Matching Supply And Demand Using Dynamic Quotation Strategies

by

Ahmet Baykal Hafizoglu

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Approved July 2012 by the
Graduate Supervisory Committee:

Esma Gel, Chair
Jesus Villalobos
Pitu Mirchandani
Pinar Keskinocak
George Runger

ARIZONA STATE UNIVERSITY

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ABSTRACT

Today's competitive markets force companies to constantly engage in the complex task of managing their demand. In make-to-order manufacturing or service systems, the demand of a product is shaped by price and lead times, where high price and lead time quotes ensure profitability for supplier, but discourage the customers from placing orders. Low price and lead times, on the other hand, generally result in high demand, but do not necessarily ensure profitability. The price and lead time quotation problem considers the trade-off between offering high and low prices and lead times. The recent practices in make-to-order manufacturing companies reveal the importance of dynamic quotation strategies, under which the prices and lead time quotes flexibly change depending on the status of the system.

In this dissertation, the objective is to model a make-to-order manufacturing system and explore various aspects of dynamic quotation strategies such as the behavior of optimal price and lead time decisions, the impact of customer preferences on optimal decisions, the benefits of employing dynamic quotation in comparison to simpler quotation strategies, and the benefits of coordinating price and lead time decisions.

I first consider a manufacturer that receives demand from spot purchasers (who are quoted dynamic price and lead times), as well as from contract customers who have agreements with the manufacturer with fixed price and lead time terms. I analyze how customer preferences affect the optimal price and lead time decisions, the benefits of dynamic quotation, and the optimal mix of spot purchaser and contract customers. These analyses necessitate the computation of expected tardiness of customer orders at the moment customer enters the system. Hence, in the second part of the dissertation, I develop methodologies to compute the expected tardiness in multi-class priority queues. For the trivial single class case, a closed formulation is obtained. For the more complex multi-class case, numerical inverse Laplace transformation algorithms are developed. In the last part of the dissertation, I model a decentralized system with two components. Marketing department determines the price quotes with the objective of maximizing revenues, and manufacturing department determines the lead time quotes to minimize lateness costs. I discuss the benefits of coordinating price and lead time decisions, and develop an incentivization scheme to reduce the negative impacts of lack of coordination.

ACKNOWLEDGMENTS

Ph.D. dissertation is a long and challenging time period, where one needs to use his knowledge and abilities to focus on an intriguing research problem in his field, analyze the main shortcomings and challenges, and work patiently to produce solid academic research. I feel lucky to have worked with great professors, surrounded by invaluable friends, and supported by a great family during this tough challenge in my life.

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CHAPTER 1

Introduction

The mismatch between demand and supply is shown to be the major cause of supply chain problems by managers. Excess supply may lead to high inventory levels, underutilized personnel and capacity in manufacturing and service systems, which in general indicates "wasted resources." Excess demand, on the other hand, results in lost sales, delayed deliveries, customer dissatisfaction, and is typically considered to be "forgone profit opportunity" [Cachon and Terwiesch, 2009]. In academic literature, a traditional method to tackle this imbalance is to match the supply to the exogenous demand that is assumed to be unalterable by the companies (inventory ordering policies, location, capacity selection, scheduling models). Although these approaches ensure supply flexibility, today's consumer purchasing behaviors and competitive business environments necessitate demand management, which is in line with the statement of Miles [2009]: "Demand volatility driven by the purchasing habits of today's fickle consumers requires flexibility in demand management like never before." Consequently, supply chain research studies should ensure demand flexibility by focusing on factors to influence demand such as price and lead time. These approaches are particularly vital for the industries where supply is inflexible such as airlines, hotels and sporting events industries [Elmaghraby and Keskinocak, 2003].

The influence of prices on the demand for goods and services can be observed on many real life cases. In 2006, Nintendo increased the Wii prices short time after the product is released to keep up with the underestimated demand. Adversely, Microsoft reduced Xbox prices to reduce the high number of products in retailer inventories [Cachon and Terwiesch, 2009]. Lead time, which is equivalent to the time in between the purchase and the promised delivery time of the order, also has significant impact on demand in manufacturing systems. For example, in the beginning of 2010, polycarbonate lead times are increased from four to six weeks to 14 to 16 weeks, due to low supply and over-commitments [Victory, 2010]. These examples indicate the critical role of price and lead times in managing the demand. By employing smart pricing and lead time quotation strategies, companies may overcome the supply-demand mismatch, and in turn, increase profits, satisfy customers and fully utilize their resources.

As stated by Littleson [2007], a key capability for manufacturers is to be able to rapidly respond to what is happening at that moment. Hence, a key aspect of the pricing and lead time quotation strategies is responsiveness to the changes in the system status. This can be achieved by dynamically quoting price and lead times based on system conditions such as inventory levels and number of orders on hand. Although such strategies require frequent changes in quotes, Elmaghraby and Keskinocak [2003] and Fleischmann *et al.* [2005] indicate that the recent developments in information technologies offer opportunities to change prices and lead times easily. Dynamic quotation strategies are widely discussed in the academic literature. Federgruen and Heching [1999], for example, consider a dynamic pricing-inventory model, where the prices are quoted based on the inventory levels. Duenyas [1995], on the other hand, discuss dynamic lead time quotation in a make-to-order manufacturing environment, where the quoted lead times are functions of the number of orders on hand. Readers are referred to Elmaghraby and Keskinocak [2003] and Keskinocak and Tayur [2004] for an extensive reviews on dynamic pricing and lead time quotation literature respectively. The benefits of dynamic quotation strategies are widely discussed particularly in pricing field (see Fleischmann *et al.* [2004, 2005], and Elmaghraby and Keskinocak [2003]).

In this dissertation, I explore the potential of dynamic price and lead time quotation strategies to match supply and demand in a make-to-order manufacturing system. The manufacturer produces single type of product that is manufactured upon the order of a customer. The manufacturer receives product inquiries from customers, and quotes a price and a delivery lead time to them. If the customer accepts the quote, the manufacturer places an order to the job queue. The price and lead time quotes are determined dynamically dependent on the status of the job queue, with the objective of maximizing manufacturer profits in the long run. I address the following questions in this dissertation:

- Given a consumer population with specific preferences, how do optimal dynamic quotation strategies behave?
- How do the optimal dynamic quotation strategies change under different customer preferences?
- What are the benefits of employing dynamic quotation in comparison to simpler quotation strategies?
- What are the benefits of coordinating price and lead time quotation decisions?

In Chapter 2, I discuss pricing and lead time quotation problem of a make-to-order company that is serving two types of customers: (i) the spot purchasers, who are arriving to the system over time, and are dynamically quoted prices and lead times; and (ii) contracted customers, who are agreed a fixed price and lead time in the beginning of time horizon. In this chapter, I first analyze the impact of price and lead time sensitivity of the spot purchasers on the optimal dynamic price and lead time quotations to spot purchasers. I next focus on the potential benefits of dynamic quotes to the spot purchasers by comparing the profits obtained when spot purchasers are quoted (i) dynamic price/lead times that may change in the number of orders on hand, and (ii) fixed price/lead times that does not change over time. Finally, the determination of price and lead time terms for contract customers are discussed. I analyze the potential benefit of achieving the optimal mixture of spot and contract customers, and develop efficient algorithms to compute optimal price and lead time terms.

Determination of optimal policy in Chapter 2 necessitates the computation of expected tardiness of orders given a lead time quote, and the number of orders in the queue. In Chapter 3, I develop methodologies to compute the expected tardiness of orders given a lead time quote, priority class of the order and number of orders in the queue, in a multi-class priority $M/M/c$ queuing model. I first focus on the single-class case, and derive a closed formulation for the expected tardiness. However, expected tardiness values cannot be evaluated using a closed formulation when there are multiple priority classes. Hence, I first derive the Laplace transform of the expected tardiness, and develop three numerical inverse Laplace transformation algorithms to approximate the expected tardiness. Two of these algorithms provide upper and lower bounds on the expected tardiness. Using these bounds, the expected tardiness in multi-class priority cases is computed with a known error bound. I finally provide a recommendation scheme for the computation of expected tardiness.

In Chapter 4, I consider a make-to-order manufacturer in two different settings. In the centralized setting, price and lead time decisions are taken by a centralized agent in the company with the objective of maximizing long run profits. In the decentralized setting, on the other hand, prices are determined by marketing department with the objective of maximizing revenues, and lead times are determined by the manufacturing department with the objective of minimizing lateness penalties. I analyze the inefficiencies caused by decentralized decision making, and develop an incentivization scheme to reduce the negative impacts of the lack of coordination. In Chapter 5, I conclude and provide a brief summary of my contributions.

CHAPTER 2

Price and Lead Time Quotation for Contract and Spot Customers

1. Introduction

Contracts formalize short-term and long-term transactions in supply chains and cultivate relationships while providing suppliers with a partial view of future demand and buyers with some guarantee of capacity availability and price stability. On the other hand, non-contractual spot purchases do not imply any future business guarantee, but provide flexibility to the suppliers to sell their excess capacity or inventory and allow buyers to meet their unexpected needs.

Often times, companies prefer to buy and/or sell through a mix of long-term contractual agreements and spot purchases to attain the benefits of both channels. For example, Hewlett Packard, one of the largest memory part buyers, has developed a procurement portfolio to meet 90% of their demand using long-term contracts, whereas the remaining 10% is met from the spot market [Feng and Pang, 2010]. A similar trend towards contractual agreements was observed in the metal products sector, where the 64% of the customers had contracts, and the remaining 36% were spot purchasers [Stundza, 2007]. Committing to a contract with, for example, a pre-determined pricing scheme or volume, could sometimes hurt the supplier and/or the buyer, if the market conditions change significantly. For instance, United Airlines announced that their loss may be as much as $544 million due to their contracts with their suppliers and falling oil prices [Demerjian, 2008]. Spot market flourishes under volatile market conditions; for example the global steel manufacturer, ArcelorMittal sells 80% of their steel in the spot market, and 20% to customers with contracts [Haksoz and Kadam, 2009].

In capacitated manufacturing systems, there is a critical trade-off between committing capacity to contract customers and reserving capacity for future spot purchasers. By offering favorable contracts, companies can attract a higher number of contract customers and increase the utilization of their capacity. For example, Wencor, an aircraft parts distributor, offers price reductions and lower lead times to contract customers to ensure some level of long-term business security [Wencor.com, 2012]. However, this leaves less flexible capacity to be used for potentially more profitable spot purchasers. Hence, offering the right contract terms and spot market deals is crucial for obtaining the most desirable customer mix and achieving profitability in the long run.

In this chapter, I address a make-to-order (MTO) manufacturer's capacity reservation trade-off by considering two main aspects of contracting and spot purchasing: price and lead time. While the prices and lead times of contract customers are set at the beginning of the planning horizon, for spot purchasers I propose the use of a Dynamic Price and Lead Time Quotation policy (DPLQ) dependent on the system conditions (i.e., congestion). I assume that all orders from contract customers must be accepted and prioritized during processing, whereas the company has the option of rejecting orders from spot purchasers, by quoting a very high price and lead time. I analyze the behavior of the optimal DPLQ for spot purchasers with varying degrees of price and lead time sensitivity, and develop insights on how customer preferences impact optimal policies. Noting the potentially undesirable effects of frequent price and/or lead time changes DPLQ policies might require, I compare the profits obtained by DPLQ policies with those obtained by fixed price/lead time quotation policies, and provide recommendations for managerial decisions. I analyze the optimal contract decisions, potential benefits of offering optimal contract terms in comparison to simple strategies and examine the impact of system parameters on the optimal customer mix.

The chapter investigates the following topics:

- the structure of the optimal dynamic price/lead time quotation policies for spot purchasers and how these policies change depending on the price and lead time sensitivity of spot purchasers (Section 3.1),
- the performance difference between dynamic versus fixed price/lead time quotation policies in the presence of spot and contract customers (Section 3.2),
- the optimal mix of spot purchasers and contract customers, and the benefits of attaining the optimal mix (Section 4).

This chapter has three main contributions that are summarized as follows. Previous studies on price and/or lead time quotation investigate the impact of customer behaviors (e.g., price/lead time sensitivity) on the optimal decisions using relatively simple models where price and lead time quotes are not dynamic, and demand functions are linear [Easton and Moodie, 1999, Ray and Jewkes, 2004, Wu *et al.*, 2011]. In this chapter, I generalize their results using a more general demand function under a dynamic quotation environment, which is indicated as a potential research direction in those papers. My theoretical analyses indicate that the optimal spot purchaser quotes heavily depend on price and lead time sensitivity of the spot purchasers. When spot purchasers are highly sensitive to price changes, it is optimal to quote zero lead times, and modify the price quotes according to the profitability of the spot purchaser. If they are highly sensitive to price changes, then it is often optimal to adjust lead time quotes, while keeping the price quotes at the lowest level.

Secondly, I analyze the benefits of DPLQ in comparison to fixed price and lead time quotation (FPLQ) with no flexibility to change price and lead time quotes. This has not been addressed in the literature, to the best of my knowledge. I analyze the profit improvements offered by DPLQ, and provide recommendations for manufacturers using an extensive numerical analysis.

Finally, I consider the impact of the contract-spot customer mix using a multi-class queuing model, which is a research problem addressed by recent studies of Haksoz and Kadam [2009] and Savasaneril *et al.* [2010]. I analyze the optimal contract terms, and develop fast algorithms that compute optimal contract terms, and provide desirable results in a relatively short time. I also analyze the profit improvements offered by the optimal customer mix, and provide recommendations.

The chapter is organized as follows. I review the literature in Section 2. In Section 3, I model and analyze the dynamic price and lead time quotation problem for spot purchasers. In Section 4, I discuss the optimal contract customer/spot purchaser mix, and the benefits of achieving the optimal mix. I summarize my major findings and conclude in Section 5.

2. Literature review

There has been a significant amount of work discussing dynamic pricing and lead time quotation separately. The readers are referred to Keskinocak and Tayur

[2004] and Elmaghraby and Keskinocak [2003] for reviews in due date management and dynamic pricing literature, respectively. The literature on joint dynamic price and lead time quotation, however, is relatively scarce (see Celik and Maglaras [2008] and Feng *et al.* [2011]).

Recent work on dynamic pricing in MTO systems focuses on admission control models with the option of accepting or rejecting price sensitive customers [Aktaran-Kalayci and Ayhan, 2009, Cil *et al.*, 2009, Yoon and Lewis, 2004]. Most of the studies in this stream employ Markov Decision Process (MDP) models, and focus on developing structural properties of optimal policies, such as the monotonicity of price quotes. Yoon and Lewis [2004] study the pricing problem in non-stationary queues considering both total discounted and average reward problems for an infinite horizon. They show that the optimal price and admission probability of an order is increasing and decreasing, respectively, in the number of orders in the system. Aktaran-Kalayci and Ayhan [2009] extend the monotonicity results to $M/M/s/K$ queues. Cil *et al.* [2009] present a general framework of queuing admission control methods and structural properties.

As stated in Savasaneril *et al.* [2010], a remarkable portion of the due date management literature focuses on sequencing and due date setting decisions assuming all arriving orders must be admitted to the system [Bertrand, 1983, Bookbinder and Noor, 1985, Spearman and Zhang, 1999, Wein, 1991]. Among the studies employing admission control with lead time quotes, Charnsirisakskul *et al.* [2004, 2006], Keskinocak *et al.* [2001], and Kapuscinski and Tayur [2007] assume deterministic processing times, whereas Ata [2006] and Ata and Olsen [2009] employ heavy-traffic approximations to overcome the complexity of MDP formulations. Similar to my setting, Keskinocak *et al.* [2001] and Kapuscinski and Tayur [2007] study settings with two customer classes, and Carr and Lovejoy [2000] consider n prioritized customer classes. Keskinocak *et al.* [2001] considers an "urgent" customer class, whose orders are processed immediately and bring more revenue than regular customer orders. In the setting of Kapuscinski and Tayur [2007], one customer class is prioritized and incurs higher delay penalties than the second customer class. In Carr and Lovejoy [2000], the demand of higher priority customer classes are served earlier, and the demand of lower priority classes may not be fulfilled. Duenyas and Hopp [1995] address a dynamic lead time quotation problem by modeling an MTO system as an $M/M/1$ queue using a MDP model. The authors show the optimality of the earliest due date scheduling method. Using a similar MDP model, Duenyas [1995] shows the monotonocity of lead time quotes in the number of orders in the system. Although Duenyas [1995] extends the work into multiple customer classes, all orders are assumed to be sequenced in first come first serve (FCFS) order, which the author indicates to be unrealistic for some real life settings [Duenyas, 1995]. Recently, Savasaneril *et al.* [2010] extend the dynamic lead time quotation problem to hybrid make-to-order/make-to-stock environments.

Joint price and lead time decisions are considered by Celik and Maglaras [2008], Easton and Moodie [1999], ElHafsi [2000], Hua *et al.* [2010], Liu *et al.* [2007], Palaka *et al.* [1998], Pekgun *et al.* [2008], Plambeck [2004], Ray and Jewkes [2004], Wu *et al.* [2011], Xiao *et al.* [2010], and Feng *et al.* [2011]. Liu *et al.* [2007] and Pekgun *et al.* [2008] consider fixed pricing and lead time decisions in decentralized supply chains.

Palaka *et al.* [1998] and Easton and Moodie [1999] consider MTO environments, where demand is a function of price and lead time, and optimize expected profits by setting price and lead times. Celik and Maglaras [2008] study the DPLQ under heavy traffic assumptions. The authors also discuss the effects of lead time flexibility, expediting (under a high cost), and dynamic pricing. Plambeck [2004] considers two customer classes, with different price and lead time sensitivities. Price decisions are taken at the beginning of the planning horizon, and lead times are quoted dynamically to the arriving customers. Hua *et al.* [2010], Liu *et al.* [2007], Pekgun *et al.* [2008] and Xiao *et al.* [2010] discuss settings where price and lead times are determined by separate entities within the company.

Feng *et al.* [2011] is the study that is closest to ours in spirit. The authors address the joint dynamic pricing and lead time quotation problem in a MTO system using an $GI/M/1$ queuing model and an MDP formulation. They define an optimal policy structure including a threshold and reward maximizing lead time quote, and show that the reward maximizing lead time is optimal under particular conditions. In contrast to Feng *et al.* [2011], I consider two customer classes with different contractual rights to processing prioritization, discuss the impact of customer preferences on the optimal policy, and quantify the benefits of DPLQ versus fixed pice and lead time policies, as well as optimizing the contract-spot customer mix.

3. Dynamic Quotation Model for Spot Purchasers

I consider an MTO system with two classes of customers, each arriving according to a Poisson process with rate λ_k, where $k = c$ and $k = s$ denote contract customers and spot purchasers, respectively. The manufacturer produces a single type of product. Hence, the service times for both customer classes are independent and identically distributed exponential random variables with rate μ. Hence, the system is modeled as an $M/M/1$ queue with two customer classes. At the beginning of the time horizon, the company sets the unit price ($p_c \geq 0$) and the lead time ($l_c \geq 0$) for the orders of contract customers. Arriving contract customer orders are always accepted and fulfilled. When a spot purchaser order arrives, a lead time ($l_s \geq 0$) and a price ($p_s \geq 0$) are quoted dynamically, based on the system state. The spot purchaser accepts the quote with probability given by the $f^S(p_s, l_s)$, which I refer to as the acceptance probability function.

Assumption 1. *(i)* $f^S(p_s, l_s)$ *is continuous, non-increasing, twice differentiable, concave in* p_s *and* l_s*, and* $\partial^2 f^S(p_s, l_s)/\partial p_s \partial l_s \leq 0$*.*

(ii) There exist a nonnegative lower bound on price, p_{sMin}*, and the lower bound on lead time is zero, such that* $f^S(p_s, l_s)$ *does not change for prices and lead times below these lower bounds. That is,* $f^S(p, l) = f^S(p_{sMin}, l)$ *for* $p < p_{sMin}$*. Furthermore,* $f^S(p_{sMin}, 0) = 1$*. Note that,* p_{sMin} *represents the minimum of the willingness-to-pay of the individuals in the spot purchaser population, and is referred to as the "accept all price" henceforth.*

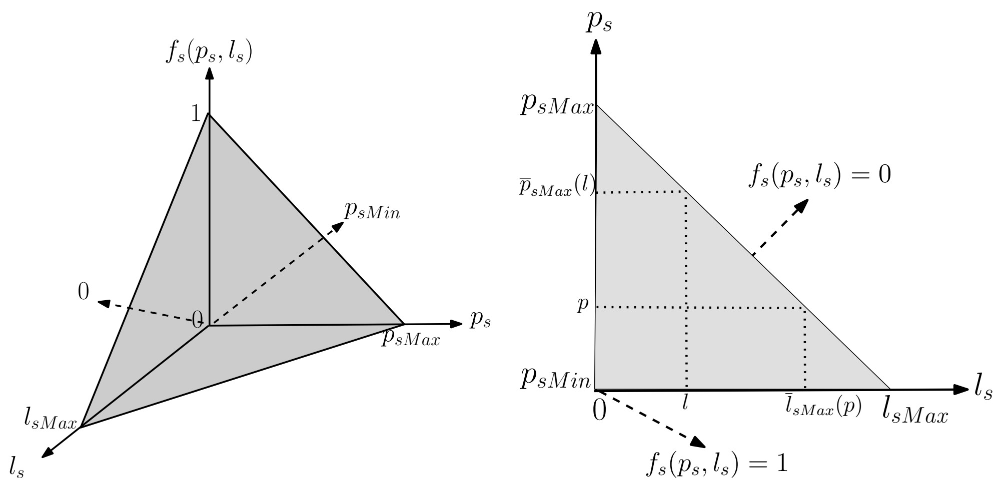
(iii) Given a lead time quote l_s*, there exists an upper bound on price, denoted by* $\overline{p}_{sMax}(l_s)$*, such that any price quote above this upper bound is definitely rejected by spot purchasers. That is,* $\overline{p}_{sMax}(l_s) = \min\{p \in \Re^+ : f^S(p, l_s) = 0\}$*.* $\overline{p}_{sMax}(l_s)$ *is nonincreasing in* l_s*, and its highest value is denoted as* p_{sMax}*, i.e.,* $p_{sMax} = \overline{p}_{sMax}(0)$*.* p_{sMax} *is referred to as the "reject all price" henceforth.*

(iv) Given a price quote p_s, there exists an upper bound on lead time, denoted by $\bar{l}_{sMax}(p_s)$, such that any price quote above this upper bound is definitely rejected. That is, $\bar{l}_{sMax}(p_s) = \min\{l \in \Re^+ : f^S(p_s, l) = 0\}$. $\bar{l}_{sMax}(p_s)$ is non-increasing in p_s, and its highest values is denoted as l_{sMax}, i.e., $l_{sMax} = \bar{l}_{sMax}(p_{sMin})$.

Assumptions 1(ii), 1(iii) and 1(iv) restrict $f^S(p_s, l_s)$ into the finite interval $[0, 1]$. Assumption 1(i) defines the general properties of the acceptance probability function, such as monotonocity and concavity in price and lead time, which is well accepted in the literature. The assumption that $\partial^2 f^S(p_s, l_s)/\partial p_s \partial l_s \leq 0$ is more restrictive than the monotonocity and concavity conditions, but fortunately this condition holds true for a wide range of demand functions. For example, consider an additive form, $f^S(p_s, l_s) = 1 - g_p(p_s) - g_d(l_s)$, and a multiplicative form, $f^S(p_s, l_s) = 1 - g_p(p_s)^n g_l(l_s)^m$, where $g_p(\cdot)$ and $g_l(\cdot)$ are nondecreasing and nonnegative functions of price and lead time, and $m \geq 1$, $n \geq 1$. Since $\partial g_p(p_s)/\partial p_s \geq 0$, and $\partial g_l(l_s)/\partial l_s \geq 0$, the condition $\partial^2 f^S(p_s, l_s)/\partial p_s \partial l_s \leq 0$ holds true for both multiplicative and additive forms.

An $f^S(p_s, l_s)$ that is linear in both p_s and l_s is illustrated in Figure 1(a) in 3-dimensions, and in Figure 1(b) in 2-dimensions. In the remainder of the chapter, I use the 2-dimensional representation to illustrate the various cases.

The objective is to maximize the long-run average expected profit per unit time, where profit from a customer is equal to the revenue minus late delivery penalties. Using the properties of $f^S(p_s, l_s)$, I first restrict the optimal price and lead times into a finite region in Observation 1.



(a) 3-dimensional representation (b) 2-dimensional representation

Fig. 1. Illustration of the acceptance probability function

Observation 1. *There exists at least one optimal solution in* θ^S*, where* θ^S *is defined as in Equation (2.1).*

$$\theta^S = \{(p, l) \in \Re^2 : p_{sMin} \leq p \leq \overline{p}_{sMax}(l), 0 \leq l \leq \bar{l}_{sMax}(p)\}. \tag{2.1}$$

Proof. Decreasing the price below p_{sMin} does not change $f^S(\cdot, \cdot)$, but decreases the expected revenue. Hence, the optimal price to quote to a spot purchaser cannot be less than p_{sMin}. Furthermore, given a lead time quote l_s, increasing prices above $\overline{p}_{sMax}(l_s)$ does not change the expected profit to be obtained from the spot purchasers. Similarly, given a price quote p_s, increasing the lead times above $\bar{l}_{sMax}(p_s)$ does not the change expected profit. Consequently, if any (p_s^*, l_s^*) satisfying $p_s^* > \overline{p}_{sMax}(l_s^*)$ or $l_s^* > \bar{l}_{sMax}(p_s^*)$ is optimal, then $(\overline{p}_{sMax}(l_s), l_s)$ is optimal as well. $\square$

In Figure 1(b), θ^S is given by the shaded area.

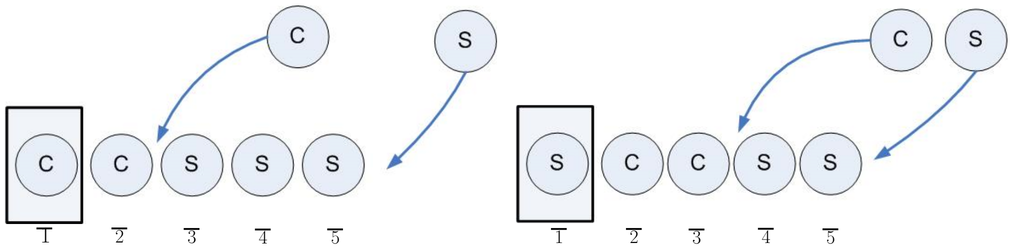
The state of the system at time t is defined by the vector $X(t) = (I(t), J(t), K(t)) \in \mathcal{S}$, where $I(t)$ and $J(t)$ denote the number of spot purchasers and contract customers in the system, respectively, and $K(t)$ denotes the class of the customer order currently being served. I assume a finite buffer of size N (due to the need to limit the state space for numerical analysis), i.e., none of the incoming customer orders are accepted when there are N orders in the system. Note that this implies the possible rejection of a contract customer order. I assume a sufficiently high buffer size in the numerical analysis, such that the probability of customer rejection due to a full buffer is negligibly small.

$\{X(t), t \in T\}$ defines a stochastic process and can be modeled as a continuous time Markov chain. I drop t from the notation, and simply use (i, j, k) to denote the state. The state space $\mathcal{S}$ can be denoted as $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \{(0, 0, 0)\}$, where $\mathcal{S}_1 = \{(i, j, k) : i + j \leq N, i \geq 1, j \geq 1, k \in \{c, s\}\}$, $\mathcal{S}_2 = \{(i, 0, s) : 1 \leq i \leq N\}$, $\mathcal{S}_3 = \{(0, j, c) : 1 \leq j \leq N\}$, and $|\mathcal{S}| = N^2 + N + 1$. Note that $\mathcal{S}_2$ and $\mathcal{S}_3$ denote the states where there are no contract and spot purchaser orders in the system, respectively. The state where there are no orders in the system is denoted by $(0, 0, 0)$.

I assume that contract customer orders are always prioritized and processed ahead of the spot purchaser orders, and preemption is not allowed. Orders within the same class follow the FCFS sequence. Hence, at any time $t \geq 0$, when the state is (i, j, k) with $i + j > 0$,

(i) an incoming spot purchaser order is placed behind $i + j$ orders in the system, to the position $i + j + 1$,

(ii) an incoming contract customer order is placed to position $j+1$, if there is a contract customer order in process; otherwise, it is placed to position $j+2$ (see Figure 2).



(a) Case (i). $i=3, j=2, k=C$ (b) Case (ii). $i=3, j=2, k=S$

Fig. 2. Illustration of the sequencing policy, where the numbers $\overline{1}, \overline{2}, \ldots$ denote the position of the order

Due date management studies, which use MDP formulations similar to this chapter, either consider a single class of customers with FCFS ordering [Duenyas, 1995, Duenyas and Hopp, 1995, Feng *et al.*, 2011, Savasaneril *et al.*, 2010], or sequencing of orders in the presence of single and multiple customer classes [Duenyas, 1995, Duenyas and Hopp, 1995]. To the best of my knowledge, this is the first study considering prioritization of a particular customer class in the dynamic lead time quotation literature.

Once the quote is accepted by the spot purchaser, the order joins the queue according to the above explained protocol. If the service completion time of an order is later than the quoted due date, a late delivery penalty, which increases linearly with the tardiness duration at rate τ^k, $k \in \{C, S\}$, is incurred. Let $L^C_{j,k}(l_c)$ denote the expected late delivery penalty incurred for the order of a contract customer who was promised a lead time of l_c and who joins the queue when there are j contract orders already in the system and a class k order is in processing. I have

$$L^C_{j,c}(l_c) = \tau^C \int_{l_c}^{\infty} (t-l_c) f^C_{j+1}(t)dt, \quad \text{and} \quad L^C_{j,s}(l_c) = \tau^C \int_{l_c}^{\infty} (t-l_c) f^C_{j+2}(t)dt, \tag{2.2}$$

where $f^C_j(t)$ denotes the probability density function (pdf) of the time-in-system (TIS) of the j^{th} job in the system, i.e., pdf of Gamma distribution with parameters j and μ. When the state is (i, j, k), the arriving spot purchaser order waits for at least $i + j + 1$ order(s) for service completion. TIS for a spot purchaser possibly increases due to contract customer orders arriving before this order starts processing. The expected late delivery penalty of a spot purchaser order that arrives when the system state is $(i, j, \cdot)$ can be expressed as

$$L^S_{i+j}(l_s) = \tau^S \int_{l_s}^{\infty} (t - l_s) f^S_{i+j+1}(t)dt, \tag{2.3}$$

for a lead time quote of l_s. In Equation (2.3), $f^S_i(t)$ denotes the pdf of the TIS of a spot purchaser order that is placed in the i^{th} position upon arrival.

The problem is formulated as an infinite-horizon MDP with the long run average expected profit per unit time criteria. Any time a state transition happens, the company determines the quote $(p_s, l_s) \in \theta^S$ for the next spot purchaser. Although this indicates that the decision epochs are all event occurrences (i.e., spot purchaser arrivals, contract customer arrivals, and service completion), in practice the quote is offered only when a spot purchaser arrives. The continuous time model is converted to an equivalent discrete time model using a uniformization rate of $\nu = \lambda^S + \lambda^C + \mu$. The Bellman's equation for the problem, referred to as `Dyna`, is presented in Equations (2.4), (2.5) and (2.6). The expression v^*_{Dyna} denotes the optimal expected average profit per unit time, and $h^*_{i,j,k}$ is the relative value of starting in state (i,j,k) under the optimal policy,

$$\texttt{Dyna}: \qquad \frac{v^*_{Dyna}}{\nu} + h^*_{i,j,k} = \max_{(p_s,l_s)\in\theta^S} \psi_{i,j,k}(p_s,l_s), \tag{2.4}$$

where

$$\psi_{i,j,k}(p_s,l_s) = \begin{cases} \frac{\lambda^S+\lambda^C}{\nu} h^*_{i,j,k} + \frac{\mu}{\nu}\overline{h}^*_{i,j,k} & \text{for } i+j=N, \\ \frac{\lambda^S}{\nu} f^S(p_s,l_s)(p_s - L^S_{i+j}(l_s) + h^*_{i+1,j,k}) & \\ \quad + \frac{\lambda^S}{\nu}(1 - f^S(p_s,l_s))h^*_{i,j,k} & \\ \quad + \frac{\lambda^C}{\nu}(p_c - L^C_{j,k}(l_c) + h^*_{i,j+1,k}) + \frac{\mu}{\nu}\overline{h}^*_{i,j,k} & \text{for } (i,j,k) \neq (0,0,0) \\ & \text{and, } i+j<N \\ \frac{\lambda^S}{\nu} f^S(p_s,l_s)(p_s - L^S_0(l_s) + h^*_{1,0,s}) & \\ \quad + \frac{\lambda^S}{\nu}(1 - f^S(p_s,l_s))h^*_{0,0,0} & \\ \quad + \frac{\lambda^C}{\nu}(p_c - L^c_{0,c}(l_c) + h^*_{0,1,c}) + \frac{\mu}{\nu}h^*_{0,0,0} & \text{for } (i,j,k) = (0,0,0). \end{cases} \tag{2.5}$$

and $\overline{h}^*_{i,j,k}$ is defined as

$$\overline{h}^*_{i,j,k} = \begin{cases} h^*_{i,j-1,c} & \text{if } i \geq 0, j \geq 2, k = c, \\ h^*_{i,0,s} & \text{if } i \geq 1, j = 1, k = c, \\ h^*_{i-1,j,c} & \text{if } i \geq 1, j \geq 1, k = s, \\ h^*_{i-1,0,s} & \text{if } i \geq 2, j = 0, k = s, \\ h^*_{0,0,0} & \text{if } (i,j,k) \in \{(1,0,s),(0,1,c)\}. \end{cases} \tag{2.6}$$

At state (i, j, k), the spot purchaser accepts the quote with probability $f^S(p_s, l_s)$, bringing an expected profit of $p_s - L^S_{i+j}(l_s)$, and changing the state to $(i+1, j, k)$. The quote is rejected with probability $1 - f^S(p_s, l_s)$ and no change occurs. When a contract customer arrives to the system, the order is immediately put into the system, transition occurs to state $(i, j+1, k)$ and an expected profit of $p_c - L^C_{j,k}(l_c)$ is obtained. When the processing of an order is completed, the system state is decreased by one depending on the class of the order currently being processed.

3.1. *Characterization of Optimal Policy*

In this section, I characterize the optimal solution to `Dyna`. The price and lead time quotes to spot purchasers are denoted as p and l to simplify notation. The optimal solution at state (i, j, k) is denoted by the $(p^*_{i,j,k}, l^*_{i,j,k})$ pair. By rearranging the terms in Equation (2.5), I observe that $(p^*_{i,j,k}, l^*_{i,j,k})$ for $(i, j, k) \in \mathcal{S}$ maximizes

$$\gamma_{i,j,k}(p, l) = f^S(p, l)(\Pi_{i,j}(p, l) - \Delta h^*_{i,j,k}) \tag{2.7}$$

where

$$\Delta h^*_{i,j,k} = \begin{cases} h^*_{i,j,k} - h^*_{i+1,j,k} & \text{for } (i, j, k) \neq (0, 0, 0), \text{ and } i + j < N \\ h^*_{0,0,0} - h^*_{1,0,S} & \text{for } (i, j, k) = (0, 0, 0), \text{ and,} \end{cases} \tag{2.8}$$

$$\Pi_{i,j}(p, l) = p - L^S_{i+j}(l). \tag{2.9}$$

$\Delta h^*_{i,j,k}$ represents the monetary burden brought on by one additional spot purchaser order if the quote is accepted at state (i, j, k). $\Pi_{i,j}(p, l)$, on the other hand, is the immediate profit earned from a spot purchaser if the quote (p, l) is accepted and the state is (i, j, k). Hence, $\Pi_{i,j}(p, l) - \Delta h^*_{i,j,k}$, which is the second term in the right hand side of Equation (2.7), denotes the long-run profit earned from a spot purchaser order (denoted by LPES) at state (i, j, k), if (p, l) is accepted.

In Theorem 1, I show that relative price/lead time sensitivity of the spot purchasers has a significant impact on the optimal solutions. I first prove two properties of $L^S_{i+j}(l)$ in Lemma 1, which are used in my theoretical analysis.

Lemma 1. *$L^S_{i+j}(l)$ is convex and non-increasing in $l \geq 0$.*

Proof. The proofs of all propositions, lemmas and theorems of Chapter 2 are included in Appendix A-1. □

I define T_1 and T_2, which are two threshold values used in Theorem 1 as follows.

$$T_1 = -\frac{\tau^S}{\partial f^S(p_{sMin}, 0)/\partial l}, \quad \text{and} \quad T_2 = -\frac{1}{\partial f^S(p_{sMin}, 0)/\partial p}. \tag{2.10}$$

Theorem 1. *The optimal policy for* `Dyna` *has the following structure.*

Case 1: *If* $\max\{T_1, T_2\} \leq \Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{i,j,k}$, *then* $(p^*_{i,j,k}, l^*_{i,j,k}) = (p_{sMin}, 0)$.

Case 2.1: *If* $T_1 < \Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{i,j,k} \leq T_2$, *then* $l^*_{i,j,k} = 0$ *and* $p^*_{i,j,k} > p_{sMin}$.

Case 2.2: *If* $T_2 < \Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{i,j,k} \leq T_1$, *then* $p^*_{i,j,k} = p_{sMin}$ *and* $l^*_{i,j,k} > 0$.

Case 3: *If* $\Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{i,j,k} < \min\{T_1, T_2\}$, *and there exists a* (p, l) *satisfying* $\Pi_{i,j}(p, l) - \Delta h^*_{i,j,k} > 0$, *then* $(p^*_{i,j,k}, l^*_{i,j,k})$ *satisfies* $0 < f^S(p^*_{i,j,k}, l^*_{i,j,k}) < 1$.

Case 4: *If* $\Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{i,j,k} < \min\{T_1, T_2\}$, *and* $\Pi_{i,j}(p, l) - \Delta h^*_{i,j,k} <= 0$, *for all* (p, l), *then* $(p^*_{i,j,k}, l^*_{i,j,k})$ *satisfies* $f^S(p^*_{i,j,k}, l^*_{i,j,k}) = 0$.

At any particular state (i, j, k), when Case 1 holds (i.e., the LPES obtained by quote $(p_{sMin}, 0)$ is greater than both T_1 and T_2), the manufacturer offers $(p_{sMin}, 0)$ to attract arriving spot purchasers. As the LPES obtained by quote $(p_{sMin}, 0)$

decreases, it first falls below either T_1 or T_2. If it is less than T_2, Case 2.1 holds, and the manufacturer quotes prices higher than p_{sMin} while keeping the lead time quote at zero. Otherwise, Case 2.2 holds, and $l^*_{i,j,k}$ increases as $p^*_{i,j,k}$ stays constant at p_{sMin}. If LPES obtained by quote $(p_{sMin}, 0)$ is less than both T_1 and T_2 but greater than 0 for any (p, l), it is optimal to quote a price and lead time pair that yields a positive acceptance probability by the spot purchaser. If obtained LPES is less than 0 for all (p, l), then the manufacturer rejects the spot purchaser by quoting sufficiently high price and lead time quotes. I next provide an example to illustrate Theorem 1.

Example 1. *Consider the case where* $\lambda^S = 0.75$, $\lambda^C = 0$, $\mu = 1$, $p_{sMin} = 60$, $p_{sMax} = 80$, $l_{sMax} = 30$, $\tau^S = \tau^C = 1$, *and* $f^S(p, l) = 1 - \frac{p-60}{20} - \frac{l}{30}$, *for* $(p, l) \in \theta^S$. *In Figure 3, I plot the change of* $\Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{0,j,0}$ *as* j *increases.*

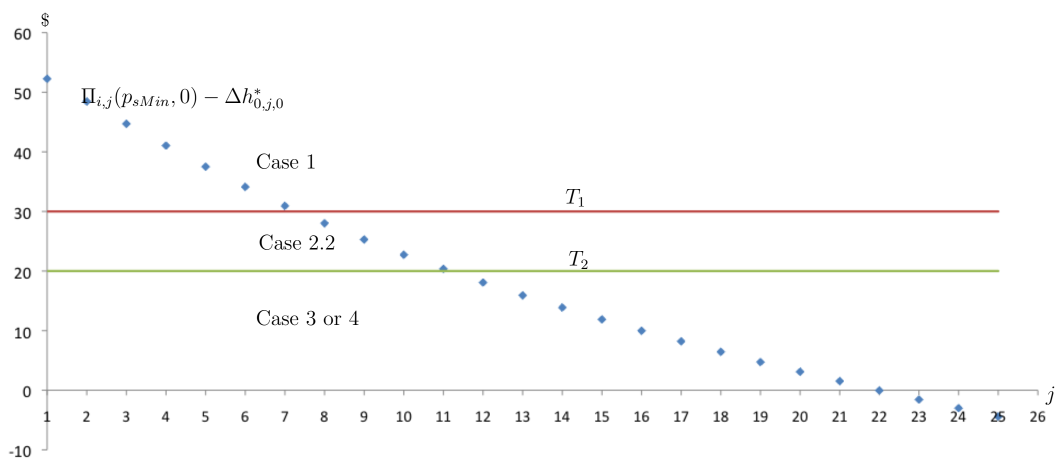


Fig. 3. Illustration of Theorem 1

Using $f^S(p, l)$, *one can simply find that* $T_1 = 30$ *and* $T_2 = 20$. *From Figure 3, I observe that* $(p^*_{0,j,0}, l^*_{0,j,0}) = (p_{sMin}, 0)$, *for* $j = \{1, 2, 3, 4, 5, 6, 7\}$, *which are the* *states with LPES above the T_1 line. Because $T_1 > T_2$, the region between T_1 and T_2 lines correspond to Case 2.2. Hence, I have $p^*_{i,j,k} = p_{sMin}$ and $l^*_{i,j,k} > 0$ for $j = \{8, 9, 10, 11\}$. For the states with $\Pi_{i,j}(p_{sMin}, 0) - \Delta h^*_{0,j,0} < 20$, Case 3 or 4 holds.*

In Corollary 1, I develop a simple rule to compare T_1 and T_2, and derive intuition about the structure of the optimal policy. Let

$$\rho^S(p,l) = \frac{\partial f^S(p,l)/\partial l}{\partial f^S(p,l)/\partial p}. \tag{2.11}$$

Corollary 1. *$T_1 < T_2$ if and only if $\rho^S(p_{sMin}, 0) > \tau^S$.*

The parameter $\rho^S(p,l)$, which represents the relative price/lead time sensitivity of the spot purchaser, gives the relative importance of due date and price changes for the spot purchaser evaluated at price-lead time quote of (p,l), where a high (low) $\rho^S(\cdot,\cdot)$ value indicates that a spot purchaser is more sensitive to lead time (price) changes. If $\rho^S(p_{sMin}, 0) \leq \tau^S$, then the spot purchaser is significantly more sensitive to price changes than lead time changes, and hence, increasing the lead time quote rather than increasing the price is preferable to mitigate the due date violation risk without discouraging the spot purchaser. If $\rho^S(p_{sMin}, 0) > \tau^S$, on the other hand, spot purchasers are less sensitive to price changes and increasing the price quote is preferable for the manufacturer to gain more revenue from spot orders while keeping the congestion under control. While Easton and Moodie [1999], Palaka *et al.* [1998], Ray and Jewkes [2004] and Wu *et al.* [2011] demonstrate similar results using non-dynamic models and linear demand functions, Theorem 1 and Corollary 1 extend their results into a more general concave demand form using a dynamic quotation model.

One observes that T_1 and T_2 denote the minimum LPES obtained by the quote $(p_{sMin}, 0)$, such that it is optimal to offer a zero lead time and accept all price, respectively for any particular state (i, j, k). If manufacturers could assess the LPES of the incoming order accurately without solving the problem explicitly, they could simply determine if the incoming order should be quoted the accept all price and/or zero lead times by evaluating T_1 and T_2. When $f^S(p, l)$ is linear in price and lead time, and additive (i.e., $\frac{\partial f^S(p,l)}{\partial p \partial l} = 0$), T_1 and T_2 can be simply evaluated as shown in Corollary 2.

Corollary 2. *$T_1 = l_{sMax}\tau^S$ and $T_2 = p_{sMax} - p_{sMin}$, when $f^S(p, l)$ is additive and linear in lead time and price, respectively.*

For example, when $f^S(p, l)$ is linear in price, T_2 is measured by the difference between the reject all and accept all prices. If an incoming spot purchaser order brings more long-run profits than this difference, then the spot purchaser should be quoted the price p_{sMin}.

Theorem 1 presents cases where $p^*_{i,j,k} = p_{sMin}$ and/or $l^*_{i,j,k} = 0$, using the values of $\Delta h^*_{i,j,k}$, which requires the optimality equations in (2.4) to be solved. In Theorem 2, I develop a simple condition ensuring that $p^*_{i,j,k} = p_{sMin}$ and/or $l^*_{i,j,k} = 0$ that does not require the problem to be solved explicitly.

Theorem 2. $(p^*_{i,j,k}, l^*_{i,j,k}) \in \theta^S_{i,j}$, *where* *(i) If $\rho^S(p,l) > \tau^S$ for all $(p,l) \in \theta^S$, then $l^*_{i,j,k} = 0$, i.e.,*

$$\theta^S_{i,j} = \{(p,0) \in \Re^2 : p_{sMin} \leq p \leq p_{sMax}\}. \tag{2.12}$$

(ii) If $\rho^S(p,l) < \tau^S \overline{F}^S_{i+j+1}(l_{sMax})$ for all $(p,l) \in \theta^S$, then $p^*_{i,j,k} = p_{sMin}$, i.e.,

$$\theta^S_{i,j} = \{(p_{sMin},l) \in \Re^2 : 0 \leq l \leq l_{sMax}\}. \tag{2.13}$$

(iii) If $\tau^S \overline{F}^S_{i+j+1}(l_{sMax}) \leq \rho^S(p,l) \leq \tau^S$ for some $(p,l) \in \theta^S$, then

$$\theta^S_{i,j} = \{(p_{sMin},l) \in \Re^2 : 0 \leq l \leq l_{sMax}\} \cup \{(p,0) \in \Re^2 : p_{sMin} \leq p \leq p_{sMax}\} \tag{2.14}$$

$$\cup \left\{(p,l) \in \theta^S : \frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)\right\}. \tag{2.15}$$

where $\overline{F}^S_i(l) = 1 - \int_0^l f^S_i(t)dt$, which is the probability that the i'th spot purchaser order is not met on time if the quoted lead time is l.

Parts (i) (Part (ii)) of Theorem 2 indicates that when the spot purchasers are sufficiently lead time (price) sensitive, then it is optimal to always quote $l^*_{i,j,k} = 0$ ($p^*_{i,j,k} = p_{sMin}$). In these cases, the manufacturer should only focus on determining optimal prices and lead times, respectively, and hence, can avoid the burden of joint price/lead time optimization.

When the spot purchasers are not highly price or lead time sensitive, from Part (iii) of Theorem 2, $p^*_{i,j,k} = p_{sMin}$ or $l^*_{i,j,k} = 0$ may not hold. When $p^*_{i,j,k} > p_{sMin}$ and $l^*_{i,j,k} > 0$, the optimal solution satisfies $1 - \rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = 1 - \overline{F}^S_{i+j+1}(l^*_{i,j,k}) = F^S_{i+j+1}(l^*_{i,j,k})$, where $F^S_{i+j+1}(l^*_{i,j,k})$ denotes the probability that the spot purchaser order is met on time when an optimal lead time is quoted. $1 - \rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S$ is analogous to the famous news-vendor critical ratio, which determines the proportion of demand to be met. When $f^S(p,l)$ is additive, I obtain a simpler optimal policy structure for Part (iii) of Theorem 2 in Proposition 1.

Proposition 1. *If $f^S(p,l)$ is additive, and $\tau^S \overline{F}^S_{i+j+1}(l_{sMax}) \leq \rho^S(p,l) \leq \tau^S$ for some $(p,l) \in \theta^S$, then the following hold:*

(i) If $\partial f^S(p_{sMin},0)/\partial l = \partial f^S(p_{sMin},0)/\partial p = 0$, then $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$,

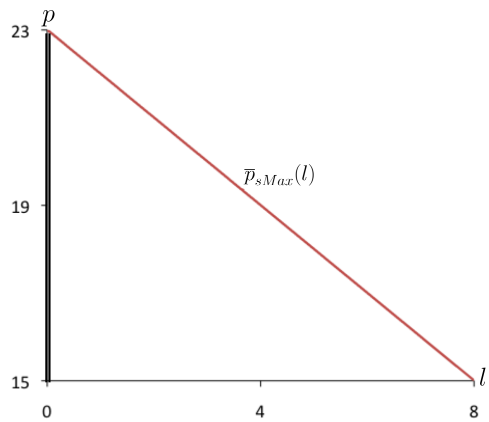
(ii) If $\partial f^S(p_{sMin},0)/\partial l < \tau^S \partial f^S(p_{sMin},0)/\partial p$, then $l^*_{i,j,k} = 0$ or $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$,

(iii) If $\partial f^S(p_{sMin},0)/\partial l > \tau^S \partial f^S(p_{sMin},0)/\partial p$, then $p^*_{i,j,k} = p_{sMin}$ or $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$.

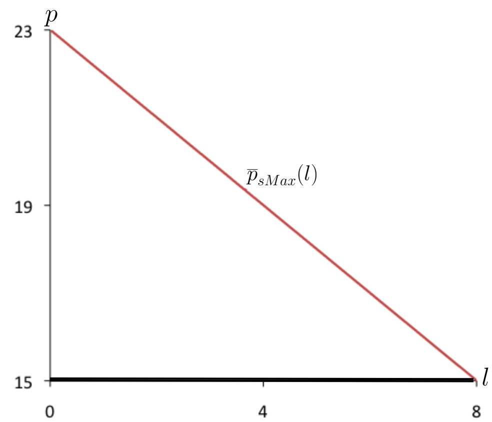
I illustrate my findings from Theorem 2 and Proposition 1, in Examples 2 and 3, respectively.

Example 2. *Consider the following setting: $\lambda^C = 0.45$, $\mu = 1$, $i + j = 3$, $p_{sMin} = 15$, $p_{sMax} = 23$, $l_{sMax} = 8$, and $f^S(p,l) = 1 - \frac{p-15}{8} - \frac{l}{8}$, for $(p,l) \in \theta^S$. Using simple algebra, I have $\rho^S(p,l) = 1$ for all $(p,l) \in \theta^S$, and $\overline{F}^S_{i+j+1}(l_{sMax}) = 0.315$. I illustrate $\theta^S_{i,j}$, for two cases: (i) $\tau^S = 0.5$, $\tau^S \overline{F}^S_{i+j+1}(l_{sMax}) = 0.16$, and (ii) $\tau^S = 5$ and $\tau^S \overline{F}^S_{i+j+1}(l_{sMax}) = 1.57$ in Figures 4(a) and 4(b), respectively.*

In Case (i), $\rho^S(p,l) > \tau^S$ holds for all $(p,l) \in \theta^S$. Thus $l^*_{i,j,k} = 0$ follows from Part (i) of Theorem 2. In Case (ii), since $\rho^S(p,l) < \tau^S \overline{F}^S_{i+j+1}(l_{sMax})$ for all $(p,l) \in \theta^S$ and Part (ii) of Theorem 2, I have $p^*_{i,j,k} = p_{sMin}$, as shown in Figure 1(b).



(a) Case (i): $\tau^S = 0.5$



(b) Case (ii): $\tau^S = 5$

Fig. 4. Demonstration of $\theta^S_{i,j}$ using Theorem 2

Example 3. *Consider the following three settings:*

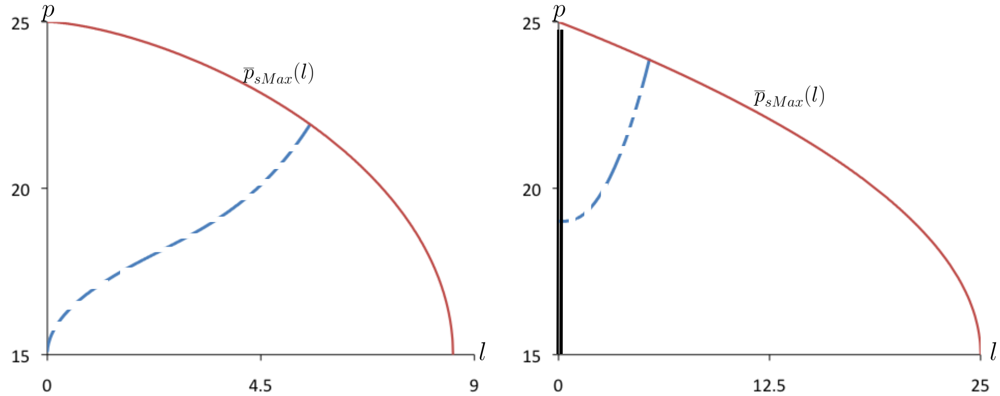
(a) $\lambda^C = 0$, $\mu = 1$, $i + j = 3$, $p_{sMin} = 15$, $p_{sMax} = 25$, $l_{sMax} = 8.55$, $\tau^S = 1.5$ *and* $f^S(p,l) = 1 - 0.01(p-15)^2 - 0.04l^{1.5}$. *Thus,* $\partial f^S(p_{sMin}, 0)/\partial l = \partial f^S(p_{sMin}, 0)/\partial p = 0$.

(b) $\lambda^C = 0$, $\mu = 1$, $i + j = 3$, $p_{sMin} = 15$, $p_{sMax} = 25$, $l_{sMax} = 25$, $\tau^S = 1.5$ *and* $f^S(p,l) = 1 - 0.01(p-15)^2 - 0.04l$. *I have* $\partial f^S(p_{sMin}, 0)/\partial l = -0.04$ *and* $\tau^S \partial f^S(p_{sMin}, 0)/\partial p = 0$.

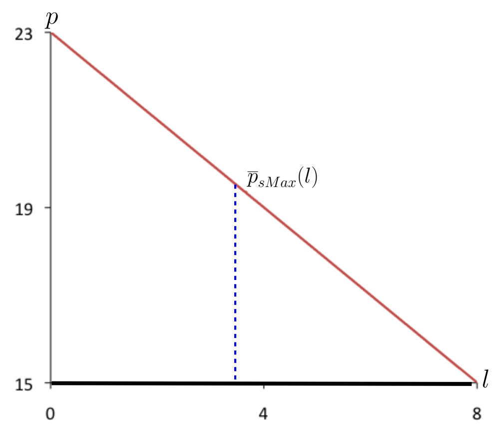
(c) $\lambda^C = 0.45$, $\mu = 1$, $i + j = 3$, $p_{sMin} = 15$, $p_{sMax} = 23$, $l_{sMax} = 8$, $\tau^S = 1.5$, $f^S(p,l) = 1 - \frac{p-15}{8} - \frac{l}{8}$, *for* $(p,l) \in \theta^S$. *Thus,* $\partial f^S(p_{sMin}, 0)/\partial l = -0.125$ *and* $\tau^S \partial f^S(p_{sMin}, 0)/\partial p = -0.1875$.

One observes that (a), (b) and (c) satisfy the conditions of Proposition 1 (i), (ii) and (iii), respectively. Reduced action spaces of $(p^*_{i,j,k}, l^*_{i,j,k})$ *are depicted in Figure*

5, for the three cases, where the dashed line indicates the set of solutions satisfying $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$.



(a) Case (i): $f^S(p,l) = 1 - 0.01(p-15)^2 - 0.04l^{1.5}$ (b) Case (ii): $f^S(p,l) = 1 - 0.01(p-15)^2 - 0.04l$



(c) Case (iii): $f^S(p,l) = 1 - \frac{p-15}{8} - \frac{l}{8}$

Fig. 5. Reduced action spaces obtained using Proposition 1

In Case (a), I have $\partial f^S(p_{sMin}, 0)/\partial l = \partial f^S(p_{sMin}, 0)/\partial p = 0$, *indicating that* $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$ *always holds and an optimal solution exists on the dashed line shown on Figure 5(a). In Cases (b) and (c), the spot purchasers* *are relatively more lead time and price sensitive, respectively. Thus, there are cases where* $l^*_{i,j,k} = 0$ *and* $p^*_{i,j,k} = p_{sMin}$*, in Cases (b) and (c), respectively. In these cases, an optimal solution may also satisfy* $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$*, similar to Case (a). For example, in case (c), I have* $\rho^S(p,l)/\tau^S = 0.667$*, for all* $(p,l) \in \theta^S$*. Solving* $\overline{F}^S_{i+j+1}(l^*_{i,j,k}) = 0.667$ *gives* $l^*_{i,j,k} = 3.67$*. Thus, the dashed line in Figure 5(c) illustrates the set of* $(p,l) \in \theta^S$ *where* $l = 3.67$*.*

Theorem 2 is beneficial in two aspects. First, it proves that joint DPLQ is not necessary when spot purchasers are highly price or lead time sensitive. Second, it provides valuable information about the behavior of the optimal solution, and hence, allows us to reduce the action space, which expedites my solution algorithms.

While the regular action space, θ^S, denotes the area below $\overline{p}_{sMax}(\cdot)$ curves in Figures 4 and 5, Theorem 2 points out that an optimal solution can be found on the indicated lines. I use this information to reduce the size of action spaces and compute the computational time savings using a numerical study. I use a relative value iteration algorithm [Bertsekas, 2001] and solve the problems with regular action space, θ^S, and the reduced action space, $\theta^S_{i,j}$ for $(i,j,k) \in \mathcal{S}$. The readers are referred to Appendix A-2 for details. My analyses conducted using 256 instances reveal that the action space reduction offers minimum, average and maximum computational time savings of 30.75%, 74.20% and 96.92%, respectively. Hence, I use the reduced action spaces to solve `Dyna` in the remainder of my computational analysis.

3.2. *Performance of Dynamic Price and Lead Time Quotation*

In this section, I discuss the optimal long-run average expected profit per unit time (denoted by OAP henceforth) improvements obtained by DPLQ in comparison to the use of fixed price and/or lead times for spot purchasers. Unlike DPLQ, the fixed price (lead time) quotation policy does not have the flexibility to change the price (lead time) quotes over time, and quotes the same price (lead time), for all $(i, j, k) \in \mathcal{S}$. I next define the problems `FixLT`, `FixP` and `Fix` that evaluates OAP when prices are dynamic and lead times are fixed, prices are fixed and lead times are dynamic; and both prices and lead times are fixed, respectively.

$$\texttt{FixLT}: \qquad v^*_{FixLT} = \max_{l_s \in [0, l_{sMax}]} v^*(l_s), \tag{2.16}$$

$$\texttt{FixP}: \qquad v^*_{FixP} = \max_{p_s \in [p_{sMin}, p_{sMax}]} v^*(p_s), \tag{2.17}$$

$$\texttt{Fix}: \qquad v^*_{Fix} = \max_{(p_s, l_s) \in \theta^S} v^*(p_s, l_s), \tag{2.18}$$

where $v^*(l_s)$, $v^*(p_s)$, and $v^*(p_s, l_s)$ are obtained by solving the problems,

$$\frac{v^*(l_s)}{\nu} + h^*_{i,j,k} = \max_{p_s \in [p_{sMin}, p_{sMax}]} \psi_{i,j,k}(p_s, l_s), \quad l_s \in [0, l_{sMax}], \tag{2.19}$$

$$\frac{v^*(p_s)}{\nu} + h^*_{i,j,k} = \max_{l_s \in [0, l_{sMax}]} \psi_{i,j,k}(p_s, l_s), \quad p_s \in [p_{sMin}, p_{sMax}], \text{ and} \tag{2.20}$$

$$\frac{v^*(p_s, l_s)}{\nu} + h^*_{i,j,k} = \psi_{i,j,k}(p_s, l_s), \quad (p_s, l_s) \in \theta^S. \tag{2.21}$$

I develop a computational study to evaluate the OAP improvement offered by `Dyna`, `FixLT` and `FixP` over `Fix`, and provide recommendations for real-life systems. The OAP improvements are evaluated by $\text{IMP}_X = (v^*_X - v^*_{\texttt{Fix}})/v^*_{\texttt{Fix}} 100\%$, $X \in \{\texttt{FixP}, \texttt{FixLT}, \texttt{Fix}\}$.

I define "Breakeven Delay" (BD), which is measured by p_{sMin}/τ^S. Hence, BD represents the maximum amount of time that the delivery of a spot purchaser order can be delayed such that the manufacturer earns non-negative profit by serving that spot purchaser at the accept all price. BD is a unique time value for each manufacturer, and can be estimated using contractual agreements. For example, purchasing conditions of Samsung requires a contract with delivery penalty in the amount of 1% of the order value for every delayed week [Samsung, 2012]. Thus, any manufacturer working with Samsung has a BD of 100 weeks if the agreed price is the accept all price for Samsung.

The unit time is the processing time for one unit, i.e., I set $\mu = 1$. I model $f^S(p,l)$ as in Equation (2.22). That is,

$$f^S(p,l) = 1 - \left(\frac{p - p_{sMin}}{p_{sMax} - p_{sMin}}\right)^{\kappa^P} - \left(\frac{l}{l_{sMax}}\right)^{\kappa^L} - \kappa^{PL}(p - p_{sMin})l, \text{ for } (p,l) \in \theta^S, \tag{2.22}$$

where $\kappa^P \geq 1$, $\kappa^L \geq 1$ and $\kappa^{PL} \geq 0$.

One verifies that used form of $f^S(p,l)$ satisfies Assumption 1 (i) using $\kappa^P \geq 1$, $\kappa^L \geq 1$ and $\kappa^{PL} \geq 0$. In addition, I have $f^S(p_{sMin}, 0) = 1$, which is sufficient for Assumption 1 (ii). I refer to Figure 18 in Appendix A-2 for the verification of Assumptions 1 (iii) and (iv).

I set $\tau^S = 1$ and test 22 levels of $p_{sMin} \in \{0.5, 0.75, 1, 1.25, 1.5, 2, 2.5, 3, 4, 5, 6, 8, 10, 15, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$, which correspond to the same 22 levels of BD. For each level of p_{sMin}, I test for (i) $\frac{\lambda^S + \lambda^C}{\mu} \in \{0.6, 0.75, 0.9\}$, (ii) $\frac{\lambda^S}{\lambda^S + \lambda^C} \in \{\frac{1}{3}, \frac{2}{3}, 1\}$, (iii) $\kappa^P \in \{1, 2\}$, (iv) $\kappa^L \in \{1, 2\}$, (v) $\kappa^{PL} \in \{0, 0.05\}$, (vi) $\frac{p_{sMax}}{p_{sMin}} \in$

$\{1.2, 2, 3\}$, and (vii) $\frac{p_{sMax}-p_{sMin}}{l_{sMax}} \in \{0.2\tau^S, 0.5\tau^S, 2\tau^S, 5\tau^S\}$, giving a total of 19872 instances.

(i)-(ii) allows us to test the impact of arrival rates. Using (iii) and (iv), I test linear ($\kappa^P = 1$, $\kappa^L = 1$), and strictly concave ($\kappa^P = 2$, $\kappa^L = 2$) forms of $f^S(p,l)$ in price and lead time, respectively. Similarly, using (v), I test additive ($\kappa^{PL} = 0$) and non-additive ($\kappa^{PL} > 0$) forms of $f^S(p,l)$. In (vi)-(vii), I test various levels of price/lead time sensitivity of the spot purchasers. Note that an increase (decrease) of $p_{sMax} - p_{sMin}$ and l_{sMax} indicate a decrease (increase) in price and lead time sensitivity, respectively. For simplicity, spot purchasers are referred to as price sensitive when $\frac{p_{sMax}-p_{sMin}}{l_{sMax}} \leq \tau^S$, and lead time sensitive otherwise. I provide a detailed discussion about the selection of parameter levels, and the discretization scheme in Appendix A-2.

When BD is quite small, the manufacturer may lose money by being in business. For example, when BD = 0.5 and $\frac{p_{sMax}}{p_{sMin}} = 1.2$, $v^*_{\texttt{Dyna}}, v^*_{\texttt{FixP}}, v^*_{\texttt{FixLT}}, v^*_{\texttt{Fix}} \leq 0$ in all instances. I observe that $v^*_{\texttt{Fix}} \leq 0$ in 69.8%, 46.1%, 25.9%, 13.9%, 7.4% and 1.9% of all instances when BD is 0,5, 0.75, 1, 1.25, 1.5 and 2, respectively. In these cases, however, implementing `Dyna` may provide positive OAP values.

Observation 2. $v^*_{\texttt{Dyna}} > 0$ *in 25% of all instances with* $v^*_{\texttt{Fix}} < 0$ *and BD* > 1*.*

While `Fix` may lead to negative profits, `Dyna` may help manufacturers to achieve positive profits in a significant proportion of all instances. This happens, in particular, when BD is longer than one time unit, where `Fix` is providing profits that are positive or slightly less than zero. Thus, improvements offered by `Dyna` lead to positive company profits.

I next analyze the average profit improvements of Dyna, FixLT and FixP in Table 1. I omit the instances where $v^*_{\texttt{Fix}} < 0$, since improvements cannot be evaluated in these settings. In addition, I ignore the instances with BD < 1 and BD > 30, because $v^*_{\texttt{Fix}} < 0$ holds in more than half of the instances, and average IMP$_{Dyna}$ is less than 1% in these settings, respectively.

	BD													
	1	1.25	1.5	2	2.5	3	4	5	6	8	10	15	20	30
IMP$_{Dyna}$	62.5%	55.2%	39.9%	25.8%	18.8%	14.8%	10.8%	8.6%	7.0%	5.1%	4.0%	2.5%	1.8%	1.2%
IMP$_{FixLT}$	59.2%	52.6%	38.2%	24.4%	17.5%	13.6%	9.7%	7.6%	6.1%	4.4%	3.3%	2.0%	1.4%	0.8%
IMP$_{FixP}$	58.5%	50.9%	38.2%	24.9%	17.8%	13.9%	10.0%	7.8%	6.3%	4.5%	3.5%	2.2%	1.5%	1.0%

Table 1. Change of average improvements in BD

Not surprisingly, Dyna provides the highest OAP improvements. As BD increases, the manufacturer earns higher profits from customers, and the improvements offered by DPLQ diminishes, where the improvements become significantly small for BD > 30. In Observations 3 and 4, I analyze the parameter spaces, where Dyna significantly improves OAP over both FixLT and FixP. For simplicity, I denote the improvement of Dyna over FixLT and FixP, as IMP$^*_{\texttt{Dyna}}$, which is evaluated as IMP$^*_{\texttt{Dyna}}$ = IMP$_{\texttt{Dyna}}$ − max{IMP$_{\texttt{FixLT}}$, IMP$_{\texttt{FixP}}$}.

Observation 3. *When spot purchasers are lead time sensitive and $f^S(p,l)$ is strictly concave in lead time, proportions of instances where IMP$^*_{\texttt{Dyna}} \geq 1\%$ are 81.1%, 56.3%, 41.4%, and 38.6% for BD levels of 0.5, 0.75, 1 and 1.25, respectively.*

The significant improvements of Dyna can be attributed to Theorem 2 and Proposition 1. Note that, IMP$^*_{\texttt{Dyna}}$ increases when both $p^*_{i,j,k}$ and $l^*_{i,j,k}$ changes frequently from state to state. When $f^S(p,l)$ is strictly concave in lead time, an optimal solution often satisfies $\rho^S(p^*_{i,j,k}, l^*_{i,j,k})/\tau^S = \overline{F}^S_{i+j+1}(l^*_{i,j,k})$ (see for example Case (a) in Figure 5), which indicates that $(p^*_{i,j,k}, l^*_{i,j,k})$ changes as the number of orders in the system changes. From a practical point of view, the flexibility in lead time decisions, offered by dynamic lead time quotation, provides higher benefits when $f^S(p,l)$ is strictly concave in lead time. In addition, from Theorem 1 dynamic pricing is useful when the spot purchaser population is lead time sensitive. Thus, when spot purchasers are lead time sensitive, and $f^S(p,l)$ is strictly concave in lead time, joint DPLQ benefits the improvements enabled by both dynamic pricing and lead time quotation, and provides significant profit improvements over both of them.

From Table 1 and Observation 3, one deduces that $\text{IMP}^*_{\texttt{Dyna}}$ is higher when BD is low, and $f^S(p,l)$ is strictly concave, respectively. In Observation 4, one observes that `Dyna` significantly improves the OAP in most cases when $f^S(p,l)$ is strictly concave in both lead time and price, and BD and p_{sMax} are sufficiently small.

Observation 4. *When $f^S(p,l)$ is strictly concave in lead time and price, $\frac{p_{sMax}}{p_{sMin}} < 2$ and BD is less than 1.5, then $IMP^*_{\texttt{Dyna}} \geq 1\%$ in 78.3% of all instances.*

The profit improvement of joint DPLQ is observed to be higher when $f^S(p,l)$ is strictly concave (see Observation 3). Furthermore, joint DPLQ is even more valuable when the attainable profit from spot purchasers is low (i.e., low p_{sMax} and p_{sMin}). In other words, even if the improvement brought by joint DPLQ is small in magnitude, it becomes quite valuable when BD is small, i.e., spot purchasers are less willing to pay.

Ideally, manufacturers should use `Dyna`, which always provides the highest OAP values. However, frequent change of price and lead times may not be desirable. Thus, `Dyna` may not be the best strategy for manufacturers if it does not provide sufficiently high profit improvements. I next provide some policy recommendations for manufacturers, given a minimum required profit improvement for dynamic quotation denoted as Δ_{DQ}. Using my comprehensive computational analysis, my recommendation scheme ensures that the manufacturer does not use dynamic quotation strategy, unless it provides OAP improvements higher than Δ_{DQ}, and (ii) the manufacturer does not sacrifice OAP improvement of more than Δ_{DQ}. Accordingly, `Dyna` is always recommended when $\Delta_{DQ} = 0\%$. The detailed explanation of my recommendation scheme is given in Appendix A-2.

In Tables 2, 3 and 4, I provide recommendations for $\frac{\lambda^S+\lambda^C}{\mu} = 0.6$, 0.75 and 0.9, respectively, for four levels of $\Delta_{DQ} \in \{0.2\%, 2\%, 10\%, 20\%\}$ in three cases: (i) lead time sensitive spot purchasers, linear $f^S(p,l)$ in lead time, (ii) lead time sensitive spot purchasers, strictly concave $f^S(p,l)$ in lead time, and (iii) price sensitive spot purchasers. I observe that the recommendations do not change significantly with the form of $f^S(p,l)$ when spot purchasers are price sensitive. Hence, I provide general recommendations when spot purchasers are price sensitive. I note that "either" indicates the parameter spaces where either `FixLT` or `FixP` is recommended. I note that dark shaded cells in Table 4 (c) indicate the cases where there is no recommendation. To obtain recommendations in these cases, one needs to conduct a more detailed analysis considering the form of $f^S(p,)$.

(a) Lead time sensitive spot purchasers, strictly concave $f^S(p,l)$ in lead time

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="14">Dyna</td><td colspan="3">FixP</td></tr>
<tr><td>2%</td><td colspan="3">Dyna</td><td colspan="5">FixLT</td><td colspan="3">Either</td><td colspan="6">Fix</td></tr>
<tr><td>10%</td><td>Dyna</td><td colspan="2">FixLT</td><td colspan="6">Either</td><td colspan="8">Fix</td></tr>
<tr><td>20%</td><td colspan="2">Either</td><td>FixLT</td><td colspan="4">Either</td><td colspan="10">Fix</td></tr>
</table>

(b) Lead time sensitive spot purchasers, linear $f^S(p,l)$ in lead time

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="14">FixLT</td><td colspan="3">Fix</td></tr>
<tr><td>2%</td><td colspan="9">FixLT</td><td colspan="2">Either</td><td colspan="6">Fix</td></tr>
<tr><td>10%</td><td colspan="3">FixLT</td><td colspan="6">Either</td><td colspan="8">Fix</td></tr>
<tr><td>20%</td><td colspan="3">FixLT</td><td colspan="4">Either</td><td colspan="10">Fix</td></tr>
</table>

(c) Price sensitive spot purchasers

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="14">Dyna</td><td colspan="3">FixP</td></tr>
<tr><td>2%</td><td>Dyna</td><td colspan="3">FixP</td><td colspan="7">Dyna</td><td colspan="2">FixP</td><td>Eit.</td><td colspan="3">Fix</td></tr>
<tr><td>10%</td><td>Dyna</td><td>FixP</td><td colspan="6">Either</td><td colspan="9">Fix</td></tr>
<tr><td>20%</td><td>Dyna</td><td colspan="5">Either</td><td colspan="11">Fix</td></tr>
</table>

Table 2. Policy recommendations for changing price/lead time sensitivity when $\frac{\lambda^S+\lambda^C}{\mu} = 0.6$

One observes a relatively significant difference between the recommendations of strictly concave and linear $f^S(p,l)$, when spot purchasers are lead time sensitive. For example, in Table 3 (a), `Dyna` is recommended for all values of BD when $\Delta_{DQ} = 0.2\%$. On the other hand, `Dyna` is never recommended in Table 3 (b). From Observation 3, `Dyna` often significantly improves the OAP for low levels of BD, when

(a) Lead time sensitive spot purchasers, strictly concave $f^S(p,l)$ in lead time

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="17">Dyna</td></tr>
<tr><td>2%</td><td colspan="3">Dyna</td><td colspan="10">FixLT</td><td colspan="4">Either</td></tr>
<tr><td>10%</td><td colspan="2">Dyna</td><td>FixLT</td><td colspan="9">Either</td><td colspan="5">Fix</td></tr>
<tr><td>20%</td><td colspan="2">Dyna</td><td>FixLT</td><td colspan="6">Either</td><td colspan="8">Fix</td></tr>
</table>

(b) Lead time sensitive spot purchasers, linear $f^S(p,l)$ in lead time

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="17">FixLT</td></tr>
<tr><td>2%</td><td colspan="13">FixLT</td><td colspan="4">Either</td></tr>
<tr><td>10%</td><td colspan="3">FixLT</td><td colspan="10">Either</td><td colspan="4">Fix</td></tr>
<tr><td>20%</td><td colspan="3">FixLT</td><td colspan="7">Either</td><td colspan="7">Fix</td></tr>
</table>

(c) Price sensitive spot purchasers

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="17">Dyna</td></tr>
<tr><td>2%</td><td colspan="2">Dyna</td><td colspan="2">FixP</td><td colspan="7">Dyna</td><td colspan="2">FixP</td><td>Eit.</td><td colspan="3">Fix</td></tr>
<tr><td>10%</td><td colspan="3">FixP</td><td colspan="7">Either</td><td colspan="7">Fix</td></tr>
<tr><td>20%</td><td colspan="2">FixP</td><td colspan="5">Either</td><td colspan="10">Fix</td></tr>
</table>

Table 3. Policy recommendations for changing price/lead time sensitivity when $\frac{\lambda^S+\lambda^C}{\mu} = 0.75$

$f^S(p,l)$ is concave in lead time. Hence, `Dyna` is recommended in part (a) of Tables 2, 3 and 4 when BD is low. On the other hand, when $f^S(p,l)$ is linear, Theorem 2 and Proposition 1 indicate that both $p^*_{i,j,k}$ and $l^*_{i,j,k}$ may not change significantly in the number of orders (see for example Figure 4 and Case (iii) in Figure 5). Thus, `Dyna` does not offer significant improvements in this case, and hence, is not recommended

(a) Lead time sensitive spot purchasers, strictly concave $f^S(p, l)$ in lead time

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="17">Dyna</td></tr>
<tr><td>2%</td><td colspan="3">Dyna</td><td colspan="10">FixLT</td><td colspan="4">Either</td></tr>
<tr><td>10%</td><td colspan="2">Dyna</td><td colspan="10">Either</td><td colspan="5">Fix</td></tr>
<tr><td>20%</td><td colspan="2">Dyna</td><td colspan="7">Either</td><td colspan="8">Fix</td></tr>
</table>

(b) Lead time sensitive spot purchasers, linear $f^S(p, l)$ in lead time

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="17">FixLT</td></tr>
<tr><td>2%</td><td colspan="13">FixLT</td><td colspan="4">Either</td></tr>
<tr><td>10%</td><td colspan="3">FixLT</td><td colspan="9">Either</td><td colspan="5">Fix</td></tr>
<tr><td>20%</td><td colspan="3">FixLT</td><td colspan="7">Either</td><td colspan="7">Fix</td></tr>
</table>

(c) Price sensitive spot purchasers

<table>
<tr><th></th><th colspan="17">BD</th></tr>
<tr><th>Δ_{DQ}</th><th>1</th><th>1.25</th><th>1.5</th><th>2</th><th>2.5</th><th>3</th><th>4</th><th>5</th><th>6</th><th>8</th><th>10</th><th>15</th><th>20</th><th>30</th><th>40</th><th>50</th><th>60</th></tr>
<tr><td>0.2%</td><td colspan="17">Dyna</td></tr>
<tr><td>2%</td><td colspan="12">Dyna</td><td></td><td colspan="4">FixP</td></tr>
<tr><td>10%</td><td></td><td colspan="2">FixP</td><td></td><td colspan="8">Either</td><td colspan="5">Fix</td></tr>
<tr><td>20%</td><td colspan="2">FixP</td><td>Either</td><td>FixP</td><td colspan="6">Either</td><td colspan="7">Fix</td></tr>
</table>

Table 4. Policy recommendations for changing price/lead time sensitivity when $\frac{\lambda^S+\lambda^C}{\mu} = 0.9$

in when $f^S(p, l)$ is linear in lead time. `FixLT` and `FixP` are recommended when spot purchasers are lead time and price sensitive, respectively, which is in line with Theorem 1.

In Table 3 (c), at $\Delta_{DQ} = 2\%$ line, `Dyna` is recommended for $2.5 \leq \text{BD} \leq 10$, whereas `FixP` is recommended for $1.5 \leq \text{BD} \leq 2$, which indicates that $\text{IMP}^*_{\texttt{Dyna}}$

may increase as BD increases. This results from increased number of states with $f^S(p^*_{i,j,k}, l^*_{i,j,k}) > 0$ (i.e., an acceptable offer is quoted), when BD is higher, which increases the portion of the state space that `Dyna` is improving profits. For example, in a particular instance I have $f^S(p^*_{i,j,k}, l^*_{i,j,k}) > 0$ only for $i + j \leq 5$ when BD is 2, whereas when BD is 10, $f^S(p^*_{i,j,k}, l^*_{i,j,k}) > 0$ for all $i + j \leq 45$.

The frequency of `Dyna` recommendations increases as the traffic intensity increases. For example, when spot purchasers are lead time sensitive and $f^S(p, l)$ is strictly concave in lead time, `Dyna` is recommended for BD ≤ 1.5 and BD ≤ 15 when $\frac{\lambda^S+\lambda^C}{\mu} = 0.75$ and 0.9, respectively for $\Delta_{DQ} = 2\%$. Similarly, `Dyna` is recommended for all tested values of BD when $\frac{\lambda^S+\lambda^C}{\mu} \geq 0.75$, whereas, it is only recommended only for BD ≤ 30 when $\frac{\lambda^S+\lambda^C}{\mu} = 0.6$ for $\Delta_{DQ} = 0.2\%$. This behavior is due to the better performance of joint DPLQ under higher traffic intensity. As the traffic intensity increases, the importance of timely lead time quotes as well. Since, joint DPLQ allows the lead time decisions to change over time, it allows higher precision lead time quotes, and joint DPLQ becomes quite necessary as the incoming traffic increases.

4. Contracting Model

In this section, I discuss the problem of selecting price and lead time terms for the contract customers to maximize the OAP, and the benefits of offering optimal contract terms so that the customer mix is optimized. I model the contract customer arrival rate, $\lambda^C(p_c, l_c)$ as a function of the contract terms p_c and l_c, with the properties outlined in Assumption 2.

Assumption 2. *(i)* $\lambda^C(p_c, l_c)$ *is continuous, non-increasing, twice differentiable, concave in* p_c *and* l_c *for* $(p_c, l_c) \in \theta^C$

(ii) There exist a maximum potential contract customer arrival rate of λ_{cMax}.

(iii) There exist nonnegative lower bounds p_{cMin} *and 0 such that decreasing* p_c *and* l_c *below these bounds does not change* $\lambda^C(p_c, l_c)$. *I have* $\lambda^C(p_{cMin}, 0) = \lambda_{cMax}$.

(iv) Given a lead time quote l_c, *there exists an upper bound on price, denoted by* $\overline{p}_{cMax}(l_c)$, *such that any price quote above this upper bound is definitely rejected.* $\overline{p}_{cMax}(l_c)$ *is nonincreasing in* l_c, *and its highest value is denoted as* $p_{cMax} = \overline{p}_{cMax}(0)$.

(v) Given a price quote p_c, *there exists an upper bound on lead time, denoted by* $\bar{l}_{cMax}(p_c)$, *such that any price quote above this upper bound is definitely rejected.* $\bar{l}_{cMax}(p_c)$ *is nonincreasing in* p_c, *and its highest value is denoted as* $l_{cMax} = \bar{l}_{cMax}(p_{cMin})$.

The set θ^C is defined as follows.

$$\theta^C = \{(p, l) \in \Re^2 : p_{cMin} \leq p \leq \overline{p}_{cMax}(l), 0 \leq l \leq \bar{l}_{cMax}(p)\}.$$

Assumption 2 defines the properties of the contract customers' arrival rate, and is analogous to Assumption 1, which defines the acceptance probability of a single customer.

Offering the contract terms (p_c, l_c), the manufacturer receives a demand rate of $\lambda^C(p_c, l_c)$ from the contract customers, where the remaining customers either leave the system, or act as spot purchasers. I assume that $1 - \gamma(p_c, l_c)$ proportion of $\lambda_{cMax} - \lambda^C(p_c, l_c)$ is lost, and the remaining $\gamma(p_c, l_c)$ proportion joins the spot purchaser stream, where $0 \leq \gamma(p_c, l_c) \leq 1$. I model the independent spot purchaser population with an arrival rate of λ_I^S. Consequently, given the contract offer of (p_c, l_c), the total arrival rate of spot purchasers and contract customers are $\lambda^S(p_c, l_c) = \lambda_I^S + \gamma(p_c, l_c)(\lambda_{cMax} - \lambda^C(p_c, l_c))$, and $\lambda^C(p_c, l_c)$, respectively. The arrival process is depicted in Figure 6.

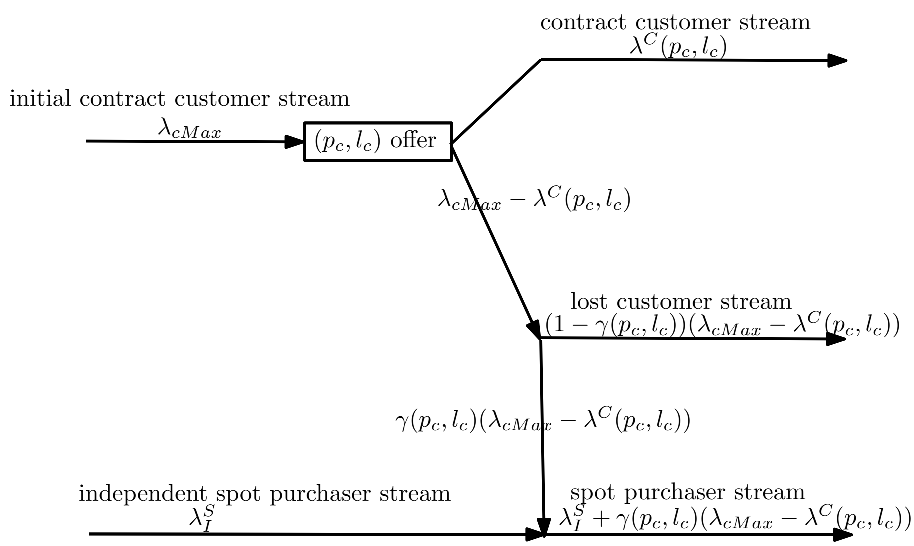


Fig. 6. Arrival processes due to contract customers and spot purchasers

The optimal contract terms, (p_c^*, l_c^*), can be determined by solving the problem

$$\texttt{Cont}: \qquad v^* = \max_{(p_c, l_c) \in \theta_c} v^*_{Dyna}(p_c, l_c), \tag{2.23}$$

where $v^* = v^*_{Dyna}(p_c^*, l_c^*)$, $\lambda^{C*} = \lambda^C(p_c^*, l_c^*)$ and $\lambda^{S*} = \lambda^S(p_c^*, l_c^*)$.

4.1. *Improving Computational Times*

One straightforward way to solve `Cont` is to discretize the $[p_{cMin}, p_{cMax}]$, $[0, l_{cMax}]$ intervals into ϵ equal intervals. However, this method, which I refer to as the discretization algorithm (denoted as DA) requires `Dyna` to be solved at least $\epsilon^2/2$ times, which results in extensive computational times when high precision required, i.e., ϵ is high. In this section, I develop (i) an algorithm that provides reliable results in less computational time, and (ii) rules to reduce the action space θ^C to speed up the DA.

To develop my algorithm, I first focus on some structural properties. My preliminary analysis indicates that while $(p^*_{i,j,k}, l^*_{i,j,k})$ is significantly affected by λ^C, the changes in (p_c, l_c) does not change $(p^*_{i,j,k}, l^*_{i,j,k})$ significantly, if λ^C remains unchanged. That is, (p_c, l_c) values within the set $\theta^C(\widetilde{\lambda}^C)$ result in similar $(p^*_{i,j,k}, l^*_{i,j,k})$ values, where $\theta^C(\widetilde{\lambda}^C)$ is defined as

$$\theta^C(\widetilde{\lambda}^C) = \{(p_c, l_c) \in \theta^C : \lambda^C(p_c, l_c) = \widetilde{\lambda}^C\}.$$

In Figure 7, I show $\theta^C(\widetilde{\lambda}^C)$ for $\widetilde{\lambda}^C = \{0, 0.2, 0.4, 0.6, 0.8\}$ in a sample instance. In this case, because $\lambda^C(p_c, l_c)$ is linear in p_c and l_c, all points that result in a particular $\widetilde{\lambda}^C$ lie on a line.

I next define Λ^C, which is the set of all $\widetilde{\lambda}^C$ which the optimal spot purchaser quotes remain the same across all elements of the set $\theta^C(\widetilde{\lambda}^C)$. That is,

$$\begin{aligned}\Lambda^C = \{\widetilde{\lambda}^C \in [0, \lambda_{cMax}] :& (p^*_{i,j,k}(p'_c, l'_c), l^*_{i,j,k}(p'_c, l'_c)) = (p^*_{i,j,k}(p''_c, l''_c), l^*_{i,j,k}(p''_c, l''_c)),\\ & \text{for all } (p'_c, l'_c), (p''_c, l''_c) \in \theta^C(\lambda^C), \text{ and } (i, j, k) \in \mathcal{S}\}.\end{aligned}$$

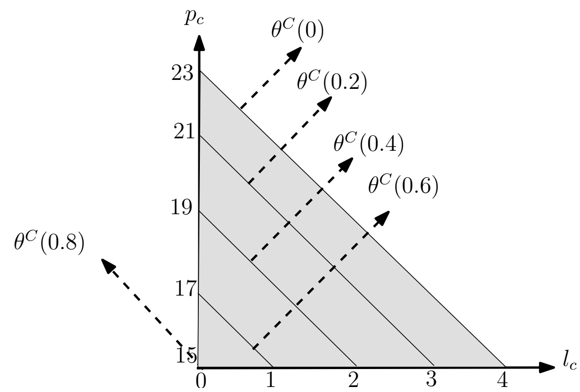


Fig. 7. Illustration of $\theta^C(\widetilde{\lambda}^C)$ for $\widetilde{\lambda}^C = 0$, 0.2, 0.4, 0.6, and 0.8

For example, if $0.4 \in \Lambda^C$, then for any given state $(i,j,k) \in \mathcal{S}$, $(p^*_{i,j,k}, l^*_{i,j,k})$ remain unchanged for all $(p_c, l_c) \in \theta^C(0.4)$ in Figure 7. In Theorem 3, I derive $(p^*_c(\widetilde{\lambda}^C), l^*_c(\widetilde{\lambda}^C))$ that maximizes $v^*_{Dyna}(p_c, l_c)$ on $\theta^C(\widetilde{\lambda}^C)$ given that $\widetilde{\lambda}^C \in \Lambda^C$, and use this result to develop my new algorithm. I define $\rho^C(\cdot,\cdot)$ similar to $\rho^S(\cdot,\cdot)$, as $\rho^C(p_c, l_c) = \frac{\partial \lambda^C(p_c,l_c)/\partial l}{\partial \lambda^C(p_c,l_c)/\partial p}$.

Theorem 3. *Let* $\widetilde{\lambda}^C \in \Lambda^C$. *Then,*

(i) if $\rho^C(p_c, l_c) \geq \tau^C$, *for all* $(p_c, l_c) \in \theta^C(\widetilde{\lambda}^C)$, *then* $l^*_c(\widetilde{\lambda}^C) = 0$,

(ii) if $\rho^C(p_c, l_c) \leq \tau^C \sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k} \overline{F}^C_{i+j+1}(l_{cMax}(1 - \frac{\widetilde{\lambda}^C}{\lambda_{cMax}}))$, *for all* $(p_c, l_c) \in \theta^C(\widetilde{\lambda}^C)$, *then* $p^*_c(\widetilde{\lambda}^C) = p_{cMin}$,

(iii) otherwise, $(p^*_c(\widetilde{\lambda}^C), l^*_c(\widetilde{\lambda}^C))$ *satisfies*

$$\frac{\rho^C(p^*_c(\widetilde{\lambda}^C), l^*_c(\widetilde{\lambda}^C))}{\tau^C} = \sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k} \overline{F}^C_{i+j+1}(l^*_c(\widetilde{\lambda}^C)),$$

where $\pi^*_{i,j,k}$ *denotes the limiting probability of state* (i,j,k) *under the optimal dynamic quotation policy.*

Theorem 3 identifies the position of $(p_c^*(\widetilde{\lambda}^C), l_c^*(\widetilde{\lambda}^C))$ on $\theta(\widetilde{\lambda}^C))$. Cases (i) and (ii) indicate the two conditions where $(p_c^*(\widetilde{\lambda}^C), l_c^*(\widetilde{\lambda}^C))$ is located on the upper-left, and lower-right extreme points of $\theta(\widetilde{\lambda}^C))$ on Figure 7, respectively. On the other hand, Case (iii) helps us find where $(p_c^*(\widetilde{\lambda}^C), l_c^*(\widetilde{\lambda}^C))$ is located in between the two extreme points.

Theorem 3 shows that the structure of optimal contract terms is analogous to that of optimal spot purchaser quotes, where both are highly dependent on the price/lead time sensitivity of the customer. Recalling that values of $(p^*_{i,j,k}, l^*_{i,j,k})$ depend on the news-vendor ratio $1 - \rho^S(\cdot,\cdot)/\tau^S$ (see Theorem 2), One observes a similar structure for $(p_c^*(\widetilde{\lambda}^C), l_c^*(\widetilde{\lambda}^C))$ in Case (iii). The term $\sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k}\overline{F}^C_{i+j+1}(l_c^*(\widetilde{\lambda}^C))$ gives the probability that a contract customer order is not met on time given the lead time quote $l_c^*(\widetilde{\lambda}^C)$ (Poisson arrivals see time averages). Thus, $1 - \rho^C(p_c^*(\lambda^C), l_c^*(\lambda^C))/\tau^C$ should be equal to the probability that the contract customer order is met on time.

I next develop an algorithm, which assumes that $\widetilde{\lambda}^C \in \Lambda^C$ for all $\widetilde{\lambda}^C \in [0, \lambda_{cMax}]$, and evaluates a solution using Theorem 3. The algorithm is denoted as T3A for simplicity.

The T3A solves (2.4) 2ϵ times. On the other hand, the DA solves (2.4) at least $\frac{\epsilon^2}{2}$ times, which presents the computational advantage of the T3A. In order to speed up the DA, I next present two conditions verifying that any solution (p'_c, l'_c) gives higher OAP than (p''_c, l'_c), i.e., $v^*_{Dyna}(p'_c, l'_c) \leq v^*_{Dyna}(p''_c, l'_c)$ in Proposition 2.

Proposition 2. *For any two solutions (p'_c, l'_c) and (p''_c, l'_c), (i) if $p'_c < p''_c$ and the inequality (2.24) holds, or (ii) $p''_c < p'_c$ and the inequality (2.25) holds, then*

Algorithm 1 Pseudocode for T3A

Choose an ϵ, and set $\widetilde{\lambda}^C = 0$, $\widetilde{\theta}^C = \emptyset$.

while $\widetilde{\lambda}^C \leq \lambda_{cMax}$ **do**

Set $l'_c = 0$, and find the corresponding p'_c by solving $\lambda^C(p'_c, 0) = \widetilde{\lambda}^C$.

Solve (2.4) using $(p_c, l_c) = (p'_c, 0)$, and obtain $\pi^*_{i,j,k}, (i,j,k) \in \mathcal{S}$.

Find the candidate solution $(p^*_c(\widetilde{\lambda}^C), l^*_c(\widetilde{\lambda}^C))$ using Theorem 3, and augment $\widetilde{\theta}^C$ by $(p^*_c(\widetilde{\lambda}^C), l^*_c(\widetilde{\lambda}^C))$.

Set $\widetilde{\lambda}^C = \widetilde{\lambda}^C + \epsilon$.

end while

Solve $\max_{(p_c,l_c)\in\widetilde{\theta}_c} v^*_{Dyna}(p_c, l_c)$.

$v^*_{Dyna}(p'_c, l'_c) \leq v^*_{Dyna}(p''_c, l'_c)$.

$$\frac{e^{-\mu l'_c}}{\mu}\left(\lambda^C(p'_c, l'_c) - \lambda^C(p''_c, l'_c)\right) \geq \lambda^S(p'_c, l'_c)p_{sMax} + \lambda^C(p'_c, l'_c)p'_c - \lambda^C(p''_c, l'_c)p''_c \quad (2.24)$$

$$L^C_{N,c}(l_{cMax})\left(\lambda^C(p'_c, l'_c) - \lambda^C(p''_c, l'_c)\right) \geq \lambda^S(p'_c, l'_c)p_{sMax} + \lambda^C(p'_c, l'_c)p'_c - \lambda^C(p''_c, l'_c)p''_c \quad (2.25)$$

I use an extensive computational analysis to analyze the performances of T3A and DA, while reducing my action set by Proposition 2. I set $\epsilon = 10$ in all of my analysis, noting that the DA runs in an average time of 76 minutes with this setting, while increasing ϵ further exponentially increases the computational times. Similar to the structure of $f^S(p, l)$ in Section 3.2, I model $\lambda^C(p_c, l_c)$ as in Equation (2.26).

$$\lambda^C(p_c, l_c) = \lambda_{cMax}\left(1 - \left(\frac{p_c - p_{cMin}}{p_{cMax} - p_{cMin}}\right)^{\kappa^P} - \left(\frac{l_c}{l_{cMax}}\right)^{\kappa^L} - \kappa^{PL}(p_c - p_{cMin})l_c\right). \quad (2.26)$$

Examples from industry indicate that while contract and spot prices are often different, the differences are not very large. For example, according to the recent survey of EnergyTrend among photovoltaic manufacturers, average contract prices for multiSi wafers and solar cells are \$2.283 per piece and \$0.853 per Watt, respectively. On the other hand, average spot prices for same products are \$2.189/piece and \$0.824/Watt, respectively [News.radio-electronics.co, 2011]. Wang [2009] presents a case where solar wafer and cell makers cancel their contracts since contract prices fell below half of the spot prices. To reflect the difference between contract and spot prices in real life settings, I assume that $p_{cMin} = \omega p_{sMin}$, where $0.5 \leq \omega \leq 2$.

Assuming that spot purchasers and contract customers are coming from the same population (see Figure 6), the structure of $\lambda^C(p_c, l_c)$ is assumed to be same as that of $f^S(p, l)$. That is, I use the same κ^P, κ^L and κ^{PL} in $\lambda^C(p_c, l_c)$ and $f^S(p, l)$. To obtain same price and lead time sensitivities for contract customers and spot purchasers, I set $\frac{p_{sMax}}{p_{sMin}} = \frac{p_{cMax}}{p_{cMin}}$ and $\frac{p_{sMax}-p_{sMin}}{l_{sMax}} = \frac{p_{cMax}-p_{cMin}}{l_{cMax}}$.

I model $\gamma(p_c, l_c)$ as in Equation (2.27).

$$\gamma(p_c, l_c) = \gamma_1 + \gamma_2 \frac{\lambda_{cMax} - \lambda^C(p_c, l_c)}{\lambda_{cMax}} + \gamma_3 \frac{\lambda^C(p_c, l_c)}{\lambda_{cMax}}, \tag{2.27}$$

where $\gamma_1, \gamma_2, \gamma_3 \geq 0$, and $0 \leq \gamma_1 + \gamma_2 + \gamma_3 \leq 1$. γ_1 denote the proportion of customers who join the spot purchaser stream independent of the contract terms (p_c, l_c). The second and third terms in Equation (2.27) denote proportions of customers joining the spot purchaser stream affected by (p_c, l_c). $\frac{\lambda_{cMax}-\lambda^C(p_c,l_c)}{\lambda_{cMax}}$ and $\frac{\lambda^C(p_c,l_c)}{\lambda_{cMax}}$, which are the proportion of rejected and accepted contracts, allow us to test $\gamma(p_c, l_c)$ that are increasing and decreasing in p_c (l_c), respectively.

Most of my testing levels are chosen according to the test given in Section 3.2. I set $\tau^S = \tau^C = \mu = 1$. I conduct a full factorial experiment with (i) $\frac{\lambda_I^S+\lambda_{cMax}}{\mu} \in \{0.6, 0.75, 0.9\}$, (ii) $\lambda_{cMax}/(\lambda_{cMax}+\lambda_I^S) \in \{\frac{1}{3}, \frac{2}{3}, 1\}$, (iii) $\kappa^P \in \{1, 2\}$, (iv) $\kappa^L \in \{1, 2\}$, (v) $\kappa^{PL} \in \{0, 0.05\}$, (vi) $p_{sMax}/p_{sMin} \in \{1.23\}$, (vii) $\frac{p_{sMax}-p_{sMin}}{l_{sMax}} \in \{0.5\tau^S, 2\tau^S\}$,

(vii) BD= $p_{sMin} \in \{2, 10, 25, 50\}$, (viii) $\omega \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1, \frac{4}{3}, \frac{3}{2}, 2\}$, and (ix) $(\gamma_1, \gamma_2, \gamma_3) \in \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ giving a total of 21504 instances.

Proposition 2 is particularly effective in reducing the action space when $\lambda_{cMax}/(\lambda_{cMax} + \lambda_I^S)$ is high, and $\lambda^C(p_c, l_c)$ is strictly concave in p_c, which results in lower values in the right hand sides of Equations (2.24) and (2.25). For example, I observe that 24.11% of all solutions are dominated when $\lambda_{cMax}/(\lambda_{cMax} + \lambda_I^S) = 1$ and $\lambda^C(p_c, l_c)$ is strictly concave in p_c. The concavity of $\lambda^C(p_c, l_c)$ indicates that increasing p_c above p_{cMin} does not result in a significant decrease in $\lambda^C(p_c, l_c)$, but, increases the profit obtained from contract customers (i.e., $p_c - L_{j,k}^C(l_c)$) significantly. This observation is most commonly observed when there is a higher contract customer population in the system (i.e., $\lambda_{cMax}/(\lambda_{cMax} + \lambda_I^S) = 1$). This helps Proposition 2 to eliminate actions with price values close or equal to p_{cMin}.

I next compute the OAPs obtained by the T3A, v^{T3A}, and DA, v^{DA}, and compare the performances of the two approaches. My analysis shows that the performance of T3A is mostly affected by the setting of $\gamma(p_c, l_c)$. Hence, I report several different statistics comparing OAPs of v^{T3A} and v^{DA} for different values of $\gamma(p_c, l_c)$ in Table 5, and report the generalized statistics in the last row. For simplicity, the percentage difference of v^{T3A} and v^{DA} is denoted as PD, i.e., PD=$(v^{DA} - v^{T3A})/v^{DA}$. I also report time time spend in T3A and DA.

Table 5 reveals that (i) T3A gives higher OAP in more than 60% of the instances, (ii) the percentage difference of v^{DA} and v^{T3A} is less than 1% in more than 80% of the instances, (iii) the average percentage difference is $(v^{DA} - v^{T3A})/v^{DA}$ is 1.21%, and (iv) T3A and DA runs with an average time of 20 and 76 minutes, respectively.

$(\gamma_1, \gamma_2, \gamma_3)$	% of inst. $v^{T3A} > v^{DA}$	Avg. PD	% of inst. PD< 1%	% of inst. PD< 10%	Avg. time DA (min.)	Avg. time T3A (min.)
(0,0,0)	56.25%	0.55%	89.32%	97.92%	74.91	19.85
(1,0,0)	60.49%	1.90%	72.14%	87.76%	77.38	20.19
(0,1,0)	59.60%	1.79%	70.57%	89.58%	76.75	20.11
(0,0,1)	64.51%	0.62%	90.36%	95.83%	76.81	19.90
general	60.21%	1.21%	80.60%	92.77%	76.46	20.01

Table 5. Performance comparison of T3A and DA

That is, T3A is performing better than DA in more than half of the instances, whereas, OAPs obtained by DA is only slightly better than T3A, with an expense of approximately quadrupled computational time on the average. The superior performance of the T3A can be attributed to either or both of the following two facts: (i) $\widetilde{\lambda}^C \in \Lambda^C$ holds for a high range of $\widetilde{\lambda}^C$ values, which indicates that Theorem 3 condition holds under most cases, (ii) $\widetilde{\lambda}^C \in \Lambda^C$ does not hold for many values of $\widetilde{\lambda}^C$, but the deviation in $(p^*_{i,j,k}, l^*_{i,j,k})$ for a given $\widetilde{\lambda}^C$ is insignificantly small, such that Theorem 3 provides close-to-optimal solutions.

T3A is performing 10% worse than DA in more than 10% of the instances when $(\gamma_1, \gamma_2, \gamma_3) \in \{(1, 0, 0), (0, 1, 0)\}$. In particular, the performance of the T3A is significantly better when $(\gamma_1, \gamma_2, \gamma_3) \in \{(0, 0, 0), (0, 0, 1)\}$. In these two cases, one observes that the proportion of customers who joins the spot purchaser stream from the initial contract customer stream is generally lower than that in the other two cases. For example, when $\lambda_{cMax} = 10$ and $\lambda^C = 3$, a rate of 0, 6, 4.9 and 2.1 joins the spot purchaser stream when $(\gamma_1, \gamma_2, \gamma_3)$ is equal to (0,0,0), (1,0,0), (0,1,0) and (0,0,1), respectively. The high value of λ^S, which is observed when $(\gamma_1, \gamma_2, \gamma_3) \in \{(1,0,0),(0,1,0)\}$, affects the performance of the T3A negatively, because it increases the impact of unsatisfied Theorem 3 conditions. Hence, I refer $(\gamma_1, \gamma_2, \gamma_3) \in \{(1,0,0),(0,1,0)\}$ and $(\gamma_1, \gamma_2, \gamma_3) \in \{(0,0,0),(0,0,1)\}$, to as the low lost customer case (LLCC), and high lost customer case (HLCC), respectively, in the remainder.

4.2. *Benefits of Offering Optimal Contract Terms*

In this section, I analyze the potential benefits obtained by offering the optimal contract terms in comparison to simple contracting strategies. To analyze the benefits of serving an optimal mix, I consider the following three schemes: (1) the optimal mix policy (MIX), where $(p_c, l_c) = (p_c^*, l_c^*)$, and $(\lambda^C, \lambda^S) = (\lambda^{C*}, \lambda^{S*})$, (2) maximal contract customer policy (MCC), where the manufacturer offers $(p_c, l_c) = (p_{cMin}, 0)$ to maximize the contract customer arrival rate, i.e., $(\lambda^C, \lambda^S) = (\lambda_{cMax}, \lambda_I^S)$, and (3) maximal spot purchaser policy (MSP), where the manufacturer offers $(p_c, l_c) = (p_{cMax}, 0)$ to maximize the spot purchaser arrival rate, i.e., $(\lambda^C, \lambda^S) = (0, (1-\gamma(p_{cMax}, 0))\lambda_{cMax} + \lambda_I^S)$. I compute the improvement of MIX over MCC and MSP, using IMP_{MCC} and IMP_{MSP}, where $\text{IMP}_{\text{MCC}} = \frac{v^*_{Dyna}(p_c^*, l_c^*) - v^*_{Dyna}(p_{cMin}, 0)}{v^*_{Dyna}(p_c^*, l_c^*)} 100\%$ and $\text{IMP}_{\text{MSP}} = \frac{v^*_{Dyna}(p_c^*, l_c^*) - v^*_{Dyna}(p_{cMax}, 0)}{v^*_{Dyna}(p_c^*, l_c^*)} 100\%$. I note that IMP_{MCC} and IMP_{MSP} can also give information about the optimal customer mix $\left(\frac{\lambda^{C*}}{\lambda^{C*}+\lambda^{s*}}, \frac{\lambda^{s*}}{\lambda^{C*}+\lambda^{s*}}\right)$. For example, $\text{IMP}_{\text{MCC}} = 0\%$ indicates that $(p_c^*, l_c^*) = (p_{cMin}, 0)$, and hence, $\left(\frac{\lambda^{C*}}{\lambda^{C*}+\lambda^{s*}}, \frac{\lambda^{s*}}{\lambda^{C*}+\lambda^{s*}}\right) = \left(\frac{\lambda_{cMax}}{\lambda_{cMax}+\lambda_I^S}, \frac{\lambda_I^S}{\lambda_{cMax}+\lambda_I^S}\right)$. Similarly, when $\text{IMP}_{\text{MSP}} = 0\%$, I get $\left(\frac{\lambda^{C*}}{\lambda^{C*}+\lambda^{s*}}, \frac{\lambda^{s*}}{\lambda^{C*}+\lambda^{s*}}\right) = (0\%, 100\%)$. Consequently, as IMP_{MCC} and IMP_{MSP} get close to 0%, optimal contract customer proportion approaches to $\frac{\lambda_{cMax}}{\lambda_{cMax}+\lambda_I^S}$ and 0%, respectively.

I use an extensive computational analysis to analyze the benefits of offering optimal contract terms. I use the same parameter settings provided in the computational analysis of Section 4.1, and use the DA to compute the optimal contracting strategy. My analysis reveals that performance of MIX is mostly affected by the setting of $\gamma(p_c, l_c)$, ω, BD and p_{sMax}/p_{sMin}. In particular, IMP_{MCC} and IMP_{MSP} differ significantly when (i) $p_{sMax}/p_{sMin} = 1.2$, and $p_{sMax}/p_{sMin} > 1.2$; and (ii) $(\gamma_1, \gamma_2, \gamma_3) \in \{(1,0,0),(0,1,0)\}$ and $(\gamma_1, \gamma_2, \gamma_3) \in \{(0,0,0),(0,0,1)\}$, i.e., LLCC, and HLCC. For simplicity, I denote the case $p_{sMax}/p_{sMin} = 1.2$, low price range case (and high price range otherwise). In Tables 6, 7, 8 and 9, I provide the change of average IMP_{MCC} and IMP_{MSP} for changing values of $(\gamma_1, \gamma_2, \gamma_3)$, ω, BD and the setting of p_{sMax}. The dark and light shaded cells demonstrate the average values below 1% and below 5%, respectively. I note that each cell is the average value obtained from 24 observations.

	Avg. IMP_{MSP}				Avg. IMP_{MCC}			
ω	BD=2	BD=10	BD=25	BD=50	BD=2	BD=10	BD=25	BD=50
1/2	0.0%	31.9%	42.7%	47.2%	291.4%	22.2%	10.3%	7.1%
2/3	0.1%	42.5%	51.0%	54.5%	227.3%	12.5%	6.3%	4.5%
3/4	1.3%	47.2%	54.5%	57.5%	201.4%	9.5%	5.0%	3.6%
1	24.8%	58.2%	62.7%	64.7%	99.8%	4.6%	2.6%	1.9%
4/3	56.7%	67.9%	70.3%	71.4%	39.7%	2.6%	1.6%	1.3%
3/2	61.7%	71.4%	73.0%	74.0%	26.4%	2.2%	1.4%	1.1%
2	78.6%	78.4%	79.0%	79.4%	9.9%	1.6%	1.2%	1.0%

Table 6. Average profit improvements of MIX for $p_{sMax}/p_{sMin} = 1.2$, $(\gamma_1, \gamma_2, \gamma_3) \in \{(0,0,0),(0,0,1)\}$

	Avg. IMP_{MSP}				Avg. IMP_{MCC}			
ω	BD=2	BD=10	BD=25	BD=50	BD=2	BD=10	BD=25	BD=50
1/2	0.0%	0.0%	0.0%	0.0%	207.9%	54.5%	43.0%	38.7%
2/3	0.0%	0.0%	0.0%	0.0%	171.9%	39.4%	30.3%	25.5%
3/4	0.0%	0.0%	0.0%	0.0%	156.0%	31.9%	24.0%	20.9%
1	0.0%	0.6%	1.0%	1.3%	99.8%	9.7%	5.9%	4.4%
4/3	18.5%	19.6%	18.4%	18.2%	41.1%	2.7%	1.7%	1.4%
3/2	32.6%	28.2%	26.0%	25.4%	26.9%	2.2%	1.5%	1.2%
2	58.4%	45.8%	42.3%	41.1%	10.0%	1.7%	1.2%	1.0%

Table 7. Average profit improvements of MIX for $p_{sMax}/p_{sMin} = 1.2$, $(\gamma_1, \gamma_2, \gamma_3) \in \{(1,0,0),(0,1,0)\}$

	Avg. IMP_{MSP}				Avg. IMP_{MCC}			
ω	BD=2	BD=10	BD=25	BD=50	BD=2	BD=10	BD=25	BD=50
1/2	25.0%	46.2%	54.9%	71.4%	185.9%	36.0%	23.1%	18.9%
2/3	38.8%	54.3%	61.2%	75.3%	142.6%	31.5%	21.4%	17.9%
3/4	44.8%	57.6%	63.8%	76.9%	126.0%	29.8%	20.7%	17.6%
1	58.3%	65.1%	69.8%	80.6%	91.6%	26.2%	19.3%	16.9%
4/3	69.4%	72.0%	75.4%	84.1%	66.8%	23.5%	18.2%	16.3%
3/2	73.1%	74.5%	77.4%	85.4%	59.1%	22.5%	17.8%	16.1%
2	80.5%	79.9%	82.0%	88.4%	45.0%	20.6%	17.1%	15.4%

Table 8. Average profit improvements of MIX for $p_{sMax}/p_{sMin} > 1.2$, $(\gamma_1, \gamma_2, \gamma_3) \in \{(0,0,0),(0,0,1)\}$

Observation 5. *MSP (MCC) performs better under LLCC (HLCC), and, when ω is low (high).*

The impact of ω on MSP and MCC is quite obvious, which indicates that the optimal mix mostly consists of spot purchasers (contract customers), when ω is low (high). The impact of $\gamma(p_c, l_c)$ on the performance of MSP and MCC can be analyzed as follows: The setting of $\gamma(p_c, l_c)$ does not affect λ^C, but affects λ^S. Thus, OAP obtained by MCC does not depend on the setting of $(\gamma_1, \gamma_2, \gamma_3)$, whereas the OAP

	Avg. IMP_{MSP}				Avg. IMP_{MCC}			
ω	BD=2	BD=10	BD=25	BD=50	BD=2	BD=10	BD=25	BD=50
1/2	1.6%	0.6%	0.3%	15.8%	143.6%	57.9%	46.7%	42.2%
2/3	2.1%	1.4%	1.6%	16.9%	126.4%	47.4%	37.6%	33.6%
3/4	2.7%	2.8%	2.9%	18.1%	117.7%	42.9%	33.5%	29.9%
1	8.4%	9.5%	9.7%	23.7%	92.7%	31.9%	24.5%	21.7%
4/3	26.6%	24.3%	24.1%	35.8%	68.6%	26.5%	21.2%	19.2%
3/2	34.5%	30.5%	30.0%	40.8%	60.6%	24.9%	20.3%	18.5%
2	51.5%	44.7%	43.5%	52.0%	46.0%	22.1%	18.6%	17.3%

Table 9. Average profit improvements of MIX for $p_{sMax}/p_{sMin} > 1.2$, $(\gamma_1, \gamma_2, \gamma_3) \in \{(1,0,0),(0,1,0)\}$

obtained MSP and MIX significantly increases (decreases) under LLCC (HLCC). As a result, as the value of $\gamma(p_c, l_c)$ increase, average IMP_{MCC} and IMP_{MSP} values increase and decrease, respectively. This is intuitive because offering low price and lead time contract terms by MCC to reduce the proportion of lost customers, is a desirable strategy under HLCC. Under LLCC, on the other hand, the manufacturer can increase p_c and l_c without risking the loss of customers due to rejected contracts.

Observation 6. *MCC performs significantly worse when BD is low.*

There are two different affects of BD on the performances of MSP and MCC: (i) Increasing BD, increases the OAPs obtained by MIX, MSP and MCC, in turn, decreasing the impact of improvements offered by MIX, and increasing IMP_{MCC} and IMP_{MSP}. (ii) my observations in Section 3.2 show that joint DPLQ brings higher profit improvements when BD is low. Because MCC only focuses on contract customers, it misses the potential profit improvement offered by joint DPLQ, and hence, performs significantly worse, as BD decreases. For example, average IMP_{MCC} values are above 100% when BD=2 and $\omega < 1$. In addition, the OAP obtained by

MCC is less than zero, in 36% of all instances when BD=2. MSP, on the other hand, benefits from the improvements offered by joint DPLQ, and performs better as BD decreases. Consequently, MCC performs better due to (i) and (ii) as BD increases. However, performance of MSP may increase or decrease due to the two conflicting effects of (i) and (ii).

Observation 7. *MIX performs better under low price range.*

Average IMP_{MCC} and IMP_{MSP} values are significantly lower when $p_{sMax}/p_{sMin} = 1.2$, i.e., low price range, and under HLCC. For example, in Table 7, 26 out of 52 cells are shaded, whereas in Table 9, 9 out of 52 cells are shaded. Better performance of MIX under high price range is due to the wide range of price opportunities (i.e., $p_c \in [p_{cMin}, p_{cMax}]$) that can be offered to contract customers. When the price range is high, the optimal price term p_c^* may be significantly different from the non-optimal price terms, p_{cMin} and p_{cMax} offered by MCC and MSP, respectively. Hence, implementing MCC or MSP may result in significant profit loss. In other words, as the range of possible price offers for contract customers increase, the manufacturers should pay more attention to determination of optimal contract terms.

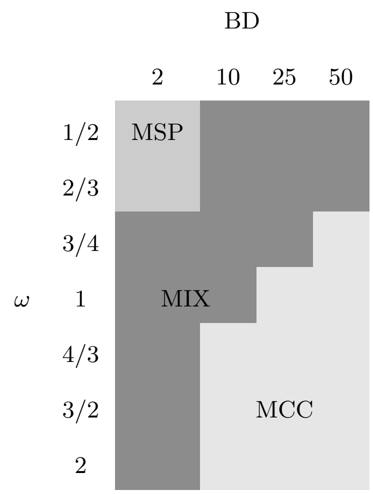
As a result, MIX significantly improves over both MCC and MSP under HLCC and high price range, as observed in Table 8, where the minimum average value of IMP_{MCC} and IMP_{MSP} is 15.4%. Under LLCC and low price range, profits offered by MSP and MCC approach to that of MIX when $\omega \leq 1$ and $\omega > 1$, respectively. In particular, maximum value of IMP_{MSP} is 0.37% when $\omega \leq 1$, indicating that MIX does not bring a significant improvement over MSP in this setting.

Although MIX typically outperforms MSP and MCC, one observes that implementing MSP and MCC may yield to close profits to that of MIX under some cases. In these cases, the manufacturers may choose to implement MSP (MCC) to focus entirely on spot purchasers (contract customers), and avoid the burden of solving `Cont`. Thus, I next provide recommendations for manufacturers, using a similar approach that is followed in Section 3.2. I define a threshold, which I refer to as the minimum profit improvement required for MIX, and denote it as Δ_{OM}. Similar to approach followed in Section 3.2, I provide recommendations ensuring the following: (i) The manufacturer does not offer the optimal mix, unless it provides OAP improvements higher than Δ_{OM}, and (ii) the manufacturer does not sacrifice OAP improvement more than Δ_{OM}. In Table 3, I provide the recommendations for $\Delta_{OM} = 10\%$. Recommendations for $\Delta_{OM} = 1\%$ and 20% is provided in the Appendix A-2. I note that Tables 10 (a), (b) and (c) are in- line Tables 6, 7 and 8 respectively, where MSP and MCC are recommended only when average IMP_{MSP} and IMP_{MCC} values are significantly small. In Table 10 (d), however, MIX is always recommended, while one observes that average MSP values may fall below 1% in Table 9. This is due to possible profit improvements of MIX that exceeds 10% when $p_{sMax}/p_{sMin} > 1.2$ and $(\gamma_1, \gamma_2, \gamma_3) \in \{(1,0,0),(0,1,0)\}$.

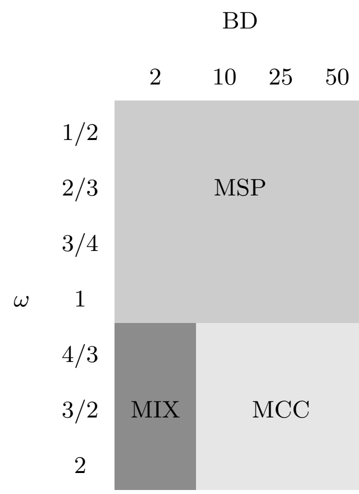
5. Conclusion

I consider the price and lead time quotation problem of a MTO manufacturer who faces a mix of contract customers and spot purchasers. Contract customers are offered a price and lead time quote at the beginning of the time horizon. Spot purchasers arrive over time, and are quoted prices and lead times dynamically. I

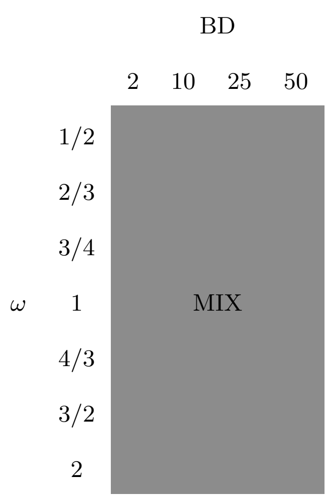
(a) $p_{sMax}/p_{sMin} = 1.2$, $\gamma \in \{(0,0,0),(0,0,1)\}$



(b) $p_{sMax}/p_{sMin} = 1.2$, $\gamma \in \{(1,0,0),(0,1,0)\}$



(c) $p_{sMax}/p_{sMin} > 1.2$, $\gamma \in \{(0,0,0),(0,0,1)\}$



(d) $p_{sMax}/p_{sMin} > 1.2$, $\gamma \in \{(1,0,0),(0,1,0)\}$

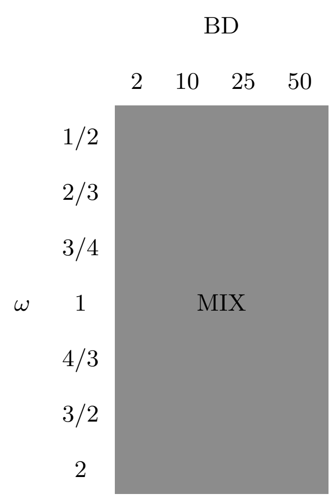


Table 10. Policy recommendations for $\Delta_{OM} = 10\%$

model the dynamic pricing and lead time quotation problem of spot purchasers as a Markov decision process over an infinite horizon, with the objective of maximizing the long run expected average profit per unit time. I analyze the structure of an optimal price and lead time policy, and examine its relation to the price and lead time sensitivities of the spot purchasers. I show that when the spot purchasers are price sensitive, it may be optimal to accept all customers by quoting a sufficiently small price, and managing congestion by increases in lead times. When the customers are lead time sensitive, on the other hand, zero lead time quotes are often optimal.

Using an extensive numerical study, I investigate the impact of several problem parameters on the profit improvements obtained by dynamic quotation when compared to fixed price and lead time strategies. In particular, I observe the superior performance of dynamic pricing (lead time quotation) over dynamic lead time quotation (pricing) for lead time (price) sensitive spot purchasers. In addition, a recommendation scheme is developed.

I finally focus on the selection of optimal price and lead time terms for contract customers to maximize long-run expected average profits. Because the problem requires excessive computational time, I focus on developing faster algorithms and reducing the action space. My computational analysis reveal that my computational algorithm performs quite satisfactorily in a limited amount of computational time.

Several extensions are possible to improve the modeling of customer behavior. One can also consider a model where unmet lead times impacts future decisions of customers over time, as a function of previous interactions with the company. These effects may incentivize the company to quote different prices and/or lead times in consideration of the future impact that current quotes may have on the customers' likelihood of accepting quotes.

CHAPTER 3

Expected Tardiness Computations for Multiclass Priority $M/M/c$ Queues

1. Introduction

In many make-to-order environments, due date quotation is a key aspect of business transactions. The due date quoted by the manufacturer impacts whether or not the customer decides to do business (i.e., place an order) with that manufacturer. For example, National Bicycle Industrial Co., a sports bicycle producer in Japan, emphasizes their ability to offer short due dates proudly in their slogan:"I can deliver a custom-made bicycle to you within two weeks [Lean-manufacturing-japan.com, 2012]. Once an order is placed, if the actual completion time of the order exceeds the manufacturer's promised due date, there may be monetary penalties as well as negative implications for future business. Savasaneril *et al.* [2010] mention several examples, where companies in various industries pay huge penalties due to late deliveries.

The late delivery penalty often (linearly) increases in the tardiness of the order, where tardiness is defined as the positive difference between the completion time of the order and the promised due date. For example, metal pipe producer Merle Blanc stipulates a contract, where suppliers pay a penalty amounting to 1% of the shipment value for each week the product is delayed [Merleblanc.de, 2012].

While quoting a due date, companies need the ability to estimate the tardiness of an order, and in turn, evaluate the late delivery penalties that can be incurred. Hence, obtaining sufficiently accurate tardiness estimates plays an important role in quoting due dates, which in turn impacts the ability to attract customer orders and maintain profitability.

Due date management literature partially addresses the issue of expected tardiness computation in the context of dynamic quotation models. The due date to quote to an arriving customer is often based on the system status, where the expected tardiness is crucial in the determination of the optimal due date. Research to date has considered relatively simple $M/M/1$ queuing models with a single customer class [Feng *et al.*, 2011, Savasaneril *et al.*, 2010], or multiple customer classes with First-Come-First-Served (FCFS) sequencing [Duenyas, 1995]. In practice, suppliers may face demand from various customer classes and prioritize the orders based on different criteria, such as the type of contract between the supplier and the customer, or whether the order comes from a long-term versus a one-time customer. For example, Motorola offers a three-day repair time with a bronze service plan, where the regular warranty guarantees ten-day repair time [Motorola, 2012]. Given that prioritization of orders is a common practice in the presence of multiple customer classes, extension of dynamic due date quotation research to multiple customer classes requires the computation of expected tardiness of orders from different priority classes.

The expected tardiness can be calculated using the distribution of the time in system (often referred to as cycle time or sojourn time in references, and denoted as TIS henceforth) of the order, and a simple conditioning argument. When the distribution of the TIS is known explicitly as, for example, in the case of a single priority class, the derivation of the expected tardiness is straightforward. In contrast, when there are multiple priority classes, a high priority order arrival may push a low priority order to the back of the queue, leading to an increase in the TIS for the low priority order. The derivation of the TIS distribution is not trivial in this case, and requires the use of Laplace transforms, which do not necessarily yield a closed-form TIS distribution [Wein, 1991, Zeltyn *et al.*, 2009].

In this chapter, I discuss the computation of expected tardiness of an order at the time of arrival for a given due date (denoted as ETA henceforth) in an $M/M/c$ queuing system. ETA is computed given (i) the quoted lead time, d (i.e., the difference between the due date and the time of arrival), (ii) the state of the queuing system at the time of arrival, and (iii) the class of the arriving order. I present my results for the cases of preemptive and non-preemptive service, and discuss a special case with a single priority class. As also argued by Zeltyn *et al.* [2009], the extension to general service times (i.e., $M/G/c$ queues) requires more complex analysis, which is beyond the scope of this chapter.

More specifically, I formulate the Laplace transform of ETA, and strive to obtain the inverse transforms. For the trivial single-class case, I obtain a closed-form expression for ETA. For the multiple-class case, however, Laplace transforms cannot be inverted into a closed-form expression, necessitating the implementation of Numerical Inverse Laplace Transformation (NILT) methods. Even though there is an abundance of general-purpose NILT algorithms (denoted as GP-NILTA henceforth) in the literature, the existing studies do not provide error bounds for the non-invertible Laplace transforms such as those in my case. The lack of error bounds may result in significant errors in late delivery penalty estimation (LPE), which may lead to erroneous due date policies. Consequently, obtaining error bounds for the NILT methods is crucial for due date management.

For the multiple classes case with non-invertible Laplace transforms, I develop three customized NILT algorithms (C-NILTA), which I refer to as *trapezoidal* ($\mathcal{Z}$), *midpoint* ($\mathcal{M}$), and *hybrid* ($\mathcal{H}$) algorithms, and show that $\mathcal{Z}$ overestimates and $\mathcal{M}$ underestimates ETA under a simple condition. This result allows us to provide lower and upper bounds for ETA, and to obtain worst-case error bounds for $\mathcal{Z}, \mathcal{M}$ and $\mathcal{H}$. Using a computational analysis, I test the precision of worst-case bounds, and observe that precision increases with computational time. Noting that high computational times may be undesirable for decision makers, I do the following: (i) test the performance of three fast and prominent GP-NILTAs presented in Abate and Whitt [2006], which provide a unification of the state-of-the-art GP-NILTAs, (ii) develop a recommendation scheme on the usage of NILT methods, and (iii) illustrate the recommendations using an example.

GP-NILTAs are first developed in the late 60's by the seminal studies of Dubner and Abate [1968], Gaver [1966], Zakian [1969], and Stehfest [1970]. Various variants of these algorithms have been developed since then. The readers are referred to Abate and Whitt [1992] for an extensive survey. The performance of GP-NILTAs has also been studied extensively, where Abate and Valko [2004], Abate and Whitt [2006], Avdis and Whitt [2007] and Hassanzadeh and Pooladi-Darvish [2007] present some recent examples. The common performance evaluation method in these papers is the comparison of algorithm results with the exact results of Laplace transforms for which the closed form inverses are readily available. For example, Abate and Whitt [2006] test the Laplace transforms in Equations (46) and (48) in their paper, whose inverse transforms are given in Equations (47) and (49), respectively. On the other hand, Hassanzadeh and Pooladi-Darvish [2007] study some non-invertible Laplace transforms, and compare the results of GP-NILTAs with each other. As also stated by Avdis and Whitt [2007], GP-NILTAs are not needed for the invertible Laplace transforms, but indispensable for non-invertible ones. In this chapter, I present an approach to evaluate the performance of GP-NILTAs in the context of expected tardiness estimation in a multi-class queue, which involves non-invertible Laplace transforms.

While results on the distributions of waiting time in the queue/system are abundant in the literature, to the best of my knowledge, this is the first study addressing the expected tardiness computation problem in a multi-class queuing setting. Earlier studies typically focus on the derivation of queuing performance criteria, e.g., TIS and waiting time distributions in priority queues. Davis [1966] derives the TIS distribution in a non-preemptive priority $M/M/c$ queue using conditioning arguments and Laplace transforms. Using a similar approach, Heyman and Sobel [2004] derive first passage time distributions in an $M/M/1$ queue. Similar to Heyman and Sobel [2004], I define the TIS of orders in Section 4. Following the approach of Davis [1966], Segal [1970] derives the Laplace transforms of TIS distributions in an $M/M/c$ queue with preemptive service for a service rate of $\mu = 1$. Unfortunately, $\mu = 1$ case does not provide a generalizable result for arbitrary values of the service rate, and hence, I revisit those results to provide a more general expression in Section 5. More recently, Zeltyn *et al.* [2009] discuss a special case with K priority classes, where P of the highest priority classes ($P < K$) can preempt the lower priority orders in an $M/M/c$ queue. The authors derive the Laplace-Stieltjes transforms of the TIS and waiting time distributions. There are some other studies considering the more general $M/G/c$ case [Jagerman and Melamed, 2003, Paterok and Ettl, 1994, Stanford and Drekic, 2000]. However, none of these studies address the ETA computations.

The early studies on due date quotation focus on the comparison of due date setting and scheduling rules using simulation analysis, with objectives or constraints on the expected (weighted) tardiness or the number of tardy jobs [Baker and Bertrand, 1981, Bertrand, 1983, Bookbinder and Noor, 1985, Hunsucker and Shah, 1992, Wein, 1991]. The reader is referred to Keskinocak and Tayur [2004] for an extensive literature survey on due date quotation. Wein [1991] proposes a due date quotation rule with the goal of minimizing the weighted average of due date quotes, subject to an upper bound on the expected tardiness. Similar to my study, the author considers prioritization under multiple customer classes, and implements Laplace transformation to compute the expected TIS of the low priority customer class in an example with two customer classes. More recently, due date quotation studies have focused on the derivation of optimal policies (with the goal of maximizing profits subject to late delivery penalties) assuming that the probability of the purchase is a function of the quoted due date to the customer [Duenyas, 1995, Duenyas and Hopp, 1995, Feng *et al.*, 2011, Savasaneril *et al.*, 2010]. Extension of these studies from FCFS settings to more realistic prioritized multi-class customer models necessitates the computation of ETA, which is the focus of this study.

In addition to being the first approach to compute ETA in the literature, my study has three major contributions in the field of NILT. First, I develop error bounds for the inverse Laplace transformation for my problem; such error bounds have not been obtained in the literature, even for a restricted class of transforms, according to Abate and Valko [2004]. Second, since the exact solutions for non-invertible Laplace transforms are not known, one cannot measure how good an algorithm performs in terms of solution quality. The error bounds I developed present a novel methodology to measure the quality of approximations obtained by GP-NILTAs using non-invertible Laplace transforms. Third, using my recommendation scheme, my C-NILTAs provide approximations within desired precision requirements, which is not necessarily achieved by GP-NILTAs.

The remainder of the chapter is organized as follows. In Section 2, I formulate the Laplace transform of the ETA. In Section 3, I discuss the case of single priority class, and provide a closed-form solution for the ETA. The Laplace transforms of the ETA for the non-preemptive and preemptive service cases are derived, in Sections 4 and 5, respectively. The upper and lower bounding algorithms for ETA are presented at the end of Section 4 for the non-preemptive case, and modified to handle the preemptive case at the end of Section 5. I conduct a computational analysis to compare the performances of NILT algorithms in Section 6, where I also develop recommendations on the use of appropriate NILT algorithm illustrating the benefits of error bounds obtained using the approach presented in this chapter. I conclude with a discussion on the contributions of the chapter in Section 7.

2. Methodology

I consider an $M/M/c$ queuing system with N prioritized customer classes, where the arrival rate is λ_i for class $i \in \{1, \ldots, N\}$, and the service rate is μ for each class. Without loss of generality, I assume that class k has higher priority than class m

where $k < m \leq N$, $k \in \{1, \ldots N-1\}$, and customer orders from the same priority class are sequenced on a FCFS basis. I assume that there are no upper bounds on the number of orders in the system, and the stability condition is met, i.e., $\sum_{i=1}^{N} \lambda_i < c\mu$. Let $V_i(t)$ and $Q_i(t)$, $i \in \{1, \ldots, N\}$, denote the number of class i orders in the system and in the queue, respectively, at time t. Let $V_0(t) = \sum_{i=1}^{N} V_i(t)$ denote the total number of orders in the system at time t. The state of the system at time t is defined by the vector $Z(t) = (V(t), Q(t))$, where $V(t) = (V_1(t), V_2(t), \ldots, V_N(t))$, and $Q(t) = (Q_1(t), Q_2(t), \ldots, Q_N(t))$. $\{Z(t), t \in T\}$ defines a stochastic process, and can be modeled as a continuous time Markov chain. My formulations provide expressions for the expected tardiness of an order that arrives at time t and is quoted a lead time of d. For simplicity, I drop t from the notation below and refer to the order for which the expected tardiness is evaluated as the "marked" order. Let $T_{z,i}(d)$ denote the ETA of a class i marked order if d is the quoted lead time and z is the system state upon the arrival of the order. Equation (3.1) provides the expression for ETA.

$$T_{z,i}(d) = \int_{d}^{\infty} (x-d) f_{z,i}(x) dx = S_{z,i} - d + \tau_{z,i}(d), \quad d \geq 0, \tag{3.1}$$

where $f_{z,i}(\cdot)$ denotes the probability density function (pdf) of the TIS, $S_{z,i}$ denotes the expected TIS, and $\tau_{z,i}(d)$ denotes the expected earliness of the marked order at the time of arrival, where earliness is defined as the positive difference between the due date and completion time of the marked order. By definition,

$$S_{z,i} = \int_{0}^{\infty} x f_{z,i}(x) dx, \quad \text{and} \quad \tau_{z,i}(d) = \int_{0}^{d} (d-x) f_{z,i}(x) dx, \quad d \geq 0. \tag{3.2}$$

I note that $T_{z,i}(d)$ depends on both the system state and the class of the marked order but it is computed for a given d that may or may not depend on the system state and the class of the marked order. For example, a manufacturer may develop a class dependent due date quotation scheme, or may quote the same due date to all customers. For a given d, the value of ETA is the same in both cases.

I evaluate $\tau_{z,i}(d)$ and $S_{z,i}$ by Laplace transforms. Following the notation of Heyman and Sobel [2004], I denote the Laplace transform of the function $f(\cdot)$ as

$$\widetilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx, \quad s \in \mathbb{C},$$

where $\mathbb{C}$ denotes the set of complex numbers.

Remark 1. *The convolution property of Laplace transforms [Doetsch, 1974, Theorem 10.1] implies,*

$$a(t) = \int_0^t b(x)c(t-x)dx \quad \Longleftrightarrow \quad \widetilde{a}(s) = \widetilde{b}(s)\widetilde{c}(s).$$

Using Laplace transforms and the convolution property I prove the following.

Proposition 3.

$$\widetilde{\tau}_{z,i}(s) = \frac{\widetilde{f}_{z,i}(s)}{s^2}, \quad s \in \mathbb{C}.$$

Proof. Note that $\tau_{z,i}(d)$ is the convolution of two functions by Equation (3.2). Letting $\gamma(x) = x$, $x \in \mathbb{R}$, the convolution can be expressed as

$$\tau_{z,i}(d) = (\gamma * f_{z,i})(d) = \int_0^d \gamma(d-x) f_{z,i}(x)dx, \quad d \geq 0.$$

Using the convolution property,

$$\widetilde{\tau}_{z,i}(s) = \widetilde{\gamma}(s)\widetilde{f}_{z,i}(s) \ . \tag{3.3}$$

The Laplace transform of $\gamma(x)$ is equal to $1/s^2$. By plugging in Equation (3.3), I reach the desired equation. $\square$

Using Equation (3.1) and Proposition (1), I have

$$T_{z,i}(d) = S_{z,i} - d + L_d^{-1}\left\{\frac{\widetilde{f}_{z,i}(s)}{s^2}\right\}, \quad d \geq 0, \tag{3.4}$$

where $L_d^{-1}\{\cdot\}$ denotes the inverse Laplace transformation operation evaluated at point $d \geq 0$. Using the properties of Laplace transforms I obtain

$$S_{z,i} = -\frac{d\widetilde{f}_{z,i}(s)}{ds}\Big|_{s=0}.$$

Hence, Equation (3.4) requires the derivation of $\widetilde{f}_{z,i}(s)$ as well as the inverse transformation of $\widetilde{\tau}_{z,i}(s) = \widetilde{f}_{z,i}(s)/s^2$. I begin the analysis with the special case of $N = 1$, i.e., one customer class.

3. Special Case: Single Customer Class

Since orders within each class are ordered in FCFS manner, $N = 1$ case is equivalent to the well-known FCFS sequencing of all customer orders. For this special case, v_0 (the number of orders in the system upon arrival of the marked order) becomes the only necessary information for the ETA evaluation. Thus, z is replaced with v_0. The index i is dropped from the notation, i.e., I aim to compute $T_{v_0}(d)$, which depends on whether all the servers are busy upon arrival of the marked order. Hence, there are two cases to consider.

Case I. $\mathbf{v_0 \leq c - 1}$: There is at least one server available at the time of the marked order's arrival, and the processing of the marked order can start immediately. Hence, the TIS of the marked order is distributed exponentially with rate μ. Using

Equation (3.1), I get

$$T_{v_0}(d) = \int_d^{\infty} (x-d)\mu e^{-\mu x} dx = \frac{e^{-\mu d}}{\mu}, \quad d \geq 0. \tag{3.5}$$

Case II. $\mathbf{v_0 > c - 1}$: Since all servers are busy, the marked order has to wait for the completion of $v_0 - c + 1$ orders before its processing can start. Thus, the TIS of the marked order is a random variable that is equal to the sum of the Waiting Time in the Queue (WTQ) and the Processing Time (PT). WTQ can be expressed as the sum of $v_0 - c + 1$ exponentially distributed random variables with rate $c\mu$, and hence, is a gamma random variable with parameters $v_0 - c + 1$ and $c\mu$. PT is an exponential random variable with rate μ, and its pdf is denoted by $h_\mu(\cdot)$. Then,

$$h_\mu(x) = \mu e^{-\mu x}, \quad x \geq 0 \ , \quad \text{and} \quad \widetilde{h}_\mu(s) = \frac{\mu}{s+\mu}, \quad s \in \mathbb{C}. \tag{3.6}$$

From the above, the pdf of the TIS, $f_{v_0}(\cdot)$, is the convolution of $h_\mu(\cdot)$ and (v_0-c+1)-fold convolution of $h_{c\mu}(\cdot)$, implying that the Laplace transform of the $f_{v_0}(\cdot)$ is equal to

$$\widetilde{f}_{v_0}(s) = \widetilde{h}_\mu(s)(\widetilde{h}_{c\mu}(s))^{v_0-c+1} = \frac{\mu}{s+\mu}\left(\frac{c\mu}{s+c\mu}\right)^{v_0-c+1}, \quad s \in \mathbb{C}.$$

Using Proposition (3), I get the Laplace transform of $\tau_{v_0}(d)$ as

$$\widetilde{\tau}_{v_0}(s) = \frac{\mu}{s^2(s+\mu)}\left(\frac{c\mu}{s+c\mu}\right)^{v_0-c+1}, \quad s \in \mathbb{C}. \tag{3.7}$$

In Theorem 4, I derive $T_{v_0}(d)$ using $\widetilde{\tau}_{v_0}(s)$.

Theorem 4.

$$T_{v_0}(d) = \begin{cases} \dfrac{e^{-d\mu}}{\mu} \displaystyle\sum_{k=0}^{v_0} (d\mu)^k \frac{v_0 + 1 - k}{k!}, & \textit{if } c = 1, \\ \\ \frac{e^{-d\mu}}{\mu} \left(\frac{c}{c-1}\right)^{v_0 - c + 1} + \dfrac{e^{-dc\mu}(dc\mu)^{v_0 - c + 1}}{c\mu(v_0 - c)!} & \\ +e^{-dc\mu} \displaystyle\sum_{k=0}^{v_0 - c} \frac{(d\mu)^k}{k!} \left[c^k \left(\frac{1}{\mu} - d + \frac{v_0 - c + 1}{c\mu} \right) - \frac{(c-1)^k}{\mu} \left(\frac{c}{c-1} \right)^{v_0 - c + 1} \right], & \textit{if } c \geq 2. \end{cases} \tag{3.8}$$

Proof. Proofs of theorems are provided in Appendix B-2. □

This result provides a closed-form solution for the expected tardiness of an order which observes v_0 orders in the system upon arrival in an $M/M/c$ queue with FCFS service discipline, when the quoted lead time is d. I next discuss the derivations for systems with multiple customer classes, under the non-preemptive and preemptive service assumptions.

4. Non-preemptive Service Discipline

In this section, I consider $N \geq 2$ customer classes, indexed in decreasing priority. Orders are sequenced on a FCFS basis within each class, and preemption of service is not allowed. That is, once the processing of an order is started, it cannot be interrupted even if an order of higher priority joins the queue. In Section 5, I consider the preemptive service case, where a higher priority order can preempt the processing of a lower priority order.

Similar to single-class case, the TIS of the marked order can be divided into two durations: (i) the time elapsed until the order starts processing, i.e., the Waiting Time in Queue (WTQ), and (ii) the processing time (PT). Any high priority order arrival during the WTQ pushes the marked order back in the queue and increases its

TIS. For a given marked order, from class i, one needs to consider two parameters for the evaluation of the ETA: (i) the number of orders in the system, v_0, and (ii) the number of orders in the queue ahead of the marked order, $\overline{q}_i$, which can be calculated from state variables q_j as

$$\overline{q}_i = \sum_{j=1}^{i} q_j \ ,$$

where i denotes the class of the marked order. To find $T_{v_0,\overline{q}_i,i}(d)$, similar to Section 3, I consider two cases based on server availability at the time of the marked order's arrival.

Case I. $\mathbf{v_0 \leq c - 1}$: This condition implies that $\overline{q}_i = 0$, and the order can start processing immediately, i.e., the WTQ of the marked order is equal to zero. Using Equation (3.5), I get

$$T_{v_0,0,i}(d) = \frac{e^{-\mu d}}{\mu}, \quad d \geq 0. \tag{3.9}$$

Case II. $\mathbf{v_0 > c - 1}$: The marked order is behind $\overline{q}_i$ orders in the queue. Thus, the parameter, v_0 is redundant and is dropped from the notation. Researchers have focused on the use of Laplace transforms to derive the TIS distribution in this case [Davis, 1966, Segal, 1970, Wein, 1991, Zeltyn *et al.*, 2009]. Let $\overline{\lambda}_i$ denote the total arrival rate of customer orders with higher priority than class i. Then,

$$\overline{\lambda}_i = \sum_{j=1}^{i-1} \lambda_j, \qquad i \in \{2, \ldots, N\},$$

where $\overline{\lambda}_1 = 0$. I can obtain $T_{\overline{q}_1,1}(d)$ by plugging $c + \overline{q}_1$ for v_0 in Equation (3.8). Hence, I focus on the calculations for lower priority classes. Since $\overline{\lambda}_i$ and $\overline{q}_i$ uniquely defines priority class of the order, I also drop i from the notation, and derive $\widetilde{\tau}_{\overline{q}_i}(s)$ for $i \in \{2, \ldots, N\}$ in Theorem 5.

Theorem 5. *For* $i \in \{2, \ldots, N\}$,

$$\widetilde{\tau}_{\overline{q}_i}(s) = \frac{\mu}{s^2(s+\mu)} \left(\widetilde{g}_{\overline{\lambda}_i, c\mu}(s)\right)^{\overline{q}_i+1}, \quad s \in \mathbb{C}, \tag{3.10}$$

where

$$\widetilde{g}_{\overline{\lambda}_i, c\mu}(s) = \frac{s + \overline{\lambda}_i + c\mu - \sqrt{(s + \overline{\lambda}_i + c\mu)^2 - 4\overline{\lambda}_i c\mu}}{2\overline{\lambda}_i}, \quad s \in \mathbb{C}.$$

Using $\widetilde{\tau}_{\overline{q}_i}(s)$, one can employ any GP-NILTA to numerically approximate $\tau_{\overline{q}_i}(d)$, although it is not clear how well any of these algorithms would perform in this setting. To obtain error bounds and a testing methodology for the GP-NILTAs, I develop two C-NILTAs approximating $\tau_j(d)$ for $j \in \{0, 1, \ldots, \overline{q}_i\}$ using an iterative approach. The main idea is to write the Laplace transform in a product form, where each component of the product is inverse transformable as in Equation (3.10), which for $\overline{q}_i = 0$ gives

$$\widetilde{\tau}_0(s) = \frac{\mu}{s^2(s+\mu)} \widetilde{g}_{\overline{\lambda}_i, c\mu}(s) \ ,$$

and,

$$\widetilde{\tau}_j(s) = \widetilde{\tau}_0(s) \left(\widetilde{g}_{\overline{\lambda}_i, c\mu}(s)\right)^j \tag{3.11}$$

for all $i \in \{2, \ldots, N\}$ and $j \in \{0, 1 \ldots, \overline{q}_i\}$. Equation (3.11) for customer class i can be written recursively as

$$\widetilde{\tau}_j(s) = \widetilde{\tau}_{j-1}(s) \widetilde{g}_{\overline{\lambda}_i, c\mu}(s), \quad j \in \{0, 1, \ldots, \overline{q}_i\} \ , \tag{3.12}$$

where $\widetilde{\tau}_{-1}(s) = \frac{\mu}{s^2(s+\mu)}$. Equation (3.12) provides a product form of the Laplace transform with inverse transformable terms. Then, using the convolution property, I can write $\tau_j(\cdot)$ as a convolution of $\tau_{j-1}(\cdot)$ and $g_{\overline{\lambda}_i, c\mu}(\cdot)$ as

$$\tau_j(d) = \int_0^d \tau_{j-1}(d-x) g_{\overline{\lambda}_i, c\mu}(x) dx, \quad j \in \{0, 1, \ldots, \overline{q}_i\} \ , \tag{3.13}$$

where $\tau_{-1}(x) = x - \frac{1}{\mu} + \frac{e^{-\mu x}}{\mu}$. Thus, using Equation (3.13), $\tau_{\overline{q}_i}(d)$ can be evaluated by recursively calculating $\tau_j(d)$ for $j \in \{0, 1, \ldots \overline{q}_i\}$. Using the results of Heyman and Sobel [2004] I get,

$$g_{\overline{\lambda}_i, c\mu}(x) = L_x^{-1}\{\widetilde{g}_{\overline{\lambda}_i, c\mu}(s)\} = e^{-(\overline{\lambda}_i + c\mu)x} I_1\left(2x\sqrt{\overline{\lambda}_i c\mu}\right)\frac{\sqrt{c\mu}}{x\sqrt{\overline{\lambda}_i}}, \qquad x \geq 0, \quad (3.14)$$

where $I_1(\cdot)$ is the Bessel function of imaginary argument and first order [Heyman and Sobel, 2004, pg. 90]. For example, $\tau_0(d)$ can be evaluated by the expression

$$\tau_0(d) = \int_0^d \left(d - x - \frac{1}{\mu} + \frac{e^{-\mu(d-x)}}{\mu}\right) e^{-(\overline{\lambda}_i + c\mu)x} I_1\left(2x\sqrt{\overline{\lambda}_i c\mu}\right)\frac{\sqrt{c\mu}}{x\sqrt{\overline{\lambda}_i}} dx, \quad d \geq 0.$$

I next approximate $\tau_j(d)$ functions for $j \in \{0, 1, \ldots \overline{q}_i\}$ using numerical integration and Equation (3.13),

$$\tau_j(d) = \lim_{w \to 0} \frac{w}{2} \sum_{k=0}^{d/w-1} \tau_{j-1}(d - kw) g_{\overline{\lambda}_i, c\mu}(kw) + \tau_{j-1}(d - kw - w) g_{\overline{\lambda}_i, c\mu}(kw + w), \quad (3.15)$$

$$d \geq 0, \quad j \in \{0, 1, \ldots, \overline{q}_i\}, \quad \text{and}$$

$$\tau_j(d) = \lim_{w \to 0} w \sum_{k=0}^{d/w-1} \tau_{j-1}\left(d - kw - \frac{w}{2}\right) g_{\overline{\lambda}_i, c\mu}\left(kw + \frac{w}{2}\right), \quad d \geq 0, j \in \{0, 1, \ldots, \overline{q}_i\}, \quad (3.16)$$

where Equations (3.15) and (3.16) follow the trapezoidal and midpoint rules respectively [Davis and Rabinowitz, 1984, Chapter 2]. Using these two rules, I develop two C-NILTAs (pseudocodes are given in Appendix B-1): (i) *trapezoidal* ($\mathcal{Z}$) and, (ii) *midpoint* ($\mathcal{M}$) to inverse transform $\widetilde{\tau}_j(s)$, whose approximations are denoted as $\mathcal{Z}_j^w(d)$ and $\mathcal{M}_j^w(d)$, given the numerical integration parameter w. I next show that $\mathcal{Z}_j^w(d)$ and $\mathcal{M}_j^w(d)$, respectively, provide an upper and lower bound for $\tau_j(d)$ under simple and mild conditions on $\overline{\lambda_i}$ and $c\mu$.

Theorem 6. *$\mathcal{M}_j^w(d) \le \tau_j(d) \le \mathcal{Z}_j^w(d)$ hold for $d \ge 0$, $j = 0, 1, \ldots,$ and $w \le d$, when the following two conditions hold: (i) $\overline{\lambda_i} + c\mu \ge 2\frac{\overline{\lambda_i}c\mu + \sqrt{\overline{\lambda_i}c\mu}}{2\sqrt{\overline{\lambda_i}c\mu}-1}$ and (ii) $\overline{\lambda_i}c\mu \ge \frac{1}{4}$.*

Theorem 6 conditions are used to show the convexity of $g_{\overline{\lambda_i},c\mu}(\cdot)$, which is a sufficient condition for the proof of Theorem 6. In Corollary 3, I provide a simple condition that implies Theorem 6 conditions.

Corollary 3. *Theorem 6 conditions hold when $\overline{\lambda_i}c\mu \ge 4$.*

$\overline{\lambda_i}c\mu \ge 4$ is a simple, albeit more restrictive condition. Figure 8 plots the parameter space in terms of $\overline{\lambda_i}$ and $c\mu$ and graphically shows that Theorem 6 conditions hold everywhere except in the small shaded region.

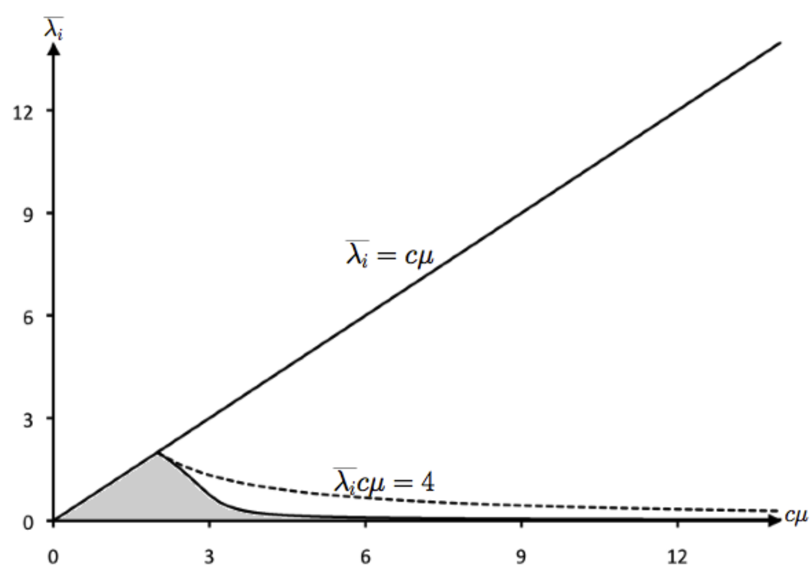


Fig. 8. Representation of the parameter space of Theorem 6 conditions, which hold everywhere except in the small shaded region

Recall that the region $\overline{\lambda_i} \ge c\mu$ is not considered due to the stability condition. $\overline{\lambda_i}c\mu \ge 4$, which is a sufficient condition for Theorem 6 conditions to hold, implies that the manufacturer's production rate times the total arrival rate of the customers with priority higher than i exceeds four. For example, a manufacturer with a production capacity of four or more per day, and receiving an average of one or more orders per day from the highest priority customer class satisfies this condition (when the lead times are quoted in daily or weekly time units, as is common practice in industry). Hence, the condition is not restrictive in practice. Under the cases where Theorem 6 conditions do not hold, I may still have convexity of $\tau_{j-1}(d-x)g_{\overline{\lambda}_i,c\mu}(x)$, which is a sufficient condition for Theorem 6 to hold.

Although numerical integration is commonly used in Laplace transformation methods [Abate and Whitt, 1992], $\mathcal{M}$ and $\mathcal{Z}$ are the first NILT algorithms that implement numerical integration methods in a recursive manner as given in Equations (3.15) and (3.16). In addition, to the best of my knowledge, they are the first algorithms providing lower and upper bounds for the inverse Laplace transform operation. The obtained lower and upper bounds enable a novel methodology in the comparison of GP-NILTAs. That is, any approximation value less than $\mathcal{M}_j^w(d)$ is definitely a worse approximation than $\mathcal{M}_j^w(d)$. Similarly, any approximation higher than $\mathcal{Z}_j^w(d)$ is a worse approximation than $\mathcal{Z}_j^w(d)$. Using these bounds, I compute worst-case error bounds for C-NILTAs in Section 6. One can achieve tighter bounds (smaller $\mathcal{Z}_j^w(d) - \mathcal{M}_j^w(d)$ values) by using lower values of w, at the expense of increased computational times due to the increased number of steps in numerical integration.

While developing my C-NILTAs, I made use of the product form of two Laplace transforms (as shown in Equation (3.11)) to obtain the closed form convolution integral. Hence, given any Laplace transform that can be inverted into a closed form inversion integral, one can implement numerical integration and develop C-NILTAs with the same structure. For example, such closed form inversion integrals are provided by Duffy [1993], who presents five Laplace transform examples with corresponding inverse transforms provided as integral equations.

To improve on the accuracy provided by the GP-NILTAs, I also present a *hybrid* C-NILTA, which applies trapezoidal/midpoint rule for odd/even j values for $j \in \{0, 1, \ldots, \overline{q}_i\}$. The *hybrid* algorithm ($\mathcal{H}$) approximates $\tau_j(d)$ by the value $\mathcal{H}_j^w(d)$ and the pseudocode of the algorithm is given in Appendix B-1.

Corollary 4. *If Theorem 6 conditions hold, then* $\mathcal{M}_j^w(d) \leq \mathcal{H}_j^w(d) \leq \mathcal{Z}_j^w(d)$ *for* $d \geq 0$, $j = 0, 1, \ldots,$ *and* $w \leq d$.

Thus, when Theorem 6 conditions hold, $\mathcal{H}$ always generates an approximation within the bounds given by $\mathcal{Z}$ and $\mathcal{M}$. Note that $\mathcal{H}_j^w(d)$ may or may not be a better approximation than $\mathcal{M}_j^w(d)$, $\mathcal{Z}_j^w(d)$, or any value obtained by any GP-NILTAs. However, it is within the derived worst-case bounds that are discussed in Section 6.

I next discuss the evaluation of $S_{\overline{q}_i}$. Wein [1991] derives the expected TIS for class-dependent service rates and a single server. I evaluate the expected TIS of the marked order by modifying the necessary parameters, e.g., service rates, as follows.

$$S_{\overline{q}_i} = \frac{\overline{q}_i + 1}{c\mu - \overline{\lambda}_i} + \frac{1}{\mu}. \tag{3.17}$$

Finally, $T_{\overline{q}_i}(d)$ can be evaluated using $\mathcal{Z}$, $\mathcal{M}$, $\mathcal{H}$, or any of GP-NILTA, and Equations (3.1) and (3.17).

5. Preemptive Priority Discipline

I now consider the case where an arriving order of higher priority preempts the lowest priority in-process order, which, upon being preempted, joins the beginning of the queue pushing back all the orders in the queue by one position. In this case, any class i order is inserted behind $\overline{v}_i = \sum_{j=1}^{i} v_j$ orders (including the orders in process), pushing back the remaining $v_0 - \sum_{j=1}^{i} v_j$ orders by one position. My goal is to evaluate $T_{\overline{v}_i,i}(d)$. In Figure 9, I illustrate a three-server case, where the numbers inside and outside the circles denote the priority class and position of the order, respectively. If a class 2 order arrives when the state is as depicted in Figure 9, it is inserted to the second position, pushing all the lower priority orders back by one position, and the processing of the class 4 order in the third position is preempted.

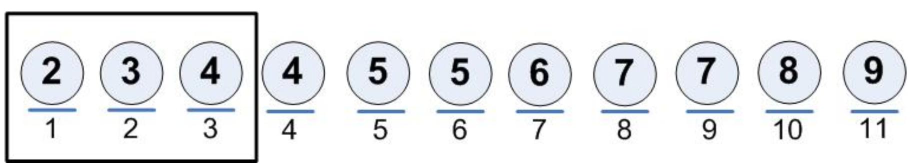


Fig. 9. Position of orders under preemptive discipline

Similar to Section 4, the highest priority orders are placed ahead of the other orders in the queue, and sequenced on a FCFS basis. Thus, the ETA for the highest priority class, $T_{\overline{v}_1,1}(d)$, can be evaluated by plugging v_1 for v_0 in Equation (3.8). For the lower priority classes, $i \in \{2, \ldots, N\}$, similar to the non-preemptive case, I have two alternatives to consider: (i) the marked order starts processing immediately

($\overline{v}_i \leq c-1$), and (ii) the marked order joins the queue ($\overline{v}_i > c-1$). As in the non-preemptive case, I drop i from the notation, and derive $\widetilde{f}_{\overline{v}_i}(s)$.

Case I. $\mathbf{\overline{v}_i > c-1}$: If this condition holds, then the marked order joins the queue by taking the position $\overline{v}_i+1$. The marked order is pushed back by the arrival of any higher priority arrival with rate $\overline{\lambda}_i$, and moves one position forward with rate $c\mu$, as long as its processing has not started yet. Thus, the Laplace transforms for $\overline{v}_i > c-1$ are obtained in Equation (3.18), similar to the non-preemption case given in Equation (3.11).

$$\widetilde{f}_{\overline{v}_i}(s) = \widetilde{f}_{c-1}(s)\left(\widetilde{g}_{\overline{\lambda}_i,c\mu}(s)\right)^{\overline{v}_i-c+1}, \quad s \in \mathbb{C}\ . \tag{3.18}$$

Case II. $\mathbf{\overline{v}_i \leq c-1}$: Laplace transforms for this case are derived in Theorem 7.

Theorem 7. *For* $\overline{v}_i \leq c-1$, $i \in \{2,\ldots,N\}$,

$$\widetilde{f}_{\overline{v}_i}(s) = \frac{\overline{\lambda}_i\widetilde{f}_{\overline{v}_i+1}(s) + \overline{v}_i\mu\widetilde{f}_{\overline{v}_i-1}(s) + \mu}{s+\overline{\lambda}_i+(\overline{v}_i+1)\mu}, \quad s \in \mathbb{C}\ . \tag{3.19}$$

Letting $\mu = 1$ in Equation (3.19), one reaches the result of Segal [1970]. However, the result in Theorem 7 cannot be derived by rescaling the time variable, i.e., setting $\overline{\lambda}_i/\mu$ for $\overline{\lambda}_i$ in Segal [1970]. To prove Theorem 7, I use Laplace-Stieltjes transforms, as presented in Appendix B-2.

By setting $\overline{v}_i = 0$, the middle term in the numerator in Equation (3.19) disappears, which gives

$$\widetilde{f}_0(s) = \frac{\overline{\lambda}_i\widetilde{f}_1(s)+\mu}{s+\overline{\lambda}_i+\mu}, \quad s \in \mathbb{C}.$$

By setting $\overline{v}_i = c-1$, using Equation (3.18) and algebraic operations I get

$$\widetilde{f}_{c-1}(s) = \frac{(c-1)\mu\widetilde{f}_{c-2}(s)+\mu}{s+\overline{\lambda}_i+c\mu-\overline{\lambda}_i\widetilde{g}_{\overline{\lambda}_i,c\mu}(s)} = \widetilde{g}_{\overline{\lambda}_i,c\mu}(s)\left((c-1)\widetilde{f}_{c-2}(s)+1\right), \quad s\in\mathbb{C}. \tag{3.20}$$

Consequently, having c unknowns and c equations by Theorem 7, Laplace transforms are solvable. The analysis simplifies vastly in the single server case, i.e., $c=1$, which is discussed below.

Single Server Case: Setting $c=1$ in Equation (3.20) and by algebraic manipulations one gets for $i\in\{2,3,\ldots,N\}$

$$\widetilde{f}_{\overline{v}_i}(s) = \left(\widetilde{g}_{\overline{\lambda}_i,\mu}(s)\right)^{\overline{v}_i+1}, \text{and} \quad \widetilde{\tau}_{\overline{v}_i}(s) = \frac{\left(\widetilde{g}_{\overline{\lambda}_i,\mu}(s)\right)^{\overline{v}_i+1}}{s^2} \quad s\in\mathbb{C}\ . \tag{3.21}$$

Thus, the recursive form in Equation (3.13) is valid for inversion of $\widetilde{\tau}_{\overline{v}_i}(s)$ given in Equation (3.21) with a modification on $\tau_{-1}(x)$ as

$$\tau_{-1}(x) = L_x^{-1}\{1/s^2\} = x.$$

Moreover,

$$S_{\overline{v}_i} = \frac{\overline{v}_i+1}{\mu-\overline{\lambda}_i}.$$

I note that, results of Theorem 6 and Corollary 1 hold for this case as well.

I derive the Laplace transforms and discuss the implementation of $\mathcal{Z}$, $\mathcal{M}$, $\mathcal{H}$ for $c=2$ and $c=3$ in Online Supplement. The useful property of Theorem 6 may or may not hold for multiple servers case due to the extensive complexity of the Laplace transforms in these cases. In the next section, I test the performances of $\mathcal{Z}$, $\mathcal{M}$, $\mathcal{H}$, as well as the three prominent GP-NILTAs for the cases where I can obtain worst-case error bounds using Theorem 6, i.e., non-preemptive and preemptive single-server cases.

6. Computational Analysis

In this section, I provide a computational study to (i) analyze the precision of approximations obtained by $\mathcal{M}, \mathcal{Z}$ and $\mathcal{H}$, and (ii) assess the performance of three prominent GP-NILTAs, namely, Gaver-Stehfest ($\mathcal{G}$), Euler ($\mathcal{E}$), and Talbot ($\mathcal{T}$). Abate and Whitt [2006] provide a general framework for GP-NILTAs and reduce the GP-NILTAs in the literature into these three main algorithms. I then provide a practical recommendation scheme for the inverse transformation of $\widetilde{\tau}_j(s)$. To evaluate the error precision, I assess the number of significant digits obtained by $\mathcal{M}, \mathcal{Z}$ and $\mathcal{H}$ in Equation (3.22) (see Equation (34) of Abate and Whitt [2006]).

$$P(X_j^w)(d) = -\log\left|\frac{X_j^w(d) - \tau_j(d)}{\tau_j(d)}\right|, \quad X \in \{\mathcal{M}, \mathcal{Z}, \mathcal{H}\}. \tag{3.22}$$

Since the exact values of $\tau_j(d)$ are not known, I define the following approximation for the number of significant digits obtained by $\mathcal{Z}$, $\mathcal{M}$, and $\mathcal{H}$.

$$\rho_j^w(d) = -\log\left|\frac{\mathcal{Z}_j^w(d) - \mathcal{M}_j^w(d)}{\mathcal{M}_j^w(d)}\right|. \tag{3.23}$$

Note that, if Theorem 6 conditions holds, I have $\rho_j^w(d) \leq P(X_j^w)(d)$, $X \in \{\mathcal{M}, \mathcal{Z}, \mathcal{H}\}$, i.e., $\rho_j^w(d)$ is a lower bound for the number of significant digits produced by $\mathcal{Z}$, $\mathcal{M}$, and $\mathcal{H}$, and hence, represents the worst-case error bounds for the three C-NILTAs developed in this chapter. The same condition applies to any GP-NILTA only when the obtained approximation value is within the interval $[\mathcal{M}_j^w(d), \mathcal{Z}_j^w(d)]$.

GP-NILTAs generally require the setting of several parameters as in the case of the Euler algorithm [Abate and Whitt, 1999], which necessitates the selection of four different parameters. Abate and Whitt [2006] overcome this burden by reducing the number of parameters in $\mathcal{G}$, $\mathcal{E}$, and $\mathcal{T}$ to a single parameter called M, which is common to all of the three algorithms. The authors recommend that M should be set to $\lceil 1.1R \rceil$, $\lceil 1.7R \rceil$, and $\lceil 1.7R \rceil$ to obtain R significant digits in $\mathcal{G}$, $\mathcal{E}$, and $\mathcal{T}$, respectively (see Sections 4, 5 and 6 in Abate and Whitt [2006]). I denote the required parameter setting to obtain R significant digits by $M_X(R), X \in \{\mathcal{G}, \mathcal{E}, \mathcal{T}\}$, i.e., $M_{\mathcal{G}}(R) = \lceil 1.1R \rceil$, $M_{\mathcal{E}}(R) = \lceil 1.7R \rceil$, $M_{\mathcal{T}}(R) = \lceil 1.7R \rceil$. The approximations obtained using these parameter settings are denoted as $X_j^R(d)$.

My testing methodology works as follows. For each instance, I first generate the approximations $\mathcal{Z}_j^w(d)$, $\mathcal{M}_j^w(d)$, and compute $\rho_j^w(d)$, for four levels of $w \in \{0.005, 0.001, 0.0005, 0.0001\}$, ten levels of $d \in D = \{1, 2, \ldots, 10\}$, and ten levels of $j \in J = \{0, 1, \ldots, 9\}$, by running $\mathcal{Z}$ and $\mathcal{M}$ with $\overline{q}_i = 9$, and $\overline{d} = 10$. I then generate approximations $\mathcal{G}_j^R(d)$, $\mathcal{E}_j^R(d)$, and $\mathcal{T}_j^R(d)$ for $d \in D$ and $j \in J$, and $R \in \{5, 7\}$. Finally, for each (w, R) pair in $\{0.005, 0.001, 0.0005, 0.0001\} \times \{5, 7\}$, I count the number of instances in which the approximations $\mathcal{G}_j^R(d)$, $\mathcal{E}_j^R(d)$, and $\mathcal{T}_j^R(d)$ are within the interval $[\mathcal{M}_j^w(d), \mathcal{Z}_j^w(d)]$, over all $d \in D$ and $j \in J$.

I use Visual C++ 6.0 compiler that allows a system precision of 18 digits, which is sufficient for my test bed. The maximum value of R is chosen as 7, since the system precision requirement for $\mathcal{G}$ exceeds 18 for $R > 8$ [Abate and Whitt, 2006]. I select a minimum value of $w = 0.0001$ to avoid excessively long computational times.

I test eight instances under the non-preemptive case with $c = 1$, $\mu = 5$, $\overline{\lambda}_i \in \{1, 2, 3, 4\}$, and $c = 3$, $\mu = 5$, $\overline{\lambda}_i \in \{3, 6, 9, 12\}$, and four instances under the preemptive case with $c = 1$, $\mu = 5$, $\overline{\lambda}_i \in \{1, 2, 3, 4\}$, to test one representative case for single and multiple servers under changing traffic intensity. Note that all of the instances comply with Theorem 6 conditions and the stability condition. The results for the non-preemptive and preemptive cases are presented in Tables 11 and 12, respectively. For each instance-w pair represented in a row, I report (i) the average, minimum, and maximum $\rho_j^w(d)$ over all $d \in D$, $j \in J$, (ii) the average, minimum and maximum $\mathcal{Z}_{j,i}^w(d) - \mathcal{M}_{j,i}^w(d)$, over all $d \in D$, $j \in J$, (iii) total CPU times of $\mathcal{Z}$ and $\mathcal{M}$, and (iv) the number of $\mathcal{G}_j^R(d)$, $\mathcal{E}_j^R(d)$, and $\mathcal{T}_j^R(d)$ that are within $[\mathcal{M}_{j,i}^w(d), \mathcal{Z}_{j,i}^w(d)]$ over all $d \in D$, $j \in J$ (i.e., out of 100 instances), for $R \in \{5, 7\}$, where R is indicated inside the parenthesis as $\mathcal{G}(R)$, $\mathcal{E}(R)$, and $\mathcal{T}(R)$. The average, minimum and maximum values of $\rho_{j,i}^w(d)$ are evaluated only for nonnegative values of $\mathcal{M}_{j,i}^w(d)$ due to the logarithm restriction in Equation (3.23). I also omit the CPU times of $\mathcal{G}$, $\mathcal{E}$, and $\mathcal{T}$ since they are negligibly small. Experiments are run on a Intel Core 2 Quad CPU PC with processors running at 2.66 GHz and 4 GB memory under Windows 7.

Observation 8. *Up to eight significant digit precision is obtained in around 20 seconds by $\mathcal{Z}$, $\mathcal{M}$ and $\mathcal{H}$, whose performances significantly improve at a cost of higher computational times.*

Since $\mathcal{Z}$, $\mathcal{M}$ and $\mathcal{H}$ provide approximations that are guaranteed to be within the bounds, I conclude that my three proposed C-NILTAs are the best NILT methods in terms of solution quality. From Tables 11 and 12, when $c = 1$, up to nine significant digits are attained by either of $\mathcal{Z}$, $\mathcal{M}$ and $\mathcal{H}$, under preemptive and non-preemptive cases. One also observes that the increase in the number of servers negatively impacts the performance of the algorithms by reducing $\rho_j^w(d)$. For example, when $w = 0.0005$, the average $\rho_j^w(d)$ ranges from 2.57 to 5.29 when $c = 1$, and from

Table 11. Performances of NILT algorithms under non-preemptive case

				$\rho_j^w(d)$			$\mathcal{Z}_j^w(d) - \mathcal{M}_j^w(d)$			CPU time		# of gen. in $[\mathcal{M}_j^w(d), \mathcal{Z}_j^w(d)]$					
w	c	μ	$\overline{\lambda}_i$	avg.	min.	max.	avg.	min.	max.	$\mathcal{M}$	$\mathcal{Z}$	$\mathcal{G}(5)$	$\mathcal{E}(5)$	$\mathcal{T}(5)$	$\mathcal{G}(7)$	$\mathcal{E}(7)$	$\mathcal{T}(7)$
0.005	1	5	1	1.73	-1.52	5.47	2.E-03	5.E-06	7.E-03	0.21	0.17	59	100	92	90	100	91
	1	5	2	1.55	-1.49	5.81	2.E-03	3.E-06	7.E-03	0.21	0.17	74	100	94	95	100	95
	1	5	3	2.33	0.43	6.20	2.E-03	2.E-06	7.E-03	0.21	0.17	90	100	94	98	100	94
	1	5	4	3.29	2.04	6.71	1.E-03	1.E-06	5.E-03	0.21	0.17	90	100	95	99	100	95
	3	5	3	1.00	-0.78	1.97	2.E-02	7.E-04	7.E-02	0.21	0.17	100	100	100	100	100	100
	3	5	6	0.94	-1.20	2.48	2.E-02	8.E-04	8.E-02	0.21	0.17	99	100	100	100	100	100
	3	5	9	0.81	-1.24	3.03	2.E-02	8.E-04	9.E-02	0.21	0.17	99	100	100	100	100	100
	3	5	12	1.05	-0.58	3.70	2.E-02	5.E-04	8.E-02	0.21	0.17	100	100	100	100	100	100
0.001	1	5	1	2.36	-1.11	6.87	8.E-05	2.E-07	2.E-04	5.37	4.39	8	87	86	39	100	85
	1	5	2	2.56	-1.25	7.21	8.E-05	1.E-07	3.E-04	5.37	4.37	23	89	91	58	100	92
	1	5	3	3.73	1.87	7.60	8.E-05	1.E-07	2.E-04	5.36	4.39	39	97	92	82	100	93
	1	5	4	4.69	3.44	8.11	7.E-05	7.E-08	2.E-04	5.36	4.39	54	98	93	91	100	93
	3	5	3	1.39	-1.58	3.37	9.E-04	2.E-05	3.E-03	5.36	4.39	82	100	100	93	100	100
	3	5	6	1.48	-1.28	3.88	1.E-03	3.E-05	3.E-03	5.35	4.39	77	100	100	99	100	100
	3	5	9	1.29	-2.30	4.43	1.E-03	3.E-05	3.E-03	5.37	4.39	88	100	100	99	100	100
	3	5	12	2.49	1.16	5.10	1.E-03	2.E-05	3.E-03	5.35	4.39	95	100	100	100	100	100
0.0005	1	5	1	2.57	-3.54	7.47	2.E-05	5.E-08	7.E-05	21.43	17.50	1	34	79	12	100	85
	1	5	2	3.18	0.09	7.81	2.E-05	4.E-08	8.E-05	21.40	17.51	3	40	90	34	100	91
	1	5	3	4.33	2.48	8.20	2.E-05	3.E-08	7.E-05	21.38	17.51	16	39	91	48	100	91
	1	5	4	5.29	4.04	8.71	1.E-05	1.E-08	5.E-05	21.38	17.51	24	44	91	71	100	93
	3	5	3	1.84	-1.21	3.97	2.E-04	7.E-06	7.E-04	21.38	17.51	61	100	99	83	100	100
	3	5	6	1.79	-0.90	4.48	2.E-04	8.E-06	8.E-04	21.38	17.53	59	100	100	86	100	100
	3	5	9	1.61	-1.36	5.03	2.E-04	8.E-06	9.E-04	21.41	17.51	60	100	100	97	100	100
	3	5	12	3.10	1.77	5.70	2.E-04	5.E-06	8.E-04	21.43	17.54	83	100	100	98	100	100
0.0001	1	5	1	3.38	-0.92	8.87	8.E-07	2.E-09	2.E-06	540.03	435.62	0	0	44	0	86	82
	1	5	2	4.58	1.59	9.21	8.E-07	1.E-09	3.E-06	538.73	435.59	0	0	52	0	88	87
	1	5	3	5.73	3.87	9.60	8.E-07	1.E-09	2.E-06	538.79	435.61	1	0	71	4	97	89
	1	5	4	6.69	5.44	10.11	7.E-07	7.E-10	2.E-06	538.74	435.64	0	0	84	13	98	90
	3	5	3	2.56	-1.06	5.37	9.E-06	2.E-07	3.E-05	538.91	435.65	12	0	90	34	100	98
	3	5	6	2.36	-1.49	5.88	1.E-05	3.E-07	3.E-05	539.06	435.63	12	0	95	19	100	100
	3	5	9	2.53	-0.63	6.43	1.E-05	3.E-07	3.E-05	539.04	435.64	2	0	100	51	100	100
	3	5	12	4.49	3.17	7.10	1.E-05	2.E-07	3.E-05	539.22	435.71	18	0	100	70	100	100

1.61 to 3.10 when $c = 3$. The traffic intensity, on the other hand, has a positive impact on the performance of the algorithms, as observed under most instances in Tables 11 and 12. Comparison of preemptive and non-preemptive cases reveals

Table 12. Performances of NILT algorithms under preemptive case

				$\rho_j^w(d)$			$\mathcal{Z}_j^w(d) - \mathcal{M}_j^w(d)$			CPU time		# of gen. in $[\mathcal{M}_j^w(d), \mathcal{Z}_j^w(d)]$					
w	c	μ	$\overline{\lambda}_i$	avg.	min.	max.	avg.	min.	max.	Mid.	Trap.	$\mathcal{G}(5)$	$\mathcal{E}(5)$	$\mathcal{T}(5)$	$\mathcal{G}(7)$	$\mathcal{E}(7)$	$\mathcal{T}(7)$
0.005	1	5	1	2.27	1.42	5.11	2.E-03	1.E-05	7.E-03	0.21	0.17	71	100	92	96	100	91
	1	5	2	2.36	1.41	5.46	2.E-03	9.E-06	8.E-03	0.21	0.17	85	100	93	97	100	93
	1	5	3	2.64	1.65	5.86	2.E-03	6.E-06	7.E-03	0.23	0.18	92	100	95	100	100	95
	1	5	4	3.29	2.30	6.38	2.E-03	4.E-06	6.E-03	0.21	0.17	94	100	95	100	100	95
0.001	1	5	1	3.67	2.82	6.51	9.E-05	5.E-07	3.E-04	5.38	4.39	18	89	90	56	100	88
	1	5	2	3.76	2.81	6.86	9.E-05	3.E-07	3.E-04	5.36	4.39	29	96	92	71	100	91
	1	5	3	4.04	3.06	7.26	9.E-05	2.E-07	3.E-04	5.36	4.38	48	98	91	93	100	92
	1	5	4	4.69	3.70	7.78	8.E-05	2.E-07	2.E-04	5.36	4.39	61	99	92	94	100	93
0.0005	1	5	1	4.27	3.43	7.11	2.E-05	1.E-07	7.E-05	21.44	17.54	0	38	83	28	100	84
	1	5	2	4.36	3.41	7.46	2.E-05	9.E-08	8.E-05	21.43	17.54	9	42	91	48	100	89
	1	5	3	4.64	3.66	7.86	2.E-05	6.E-08	7.E-05	21.43	17.54	17	43	90	73	100	91
	1	5	4	5.29	4.30	8.38	2.E-05	4.E-08	6.E-05	21.42	17.54	29	50	91	90	100	92
0.0001	1	5	1	5.67	4.82	8.51	9.E-07	5.E-09	3.E-06	545.22	437.12	0	0	49	1	88	80
	1	5	2	5.76	4.81	8.86	9.E-07	3.E-09	3.E-06	544.21	437.03	2	0	59	2	89	85
	1	5	3	6.04	5.06	9.26	9.E-07	2.E-09	3.E-06	544.13	436.96	0	0	85	3	98	88
	1	5	4	6.69	5.70	9.78	8.E-07	2.E-09	2.E-06	544.72	437.09	0	0	88	19	99	90

that slightly higher $\rho_j^w(d)$ values are obtained for the preemptive case. This is due to the lower ETA values under preemptive case resulting in lower $\mathcal{M}_j^w(d)$ values, while $\mathcal{Z}_j^w(d) - \mathcal{M}_j^w(d)$ differences do not differ significantly under preemptive and non-preemptive cases. Moreover, I observe that lower w values lead to an increase in both CPU times and $\rho_j^w(d)$.

Observation 9. *$\mathcal{E}$ outperforms $\mathcal{G}$ and $\mathcal{T}$.*

Results of Observation 9 are seen on the right-most six columns of Tables 11 and 12, with some exceptions for $R = 5$, $w \in \{0.0005, 0.0001\}$, where $\mathcal{T}$ performs better. Using the parameter setting $M_{\mathcal{E}}(7)$, $\mathcal{E}$ generates approximations 100% within the bounds until the maximums of $\rho_j^w(d)$ exceed 8.5 (see the results for $w = 0.0001$). Similarly, under the non-preemptive case, $\mathcal{E}$ fails to generate approximations 100%

within the bounds only when $\rho_j^w(d)$ maximums reach 8.87 and 6.87 respectively for the settings with $M_{\mathcal{E}}(7)$, and $M_{\mathcal{E}}(5)$ (see the first row for $w = 0.0001$, and the fifth row for $w = 0.001$ in Table 11, respectively). I conclude that $\mathcal{E}$ approximations are reliable for $R \geq \lceil \rho_j^w(d) \rceil - 1$, when the parameter setting $M_{\mathcal{E}}(R)$ is used.

$\mathcal{G}$ performs surprisingly poorly in some instances. For example, consider the non-preemptive case results in Table 11 for $w = 0.001$. When it is executed with the parameter setting $M_{\mathcal{G}}(5)$, 92% of the approximations are out of the bounds, where the average of $\rho_j^w(d)$ is only 2.36. This may occur due to two main reasons: (i) the significant digits produced by $\mathcal{Z}$ and $\mathcal{M}$ are higher than 5, (recall that $\rho_j^w(d)$ gives a lower bound), (ii) the parameter setting $M_{\mathcal{G}}(5)$ gives fewer significant digits than 5, probably even less than 2.36. The performances of $\mathcal{E}$ and $\mathcal{T}$ for the same instance, which are 87%, and 88%, respectively, imply that case (i) is not possible. Hence, I conclude that the precision analysis of Abate and Whitt [2006] does not hold for this particular instance.

In summary, $\mathcal{Z}$, $\mathcal{M}$ and $\mathcal{H}$ perform best in terms of solution quality, whereas $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{T}$ run faster but at the expense of lower solution quality. Although I cannot determine which one of $\mathcal{Z}$, $\mathcal{M}$ and $\mathcal{H}$ gives better solution quality on the average, I recommend the use of $\mathcal{H}$ which is the hybrid version of the other two algorithms. The algorithmic structures reveal that under the same settings, $\mathcal{H}$ runs longer than $\mathcal{Z}$ and shorter than $\mathcal{M}$. On the other hand, $\mathcal{E}$ is the best NILT algorithm when fast solutions are needed, and the precision requirements are relaxed.

6.1. *NILT Algorithm Recommendations*

In this section, I obtain an upper bound on the expected late delivery penalty estimation (LPE) error, and develop a recommendation scheme for the use of an appropriate NILT algorithm. Consider the moment when a class i customer arrives to the system when there are j orders in the queue. Assuming that the customer is quoted a lead time of d, and the late delivery penalty cost per unit time is k dollars per unit time, the expected late delivery penalties to be incurred can be expressed as $kT_{j,i}(d)$. Using Theorem 6 and Equation (3.1), $kT_{j,i}(d)$ can be bounded as

$$k\left(S_{z,i} - d + \mathcal{M}^{w}_{j,i}(d)\right) \leq kT_{j,i}(d) \leq k\left(S_{z,i} - d + \mathcal{Z}^{w}_{j,i}(d)\right) \tag{3.24}$$

where I include the priority class index i back into the notation to avoid confusion. Using Equation (3.24), $kT_{j,i}(d)$ can be estimated with a maximum error of $k\left(\mathcal{Z}^{w}_{j,i}(d) - \mathcal{M}^{w}_{j,i}(d)\right)$. Let $C(\mathcal{H}^{w}_{j,i}(d))$ denote the computational time of $\mathcal{H}$ for approximating $\tau_{j,i}(d)$, within the LPE error tolerance, $\overline{K}$, and maximum allowable computational time for NILT of the manufacturer, $\overline{C}$, when $w \in W_1$ and $w \in W_2$ respectively. That is,

$$W_1 \in \{w : \max_{d\in\overline{D}, j\in\overline{J}, i\in\overline{I}} k\left(\mathcal{Z}^{w}_{j,i}(d) - \mathcal{M}^{w}_{j,i}(d)\right) \leq \overline{K}\}, \text{ and}$$

$$W_2 \in \{w : \max_{d\in\overline{D}, j\in\overline{J}, i\in\overline{I}} C(\mathcal{H}^{w}_{j,i}(d)) \leq \overline{C}\},$$

where $\overline{D}$, $\overline{J}$ and $\overline{I}$ denote the set of d, j and i values, respectively, under consideration for due date quotation. The NILT recommendation scheme is as follows:

- If there are no computational time restrictions, implement $\mathcal{H}$ with setting $w \in W_1$,

- if there are computational time restrictions, and $W_1 \cap W_2 \neq \emptyset$, implement $\mathcal{H}$ with setting $w \in W_1 \cap W_2$,

- if there are computational time restrictions, and $W_1 \cap W_2 = \emptyset$, find a $w^* \in W_1$. Then, implement $\mathcal{E}$ with setting R such that $\mathcal{E}^R_{j,i}(d) \in [\mathcal{M}^{w^*}_{j,i}(d), \mathcal{Z}^{w^*}_{j,i}(d)]$ for all $d \in \overline{D}$, $j \in \overline{J}$, and $i \in \overline{I}$.

The first two cases identify the conditions, where $\mathcal{H}$ is recommended due to its superior solution quality. In the third case, due to tight computational time restrictions, $\mathcal{E}$ is recommended with a guarantee that the estimation is within the error tolerances. Therefore, the presented recommendation scheme ensures that the selected NILT algorithm gives approximations within LPE error tolerance and acceptable computation times.

As discussed in Section 4, for the cases where the sufficient conditions for $\mathcal{M}^w_j(d) \leq \tau_j(d) \leq \mathcal{Z}^w_j(d)$ do not hold, the bounds are still likely to be valid. Hence, I suggest the use of the above recommendation scheme in general.

6.2. *An Example*

Consider a make-to-order manufacturer, who promises to pay k dollars to its customers per each day the product is delivered late, and wants to analyze the impact of lead time quotes on late delivery penalty costs. Assume that there is a single server working at a rate of 5 per day, three priority classes with arrival rates of $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 1$ per day, preemption is not allowed, and the manufacturer considers only ten lead time options as $\overline{D} = \{1, 2, \ldots, 10\}$ days, and quotes due dates to newly arriving customers when there are less than 10 orders in the system, i.e., $\overline{J} = \{0, 1, \ldots, 9\}$.

In Table 13, I present the maximum LPE errors and computational times obtained by using $\mathcal{H}$ with $w \in \{0.005, 0.001, 0.0005, 0.0001\}$ over all $j \in \overline{J}$, $d \in \overline{D}$, and $i \in \{2, 3\}$, for $k \in \{1000\$/\text{day}, 10000\$/\text{day}\}$. I note that the LPE for priority class 1 can be obtained with zero error using Theorem 4, and $\overline{\lambda_2} = \lambda_1 = 2$, $\overline{\lambda_3} = \lambda_1 + \lambda_2 = 3$.

Table 13. Maximum expected lateness penalty cost estimation errors and CPU times obtained by $\mathcal{H}$

		Priority Class 2		Priority Class 3	
k	w	max. LPE	max. CPU time	max. LPE	max. CPU time
$1000/day	0.005	$7	less than 1 sec.	$7	less than 1 sec.
	0.001	$0.3	around 5 sec.	$0.2	around 5 sec.
	0.0005	$0.08	around 20 sec.	$0.07	around 20 sec.
	0.0001	$0.003	around 8 min.	$0.002	around 8 min.
$10000/day	0.005	$70	less than 1 sec.	$70	less than 1 sec.
	0.001	$3	around 5 sec.	$2	around 5 sec.
	0.0005	$0.8	around 20 sec.	$0.7	around 20 sec.
	0.0001	$0.03	around 8 min.	$0.02	around 8 min.

Using the information from Table 13 and the recommendation scheme in Section 6.1, I develop NILT algorithm recommendations for three levels of $\overline{K} \in \{\$10, \$1, \$0.1\}$, and $\overline{C} \in \{10 \text{ sec.}, 1 \text{ min.}, 10 \text{ min.}\}$, for two delay penalty levels, $k \in \{\$1000/\text{day}, \$10000/\text{day}\}$ in Tables 14 and 15, respectively. I select the smallest w in case there are multiple $w \in W_1 \cap W_2$.

Table 14. NILT recommendations for $k = \$1000$/day

$\overline{K}$	$\overline{C}$	w within W_1	w within W_2	Recommendation
\$10	10 min.	{0.005, 0.001, 0.0005, 0.0001}	{0.005, 0.001, 0.0005, 0.0001}	$\mathcal{H}$ with $w = 0.0001$
\$10	1 min.	{0.005, 0.001, 0.0005, 0.0001}	{0.005, 0.001, 0.0005}	$\mathcal{H}$ with $w = 0.0005$
\$10	10 sec.	{0.005, 0.001, 0.0005, 0.0001}	{0.005, 0.001}	$\mathcal{H}$ with $w = 0.001$
\$1	10 min.	{0.001, 0.0005, 0.0001}	{0.005, 0.001, 0.0005, 0.0001}	$\mathcal{H}$ with $w = 0.0001$
\$1	1 min.	{0.001, 0.0005, 0.0001}	{0.005, 0.001, 0.0005}	$\mathcal{H}$ with $w = 0.0005$
\$1	10 sec.	{0.001, 0.0005, 0.0001}	{0.005, 0.001}	$\mathcal{H}$ with $w = 0.001$
\$0.1	10 min.	{0.0005, 0.0001}	{0.005, 0.001, 0.0005, 0.0001}	$\mathcal{H}$ with $w = 0.0001$
\$0.1	1 min.	{0.0005, 0.0001}	{0.005, 0.001, 0.0005}	$\mathcal{H}$ with $w = 0.0005$
\$0.1	10 sec.	{0.0005, 0.0001}	{0.005, 0.001}	$\mathcal{E}$ with $R = 7$

Table 15. NILT recommendations for $k = \$10000$/day

$\overline{K}$	$\overline{C}$	w within W_1	w within W_2	Recommendation
\$10	10 min.	{0.001, 0.0005, 0.0001}	{0.005, 0.001, 0.0005, 0.0001}	$\mathcal{H}$ with $w = 0.0001$
\$10	1 min.	{0.001, 0.0005, 0.0001}	{0.005, 0.001, 0.0005}	$\mathcal{H}$ with $w = 0.0005$
\$10	10 sec.	{0.001, 0.0005, 0.0001}	{0.005, 0.001}	$\mathcal{H}$ with $w = 0.001$
\$1	10 min.	{0.0005, 0.0001}	{0.005, 0.001, 0.0005, 0.0001}	$\mathcal{H}$ with $w = 0.0001$
\$1	1 min.	{0.0005, 0.0001}	{0.005, 0.001, 0.0005}	$\mathcal{H}$ with $w = 0.0005$
\$1	10 sec.	{0.0005, 0.0001}	{0.005, 0.001}	$\mathcal{E}$ with $R = 7$
\$0.1	10 min.	{0.0001}	{0.005, 0.001, 0.0005, 0.0001}	$\mathcal{H}$ with $w = 0.0001$
\$0.1	1 min.	{0.0001}	{0.005, 0.001, 0.0005}	$\mathcal{E}$ with $R > 7$
\$0.1	10 sec.	{0.0001}	{0.005, 0.001}	$\mathcal{E}$ with $R > 7$

Note that when $k = \$10000$/day and $\overline{K} = \$0.1$, only $w = 0.0001$ gives results within error tolerances, but it does not satisfy the computational time restriction when $\overline{C} \leq 1$ min; hence, the use of $\mathcal{E}$ is recommended. Since, the condition $\mathcal{E}_{j,i}^{R}(d) \in$

$[\mathcal{M}_{j,i}^{w^*}(d), \mathcal{Z}_{j,i}^{w^*}(d)]$ does not hold for $d \in \overline{D}, j \in \overline{J}, i \in \overline{I}$, $R \in \{5, 7\}$, and $w^* = 0.0001$, $\mathcal{E}$ should be executed with a parameter setting giving better solution quality, i.e., $R > 7$.

7. Conclusions

In this study, I discuss the computation of the expected tardiness of an order as a function of the system state at the moment of the order's arrival in an $M/M/c$ queue with N priority classes. The study is motivated by the need for tardiness computations in dynamic due date quotation models considering multiple priority classes. I first formulate the Laplace transform of expected tardiness and discuss the numerical inverse Laplace transformation methodologies to approximate the expected tardiness. I develop two customized numerical inverse Laplace transformation algorithms giving upper and lower bounds for the expected tardiness, and another hybrid algorithm that is guaranteed to give approximations well inside the bounds. Using the two bounds, I obtain worst-case error bounds for the developed algorithms, and test them using a computational study. Noting that these customized algorithms may require long computational times, I also test the performance of three prominent general purpose numerical inverse transformation algorithms which run faster. I finally develop a recommendation scheme on the selection of the numerical inverse transformation algorithm given an error tolerance and allowable computational time, and illustrate the recommendation scheme on an example.

My study is unique in a couple of aspects. First, to the best of my knowledge, it is the first study addressing the evaluation of expected tardiness at the moment of order arrival (ETA) in a multi-class queuing environment. Second, it presents three numerical inverse Laplace transformation algorithms with worst-case error bounds. Third, it provides a novel methodology for performance evaluation of numerical inverse Laplace transformation methods. Fourth, the Laplace transforms for the preemptive priority settings are rederived to generalize the previous results for $\mu = 1$ to any service rate.

My research is expected to contribute to the dynamic due date quotation literature to extend those studies to multi-server and multi-class cases. A worthwhile extension to my study is the consideration of class-dependent service rates, i.e., μ_i for class i. This case is more complex since the total service rate depends on the class of orders being processed, and the class of the higher priority arrival joining in front of the order affects the ETA. Extensions to more general $M/G/c$ or $G/G/c$ models are more difficult to handle, however, approximations based on the results outlined in this chapter may provide plausible solutions.

CHAPTER 4

Centralized and Decentralized Dynamic Price and Lead Time Quotation

1. Introduction

Recent business trends and advances in customer behavior modeling reveal the critical role of price and lead time decisions on the demand of goods and services. In 2006, Nintendo increased the Wii prices a short time after the product was released to keep up with the underestimated demand. On the other hand, Microsoft reduced Xbox prices to reduce the high number of products in retailer inventories [Cachon and Terwiesch, 2009]. The quoted lead time also has significant impact on demand of products in real-life systems. For example, in the beginning of 2010, polycarbonate lead times were increased from 4-6 weeks to 14-16 weeks, due to supply shortages and over-commitments [Victory, 2010]. By employing smart pricing and lead time quotation strategies, companies can manage their demand, increase profits, and satisfy customers.

Dynamic pricing and lead time quotation (DPLQ), where companies determine price and lead time quotes depending on the status of the company (e.g., congestion, workload, inventory level) has been shown to be a quite profitable strategy. Sahay [2007] indicates that companies from various sectors, such as apparel, automative, and telecommunication, have recently increased their profits and revenues by implementing dynamic pricing strategies. According to estimations of Accenture, the improvement in gross margins of major retailers can increase up to $20 billion by implementing dynamic pricing [Fleischmann *et al.*, 2005]. Benefits of joint DPLQ are addressed in Chapter 2, where I observe that up to 100% profit improvement can be attained using joint DPLQ.

DPLQ requires a high level of coordination between marketing department (which generally sets prices, and the manufacturing department (which generally determines lead times). However, the incoordination of marketing and manufacturing departments has been a major issue of today's companies, which has been widely discussed in trade publications and academic literature [Balasubramanian and Bhardwaj, 2004, Crittenden, 1992, Omurgonulsen and Surucu, 2008, Pekgun *et al.*, 2008, Shapiro, 1977, Shaw *et al.*, 1999]. For instance, Omurgonulsen and Surucu [2008] state that 63.2% of the manufacturing employees in Turkish manufacturing industry admit that they have poor coordination with marketing department. Technology and Manufacturing Group director of Intel Corporation indicates that the negative impact of the incoordination between marketing and manufacturing departments on their business is about 100M$ [Pekgun *et al.*, 2008].

In this chapter, my goal is to analyze the inefficiencies of marketing-manufacturing incoordination in a make-to-order company whose prices and lead times are dynamically quoted. I first consider a fully-coordinated centralized setting, where dynamic prices and lead times are determined by a central agent with the objective of maximizing expected profits, which is defined as revenues minus late delivery penalties. I next discuss three decentralized settings, where dynamic price quotes are determined by marketing department with the objective of maximizing expected revenues, and dynamic lead times are determined by manufacturing department with the objective of maximizing expected profits of the department, which is the incentives given to the manufacturing department minus late delivery penalties. The first decentralized setting assumes that price and lead time decisions are simultaneously determined, whereas the second and third settings considers the cases manufacturing and marketing departments act in advance, respectively.

Assuming stochastic arrivals and deterministic processing times, I model the centralized setting as finite horizon stochastic dynamic program with the objective of maximization of total discounted expected profit over a finite horizon. The decentralized settings, on the other hand, are modeled as repeated Cournot and Stackelberg games. I obtain a closed form solution for optimal dynamic price and lead times centralized setting, and obtain the Markov perfect equilibrium for the marketing and manufacturing departments under all decentralized setting. Using these findings, I finally compare the profits obtained under the two settings.

The remainder of the chapter is organized as follows. In Section 2 I survey the literature. I present the models for centralized and decentralized settings, and derive the closed form solutions for them in Section 3. In Section 4, I compare the performances of the two settings, analyze the inefficiencies of decentralized setting. In Section 5, I conclude.

2. Literature Survey

The literature on joint price and lead time quotation is mostly focused on determination of single values of price and lead times, i.e., fixed price and lead time quotation (FPLQ) to maximize company profits in make-to-order systems. Although, both centralized and decentralized settings are analyzed by FPLQ literature, DPLQ is only discussed under centralized settings. In Table 16 I categorize the studies that discuss price and lead time decisions according to two criteria: (i) the price and lead time quotation strategy, i.e., FPLQ or DPLQ, and (ii) whether they consider decentralized settings.

Table 16. Summary of literature review

	Centralized only	Centralized and Decentralized
FPLQ	Palaka *et al.* [1998], So and Song [1998], Easton and Moodie [1999], Boyaci and Ray [2003], Ray and Jewkes [2004]	Liu *et al.* [2007], Pekgun *et al.* [2008], Hua *et al.* [2010], Xiao *et al.* [2010]
DPLQ	Plambeck [2004], Charnsirisakskul *et al.* [2006], Celik and Maglaras [2008], Ata and Olsen [2009], Feng *et al.* [2011]	

Studies implementing FPLQ typically discuss the steady state behavior of the system to maximize company profits. Palaka *et al.* [1998] consider the problem of capacity, price and lead time selection to maximize the expected profit of the company, which is defined as revenue minus congestion cost and late delivery penalty, subject to a service level requirement, under an M/M/1 queuing model. This study is extended to multiple product settings by Boyaci and Ray [2003], who consider the substitution effect of two different products. Both of these studies assume linear demand forms similar to my study. So and Song [1998], on the other hand, consider the same problem under a log-linear demand function. Easton and Moodie [1999] discuss a different setting where companies compete on price and lead time bids that are offered to customers.

DPLQ literature mostly aims to maximize long run profits considering a finite or infinite horizon. Plambeck [2004] considers dynamic lead time and fixed price quotation, where the company also has the flexibility to choose the capacity, in order to maximize profits such that the customer orders are processed on time. Charnsirisakskul *et al.* [2006] discuss the dynamic price and fixed lead time quotation problem under deterministic processing rate similar to my study. Recently, joint DPLQ has been implemented by Celik and Maglaras [2008] and Feng *et al.* [2011]. While Celik and Maglaras [2008] investigate the expediting and sequencing policies considering a diffusion model, Feng *et al.* [2011] model the problem as a Markov decision processes and focus on the derivation of optimal policies and benefits of DPLQ, respectively.

Decentralization of price and lead time decisions are first considered by Liu *et al.* [2007] and Pekgun *et al.* [2008]. According to Pekgun *et al.* [2008], marketing departments, who determine price quotes, work as a revenue center with the objective of maximizing revenues. In contrast, manufacturing departments, who are in charge of lead time decisions, work as a cost center and strive to minimize them. In the setting of Liu *et al.* [2007], the manufacturing department (that acts as a supplier) charges the marketing department (that acts as a retailer) a wholesale price per order, and hence, aims to maximize its own profit. In my study, I adopt the setting of Pekgun *et al.* [2008], which is in line with the setting of Balasubramanian and Bhardwaj [2004] as well. Recently Hua *et al.* [2010] and Xiao *et al.* [2010] extend Liu *et al.* [2007] into a dual sourcing model where the supplier (i.e., the manufacturing department in my setting) has the option to make independent sales, and a setting where there exists a subcontractor between the supplier and the retailer (i.e., the marketing department in my setting), respectively.

My study aims to fill the void for the implementation of DPLQ under centralized and decentralized settings. Similar to Liu *et al.* [2007], Pekgun *et al.* [2008] and Hua *et al.* [2010], I analyze the centralized and decentralized settings separately, and assess the inefficiencies of decentralized setting.

3. Model

I consider a make-to-order (MTO) manufacturing company, who produces a single type of product. Customers arrive to the system according to a stochastic process, and there is a single server working with a deterministic processing time of ν. At the moment the customer arrives, she is quoted a price p, and a lead time l by the company. The customer accepts this quote with probability $f(p, l)$, which is assumed to be linear and non-increasing in p and l. If the quote is rejected, the customer leaves the system. I define the general properties of $f(p, l)$ in Assumption 3, which is similar to Assumption 1 in Chapter 2.

Assumption 3. *(i) There exist nonnegative lower bounds on price and lead time, such that decreasing price and lead times below these lower bounds does not change $f(p, l)$. The lower bound on price is p_{min}, whereas the lower bound on lead time is zero. Furthermore, given a lead time quote l, there exists an upper bound on price (denoted by $p_{max}(l)$), such that any price quote above that level is definitely rejected. Similarly, given the price quote p, there exists an upper bound on lead time (denoted by $l_{max}(p)$), such that any price quote above that level is definitely rejected. $p_{max}(l)$ and $l_{max}(p)$ are nonincreasing in l and p, respectively. Furthermore, the maximum values of $p_{max}(l)$ and $l_{max}(p)$*

are denoted as p_{max} *and* l_{max}*, respectively. Hence, I have,* $l_{max} = \bar{l}_{max}(p_{min})$*, and* $p_{max} = \overline{p}_{max}(0)$*.*

(ii) $f(p,l)$ *is linear and decreasing in* p *and* l *within* θ *that is defined as follows.*

$$f(p,l) = 1 - \frac{p - p_{min}}{p_{max} - p_{min}} - \frac{l}{l_{max}}, \quad \textit{for } (p,l) \in \theta, \textit{where} \tag{4.1}$$

$$\theta = \{(p,l) \in \Re^2 : p_{min} \le p \le p_{max}(l), 0 \le l \le l_{max}\}. \tag{4.2}$$

Assumption 3 (i) confines $f(p,l)$ into $[0,1]$ interval. Assumption 3 (ii) defines general properties of $f(p,l)$ such as monotonocity in price and lead time, and linearity. Although linearity assumption is restrictive for the structure of $f(p,l)$, it is commonly implemented in the literature due to its analytical tractibility [Boyaci and Ray, 2003, Hua *et al.*, 2010, Liu *et al.*, 2007, Palaka *et al.*, 1998, Pekgun *et al.*, 2008]. Similar to Chapter 2, I denote p_{min} as the accept all price in the remainder.

If the customer accepts the quote, then the order is placed in the queue, using customer orders are sequenced in first-come-first-serve (FCFS) discipline. If the order is processed later than the quoted due date, a lateness penalty is paid which increases linearly with the lateness duration at a rate of τ is incurred by the manufacturer.

I consider four decision making schemes: centralized setting, where a central agent makes both price and lead time decisions, and three decentralized settings where the prices are determined by the marketing department (referred to as marketing henceforth), whereas lead times are determined by the manufacturing (referred to as production henceforth) department.

3.1. *Centralized Setting (C)*

I model the problem as a finite-horizon stochastic dynamic program, maximizing the long-run expected total discounted profit of the company, with a discount rate of $0 < \alpha \leq 1$, where profit is defined as revenue minus lateness penalties. Note that, the formulation in Chapter 2 does not include a discount factor unlike the formulation in this chapter. This is due to the average reward formulation that does not require a discount factor. In this chapter, I model the problem using a finite horizon formulation that allows both discounted and non-discounted formulations. In my problem, I allow $\alpha = 1$, which transforms the problem into the non-discounted cases similar to the one modeled in Chapter 2. I also note that, all the forthcoming theoretical results hold for non-discounted case, $\alpha = 1$.

To facilitate the analysis, I divide the planning horizon into equal-sized intervals of arbitrarily small length, in which the probability of a customer arrival is β, and more than one customer arrival is negligibly small, i.e., assumed to be zero (see Bitran and Mondschein [1995] for another example who implement the same approach). The unit time is the duration of the interval. The stages correspond to each of these periods and the state of the system is defined as the workload of the manufacturer expressed in terms of unit time.

I assume a finite buffer of N time units of workload, and consider a finite planning horizon of T periods. The revenues are immediately obtained, and delivery penalties are paid at the time that the order is delivered to the customer. Let $\Pi_{t,i}^{C*}$ denote the long-run optimal discounted total expected profit for periods $t, t+1, \ldots, T$, given an initial workload of i time units, under the centralized setting **C**. The optimization problem under **C** is given in Equations (4.3), (4.4), and (4.5).

$$\Pi_{t,i}^{C*} = \max_{(p,l)\in\theta} \Pi_{t,i}^{C}(p,l), \tag{4.3}$$

where

$$\Pi_{t,i}^{C}(p,l) = \begin{cases} \beta f(p,l)(p - \tau\alpha^{i+\nu}(i+\nu-l)^{+} + \alpha\Pi_{t+1,i+\nu-1}^{C*}), & \text{if } 0 \le i \le N, \\ + (1-\beta f(p,l))\,\alpha\underline{\Pi}_{t+1,i}^{C*} & \\ \alpha\Pi_{t+1,i-1}^{C*}, & \text{if } i > N, \end{cases} \tag{4.4}$$

$x^{+} = \max\{0, x\}$, and

$$\underline{\Pi}_{t+1,i}^{C*} = \begin{cases} \Pi_{t+1,i-1}^{C*}, & i > 0, \\ \Pi_{t+1,0}^{C*}, & i = 0, \end{cases} \tag{4.5}$$

Given an initial workload of i ($i \le N$), and the price-lead time quote pair (p,l), at the beginning of period t, the customer arrives and places an order with probability $\beta f(p,l)$, which brings an expected discounted profit of $p - \tau\alpha^{i+\nu}(i+\nu-l)^{+}$, and increases the workload level to $i+\nu-1$ at time $t+1$ (see the first case in Equation (4.4)). I note that $l(i+\nu-l)^{+}$ denotes the lateness penalty to be paid at time $t+i+\nu$, which is discounted to the time t with a discount rate of α, giving a present value of $\alpha^{i+\nu}(i+\nu-l)^{+}$. Otherwise (i.e., either the customer arrives and rejects the quote, or does not arrive), the workload decreases by one unit at time $t+1$, given that $i > 0$. When $i > N$, no customer orders are accepted. I finally add the following boundary condition.

$$\Pi_{T,i}^{C*} = 0, \quad \text{for } 0 \le i \le N. \tag{4.6}$$

For notational simplicity, I define the following parameters.

$$\rho = \frac{\partial f(p,l)/\partial l}{\partial f(p,l)/\partial p} = \frac{p_{max} - p_{min}}{l_{max}}, \text{ and, } \quad \Delta\Pi_{t,i}^{C} = \alpha\underline{\Pi}_{t+1,i}^{C*} - \alpha\Pi_{t+1,i+\nu-1}^{C*}. \qquad (4.7)$$

The expression ρ is the ratio of lead time sensitivity to price sensitivity of the customers, where high (low) ρ values indicate relatively high lead time (price) sensitive customers. $\Delta\Pi_{t,i}^{C}$ is the additional burden brought by one customer order under **C** in period t given a workload of i time units. The optimal solution to Equation (4.3) is denoted as $(p_{t,i}^{C*}, l_{t,i}^{C*})$, which is derived in Theorem 8.

Theorem 8. $(p_{t,i}^{C*}, l_{t,i}^{C*})$ *can be found as follows.*

(i) If $\rho > \tau\alpha^{i+\nu}$*, then,*

$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{C} \leq 2p_{min} - p_{max} - \tau\alpha^{i+\nu}(i+\nu) \\ \left(\frac{\Delta\Pi_{t,i}^{C} + p_{max} + \tau\alpha^{i+\nu}(i+\nu)}{2}, 0\right), & \text{if } 2p_{min} - p_{max} - \tau\alpha^{i+\nu}(i+\nu) < \\ & \Delta\Pi_{t,i}^{C} < p_{max} - \tau\alpha^{i+\nu}(i+\nu), \\ (p_{max}, 0), & \text{if } p_{max} - \tau\alpha^{i+\nu}(i+\nu) < \Delta\Pi_{t,i}^{C}. \end{cases} \qquad (4.8)$$

(ii) If $\rho < \tau\alpha^{i+\nu}$*, and* $i + \nu \leq l_{max}$ *then,*

$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{C} \leq p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu), \\ \left(p_{min}, \frac{\Delta\Pi_{t,i}^{C} - p_{min}}{2\tau\alpha^{i+\nu}} + \frac{i+\nu+l_{max}}{2}\right), & \text{if } p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu) < \Delta \\ & \Pi_{t,i}^{C} \leq p_{min} - \rho(l_{max} - (i+\nu)), \\ (p_{min}, i+\nu), & \text{if } p_{min} - \rho(l_{max} - (i+\nu)) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{min} - \tau\alpha^{i+\nu}(l_{max} - (i+\nu)), \\ \left(\frac{\Delta\Pi_{t,i}^{C} + p_{max} - \rho(i+\nu)}{2}, i+\nu\right), & \text{if } p_{min} - \tau\alpha^{i+\nu}(l_{max} - (i+\nu)) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{max} - \rho(i+\nu), \\ (p_{max}, 0), & \text{if } p_{max} - \rho(i+\nu) < \Delta\Pi_{t,i}^{C}. \end{cases} \qquad (4.9)$$

(iii) If $\rho < \tau\alpha^{i+\nu}$, and $i+\nu > l_{max}$ then,

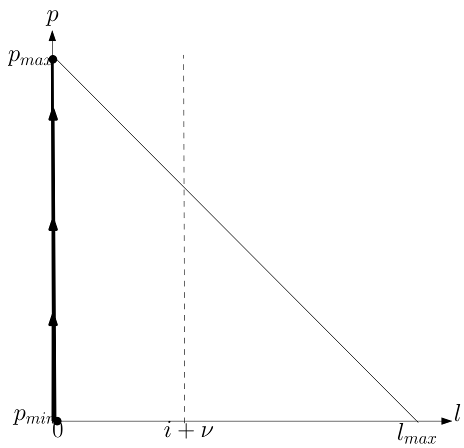
$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{C} \leq p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu), \\ \left(p_{min}, \frac{\Delta\Pi_{t,i}^{C} - p_{min}}{2\tau\alpha^{i+\nu}} + \frac{i+\nu+l_{max}}{2}\right), & \text{if } p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{min} + \tau\alpha^{i+\nu}(l_{max} - (i+\nu)), \\ (p_{max}, 0), & \text{if } p_{max} - \rho(i+\nu) < \Delta\Pi_{t,i}^{C}. \end{cases} \tag{4.10}$$

(iv) If $\rho = \tau\alpha^{i+\nu}$, then $(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_{max}, 0)$.

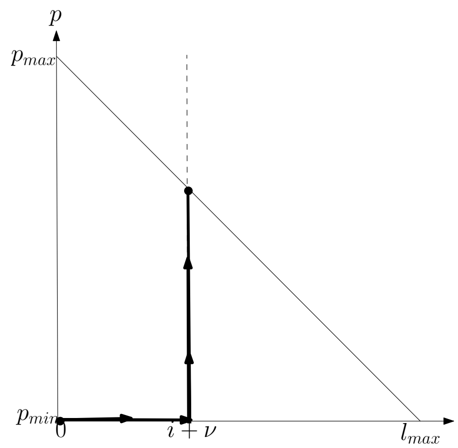
Proof. All of the proofs in this chapter are given in Appendix C. □

I depict the change of optimal solution in response to the increase of $\Delta\Pi_{t,i}^{C}$ in Figure 10. Note that the change of $(p_{t,i}^{C*}, l_{t,i}^{C*})$ depends on the value of ρ, i.e., whether $\rho > \tau\alpha^{i+\nu}$ or $\rho < \tau\alpha^{i+\nu}$. When $\Delta\Pi_{t,i}^{C}$ is sufficiently small, optimal solution is $(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_{min}, 0)$ under both cases. Hence, it is optimal to quote "sufficiently" low price and lead times such that the customer definitely accepts the quote. When $\rho > \tau\alpha^{i+\nu}$, $p_{t,i}^{C*}$ increases with the increase of $\Delta\Pi_{t,i}^{C}$, until it reaches p_{max} (see Figure 10(a)), which indicates that the customer definitely rejects the quote. On the other hand, when $\rho < \tau\alpha^{i+\nu}$, as $\Delta\Pi_{t,i}^{C}$ increases, $p_{t,i}^{C*}$ remain stable at p_{min} while $l_{t,i}^{C*}$ increases until it reaches $i+\nu$ (see Figure 10(b)). When, $l_{t,i}^{C*}$ reaches $i+\nu$, optimal lead time quote does not increase further, since the late delivery penalties will hit zero, however, decreases the acceptance probability of the quote. When $l_{t,i}^{C*} = i+\nu$, $p_{t,i}^{C*}$ increases with increase of $\Delta\Pi_{t,i}^{C}$.

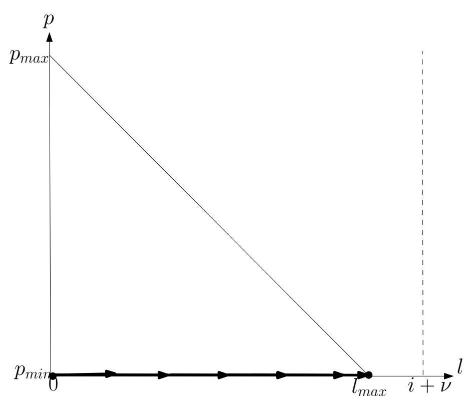
Theorem 8 is analogous to Theorems 1 and 2 in Chapter 2, where it is shown that optimal price and lead time quotes depend highly on price/lead time sensitivity of the customers. In particular, Cases (i) and (iii) provide the same results with



(a) Case I: $\rho > \tau\alpha^{i+\nu}$



(b) Case II: $\rho < \tau\alpha^{i+\nu}$, $i+\nu \leq l_{max}$



(c) Case III: $\rho < \tau\alpha^{i+\nu}$, $i+\nu > l_{max}$

Fig. 10. Change of $(p^{C*}_{t,i}, l^{C*}_{t,i})$ as $\Delta\Pi^{C}_{t,i}$ increases

Cases (i) and (ii) of Theorem 2, where it is shown that the optimal lead time quote is zero, and price quote is the accept all price, respectively. In the remainder, the customer is denoted as lead time sensitive if $\rho \geq l$, and price sensitive otherwise.

3.2. *Decentralized Settings*

I now assume that the marketing department works as a revenue center that determines the price with the objective of maximizing the long-run total discounted expected revenues with respect to price, denoted as $\Pi_{t,i}^{M}(p,l)$ and given in Equation (4.11).

$$\Pi_{t,i}^{M}(p,l) = \begin{cases} \beta f(p,l)\left(p + \alpha \Pi_{t+1,i+\nu-1}^{M*}\right) + (1-\beta f(p,l))\, \alpha \underline{\Pi}_{t+1,i}^{M*}, & i \le N, \\ \alpha \Pi_{t+1,i-1}^{M*}, & i > N, \end{cases} \tag{4.11}$$

where,

$$\underline{\Pi}_{t+1,i}^{M*} = \begin{cases} \Pi_{t+1,i-1}^{M*}, & i > 0, \\ \Pi_{t+1,0}^{M*}, & i = 0, \end{cases} \tag{4.12}$$

and $\Pi_{t,i}^{M*}$ denotes the optimal total discounted revenues of marketing department earned for periods for $t, t+1, \ldots, T$, given the initial workload of i.

The production department, on the other hand, works as the cost center that determines lead times, with the objective of minimization of total discounted expected costs. However, as stated by Pekgun *et al.* [2008], if this problem were a solely cost minimization problem, a trivial solution to this problem would be to quote l_{max}, which would ensure zero cost by rejecting customers. Hence, similar to Pekgun *et al.* [2008], I assume a γ incentive given to production department for each unit of production by the company. Consequently, the objective of production department is to maximize $\Pi_{t,i}^{P}(p,l)$, where,

$$\Pi_{t,i}^{P}(p,l) = \begin{cases} \beta f(p,l)(\gamma - \tau \alpha^{i+\nu}(i+\nu-l)^{+} + \alpha \Pi_{t+1,i+\nu-1}^{P*}) & \\ + (1-\beta f(p,l))\, \alpha \underline{\Pi}_{t+1,i}^{P*}, & i \le N, \\ \alpha \Pi_{t+1,i-1}^{P*}, & i > N, \end{cases} \tag{4.13}$$

$$\underline{\Pi}^{P*}_{t+1,i} = \begin{cases} \Pi^{P*}_{t+1,i-1}, & i > 0, \\ \\ \Pi^{P*}_{t+1,0}, & i = 0. \end{cases} \tag{4.14}$$

and $\Pi^{P*}_{t,i}$ denotes the optimal long-run expected total discounted profit of the production department for periods $t+1, t+2, \ldots, T$, given an initial workload of i time units. The boundary conditions are given as

$$\Pi^{P*}_{T,i} = \Pi^{M*}_{T,i} = 0, \quad \text{for } 0 \leq i \leq N. \tag{4.15}$$

I next model the sequence of price and lead time decisions under three settings that correspond to three different repeated games.

(i) **Simultaneous decision making setting (S)**: Marketing and Production determines the price and lead times simultaneously.

(ii) **Production leader setting (P)**: Production first chooses a lead time that is followed by the price decision of the marketing department.

(iii) **Marketing leader setting (M)**: Marketing first chooses a price that is followed by the lead time decision of the production department.

In all of the above games, the state i, the value functions $\Pi^{P*}_{t+1,i}$ and $\Pi^{M*}_{t+1,i}$, and objective functions of marketing and production are common knowledge to both departments. I model **S** as a repeated Cournot game, whereas **P** and **M** are modeled as a repeated Stackelberg games [Gibbons, 1992]. In all of the games I find (i) the price $p^{M*}_{t,i}(l)$ that maximizes $\Pi^{M}_{t,i}(p,l)$ given the lead time value l, and (ii) the lead time $l^{M*}_{t,i}(p)$ that maximizes $\Pi^{P}_{t,i}(p,l)$ given the lead time value p. In the Cournot game, which is used to model **S**, I solve $(p^{M*}_{t,i}(l), l) = (p, l^{P*}_{t,i}(p))$ to find the equilibrium. In **P**, I find the $(l_{t,i}^{P*}(p^*))$, where $p^* = p_{t,i}^{M*}(l_{t,i}^{P*}(p^*)))$. That is, production chooses the lead time decisions with the information that the marketing department will choose the price that will maximize its revenue given the lead time from production department. On the other hand, in **M**, I find the $(p_{t,i}^{M*}(l^*))$, where $l^* = l_{t,i}^{P*}(p_{t,i}^{M*}(l^*)))$. That is, marketing chooses the price decisions with the information that the production will choose the lead time that will maximize its profit given the price from marketing department. The above equilibrium is solved for each $t = 1, 2, \ldots, T$ and $i = 0, 1, \ldots, N$. I assume that in case there are multiple equilibria, Marketing and Production chooses an equilibrium that brings positive immediate profit, (i.e., p for Marketing, and $\gamma - \tau\alpha^{i+\nu}(i + \nu - l)^+$ for production) to both sides, which I denote as the positive immediate profit Markov perfect equilibrium (PIPE). If there is no PIPE, then price and lead time decisions are chosen such that the customer definitely rejects the quote, i.e., $(p_{t,i}^{M*}, l_{t,i}^{P*}) = (p_{max}, 0)$.

I solve the problems using backward recursion, where in each state i and period t $(p_{t,i}^{M*}, l_{t,i}^{P*})$ is determined, and the optimal expected total discounted cost and profits are computed as in Equation (4.16).

$$\Pi_{t,i}^{M*} = \Pi_{t,i}^{M}(p_{t,i}^{M*}, l_{t,i}^{P*}), \text{ and } \Pi_{t,i}^{P*} = \Pi_{t,i}^{P}(p_{t,i}^{M*}, l_{t,i}^{P*}). \tag{4.16}$$

In Theorem 9, I derive $(p_{t,i}^{M*}, l_{t,i}^{P*})$ for **S** and **P**, whereas the optimal solution for **M** is derived in Theorem 10. For simplicity, I define $\Delta\Pi_{t,i}^{M}$ and $\Delta\Pi_{t,i}^{P}$ that are analogously to $\Delta\Pi_{t,i}^{C}$ along with several other parameters and functions. Furthermore, I provide a list of parameters in Table 17, including all parameters used in Theorems 9 and 10.

$$\Delta\Pi_{t,i}^{M} = \alpha\underline{\Pi}_{t+1,i}^{M*} - \alpha\Pi_{t+1,i+\nu-1}^{M*}, \quad \Delta\Pi_{t,i}^{P} = \alpha\underline{\Pi}_{t+1,i}^{P*} - \alpha\Pi_{t+1,i+\nu-1}^{P*}, \tag{4.17}$$

$$p^*_{t,i}(l) = \frac{1}{2}\left(\Delta\Pi^M_{t,i} + p_{max} - \rho l\right), \quad \Theta_{t,i} = \frac{\Delta\Pi^P_{t,i} - \gamma}{\tau\alpha^{i+\nu}} + i + \nu, \tag{4.18}$$

$$\Delta_{t,i}(x,y,z) = \gamma + \tau\alpha^{i+\nu}\left(x\frac{\Delta\Pi^M_{t,i} - z}{\rho} + y - (i+\nu)\right), \tag{4.19}$$

$$\Omega_{t,i}(x,y,z) = x\rho\Theta - y\rho(i+\nu) + z. \tag{4.20}$$

Table 17. Defined parameters for Theorems 9 and 10

General Parameters

$\psi_1 = 2p_{min} - p_{max}$	$\kappa_1 = \Delta_{t,i}(0, -l_{max}, 0)$
$\psi_2 = \rho(i+\nu) + 2p_{min} - p_{max}$	$\kappa_2 = \Delta_{t,i}(0, 2(i+\nu) - l_{max}, 0)$
$\psi_3 = p_{min}$	$\kappa_3 = \Delta_{t,i}(0, i+\nu, 0)$
$\psi_4 = p_{max} - \rho(i+\nu)$	$\kappa_4 = \Delta_{t,i}(0, l_{max}, 0)$
$\psi_5 = p_{max}$	

Specific Parameters

	S	**P**	**M**
κ_A	$\Delta_{t,i}(0.5, 0, p_{max})$	$\Delta_{t,i}(1, 0, p_{max})$	–
κ_B	$\Delta_{t,i}(2, l_{max}, p_{min})$	$\Delta_{t,i}(3, l_{max}, p_{min})$	–
κ_C	$\Delta_{t,i}(2, l_{max}, p_{min})$	$\Delta_{t,i}(2, l_{max}, p_{min})$	–
κ_D	$\Delta_{t,i}(0.5, 1.5(i+\nu), p_{max})$	$\Delta_{t,i}(1, 2(i+\nu), p_{max})$	–
κ_E	$\Delta_{t,i}(-1, 0, p_{max})$	$\Delta_{t,i}(-1, 0, p_{max})$	–
ψ_A	–	–	$\Omega_{t,i}(1, 0, 2p_{min} - p_{max})$
ψ_B	–	–	$\Omega_{t,i}(3, 0, p_{max})$
ψ_C	–	–	$\Omega_{t,i}(2, 0, p_{max})$
ψ_D	–	–	$\Omega_{t,i}(-1, 0, p_{max})$
ψ_E	–	–	$\Omega_{t,i}(2, 3, p_{max})$
ψ_F	–	–	$\Omega_{t,i}(3, 4, p_{max})$
l^*_1	$\frac{1}{2}(\Theta_{t,i} + l_{max})$	$\frac{1}{2}(\Theta_{t,i} + l_{max})$	$\frac{1}{2}(\Theta_{t,i} + l_{max})$
l^*_2	$\frac{2}{3}\left(\Theta_{t,i} + \frac{p_{max} - \Delta\Pi^M_{t,i}}{2\rho}\right)$	$\frac{1}{2}\left(\Theta_{t,i} + \frac{p_{max} - \Delta\Pi^M_{t,i}}{\rho}\right)$	$\frac{1}{4}\left(3\Theta_{t,i} + \frac{p_{max} - \Delta\Pi^M_{t,i}}{\rho}\right)$
l^*_3	$\frac{1}{\rho}\left(\Delta\Pi^M_{t,i} + p_{max} - 2p_{min}\right)$	$\frac{1}{\rho}\left(\Delta\Pi^M_{t,i} + p_{max} - 2p_{min}\right)$	–
p^*_4	–	–	$\Omega_{t,i}(1, 0, p_{max})$
p^*_5	–	–	$\Omega_{t,i}(1, 2, p_{max})$

Theorem 9. *Unique Markov perfect equilibrium of* $(p_{t,i}^{M*}, l_{t,i}^{P*})$ *under settings* $\boldsymbol{S}$ *and* $\boldsymbol{P}$ *can be found as follows.*

(i) If $i + \nu \le l_{max}$ *then*

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{M} \le \psi_1, \Delta\Pi_{t,i}^{P} \le \kappa_1, \\ (p_{t,i}^{*}(0), 0), & \text{if } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_5, \Delta\Pi_{t,i}^{P} \le \kappa_A, \\ (p_{min}, l_1^{*}), & \text{if } \Delta\Pi_{t,i}^{M} \le \psi_1, \kappa_1 < \Delta\Pi_{t,i}^{P} \le \kappa_2, \\ & \text{ or } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_2, \kappa_C < \Delta\Pi_{t,i}^{P} \le \kappa_2, \\ (p_{min}, l_3^{*}), & \text{if } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_2, \kappa_B^{P} < \Delta\Pi_{t,i}^{P} \le \kappa_C^{P}, \\ (p_{min}, i+\nu), & \text{if } \Delta\Pi_{t,i}^{M} \le \psi_2, \kappa_2 < \Delta\Pi_{t,i}^{P} \le \kappa_3, \\ (p_{t,i}^{*}(l_2^{*}), l_2^{*}), & \text{if } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_2, \kappa_A < \Delta\Pi_{t,i}^{P} \le \kappa_B, \\ & \text{ or } \psi_2 < \Delta\Pi_{t,i}^{M} \le \psi_4, \kappa_A < \Delta\Pi_{t,i}^{P} \le \kappa_D, \\ & \text{ or } \psi_4 < \Delta\Pi_{t,i}^{M} \le \psi_5, \kappa_A < \Delta\Pi_{t,i}^{P} \le \kappa_E, \\ (p_{t,i}^{*}(i+\nu), i+\nu), & \text{if } \psi_2 < \Delta\Pi_{t,i}^{M} \le \psi_4, \kappa_D < \Delta\Pi_{t,i}^{P} \le \kappa_3, \\ (p_{max}, 0), & \text{otherwise.} \end{cases} \tag{4.21}$$

(ii) If $i + \nu > l_{max}$ *then*

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_{min}, 0), & \textit{if } \Delta\Pi_{t,i}^M \le \psi_1, \Delta\Pi_{t,i}^P \le \kappa_1, \\ (p_{t,i}^*(0), 0), & \textit{if } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_5, \Delta\Pi_{t,i}^P \le \kappa_A, \\ (p_{min}, l_1^*), & \textit{if } \Delta\Pi_{t,i}^M \le \psi_1, \kappa_1 < \Delta\Pi_{t,i}^P \le \kappa_4, \\ & \quad \textit{or } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_3, \kappa_C < \Delta\Pi_{t,i}^P \le \kappa_4, \\ (p_{min}, l_3^*), & \textit{if } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_3, \kappa_B^P < \Delta\Pi_{t,i}^P \le \kappa_C^P, \\ (p_{t,i}^*(l_2^*), l_2^*), & \textit{if } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_3, \kappa_A < \Delta\Pi_{t,i}^P \le \kappa_B, \\ & \quad \textit{or } \psi_3 < \Delta\Pi_{t,i}^M \le \psi_5, \kappa_A < \Delta\Pi_{t,i}^P \le \kappa_E \\ (p_{max}, 0), & \textit{otherwise.} \end{cases} \tag{4.22}$$

In Figure 11, I depict the change of $(p_{t,i}^{M*}, l_{t,i}^{P*})$ for $i + \nu \le l_{max}$, depending on the values of $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$, where changes of $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$ are depicted on horizontal and vertical axes, respectively. I note that whenever $i + \nu \le l_{max}$, I have $\kappa_1 \le \kappa_2 \le \kappa_3$, and $\psi_1 \le \psi_2 \le \psi_4 \le \psi_5$ as shown in the sorted order in Figure 11. Furthermore, I have $\kappa_A \le \kappa_B \le \kappa_C \le \kappa_2$ whenever $\psi_1 < \Delta\Pi_{t,i}^M \le \psi_2$, $\kappa_A \le \kappa_D \le \kappa_3$ whenever $\psi_2 < \Delta\Pi_{t,i}^M \le \psi_4$, and $\kappa_A \le \kappa_E$ whenever $\psi_4 < \Delta\Pi_{t,i}^M \le \psi_5$, as illustrated.

In Theorem 9, I observe a similar structure to the one found in Theorem 8. If the burden values $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$ are sufficiently small, the customer is quoted $(p_{min}, 0)$ to guarantee the acceptance of the order. As $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$ increase, $p_{t,i}^{M*}$ and/or $l_{t,i}^{P*}$ increase until the point where the customer definitely rejects the order. If $\Delta\Pi_{t,i}^M$ is increased solely, $p_{t,i}^{M*}$ increases from p_{min} to $p^*(0)$, and reaches p_{max} eventually, while $l_{t,i}^{M*}$ is kept constant at zero. Similarly as $\Delta\Pi_{t,i}^P$ increases,

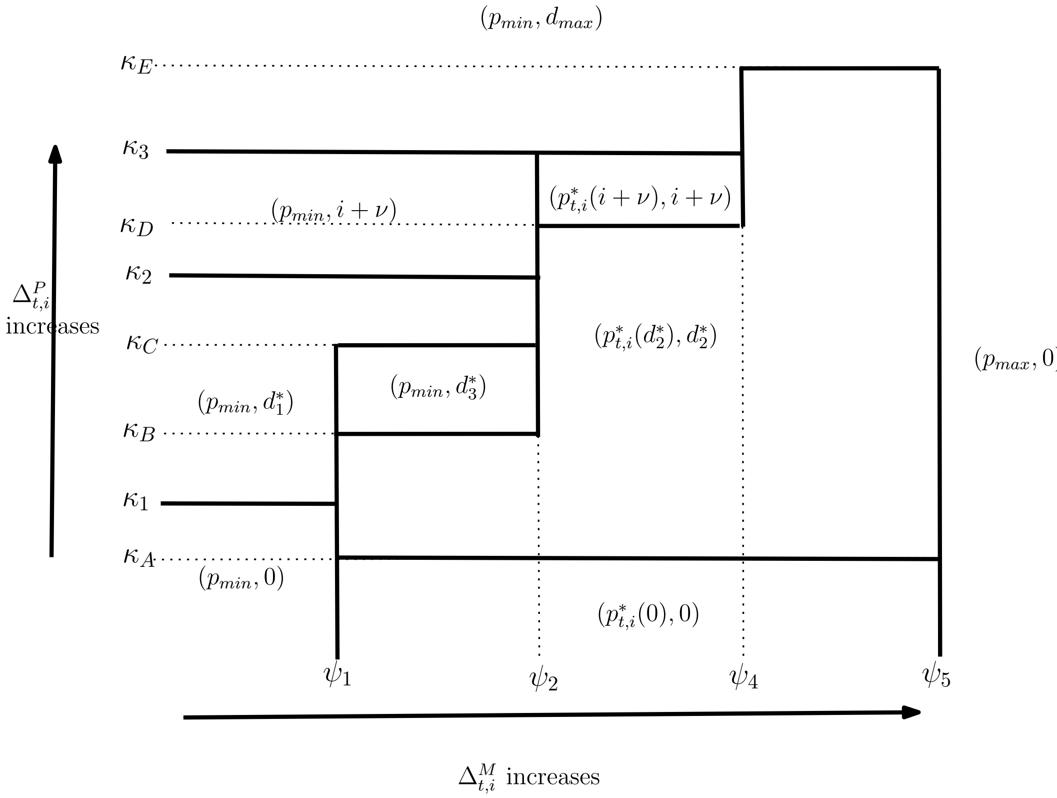


Fig. 11. Change of $(p^{M*}_{t,i}, l^{P*}_{t,i})$ as a function of the $\Delta\Pi^M_{t,i}$ and $\Delta\Pi^P_{t,i}$ values under settings **S** and **P** when $i + \nu \leq l_{max}$.

$l^{M*}_{t,i}$ increases from zero to $i + \nu$ while $p^{M*}_{t,i}$ remains constant at p_{min}. I conclude that, increasing of $\Delta\Pi^M_{t,i}$ and $\Delta\Pi^P_{t,i}$ increases $p^{M*}_{t,i}$ and $l^{P*}_{t,i}$, respectively.

Theorem 10. *Unique Markov perfect equilibrium of* $(p^{M*}_{t,i}, l^{P*}_{t,i})$ *under setting* ***M*** *can be found as follows.*

(i) If $i+\nu \leq \frac{l_{max}}{2}$ *and* $\Delta\Pi^P_{t,i} \leq \kappa_2$*, then*

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa_1, \Delta\Pi^M_{t,i} \leq \psi_1, \\ (p_{min}, l^*_1), & \text{if } \kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2, \Delta\Pi^M_{t,i} \leq \psi_A, \\ (p^*_{t,i}(0), 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa_1, \psi_1 < \Delta\Pi^M_{t,i} \leq \psi_5, \\ & \text{ or } \kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2, \psi_C < \Delta\Pi^M_{t,i} \leq \psi_5, \\ (p^*_{t,i}(\Theta_{t,i}), l^*_2), & \text{if } \kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2, \psi_A < \Delta\Pi^M_{t,i} \leq \psi_B, \\ (p^*_4, 0), & \text{if } \kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2, \psi_B < \Delta\Pi^M_{t,i} \leq \psi_C. \\ (p_{max}, 0), & \text{otherwise.} \end{cases} \tag{4.23}$$

(ii) If $\frac{l_{max}}{2} < i+\nu \leq l_{max}$ *and* $\Delta\Pi^P_{t,i} \leq \kappa_2$*, or if* $l_{max} < i+\nu$ *and* $\Delta\Pi^P_{t,i} \leq \kappa_4$*, then*

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_{min}, l^*_1), & \text{if } \Delta\Pi^M_{t,i} \leq \psi_A, \\ (p^*_{t,i}(\Theta_{t,i}), l^*_2), & \text{if } \psi_A < \Delta\Pi^M_{t,i} \leq \psi_D, \\ (p_{max}, 0), & \text{otherwise.} \end{cases} \tag{4.24}$$

(iii) If $\frac{l_{max}}{2} < i+\nu \leq l_{max}$ *and* $\kappa_2 < \Delta\Pi^P_{t,i} \leq \kappa_3$*, then*

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \text{argmax}\{\Pi^M_{t,i}(p^*_A, l^*_A), \Pi^M_{t,i}(p^*_B, l^*_B)\}, \tag{4.25}$$

where

$$(p^*_A, l^*_A) = \begin{cases} (p_{min}, i+\nu), & \text{if } \Delta\Pi^M_{t,i} \leq \psi_2, \\ (p^*_{t,i}(i+\nu), i+\nu), & \text{if } \psi_2 < \Delta\Pi^M_{t,i} \leq \psi_E, \\ (p^*_5, i+\nu), & \text{if } \psi_E < \Delta\Pi^M_{t,i}, \end{cases} \tag{4.26}$$

$$(p_B^*, l_B^*) = \begin{cases} (p_5^*, i+\nu), & \textit{if } \Delta\Pi_{t,i}^M \leq \psi_F, \\ (p_{t,i}^*(\Theta_{t,i}), l_2^*), & \textit{if } \psi_F < \Delta\Pi_{t,i}^M \leq \psi_D, \\ (p_{max}, 0) & \textit{if } \psi_D < \Delta\Pi_{t,i}^M. \end{cases} \tag{4.27}$$

(iv) If $i + \nu \leq \frac{l_{max}}{2}$ *and* $\kappa_2 < \Delta\Pi_{t,i}^P \leq \kappa_3$ *then*

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \text{argmax}\{\Pi_{t,i}^M(p_A^*, l_A^*), \Pi_{t,i}^M(p_C^*, l_C^*)\}, \tag{4.28}$$

where

$$(p_C^*, l_C^*) = \begin{cases} (p_5^*, i+\nu), & \textit{if } \Delta\Pi_{t,i}^M \leq \psi_F, \\ (p_{t,i}^*(\Theta_{t,i}), l_2^*), & \textit{if } \psi_F < \Delta\Pi_{t,i}^M \leq \psi_B, \\ (p_4^*, 0), & \textit{if } \psi_B < \Delta\Pi_{t,i}^M \leq \psi_C, \\ (p_{t,i}^*(0), 0), & \textit{if } \psi_C < \Delta\Pi_{t,i}^M \leq \psi_5, \\ (p_{max}, 0), & \textit{if } \psi_5 < \Delta\Pi_{t,i}^M. \end{cases} \tag{4.29}$$

The change of $(p_{t,i}^{M*}, l_{t,i}^{P*})$ for case (i) of Theorem 10 is depicted in Figure 12

Similar to my observations from Figure 11, $p_{t,i}^{M*}$ and $l_{t,i}^{P*}$ increase with the increases in $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$. In contrast, however, I observe that, the increase of $\Delta\Pi_{t,i}^M$ may lower the optimal lead time quotes. For example, when $\kappa_1 < \Delta\Pi_{t,i}^P \leq \kappa_2$, $l_{t,i}^{P*}$ is strictly positive if $\Delta\Pi_{t,i}^M < \psi_B$, and zero otherwise.

4. Comparison of Centralized and Decentralized Settings

In this section, I conduct a numerical study to analyze the total discounted expected profits (TDEP) earned by the company under **C**, **S**, **P**, and **M** settings for

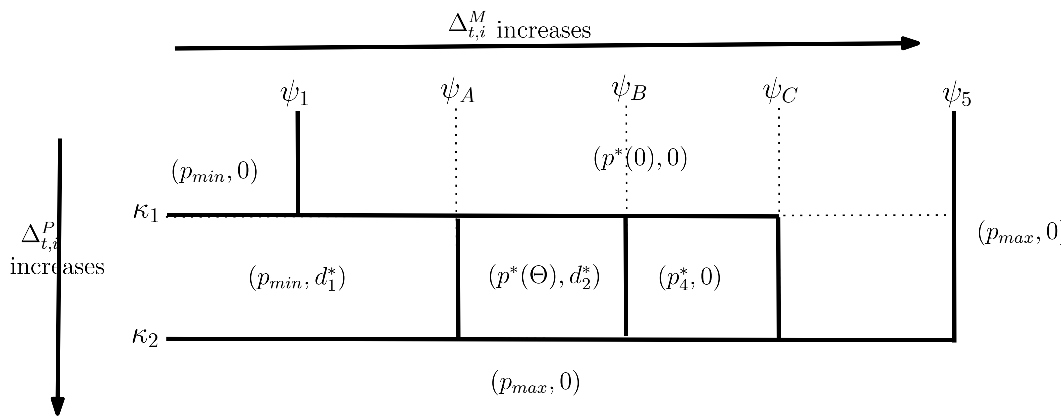


Fig. 12. Change of $(p^{M*}_{t,i}, l^{P*}_{t,i})$ as a function of $\Delta\Pi^M_{t,i}$ and $\Delta\Pi^P_{t,i}$ for part (i) of Theorem 10

a length of T periods. Noting that γ can be determined by the company, TDEP's are denoted as $\Pi^C, \Pi^S(\gamma), \Pi^P(\gamma)$ and $\Pi^M(\gamma)$, for **C**, **S**, **P**, and **M**, respectively.

I first analyze the incentive values that maximize the performance of the decentralized settings, i.e., $\Pi^S(\gamma), \Pi^P(\gamma)$ and $\Pi^M(\gamma)$ (I refer such an incentive value as the optimal incentive, and denote it as γ^*), and measure the difference of $\Pi^S(\gamma^*), \Pi^P(\gamma^*)$ and $\Pi^M(\gamma^*)$ with Π^C. In the second part of my analysis, I focus on the impact of three fundamental parameters on the TDEP: (i) traffic intensity, which is quantified by $\beta\nu$, i.e., the ratio of the arrival rate to the processing rate, (ii) accept all price, p_{min}, and (iii) price/lead time sensitivity of the customers obtained by the comparison of ρ and τ.

Assuming that the company starts the business with zero inventory at period $t = 0$, Π^C is computed by evaluating $\Pi^{C*}_{0,0}$, using the closed-form solutions of Theorem 8 and backward recursion. To evaluate $\Pi^S(\gamma), \Pi^P(\gamma)$ and $\Pi^M(\gamma)$, I first compute $(p^{M*}_{t,i}, l^{P*}_{t,i})$, $i \in \{0, 1, \ldots, N\}$, $t \in \{0, 1, \ldots, T\}$ for **S**, **P** and **M** using the closed form solutions from Theorems 9 and 10, and backward recursion. Given these sets of optimal solutions, I compute company TDEPs similar to the centralized case, except for the fact that $\Pi^{C*}_{t,i}$ is computed by $\Pi^{C}_{t,i}(p^{M*}_{t,i}, l^{P*}_{t,i})$. One clearly observes that setting **C** is guaranteed to yield highest TDEPs, ensuring that $\Pi^C \geq \Pi^X(\gamma), X \in \{S, P, M\}$. I report the TDEP percentage differences between centralized and decentralized settings (which are denoted as $\Delta^X(\gamma), X \in \{S, P, M\}$) computed as

$$\Delta^X(\gamma) = \frac{\Pi^C - \Pi^X(\gamma)}{\Pi^C} 100\%, \quad X \in \{S, P, M\}. \tag{4.30}$$

I next provide a discussion about the selection of parameter ranges used throughout the numerical analysis, and illustrate the practical meaning of my selections using an example case.

- Noting that each period is taken to be absolutely short, β and $1/\nu$ are chosen to be sufficiently small. I fix $\nu = 20$ in all instances, and vary β in the range [0.030,0.045] to obtain the traffic intensity levels ranging between 60% and 90%.

- p_{min} is tested in a range of [10,50].

- τ and ρ are varied in a range of [0.025,0.125] to represent price and lead time sensitive customer cases.

- l_{max} is tested in a range of [60,140].

- I set $p_{max} = \rho l_{max} + p_{min}$ for changing values of p_{min}, ρ and l_{max} following Equation (4.7).

- α is set to 0.9999.

- Considering that α is set to 0.9999, the future value of one dollar decreases down to \$0.01 in approximately 46,000 periods. Hence, I set $T = 46,000$, noting that, increasing T further does not affect the TDEPs significantly, while drastically increasing the computational burden.

- To determine the value of N, I conduct a preliminary analysis and determine a sufficiently big N value that minimizes the impact of buffer on TDEPs. I test $N \in \{150, 200, \ldots, 400\}$ using three instances, and observe that the impact of increasing N from 250 to 300 on the TDEP values is less than 0.01 on average. Hence, I set $N = 250$.

4.1. *Optimal Incentives*

In this section, I seek for the value of γ^{*X} that minimizes $\Delta^X(\gamma)$ for each $X \in \{S, P, M\}$ (i.e., maximizes $\Pi^X(\gamma)$) using the numerical analysis settings in Table 18.

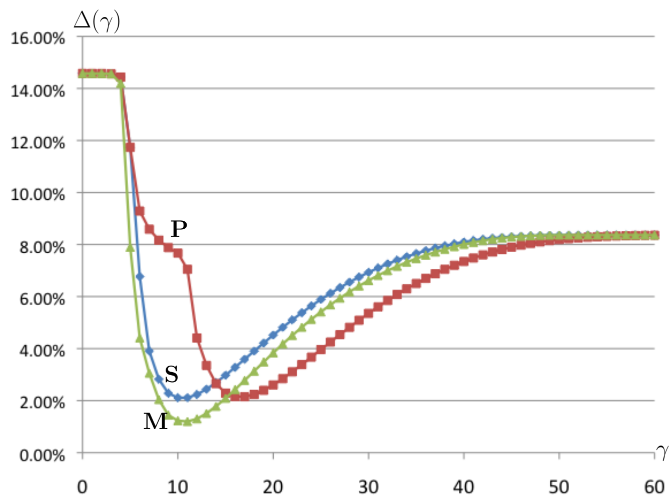
Table 18. Numerical analysis settings in Section 4.1

Traffic intensity	Accept All Price	Price/lead time sensitivity
$\beta = 0.0375$	$p_{min} \in \{10, 30, 50\}$	$\rho, \tau \in \{0.025, 0.05, 0.075, 0.1, 0.125\}$ $l_{max} \in \{60, 100, 140\}$

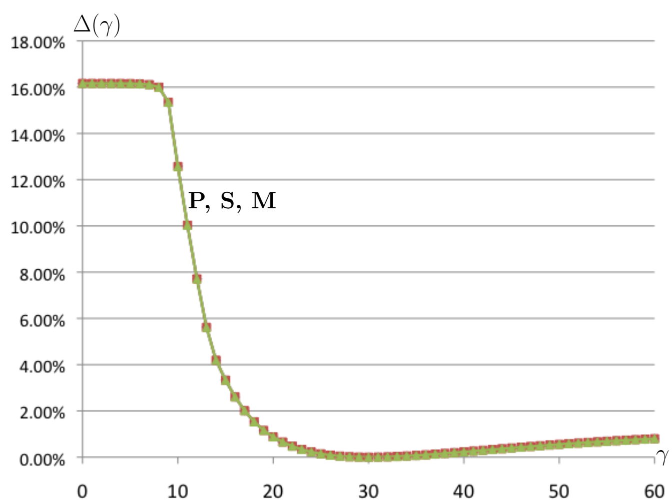
I test three levels of p_{min}, a single level of traffic intensity, and five levels of each ρ and τ. In Sections 4.2 and 4.3, I analyze the impact of traffic intensity and p_{min} in more detail. For all instances, I evaluate $\Delta^X(\gamma)$ for integer values of γ, and find γ^{*X}. My preliminary analysis indicates that under the parameter settings listed in

Table 18, increasing γ above a value of 60 does not decrease $\Delta^X(\gamma)$. Hence, I test for all $\gamma \in \{0, 1, 2, \ldots, 59, 60\}$.

In Figures 13(a) and 13(b), I plot the change of $\Delta^X(\gamma)$ for all $X \in \{S, P, M\}$ in γ, for two sample instances.



(a) Instance 1: $p_{min} = 10, l_{max} = 100, l = 0.05, \rho = 0.1$



(b) Instance 2: $p_{min} = 30, l_{max} = 100, l = 0.1, \rho = 0.05$

Fig. 13. Change of $\Delta^X(\gamma)$ with respect to γ

In both sample instances, I observe decrease in $\Delta^X(\gamma)$ up to a point, which is a function of p_{min}, and the increase in $\Delta^X(\gamma)$ beyond this γ value. In the first instance, $\Delta^P(\gamma)$, $\Delta^S(\gamma)$ and $\Delta^M(\gamma)$ curves differ significantly, where I have $\gamma^{*P} = 17$, $\gamma^{*S} = 10$, $\gamma^{*M} = 11$, $\Delta^P(\gamma^{*P}) = 2.15\%$, $\Delta^S(\gamma^{*S}) = 2.11\%$, and $\Delta^M(\gamma^{*M}) = 1.20\%$. On the other hand, all three curves behave quite similarly in the second instance, where I obtain $\gamma^{*P} = \gamma^{*S} = \gamma^{*M} = 30$, and $\Delta^P(\gamma^{*P}) = \Delta^S(\gamma^{*S}) = \Delta^M(\gamma^{*M}) = 0$.

Figures 13(a) and 13(b) reveal three crucial observations: (i) Optimal incentive values are typically quite close to p_{min}, with an exception in the first instance for setting **P**, (ii) using the optimal incentives, $\Delta^X(\gamma), X \in \{S, P, M\}$ can be decreased down to 1 to 2%, and (iii) the accept all price, p_{min}, have significant impact on $\Delta^X(\gamma^*)$. I analyze (iii) in Section 4.3, and now discuss (i) and (ii) in more detail.

To observe the value of offering optimal incentives, in Tables 19 and 20, I provide the average $\Delta^X(\gamma^*)$ and $\Delta^X(0)$ values for $\rho, \tau \in \{0.025, 0.05, 0.075, 0.1, 0.125\}$ and $X \in \{S, P, M\}$, respectively.

Observation 10. *Offering optimal incentives improves the performances of all decentralized settings significantly.*

As observed in Table 19, by offering optimal incentives $\Delta^X(\cdot)$ can be decreased down to 1% on the average. On the other hand, when no incentives are offered $\Delta^X(\cdot)$ averages are above 13%, and reach up to 27% (see Table 20). This reveals the importance of offering optimal incentives to maximize the performance of all decentralized settings. In order to analyze the performance of offering p_{min} as the incentive, I present the average $\Delta^X(p_{min})$ values in Table 21.

Table 19. Average $\Delta^X(\gamma^*)$ Values

(a) $\Delta^S(\gamma^*)$

$\rho \backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	0.00%	0.00%	0.00%	0.00%	0.00%
0.05	0.13%	0.01%	0.00%	0.00%	0.00%
0.075	0.24%	0.45%	0.02%	0.00%	0.00%
0.1	0.25%	0.58%	0.66%	0.02%	0.00%
0.125	0.22%	0.58%	0.86%	0.83%	0.03%

(b) $\Delta^P(\gamma^*)$

$\rho \backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	0.00%	0.00%	0.00%	0.00%	0.00%
0.05	0.13%	0.01%	0.00%	0.00%	0.00%
0.075	0.24%	0.45%	0.02%	0.00%	0.00%
0.1	0.25%	0.59%	0.68%	0.02%	0.00%
0.125	0.22%	0.58%	0.86%	0.86%	0.03%

(c) $\Delta^M(\gamma^*)$

$\rho \backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	0.00%	0.00%	0.00%	0.00%	0.00%
0.05	0.05%	0.03%	0.13%	0.00%	0.00%
0.075	0.14%	0.21%	0.04%	0.11%	0.00%
0.1	0.17%	0.30%	0.31%	0.08%	0.29%
0.125	0.16%	0.33%	0.39%	0.37%	0.08%

Observation 11. *Offering p_{min} incentives in decentralized settings often yields TDEP values that are close to the case where optimal incentives are offered.*

Table 20. Average $\Delta^X(0)$ values

$\rho\backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	26.58%	22.16%	18.58%	15.87%	13.65%
0.05	26.69%	22.17%	18.58%	15.87%	13.65%
0.075	26.98%	22.85%	18.60%	15.87%	13.65%
0.1	27.45%	23.70%	19.85%	15.89%	13.65%
0.125	27.95%	24.51%	21.00%	17.40%	13.68%

One observes that average $\Delta^X(p_{min})$ values are typically below 2%, with some exceptions for setting **P** where $\rho = 0.125$. In particular, $\Delta^S(p_{min})$ and $\Delta^S(\gamma^*)$ values are quite close to each other where the largest observed difference is 0.54% when $\rho = 0.125$ and $l = 0.075$. The differences between $\Delta^P(p_{min})$ and $\Delta^P(\gamma^*)$, on the other hand, are larger and may exceed 2%. However, one still observes a huge difference between the averages of $\Delta^X(p_{min})$ and $\Delta^X(0)$ in accordance with Observation 11.

In summary, my analysis reveals that average $\Delta^X(\gamma^*)$ values are below 1%, and average $\Delta^X(p_{min})$ values are below 3%. However, one notes that the numerical analysis in this section are conducted using three values of p_{min} and single value of traffic intensity. In Sections 4.2 and 4.3, I explore wider ranges of these parameters in an attempt to generalize my observations.

I next analyze the performances of decentralized settings with respect to ρ and τ.

Observation 12. ***S*** *and* ***P*** *perform better than* ***M*** *when customers are price sensitive.* ***M*** *performs better than* ***S*** *and* ***P*** *when customers are lead time sensitive.*

Table 21. Average $\Delta^X(p_{min})$ Values

(a) $\Delta^S(p_{min})$

$\rho\backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	0.00%	0.00%	0.00%	0.00%	0.00%
0.05	0.13%	0.01%	0.00%	0.00%	0.00%
0.075	0.25%	0.47%	0.02%	0.00%	0.00%
0.1	0.29%	0.59%	0.87%	0.02%	0.00%
0.125	0.25%	0.59%	1.07%	1.37%	0.03%

(b) $\Delta^P(p_{min})$

$\rho\backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	0.00%	0.00%	0.00%	0.00%	0.00%
0.05	0.13%	0.01%	0.00%	0.00%	0.00%
0.075	0.41%	0.95%	0.02%	0.00%	0.00%
0.1	0.35%	2.34%	1.38%	0.02%	0.00%
0.125	0.23%	3.04%	2.84%	1.61%	0.03%

(c) $\Delta^M(p_{min})$

$\rho\backslash l$	0.025	0.05	0.075	0.1	0.125
0.025	0.00%	0.00%	0.00%	0.00%	0.00%
0.05	0.06%	0.21%	1.06%	0.00%	0.00%
0.075	0.16%	0.29%	0.54%	0.85%	0.01%
0.1	0.22%	0.32%	0.58%	1.10%	1.38%
0.125	0.21%	0.33%	0.61%	1.24%	0.79%

Recalling that customers are price (lead time) sensitive when $\rho < \tau$ ($\rho > \tau$) the results given in the upper-right (lower-left) triangles, in each $5 \times 5 = 25$ cells presented in Tables 19 and 21. One observes that in both tables, all of the average $\Delta^X(\cdot)$ values in the upper-right triangles of **S** and **P** settings are lower than that are in the upper-right triangles of **M**. For example, all of the values observed in upper-right triangles of **S** and **P** in Table 21 are 0.00%, whereas I observe average $\Delta^X(\cdot)$ values of 1.06%, 0.85% in the upper-right triangles of **M**. In contrast, the lower-left triangles point to the superior performance of **M**, where each average $\Delta^M(\cdot)$ is lower than either of the corresponding $\Delta^S(\cdot)$ and $\Delta^P(\cdot)$ averages in Tables 19 and 21. As a result, I reach the conclusion in Observation 12.

A more detailed investigation on Tables 19 and 21 reveals two more crucial observations: (i) average $\Delta^P(\gamma^{*P})$ and $\Delta^S(\gamma^{*S})$ values are quite close to each other, and (ii) average $\Delta^S(p_{min})$ values are always less than $\Delta^P(p_{min})$. In my analysis in Sections 4.2 and 4.3, I only compute $\Delta^X(p_{min})$ for all $X \in \{S, P, M\}$ to observe the performance of p_{min} choice as the incentive. In addition, to improve the performance of incentivization scheme, I propose a simple decision rule, which is denoted as **DR** and implements **M** when customers are lead time sensitive (i.e., $\rho > \tau$), and **S** otherwise (i.e., $\rho > \tau$).

4.2. *Impact of Traffic Intensity*

In this section, I analyze the impact of $\beta\nu$ on the performances of **S**, **P**, **M** and **DR**. I compute $\Delta^X(p_{min})$ for all $X \in \{S, P, M, DR\}$ using the numerical analysis setting given in Table 22 that tests traffic intensity levels of $0.6, 0.62, \ldots, 0.9$.

Table 22. Numerical analysis settings in Section 4.2

Traffic intensity	Accept All Price	Price/lead time sens.
$\beta \in \{0.030, 0.031, \ldots, 0.045\}$	$p_{min} \in \{10, 30, 50\}$	$\rho, \tau \in \{0.05, 0.075, 0.1\}$ $l_{max} \in \{60, 100, 140\}$

The average $\Delta^X(p_{min})$ for $X \in \{S, P, M, DR\}$ values for $\beta \in \{0.030, 0.031, \ldots, 0.045\}$ are plotted in Figure 14.

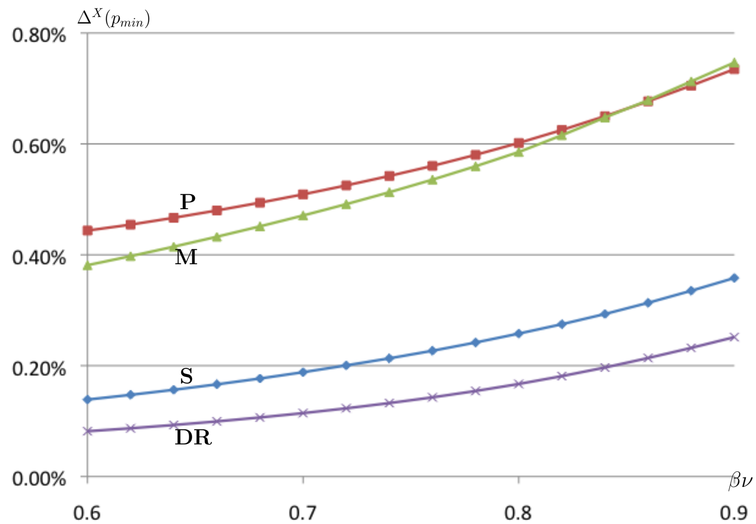


Fig. 14. The impact of traffic intensity on $\Delta^X(p_{min})$ for $X \in \{S, P, M, DR\}$

Observation 13. *Decentralized settings perform significantly worse under higher traffic intensity. Furthermore,* ***DR*** *outperforms all decentralized settings, when p_{min} is offered as the incentive.*

As observed in Figure 14, $\Delta^X(p_{min})$ averages increase slightly as the traffic intensity increases. Although **S** performs better than both **M** and **P**, all decentralized settings are outperformed by **DR**, which gives a maximum average $\Delta^{DR}(p_{min})$ of 0.25% when traffic intensity is 0.90%. The superior performance of **DR** can be attributed to improved performance of **M** and **S** for lead time and price sensitive customer cases, respectively. In addition, I note that $\Delta^{DR}(p_{min})$ is an upper bound on $\Delta^{DR}(\gamma^{*DR})$. Hence, one can obtain even better results by implementing optimal incentive values.

4.3. *Impact of the Accept All Price*

I now analyze the impact of p_{min} on $\Delta^X(p_{min})$ using the levels given in Table 23.

Table 23. Numerical analysis settings used in Section 4.3

Traffic intensity	Accept All Price	Price/lead time sens.
$\beta = 0.0375$	$p_{min} \in \{5, 6, \dots, 50\}$	$\rho, \tau \in \{0.05, 0.075, 0.1\}$ $l_{max} \in \{60, 100, 140\}$

Averaged $\Delta^X(p_{min})$ values for p_{min} values changing from 5 to 50 are illustrated in Figure 15.

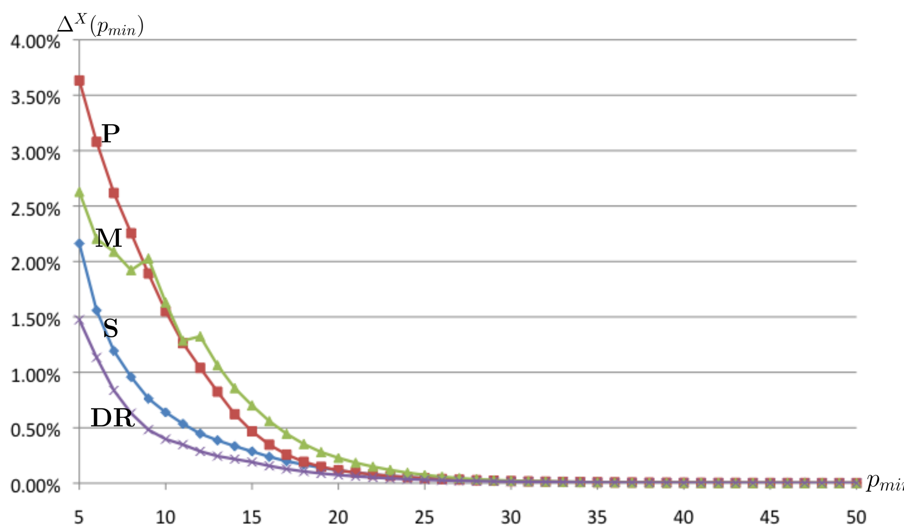


Fig. 15. The impact of accept all price on $\Delta^X(p_{min})$ for $X \in \{S, P, M, DR\}$

Observation 14. *When p_{min} is offered as the incentive, decentralized settings perform better as the accept all price increases.*

The averaged $\Delta^X(p_{min})$ values decrease in p_{min} with some exceptions under **M**. Unlike the previous observations, I observe that the averaged $\Delta^X(p_{min})$ values are greater than 1% in particular when p_{min} is relatively small. For example, for $p_{min} = 5$, I have averaged $\Delta^X(p_{min})$ values of 3.63%, 2.63%, 2.16% and 1.47% for **P**, **M**, **S** and **DR**, respectively. These results reveal the superior performance of **DR**, which provides an improvement of 31.8% over the performance of **S** when $p_{min} = 5$.

5. Conclusion

In this chapter, I discuss the impact of centralization of price and lead time decisions in a make-to-order company, whose price and lead time decisions are dynamically determined. Assuming deterministic processing times and stochastic arrivals, I model the centralized setting using a finite horizon stochastic dynamic programming formulation with the objective of maximizing total discounted expected profits, where profit is defined as revenues minus late delivery penalties. In the decentralized setting, marketing and manufacturing departments aim to maximize total discounted expected revenues and profits, respectively, where the profit of manufacturing department is defined as the incentive given for each unit of production minus the late delivery penalty. I consider three decision making sequences. In the marketing leader game, which is modeled using a repeated Stackelberg game, price decisions are determined in advance by the marketing department, which is followed by the lead time decision of the manufacturing department. In the manufacturing leader game, which is modeled similarly using a repeated Stackelberg game, lead decisions of manufacturing are followed by the price decisions of the marketing department. On the other hand, simultaneous setting represents the case where both departments act simultaneously, and is modeled using a repeated Cournot game. I derive optimal solutions for the centralized setting, and Markov perfect equilibria for all decentralized settings.

Using the derived optimal solutions, I conduct a numerical analysis to compute the total discounted expected profit differences of centralized and decentralized settings that yields several crucial findings: (i) using the optimal incentives, the profit differences can be decreased below 1% on the average, (ii) offering the accept all price (p_{min}) as the incentive, the profit differences can be decreased to below 1% when the accept all price is sufficiently high. (iii) marketing leader setting outperforms other decentralized settings when the customers are lead time sensitive, (iv) production leader setting and simultaneous setting perform better than the marketing-leader setting when customers are price sensitive, (v) increase of the accept all price and traffic intensity decreases and increases the profit differences, respectively. Using these results, I develop a simple decision rule that chooses accept all price as the incentive, marketing leader (simultaneous) setting when customers are lead time (price) sensitive, which is shown to outperform all decentralized settings.

As the future work, I plan to focus on two different directions. First, I will consider the decentralized settings where the marketing and manufacturing departments do not have complete information about each other's objective function, and objective function values. I will focus on the derivation of the perfect Markov equilibrium, and analyze the profits obtained under this imperfect information case. Second, I will focus on the cases where the price and lead times are not dynamically quoted, similar to analysis conducted in Chapter 2. I plan to answer the research question “Should companies focus on the coordination of the price and lead time decisions, or dynamic quotation strategies in order to improve their profits?”

CHAPTER 5

Conclusions

1. Summary of Results and Contributions

In this dissertation, I discuss the dynamic pricing and lead time quotation problem that is faced by make-to-order manufacturers. The manufacturer produces a single type of product, receives inquiries from customers, and quotes a price for the product and a delivery lead time to them. Late deliveries are penalized. The problem seeks for the optimal price and lead time decisions of the manufacturers that maximizes the company profits in the long run.

In Chapter 2, I discuss the problem of price and lead time quotation in a make-to-order system with two customer classes: (1) contract customers whose orders are practically always accepted and fulfilled based on a contract price and lead time set up at the beginning of the time horizon, and (2) spot purchasers who arrive over time and are quoted a price and lead time pair dynamically. The objective is to maximize the long run expected average profit per unit time, where profit from a customer is defined as revenues minus lateness penalties incurred due to lead time violations. I first model the dynamic quotation problem of the spot purchasers as an infinite horizon Markov decision process, given a fixed price and lead time for contract customers. I analyze the impact of price and lead time sensitivity of customers on the optimal price and lead time decisions for spot purchasers, and characterize the optimal policy. I explore the benefits of dynamic quotation in comparison to the use of fixed price and lead times, and provide recommendations for manufacturers. I finally analyze the optimal contract terms given the dynamic quotation strategy for spot purchasers, and discuss the profit improvements offered by the optimal mix of spot and contract customers.

There are two main theoretical contribution of this chapter. First is the theorems demonstrating the impact of price and lead time sensitivity on the optimal price and lead time decisions. Although price and lead time quotation literature demonstrate similar results, the papers in the literature use simple non-dynamic models with relatively simple linear demand functions. Second, I provide several theorems that reduce the action spaces in the MDP formulations, and in turn, allow huge computational time savings. This is a significant contribution to the MDP literature that typically suffer from the high computational effort due to curse of dimensionality. As the practical contribution, this chapter provides useful recommendations for manufacturers about whether the dynamic price and/or lead time quotation strategies should be implemented, and whether the manufacturers should spend effort in optimizing their customer mix, or simply focus on contract customers (spot purchasers).

My model formulation in Chapter 2, requires computation of expected lateness in multi-class priority queuing systems. Hence, in Chapter 3, I discuss the evaluation of expected tardiness of an order at the time of arrival in an $M/M/c$ queuing system with N priority classes, considering both nonpreemptive and preemptive service disciplines. Upon arrival, a customer order is quoted a lead time, and placed in the queue according to the priority class of the customer. Orders within the same priority class are processed on a first-come, first-served basis. I derive the Laplace transforms of the expected tardiness of the order given the quoted lead time, priority class of the order, and system status. For the special case of single priority class, the Laplace transform can be inverted into a closed-form expression. For the case with multiple priority classes, a closed-form expression cannot be obtained, hence, I develop three customized numerical inverse Laplace transformation algorithms. Two of these algorithms provide upper and lower bounds for the expected tardiness under a simple condition on system parameters. Using this property, I obtain error bounds for my customized algorithms; such bounds are not available for general purpose numerical inversion algorithms in the literature. Next, I develop a novel methodology to compare the precision of general purpose numerical inversion algorithms and analyze the performances of three algorithms from the literature. Finally, I provide a recommendation scheme given computational time and error tolerances of the decision maker.

This chapter contributes to the academic literature in two ways. First, it develops a methodology for the accurate estimation of expected tardiness that establishes an important step toward developing due date quotation policies in a multiclass queue. This contributes to the due date quotation literature that has been largely focused on single-class queues. Second, the novel numerical inverse Laplace transformation algorithm comparison methodology allows the comparison of algorithms using non-invertible Laplace transforms, which has not been achieved in Laplace transformation literature to the best of my knowledge. Similar to Chapter 2, this chapter provides a recommendation scheme for manufacturers as a significant practical contribution. This scheme helps the manufacturers choose their tardiness computation algorithm given a computational time and late delivery penalty estimation error tolerance. This recommended tardiness computation algorithm allows manufacturers to determine accurate and timely due date decisions.

In Chapter 4 I discuss the dynamic price and lead time quotation problem in a make-to-order system under two decision making settings: (1) Centralized setting considers a central agent determining price and lead times with the objective of maximizing total discounted expected profits, and (2) decentralized setting assumes that price and lead time decisions are taken respectively by marketing and manufacturing departments. The objective of marketing department is to maximize total discounted expected revenues, whereas manufacturing department strives to maximize total discounted expected profits of the department, which is defined as the incentive offered for each unit of production minus late delivery penalties. I consider three decision making sequences under decentralized setting: (1) marketing leader setting, where marketing determines the prices in advance, (2) manufacturing leader setting, where manufacturing determines lead times in advance, and (3) simultaneous setting, where both departments take decisions simultaneously. I derive closed form solutions for all settings, and discuss the impact of price and lead time sensitivity of customers on optimal solutions. I next analyze the inefficiencies of the decentralized setting in comparison to centralized decision making. Numerical analysis results reveal that performance of decentralized settings can be significantly increased by the choice of right incentives. Furthermore, production leader and simultaneous settings perform better than marketing leader setting when customers are price sensitive, whereas marketing leader setting outperforms the other two settings when the customers are lead time sensitive.

Although dynamic price and lead time quotation, and decentralization of price and lead time decisions are widely discussed in the literature, to the best of my knowledge, Chapter 4 is the first study that considers the decentralization of price and lead time decisions under a dynamic quotation environment. I provide theoretical results to determine optimal price and lead time decisions under the decentralized setting, as well as the centralized setting where decisions are coordinated similar to Chapter 2. For the practical contribution, an incentivization scheme is recommended. This scheme minimizes the negative impacts of the lack of coordination, often times yielding to close profits to the fully coordinated centralized case.

2. Future Work

This dissertation can be extended in several directions. First, one can relax the assumption of exponentially distributed inter-arrival times and processing times, and focus on the research question "how do the optimal price and lead time decisions found in Chapters 2 and 4 perform in an environment where inter-arrival time and processing time distributions are not exponential?" However, in this case, obtaining the optimal price and lead time decisions requires an MDP model with additional state variables to record the amount of time the product is under process, and the amount of time since the last order arrival. Hence, the MDP formulation would be unsolvable due to the huge state space. One way to overcome this complication is to develop a simulation study. Using the simulation study, one can compare the performances of optimal price and lead time decisions obtained in this dissertation, and simple price and lead time quotation strategies.

In Chapters 2 and 4, the models assume that arrival rate and acceptance probability of the customers does not change over time. Second, one can consider the case, where the customers keep a memory of previously quoted prices and lead times. Thus, high (low) prices may decrease (increase) the future arrival rate of the customers. Similarly, unsatisfied (satisfied) lead times may decrease (increase) the acceptance probability of a given quote. This problem can be modeled as an MDP similar to Chapter 2 and 4, however, the extension requires state parameters to keep the memory of previously quoted prices and lead times. Using this research problem one can answer the following research questions: What is the impact of customer memory on the optimal price and lead time decisions? What are the negative consequences of ignoring customer memory? What is the positive impact of customer memory consideration on the service levels, i.e., proportion of orders met on time?

Third, price and lead time quotation problem can be extended to multiple product case in several ways. One interesting research direction is the case where there are multiple products that have different demand functions (i.e., acceptance probability functions) and are produced by the same server. These products may also be substitute or complementary of each other, and thus, sale of any particular product increases or decreases of the demand of the other, respectively. The intriguing research questions in this problem are: what is the optimal capacity allocation between the two products? what is the optimal production schedule of the server?, how do the substitution effects impact the optimal price, lead time and scheduling decisions?

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APPENDIX A

PROOFS AND NUMERICAL ANALYSIS SETTINGS OF CHAPTER 2

In this Appendix, I provide the proofs my Propositions, Lemmas and Theorems, two additional Theorems, and detailed discussion about my numerical analysis settings and results.

1. Proofs of Propositions, Lemmas and Theorems in Chapter 2

Proof of Lemma 1. Using Equation (2.3) and Leibnitz's derivation rule under the integral sign, I get

$$\frac{\partial L^S_{i+j}(l)}{\partial l} = \tau^S \int_l^\infty \frac{\partial (t-l) f_{i+j+1}(t)}{\partial l} dt = -\tau^S \int_l^\infty f_{i+j+1}(t) dt = \tau^S (F_{i+j+1}(l) - 1). \tag{A.1}$$

where $F_{i+j+1}(\cdot)$ denotes the cumulative distribution function of TIS for a spot purchaser order, who upon arrival finds the $i+j$ orders in the system. Since, $F_{i+j+1}(l) < 1$ for all $d \geq 0$, $\partial L^S_{i+j}(l)/\partial l$ is negative and $L^S_{i+j}(l)$ is hence decreasing in l. To prove convexity, I differentiate one more time to obtain

$$\frac{\partial^2 L^S_{i+j}(l)}{\partial l^2} = \tau^S f_{i+j+1}(l) \geq 0, \quad \text{for all } l \geq 0, \tag{A.2}$$

which ensures that $L^S_{i+j}(l)$ is convex. □

Proof of Theorem 1. The proof of Theorem 1 requires the following Lemma.

Lemma 2. *If $p \geq L^S_{i+j}(l) + \Delta h^*_{i,j,k}$ then $\partial^2 \gamma_{i,j,k}(p,l)/\partial p \partial l \leq 0$, $\partial^2 \gamma_{i,j,k}(p,l)/\partial p^2 \leq 0$ and $\partial^2 \gamma_{i,j,k}(p,l)/\partial l^2 \leq 0$ hold.*

Proof. By differentiating $\gamma_{i,j,k}(p,l)$, I get the following,

$$\frac{\partial^2 \gamma_{i,j,k}(p,l)}{\partial p^2} = \frac{\partial^2 f^S(p,l)}{\partial p^2}(p - L^S_{i+j}(l) - \Delta h^*_{i,j,k}) + 2\frac{\partial f^S(p,l)}{\partial p}, \tag{A.3}$$

$$\frac{\partial^2 \gamma_{i,j,k}(p,l)}{\partial l^2} = \frac{\partial^2 f^S(p,l)}{\partial l^2}(p - L^S_{i+j}(l) - \Delta h^*_{i,j,k}) - 2\frac{\partial f^S(p,l)}{\partial l}\frac{\partial L^S_{i+j}(l)}{\partial l} - f^S(p,l)\frac{\partial^2 L^S_{i+j}(l)}{\partial l^2}, \tag{A.4}$$

$$\frac{\partial^2 \gamma_{i,j,k}(p,l)}{\partial p \partial l} = \frac{\partial^2 f^S(p,l)}{\partial p \partial l}(p - L^S_{i+j}(l) - \Delta h^*_{i,j,k}) + \frac{\partial f^S(p,l)}{\partial l} - \frac{\partial L^S_{i+j}(l)}{\partial l}\frac{\partial f^S(p,l)}{\partial p}. \tag{A.5}$$

I have $\frac{\partial^2 f^S(p,l)}{\partial l^2} \leq 0$, $\frac{\partial^2 f^S(p,l)}{\partial p^2} \leq 0$, and $p \geq L^S_{i+j}(l) + \Delta h^*_{i,j,k}$ from the assumption of the lemma, $\frac{\partial f^S(p,l)}{\partial l} \leq 0$, $\frac{\partial f^S(p,l)}{\partial p} \leq 0$, $f^S(p,l) \geq 0$, and $\frac{\partial^2 f^S(p,l)}{\partial p \partial l} \leq 0$ from Assumption 1, $\frac{\partial L^S_{i+j}(l)}{\partial l} \leq 0$, and $\frac{\partial^2 L^S_{i+j}(l)}{\partial l^2} \geq 0$ from Lemma 1. Thus, the desired conditions are satisfied using straightforward algebra. □

Proof of Theorem 1. I show that

(i) If $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l}$, then $l^*_{i,j,k} = 0$,

(ii) If $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{1}{\partial f^S(p_{sMin},0)/\partial p}$, then $p^*_{i,j,k} = p_{sMin}$,

(iii) if $p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l} < L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq T^2_{i,j}$, then $l^*_{i,j,k} > 0$, and

(iv) if $p_{sMin} + \frac{1}{\partial f^S(p_{sMin},0)/\partial p} < L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l}$ then $p^*_{i,j,k} > p_{sMin}$,

which complete the proof for Cases 1, 2.1 and 2.2. I omit the proofs of Cases 3 and 4, because they are straightforward.

(i) Assume $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l}$ holds. Then, I have

$$L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l} \leq p_{sMin}. \tag{A.6}$$

From $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin}$, and concavity of $f^S(p,l)$ in p and l, all conditions for Lemma 2 hold for $(p,l) \in \theta^S$. From $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l}$, $f^S(p_{sMin},0)/\partial l \leq 0$, and by simple algebraic operations, the following inequality holds.

$$\left(p_{sMin} - L^S_{i+j}(0) - \Delta h^*_{i,j,k}\right)\frac{\partial f^S(p_{sMin},0)}{\partial l} + \tau^S \leq 0. \tag{A.7}$$

Since $f^S(p_{sMin}, 0) = 1$ and $\tau^S = -\partial L^S_{i+j}(0)/\partial l$ (see Lemma 1), the inequality can be rewritten as

$$\left(p_{sMin} - L^S_{i+j}(0) - \Delta h^*_{i,j,k}\right) \frac{\partial f^S(p_{sMin}, 0)}{\partial l} - f^S(p_{sMin}, 0) \frac{\partial L^{S+}_{i+j}(0)}{\partial l} \leq 0. \quad (A.8)$$

Note that the left hand side in Equation (A.8) is the derivative of $\gamma_{i,j,k}(p, l)$ in l evaluated at the point $(p_{sMin}, 0)$. Thus, I reach $\frac{\partial \gamma_{i,j,k}(p_{sMin},0)}{\partial l} \leq 0$.

From Lemma 2, $\frac{\partial \gamma_{i,j,k}(p,l)}{\partial l}$ is nonincreasing in both p and l. Thus, $\frac{\partial \gamma_{i,j,k}(p,l)}{\partial l} \leq 0$ holds for $(p, l) \in \theta^S$, and the maximum is achieved at $l^*_{i,j,k} = 0$.

(ii) Assume $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{1}{\partial f^S(p_{sMin},0)/\partial p}$ holds. Then, I have

$$L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{1}{\partial f^S(p_{sMin}, 0)/\partial p} \leq p_{sMin}. \quad (A.9)$$

From $L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin}$ and concavity of $f^S(p, l)$ in p and l, all conditions for Lemma 2 hold. Proceeding similarly to case (i), I obtain $\frac{\partial \gamma_{i,j,k}(p_{sMin},0)}{\partial p} \leq 0$. From Lemma 2, $\frac{\partial \gamma_{i,j,k}(p,l)}{\partial p}$ is nonincreasing in both p and l. Thus, $\frac{\partial \gamma_{i,j,k}(p,l)}{\partial p} \leq 0$ holds for $(p, l) \in \theta^S$, and the maximum is achieved at $p^*_{i,j,k} = p_{sMin}$.

(iii) Assume that $p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l} < L^S_{i+j}(0) + \Delta h^*_{i,j,k} \leq p_{sMin} + \frac{1}{\partial f^S(p_{sMin},0)/\partial p}$. Proceeding similarly to case (i), and using the inequality $p_{sMin} + \frac{\tau^S}{\partial f^S(p_{sMin},0)/\partial l} < L^S_{i+j}(0) + \Delta h^*_{i,j,k})$, one reaches the following inequality.

$$\frac{\partial \gamma_{i,j,k}(p_{sMin}, 0)}{\partial l} > 0. \quad (A.10)$$

Since $\gamma_{i,j,k}(p, l)$ is concave in l, there exists an optimal $l^* > 0$ such that $\frac{\partial \gamma_{i,j,k}(p_{sMin},l^*)}{\partial l} = 0$.

(iv) The proof for this case is similar to case (ii), and hence, is omitted. □

Proof of Theorem 2 I define the following,

$$\theta^{Sp} = \{(p,l) \in \theta^S : \partial\gamma_{i,j,k}(p,l)/\partial p = 0\}, \quad \theta^{Sl} = \{(p,l) \in \theta^S : \partial\gamma_{i,j,k}(p,l)/\partial l = 0\},$$

$$\underline{\theta}^{Sp} = \{(p,l) \in \theta^S : \partial\gamma_{i,j,k}(p,l)/\partial p < 0\}, \quad \underline{\theta}^{Sl} = \{(p,l) \in \theta^S : \partial\gamma_{i,j,k}(p,l)/\partial l < 0\}.$$

I make use of the following fmy cases that can be proven using the concavity of $\gamma_{i,j,k}(p,l)$.

(A) If there exists a (p^*, l^*) such that $(p^*, l^*) \in \theta^{Sp} \cap \theta^{Sl}$, then $(p^*_{i,j,k}, l^*_{i,j,k}) = (p^*, l^*)$,

(B) if $\theta^{Sp} \subset \underline{\theta}^{Sl}$, then $l^*_{i,j,k} = 0$,

(C) if $\theta^{Sl} \subset \underline{\theta}^{Sp}$, then $p^*_{i,j,k} = p_{sMin}$, and

(D) if $\underline{\theta}^{Sp} = \underline{\theta}^{Sl} = \theta^S$, then $(p^*_{i,j,k}, l^*_{i,j,k}) = (p_{sMin}, 0)$

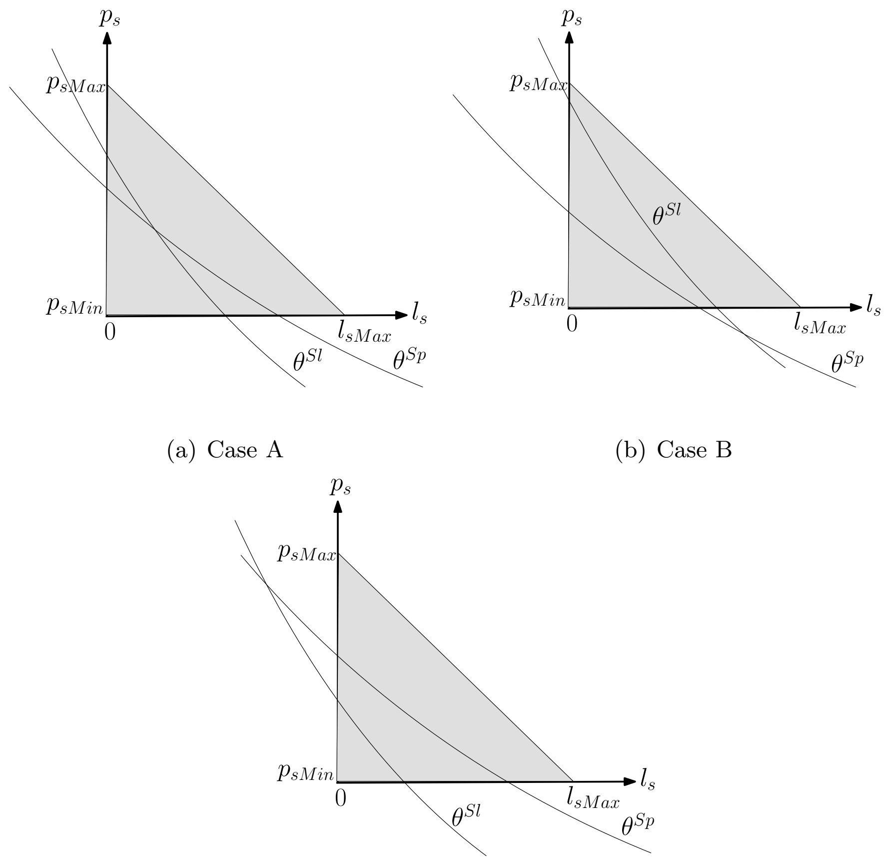
In Figure 16, I plot representative θ^{Sp} and θ^{Sl} curves for the first three cases. I note that from concavity of $\gamma_{i,j,k}(p,l)$, areas below θ^{Sp} and θ^{Sl} are $\underline{\theta}^{Sp}$ and $\underline{\theta}^{Sl}$, respectively.

When $(p^*_{i,j,k}, l^*_{i,j,k}) \neq (p_{sMax}, 0)$, either of the following above fmy cases should hold. I next prove the following Lemma that is used in the proof.

Lemma 3. *(i)* $p^*_{i,j,k} > p_{sMin}$, $l^*_{i,j,k} > 0$ *if and only if Equation A.11 holds.*

$$\rho^S(p^*_{i,j,k}, l^*_{i,j,k}) = \tau^S \overline{F}_{i+j+1}(l^*_{i,j,k}), \tag{A.11}$$

holds.



(a) Case A (b) Case B

(c) Case C

Fig. 16. Illustration of Cases A, B and C. Cases A represents the case where $p^*_{i,j,k} > p_{sMin}$, $l^*_{i,j,k} > 0$, whereas Cases B and C represent cases where $l^*_{i,j,k} = 0$ and $p^*_{i,j,k} = p_{sMin}$, respectively

(ii) If there is no solution satisfying Equation A.11 in θ^S*, then at least one of the followings hold (i)* $p^*_{i,j,k} > p_{sMin}$*, (ii)* $l^*_{i,j,k} > 0$.

Proof of Lemma 3 I first note that any optimal solution $p^*_{i,j,k} > p_{sMin}$, $l^*_{i,j,k} > 0$ can only be obtained by Case A. Solving for $\partial\gamma_{i,j,k}(p^*, l^*)/\partial l = \partial\gamma_{i,j,k}(p^*, l^*)/\partial p = 0$, I obtain the Equality in Equation A.11, which completes the proof of (i). Furthermore, if (ii) holds, then I have no solution for Case A. Thus, B, C or D should hold. □

Proof of Theorem 2 I now prove Theorem 2 case by case.

(i) If $\rho^S(p,l) > \tau^S$ for $(p,l) \in \theta^S$ holds, then (i) there is no solution for Equation A.11 for $(p,l) \in \theta^S$, and (ii) $\tau^S \leq \rho^S(p,0)$, for $p_{sMin} \leq p \leq p_{sMax}$. From (i), Case A does not hold. Hence, as long as $(p^*_{i,j,k}, l^*_{i,j,k}) \neq (p_{sMax}, 0)$, in the remainder of my analysis either of B, C or D should hold. Consider the point $(p_1, 0)$ where $\partial\gamma_{i,j,k}(p_1,0)/\partial l = 0$. Taking derivatives, I obtain

$$\frac{\partial\gamma_{i,j,k}(p_1,0)}{\partial l} = \left(p_1 - L^S_{i+j}(0) - \Delta h^*_{i,j,k}\right)\frac{\partial f^S(p_1,0)}{\partial l} + \tau^S f^S(p_1,0) = 0. \quad \text{(A.12)}$$

Rearranging the terms gives

$$p_1 - L^S_{i+j}(0) - \Delta h^*_{i,j,k} = -\frac{\tau^S f^S(p_1,0)}{\partial f^S(p_1,0)/\partial l} \quad \text{(A.13)}$$

Evaluating $\frac{\partial\gamma_{i,j,k}(p_1,0)}{\partial p}$, and plugging in $p_1 - L^S_{i+j}(0) - \Delta h^*_{i,j,k}$ I obtain

$$\begin{aligned}\frac{\partial\gamma_{i,j,k}(p_1,0)}{\partial p} &= \left(p_1 - L^S_{i+j}(0) - \Delta h^*_{i,j,k}\right)\frac{\partial f^S(p_1,0)}{\partial p} + f^S(p_1,0),\\ &= -\frac{\tau^S f^S(p_1,0)}{\partial f^S(p_1,0)/\partial l}\frac{\partial f^S(p_1,0)}{\partial p} + f^S(p_1,0),\\ &= f^S(p_1,0)\left(1 - \frac{\tau^S}{\rho^S(p_1,0)}\right). \quad \text{(A.14)}\end{aligned}$$

Since $\tau^S \leq \rho^S(p,0)$, for $p_{sMin} \leq p \leq p_{sMax}$, right hand side of Equation (A.14) is non-negative, and hence, $\frac{\partial\gamma_{i,j,k}(p_1,0)}{\partial p} \geq 0$. One notes that $(p_1,0) \in \theta^{Sl}$, and $(p_1,0) \notin \underline{\theta}^{Sp}$. Hence, Case C cannot hold, and only remaining possible Cases are B and D, i.e., $l^*_{i,j,k} = 0$ (even the optimal solution is rejection of customer, the optimal solution is $(p^*_{i,j,k}, l^*_{i,j,k}) = (p_{sMax}, 0)$).

(ii) Similar to the proof of part (i), if $\rho^S(p,l) < \tau^S \overline{F}^S_{i+j+1}(l_{sMax})$ for $(p,l) \in \theta^S$, then I have the following: (i) there is no solution for Equation A.11 for $(p,l) \in \theta^S$, and (ii) $\rho^S(p,0) < \tau^S$, for $p_{sMin} \leq p \leq p_{sMax}$. The proof follows similar to the proof of part (i).

(iii) The proof of part (iii) immediately follows from Lemma 3. □

Proof of Proposition 1. From Lemma 1 and Assumption 1 for any $l_1 \geq l_2$, I have (i) $\overline{F}^S_{i+j+1}(l_1) \leq \overline{F}^S_{i+j+1}(l_2)$, (ii) $-\partial f^S(p,l_1)/\partial p \geq -\partial f^S(p,l_2)/\partial p$. Using (i), (ii) and additivity of $f^S(p,l)$, one obtains

$$\frac{\partial f^S(p,l_2)/\partial l}{\overline{F}^S_{i+j+1}(l_2)} \leq \frac{\partial f^S(p,l_1)/\partial l}{\overline{F}^S_{i+j+1}(l_1)}, \tag{A.15}$$

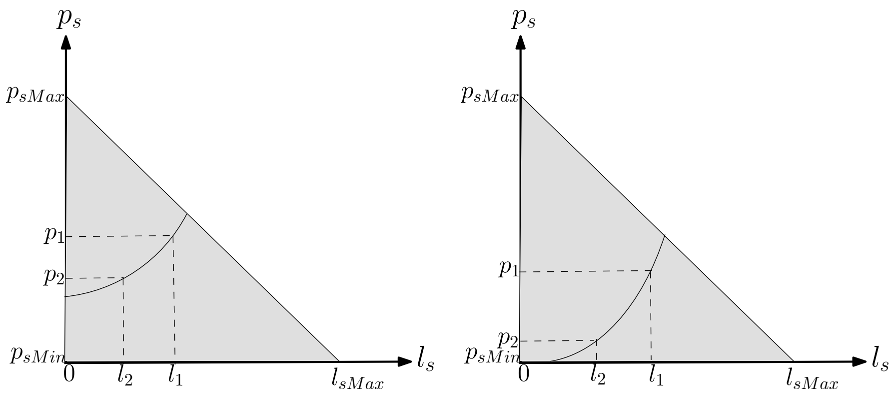
for any $p_{sMin} \leq p \leq p_{sMax}$.

Assume that (p_1,l_1) and (p_2,l_2) are two pairs of solutions satisfying $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$, where $l_1 \geq l_2$. Using Inequality (A.15), I have,

$$-\tau^S \partial f^S(p_2,l_2)/\partial p = \frac{\partial f^S(p_2,l_2)/\partial l}{\overline{F}^S_{i+j+1}(l_2)} \leq \frac{\partial f^S(p_1,l_1)/\partial l}{\overline{F}^S_{i+j+1}(l_1)} = -\tau^S \partial f^S(p_1,l_1)/\partial p, \tag{A.16}$$

which gives $-\tau^S \partial f^S(p_2,l_2)/\partial p \leq -\tau^S \partial f^S(p_1,l_1)/\partial p$. Thus, from Assumption 1, I have $p_1 \geq p_2$. Consequently, the set of p satisfying $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$ is non-decreasing in l. I illustrate two representative set of (p,l) satisfying $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$ in Figure 17.

First consider case (ii). From $\partial f^S(p_{sMin},0)/\partial l < \tau^S \partial f^S(p_{sMin},0)/\partial p$, one clearly obtains $T_1 < T_2$, which indicates that I have $p^*_{i,j,k} > p_{sMin}$ and $l^*_{i,j,k} = 0$ for some value of $\Delta h^*_{i,j,k}$ from Theorem 1. In addition, I know that the set of p satisfying $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$ is non-decreasing in l, which indicates that the curve



(a) Case 1

(b) Case 2

Fig. 17. Two representation set of (p,l) satisfying $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$

satisfying $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$ looks like the one depicted in Case 1 of Figure 17. Thus, $p^*_{i,j,k} = p_{sMin}$ cannot hold. In case (iii), similarly I have $T_2 < T_1$, and hence, $p^*_{i,j,k} = p_{sMin}$ and $l^*_{i,j,k} > 0$ for some value of $\Delta h^*_{i,j,k}$ from Theorem **??**. Thus $p^*_{i,j,k} > p_{sMin}$ and $l^*_{i,j,k} = 0$ cannot hold. In case (i), on the other hand, I have that both $T_1 \to -\infty$ and $T_2 \to -\infty$ indicating that neither $T_1 < T_2$, nor $T_1 \geq T_2$ hold. Thus, neither $p^*_{i,j,k} = p_{sMin}$, nor $l^*_{i,j,k} = 0$ holds, implying that $\frac{\rho^S(p,l)}{\tau^S} = \overline{F}^S_{i+j+1}(l)$ always holds. □

Proof of Theorem 3.

To facilitate my analysis, $v^*_{Dyna}(p_c, l_c)$ is rewritten in Equation (A.17).

$$v^*_{Dyna}(p_c,l_c) = \lambda^S(p_c,l_c) \sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k}) f^S(p^*_{i,j,k}, l^*_{i,j,k}) \Omega^{S*}_{i,j,k}(p_c,l_c) + \lambda^C(p_c,l_c) \sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k}) \Omega^{C}_{i,j,k}(p_c,l_c), \tag{A.17}$$

where $\pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k})$ denotes the steady state probability of being at state $(i,j,k) \in \mathcal{S}$ given the set of optimal decisions $(p^*_{i,j,k}, l^*_{i,j,k})$, $\Omega^{S*}_{i,j,k}(p_c,l_c) = p^*_{i,j,k} -$

$L^S_{i+j}(l^*_{i,j,k})$ (i.e., the optimal expected profit obtained from a spot purchaser at state (i,j,k)). $\pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k}) f^S(p^*_{i,j,k}, l^*_{i,j,k}) \Omega^{S*}_{i,j,k}(p_c, l_c)$ and $\pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k}) \Omega^C_{i,j,k}(p_c, l_c)$ denote the average optimal expected profit obtained from a spot purchaser and contract customer arriving in state $(i,j,k) \in \mathcal{S}$ when the offered contract terms are (p_c, l_c), respectively.

Given that $\lambda^C \in \Lambda^C$, changing $p_c(\lambda^C)$ (or equivalently $l_c(\lambda^C)$) only impacts $\Omega^C_{i,j,k}(p_c, l_c) = p_c - L^C_{j,k}(l_c)$ in (A.17), whereas the other terms do not change since λ^C, λ^S, $p^*_{i,j,k}$ and $l^*_{i,j,k}$ do not change. Since $\Omega^C_{i,j,k}(p_c, l_c)$ is concave in p_c and l_c, one can obtain the maximizer of $v^*_{Dyna}(p_c, l_c)$ using first order conditions. Plugging $p_c(\lambda^C)$ in Equation (A.17) for p_c and taking the first derivative w.r.t. $l_c(\lambda^C)$, I obtain,

$$\frac{\partial v^*_{Dyna}(p_c, l_c(\lambda^C))}{\partial l_c(\lambda^C)} = \lambda^C \sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k}) \left[-\rho^C + \tau^C \overline{F}^C_{j,k}(l_c(\lambda^C))\right] \quad \text{(A.18)}$$

Since, $\frac{\partial v^*_{Dyna}(p_c, l_c(\lambda^C))}{\partial l_c(\lambda^C)}$ is non-increasing in $l_c(\lambda^C)$, I have the following three conditions: (i) If $\frac{\partial v^*_{Dyna}(p_c,0)}{\partial l_c(\lambda^C)} \le 0$ then $l^*_c(\lambda^C) = 0$, (ii) if $\frac{\partial v^*_{Dyna}(p_c, l_{cMax}(1-\frac{\lambda^C}{\lambda_{cMax}}))}{\partial l_c(\lambda^C)} \ge 0$ then $l^*_c(\lambda^C) = l_{cMax}(1-\frac{\lambda^C}{\lambda_{cMax}})$ (note that the maximum value of $l_c(\lambda^C)$ is $l_{cMax}(1-\frac{\lambda^C}{\lambda_{cMax}})$), (iii) if neither (i), nor (ii) holds, then $\frac{\partial v^*_{Dyna}(p_c, l^*_c(\lambda^C))}{\partial l_c(\lambda^C)} = 0$. If (i) holds, then from $\overline{F}^C_{j,k}(0) = 1$ and $\sum_{(i,j,k)\in\mathcal{S}} \pi^*_{i,j,k}(p^*_{i,j,k}, l^*_{i,j,k}) = 1$, it follows that $-\rho^C + \tau^C \le 0$, which completes the proof for Case (i). The proofs for Cases (ii) and (iii) follow immediately from (ii) and (iii). □

Proof of Proposition 2. Assume that the given inequality holds. From Equation (3.9), I have $L^C_{0,0}(l'_c) = \frac{e^{-\mu l'_c}}{\mu}$. In addition, from $p''_c \ge p'_c$, I have $\lambda^C(p''_c, l'_c) \le$

$\lambda^C(p'_c, l'_c)$, which ensures that the following holds.

$$\lambda^S(p'_c, l'_c)p_{sMax} + \lambda^C(p'_c, l'_c)p'_c - \lambda^C(p''_c, l'_c)p''_c \leq L^C_{0,0}(l'_c)\left(\lambda^C(p'_c, l'_c) - \lambda^C(p''_c, l'_c)\right) \tag{A.19}$$

In addition, I have $L^C_{0,0}(l_c) \leq L^C_{j,k}(l_c)$ for any j and k, that gives the following:

$$\lambda^S(p'_c, l'_c)p_{sMax} + \lambda^C(p'_c, l'_c)p'_c - \lambda^C(p''_c, l'_c)p''_c \leq L^C_{j,k}(l'_c)\left(\lambda^C(p'_c, l'_c) - \lambda^C(p''_c, l'_c)\right), \quad (i,j,k) \in \mathcal{S}, \tag{A.20}$$

where by rearranging the terms I get

$$\lambda^S(p'_c, l'_c)p_{sMax} + \lambda^C(p'_c, l'_c)\Omega^C_{i,j,k}(p'_c, l'_c) \leq \lambda^C(p''_c, l'_c)\Omega^C_{i,j,k}(p''_c, l'_c), \quad (i,j,k) \in \mathcal{S}. \tag{A.21}$$

Note that, the following inequalities trivially hold

$$p_{sMax} \geq f^S(p^*_{i,j,k}(p_c, l_c), l^*_{i,j,k}(p_c, l_c)\Omega^{S*}_{i,j,k}(p_c, l_c), \quad (i,j,k) \in \mathcal{S}, \tag{A.22}$$

$$0 \leq f^S(p^*_{i,j,k}(p_c, l_c), l^*_{i,j,k}(p_c, l_c)\Omega^{S*}_{i,j,k}(p_c, l_c), \quad (i,j,k) \in \mathcal{S}. \tag{A.23}$$

where I write $(p^*_{i,j,k}, l^*_{i,j,k})$ in terms of (p_c, l_c) to avoid confusion. Combining inequalities (A.21), (A.22) and (A.23), I obtain

$$\begin{aligned} &\lambda^S(p'_c, l'_c)f^S(p^*_{i,j,k}(p'_c, l'_c), l^*_{i,j,k}(p'_c, l'_c))\Omega^{S*}_{i,j,k}(p'_c, l'_c) + \lambda^C(p'_c, l'_c)\Omega^C_{i,j,k}(p'_c, l'_c) \\ &\qquad \leq \lambda^S(p''_c, l'_c)f^S(p^*_{i,j,k}(p''_c, l'_c), l^*_{i,j,k}(p''_c, l'_c))\Omega^{S*}_{i,j,k}(p''_c, l'_c) + \lambda^C(p''_c, l'_c)\Omega^C_{i,j,k}(p''_c, l'_c) \end{aligned} \tag{A.24}$$

From Equation (A.17), one observes that (A.24) is a sufficient condition for $v^*_{Dyna}(p'_c, l'_c) \leq v^*_{Dyna}(p''_c, l'_c)$. □

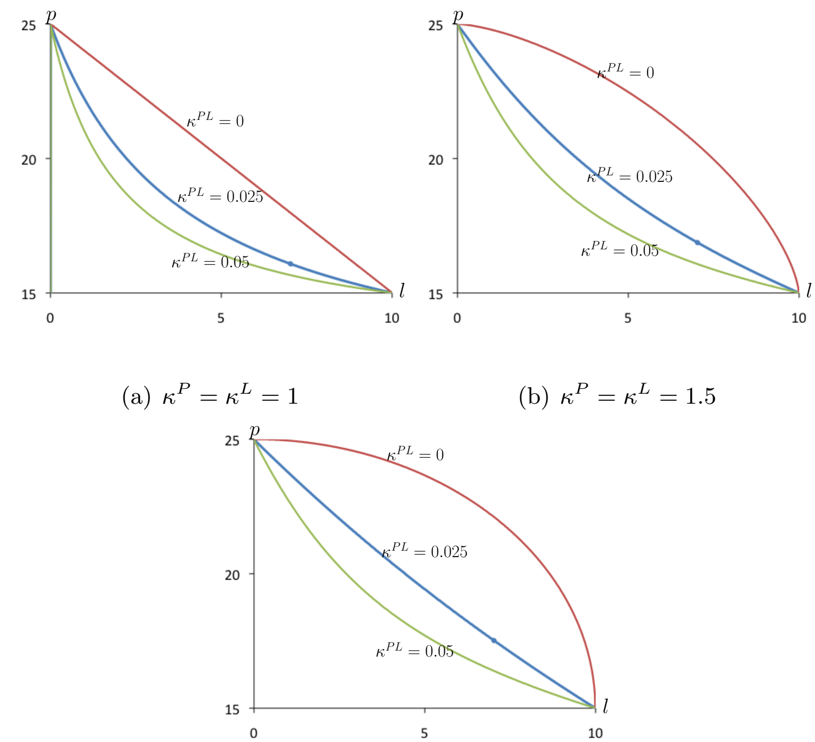
2. Numerical Analysis Settings in Chapter 2

In this section, I discuss setting of my numerical analysis. I use a relative value iteration algorithm to solve `Cont`, `Dyna`, `Fix`, `FixP` and `FixLT` [Bertsekas, 2001]. The late delivery penalties $L^S_{i+j}(l_s)$ and $L^C_{j,k}(l_c)$ are computed using Chapter 3, where it is indicated that a closed form solution for $L^S_{i+j}(l_s)$ cannot be obtained, necessitating approximation approaches based on numerical inverse Laplace transformation algorithms. I approximate $L^S_{i+j}(l_s)$ allowing an error tolerance of 0.001 using the Hybrid Algorithm in Appendix B-1. My computational results indicate that using an error tolerance lower than 0.001 does not change the optimal quotation decisions.

The buffer size is selected sufficiently high to minimize the probability of contract customer rejections (PCR). Maximum possible value of PCR can be evaluated by the $M/M/1/N$ queue limiting probability for state N, i.e., π_N. Under a traffic intensity (i.e., $\frac{\lambda^C+\lambda^S}{\mu}$) of 0.9, $N = 80$ gives the limiting probability of $\pi_{80} = 2.18 \times 10^{-5}$ which is sufficiently small [Ross, 2007]. Hence, I set $N = 80$, and the tested traffic intensity levels are bounded above by 0.9. Note that PCR decreases further at lower traffic intensity levels.

I model $f^S(p,l)$ as in Equation (2.22), and plot $p_{sMax}(l)$ for several tested forms of $f^S(p,l)$ in Figure 18, where $p_{sMin} = 15$, $p_{sMax} = 25$ and $l_{sMax} = 10$.

One verifies that used form of $f^S(p,l)$ satisfies Assumption 1 (i) using $\kappa^P \geq 1$, $\kappa^L \geq 1$ and $\kappa^{PL} \geq 0$. In addition, I have $f^S(p_{sMin}, 0) = 1$, which is sufficient for Assumption 1 (ii). In Figure 18, I observe that $p_{sMax}(l)$ and $l_{sMax}(p)$ is nonincreasing in l and p, respectively. Thus, Assumptions 1 (iii) and (iv) hold for the tested values as well.



(a) $\kappa^P = \kappa^L = 1$ (b) $\kappa^P = \kappa^L = 1.5$

(c) $\kappa^P = \kappa^L = 2$

Fig. 18. Illustration of $p_{sMax}(l)$ for $\kappa^P = \kappa^L = 1, 1.5, 2$ and $\kappa^{PL} = 0, 0.025, 0.05$

The tested values of $\kappa^P \in \{1, 1.5, 2\}$, $\kappa^L \in \{1, 1.5, 2\}$ and $\kappa^{PL} \in \{0, 0.025, 0.05\}$ allow us to test linear/strictly concave, and additive/non-additive forms of $f^S(p, l)$. As observed in Figure 18 (a), when $\kappa^{PL} = 0.05$ the action space (i.e., θ^S) is significantly smaller than the case where $\kappa^{PL} = 0$. Increasing κ^{PL} further results in an even smaller action space. Hence the highest tested value of κ^{PL} is set to 0.05.

In all my numerical analysis, I set $\mu = 1$. The tested levels of traffic intensity, $\frac{\lambda^S+\lambda^C}{\mu} \in \{0.6, 0.75, 0.9\}$, and spot purchaser proportion, $\frac{\lambda^S}{\lambda^S+\lambda^C} \in \{\frac{1}{3}, \frac{2}{3}, 1\}$ indicate low, medium and high levels, respectively.

I note that changing $\tau^S, \tau^C, p_{sMax}, p_{sMin}, p_{cMax}$ and p_{cMin} in the same proportion changes the OAP values in the same proportion. Thus, I set $\tau^S = \tau^C = 1.5$, and change the values of other parameters. I change p_{sMax} proportional to p_{sMin}. The values of l_{sMax} are chosen considering the price/lead time sensitivity of the spot purchasers. When $f^S(p,l)$ is additive and linear in price and lead time, I have $\rho^S(p,l) = \frac{p_{sMax} - p_{sMin}}{l_{sMax}}$. I test for $\rho^S(p,l) = \{\frac{1}{5}\tau^S, \frac{1}{2}\tau^S, 2\tau^S, 5\tau^S\}$ to analyze changing levels of price/lead time sensitivity, and change the values of l_{sMax} considering the values of p_{sMax} and l_{sMin}. Below, I discuss the settings in each Section in detail.

2.1. *Settings in Section 3.1*

To test the computational time savings offered by action space reduction, I conduct a preliminary experiment to determine the appropriate discretization scheme for price and lead time decisions. The $[p_{sMin}, p_{sMax}]$ and $[0, l_{sMax}]$ intervals are tested for 5, 10, 20, 40, 60 and 80 equal-sized intervals leading to 6, 11, 21, 41, 61 and 81 discrete p and l alternatives, respectively. I observe that increasing the discrete alternatives from 61 to 81 increases the computational times significantly, whereas the computational time improvements obtained by the reduced action space increases less than 1%. Hence, I discretize the action space into 60 equal-sized intervals.

I choose the parameter levels as follows: My analyses from real life cases reveal that contract prices are often close to spot prices [Macyel.com, 2011, News.radio-electronics.co, 2011]. Hence I set $p_c = \frac{p_{sMax}+p_{sMin}}{2}$. Similarly l_c is set to $\frac{l_{sMax}}{2}$. I test (i) two levels of $\frac{\lambda^S+\lambda^C}{\mu} \in \{0.6, 0.9\}$, (ii) two levels of spot purchaser proportion, $\frac{\lambda^S}{\lambda^S+\lambda^C} \in \{\frac{1}{3}, 1\}$, (iii) two levels of $\kappa^P \in \{1, 2\}$, (iv) two levels of $\kappa^L \in \{1, 2\}$,

(v) two levels of $\kappa^{PL} \in \{0, 0.05\}$, (vi) two levels of $p_{sMin} \in \{5, 25\}$, (vii) two levels of $p_{sMax} \in \{2p_{sMin}, 4p_{sMin}\}$, (viii) two levels of $\rho^S(p,l) = \{\frac{1}{2}\tau^S, 2\tau^S,\}$, giving $l_{sMax} \in \{\frac{p_{sMax}-p_{sMin}}{0.75}, \frac{p_{sMax}-p_{sMin}}{3}\}$. I observe a minimum computational time improvement of 93% using this test bed, which is sufficient to demonstrate the substantial computational time improvement of the reduced action space. Thus, I use this 256 instance test bed in this Section, unlike Section 3.2, where I test 19880 instances.

2.2. *Settings in Section 3.2*

I conduct a preliminary experiment to determine the appropriate discretization scheme similar to the one used in Section 3.1. I observe that increasing the discrete alternatives from 21 to 41 increases obtained IMP. values less than 0.01% on the average, however quadruples the computational times. Hence, I discretize the action space into 20 equal-sized intervals.

Similar to the settings in Section 3.1, I set $p_c = \frac{p_{sMax}+p_{sMin}}{2}$ and $l_c = \frac{l_{sMax}}{2}$. For notational simplicity, I denote $\text{IMP}^*_{\texttt{FixP}} = \text{IMP}_{\texttt{FixP}} - \text{IMP}_{\texttt{FixLT}}$. Given a particular parameter setting Θ (e.g., BD=2, price sensitive spot purchasers), recommendations scheme works as follows:

(i) If $\max_\Theta \text{IMP}^*_{\texttt{Dyna}} \geq \Delta_{DQ}$, then recommend `Dyna`.

(ii) If `Dyna` is not recommended, $\max_\Theta \text{IMP}_{\texttt{FixLT}} \leq \Delta_{DQ}$, and $\max_\Theta \text{IMP}_{\texttt{FixP}} \leq \Delta_{DQ}$, then recommend `Fix`.

(iii) If `Dyna` and `Fix` are not recommended, $\max_\Theta \text{IMP}^*_{\texttt{FixP}} \geq \Delta_{DQ}$, and $\min_\Theta \text{IMP}^*_{\texttt{FixP}} \geq -\Delta_{DQ}$, then recommend `FixP`.

(iv) If Dyna and Fix are not recommended, $\max_\Theta \text{IMP}^*_{\texttt{FixP}} < \Delta_{DQ}$, and $\min_\Theta \text{IMP}^*_{\texttt{FixP}} \leq -\Delta_{DQ}$, then recommend FixLT.

(v) If Dyna and Fix are not recommended, $\max_\Theta \text{IMP}^*_{\texttt{FixP}} < \Delta_{DQ}$, and $\min_\Theta \text{IMP}^*_{\texttt{FixP}} > -\Delta_{DQ}$, then recommend either FixLT or FixP.

(vi) Otherwise, no recommendation.

APPENDIX B

PROOFS AND ALGORITHMS OF CHAPTER 3

I present the pseudo-codes for trapezoidal, midpoint and hybrid algorithms in Section 1, proofs of my theorems in Section 2, and Laplace transforms for the pre-emptive multi-server case in Section 3.

1. Proposed NILT Algorithms

Algorithm 2 The trapezoidal algorithm approximating $\tau_j(d)$ for $j \in \{0, 1, \ldots \overline{q}_i\}$, and $d = 0, w, 2w, \ldots, \overline{d}$

Set $j = 0$, and $\mathcal{Z}^w_{-1}(d) = \tau_{-1}(d)$ for $d = 0, w, 2w, \ldots \overline{d}$

while $j \leq \overline{q}_i$ **do**

  $d = w$

  while $d \leq \overline{d}$ **do**

    Set $\psi(d, x) = \mathcal{Z}^w_{j-1}(d - x) g_{\overline{\lambda}_i, c\mu}(x)$ for $x = w, 2w, \ldots d$; and $\psi(d, 0) = \mathcal{Z}^w_{j-1}(d) c\mu$.

    Compute $\mathcal{Z}^w_j(d) = \dfrac{w}{2} \displaystyle\sum_{k=0}^{d/w-1} \psi(d, kw) + \psi(d, kw + w)$

    $d = d + w$

  end while

  $j = j + 1$

end while

Remark 2. *The trapezoidal algorithm evaluates $\mathcal{Z}^w_j(d)$ for $d = 0, w, 2w, \ldots, \overline{d}$, $j = 0, 1, \ldots \overline{q}_i$, whereas $\mathcal{M}^w_j(d)$ is computed for $d = 0, w/2, w, 3w/2, \ldots, \overline{d}$, $j = 0, 1, \ldots \overline{q}_i$. The difference is due to the structural difference of two algorithms (see Equations (3.15) and (3.16)), and results in higher computational times for the midpoint algorithm.*

Remark 3. *In the trapezoidal algorithm, the evaluation of $\psi(d, x)$ is handled separately for $x = 0$ and $x > 0$, due to the division problems at $x = 0$ in programming languages. However, observing the Bessel expansion in Equation (B.1)*

Algorithm 3 The midpoint algorithm approximating $\tau_j(d)$ for $j \in \{0, 1, \ldots \overline{q}_i\}$, and $d = 0, w/2, w, \ldots, \overline{d}$

Set $j = 0$, and $\mathcal{M}_{-1}^{w}(d) = \tau_{-1}(d)$ for $d = 0, w/2, w, \ldots \overline{d}$

while $j \leq \overline{q}_i$ **do**

  $d = w/2$

  while $d \leq \overline{d}$ **do**

    Set $\psi(d, x) = \mathcal{M}_{j-1}^{w}(d - x) g_{\overline{\lambda}_i, c\mu}(x)$ for $x = w/2, 3w/2, \ldots, d - w/2$.

    Compute $\mathcal{M}_j^w(d) = w \sum_{k=0}^{\lceil d/w \rceil - 1} \psi(d, kw + w/2)$

    $d = d + w/2$

  end while

  $j = j + 1$

end while

(Korenev [2002] Equation 4.3), one can clearly see that $g_{\overline{\lambda}_i, c\mu}(0) = c\mu$*, and thus,* $\psi(d, 0) = \mathcal{Z}_{j-1,i}^{w}(d) g_{\overline{\lambda}_i, c\mu}(0)$.

$$I_1(x) = \frac{x}{2}\left[1 + \frac{(x/2)^2}{2(1!)^2} + \frac{(x/2)^4}{3(2!)^2} + \ldots\right]. \tag{B.1}$$

2. Proofs of theorems

Proof of Theorem 4. I use the convolution property to inverse transform $\widetilde{\tau}_{v_0}(s)$ with the following inverse transforms,

$$L_x^{-1}\left\{\frac{\mu}{s^2(s+\mu)}\right\} = x - \frac{1}{\mu} + \frac{e^{-\mu x}}{\mu}, L_x^{-1}\left\{\left(\frac{c\mu}{s + c\mu}\right)^{v_0 - c + 1}\right\} = \frac{(c\mu)^{v_0 - c + 1} x^{v_0 - c}}{(v_0 - c)!} e^{-c\mu x}.$$

Note that $L_x^{-1}\left\{\left(\frac{c\mu}{s+c\mu}\right)^{v_0-c+1}\right\}$ is the pdf of a gamma random variable. Using convolution property, the Laplace transform in Equation (3.7) can be inverse transformed as

$$\tau_{v_0}(d) = \int_0^d \left(d - x - \frac{1}{\mu} + \frac{e^{-\mu(d-x)}}{\mu}\right) \frac{(c\mu)^{v_0-c+1} x^{v_0-c}}{(v_0 - c)!} e^{-c\mu x} dx, \quad d \geq 0.$$

Algorithm 4 The hybrid algorithm approximating $\tau_j(d)$ for $j \in \{0, 1, \ldots \overline{q}_i\}$, and $d = 0, w, \ldots, \overline{d}$

Set $j = 0$, and $\mathcal{H}^w_{-1}(d) = \tau_{-1}(d)$ for $d = 0, w/2, w, \ldots \overline{d}$

while $j \leq \overline{q}_i$ **do**

  if j is even **then**

    $d = w/2$

    while $d \leq \overline{d}$ **do**

      Set $\psi(d, x) = \mathcal{H}^w_{j-1}(d - x) g_{\overline{\lambda}_i, c\mu}(x)$ for $x = w/2, 3w/2, \ldots, d - w/2$.

      Compute $\mathcal{H}^w_j(d) = w \sum_{k=0}^{\lceil d/w \rceil - 1} \psi(d, kw + w/2)$

      $d = d + w/2$

    end while

  else

    $d = w$

    while $d \leq \overline{d}$ **do**

      Set $\psi(d, x) = \mathcal{H}^w_{j-1}(d - x) g_{\overline{\lambda}_i, c\mu}(x)$ for $x = w, 2w, \ldots d$; and $\psi(d, 0) = \mathcal{H}^w_{j-1}(d) c\mu$.

      Compute $\mathcal{H}^w_j(d) = \frac{w}{2} \sum_{k=0}^{d/w-1} \psi(d, kw) + \psi(d, kw + w)$

      $d = d + w$

    end while

  end if

  $j = j + 1$

end while

I next find a closed-form solution for the integral. Let $V(n) = \int_0^d e^{-x\alpha} x^n dx$. Given $\alpha \neq 0$, by partial integration,

$$V(n) = -\frac{d^n e^{-d\alpha}}{\alpha} + \frac{n}{\alpha} V(n-1), \quad n \in \{0, 1, 2, \ldots\},$$

and using the initial condition of

$$V(0) = \frac{1}{\alpha} - \frac{e^{-d\alpha}}{\alpha},$$

$V(n)$'s can be solved recursively. So I get,

$$\frac{V(n)}{n!} = \frac{1}{\alpha^{n+1}} - e^{-d\alpha} \sum_{k=0}^{n} \frac{d^k}{k!\alpha^{n-k+1}}.$$

If $\alpha = 0$ then, the integral can be evaluated as,

$$V(n) = \int_0^d e^{-x\alpha} x^n dx = \frac{d^{n+1}}{n+1}.$$

I can now derive the expression for $\tau_{v_0}(d)$ as

$$\begin{aligned}
\tau_{v_0}(d) &= \int_0^d \left(d - x - \frac{1}{\mu} + \frac{e^{-\mu(d-x)}}{\mu}\right) \frac{(c\mu)^{v_0-c+1} x^{v_0-c}}{(v_0-c)!} e^{-c\mu x} dx \\
&= (c\mu)^{v_0-c+1} \frac{d - \frac{1}{\mu}}{(v_0-c)!} \int_0^d x^{v_0-c} e^{-xc\mu} dx \\
&\quad -(c\mu)^{v_0-c+1} \frac{1}{(v_0-c)!} \int_0^d x^{v_0-c+1} e^{-xc\mu} dx \\
&\quad +(c\mu)^{v_0-c+1} \frac{e^{-d\mu}}{\mu(v_0-c)!} \int_0^d x^{v_0-c} e^{-x(c-1)\mu} dx.
\end{aligned}$$

The cases with $c = 1$ and $c \geq 2$ need to be handled separately.

Case I. $c = 1$

$$\begin{aligned}
\tau_{v_0}(d) &= \mu^{v_0} \left(d - \frac{1}{\mu}\right) \left(\frac{1}{\mu^{v_0}} - e^{-d\mu} \sum_{k=0}^{v_0-1} \frac{d^k}{k!\mu^{v_0-k}}\right) \\
&\quad -\mu^{v_0} v_0 \left(\frac{1}{\mu^{v_0+1}} - e^{-d\mu} \sum_{k=0}^{v_0} \frac{d^k}{k!\mu^{v_0-k+1}}\right) \\
&\quad +\mu^{v_0} \frac{e^{-d\mu}}{\mu(v_0-1)!} \frac{d^{v_0}}{v_0} \\
&= d - \frac{v_0+1}{\mu} + e^{-d\mu} \left(\frac{1}{\mu} - d\right) \sum_{k=0}^{v_0-1} \frac{(d\mu)^k}{k!} + \frac{e^{-d\mu} v_0}{\mu} \sum_{k=0}^{v_0} \frac{(d\mu)^k}{k!} + \frac{e^{-d\mu}(d\mu)^{v_0}}{\mu(v_0)!} \\
&= d - \frac{v_0+1}{\mu} + (v_0+1) \frac{e^{-d\mu}}{\mu} \sum_{k=0}^{v_0} \frac{(d\mu)^k}{k!} - \frac{e^{-d\mu}}{\mu} \sum_{k=0}^{v_0-1} \frac{(d\mu)^{k+1}}{k!} \\
&= d - \frac{v_0+1}{\mu} + \frac{e^{-d\mu}}{\mu} \left(v_0 + 1 + \sum_{k=1}^{v_0} (d\mu)^k \left(\frac{v_0+1}{k!} - \frac{1}{(k-1)!}\right)\right) \\
&= d - \frac{v_0+1}{\mu} + \frac{e^{-d\mu}}{\mu} \sum_{k=0}^{v_0} (d\mu)^k \frac{v_0+1-k}{k!}.
\end{aligned}$$

Case II. $c \geq 2$

$$\begin{aligned}
\tau_{v_0}(d) &= (c\mu)^{v_0-c+1}\left(d-\frac{1}{\mu}\right)\left(\frac{1}{(c\mu)^{v_0-c+1}} - e^{-dc\mu}\sum_{k=0}^{v_0-c}\frac{d^k}{k!(c\mu)^{v_0-c-k+1}}\right)\\
&\quad - (v_0-c+1)(c\mu)^{v_0-c+1}\left(\frac{1}{(c\mu)^{v_0-c+2}} - e^{-dc\mu}\sum_{k=0}^{v_0-c+1}\frac{d^k}{k!(c\mu)^{v_0-c-k+2}}\right)\\
&\quad + (c\mu)^{v_0-c+1}\frac{e^{-d\mu}}{\mu}\left(\frac{1}{((c-1)\mu)^{v_0-c+1}} - e^{-d(c-1)\mu}\sum_{k=0}^{v_0-c}\frac{d^k}{k!((c-1)\mu)^{v_0-c-k+1}}\right)\\
&= d - \frac{1}{\mu} - \frac{v_0-c+1}{c\mu} + \frac{e^{-d\mu}}{\mu}\left(\frac{c}{c-1}\right)^{v_0-c+1} + \left(\frac{1}{\mu}-d\right)e^{-dc\mu}\sum_{k=0}^{v_0-c}\frac{(dc\mu)^k}{k!}\\
&\quad + \frac{(v_0-c+1)e^{-dc\mu}}{c\mu}\sum_{k=0}^{v_0-c+1}\frac{(dc\mu)^k}{k!} - \frac{e^{-dc\mu}}{\mu}c^{v_0-c+1}\sum_{k=0}^{v_0-c}\frac{(dc\mu)^k}{k!}\frac{1}{(c-1)^{v_0-c-k+1}}\\
&= d - \frac{1}{\mu} - \frac{v_0-c+1}{c\mu} + \frac{e^{-d\mu}}{\mu}\left(\frac{c}{c-1}\right)^{v_0-c+1} + \frac{(v_0-c+1)e^{-dc\mu}}{c\mu(v_0-c+1)!}(dc\mu)^{v_0-c+1}\\
&\quad + \left(\frac{1}{\mu} - d + \frac{v_0-c+1}{c\mu}\right)e^{-dc\mu}\sum_{k=0}^{v_0-c}\frac{(dc\mu)^k}{k!} - \frac{e^{-dc\mu}}{\mu}\left(\frac{c}{c-1}\right)^{v_0-c+1}\sum_{k=0}^{v_0-c}\frac{(d(c-1)\mu)^k}{k!}\\
&= d - \frac{1}{\mu} - \frac{v_0-c+1}{c\mu} + \frac{e^{-d\mu}}{\mu}\left(\frac{c}{c-1}\right)^{v_0-c+1} + \frac{e^{-dc\mu}(dc\mu)^{v_0-c+1}}{c\mu(v_0-c)!}\\
&\quad + e^{-dc\mu}\sum_{k=0}^{v_0-c}\frac{(d\mu)^k}{k!}\left[c^k\left(\frac{1}{\mu} - d + \frac{v_0-c+1}{c\mu}\right) - \frac{(c-1)^k}{\mu}\left(\frac{c}{c-1}\right)^{v_0-c+1}\right].
\end{aligned}$$

The expected TIS, denoted by S_{v_0}, is simply the summation of expectations of $v_0 - c + 1$ exponential random variables with rate $c\mu$ plus the expected service time:

$$S_{v_0} = \frac{v_0-c+1}{c\mu} + \frac{1}{\mu}.$$

Thus, using Equation (3.1) and S_{v_0}, $\tau_{v_0}(d)$ formulas, $T_{v_0}(d)$ is obtained. □

Proof of Theorem 5. The PT of the marked order is distributed exponentially with pdf $h_\mu(\cdot)$. Thus, I focus on deriving expressions for the WTQ of the marked order using first passage times. Let $T^{\lambda,\mu}_{m,n}$ denote the first passage time from state m to n in a birth-death process with the constant arrival and service rates of λ and μ respectively for all states. The WTQ of a marked order that arrives when there are $\overline{q}_i$ orders in the system ahead of it in the queue can be expressed by the random variable, $T^{\overline{\lambda}_i,c\mu}_{\overline{q}_i+1,0}$. Recall that in a birth-death process the length of the queue ahead of the marked order increases with the arrival rate $\overline{\lambda}_i$, and decreases with the service rate $c\mu$ for all states. Let $g^{\lambda,\mu}_{m,n}(\cdot)$ denote the pdf of the random variable $T^{\lambda,\mu}_{m,n}$. The distribution of TIS of a class i order can be calculated through the convolution of $g^{\overline{\lambda}_i,c\mu}_{\overline{q}_i+1,0}(\cdot)$ and $h_\mu(\cdot)$. Using the convolution property, the Laplace transform of the TIS distribution is

$$\widetilde{f}_{\overline{q}_i,i}(s) = \widetilde{h}_\mu(s)\widetilde{g}^{\overline{\lambda}_i,c\mu}_{\overline{q}_i+1,0}(s), \quad s \in \mathbb{C}, \tag{B.2}$$

for all $i \in \{2, \dots, N\}$. Using the analysis of Heyman and Sobel [2004, pg. 87-89], I have,

$$T^{\lambda,\mu}_{j,0} = \sum_{k=1}^{j} T^{\lambda,\mu}_{k,k-1} \ ,$$

and $T^{\overline{\lambda}_i,c\mu}_{j,j-1}$'s are independent and identically distributed, i.e., $\widetilde{g}^{\overline{\lambda}_i,c\mu}_{j,j-1}(s)$ are the same for $j \in \{1, 2, \dots\}$; for simplicity, the subscripts are dropped and I use $\widetilde{g}_{\overline{\lambda}_i,c\mu}(s)$ to denote $\widetilde{g}^{\overline{\lambda}_i,c\mu}_{j,j-1}(s)$. Using the rightward convolution property I obtain

$$\widetilde{g}^{\overline{\lambda}_i,c\mu}_{\overline{q}_i+1,0}(s) = \left(\widetilde{g}_{\overline{\lambda}_i,c\mu}(s)\right)^{\overline{q}_i+1}, \quad s \in \mathbb{C}, \tag{B.3}$$

for all $i \in \{2, \dots, N\}$. $\widetilde{g}_{\overline{\lambda}_i,c\mu}(s)$ was first derived by Davis [1966] as

$$\widetilde{g}_{\overline{\lambda}_i,c\mu}(s) = \frac{s + \overline{\lambda}_i + c\mu - \sqrt{(s + \overline{\lambda}_i + c\mu)^2 - 4\overline{\lambda}_i c\mu}}{2\overline{\lambda}_i}, \quad s \in \mathbb{C}. \tag{B.4}$$

$\widetilde{\tau}_{\overline{q}_i,i}(s)$ is obtained by applying Proposition (1) and Equations (3.6), (B.2), and (B.3). □

Proof of Theorem 6. I need Lemmas 4 and 5 to prove this theorem.

Lemma 4. *$\tau_j(d-x)$ is nonnegative, nondecreasing and convex in x for $j = 0, 1, \dots,$ $0 \le x \le d$.*

Proof. From Equation (3.2), I have, $\tau_j(d-x) = \int_0^{d-x}(d-x-y)f_j(y)dy$ for $x \leq d$. $\tau_j(d-x)$ is clearly nonnegative since $f_j(\cdot) \geq 0$. By differentiating both sides twice using the Leibniz integral rule I get,

$$-\tau_j'(d-x) = -\int_0^{d-x} f_j(y)dy = -F_j(d-x), \quad x \leq d,$$

$$\tau_j''(d-x) = f_j(d-x), \quad x \leq d,$$

where $F_j(\cdot)$ is the corresponding cumulative distribution function. Since $0 \leq F_j(d-x) \leq 1$, and $f_j(d-x) \geq 0$ for $x \leq d$ the result is obvious. □

Lemma 5. *If* $\overline{\lambda_i} + c\mu \geq 2\frac{\overline{\lambda_i}c\mu + \sqrt{\overline{\lambda_i}c\mu}}{2\sqrt{\overline{\lambda_i}c\mu}-1}$ *and* $\overline{\lambda_i}c\mu \geq \frac{1}{4}$ *then* $g_{\overline{\lambda}_i,c\mu}(x)$ *is nonnegative, nonincreasing and convex in* $x \geq 0$.

Proof. For simplicity let $p = 2\sqrt{\overline{\lambda_i}c\mu}$ and $k = \overline{\lambda_i} + c\mu$. By simple algebra I get, $p \leq k$, and assuming, $\overline{\lambda_i} + c\mu \geq 2\frac{\overline{\lambda_i}c\mu + \sqrt{\overline{\lambda_i}c\mu}}{2\sqrt{\overline{\lambda_i}c\mu}-1}$ and $\overline{\lambda_i}c\mu \geq \frac{1}{4}$, I reach the following inequalities

$$k \geq p \geq 1, \text{ and } \quad 2pk \geq p^2 + 2k + 2p$$

Rewriting Equation (3.14) in terms of p and k, I get

$$g_{\overline{\lambda}_i,c\mu}(x) = \frac{pe^{-kx}}{2\overline{\lambda}x} I_1(px).$$

Since $I_1(px) \geq 0$, $g_{\overline{\lambda}_i,c\mu}(x)$ is nonnegative. Differentiating twice by using the well known differentiation rule, $I_n'(x) = I_{n-1}(x) - n/xI_n(x)$, I reach,

$$g'_{\overline{\lambda}_i,c\mu}(x) = \frac{pe^{-kx}}{2\overline{\lambda}_i x^2}\left[pxI_0(px) - (kx+p+1)I_1(px)\right],$$

$$\begin{aligned} g''_{\overline{\lambda}_i,c\mu}(x) =& \frac{pe^{-kx}}{2\overline{\lambda}_i x^3}\Big[I_1(px)((k^2+p^2)x^2 + (2pk+2k)x + 3p + 3) \\ & - I_0(px)(2kpx^2 + (p^2+2p)x)\Big]. \end{aligned} \tag{B.5}$$

I first show that $g'_{\overline{\lambda}_i,c\mu}(x) \leq 0$. The inequality (B.6) gives a sufficient condition for $g'_{\overline{\lambda}_i,c\mu}(x) \leq 0$.

$$pxI_0(px) \leq (kx + p + 1)I_1(px), \quad x \geq 0. \tag{B.6}$$

Using 2.21 in Ifantis and Siafarikas [1990], $pxI_0(px)$ can be bounded above as:

$$pxI_0(px) \leq (2 + px)I_1(px). \tag{B.7}$$

Since $I_1(px) \geq 0$ and $k \geq p \geq 1$ for $x \geq 0$, I easily get

$$(2 + px)I_1(px) \leq (kx + p + 1)I_1(px). \tag{B.8}$$

Hence by combining inequalities (B.7) and (B.8), I get inequality (B.6), and complete the first part of the proof. For $g''_{\overline{\lambda}_i,c\mu}(x) \geq 0$, I need the following inequality.

$$I_1(px)((k^2+p^2)x^2+(2pk+2k)x+3p+3) \geq I_0(px)(2kpx^2+(p^2+2p)x), \quad x \geq 0. \tag{B.9}$$

By following the same approach in (B.7) I reach

$$\begin{aligned}(2kpx^2 + (p^2 + 2p)x)I_0(px) \leq &\frac{(2 + px)(2kpx^2 + (p^2 + 2p)x)}{px} I_1(px) \\ &= (2kpx^2 + (p^2 + 2p + 4k)x + 2p + 4)I_1(px).\end{aligned} \tag{B.10}$$

I have $k^2 + p^2 \geq 2kp$, $3p + 3 \geq 2p + 4$ from $p > 1$, and $2pk + 2k \geq p^2 + 2p + 4k$ from $2pk \geq p^2 + 2k + 2p$. Thus, the following holds.

$$(2kpx^2 + (p^2 + 2p + 4k)x + 2p + 4)I_1(px) \leq ((k^2 + p^2)x^2 + (2pk + 2k)x + 3p + 3)I_1(px). \tag{B.11}$$

Combining inequalities (B.10) and (B.11), I reach inequality (B.9), which completes the second part of the proof. □

For simplicity let $\gamma_j(d,x) = \tau_{j-1}(d-x)g_{\overline{\lambda}_i,c\mu}(x)$. First note that $\gamma_j(d,x)$ is continuous and obtains continuous first and second derivatives for $0 \le x \le d$. Moreover, given $\overline{\lambda}_i c\mu \ge 4$, it can be trivially shown to be convex in $0 \le x \le d$ by taking the second derivative and combining the results of Lemmas 4 and 5.

By induction: $\gamma_0(d,x)$ possesses continuous second order derivatives and is convex in $0 \le x \le d$. Since I have $\tau_0(d) = \int_0^d \gamma_0(d,x)$, using the bracketing property given in the Corollary of Davis and Rabinowitz [1984, pg. 52], the trapezoidal and midpoint rules respectively over and under estimates $\tau_0(d)$, and I get

$$\mathcal{M}_0^w(d) \le \tau_0(d) \le \mathcal{Z}_0^w(d).$$

Now, assume that for some $j = 1, 2, \ldots$, $\mathcal{M}_{j-1}^w(d) \le \tau_{j-1}(d) \le \mathcal{Z}_{j-1}^w(d)$ holds. Since $\tau_{j-1}(d-x)g_{\overline{\lambda}_i,c\mu}(x)$ possesses continuous second order derivatives and is convex in $0 \le x \le d$ as well, the bracketing property holds. Thus, I get

$$\begin{aligned} w \sum_{k=0}^{\lceil d/w \rceil - 1} & \tau_{j-1}(d - kw - w/2) g_{\overline{\lambda}_i,c\mu}(kw + w/2) \\ & \le \tau_j(d) \\ & \le \frac{w}{2} \sum_{k=0}^{d/w-1} \tau_{j-1}(d-kw) g_{\overline{\lambda}_i,c\mu}(kw) + \tau_{j-1}(d-kw-w) g_{\overline{\lambda}_i,c\mu}(kw + w) \end{aligned} \tag{B.12}$$

Since $g_{\overline{\lambda}_i,c\mu}(x) \ge 0$ for $0 \le x \le d$, inequalities (B.13) and (B.14) hold.

$$\begin{aligned} \mathcal{M}_j(d) &= w \sum_{k=0}^{\lceil d/w \rceil - 1} \mathcal{M}_{j-1}(d - kw - w/2) g_{\overline{\lambda}_i,c\mu}(kw + w/2) \\ &\le w \sum_{k=0}^{\lceil d/w \rceil - 1} \tau_{j-1}^w(d - kw - w/2) g_{\overline{\lambda}_i,c\mu}(kw + w/2) \end{aligned} \tag{B.13}$$

$$\frac{w}{2}\sum_{k=0}^{d/w-1}\tau_{j-1}(d-kw)g_{\overline{\lambda}_i,c\mu}(kw)+\tau_{j-1}(d-kw-w)g_{\overline{\lambda}_i,c\mu}(kw+2)$$
$$\leq \frac{w}{2}\sum_{k=0}^{d/w-1}\mathcal{Z}_{j-1}(d-kw)g_{\overline{\lambda}_i,c\mu}(kw)+\mathcal{Z}_{j-1}(d-kw-w)g_{\overline{\lambda}_i,c\mu}(kw+2)=\mathcal{Z}_j(d). \tag{B.14}$$

Combining the inequalities (B.12), (B.13), and (B.14) the proof is completed. □

Proof of Theorem 7. $\widetilde{f}_{\overline{v}_i}(s)$ for $\overline{v}_i \leq c-1$ derivation is done by Laplace-Stieltjes Transforms (LST) of the cumulative distribution functions, $F_{\overline{v}_i}(\cdot)$. The LST of $F_{\overline{v}_i}(\cdot)$ is equivalent to the sought Laplace Transform, $f_{\overline{v}_i}(\cdot)$ by the following LST definition [see pg. 523 of Heyman and Sobel, 2004].

$$\int_0^\infty e^{-sx}dF_{\overline{v}_i}(x)dx=\int_0^\infty e^{-sx}f_{\overline{v}_i}(x)dx=\widetilde{f}_{\overline{v}_i}(s), \quad s\in\mathbb{C}.$$

My purpose is to obtain recursive equations of the Laplace transforms $\widetilde{f}_{\overline{v}_i}(s)$ by taking the LST of the $F_{\overline{v}_i}(\cdot)$ and using a conditioning argument, similar to the approach presented in Heyman and Sobel [2004, pg. 88]. Consider the marked order observing $\overline{v}_i$ orders in front upon arrival, where $\overline{v}_i \leq c-1$. One can rewrite $F_{\overline{v}_i}(\cdot)$ by conditioning on the next event, considering three possibilities:

1. Arrival of a higher priority order with rate $\overline{\lambda}_i$, pushing the marked order one position back,

2. Service completion of an order in a lower position with rate $\overline{v}_i\mu$, pulling the marked order one position to the front, and

3. Service completion of the marked order with rate μ.

Hence, by conditioning on the time and type of the next event I reach,

$$\begin{aligned}F_{\overline{v}_i}(d) =&\frac{\overline{\lambda}_i}{\overline{\lambda}_i+(\overline{v}_i+1)\mu}\int_0^d F_{\overline{v}_i+1}(d-x)(\overline{\lambda}_i+(\overline{v}_i+1)\mu)e^{-(\overline{\lambda}_i+(\overline{v}_i+1)\mu)x}dx\\&+\frac{\overline{v}_i\mu}{\overline{\lambda}_i+(\overline{v}_i+1)\mu}\int_0^d F_{\overline{v}_i-1}(d-x)(\overline{\lambda}_i+(\overline{v}_i+1)\mu)e^{-(\overline{\lambda}_i+(\overline{v}_i+1)\mu)x}dx\\&+\frac{\mu}{\overline{\lambda}_i+(\overline{v}_i+1)\mu}\left(1-e^{-(\overline{\lambda}_i+(\overline{v}_i+1)\mu)d}\right),\quad d\geq 0.\end{aligned}$$

Conditions 1, 2, and 3 are reflected in the first, second and third term of the right hand side respectively. Note that, the integrals in the first and second terms are convolutions of the exponential distribution, and the conditional cumulative TIS distributions. Taking the LST's of both sides and using the convolution property, I get

$$\begin{aligned}\widetilde{f}_{\overline{v}_i}(s) =&\frac{\overline{\lambda}_i}{\overline{\lambda}_i+(\overline{v}_i+1)\mu}\widetilde{f}_{\overline{v}_i+1}(s)\frac{\overline{\lambda}_i+(\overline{v}_i+1)\mu}{s+\overline{\lambda}_i+(\overline{v}_i+1)\mu}\\&+\frac{\overline{v}_i\mu}{\overline{\lambda}_i+(\overline{v}_i+1)\mu}\widetilde{f}_{\overline{v}_i-1}(s)\frac{\overline{\lambda}_i+(\overline{v}_i+1)\mu}{s+\overline{\lambda}_i+(\overline{v}_i+1)\mu}\\&+\frac{\mu}{\overline{\lambda}_i+(\overline{v}_i+1)\mu}\frac{\overline{\lambda}_i+(\overline{v}_i+1)\mu}{s+\overline{\lambda}_i+(\overline{v}_i+1)\mu},\quad s\in\mathbb{C}.\end{aligned}$$

By arranging the terms I reach the recursive form. □

3. Laplace Transform Derivations for Multi-Server Cases

Derivation of $\widetilde{f}_{\overline{v}_i}(s)$ for $c \geq 2$ is more complicated, also burdening the evaluation of expected TIS. However, after solving $\widetilde{f}_{\overline{v}_i}(s)$ for $\overline{v}_i \leq c-1$, one easily reaches the useful product form of $\widetilde{f}_{\overline{v}_i}(s)$ for $\overline{v}_i > c-1$ through Equation (3.19). Thus, $\mathcal{Z}$, $\mathcal{M}$, and $\mathcal{H}$ can also be applied for the preemptive case for $\overline{v}_i > c-1$ given $\widetilde{f}_{c-1}(s)$ and $f_{c-1}(\cdot)$. I next derive $\widetilde{f}_0(s)$ and $\widetilde{f}_1(s)$ for $c = 2$ in Equation (B.15).

$$\widetilde{f}_0(s) = \frac{\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,2\mu}(s) + \mu}{s + \overline{\lambda}_i + \mu - \overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,2\mu}(s)}, \quad \widetilde{f}_1(s) = \widetilde{g}_{\overline{\lambda}_i,2\mu}(s) \frac{s + \overline{\lambda}_i + 2\mu}{s + \overline{\lambda}_i + \mu - \overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,2\mu}(s)}, \quad s \in \mathbb{C}\,. \tag{B.15}$$

Following the same approach, I derive $\widetilde{f}_0(s)$, $\widetilde{f}_1(s)$ and $\widetilde{f}_2(s)$ for $c = 3$.

$$\widetilde{f}_0(s) = \frac{\overline{\lambda}_i(\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,3\mu}(s) + \mu) + \mu(s + \overline{\lambda}_i + 2\mu - 2\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,3\mu}(s))}{(s + \overline{\lambda}_i + \mu)(s + \overline{\lambda}_i + 2\mu - 2\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,3\mu}(s)) - \overline{\lambda}_i \mu}, \quad s \in \mathbb{C}\,.$$

$$\widetilde{f}_1(s) = \frac{(s + \overline{\lambda}_i + \mu)(\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,3\mu}(s) + \mu) + \mu^2}{(s + \overline{\lambda}_i + \mu)(s + \overline{\lambda}_i + 2\mu - 2\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,3\mu}(s)) - \overline{\lambda}_i \mu}, \quad s \in \mathbb{C}\,.$$

$$\widetilde{f}_2(s) = \widetilde{g}_{\overline{\lambda}_i,2\mu}(s) \frac{(s + \overline{\lambda}_i + \mu)(s + \overline{\lambda}_i + 4\mu) + 2\mu^2 - \overline{\lambda}_i \mu}{(s + \overline{\lambda}_i + \mu)(s + \overline{\lambda}_i + 2\mu - 2\overline{\lambda}_i \widetilde{g}_{\overline{\lambda}_i,3\mu}(s)) - \overline{\lambda}_i \mu}, \quad s \in \mathbb{C}\,.$$

APPENDIX C

PROOFS OF CHAPTER 4

In this Appendix, I provide proofs for Theorems 8 and 9. I first define simple properties facilitating my proofs.

Remark 4. *Let $f(x)$ be a concave function in $x \in \Re$ with the global maximum of x^*, i.e., $x^* = \operatorname{argmax}_{x \in \Re} f(x)$. Let a and b two real numbers with $a < b$. I have the following three properties from the concavity of $f(x)$.*

(i) If $x^* \leq a$ then $\operatorname{argmax}_{a \leq x \leq b} f(x) = a$ (R1.1).

(ii) If $a < x^* \leq b$ then $\operatorname{argmax}_{a \leq x \leq b} f(x) = x^*$ (R1.2).

(iii) If $b < x^*$ then $\operatorname{argmax}_{a \leq x \leq b} f(x) = b$ (R1.3).

Remark 5. *Let $f_1(x)$ and $f_2(x)$ be concave functions in $x \in \Re$, with the global maximums of x_1^* and x_2^*, respectively. Let $g(x)$ be the piecewise function defined as follows.*

$$g(x) = \begin{cases} f_1(x) & \text{for } a \leq x < b \\ f_2(x) & \text{for } b \leq x < c \end{cases} \tag{C.1}$$

Using Remark 1, I obtain the following results.

(i) If $x_1^* \leq a$ and $x_2^* \leq b$ then $\operatorname{argmax}_{a \leq x \leq c} g(x) = a$ (R2.1).

(ii) If $a < x_1^* \leq b$ and $x_2^* \leq b$ then $\operatorname{argmax}_{a \leq x \leq c} g(x) = x_1^*$ (R2.2).

(iii) If $b < x_1^*$ and $x_2^* \leq b$ then $\operatorname{argmax}_{a \leq x \leq c} g(x) = b$ (R2.3).

(iv) If $b < x_1^*$ and $b < x_2^* \leq c$ then $\operatorname{argmax}_{a \leq x \leq c} g(x) = x_2^*$ (R2.4).

(v) If $b < x_1^*$ and $c < x_2^*$ then $\operatorname{argmax}_{a \leq x \leq c} g(x) = c$

Note that, concavity of the piecewise elements is the sufficient condition for Remark 1 and 2. In the remainder, at any point Remark 1 and 2 are implemented, I already know that this sufficient condition is satisfied. Hence I do not provide any discussion about concavity of the functions.

Proof of Theorem 8. In this proof, I follow the following roadmap. I first derive the optimal price, $p^{C*}_{t,i}(l)$, given a fixed lead time l, and the optimal lead time $l^{C*}_{t,i}(p)$, given a fixed price that are defined in Equation (C.2), and then use these two curves to derive $(p^{C*}_{t,i}, l^{C*}_{t,i})$. $p^{C*}_{t,i}(l)$ and $l^{C*}_{t,i}(p)$ are defined as follows.

$$\begin{aligned} p^{C*}_{t,i}(l) &= \operatorname*{argmax}_p \Pi^C_{t,i}(p,l), \quad d \in [0, l_{max}], \text{ and} \\ l^{C*}_{t,i}(p) &= \operatorname*{argmax}_l \Pi^C_{t,i}(p,l), \quad p \in [p_{min}, p_{max}] \end{aligned} \tag{C.2}$$

Derivation of $p^{C*}_{t,i}(l)$ ***and*** $l^{C*}_{t,i}(p)$

First, note that $\Pi^C_{t,i}(p,l)$ is concave in p and l. Hence, I can find $p^{C*}_{t,i}(l)$ and $l^{C*}_{t,i}(p)$ using first order conditions. Taking derivatives of $\Pi^C_{t,i}(p,l)$ in p and l respectively gives,

$$\frac{\partial \Pi^C_{t,i}(p,l)}{\partial p} = \begin{cases} \beta \left[\frac{-1}{p_{max}-p_{min}} \left(p - \tau\alpha^{i+\nu}(i+\nu-l) - \Delta\Pi^C_{t,i} \right) + f(p,l) \right] & \text{if } l \le i+\nu \\ \beta \left[\frac{-1}{p_{max}-p_{min}} \left(p - \Delta\Pi^C_{t,i} \right) + f(p,l) \right] & \text{if } l > i+\nu. \end{cases} \tag{C.3}$$

$$\frac{\partial \Pi^C_{t,i}(p,l)}{\partial l} = \begin{cases} \beta \left[\frac{-1}{l_{max}} \left(p - \tau\alpha^{i+\nu}(i+\nu-l) - \Delta\Pi^C_{t,i} \right) + \tau\alpha^{i+\nu} f(p,l) \right] & \text{if } l \le i+\nu \\ \beta \left[\frac{-1}{l_{max}} \left(p - \Delta\Pi^C_{t,i} \right) \right] & \text{if } l > i+\nu. \end{cases} \tag{C.4}$$

I obtain the first order condition in p for $(p, l) \in \theta$, using Equation (C.3) in Equation (C.5).

$$p(l) = \frac{1}{2}\left(\Delta\Pi_{t,i}^{C} + p_{max} - \rho l + \tau\alpha^{i+\nu}(i+\nu-l)^{+}\right). \tag{C.5}$$

Note that, $p_{t,i}^{C*}(l) = p(l)$ only for $(p, l) \in \theta$, i.e., $p_{min} < p(l) \leq p_{max}(l)$ (R1.2). If $p(l) \leq p_{min}$, then I get $p_{t,i}^{C*}(l) = p_{min}$ (R1.1). If $p(l) > p_{max}(l)$, on the other hand, I obtain, $p_{t,i}^{C*}(l) = p_{max}(l)$ (R1.3). Consequently, I get,

$$p_{t,i}^{C*}(l) = \begin{cases} p_{min} & \text{if } p(l) \leq p_{min}, \\ p(l) & \text{if } p_{min} < p(l) \leq p_{max}(l), \\ p_{max}(l) & \text{if } p_{max}(l) < p(l). \end{cases} \tag{C.6}$$

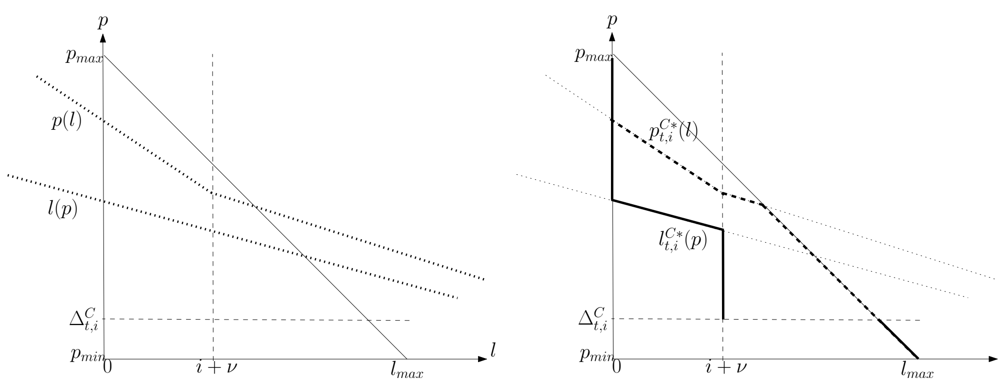
I next obtain the first order condition in l for $l \leq i + \nu$ and $(p, l) \in \theta$, using Equation (C.4) in Equation (C.7).

$$l(p) = \frac{1}{2}\left(\frac{\Delta\Pi_{t,i}^{C} - p}{\tau\alpha^{i+\nu}} + i + \nu + l_{max} - \frac{p - p_{min}}{\rho}\right). \tag{C.7}$$

Note that, $l_{t,i}^{C*}(p) = l(p)$, only when $0 \leq l(p) < \min\{i+\nu, l_{max}(p)\}$ (R1.2). Similar to above case, If $l(p) < 0$, then $l_{t,i}^{C*}(p) = 0$ (R1.1). If $l(p) > i+\nu$, where $i+\nu \leq l_{max}(p)$, then I know that $l_{t,i}^{C*}(p) \geq i + \nu$ (R1.3). In this case there are two possibilities: (i) if $p \geq \Delta\Pi_{t,i}^{C}$, then $\frac{\partial\Pi_{t,i}^{C}(p,l)}{\partial l} \leq 0$ for $l > i + \nu$ from Equation (C.4), and hence $l_{t,i}^{C*}(p) = i + \nu$, (ii) If $p \leq \Delta\Pi_{t,i}^{C}$, then $\frac{\partial\Pi_{t,i}^{C}(p,l)}{\partial l} > 0$, which means $\Pi_{t,i}^{C}(p, l)$ is increasing as l increases for $l > i + \nu$. Thus, I get $l_{t,i}^{C*}(p) = l_{max}(p)$. In summary, when $i + \nu \leq l_{max}$ I obtain,

$$l_{t,i}^{C*}(p) = \begin{cases} 0 & \text{if } l(p) \leq 0, \\ l(p) & \text{if } 0 \leq l(p) \leq \min\{i+\nu, l_{max}(p)\}, \\ i+\nu & \text{if } i+\nu \leq l_{max}(p), i+\nu \leq l(p), \text{and } p \geq \Delta\Pi_{t,i}^{C} \\ l_{max}(p) & \text{otherwise .} \end{cases} \tag{C.8}$$

An illustration of $l_{t,i}^{C*}(p)$ and $p_{t,i}^{C*}(l)$ are given in Figure 19(b), for the corresponding $l(p)$, $p(l)$ given in Figure 19(a).



(a) Sample $l(p)$, $p(l)$ lines are denoted as dotted lines

(b) $l_{t,i}^{C*}(p)$ and $p_{t,i}^{C*}(l)$ curves are plotted for the corresponding $l(p)$, $p(l)$ lines. $p_{t,i}^{C*}(l)$ is denoted as a dashed line, and $l_{t,i}^{C*}(p)$ is plotted using a solid line

Fig. 19. Illustration of $l^*(p)$ and $p^*(l)$

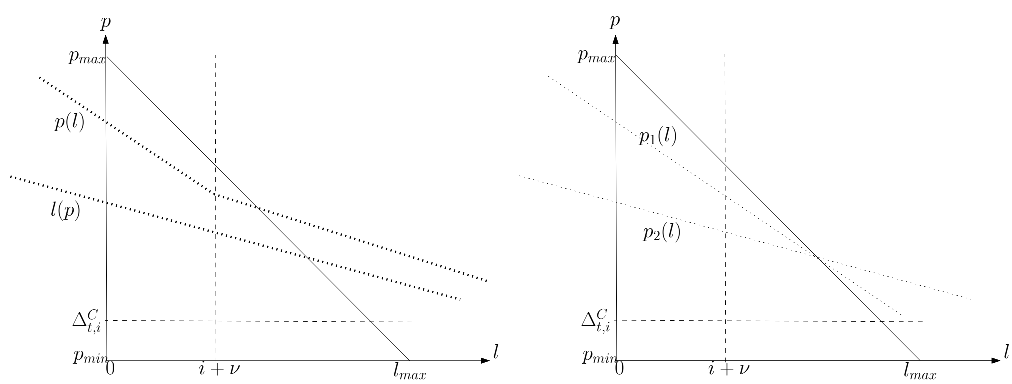
Derivation of $(p_{t,i}^{C*}, l_{t,i}^{C*})$

Note that $(p_{t,i}^{C*}, l_{t,i}^{C*})$ can be found on the intersections of $p_{t,i}^{C*}(l)$ and $l_{t,i}^{C*}(p)$. I next derive $(p_{t,i}^{C*}, l_{t,i}^{C*})$ considering the possible intersections. I first define $p_1(l)$, which is the $d \leq i + \nu$ part of $p(l)$ line curve, and $p_2(l)$, which is the inverse of $l_2(p)$, i.e., $p_2(l) = p$, where $l_2(p) = d$.

$$p_1(l) = \frac{1}{2}\left(\Delta\Pi^C_{t,i} + p_{max} - \rho l + \tau\alpha^{i+\nu}(i+\nu-l)\right) \tag{C.9}$$

$$p_2(l) = \frac{\rho\Delta\Pi^C_{t,i} + \rho\tau\alpha^{i+\nu}(i+\nu+l_{max}-2d) + \tau\alpha^{i+\nu}p_{min}}{\rho + \tau\alpha^{i+\nu}} \tag{C.10}$$

In Figure 20(b), I plot $p_1(l)$ and $p_2(l)$, for the same $l(p)$ and $p(l)$ curves given in Figure 19(a) that are also plotted in Figure 20(a).



(a) Sample $l(p)$, $p(l)$ lines are denoted as dotted lines

(b) $p_1(l)$ and $p_2(l)$ lines are plotted for the corresponding $l(p)$, $p(l)$ lines. Note that $p_2(l)$ corresponds to the same line with $l(p)$, whereas $p_1(l)$ is different than $p(l)$ for $l > i + \nu$

Fig. 20. Illustration of $p_1(l)$ and $p_2(l)$

I note three properties that are shown on Figure 20(b): (i) $p_1(l)$, $p_2(l)$ and $p_{max}(l)$ intersect at the same point, (ii) $p_1(l)$ line slopes down faster than $p_2(l)$, and (iii) $p_{max}(l)$ slopes down faster than both $p_1(l)$ and $p_2(l)$. I show that properties (i) and (ii) always hold, whereas property (iii) holds under a condition:

(i) Solving $p_1(l^*) = p_2(l^*) = p_{max}(l^*)$, I get,

$$l^* = \frac{p_{max} - \Delta\Pi^C_{t,i} - \tau\alpha^{i+\nu}(i+\nu)}{\rho - \tau\alpha^{i+\nu}}, \tag{C.11}$$

which gives the point $p_1(l)$, $p_2(l)$ and $p_{max}(l)$ intersect.

(ii) The slopes of $p_1(l)$ and $p_2(l)$ are respectively given as $-\frac{\rho+\tau\alpha^{i+\nu}}{2}$ and $-\frac{2\rho\tau\alpha^{i+\nu}}{\rho+\tau\alpha^{i+\nu}}$. Using simple algebra one can clearly obtain the following inequality,

$$\frac{\rho + \tau\alpha^{i+\nu}}{2} \geq \frac{2\rho\tau\alpha^{i+\nu}}{\rho + \tau\alpha^{i+\nu}}. \tag{C.12}$$

and, I get $-\frac{\rho+\tau\alpha^{i+\nu}}{2} \leq -\frac{2\rho\tau\alpha^{i+\nu}}{\rho+\tau\alpha^{i+\nu}}$, which gives the desired property.

(iii) The slope of $p_{max}(l)$ is $-\rho$. First assume that $\rho < \tau\alpha^{i+\nu}$. Then, I simply obtain

$$\rho < \frac{2\rho\tau\alpha^{i+\nu}}{\rho + \tau\alpha^{i+\nu}} < \frac{\rho + \tau\alpha^{i+\nu}}{2}. \tag{C.13}$$

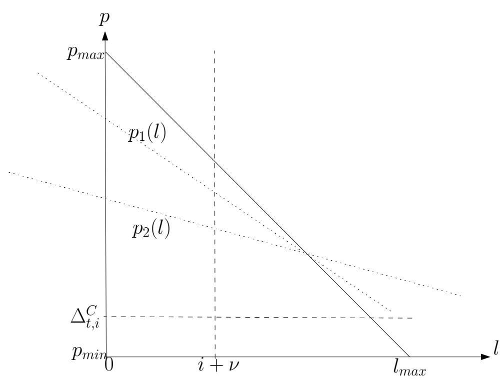
Now assume that $\rho > \tau\alpha^{i+\nu}$. Now I get,

$$\frac{2\rho\tau\alpha^{i+\nu}}{\rho + \tau\alpha^{i+\nu}} < \frac{\rho + \tau\alpha^{i+\nu}}{2} < \rho. \tag{C.14}$$

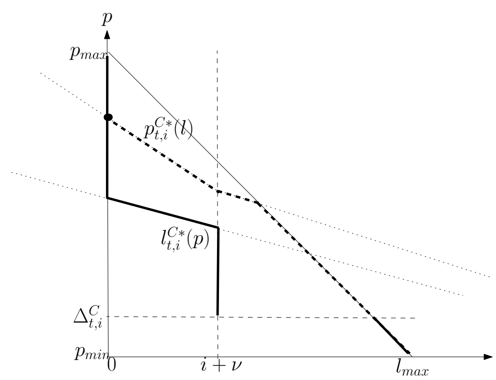
In summary, when $\rho < \tau\alpha^{i+\nu}$ then both $p_1(l)$ and $p_2(l)$ slope down faster than $p_{max}(l)$, whereas the opposite holds when $\rho > \tau\alpha^{i+\nu}$, which is the case depicted in Figure 20(b).

I next analyze the optimal solution under various different cases, where proofs for two cases are illustrated and given in detail. I consider the case with $i+\nu \leq l_{max}$, and the proof for complementary condition follows from the derived results. I first analyze the condition $\rho > \tau\alpha^{i+\nu}$ under two cases. I also note that $p_{max}(l)$ slopes down faster than both $p_1(l)$ and $p_2(l)$ in this case.

Case I $\rho > \tau\alpha^{i+\nu}$**,** $l^* > 0$**:** I plot the $p_1(l)$ and $p_2(l)$ lines in Figure 21(a), and the corresponding $l_{t,i}^{C*}(p)$, $p_{t,i}^{C*}(l)$ curves in Figure 21(b). As depicted in Figure



(a) Representative $p_1(l)$ and $p_2(l)$ lines



(b) The corresponding $l_{t,i}^{C*}(p)$, $p_{t,i}^{C*}(l)$ curves

Fig. 21. Illustration of $(p_{t,i}^{C*}, l_{t,i}^{C*})$, when $l^* > 0$, $\rho > \tau\alpha^{i+\nu}$, $\Delta\Pi_{t,i}^C \geq p_{min}$, where $(p_{t,i}^{C*}, l_{t,i}^{C*})$ is shown using a circular mark

21(b), $l_{t,i}^{C*}(p)$ and $p_{t,i}^{C*}(l)$ intersects at a point where $l_{t,i}^{C*} = 0$ and $p_{t,i}^{C*} < p_{max}$. Furthermore, if $p_1(0) > p_{min}$, as depicted in Figure 22, then I obtain $p_{t,i}^{C*} = p_1(0)$ (R1.2). Otherwise, I get $p_{t,i}^{C*} = p_{min}$ (R1.1). Hence, by plugging in the value of $p_1(0)$, I have

$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = \begin{cases} \left(\frac{\Delta\Pi_{t,i}^C + p_{max} + \tau\alpha^{i+\nu}(i+\nu)}{2}, 0\right), & \text{if } p_1(0) > p_{min}, \\ (p_{min}, 0), & \text{if } p_1(0) \leq p_{min}. \end{cases} \tag{C.15}$$

Case II $\rho > \tau\alpha^{i+\nu}$**,** $l^* \leq 0$**:** In this case, $p_1(l)$ and $p_2(l)$ intersects at a point smaller than $d = 0$. Hence, I get $p_1(l) > p_{max}(l)$ and $p_2(l) > p_{max}(l)$ for $0 \leq l \leq i+\nu$, and

$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_{max}, 0)\,. \tag{C.16}$$

Summary of Case I and II:

In order to simplify my analysis, I describe the derivation of some inequalities and equalities using derivation tables in the remainder. In each derivation table the equalities (inequalities) on the left side explains how the equality (inequality) on right hand side is acquired using simple algebra. For example, in Table 24 the inequality $\Delta\Pi^C_{t,i} < p_{max} - \tau\alpha^{i+\nu}(i+\nu)$ is derived from the inequalities $\rho > \tau\alpha^{i+\nu}$ and $l^* > 0$.

Table 24. Derivation Table for Centralized Setting for Cases I and II

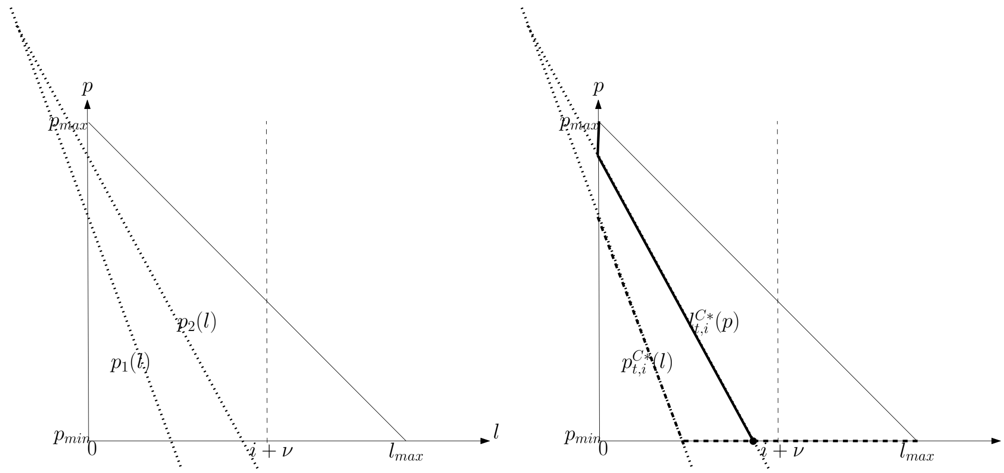
$\rho > \tau\alpha^{i+\nu}$, $l^* > 0$	$\Delta\Pi^C_{t,i} < p_{max} - \tau\alpha^{i+\nu}(i+\nu)$
$\rho > \tau\alpha^{i+\nu}$, $p_1(0) \leq p_{min}$	$\Delta\Pi^C_{t,i} \leq 2p_{min} - p_{max} - \tau\alpha^{i+\nu}(i+\nu)$

Using the results from Table 24, I combine the results of Cases I and II, and obtain

$$(p^{C*}_{t,i}, l^{C*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi^C_{t,i} \leq 2p_{min} - p_{max} - \tau\alpha^{i+\nu}(i+\nu) \\ \left(\frac{\Delta\Pi^C_{t,i} + p_{max} + \tau\alpha^{i+\nu}(i+\nu)}{2}, 0\right), & \text{if } 2p_{min} - p_{max} - \tau\alpha^{i+\nu}(i+\nu) < \Delta\Pi^C_{t,i} \\ & < p_{max} - \tau\alpha^{i+\nu}(i+\nu), \\ (p_{max}, 0), & \text{if } p_{max} - \tau\alpha^{i+\nu}(i+\nu) < \Delta\Pi^C_{t,i} \end{cases} \tag{C.17}$$

I next analyze three different cases under $\rho < \tau\alpha^{i+\nu}$. Note that, in this case $p_{max}(l)$ slopes down slower than both $p_1(l)$ and $p_2(l)$.

Case III $\rho < \tau\alpha^{i+\nu}$, $l^* < i+\nu$, $p_2(i+\nu) \leq p_{min}$: I plot this case in Figure 22.



(a) Representative $p_1(l)$ and $p_2(l)$ lines (b) The corresponding $p^{C*}_{t,i}(l)$, $l^{C*}_{t,i}(p)$ curves

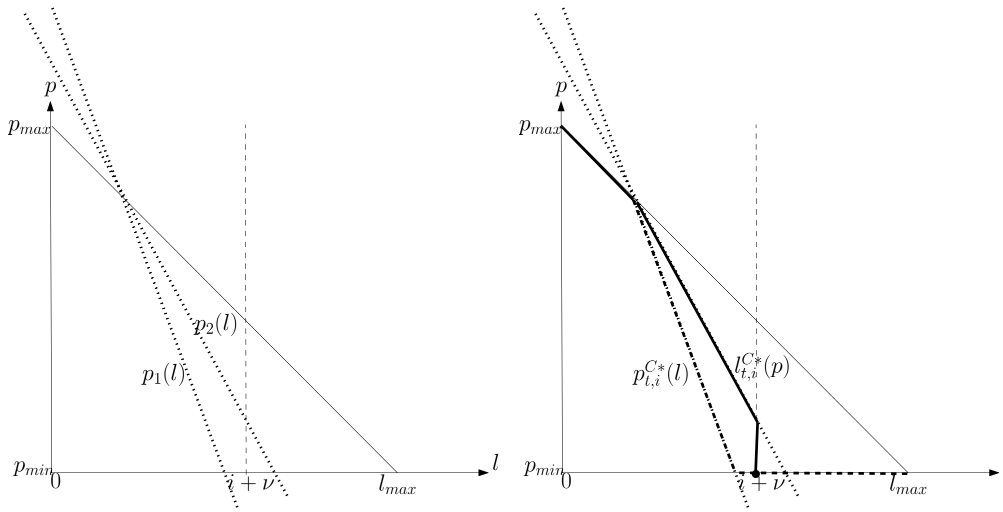
Fig. 22. Illustration of $(p^{C*}_{t,i}, l^{C*}_{t,i})$, when $\rho < \tau\alpha^{i+\nu}$, $l^* < i+\nu$, $p_1(i+\nu) \leq p_{min}$, where $(p^{C*}_{t,i}, l^{C*}_{t,i})$ is shown using a circular mark

In this case, I have $p^{C*}_{t,i} = p_{min}$, and the value of $l^{C*}_{t,i}$ depends on the point l^{**}, where $p_2(l^{**}) = p_{min}$, i.e., $l^{**} = \frac{\Delta\Pi^C_{t,i} - p_{min}}{2\tau\alpha^{i+\nu}} + \frac{i+\nu+l_{max}}{2}$. If $l^{**} > 0$ then $l^{C*}_{t,i} = l^{**}$ (R1.2). Otherwise, I have $l^{C*}_{t,i} = 0$ (R1.1). In summary, I have

$$(p^{C*}_{t,i}, l^{C*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } l^{**} \leq 0, \\ (p_{min}, l^{**}), & \text{if } l^{**} > 0. \end{cases} \tag{C.18}$$

Case IV $\rho < \tau\alpha^{i+\nu}$, $l^* < i+\nu$, $p_2(i+\nu) > p_{min}$, $p_1(i+\nu) \leq p_{min}$: I illustrate the case in Figure 23.

In this case, the possible intersection occurs at the point $(p_{min}, i+\nu)$. However, recall that $l^{C*}_{t,i}(p_{min}) = i+\nu$ only if $p_{min} \geq \Delta\Pi^C_{t,i}$ (see Equation (C.8)). Note that, I have, $p_1(i+\nu) = \frac{1}{2}\left(\Delta\Pi^C_{t,i} + p_{max} - \rho(i+\nu)\right)$. From $p_1(i+\nu) \leq p_{min}$, one can



(a) Representative $p_1(l)$ and $p_2(l)$ lines (b) The corresponding $p_{t,i}^{C*}(l)$, $l_{t,i}^{C*}(p)$ curves

Fig. 23. Illustration of $(p_{t,i}^{C*}, l_{t,i}^{C*})$, when $\rho < \tau\alpha^{i+\nu}$, $l^* < i+\nu$, $p_2(i+\nu) > p_{min}$, $p_1(i+\nu) \leq p_{min}$, where $(p_{t,i}^{C*}, l_{t,i}^{C*})$ is shown using a circular mark

obtain the following inequality

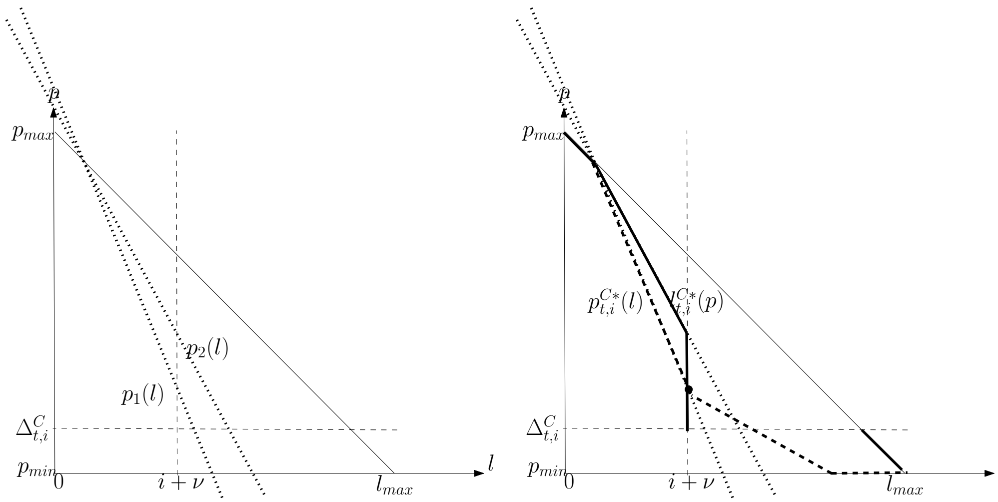
$$\Delta\Pi_{t,i}^{C} \leq p_{min} - \rho(l_{max} - (i+\nu)), \tag{C.19}$$

which clearly satisfies the desired condition, and I obtain $(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_{min}, i+\nu)$.

Case V $\rho < \tau\alpha^{i+\nu}$**,** $l^* < i+\nu$**,** $p_1(i+\nu) > p_{min}$**:** I illustrate the case in Figure 24.

In this case, the possible intersection occurs on line $(p_1(i+\nu), i+\nu)$. However, as depicted in Figure 24, one should also ensure that $p_1(i+\nu) > \Delta\Pi_{t,i}^{C}$. Note that from $l^* < i+\nu$, I obtain $\Delta\Pi_{t,i}^{C} < p_{max} - \rho(i+\nu)$, and the desired condition is trivially satisfied. Consequently, I obtain $(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_1(i+\nu), i+\nu)$.

Case VI $l^* \geq i+\nu$**,** $\rho < \tau\alpha^{i+\nu}$**:** Similar to Case II, I have $p_1(l) > p_{max}(l)$ and $p_2(l) > p_{max}(l)$ for $0 \leq l \leq i+\nu$, and hence $(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_{max}, 0)$.



(a) Representative $p_1(l)$ and $p_2(l)$ lines (b) The corresponding $p^{C*}_{t,i}(l)$, $l^{C*}_{t,i}(p)$ curves

Fig. 24. Illustration of $(p^{C*}_{t,i}, l^{C*}_{t,i})$, when $\rho < \tau\alpha^{i+\nu}$, $l^* < i+\nu$, $p_1(i+\nu) > p_{min}$, where $(p^{C*}_{t,i}, l^{C*}_{t,i})$ is shown using a circular mark

Summary of Cases III, IV, V and VI: Using Equations (C.7), (C.9) and (C.11) I get Table 25.

Table 25. Derivation Table for Centralized Setting for Cases III, IV, V and VI

$\rho < \tau\alpha^{i+\nu}$, $l^* \leq i+\nu$	$\Delta\Pi^C_{t,i} \leq p_{max} - \rho(i+\nu)$
$\rho < \tau\alpha^{i+\nu}$, $l^{**} \leq 0$	$\Delta\Pi^C_{t,i} \leq p_{min} - \tau\alpha^{i+\nu}(l_{max}+i+\nu)$
$\rho < \tau\alpha^{i+\nu}$, $p_2(i+\nu) \leq p_{min}$	$\Delta\Pi^C_{t,i} \leq p_{min} - \tau\alpha^{i+\nu}(l_{max}-(i+\nu))$
$\rho < \tau\alpha^{i+\nu}$, $p_1(i+\nu) \leq p_{min}$	$\Delta\Pi^C_{t,i} \leq p_{min} - \rho(l_{max}-(i+\nu))$

Combining the results of Cases III, IV, V and VI, I obtain

$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{C} \leq p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu), \\ \left(p_{min}, \frac{\Delta\Pi_{t,i}^{C} - p_{min}}{2\tau\alpha^{i+\nu}} + \frac{i+\nu+l_{max}}{2}\right), & \text{if } p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{min} - \rho(l_{max} - (i+\nu)), \\ (p_{min}, i+\nu), & \text{if } p_{min} - \rho(l_{max} - (i+\nu)) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{min} - \tau\alpha^{i+\nu}(l_{max} - (i+\nu)), \\ \left(\frac{\Delta\Pi_{t,i}^{C} + p_{max} - \rho(i+\nu)}{2}, i+\nu\right), & \text{if } p_{min} - \tau\alpha^{i+\nu}(l_{max} - (i+\nu)) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{max} - \rho(i+\nu), \\ (p_{max}, 0), & \text{if } p_{max} - \rho(i+\nu) < \Delta\Pi_{t,i}^{C}. \end{cases} \tag{C.20}$$

When $i + \nu > l_{max}$, the possible intersection occurs on $(0, l^{**})$. Note that, given $0 \leq l^{**} < l_{max}$, I obtain $p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu) < \Delta\Pi_{t,i}^{C} \leq p_{min} + \tau\alpha^{i+\nu}(l_{max} - (i+\nu))$. Consequently, implementing R1.1, R1.2 and R1.3 I have

$$(p_{t,i}^{C*}, l_{t,i}^{C*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{C} \leq p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu), \\ \left(p_{min}, \frac{\Delta\Pi_{t,i}^{C} - p_{min}}{2\tau\alpha^{i+\nu}} + \frac{i+\nu+l_{max}}{2}\right), & \text{if } p_{min} - \tau\alpha^{i+\nu}(l_{max} + i + \nu) < \Delta\Pi_{t,i}^{C} \\ & \leq p_{min} + \tau\alpha^{i+\nu}(l_{max} - (i+\nu)), \\ (p_{max}, 0), & \text{if } p_{max} - \rho(i+\nu) < \Delta\Pi_{t,i}^{C}. \end{cases} \tag{C.21}$$

Furthermore, note that when $\rho = \tau\alpha^{i+\nu}$, I obtain $p_1(l) = p_2(l) = p_{max}(l)$, and hence $(p_{t,i}^{C*}, l_{t,i}^{C*}) = (p_{max}, 0)$. □

Proof of Theorem 9. In this proof, I first find the best response curves for **M** and **P** that are denoted to as $p_{t,i}^{M*}(l)$ and $l_{t,i}^{P*}(p)$, respectively, and find the equilibrium under three different settings.

Derivation of $p_{t,i}^{M*}(l)$ ***and*** $l_{t,i}^{P*}(p)$

Note that $\Pi_{t,i}^{P}(p,l)$ and $\Pi_{t,i}^{M}(p,l)$ are concave in l and p respectively. $p_{t,i}^{M*}(l)$ is derived using first order conditions in Equation (C.22) similar to the derivation of $p^*(l)$ in the proof of Theorem 8.

$$p_{t,i}^{M*}(l) = \begin{cases} p_{min} & \text{if } p_1(l) \leq p_{min}, \\ p_1(l) & \text{if } p_{min} \leq p_1(l) \leq p_{max}(l), \\ p_{max}(l) & \text{if } p_{max}(l) \leq p_1(l), \end{cases} \tag{C.22}$$

where,

$$p_1(l) = \frac{1}{2}\left(\Delta\Pi_{t,i}^{M} + p_{max} - \rho l\right). \tag{C.23}$$

To derive $l_{t,i}^{P*}(p)$, I first differentiate $\Pi_{t,i}^{P}(p,l)$ w.r.t. l.

$$\frac{\partial \Pi_{t,i}^{P}(p,l)}{\partial l} = \begin{cases} \beta\left[\frac{-1}{l_{max}}\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l) - \Delta\Pi_{t,i}^{P}\right) + \tau\alpha^{i+\nu} f(p,l)\right] & \text{if } l \leq i+\nu, \\ \frac{-\beta}{l_{max}}\left(\gamma - \Delta\Pi_{t,i}^{P}\right) & \text{if } l > i+\nu. \end{cases} \tag{C.24}$$

The first order condition in l for $l \leq i+\nu$ is derived in Equation (C.25).

$$l(p) = \frac{1}{2}\left(\frac{\Delta\Pi_{t,i}^{P} - \gamma}{\tau\alpha^{i+\nu}} + i + \nu + \frac{p_{max} - p}{\rho}\right). \tag{C.25}$$

I next derive $l_{t,i}^{P*}(p)$ similar to $l_{t,i}^{C*}(p)$ in the proof of Theorem 8.

$$l_{t,i}^{P*}(p) = \begin{cases} 0 & \text{if } l(p) \leq 0, \\ l(p) & \text{if } 0 \leq l(p) \leq \min\{i+\nu, l_{max}(p)\}, \\ i+\nu & \text{if } i+\nu \leq l(p), i+\nu \leq l_{max}(p), \text{and } \Delta\Pi_{t,i}^{P} \leq \gamma \\ l_{max}(p) & \text{otherwise .} \end{cases} \quad \text{(C.26)}$$

I invert $l(p)$ curve to simplify my analysis, and obtain

$$p_2(l) = \rho\Theta - 2\rho l + p_{max}, \quad \text{(C.27)}$$

where

$$\Theta = \frac{\Delta\Pi_{t,i}^{P} - \gamma}{\tau\alpha^{i+\nu}} + i + \nu. \quad \text{(C.28)}$$

I note that slopes of $p_1(l)$, $p_{max}(l)$ and $p_2(l)$ are $-\rho/2$, $-\rho$ and -2ρ, respectively, as precisely drawn in Figure 25. Hence $p_1(l)$ slopes down slower than $p_{max}(l)$, whereas $p_2(l)$ slopes down faster than $p_{max}(l)$.

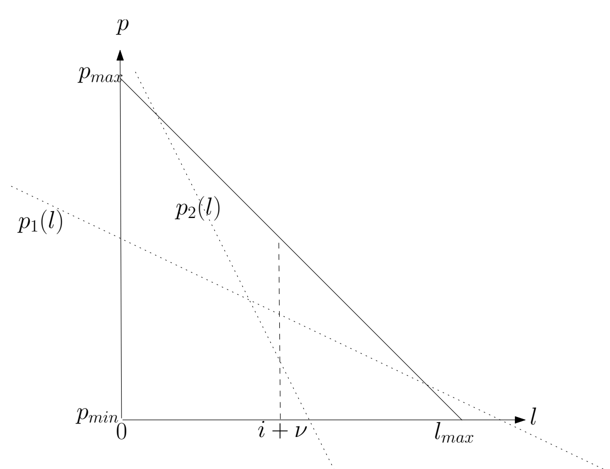


Fig. 25. Illustration of $p_1(l)$ and $p_2(l)$

Definition of Parameters

In this section, I define various parameters that are used in the proofs for settings S, P and M. First, note that, $p_1(l)$ and $p_2(l)$ are also functions of $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$. To facilitate my analysis, I redefine them in Equations (C.29) and (C.30).

$$p_1(d,K) = \frac{1}{2}\left(K + p_{max} - \rho l\right), \tag{C.29}$$

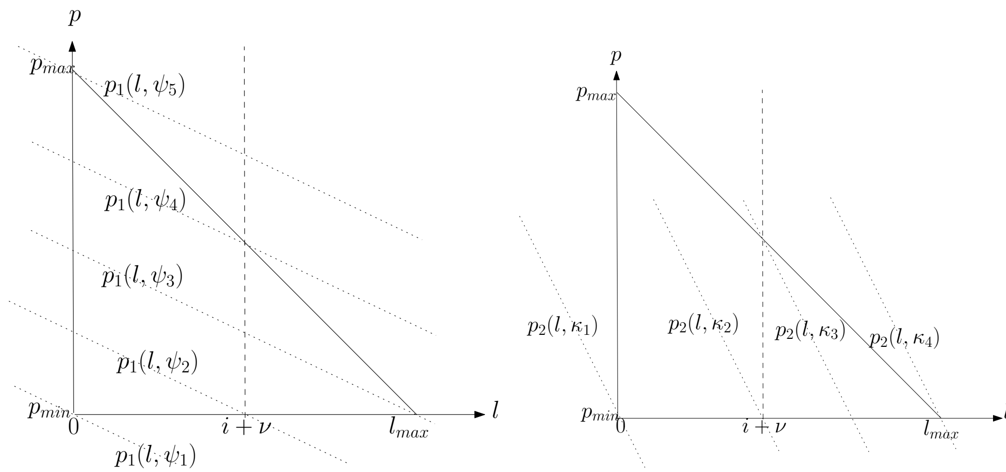
$$p_2(d,K) = \rho\left[\frac{K-\gamma}{\tau\alpha^{i+\nu}} + i + \nu\right] - 2\rho l + p_{max}. \tag{C.30}$$

In the remainder, I use $p_i(d,K)$ and $p_i(l)$ $i = 1,2$ interchangeably. Using the new definitions, I define various parameters in Table 26 that are functions of $\Delta(\cdot,\cdot,\cdot)$ (see Equation (4.18)).

Table 26. Derivation Table for parameters $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \kappa_1, \kappa_2, \kappa_3, \kappa_4$

$p_1(0,\psi_1) = p_{min}$	$\psi_1 = 2p_{min} - p_{max}$
$p_1(i+\nu,\psi_2) = p_{min}$	$\psi_2 = \rho(i+\nu) + 2p_{min} - p_{max}$
$p_1(l_{max},\psi_3) = p_{min}$	$\psi_3 = p_{min}$
$p_1(i+\nu,\psi_4) = p_{max}(i+\nu)$	$\psi_4 = p_{max} - \rho(i+\nu)$
$p_1(0,\psi_5) = p_{max}$	$\psi_5 = p_{max}$
$p_2(0,\kappa_1) = p_{min}$	$\kappa_1 = \Delta(0,-l_{max},0)$
$p_2(i+\nu,\kappa_2) = p_{min}$	$\kappa_2 = \Delta(0,2(i+\nu)-l_{max},0)$
$p_2(i+\nu,\kappa_3) = p_{max}(i+\nu)$	$\kappa_3 = \Delta(0,i+\nu,0)$
$p_2(l_{max},\kappa_4) = p_{min}$	$\kappa_4 = \Delta(0,l_{max},0)$

Using ψ_i $i = 1,2,3,4,5$, and κ_i, $i = 1,2,3,4$, one can determine the location of $p_1(d,\cdot)$ and $p_2(d,\cdot)$ that will help us to find the optimal decisions. I illustrate various cases of $p_1(d,\cdot)$ and $p_2(d,\cdot)$ in Figure 26.



(a) Illustration of $p_1(d,\psi_1)$, $p_1(d,\psi_2)$, $p_1(d,\psi_3)$, $p_1(d,\psi_4)$, and $p_1(d,\psi_5)$

(b) Illustration of $p_2(d,\kappa_1)$, $p_2(d,\kappa_2)$, $p_2(d,\kappa_3)$, and $p_2(d,\kappa_4)$

Fig. 26. Various cases of $p_1(d,\cdot)$ and $p_2(d,\cdot)$

In the remainder of this proof, I analyze the PIPE considering the different positions of $p_1(l)$ and $p_2(l)$ as a function of $\Delta\Pi^M_{t,i}$ and $\Delta\Pi^P_{t,i}$, respectively. For example, $\psi_1 < \Delta\Pi^M_{t,i} \leq \psi_2$ indicates that $p_1(l)$ cuts l axis between 0 and $i+\nu$. Similarly, $\kappa_2 < \Delta\Pi^P_{t,i} \leq \kappa_3$ implies that $p_{min} < p_1(i+\nu) \leq p_{max}(i+\nu)$. As observed from Figure 26, whenever $i+\nu < l_{max}$, the following inequality hold,

$$\psi_1 < \psi_2 < \psi_3 < \psi_4 < \psi_5. \tag{C.31}$$

Whereas, when $i+\nu > l_{max}$, I have

$$\psi_1 < \psi_3 < \psi_5. \tag{C.32}$$

Similarly when $i+\nu < l_{max}$, I obtain

$$\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4. \tag{C.33}$$

In addition, I define fmy parameters in Table 27 using Equations (C.23) and (C.27).

Table 27. Derivation Table for parameters l_1^*, l_2^*, l_3^* and l_4^*

$p_2(l_1^*) = p_{min}$	$l_1^* = \frac{1}{2}\left(\Theta + l_{max}\right)$
$p_2(l_2^*) = p_1(l_2^*)$	$l_2^* = \frac{2}{3}\left(\Theta + \frac{p_{max} - \Delta\Pi_{t,i}^M}{2\rho}\right)$
$p_1(l_3^*) = p_{min}$	$l_3^* = \frac{1}{\rho}\left(\Delta\Pi_{t,i}^M + p_{max} - 2p_{min}\right)$
$p_2(l_4^*) = p_{max}(l_4^*)$	$l_4^* = \Theta$

For example, l_2^* denotes the intersection of $p_1(l)$ and $p_2(l)$, and l_3^* is the point where $p_1(\cdot)$ intersects p_{min}. I demonstrate l_1^*, l_2^*, l_3^* and l_4^* on sample $p_1(l)$ and $p_2(l)$ lines in Figure 27. □

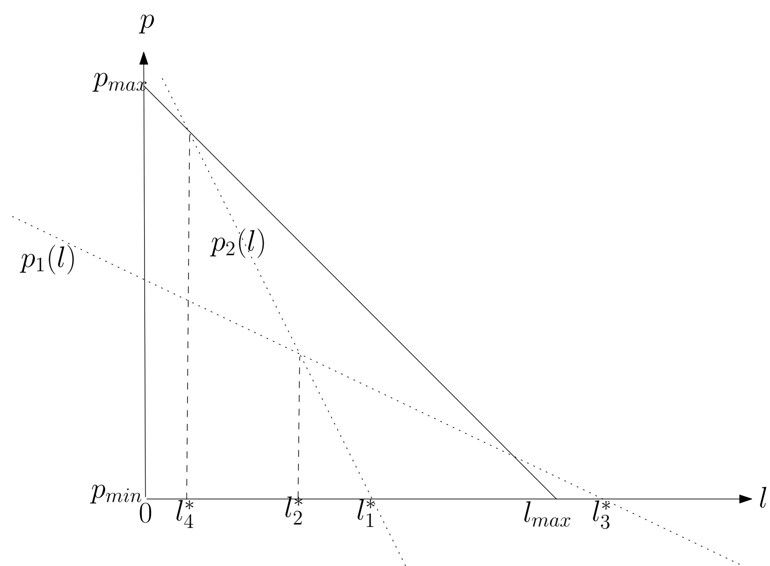


Fig. 27. Illustration of l_1^*, l_2^*, l_3^* and l_4^*

Proof of Theorem 9 for setting S The Markov perfect equilibria can be found by the intersections of $p_{t,i}^{M*}(l)$ and $l_{t,i}^{P*}(p)$ for the repeated Cournot game [Fudenberg and Tirole, 1991]. In this proof, I derive the Markov equilibrium, $(p_{t,i}^{M*}, l_{t,i}^{P*})$, using a similar approach to the proof of Theorem 8. In particular, I determine the possible $p_{t,i}^{M*}(l)$ and $l_{t,i}^{P*}(p)$ curves for all possible cases of $p_1(l)$ and $p_2(l)$, and find the intersection of $p_{t,i}^{M*}(l)$ and $l_{t,i}^{P*}(p)$ to obtain $(p_{t,i}^{M*}, l_{t,i}^{P*})$ like in the examples given in Figures 22 and 23.

I define fmy new parameters in Table 28.

Table 28. Derivation Table for parameters κ_A^S, κ_B^S, κ_l^S and κ_E^S

$p_1(0, \Delta\Pi_{t,i}^M) = p_2(0, \kappa_A^S)$, i.e., $l_2^* = 0$	$\kappa_A^S = \Delta(0.5, 0, p_{max})$
$p_1(l_3^*, \Delta\Pi_{t,i}^M) = p_2(l_3^*, \kappa_B^S)$, i.e., $l_2^* = l_3^*$	$\kappa_B^S = \Delta(2, l_{max}, p_{min})$
$p_1(i+\nu, \Delta\Pi_{t,i}^M) = p_2(i+\nu, \kappa_l^S)$, i.e., $l_2^* = i+\nu$	$\kappa_l^S = \Delta(0.5, 1.5(i+\nu), p_{max})$
$p_1(l_4^*, \Delta\Pi_{t,i}^M) = p_2(l_4^*, \kappa_E^S)$, i.e., $l_2^* = l_4^*$	$\kappa_E^S = \Delta(-1, 0, p_{max})$

In the analysis below, I follow the following roadmap: (i) Determine the possible regions of $p_1(l)$ and $p_2(l)$ given the possible values of $\Delta\Pi_{t,i}^M$ and $\Delta\Pi_{t,i}^P$, (ii) determine the $p_{t,i}^{M*}(l)$ and $l_{t,i}^{P*}(p)$ curves given $p_1(l)$ and $p_2(l)$ curves, respectively, and (iii) find $(p_{t,i}^{M*}, l_{t,i}^{P*})$ by finding the intersection of $p_{t,i}^{M*}(l)$ and $l_{t,i}^{P*}(p)$. I note that in case of multiple equilibria, I choose PIPE, i.e., any equilibrium that is not on $p_{max}(l)$ line, and hence, gives positive immediate profit. I first discuss the evaluation of $(p_{t,i}^{M*}, l_{t,i}^{P*})$ for $i + \nu \leq l_{max}$.

Case I. $\Delta\Pi_{t,i}^M \leq \psi_1$:

In Figure 28, I illustrate a sample $p_1(l)$, and fmy regions where $p_2(l)$ can possibly exist depending on the value of $\Delta\Pi_{t,i}^P$. For example, when $\kappa_1 < \Delta\Pi_{t,i}^P \leq \kappa_2$, $p_2(l)$ intersects with p_{min} line at a value ranged in between $0 < l \leq i + \nu$. I also note that $p_1(0) \leq p_{min}$ due to $\Delta\Pi_{t,i}^M \leq \psi_1$ (see Figure 26(a) and observe that $p_1(\cdot)$ is increasing in $\Delta\Pi^M_{t,i}$). Hence, $p^{M*}_{t,i}(l)$ curve lies on p_{min} as depicted as a solid line in Figure 28.

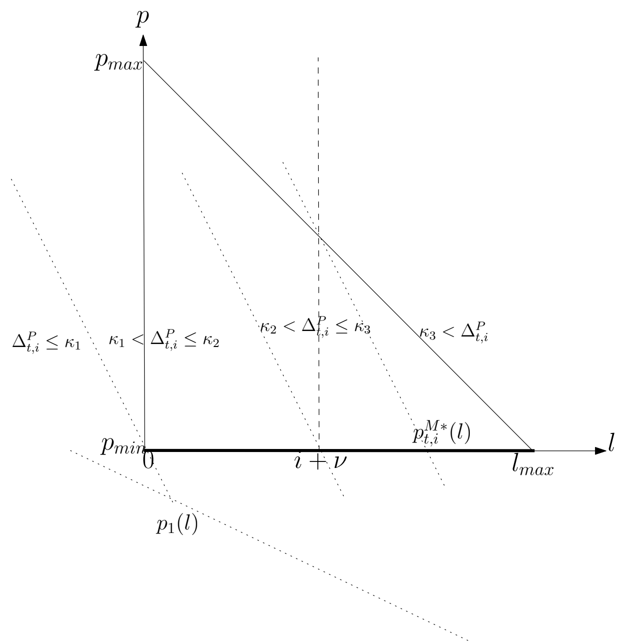


Fig. 28. Possible regions of $p_2(l)$ and $p^{M*}_{t,i}(l)$ curve under Case I.

I have fmy different optimal solutions depending on the range of $p_2(l)$ that are separately analyzed in Table 29. In this derivation table, I demonstrate my analysis in three columns, where the inequality in second column is obtained using the inequality in first column, and the result shown in third column is derived using the inequality in the second column using the indicated property. For example using $\Delta\Pi^P_{t,i} \leq \kappa_1$, one obtains $l^*_1 \leq 0$ which indicates that $l^{P*}_{t,i} = 0$ by Remark 1.1.

Table 29. Derivation Table for Case I under setting S

$\Delta\Pi^P_{t,i} \leq \kappa_1$	$l_1^* \leq 0$	$l^{P*}_{t,i} = 0$ (R1.1), and $p^{M*}_{t,i} = p_{min}$
$\kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2$	$0 < l_1^* \leq i + \nu$	$l^{P*}_{t,i} = l_1^*$ (R1.2), and $p^{M*}_{t,i} = p_{min}$
$\kappa_2 < \Delta\Pi^P_{t,i} \leq \kappa_3$	$\Delta\Pi^P_{t,i} \leq \gamma$	$l^{P*}_{t,i} = i + \nu$ (R1.3), and $p^{M*}_{t,i} = p_{min}$
$\kappa_3 < \Delta\Pi^P_{t,i}$ and $\Delta\Pi^P_{t,i} > \gamma$	no PIPE	$l^{P*}_{t,i} = 0$, and $p^{M*}_{t,i} = p_{max}$

In Equation (C.34), I summarize my findings.

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa_1, \\ (p_{min}, l_1^*), & \text{if } \kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2, \\ (p_{min}, i+\nu), & \text{if } \kappa_2 < \Delta\Pi^P_{t,i} \leq \kappa_3, \\ (p_{max}, 0), & \text{if } \kappa_3 < \Delta\Pi^P_{t,i}. \end{cases} \tag{C.34}$$

Case II. $\psi_1 < \Delta\Pi^M_{t,i} \leq \psi_2$:

From $\psi_1 < \Delta\Pi^M_{t,i} \leq \psi_2$, I have $0 \leq l_3^* < i + \nu$, $\kappa^S_A < \kappa^S_B < \kappa_2$, and $p^{M*}_{t,i}(l)$ curve takes the form depicted in Figure 29. Similar to Figure 28, several ranges of $p_2(l)$ are depicted as well.

Note that results for $\kappa^S_B < \Delta\Pi^P_{t,i} \leq \kappa_2$, and $\kappa_2 < \Delta\Pi^P_{t,i}$ are identical to $\kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2$, and $\kappa_2 < \Delta\Pi^P_{t,i}$ with those derived in Case I., respectively. Hence, I focus on the remaining two cases in Table 30.

Table 30. Derivation Table for Case II under setting S

$\Delta\Pi^P_{t,i} \leq \kappa^S_A$	$l_2^* \leq 0$	$l^{P*}_{t,i} = 0$ (R1.1), and $p^{M*}_{t,i} = p_1(0)$
$\kappa^S_A < \Delta\Pi^P_{t,i} \leq \kappa^S_B$	$0 < l_2^* \leq l_3^*$	$l^{P*}_{t,i} = l_2^*$ (R1.2), and $p^{M*}_{t,i} = p_1(l_2^*)$

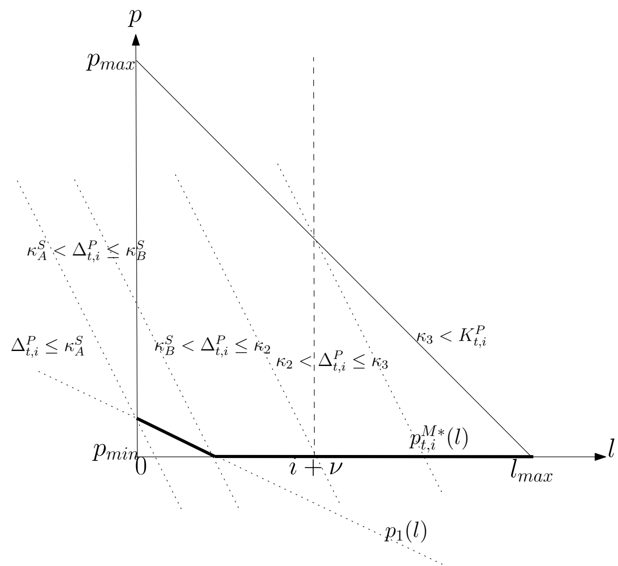


Fig. 29. Possible regions of $p_2(l)$ and $p_{t,i}^{M*}(l)$ curve under Case II.

I summarize all possible conditions under Case II. in Equation (C.35).

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi_{t,i}^P \leq \kappa_A^S, \\ (p_1(l_2^*), l_2^*), & \text{if } \kappa_A^S < \Delta\Pi_{t,i}^P \leq \kappa_B^S, \\ (p_{min}, l_1^*), & \text{if } \kappa_B^S < \Delta\Pi_{t,i}^P \leq \kappa_2, \\ (p_{min}, i+\nu), & \text{if } \kappa_2 < \Delta\Pi_{t,i}^P \leq \kappa_3, \\ (p_{max}, 0), & \text{if } \kappa_3 < \Delta\Pi_{t,i}^P. \end{cases} \tag{C.35}$$

Case III. $\psi_2 < \Delta\Pi_{t,i}^M \leq \psi_4$: In this case, I have $i+\nu < l_3^*$ (or equivalently $p_1(i+\nu) > p_{min}$) and $p_1(i+\nu) \leq p_{max}(i+\nu)$. The results for various cases of $p_2(l)$ line are illustrated in Figure 30.

$(p_{t,i}^{M*}, l_{t,i}^{P*})$ are derived using the results of the Case II in Table 31.

Hence I obtain the following PIPE.

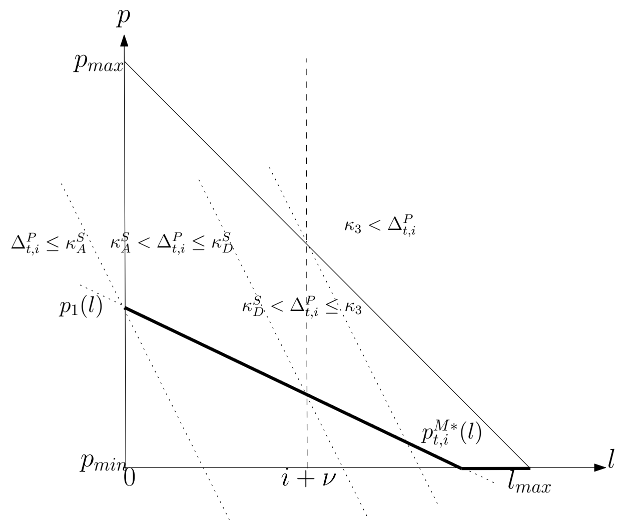


Fig. 30. Possible regions of $p_2(l)$ and $p^{M*}_{t,i}(l)$ curve under Case III.

Table 31. Derivation Table for Case III under setting S

$\Delta\Pi^P_{t,i} \leq \kappa^S_A$	$l^*_2 \leq 0$	$l^{P*}_{t,i} = 0$ (R1.1) and $p^{M*}_{t,i} = p_1(0)$
$\kappa^S_A < \Delta\Pi^P_{t,i} \leq \kappa^S_l$	$0 < l^*_2 \leq i + \nu$	$l^{P*}_{t,i} = l^*_2$ (R1.2) and $p^{M*}_{t,i} = p_1(l^*_2)$
$\kappa^S_l < \Delta\Pi^P_{t,i} \leq \kappa_3$	$\Delta\Pi^P_{t,i} \leq \gamma$	$l^{P*}_{t,i} = i + \nu$ (R1.3) and $p^{M*}_{t,i} = p_1(i + \nu)$

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa^S_A, \\ (p_1(l^*_2), l^*_2), & \text{if } \kappa^S_A < \Delta\Pi^P_{t,i} \leq \kappa^S_l, \\ (p_1(i+\nu), i+\nu), & \text{if } \kappa^S_l < \Delta\Pi^P_{t,i} \leq \kappa_3, \\ (p_{max}, 0), & \text{if } \kappa_3 < \Delta\Pi^P_{t,i}. \end{cases} \tag{C.36}$$

Case IV. $\psi_4 < \Delta\Pi^M_{t,i} \leq \psi_5$: I now have $p_1(i + \nu) > p_{max}(i + \nu)$, $p_1(0) \leq p_{max}$, and $\kappa^S_A < \kappa^S_E$. The results for various cases of $p_2(l)$ line are illustrated in Figure 31.

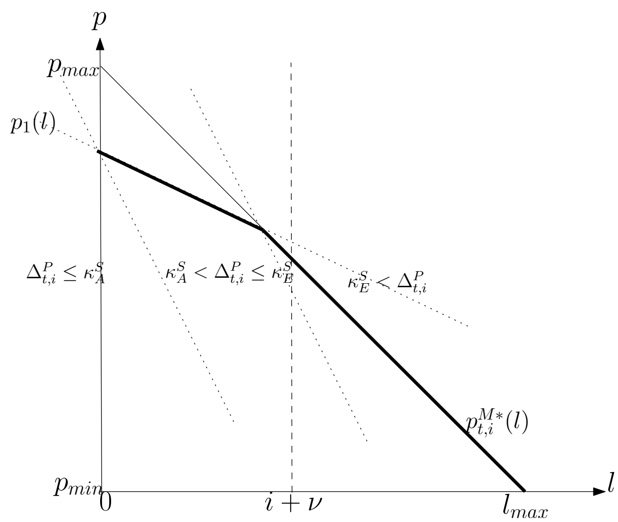


Fig. 31. Possible regions of $p_2(l)$ and $p^{M*}_{t,i}(l)$ curve under Case IV.

Note that, the results for this case differs from the previous cases only for the optimality region of $(p_1(l^*_2), l^*_2)$ for $0 < l^*_2 \leq l^*_4$, i.e., $\kappa^S_A < \Delta\Pi^P_{t,i} \leq \kappa^S_E$. Solution is given in Equation (C.37).

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa^S_A, \\ (p_1(l^*_2), l^*_2), & \text{if } \kappa^S_A < \Delta\Pi^P_{t,i} \leq \kappa^S_E, \\ (p_{max}, 0), & \text{if } \kappa^S_E < \Delta\Pi^P_{t,i}. \end{cases} \tag{C.37}$$

Case V. $\psi_4 < \Delta\Pi^M_{t,i}$: In this case $p_1(0) > p_{max}$, and no PIPE exists, i.e., $(p^{M*}_{t,i}, l^{P*}_{t,i}) = (p_{max}, 0)$.

In summary, I have

$$
(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases}
(p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^M \leq \psi_1, \Delta\Pi_{t,i}^P \leq \kappa_1, \\
(p_1(0), 0), & \text{if } \psi_1 < \Delta\Pi_{t,i}^M \leq \psi_5, \Delta\Pi_{t,i}^P \leq \kappa_A^S, \\
(p_{min}, l_1^*), & \text{if } \Delta\Pi_{t,i}^M \leq \psi_1, \kappa_1 < \Delta\Pi_{t,i}^P \\
& \leq \kappa_2, \text{ or } \psi_1 < \Delta\Pi_{t,i}^M \leq \psi_2, \kappa_B^S < \Delta\Pi_{t,i}^P \leq \kappa_2, \\
(p_{min}, i+\nu), & \text{if } \Delta\Pi_{t,i}^M \leq \psi_2, \kappa_2 < \Delta\Pi_{t,i}^P \leq \kappa_3, \\
(p_1(l_2^*), l_2^*), & \text{if } \psi_1 < \Delta\Pi_{t,i}^M \leq \psi_2, \kappa_A^S < \Delta\Pi_{t,i}^P \\
& \leq \kappa_B^S, \text{ or } \psi_2 < \Delta\Pi_{t,i}^M \leq \psi_4, \kappa_A^S < \Delta\Pi_{t,i}^P \leq \kappa_l^S, \\
& \text{ or } \psi_4 < \Delta\Pi_{t,i}^M \leq \psi_5, \kappa_A^S < \Delta\Pi_{t,i}^P \leq \kappa_E^S, \\
(p_1(i+\nu), i+\nu), & \text{if } \psi_2 < \Delta\Pi_{t,i}^M \leq \psi_4, \kappa_l^S < \Delta\Pi_{t,i}^P \leq \kappa_3, \\
(p_{max}, 0), & \text{otherwise.}
\end{cases} \tag{C.38}
$$

I now discuss the case for $i + \nu > l_{max}$.

Case VI. $\Delta\Pi_{t,i}^M \leq \psi_1$:

The result for this case partially follows from Case I. Note that since $i+\nu > l_{max}$, (p_{min}, l_1^*) is the PIPE for $\kappa_1 < \Delta\Pi_{t,i}^P \leq \kappa_4$ (see Figure 26(b)). Hence I obtain,

$$
(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases}
(p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^P \leq \kappa_1, \\
(p_{min}, l_1^*), & \text{if } \kappa_1 < \Delta\Pi_{t,i}^P \leq \kappa_4, \\
(p_{max}, 0), & \text{if } \kappa_4 < \Delta\Pi_{t,i}^P.
\end{cases} \tag{C.39}
$$

Case VII. $\psi_1 < \Delta\Pi_{t,i}^M \leq \psi_3$:

The result for this case partially follows from Case II. as well.

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi_{t,i}^P \le \kappa_A^S, \\ (p_1(l_2^*), l_2^*), & \text{if } \kappa_A^S < \Delta\Pi_{t,i}^P \le \kappa_B^S, \\ (p_{min}, l_1^*), & \text{if } \kappa_B^S < \Delta\Pi_{t,i}^P \le \kappa_4, \\ (p_{max}, 0), & \text{if } \kappa_4 < \Delta\Pi_{t,i}^P. \end{cases} \tag{C.40}$$

Case VIII. $\psi_3 < \Delta\Pi_{t,i}^M \le \psi_5$:

The result for this case is identical to that of Case IV.

In summary, I obtain

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^M \le \psi_1, \Delta\Pi_{t,i}^P \le \kappa_1, \\ (p_1(0), 0), & \text{if } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_5, \Delta\Pi_{t,i}^P \le \kappa_A^S, \\ (p_{min}, l_1^*), & \text{if } \Delta\Pi_{t,i}^M \le \psi_1, \kappa_1 < \Delta\Pi_{t,i}^P \le \kappa_4, \\ & \text{ or } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_3, \kappa_B^S < \Delta\Pi_{t,i}^P \le \kappa_4, \\ (p_1(l_2^*), l_2^*), & \text{if } \psi_1 < \Delta\Pi_{t,i}^M \le \psi_3, \kappa_A^S < \Delta\Pi_{t,i}^P \le \kappa_B^S, \text{ or} \\ & \psi_3 < \Delta\Pi_{t,i}^M \le \psi_5, \kappa_A^S < \Delta\Pi_{t,i}^P \le \kappa_E^S, \\ (p_{max}, 0), & \text{otherwise.} \end{cases} \tag{C.41}$$

□

Proof of Theorem 9 for setting P Since production moves first, I first solve the marketing problem, and use their solution as an input to the production's problem. Note that, given a lead time decision l, the optimal decision of marketing is given by

$p^{M*}_{t,i}(l)$ in Equation (C.22). Hence, given the production's lead time decision of l, I derive the optimal acceptance probability function, $f(p^{M*}_{t,i}(l), l)$, in Equation (C.42), using Equations (C.22) and (C.23).

$$f(p^{M*}_{t,i}(l), l) = \begin{cases} 1 - \frac{l}{l_{max}} & \text{if } p_1(l) \leq p_{min}, \\ \frac{p_{max} - \Delta\Pi^M_{t,i} - \rho l}{2\rho l_{max}} & \text{if } p_{min} \leq p_1(l) \leq p_{max}(l), \\ 0 & \text{if } p_{max}(l) \leq p_1(l), \end{cases} \tag{C.42}$$

Then, by plugging in $p^{M*}_{t,i}(l)$, $\Delta\Pi^P_{t,i}$ and rewriting $\Pi^P_{t,i}(p, l)$ for $i \leq N$, I obtain,

$$\Pi^P_{t,i}(p^{M*}_{t,i}(l), l) = \beta f(p^{M*}_{t,i}(l), l)(\gamma - \tau\alpha^{i+\nu}(i+\nu-l)^+ - \Delta\Pi^P_{t,i}) + \alpha\underline{\Pi}^{P*}_{t,i}, \tag{C.43}$$

and I reach

$$\Pi^P_{t,i}(p^{M*}_{t,i}(l), l) = \begin{cases} \left(1 - \frac{l}{l_{max}}\right)\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l)^+ - \Delta\Pi^P_{t,i}\right) + \alpha\underline{\Pi}^{P*}_{t,i}, & \text{if } p_1(l) \leq p_{min}, \\ \frac{p_{max} - \Delta\Pi^M_{t,i} - \rho l}{2\rho l_{max}}\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l)^+ - \Delta\Pi^P_{t,i}\right) + \alpha\underline{\Pi}^{P*}_{t,i} & \text{if } p_{min} \leq p_1(l) \leq p_{max}(l), \\ \alpha\underline{\Pi}^{P*}_{t,i} & \text{if } p_{max}(l) \leq p_1(l). \end{cases} \tag{C.44}$$

The optimal lead time decision, $l^{P*}_{t,i}$, can be found by

$$l^{P*}_{t,i} = \underset{0 \leq l \leq l_{max}}{\operatorname{argmax}} \Pi^P_{t,i}(p^{M*}_{t,i}(l), l). \tag{C.45}$$

After evaluating $l^{P*}_{t,i}$ the optimal decision of marketing can be found by

$$p^{M*}_{t,i} = p^{M*}_{t,i}(l^{P*}_{t,i}). \tag{C.46}$$

To solve the maximization problem in (C.45), I first find the maximizers of the piecewise elements of $\Pi^P_{t,i}(p^{M*}_{t,i}(l), l)$ that will help us in derivation of $l^{P*}_{t,i}$. The local maximizers of first and second piecewise elements are denoted as l^*_5 and l^*_6, respectively, and are derived in Table 32.

Table 32. Derivation Table for parameters l_5^* and l_6^*

$l_5^* = \text{argmax}_d \left(1 - \frac{l}{l_{max}}\right)\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l) - \Delta\Pi_{t,i}^P\right)$ $l_6^* = \text{argmax}_d \frac{p_{max}-\Delta\Pi_{t,i}^M-\rho l}{2\rho l_{max}}\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l) - \Delta\Pi_{t,i}^P\right)$	$l_5^* = \frac{1}{2}\left(\Theta + l_{max}\right)$ $l_6^* = \frac{1}{2}\left(\Theta + \frac{p_{max}-\Delta\Pi_{t,i}^M}{\rho}\right)$

Since l_5^* and l_6^* are functions of $\Delta\Pi_{t,i}^P$, I redefine them as follows.

$$l_5^*(\kappa) = \frac{1}{2}\left(\frac{\kappa-\gamma}{\tau\alpha^{i+\nu}} + i + \nu + l_{max}\right), \text{ and} \tag{C.47}$$

$$l_6^*(\kappa) = \frac{1}{2}\left(\frac{p_{max}-\Delta\Pi_{t,i}^M}{\rho} + \frac{\kappa + \tau\alpha^{i+\nu}(i+\nu)-\gamma}{\tau\alpha^{i+\nu}}\right). \tag{C.48}$$

I follow the approach used in the proof for the setting S, and define new parameters in Table 33.

Table 33. Derivation Table for parameters κ_A^P, κ_B^P, κ_C^P, κ_l^P and κ_E^P

$l_6^*(\kappa_A^P) = 0$	$\kappa_A^P = \Delta(1, 0, p_{max})$
$p_1(l_6^*(\kappa_B^P)) = p_{min}$	$\kappa_B^P = \Delta(3, l_{max}, p_{min})$
$p_1(l_5^*(\kappa_C^P)) = p_{min}$	$\kappa_C^P = \Delta(2, l_{max}, p_{min})$
$l_6^*(\kappa_l^P) = i+\nu$	$\kappa_D = \Delta(1, 2(i+\nu), p_{max})$
$p_1(l_5^*(\kappa_E^P)) = p_{max}(l_6^*(\kappa_E^P))$	$\kappa_E^P = \Delta(-1, 0, p_{max})$

I only derive $(p_{t,i}^{M*}, l_{t,i}^{P*})$ for $i+\nu \le l_{max}$ under several cases listed below. The proof for $i+\nu > l_{max}$ follows from $i+\nu \le l_{max}$ and is omitted.

Case I. $\Delta\Pi_{t,i}^M \le \psi_1$:

As depicted in Figure 28, I have $p_1(l) \le p_{min}$ and $\Pi_{t,i}^P(p_{t,i}^{M*}(l), l) = \left(1 - \frac{l}{l_{max}}\right)\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l)^+ - \Delta\Pi_{t,i}^P\right) + \alpha\underline{\Pi}_{t,i}^{P*}$ for $0 \le l \le l_{max}$. Recalling that l_5^* is the maximizer of $l_{t,i}^{M*}(p)$ in this case (see Table 32), I analyze the possible values of l_5^* to determine the optimal decision in Table 34

Table 34. Derivation Table for Case I under setting P

$\Delta\Pi^P_{t,i} \leq \kappa_1$	$l_5^* \leq 0$	$l^{P*}_{t,i} = 0$ (R1.1), and $p^{M*}_{t,i} = p_{min}$
$\kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2$	$0 < l_5^* \leq i+\nu$	$l^{P*}_{t,i} = l_5^*$ (R1.2), and $p^{M*}_{t,i} = p_{min}$
$\kappa_2 < \Delta\Pi^P_{t,i} \leq \kappa_3$	$\Delta\Pi^P_{t,i} \leq \gamma$	$l^{P*}_{t,i} = i+\nu$ (R1.3), and $p^{M*}_{t,i} = p_{min}$
$\kappa_3 < \Delta\Pi^P_{t,i}$ and $\Delta\Pi^P_{t,i} > \gamma$	no PIPE	$l^{P*}_{t,i} = 0$, and $p^{M*}_{t,i} = p_{max}$

Note that, obtained optimal solution is same with the one derived for setting S Case I. The findings are summarized in Equation (C.49).

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa_1, \\ (p_{min}, l_1^*), & \text{if } \kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2, \\ (p_{min}, i+\nu), & \text{if } \kappa_2 < \Delta\Pi^P_{t,i} \leq \kappa_3, \\ (p_{max}, 0), & \text{if } \kappa_3 < \Delta\Pi^P_{t,i}. \end{cases} \tag{C.49}$$

Case II. $\psi_1 < \Delta\Pi^M_{t,i} \leq \psi_2$:

$p^{M*}_{t,i}(l)$ is depicted in Figure 32.

Note that in this case $\Pi^P_{t,i}(p^{M*}_{t,i}(l), l)$ is a piecewise function given as follows.

$$\Pi^P_{t,i}(p^{M*}_{t,i}(l), l) = \begin{cases} \frac{p_{max} - \Delta\Pi^M_{t,i} - \rho l}{2\rho l_{max}} \left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l)^+ - \Delta\Pi^P_{t,i}\right) + \alpha\underline{\Pi}^{P*}_{t,i}, & \text{for } 0 \leq d < l_3^*, \\ \left(1 - \frac{l}{l_{max}}\right)\left(\gamma - \tau\alpha^{i+\nu}(i+\nu-l)^+ - \Delta\Pi^P_{t,i}\right) + \alpha\underline{\Pi}^{P*}_{t,i}, & \text{if } l_3^* \leq d \leq l_{max}. \end{cases} \tag{C.50}$$

One observes that the first and second elements of $\Pi^P_{t,i}(p^{M*}_{t,i}(l), l)$ are maximized by l_6^* and l_5^*, respectively. Hence, I analyze the possible values of l_5^* and l_6^* and use the findings of Remark 2 to derive the optimal solution in Table 35. Furthermore, I use the inequality $\kappa^P_A < \kappa^P_B < \kappa^P_C < \kappa_2$, which is simply acquired from $\psi_1 < \Delta\Pi^M_{t,i} \leq \psi_2 < \psi_3$.

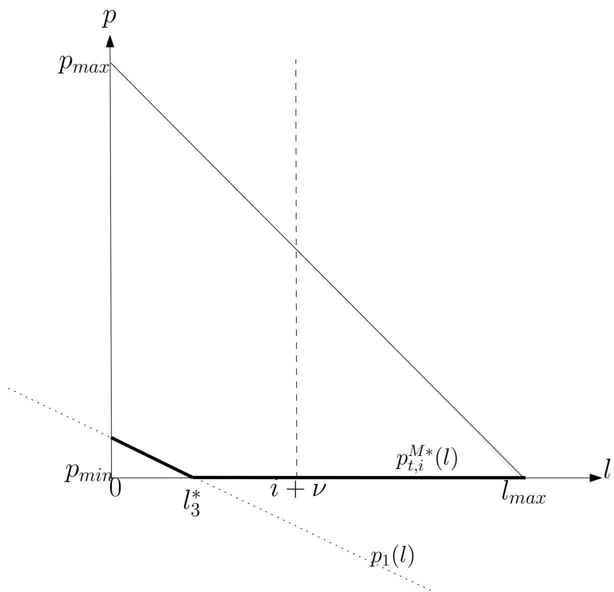


Fig. 32. Illustration of Case II.

Table 35. Derivation Table for Case II under setting P

$\Delta\Pi_{t,i}^P \leq \kappa_A^P < \kappa_C^P$	$l_6^* \leq 0$ and $l_5^* \leq l_3^*$	$l_{t,i}^{P*} = 0$ (R2.1), and $p_{t,i}^{M*} = p_1(0)$
$\kappa_A^P < \Delta\Pi_{t,i}^P \leq \kappa_B^P < \kappa_C^P$	$0 \leq l_6^* \leq l_3^*$ and $l_5^* \leq l_3^*$	$l_{t,i}^{P*} = l_6^*$ (R2.2) and $p_{t,i}^{M*} = p_1(l_2^*)$
$\kappa_B^P < \Delta\Pi_{t,i}^P \leq \kappa_C^P$	$l_3^* < l_6^*$ and $l_5^* \leq l_3^*$	$l_{t,i}^{P*} = l_3^*$ (R2.3) and $p_{t,i}^{M*} = p_{min}$
$\kappa_C^P < \Delta\Pi_{t,i}^P \leq \kappa_2$	$l_3^* < l_6^*$ and $l_3^* \leq l_5^* \leq i+\nu$	$l_{t,i}^{P*} = l_5^*$ (R2.4) and $p_{t,i}^{M*} = p_{min}$
$\kappa_2 < \Delta\Pi_{t,i}^P \leq \kappa_3$	$l_3^* < l_6^*$ and $i+\nu < l_5^*$	$l_{t,i}^{P*} = i+\nu$ (R2.5) and $p_{t,i}^{M*} = p_{min}$

Consequently, I obtain

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi_{t,i}^P \leq \kappa_A^P, \\ (p_1(l_6^*), l_6^*), & \text{if } \kappa_A^P < \Delta\Pi_{t,i}^P \leq \kappa_B^P, \\ (p_{min}, l_3^*), & \text{if } \kappa_B^P < \Delta\Pi_{t,i}^P \leq \kappa_C^P, \\ (p_{min}, l_5^*), & \text{if } \kappa_C^P < \Delta\Pi_{t,i}^P \leq \kappa_2, \\ (p_{min}, i+\nu), & \text{if } \kappa_2 < \Delta\Pi_{t,i}^P \leq \kappa_3, \\ (p_{max}, 0), & \text{if } \kappa_3 < \Delta\Pi_{t,i}^P. \end{cases} \tag{C.51}$$

Case III. $\psi_2 < \Delta\Pi^M_{t,i} \leq \psi_4$:

The results for this case follows from Case III for setting S, where l_2^*, κ_A^S and κ_l^S are replaced by l_6^*, κ_A^P and κ_l^P, respectively.

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi^P_{t,i} \leq \kappa^P_A, \\ (p_1(l_6^*), l_6^*), & \text{if } \kappa^P_A < \Delta\Pi^P_{t,i} \leq \kappa^P_l, \\ (p_1(i+\nu), i+\nu), & \text{if } \kappa^P_l < \Delta\Pi^P_{t,i} \leq \kappa_3, \\ (p_{max}, 0), & \text{if } \kappa_3 < \Delta\Pi^P_{t,i}. \end{cases} \tag{C.52}$$

Case IV. $\psi_4 < \Delta\Pi^M_{t,i} \leq \psi_5$:

This case is depicted in Figure 33.

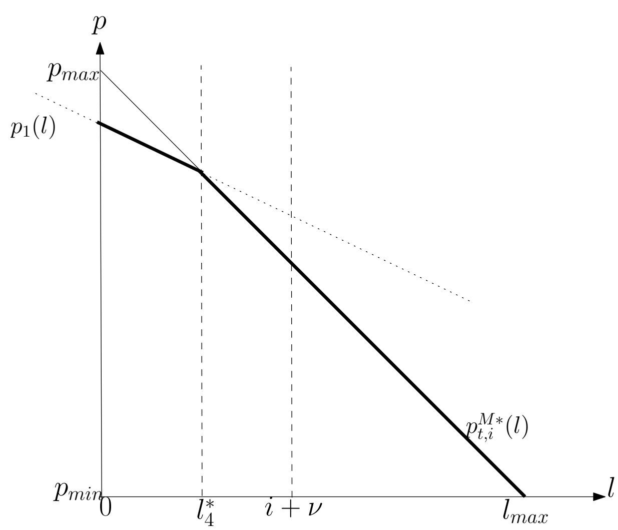


Fig. 33. Illustration of Case IV.

Similarly, the results for this case is identical to that of Case IV for setting S, where l_2^*, κ_A^S and κ_E^S are replaced by l_6^*, κ_A^P and κ_E^P, respectively.

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_1(0), 0), & \text{if } \Delta\Pi^P_{t,i} \le \kappa^P_A, \\ (p_1(l^*_6), l^*_6), & \text{if } \kappa^P_A < \Delta\Pi^P_{t,i} \le \kappa^P_E, \\ (p_{max}, 0), & \text{if } \kappa^P_E < \Delta\Pi^P_{t,i}. \end{cases} \quad \text{(C.53)}$$

Case V. $\psi_4 < \Delta\Pi^M_{t,i}$: In this case $p_1(0) > p_{max}$, and no PIPE exists, i.e., $(p^{M*}_{t,i}, l^{P*}_{t,i}) = (p_{max}, 0)$.

In summary, I get the following.

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi^M_{t,i} \le \psi_1, \Delta\Pi^P_{t,i} \le \kappa_1, \\ (p_1(0), 0), & \text{if } \psi_1 < \Delta\Pi^M_{t,i} \le \psi_5, \Delta\Pi^P_{t,i} \le \kappa^P_A, \\ (p_{min}, l^*_5), & \text{if } \Delta\Pi^M_{t,i} \le \psi_1, \kappa_1 < \Delta\Pi^P_{t,i} \le \kappa_2, \text{ or} \\ & \psi_1 < \Delta\Pi^M_{t,i} \le \psi_2, \kappa^P_C < \Delta\Pi^P_{t,i} \le \kappa_2, \\ (p_{min}, l^*_3), & \text{if } \psi_1 < \Delta\Pi^M_{t,i} \le \psi_2, \kappa^P_B < \Delta\Pi^P_{t,i} \le \kappa^P_C, \\ (p_{min}, i+\nu), & \text{if } \Delta\Pi^M_{t,i} \le \psi_2, \kappa_2 < \Delta\Pi^P_{t,i} \le \kappa_3, \\ (p_1(l^*_6), l^*_6), & \text{if } \psi_1 < \Delta\Pi^M_{t,i} \le \psi_2, \kappa^P_A < \Delta\Pi^P_{t,i} \le \kappa^P_B, \text{ or} \\ & \psi_2 < \Delta\Pi^M_{t,i} \le \psi_4, \kappa^P_A < \Delta\Pi^P_{t,i} \le \kappa^P_l, \\ & \text{ or } \psi_4 < \Delta\Pi^M_{t,i} \le \psi_5, \kappa^P_A < \Delta\Pi^P_{t,i} \le \kappa^P_E, \\ (p_1(i+\nu), i+\nu), & \text{if } \psi_2 < \Delta\Pi^M_{t,i} \le \psi_4, \kappa^P_l < \Delta\Pi^P_{t,i} \le \kappa_3, \\ (p_{max}, 0), & \text{otherwise.} \end{cases}$$

(C.54)

Following the results for $i+\nu \le l_{max}$, I obtain the optimal solution for $i+\nu > l_{max}$ in Equation (C.55).

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{M} \le \psi_1, \Delta\Pi_{t,i}^{P} \le \kappa_1, \\ (p_1(0), 0), & \text{if } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_5, \Delta\Pi_{t,i}^{P} \le \kappa_A^P, \\ (p_{min}, l_5^*), & \text{if } \Delta\Pi_{t,i}^{M} \le \psi_1, \kappa_1 < \Delta\Pi_{t,i}^{P} \le \kappa_4, \text{ or} \\ & \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_3, \kappa_C^P < \Delta\Pi_{t,i}^{P} \le \kappa_4, \\ (p_{min}, l_3^*), & \text{if } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_3, \kappa_B^P < \Delta\Pi_{t,i}^{P} \le \kappa_C^P, \\ (p_1(l_6^*), l_6^*), & \text{if } \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_3, \kappa_A^P < \Delta\Pi_{t,i}^{P} \le \kappa_B^P, \text{ or} \\ & \psi_3 < \Delta\Pi_{t,i}^{M} \le \psi_5, \kappa_A^P < \Delta\Pi_{t,i}^{P} \le \kappa_E^P, \\ (p_{max}, 0), & \text{otherwise.} \end{cases} \tag{C.55}$$

□

Proof of Theorem 9 for setting M

I first solve the optimization problem of production department, and use the solution as an input to the marketing's problem. I derive the optimal acceptance probability function, $f(p, l_{t,i}^{M*}(p))$, in Equation (C.56), using Equations (C.8) and (C.7) for $\Delta\Pi_{t,i}^{P} \le \kappa_3$. Note that $f(p, l_{t,i}^{M*}(p)) = 0$ when $\Delta\Pi_{t,i}^{P} > \kappa_3$.

$$f(p, l_{t,i}^{M*}(p)) = \begin{cases} \frac{p_{max}-p}{\rho l_{max}} & \text{if } l(p) \leq 0, \\ \frac{p_{max}-p-\rho\Theta}{2\rho l_{max}} & 0 \leq l(p) \leq \min\{i+\nu, l_{max}(p)\}, \\ \frac{p_{max}-p-\rho(i+\nu)}{\rho l_{max}} & \text{if } i+\nu \leq l(p), i+\nu \leq l_{max}(p), \\ & \text{and } \Delta\Pi_{t,i}^{P} \leq \gamma, \\ 0 & \text{otherwise .} \end{cases} \tag{C.56}$$

Then, by plugging in $f(p, l_{t,i}^{M*}(p))$, $\Delta\Pi_{t,i}^{M}$ and rewriting $\Pi_{t,i}^{M}(p,l)$ for $i \leq N$ I obtain,

$$\Pi_{t,i}^{M}(p, l_{t,i}^{M*}(p)) = \beta f(p, l_{t,i}^{M*}(p))(p - \Delta\Pi_{t,i}^{M}) + \alpha\underline{\Pi}_{t,i}^{M*}, \tag{C.57}$$

and I reach

$$\Pi_{t,i}^{M}(p, l_{t,i}^{M*}(p)) = \begin{cases} \frac{p_{max}-p}{\rho l_{max}}(p - \Delta\Pi_{t,i}^{M}) + \alpha\underline{\Pi}_{t,i}^{M*}, & \text{if } l(p) \leq 0, \\ \frac{p_{max}-p-\rho\Theta}{2\rho l_{max}}(p - \Delta\Pi_{t,i}^{M}) + \alpha\underline{\Pi}_{t,i}^{M*}, & 0 \leq l(p) \leq \min\{i+\nu, l_{max}(p)\}, \\ \frac{p_{max}-p-\rho(i+\nu)}{\rho l_{max}}(p - \Delta\Pi_{t,i}^{M}) + \alpha\underline{\Pi}_{t,i}^{M*}, & \text{if } i+\nu \leq l(p), i+\nu \leq l_{max}(p), \\ & \text{and } \Delta\Pi_{t,i}^{P} \leq \gamma, \\ 0 & \text{otherwise .} \end{cases} \tag{C.58}$$

The optimal lead time decision, $l_{t,i}^{P*}$, can be found by

$$l_{t,i}^{P*} = \operatorname*{argmax}_{0 \leq l \leq l_{max}} \Pi_{t,i}^{P}(p_{t,i}^{M*}(l), l) \tag{C.59}$$

I first evaluate the optimal price decisions that maximizes each piecewise function in Equation (C.73). Using simple algebra, one obtains

$$p_1(0) = \operatorname*{argmax}_{p} \frac{p_{max} - p}{\rho l_{max}}(p - \Delta\Pi_{t,i}^{M}). \tag{C.60}$$

$$p_1(\Theta) = \underset{p}{\text{argmax}} \frac{p_{max} - p - \rho\Theta}{2\rho l_{max}}(p - \Delta\Pi^M_{t,i}). \quad \text{(C.61)}$$

$$p_1(i+\nu) = \underset{p}{\text{argmax}} \frac{p_{max} - p - \rho(i+\nu)}{\rho l_{max}}(p - \Delta\Pi^M_{t,i}). \quad \text{(C.62)}$$

Given these pricing decisions, $l^{P*}_{t,i}$ takes values of 0, $\frac{1}{4}(3\Theta + \frac{p_{max}-\Delta\Pi^M_{t,i}}{\rho})$, and $i+\nu$, respectively. For simplicity I let,

$$l^*_7 = \frac{1}{4}\left(3\Theta + \frac{p_{max} - \Delta\Pi^M_{t,i}}{\rho}\right). \quad \text{(C.63)}$$

Following the approaches implemented in the previous proofs, I define new parameters in Table 36 using the definition of $\Omega(\cdot)$ in Equation (4.20).

Table 36. Derivation Table for parameters ψ_A, ψ_B, ψ_C, ψ_D, ψ_E and ψ_F

$p_1(\Theta, \psi_A) = p_{min}$	$\psi_A = \Omega(1, 0, 2p_{min} - p_{max})$
$p_1(\Theta, \psi_B) = p_2(0)$	$\psi_B = \Omega(3, 0, p_{max})$
$p_1(0, \psi_C) = p_2(0)$	$\psi_C = \Omega(2, 0, p_{max})$
$p_1(\Theta, \psi_D) = p_2(l^*_4)$	$\psi_D = \Omega(-1, 0, p_{max})$
$p_1(i+\nu, \psi_E) = p_2(i+\nu)$	$\psi_E = \Omega(2, 3, p_{max})$
$p_1(\Theta, \psi_F) = p_2(i+\nu)$	$\psi_F = \Omega(3, 4, p_{max})$

Similar to my previous analysis, I plot various cases of $p_2(l)$, and find the maximizer of $\Pi^M_{t,i}(p, l^{M*}_{t,i}(p))$ in each case. I note that the slope of $p_2(l)$ is -2ρ, which indicates that it slopes down two times faster than $p_{max}(l)$. Any $p_2(l)$ line passing through $(p_{max}, 0)$, also passes from $(p_{min}, \frac{l_{max}}{2})$ (which is also $p_2(d, \frac{\kappa_1+\kappa_4}{2})$) as shown in Figure 34.

Consequently, I may obtain different solutions for the case with $i + \nu \leq \frac{l_{max}}{2}$ and $i + \nu > \frac{l_{max}}{2}$, as well as $\Delta\Pi^P_{t,i} \leq \frac{\kappa_1+\kappa_4}{2}$ and otherwise. I consider these cases

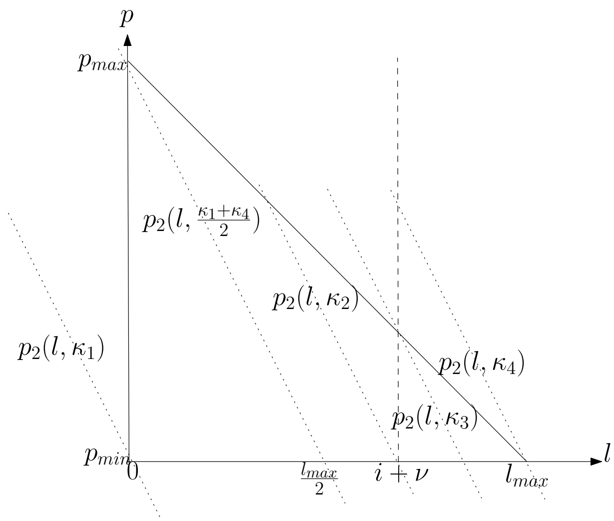


Fig. 34. Illustration of $p_2(d, \frac{\kappa_1+\kappa_4}{2})$

separately. In each figure below, I plot $p_2(l)$ and the corresponding $l_{t,i}^{M*}(p)$ using dashed and solid lines respectively.

Case I. $\Delta\Pi_{t,i}^{P} \leq \kappa_1$:

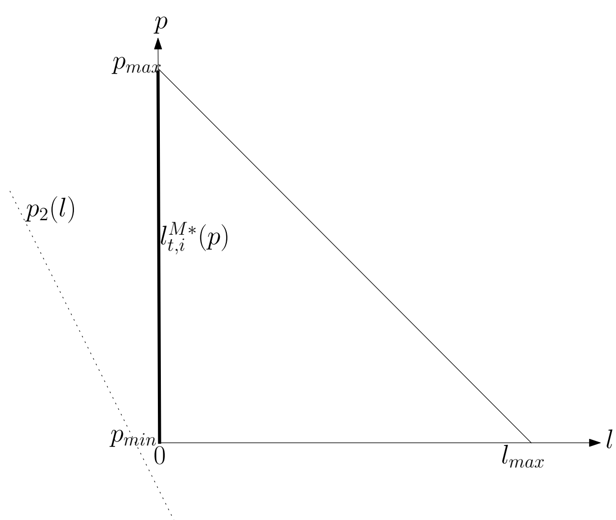


Fig. 35. Representative $p_2(l)$ and $l_{t,i}^{M*}(p)$ curve under Case I.

As observed in Figure 31, I have

$$\Pi^M_{t,i}(p, l^{M*}_{t,i}(p)) = \frac{p_{max} - p}{\rho l_{max}}(p - \Delta\Pi^M_{t,i}) + \alpha\underline{\Pi}^{M*}_{t,i}, \quad p_{min} \leq p \leq p_{max}$$

Recalling that $p_1(0)$ is the maximizer of $l^{M*}_{t,i}(p)$ in this case (see Equation (C.60)), I analyze the possible values of $p_1(0)$ to determine the optimal decision in Table 37.

Table 37. Derivation Table for Case I under setting M

$\Delta\Pi^M_{t,i} \leq \psi_1$	$p_1(0) \leq p_{min}$	$p^{M*}_{t,i} = p_{min}$ (R1.1), and $l^{P*}_{t,i} = 0$
$\psi_1 < \Delta\Pi^M_{t,i} \leq \psi_5$	$p_{min} < p_1(0) \leq p_{max}$	$p^{M*}_{t,i} = p_1(0)$ (R1.2), and $l^{P*}_{t,i} = 0$
$\psi_5 < \Delta\Pi^M_{t,i}$	$p_{max} < p_1(0)$	$p^{M*}_{t,i} = p_{max}$ (R1.3), and $l^{P*}_{t,i} = 0$

Hence, I obtain,

$$(p^{M*}_{t,i}, l^{P*}_{t,i}) = \begin{cases} (p_{min}, 0), & \text{if } \Delta\Pi^M_{t,i} \leq \psi_1, \\ (p_1(0), 0), & \text{if } \psi_1 < \Delta\Pi^M_{t,i} \leq \psi_5, \\ (p_{max}, 0), & \text{if } \psi_5 < \Delta\Pi^M_{t,i}. \end{cases} \tag{C.64}$$

Case II. $\kappa_1 < \Delta\Pi^P_{t,i} \leq \kappa_2$ and $\Delta\Pi^P_{t,i} \leq \frac{\kappa_1+\kappa_4}{2}$:

In this case $\Pi^M_{t,i}(p, l^{M*}_{t,i}(p))$ is in a piecewise form as observed in Figure 36, which is given as follows.

$$\Pi^M_{t,i}(p, l^{M*}_{t,i}(p)) = \begin{cases} \frac{p_{max}-p-\rho\Theta}{2\rho l_{max}}(p - \Delta\Pi^M_{t,i}) + \alpha\underline{\Pi}^{M*}_{t,i} & \text{if } p_{min} \leq p \leq p_2(0), \\ \frac{p_{max}-p}{\rho l_{max}}(p - \Delta\Pi^M_{t,i}) + \alpha\underline{\Pi}^{M*}_{t,i} & \text{if } p_2(0) < p \leq p_{max}, \end{cases} \tag{C.65}$$

where the maximizers of the piecewise functions are $p_1(\Theta)$ and $p_1(0)$ (see Equations (C.60) and (C.61)), respectively. Furthermore, using the inequalities $\kappa_1 < \Delta\Pi^P_{t,i} \leq$

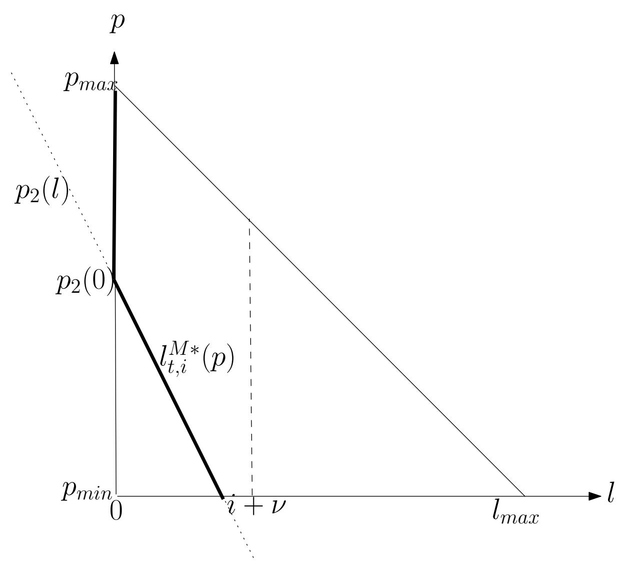


Fig. 36. Representative $p_2(l)$ and $l_{t,i}^{M*}(p)$ curve under Case II.

κ_2 and $i + \nu \leq \frac{l_{max}}{2}$, one obtains

$$\psi_A \leq \psi_B \leq \psi_C \leq \psi_5 \tag{C.66}$$

I analyze each case separately in Table 38.

Table 38. Derivation Table for Case II under setting M

$\Delta\Pi_{t,i}^M \leq \psi_A < \psi_C$	$p_1(\Theta) \leq p_{min}$ and $p_1(0) \leq p_2(0)$	$p_{t,i}^{M*} = p_{min}$ (R2.1), and $l_{t,i}^{P*} = l_1^*$
$\psi_A < \Delta\Pi_{t,i}^M \leq \psi_B < \psi_C$	$p_{min} < p_1(\Theta) \leq p_2(0)$ and $p_1(0) \leq p_2(0)$	$p_{t,i}^{M*} = p_1(\Theta)$ (R2.2), and $l_{t,i}^{P*} = l_7^*$
$\psi_B < \Delta\Pi_{t,i}^M \leq \psi_C$	$p_2(0) < p_1(\Theta)$ and $p_1(0) \leq p_2(0)$	$p_{t,i}^{M*} = p_2(0)$ (R2.3), and $l_{t,i}^{P*} = 0$
$\psi_B < \psi_C < \Delta\Pi_{t,i}^M < \psi_5$	$p_2(0) < p_1(\Theta)$ and $p_2(0) < p_1(0) \leq p_{max}$	$p_{t,i}^{M*} = p_1(0)$ (R2.4), and $l_{t,i}^{P*} = 0$
$\psi_B < \psi_5 < \Delta\Pi_{t,i}^M$	$p_2(0) < p_1(\Theta)$ and $p_{max} < p_1(0)$	$p_{t,i}^{M*} = p_{max}$ (R2.5), and $l_{t,i}^{P*} = 0$

I have,

$$
(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_{min}, l_1^*), & \text{if } \Delta\Pi_{t,i}^M \leq \psi_A, \\ (p_1(\Theta), l_7^*), & \text{if } \psi_A < \Delta\Pi_{t,i}^M \leq \psi_B, \\ (p_2(0), 0), & \text{if } \psi_B < \Delta\Pi_{t,i}^M \leq \psi_C. \\ (p_1(0), 0), & \text{if } \psi_C < \Delta\Pi_{t,i}^M \leq \psi_5, \\ (p_{max}, 0), & \text{if } \psi_5 < \Delta\Pi_{t,i}^M. \end{cases} \tag{C.67}
$$

Case III. $i + \nu \leq l_{max}$, $\kappa_1 < \Delta\Pi_{t,i}^P \leq \kappa_2$ and $\Delta\Pi_{t,i}^P > \frac{\kappa_1+\kappa_4}{2}$:

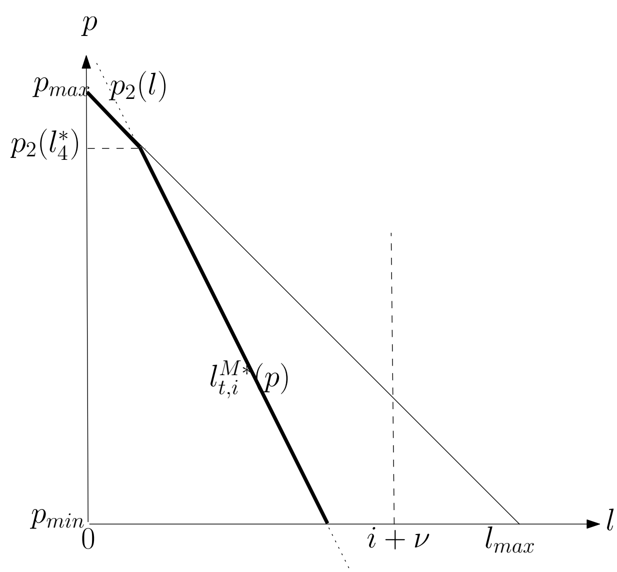


Fig. 37. Representative $p_2(l)$ and $l_{t,i}^{M*}(p)$ curve under Case III.

As shown in Figure 37, I now have

$$
\Pi_{t,i}^M(p, l_{t,i}^{M*}(p)) = \frac{p_{max} - p - \rho\Theta}{2\rho l_{max}}(p - \Delta\Pi_{t,i}^M) + \alpha\underline{\Pi}_{t,i}^{M*}, \quad p_{min} \leq p \leq p_2(l_4^*),
$$

whose maximum is achieved by $p_1(\Theta)$. I derive the solution in Table 39 and Equation (C.68), respectively.

Table 39. Derivation Table for Case III under setting M

$\Delta\Pi_{t,i}^{M} \leq \psi_A$	$p_1(\Theta) \leq p_{min}$	$p_{t,i}^{M*} = p_{min}$ (R1.1), and $l_{t,i}^{P*} = l_1^*$
$\psi_A < \Delta\Pi_{t,i}^{M} \leq \psi_D$	$p_{min} < p_1(\Theta) \leq p_2(l_4^*)$	$p_{t,i}^{M*} = p_1(\Theta)$ (R1.2), and $l_{t,i}^{P*} = l_7^*$
$\psi_D < \Delta\Pi_{t,i}^{M}$	$p_2(l_4^*) < p_1(\Theta)$	$p_{t,i}^{M*} = p_{max}$ (R1.3), and $l_{t,i}^{P*} = 0$

$$(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases} (p_{min}, l_1^*), & \text{if } \Delta\Pi_{t,i}^{M} \leq \psi_A, \\ (p_1(\Theta), l_7^*), & \text{if } \psi_A < \Delta\Pi_{t,i}^{M} \leq \psi_D, \\ (p_{max}, 0), & \text{if } \psi_D < \Delta\Pi_{t,i}^{M}. \end{cases} \tag{C.68}$$

Case IV. $i + \nu \leq l_{max}$, $\kappa_2 < \Delta\Pi_{t,i}^{P} \leq \kappa_3$ and $\Delta\Pi_{t,i}^{P} > \frac{\kappa_1+\kappa_4}{2}$:

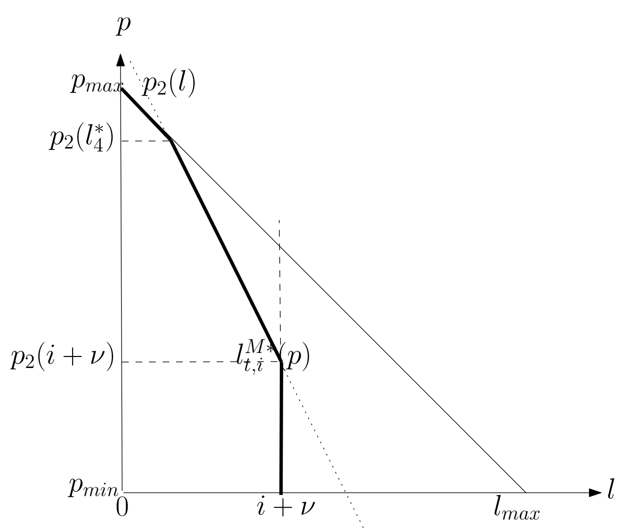


Fig. 38. Representative $p_2(l)$ and $l_{t,i}^{M*}(p)$ curve under Case III.

I now have,

$$\Pi_{t,i}^{M}(p, l_{t,i}^{M*}(p)) = \begin{cases} \frac{p_{max}-p-\rho\Theta}{2\rho l_{max}}(p - \Delta\Pi_{t,i}^{M}) + \alpha\underline{\Pi}_{t,i}^{M*}, & \text{if } p_{min} \leq p \leq p_2(i+\nu), \\ \frac{p_{max}-p-\rho(i+\nu)}{\rho l_{max}}(p - \Delta\Pi_{t,i}^{M}) + \alpha\underline{\Pi}_{t,i}^{M*}, & \text{if } p_2(i+\nu) \leq p \leq p_2(l_4^*). \end{cases} \tag{C.69}$$

Unlike previous piecewise cases, unfortunately conditions of Remark 2 does not hold in this case. Hence, I analyze $p_{min} \leq p \leq p_2(i+\nu)$ and $p_2(i+\nu) \leq p \leq p_2(l_4^*)$ separately and find the local maximizers in each case that are denoted as p_1^* and p_2^*, respectively. I have,

$$p_1^* = \begin{cases} p_{min}, & \text{if } \Delta\Pi_{t,i}^{M} \leq \psi_2, \\ p_1(i+\nu), & \text{if } \psi_2 < \Delta\Pi_{t,i}^{M} \leq \psi_E, \\ p_2(i+\nu), & \text{if } \psi_E < \Delta\Pi_{t,i}^{M}, \end{cases} \tag{C.70}$$

$$p_2^* = \begin{cases} p_2(i+\nu), & \text{if } \Delta\Pi_{t,i}^{M} \leq \psi_F, \\ p_1(\Theta), & \text{if } \psi_F < \Delta\Pi_{t,i}^{M} \leq \psi_D, \\ p_{max}, & \text{if } \psi_D < \Delta\Pi_{t,i}^{M}, \end{cases} \tag{C.71}$$

where, the derivations are given in Table 40.

Finally, the optimal decision can be found by the local maximum that gives higher revenues, i.e.,

$$p_{t,i}^{M*} = \text{argmax}\{\Pi_{t,i}^{M}(p_1^*, l_{t,i}^{M*}(p_1^*)), \Pi_{t,i}^{M}(p_2^*, l_{t,i}^{M*}(p_2^*))\}. \tag{C.72}$$

Case V. $i+\nu \leq l_{max}$, $\kappa_2 < \Delta\Pi_{t,i}^{P} \leq \kappa_3$ and $\Delta\Pi_{t,i}^{P} \leq \frac{\kappa_1+\kappa_4}{2}$:

Table 40. Derivation Table for Case IV under setting M

$\Delta\Pi_{t,i}^M \le \psi_2$	$p_1(i+\nu) \le p_{min}$	$p_1^* = p_{min}$ (R1.1)
$\psi_2 < \Delta\Pi_{t,i}^M \le \psi_E$	$p_{min} < p_1(i+\nu) \le p_2(i+\nu)$	$p_1^* = p_1(i+\nu)$ (R1.2)
$\psi_E < \Delta\Pi_{t,i}^M$	$p_2(i+\nu) < p_1(i+\nu)$	$p_1^* = p_{max}$ (R1.3)
$\Delta\Pi_{t,i}^M \le \psi_F$	$p_1(\Theta) \le p_2(i+\nu)$	$p_2^* = p_2(i+\nu)$ (R1.1)
$\psi_F < \Delta\Pi_{t,i}^M \le \psi_D$	$p_2(i+\nu) < p_1(\Theta) \le p_2(l_4^*)$	$p_2^* = p_1(\Theta)$ (R1.2)
$\psi_D < \Delta\Pi_{t,i}^M$	$p_2(l_4^*) < p_1(\Theta)$	$p_2^* = p_{max}$ (R1.3)

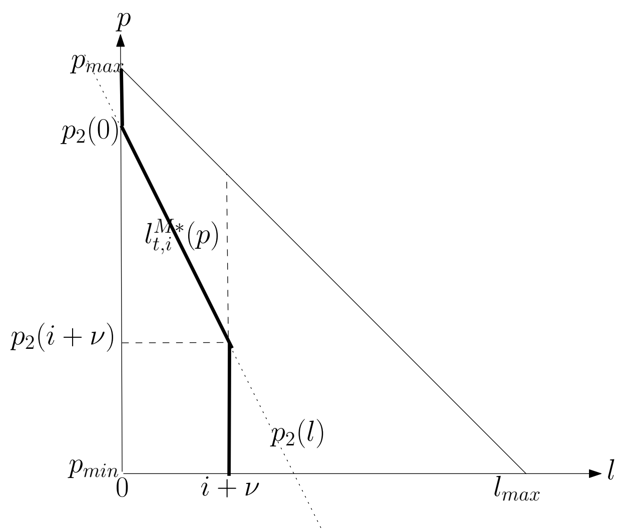


Fig. 39. Representative $p_2(l)$ and $l_{t,i}^{M*}(p)$ curve under Case III.

As shown in Figure 39, I now have three piecewise functions, and

$$\Pi_{t,i}^M(p, l_{t,i}^{M*}(p)) = \begin{cases} \frac{p_{max}-p-\rho(i+\nu)}{\rho l_{max}}(p - \Delta\Pi_{t,i}^M) + \alpha\underline{\Pi}_{t,i}^{M*}, & p_{min} \le p < p_2(i+\nu), \\ \frac{p_{max}-p-\rho\Theta}{2\rho l_{max}}(p - \Delta\Pi_{t,i}^M) + \alpha\underline{\Pi}_{t,i}^{M*}, & p_2(i+\nu) \le p \le p_2(0), \\ \frac{p_{max}-p}{\rho l_{max}}(p - \Delta\Pi_{t,i}^M) + \alpha\underline{\Pi}_{t,i}^{M*}, & \text{if } p_2(0) < p \le p_{max}. \end{cases} \tag{C.73}$$

I follow a similar approach to that of Case IV. I know the local maximizers for the range $p_{min} \leq p < p_2(i+\nu)$ and $p_2(i+\nu) \leq p < p_{max}$. The local maximizer for the former is shown to be p_1^* (see Equation (C.70)). The local maximizer of the latter is denoted as p_3^* and given in Equation (C.74).

$$p_3^* = \begin{cases} p_2(i+\nu), & \text{if } \Delta\Pi_{t,i}^M \leq \psi_F, \\ p_1(\Theta), & \text{if } \psi_F < \Delta\Pi_{t,i}^M \leq \psi_B, \\ p_2(0), & \text{if } \psi_B < \Delta\Pi_{t,i}^M \leq \psi_C, \\ p_1(0), & \text{if } \psi_C < \Delta\Pi_{t,i}^M \leq \psi_5, \\ p_{max}, & \text{if } \psi_5 < \Delta\Pi_{t,i}^M. \end{cases} \tag{C.74}$$

Finally, the optimal decision can be found by (C.75).

$$p_{t,i}^{M*} = \text{argmax}\{\Pi_{t,i}^M(p_1^*, l_{t,i}^{M*}(p_1^*)), \Pi_{t,i}^M(p_3^*, l_{t,i}^{M*}(p_3^*))\}. \tag{C.75}$$

Case VI. $i+\nu > l_{max}$, $\kappa_1 < \Delta\Pi_{t,i}^P \leq \kappa_4$ and $\Delta\Pi_{t,i}^P > \frac{\kappa_1+\kappa_4}{2}$:

The results for this case follows from that of Case III and the optimal solution is given in Equation (C.68).

I sum up the results in three general cases. If $i+\nu \leq \frac{l_{max}}{2}$ and $\Delta\Pi_{t,i}^P \leq \kappa_2$, then Cases I and II are considered, and I obtain.

$$
(p_{t,i}^{M*}, l_{t,i}^{P*}) = \begin{cases}
(p_{min}, 0), & \text{if } \Delta\Pi_{t,i}^{P} \le \kappa_1, \Delta\Pi_{t,i}^{M} \le \psi_1, \\
(p_{min}, l_1^*), & \text{if } \kappa_1 < \Delta\Pi_{t,i}^{P} \le \kappa_2, \Delta\Pi_{t,i}^{M} \le \psi_A, \\
(p_1(0), 0), & \text{if } \Delta\Pi_{t,i}^{P} \le \kappa_1, \psi_1 < \Delta\Pi_{t,i}^{M} \le \psi_5, \text{ or} \\
 & \kappa_1 < \Delta\Pi_{t,i}^{P} \le \kappa_2, \psi_C < \Delta\Pi_{t,i}^{M} \le \psi_5, \\
(p_1(\Theta), l_7^*), & \text{if } \kappa_1 < \Delta\Pi_{t,i}^{P} \le \kappa_2, \psi_A < \Delta\Pi_{t,i}^{M} \le \psi_B, \\
(p_2(0), 0), & \text{if } \kappa_1 < \Delta\Pi_{t,i}^{P} \le \kappa_2, \psi_B < \Delta\Pi_{t,i}^{M} \le \psi_C. \\
(p_{max}, 0), & \text{otherwise.}
\end{cases} \tag{C.76}
$$

The optimal solution can be found as follows in the remaining cases:

- If $i + \nu \le \frac{l_{max}}{2}$ and $\kappa_2 < \Delta\Pi_{t,i}^{P} \le \kappa_3$ then the optimal solution can be found as given in Case V.

- If $\frac{l_{max}}{2} < i + \nu \le l_{max}$ and $\Delta\Pi_{t,i}^{P} \le \kappa_2$, then the optimal solution can be found as given in Case III.

- If $\frac{l_{max}}{2} < i + \nu \le l_{max}$ and $\kappa_2 < \Delta\Pi_{t,i}^{P} \le \kappa_3$, then the optimal solution can be found as given in Case IV.

- If $i + \nu > l_{max}$ then the optimal solution can be found as given in Case VI.

□