{\infty} \frac{\sqrt{(2p)!}}{2^p p!} \frac{\left(\dfrac{1-\beta_0^2}{2} + i\alpha_0\right)^p}{\left(\dfrac{1+\beta_0^2}{2} - i\alpha_0\right)^{p+1/2}}\, e^{-i(2p+1/2)t}\, \Psi_{2p}(x) \end{aligned} \tag{4.73}$$

provided that $\mu_0\beta_0 = 1$. The probability distribution is restricted to the even states and given by

$$P_{m=2p} = \frac{(2p)!}{\left(\sigma_p + \sigma_x + 1\right)^{1/2} 2^{2p-1/2} \left(p!\right)^2} \left(\frac{\sigma_p + \sigma_x - 1}{\sigma_p + \sigma_x + 1}\right)^p \tag{4.74}$$

in terms of the variances (4.25). This is a special case of the negative binomial, or Pascal, distribution.

A vacuum squeezed state of ionic motion was created in the NIST group experiments [146] by a parametric drive at 2ν (see also [86], [129] and the references therein). The data were fitted to the vacuum state distribution (4.74) with $\sigma_p + \sigma_x = 40 \pm 10$ and $\alpha_0 = 0$ (corresponding to a noise level 16 dB below the zero-point variance in the squeezed quadrature component; see [129] and [146] for more experimental details).

A vacuum squeezed state of motion of neutral Cs atoms was also generated in [156]. Here, the cold atom sample containes about 10^5 atoms. Therefore a single image provides the full velocity distribution of the quantum state and the squeezing can be readily visualized — a set of images gives the state's time evolution [156].

In a similar fashion, for the squeezed Fock state with $n = 1$ and $\gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$, expansion (4.71) simplifies to

$$\psi_1(x,t) = \sqrt{\frac{2}{\mu(t)\sqrt{\pi}}} e^{i\left(\alpha(t)x^2 + 3\gamma(t)\right)} \beta(t)\, x e^{-\beta^2(t)x^2/2} \tag{4.75}$$

$$= \frac{\beta_0}{\sqrt{\pi\mu_0}} \sum_{p=0}^{\infty} \frac{2^{p+1}\Gamma(p+3/2)}{\sqrt{(2p+1)!}} \frac{\left(\frac{1-\beta_0^2}{2} + i\alpha_0\right)^p}{\left(\frac{1+\beta_0^2}{2} - i\alpha_0\right)^{p+3/2}} e^{-i(2p+3/2)t}\, \Psi_{2p+1}(x).$$

The corresponding Pascal distribution for the odd states is given by

$$P_{m=2p+1} = \frac{2^{3/2}(3/2)_p}{(\sigma_p + \sigma_x + 1)^{3/2}\, p!} \left(\frac{\sigma_p + \sigma_x - 1}{\sigma_p + \sigma_x + 1}\right)^p, \tag{4.76}$$

where $(3/2)_0 = 1$ and $(3/2)_p = (3/2)(5/2)\cdots(1/2+p)$. These squeezed Fock states were generated in [19] and their dynamics was analyzed in [156]. When $\varepsilon_0 \neq 0$, displaced Fock states of the electromagnetic field, have been synthesized in [139] (see also the references therein).

Moreover, even/odd oscillations in the photon number distribution of the *squeezed vacuum* state, which are consequence of pair-wise generation of photon, were observed in [21], [191]. For an ideal minimum-uncertainty squeezed state zero probabilities for odd n are expected, since the Hamiltonian describing the parametric process occurring inside the nonlinear crystal is quadratic in the creation and annihilation operators [58]. [186]. However, the probabilities for odd photon numbers are nonzero because the squeezed state detected there is a mixed state having undergone losses inside the resonator and during the detection process which cause the distribution

to smear out (see [58] and [191] for more details). The corresponding Pascal distributions (4.74) and (4.76) have different parameter values for even and odd states, which is consistent with the result of these experiments. Further details will be discussed elsewhere.

Engineering Mixed Squeezed States

Generation of a coherent superposition of the ground state and the first excited states of motion of cold Cs atoms in the harmonic microtraps, namely,

$$\psi(x,t) = c_0\psi_0(x,t) + c_1\psi_1(x,t), \tag{4.77}$$

where $c_0 = 2^{-1/2}$ and $c_1 = 2^{-1/2}e^{i\phi}$, was reported in [156] and the corresponding time evolution had been experimentally observed. This evolution is obviously nonclassical and contrasts with that of a coherent state which oscillates as a classical particle without deformation (see [156] for more details). This dynamics is consistent with the expansion (4.72) but details of these calculations will appear elsewhere.

4.8 An Application: Cavity QED and Quantum Optics

Foundations of quantum electrodynamics and quantum optics are presented in many excellent books and articles [2], [14], [15], [18], [48], [52], [60], [68], [69], [70], [71], [72], [80], [87], [99], [106], [114], [115], [136], [179], [190], [196], [199], [223], [224], and [225]. Here, a modification of the radiation field operators in a perfect cavity is suggested in order to incorporate the Schrödinger symmetry group into the second quantization. The approach presented gives a natural description of squeezed photons that can be created as a result of parametric amplification of quantum fluctuations in the dynamic Casimir effect [123], [227] and are registered in quantum optics [21], [142], [166].

Radiation Field Quantization in a Perfect Cavity

In the formalism of second quantization, one expands electromagnetic fields in terms of resonant modes of the particular cavity under consideration [60], [106], [186], [199]. The cavity is represented by a volume V, bounded by a closed surface. Let $\mathbf{E}_\alpha(\mathbf{r})$, $k_\alpha^2 = \omega_\alpha^2/c^2$ be the eigenfunctions and the eigenvalues of the corresponding boundary-value problem:

$$\begin{aligned} \nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E} &= \mathbf{0} \qquad \text{in } V \\ \mathbf{n}\times\mathbf{E} &= \mathbf{0} \qquad \text{on } S, \end{aligned} \tag{4.78}$$

where $\mathbf{n}$ is a unit normal vector to S. The vector functions $\mathbf{H}_\alpha(\mathbf{r})$ are related to $\mathbf{E}_\alpha(\mathbf{r})$ by

$$\nabla \times \mathbf{E}_\alpha = k_\alpha \mathbf{H}_\alpha, \qquad \nabla \times \mathbf{H}_\alpha = k_\alpha \mathbf{E}_\alpha. \tag{4.79}$$

The eigenfunctions are orthonormal in V :

$$\int_V \mathbf{E}_\alpha \cdot \mathbf{E}_\beta \, dV = \delta_{\alpha\beta}, \qquad \int_V \mathbf{H}_\alpha \cdot \mathbf{H}_\beta \, dV = \delta_{\alpha\beta}. \tag{4.80}$$

The electric and magnetic fields are expanded in the following forms

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= -\sqrt{4\pi}\sum_\alpha p_\alpha(t)\,\mathbf{E}_\alpha(\mathbf{r}), \\ \mathbf{H}(\mathbf{r},t) &= \sqrt{4\pi}\sum_\alpha \omega_\alpha q_\alpha(t)\,\mathbf{H}_\alpha(\mathbf{r}). \end{aligned} \tag{4.81}$$

The total energy is given by

$$\mathscr{H} = \int \frac{\mathbf{H}^2 + \mathbf{E}^2}{8\pi} \, dV = \frac{1}{2}\sum_\alpha \left(p_\alpha^2 + \omega_\alpha^2 q_\alpha^2\right) \tag{4.82}$$

and the Maxwell equations,

$$\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{4.83}$$

are equivalent to the canonical Hamiltonian equations,

$$\frac{dq_\alpha}{dt} = \frac{\partial \mathscr{H}}{\partial p_\alpha} = p_\alpha, \qquad \frac{dp_\alpha}{dt} = -\frac{\partial \mathscr{H}}{\partial q_\alpha} = -\omega_\alpha^2 q_\alpha, \tag{4.84}$$

respectively.

In the second quantization, one replaces canonically conjugate coordinates and momenta by time-dependent operators $q_\alpha(t)$ and $p_\alpha(t)$ that satisfy the commutation rules

$$\left[q_\alpha(t),\, q_\beta(t)\right] = \left[p_\alpha(t),\, p_\beta(t)\right] = 0, \qquad \left[q_\alpha(t),\, p_\beta(t)\right] = i\hbar\delta_{\alpha\beta}. \tag{4.85}$$

The time-evolution is determined by the Heisenberg equations of motion [87]:

$$\frac{d}{dt}p_\alpha(t) = \frac{i}{\hbar}\left[p_\alpha(t),\, \mathscr{H}\right], \qquad \frac{d}{dt}q_\alpha(t) = \frac{i}{\hbar}\left[q_\alpha(t),\, \mathscr{H}\right], \tag{4.86}$$

with appropriate initial conditions. It is worth to mention that the standard form of Heisenberg's equations can be obtained by the time reversal $t \to -t$ (with $\alpha_0 \to -\alpha_0$, $\gamma_0 \to -\gamma_0$, $\delta_0 \to -\delta_0$, and $\kappa_0 \to -\kappa_0$; see below). From now on, it will be considered a single photon cavity mode, say α, with frequency $\omega_\alpha = 1$ and use the units $c = \hbar = 1$.

Explicit solution of equations (4.86) for squeezed states can be found as follows

$$p(t)=\frac{\widehat{b}(t)-\widehat{b}^{\dagger}(t)}{i\sqrt{2}},\qquad q(t)=\frac{\widehat{b}(t)+\widehat{b}^{\dagger}(t)}{\sqrt{2}}. \tag{4.87}$$

The time-dependent annihilation $\widehat{b}(t)$ and creation $\widehat{b}^{\dagger}(t)$ operators are given by [118]

$$\begin{aligned}\widehat{b}(t)&=\frac{e^{-2i\gamma}}{\sqrt{2}}\left(\beta x+\varepsilon+i\frac{p-2\alpha x-\delta}{\beta}\right),\\ \widehat{b}^{\dagger}(t)&=\frac{e^{2i\gamma}}{\sqrt{2}}\left(\beta x+\varepsilon-i\frac{p-2\alpha x-\delta}{\beta}\right)\end{aligned} \tag{4.88}$$

in terms of solutions (4.6)–(4.11) of the corresponding Ermakov-type system. The time-independent operators x and p obey the canonical commutation rule $[x,p]=i$ in an abstract Hilbert space. At all times,

$$\widehat{b}(t)\widehat{b}^{\dagger}(t)-\widehat{b}^{\dagger}(t)\widehat{b}(t)=1. \tag{4.89}$$

By back substitution, operators $\widehat{b}(t)$ and $\widehat{b}^{\dagger}(t)$ are solutions of the Heisenberg equation:

$$\frac{d}{dt}\widehat{b}(t)=i\left[\,\widehat{b}(t)\,,\,H\right],\qquad \frac{d}{dt}\widehat{b}^{\dagger}(t)=i\left[\,\widehat{b}^{\dagger}(t)\,,\,H\right], \tag{4.90}$$

with the standard Hamiltonian

$$H=\frac{1}{2}\left(p^2+x^2\right) \tag{4.91}$$

subject to the following initial conditions

$$\begin{aligned}\widehat{b}(0)&=\frac{e^{-2i\gamma_0}}{\sqrt{2}}\left(\beta_0 x+\varepsilon_0+i\frac{p-2\alpha_0 x-\delta_0}{\beta_0}\right),\\ \widehat{b}^{\dagger}(0)&=\frac{e^{2i\gamma_0}}{\sqrt{2}}\left(\beta_0 x+\varepsilon_0-i\frac{p-2\alpha_0 x-\delta_0}{\beta_0}\right).\end{aligned} \tag{4.92}$$

The creation and annihilation operators (4.88) allow to incorporate the Schrödinger group of harmonic oscillator, originally found in the coordinate representation [161], into a more abstract Heisenberg picture — the classical case occurs when $\beta_0=1$ and $\alpha_0=\gamma_0=\delta_0=\varepsilon_0=\kappa_0=0$. (For the sake of simplicity, this works have been restricted to the case of a single photon mode with frequency $\omega=1$.)

The time-dependent quadratic invariant,

$$\begin{aligned}\widehat{E}(t) &= \frac{1}{2}\left[\frac{(p-2\alpha x-\delta)^2}{\beta^2}+(\beta x+\varepsilon)^2\right] \\ &= \frac{1}{2}\left[\widehat{b}(t)\widehat{b}^{\dagger}(t)+\widehat{b}^{\dagger}(t)\widehat{b}(t)\right], \qquad \frac{d}{dt}\langle\widehat{E}(t)\rangle = 0\end{aligned} \tag{4.93}$$

with

$$\frac{\partial\widehat{E}}{\partial t}+i^{-1}\left[\widehat{E},H\right]=0, \qquad H=\frac{1}{2}\left(p^2+x^2\right), \tag{4.94}$$

extends the standard Hamiltonian/Number operator H for any given real values of the "hidden" parameters/integrals of motion in the description above of the squeezed photon state. The oscillator-type spectrum,

$$\widehat{E}(t)\,|\psi_n(t)\rangle=\left(n+\frac{1}{2}\right)|\psi_n(t)\rangle, \tag{4.95}$$

can be obtained by using the modified creation and annihilation operators [2]:

$$\begin{aligned}\widehat{b}(t)\,|\psi_n(t)\rangle &= \sqrt{n}\;|\psi_{n-1}(t)\rangle, \\ \widehat{b}^{\dagger}(t)\,|\psi_n(t)\rangle &= \sqrt{n+1}\;|\psi_{n+1}(t)\rangle.\end{aligned} \tag{4.96}$$

For the "minimum-uncertainty squeezed states", one gets

$$\widehat{b}(t)\,|\psi_0(t)\rangle=0 \tag{4.97}$$

with

$$\langle\psi_0(t)\,|H|\,\psi_0(t)\rangle=\frac{1+4\alpha_0^2+\beta_0^4}{4\beta_0^2}+\frac{(2\alpha_0\varepsilon_0-\beta_0\delta_0)^2+\varepsilon_0^2}{2\beta_0^2}\geq\frac{1}{2} \tag{4.98}$$

in the Schrödinger picture. The generalized coherent (or TCS's) states are given by

$$\widehat{b}(t)\,|\psi(t)\rangle=\zeta\,|\psi(t)\rangle \tag{4.99}$$

for an arbitrary complex $\zeta\neq 0$.

The noncommuting electric $\mathbf{E}(\mathbf{r},t)$ and magnetic $\mathbf{H}(\mathbf{r},t)$ field operators are given by equations (4.81) and (4.87)–(4.88) for a squeezed photon in the Heisenberg picture, which provides a more direct analogy between quantum and classical physics [85]. The electromagnetic radiation mode in a cavity resonator is analogous to a harmonic oscillator [88]. In the Schrödinger picture, all previous results on the minimum-uncertainty squeezed states can be reproduced for the field oscillators in an operator QED-style. For a single mode with $\omega_\alpha = 1$,

$$\begin{aligned} \langle \mathbf{E}(\mathbf{r},t) \rangle &= -\sqrt{4\pi}\mathbf{E}_\alpha(\mathbf{r}) \langle \psi_n(t) | p | \psi_n(t) \rangle, \\ \langle \mathbf{H}(\mathbf{r},t) \rangle &= \sqrt{4\pi}\mathbf{H}_\alpha(\mathbf{r}) \langle \psi_n(t) | x | \psi_n(t) \rangle, \end{aligned} \tag{4.100}$$

where equations (4.18)–(4.19) hold. The corresponding variances are given (up to a normalization) by equations (A.4)–(A.5) of Ref. [135].

The minimum-uncertainty squeezed states are identified in quantum optics [49], [88], [82], [100] [129], [197], [200], [174], [179], [235] and in state tomography [28], [63], [130], [142]. They are also important in the dynamical Casimir effect [50], [51], [52], [60], [76], [118], [123], [145], [227], and [234], where the photon squeezing occurs as a result of a "parametric excitation" of vacuum oscillations.

4.9 An Important Variance

The Hamiltonian $H = \left(p^2 + x^2\right)/2$ can be rewritten in terms of the creation and annihilation operators (4.15) as follows:

$$\begin{aligned} H &= \left(\frac{4\alpha^2 - \beta^4 + 1}{4\beta^2} - i\alpha\right)\widehat{a}^2(t) + \left(\frac{4\alpha^2 - \beta^4 + 1}{4\beta^2} + i\alpha\right)\widehat{a}^\dagger(t)^2 \\ &+ \frac{4\alpha^2 + \beta^4 + 1}{4\beta^2}\left[\widehat{a}(t)\widehat{a}^\dagger(t) + \widehat{a}^\dagger(t)\widehat{a}(t)\right] \\ &+ \sqrt{2}\left[\frac{\alpha}{\beta}\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right) - \frac{\varepsilon}{2\beta^2} - \frac{i\beta}{2}\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)\right]\widehat{a}(t) \\ &+ \sqrt{2}\left[\frac{\alpha}{\beta}\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right) - \frac{\varepsilon}{2\beta^2} + \frac{i\beta}{2}\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)\right]\widehat{a}^\dagger(t) \\ &+ \frac{1}{2}\left(\delta - \frac{2\alpha\varepsilon}{\beta}\right)^2 + \frac{\varepsilon^2}{2\beta^2} \end{aligned} \tag{4.101}$$

and by definition:

$$\text{Var}\, H = \langle (H - \langle H \rangle)^2 \rangle = \langle H^2 \rangle - \langle H \rangle^2. \tag{4.102}$$

Then a direct *Mathematica* calculation results in

$$\begin{aligned} \text{Var}\, H = {} & \frac{\left(4\alpha_0^2 + \left(\beta_0^2+1\right)^2\right)\left(4\alpha_0^2 + \left(\beta_0^2-1\right)^2\right)}{8\beta_0^4}\left[\left(n+\frac{1}{2}\right)^2 + \frac{3}{4}\right] \\ & + \left[\frac{\left(4\alpha_0^2+\beta_0^4+1\right)\left(\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 + \varepsilon_0^2\right)}{\beta_0^4} - \left(\varepsilon_0^2 + \frac{\delta_0^2}{\beta_0^2}\right)\right]\left(n+\frac{1}{2}\right) \end{aligned} \tag{4.103}$$

for the wave functions (4.36) in terms of the invariants (4.62)–(4.63). (These calculations can be performed in pure operator form with the help of standard relations (1.15) of Ref. [135]; see also (4.96).) In terms of the variances,

$$\begin{aligned} \text{Var}\, H \quad = \quad & \frac{1}{2}\left[\left(\sigma_p + \sigma_x\right)^2 - 1\right]\left[\left(n+\frac{1}{2}\right)^2 + \frac{3}{4}\right] \\ & + 2\left[\sigma_p \langle p \rangle^2 + 2\sigma_{px}\langle p\rangle\langle x\rangle + \sigma_x \langle x\rangle^2\right]\left(n+\frac{1}{2}\right), \end{aligned} \tag{4.104}$$

where σ_p, σ_x, and σ_{px} are given by (4.24). When $n = 0$, this formula is consistent with the variance of the number operator derived for a generic Gaussian Wigner function in Ref. [58]. A similar expression holds for the TCS states.

4.10 Concluding Remarks

In this chapter, the nonclassical states of harmonic motion, which were originally found in [147] (in the coordinate representation) and have been rediscovered recently in [135], in a form which is convenient in applications to cavity QED and quantum optics, are reviewed. In particular, the minimum-uncertainty squeezed states are studied in detail. Expansions in the Fock states are established and their relations with experimentally observed photon statistics are briefly discussed. In the method of second quantization, a modification of the radiation field operators for squeezed photons in a perfect cavity is suggested with the help of a nonstandard solution of Heisenberg's equation of motion. These results should be of interest to everyone who studies introductory quantum mechanics and quantum optics.

4.11 Appendix A: Complex Parametrization of the Schrödinger Group

Consider the following complex-valued function:

$$z = c_1 e^{it} + c_2 e^{-it}, \qquad z'' + z = 0, \tag{4.105}$$

where

$$c_1 = \left(1+\beta_0^2\right)/2 - i\alpha_0, \qquad c_2 = \left(1-\beta_0^2\right)/2 + i\alpha_0 \tag{4.106}$$
$$\left(c_1 + c_2 = 1, \qquad |c_1|^2 - |c_2|^2 = \beta_0^2\right),$$

and

$$c_3 = \frac{\delta_0}{\beta_0} - i\varepsilon_0. \tag{4.107}$$

Then equations (4.5)–(4.11) can be rewritten in a compact form in terms of the complex parameters c_1, c_2, and c_3. With the help of identities (4.67)–(4.70), one gets

$$\frac{\mu}{\mu_0} = |z| = \left(|c_1|^2 + c_1 c_2^* e^{2it} + c_1^* c_2 e^{-2it} + |c_2|^2\right)^{1/2} \tag{4.108}$$

and

$$\alpha = i\frac{c_1 c_2^* e^{2it} - c_1^* c_2 e^{-2it}}{2|z|^2}, \tag{4.109}$$

$$\beta = \frac{\beta_0}{|z|} = \pm\frac{\sqrt{|c_1|^2 - |c_2|^2}}{|z|}, \tag{4.110}$$

$$\gamma = \gamma_0 - \frac{1}{2}\arg z, \tag{4.111}$$

$$\delta = \frac{\beta_0}{2|z|}\left(c_3 e^{i\arg z} + c_3^* e^{-i\arg z}\right), \tag{4.112}$$

$$\varepsilon = \frac{i}{2}\left(c_3 e^{i\arg z} - c_3^* e^{-i\arg z}\right), \tag{4.113}$$

$$\kappa = \kappa_0 - \frac{i}{8}\left[c_3^2\left(1 - e^{2i\arg z}\right) - c_3^{*\,2}\left(1 - e^{-2i\arg z}\right)\right]. \tag{4.114}$$

The inverse relations between the essential, real and complex, parameters are given by

$$\alpha_0 = \frac{i}{2}\left(c_1 c_2^* - c_1^* c_2\right), \qquad \beta_0 = \pm\sqrt{|c_1|^2 - |c_2|^2}, \tag{4.115}$$

$$\delta_0 = \pm\frac{1}{2}\sqrt{|c_1|^2 - |c_2|^2}\,(c_3 + c_3^*), \qquad \varepsilon_0 = \frac{i}{2}\left(c_3 - c_3^*\right). \tag{4.116}$$

These formulas (4.109)–(4.114) provide a complex parametrization of the Schrödinger group of "hidden" symmetry for the simple harmonic oscillator found in Ref. [161] (see also the explicit action of this group in [135], specifically equation (32) there). A similar parametrization for the wave functions (4.36) was used in Ref. [55] (see also [84]).

Chapter 5

CONCLUSION

For centuries, the study of evolution equations has been of great interest for many mathematicians. Moreover, these equations constitute a substantial portion of the current frontier in the advancement of the theory of differential equations. The construction of analytic solution methods for these equations have attracted much attention both for their broad range of applicability and for the techniques developed. The complexity and challenges in their theoretical study have attracted much interest from many mathematicians and scientists. Motivated by such complexities and challenges, this dissertation discussed several analytical methods to solve the initial value problem for the following fundamental evolution equations: Heat equation, Burgers equation and the Schrödinger equation.

A method to construct the fundamental solution for a class of nonautonomous and inhomogeneous linear diffusion-type equation with variable coefficients on the etire real line was discussed in **Chapter 2**. This method involves explicit transformations to reduce the evolution equation under study to their corresponding standard forms emphasizing on natural relations with certain Riccati(and/or Ermakov)-type systems. These relations give solvability results for the Cauchy problem of the parabolic equation considered. The superposition principle allowed to solve formally this problem from an unconventional point of view. The diffusion-type equation was first reduced to the standard heat equation by means of an exponential transformation subject to the solvable Riccati-system. Then a general solution of the Riccati system was presented in terms of its particular solution. This general solution was inverted and the asymptotics of this inversion resulted in the variable coefficients of the exponential transformation. A direct substitution of these asymptotics allows one to construct the fundamental solution for the master evolution equation considered following similar published work on the case of the corresponding Schrödinger equation. An eigenfunction expansion approach was also considered for this nonautonomous diffusion equation. Several examples were considered in order to confirm the efficacy of both proposed solution methods. Among these examples is worth to mention the Fokker-Planck equation, the Black-Scholes model and the one-factor Gaussian Hull-White model. The fundamental solution for the one-factor Gaussian Hull-White,

which is missing in the available literature, was presented here in the most closed form. The symmetries of the nonautonomous diffusion-type equation were also evaluated from the point of view of an exponential transformation.

For **Chapter 3** a study of certain Burgers-type equations and the corresponding relations with the master diffusion-type (2.6) equation was stressed. The results obtained for the diffusion-type equation with variable coefficients were used to solve the Cauchy initial value problem for a certain nonautonomous and inhomogeneous Burgers-type equation. This Burgers-type equation was associated to the diffusion-type equation analyzed in the second chapter by means of the celebrated Cole-Hopf transformation, thus constructing the Kernel from the master equation. The connections between the linear (the Diffusion-type) and nonlinear (Burgers-type) parabolic equations were also investigated in order to establish the existing relation among them. The results from the second chapter were the key to establish such relations.Traveling wave solutions of a nonautonomous Burgers equation were also explored in terms of the Riccati-type system. The integrability of these Burgers-type equations with variable coefficients is not clear yet, however the methods presented herein gave some ideas on this direction. The results presented in the third chapter justify the utility and efficacy of the solution methods for the proposed generalized diffusion-type equations presented in the second chapter. The author of this dissertation believe that the explicit results of this chapter could be also used to corroborate the efficacy of numerical algorithms to solve familiar systems.

The **Chapter 4** was devoted to construct explicitly the minimum-uncertainty squeezed states for quantum harmonic oscillators in the most simple closed form. These states were derived by the action of corresponding maximal kinematical invariance group on the standard ground state solution. It was shown that the product of the variances attains the required minimum value $1/4$ only at the instances that one variance is a minimum and the other is a maximum, when the squeezing of one of the variances occurs. This explicit construction was possible due to the relation between the diffusion-type equation studied in the first part and the time-dependent Schrödinger equation. The approach adopted here reveals the "hidden" quantum numbers/integrals of motion of the squeezed states in terms of the solution of certain Ermakov-type system. The generalized coherent states were also explicitly constructed and their Wigner function was studied. The overlap coefficients between the squeezed, or generalized harmonic, and the Fock states were explicitly evaluated in

terms of hypergeometric functions.The corresponding oscillating photons statistics were discussed and an application to quantum optics and cavity quantum electrodynamics was mentioned. Their experimental observations in cavity QED and quantum optics were briefly reviewed. Moreover, the radiation field operators of squeezed photons, which can be created from the QED vacuum, were introduced by second quantization with the aid of the hidden symmetry of the harmonic oscillator problem in the Heisenberg picture. A modification of the radiation field operators for squeezed photons in a perfect cavity was also suggested with the help of a nonstandard solution of Heisenberg's equation of motion. Explicit solutions to the Heisenberg equations for radiation field operators with squeezing were also presented. The analytical solutions presented in this chapter regarding photon statistics were experimentally realized in the works of David Wineland, NIST group, 2012 Nobel Prize winners in physics, and by the French group Breitenbach-Schiller-Mlynek. The theoretical consideration presented herein complements all of these advanced experimental techniques by identifying the state quantum numbers from first principles. This approach may provide a guidance for engineering more advanced nonclassical states.

To sum up, the solution methods to the Cauchy initial value problem of the evolution equations presented on this dissertation discussed several techniques that can shed light into other areas of science. The association of these equations to certain Riccati/Ermakov-type systems is one of the most important key steps in order to obtain the desired results. The applicability of the obtained results seems to be endless. There are still several open questions about the possibility of using the methods presented on chapter four to study the coherent states for more general harmonic oscillators. Therefore a detailed investigation on the construction of the coherent states solution for time dependent Schrödinger equation with the most general variable could be an interest topic for further research.

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BIOGRAPHICAL SKETCH

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