On-line Coloring of Partial Orders, Circular Arc Graphs, and Trees

by

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ABSTRACT

A central concept of combinatorics is partitioning structures with given constraints. Partitions of on-line posets and on-line graphs, which are dynamic versions of the more familiar static structures posets and graphs, are examined. In the on-line setting, vertices are continually added to a poset or graph while a chain partition or coloring (respectively) is maintained.

Both upper and lower bounds for the optimum of the number of chains needed to partition a width w on-line poset exist. Kierstead's upper bound of $\frac{5^w-1}{4}$ was improved to $w^{14 \lg w}$ by Bosek and Krawczyk. This is improved to $w^{3+6.5 \lg w}$ by employing the First-Fit algorithm on a family of restricted posets (expanding on the work of Bosek and Krawczyk) . Namely, the family of ladder-free posets where the m-ladder is the transitive closure of the union of two incomparable chains $x_1 \leq \cdots \leq x_m$, $y_1 \leq \cdots \leq y_m$ and the set of comparabilities $\{x_1 \leq y_1, \ldots, x_m \leq y_m\}$.

No upper bound on the number of colors needed to color a general on-line graph exists. To lay this fact plain, the performance of on-line coloring of trees is shown to be particularly problematic. There are trees that require n colors to color on-line for any positive integer n. Furthermore, there are trees that usually require many colors to color on-line even if they are presented without any particular strategy.

For restricted families of graphs, upper and lower bounds for the optimum number of colors needed to maintain an on-line coloring exist. In particular, circular arc graphs can be colored on-line using less than 8 times the optimum number from the static case. This follows from the work of Pemmaraju, Raman, and Varadarajan in on-line coloring of interval graphs.

DEDICATION

Studying mathematics is engaging in endless puzzle solving. Richard Bates taught me, and countless other young people, this.

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I owe my parents a debt beyond words and the same is true for my brother and sister. Thanks and love to K$, DDar, J, and Char.

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I shouldn't try to list friends who have been kind to me; there are too many names and too many reasons. Though, in particular I will say thank you to Heather Sollom for your love and support and Sam Leshner for a near lifetime of friendship.

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INTRODUCTION AND NOTATION

1.1 Basic Notation

The sets of integers, positive integers, nonnegative integers, and real numbers will be denoted $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, and $\mathbb{R}$. For $i, j \in \mathbb{Z}^+$ with $i < j$, we let $[j] = \{1, 2, 3, \ldots, j\}$ and $[i, j] = [j] \setminus [i-1]$. We take $[0]$ to be the empty set $\emptyset$. Let U and V be sets. For $k \in \mathbb{N}$, we use $\binom{U}{k} = \{S \subseteq U : |S| = k\}$ and $\binom{n}{k} = \left|\binom{[n]}{k}\right|$ (the usual binomial coefficient). For a singleton $\{x\}$, we use the notation $U + x$ to mean $U \cup \{x\}$. Similarly, we let $U - x$ stand for $U \setminus \{x\}$. Let f be a function $f : U \to V$. For $y \in V$, we set $f^{-1}(y) = \{x \in U : f(x) = y\}$. For subsets $A \subseteq U$ and $B \subseteq V$, we use $f(A) = \{f(u) : u \in A\}$ and $f^{-1}(B) = \{u \in U : f(u) \in B\}$. The binary logarithm will be written as lg.

1.2 Posets and Chain Partitions

An ordered pair $P = (V, \leq_P)$ is a *poset* (or *partially ordered set*) if V is a set and $\leq_P$ is a relation on V so that for any $x, y, z \in V$, $\leq_P$ is

(1) reflexive (i.e.: if $x \leq_P x$),

(2) transitive (i.e.: if $x \leq_P y$ and $y \leq_P z$, then $x \leq_P z$),

(3) and antisymmetric (i.e.: if $x \leq_P y$ and $y \leq_P x$, then $x = y$).

We call the set V the *vertices* of P and $\leq_P$ the *partial order* of P. Unless otherwise explicitly stated, we assume the set of vertices to be nonempty and finite. When discussing the order of a poset P, we will use $\leq_P$ and reserve $\leq$ to indicate the usual order on $\mathbb{R}$. We will rarely make mention of V. We write $u \in P$ to mean u is a vertex of P and $|P|$ to mean the number of vertices in P.

Let $u, v \in P$. We use $u <_P v$ to mean $u \leq_P v$ and $u \neq v$. If we have $u \leq_P v$ or $v \leq_P u$, we say u and v are *comparable* and write $u \bowtie_P v$. If u and v are not comparable, we say u and v are *incomparable* and write $u \parallel_P v$. A set $U \subseteq V$ is a *chain* if for all $u, v \in U$, we have $u \bowtie_P v$. We denote an n vertex chain by $\mathbf{n}$. The set is an *antichain* if for all $u.v \in U$, we have $u \parallel_P v$. The *width* of P (denoted $\mathrm{width}(P)$) is the cardinality of the largest antichain in P.

A function $f : V \to [n]$ is an *n-chain partition* of P if for each $k \in [n]$, $f^{-1}(k)$ is a chain. If n is unknown or unimportant, we may simply refer to a chain partition. In some contexts, we will refer to f as an *n-coloring* or *coloring* (for reasons we explain in the next section). We call the elements of $[n]$ *chains* or *colors*. One of the foundational theorems of Order Theory is Dilworth's Theorem, which characterizes the smallest n so that P has an n-chain partition.

Theorem 1.1 (Dilworth [13])**.** *For any poset P so that* $\mathrm{width}(P) = w$ *is finite, there is a w-chain partition of P. Furthermore, there is no n-chain partition of P for $n < w$.*

We should note that the theorem does not require $|P|$ to be finite, only $\mathrm{width}(P)$ to be finite. A $\mathrm{width}(P)$-chain partition of P is a *Dilworth partition* of P. If there is a Dilworth partition of P so that vertices u and v are in the same chain (i.e: $f(u) = f(v)$), then we say uv is a *Dilworth edge.*

Let $P = (V, \leq_P)$ be a poset with $u, v \in P$. If, we use $P - u$ to mean $(V - u, \leq_P |_{V-u})$. The *upset of u in P* is $U_P(u) = \{v : u <_P v\}$, the *downset* *of u in P* is $D_P(u) = \{v : v <_P u\}$, and the *incomparability set of u in P* is $I_P(u) = \{v : v \parallel_P u\}$. The *closed upset* and *closed downset of u in P* are, respectively, $U_P[u] = U_P(u) + u$ and $D_P[u] = D_P(u) + u$. We also define $[u, v]_P = U_P[u] \cap D_P[v]$. For $U \subseteq V$, use similarly define $D_P(U) = \bigcup_{u \in U} D_P(u)$ and $U_P(U) = \bigcup_{u \in U} U_P(u)$, as well as $D_P[U] = D_P(U) \cup U$ and $U_P[U] = U_P(U) \cup U$. If $U' \subseteq V$, we take $[U, U']_P = U_P[U] \cap D_P[U']$. The *subposet of P induced by U* is the poset $(U, \leq_P|_U)$. We also denote this by $P[U]$. If $U_P(u) = \emptyset$, then u is *maximal.* If $D_P(u) = \emptyset$, then u is *minimal.* If $D_P[u] = P$, then u is *maximum*, *greatest*, or *largest.* If $U_P[u] = P$, then u is *minimum*, *least*, or *smallest.* Let $\mathrm{Max}_P(U)$ be the set of maximal vertices in $P[U]$ and $\mathrm{Min}_P(U)$ be the set of minimal vertices in $P[U]$. In an abuse of notation, we use $\mathrm{Max}_P(P)$ and $\mathrm{Min}_P(P)$ to represent $\mathrm{Max}_P(V)$ and $\mathrm{Min}_P(V)$, respectively.

Let $Q = (W, \leq_Q)$ be a poset. If there is a bijection $g : V \to W$ so that $u \leq_P v$ if and only if $g(u) \leq_Q g(v)$, then we say P and Q are *poset isomorphic* (or simply *isomorphic*). We consider isomorphic posets to be indistinguishable and so we write $P = Q$ if P and Q are isomorphic. It should be noted that this is not a universally accepted convention. If $Q = P[U]$ for some $U \subseteq V$, we say Q is a *subposet* of P (some conventions use the term *induced subposet*).

To represent posets visually, we will use Hasse diagrams. We refer the reader to [51] for details regarding these diagrams.

1.3 Graphs and Coloring

An ordered pair of sets $G = (V, E)$ is a *graph* if $E \subseteq \binom{V}{2}$. We refer to V as the *vertices* of G and E as the *edges* of G. As with posets, V will assumed to be finite and nonempty unless explicitly stated otherwise. We set $|G| = |V(G)|$ and $||G|| = |E(G)|$. When referring to an edge, we omit the braces and comma of set notation and write $uv \in E$ to mean $\{u, v\} \in E$. We will assume that $uu \notin E(G)$. In some settings, a graph can be more broadly defined and the structure we define here is called a *simple graph*. Given an arbitrary graph H, we use $V(H)$ to represent the set of vertices of H and $E(H)$ to represent the set of edges of H.

Let $U \subseteq V$. The *subgraph of G induced by U* is $\left(U, \binom{U}{2} \cap E\right)$, denoted $G[U]$. If $||G[U]|| = \binom{|U|}{2}$ then U is a *clique*. If $||G[U]|| = 0$, then U is a *coclique* (also commonly called an *independent set* or a *stable set*). The *clique number* of G, denoted $\omega(G)$, is the cardinality of the largest clique in G. Similarly, the *coclique number* of G, denoted $\alpha(G)$, is the cardinality of the largest coclique in G. In the case of $U = V$, we may refer to G itself as a clique or coclique.

A function $f : V \to [n]$ is a *proper n-coloring of G* (or simply an *n-coloring* or *coloring* if n is unknown or unimportant) if for each $k \in [n]$ the set $f^{-1}(k)$ is a coclique. A more common equivalent definition is that for each $uv \in E$ we have $f(u) \neq f(v)$. A coloring is a partitioning the vertices of G into cocliques. However, the term "coloring" is used rather than "partitioning" for the historical roots of the problem. In fact, the conjecture that a mere four colors are needed to color any map so that no two adjacent countries share a color is perhaps the second widely studied problem of graphs, dating back to at least 1852 [21, 42] in a paper published under the mysterious name F.G. [1] For this reason, we call the elements of $[n]$ *colors*.

[1] Euler's famous paper on the Seven Bridges of Königsberg in 1736 is regarded to be the first graph theory publication [21].

The smallest $n \in \mathbb{Z}^+$ so that G has an n-coloring is the *chromatic number* of G, denoted $\chi(G)$. In contrast to chain partitioning posets, there is no parameter for graphs that describes $\chi(G)$ exactly for a general graph G. There are theorems that provide upper and lower bounds for $\chi(G)$, such as Brooks' Theorem [10], but the gap between the bounds is usually very large when G is an arbitrary graph. There are even more theorems that show much naïve intuition regarding $\chi(G)$ is false. We refer the interested reader to [12, 53] for an introduction to such theorems. As a simple lower bound, we can see that if U is a clique in G, then each vertex of U must have a different color. Hence $\omega(G) \leq \chi(G)$. As we will see, equality does not always hold. If $\omega(G[U]) = \chi(G[U])$ for all $U \subseteq V$, we call G a *perfect graph*.

Although G is an ordered pair, not a set, we will use notation for adding or removing vertices or edges similar to that of adding or removing singletons from a set. For $uv \in E$, let $G - uv = (V, E - uv)$ (i.e.: the graph G with edge uv removed). For $u, v \in V$ we define $G + uv = (V, E + uv)$. Note that if $uv \in E$, then $G + uv = G$. Let $v \in V$ and set $D = \{uv : uv \in E\}$, we use $G - v = (V - v, E \setminus D)$.

For $v \in V$, the *neighborhood* of v is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) + v$. If $U \subseteq V$, we extend these conventions to $N_G(U) = \bigcup_{v \in U} N_G(v)$ and $N_G[U] = N_G(U) \cup U$. If the graph is clear from context, we will omit the subscripts. If $u \in N(v)$, then we say u and v are *adjacent* or *neighbors*.

Take $G = (V, E)$ and $H = (W, F)$ to be graphs. If there is a bijection $f : V \to W$ so that $uv \in E$ if and only if $f(u)f(v) \in F$, then G and H are *isomorphic*. As with posets, we will treat two isomorphic graphs as the same graph and write $G = H$ (again, this is not a universally accepted convention). The graph H is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. For short, we sometimes say H is *in* G, G *has* H, or $H \subseteq G$. The subgraph is *induced* if $F = E \cap \binom{W}{2}$. In this case, we denote H by $G[W]$. If H' is isomorphic to H, we observe the same conventions in referring to H' as a subgraph of G without mentioning the isomorphism.

We define a few special families of graphs. The graph G is a *path* if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_i v_{i+1} : i \in [n-1]\}$. The *endpoints* (or simply *ends*) of the path are v_1 and v_n. Sometimes we say this is a path *from* v_1 *to* v_n or *from* v_n *to* v_1. The *length of a path* is $||G|| = |G| - 1$. We say G is a *cycle* if $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $n \geq 3$ and $E(G) = \{v_i v_{i+1} : i \in [n-1]\} + v_n v_1$. The *length of a cycle* is $||G|| = |G|$. If G has no cycles, we say G is a *forest*. If for any $u, v \in V$ there is a path in G with u and v as its endpoints, then G is *connected*. If G is a connected forest, then G is a *tree*. If v is a vertex in a tree G so that $|N_G(v)| = 1$, then v is a *leaf*. If G is a subgraph of H and G is path or tree, then we say G is a *subpath* or *subtree* (respectively) of H. A *component* of a graph H is a maximal subset of $U \subseteq V(H)$ so that $H[U]$ is connected.

Trees are a very special family of graphs with a simple elegance that lends them to many uses. These include applications to data organization and optimization in network problems, enumeration of various structures, and linguistic analysis. For our purposes, trees serve as a “simplest case;” we use trees to examine our ideas for general graphs. The following five propositions can be found in any introductory Graph Theory text. They are so central to the structure of trees we use them without mention.

Proposition 1.2. *If G is a tree and $|G| > 1$, then G has at least 2 leaves.*

Proposition 1.3. *If G is a tree, $|G| > 1$, and v is a leaf, then $G - v$ is a tree.*

Proposition 1.4. *If G is a tree and $|G| > 1$, then $\chi(G) = 2$.*

Proposition 1.5. *If G is a tree, $|G| > 1$, and $u, v \in V(G)$, then there is a unique path in G with endpoints u and v.*

Proposition 1.6. *If G is a tree, $|G| > 1$, and $xy \in E(G)$, then $G - xy$ is a forest with two components.*

We have hinted at a connection between graphs and posets in the shared use of the terms "vertices" and "colors." For a poset P, the *cocomparability graph* of P is the graph G^P where $V(G^P)$ is the set of vertices of P and $uv \in E(G^P)$ if $u \parallel_P v$. We can see that any antichain in P is a clique in G^P and any chain is a coclique in G^P. Hence, $\text{width}(P) = \omega(G^P)$ and any n-chain partition of P is also an n-coloring of G^P. The equivalence of chain partitioning P and coloring G^P lets us use the terms interchangeably without too much abuse of terminology. Applying Theorem 1.1 to the subposets of P, we see that G^P is a perfect graph.

1.4 Digraphs

An ordered pair of sets $G = (V, A)$ is a *digraph* (or *directed graph*) if A is a subset of ordered pairs of elements of V. We call V the *vertices* of G and A the *arrows* of G (to help distinguish directed graphs from simple graphs where edges are subsets of V). When referring to an arrow, we write $\overrightarrow{uv} \in A$ or $\overleftarrow{vu} \in A$ to mean $(u, v) \in A$. We assume that $\overrightarrow{uu}$ is not an arrow in G. Most terminology for digraphs is the same as that used for simple graphs (if we substitute "arrows" for "edges") and does bear repeating here. However, we do address paths and cycles.

A digraph G is a *directed path* if the vertices of G are $\{v_1, v_2, \ldots, v_n\}$ and the arrows are $\{\overrightarrow{v_i v}_{i+1} : i \in [n-1]\}$. We refer to v_1 as the *start* and v_n as the *end* of the path. We can say this is a path *from* v_1 *to* v_n, but not vice versa. The *length of a directed path* is $|G|$. A digraph G is a *directed cycle* if the vertices of G are $\{v_1, v_2, \ldots, v_n\}$ and the arrows are $\{\overrightarrow{v_i v}_{i+1} : i \in [n-1]\} + \overrightarrow{v_n v}_1$. The *length of a directed cycle* is $|G|$. Let G be a directed cycle and $u, v \in V(G)$. We use uGv to mean the directed subpath of length at least 1 of G with start u and end v. Note that $uGu = G$. When speaking of paths or cycles in the context of a directed graph, we will assume we are speaking of directed paths and directed cycles.

We will only use digraphs built upon simple graphs. That is, we will start with a simple graph and build a digraph by assigning a direction to the edges to form arrows. We call this an *orientation* of a (simple) graph.

Chapter 2

ON-LINE PARTITIONING AND COLORING

2.1 General Algorithms and First-Fit

At first glance, Dilworth's Theorem looks to be the final word in poset partitioning. In many ways it is, but it is an existential theorem; it does not indicate how one would find a Dilworth partition nor the complexity of such a task. If $|P|$ is finite, there are several algorithms (which are polynomial in $|P|$) to find a Dilworth partition. If $|P|$ is not finite but width(P) is finite, we may ask, is there an algorithm to create or maintain a chain partition using a finite number of chains? In other words, can we maintain a partition of a poset P if new vertices are continually appearing? To study this question, we define a new structure. An *on-line poset* $P^{\prec}$ is a poset $P = (V, \leq_P)$, where $\prec$ is a total order on V called a *presentation*.

An *on-line chain partitioning algorithm* is a deterministic algorithm $\mathcal{A}$ that assigns the vertices $v_1 \prec v_2 \prec \cdots \prec v_n$ of $P^{\prec}$ to disjoint chains $C_1, C_2, \ldots, C_t$ so that for each v_i, the chain C_j to which v_i is assigned is determined solely by the subposet $P[\{v_1, v_2, \ldots, v_i\}]$ (i.e: the first i vertices of the presentation). Let $\chi_{\mathcal{A}}(P^{\prec})$ denote the number of (nonempty) chains that $\mathcal{A}$ uses to partition $P^{\prec}$, and $\chi_{\mathcal{A}}(P) = \max_{\prec}(\chi_{\mathcal{A}}(P^{\prec}))$ over all presentations $\prec$ of P. For a family of posets $\mathcal{P}$, let $\mathrm{val}_{\mathcal{A}}(\mathcal{P}) = \max_{P \in \mathcal{P}}(\chi_{\mathcal{A}}(P))$ and $\mathrm{val}(\mathcal{P}) = \min_{\mathcal{A}}(\mathrm{val}_{\mathcal{A}}(\mathcal{P}))$ over all on-line chain partitioning algorithms $\mathcal{A}$. Let $\mathcal{P}_w$ be the family of posets of width w. For the sake of brevity, we will abuse notation and use $\mathrm{val}_{\mathcal{A}}(w)$ and $\mathrm{val}(w)$ to mean $\mathrm{val}_{\mathcal{A}}(\mathcal{P}_w)$ and $\mathrm{val}(\mathcal{P}_w)$, respectively. Traditional chain partitioning, where an algorithm views all the vertices of P before making a partition or is allowed to make changes to earlier assignments, will be called *off-line partitioning.* Again, we will use the terms partitioning and coloring interchangeably.

Informally, we may think of on-line partitioning as a game where players Spoiler and Algorithm take turns. A positive integer w is selected by Spoiler. In each round, Spoiler adds a vertex to a poset and reveals all comparabilities to previously added vertices. The width of the poset must not exceed w. Algorithm then uses a fixed set of instructions (an algorithm) to assign the new vertex to chain in a partition of the poset that he maintains throughout the game. Algorithm wants to maintain a partition with few chains and Spoiler wants to force Algorithm to use many chains. In this setting, we can think of $\text{val}(w)$ as the largest integer m so that Spoiler has a poset of width at most w and order of revealing the vertices that forces Algorithm to use at least m chains. Dually, it is the smallest integer n so that Algorithm may play the game indefinitely using only n chains for any poset of width w and for any order in which the vertices are revealed. Spoiler is not forced to decide on $P^{\prec}$ before the game starts. He may change his mind about either $\prec$ or P at will, so long at the width is at most w at each round of the game and he does not alter the previously revealed portion of the poset.

For each $w \geq 2$, it is easy to see $\text{val}(w) > w$. To demonstrate this, we offer a simple strategy for Spoiler to force any on-line partitioning algorithm $\mathcal{A}$ to use $w+1$ chains while presenting a poset of width w. For the first w rounds, Spoiler presents a w vertex antichain $a_1, a_2, \ldots, a_w$. In each of these rounds, Algorithm must assign each of these vertices to a distinct chain. In round $w+1$, Spoiler reveals a maximum vertex u. As Algorithm wishes to use only w chains, u must be assigned to a chain with some vertex from the antichain, say a_i. Spoiler then reveals v where v is greater than a_i and incomparable to all other vertices. As v is incomparable to a vertex in each of Algorithm's w chains, Algorithm must add another chain to his partition. In Figure 2.1, we show seven rounds of this strategy with $w = 5$. From top to bottom: the result of rounds 1-5, round 6, and round 7.

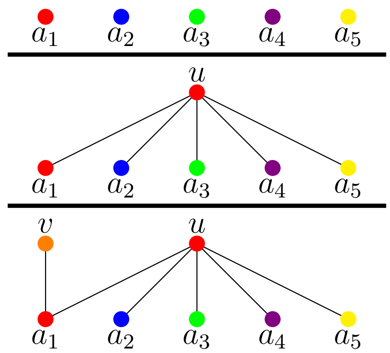

Figure 2.1: Spoiler's strategy for $w = 5$.

Even though a Dilworth partition cannot be maintained, we still ask, can we maintain some partition using a finite number of chains? If so, how many chains would we need? In 1981, Kierstead [30] proved that $4w - 3 \leq \mathrm{val}(w) \leq \frac{5^w-1}{4}$, and asked whether $\mathrm{val}(w)$ is polynomial in w. It was also noted that the arguments could be modified to provide a superlinear lower bound. Shortly after, Szemerédi proved a quadratic lower bound (see [31]) of $\binom{w+1}{2} \leq \mathrm{val}(w)$. In 1997 Felsner [18] proved that $\mathrm{val}(2) \leq 5$, and in 2008 Bosek [2] proved that $\mathrm{val}(3) \leq 16$. Bosek, et al. [3] improved the lower bound to $(2 - o(1))\binom{w+1}{2}$. In 2010, Bosek and Krawczyk made a major advance in the upper bound.

Theorem 2.1 (Bosek & Krawczyk [4]). $\mathrm{val}(w) \leq w^{14 \lg w}$.

If we require the presented poset to be from a certain family or $\prec$ to have certain restrictions, there are further results. We refer the reader to [3] for a survey.

Perhaps the simplest on-line chain partitioning algorithm is First-Fit (FF). It assigns each new vertex v_i to the chain C_j with the least index $j \in \mathbb{Z}^+$ such that for all $h < i$ if $v_h \in C_j$ then $v_h \bowtie_P v_i$. It is easy to see the result of FF is a chain partition. It was observed in [30] that $\mathrm{val}_{\mathrm{FF}}(w)$ is unbounded (see [31] for details). The poset that Kierstead used to obtain this result in now part of the folklore of Order Theory and will play and important part in much our work to come, so it bears repeating here.

Lemma 2.2. *For every positive integer n there exists an on-line poset $R_n^{\prec}$ with width 2 such that $\chi_{\mathrm{FF}}(R_n^{\prec}) = n$.*

Proof. We define the on-line poset $R_n^{\prec}$ with $R_n = (X, \leq_R)$ as follows. The poset R_n consists of n chains $X^1, \ldots, X^n$ with

$$X^k = x_k^k \leq_R x_{k-1}^k \leq_R \cdots \leq_R x_2^k \leq_R x_1^k$$

and the additional comparabilities and incomparabilities given by:

$$x_i^k \geq_R X^1 \cup X^2 \cup \cdots \cup X^{k-2} \cup \{x_k^{k-1}, x_{k-1}^{k-1}, \ldots, x_i^{k-1}\}$$

$$x_i^k \parallel_R \{x_1^{k-1}, x_2^{k-1}, \ldots, x_{i-1}^{k-1}\}.$$

Note that the superscript of a vertex indicates to which chain X^k it belongs and the subscript is its index within that chain. We illustrate R_5 in Figure 2.2. The presentation $\prec$ is given by

$$X^1 \prec \cdots \prec X^n,$$

where the order $\prec$ on the vertices of X^k does not matter, but we let $\prec$ be the same as $\leq_R$ on X^k.

By induction on k, it is easy to show that the width of R_n is 2, and each vertex x_i^k is assigned to chain C_i. $\square$

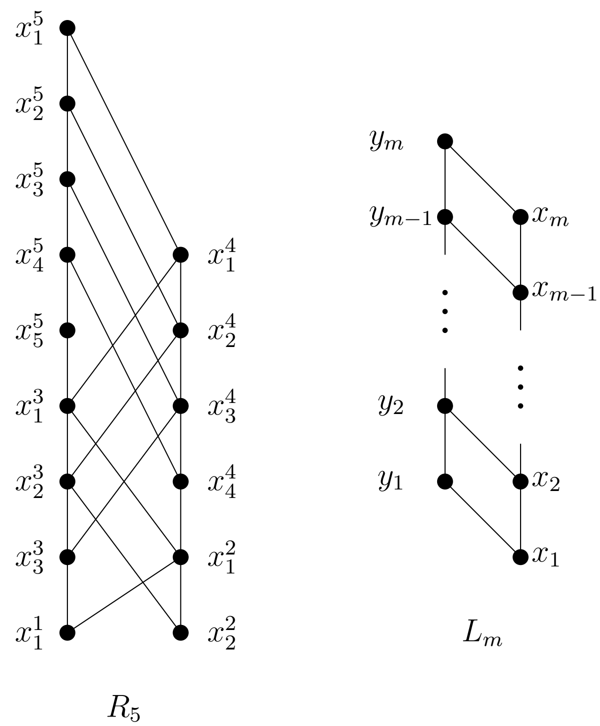

Figure 2.2: Hasse diagrams of R_5 and L_m.

Despite Lemma 2.2, the analysis of the performance of First-Fit on restricted families of posets has proved very useful and interesting. For posets P and Q, we say P is *Q-free* if Q is not isomorphic to any induced subposet of P. Let $\mathrm{Forb}(Q)$ denote the family of Q-free posets, and $\mathrm{Forb}_w(Q)$ denote the family of Q-free posets with width at most w. Slightly abusing notation, we write $\mathrm{val}_{\mathrm{FF}}(Q, w)$ for $\mathrm{val}_{\mathrm{FF}}(\mathrm{Forb}_w(Q))$. In 2010 Bosek, Krawczyk, and Matecki proved the following:

Theorem 2.3 (Bosek, Krawczyk & Matecki [5]). *For every width* 2 *poset* Q *there exists a function* f_Q *such that* $\mathrm{val}_{\mathrm{FF}}(Q, w) \leq f_Q(w)$.

Lemma 2.2 shows that the theorem cannot be extended to posets Q with width greater than 2. For general Q the proof of the theorem gives an exponential function f_Q. However, in many cases were Q is specified, $\mathrm{Forb}(Q, w)$ is either tightly bounded or known exactly. The case of $Q = \mathbf{s} + \mathbf{t}$ is especially interesting and well-studied. We will soon discuss it further.

Bosek and Krawczyk focused attention on the family of *ladders*. For a positive integer m, we say poset L is an m-ladder (or $L = L_m$) if its vertices are two disjoint chains $x_1 <_L x_2 <_L \cdots <_L x_m$ and $y_1 <_L y_2 <_L \cdots <_L y_m$ with $x_i <_L y_i$ for all $i \in [m]$ and $y_i \parallel_L x_j$ if $i \leq j \leq m$. We provide a Hasse diagram of $L = L_m$ in Figure 2.2. Notice that for two consecutive chains X^i and X^{i+1} of R_n, the set $X^i \cup (X^{i+1} - x^{i+1}_{i+1})$ induces the ladder L_i in R_n. The vertices $x_1, x_2, \ldots, x_m$ are the *lower leg* and the vertices $y_1, y_2, \ldots, y_m$ are the *upper leg* of L_m. The edge $x_i y_i$ is called the *ith rung* of L_m. We denote the poset (ladder) P with two disjoint chains $x_1 <_P x_2 <_P \cdots <_P x_m$ and $y_1 <_P y_2 <_P \cdots <_P y_m$ such that the subposet induced by these chains is isomorphic to L_m with $x_m <_P y_m$ by $L_m(x_1 x_2 \ldots x_m; y_1 y_2 \ldots y_m)$. Based on extensive work, the following observation was made during the proof of Theorem 2.1.

Observation 2.4 (Bosek & Krawczyk [4]). *If* $\mathrm{val}_{\mathrm{FF}}(L_m, w)$ *is bounded from above by a function* $f(m, w)$ *then* $\mathrm{val}(w)$ *is bounded from above by* $w \cdot f(2w^2 + 1, w)$.

Motivated by this observation, we will prove the following three theorems.

Theorem 2.5 (Kierstead & MES, Krawczyk)**.** $\mathrm{val}_{\mathrm{FF}}(L_2, w) = w^2$.

Theorem 2.6 (Kierstead & MES)**.** $m - 1 \leq \mathrm{val}_{\mathrm{FF}}(L_m, 2) \leq 2m$.

Theorem 2.7 (Bosek, Kierstead, Krawczyk, Matecki & MES)**.**
$w^{\lg(m-1)} \leq \mathrm{val}_{\mathrm{FF}}(L_m, w) \leq w^{2.5 \lg w + 2 \lg m}$.

Theorem 2.7 is used to prove the following theorem, offering a somewhat improved upper bound from Theorem 2.1 with a significantly simplified proof.

Theorem 2.8 (Bosek, Kierstead, Krawczyk & MES)**.** $\mathrm{val}(w) \leq w^{3+6.5 \lg w}$.

One may ask a similar question for graphs. If a graph is revealed one vertex at a time, is it possible to maintain a coloring using a finite number of colors? An *on-line graph* $G^{\prec}$ is a graph $G = (V, E)$ is a graph and $\prec$ is a total order on V called a *presentation*. On-line graph coloring is defined in a parallel way to on-line chain partitioning. An *on-line coloring algorithm* is a deterministic algorithm $\mathcal{A}$ that assigns the vertices $v_1 \prec v_2 \prec \cdots \prec v_n$ of $G^{\prec}$ to disjoint cocliques $C_1, C_2, \ldots, C_t$ so that for each v_i, the coclique C_j to which v_i is assigned is determined solely by the subposet $G[\{v_1, v_2, \ldots, v_i\}]$ (i.e: the first i vertices of the presentation). Let $\chi_{\mathcal{A}}(G^{\prec})$ denote the number of (cocliques) chains that $\mathcal{A}$ uses to color $G^{\prec}$, and $\chi_{\mathcal{A}}(G) = \max_{\prec}(\chi(G^{\prec}))$ over all presentations $\prec$ of G. For a given graph G, define $\chi_{\mathrm{OL}}(G) = \min_{\mathcal{A}}(\chi_{\mathcal{A}}(G))$ over all on-line coloring algorithms. Traditional coloring, where an algorithm views all the vertices of G before making a partition or is allowed to make changes to earlier assignments, will be called *off-line coloring.*

For posets, we found the number of chains needed to partition $P^{\prec}$ can be bounded by a function of width(P). So, it is natural to ask is $\chi_{\mathrm{OL}}(G)$ bounded by a function of $\chi(G)$? In general, this is not true. In fact, as we will demonstrate in Chapter 6, for each $n \in \mathbb{Z}^+$ there is a tree T so that $\chi_{\mathrm{OL}}(T) > n$. As all trees with at least two vertices require only two colors in the off-line setting, this removes any hope of bounding $\chi_{\mathrm{OL}}(G)$ in terms of $\chi(G)$. However, if we place restrictions on G, we can bound $\chi_{\mathrm{OL}}(G)$.

Again echoing the notation for posets, for graphs G and H, we say G is *H-free* if H is not isomorphic to any induced subgraph of G. Let Forb(H) denote the family of H-free graphs. If $\mathcal{H}$ is a set of graphs, then Forb$(\mathcal{H})$ is the family of graph that are H-free for all $H \in \mathcal{H}$. Note that $\mathcal{H}$ need not be finite. The First-Fit algorithm will be used in this setting as well. In the graph setting, FF assigns each new vertex v_i to the coclique C_j, with the least index $j \in \mathbb{Z}^+$ such that for all $h < i$ if $v_h \in C_j$ then $v_h v_i$ is not an edge in $G[\{v_1, v_2, \ldots, v_i\}]$. Again, we can see the result of FF is a proper coloring. In 1978, McDiarmid [41] showed that $\chi_{\mathrm{FF}}(G) = 2\chi(G) + \varepsilon$ for almost all graphs G. This proof is asymptotic, so there are infinitely many graphs for which this is not true and so the performance of FF in on-line coloring remains an active area of research.

Gyárfás [24], and independently Sumner [52], studied Forb$(\{T, K_{t+1}\})$ and conjectured that if T is a tree then $\chi_{\mathrm{OL}}(G)$ is bounded for all $G \in$ Forb$(\{T, K_{t+1}\})$. Gyárfás, Szemerédi, and Tuza [27] proved this conjecture in the case that T has radius 2 and $t = 2$, Kierstead and Penrice [34] extended their proof to all t and radius 2 trees T, and Kierstead, Penrice and Trotter [35] proved that there is an on-line algorithm that colors every graph in Forb$(\{T, K_{t+1}\})$ using a bounded number of colors.

Of particular interest is the family of cocomparability graphs. Suppose G is a graph so that $G = G^P$ for some poset P. The First-Fit coloring algorithm performs exactly the same on G as the First-Fit chain partitioning algorithm performs on P. However the input $P[\{v_1, v_2, \ldots, v_i\}]$ to an on-line chain partitioning algorithm provides more information than $G[\{v_1, v_2, \ldots, v_i\}]$. This extra information is needed for the on-line chain partitioning algorithms in [30, 4] as well as the algorithm used to prove Theorem 2.8. However, cocomparability graphs are a subfamily of $\text{Forb}(\{T, K_{t+1}\})$. Hence, the results of [34] allowed the authors to build an on-line coloring algorithm that uses a bounded number of colors on any cocomparability graph with bounded clique number. The number of colors used is large enough to be called astronomical. It remains an interesting open question to find good bounds for this on-line graph coloring problem.

A *circular arc graph* is the intersection graph of subpaths of a cycle. Formally, if $A_1, A_1, \ldots, A_n$ are subpaths of a cycle C, the corresponding circular arc graph is $G = (V, E)$ where $V = \{A_1, A_2, \ldots, A_n\}$ with $AB \in E$ if and only if $V(A) \cap V(B) \neq \emptyset$. We will refer to the any element of $\{A_1, A_2, \ldots, A_n\}$ as a path rather than a subpath and use $A \cap B$ to mean $V(A) \cap V(B)$ when referring to the intersection of paths. The cycle C is the *base cycle* and, to avoid confusion, we refer to $V(C)$ as *nodes* and $E(C)$ as *links*. The *overlap number* of G, denoted $\iota(G)$ is $\max\{|U| : U \subseteq V(G), \bigcap_{A \in U} A \neq \emptyset\}$ (i.e.: the cardinality of the largest collection of arcs that contain a common node).

We should note that the traditional definition of circular arc graphs is geometric; arcs along the circumference of a given circle are the vertices and their intersections determine the edges in the same fashion as our base cycle definition. Given a geometrically defined (finite) circular arc graph, we may find an isomorphic circular arc graph on a base cycle using subpaths by selecting a cycle with a node for each distinct endpoint of the set of arcs and then mapping the arcs to subpaths in the obvious way.

A well-studied subfamily of circular arc graphs is the family of *interval graphs*. If $I_1, I_2, \ldots, I_n$ is a set of subpaths of a path P, the corresponding interval graph is $G = (V, E)$ where $V = \{I_1, I_2, \ldots, I_n\}$ with $IJ \in E$ if and only if $I \cap J \neq \emptyset$. These graphs are also more traditionally thought of as geometric intersection graphs; in this case, as intersections of intervals on the real line. To see that this is indeed a subfamily of the circular arc graph, if we add a link to between the endnodes of P, we have a base cycle without altering the structure of G.

Despite the similar definitions of circular arc and interval graphs, they have very different properties. First, each interval graph is a cocomparability graph of an *interval order*. Roughly speaking, given an interval graph G, we may think of a base path drawn from left to right and build an interval order P based on G. Each vertex of the graph is a vertex in the poset. For vertices u and v we have $u \leq_P v$ if the right endpoint of u is to the left of the left endpoint of u. If $v \cap u \neq \emptyset$, the $u \parallel_P v$. We leave it to the reader to verify that cycles with an odd number of vertices are not cocomparability graphs. Any cycle may be represented as a circular arc graph and hence circular arc graphs (as a family) are not cocomparability graphs. Furthermore, the chromatic number, clique number, and overlap number of the graphs of these respective families have different interactions. Suppose G is an interval graph. The following is a well-known fact:

$$\iota(G) = \omega(G) = \chi(G). \tag{2.1}$$

When one is interested in only interval graphs, $\iota(G)$ is rarely mentioned. Circular arc graphs, however, are not perfect. If C_3 and C_7 are respectively a cycle of length 3 and a cycle of length 7, we have $2 = \iota(C_3) < \omega(C_3) = 3$ and $2 = \omega(C_7) < \chi(C_7) = 3$ while both are circular arc graphs (see Figure 2.3). From this, we see circular arc graphs satisfy

$$\iota(G) \leq \omega(G) \leq \chi(G). \tag{2.2}$$

Additionally, the chromatic number of an interval graph may be found in polynomial time where the same problem NP-hard for circular arc graphs, as shown by Garey, et al [22].

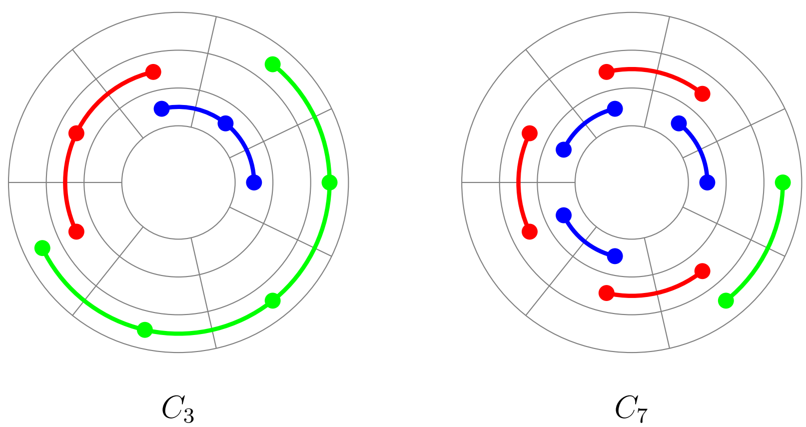

Figure 2.3: $\iota(C_3) < \omega(C_3)$ and $\omega(C_7) < \chi(C_7)$.

Despite these differences, interval and circular arc graphs show remarkable similarity with regard to on-line coloring. In 1981, Kierstead and Trotter [39] showed $\chi_{\mathrm{OL}}(G) = 3\chi(G) - 2$ for any interval graph G. Furthermore, the algorithm that provides the upper bound is easily understood and used. A more accessible version of this proof can be found in [33] or [47]. Marathe, Hunt, and Ravi [40] demonstrated an algorithm $\mathcal{A}$ for which $\chi_{\mathcal{A}}(G) \leq 4\chi(G)$ for any circular arc graph G. In 1995 Ślusarek [49] applied a slightly modified version of the algorithm from [39] to circular arc graphs to obtain $\chi_{\mathrm{OL}}(G) = 3\chi(G) - 2$ for any circular arc graph G. Hence, there is a strategy for coloring both interval graphs and circular arc graphs on-line with identical performance. This stands in stark contrast to the many differences in coloring these two families off-line. Of key interest is the role of $\iota(G)$ in the upper bound: the proof of the upper bounds on $\chi_{\mathrm{OL}}(G)$ for both circular arc and interval graphs use $\iota(G)$ in their algorithms, even though $\iota(G) \leq \chi(G)$ in circular arc graphs.

It is well-known and much-used that the family of interval orders is equal to $\mathrm{Forb}(\mathbf{2}+\mathbf{2})$ [20]. Recalling that poset P offers more information than its cocomparability graph G^P, we have $\mathrm{val}(\mathrm{Forb}_w(\mathbf{2}+\mathbf{2})) \leq 3w - 2$, using the results of [39]. Furthermore, an example from the same paper is easily adapted to show equality. First-Fit coloring of interval orderings has applications to polynomial time approximation algorithms [32, 33] and Max-Coloring [44], hence, it has been an area of great interest. In 1988, Kierstead [32] proved that $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w) \leq 40w$. This was improved in $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w) \leq 25.72w$ [36]. In 2004 Pemmaraju, Raman and K. Varadarajan [44] introduced a beautiful new technique to show $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w) \leq 10w$. This was quickly improved to $\mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w) \leq 8w$ [9, 43] with minor modifications. We will show that the proof from [44] can be adapted to show the following.

Theorem 2.9 (Kierstead & MES)**.** *For any circular arc graph G, $\chi_{\mathrm{FF}}(G) < 8\chi(G)$.*

In 1976 Witsenhausen [54] proved $4 \leq \mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w)$ (Chrobak and Ślusarek [11] independently found the same result). In 1993, Ślusarek [48] improved this to $4.45 \leq \mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w)$. In 2010 Kierstead, D. Smith and Trotter [50, 37] proved $5(1-o(1))w \leq \mathrm{val}_{\mathrm{FF}}(\mathbf{2}+\mathbf{2}, w)$.

In a natural generalization of $\mathrm{Forb}(\mathbf{2}+\mathbf{2})$, there has been research into $\mathrm{Forb}(\mathbf{s}+\mathbf{t})$ for $s, t \in \mathbb{Z}^+$. In 2010 Bosek, Krawczyk, and Szczypka [6] proved that $\mathrm{val}_{\mathrm{FF}}(\mathbf{t}+\mathbf{t}) \leq 3tw^2$. This result plays an important role in the proof of Theorem 2.1. Joret and Milans [29] improved this to $\mathrm{val}_{\mathrm{FF}}(\mathbf{s}+\mathbf{t}, w) \leq 8(s-1)(t-1)w$. Very recently, Dujmović, Joret, and Wood [15] proved $\mathrm{val}_{\mathrm{FF}}(\mathbf{t}+\mathbf{t}, w) \leq 16tw$. Another generalization of interval graphs and interval orders is *tolerance graphs*. We refer the reader to [23] for definitions and examples. Kierstead and Saoub [38, 46] established linear upper bounds on the performance of FF on certain families of tolerance graphs.

We have now discussed graphs where $\chi_{\mathrm{OL}}(G)$ cannot be bounded by $\chi(G)$ and families where $\chi_{\mathrm{FF}}(G)$ is bounded (in some cases very closely) by $\chi(G)$. As mentioned before, for each $n \in \mathbb{Z}^+$, there is a tree T so that $\chi_{\mathrm{FF}}(T) \geq \chi_{\mathrm{OL}}(T) > n$. Recalling $\chi_{\mathrm{FF}}(T) \geq \chi_{\mathrm{FF}}(T^{\prec})$ for any presentation $\prec$, we might think of $\chi_{\mathrm{FF}}(T)$ as looking at a worst case. We might wonder, for a tree with $\chi_{\mathrm{FF}}(T) = n$, how likely is it $\chi_{\mathrm{FF}}(T^{\prec}) = k$ for some $k \in [n]$ and randomly chosen presentation $\prec$?

For a fixed forest T, we define a probability space $(\Omega_T, \mathcal{F}_T, \mathrm{Pr})$ where $\Omega_T = \{\chi_{\mathrm{FF}}(T^{\prec}) : \prec \text{ is a presentation}\}$, $\mathcal{F}_T$ is the power set of Ω_T, and Pr is the probability measure. Define $\dot{A}_T^n$ to be the the event $\chi_{\mathrm{FF}}(T^{\prec}) = n$ and A_T^n to be the the event $\chi_{\mathrm{FF}}(T^{\prec}) \geq n$. If n is large in comparison to $|T|$, we expect the probability of A_T^n occurring (we will use $\mathrm{Pr}(A_T^n)$ to denote this) to be small. Somewhat counter to this intuition, we provide the following theorem.

Theorem 2.10 (Kierstead, MES & Winkler)**.** *For any $n \in \mathbb{Z}^+$ there exists a tree T with $Pr(A_T^n) \geq 1/2$ so that $n \geq \sqrt{1 + \lg |T|}$.*

2.2 Grundy Colorings

For an on-line graph $G^{\prec}$ or poset $P^{\prec}$, examining the performance of FF in coloring $G^{\prec}$ or partitioning $P^{\prec}$, the presentation often becomes cumbersome. To avoid this annoyance, we introduce the following colorings.

Definition 2.11. *Let G be a graph and $n \in \mathbb{Z}^+$. A function $\mathfrak{g} : V(G) \to [n]$ is an n-Grundy coloring of G if the following three conditions hold.*

(G1) For each $i \in [n]$, the set $\{u \in V(G) : \mathfrak{g}(u) = i\}$ is a coclique in G (i.e.: $\mathfrak{g}$ is an n-coloring of G).

(G2) For each $i \in [n]$, there is some $u \in V(G)$ so that $\mathfrak{g}(u) = i$ (i.e.: $\mathfrak{g}$ is surjective).

(G3) If $v \in V(G)$ with $\mathfrak{g}(v) = j$, then for all $i \in [j-1]$ there is some $u \in N_G(v)$ such that $\mathfrak{g}(u) = i$.

If $u \in V(G)$ and $\mathfrak{g}(u) = i$, we will say u is *colored with i*. Let the *color class i* be the coclique $V_i^{\mathfrak{g}}(G) = \{u \in V(G) : \mathfrak{g}(u) = i\}$. We will omit $\mathfrak{g}$ and G if they are clear from context. If H is a subgraph of G and $V(H) \cap V_i(G) \neq \emptyset$, then color i *appears* in H.

Let $u, v \in V(G)$. If $uv \in E(G)$ and $\mathfrak{g}(u) < \mathfrak{g}(v)$, we will say u is a $\mathfrak{g}(u)$*-witness for v under* $\mathfrak{g}$. If we are only concerned with one coloring function, this will be shortened to $\mathfrak{g}(u)$*-witness*. If we are not concerned with a specific color, we will simply say u is a *witness* for v. If H is a subgraph of G and $\mathfrak{g}$ is an n-Grundy coloring of G, we will yet again abuse notation and use $\mathfrak{g}$ for the function $\mathfrak{g}|_{V(H)} : V(H) \to [n]$ (i.e.: the function $\mathfrak{g}$ with domain restricted to H). Note that $\mathfrak{g}$ might not be an n-Grundy coloring of H.

In the following lemma, we see how a Grundy coloring allows us to ignore presentations in examining the $\chi_{\mathrm{FF}}(G)$.

Lemma 2.12. *If $G = (V, E)$ is a graph, then G has an n-Grundy coloring if and only if G has a presentation $\prec$ so that $\chi_{\mathrm{FF}}(G^{\prec}) = n$. Consequently, $\chi_{\mathrm{FF}}(G)$ is equal to the largest n so that G has an n-Grundy coloring.*

Proof. Let G be a graph and $\mathfrak{g}$ be an n-Grundy coloring of G. We build presentation $\prec$ based on $\mathfrak{g}$. Let $V_1, V_2, \ldots, V_n$ be the color classes of $\mathfrak{g}$. Set $V_1 \prec V_2 \prec \cdots \prec V_n$. For each $i \in [n]$, the order $\prec$ on the vertices of V_i may be chosen arbitrarily. If $\mathfrak{g}(v) = j$, First-Fit will assign v to C_j because, for each $i < j$, there is some vertex $u \in C_i$ so that $uv \in E$ with $u \prec v$. Hence, $\chi_{\mathrm{FF}}(G^{\prec}) = n$.

Suppose we have presentation $\prec$ so that $\chi_{\mathrm{FF}}(G^{\prec}) = n$. For each $u \in V$, let $\mathfrak{g}(u)$ be the index of the coclique to which u is assigned by FF. We will show the conditions of Definition 2.11 are satisfied. The result of FF is a coloring, so (G1) holds. Each coclique used by FF is nonempty, so (G2) is satisfied. Suppose $\mathfrak{g}(v) = j$. Then v was assigned to coclique C_j. By the definition of FF, for each $i < j$, there is some $u \in C_i$ so that $uv \in E$ with $u \prec v$. By our choice of $\mathfrak{g}$, we see that u is an i-witness and so (G3) holds as well. $\square$

Let P be a poset. To analyze the performance of FF in coloring P on-line, we will use Grundy colorings of G^P. Although we could simply speak of a Grundy coloring of cocomparability graphs, it is worth the time to define this type of coloring explicitly for posets.

Definition 2.13. *Let P a poset and n a positive integer. The function $\mathfrak{g} : P \to [n]$ is an n-Grundy coloring of P if the following three conditions hold.*

(P1) For each $i \in [n]$, the set $\{u \in P : \mathfrak{g}(u) = i\}$ is a chain in P (i.e.: $\mathfrak{g}$ is an n-coloring of P).

(P2) For each $i \in [n]$, there is some $u \in P$ so that $\mathfrak{g}(u) = i$ (i.e.: $\mathfrak{g}$ is surjective).

(P3) If $v \in P$ with $\mathfrak{g}(v) = j$, then for all $i \in [j-1]$ there is some $u \in I_P(v)$ such that $\mathfrak{g}(u) = i$.

Most of our terms for Grundy colorings of posets are parallel to those used for Grundy colorings of graphs, but we take the time to explicitly define them here. If $u \in P$ and $\mathfrak{g}(u) = i$, we will say u is *colored with* i. Let the *color class* i be the chain $P_i(\mathfrak{g}) = \{u \in P : \mathfrak{g}(u) = i\}$. If we are only concerned with one coloring function, we will shorten this to P_i. If Q is a subposet of P and $Q \cap P_i \neq \emptyset$, then color i *appears* on Q.

Let $u, v \in P$. If $u \parallel_P v$ and $\mathfrak{g}(u) < \mathfrak{g}(v)$, we will say u is a $\mathfrak{g}(u)$-*witness for* v *under* $\mathfrak{g}$. If we are only concerned with one coloring function, this will be shortened to $\mathfrak{g}(u)$-*witness*. If we are not concerned with a specific color, we will simply say u is a *witness* for v. If Q is a subposet of P and $\mathfrak{g}$ is an n-Grundy coloring of P, we will abuse notation and use $\mathfrak{g}$ for the function $\mathfrak{g}|_Q : Q \to [n]$ (i.e.: the function $\mathfrak{g}$ with domain restricted to Q). Note that $\mathfrak{g}$ might not be an n-Grundy coloring of Q.

For a poset P, we apply Lemma 2.12 to G^P and arrive at the following lemma.

Lemma 2.14. *If P is a poset, then P has an n-Grundy coloring if and only if P has a presentation $\prec$ so that $\chi_{\mathrm{FF}}(P^{\prec}) = n$. Consequently, $\chi_{\mathrm{FF}}(P)$ is equal to the largest n so that P has an n-Grundy coloring.*

Chapter 3

ON-LINE COLORING OF LADDER-FREE POSETS

3.1 Proof of Theorem 2.5

We first bound the performance of First-Fit on width w posets in $\mathrm{Forb}(L_2)$, and then provide examples to show that our bound is tight.

Lemma 3.1. *Every poset $P \in \mathrm{Forb}_w(L_2)$ satisfies $\chi_{\mathrm{FF}}(P) \leq w^2$.*

Proof. Let P be a poset of width at most w in the family $\mathrm{Forb}(L_2)$ and $\mathfrak{g}$ be an n-Grundy coloring of P. Furthermore, let P be minimal with respect to $|P|$, i.e.: there is no $x \in P$ so that $P - x$ has an n-Grundy coloring. Fix a Dilworth partition $\mathcal{C} = \{C_1, C_2, \ldots, C_w\}$ of P. We will show $n \leq w^2$.

Claim 3.2. *For any chain $C \in \mathcal{C}$, there is at most one color j such that $P_j \subseteq C$.*

Proof. Let j (if it exists) be the largest color such that $P_j \subseteq C$, and let $v \in P_j$. Then $P_i \nsubseteq C$ for any color $i > j$; if $i < j$ then by Definition 2.13(P3), $x \parallel_P v$ for some $x \in P_i$, and so again $P_i \nsubseteq C$. $\square$

Claim 3.3. $|P_n| = 1$.

Proof. If $u, v \in P_n$ are distinct vertices, then the poset $P' = P - u$ is in $\mathrm{Forb}(L_2)$, $\mathfrak{g}$ is an n-Grundy coloring of P' and $|P'| < |P|$. This contradicts the hypothesis that P is minimal. $\square$

Now, we will focus on the colors appearing on more than one chain of $\mathcal{C}$.

Claim 3.4. *For every $k \in [n]$ and $u \in P_k$, there exists $x \in I_P(u)$ such that $P_k - u + x$ is a chain.*

Proof. Let $u \in P_k$, $P'_k = P_k - u$, and $P' = P - u$. Of course $P' \in \mathrm{Forb}(L_2)$. So the minimality of P implies that $\mathfrak{g}$ is not an n-Grundy coloring of P'. If $P'_k = \emptyset$, we are done; so assume $P'_k \neq \emptyset$. Thus Definition 2.13(P1) holds for $\mathfrak{g}$, and 2.13(P2) holds trivially. So 2.13(P3) fails. Thus u is the only k-witness for some $x \in P_i$ with $k < i \in [n]$. So $x \in I_P(u)$ and $P_k - u + x$ is a chain. □

Claim 3.5. *For any color k, $|P_k| \leq 2$.*

Proof. Suppose the chain P_k has distinct vertices $t <_P u <_P v$. By Lemma 3.4, there exists $x \in I_P(u)$ so that $x \bowtie_P t$ and $x \bowtie_P v$. As $x \parallel_P u$, we must have $t <_P x <_P v$. So $P[t, u, v, x] = L_2(t, u; x, v)$ is an induced 2-ladder (see Figure 3.1). This contradicts the hypothesis $P \in \mathrm{Forb}(L_2)$.

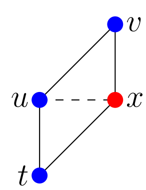

Figure 3.1: The subposet induced by t, u, v, and x.

□

Claim 3.6. *For all chains $A, B \in \mathcal{C}$, at most two colors appear on both A and B.*

Proof. Let S be the set of the colors that appear on both A and B. For a contradiction, assume three distinct colors $i < j < k$ are contained in S. Let $P' = P[P_i \cup P_j \cup P_k]$. By Claim 3.5, each of the colors in $\{i, j, k\}$ appears at most twice. So P' is contained in the chains A and B. Consider the vertices a_i, a_j, a_k, b_i, b_j, and b_k where $a_\gamma \in A$, $b_\gamma \in B$, and $\mathfrak{g}(a_\gamma) = \mathfrak{g}(b_\gamma) = \gamma$ for $\gamma \in \{i, j, k\}$. As $\mathfrak{g}$ is a Grundy coloring of P, a_k and b_k must have i- and j-witnesses; so a_k is incomparable to b_i and b_j, and b_k is incomparable to a_j and a_k. Similarly, $a_j \parallel_P b_i$ and $b_j \parallel_P a_i$.

We also note the vertices a_i, a_j, and a_k are pairwise comparable as they belong to the same chain, A. Similarly, b_i, b_j, and b_k are pairwise comparable. As $\mathfrak{g}$ is a Grundy coloring, for $\gamma \in \{i, j, k\}$, we have $a_\gamma \bowtie_P b_\gamma$.

Without loss of generality, we may take $a_j <_P a_k$. As $b_k \parallel_P a_j$ and $b_k \bowtie_P a_k$, we must have $b_k <_P a_k$. Also $b_j \parallel_P a_k$, so we must have $a_j <_P b_j$. We depict this in Figure 3.2.

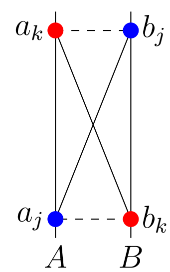

Figure 3.2: Hasse diagram of chains A and B.

Since $b_j, b_k \in I_P(a_i)$, we have $a_j <_P a_i <_P a_k$. Similarly, we find $b_k <_P b_i <_P b_j$. Recall that a_i and b_i are comparable (see Figure 3.3). If $a_i <_P b_i$ then $a_i < b_j$, a contradiction; otherwise $b_i < a_i$, and so $b_i < a_k$, another contradiction. $\square$

By Claim 3.5, colors appear on either one or two chains in $\mathcal{C}$. By Claim 3.2, there are at most w colors that appear on exactly one chain in $\mathcal{C}$. By Claim 3.6, any pair of chains in $\mathcal{C}$ share at most two colors. From this, we

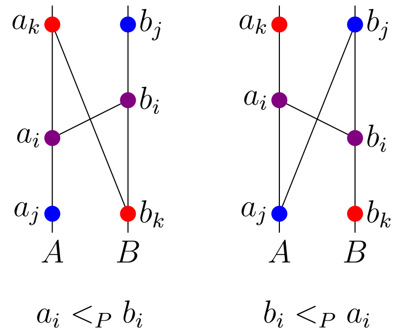

Figure 3.3: Hasse diagrams of the possible orderings of a_i and b_i.

conclude

$$n \leq w + 2\binom{w}{2} = w^2.$$

So $\chi_{\mathrm{FF}}(P) \leq w^2$, as desired. $\square$

Now we prove a matching lower bound for Lemma 3.1.

Lemma 3.7. *For each positive integer w, there exists a poset $P \in \mathrm{Forb}(L_2)$ with $\mathrm{width}(P) = w$ that satisfies $\chi_{\mathrm{FF}}(P) = w^2$.*

Proof. It suffices to build the desired poset $P \in \mathrm{Forb}(L_2)$ with $\mathrm{width}(P) = w$ and a w^2-Grundy coloring $\mathfrak{g}$ of P. Arguing by induction on w, we will construct P and $\mathfrak{g}$ satisfying the following.

(I1) $P \in \mathrm{Forb}(L_2)$.

(I2) $\mathrm{width}(P) = w$.

(I3) P has w minimal and w maximal vertices.

(I4) $\mathfrak{g}$ is a w^2-Grundy coloring of P.

The case for $w = 1$ is simple: P is the poset with one vertex. Clearly, $P \in \mathrm{Forb}(L_2)$, P has width one, P has one minimal and one maximal vertex, and P has a 1-Grundy coloring. Now, assume the inductive hypothesis holds for all cases smaller than w.

Define a poset H and a coloring $\dot{\mathfrak{g}}$ of H as follows (see Figure 3.4).

(H1) The vertices of H are two $w-1$ vertex antichains, $A = \{a_1, \ldots, a_{w-1}\}$ and $B = \{b_1, \ldots, b_{w-1}\}$, and a $2w-1$ vertex chain $C = c_{2w-1} <_H c_{2w-2} <_H \cdots <_H c_2 <_H c_1$.

(H2) For $i \in [w-1]$, $c_{2w-i} <_H a_i$.

(H3) For $i \in [w-1]$, $b_i <_H c_i$.

(H4) For all $i, j \in [w-1]$, $b_i <_H a_j$.

(H5) There are no other comparabilities, except those implied by transitivity and reflexivity.

(H6) $\dot{\mathfrak{g}} : H \to [2w-1]$ by:

$$\dot{\mathfrak{g}}(u) = \begin{cases} i & \text{if } u = b_i \text{ for } i \in [w-1] \\ w+i-1 & \text{if } u = a_i \text{ for } i \in [w-1] \\ i & \text{if } u = c_i \text{ for } i \in [w-1] \\ 3w-i-1 & \text{if } u = c_i \text{ for } i \in [w, 2w-1] \end{cases}.$$

We now show that H satisfies (I1–3).

Claim 3.8. *$H \in \mathrm{Forb}(L_2)$, $\mathrm{width}(H) = w$, and H has w minimal and w maximal vertices.*

Proof. Set $C_i = \{a_i, b_i\}$ for $i \in [w-1]$. Then $\{C, C_1, C_2, \ldots, C_{w-1}\}$ is a chain partition of H, so it has width at most w. The set $B + c_{2w-1}$ shows that there are w minimal vertices and that the width is exactly w. The set $A + c_1$

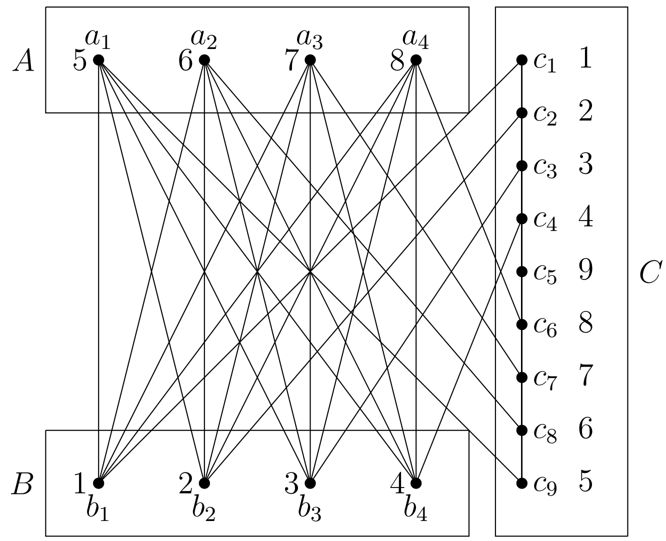

Figure 3.4: Hasse diagram of H along with coloring $\dot{\mathfrak{g}}$.

shows that there are w maximal vertices. Now, assume $L_2(pq; rs)$ is in H, as in Figure 3.5. Note that q and r cannot both be from C, as C is a chain.

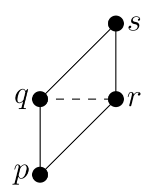

Figure 3.5: Labeling of L_2.

Thus, one of q or r, say q, must be in $A \cup B$. If $q \in A$, then q is maximal in H, and so there is no vertex s in H with $q \leq_H s$. If $q \in B$, then q is minimal in H, and so there is no vertex p in H with $p \leq_H q$. Thus $H \in \text{Forb}(L_2)$. $\square$

Claim 3.9. $\dot{\mathfrak{g}}$ *is a* $(2w-1)$*-Grundy coloring of* H*.*

Proof. The definitions of H and $\dot{\mathfrak{g}}$ show $\dot{\mathfrak{g}}$ is surjective. It is also easy to verify that each color class is a chain in H. We will now verify that each vertex $x \in H$ has a j-witness for each $j \in [\dot{\mathfrak{g}}(x) - 1]$.

For each $i \in [w-1]$, we have $\dot{\mathfrak{g}}(b_i) = i$. As $b_i \parallel_H b_j$ for $j < i$, each $b_i \in B$ has a j-witnesses for all $j \in [i-1]$. If $i \in [w-1]$, then $\dot{\mathfrak{g}}(c_i) = i$. We have $c_i \parallel_H b_j$ for $j < i$ and so each c_i has j-witnesses for all $j \in [i-1]$. For $a_i \in A$, $\dot{\mathfrak{g}}(a_i) = w + i - 1$. We have $a_i \parallel_H c_j$ for $j \in [w-1]$, so a_i has j witnesses for $j \in [w-1]$. Also, for $i \neq 1$, $a_i \parallel_H a_j$ for $j < i$ so a_i has j-witnesses for $j \in [w, w+i-2]$. If $i \in [w, 2w-1]$, then $\dot{\mathfrak{g}}(c_i) = 3w - i - 1$. We have $c_i \parallel_H B$, so c_i has j-witnesses for $j \in [w-1]$. Also, for $i \neq 2w-1$, $c_i \parallel_H a_j$ for $j < 2w - i + 1$, so c_i has j-witnesses for $j \in [w, (2w-1) - (i-w)] = [w, 3w-i-1]$. Thus, $\dot{\mathfrak{g}}$ is a Grundy coloring of H. □

We are now ready to build the desired poset P and the Grundy coloring $\mathfrak{g}$ (see Figure 3.6). By the inductive hypothesis, there is a poset P' and a Grundy coloring $\mathfrak{g}'$ satisfying (I1–4). By Dilworth's Theorem and (I2), there is a chain partitioning $\mathcal{C}' = \{C'_1, C'_2, \ldots, C'_{w-1}\}$ of P'. By (I3) P' has $w-1$ minimal and $w-1$ maximal vertices. Thus every chain C'_i contains a minimal vertex b'_i and a maximal vertex a'_i of P'. Define P and $\mathfrak{g}$ to be the poset and coloring formed by combining H, P', $\mathfrak{g}'$ and $\dot{\mathfrak{g}}$ as follows (see Figure 3.6):

(1) The disjoint union of the vertices of H and P' are the vertices of P.

(2) For $i \in [w-1]$, $a'_i <_P a_i$.

(3) For $i \in [w-1]$, $b_i <_P b'_i$.

(4) There are no other comparabilities, except the order relations from H and P' and those implied by transitivity.

(5) $\mathfrak{g} : P \to [w^2]$ by:

$$\mathfrak{g}(u) = \begin{cases} \dot{\mathfrak{g}}(u) & \text{if } u \in H \\ \mathfrak{g}'(u) + 2w - 1 & \text{if } u \in P' \end{cases}$$

From now on we refer to H and P' as subposets of P.

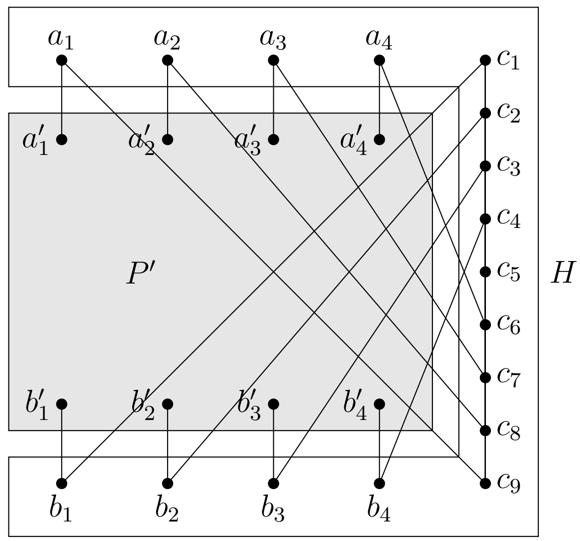

Figure 3.6: Partial Hasse diagram of P.

The next three claims complete the proof.

Claim 3.10. *(I2-3)* $\text{width}(P) = w = |\operatorname{Max}_P(P)| = |\operatorname{Min}_P(P)|$.

Proof. For $i \in [w-1]$, set $C_i = C'_i + a_i + b_i$. Since

$$b_i <_P b'_i \leq_P C'_i \leq_P a'_i <_P a_i,$$

we see C_i is a chain. Thus $\mathcal{C} = \{C_1, \ldots, C_{w-1}, C\}$ is a chain partition of P. It follows that

$$w = \text{width}(H) \leq \text{width}(P) \leq w \text{ and}$$

$$|\operatorname{Min}_P(P)| = |\operatorname{Min}_P(H)| = w = |\operatorname{Max}_P(H)| = |\operatorname{Max}_P(P)|.$$

$\square$

Claim 3.11. *(I1)* $P \in \mathrm{Forb}(L_2)$.

Proof. Assume $L_2(pq;rs)$ is contained in P (see Figure 3.5). By the inductive hypothesis and Claim 3.8, at least one of the vertices p,q,r,s is in P' and at least one is in H. As in the proof of Claim 3.8 the vertices of A (respectively, B) are maximal (minimal) in P. Thus we cannot have q or r in antichain A (antichain B). Also, we cannot have both q and r in chain C.

First suppose q and r are in P'. Because the vertices of P' are incomparable to the vertices of C, we must have $p \in B$ or $s \in A$. If $p \in B$, then $p = b_i$ for some $i \in [w-1]$; set $p' = b_i'$. As $U_P[b_i] - b_i = U_P[b_i']$, we must have $p' = b_i' <_P q$ and $p' = b_i' <_P r$. If $s \in A$, then $s = a_i$ for some $i \in [w-1]$; set $s' = a_i'$. As $D_P[a_i] - a_i = D_P[a_i']$, we must have $q <_P a_i' = p'$ and $r <_P a_i' = s'$. From this, we see $L_2(p'q;rs)$, $L_2(pq;rs')$, or $L_2(p'q;rs')$ is in P', contradicting the inductive hypothesis.

Otherwise, we may assume $q \in P'$ and $r \in C$. If $r = c_i$ with $i \in [w]$, then q and r have no common greater vertex s. If $r = c_i$ with $i \in [w, 2w-1]$, then q and r have no common lesser vertex p. Hence, we cannot have $L_2(pq;rs)$ with $r \in C$ and $q \in P'$.

These contradictions complete the proof of this claim. □

Claim 3.12. *(I4)* $\mathfrak{g}$ *is a* w^2*-Grundy coloring of* P.

Proof. We can see $\mathfrak{g}$ is surjective. Each color class of $\mathfrak{g}$ is a color class in $\dot{\mathfrak{g}}$ or $\mathfrak{g}'$. By the inductive hypothesis and Claim 3.9, they are chains. So $\mathfrak{g}$ satisfies Definition 2.13(P1,P2).

Now consider Definition 2.13(P3). First suppose $u \in H$. We have $\mathfrak{g}(u) = \dot{\mathfrak{g}}(u)$. By Claim 3.9, u has an i-witness for each $i \in [\mathfrak{g}(u) - 1]$. Now suppose $u \in P'$. Note that $u \parallel_P C$. For each $i \in [2w-1]$, there is some $c \in C$ so that $\mathfrak{g}(u) = i$. Let $j \in [2w, \mathfrak{g}(u)-1]$. By the inductive hypothesis, u has a $(j-2w+1)$-witness w under $\mathfrak{g}'$. It follows that w is a j-witness under $\mathfrak{g}$. $\square$

This completes the proof of Lemma 3.7. $\square$

Proof of Theorem 2.5. By Lemma 3.1, every poset $P \in \mathrm{Forb}(L_2)$ with $\mathrm{width}(P) = w$ satisfies $\chi_{\mathrm{FF}}(P) \leq w^2$. By Lemma 3.7 there is a poset $P \in \mathrm{Forb}(L_m)$ with $\mathrm{width}(P) = w$ and $\chi_{\mathrm{FF}}(P) \geq w^2$. Hence, $\mathrm{val}_{\mathrm{FF}}(L_2, w) = w^2$. $\square$

3.2 Proof of Theorem 2.6

In this section we consider posets in $\mathrm{Forb}(L_m)$ with $\mathrm{width}(P) = 2$.

Proof of Theorem 2.6. The claim is trivial if $m = 1$ as the posets of $\mathrm{Forb}(L_1)$ are antichains, and a two vertex antichain trivially has a 2-Grundy coloring. The result for $m = 2$ follows from Theorem 2.5. Furthermore, the result is tight for these cases.

Fix a positive integer $m > 2$. Let P be a width 2 poset in the family $\mathrm{Forb}(L_m)$ and let $\mathfrak{g}$ be an n-Grundy coloring of P. Let $v \in P_n$ and select a chain partition $\{A, B\}$ of P so that $v \in A$. For each $i \in [n-1]$, v has an i-witness. Select one witness for each color and denote the set of these witnesses as X. Index the vertices of X as $x_1, x_2, \ldots, x_{n-1}$ so that $x_i <_P x_{i+1}$ for each $i \in [n-2]$. Note that the subscripts of the vertices in X denote their order in the poset, not their color under $\mathfrak{g}$. For $i \in [n-2]$, let $\mathfrak{A} = \{i : \mathfrak{g}(x_i) < \mathfrak{g}(x_{i+1})\}$ and $\mathfrak{D} = \{i : \mathfrak{g}(x_i) > \mathfrak{g}(x_{i+1})\}$.

Lemma 3.13. *If $|\mathfrak{A}| = p$ or $|\mathfrak{D}| = p$, then L_p is an induced subposet of P.*

Proof. Suppose $\mathfrak{A} = \{a_1, a_2, \ldots, a_p\}$ where $a_i < a_{i+1}$ for $i \in [p-1]$. Let $\kappa \in \mathfrak{A}$. By the definition of $\mathfrak{A}$, we have $\mathfrak{g}(x_\kappa) < \mathfrak{g}(x_{\kappa+1})$ and so $x_{\kappa+1}$ must have a $\mathfrak{g}(x_\kappa)$-witness, y_κ.

Claim 3.14. *For $\kappa \in \mathfrak{A}$, we have the following:*

(1) $x_\kappa <_P y_\kappa$,

(2) $v <_P y_\kappa$,

(3) $y_\kappa <_P y_\lambda$, *with* $\kappa < \lambda \in \mathfrak{A}$,

(4) $y_\kappa \parallel_P x_\lambda$, *with* $\kappa < \lambda \in \mathfrak{A}$.

Proof. See Figure 3.7. Because x_κ and y_κ have the same color, and y_κ is a witness for $x_{\kappa+1}$ we have $y_\kappa \bowtie_P x_\kappa <_P x_{\kappa+1} \parallel_P y_\kappa$. It follows that $x_\kappa < y_\kappa$. This shows (1).

Because width$(P) = 2$ and $v \in A$, each $x_\kappa \in B$; thus each $y_\kappa \in A$. Using (1) and $v, y_\kappa \in A$, we have $x_\kappa <_P y_\kappa \bowtie_P v \parallel_P x_\kappa$. Thus $v <_P y_\kappa$, showing (2).

As $y_\kappa, y_\lambda \in A$, we have $y_\kappa \bowtie_P y_\lambda$. As $x_{\kappa+1} \leq_P x_\lambda <_P y_\lambda \bowtie_P y_\lambda \parallel_P x_{\kappa+1}$, it follows $y_\kappa <_P y_\lambda$. This shows (2).

Since $y_\kappa \parallel_P x_{\kappa+1} \leq x_\lambda$, $x_\lambda \not<_P y_\kappa$; since by (2), $x_\lambda \parallel_P v <_P y_\kappa$, $y_\kappa \not<_P x_\lambda$. Thus $x_\lambda \parallel_P y_\kappa$, and (4) holds. $\square$

The comparabilities and incomparabilities established by Claim 3.14, show that the chains $x_{a_1} <_P x_{a_2} <_P \cdots <_P x_{a_p}$ and $y_{a_1} <_P y_{a_2} <_P \cdots <_P y_{a_p}$ induce the ladder $L_p(x_{a_1} \ldots x_{a_p}; y_{a_1} \ldots y_{a_p})$ in P.

A dual argument for $\mathfrak{D} = \{d_1, d_2, \ldots, d_p\}$ completes the lemma. $\square$

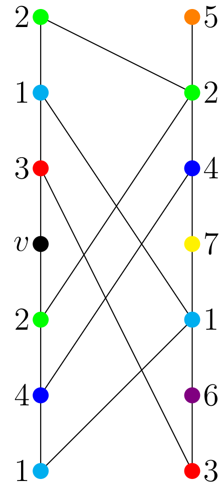

Figure 3.7: Hasse diagram with vertices labeled by their color under $\mathfrak{g}$.

By our selection of X, each color in $[n-1]$ appears exactly once; so $\mathfrak{g}(x_i) \neq \mathfrak{g}(x_{i+1})$. From this, we conclude

$$\mathfrak{A} \cap \mathfrak{D} = \emptyset, \tag{3.1}$$

$$\mathfrak{A} \cup \mathfrak{D} = [n-2]. \tag{3.2}$$

As $P \in \mathrm{Forb}(L_m)$, Lemma 3.13 tells us $|\mathfrak{A}| \leq m-1$ and $|\mathfrak{D}| \leq m-1$. From (3.1) and (3.2), we have

$$n-2 = |\mathfrak{A}| + |\mathfrak{D}| \leq m-1+m-1 = 2m-2.$$

From this, we see $n \leq 2m$.

To provide the lower bound, first, we note that if L is an m-ladder with lower leg $a_1 <_L a_2 <_L \cdots <_L a_m$ and upper leg $b_1 <_L b_2 <_L \cdots <_L b_m$, then $|I_L(a_m)| = |I_L(b_1)| = m-1$. Let $R = R_{m-1}$ be the poset used in Lemma 2.2. By construction, for any $k \in [m-1]$ and $i \in [k]$, $I_R(x_i^k) = \{x_1^{k-1}, x_2^{k-1}, \ldots, x_{i-1}^{k-1}\}$. Hence, $\max_{x \in R} |I_R(x)| = m-2$. From this, we conclude $R \in \mathrm{Forb}(L_m)$. As demonstrated in Lemma 2.2, $\chi_{\mathrm{FF}}(R) \geq m-1$. $\square$

3.3 Proof of Theorem 2.7

Before we begin, we recall a well-known fact.

Proposition 3.15. *Let A be an antichain in poset P. Then $U_P[A] \cap D_P[A] = A$.*

Proof of Theorem 2.7. From Theorem 2.5, we need only consider $m \geq 3$. We argue by induction on $w = \text{width}(P)$. The base step $w = 1$ is trivial. For the induction step, assume the theorem holds for all posets of width less than w. Consider any poset $P \in \text{Forb}(L_m)$ with $\text{width}(P) = w$, and let $\mathfrak{g}$ be an n-Grundy coloring of P. We must show that $n \leq w^{2.5 \lg w + 2 \lg m}$.

Let $\mathcal{A}$ be the set of maximum antichains in P. Select $A \in \mathcal{A}$ so that

$$\min_{a \in A} \mathfrak{g}(a) = \max_{B \in \mathcal{A}} \min_{b \in B} \mathfrak{g}(b).$$

In other words, A is a maximum antichain so that smallest color of a vertex in A is as large as possible. Set

$$N = \min_{a \in A} \mathfrak{g}(a).$$

By our choice of A, the subposet of P induced by the vertices in $P_{N+1} \cup P_{N+2} \cup \cdots \cup P_n$ has width at most $w - 1$. By the inductive hypothesis, at most $\text{val}_{\text{FF}}(L_m, w-1)$ colors appear on this subposet. Hence,

$$n \leq N + \text{val}_{\text{FF}}(L_m, w-1). \tag{3.3}$$

Consider any $i \in [N-1]$ and let x be the greatest vertex of the chain P_i. Since A is a maximum antichain, x is comparable to some $a \in A$. As $\mathfrak{g}$ is a Grundy coloring, and $\mathfrak{g}(a) > i$, there exists a witness $u \in P_i$ with $u \parallel_P a$.

Thus $a \parallel_P u <_P x \bowtie_P a$, and so $a <_P x$. Similarly, the least vertex of P_i is less than some vertex of A. We conclude

$$U_P(A) \cap P_i \neq \emptyset \text{ and } D_P(A) \cap P_i \neq \emptyset. \tag{3.4}$$

For $i \in [N-1]$, define $q_i^{\downarrow}$ to be the greatest $q \in P_i \cap D_P(A)$ and $q_i^{\uparrow}$ to be the smallest $q \in P_i \cap U_P(A)$. They exist by (3.4), and are distinct, since A is an antichain disjoint from P_i. Moreover, if the i-witness for $a \in A$ is in $D_P(A)$ then $a \parallel_P q_i^{\downarrow}$, and if the i-witness for $a \in A$ is in $U_P(A)$ then $a \parallel_P q_i^{\uparrow}$. It follows that:

$$A \subseteq I(q_i^{\downarrow}) \cup I(q_i^{\uparrow}). \tag{3.5}$$

We say a vertex $x \in P$ has property $(\star)$ if $|I(x) \cap A| \geq w/2$. By (3.5) and the pigeonhole principle, at least one of $q_i^{\downarrow}$ or $q_i^{\uparrow}$ has property $(\star)$; select one that does, denote it by q_i, and call it the *near witness for color i*. Let $Q = \{q_1, q_2, \ldots, q_{N-1}\}$. If $q_i \in D_P(A)$ then let r_i be the smallest vertex of the chain P_i with property $(\star)$; otherwise (when $q_i \in U_P(A)$) let r_i be the greatest vertex of P_i with property $(\star)$. Call r_i the *far witness for color i*. Let $R = \{r_1, r_2, \ldots, r_{N-1}\}$. The pair (q_i, r_i) is called the *corresponding pair for color i*. Note that it is possible that $q_i = r_i$.

Let $\mathcal{C}$ be a chain partition of P into w chains. We will need the classic Erdős-Szekeres Theorem [16]: for natural numbers k and ℓ, every sequence of $k\ell + 1$ totally ordered terms contains a strictly increasing subsequence of $k+1$ terms or a strictly decreasing subsequence of $\ell + 1$ terms.

Lemma 3.16. *For any $C \in \mathcal{C}$, we have* $|R \cap C| \leq wm^2(w-1)^2 \operatorname{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$.

Proof. Take $C \in \mathcal{C}$. We begin by analyzing $R \cap C \cap U_P(A)$; later we will use duality to draw similar conclusions about $R \cap C \cap D_P(A)$. Label the vertices of $R \cap C \cap U_P(A)$ as $s_1, s_2, \ldots, s_\alpha$ so that $\mathfrak{g}(s_i) < \mathfrak{g}(s_{i+1})$ for all $i \in [\alpha - 1]$. This yields a sequence $S = s_1 s_2 \ldots s_\alpha$ of length α. Let $T = t_1 t_2 \ldots t_\beta$ be any monotonic subsequence of S with respect to the order $\leq_P$. For each $i \in [\beta - 1]$, t_{i+1} has a $\mathfrak{g}(t_i)$-witness y_i, since $\mathfrak{g}$ is a Grundy coloring and $\mathfrak{g}(t_i) < \mathfrak{g}(t_{i+1})$.

Claim 3.17. *If T is ascending then for each $i \in [\beta - 1]$ and $j \in [i+1, \beta]$:*

(C1) $t_i <_P y_i$

(C2) $t_j \parallel_P y_i$

(C3) If $j \in [\beta - 1]$ (and so y_j is defined) then $y_j \not\leq_P y_i$.

Proof. (See Figure 3.8.) Fix $i \in [\beta - 1]$ and $j \in [i+1, \beta]$. Then

$$t_{i+1} \parallel_P y_i \bowtie_P t_i <_P t_{i+1} \leq_P t_j. \tag{3.6}$$

This implies $t_i <_P y_i$ and so (C1) holds. Both t_i and t_j have property $(\star)$. Since $\mathfrak{g}(t_i) = \mathfrak{g}(y_i)$, $t_i <_P y_i$ by (C1), and t_i is the greatest vertex in its color class with property $(\star)$, it follows that y_i does not have property $(\star)$. If $y_i <_P t_j$, then $D_P(y_i) \subseteq D_P(t_j)$, and so $|A \cap D_P(t_j)| < w/2$; this means that y_i has property $(\star)$, a contradiction. So $y_i \not<_P t_j$. By (3.6), $t_j \not<_P y_i$. Thus (C2) holds. Finally, suppose $j \in [\beta - 1]$. If $y_j \leq_P y_i$ then by (C1) we have $t_{i+1} \leq_P t_j <_P y_j \leq_P y_i \parallel_P t_{i+1}$, a contradiction. So (C3) holds. ☐

Claim 3.18. *If T is an ascending subsequence of S then its length β is at most $m(w-1)$.*

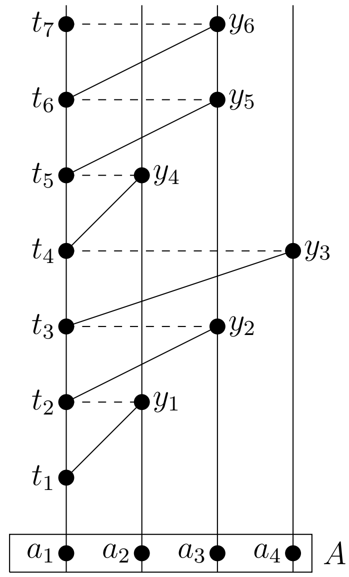

Figure 3.8: Hasse diagram of T, A, and witnesses for T.

Proof. Since $y_i \parallel_P t_{i+1} \in C$ for each $i \in [\beta - 1]$, $y_i \notin C$. Hence, all of $y_1, y_2, \ldots, y_{\beta-1}$ are contained in the $w-1$ chains of $\mathcal{C} - C$. By the pigeonhole principle, at least $(\beta-1)/(w-1)$ of them must be on the same chain, say D. Set $\beta' = \lceil(\beta-1)/(w-1)\rceil$ and let $T' = t'_1 t'_2 \ldots t'_{\beta'}$ be a subsequence of T so that y'_i (the vertex so that $\mathfrak{g}(t'_i) = \mathfrak{g}(y'_i)$ and y'_i is a witness for the vertex after t'_i in T) is on D. Here we have two disjoint chains $t'_1 <_P t'_2 <_P \cdots <_P t'_{\beta'}$ and $y'_1 <_P y'_2 <_P \cdots <_P y'_{\beta'}$. The comparabilities between these chains, obtained from Claim 3.17, show that $L_{\beta'}(t'_1 \ldots t'_{\beta'}; y'_1 \ldots y'_{\beta'})$ is an induced subgraph of P. Since $P \in \mathrm{Forb}(L_m)$, we must have $\beta' < m$, and thus $\beta \leq m(w-1)$. $\square$

Claim 3.19. *If T is descending then its length β is at most $\frac{w}{2} m(w-1)\,\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$.*

Proof. Let P' be the subposet of P induced by $D_P[t_1] \cap U_P[A]$. Let B be a maximum antichain in P' and let $A' = D_P[B] \cap A$. Because $D_P[B] \subseteq D_P[t_1]$ and t_1 has property $(\star)$, we must have $|A'| \leq w/2$. If $|A'| < |B|$, then the antichain $(A \setminus A') \cup B$ has more than w vertices, which is impossible as width$(P) = w$. So we have width$(P') = |B| \leq |A'| \leq w/2$.

For each $i \in [\beta]$, let (v_i, t_i) be the corresponding pair for color $\mathfrak{g}(t_i)$, and note that $v_i <_P t_i \leq_P t_1$. By Dilworth's Theorem and the pigeonhole principle, at least $\beta' = \lceil 2\beta/w \rceil$ of the vertices of $V = \{v_1, v_2, \ldots, v_\beta\}$ must form a chain D. Let $V' = v'_1 v'_2 \ldots v'_{\beta'}$ be a sequence with $v'_i \in V \cap D$ and $\mathfrak{g}(v'_i) < \mathfrak{g}(v'_{i+1})$ for each $i \in [\beta']$. Consider any monotonic subsequence $X = x_1 x_2 \ldots x_\gamma$ of V' with respect to $\leq_P$. For each $i \in [\gamma - 1]$, the vertex x_{i+1} has a $\mathfrak{g}(x_i)$-witness y_i, since $\mathfrak{g}$ is a Grundy coloring and $\mathfrak{g}(x_i) < \mathfrak{g}(x_{i+1})$.

Subclaim 3.20. *If X is descending then for each $i \in [\gamma - 1]$ and $j \in [i+1, \gamma]$:*

(C1) $y_i <_P x_i$;

(C2) $y_i \in D_P(A)$;

(C3) $y_i \parallel_P x_j$;

(C4) if $j \in [\gamma - 1]$ (so y_j is defined) then $y_i \not\leq_P y_j$.

Proof. Fix $i \in [\gamma - 1]$ and $j \in [i+1, \gamma]$. Set $\iota = \mathfrak{g}(x_i)$. Then

$$x_j \leq_P x_{i+1} \leq_P x_i \bowtie_P y_i \parallel_P x_{i+1}, \tag{3.7}$$

and (C1) follows. Recall x_i is the $\leq_P$-least vertex of $P_\iota \cap U_P(A)$. As $y_i \in P_\iota$, and by (C1) $y_i <_P x_i$, we have $y_i \in D_P(A)$; so (C2) holds.

Suppose $y_i \bowtie_P x_j$. Recall that $x_j \in U_P(A)$, and by (C2) $y_i \in D_P(A)$. Since A is an antichain, $y_i \leq_P x_j$. By (3.7), $y_i \leq_P x_j \leq_P x_{i+1} \parallel_P y_i$, a contradiction. So (C3) holds.

Suppose $j \in [\gamma - 1]$. If $y_i \leq_P y_j$ then by (C1) and (3.7), $y_i \leq_P y_j <_P x_j \leq_P x_{i+1} \parallel_P y_i$, a contradiction. So (C4) holds. $\square$

Subclaim 3.21. *If X is descending then its length γ is at most $m(w-1)$.*

Proof. Now, as $y_i \parallel_P x_{i+1}$ for each $i \in [\gamma - 1]$, we see that y_i is not on D. Hence, all of $y_1, y_2, \ldots, y_{\gamma-1}$ are contained in the $w-1$ remaining chains of $\mathcal{C} - D$. By the pigeonhole principle, at least $(\gamma-1)/(w-1)$ of them must form a chain, E. Set $\gamma' = \lceil (\gamma-1)/(w-1) \rceil$ and let $X' = x'_1 x'_2 \ldots x'_{\gamma'}$ be a subsequence of X so that y'_i (the vertex so that $\mathfrak{g}(x'_i) = \mathfrak{g}(y'_i)$ and y'_i is a witness for the vertex after x'_i in X') is on E. Here we have two disjoint chains $x'_{\gamma'} <_P x'_{\gamma'-1} <_P \cdots <_P x'_1$ and $y'_{\gamma'} <_P y'_{\gamma'-1} <_P \cdots <_P y'_1$. By Subclaim 3.20, $L_{\gamma'}(y'_{\gamma'}, \ldots y'_1; x'_{\gamma'} \ldots x'_1)$ is an induced subposet of P. Since $P \in \mathrm{Forb}(L_m)$, we must have $\gamma' < m$. So $\gamma \leq m(w-1)$. $\square$

Subclaim 3.22. *If X is ascending then its length γ is at most $\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$.*

Proof. For each $i \in [\gamma]$, let (x_i, t'_i) be the corresponding pair for color $\mathfrak{g}(x_i)$. Define a function $c : [\gamma] \to [N]$ by $c(i) = \mathfrak{g}(x_i)$. Then c is increasing by the definition of V'. Let Q be the subposet of P' induced by

$$U = ([x_1, t'_1]_P \cap P_{c(1)}) \cup \cdots \cup ([x_\gamma, t'_\gamma]_P \cap P_{c(\gamma)}).$$

Finally, define $\mathfrak{g}' : U \to [\gamma]$ by $\mathfrak{g}'(u) = c^{-1} \circ \mathfrak{g}(u)$. We claim that $\mathfrak{g}'$ is a Grundy coloring of Q. Clearly $\mathfrak{g}'$ is surjective.

By our definitions of near and far witnesses, we have $x_\gamma \leq_P t'_\gamma$. Recalling that T is descending under $\leq_P$, we have

$$x_1 <_P x_2 <_P \cdots <_P x_\gamma \leq_P t'_\gamma <_P t'_{\gamma-1} <_P \cdots <_P t'_1.$$

Consider $u \in U$. By the definition of U, there exist $i \in [\gamma]$ such that $u \in [x_i, t'_i]_P$ and $\mathfrak{g}(u) = \mathfrak{g}(x_i)$. Thus $\mathfrak{g}'(u) = c^{-1} \circ \mathfrak{g}(x_i) = i$. If $i \in [2, \gamma]$ and $k \in [i-1]$ then we must show there exists $z \in I_Q(u)$ with $\mathfrak{g}'(z) = k$. As $\mathfrak{g}$ is a Grundy coloring of P and $\mathfrak{g}(x_k) < \mathfrak{g}(x_i)$, u has a $\mathfrak{g}(x_k)$-witness, y_k. Since

$$u \parallel_P y_k \bowtie_P x_k <_P x_i <_P u <_P t'_i <_P t'_k \bowtie_P y_k \parallel_P u,$$

we have $y_k \in [x_k, t'_k]_P$. Thus $y_k \in U$. Set $z = y_k$. Then

$$\mathfrak{g}'(z) = c^{-1} \circ \mathfrak{g}(y_k) = c^{-1} \circ \mathfrak{g}(x_k) = k.$$

This shows that u has a k-witness in Q for every $k \in [i-1]$.

As $P \in \mathrm{Forb}(L_m)$, we have $Q \in \mathrm{Forb}(L_m)$. By the inductive hypothesis, at most $\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$ colors appear on Q. Thus $\gamma \leq \mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$. □

By Subclaims 3.22 and 3.21, the length of every ascending subsequence of V' is at most $\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$, and every descending subsequence has length at most $m(w-1)$. By the Erdős-Szekeres Theorem we have $\beta' \leq m(w-1)\,\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2t \rfloor)$. Thus $\beta \leq \frac{w}{2} m(w-1)\,\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$, proving the claim. □

By the Erdős-Szekeres Theorem and Claims 3.18 and 3.19, S has length at most

$$\frac{w}{2} m^2 (w-1)^2\,\mathrm{val}_{\mathrm{FF}}\left(L_m, \left\lfloor \frac{w}{2} \right\rfloor\right).$$

By duality, we have $|R \cap C \cap D(A)_P| \leq \frac{w}{2} m^2 (w-1)^2\,\mathrm{val}_{\mathrm{FF}}(L_m, \lfloor w/2 \rfloor)$ as well. As $R \cap C = (R \cap C \cap U_P(A)) \cup (R \cap C \cap D_P(A))$, we have completed the proof of the lemma. □

Since $R = \bigcup_{C \in \mathcal{C}} R \cap C$, Lemma 3.16 yields

$$N \le m^2 w^4 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w}{2} \right\rfloor \right).$$

By (3.3), we now have

$$n \le m^2 w^4 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w}{2} \right\rfloor \right) + \operatorname{val}_{\mathrm{FF}}(L_m, w-1).$$

Applying this recursion repeatedly yields

$$\begin{aligned} n &\le m^2 w^4 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w}{2} \right\rfloor \right) + m^2 (w-1)^4 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w-1}{2} \right\rfloor \right) + \operatorname{val}_{\mathrm{FF}}(L_m, w-2) \\ &\vdots \\ &\le \sum_{0 \le k \le w-2} m^2 (w-k)^4 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w-k}{2} \right\rfloor \right) \\ &\le w m^2 w^4 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w}{2} \right\rfloor \right). \end{aligned}$$

Recalling $n = \operatorname{val}_{\mathrm{FF}}(L_m, w)$, we see

$$\begin{aligned} \operatorname{val}_{\mathrm{FF}}(L_m, w) &\le m^2 w^5 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w}{2} \right\rfloor \right) \\ &\le m^2 w^5 m^2 \left(\frac{w}{2} \right)^5 \operatorname{val}_{\mathrm{FF}} \left(L_m, \left\lfloor \frac{w}{4} \right\rfloor \right) \\ &\vdots \\ &\le m^{2 \lg w} \left(w^{\lg w} / 2^{\sum_{0 \le k \le \lg w} k} \right)^5 \\ &\le m^{2 \lg w} \left(w^{\lg w} / 2^{(\lg w)(1 + \lg w)/2} \right)^5 \\ &= m^{2 \lg w} \left(w^{\lg w} / w^{(1 + \lg w)/2} \right)^5 \\ &\le m^{2 \lg w} w^{2.5 \lg w}. \end{aligned}$$

Hence $\operatorname{val}_{\mathrm{FF}}(L_m, w) \le w^{2.5 \lg w + 2 \lg m}$. The lower bound will be provided in Lemma 3.23. □

For posets P and Q, we define the *lexicographical product* $P \cdot Q$ to be the poset with vertices $\{(p, q) : p \in P, q \in Q\}$ and order $\le_{P \cdot Q}$ where

$(p_1, q_1) \leq_{P \cdot Q} (p_2, q_2)$ if either $p_1 <_P p_2$ or $p_1 = p_2$ and $q_1 \leq_Q q_2$. Informally, we may think of $P \cdot Q$ as a the poset P where each vertex has been "inflated" to a copy of Q. Any antichain $\{(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)\}$ in $P \cdot Q$ must have either $p_i \parallel_P p_j$ or $p_i = p_j$ and $q_i \parallel_Q q_j$. So a maximum antichain in $P \cdot Q$ is $\{(p, q) : p \in A, q \in B\}$ where A is a maximum antichain in P and B is a maximum antichain in Q. From this, we can see

$$\text{width}(P \cdot Q) = \text{width}(P)\,\text{width}(Q). \tag{3.8}$$

For $p, r \in P$ and $u, v, s \in Q$ we have the following.

$$\text{If } (p, u) \bowtie_{Q \cdot P} (r, s) \parallel_{Q \cdot P} (p, v), \text{then } p = r. \tag{3.9}$$

To see this, note that if $p \neq r$, we would have $p \bowtie_P r \parallel_P p$, which is impossible.

$$\text{If } (p, u) \leq_{Q \cdot P} (r, s) \leq_{Q \cdot P} (p, v), \text{then } p = r. \tag{3.10}$$

By contradiction: if $p \neq r$, then $p <_P r$ and $r <_P p$. As P is a poset, we see this cannot happen.

Define the *disjoint sum* $P + Q$ to be the poset with vertices $\{p \in P\} \cup \{q \in Q\}$ and order $\leq_{P+Q}$ where $r_1 \leq_{P+Q} r_2$ if either $r_1, r_2 \in P$ and $r_1 \leq_P r_2$ or $r_1, r_2 \in Q$ and $r_1 \leq_Q r_2$. Informally, we may think of $P + Q$ as a copy of P next to a copy of Q with no additional comparabilities. If A is an antichain in P and B is an antichain in Q, then $A \cup B$ is an antichain in $P + Q$. It is easy to see

$$\text{width}(P + Q) = \text{width}(P) + \text{width}(Q). \tag{3.11}$$

We leave it to the reader to verify that both the lexicographical product and disjoint sum are indeed posets. We will use these combinations of posets in the next lemma.

Lemma 3.23 (Bosek & Matecki [7]). *For $m, w \in \mathbb{Z}^+$,*
$\mathrm{val}_{\mathrm{FF}}(L_m, w) \geq w^{\lg(m-1)}$.

Proof. Fix $m \in \mathbb{Z}^+$. The results of Lemma 2.5 allow us to take $m > 2$. We will build a poset $Q_w \in \mathrm{Forb}(L_m)$ and n-Grundy coloring $\mathfrak{h}$ so that $\mathrm{width}(Q_w) = w$ and $n \geq w^{\lg(m-1)}$. Let R_{m-1} be the poset from Lemma 2.2. Define P to be R_{m-1} with added minimum vertex $\hat{0}$. We will treat R_{m-1} as a subposet of P. By Lemma 2.2, $\mathrm{width}(R_{m-1}) = 2$. Any antichain in R is an antichain in P. If we have a Dilworth partition of R (with two chains), we can add $\hat{0}$ to either chain to form a 2-chain partition of P. So, $\mathrm{width}(P) = 2$. As $I_P(\hat{0}) = \emptyset$ and $I_P(x) = I_R(x)$ for $x \in R$, following the reasoning employed in Theorem 2.5, we see that

$$P \in \mathrm{Forb}(L_m). \tag{3.12}$$

In the construction of R_{m-1}, we see x_{m-1}^{m-1} is a maximum vertex. In our construction of P, we defined $\hat{0}$ to be a minimum vertex, so

$$P \text{ has a maximum vertex and a minimum vertex.} \tag{3.13}$$

From Lemma 2.2, we know R has an $m-1$-Grundy coloring $\mathfrak{f}'$. We extend $\mathfrak{f}'$ to an $m-1$-Grundy coloring $\mathfrak{f}$ of P by specifying $\mathfrak{f}(\hat{0}) = 1$ and $\mathfrak{f}(x) = \mathfrak{f}'(x)$ for all other vertices. It is easy to see this is an $m-1$-Grundy coloring. As $\hat{0}$ cannot be a witness for any vertex in P, we see $\chi_{\mathrm{FF}}(R) \not< \chi_{\mathrm{FF}}(P)$. Hence,

$$\chi_{\mathrm{FF}}(P) = \chi_{\mathrm{FF}}(R) = m - 1. \tag{3.14}$$

We will use P to build the desired poset. For $w \geq 2$, define Q_w as follows (see Figure 3.9).

(Q1) Q_1 is a single vertex z.

(Q2) If $w \geq 2$ is even, $Q_w = P \cdot Q_{w/2}$.

(Q3) If $w \geq 3$ is odd, $Q_w = P \cdot Q_{(w-1)/2} + Q_1$ along with a minimum vertex $\hat{0}_w$ and a maximum vertex $\hat{1}_w$.

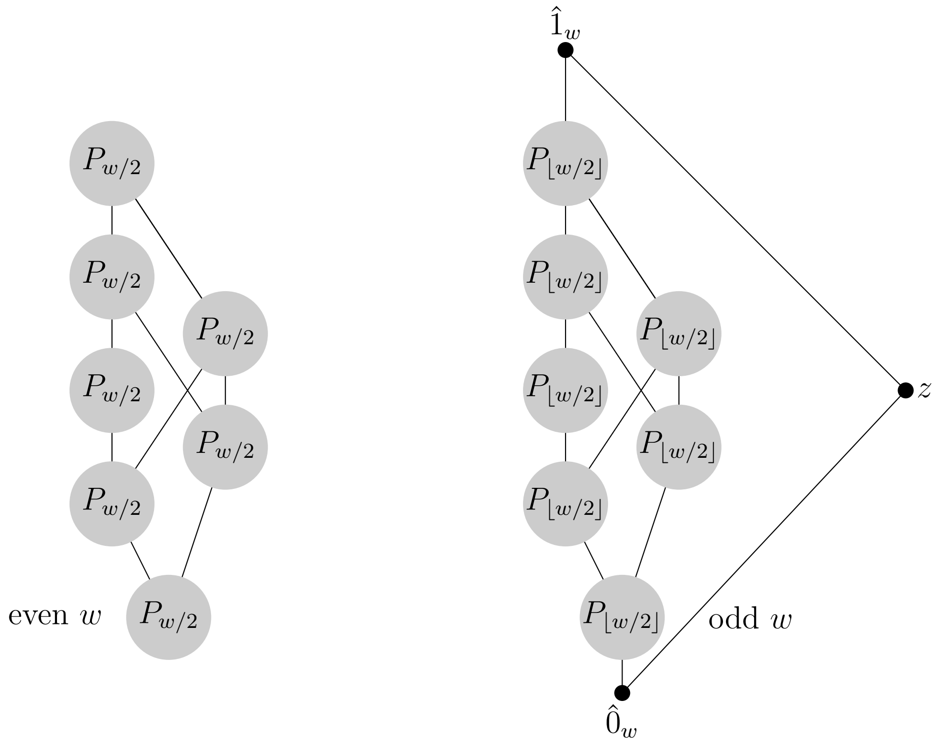

Figure 3.9: Simplified Hasse diagrams of Q_w with $m = 3$.

Note that $Q_2 = P$ and so we will treat Q_2 as P. For $w > 2$, the vertices of Q_w (other than z, $\hat{0}_w$, or $\hat{1}_w$ in the case that w is odd) have the form (p, q) where $p \in P$ and $q \in Q_{\lfloor w/2 \rfloor}$. To avoid confusion, we refer to (p, q) as a vertex, p as the first coordinate, and q as the second coordinate. In the case that n is odd, we refer to z as the isolated vertex. We examine the properties of Q_w in the following claims.

Claim 3.24. *For each $w \in \mathbb{Z}^+$, Q_w has a minimum vertex and a maximum vertex.*

Proof. We use induction on w. For $w = 1$ the claim is trivial and for $w = 2$ we recall (3.13). Now take $w > 2$ and suppose the claim holds for all smaller cases. If w is odd, the claim follows directly from (Q3). If w is even, the inductive hypothesis tells us $Q_{w/2}$ has a minimum vertex, say x. The vertex $\hat{0}$ is a minimum vertex in P, so $(\hat{0}, x)$ is the minimum in $P \cdot Q_{w/2}$. Similar reasoning using a maximum vertex from $Q_{w/2}$ and x_{m-1}^{m-1} from P shows Q_w has a maximum vertex. □

Claim 3.25. *For each* $w \in \mathbb{Z}^+$*,* $\mathrm{width}(Q_w) = w$.

Proof. This follows immediately from (3.8) and (3.11). □

Claim 3.26. *For each* $w \in \mathbb{Z}^+$*,* $Q_w \in \mathrm{Forb}(L_m)$.

Proof. We will use induction on w. For our bases, we see $w = 1$ is trivial and $w = 2$ is established by (3.12). Take $w > 2$ and suppose the inductive hypothesis holds for all smaller cases. Assume L is an m-ladder in Q_w with lower leg $(a_1, u_1) <_{Q_n} (a_2, u_2) <_{Q_n} \cdots <_{Q_n} (a_m, u_m)$ and upper leg $(b_1, v_1) <_{Q_w} (b_2, v_2) <_{Q_w} \cdots <_{Q_w} (b_m, v_m)$. Note that z (the isolated vertex) cannot be part of the ladder; the longest chain z belongs to has three vertices and each vertex of an m ladder is in an $m+1$ vertex chain. If there are $2m$ distinct first coordinates in the vertices of L, then these vertices would induce an m-ladder in P, which violates (3.12). Hence, at least two vertices of L share a first coordinate, say $p \in P$. Define $Q' = \{(p, q) : q \in Q_{\lfloor w/2 \rfloor}\}$. It is easy to see $Q' = Q_{\lfloor w/2 \rfloor}$. Set x^* and y^* to be to be the minimum and maximum, respectively, vertices of Q' (which exist by Claim 3.24).

Assume only vertices of the lower leg of L are in Q'. Then there are $i < j \in [m]$ so that $(a_i, u_i), (a_j, u_j) \in Q'$. From the definition of a ladder, we know $(a_i, u_i) \leq_{Q_w} (b_i, u_i) \parallel_{Q_w} (a_j, u_j)$. From (3.9), $(b_i, v_i) \in Q'$. Similar reasoning shows that Q' does not contain only vertices from the upper leg. Take $(a_i, u_i), (b_j, v_j) \in Q'$. We see $(a_i, u_i) \leq_{Q_w} (a_m, u_m) \parallel_{Q_w} (b_j, v_j)$ (if $j < m$) or $(a_i, u_i) \leq_{Q_w} (a_m, u_m) \leq_{Q_w} (b_j, v_j)$ (if $j = m$). In the former case (3.9) shows $(a_m, u_m) \in Q'$ and in the latter case (3.10) shows $(a_m, u_m) \in Q'$. Similarly, $(a_i, u_i) \parallel_{Q_2} (b_1, v_1) \leq_{Q_w} (b_j, v_j)$ (if $i > 1$) or $(a_i, u_i) \leq_{Q_2} (b_1, v_1) \leq_{Q_w} (b_j, v_j)$ (if $i = 1$). Again using (3.9) and (3.10), we have $(b_1, v_1) \in Q'$.

For any vertex (r, s) in L so that $(r, s) \notin \{(a_1, u_1), (b_m, v_m)\}$, we have either $(b_1, v_1) \leq_{Q_w} (r, s) \parallel_{Q_w} (a_m, u_m)$ or $(b_1, v_1) \parallel_{Q_w} (r, s) \leq_{Q_w} (a_m, u_m)$. By (3.9), $(r, s) \in Q$. Now, the vertices

$$\{x^*, (a_2, u_2), (a_3, u_3), \ldots, (a_m, u_m), (b_1, v_1), (b_2, v_2), \ldots, (b_{m-1}, v_{m-1}), y^*\} \subseteq Q'$$

induce an m-ladder in Q', which contradicts the inductive hypothesis, proving the claim. □

Claim 3.27. *For all $w \in \mathbb{Z}^+$, $\chi_{\mathrm{FF}}(Q_w) \geq (m-1)\chi_{\mathrm{FF}}(Q_{\lfloor w/2 \rfloor})$.*

Proof. Let $\mathfrak{f}$ be an $m-1$-Grundy coloring of P (which exists by (3.14)) and $\mathfrak{g}$ be a k-Grundy coloring of $Q_{\lfloor w/2 \rfloor}$. For now, let us suppose w is even. Define $\mathfrak{h} : Q_w \to [(m-1)k]$ by $\mathfrak{h}((p, q)) = (\mathfrak{f}(p) - 1)k + \mathfrak{g}(q)$.

We will show $\mathfrak{h}$ is a $(m-1)k$-Grundy coloring. Take $i \in [(m-1)k]$. We can find $a \in [m-1]$ and $b \in [k]$ so that $i = (a-1)k + b$. As $\mathfrak{f}$ and $\mathfrak{g}$ satisfy Definition 2.11(G1), we have some $p \in P$ and $q \in Q_{w/2}$ so that $\mathfrak{f}(p) = a$ and $\mathfrak{g}(q) = b$. And so $\mathfrak{h}(p, q) = (\mathfrak{f}(p) - 1)k + \mathfrak{g}(q) = (a-1)k + b = i$. Thus $\mathfrak{h}$ satisfies (G1). Now, suppose $\mathfrak{h}((p, q)) = \mathfrak{h}((r, s))$. This gives us $(\mathfrak{f}(p) - 1)k + \mathfrak{g}(q) = (\mathfrak{f}(r) - 1)k + \mathfrak{g}(s)$ and so $(\mathfrak{f}(p) - \mathfrak{f}(r))k = \mathfrak{g}(s) - \mathfrak{g}(q)$. As

$\mathfrak{g}(q), \mathfrak{g}(s) \in [k]$, we see this is only possible if $\mathfrak{f}(p) - \mathfrak{f}(r) = \mathfrak{g}(s) - \mathfrak{g}(q) = 0$. This gives us $\mathfrak{f}(p) = \mathfrak{f}(r)$ and $\mathfrak{g}(q) = \mathfrak{g}(s)$. Because $\mathfrak{f}$ and $\mathfrak{g}$ are Grundy colorings we have $p \bowtie_P r$ and $q \bowtie_{Q_{w/2}} s$. By the definition of the lexicographical product we have $(p, q) \bowtie_{Q_w} (r, s)$ and so (G2) holds for $\mathfrak{h}$.

Now, take (r, s) so that $\mathfrak{h}(r, s) = j > 1$. We have unique integers c, d so $c \in [m-1]$ and $d \in [k]$ with $j = (c-1)k + d$. Hence $\mathfrak{f}(r) = c$ and $\mathfrak{g}(s) = d$. Let $i < j$. We will show (r, s) has an i-witness. Again, we have $a \in [m-1]$ and $b \in [k]$ so that $i = (a-1)k + b$. As $i < j$, we must have $a \leq c$. Suppose $a = c$, then $b < d$. As $\mathfrak{g}$ is a Grundy coloring, there is some $q \in Q_{w/2}$ so that $\mathfrak{g}(q) = b$ and $q \parallel_{Q_{w/2}} s$. By the definition of lexicographical product, $(r, q) \parallel_{Q_w} (r, s)$. As $\mathfrak{h}((r, q)) = (\mathfrak{f}(p) - 1)k - \mathfrak{g}(q) = (c-1)k + b = i$, we see (r, q) is the desired witness. Now, suppose $a < c$. As $\mathfrak{f}$ and $\mathfrak{g}$ are Grundy colorings, there is some $p \in P$ so that $\mathfrak{f}(p) = a$ and $p \parallel_P r$ and some $q \in Q_{w/2}$ so that $\mathfrak{g}(q) = b$. Now, $(p, q) \parallel_{Q_w} (r, s)$ by the definition of lexicographical product. Finally, we note $\mathfrak{h}((p, q)) = (\mathfrak{f}(p) - 1) + \mathfrak{g}(q) = (a-1)k + b = i$ so (p, q) is the desired witness in this case. Hence, (G3) holds as well.

In the case that w is odd, we define $\mathfrak{h}$ by $\mathfrak{h}(\{\hat{0}_w, z, \hat{1}_w\}) = 1$ and $\mathfrak{h}(p, q) = k(\mathfrak{f}(p) - 1) + \mathfrak{g}(q) = ak + b = i$ for all other vertices. Combining our work for the case were w is even along with the fact $\hat{0}_w <_{Q_w} z <_{Q-w} \hat{1}_w$, we are done. $\square$

It is easy to see $\chi_{\mathrm{FF}}(Q_1) = 1$. Taking this with (3.14) and Claim 3.27, we have $\mathrm{val}_{\mathrm{FF}}(L_m, w) \geq w^{\lg(m-1)}$. $\square$

3.4 Concluding Remarks

The gap between the upper and lower bounds for $\text{val}_{\text{FF}}(L_m, 2)$ can probably to closed by lowering the upper bound to $m + c$ for some constant c. This will require an advancement in the sophistication of our current methods. An improvement in gap between the upper and lower bounds for $\text{val}_{\text{FF}}(L_m, w)$ will probably require changes to both bounds. From Theorem 2.6, we see that the method used in the Lemma 3.23 cannot be used to force the lower bound on $\text{val}(L_m, w)$ over $w^{1+\lg m}$ as the construction of Q_w starts with a width two poset and its Grundy coloring. The upper bound can be quickly improved if a method for restricting the the size of $N - n$ is found.

Chapter 4

ON-LINE COLORING OF GENERAL POSETS

4.1 The Lattice of Maximum Antichains

We provide some background that will allow us to more clearly define a new family of posets.

A poset P is a *lattice* if for any $x, y \in P$, the sets $D_P[x] \cap D_P[y]$ and $U_P[x] \cap U_P[y]$ have a maximum and minimum vertex, respectively, The (unique) maximum vertex of $D_P[x] \cap D_P[y]$ is the *meet of x and y*, denoted $x \wedge y$. The (unique) minimum vertex of $U_P[x] \cap U_P[y]$ is the *join of x and y*, denoted $x \vee y$.

Let P be a finite partial order and let $\mathcal{M}$ be the set of maximum antichains in P. Let $A, B \in \mathcal{M}$ and define the relation $\sqsubseteq_P$ by $A \sqsubseteq_P B$ if $A \subseteq D_P[B]$ (note that this is equivalent to $B \subseteq U_P[A]$). If $A \sqsubseteq_P B$ and $A \neq B$, we write $A \sqsubset_P B$. In [14] Dilworth showed that

$$(\mathcal{M}, \sqsubseteq_P) \text{ is a lattice.} \tag{4.1}$$

We will abuse notation and refer to the lattice $(\mathcal{M}, \sqsubseteq_P)$ as $\mathcal{M}$. Let A and B be antichains in $\mathcal{M}$. As $\mathcal{M}$ is a lattice, the meet and join of these antichains are defined. We note that

$$A \wedge B = \mathrm{Min}_P\{A \cup B\} \tag{4.2}$$

and

$$A \vee B = \mathrm{Max}_P\{A \cup B\}. \tag{4.3}$$

4.2 Regular Posets

In this section, we explore regular posets, a family of posets introduced and studied by Bosek and Krawczyk [9] (the roots of regular posets are in the local game used by Bosek in [2] to show val(3) $\leq$ 16).

We say a poset P is a *core* if its vertices can be partitioned into two disjoint maximum antichains, A and B, so that $A \sqsubset_P B$ and for any comparable pair $x \leq_P y$ with $x \in A$ and $y \in B$, xy is a Dilworth edge.

Definition 4.1. *Let $w, n \in \mathbb{Z}^+$ and $P = (V, \leq_P)$ be a width w poset. Let $X = \{x_1, x_2, \ldots, x_n\} \subseteq V$ be a set of n distinct vertices, and let $\mathcal{A} = A_1, A_2, \ldots, A_n$ be a sequence of w vertex antichains under $\leq_P$ with $A_i \neq A_j$ for $i \neq j \in [n]$. Then P is a regular on-line poset if it satisfies the following properties.*

(R1) $V = A_1 \cup A_2 \cup \cdots \cup A_n$.

(R2) For $i \in [n]$, $x_i \in A_i$ *and* $x_i \notin A_j$ *with* $j \in [i-1]$.

(R3) The set $\{A_1, A_2, \ldots, A_n\}$ *is linearly ordered under* $\sqsubseteq_P$.

Note that (R3) does not imply $A_1 \sqsubseteq_P A_2 \sqsubseteq_P \cdots \sqsubseteq_P A_n$. For each $i \in [n]$, define $A_{p(i)}$ to be the $\sqsubseteq_P$-greatest element in $\{A_1, A_2, \ldots, A_{i-1}\}$ so that $A_{p(i)} \sqsubset_P A_i$. Similarly, define $A_{s(i)}$ to be the $\sqsubseteq_P$-least element in $\{A_1, A_2, \ldots, A_{i-1}\}$ so that $A_i \sqsubset_P A_{s(i)}$. Note that $A_{p(i)}$ does not exist if A_i is minimal in $\sqsubseteq_P$ and $A_{s(i)}$ does not exist if A_i is maximal in $\sqsubseteq_P$.

(R4) For $i \neq j \in [n]$, $A_i \cap A_j = \emptyset$.

(R5) $P[A_i \cup A_{s(i)}]$ *is a core (respectively,* $P[A_{p(i)} \cup A_i]$ *is a core), provided* $A_{s(i)}$ *exists (*$A_{p(i)}$ *exists).*

(R6) Suppose $x <_P y$*. If* $x \in A_i$*, then there is some* $z \in A_{s(i)}$ *so that* $x <_P z \leq_P y$*. If* $y \in A_i$ *then there is some* $z \in A_{p(i)}$ *so that* $x \leq_P z <_P y$*.*

We provide an illustration of a regular posets in Figure 4.1. We call X the *index vertices* and $\mathcal{A}$ as the *antichain presentation*. These structures will be mentioned if they will be used and omitted if they are not needed. Informally, we will think of P as being presented one antichain at a time. When the antichains $A_1, A_2, \ldots, A_i$ have been presented (for $i \in [n]$) we say we are in the *i-th round*. For a vertex $x \in P$, we define $A(x)$ to be the unique $A \in \mathcal{A}$ so that $x \in A$. We now discuss some of the properties of regular posets.

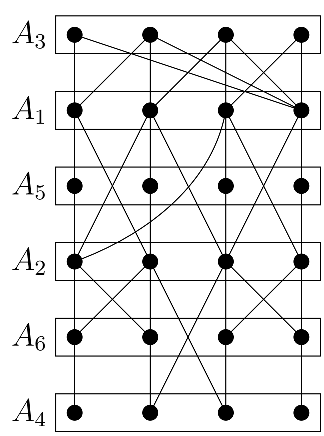

Figure 4.1: A regular poset.

Claim 4.2. *Let* P *be a width* w *regular on-line poset with antichain presentation* $\mathcal{A} = A_1, A_2, \ldots, A_n$*. Take* $i, j \in [n]$ *with* $i \neq j$ *and* $A_i \sqsubseteq_P A_j$*. We have the following:*

(R7) $P[A_i \cup A_j]$ *is a core.*

(R8) *Let* $t \in [n]$ *be the least integer so that* $A_i \sqsubseteq_P A_t \sqsubseteq_P A_j$. *If* $x \leq_P y$ *with* $x \in [A_i, A_t]_P$ *and* $y \in U_P[A_t]$ *(or* $y \in [A_t, A_j]_P$ *and* $x \in D_P[A_t]$*), then there is some vertex* $z \in A_t$ *so that* $x \leq_P z \leq_P y$.

Proof. We will prove both claims with $i < j$ and leave it to the reader to use similar reasoning to verify the claims for $i > j$.

To prove (R7), we employ induction on j. For the base, we take $j = 2$. This implies $i = 1$ and $p(j) = 1$. By (R5), the claim holds. Suppose (R7) holds in all smaller cases and take $j > 2$. We have $A_i \sqsubseteq_P A_{p(j)} \sqsubset_P A_j$. Take $x \in A_i$ and $y \in A_j$ with $x \leq_P y$ (such a pair must exist as width$(P) = w$ and A_i and A_j are w vertex antichains). By (R6), there is some $z \in A_{p(j)}$ so that $x \leq_P z \leq_P y$. By the inductive hypothesis, $P[A_i \cup P_{p(j)}]$ is a core and by (R5) $P[A_{p(j)} \cup A_j]$ is a core. Let $\mathcal{C}$ be a Dilworth partition of $P[A_i \cup P_{p(j)}]$ with x and z in the same chain and $\mathcal{D}$ be a Dilworth partition of $P[A_{p(j)} \cup A_j]$ with z and y in the same chain. It is easy to see that

$$\{C \Delta D : C \in \mathcal{C},\ D \in \mathcal{D},\ C \cap D \neq \emptyset\}$$

is a Dilworth partition of $P[A_i \cup A_j]$ with x and y in the same chain. Hence, $P[A_i \cup A_j]$ is a core, proving (R7).

We turn our attention to (R8) and again use induction on j. In the case of $j = 2$, we have $i = 1$ and $A_1 = A_{p(j)} = A_t$ and (R8) is trivial. Now suppose $j > 2$ and (R8) holds for all smaller cases. Note that $A_t \sqsubseteq A_{p(j)}$. If $A_t = A_{p(j)}$, the claim holds by (R6), so we take $A_t \sqsubset A_{p(j)}$. If $y \notin A_j$, the claim holds by the inductive hypothesis, so we take $x \in D_P[A_t]$ and $y \in A_j$. By (R6), there is some $z' \in A_{p(j)}$ so that $x \leq_P z' \leq_P y$. By hypothesis, t is the least integer in $[n]$ so that $A_i \sqsubseteq_P A_t \sqsubseteq_P A_{p(j)}$. By the inductive hypothesis, we have $z \in A_t$ so that $x \leq_P z \leq_P z'$. By transitivity, we have $x \leq_P z \leq_P y$. $\square$

We know examine ladders in regular on-line posets. Suppose L_m is a ladder with lower leg $x_1, x_2, \ldots, x_m$ and upper leg $y_1, y_2, \ldots, y_m$. We say that L_m is *canonical* if $A(y_i) \sqsubseteq_P A(x_{i+1})$ for each $i \in [m-1]$.

Claim 4.3. *Let P be a width w regular on-line poset with antichain presentation $\mathcal{A} = A_1, A_2, \ldots, A_n$. If L is a subposet of P and a canonical m-ladder, then $m \leq w$.*

Proof. Let the lower leg of L be $x_1, x_2, \ldots, x_m$ and upper leg be $y_1, y_2, \ldots, y_m$. We remind ourselves that $y_1 \parallel_P x_i$ for $i \in [2, m]$. As a subclaim, we will show that $|U_P[y_1] \cap A(y_i)| \geq i$ by induction on i. For $i = 1$, we have $U_P[y_1] \cap A(y_1) = \{y_1\}$ and the subclaim holds. Now, let $i > 1$ and suppose the subclaim holds. As L is a canonical ladder, we have $A(y_{i-1}) \sqsubseteq_P A(x_i)$. We must have $D_P[x_i] \cap A(y_{i-1}) \neq \emptyset$, or else we would have a $w+1$ element antichain $A(y_{i-1}) + x_i$. Let $z \in D_P[x_i] \cap A(y_{i-1})$. By transitivity, we have $z \leq_P y_i$. Because $z \in A(y_{i-1})$ and $A(y_1) \sqsubseteq_P A(y_{i-1})$, we have $z \notin D_P(y_1)$. We also cannot have $z \in U_P(y_1)$, or else we would have $y_1 \leq_P z \leq_P x_i$. This gives us $z \parallel_P y_1$. Let $S = U_P[y_1] \cap A(y_{i-1})$. By the inductive hypothesis, we have $|S| \geq i - 1$. By (R7), we know that $P[A(y_{i-1}) \cup A(y_i)]$ is a core.

Let $\mathcal{C}$ be a Dilworth partition of $P[A(y_{i-1}) \cup A(y_i)]$ with z and y_i in the same chain. Each vertex of S is matched to a distinct vertex of $A(y_i)$ in $\mathcal{C}$ that is not y_i (see Figure 4.2). We now have $|U_P[y_1] \cap A(y_i)| \geq |S| + 1 \geq i - 1 + 1 = i$, proving the subclaim.

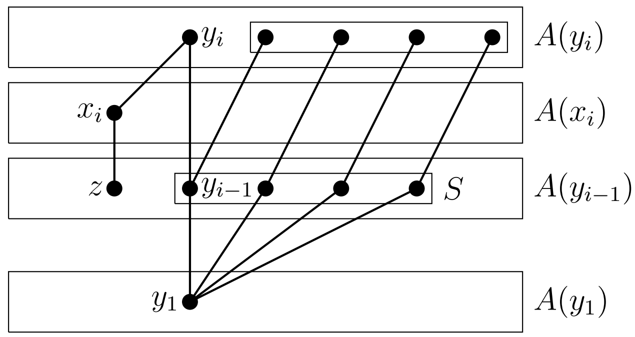

Figure 4.2: The intersections of $U_P[y_1]$ with $A(y_{i-1})$ and $A(y_i)$.

We cannot have x_{w+1}, because $A(y_w) \sqsubseteq_P A(x_{w+1})$ and, by our subclaim, $A(y_w) \subset U_P[y_1]$. So, we would have $y_1 \leq_P x_{w+1}$ which is impossible as L is a ladder. This shows $m < w + 1$. $\square$

Claim 4.4. *Let P be a width w regular on-line poset with antichain presentation $\mathcal{A} = A_1, A_2, \ldots, A_n$. Let $\mathcal{C}$ be a Dilworth partition of P. Suppose L is a subposet of P and a m-ladder with upper leg $x_1, x_2, \ldots, x_m$ and lower leg $y_1, y_2, \ldots, y_m$. If there are integers $i < j \in [m]$ so that $A(x_j) \sqsubset_P A(y_i)$, then there is some $A \in \mathcal{A}$ so that $|A \cap [x_1, y_m]_P| \geq \frac{1}{2}(j - i)$.*

Proof. Define $k = j - \lceil \frac{1}{2}(j - i) \rceil$ and $k' = i + \lceil \frac{1}{2}(j - i) \rceil$. Let $t \in [n]$ be the least integer so that $A(x_k) \sqsubseteq_P A_t \sqsubseteq_P A(y_{k'})$. Note that for $r \in [k, j]$ and $s \in [i, k']$, the antichains $A(x_r)$ and $A(y_s)$ are presented on or after the t-th round. By the linear order under $\sqsubseteq_P$, we must have $A(x_j) \sqsubset_P A_t$ or $A_t \sqsubset_P A(y_i)$.

Suppose we have $A(x_j) \sqsubset_P A_t$ (see Figure 4.3). Consider the comparabilities $x_r \leq_P y_r$ for $r \in [k, j]$. We have $x_r \in [A(x_k), A_t]_P$ and $y_r \in U_P[A_t]$. By the choice of t and (R8), there is some $z_r \in A_t$ so that $x_r \leq_P z_r \leq_P y_r$. Take $r < r' \in [k, j]$. If $z_r = z_{r'}$, then transitivity tells us

$x_{r'} \leq_P y_r$ which is impossible as L is a ladder. Hence, we have $\lceil \frac{1}{2}(j-i) \rceil$ distinct elements in $Z = \{z_k, z_{k+1}, \ldots z_j\} \subseteq A_t$.

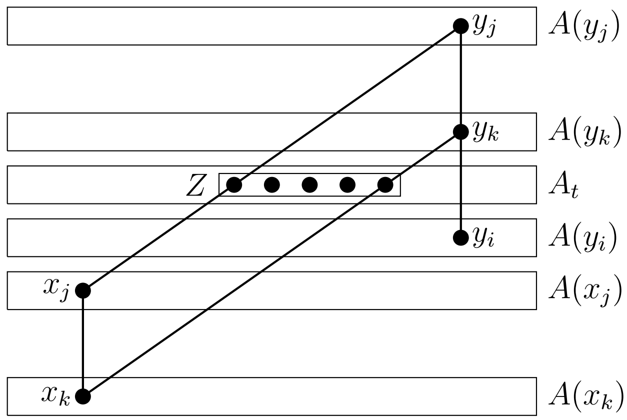

Figure 4.3: Comparabilities of the ladder L and A_t.

As $x_1 \leq_P x_r \leq_P z_r$ and $z_r \leq_P y_r \leq_P y_m$ for each $r \in [k, j]$, we have $Z \subseteq [x_1, y_m]_P$, proving the claim. the second, for each $z_r \in Z$, select $C_r \in \mathcal{C}$ so that $z_r \in C_r$ and set $\mathcal{D} = \{C_k, C_{k+1}, \ldots C_j\}$. As Z is an antichain, the elements of $\mathcal{D}$ are pairwise distinct and we have $|\mathcal{D}| \geq \frac{1}{2}(j-i)$. Suppose $D \in \mathcal{D}$ and $d \in D$. We must have $d \bowtie_P z$ for some $z \in Z$. If $d \leq_P z$, then we have $d \in D_P[y_m]$. If $z \leq_P d$, then we have $d \in U_P[x_1]$. Hence, $D \subseteq U_P[x_1] \cup D_P[y_m]$. This proves the second claim.

In the case that $A_t \sqsubset_P A(y_i)$, examination of the comparabilities $x_s \leq_P y_s$ for $s \in [i, k']$ yields the same conclusion. $\square$

These claims will now be used to limit the number of rungs on a ladder in a regular on-line poset.

Lemma 4.5. *Suppose P is a width w regular on-line poset with antichain presentation $\mathcal{A} = A_1, A_2, \ldots, A_n$. Then $P \in \mathrm{Forb}(L_{2w^2+1})$.*

Proof. Assume $L_{2w(w-h)+1}$ is a ladder in P with lower leg $x_1, x_2, \ldots, x_{2w^2+1}$, and upper leg $y_1, y_2, \ldots, y_{2w^2+1}$. By Claim 4.4, for any $i < j \in [2w^2]$ with $j - i > 2w$, we must have $A(y_i) \sqsubseteq_P A(x_j)$ or else we would have $|A \cap [x_1, y_{2w^2+1}]_P| > w$ which is impossible. Thus, the subposet induced by the vertices

$$\bigcup_{0 \le i \le w} \{x_{2wi+1}, y_{2wi+1}\}$$

is a canonical ladder with $w + 1$ rungs, which contradicts Claim 4.3, proving the lemma. $\square$

Although L_{2w^2+1} is not a subposet of any width w regular poset, we will show that there exist width w regular posets with ladders whose number of rungs is quadratic in w.

Lemma 4.6. *For each integer $w \ge 2$, there is a regular poset Q so that* $\mathrm{width}(Q) = w$ *and* $Q \notin \mathrm{Forb}(L_{w\lfloor (w+2)/2 \rfloor})$.

Proof. Fix a positive integer $w \ge 2$ and define $h = \lfloor (w+2)/2 \rfloor$. For $k \in [w]$, we define the cores I, S_k, and T_k as follows. The vertices of each of these cores are two disjoint antichains $U = \{u_1, u_2, \ldots, u_w\}$ and $V = \{v_1, v_2, \ldots, v_w\}$. In I, the only comparabilities are $u_i \le_I v_i$ for all $i \in [w]$. For $k = 1$, we take $S_1 = T_1 = I$. Now fix $k \in [2, w]$. Take $i \in [2, k]$ and $j \in [k+1, w]$ (note that j is not defined for $k = w$). The only comparabilities of S_k are

$$u_1 \le_{S_k} \{v_1, v_2, \ldots, v_k\},$$

$$u_i \le_{S_k} \{v_{i-1}, v_i\},$$

$$u_j \le_{S_k} v_j,$$

provided that j exists. Now, take $i \in [w-k+1, w-1]$ and $j \in [1, w-k]$ (again, j is not defined for $k = w$). The only comparabilities of T_k are

$$\{u_{w-k+1}, u_{w-k+2}, \ldots, u_w\} \leq_{T_k} v_w,$$

$$\{u_i, u_{i+1}\} \leq_{T_k} v_i,$$

$$u_j \leq_{T_k} v_j,$$

provided that j exists. See Figure 4.4 for examples. It is clear that I is a core and it is straightforward to use induction on k to verify that S_k and T_k are cores.

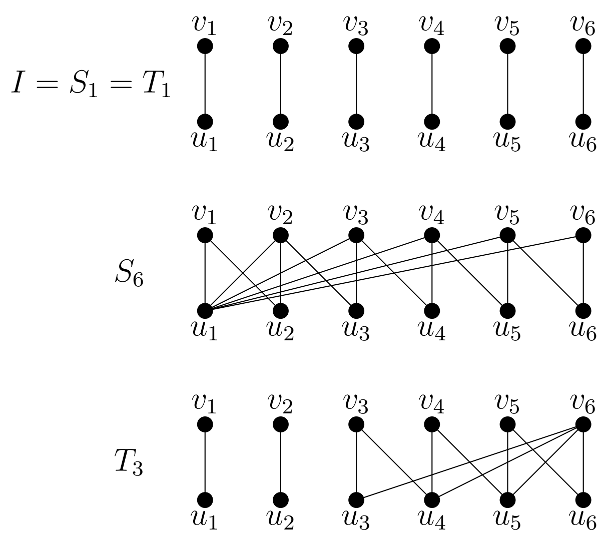

Figure 4.4: Hasse diagrams of I, S_6, and T_4 for $w = 6$.

We will now build Q with antichain presentation $\mathcal{A} = A_1, A_2, \ldots A_{(2w+1)h}$ with properties (Q1-8) as follows. To offer some relief in dealing with subscripts, define $f(i) = (2w+1)(i-1)+1$. To help us keep track of comparabilities, for each $A_i \in \mathcal{A}$, label the vertices $A_i = \{a_1^i, a_2^i, \ldots, a_w^i\}$.

(Q1) For each $i \in [h]$,

$$A_{f(i)+1} \sqsubseteq A_{f(i)+2} \sqsubseteq \cdots \sqsubseteq A_{f(i)+w} \sqsubseteq A_{f(i)} \sqsubseteq A_{f(i)+2w} \sqsubseteq A_{f(i)+2w-1} \sqsubseteq \cdots \sqsubseteq A_{f(i)+w+1}.$$

(Q2) For each $i \in [h-1]$, $A_{f(i)+w+1} \sqsubseteq A_{f(i+1)+1}$.

(Q3) For each $i \in [h]$ and $j \in [w]$, $Q[A_{f(i)+j} \cup A_{f(i)}] = S_{w-j+1}$ with $a_k^{f(i)+j} = u_k$ and $a_k^{f(i)} = v_k$ for all $k \in [w]$.

(Q4) For each $i \in [h]$ and $j \in [2, w]$, $Q[A_{f(i)+j-1} \cup A_{f(i)+j}] = I$ with $a_k^{f(i)+j-1} = u_k$ and $a_k^{f(i)+j} = v_k$ for all $k \in [w]$.

(Q5) For each $i \in [h]$ and $j \in [w+2, 2w]$, $Q[A_{f(i)+j} \cup A_{f(i)+j-1}] = I$ with $a_k^{f(i)+j} = u_k$ and $a_k^{f(i)+j-1} = v_k$ for all $k \in [w]$.

(Q6) For each $i \in [h]$ and $j \in [w+1, 2w]$, $Q[A_{f(i)} \cup A_{f(i)+j}] = T_{2w-j+1}$ with $a_k^{f(i)} = u_k$ and $a_k^{f(i)+j} = v_k$ for all $k \in [w]$.

(Q7) For each $i \in [h-1]$, $Q[A_{f(i)+w+1} \cup A_{f(i+1)}] = S_w$ with $a_k^{f(i)+w+1} = u_k$ and $a_k^{f(i+1)} = v_k$ for all $k \in [w]$.

(Q8) For each $i \in [h-1]$, $Q[A_{f(i)+w+1} \cup A_{f(i+1)+1}] = I$ with $a_k^{f(i)+w+1} = u_k$ and $a_k^{f(i+1)+1} = v_k$ for all $k \in [w]$.

See Figure 4.5. Although it is tedious to verify that Q is indeed a width w regular on-line poset (we may create an index set by selecting an arbitrary vertex from each antichain in $\mathcal{A}$), it is straightforward so we leave the task to the reader.

Now, we will show L_{wh} is an induced subposet of Q. The vertices $x_1, x_2, \ldots, x_{wh}$ of the lower leg and $y_1, y_2, \ldots, y_{wh}$ of the upper leg are defined by the following rules.

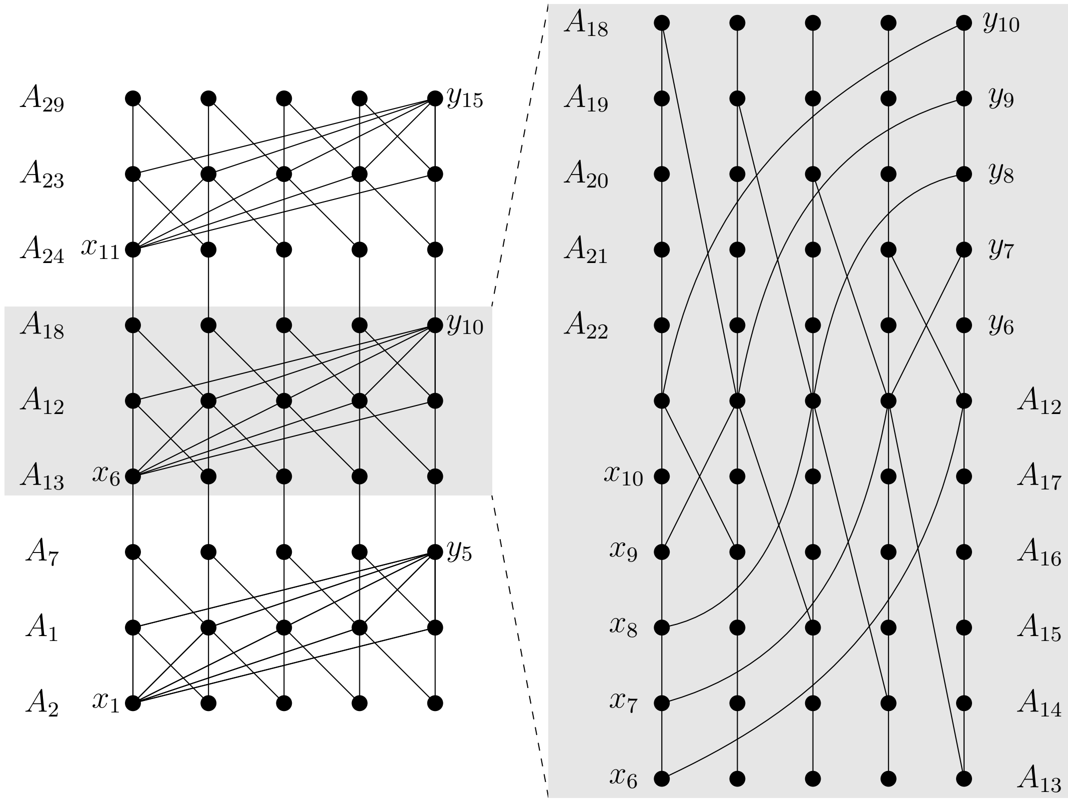

Figure 4.5: Q with $w = 5$.

(X1) For $i \in [h]$ and $j \in [w]$, $x_{w(i-1)+j} = a_1^{f(i)+j}$.

(Y1) For $i \in [h]$ and $j \in [w]$, $y_{w(i-1)+j} = a_w^{f(i+1)-j}$.

Part of this labeling can be seen in Figure 4.5. We must show the subposet induced the the lower and upper leg vertices just defined is a wh-ladder.

Claim 4.7. *For each $k \in [wh]$, we have $x_k \leq_Q y_k$.*

Proof. There are integers $i \in [h]$ and $j \in [w]$ so that $k = w(i-1)+j$. Hence, $x_k = a_1^{f(i)+j}$ and $y_k = a_w^{f(i+1)-j}$. Note that $f(i+1) = f(i) + 2w + 1$. Let $\ell = 2w + 1 - j$. Now, we have $f(i+1) - j = f(i) + \ell$ with $\ell \in [w+1, 2w]$.

By the construction of Q (specifically (Q6)),

$$Q[A_{f(i)} \cup A_{f(i+1)-j}] = Q[A_{f(i)} \cup A_{f(i)+\ell}] = T_{2w-\ell+1} = T_j$$

with $a^{f(i)}_{w-j+1} \leq_Q a^{f(i+1)-j}_w$. By our construction (specifically (Q3)), we have $Q[A_{f(i)+j} \cup A_{f(i)}] = S_{w-j+1}$ with $a^{f(i)+j}_1 \leq_Q a^{f(i)}_{w-j+1}$. By the transitivity of $\leq_Q$, we have proven our claim. $\square$

From inspection of our construction (and looking back at Figure 4.5), we see that for any antichains $A_i \sqsubset A_j \in \mathcal{A}$, we have $a^i_k <_Q a^j_k$ for any $k \in [w]$. From our choices of the ladder vertices, the following claim is clear.

Claim 4.8. *We have $x_1 <_Q x_2 <_Q \cdots <_Q x_{wh}$ and $y_1 <_Q y_2 <_Q \cdots <_Q y_{wh}$.*

We turn our attention to the incomparabilities of the ladder.

Claim 4.9. *For each $i \in [h]$, we have $D_Q(x_{w(i-1)+1}) \supseteq D_Q(x_{w(i-1)+j})$ and $U_Q(y_{w(i-1)+j}) \subseteq U_Q(y_{w(i-1)+w})$ for any $j \in [w]$.*

Proof. From (X1), we see $A(x_{w(i-1)+j}) = A_{f(i)+j}$ and $x_{w(i-1)+j} = a^{f(i)+j}_1$. From our construction of Q (in particular (Q4)), we have $A_{p(f(i)+j)} = A_{f(i)+j-1}$ and $D_Q(x_{w(i-1)+j}) \cap A_{f(i)+j-1} = \{x_{w(i-1)+j-1}\}$ for all $j \in [2, w]$. By (R6), we have $z \in D_Q(x_{w(i-1)+j})$ if and only if $z \in D_Q(x_{w(i-1)+1})$. From this, we can see $D_Q(x_{w(i-1)+1}) \supseteq D_Q(x_{w(i-1)+2}) \supseteq \cdots \supseteq D_Q(x_{w(i-1)+w})$. Similar reasoning proves the second part of the claim. $\square$

Claim 4.10. *If $m < n \in [wh]$, then $y_m \parallel_Q x_n$.*

Proof. By our construction, we have $y_m \neq x_n$. Let us assume $y_m \bowtie_Q x_n$. Suppose $x_n <_Q y_m$. Then we must have $A(x_n) \sqsubseteq A(y_m)$ and $n - m < w$.

Hence, there there are integers $i \in [h]$ and $j, k \in [w]$ with $j < k$ so that $m = w(i-1) + j$ and $n = w(i-1) + k$. As in the proof of Claim 4.7, set $\ell = 2w + 1 - j$ so that $f(i+1) - j = f(i) + \ell$. From (X1) and (Y1), we have $A(x_n) = A_{f(i)+k}$ and $A(y_m) = A_{f(i)+\ell}$. Our construction shows that $A_{s(f(i)+k)} = A_{f(i)} = A_{p(f(i)+\ell)}$. We recall Q is a regular poset, so (R6) tells us there is some $z \in A_{f(i)}$ so that $x_n \leq_Q z \leq_Q y_m$. Following the methods of Claim 4.7, (Q3), and (Q6) show us

$$A_{f(i)} \cap U_Q(x_{w(i-1)+k}) = \left\{a_1^{f(i)}, a_2^{f(i)}, \ldots, a_{w-k+1}^{f(i)}\right\}$$

and

$$A_{f(i)} \cap D_Q(y_{w(i-1)+j}) = \left\{a_{w-j+1}^{f(i)}, a_{j+1}^{f(i)}, \ldots, a_w^{f(i)}\right\}.$$

As $w - k + 1 < w - j + 1$, we see these sets do not overlap. Hence, our desired z does not exist and $y_m \parallel_Q x_n$ in this case.

Suppose $y_m <_Q x_n$. We have $A(y_m) \sqsubseteq A(x_n)$. Also, this would imply $y_1 \leq_Q y_m \leq_Q x_n$. We will show $x_n \notin U_Q(y_1)$ to provide our contradiction. By Claim 4.9, it is sufficient to show for $i \in [2, h]$ that $x_{w(i-1)+1} \notin U_Q[y_w]$. Recall that $x_{w(i-1)+1} = a_1^{f(i)+1}$. So, we will show

$$U_Q[y_w] \cap A_{f(i)+w+1} = \left\{a_{w-2(i-1)}^{f(i)+w+1}, a_{w-2(i-1)+1}^{f(i)+w+1}, \ldots, a_w^{f(i)+w+1}\right\}$$

for $i \in [h-1]$. From (Q8), we see $A_{p(f(i+1)+1)} = A_{f(i)+w+1}$ and (by our choice of comparabilities) $a_k^{f(i+1)+1} \in U_Q[y_w] \cap A_{f(i+1)+1}$ if and only if $a_k^{f(i)+w+1} \in U_Q[y_w] \cap A_{f(i)+w+1}$. If $i \in [h-1]$, then $1 < w - 2(i-1)$ so this will give us the desired result. We will use induction on i. We have $U_Q[y_w] \cap A_{f(1)+w+1} = \{a_w^{f(1)+w+1}\}$, establishing our base.

Suppose the inductive hypothesis holds for all cases less than i. From (Q7), we have $A_{p(f(i))} = A_{f(i-1)+w+1}$ with $A_{s(f(i-1)+w+1)}$ undefined. From our choice of comparabilities, we have

$$U_Q[y_w] \cap A_{f(i)} = \left\{a^{f(i)}_{w-2(i-2)-1}, a^{f(i)}_{w-2(i-2)}, \ldots, a^{f(i)}_{w}\right\}.$$

From (Q6), we have $A_{p(f(i)+w+1)} = A_{f(i)}$ with $A_{s(f(i))}$ undefined. From our choice of comparabilities, we have

$$U_Q[y_w] \cap A_{f(i)+w+1} = \left\{a^{f(i)+w+1}_{w-2(i-2)-2}, a^{f(i)+w+1}_{w-2(i-2)-1}, \ldots, a^{f(i)+w+1}_{w}\right\}.$$

Because $w - 2(i-2) - 2 = w - 2(i-1)$, our inductive hypothesis holds. And so in this case, we again have $y_n \parallel_Q x_n$. $\square$

The comparabilities shown in Claims 4.7, 4.8, and 4.10 prove the lemma. $\square$

4.3 Proof of Theorem 2.8

From Lemma 4.5 and Theorem 2.7, we know any regular poset can be colored using FF using a bounded number of colors. However, to bound val(w), we need to address general posets. Using methods developed by Bosek and Krawczyk, we will show it is possible to color an arbitrary on-line poset by maintaining and coloring an auxiliary regular poset on-line.

Proof of Theorem 2.8. We will proceed by induction on w the width of P. If width$(P) = 1$, we are done. Suppose we have a strategy to color any on-line poset of width less that w at most $w^{3+6.5 \lg w}$ colors. Let P be a width w poset with arbitrary presentation $\prec$. We will provide an algorithm to color $P^{\prec}$ on-line using several auxiliary structures.

Select $k \in \mathbb{Z}^+$ and let P be the on-line poset after $k-1$ vertices have been presented. For any variable a that represents a structure after $k-1$

vertices have been presented, we will use a^+ to represent the structure after the k-th vertex has been presented. Suppose x is the k-th vertex presented that extends P to P^+.

First, we partition the vertices of $P^{\prec}$ into $\dot{X}$ and X using Algorithm 4.1:

Algorithm 4.1 Extend $\dot{X}$ and X to $\dot{X}^+$ and X^+.

---

if $\text{width}(P^+[\dot{X} + x]) = w$ **then**
    $\dot{X}^+ = \dot{X}$
    $X^+ = X + x$
    $A_x =$ an antichain of size w in $P^+[\dot{X} + x]$ containing x
else
    $\dot{X}^+ = \dot{X} + x$
    $X^+ = X$
end if

For $\dot{X}$, we have only one property that we are interested in. From Algorithm 4.1, it is clear

$$\text{width}(P[\dot{X}]) < w \tag{4.4}$$

at each stage in the presentation of P. Hence, the the vertices of $\dot{X}$ can be colored using the inductive hypothesis. All we must do is color the vertices of X. Set $|X| = \ell - 1$ (so we have $|X^+| = \ell$ if $x \in X^+$).

To color the vertices of X, we will build an auxiliary poset in multiple stages. Our goal is to build a regular on-line poset, with an index set and antichain presentation. In the next algorithm, we will recall $\mathcal{M}$, the lattice of maximum antichains in P. We will select a set of vertices U, and a sequence $\mathcal{A} = A_1, A_1, \ldots, A_{|X|}$ of maximum antichains that are linearly ordered under $\sqsubseteq_P$.

Algorithm 4.2 Select an antichain to extend $\mathcal{A}$ and U.

if $|X^+| > |X|$ **then**
    if $x \in D_P(A)$ for some $A \in \mathcal{A}$ **then**
        $A_u = \min_{\sqsubseteq_P}\{A \in \mathcal{A} : x \in D_P(A)\}$
    else
        $A_u = A_x$
    end if
    if $x \in U_P(A)$ for some $A \in \mathcal{A}$ **then**
        $A_d = \max_{\sqsubseteq_P}\{A \in \mathcal{A} : x \in U_P(A)\}$
    else
        $A_d = A_x$
    end if
    $A_\ell = (A_d \vee (A_u \wedge A_x))$
    $\mathcal{A}^+ = A_1, A_2, \ldots, A_{\ell-1}, A_\ell$
    $U^+ = U \cup A_\ell$
else
    $\mathcal{A}^+ = \mathcal{A}$
    $U^+ = U$
end if

We illustrate Algorithm 4.2 in Figure 4.6. By inspection, we can verify the following proposition.

Proposition 4.11. *The elements of $\mathcal{A}$ are w vertex antichains which are linearly ordered under $\sqsubseteq_P$.*

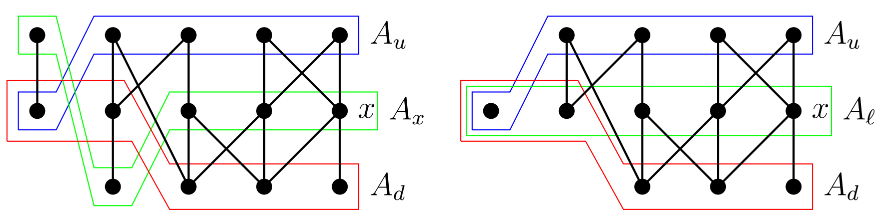

Figure 4.6: Finding A_ℓ from A_x.

The set X forms the base for the index set and $\mathcal{A}$ forms the base for the antichain presentation for our auxiliary regular on-line poset. However, they need to be modified to meet the required properties. Recall that x is the most recently presented vertex added to X.

Algorithm 4.3 Extend V, Y, and $\mathcal{B}$ and to V^+, Y^+, and $\mathcal{B}^+$.

if $|X^+| > |X|$ **then**
  $B_\ell = \{(u, A_\ell) : u \in A_\ell\}$
  $\mathcal{B}^+ = B_1, B_2, \ldots, B_{\ell-1}, B_\ell$
  $V^+ = V \cup B_\ell$
  $Y^+ = Y + (x, A_\ell)$
else
  $\mathcal{B}^+ = \mathcal{B}$
  $V^+ = V$
  $Y^+ = Y$
end if

Define the poset $Q = (V, \leq_Q)$ where $(v, B) \leq_Q (v', B')$ if $v \leq_P v'$ and $B \sqsubseteq_P B'$. It is easy to verify that $\leq_Q$ defines a partial order on V. This will simplify the next algorithm, where we build the regular on-line poset R with order $\leq_R$.

Algorithm 4.4 Maintaining the on-line poset R and order $\leq_R$.

if $|Y^+| > |Y|$ **then**
  for all $u, v \in V$ **do**
    if $u \leq_R v$ **then**
      $u \leq_R^+ v$
    end if
  end for
  for all $u \in B_\ell$ **do**
    $u \leq_R^+ u$
    if $A_{s(\ell)}$ exists **then**
      $U_{R^+}(u) = \bigcup U_R[v]$ where uv is a Q-Dilworth edge in $Q[B_\ell \cup B_{s(\ell)}]$
    end if
    if $A_{p(\ell)}$ exists **then**
      $D_{R^+}(u) = \bigcup D_R[v]$ where vu is a Q-Dilworth edge in $Q[B_{p(\ell)} \cup B_\ell]$
    end if
    $R^+ = (V^+, \leq_R^+)$
  end for
else
  $\leq_R^+ = \leq_R$
  $R^+ = R$
end if

We illustrate Algorithm 4.4 in Figure 4.7 (compare this to Figure 4.6). We examine the output of these algorithms. Some properties can be verified by inspection.

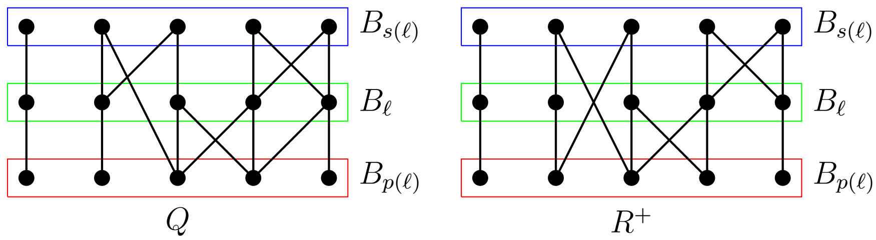

Figure 4.7: Hasse diagrams of $Q[B_{p(\ell)} \cup B_\ell \cup B_{s(\ell)}]$ and $R^+[B_{p(\ell)} \cup B_\ell \cup B_{s(\ell)}]$.

Proposition 4.12. *For each $B \in \mathcal{B}$, B is an antichain under $\leq_R$.*

Proof. This follows from inspection of Algorithm 4.4. $\square$

Claim 4.13. *The relation is $\leq_R$ is on-line. That is, after each iteration of Algorithm 4.4, we have $\leq_R^+ = \leq_R$ on $R^+[V]$.*

Proof. Suppose $V \neq V^+$ and $u, v \in V$. Suppose $u \bowtie_R v$. From Line 4 in Algorithm 4.4, we see we have $u \bowtie_R^+ v$. Suppose we have $u \parallel_R v$. By inspection of Algorithm 4.4, we see the only comparabilities added to form $\leq_R^+$ a between vertices of B_ℓ and V. As both $u, v \in V$ and $B_\ell \cap V = \emptyset$, we have $u \parallel_R^+ v$. $\square$

Claim 4.14. *Let $i, j \in [\ell]$. If $(y, A_i), (z, A_j) \in V$ with $(y, A_i) \leq_R (z, A_j)$, then $(y, A_i) \leq_Q (z, A_j)$.*

Proof. We proceed by induction on ℓ. If $\ell = 1$ then we have $i = j = 1$ and so $(y, A_1) \leq_R (z, A_1)$. Proposition 4.12 shows $(y, A_1) = (z, A_1)$. Let $\ell > 1$ and suppose the claim holds for all smaller cases. If $(y, A_i) = (z, A_j)$, we are done, so we take $(y, A_i) \neq (z, A_j)$. If $i = j$, we are done by Proposition 4.12, so we take $i \neq j$. If $\ell \notin \{i, j\}$, by Claim 4.13 and the inductive hypothesis, we are done. Suppose $j = \ell$. We see from Line 13 in Algorithm 4.4 there is some $(u, A_{p(\ell)}) \in B_{p(\ell)}$ so that $(y, A_i) \in D_R[(u, A_{p(\ell)})]$ and $(u, A_{s(\ell)}) \leq_Q (z, A_\ell)$. The inductive hypothesis shows that $(y, A_i) \leq_Q (u, A_{p(\ell)})$ and the transitivity of $\leq_Q$ shows $(y, A_i) \leq_Q (z, A_j)$. Similar reasoning shows the claim holds in the case $i = \ell$. □

Claim 4.15. *Let $i, j \in [\ell]$. If $(y, A_i), (z, A_j) \in V$ with $(y, A_i) \leq_R (z, A_j)$, then $y \leq_P z$.*

Proof. We proceed by induction on ℓ. If $\ell = 1$ then we have $i = j = 1$ and so $(y, A_1) \leq_R (z, A_1)$. Proposition 4.12 and the definition of $\leq_Q$ show $y = z$. Let $\ell > 1$ and suppose the claim holds for all smaller cases. If $(y, A_i) = (z, A_j)$, we are done, so we take $(y, A_i) \neq (z, A_j)$. If $i = j$, we are done by Proposition 4.12, so we take $i \neq j$. If $\ell \notin \{i, j\}$, by Claim 4.13 and the inductive hypothesis, we are done. Suppose $j = \ell$. We see from Line 13 in Algorithm 4.4 there is some $(u, A_{p(\ell)}) \in B_{p(\ell)}$ so that $(y, A_i) \in D_R[(u, A_{p(\ell)})]$ and $(u, A_{p(\ell)}) \leq_Q (z, A_\ell)$. The inductive hypothesis shows that $y \leq_P u$ and the definition of $\leq_Q$ shows $u \leq_P z$. The transitivity of $\leq_P$ proves the claim. Similar reasoning shows the claim hold in the case $i = \ell$. □

Lemma 4.16. *After each iteration of Algorithm 4.4, the order is $\leq_R$ is reflexive, antisymmetric, and transitive.*

Proof. We use induction on ℓ. When $\ell = 1$, we have $\mathcal{B} = B_1$. From Proposition 4.12 and inspection of Algorithm 4.4, we see that R is a w vertex antichain, thus establishing our base. Now let $\ell > 1$ and assume the lemma holds for $\leq_R$. We will show it holds for $\leq_R^+$. By the inductive hypothesis, Claim 4.13, and Line 8 of Algorithm 4.4, we see $\leq_R^+$ is reflexive. By Proposition 4.14, we see $\leq_R^+$ is antisymmetric.

It remains to show $\leq_R^+$ is transitive. Suppose we have $u, y, z \in V^+$ with $u \leq_R^+ y$ and $y \leq_R^+ z$. If $|\{u, y, z\}| < 3$ we are done so we will assume the vertices are distinct. If $\{u, y, z\} \cap B_\ell = \emptyset$, we are done by Proposition 4.13 and the inductive hypothesis. Suppose $u \in B_\ell$. By Line 10 in Algorithm 4.4, there is some $v \in B_{s(\ell)}$ so that $y \in U_R[v]$. As $y \leq_R z$, by the inductive hypothesis, we have $z \in U_R[v]$. Again by Line 10 in Algorithm 4.4, we have $u \leq_R^+ z$, as desired. Similar reasoning shows $u \leq_R^+ z$ if $z \in B_\ell$.

Suppose $y \in B_\ell$. By Lines 10 and 13 in Algorithm 4.4, there are vertices $u' \in B_{p(\ell)}$ so that $u \in D_R[u']$ where $u'y$ is a Dilworth edge in $Q[B_{s(\ell)} \cup B_\ell]$ and $z' \in B_{s(\ell)}$ so that $z \in U_R[z']$ where yz' is a Dilworth edge in $Q[B_\ell \cup B_{s(\ell)}]$. If $u' \leq_R z'$, then we are done by the inductive hypothesis. Suppose $p(\ell) < s(\ell)$. Note that this implies $p(\ell) = p(s(\ell))$. Let $\mathcal{C}$ be a Dilworth partition of $Q[B_{s(\ell)} \cup B_\ell]$ with u and u' in the same chain and $\mathcal{D}$ be a Dilworth partition of $Q[B_\ell \cup B_{p(\ell)}]$ with z' and z in the same chain. The set

$$\{C \Delta D : C \in \mathcal{C},\ D \in \mathcal{D},\ C \cap D \neq \emptyset\}$$

is a Dilworth partition of $Q[B_{p(\ell)} \cup B_{s(\ell)}]$ with u' and z' in the same chain. Hence, $u'z'$ is a Dilworth edge in $Q[B_{p(\ell)} \cup B_{s(\ell)}]$. Recalling $p(\ell) = p(s(\ell))$, we see that Line 13 in Algorithm 4.4 sets $u' \leq_R z'$, as desired. Similar reasoning shows $u' \leq_R z'$ when $p(\ell) > s(\ell)$. $\square$

Now that we have established $R = (V, \leq_R)$ is a partial order, we will show we have a regular poset.

Lemma 4.17. *R is a width w regular on-line poset with index set Y, and antichain presentation $\mathcal{B} = B_1, B_2, \ldots, B_\ell$.*

Proof. We will show $R = (V, \leq_R)$ has properties (R1-6) from Definition 4.1. From the construction of V (in Line 4 in Algorithm 4.3), we see (R1) holds. The construction of Y (in Line 5 in Algorithm 4.3) shows $y_i \notin B_j$ for $j < i \in [\ell]$ (establishing (R2)). From Proposition 4.11, the definition of $\leq_Q$, and Claim 4.14, we see $\mathcal{B}$ is linearly ordered under $\sqsubseteq_R$ (establishing (R3)). The construction of $\mathcal{B}$ (in Line 2 in Algorithm 4.3) establishes (R4). Property (R5) is clear from Lines 10 and 13 in Algorithm 4.4; a comparability in exist in $R[B_i \cup B_{s(i)}]$ or $R[B_{p(i)} \cup B_i]$ only if it is a Dilworth edge. Hence, both are cores. The same lines also establish (R6).

To show R has width w, first we note each $B \in \mathcal{B}$ is an antichain with w vertices showing the width is at least w. To bound it from above, we will show that for any antichain A, there is an antichain A' so that $A' \subseteq B$ where $B \in \mathcal{B}$, $|A| = |A'|$, $A' \subseteq U_R[A]$, and $B = \max_{\sqsubseteq_R}\{B(a) : a \in A\}$. We show this by induction on $|A|$. When $|A| = 1$, then A is the desired antichain A'. Suppose $|A| > 1$ and the claim hold for all smaller antichains. Select $z \in A$ so that $B(z)$ is $\sqsubseteq_R$-maximal for all $z \in A$. By the inductive hypothesis, we have the A'', an antichain in B' where $B' \in \mathcal{B}$ for some $B \in \mathcal{B}$, $|A''| = |A - x|$, $A'' \subseteq U_R[A - z]$ and $B' \sqsubseteq_R B(z)$. As A is an antichain and by the transitivity of $\leq_R$, we see $z \notin U_R[A'']$. If $B' = B(z)$, then $A' = A'' + z$ is the desired antichain. Suppose $B' \sqsubset A(z)$. By (R7), $R[B' \cup B(z)]$ is a core. The set of vertices matched to A'' together with z form the desired antichain A'. As $|B| = w$ for each $B \in \mathcal{B}$, we see a $|A| \leq |B|$. This shows width$(R) = w$. $\square$

To color of P, we color each new point x with the inductive hypothesis if Algorithm 4.1 assigns x to $\dot{X}$ and we color x using the color First-Fit assigns to the vertex (x, B_ℓ) in R if Algorithm 4.1 assigns x to X. By Claim 4.15, any chain in R is a chain in P. Therefore we have a proper coloring of P. As R is a regular poset of width w, by Lemma 4.5, we have $R \in \mathrm{Forb}(L_{2w^2+1})$. By Theorem 2.7, at most $w^{2.5 \lg w + 2\lg(2w^2+1)}$. As

$$\sum_{k \in [w]} k^{2.5 \lg k + 2 \lg(2k^2+1)} \leq w^{3+6.5 \lg w},$$

we have a strategy to properly color P on-line using at most $w^{3+6.5 \lg w}$ colors. □

4.4 Concluding Remarks

Roughly speaking, our current upper bound on $\mathrm{val}(w)$ relies on using simple FF on the result of Algorithms 4.1, 4.2, 4.3, and 4.4. The number of colors used by FF is bounded because the poset resulting from the algorithms is in the family $\mathrm{Forb}(L_{2w^2+1})$. However, the results of Lemmas 3.23 and 4.6 show that these methods, as they stand, cannot bring the upper bound below $w^{\lg w}$. Perhaps using a more sophisticated on-line algorithm on the auxiliary regular poset R will yield an improvement.

Chapter 5

FIRST-FIT COLORING OF CIRCULAR ARC GRAPHS

5.1 Labeling Nodes in Base Cycles

Let G be a circular arc graph with base cycle C. Select a direction for the links of C so that G is a directed cycle. Even though C is now a directed cycle, the structure of G will not be changed. Each path A in $V(G)$ can be identified with a unique pair of nodes α, β so that $A = \alpha C \beta$. Let α and β be nodes in C. The *positive distance from α to β*, denoted $\alpha \overrightarrow{C} \beta$, is the length of $\alpha C \beta$ and the *negative distance from α to β*, denoted $\alpha \overleftarrow{C} \beta$, is the length of $\beta C \alpha$. In the case that $\alpha = \beta$, we have $\alpha \overrightarrow{C} \alpha = \alpha \overleftarrow{C} \alpha = |C| - 1$. In Figure 5.1, we have $\alpha \overrightarrow{C} \beta = 2$, $\alpha \overleftarrow{C} \beta = 7$, and highlighted path $\gamma C \delta$.

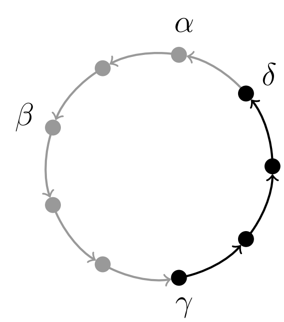

Figure 5.1: Directing the base cycle of a circular arc graph.

5.2 Proof of Theorem 2.9

Proof of Theorem 2.9. Let $G = (V, E)$ be a circular arc graph with base cycle C directed as in the previous section and let $\mathfrak{g}$ be a n-Grundy coloring of G. To each node α in C, we assign a list called a ray R_α. We say R_α is the ray corresponding to node α. In stages, we maintain a set of surviving nodes. If R_α is a surviving ray at stage i, then $R_\alpha(i) \in \{H, N, D\}$. Once we have finished building the rays, we will use them to form an upper bound on n in terms of $\iota(G)$. At stage i, the set of surviving nodes is S_i. For $\alpha \in S_i$, define the positive i neighbor of α, denoted $N_i^+(\alpha)$, to be $\beta \in S_i$ so that $\alpha \overrightarrow{C} \beta$ is as small as possible. Similarly, define the negative i neighbor of α, denoted $N_i^-(\alpha)$, to be $\beta \in S_i$ so that $\alpha \overleftarrow{C} \beta$ is as small as possible. Note that it is possible that $N_i^+(\alpha) = N_i^-(\alpha)$ or even, in the case that $S_i = \{\alpha\}$, $N_i^+(\alpha) = \alpha = N_i^-(\alpha)$.

Let R_α be a ray and $1 \leq i \leq j$. We define $H_\alpha(i,j) = |\{\ell \in [i,j] : R_\alpha(\ell) = H\}|$ (i.e.: the number of H entries in the ray between entry i and entry j, inclusive). Similarly, we set $N_\alpha(i,j) = |\{\ell \in [i,j] : R_\alpha(\ell) = N\}|$ and $D_\alpha(i,j) = |\{\ell \in [i,j] : R_\alpha(\ell) = D\}|$.

The set of surviving nodes and entries for the corresponding rays are maintained in stages as follows, starting at stage 1 and ending when the set of surviving nodes is empty. At stage 1, for each node α, if $\alpha \in A$ for some $A \in V_1$, then $R_\alpha(1) = H$ and $\alpha \in S_1$. At the start of stage j, suppose we have S_{j-1}. If $S_{j-1} \neq \emptyset$, we use Algorithm 5.1 to build $S_j \subseteq S_{j-1}$ and assign labels to some rays.

Algorithm 5.1 Add nodes to S_j and assign entries for corresponding rays.

$S_j = \emptyset$
for $\alpha \in S_{j-1}$ **do**
  if there is some $A \in V_j$ so that $\alpha \in A$ **then**
    $R_\alpha(j) = H$
    $S_j = S_j + \alpha$
  end if
end for
for $\alpha \in S_{j-1} \setminus S_j$ **do**
  $\beta = N^+_{j-1}(\alpha)$
  $\gamma = N^-_{j-1}(\alpha)$
  if $R_\beta(j) = H$ or $R_\gamma(j) = H$ **then**
    $R_\alpha(j) = N$
    $S_j = S_j + \alpha$
  end if
end for
for $\alpha \in S_{j-1} \setminus S_j$ **do**
  if there is some $i \in [j-1]$ so that $4H_\alpha(i, j-1) > j - i$ **then**
    $R_\alpha(j) = D$
    $S_j = S_j + \alpha$
  end if
end for

We see that if $\alpha \notin S_j$, then $R_\alpha(j)$ is undefined. We will simply think of undefined entries as being empty. The quantities $H_\alpha(i,j)$, $N_\alpha(i,j)$, and $D_\alpha(i,j)$ are defined for any $1 \leq i \leq j$ and node α and do not depend on α being a surviving node at a given stage.

The loops that assign entries H, N, and D are ordered so that, roughly speaking, Algorithm 5.1 prefers to assign an H entry versus an N entry, an N entry versus a D entry, and a D entry versus an empty entry. Furthermore, the algorithm prefers to add a node to the next surviving set versus leaving it out. From this, we have the following claim.

Claim 5.1. *For $j \geq 1$ and node $\alpha \in S_j$, $R_\alpha(j) = H$ if and only if $\alpha \in A$ for some $A \in V_j$.*

This claim immediately tells us that if $H_\alpha(1, j) = i$, then there is some $U \subseteq V$ so that $|U| = i$ and $\alpha \in \bigcap_{A \in U} A$. Hence, the following claim.

Claim 5.2. *For any $j \geq 1$ and node α, $\iota(G) \geq H_\alpha(1, j)$.*

Now, we will examine the connection between the rays (and their entries) and n-Grundy coloring $\mathfrak{g}$.

Claim 5.3. *Take $i \leq j \in [n]$. For any $A \in V_j$ we have $A \cap S_i \neq \emptyset$.*

Proof. We prove this by double induction where our primary induction is on j and secondary on i. The base of $j = 1$ follows from from the construction of S_1; the nodes in S_1 are exactly the nodes in the paths of V_1. Suppose the primary induction holds for all cases less than j. Select $A \in V_j$. We establish base of $i = 1$ of the secondary induction by using property (G3) of $\mathfrak{g}$: there is some path $B \in V_1$ so that $A \cap B \neq \emptyset$. As the construction of S_1 adds all nodes in B to S_1, we have our base. Now, suppose the secondary hypothesis holds for cases less than i. By property (G3) of $\mathfrak{g}$, there is some $B \in V_i$ so that $A \cap B \neq \emptyset$. By our primary inductive hypothesis, we have $S' = B \cap S_i \neq \emptyset$. If $A \cap S' \neq \emptyset$, we are done.

Suppose $A \cap S' = \emptyset$. Select nodes γ and δ so that $\gamma C \delta = A$. We must have exactly one of $\gamma \in B$ or $\delta \in B$; if this were not true, we would have either $A \cap B = \emptyset$ or $S' \subseteq B \subseteq A$, both of which violate our hypotheses. Suppose $\delta \in B$. By the secondary inductive hypothesis, $A \cap S_{i-1} \neq \emptyset$. Select $\alpha \in A \cap S_{i-1}$ and $\beta \in S'$ so that $\alpha \overrightarrow{C} \beta$ is as small as possible. We must have $\delta \in V(\alpha C \beta) \subseteq A \cup B$. From our selection of α and β, we have $V(\alpha C \beta) \cap S_{i-1} = \{\alpha, \beta\}$. Thus $\beta = N^+_{i-1}(\alpha)$. Because $R_\beta(i) = H$ (by Claim 5.1), Algorithm 5.1 adds α to S_i and sets $R_\alpha(i) = N$.

We see the claim holds if $\gamma \in B$ using similar reasoning, this time selecting $\alpha \in A \cap S_{i-1}$ and $\beta \in S'$ so that $\alpha \overleftarrow{C} \beta$ is as small as possible. $\square$

We show that our ray construction halts at some point. Recall that n is the largest color used by $\mathfrak{g}$.

Claim 5.4. *There exists an integer $m > n$ so that $S_m \neq \emptyset$ and $S_{m+1} = \emptyset$.*

Proof. From Claim 5.3, $S_n \neq \emptyset$. Take $\alpha \in S_n$ and $A \in V_n$ so that $\alpha \in A$. By Claim 5.1, we have $R_\alpha(n) = H$. As $4H_\alpha(n, n) = 4 > n - n + 1 = 1$, Algorithm 5.1 adds α to S_{n+1}. Hence $S_{n+1} \neq \emptyset$.

Take $m > n$ so that $S_m \neq \emptyset$ and let $\alpha \in S_m$. Algorithm 5.1 will try to add α to S_{m+1}. As $V_{m+1} = \emptyset$, Claim 5.1 shows we cannot have $R_\alpha(m+1) = H$. Also, $R_\alpha(m+1) \neq N$ as each N entry in stage $m+1$ requires the positive m neighbor or negative m neighbor with an H entry in stage $m+1$, which we have just shown is impossible. We must have $R_\alpha(m+1) = D$. Recalling $V_k = \emptyset$ for any $k \in [n, m]$, $R_\alpha(k) \neq H$. Hence, $\max_i H_\alpha(i, n) = \max_i H_\alpha(i, m)$. Fix i so that $H_\alpha(i, n)$ is as large as possible. If $m = 4H_\alpha(i, n) + i + 1$, Algorithm 5.1 will not add α to S_{m+1}. Hence, $S_{m+1} = \emptyset$. $\square$

For the rest of the proof, set m so that $S_m \neq \emptyset$ and $S_{m+1} = \emptyset$. For any $\alpha \in S_m$, we have $n \leq m = H_\alpha(1, m) + N_\alpha(1, m) + D_\alpha(1, m)$. We now seek to provide an upper bound on n in terms of $H_\alpha(1, m)$. Together with Claim 5.2, we will have an upper bound on n in terms of $\iota(G)$. First, we bound $D_\alpha(1, m)$ in terms of $H_\alpha(1, m)$.

Claim 5.5. *For any $j \geq 1$, if $\alpha \in S_j$, then $3H_\alpha(1, j) \geq D_\alpha(1, j)$.*

Proof. We use induction on j. The construction of S_1 shows the base of $j = 1$ holds. Suppose the claim holds for classes less than j. If $R_\alpha(j) \in \{H, N\}$, then $D_\alpha(1, j) = D_\alpha(1, j-1)$ and $H_\alpha(1, j) \geq H_\alpha(1, j-1)$. From these bounds with the inductive hypothesis, we have

$$3H_\alpha(1, j) \geq 3H_\alpha(1, j-1) \geq D_\alpha(1, j-1) = D_\alpha(1, j).$$

So the claim holds in this case.

Suppose $R_\alpha(j) = D$. In Algorithm 5.1, we see there is some $i \in [j-1]$ so that $4H_\alpha(i, j-1) > j - i$, which tells us $4H_\alpha(i, j-1) \geq j - i + 1$. Because $R_\alpha(j) = D$, we have $H_\alpha(i, j-1) = H_\alpha(i, j)$. Hence, $4H_\alpha(i, j) \geq j - i + 1$. There are $j - i + 1$ entries in R_α between i and j (inclusive), so $D_\alpha(i, j) + H_\alpha(i, j) \leq j - i + 1 \leq 4H_\alpha(i, j)$. We now have $D_\alpha(i, j) \leq 3H_\alpha(i, j)$. By the inductive hypothesis, we have $3H_\alpha(1, i-1) \geq D_\alpha(1, i-1)$. From this, we find

$$3H_\alpha(1, j) = 3H_\alpha(1, i-1) + 3H_\alpha(i, j) \geq D_\alpha(1, i-1) + D_\alpha(i, j) = D_\alpha(1, j),$$

proving the claim. □

Now, we bound the number of N entries in a ray in S_m.

Claim 5.6. *If $\beta \in S_m$, then $N_\beta(m) < m/2$.*

Proof. As in the proof of Claim 5.4 we note $R_\beta(m) = D$. If $R_\beta(m) = H$, then $4H_\beta(m, m) = 4 > m - m + 1 = 1$. Algorithm 5.1 would then add β to S_{m+1}. If $R_\beta(m) = N$, there is some $\alpha \in S_m$ so that $\alpha \in \{N_\beta^+(m), N_\beta^-(m)\}$ and $R_\alpha(m) = H$. Again, this would imply $S_{m+1} \neq \emptyset$. Hence $N_\beta(m) = N_\beta(m-1)$.

Take $j \in [2, m-2]$ and set $\alpha = N_{j-1}^+(\beta)$ and $\gamma = N_{j-1}^-(\beta)$. From inspection of Algorithm 5.1, we see if $R_\beta(j) = N$, then either $R_\alpha(j) = H$ or

$R_\gamma(j) = H$. Thus, each N in R_β requires an H entry in one of the rays corresponding to a node in the set

$$\{N_2^+(\beta), N_3^+(\beta), \ldots, N_{m-1}^+(\beta), N_2^-(\beta), N_3^-(\beta), \ldots, N_{m-1}^-(\beta)\}.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_a$ be nodes and $0 = s_0 < s_1 < s_2 < \cdots < s_a < m$ be integers so that $\alpha_i = N_\beta^+(\ell)$ for $\ell \in [s_{i-1}+1, s_i]$ and each s_i is as large as possible. Similarly, let $\gamma_1, \gamma_2, \ldots, \gamma_b$ be nodes and $0 = t_0 < t_1 < t_2 < \cdots < t_b < m$ be integers so that $\gamma_j = N_\beta^-(\ell)$ for $\ell \in [t_{j-1}+1, t_j]$ and each t_j is as large as possible (see Figure 5.2). We should note that it is possible that $\alpha_a = \gamma_b$ or $\alpha_a = \beta = \gamma_b$.

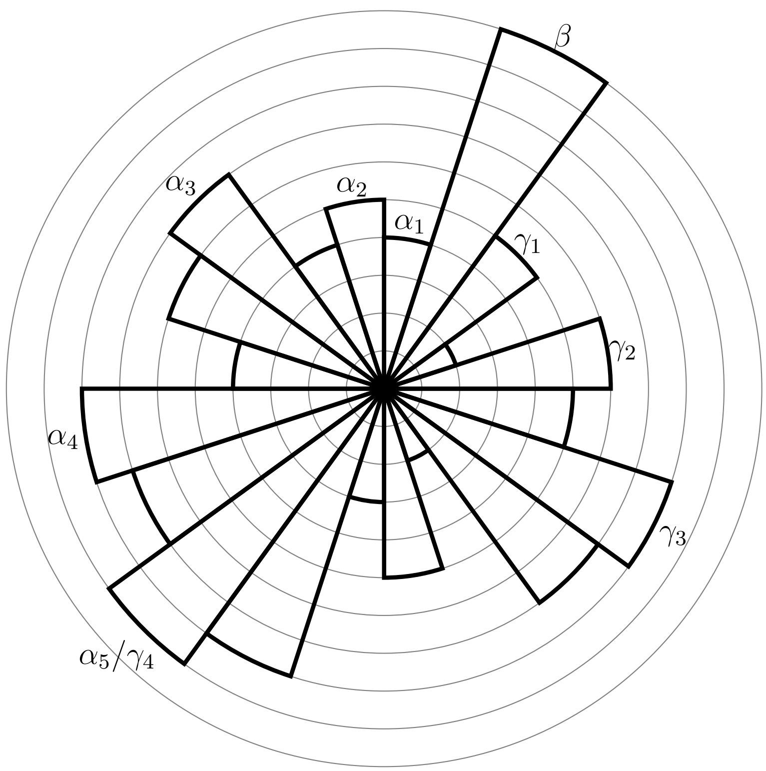

Figure 5.2: Selection of $\alpha_1, \alpha_2, \ldots, \alpha_a$ and $\gamma_1, \gamma_2, \ldots, \gamma_b$.

From this, we have

$$N_\beta(1, m) \leq \sum_{i \in [a]} H_{\alpha_i}(s_{i-1}+1, s_i) + \sum_{j \in [b]} H_{\gamma_j}(t_{j-1}+1, t_j).$$

It does not matter if $\alpha_a = \gamma_b$ or $\alpha_a \neq \gamma_b$. For $i \in [a-1]$, $\alpha_i \notin S_{s_{i+1}}$ (or else we would have chosen larger s_i). Hence we have $4H_{\alpha_i}(s_{i-1}+1, s_i) \leq s_i - s_{i-1}$ or else Algorithm 5.1 would add α_i to S_{s_i+1} with $R_{\alpha_i}(s_i+1) = D$. Similarly, for $j \in [b-1]$, we have $4H_{\gamma_j}(t_{j-1}+1, s_j) \leq t_j - t_{j-1}$. We now have

$$N_\beta(m) \leq \frac{1}{4}\sum_{i\in[a]} s_i - s_{i-1} + \frac{1}{4}\sum_{j\in[b]} t_j - t_{j-1}.$$

These sums telescope and simplify so that we have $N_\beta(m) \leq s_a/4 + t_b/4$. As $s_a < m$ and $t_b < m$, we have $N_\beta(m) < m/2$. □

We are ready to prove the theorem. Take $\beta \in S_m$. We have $H_\beta(m) + N_\beta(m) + D_\beta(m) = m$. By Claim 5.6 we have $H_\beta(m) + D_\beta > m/2$. By Claim 5.5, we have $4H_\beta(m) > m/2$. Finally, by Claim 5.2 and Claim 5.4 we have $\iota(G) \geq H_\beta(m) > m/8 \geq n/8$. From (2.2) and Claim 2.12, we have $\chi_{\mathrm{FF}}(G) < 8\iota(G) \leq 8\chi(G)$. □

5.3 Concluding Remarks

One might ask if we can extend the methods of Pemmaraju, Raman, and Varadarajan to a larger family of graphs. Recall our notation for circular arc graphs from Chapter 2. Suppose $V_1, V_2, \ldots, V_n$ are connected subgraphs of graph H, our base graph. The corresponding *intersection graph* is $G = (V, E)$ where $V = \{V_1, V_2, \ldots, V_n\}$ and $UV \in E$ if and only if $U \cap V \neq \emptyset$. Can we bound the number of colors used by FF on such a graph? In general, the answer is no. Trees can be expressed as intersection graphs and, as we will see in the next section, the performance of FF on trees cannot be bounded from above. However, we have seen that restricting H to paths and cycles does bound the performce of FF. Perhaps we can find another family of base graphs for which the performance of FF is also bounded.

It is also possible to alter the rays and their labeling. As the method stands, the labels are determined in an on-line manner; that is, we assign the labels once and do not alter them. Perhaps we can modify the algorithm that assigns labels to improve our bound or apply the method to a larger family of graphs. Currently, we only use the labels $\{H, N, D\}$. It is possible that using a larger family of labels can open this method up to improvement or larger application.

Chapter 6

FIRST-FIT COLORING OF TREES

6.1 On-Line Coloring of Trees

Trees have a relationship with Grundy colorings that allow us to exploit inductive methods. For a graph G and Grundy coloring $\mathfrak{g}$, define $\overrightarrow{G}_{\mathfrak{g}}$ to be the oriented graph with vertices $V(G)$ where $\overrightarrow{xy}$ is an arrow if $xy \in E(G)$ and $\mathfrak{g}(x) > \mathfrak{g}(y)$. The following lemma is helpful in studying Grundy colorings of trees.

Lemma 6.1. *Suppose T is a forest and $\mathfrak{g}$ is an n-Grundy coloring of T. Let $\overrightarrow{xy}$ be an arrow in $\overrightarrow{T}_{\mathfrak{g}}$. If S be the component of $T - xy$ containing y, then $\mathfrak{g}$ is a k-Grundy coloring of S where $k = \max_{v \in V(S)} \mathfrak{g}(v)$.*

Proof. Fix $v \in V(S)$ so that $\mathfrak{g}(v) = \max_{u \in V(S)} \mathfrak{g}(u)$. For each $i \in [k]$, we have $V_i(S) \subseteq V_i(T)$. By hypothesis, $V_i(T)$ is a coclique so Definition 2.11(G1) holds. We have $N_S(y) = N_T(y) - x$. Recalling $\mathfrak{g}$ is a Grundy coloring of T, we have $\mathfrak{g}(N_T(y)) \supseteq [k-1]$. As $\mathfrak{g}(x) > k$, we see $\mathfrak{g}(N_S(y)) \supseteq [k-1]$. Thus, (G2) holds and (G3) holds for y. For each $u \in V(S) - y$, we have $N_S(u) = N_T(u)$ so (G3) holds for these vertices as well. $\square$

We should note that Lemma 6.1 does not hold for general graphs.

For a tree T and a vertex v, we let $\overrightarrow{T}(v)$ be the tree T with edges oriented away from v. Formally, $\overrightarrow{xy}$ is an arrow in $\overrightarrow{T}(v)$ if $xy \in E(T)$ and the distance from v to x is less than the distance from v to y. The *depth* of $\overrightarrow{T}(v)$ is the number of vertices in the longest directed path in the directed graph.

Observation 6.2. *Any path that witnesses the depth of $\overrightarrow{T}(v)$ starts at vertex v.*

Although our definition of depth differs from our earlier definition of the length of a path, the next lemma shows why we make this choice.

Lemma 6.3. *Suppose T is a tree, v is a vertex of T, and $\mathfrak{g}$ is a Grundy coloring of T. If $\mathfrak{g}(v) = k$, then the depth of $\overrightarrow{T}(v)$ is at least k.*

Proof. We prove this by induction on k. For $k = 1$, the lemma is trivially true. Now suppose the lemma holds for $\mathfrak{g}(v) < k$. Let v be a vertex in T so that $\mathfrak{g}(v) = k$. As $\mathfrak{g}$ is a Grundy coloring, there is a vertex $u \in N(v)$ so that $\mathfrak{g}(u) = k - 1$. Let S be the component containing u in $T - vu$. As $\overrightarrow{vu}$ is an arrow in $\overrightarrow{T}_{\mathfrak{g}}$, $\mathfrak{g}$ is an m-Grundy coloring of S for some $m \geq n - 1$. Let P be a maximum path in $\overrightarrow{S}(u)$ (which, by Observation 6.3, starts at u). By the inductive hypothesis, P has at least $k - 1$ vertices. Note that $\overrightarrow{S}(u)$ is isomorphic to the subtree of $\overrightarrow{T}(v)$ induced by the vertices $V(S)$. Hence, $v + P$ is a path in $\overrightarrow{T}_v$ on at least k vertices. $\square$

First, we will show that for any $n \in \mathbb{Z}^+$, there is a tree T with a n-Grundy coloring.

Construction 6.4. *The tree Ψ_n with root ψ_n is defined recursively (see Figure 6.1):*

(1) Ψ_1 *is a single vertex* ψ_1.

(2) Ψ_n *is* $\psi_n + \Psi_{n-1} + \Psi_{n-2} + \cdots + \Psi_1$ *along with edges* $\psi_i\psi_n$ *for each* $i \in [n-1]$.

One may easily verify that Ψ_n is a tree. This construction is referred to as a broadcast tree by Farley, et al in [17] and [45]. Its recursive

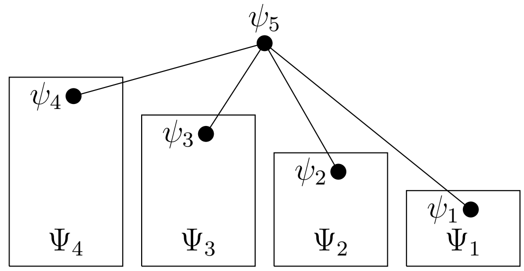

Figure 6.1: Building Ψ_5 earlier iterations.

construction motivates the proofs of the bounds of the performance of on-line algorithms in coloring trees.

Lemma 6.5. *For each $n \in \mathbb{Z}^+$, $\chi_{\mathrm{FF}}(\Psi_n) \geq n$ and $|\Psi_n| = 2^{n-1}$.*

Proof. We employ induction on n and use a slightly stronger inductive hypothesis: for each $n \in \mathbb{Z}^+$, $|\Psi_n| = 2^{n-1}$ and there is an n-Grundy coloring $\mathfrak{g}$ of Ψ_n so that $\mathfrak{g}(\psi_n) = n$. For $n = 1$, the lemma is trivial. Suppose the lemma holds for all positive integers less than n. We will treat $\Psi_1, \Psi_2, \ldots, \Psi_{n-1}$ as subtrees of Ψ_n. We have

$$|\Psi_n| = 1 + \sum_{i \in [n-1]} |\Psi_i| = 1 + \sum_{i \in [n-1]} 2^{i-1} = 2^{n-1}.$$

From the inductive hypotheses, for each $i \in [n-1]$ set $\mathfrak{g}_i$ to be an i-Grundy coloring of Ψ_i with $\mathfrak{g}_i(\psi_i) = i$. Define $\mathfrak{g} : V(\Psi_n) \to [n]$ by

$$\mathfrak{g}(v) = \begin{cases} \mathfrak{g}_i(v) & \text{if } v \in V(\Psi_i) \\ n & \text{if } v = \psi_n \end{cases}.$$

There are no edges between Ψ_i and Ψ_j for $i \neq j$, so the inductive hypothesis tells us $V_i^{\mathfrak{g}}$ is a union of cocliques for each $i \in [n-1]$. Noting that $V_n^{\mathfrak{g}} = \{\psi_n\}$, we see Definition 2.11(G1) holds. Because $\mathfrak{g}(N(\psi_n)) = [n]$, we see (G2) holds and (G3) holds for ψ_n. The inductive hypothesis shows (G3) holds for all $u \in V(\Psi_n) - \psi_n$. This gives us $\chi_{\mathrm{FF}}(\Psi_n) \geq n$. $\square$

Now that we have shown there are trees that force FF to use arbitrarily many colors, we ask how many vertices are needed in a tree to force a given number of colors.

Theorem 6.6. *For any forest T, $\chi_{\mathrm{FF}}(T) \leq \lg 2|T|$. Furthermore, this bound is tight; there is a forest T so that $\chi_{\mathrm{FF}}(T) \geq \lfloor \lg 2|T| \rfloor$.*

Proof. It suffices to prove the theorem for trees and apply the proof to the components of a forest. Set $n = \chi_{\mathrm{FF}}(T)$. To demonstrate the upper bound on n, we prove the equivalent inequality $2^{n-1} \leq |T|$. We proceed by induction on n. The bound is trivial for $n = 1$. Now suppose the bound holds for all positive integers less than n. Select a tree T so that $\chi_{\mathrm{FF}}(T) = n$ and an n-Grundy coloring $\mathfrak{g}$ of T. Select $v \in V(T)$ so that $\mathfrak{g}(v) = n$. As $\mathfrak{g}$ is a Grundy coloring, we have a set of vertices $\{u_1, u_2, \ldots, u_{n-1}\} \subseteq N_T(v)$ so that $\mathfrak{g}(u_i) = i$ for each $i \in [n-1]$. We examine F, the forest created from T by removing the edges $vu_1, vu_2, \ldots, vu_{n-1}$. For each $i \in [n-1]$, let S_i be the component of F containing u_i. As each edge of a tree is a cut-edge, these components are distinct. Because $\overrightarrow{vu_i}$ is an arrow in $\overrightarrow{T}_{\mathfrak{g}}$, Lemma 6.1 shows that $\mathfrak{g}$ is a m_i-Grundy coloring of S_i where $m_i \geq i$. By the inductive hypothesis, we have $|S_i| \geq 2^{m_i-1}$. Noting that $V(T) \supseteq v + V(S_1) \cup V(S_2) \cup \cdots \cup V(S_{n-1})$ with $V(S_i) \cap V(S_j) = \emptyset$ when $i \neq j$, we have

$$|T| \geq 1 + \sum_{i \in [n-1]} |S_i| \geq 1 + \sum_{i \in [n-1]} 2^{m_i-1} \geq 1 + \sum_{i \in [n-1]} 2^{i-1} = 2^{n-1},$$

proving the upper bound.

The tree Ψ_n and its properties shown in Lemma 6.5 complete the theorem. $\square$

There is more we can say about the tree that provides the lower bound in Theorem 6.6. Although the following construction is not strictly necessary, it is simple and provides insight into the lower bound that leads to Lemma 6.10.

Construction 6.7. *The tree Υ_n is defined recursively (see Figure 6.2):*

(1) *Υ_1 is a single vertex.*

(2) *Υ_n is Υ_{n-1} with a leaf attached to each vertex.*

One may quickly verify that Υ_n is a tree.

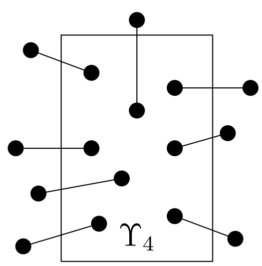

Figure 6.2: Building Υ_5 from Υ_4.

Lemma 6.8. *For each $n \in \mathbb{Z}^+$, $\chi_{\mathrm{FF}}(\Upsilon_n) = n$ and $|\Upsilon_n| = 2^{n-1}$.*

Proof. We use induction on n. The case of $n = 1$ is trivial. Assume the lemma holds for all positive integers less than n. We think of Υ_{n-1} as a subgraph of Υ_n. By the construction of Υ_n, we have $|\Upsilon_n| = 2|\Upsilon_{n-1}| = 2 \cdot 2^{n-2} = 2^{n-1}$. Let $\mathfrak{g}'$ be an $n-1$-Grundy coloring of Υ_{n-1}. Define $\mathfrak{g} : V(\Upsilon_n) \to [n]$ by

$$\mathfrak{g}(v) = \begin{cases} \mathfrak{g}'(v) + 1 & \text{if } v \in V(\Upsilon_{n-1}) \\ 1 & \text{if } v \in V(\Upsilon_n) \setminus V(\Upsilon_{n-1}) \end{cases}.$$

The leaves added to Υ_n are a coclique, so $V_1^{\mathfrak{g}}(\Upsilon_n)$ is a coclique. For $i \in [2, n]$ we have $V_i^{\mathfrak{g}}(\Upsilon_n) = V_{i-1}^{\mathfrak{g}'}(\Upsilon_{n-1})$, so $V_i^{\mathfrak{g}}(\Upsilon_n)$ is a coclique. We have established (G1). By the inductive hypothesis, $\mathfrak{g}'$ is surjective on $[n-1]$. By inspection of the definition of $\mathfrak{g}$, we see that $\mathfrak{g}$ is surjective on $[n]$. Hence, (G2) holds. Take $v \in V(\Upsilon_n)$. If $v \in V(\Upsilon_n) \setminus V(\Upsilon_{n-1})$, then (G3) is trivial. Suppose $v \in V(\Upsilon_{n-1})$ with $\mathfrak{g}(v) = j$. Fix $i \in [2, j-1]$. There is some $u \in N_{\Upsilon_{n-1}}(v)$ so that $\mathfrak{g}'(u) = i - 1$. Hence, we have $u \in N_{\Upsilon_n}(v)$ with $\mathfrak{g}(u) = i$. By construction, there is a leaf adjacent to v colored 1 by $\mathfrak{g}$. From this, we see (G3) holds for all vertices of Υ_n.

Now, we have $\chi_{\mathrm{FF}}(\Upsilon_n) \geq n$. Theorem 6.6 shows $\chi_{\mathrm{FF}}(\Upsilon_n) = n$. $\square$

The construction of the n-Grundy coloring in Lemma 6.8 provides a hint at the following proposition.

Proposition 6.9. *Suppose T is a tree with n-Grundy coloring $\mathfrak{g}$ so that $|T| = 2^{n-1}$ and $n > 1$. Then a vertex $u \in V(T)$ is a leaf if and only if $\mathfrak{g}(u) = 1$.*

Proof. Suppose $u \in V(T)$ is a leaf. Because $|N_T(u)| = 1$, we must have $\mathfrak{g}(u) \in \{1, 2\}$ by Definition 2.11(G3). Assume $\mathfrak{g}(u) = 2$. Let v be the unique neighbor of u. We must have $\mathfrak{g}(v) = 1$. Let T' be the component of the forest $T - uv$ containing v. Note that T' contains all the vertices of T other than u. Because $\overrightarrow{uv}$ is an arrow in $\overrightarrow{T}_{\mathfrak{g}}$, Lemma 6.1 tells us $\mathfrak{g}$ is an n-Grundy coloring of T'. As $|T'| = 2^{n-1} - 1$, Theorem 6.6 tells us T' cannot have an n-Grundy coloring; this shows $\mathfrak{g}(u) = 1$.

Now suppose $\mathfrak{g}(u) = 1$. Take $w \in V_n$. As T is a tree there is a unique vertex $v \in N_T(u)$ so that v is on the unique path from u to w (note that because $n > 1$, $u \neq w$). Assume there is some $v' \in N_T(u)$ so that $v \neq v'$. We again use Lemma 6.1 to see that $\mathfrak{g}$ is an n-Grundy coloring of the component of $T - uv'$ that contains w. This component has fewer that 2^{n-1} vertices and Theorem 6.6 shows we have arrived at a contradiction. Hence, there is no such v' and we see u is a leaf. $\square$

The new construction and the preceding proposition allow us to show the following lemma:

Lemma 6.10. *For each $n \in \mathbb{Z}^+$ there is a unique tree T so that $|T| = 2^{n-1}$ and $\chi_{\mathrm{FF}}(T) = n$.*

Proof. This lemma is equivalent to claiming $T = \Upsilon_n$. We use induction on n. In the case of $n = 1$, the tree on one vertex is the unique tree that satisfies our conditions. Now assume the lemma holds for all cases smaller than n. Assume we have trees S and T with respective n-Grundy colorings $\mathfrak{g}$ and $\mathfrak{h}$ so that $|S| = |T| = 2^{n-1}$ and $S \neq T$.

Take $u \in V_i^{\mathfrak{g}}(S)$ where $i \in [2, n]$. By (G3), u has a 1-witness and so, by Proposition 6.9, u is adjacent to a leaf. Similarly, we see each vertex $v \in V(T)$ with $\mathfrak{h}(v) > 1$ is adjacent to a leaf. Let S' to be the subtree of S induced by all the non-leaf vertices. Similarly, let T' be the subtree of T induced by all the non-leaf vertices. Define $\mathfrak{g}' : V(S') \to [n-1]$ and $\mathfrak{h}' : V(T') \to [n-1]$ by $\mathfrak{g}'(u) = \mathfrak{g}(u) - 1$ and $\mathfrak{h}'(v) = \mathfrak{h}(v) - 1$. As only vertices colored 1 by $\mathfrak{g}$ or $\mathfrak{h}$ were removed, it is straightforward to verify $\mathfrak{g}'$ and $\mathfrak{h}'$ are $n-1$-Grundy colorings of S' and T'. Also, only leaves were removed so both S' and T' are trees. By our inductive hypothesis, we have $S' = T' = \Upsilon_{n-1}$. We see S and T are obtained from S' and T' in the same way Υ_n is obtained from Υ_{n-1}, so $S = T$. $\square$

Using terminology of Gyárfás and Lehel [26], for each $n \in \mathbb{Z}^+$, let the unique tree $\Upsilon_n = \Psi_n$ be the *canonical n-tree*. In 1982, the canonical n-tree was used by Hedetniemi, et al in [28] to show $\chi_{\mathrm{FF}}(T)$ can be determined in linear time for any tree T. The properties of the canonical n-tree highlighted in Constructions 6.4 and 6.7 are used serval times in [26], including the proof of a version of the following lemma. We present a slightly strengthened version here.

Lemma 6.11. *Let T be a forest and let $n \geq 2$ be an integer. Then T has an n-Grundy coloring if and only if the canonical n-tree is a subtree of T. Consequently, $\chi_{\mathrm{FF}}(T) = n$ if and only if n is the maximum integer so that the canonical n-tree is a subtree of T.*

Proof. It suffices to prove the lemma for trees and apply the result to the individual components of a forest. Let T_n be the canonical n-tree and set $U = V(T_n)$.

First, suppose $T_n \subseteq T$. We show T has an n-Grundy coloring. Let $\mathfrak{g}'$ be an n-Grundy coloring of T_n. Define $V_1 = N_T(U) \setminus U$ and for $i > 1$ set

$$V_i = N_T(U \cup V_1 \cup V_2 \cup \cdots \cup V_{i-1}) \setminus U \cup V_1 \cup V_2 \cup \cdots \cup V_{i-1}.$$

See Figure 6.3. Let j be the least positive integer so that $V_{j+1} = \emptyset$. For $i \in [j]$, define $W_i = U \cup V_1 \cup \cdots \cup V_i$ and $W_0 = U$.

We extend $\mathfrak{g}'$ to $\mathfrak{g}$ through a sequence of n-Grundy colorings $\mathfrak{g}' = \mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_j = \mathfrak{g}$ where $\mathfrak{g}_i$ is an n-Grundy coloring of $T[W_i]$. For $i \in [j]$, we define $\mathfrak{g}_i$ as follows. For each $u \in W_{i-1}$ set $\mathfrak{g}_i(u) = \mathfrak{g}_{i-1}(u)$. For $v \in V_i$, set $\mathfrak{g}_i(v)$ to be the smallest positive integer so that $\mathfrak{g}_i$ is a proper coloring of $T[W_i]$. As $T[W_{i-1}]$ is a tree and by the definition of V_i, $v \in V_i$ has a unique

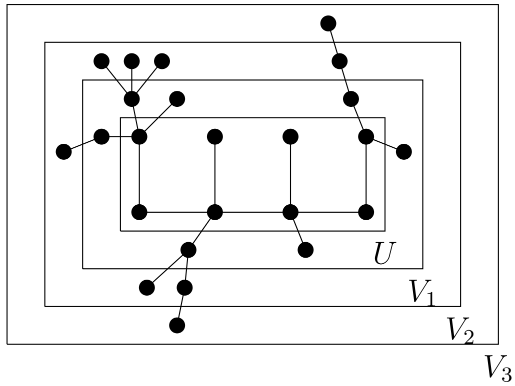

Figure 6.3: Partitioning $V(T)$ in $U, V_1, \dots, V_j$.

neighbor in $T[W_{i-1}]$ and no neighbors in V_i. Thus, $\mathfrak{g}_i(v) \in \{1,2\}$. It is easy to see $\mathfrak{g}_i$ is an n-Grundy coloring of $T[W_i]$.

Now suppose T has an n-Grundy coloring. We will show T_n is a subtree of T by induction on $|T|$. By Theorem 6.6, the inductive base is $|T| = 2^{n-1}$ and Lemma 6.10 gives us $T = T_n$, establishing our base. Now, assume there is some n so that T has an n-Grundy coloring $\mathfrak{g}$ but T_n is not a subtree of T. Select T so that $|T|$ is as small as possible. Take $S = \overrightarrow{T}_{\mathfrak{g}}$. For simplicity, we will use $V(S)$ and $V(T)$ interchangeably. First, we note $|V_n| = 1$. If $x, y \in V_n$ are distinct vertices, Lemma 6.1 tells $\mathfrak{g}$ is an n-Grundy coloring of the component containing x in the forest $T - y$ as $\overrightarrow{yu}$ is an arrow in S for all $u \in N(T)$. Let x be the unique vertex in V_n.

Assume we have some $u \in V(S)$ with $\mathfrak{g}(u) < n$ so that $N_S^-(u) = \emptyset$ (i.e.: $\overrightarrow{uv}$ is an arrow in S for all $v \in N(u)$). Lemma 6.1 shows that $\mathfrak{g}$ is an n-Grundy coloring of the the component in $T - u$ containing x, again violating our hypothesis of the properties of T. Hence, we have the following.

$$\text{If } u \in V(S) \text{ with } \mathfrak{g}(u) < n, \text{ then } N_S^-(u) \neq \emptyset. \tag{6.1}$$

Select $i \in [n-1]$ to be as large as possible so that $|V_i| > 2^{n-i-1}$. Such a color class must exist or else we would have

$$|T| = |V_n| + \sum_{i\in[n-1]} |V_i| \leq 1 + \sum_{i\in[n-1]} 2^{n-i-1} = 1 + \sum_{\ell\in[n-1]} 2^{\ell-1} = 2^{n-1}.$$

By our inductive base, this is impossible. By our choice of i, we have $|V_j| \leq 2^{n-j-1}$ for $j \in [i+1, n-1]$. As $V_{i+1}, V_{i+2}, \ldots, V_n$ are pairwise disjoint, we have

$$\bigcup_{j\in[i+1,n]} |V_j| \leq 1 + \sum_{j\in[i+1,n-1]} 2^{n-j-1} = 1 + \sum_{\ell\in[n-i-1]} 2^{\ell-1} = 2^{n-i-1}.$$

By property (6.1) and the pigeonhole principle, there is some $j \in [i+1, n]$ with $w \in V_j$ so that $|N_T(w) \cap V_i| \geq 2$. Suppose $u, v \in N_T(w) \cap V_i$. Take u to be the unique vertex in $N_T(w)$ on the path from x to w and let T' be the component of $T - v$ containing x. We claim $\mathfrak{g}$ is an n-Grundy coloring of T'. For each $k \in [n]$ we have $V_k(T') \subseteq V_k(T)$, so (G1) holds. For all $y \in V(T') - w$, we have $N_{T'}(y) = N_T(y)$. This establishes (G3) for all vertices other than w. To show (G3) holds for w, note that $N_{T'}(w) = N_T(w) - v$, and by our choice of w, we have $u \in N_T(w)$ with $\mathfrak{g}(u) = \mathfrak{g}(v)$. The neighborhood of x shows (G2) holds. By the inductive hypothesis, T' has the desired subtree. We have our desired contradiction and we see T must have T_n as a subtree. □

Now, we turn our attention to algorithms other than FF. In 1988, Gyárfás and Lehel [25], showed for any $n \in \mathbb{Z}^+$ and any on-line coloring algorithm, there is a tree that requires n colors. In 1990, the same authors strengthened the result in [26] to show results equivalent to the following lemma and theorem. We offer a different proof.

Lemma 6.12 (Gyárfás & Lehel [26]). *For any on-line coloring algorithm $\mathcal{A}$, there is a subforest T of the canonical n-tree so that $\chi_{\mathcal{A}}(T) \geq n$.*

Proof. Using the terminology of Spoiler and Algorithm from our introduction, we will provide a strategy for Spoiler to present a subforest of Ψ_n so that Algorithm uses n colors. Suppose $\mathcal{A}$ is an on-line coloring algorithm. We will use induction on n with a strengthened inductive hypothesis as follows: For positive integer n and arbitrary set of colors C^n so that $|C| = n - 1$, Spoiler has a strategy $\mathcal{S}_n(C^n)$ that presents a subforest of Ψ_n, halts with $\mathcal{A}$ uses a color not in C^n, and the final vertex presented may be taken as ψ_n (the root of Ψ_n). Without loss of generality, we make take $C = [k-1]$. We may think of this as Spoiler relabeling the colors used by $\mathcal{A}$ by the order they are introduced.

For $n = 1$, the strategy $\mathcal{S}_1([0])$ is presenting a single vertex. As $\mathcal{A}$ must use a color (which Spoiler relabels as 1) and Ψ_1 is a single vertex, we have established our base. Now, suppose the hypothesis holds for all cases smaller than n. To form $\mathcal{S}_n([n-1])$, Spoiler builds a forest $F_1 + F_2 + \cdots + F_{n-1}$ so that for each $i \in [n-1]$, F_i is formed using $\mathcal{S}_i([i-1])$. By the inductive hypothesis, each F_i is a subforest of Ψ_i. Furthermore, each root vertex ψ_i is colored i. To complete $\mathcal{S}_n([n-1])$, Spoiler presents a vertex ψ_n adjacent to $\psi_1, \psi_2, \ldots, \psi_{n-1}$. As each F_i is a subforest of Ψ_i and ψ_n is adjacent to only ψ_i, from Construction 6.14 we see we have a subforest of Ψ_n. Furthermore, for each $i \in [n-1]$, ψ_n is adjacent to a vertex $\mathcal{A}$ has colored i and so ψ_n must be colored with a new color, which without loss of generality we take to be n. $\square$

Lemmas 6.11 and 6.12 provide a theorem:

Theorem 6.13 (Gyárfás & Lehel [26]). *For any tree T, we have $\chi_{\mathrm{FF}}(T) \leq \min_{\mathcal{A}} \chi_{\mathcal{A}}(T) = \chi_{\mathrm{OL}}(T)$.*

6.2 Proof of Theorem 2.10

Take T to be a forest and $n \in \mathbb{Z}^+$. If n is large in comparison to $|T|$, we expect the probability of A_T^n (recall the definition of A_T^n from Chapter 2) to be small. To lend credence to our intuition, we look deeper into the canonical n-tree with a third construction.

Construction 6.14. *The tree Φ_n with root ϕ_n and semiroot $\bar{\phi}_n$ is defined recursively (see Figure 6.4):*

(1) Φ_1 is the single vertex ϕ_1.

(2) Φ_2 is the edge $\phi_2\bar{\phi}_2$.

(3) Φ_n is $\Phi_{n-1} + \Phi'_{n-1}$ (two disjoint copies of Φ_{n-1}) along with edge $\phi_{n-1}\phi'_{n-1}$. Set $\phi_n = \phi_{n-1}$ and $\bar{\phi}_n = \phi'_{n-1}$.

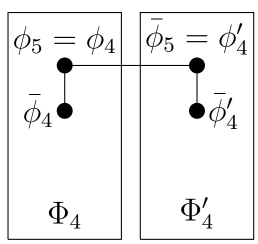

Figure 6.4: Building Φ_5 from Φ_4 and Φ'_4.

Again, it is easy to show that Φ_n is a tree. Although Gyárfás and Lehel did not explicitly use this construction, they employed its "left/right" structure in some of their proofs in [26]. As we will see, this structure highlights the fact that in any n-Grundy coloring of a canonical n-tree, either ϕ_n or $\bar{\phi}_n$ can be colored n. We go slightly further and offer the following lemma to classify all n-Grundy colorings of the canonical n-tree.

Lemma 6.15. *For each integer $n \geq 2$, we have $|\Phi_n| = 2^{n-1}$ and there are exactly two distinct n-Grundy colorings $\mathfrak{g}$ and $\mathfrak{h}$ of Φ_n. Furthermore, we may specify $V_n^{\mathfrak{g}} = \{\phi_n\}$, $V_{n-1}^{\mathfrak{g}} = \{\bar{\phi}_n\}$, $V_n^{\mathfrak{h}} = \{\bar{\phi}_n\}$, and $V_{n-1}^{\mathfrak{h}} = \{\phi_n\}$.*

Proof. We will use induction on n. The base of $n = 2$ is trivial. Suppose the lemma holds for all integers smaller than c. From Construction 6.14, we see $|\Phi_n| = |\Phi_{n-1} + \Phi'_{n-1}| = 2^{n-2} + 2^{n-2} = 2^{n-1}$. Again, we treat Φ_{n-1} and Φ'_{n-1} as subgraphs of Φ_n. Let $\hat{\mathfrak{g}}$ and $\check{\mathfrak{g}}$ be $n-1$-Grundy colorings of Φ_{n-1} and Φ'_{n-1} (respectively) provided by the inductive hypothesis satisfying

(1) $\hat{\mathfrak{g}}(\phi_{n-1}) = \check{\mathfrak{g}}(\phi'_{n-1}) = n - 1$,

(2) $\hat{\mathfrak{g}}(\bar{\phi}_{n-1}) = \check{\mathfrak{g}}(\bar{\phi}'_{n-1}) = n - 2$.

The inductive hypothesis tells us the color classes n and $n-1$ for both $\hat{\mathfrak{g}}$ and $\check{\mathfrak{g}}$ consist of one vertex each. Define $\mathfrak{g}$ and $\mathfrak{h}$ as follows:

$$\mathfrak{g}(v) = \begin{cases} \hat{\mathfrak{g}}(v) & \text{if } v \in V(\Phi_{n-1}) - \phi_{n-1} \\ n & \text{if } v = \phi_{n-1} = \phi_n \\ \check{\mathfrak{g}}(v) & \text{if } v \in V(\Phi'_{n-1}) \end{cases},$$

$$\mathfrak{h}(v) = \begin{cases} \check{\mathfrak{g}}(v) & \text{if } v \in V(\Phi'_{n-1}) - \phi'_{n-1} \\ n & \text{if } v = \phi'_{n-1} = \bar{\phi}_n \\ \hat{\mathfrak{g}}(v) & \text{if } v \in V(\Phi_{n-1}) \end{cases}.$$

We will show that $\mathfrak{g}$ is an n-Grundy coloring Assume we have $uv \in E(\Phi_n)$ so that $\mathfrak{g}(u) = \mathfrak{g}(v)$. We cannot have $\phi_n \in \{u, v\}$ as $V_n^{\mathfrak{g}}(\Phi_n) = \{\phi_n\}$. If $uv \in E(\Phi_{n-1})$ (where $\phi_n \notin \{u, v\}$) this contradicts the hypothesis that $\hat{\mathfrak{g}}$ is a Grundy coloring. We arrive at the same contradiction if $uv \in E(\Phi'_{n-1})$. This eliminates all possible edges in Φ_n, so our assumption is false and we see (G1) holds. As $\check{\mathfrak{g}}$ is surjective on $[n-1]$ and $\mathfrak{g}(\phi_n) = n$, we see (G2) holds.

We have $\{u_1, u_2, \ldots, u_{n-2}\} \subseteq N_{\Phi_{n-1}}(\phi_{n-1})$ so that $\hat{\mathfrak{g}}(u_i) = \mathfrak{g}(u_i) = i$ for each $i \in [n-2]$. We can see $\bar{\phi}_n$ is an $n-1$-witness for ϕ_n under $\mathfrak{g}$, so ϕ_n has a witness for each color in $[n-1]$. Note that $\phi_n = \phi_{n-1}$ is not a witness for any vertex in any of the colorings $\hat{\mathfrak{g}}$, $\check{\mathfrak{g}}$, or $\mathfrak{g}$. Furthermore, no other colorings have been altered and so (G3) holds for all other vertices of Φ_n from the inductive hypothesis. We leave it to the reader to follow the same reasoning to see $\mathfrak{h}$ is an n-Grundy coloring as well.

Now, suppose $\mathfrak{f}$ is an n-Grundy coloring of Φ_n. Let $u \in V^{\mathfrak{f}}_{n-1}$ and $v \in V^{\mathfrak{f}}_n$ so that $uv \in E(\Phi_n)$ (such a pair of vertices must exist as $\mathfrak{f}$ is surjective and any every vertex in $V^{\mathfrak{f}}_n$ has an $n-1$-witness). Let S_u and S_v be the components of $T - uv$ containing u and v (respectively). Because $\overrightarrow{uv}$ is an arrow in $\overrightarrow{T}_{\mathfrak{f}}$, Lemma 6.1 tells us $\mathfrak{f}^u$ is an $n-1$-Grundy coloring of S_u. As $|S_v| < 2^{n-1}$, Theorem 6.6 tells us $\mathfrak{f}$ is not an n-Grundy coloring of S_v. We can see this is because v has no $n-1$ witness. Define $\mathfrak{f}^u : V(S_u) \to [n-1]$ and $\mathfrak{f}^v : V(S_u) \to [n-1]$ by

$$\mathfrak{f}^u(x) = \mathfrak{f}(x),$$

$$\mathfrak{f}^v(x) = \begin{cases} \mathfrak{f}(x) & \text{if } x \neq v \\ n-1 & \text{if } x = v \end{cases}.$$

We have seen that Lemma 6.1 shows $\mathfrak{f}^u$ is an $n-1$-Grundy coloring fo S_u. We leave it to the reader to verify that $\mathfrak{f}^v$ is an $n-1$-Grundy coloring of S_v.

By Theorem 6.6, we must have $|S_u|, |S_v| \geq 2^{n-2}$. So, we have $|S_u| = |S_v| = 2^{n-2}$ and $S_u = S_v = \Phi_{n-1}$. Without loss of generality, we may take $S_u = \Phi_{n-1}$ with $u = \phi'_{n-1}$ and $S_v = \Phi_{n-1}$ with $v = \phi_{n-1}$. From our inductive hypothesis, we have $\{\mathfrak{f}^u, \mathfrak{f}^v\} = \{\hat{\mathfrak{g}}, \check{\mathfrak{g}}\}$. By the constructions of $\mathfrak{g}$ and $\mathfrak{h}$, we see $\mathfrak{f} \in \{\mathfrak{g}, \mathfrak{h}\}$. $\square$

Now that we have a firm grasp of not only the structure of a canonical n-tree but also any accompanying n-Grundy coloring, we can examine the probability that FF uses n colors on this tree in a random presentation of the vertices.

Lemma 6.16. *For a positive integer $n \geq 2$, if T is the canonical n-tree, then $Pr(A_T^n) = (1/2)^{2^{n-1}-2}$.*

Proof. By Lemma 6.10, we may focus on Φ_n and use the results of Lemma 6.15. Let B_n be the event that FF colors ϕ_n with n and let $\bar{B}_n$ be the event that FF colors $\bar{\phi}_n$ with n in $\Phi_n^{\prec}$ (for some arbitrarily chosen $\prec$). We will use B'_n for the event that FF colors ϕ'_n with n and $\bar{B}'_n$ be the event that FF colors $\bar{\phi}'_n$ with n in $\Phi_n'^{\prec}$ (for some arbitrarily chosen $\prec$).

In the proof of Lemma 6.15, we saw that and $n+1$-Grundy coloring of Φ_{n+1} requires either ϕ_{n+1} or ϕ'_{n+1} to be colored $n+1$. We conclude $\Pr(A_T^{n+1}) = \Pr(B_{n+1} \cup \bar{B}_{n+1})$. As demonstrated in the proof of Lemma 6.15, if the edge $\phi_{n+1}\bar{\phi}_{n+1}$ is removed form Φ_{n+1} then an $n+1$-Grundy coloring $\mathfrak{g}$ of Φ_{n+1} can be used to create n-Grundy colorings of Φ_n and Φ'_n so that both ϕ_n and ϕ'_n are colored n. Similarly, if we have n-Grundy colorings of Φ_n and Φ'_n so that both ϕ_n and ϕ'_n are colored n, we can build two $n+1$-Grundy colorings of Φ_{n+1}, one where ϕ_{n+1} is colored $n+1$ and one where $\bar{\phi}_{n+1}$ is colored $n+1$. From this, we can see $\Pr(B_{n+1} \cup \bar{B}_{n+1}) = \Pr(B_n \cap B'_n)$. Furthermore, as Φ_n and Φ'_n have no edges between them, the events B_n and B'_n are independent. Hence, $\Pr(A_T^{n+1}) = \Pr(B_n)\Pr(B'_n)$.

We examine the relationships between B_{n+1}, B_n, and B'_n. First, as $\Phi_n = \Phi'_n$, we have $\Pr(B_n) = \Pr(B'_n)$. Suppose both ϕ_n and ϕ'_n are colored n by FF in a presentation $\prec$ of $\Phi_{n+1} - \phi_n\phi'_n$ (recall $\phi_{n+1}\bar{\phi}_{n+1} = \phi_n\phi'_n$), then one of ϕ_{n+1} or $\bar{\phi}_{n+1}$ is colored $n+1$ in $\Phi^{\prec}_{n+1}$. We have ϕ_{n+1} colored n if and only if $\bar{\phi}_{n+1} \prec \phi_{n+1}$ and $\bar{\phi}_{n+1}$ colored n if and only if $\phi_{n+1} \prec \bar{\phi}_{n+1}$. There is a natural bijection between the presentations of the former and the latter cases. From this we see $\Pr(B_{n+1}) = (1/2)\Pr(B_n)\Pr(B'_n) = (1/2)\Pr(B_n)^2$. Knowing that $\Pr(B_1) = 1$, we solve this recurrence to find $\Pr(B_n) = (1/2)^{2^{n-1}-1}$. Recalling $\Pr(A_T^{n+1}) = \Pr(B_n)\Pr(B'_n) = \Pr(B_n)^2$, and (making adjustments for indices) we have completed the proof. □

The preceding lemma is perhaps not surprising. We can see that if u is a leaf in the canonical n-tree T and v is a unique neighbor FF will use fewer than n colors on $T^{\prec}$ if $v \prec u$. This restriction alone means $\chi_{\mathrm{FF}}(T^{\prec}) < n$ for a large number of presentations of T. However, we will show there are trees with a structure that forces FF to use many colors over a larger set of presentations.

Proof of Theorem 2.10. First, we recursively build a tree S_n with root r_n. Informally speaking, we desire this tree to have the property that if r_n is $\prec$-greater than most of the vertices of S_n (for some arbitrarily chosen $\prec$), there is a probability bounded from below that FF colors r_n with n. Using discrete probability could become cumbersome, so we opt to determine $\prec$ using continuous probabilities. Let $f : V(S_n) \to [0,1]$ be a uniform distribution. We will take $u \prec v$ if $f(u) < f(v)$. We say such a presentation is induced by f. We can see each $\prec$ is equally likely to be selected by f.

To build S_n and investigate its properties, we will need the quantities

$$\varepsilon_n = \frac{1}{2^{n-1}}, \qquad p_n = \frac{1}{2^{n-1}}, \qquad \text{and,} \qquad m_n = 2^{2n-3}.$$

We start by defining S_1 to be a single vertex. For $n \geq 2$ set $m = m_{n+1} - 1$ and define S_{n+1} to be m_{n+1} disjoint copies of S_n which are labeled

$S_n^0, S_n^1, S_n^2, \ldots, S_n^m$ with respective roots $r_n^0, r_n^1, r_n^2, \ldots, r_n^m$ along with edges $r_n^0 r_n^1, r_n^0 r_n^2, \ldots, r_n^0 r_n^m$. We set $r_{n+1} = r_n^0$ (see Figure 6.5). This construction makes it easy to see S_{n+1} is a a tree. We will also use $\bar{S}_n$ to be the forest $S_n - r_n$.

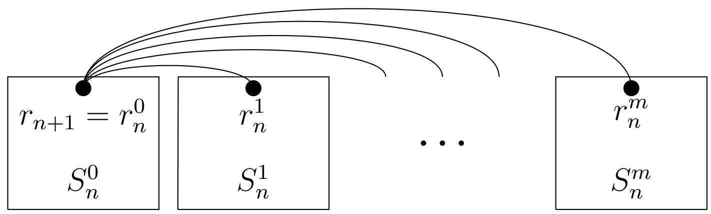

Figure 6.5: Building S_{n+1} from m copies of S_n.

Take f be a uniform distribution of the vertices of S_n to $[0, 1]$. Let X_n be the event that in the presentation of $\bar{S}_n$ induced by f there is a set $W_n = \{w_1, w_2, \ldots, w_{n-1}\} \subseteq N_{S_n}(r_n)$ of distinct vertices so that:

(W1) $f(W_n) \subseteq [1, 1 - \varepsilon_n)$.

(W2) For each $i \in [n-1]$, FF assigns w_i color i.

Let Y_n be the event that $f(r_n) \geq 1 - \varepsilon_n$. In X_n, it may seem strange focus on $\bar{S}_n$ rather than S_n. If $f(r_n) < 1 - \varepsilon_{n-1}$ then the colors of the vertices in W_n might be altered. Our choice to work with $\bar{S}_n$ makes X_n and Y_n independent events, removing a layer of complexity in calculating probabilities. Informally, we may think of FF ignoring r_n when looking at event X_n.

By induction on n, we will show $\Pr(X_n) \geq p_n$ and the depth of $\overrightarrow{S}_n(r_n)$ is n. The trivial case of $n = 1$ and the slightly less trivial case of $n = 2$ establish our base. Suppose the hypothesis holds for all cases smaller than $n + 1$. Let $f : V(S_{n+1}) \to [0, 1]$ be a uniform distribution. The inductive hypothesis tells us there is probability at least p_n that there exists a set of distinct vertices $W'_{n+1} = \{w_1, w_2, \ldots, w_{n-1}\} \subseteq N_{S_n^0}(r_{n+1})$ so that $f(W'_{n+1}) \subseteq [1, 1-\varepsilon_n)$ and w_i is assigned color i for each $i \in [n-1]$ when FF ignores r_{n+1}, as in the definition of X_n. If there is some r_n^i (with $i \in [m]$) assigned color n with $f(r_n^i) \in [1, 1-\varepsilon_{n+1})$, then we may add this vertex to W'_{n+1} to form the desired set W_{n+1}. Here, we should note that the inductive hypothesis and Lemma 6.3, the maximum color FF can assign r_n^i in $\bar{S}_n$ is n.

To express this mathematically, for each $i \in [m]$, we let Z^i be the event FF assigns r_n^i color n (where FF still ignores r_{n+1}) and $f(r_n^i) < \varepsilon_{n+1}$. If one of $Z^1, Z^2, \ldots, Z^m$ occurs, we have the desired r_n^i to add to W'_{n+1}. The events $Z^1, Z^2, \ldots, Z^m$ are independent trials (there are no edges between S_n^i and S_n^j when $i \neq j$). We can also note $\Pr(Z^i) = \Pr(Z^j)$ (because $S_n^i = S_n^j$) for $i, j \in [m]$. Hence, the probability of at least one of $Z^1, Z^2, \ldots, Z^m$ occurring is

$$1 - \left(1 - \Pr(Z^1)\right)\left(1 - \Pr(Z^2)\right) \cdots (1 - \Pr(Z^m)) = 1 - \left(1 - \Pr(Z^1)\right)^m.$$

The probability of the existence of W'_{n+1} and the occurrence of any Z^i are independent events. This gives us $\Pr(X_{n+1}) \geq p_n \left(1 - (1 - \Pr(Z^1))^m\right)$.

Let Y^i be the event $f(r_n^i) \in [1-\varepsilon_n, 1-\varepsilon_{n+1})$. As f is uniform, $Y^1, Y^2, \ldots, Y^m$ are independent and $\Pr(Y^i) = \Pr(Y^j) = (1-\varepsilon_{n+1}) - (1-\varepsilon_n) = \varepsilon_{n+1}$. Echoing our earlier notation, for $i \in [m]$, we take X^i to be the event X_n occurs in S_n^i (recalling that FF is ignoring r_n^i). If events X^i and Y^i occur, then Z^i occurs. Hence, $\Pr(Z^i) \geq \Pr(X^i \cap Y^i)$. Noting that FF ignores r_n^i in X_n, the events X^i and Y^i are independent. We have $\Pr(Z^i) \geq \Pr(X^i \cap Y^i) = \Pr(X^i)\Pr(Y^i) = p_n\varepsilon_n$.

We have

$$1-\left(1-\Pr(Z^1)\right)^m \geq 1-(1-p_n\varepsilon_n)^m$$

and so $\Pr(X_{n+1}) \geq p_n\left(1-(1-p_n\varepsilon_n)^m\right)$. Substituting our definitions for ε_{n+1}, p_n, and $m = m_{n+1}-1$, we have

$$\Pr(X_{n+1}) \geq \frac{1}{2^{n-1}}\left[1-\left(1-\frac{1}{2^n}\frac{1}{2^{n-1}}\right)^{2^{2n-1}-1}\right] = \frac{1}{2^{n-1}}\left[1-\left(1-\frac{1}{2^{2n-1}}\right)^{2^{2n-1}-1}\right].$$

If the quantity $\left(1-\frac{1}{2^{2n-1}}\right)^{2^{2n-1}-1}$ is at most $1/2$ for $n \in \mathbb{Z}^+$, then we have $Pr(X_{n+1}) \geq \frac{1}{2}p_n = p_{n+1}$. To show this is true, define

$$F(n) = \left(1-\frac{1}{2^{2n-1}}\right)^{2^{2n-1}-1}, G(n) = \left(1-\frac{1}{2^{2n-1}}\right)^{2^{2n-1}}, \text{ and } H(n) = \left(1-\frac{1}{2^{2n-1}}\right)^{-1}.$$

We verify $F(1) = 1/2$. We take $D = \{n \in \mathbb{R} : n \geq 2\}$. We will use basic methods of calculus. We have

$$\max_{n\in D} F(n) = \max_{n\in D} G(n)H(n) \leq \left(\max_{n\in D} G(n)\right)\left(\max_{n\in D} H(n)\right).$$

Through straightforward but tedious calculations, we see $G'(n) > 0$ and $H'(n) < 0$ for all $n \in D$. So $G(n)$ is increasing on D. Recalling $\lim_{h\to\infty}\left(1-\frac{1}{h}\right)^h = \frac{1}{e}$, we see $G(n)$ is bounded from above by $\frac{1}{e}$. As $H(n)$ is decreasing on D, its maximum is at the least element of D and so $\max_{n\in D} H(n) = H(2) = 8/7$. Thus $\max_{n\in D} F(n) \leq \frac{8/7}{e} \approx .42043 < \frac{1}{2}$. This proves the desired bound for $Pr(X_{n+1})$.

By the inductive hypothesis, the depth of $\overrightarrow{S}^i_n(r^i_n)$ is n for $0 \leq i \leq m$. Fix some $i \in [m]$. Let P be a path on n vertices in $\overrightarrow{S}^i_n(r^i_n)$ starting at r^i_n. Noting that $\overrightarrow{S}^i_n(r^i_n)$ is isomorphic to the subtree of $\overrightarrow{S}_{n+1}(r_{n+1})$ induced by the vertices of S^i_n, we see that $r_{n+1}+P$ is a directed path on $n+1$ vertices. Hence, the depth of $\overrightarrow{S}_{n+1}(r_{n+1})$ is at least $n+1$. To show the depth is exactly $n+1$, assume there is a path Q in $\overrightarrow{S}_{n+1}(r_{n+1})$ on $n+2$ vertices. By the construction of S_{n+1}, this path must be contained entirely in S_n^0 or have $n+1$ vertices in S_n^i for some $i \in [m]$. The first case is impossible by our inductive hypothesis. The second case would require $\overrightarrow{S}_n^i(r_n^i)$ to have a path on $n+1$ vertices and this would violate our inductive hypothesis. This shows our desired hypotheses hold for all n.

We are now ready to use the tree S_n to build the desired trees to prove the theorem. For some uniform distribution $f : V(S_n) \to [0, 1]$. If r_n has distinct neighbors $u_1, u_2, \ldots, u_{n-1}$ so that $f(u_i) < f(r_n)$ and u_i is assigned color i for each $i \in [n-1]$, then FF assigns r_n color n. By our earlier work, we see the probability of this occurring is at least $\Pr(X_n \cap Y_n) = p_n\varepsilon_n$.

Let $t_n = 2^{2n-2}$. Take n to be a fixed positive integer greater than 1. Define $\bar{T}$ to be the disjoint union of $t = t_n$ copies of S_n labeled $S_n^1, S_n^2, \ldots, S_n^t$ with respective roots $r_n^1, r_n^2, \ldots, r_n^t$. Define T to be $\bar{T}$ along with edges $r_n^i r_n^{i+1}$ for $i \in [t-1]$ (see Figure 6.6). Again, it is easy to see from the construction that T is also a tree.

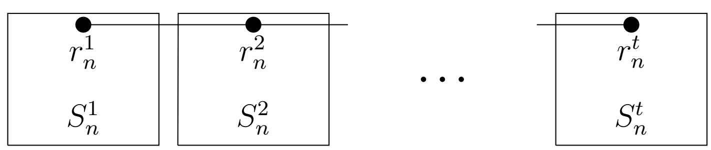

Figure 6.6: Building T from t copies of S_n.

Let $f : V(\bar{T}) \to [0, 1]$ be a uniform distribution. For each $i \in [t]$, let Z_i be the event that FF assigns r_n^i color n. We will bound the probability that at least one of the evens $Z_1, Z_2, \ldots, Z_t$ occurs. As $\bar{T}$ is a collection of disjoint trees, we can see the events $Z_1, Z_2, \ldots, Z_t$ are independent. Hence

$$1 - (1 - \Pr(Z_1))(1 - \Pr(Z_2)) \cdots (1 - \Pr(Z_t)) = 1 - (1 - \Pr(Z_1))^t .$$

From our work with S_n, we see

$$1-(1-\Pr(Z_1))^t \geq 1-(1-\varepsilon_n p_n)^t = 1-\left(1-\frac{1}{2^{2n-2}}\right)^{2^{2n-2}}.$$

As before, we recognize $(1-\frac{1}{2^{2n-2}})^{2^{2n-2}} \leq \frac{1}{e} < \frac{1}{2}$. Hence, $\Pr(A^n_{\bar{T}}) \geq 1/2$.

Suppose FF assigns color n to vertex $r^i_n \in V(\bar{T})$ in the presentation $\prec$. We claim FF uses color n when T is presented using $\prec$. If r^i_n is colored with n, we are done. Suppose r^i_n is not colored with n. We see that this must be because r^{i-1}_n or r^{i+1}_n (if they exist) are assigned color n. In either case, color n is used in coloring T. Hence $\Pr(A^n_T) \geq \Pr(A^n_{\bar{T}}) \geq 1/2$.

To complete the theorem, we examine $|T|$. Let $s_n = |S_n|$. The construction of S_n shows $s_n = (2^{2n-3})s_{n-1}$ for $n > 1$ with $s_1 = 1$. Solving this recurrence shows $s_n \leq 2^{(n-1)^2}$. The construction of T shows $|T| \leq s_n t_n = 2^{(n-1)^2} 2^{2n-2} = 2^{n^2-1}$. Solving for n, we see $n \geq \sqrt{1+\lg|T|}$. $\square$

6.3 Concluding Remarks

When coloring trees on-line the "worst cases" are thoroughly understood, but there are still many questions when it comes to the "random" setting. For a tree T, define

$$\mathrm{Avg}_{\mathrm{FF}}(T) = \frac{1}{n!}\sum_{\prec} \chi_{\mathrm{FF}}(T^{\prec})$$

where the sum ranges over all presentations $\prec$ of $V(T)$, which is the average number of colors used by FF on T. Is there a parameter that would let us find either lower or upper bounds for $\mathrm{Avg}_{\mathrm{FF}}(T)$? Or perhaps there is a specific subtree analogous to the canonical n-tree? Although trees are simple structures, $\mathrm{Avg}_{\mathrm{FF}}(T)$ offers many puzzles.

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