Mathematical Development: The Role of Broad Cognitive Processes

by

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ABSTRACT

This study investigated the role of broad cognitive processes in the development of mathematics skills among children and adolescents. The participants for this study were a subsample of a nationally representative sample used in the standardization of the Woodcock-Johnson III Tests of Cognitive Abilities and the Woodcock-Johnson III Tests of Achievement, Normative Update (Woodcock, McGrew, & Mather, 2007). Participants were between 5 years old and 18 years old (N = 4721; mean of 10.98 years, median of 10.00 years, standard deviation of 3.48 years), and were 50.7% male and 49.3% female. Structural equation models supported the theoretical suggestion that broad cognitive processes play significant and specific roles in the development of mathematical skills among children and adolescents. Implications for school psychology researchers and practitioners are discussed.

DEDICATION

To my wife, Brenda

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TABLE OF CONTENTS

CHAPTER Page

CHAPTER Page

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

Mathematics is a system for representing and thinking about quantitative information. The foundations of mathematics are the concepts of number and arithmetic operations (Gallistel & Gelman, 2005). Indeed, the importance of mathematics in human life cannot be overstated. It is one of the most formalized human systems, serving as the basis for many activities ranging from simple daily living skills, such as counting and trading, to highly specialized activities, such as scientific research and technology development.

It is reasonable to speculate that the origins of the skills necessary for human mathematics are embedded in our evolutionary history, biologically and culturally, along with the development of language and abstract thought. Basic mathematical abilities are shared by a number of mammals, particularly primates (Gallistel & Gelman, 2005). Human mathematical abilities emerge during infancy, and continue to develop throughout childhood (Gallistel & Gelman, 2005). Some aspects of mathematical conventions are nearly universal among modern civilizations, such as the use of Arabic numerals and the decimal system, but other aspects are more culture-specific, such as the names of numerals (e.g., the numeral 80 in French is called “quatre-vingts,” literally four-twenties) and geometric figures (e.g., the Chinese name for triangle is literally “three corner shape”). Cultural affordances (as defined by Kitayama and Markus, 1999) may present learning opportunities that may impact the development of mathematical skills. Thus, the development of mathematical skills is probably deeply interconnected with both ontogenetic factors, as well as cultural ones.

The study of mathematical development is a relatively new area of scientific research. However, the importance of this subject will only increase, as human societies are increasingly dependent on information and mathematics-based technologies. It is in the best interest of societies to prepare all students with foundational mathematics skills and to serve the needs of a widely diverse student body. On the one hand, educators need to provide opportunities for exceptionally talented students (e.g., mathematically gifted students) to pursue advanced, fast paced studies in mathematics. On other hand, educators need to provide opportunities for students with disabilities to overcome or compensate specific deficits in order to achieve their potential in mathematics and other areas. In the United States, there are approximately 50 million children and adolescents enrolled in public schools, with 44 million of them attending school on any given day (National Center for Education Statistics, 2005). These figures do not include youth enrolled in private schools. According to the National Institute of Mental Health (2004), approximately 10% of students in the United States will suffer from a psychological disorder affecting their learning experiences at some point during their school years. The cumulative incidence of mathematics learning disabilities in children up to 19 years is as high as 9.8% using aptitude-achievement discrepancy definitions, with a male-female ratio of 2:1 (Shalev, 2007). Thus, the study of mathematical development is relevant both as a normative issue, as well as a special case of psychoeducational disorders.

Chapter 2

Background Literature

Genetic Factors

Mathematical developmental trajectories and mathematical achievement are related to genetic, neuropsychological, and cognitive factors (Geary, 1993; Fuchs et al., 2010). Possible hereditary influences on mathematics achievement were documented as early as the 1950s. However, the first empirical studies on genetic influences on mathematics achievement were not conducted until the mid 1990s (Gersten, Clarke, & Mazzocco, 2007). Quantitative genetic analysis allows researchers to examine the magnitude of genetic and environmental influences on individual differences in mathematics performance. This method assumes that mathematics ability influences can be divided into genetic and environmental variance components by comparing family members who vary in their "relatedness" degree, as well as their environment (Hart, Petrill, Thompson, & Plomin, 2009). For example, monozygotic twins are genetically identical (i.e., they share 100% of their genetic variance), whereas dizygotic twins share approximately 50% of their genetic variance. If monozygotic twins are more similar in mathematics performance, relative to dizygotic twins, then a genetic factor may be supported as an explanation for the difference. Interestingly, even monozygotic twins sharing 100% of their genetic variance, as well as very similar environments, will not share 100% of their achievement variance, supporting a "non-shared" environmental role explanation.

Geneticists abbreviate “heritability” as h^2, which is a term that denotes the magnitude of a genetic influence on a phenotypic outcome. “Shared environment” is abbreviated as c^2, which is a term that denotes the proportion of variance in an outcome that is due to shared environments. These shared environments can include similar experiences at home, school, or even in the womb. “Non-shared environment” is abbreviated as e^2, which refers to different environmental factors that result in differential effects on family members. For example, siblings may grow up experiencing different family environments (e.g., one sibling grows up with both parents, while another sibling grows up with only one parent). In the case of mathematics achievement, the extant empirical literature suggests that heritability and environmental influences are approximately the same for the general population and for special subpopulations at either end of the normal distribution (Kovas, Haworth, Petrill, & Plomin, 2007). In essence, mathematics difficulties and disabilities fall on the far-left end point of the normal distribution, while exceptionally high mathematical abilities fall on the far-right end point of the normal distribution. The proportion of genetic influences (h^2), following this quantitative methodology, has been estimated to range from .4 to .7 approximately (Kovas et al., 2007).

It is important to note that the literature does not imply that there will be a single gene found to “cause” mathematics abilities or disabilities. Rather, mathematical development, like other complex developmental phenomena, is probably the result of polygenic influences, multiple environmental influences, and interactions among these variables.

Neuropsychological Factors

The study of neuroanatomy and its relation to cognition emerged from brain-behavior studies in the early ninetieth century (Hallahan & Mercer, 2001). In 1802, Gall explained relationships between brain injuries and mental impairments in living patients (Hallahan & Mercer, 2001). He hypothesized that the brain consisted of three independent parts, which controlled, respectively, movement and sensation, morality (what modern psychologists may call executive functions such as self-control), and intellect. The notion of specific cognitive disabilities emerged from physicians documenting cases of patients with normal intellectual abilities, but with difficulties in specific cognitive tasks. For example, in the 1860s, Broca concluded that a section in the inferior left frontal lobe was responsible for speech abilities (Hallahan & Mercer, 2001). This section became known as Broca's area. Damage to Broca's area often results in slow, laborious, dysfluent speech, a condition termed Broca's aphasia. However, research on the neuropsychology of mathematics had to wait until the twentieth century (Gersten, Clarke, & Mazzocco, 2007).

Mathematics disabilities (named "acalculia" by Henschen in 1919, particularly in the context of "acquired" mathematics disabilities) is often the result of disruptions within several brain regions, including the frontal, temporal, and parietal lobes in the left, as well as the right hemisphere (Kahn & Whitaker, 1991). Henschen's studies demonstrated that some patients' mathematics disabilities were independent of their linguistic abilities, while other patients' mathematics disabilities seemed to be related to their linguistic abilities.

Henschen's neuropsychological studies paved the way for theoretical debates regarding the functional independence/interdependence between language and mathematics abilities at the neuroanatomical level (Gersten et al., 2007). Cognitive and neuropsychological research is converging on the conclusion that both developmental and acquired acalculia stem from similar neurological disruptions, which impair mathematical cognitive processes (Geary, 1993).

Domain Specific Cognitive Processes

Mathematical development is related to domain specific cognitive processes (cognitive processes that are specific to mathematics), as well as broad cognitive processes such as working memory, language, and reasoning (Fuchs et al., 2010). Domain specific cognitive processes include: Subitizing or numerosity (i.e., the ability to automatically and accurately determine the quantity of sets of up to three or four items; Wynn, Bloom, & Chiang, 2002); magnitude estimation (i.e., the inexact, but quick estimation of quantities larger than 3 or 4; Pica et al., 2004); ordinality (i.e., understanding of the concepts "more than" and "less than"; Feigenson, Carey, & Hauser, 2002); counting (i.e., understanding of counting principles such as one to one correspondence; Gelman & Gallistel, 1978); arithmetic sensitivity (i.e., sensitivity to increases and decreases in the quantity of small sets of items; Kobayashi, Hiraki, Mugitani, & Hasegawa, 2004); and geometry (i.e., basic understanding of spatial relations; Dehaene, Izard, Pica, & Spelke, 2006).

These domain specific processes (and perhaps others currently unknown) are the foundation for early aspects of mathematical learning in school (Von Aster

& Shalev, 2007). Children transition from these automatic, basic quantitative competencies (biological evolution-based; Geary, 2007) to formal, complex mathematical competencies (culturally-based), such as counting words (vocabulary), Arabic numerals, the decimal system, and the rest of mathematical theory through broad, domain-general cognitive processes, including reasoning, language, memory, visual and auditory processes, among others (Geary, Hoard, Nugent, & Byrd-Craven, 2008). However, the specific contribution of broad cognitive processes on mathematics achievement is not clearly understood, particularly from a developmental perspective (Fuchs et al., 2010).

Broad Cognitive Processes

General, fluid, and crystallized intelligence. Psychologists have documented the predictive utility of general intelligence (commonly known today as IQ – intellectual quotient) for over a hundred years (see Spearman, 1904). General intelligence has been correlated to academic achievement (including mathematics), level of education attained, socio-economic status, income, longevity, health-related behaviors, among other life outcomes (e.g., Brody, 1997). Empirical evidence suggests that general intelligence is related to both genetic, as well as environmental factors (i.e., the answer to the "nature versus nurture" question appears to be "both"; Sternberg, Grigorenko, & Kidd, 2005). General intelligence or IQ is a complex psychological construct described as an "incomplete definition of intelligence" by leading intelligence researchers, including those who accept its predictive utility (Carroll, 1993; Sternberg, Grigorenko, & Kidd, 2005). However, intelligence tests are widely accepted as an important technological achievement and as a powerful tool because of their utility (Carroll, 1993), despite a history of misuse (see Cooper, 2005 in the American Psychologist special issue on genetics, race, and psychology), and despite the enormous philosophical and empirical difficulty of defining intelligence and cognition.

Indeed, cognition may be one of three fundamentally unsolvable philosophical problems. The first unsolvable problem is the existence of matter (i.e., "why does the universe go to all the bother of existing," in the words of Hawking, 1988), although recently, Krauss (2012) has argued that the problem of "something out of nothing" is a solvable empirical problem. The second unsolvable problem is the existence of universal properties or constants. For example, why is the speed of light approximately 300,000,000 meters per second (under specific circumstances), and not twice, or half, that amount (under the same circumstances). The third unsolvable problem is cognition: How does cognition occur (i.e., how is matter-energy transformed into sensory information, consciousness, the experience of qualia, etc.)? Thus, cognition will be practically defined here as brain-based functions that allow animals (in this case, humans) to solve problems relevant to their environment (in this case, mathematics).

General intelligence is usually conceptualized today as a higher order construct related to subordinate, specific cognitive processes (Carroll, 1993). Spearman argued that IQ is a unitary construct, which he termed *g* for general intelligence. Subsequently, Cattell and Horn argued that Spearman's *g* should be divided into two equally important, but distinct cognitive abilities (Cattell, 1963;

Horn, 1968; Horn & Cattell, 1966). The first ability is called crystallized intelligence (Gc) and it deals with information that is the result of repeated experiences, such as schooling and acculturation. Gc manifests itself primarily through over-learned skills such as vocabulary and knowledge. The second ability is called fluid intelligence (Gf), and it deals with information that is the result of novel experiences, which require inductive reasoning (finding patterns and creating concepts) and deductive reasoning (solving problems through logical, sequential steps).

The ability to solve mathematics problems involving verbal and general information (e.g., solving the problem, "If Joe has twice as much money as Jane, and Jane has three times as much money as Pedro, how many books can Joe buy, if Pedro has $1.00, and each book costs $2.50?) has been consistently associated with Gc and Gf (McGrew & Wendling, 2010). This is not surprising given that Gc is a measure of crystallized knowledge (i.e., verbal and general information), and Gf is a measure of analytical and logical skills (i.e., reasoning), which allow for solving problems involving relatively novel information (a "word problem" as opposed to a given equation). Interestingly, the association between problem solving and Gc has been observed to increase with age, while the association between problem solving and Gf has been observed to decrease with age (McGrew & Wendling, 2010). This may be due to the fact that, as children and adolescents are exposed to formerly "novel" problems, with time they develop strategies and procedures (mathematical schemas) to solve logically similar word problems, which become readily accessible via Gc, requiring less use of Gf abilities. Conversely, younger children (children who have less experiential knowledge to draw from, less vocabulary, and less acculturation in general) will resort to reasoning abilities relatively more. This differential Gf-Gc trend may be explained by Cattell's investment hypothesis (Cattell, 1987): Individual differences in acquisition of knowledge and skills are partly the result of investment of fluid intelligence (Gf) in learning situations requiring insights in complex relations.

Working memory, auditory, and visual processing. Working memory is the ability to hold information in immediate awareness, manipulate it, and retrieve a product (e.g., hold two quantities in immediate awareness, add them, and produce a result). Baddeley and Hitch (1974) conceptualized working memory as a multicomponent cognitive device comprised of three systems: the central executive (the "central processor," which carries out the operations), the phonological loop (the "auditory processor," which organizes verbal information and feeds it to the central processor), and the visual-spatial sketchpad (the "visual processor," which organizes visual information and feeds it to the central processor). Calculation skills and problem solving have been theorized to depend, in part, on the working memory central executive, the phonological loop, and the visual-spatial sketchpad (Geary & Widaman, 1992; Hitch, 1978; Swanson, Cooney, & Brock, 1993). It is reasonable to hypothesize that the working memory central executive is engaged during translation of word problem sentences into equations, and in executing arithmetic steps.

Empirically, working memory has been consistently associated with calculation skills (arithmetic and equation problem solving) and problem solving at all ages (McGrew & Wendling, 2010). Phonological processing has been conceptualized as an independent cognitive process (parallel to working memory; Carroll, 1993), and it has been associated with calculation skills at ages 6-13, and with problem solving at ages 6-19. However, this latter association becomes less consistent as age increases (McGrew & Wendling, 2010). The decrease of phonological processing in problem solving may be due to a process similar to the one described earlier, where Gc skills are increasingly used to solve cognitive problems that required other cognitive processes at younger ages (e.g., as vocabulary develops, processing individual phonemes accurately and efficiently may become less important; if true, this phenomenon may be related to the empirical finding that people lose the ability to distinguish sounds not used in their first language as they age).

Visual-spatial processing (as a component of working memory, or as an independent cognitive process) remains a factor theorized to support basic calculation skills, as well as advanced mathematics and geometry; however, empirical evidence is lacking or contradictory (Geary, 1993; Fuchs et al., 2010; McGrew & Wendling, 2010). From an evolutionary perspective, visual-spatial cognitive processes probably precede auditory cognitive processes, and they are most likely highly elaborated and robust given the importance of visual information for primates. Sophisticated linguistic abilities are a relatively new phenomenon, and therefore, it may be easier, statistically speaking, to encounter auditory cognitive disruptions than visual cognitive disruptions among humans.

Learning and long-term retrieval. The ability to integrate new information with previously learned information, and store it in long-term memory (associative memory), as well as to retrieve previously learned information efficiently to solve problems (retrieval fluency) has been associated with calculation and equation problem solving among children and adolescents (Floyd, Evans, & McGrew, 2003). In order to solve calculations, from basic arithmetic and fractions, to algebra and calculus, students must learn new mathematical concepts and procedures, integrate new information with the rest of their knowledge, and retrieve that knowledge to solve subsequent problems. Furthermore, children with associative memory and/or long-term retrieval dysfunctions may have difficulty performing the transition described by Geary (2007), from basic quantitative skills (e.g., subitizing) to formalized mathematics skills taught in school, resulting in a profound learning developmental disability. For example, whereas deficits in working memory may be supported with memory aids during calculation operations, and deficits in reasoning abilities may be circumvented by memorizing procedural "cheat sheets," it is reasonable to speculate that a dysfunction in associative memory and/or long-term retrieval may result in comorbid learning disabilities in mathematics, reading, and other academic skills, which may require comprehensive, intensive interventions and supports.

Processing speed. Processing speed is defined as the efficiency to fluently perform cognitive tasks (Geary, 1993). Commonly used indicators of processing speed include perceptual processing speed (e.g., asking subjects to look at a series of digits and circle the two digits that are identical as quickly as they can), and semantic processing speed (e.g., asking subjects to look at a series of object drawings and circle the two drawings that "go together," which requires the identification of common semantic categories such as "food," "clothes," "things in the sky," etc.). Processing speed has been empirically linked to domain specific quantitative skills related to counting (Geary, 1993), the amount of time required to solve calculations, and problem solving (Fuchs et al., 2010; McGrew & Hessler, 1995; McGrew & Wendling, 2010).

The Cattell-Horn-Carroll model of cognitive processes. One of the most comprehensive and empirically supported models of cognitive processes is the Cattell-Horn-Carroll (CHC) model. The CHC model integrates the Cattell-Horn fluid intelligence/crystallized intelligence (Gf-Gc) model of cognitive abilities (Cattell, 1963; Horn, 1968; Horn & Cattell, 1966) and the Carroll three-stratum model of cognitive abilities (Carroll, 1993). Carroll expanded on the Cattell-Horn Gf-Gc model and proposed a three-stratum model, which contains more than 70 narrow cognitive abilities in the first stratum (e.g., inductive reasoning, deductive reasoning, associative memory, etc.), at least seven second-order broad factors in the second stratum (e.g., Gf, Gc, etc.), and one general intelligence third-order factor (i.e., G). Although the number of accepted broad second-order factors varies from 7 to 10, depending on slightly different conceptualizations (see Carroll, 1993; McGrew, 2005; McGrew, 2009; McGrew & Wendling, 2010), the following seven broad cognitive factors (and their corresponding narrow abilities in parentheses) are usually measured in the context of cognitive evaluations: Fluid reasoning (Gf; inductive and deductive reasoning), crystallized knowledge (Gc; lexical and general knowledge), short-term memory (Gsm; working memory and memory span), auditory processing (Ga; phonetic coding synthesis and speech-noise discrimination), visual-spatial processing (Gv; spatial operations and visual memory), long-term retrieval (Glr; associative memory and retrieval fluency), and processing speed (Gs; perceptual processing speed and semantic processing speed). Additional CHC broad factors (Woodcock, Mather, & McGrew, 2001) include reading and writing ability (Grw) and quantitative knowledge (Gq). However, these two factors are usually considered achievement outcomes in the context of cognitive-academic assessment and research, rather than cognitive factors (McGrew & Wendling, 2010). Lastly, an additional factor termed reaction time (Gt) refers to an individual's reaction time to the onset of a visual or auditory stimulus (Carroll, 1993). Reaction time is not widely used in clinical assessment, but it is commonly used in certain basic research areas, such as social psychology (e.g., Schmidt & Nosek, 2010). Additional CHC factors have been recently postulated by McGrew (see McGrew, 2009).

The CHC Model and Mathematics Achievement

To date, some of the most comprehensive exploratory studies to assess the relationships between CHC factors and mathematics achievement across the school-age years include Floyd, Evans, and McGrew (2003); McGrew and Hessler (1995); and Taub, Floyd, Evans, and McGrew (2008). Floyd et al. found that, in a nationally representative sample of students of ages 6 to 19, crystallized knowledge (Gc) demonstrated moderate relations with calculation skills, and moderate to strong relations with mathematics reasoning (increasing with age). Fluid reasoning (Gf) demonstrated moderate relations with calculation skills and mathematics reasoning (increasing with age, then decreasing in latter age groups). Short-term memory (Gsm), and more specifically, working memory, generally demonstrated moderate relations with calculation skills and mathematics reasoning (constant across age groups). Processing speed (Gs) demonstrated moderate to strong relations with calculation skills (generally constant across age groups), and moderate relations with mathematics reasoning during the elementary school years only. Long-term retrieval (Glr) demonstrated moderate relations with calculation skills and mathematics reasoning during the early school years. Auditory processing (Ga) demonstrated moderate relations with calculation skills during the early school years. Visual processing (Gv) generally demonstrated nonsignificant relations with calculation skills and mathematics reasoning.

This is an important, landmark study. However, the study utilized cluster scores (e.g., calculation skills) as dependent variables rather than individually observed variables (e.g., calculation complexity and calculation fluency). It is possible that this may have obscured some variable associations in the analyses. Similarly, CHC factor cluster scores were used as independent variables in the regression equations. Structural equation modeling may be employed in future studies to estimate each CHC factor as a latent variable derived from individually observed cognitive variables (e.g., a latent variable of Gc can be estimated from the observed variables of lexical knowledge and general knowledge). Such an analytic approach (using structural equation modeling) was conducted by Taub, Floyd, Evans, and McGrew (2008). These researchers utilized more than the two commonly measured narrow cognitive abilities to estimate each CHC broad factor. For example, Gc was estimated using measures of lexical knowledge, general knowledge, academic knowledge, oral comprehension, picture vocabulary, and story recall. However, the dependent variable, mathematics achievement, consisted of a single latent variable estimated from one measure of calculation complexity and one measure of problem solving. This approach mixes basic calculation skills with mathematics reasoning skills. Future studies should distinguish basic calculation skills and mathematics reasoning skills in order to better understand the associations between cognitive processes and mathematics development.

Last, a study by Proctor, Floyd, and Shaver (2005) examined the CHC cognitive profiles of students with low mathematics achievement. The study found that approximately half of the children with normative delays in mathematics reasoning exhibited commensurate normative delays in one or more cognitive abilities, most often including fluid reasoning and crystallized knowledge. This is a seminal study in the CHC-based mathematics learning disability diagnosis literature.

Mathematical Development and Cognitive Development

One of the best established theories of cognitive development is Piaget's theory of genetic epistemology (Piaget, 1961). Piaget's theoretical perspective provides a useful framework for the study of mathematical development (Ojose, 2008). Piaget postulated four primary stages of cognitive development: Sensorimotor (birth to 2 years old), preoperational (2 to 7 years old), concrete operational (7 to 11 years old), and formal operational (11 years old and older). Regarding mathematics, during the sensorimotor stage, infants start to display domain-specific mathematical abilities, such as displaying some understanding of the concept of number and counting. In the preoperational stage, children's language abilities allow them to make concept associations (with over-generalizations) and to begin to engage in symbolic thought. However, there is a lack of reversibility at this stage (e.g., children can add two plus three, but they cannot subtract three from five). Further, preoperational children can consider one dimension at a time only (e.g., in a classical experiment consisting of transferring a certain amount of liquid from a short, wide container into a long, thin container, preoperational children concluded that the amount of the liquid increased, given that the height of the liquid increased in the long, thin container, relative to the short, wide container). During the concrete operational stage, children's reasoning and language skills increase dramatically. Reversibility is achieved, and more than one dimension can be considered simultaneously. Children in this stage rely on their senses in order to know (i.e., they engage in concrete reasoning). Lastly, during the formal operational stage, adolescents achieve the capacity for abstract reasoning (e.g., they do not need concrete examples in order to solve problems, they can make logical inferences, they can evaluate and apply information, etc.).

Thus, in general, children in the sensorimotor stage will probably display mostly domain-specific mathematics skills. Children in the preoperational stage can be expected to rely on associative memory and long-term retrieval skills (Glr) in order to associate concepts and learn basic calculation skills. There probably is a significant increase in Gc (mostly due to language development) and Gf skills (including qualitative changes in reasoning described by Piaget, such as reversibility) among concrete operational children. Because their reasoning is mostly concrete, concrete operational children will rely on their visual and auditory processing skills during calculation tasks. Last, adolescents in the formal operational stage will continue to rely on Gc and Gf skills, but Gf skills may become less significant as Gc skills (i.e., over-learned skills) start to take over Gf skills (i.e., skills related to reasoning with novel information or situations). This hypothesis is consistent with the observations of Floyd, Evans, and McGrew (2003), who documented relatively weaker associations between Gf and mathematics reasoning among late formal operational adolescents (ages 17 to 19), relative to early formal operational adolescents (ages 12 to 16).

Recent developmental models of mathematics learning (Fuchs et al., 2010; Geary, 2007; Geary, Hoard, Nugent, & Bailey, 2012) emphasize the role of Gf, Gc, Gsm (working memory central executive in particular), Gs, Glr, and Gv (as a subcomponent of working memory) in mathematics achievement. Geary (2007) made the distinction between primary (biological) mathematical competencies and secondary (cultural) mathematical competencies learned through acculturation and schooling. Geary et al. (2012) provided empirical evidence suggesting that differences between low mathematics achievers and students diagnosed with mathematics learning disabilities are mediated by Glr (retrieval fluency in particular) and Gsm (central executive in particular). Last, Fuchs et al. (2008) provided empirical evidence suggesting that both mathematics-specific and broad cognitive factors are related to mathematics developmental trajectories.

Proposed Model of Broad Mathematical Cognition

General Model Hypotheses

Given the extant empirical and theoretical literature, the following model of mathematical cognition was proposed (Figure 1). The development of basic calculation skills in general was hypothesized to be related to associative memory and retrieval fluency (Glr as a broad cognitive factor), processing speed (specifically, perceptual processing speed), short-term memory (specifically, working memory), auditory processing (specifically, phonetic coding synthesis), and visual-spatial processing (specifically, spatial operations; Geary, 1993; Fuchs et al., 2010 ; McGrew & Wendling, 2010). More specifically, the *complexity* of calculations one is able to solve was hypothesized to be related to associative memory and retrieval fluency (Glr), and working memory. Associative memory was hypothesized to be related to the level of learning achieved (simple arithmetic, fractions, algebra, calculus), while retrieval fluency and working memory were hypothesized to be related to calculation performance (correct responses). The *fluency* or speed with which one is able to solve calculations was hypothesized to be related to perceptual processing speed, as well as phonetic coding synthesis and spatial operations, which support the ability to carry out operations by working memory (Fuchs et al., 2010). Calculation complexity and calculation fluency were hypothesized to be intercorrelated given that working memory and perceptual processing speed tend to be associated (Fuchs et al., 2010), as well as the fact that calculation fluency is based, in part, on basic calculation facts mastery. The development of problem solving skills was hypothesized to be related to fluid reasoning (Gf) and crystallized knowledge (Gc) as broad cognitive factors, as well as working memory and perceptual processing speed (Geary 2007; McGrew & Wendling, 2010). Problem solving was hypothesized to be correlated with calculation complexity and calculation fluency due to their shared dependency on working memory processes, as well as the fact that mathematics problem solving is based, in part, on basic calculation skills (Fuchs et al., 2010).

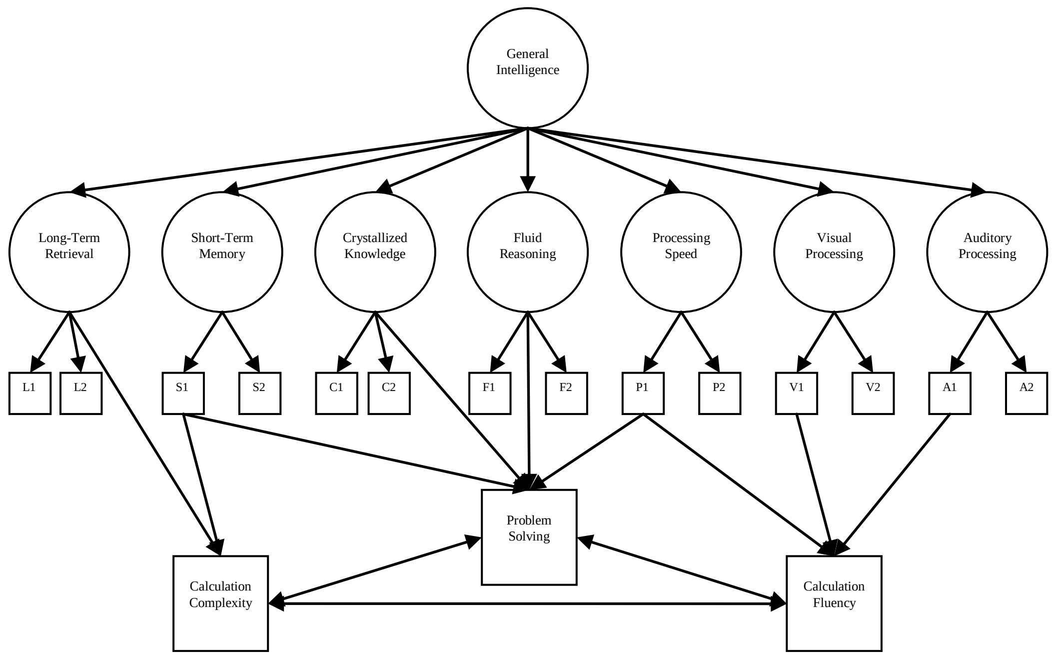

Figure 1. Model of broad mathematical cognitive processes. L1: associative memory. L2: retrieval fluency. S1: working memory. S2: memory span. C1: lexical knowledge. C2: general knowledge. F1: inductive reasoning. F2: deductive reasoning. P1: perceptual processing speed. P2: semantic processing speed. V1: spatial operations. V2: visual memory. A1: phonetic coding synthesis. A2: speech-noise discrimination.

Developmental Hypotheses

1. The association between crystallized knowledge (Gc) and problem solving increases as age increases.
2. The association between fluid reasoning (Gf) and problem solving decreases as age increases.
3. The association between working memory and problem solving remains constant across the age span.

4. The association between perceptual processing speed and problem solving remains constant across the age span.
5. The association between long-term retrieval (Glr) and calculation complexity remains constant across the age span, or slightly decreases as age increases.
6. The association between working memory and calculation complexity remains constant across the age span.
7. The association between perceptual processing speed and calculation fluency remains constant across the age span.
8. The association between phonetic coding synthesis and calculation fluency decreases as age increases.
9. The association between spatial operations and calculation fluency remains constant across the age span, or slightly decreases as age increases.

Chapter 3

Method

Participants

The participants for this study were a subsample of a nationally representative sample used in the standardization of the Woodcock-Johnson III Tests of Cognitive Abilities and the Woodcock-Johnson III Tests of Achievement, Normative Update (Woodcock, McGrew, & Mather, 2007). The standardization sample was stratified according to race, ethnicity, gender, geographic region, education, and age to ensure that the sample mirrored the population characteristics of children, adolescents, and adults in the United States, as described by the United States Census projections for the year 2000. Participants in the current study consisted of that portion of the standardization sample between 5 years old and 18 years old (N = 4721; mean of 10.98 years, median of 10.00 years, standard deviation of 3.48 years), and were 50.7% male and 49.3% female. The racial composition of the sample was: 78.3% European American, 14.4% African American, 5.1% Asian American, and 2.0% Native American. The ethnic composition of the sample was: 87.9% Non-Hispanic, and 12.1% Hispanic, who can be of any race.

Procedure

Selected subtests from the Woodcock-Johnson Tests of Cognitive Ability - Third Edition (WJ-III COG; Woodcock, Mather, & McGrew, 2001) and the Woodcock-Johnson Tests of Academic Achievement - Third Edition (WJ-III ACH; McGrew & Woodcock, 2001) were used in this study. The hypothesized model of mathematics cognitive processes was tested through structural equation modeling (SEM).

Measures

Woodcock-Johnson Tests of Cognitive Ability. The WJ-III COG is an individually administered test of intelligence that was developed for individuals aged 2 years to 90 years (Woodcock, Mather, & McGrew, 2001). The measure contains 7 standard and 14 supplemental subtests with a mean of 100 and a standard deviation of 15. The measure produces a global IQ score, 7 broad cognitive scores, 14 narrow cognitive scores, and 7 clinical cluster scores. The 7 broad cognitive areas are each comprised of two qualitatively different narrow cognitive processes described in the CHC model. The 7 broad areas and their corresponding narrow areas are:

1. Fluid reasoning (Gf) – the ability to reason, form concepts, and solve problems with novel tasks. Narrow areas: Inductive reasoning and deductive reasoning.
2. Crystallized knowledge (Gc) – the ability to use previously learned procedures (breadth and depth of a person's knowledge of a culture), particularly verbally. Narrow areas: Lexical knowledge and general knowledge.
3. Short-term memory (Gsm) – the ability to hold information in immediate awareness and use it within a few seconds. Narrow areas: Working memory and memory span.

4. Visual-spatial processing (Gv) – the ability to analyze, synthesize, and manipulate visual information. Narrow areas: Spatial operations and visual memory.
5. Auditory processing (Ga) – the ability to analyze, synthesize, and manipulate auditory information. Narrow areas: Phonetic coding synthesis and speech-noise discrimination.
6. Long-term retrieval (Glr) – the ability to store information in long-term memory, and to retrieve it later. Narrow areas: Associative memory and retrieval fluency.
7. Processing speed (Gs) – the speed and efficiency to perform cognitive tasks. Narrow areas: Perceptual processing speed and semantic processing speed.

The complete set of 14 narrow cognitive processes was used in order to estimate the 7 broad cognitive latent variables and test the broad mathematical cognition model through structural equation modeling.

Woodcock-Johnson Tests of Academic Achievement. The WJ-III ACH is an individually administered achievement test co-normed with the WJ-III COG (McGrew & Woodcock, 2001). The WJ-III ACH measures reading, mathematics and written language achievement. Three mathematics achievement tests were used to test the proposed model:

1. Calculation complexity was measured with the WJ-III ACH Calculation subtest: Calculation measures the ability to perform mathematical computations of increasing complexity. It starts with requiring the subject

to write individual numerals. The test progresses to addition, subtraction, multiplication, division, combinations of these operations, decimals, fractions, algebra, logarithms, and calculus.

2. Calculation fluency was measured with the WJ-III ACH Mathematics Fluency subtest: Mathematics Fluency measures the ability to solve simple addition, subtraction, and multiplication quickly. The test has a 3-minute time limit.
3. Problem solving was measured with the WJ-III ACH Applied Problems subtest: Applied Problems measures the ability to solve mathematics problems involving language, general information, and hypothetical scenarios.

Chapter 4

Results

Preliminary Analyses

Descriptive analyses were conducted on all of the observed variables to assess their distribution. All of the observed variables had normal distributions, with skewness values between -.324 and .285, and kurtosis values between .242 and 1.632 (see Table 1). Skewness values between -1.0 and 1.0, and kurtosis values of less than 3.0 are considered to be within normal parameters.

Table 1

Descriptive Statistics of Observed Variables

Variable	Min.	Max.	Mean	S.D.	Skew.	Kurt.
Inductive Reasoning (Gf)	35	149	100.1	15.5	-.252	.331
Deductive Reasoning (Gf)	19	158	100.4	15.3	-.251	.820
Lexical Knowledge (Gc)	44	158	100.9	14.7	-.203	.242
General Knowledge (Gc)	36	166	100.7	14.8	-.308	.745
Associative Memory (Glr)	50	173	99.9	15.3	.075	.392
Retrieval Fluency (Glr)	14	149	100.4	14.5	-.290	.710
Spatial Operations (Gv)	36	161	100.1	15.0	.006	.482
Visual Memory (Gv)	33	157	100.3	14.8	-.091	.467
Phonetic Coding Synthesis (Ga)	38	153	99.5	14.6	.098	.365
Speech-Noise Discrimination (Ga)	11	166	99.5	16.1	-.324	1.632
Working Memory (Gsm)	34	154	99.9	15.5	-.230	.575
Memory Span (Gsm)	45	154	100.8	15.4	-.117	.414
Perceptual Processing Speed (Gs)	13	153	99.7	14.8	-.270	1.466
Semantic Processing Speed (Gs)	37	159	99.8	15.4	-.039	.513
Calculation Complexity	36	169	100.1	16.1	-.201	.690
Calculation Fluency	48	166	99.8	14.9	.285	.556
Problem Solving	47	150	100.8	14.7	-.121	.351

Main Analyses

The hypothesized model of mathematics cognitive processes was tested through SEM using Mplus 3.0 (Muthén & Muthén, 2004) with maximum likelihood estimation for missing data as recommended by Baraldi and Enders (2010). Model fit was assessed considering the following standards (Hu & Bentler, 1999; Kline, 1998; Weston & Gore, 2006): The Comparative Fit Index (CFI) is greater than or equal to .95 (or .90 for adequate fit), the Root Mean Square Error of Approximation (RMSEA) is less than or equal to .06 (or .08 for adequate fit), and the Standardized Root Mean Square Residual (SRMR) is less than or equal to .08 (or .10 for adequate fit). The model obtained adequate fit indices: χ^2 (103) = 1509.665, p > .00, CFI=.933; RMSEA=.054; SRMR=.049. Modification indices indicated that the following modifications would improve the fit of the model significantly: a) replacing the path from the observed variable of visual operations (V1) to calculation fluency with a path from the latent variable of visual processing (Gv) to calculation fluency; b) adding a correlation path between calculation complexity and perceptual processing speed (P1); and c) adding a correlation path between calculation fluency and working memory (S1). After implementing these modifications, the model obtained better fit indices: χ^2 (101) = 803.391, p > .00, CFI=.949; RMSEA=.047; SRMR=.039. Because these model modifications represented minor and reasonable conceptual modifications consistent with the theorized model, this modified version was utilized in subsequent analyses. The structural equation model of broad mathematical cognition is presented in Figure 2.

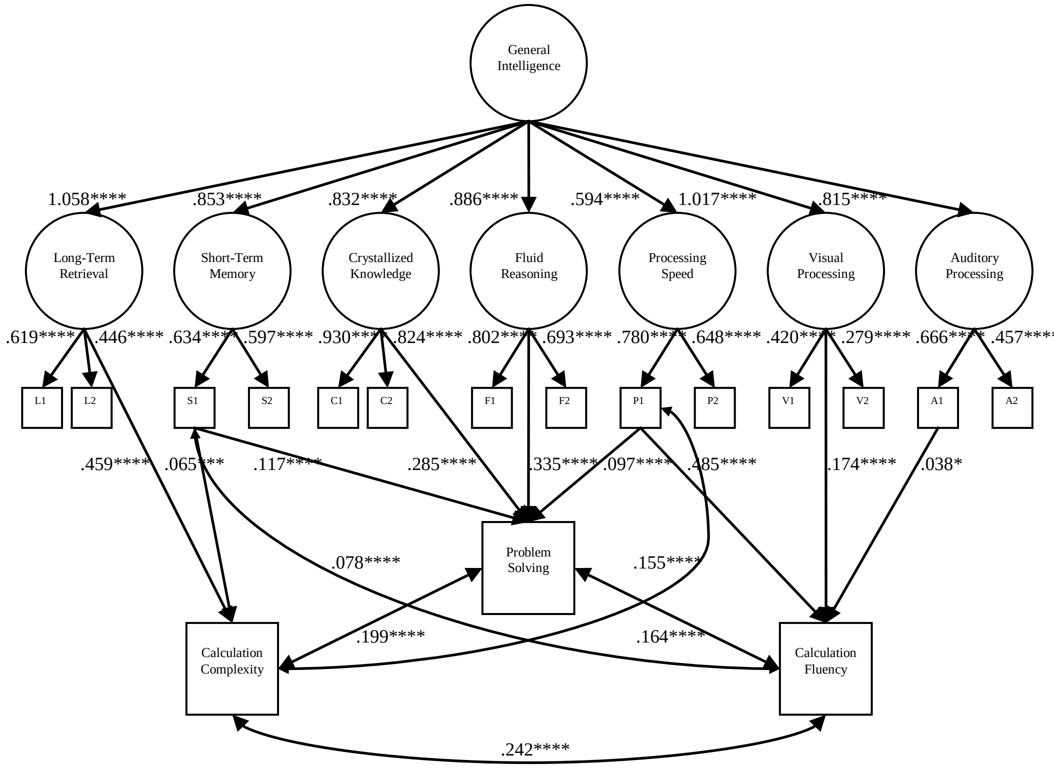

Figure 2. Structural equation model of broad mathematical cognitive processes with standardized path coefficients. L1: associative memory. L2: retrieval fluency. S1: working memory. S2: memory span. C1: lexical knowledge. C2: general knowledge. F1: inductive reasoning. F2: deductive reasoning. P1: perceptual processing speed. P2: semantic processing speed. V1: spatial operations. V2: visual memory. A1: phonetic coding synthesis. A2: speech-noise discrimination. **p* < .05, ***p* < .01, ****p* < .001, *****p* < .0001.

As hypothesized, calculation complexity was predicted by long-term retrieval (Glr), and working memory (S1). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), phonetic coding synthesis (A1), and visual processing (Gv). Calculation fluency was correlated with working memory (S1). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency. All indirect effects in the model (i.e., all single mediator and multiple mediator pathways implied in the model) were statistically significant at the $p < .05$ or lower level. General intelligence had indirect effects on calculation complexity, calculation fluency, and problem solving via the broad and narrow cognitive processes specified in the model. Table 2 presents all indirect effect standardized path coefficients tested in the model.

Table 2. *Indirect and total effects of broad cognitive processes on calculation complexity, calculation fluency, and problem solving.*

	Complexity	Fluency	Problem solving
General Intelligence			
Indirect Effects			
Via Long-Term Retrieval	.486*		
Via Short-Term Memory	.035*		
Total Effect:	.521*		
Via Processing Speed		.225*	
Via Visual Processing		.177*	
Via Auditory Processing		.021*	
Total Effect:		.423*	
Via Fluid Reasoning			.297*
Via Crystallized Knowledge			.237*
Via Short-Term Memory			.063*
Via Processing Speed			.045*
Total Effect:			.642*

*$p < .05$

Moderation by Age

In order to test the developmental hypotheses, a series of multi-group structural equation models were computed (Calderón-Tena, Knight, & Carlo, 2011 provide examples of this procedure). Four age groups were created by splitting the sample according to developmental stages, considering Piaget's theory of cognitive development (Piaget, 1961): preoperational (ages 5-6 years, N = 493); concrete operational (ages 7-10 years, N = 1878); early formal operational (ages 11-15 years, N = 1693); and late formal operational (ages 16-18 years, N = 657). A chi-square difference test was used to determine whether the model fit the data differently for different age groups, and path coefficient value changes were used to support or reject each alternative hypothesis. The first multi-group model constrained the path coefficients to be equal across all four age groups and yielded the following fit indices: χ^2 (473) = 1929.742, p > .00**,** CFI = .891, RMSEA = .077, SRMR = .071. The second model allowed the path coefficients to vary across age groups (i.e., was unconstrained) and yielded the following fit indices: χ^2 (425) = 1201.565, p > .00**,** CFI = .942, RMSEA = .059, SRMR = .051. A significant chi-square difference test [$_{\Delta}\chi^2$ (48) = 728.177, p < .0001] and the better fit indices of the unconstrained model indicates that age tended to moderate the path coefficients, and that the model tended to fit the data differently for each age group. Figures 3 to 6 present the standardized path coefficients for students in the preoperational, concrete operational, early formal operational, and late formal operational age groups, respectively.

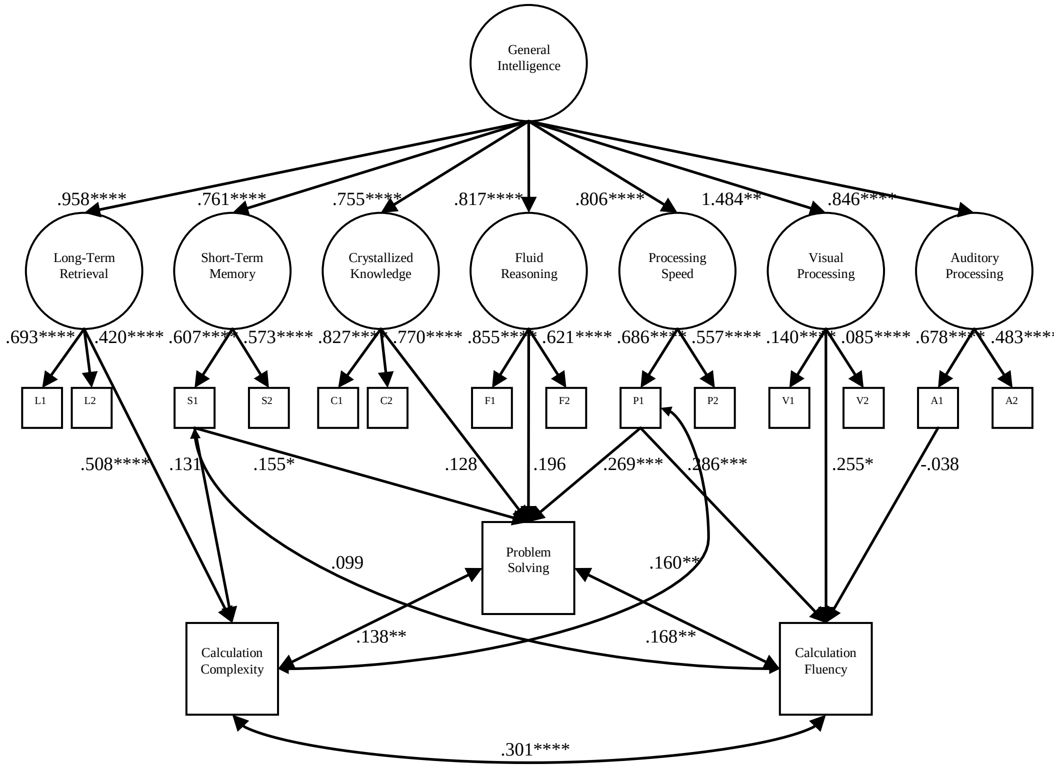

Figure 3. Preoperational group (ages 5-6) in the multi-group structural equation model of broad mathematical cognitive processes with standardized path coefficients. L1: associative memory. L2: retrieval fluency. S1: working memory. S2: memory span. C1: lexical knowledge. C2: general knowledge. F1: inductive reasoning. F2: deductive reasoning. P1: perceptual processing speed. P2: semantic processing speed. V1: spatial operations. V2: visual memory. A1: phonetic coding synthesis. A2: speech-noise discrimination. *p < .05, **p < .01, ***p < .001, ****p < .0001.

In the preoperational group, calculation complexity was predicted by long-term retrieval (Glr); calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Problem solving was predicted by working memory (S1), and perceptual processing speed (P1);

problem solving was correlated with calculation complexity and calculation fluency.

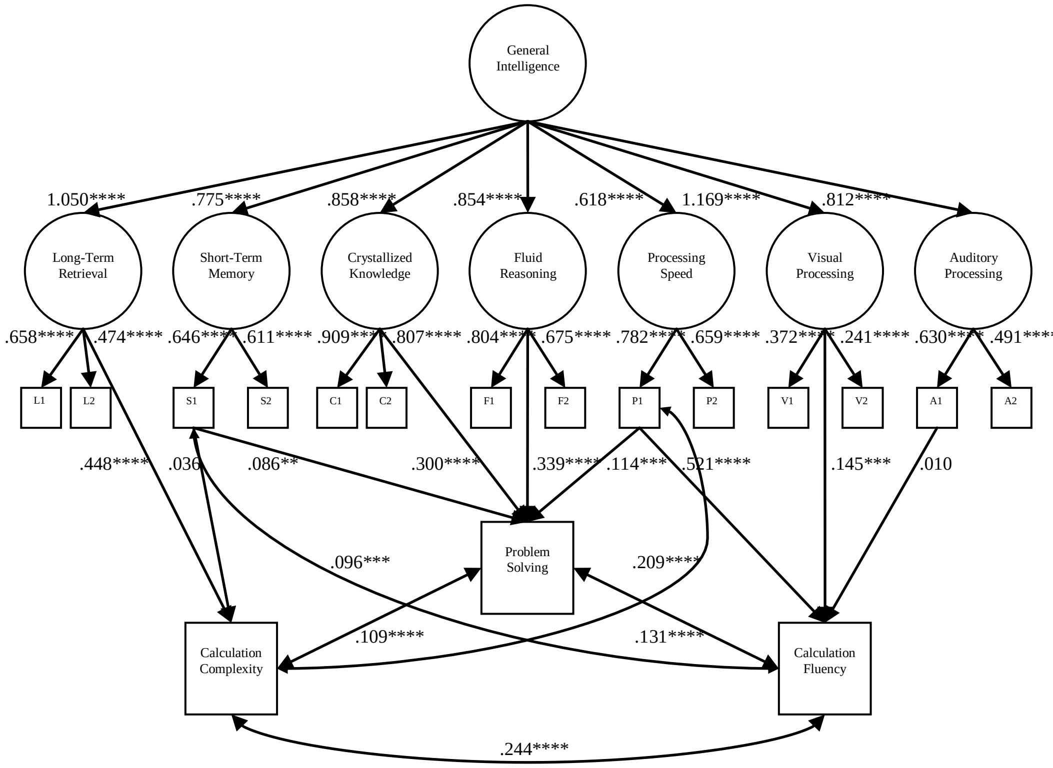

Figure 4. Concrete operational group (ages 7-10) in the multi-group structural equation model of broad mathematical cognitive processes with standardized path coefficients. L1: associative memory. L2: retrieval fluency. S1: working memory. S2: memory span. C1: lexical knowledge. C2: general knowledge. F1: inductive reasoning. F2: deductive reasoning. P1: perceptual processing speed. P2: semantic processing speed. V1: spatial operations. V2: visual memory. A1: phonetic coding synthesis. A2: speech-noise discrimination. *$p < .05$, **$p < .01$, ***$p < .001$, ****$p < .0001$.

In the concrete operational group, calculation complexity was predicted by long-term retrieval (Glr); calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency.

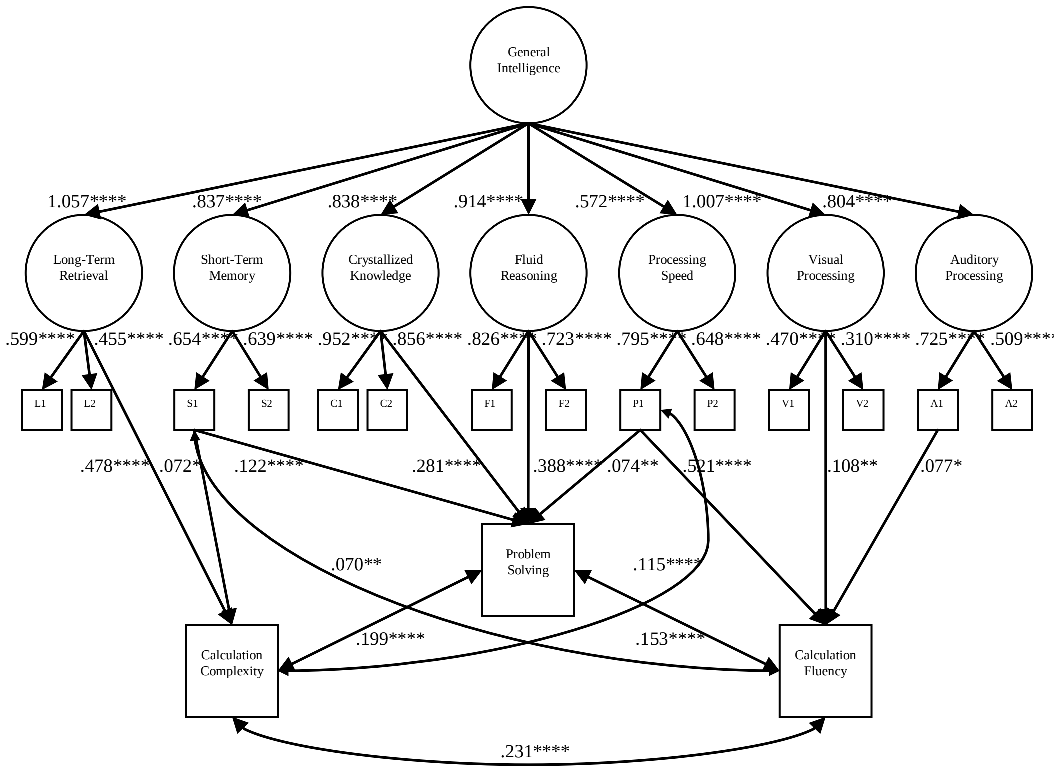

Figure 5. Early formal operational group (ages 11-15) in the multi-group structural equation model of broad mathematical cognitive processes with standardized path coefficients. L1: associative memory. L2: retrieval fluency. S1: working memory. S2: memory span. C1: lexical knowledge. C2: general knowledge. F1: inductive reasoning. F2: deductive reasoning. P1: perceptual processing speed. P2: semantic processing speed. V1: spatial operations. V2: visual memory. A1: phonetic coding synthesis. A2: speech-noise discrimination. *$p < .05$, **$p < .01$, ***$p < .001$, ****$p < .0001$.

In the early formal operational group, calculation complexity was predicted by long-term retrieval (Glr), and working memory (S1); calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), phonetic coding synthesis (A1), and visual processing (Gv). Calculation fluency was correlated with working memory (S1). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency.

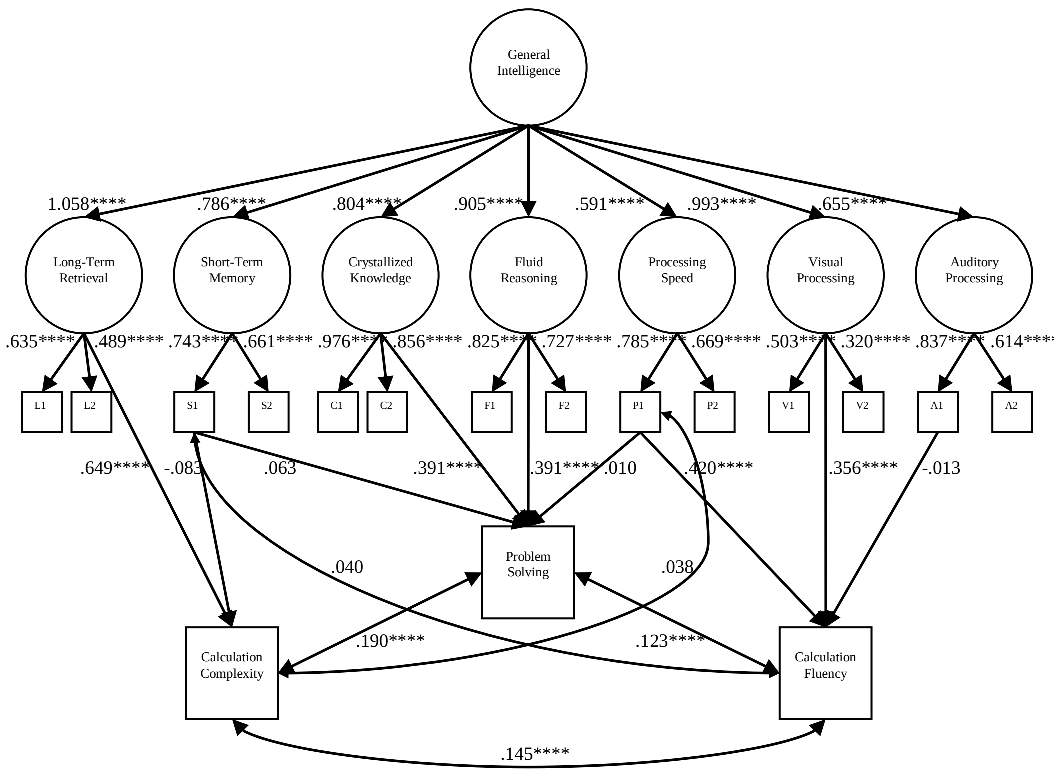

Figure 6. Late formal operational group (ages 16-18) in the multi-group structural equation model of broad mathematical cognitive processes with standardized path coefficients. L1: associative memory. L2: retrieval fluency. S1: working memory. S2: memory span. C1: lexical knowledge. C2: general knowledge. F1: inductive reasoning. F2: deductive reasoning. P1: perceptual processing speed. P2: semantic processing speed. V1: spatial operations. V2: visual memory. A1: phonetic coding synthesis. A2: speech-noise discrimination. **p* < .05, ***p* < .01, ****p* < .001, *****p* < .0001.

In the late formal operational group, calculation complexity was predicted by long-term retrieval (Glr); calculation complexity was correlated with calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Problem solving was predicted by fluid reasoning (Gf), and crystallized knowledge (Gc). Problem solving was correlated with calculation complexity and calculation fluency.

Together, these results provide evidence to assess the developmental hypotheses proposed. In general, the path coefficient from Gc to problem solving tended to increase with age, from a non-significant standardized path coefficient of .128 among preoperational students, to a significant standardized path coefficient of .391 among late formal operational students, supporting hypothesis 1. Similarly, the path coefficient from Gf to problem solving tended to increase with age, from a non-significant standardized path coefficient of .196 among preoperational students, to a significant standardized path coefficient of .391 among late formal operational students, disconfirming hypothesis 2. In general, the path coefficient from working memory to problem solving indicated a modest association across all age groups, except in the late formal operational students, partially supporting hypothesis 3. The path from perceptual processing speed to problem solving tended to decrease with age, from a significant standardized path coefficient of .269 among preoperational students, to a non-significant standardized path coefficient of .010 among late formal operational students, disconfirming hypothesis 4.

The path coefficient from Glr to calculation complexity indicated a significant association across all age groups, partially supporting hypothesis 5. The path coefficient from working memory to calculation complexity was inconsistent (fluctuated) across age groups, disconfirming hypothesis 6. The path coefficient from perceptual processing speed to calculation fluency was significant across all age groups, but it tended to increase (rather than remain constant), partially supporting hypothesis 7. The path coefficient from phonetic coding synthesis to calculation fluency remained constant (non-significant) across age groups, except among early formal operational students, disconfirming hypothesis 8. Last, the path coefficient from Gv to calculation fluency was significant across all age groups, but it tended to increase (rather than remain constant), partially supporting hypothesis 9.

Subsequently, the developmental hypotheses were tested using a more rigorous method than the one described earlier: A new set of multi-group structural equation models were computed in which one model is partially unconstrained and the other model is fully constrained. The partially unconstrained model allows only one path to vary across age groups, according to the hypothesis being tested (e.g., in order to test hypothesis 1, a partially unconstrained model allows the path from Gc to problem solving to vary across groups), then this model is compared against the fully constrained model using a chi-square difference test.

The fully constrained model obtained the following fit indices: χ^2 (473) = 1929.742, $p > .00$, CFI = .891, RMSEA = .077, SRMR = .071. The partially unconstrained models obtained the following fit indices:

Hypothesis 1 (Gc to problem solving): χ^2 (469) = 1806.243, $p > .00$, CFI = .900, RMSEA = .074, SRMR = .069.

Hypothesis 2 (Gf to problem solving): χ^2 (469) = 1815.054, $p > .00$, CFI = .899, RMSEA = .074, SRMR = .068.

Hypothesis 3 (Gsm1 to problem solving): χ^2 (469) = 1920.347, $p > .00$, CFI = .892, RMSEA = .077, SRMR = .070.

Hypothesis 4 (Gs1 to problem solving): χ^2 (469) = 1813.024, $p > .00$, CFI = .900, RMSEA = .074, SRMR = .069.

Hypothesis 5 (Glr to calculation complexity): χ^2 (469) = 1801.232, $p > .00$, CFI = .900, RMSEA = .074, SRMR = .062.

Hypothesis 6 (Gsm1 to calculation complexity): χ^2 (469) = 1799.854, $p > .00$, CFI = .901, RMSEA = .074, SRMR = .063.

Hypothesis 7 (Gs1 to calculation fluency): χ^2 (469) = 1596.889, $p > .00$, CFI = .916, RMSEA = .068, SRMR = .070.

Hypothesis 8 (Ga1 to calculation fluency): χ^2 (469) = 1616.163, $p > .00$, CFI = .914, RMSEA = .069, SRMR = .065.

Hypothesis 9 (Gv to calculation fluency): χ^2 (469) = 1919.909, $p > .00$, CFI = .892, RMSEA = .077, SRMR = .072.

The chi-square difference tests yielded the following results:

Hypothesis 1 (Gc to problem solving): $\Delta\chi^2$ (4) = 123.499, $p < .0001$.

Hypothesis 2 (Gf to problem solving): $\Delta\chi^2$ (4) = 114.688, $p < .0001$.

Hypothesis 3 (Gsm1 to problem solving): $\Delta\chi^2$ (4) = 9.395, $p < .0520$.

Hypothesis 4 (Gs1 to problem solving): $\Delta\chi^2$ (4) = 116.718, $p < .0001$.

Hypothesis 5 (Glr to calculation complexity): $\Delta\chi^2$ (4) = 128.51, $p < .0001$.

Hypothesis 6 (Gsm1 to calculation complexity): $\Delta\chi^2$ (4) = 129.888, $p < .0001$.

Hypothesis 7 (Gs1 to calculation fluency): $\Delta\chi^2$ (4) = 332.853, $p < .0001$.

Hypothesis 8 (Ga1 to calculation fluency): $\Delta\chi^2$ (4) = 313.579, $p < .0001$.

Hypothesis 9 (Gv to calculation fluency): $\Delta\chi^2$ (4) = 9.833, $p < .0433$.

The chi-square difference tests based on partially unconstrained models supported hypotheses 1 (the association between Gc and problem solving is significantly different across age groups; in general it increases); and 3 (the association between working memory and problem solving is not significantly different across age groups).

The chi-square difference tests based on partially unconstrained models did not support hypotheses 2 (the association between Gf and problem solving is significantly different across age groups; in general it decreases); 4 (the association between perceptual processing speed and problem solving is not significantly different across age groups); 5 (the association between Glr and calculation complexity is not significantly different across age groups); 6 (the association between working memory and calculation complexity is not significantly different across age groups); 7 (the association between perceptual processing speed and calculation fluency is not significantly different across age groups); 8 (the association between phonetic coding synthesis and calculation fluency is significantly different across age groups; in general it decreases); and 9 (the association between Gv and calculation fluency is not significantly different across age groups).

These chi-square difference tests based on partially unconstrained models corroborated the initial results regarding the relationships between Gc, Gf, working memory, and perceptual processing speed, with problem solving; the relationship between Glr and calculation complexity; and the relationships between perceptual processing speed, and Gv, with calculation fluency. However, no clear patterns emerged (i.e., increasing with age, decreasing with age, or remaining constant) regarding the relationships between working memory and calculation complexity, and phonetic coding synthesis and calculation fluency.

A final, partially unconstrained model was estimated in which the paths corresponding to hypotheses 1, 2, and 8 were unconstrained, and the paths corresponding to hypotheses 3, 4, 5, 6, 7, and 9 were constrained to be equal across groups. This model obtained the following fit indices: χ^2 (446) = 1447.883, $p > .00$, CFI = .925, RMSEA = .066, SRMR = .062. The chi-square difference test (against the fully constrained model) yielded the following result: $\Delta\chi^2$ (27) = 481.859, $p < .0001$.

Moderation by Gender

Given that the incidence of mathematics learning disabilities in children and adolescents has exhibited a male-female ratio of approximately 2:1 (Shalev, 2007), it is reasonable to expect gender differences in mathematics development. In order to examine whether gender may moderate the relationships between variables, a series of exploratory multi-group structural equation models were computed by splitting the sample according to gender (male N = 2394; female N = 2327). The first multi-group model constrained the path coefficients to be equal across gender groups and yielded the following fit indices: χ^2 (229) = 1506.355, *p* > .00**,** CFI = .903, RMSEA = .073, SRMR = .063. The second model allowed the path coefficients to vary across groups (i.e., was unconstrained) and yielded the following fit indices: χ^2 (209) = 880.945, *p* > .00**,** CFI = .949, RMSEA = .059, SRMR = .044. A significant chi-square difference test [$_{\Delta}\chi^2$ (20) = 625.41, *p* < .0001] and the better fit indices of the unconstrained model indicates that gender tended to moderate the path coefficients, and that the model tended to fit the data differently for each gender group. Therefore, a new series of structural equation models were computed by splitting the sample according to gender at each age group level (i.e., preoperational: 271 male, 222 female; concrete operational: 912 male, 966 female; early formal operational: 884 male, 809 female; late formal operational: 327 male, 330 female).

Interaction of Age and Gender

Preoperational Male Group. The preoperational male model obtained the following fit indices: χ^2 (101) = 141.587, p > .0048, CFI=.963; RMSEA=.039; SRMR=.052. Calculation complexity was predicted by long-term retrieval (Glr). Calculation complexity was correlated with calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1). Problem solving was predicted by fluid reasoning (Gf), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency. Tables 3 to 5 indicate the statistical significance of variable relationships by age and gender group.

Preoperational Female Group. The preoperational female model obtained the following fit indices: χ^2 (101) = 182.998, p > .00, CFI=.908; RMSEA=.060; SRMR=.061. Calculation complexity was predicted by long-term retrieval (Glr). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Problem solving was predicted by crystallized knowledge (Gc). Problem solving was correlated with calculation complexity.

Concrete Operational Male Group. The concrete operational male model obtained the following fit indices: χ^2 (101) = 310.447, p > .00, CFI=.940; RMSEA=.047; SRMR=.051. Calculation complexity was predicted by long-term retrieval (Glr), and working memory (S1). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and phonetic coding synthesis (A1). Calculation fluency was correlated with working memory (S1). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency.

Concrete Operational Female Group. The concrete operational female model obtained the following fit indices: χ^2 (101) = 285.015, p > .00, CFI=.950; RMSEA=.043; SRMR=.045. Calculation complexity was predicted by long-term retrieval (Glr). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Calculation fluency was correlated with working memory (S1). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency.

Early Formal Operational Male Group. The early formal operational male model obtained the following fit indices: χ^2 (101) = 312.276, p > .00, CFI=.955; RMSEA=.049; SRMR=.040. Calculation complexity was predicted by long-term retrieval (Glr), and working memory (S1). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and phonetic coding synthesis (A1). Calculation fluency was correlated with working memory (S1). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency.

Early Formal Operational Female Group. The early formal operational female model obtained the following fit indices: χ^2 (101) = 287.255, p > .00, CFI=.953; RMSEA=.048; SRMR=.040. Calculation complexity was predicted by long-term retrieval (Glr). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Calculation fluency was correlated with working memory (S1). Problem solving was predicted by fluid reasoning (Gf), crystallized knowledge (Gc), working memory (S1), and perceptual processing speed (P1). Problem solving was correlated with calculation complexity and calculation fluency.

Late Formal Operational Male Group. The late formal operational male model obtained the following fit indices: χ^2 (101) = 185.977, p > .00, CFI=.955; RMSEA=.051; SRMR=.045. Calculation complexity was predicted by long-term retrieval (Glr). Calculation complexity was correlated with calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Problem solving was predicted by fluid reasoning (Gf), and crystallized knowledge (Gc). Problem solving was correlated with calculation complexity and calculation fluency.

Late Formal Operational Female Group. The late formal operational female model obtained the following fit indices: χ^2 (101) = 228.414, p > .00, CFI=.936; RMSEA=.062; SRMR=.055. Calculation complexity was predicted by long-term retrieval (Glr). Calculation complexity was correlated with perceptual processing speed (P1) and calculation fluency. Calculation fluency was predicted by perceptual processing speed (P1), and visual processing (Gv). Problem solving was predicted by fluid reasoning (Gf), and crystallized knowledge (Gc). Problem solving was correlated with calculation complexity and calculation fluency.

Table 3

Statistical Significance of the Relationships between Latent and Observed Cognitive Variables and Calculation Complexity at the .05 or Lower Level

Long-Term Retrieval and Calculation Complexity, Controlling for Working Memory		
	Male	Female
Preoperational	Significant	Significant
Concrete Operational	Significant	Significant
Early Formal Operational	Significant	Significant
Late Formal Operational	Significant	Significant
Working Memory and Calculation Complexity, Controlling for Long-Term Retrieval		
	Male	Female
Preoperational	Non-significant	Non-significant
Concrete Operational	Significant	Non-significant
Early Formal Operational	Significant	Non-significant
Late Formal Operational	Non-significant	Non-significant

Long-term retrieval (a latent variable predicting associative memory and retrieval fluency) was consistently associated with calculation complexity among male and female students at all age levels, when controlling for working memory (controlling for working memory as specified in the a priori model; see figure 2). Working memory was associated with calculation complexity among male students in the concrete operational and early formal operational age groups, when controlling for Glr as specified in the model (Table 3).

Table 4

Statistical Significance of the Relationships between Latent and Observed Cognitive Variables and Calculation Fluency at the .05 or Lower Level

Perceptual Processing Speed and Calculation Fluency, Controlling for Visual Processing and Phonetic Coding Synthesis		
	Male	Female
Preoperational	Significant	Significant
Concrete Operational	Significant	Significant
Early Formal Operational	Significant	Significant
Late Formal Operational	Significant	Significant
Visual Processing and Calculation Fluency, Controlling for Phonetic Coding Synthesis and Perceptual Processing Speed		
	Male	Female
Preoperational	Non-significant	Significant
Concrete Operational	Non-significant	Significant
Early Formal Operational	Non-significant	Significant
Late Formal Operational	Significant	Significant
Phonetic Coding Synthesis and Calculation Fluency, Controlling for Visual Processing and Perceptual Processing Speed		
	Male	Female
Preoperational	Non-significant	Non-significant
Concrete Operational	Significant	Non-significant
Early Formal Operational	Significant	Non-significant
Late Formal Operational	Non-significant	Non-significant

Perceptual processing speed was consistently associated with calculation fluency among male and female students at all age levels, controlling for phonetic coding synthesis and Gv. Visual processing (a latent variable predicting spatial operations and visual memory) was associated with calculation fluency among female students at all age levels, and among male students in the late formal operational group, controlling for phonetic coding synthesis and perceptual processing speed. Last, phonetic coding synthesis was associated with calculation fluency among male students in the concrete operational and early formal operational age groups, when controlling for Gv and perceptual processing speed (Table 4).

Table 5

Statistical Significance of the Relationships between Latent and Observed Cognitive Variables and Problem Solving at the .05 or Lower Level

Fluid Reasoning and Problem Solving, controlling for Crystallized Knowledge, Working Memory, and Perceptual Processing Speed		
	Male	Female
Preoperational	Significant	Non-significant
Concrete Operational	Significant	Significant
Early Formal Operational	Significant	Significant
Late Formal Operational	Significant	Significant
Crystallized Knowledge and Problem Solving, controlling for Fluid Reasoning, Working Memory, and Perceptual Processing Speed		
	Male	Female
Preoperational	Non-significant	Significant
Concrete Operational	Significant	Significant
Early Formal Operational	Significant	Significant
Late Formal Operational	Significant	Significant
Working Memory and Problem Solving, controlling for Fluid Reasoning, Crystallized Knowledge, and Perceptual Processing Speed		
	Male	Female
Preoperational	Significant	Non-significant
Concrete Operational	Non-significant	Significant
Early Formal Operational	Significant	Significant
Late Formal Operational	Non-significant	Non-significant
Perceptual Processing Speed and Problem Solving, controlling for Fluid Reasoning, Crystallized Knowledge, and Working Memory		
	Male	Female
Preoperational	Significant	Non-significant
Concrete Operational	Significant	Significant
Early Formal Operational	Significant	Significant
Late Formal Operational	Non-significant	Non-significant

Fluid reasoning (a latent variable predicting inductive and deductive reasoning) was consistently associated with problem solving among male students at all age levels, and among female students at all age levels except preoperational, controlling for Gc, working memory, and perceptual processing speed. Crystallized knowledge (a latent variable predicting lexical and general knowledge) was consistently associated with problem solving among female students at all age levels, and among male students at all age levels except preoperational, controlling for Gf, working memory, and perceptual processing speed. No clear patterns regarding the associations between working memory and perceptual processing speed with problem solving emerged (Table 5).

Chapter 5

Discussion

This study set out to examine the role of broad cognitive processes in the development of mathematics skills among child and adolescent students. This is an important, yet understudied topic. This study represents an attempt to unify basic cognitive developmental research literature (e.g., Geary, 1993; Fuchs et al., 2010) with applied research literature from school psychology (e.g., McGrew & Wendling, 2010).

Cognitive psychologists have identified a number of domain-specific cognitive processes involved in mathematical development. These include abilities such as subitizing, which consists of spontaneously identifying quantities of 1, 2, or 3 objects without counting, among various others. Children transition from these automatic, basic competencies (evolution based; Geary, 2007) to formal, complex mathematical competencies (acculturation based), including mathematical vocabulary and theory through broad (i.e., domain general) cognitive processes (Geary, Hoard, Nugent, & Byrd-Craven, 2008).

A comprehensive exploratory study of CHC factors and mathematics achievement was conducted by Floyd, Evans, and McGrew (2003) using a nationally representative sample. Subsequently, a preliminary study of the associations between broad cognitive processes and mathematics achievement was made by Bacal, Caterino, Dial, and Kube (2008), using a clinical sample and the CHC framework. Taub, Floyd, Keith, and McGrew (2008) expanded on Floyd et al. (2003), and published an exploratory study using structural equation modeling. Subsequently, Calderón-Tena, Caterino, and Felicetta (2012) replicated Taub et al.'s results using a clinical sample. Recently, Calderón-Tena (2011) presented a confirmatory study (a model proposed a priori) using a clinical sample and multi-group structural equation modeling looking at age differences. To this author's knowledge, this is the first study to test a hypothesized model of broad mathematical cognition through multi-group structural equation modeling using a normative sample, looking at age and gender differences. This a priori model was primarily based on Floyd et al. (2003) empirical study, and the theoretical framework of Geary (1993, 2007).

The general model of mathematical cognition (complete sample, single group analysis) fit the data adequately, and all of the hypothesized paths between cognitive processes and mathematics achievement were statistically significant. A series of multi-group structural equation models were used to test the developmental hypotheses. The results indicated that Gf and Gc became stronger predictors of mathematics problem solving as age increased; working memory remained a constant, weak predictor of problem solving skills (when controlling for Gf and Gc); and perceptual processing speed became a weaker predictor of mathematics problem solving as age increased (when controlling for Gf and Gc). Glr became a stronger predictor of calculation complexity as age increased; and working memory was an inconsistent, weak predictor of calculation complexity (when controlling for Glr). Perceptual processing speed became a stronger predictor of calculation fluency as age increased (when controlling for phonetic coding synthesis and Gv); phonetic coding synthesis was an inconsistent, weak predictor of calculation fluency (when controlling for perceptual processing speed and Gv); and Gv became a stronger predictor of calculation fluency as age increased (when controlling for perceptual processing speed and phonetic coding synthesis). It is also important to note that the most meaningful changes (i.e., path coefficient changes that were not only statistically significant, but meaningfully large) across age groups occurred among the relationships involving Gf, Gc, Glr, and Gs.

A series of exploratory multi-group structural equation models simultaneously assessed the impact of age and gender as moderators for each of the variable relationships proposed in the model. Final structural equation models of each gender group at each age group level supported the notion that the relation between broad cognitive processes and mathematics achievement is better understood within a developmental framework that considers gender differences. In other words, mathematical development is a function of a three way interaction between broad cognitive factors, developmental status, and gender. The results of these analyses are summarized next.

Fluid Reasoning and Crystallized Knowledge (Gf and Gc)

Fluid reasoning is the ability to solve novel problems, using inductive and deductive reasoning processes. This ability was consistently associated with problem solving skills among male and female students at all age levels (preoperational, concrete operational, early formal operational, and late formal operational), except for female students in the preoperational group. On the other hand, crystallized knowledge, the ability to solve problems using over-learned skills, mainly language and general knowledge, was consistently associated with problem solving skills among male and female students at all age levels, except for male students in the preoperational group. This suggests an interesting gender difference, which may be related to different socialization experiences. It is reasonable to speculate that preoperational girls are given more opportunities to practice language skills, relative to preoperational boys. A meta-analysis on gender differences in verbal abilities by Hyde and Linn (1988) provides some support for this notion. Although the authors conclude that the magnitude of gender differences across the life span is so small that it can be considered negligible in general, the largest positive effect size reported in their study (d = .31) was for girls of age 5 and younger (preoperational) in reading comprehension. Last, the results of this study suggest that fluid reasoning and crystallized knowledge tend become stronger predictors of problem solving skills among both male and female students as age increases.

Long-Term Retrieval (Glr)

Long-term retrieval is the broad cognitive process associated with learning and information retrieval from long-term memory. This broad cognitive process was significantly associated with calculation complexity among male and female students across all age levels. Long-term retrieval should be studied in more depth in future studies, both at the broad level (Glr) and at the narrow cognitive level (associative memory and retrieval fluency). Given the magnitudes of the standardized path coefficients obtained in this study, it appears that Glr may be a key broad cognitive factor in the development of mathematical skills, along with

Gf and Gc (perhaps the two most well established broad cognitive processes). It is also reasonable to postulate that Glr deficits may be associated with learning problems in other academic areas besides mathematics, given the broad nature of Glr processes. In fact, McGrew (1993) has documented a significant relationship between Glr, and basic reading skills and reading comprehension.

Processing Speed (Gs)

Processing speed was examined in the present study through its narrow ability of perceptual processing speed. This narrow ability was significantly associated with calculation fluency among male and female students across all age levels. The results indicated that perceptual processing speed was the single best indicator of calculation fluency, relative to visual processing and phonetic coding synthesis. The association between perceptual processing speed and problem solving was less clear (it varied across age and gender groups without a clear pattern) and its standardized path coefficients were much smaller, relative to calculation fluency. The results indicated that perceptual processing speed is not a strong indicator of problem solving skills when Gf and Gc data is considered.

Visual-Spatial Processing and Auditory Processing (Gv and Ga)

The association between visual-spatial processing and calculation fluency across gender groups was interesting: Gv was significantly related to calculation fluency among female students at all age levels, but it was significantly related to calculation fluency among male students in the late formal operational age group only. In contrast, the association between phonetic coding synthesis (a narrow indicator of auditory processing) was significantly related to calculation fluency among male students in the concrete operational and early formal operational age groups, but not among any of the female age groups. It is reasonable to speculate that, to the extent that visual-spatial processes are related to simultaneous processing (i.e., perceiving several visual components at once), and auditory processes are related to sequential processing (i.e., processing individual phonemes in a sequential fashion), this Gv/Ga gender difference may indicate a subtle, but statistically significant difference in the way male and female students process information. That is, it may be that visual (simultaneous) processing is more important for female calculation processing, while auditory (sequential) processing is more important for male calculation processing. Stated differently, female students with significant visual (simultaneous) processing deficits will tend to exhibit lower calculation fluency, while female students with significant visual (simultaneous) processing abilities will tend to exhibit higher calculation fluency, and likewise for male students and auditory (sequential) processing.

Short-Term Memory (Gsm)

Short-term memory was examined in the present study through its narrow ability of working memory. This narrow ability did not emerge as having a clear developmental pattern among male and female students across age levels, except that is was not significant among female students at any age group level when predicting calculation complexity, controlling for long-term retrieval. It is possible that the broad cognitive processes of Gf, Gc, and Glr actually involve working memory processes (e.g., the narrow ability of associative memory, a component of Glr, requires that the individual maintain a piece of information in the immediate awareness, while a second piece of information is presented as a related component or piece of information; the narrow ability of deductive reasoning, a component of Gf, requires that the individual maintain a rule or set of rules in the immediate awareness, while solving a problem; etc.) Thus, it is possible that the lack of associations between working memory and calculation complexity and problem solving is due to the fact that other broad cognitive processes accounted for its effects. In fact, Baddeley and Bristol (2001) have suggested that working memory can be understood as an executive processor directly linked to fluid cognitive systems (Gv, Ga, Gs, Gf) and mediately linked to crystallized cognitive systems (Gc and Glr) via fluid systems. In other words, working memory supports fluid systems directly, and crystallized systems indirectly.

Indirect and Total Effects of Broad Cognition

General intelligence (IQ) had indirect effects on calculation complexity, calculation fluency, and problem solving via the broad and narrow cognitive processes specified in the model. The total effects of IQ were greatest for problem solving, followed by calculation complexity, followed by calculation fluency. Indirect effects of broad cognitive factors via narrow processes specified in the model were very small in general, suggesting that broad cognitive factors tend to have more explanatory power than their individual narrow indicators.

Conclusion

The results of this study supported the theoretical suggestion that domain-general cognitive processes play significant and specific roles in the development of mathematical skills among children and adolescents. Consistent with McArdle, Ferrer-Caja, Hamagami, and Woodcock (2002), most broad cognitive processes examined in this study became increasingly associated with mathematical development with age. McArdle et al. documented that broad cognitive processes do not reach a developmental peak until at least age 18. Specifically, Gf peaks at age 22; Gc at age 35; Glr at age 18; Gsm at age 24; Gs at age 25; Ga at age 22; and Gv at age 24. Therefore, it is not surprising that most broad cognitive processes become better predictors of mathematics achievement through the school age years.

These results will contribute, not only to the empirical literature, but will have the potential to support practitioners in the development of cognitive-based mathematics disability diagnoses and interventions. For example, interventions targeting long-term retrieval may emphasize repetition and over-learning; interventions targeting short-term memory may emphasize practice re-telling stories; and so on with the rest of broad cognitive factors. This cognitive-based approach is not meant to replace prevention, universal screening, or response to intervention monitoring. Rather, cognitive research-based screening, assessment, diagnosis, and intervention represents one of the key components of a multi-tiered, prevention and intervention-focused model of school psychology practice (screening and intervention based on information related to specific cognitive processes involved in specific academic needs), which is in line with ethical standards and best practices of the National Association of School Psychologists, and the American Psychological Association. In particular, cognitive research-based assessment and intervention is a timely (and urgently needed) response to unsupported practices, such as ability-achievement discrepancy analysis, which has weak theoretical validity, and limited empirical support (see Flanagan, Fiorello, & Ortiz, 2010, for a discussion on the application of cognitive research in school psychology practice, particularly in the context of specific learning disabilities). Additionally, these results support the notion of “intelligent testing” (McGrew & Wendling, 2010), which suggests that psychologists should design assessment plans that include specific narrow and/or broad cognitive factors, rather than “one size fits all” approaches (i.e., administering the same whole battery of tests for all evaluations).

Although the present study represents an important contribution to the school psychology literature, future studies should address at least three important limitations. First, the concurrent validity of the model should be assessed using different cognitive and achievement instruments. Second, future studies should integrate both broad and domain-specific cognitive processes, in order to test the broader theoretical model proposed by Geary (1993; 2007). Third, future studies should cross-validate the model by using a split-sample model approach similar to the one described by Taub, Floyd, Evans, and McGrew (2008). However, independent corroboration of this model is available from Calderón-Tena (2011) study on mathematical development among students who had been referred for a psychoeducational evaluation. That study found similar results with an independent, clinical sample from an elementary school district.

Despite the limitations noted, the present study provides a number of significant contributions. First, this is one of the first studies to test a mathematics developmental model using a nationally representative sample. Second, the size of the sample allowed for the examination, not only a general model, but it allowed for comparisons between age and gender groups in detail. Third, various researchers have investigated the role of a limited number of cognitive processes (e.g., working memory, processing speed, fluid reasoning). However, this study investigated the role of all broad cognitive processes within the CHC framework, all of which have been identified in the mathematics developmental literature by researchers conducting studies independently from CHC research.

The theoretical and empirical basis of school psychology is becoming increasingly multi-disciplinary (i.e., school psychology is becoming informed and influenced by other areas of psychology), and this will serve to enhance the validity and applicability of school psychologists' tools (e.g., school psychology tools and methods will be more relevant in areas such as pediatric neuropsychology, to the extent that school psychology is informed by basic cognitive and developmental science). This empirical study is a direct attempt to take a step in that direction by integrating research by cognitive, developmental, and school psychologists.

The present findings have the potential to inform regular and special education teachers' curricula by taking into consideration the developmental nature of cognition and mathematics learning. School psychologists working with students with developmentally-based learning disabilities in mathematics (and perhaps, with acquired acalculia if this model is tested among students with traumatic brain injuries, for example) will have a theoretical framework that is completely approachable with current major diagnostic instruments. Last, this study represents a timely contribution for school psychology scientist-practitioners. The ultimate goal of this study is to strengthen the scientific research foundation of school psychology.

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APPENDIX A

INSTITUTIONAL REVIEW BOARD EXEMPTION

ARIZONA STATE UNIVERSITY
Research Compliance Office
Office for Research & Sponsored Projects Administration
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To:	Linda Caterino Kulhavy EDB 308K
From:	Albert Kagan, Chair Institutional Review Board
Date:	05/16/2006
Committee Action:	**Exemption Granted**
IRB Action Date:	05/16/2006
IRB Protocol #:	0605000839
Study Title:	The Contribution of the Woodcock-Johnson III Test of Cognitive Ability in Identifying Educational Di:

The above-referenced protocol is considered exempt after review by the Institutional Review Board pursuant to Federal regulations, 45 CFR Part 46.101(b)(4) .

This part of the federal regulations requires that the information be recorded by investigators in such a manner that subjects cannot be identified, directly or through identifiers linked to the subjects. It is necessary that the information obtained not be such that if disclosed outside the research, it could reasonably place the subjects at risk of criminal or civil liability, or be damaging to the subjects' financial standing, employability, or reputation.

You should retain a copy of this letter for your records.

BIOGRAPHICAL SKETCH

Carlos Orestes Calderón Tena was born in Mexico City and grew up in Mexicali, Baja California. He attended La Sierra University in Riverside, California, where he graduated with a bachelor's degree in psychology, with minors in music (piano) and Spanish (literature). Before becoming a doctoral student in educational psychology at Arizona State University (ASU), Carlos graduated with a master's degree in psychology at ASU. Carlos' research has focused on social, cultural, and cognitive aspects of development.