Symplectic Topology and Geometric Quantum Mechanics

by

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ABSTRACT

The theory of geometric quantum mechanics describes a quantum system as a Hamiltonian dynamical system, with a projective Hilbert space regarded as the phase space. This thesis extends the theory by including some aspects of the symplectic topology of the quantum phase space. It is shown that the quantum mechanical uncertainty principle is a special case of an inequality from J-holomorphic map theory, that is, J-holomorphic curves minimize the difference between the quantum covariance matrix determinant and a symplectic area. An immediate consequence is that a minimal determinant is a topological invariant, within a fixed homology class of the curve. Various choices of quantum operators are studied with reference to the implications of the J-holomorphic condition. The mean curvature vector field and Maslov class are calculated for a lagrangian torus of an integrable quantum system. The mean curvature one-form is simply related to the canonical connection which determines the geometric phases and polarization linear response. Adiabatic deformations of a quantum system are analyzed in terms of vector bundle classifying maps and related to the mean curvature flow of quantum states. The dielectric response function for a periodic solid is calculated to be the curvature of a connection on a vector bundle.

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Chapter 1

INTRODUCTION

The primary objective of this thesis is to describe an approach to quantum mechanics that is based in a framework of differential geometry and Hamiltonian dynamical systems. Topology and geometry of symplectic manifolds are fundamental to this approach. The guiding purpose of the thesis is to formulate a geometric description of condensed matter physics by application of geometric quantum mechanics to many-body systems. The project was originally motivated by developments in condensed matter physics that involve a geometric perspective on the electronic properties of matter, particularly where holonomy and characteristic classes arise. An important feature to understand in this perspective is the geometric phase appearing in theories of the polarization and transport properties of matter.[95, 82] According to the modern theory of polarization,[82, 88] the information about the macroscopic polarization of an extended insulating system such as a periodic crystalline solid, is not in the charge density, but in the wave function. Also related to the geometric underpinnings of our understanding of interactions in matter, is pre-metric electromagnetism which features the significance of the constitutive relation.[47, 46, 35, 36]

During the same period of time that intrinsic geometric structures of matter have been recognized in physics, a new field of mathematics has emerged: symplectic topology, which is the study of the *global* structure of a symplectic manifold.[62] A foundational result is Gromov's[43] nonsqueezing theorem, which states that if there is an embedding of the closed symplectic Euclidean ball $B^{2n}(r)$ of radius r into the symplectic cylinder $B^2(R) \times \mathbb{R}^{2n-2}$ that preserves the symplectic form, then $r \leq R$. It shows that there is a basic property of the ball and the cylinder that is invariant under symplectomorphisms, which is two-dimensional. The symplectic capacities[50, 62] are defined to describe this symplectic area invariance. New methods have been developed to study the group of symplectomorphisms of a symplectic manifold, and its relation to the group of volume-preserving diffeomorphisms. The method of J-holomorphic curves, introduced by Gromov[43], has proved to be an especially powerful and useful tool.

The importance of symplectic geometry as a key to understanding the dynamics of physical systems is well established.[6, 61] Could the new insights and methods of symplectic topology be applied to the recent physical theories of matter that emphasize a synthesis of topological and geometric perspectives? This question, and the attempt to answer it, permeate the present work.

Quantum mechanics has traditionally been considered from an algebraic point of view. In this view, a quantum system is described with a complex separable Hilbert space $\mathcal{H}$. The observables, or measurable quantities of the system, are represented by self-adjoint linear operators on $\mathcal{H}$, while the dynamics of the system is given by a family of unitary operators. In contrast, classical mechanics is often described within a geometric perspective. The states of a classical mechanical system are elements of a symplectic manifold, $\mathcal{M}_{cl}$, called the phase space of the system. The classical observables are functions on $\mathcal{M}_{cl}$, while the dynamics is governed by the symplectic form and a preferred Hamiltonian vector field on $\mathcal{M}_{cl}$. The counterintuitive observation that such a nonlinear description of mechanics arises in the classical approximation to purely linear interactions within the more accurate quantum theory has prompted investigations into a more unified geometric view of mechanics; one of the goals of such an approach is to clarify the relationship between the classical and quantum theories. This approach has come to be known as geometric quantum mechanics. Its foundations were laid in early work by Chernoff and Marsden[24] and by Kibble[55] and was developed by others[8, 22, 13, 28, 48, 61, 83].

These investigations resulted in a geometric view of quantum mechanics that shows it to be a Hamiltonian dynamical system on a symplectic manifold; the phase space is the projective Hilbert space, $P(\mathcal{H})$. Schrödinger's equation on $P(\mathcal{H})$ is equivalent to Hamilton's equations, determined by the natural symplectic structure arising from the Hermitian scalar product on $\mathcal{H}$. The geometric point of view emphasizes that the quantum phase space is endowed with an extra structure not found in classical mechanics. The Kähler structure of $P(\mathcal{H})$ furnishes the space of quantum states with a Riemannian metric in addition to the symplectic structure characteristic of Hamiltonian mechanics. This additional structure can be viewed as the source of features in the quantum theory that are distinctly different from classical mechanics.

One of the aims of geometric quantum theory has been to describe a quantum system entirely on the Kähler manifold $P(\mathcal{H}) \cong \mathbb{C}P^\infty$ without resource to the Hilbert space $\mathcal{H}$ at all, and this effort has been largely successful. On the other hand, it is commonly known that macroscopic quantum effects can arise as manifestations of the geometric phase, which is the holonomy of a closed path in the space of states $P(\mathcal{H})$. The relationship of vectors in $\mathcal{H}$ to quantum states in $P(\mathcal{H})$ can be viewed in terms of the principal bundle $U(1) \hookrightarrow S(\mathcal{H}) \to P(\mathcal{H})$, where $S(\mathcal{H}) = \{\psi \in \mathcal{H} : |\psi|^2 = 1\}$. A distinguished connection on this bundle defines the horizontal subspace of the tangent space $T_\psi(\mathcal{H})$ at a point $\psi \in \mathcal{H}$ to be the orthogonal complement to the fiber at $[\psi] \in P(\mathcal{H})$, where orthogonality is determined by the given Hermitian scalar product on $\mathcal{H}$. The curvature of this connection is directly related to the symplectic form Ω on $P(\mathcal{H})$ which is determined as the imaginary part of the Fubini-Study Hermitian metric. The cohomology information contained in the expanded view which includes this extra bundle structure is somehow embodied in the special structure of the quantum phase space as a homogeneous (Hermitian symmetric) space.

An aspect of the special structure of quantum mechanics is evident in the following notable feature. A dynamical system on $P(\mathcal{H})$ determined by a time-independent Hamiltonian function is completely integrable. In the finite-dimensional case, the expectation values of the eigenspace projection operators $\hat{P}_j$ are n constants of the motion. This fact is equivalent to the existence of a torus action on $P(\mathcal{H})$, providing a foliation of the quantum phase space by tori $\mathcal{L} = \mathbb{T}^n \subset P(\mathcal{H})$ with the *lagrangian* property that the symplectic form vanishes on $\mathcal{L}$, that is, $\Omega|_{\mathcal{L}} = 0$. Moreover, if we define a certain symplectic vector bundle $\pi : E \to P(\mathcal{H})$, the Lagrangian subbundle $\mathcal{L}_E \to P(\mathcal{H})$ is a flat bundle, that is, the curvature of the connection vanishes on $\mathcal{L}_E$.

The space $\Lambda(\mathcal{L})$ of lagrangian submanifolds of $P(\mathcal{H})$ may be associated with a family of integrable systems. Moreover, the dynamics determined by a Hamiltonian function on $P(\mathcal{H})$ that depends adiabatically on time can be viewed as a path in $\Lambda(\mathcal{L})$.

The mean curvature vector field H_i of an immersion $i : \mathcal{L} \to \mathcal{M}$ of a lagrangian submanifold $\mathcal{L}$ in $\mathcal{M}$ is the trace of the second fundamental form of $\mathcal{L}$. As such, it is a section of the normal bundle to $\mathcal{L}$ in $\mathcal{M}$, and generates a local flow on $\mathcal{L}$. A family of submanifolds evolves under mean curvature flow if the velocity at each point of the submanifold is given by the mean curvature vector at that point. In addition, the mean curvature one-form $\alpha_{H_i} := \frac{1}{\pi}\iota_{H_i}\Omega$ represents the Maslov class $\mu \in H^1(\mathcal{L}, \mathbb{Z})$.[64] Since the quantum phase space $P(\mathcal{H})$ is Einstein-Kähler, the one form α_{H_i} on $\mathcal{L} = \mathbb{T}^n \subset P(\mathcal{H})$ is closed[31], and so defines a real cohomology class in $H^1(\mathcal{L}, \mathbb{R})$. In particular, this means that lagrangian submanifolds stay lagrangian under the mean curvature flow on $P(\mathcal{H})$. Moreover, in this case, under global Hamiltonian isotopy, the one forms α_{H_i} represent the same cohomology class.

A central theme of the thesis is the study of deformations of a quantum system; one type of such a deformation is time evolution. The investigation proceeds within an extended framework for geometric quantum mechanics that incorporates some recent developments in the emerging field of symplectic topology, which allows clarification of the relationship between the notions of symplectic capacitance, the geometric phase, and the uncertainty principle.

Chapter 2 provides a review of the theory of geometric quantum mechanics. Chapter 3 shows that the quantum mechanical uncertainty principle is a special case of an inequality from J-holomorphic map theory. The inequality can be viewed as a comparison of metric and symplectic areas. It is shown that the quantum covariance matrix determinant $D(\hat{A}, \hat{B}, z) = (\Delta\hat{A})^2_z\,(\Delta\hat{B})^2_z - (C(\hat{A}, \hat{B})_z)^2$ is equal to the *harmonic energy* of a complex map $u : \Sigma \to \mathcal{H}$. Moreover, J-holomorphic curves minimize the difference between the quantum covariance matrix determinant $D(\hat{A}, \hat{B}, z)$ and the square of the symplectic area $\Omega(\hat{A}z, \hat{B}z)$. When equality is achieved, the off-diagonal part $C(\hat{A}, \hat{B})_z$ of the covariance matrix vanishes and the product of the variances $(\Delta\hat{A})_z\,(\Delta\hat{B})_z$ is a topological invariant, within a fixed homology class of the curve. Among the examples considered is the choice of $\hat{A} = \hat{X}$ and $\hat{B} = \hat{P}$, the position and momentum operators. In this case, $\Omega(\hat{A}z, \hat{B}z) = \hbar$ represents a symplectic area in the quantum phase space.

In Chapter 4, the mean curvature vector field is calculated for a finite dimensional quantum system, viewed as a Hamiltonian dynamical system on $P(\mathcal{H})$, the Maslov class representative is thereby determined and related to the connection one form and holonomy group on $U(1) \hookrightarrow S(\mathcal{H}) \to P(\mathcal{H})$. Of interest is the time evolution problem determined by the mean curvature flow of the quantum system. If the initial state is a lagrangian immersion, then the closedness of the mean curvature form guarantees that the deformation is lagrangian; that is, the mean curvature flow may be regarded as a smooth family of integrable systems. The relationship between adiabatic time evolution and the mean curvature flow of a quantum system is studied and compared to the isodrastic condition of stationary action given by Weinstein [93].

In Chapter 5, a study of the symplectic structure of a periodic solid is begun, and the dielectric function for a periodic solid is calculated to be the curvature of a connection on a vector bundle. Chapter 6 serves as an appendix to provide fundamentals of differential and symplectic geometry and includes the numbered examples.

Chapter 2

A GEOMETRIC VIEW OF QUANTUM MECHANICS

2.1 Hamiltonian Dynamics

A Hamiltonian dynamical system consists of a phase space, which is a smooth manifold $\mathcal{M}$ equipped with a symplectic form, and a Hamiltonian vector field $X_E : \mathcal{M} \to T(\mathcal{M})$.

Recall that a two-form ω on $\mathcal{M}$ is symplectic if ω is closed and nondegenerate. That is, 1) $d\omega = 0$ and 2) for each $p \in \mathcal{M}$, $\omega : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}$ such that $\omega(u, v) = 0$ for all $v \in T_p\mathcal{M}$ only when $u = 0$.

Nondegeneracy of ω implies that the contraction map of a vector field $X \in \mathfrak{X}(\mathcal{M})$ with ω

$$\iota : T_p(\mathcal{M}) \to T_p^*(\mathcal{M}) \quad X \mapsto \iota_X\omega \quad \mathrm{d}efined\ by\ \ \iota_X(\cdot) = \omega(X, \cdot) \tag{2.1}$$

is injective.

A vector field X on $\mathcal{M}$ is called *symplectic* if $\iota_X\omega$ is closed. A vector field X on $\mathcal{M}$ is called *hamiltonian* if $i_X\omega$ is exact. Equivalently, a vector field $X : \mathcal{M} \to T(\mathcal{M})$ is hamiltonian if there is a C^1 function $E : \mathcal{M} \to \mathbb{R}$ such that

$$\iota_X\omega = dE\,. \tag{2.2}$$

Locally on every contractible open set, every symplectic vector field is hamiltonian. If the first de Rham cohomology group is trivial, then globally every symplectic vector field is hamiltonian. When (2.2) holds, we write $X = X_E$ and call E an energy function for the vector field X_E. $(\mathcal{M}, \omega, X_E)$ is called a *Hamiltonian system*.

A hamiltonian vector field X_F generates the one-parameter group of diffeomorphisms $\{\phi_t^F\} : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, where $\phi_t^F \in \mathrm{D}iff\mathcal{M}$ satisfies $\frac{d}{dt}\phi_t^F = X_F \circ \phi_t^F$, $\phi_0^F = \mathrm{i}d$. In addition, as a one-parameter group $\{\phi_t^F\}$ satisfies the flow property

$$\phi_{t+s}^F(p) = \phi_t^F(\phi_s^F(p))\ \mathrm{f}or\,\mathrm{all}\,t, s \in \mathbb{R}\,\mathrm{a}nd\,p \in \mathcal{M}\,. \tag{2.3}$$

Now consider the dynamics of the Hamiltonian system $(\mathcal{M}, \omega, X_E)$. Each point of the symplectic manifold $\mathcal{M}$ corresponds to a *state* of the system, and $\mathcal{M}$ is called the state space or the phase space. The *observables*, or measurable quantities, in Hamiltonian mechanics are real-valued functions on $\mathcal{M}$. Any function $G : \mathcal{M} \to \mathcal{M}$ is constant along the orbits of the flow of X_F if and only if the *Poisson bracket*

$$\{G, F\}_z := \omega_z(X_G, X_F) = dG(X_F)(z) \tag{2.4}$$

vanishes for all $z \in \mathcal{M}$.

We assume that, with respect to a given choice of time axis, the time evolution of each state can be represented by a path in $\mathcal{M}$. Let $m \in \mathcal{M}$ and let $t \mapsto \phi_t$ be the local 1-parameter group generated by the vector field X_E in a neighborhood of m. If the initial state is m, then the state follows the orbit $z_m : \mathbb{R} \to \mathcal{M}$ defined by $z_m(t) = \phi_t(m)$ with $z_m(0) = m$. When $\mathcal{M}$ is compact, the path z_m is an integral curve of X_E starting at m. In other words, given the Hamiltonian vector field X_E on $\mathcal{M}$ and a specified initial condition $z_m(0) = m$, the trajectory $z_m(t)$ is uniquely determined by *Hamilton's equations*, $z' = X_E(z)$, or

$$z'_m(t) = (X_E)_{z_m(t)} \, . \tag{2.5}$$

The flow property of ϕ_t says that $z_p(t+s) = \phi_t(z_p(s))$. ϕ_t preserves the symplectic structure: $\phi_t^* \omega = \omega$, since

$$\mathcal{L}_{X_E}\omega = \iota_{X_E} d\omega + d(\iota_{X_E}\omega) = 0 + ddE = 0 \, , \tag{2.6}$$

where $\mathcal{L}_{X_E}$ is the Lie derivative with respect to X_E.

Observe that X_E is tangent to the level sets of E, since $dE(X_E) = \iota_{X_E}\omega(X_E) = \omega(X_E, X_E) = 0$.

Theorem 1 *(conservation of energy) If $(\mathcal{M}, \omega, X_E)$ is a Hamiltonian system and $\alpha(t)$ is an integral curve of X_E, then $E(\alpha(t))$ is constant for all $t \in I$. If ϕ_t denotes the flow of E, then $E \circ \phi_t = E$.*

Proof: By using the chain rule, Hamilton's equations (2.5), and the definition (2.2), we have

$$\begin{aligned}\frac{d}{dt}E(\alpha(t)) &= dE_{\alpha(t)}(\alpha'(t)) = dE_{\alpha(t)}((X_E)_{\alpha(t)}) \\ &= \omega((X_E)_{\alpha(t)}, (X_E)_{\alpha(t)}) = 0\,. \end{aligned} \tag{2.7}$$

□

When the Hamiltonian function depends on time, $H : \mathcal{M} \times [0,1] \to \mathbb{R}$, $H(z,t) = H_t(z)$, one can define the time dependent vector field, X_{H_t} as above. A *Hamiltonian isotopy* is a family of symplectomorphisms $\{\phi_t^E\}$ with $\phi_0^E = \mathrm{i}d$, which satisfies

$$\frac{d}{dt}(\phi_t^E(z)) = X_{E_t}(\phi_t^E(z))\,, \tag{2.8}$$

which are tangent to X_{H_t} at $z(t)$. In the time-dependent case, the family $\{\phi_t^E\}$ will not generally satisfy the flow property (2.3).

Let (V,ω) be a 2n-dimensional vector space and let

$$(q,p) = (q_1, \cdots, q_n, p_1, \cdots, p_n) \tag{2.9}$$

be canonical coordinates with respect to which Ω has matrix J as in (6.21). In this coordinate system, $X_H : V \to V$ is given by

$$X_H = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}) = J \cdot \nabla H\,. \tag{2.10}$$

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i\,, \tag{2.11}$$

and the solution curves $(q_i(t), p_i(t)) = \phi_t((q(0), p(0))$ satisfy Hamilton's equations are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,. \tag{2.12}$$

Examples include classical dynamics on the cotangent bundle of space or spacetime.

By Darboux's theorem, if $\mathcal{M}$ is a $2n$-dimensional symplectic manifold, then one can choose local coordinates in some neighborhood of each point of $\mathcal{M}$ as in (2.9), so that the symplectic form takes the canonical form (2.11).

Solutions to Hamilton's equations $\dot{z} = X_H(t,z)$ obey a variational principle known as the principle of stationary action. In classical mechanics, the variational problem is often expressed in terms of a Lagrangian function, $\mathcal{L}$, defined on a configuration space, $\mathcal{C}$ (usually isomorphic to $\mathbb{R}^n$); the Euler-Lagrange solutions correspond to critical points of an integral of $\mathcal{L}$ for variations over a set of paths $[t_0, t_1] \to \mathcal{C}$. When a Legendre transformation of variables is made, the variational problem can be expressed in terms of a particular 1-form on the phase space $\mathcal{M} = T^*(\mathcal{C})$. Let $z : [t_0, t_1] \to \mathcal{M} \cong \mathbb{R}^{2n}$ with $z(t) = (x(t), y(t))$. Define the *action form* as

$$\lambda_H = \sum_{j=1}^{n} y_j dx_j - H dt \tag{2.13}$$

and the action integral as

$$\Phi_H(z) = \int_{t_0}^{t_1} (\langle y, \dot{x} \rangle - H(t,x,y))\, dt\,. \tag{2.14}$$

The following well known lemma[62] shows that the Euler-Lagrange equations of the action integral are the Hamilton equations.

Lemma 2 *A curve $z : [t_0, t_1] \to \mathbb{R}^{2n}$ is a critical point of Φ_H (with respect to fixed endpoints) if and only if it satisfies Hamilton's equations.*

2.2 Quantum Mechanics as a Hamiltonian Dynamical System

To see how quantum mechanics may be viewed as a Hamiltonian system, we must show the equivalence to the traditional algebraic point of view. In this view, a quantum system is described with a complex separable Hilbert space $\mathcal{H}$, while the observables, or measurable quantities of the system, are represented by self-adjoint linear operators on $\mathcal{H}$. A special role is played by the linear self-adjoint operator $\hat{H}$, known as the *Hamiltonian* operator, whose eigenvalues are the energies of the system: $\hat{H}\psi_\lambda = E_\lambda \psi_\lambda$. The dynamics of the elements of $\mathcal{H}$ is governed by solutions of Schrödinger's equation,

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t)\,. \tag{2.15}$$

To compare with Hamiltonian dynamics, let us first identify the phase space. Observe that the result of a measurement of an operator $\hat{O}$ on $\psi \in \mathcal{H}$ is given by the expectation value $\langle \psi, \hat{O}\psi \rangle / \langle \psi, \psi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Hermitian scalar product on $\mathcal{H}$. Notice that, for any nonzero $c \in \mathbb{C}$, $c\psi$ yields the same measured value as does ψ. A physical state at any time is determined by measuring a complete set of commuting elements of the collection of (densely defined) self-adjoint linear operators on $\mathcal{H}$, which defines a one-dimensional subspace of $\mathcal{H}$, called a *ray*. A ray is an equivalence class of vectors in $\mathcal{H}$: two vectors in $\mathcal{H}$ are equivalent if and only if one is a nonzero complex scalar multiple of the other. As such, physical quantum states are elements of the complex projective Hilbert space, $P(\mathcal{H})$. Thus, a quantum system may be described with the principal bundle $\mathbb{C}^* \hookrightarrow (\mathcal{H} - \{0\}) \xrightarrow{\pi} P(\mathcal{H})$ (Example 9), with state vectors in $\mathcal{H}$ and physical states in $P(\mathcal{H})$. Invariance under unitary transformations requires that state vectors be normalized to unit length; hence we frequently restrict to the subbundle $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$, where $S(\mathcal{H}) = \{z \in \mathcal{H} : |z|^2 = 1\}$ as in Example 10.

Recall from Example 19 that $P(\mathcal{H})$ is a Kähler manifold with the Fubini-Study Hermitian metric. In particular, $P(\mathcal{H})$ is equipped with a symplectic form which we denote by $\Omega_{P(\mathcal{H})}$. In the geometrical formulation of quantum mechanics,[8, 22, 55, 61] $P(\mathcal{H})$ is regarded as the state space of a Hamiltonian system on the symplectic manifold $(P(\mathcal{H}), \Omega_{P(\mathcal{H})})$. In fact, within the geometric perspective taken by Kibble and others, the dynamics of a quantum mechanical system may be described entirely in terms of the symplectic geometry of the state manifold $P(\mathcal{H}) \cong \mathbb{C}P_n$ (n may be ∞). Moreover, the Kähler structure on $\mathbb{C}P_n$ provides the state space of a quantum system with a Riemannian metric, which gives the system the probabilistic character that distinguishes it from a classical one.

The dynamics on $P(\mathcal{H})$ is related to the dynamics on $\mathcal{H}$ due to the fact that the preferred Hamiltonian vector field on $P(\mathcal{H})$ is the projection by π_* of the corresponding preferred Hamiltonian vector field on $\mathcal{H}$. To see how this works, it is helpful to recognize that the linear space $\mathcal{H}$ itself has a natural Kähler structure arising from the Hermitian inner product; in particular, $\mathcal{H}$ has a symplectic structure. Begin by decomposing the Hermitian inner product on $\mathcal{H}$ into real and imaginary parts.

$$\langle \psi, \phi \rangle := \frac{1}{2\hbar} g_{\mathcal{H}}(\psi, \phi) + \frac{i}{2\hbar} \Omega_{\mathcal{H}}(\psi, \phi) \tag{2.16}$$

The real part $g_{\mathcal{H}}$ is a Riemannian structure. Define the symplectic structure $\Omega_{\mathcal{H}}$ on $\mathcal{H}$ as in (6.31):

$$(\Omega_{\mathcal{H}})_p(v_p, w_p) := 2\,\mathrm{Im}\langle v, w\rangle\,, \tag{2.17}$$

for each $p \in \mathcal{H}$ and $v_p, w_p \in T_p(\mathcal{H})$. (needs more discussion in text.)(Track down $2\hbar$ needed to yield Schrödinger's equation.)

The expectation value function $A : \mathcal{H} \to \mathbb{R}$ for the linear self-adjoint operator $\hat{A}$ on $\mathcal{H}$ is defined by

$$A(\psi) := \frac{\langle \psi, \hat{A}\psi\rangle}{\langle \psi, \psi\rangle}\,. \tag{2.18}$$

Since $A(\psi)$ is real-valued,

$$A(\psi) = \frac{1}{2\hbar} g_{\mathcal{H}}(\psi, \hat{A}\psi)\,. \tag{2.19}$$

(normalization?) Associated to the function A is the Hamiltonian vector field $X_A \in \mathfrak{X}(\mathcal{H})$ defined by $\iota_{X_A}\Omega_{\mathcal{H}} = dA$, where ι_{X_A} denotes contraction with X_A, as in (2.1). By the identification of $T_p(\mathcal{H})$ with $\mathcal{H}$ for each $p \in \mathcal{H}$ (see Example 18), the definition of a vector field X as a map $\mathcal{H} \to T_p(\mathcal{H})$, $p \mapsto X_p$ reduces to a map $X : \mathcal{H} \to \mathcal{H}$. In this way, a linear operator acts like a vector field. As such, define

$$(X_A)_\phi = -\frac{i}{\hbar}\hat{A}\phi. \tag{2.20}$$

Using the definitions in Example 18 gives for $u \in T_z(\mathcal{H})$,

$$\begin{aligned} uA(z) = (dA)_z(u) &= \frac{d}{dt}\langle z + tu, \hat{A}(z + tu)\rangle|_{t=0} \\ &= 2\mathrm{Re}\langle u, \hat{A}z\rangle = \Omega_{\mathcal{H}}(-i\hat{A}z, u) \\ &= (\iota_{X_A}\Omega_{\mathcal{H}})_z(u)\,. \end{aligned} \tag{2.21}$$

Thus, each observable $A = \langle \cdot, \hat{A}\cdot\rangle$, with $\hat{A}$ self-adjoint (possibly unbounded), generates the 1-parameter group $\phi_t : \mathcal{H} \to \mathcal{H}$ with $\phi_t(z) = \exp(iAt)z$. The flow of a Hamiltonian vector field on $\mathcal{H}$ consists of *linear* symplectic transformations.

The quantum dynamics can be described in terms of the flow along the Hamiltonian vector field X_H defined by $\iota_{X_H}\Omega_{\mathcal{H}} = dH$ and $(X_H)_\psi = \frac{-i}{\hbar}\hat{H}\psi$. Thus, on the Hilbert space,

$\mathcal{H}$, Schrödinger's equation (2.15) take the form of Hamilton's equations (2.5)

$$\frac{d\psi(t)}{dt} = X_H \psi(t)\,. \tag{2.22}$$

With $z =$(Re ψ,Im ψ) in (2.13), the action form $\lambda_H \in T^*(\mathcal{H})$

$$\lambda_H = \frac{1}{2}\sum(\psi^* d\psi - \psi d\psi^*) - H dt\,. \tag{2.23}$$

One can then show that Schrödinger's equation arises from a variational principle, as in [58].

To study the dynamics on the quantum phase space, it is helpful to make use of the bijection between points of $P(\mathcal{H})$ and rank one projection operators on $\mathcal{H}$. For each $\phi \in \mathcal{H} - \{0\}$, let $[\phi]$ be the one-dimensional subspace of $\mathcal{H}$ generated by ϕ. Viewing $[\phi]$ as an element of $P(\mathcal{H})$, define

$$V_\phi \;:=\; \{[x] \in P(\mathcal{H}) \mid \langle \phi, x\rangle \neq 0\}; \tag{2.24}$$

$$\{\phi\}^\perp \;:=\; \{x \in \mathcal{H} \mid \langle \phi, x\rangle = 0\} \tag{2.25}$$

$$b_\phi : V_\phi \;\to\; \{\phi\}^\perp,\;\; [x] \mapsto b_\phi([x]) := \frac{x}{\langle \phi, x\rangle} - \phi \tag{2.26}$$

The collection $\{V_\phi, b_\phi, \{\phi\}^\perp\}(\phi \in \mathcal{H}, ||\phi|| = 1)$ is a holomorphic atlas for $P(\mathcal{H})$. Observe that $\{\phi\}^\perp$ is a closed subspace of $\mathcal{H}$, and inherits the Hermitian product from $\mathcal{H}$. Thus to each point $[\phi] \in P(\mathcal{H})$ is associated the orthogonal subspace $\{\phi\}^\perp$. The map b_ϕ is, up to a scale factor, the projection onto this orthogonal subspace. Identifying each $[\phi] \in P(\mathcal{H})$ with the corresponding one-dimensional projection, $P(\mathcal{H})$ becomes the boundary of the positive part of the unit ball of the Banach space of trace class operators on $\mathcal{H}$.[27] Technical issues raised in the case of infinite dimensional $\mathcal{H}$ have been dealt with carefully in several works.[28, 24]

One can obtain the Fubini-Study Hermitian metric on $P(\mathcal{H})$ by pulling back the Killing-Cartan metric: $\langle A, B\rangle := -\frac{1}{2}\mathrm{Tr}(AB)$ for $A, B \in u(\mathcal{H})$ by the map b_ϕ[44, 13] or by directly using the isomorphism discussed in Example 18.

$$\langle V_1, V_2\rangle_{P(\mathcal{H})} = 2\hbar \frac{\langle u_1, u_2\rangle\langle z, z\rangle - \langle u_1, z\rangle\langle z, u_2\rangle}{\langle z, z\rangle^2} \tag{2.27}$$

for any two vectors $u_1, u_2 \in T_z(\mathcal{H})$ satisfying $\pi_* u_k = V_k \in T_{\pi z}(P(\mathcal{H}))$ for $k = 1, 2$.

With the normalization of the metric given in (2.27), the Riemannian scalar curvature (holomorphic sectional curvature) is[28, 48]

$$c = \frac{2}{\hbar}\,. \tag{2.28}$$

Now consider the dynamics on the phase space $P(\mathcal{H})$. Observe that the definition (2.18) for $A : \mathcal{H} \to \mathbb{R}$ does not depend on the representative $\psi \in [\psi]$, and thus provides a definition of the function $a : P(\mathcal{H}) \to \mathbb{R}$ by $a([\psi]) = A(\psi)$. The symplectic form $\Omega_{P(\mathcal{H})}$ which governs the dynamics on $P(\mathcal{H})$ is defined the imaginary part of the Fubini-Study metric (2.27) on $P(\mathcal{H}) \simeq \mathbb{C}P_n$ (see also (6.33). Thus, we may define the preferred Hamiltonian vector field X_h on $P(\mathcal{H})$ by $\iota_{X_h}\Omega_{P(\mathcal{H})} = dh$, where $h([\psi]) = H(\psi)$. The vector field X_h gives rise to Hamilton's equations on $P(\mathcal{H})$,

$$\frac{d[\psi](t)}{dt} = (X_h)_{[\psi(t)]}\,. \tag{2.29}$$

Show $X_h = \pi_* X_H$ (see [55]).

Thus, from the geometric point of view, quantum observables are represented by real-valued functions on the quantum phase space $P(\mathcal{H})$, and Schrödinger evolution can be viewed as the symplectic flow of a Hamiltonian function on $P(\mathcal{H})$. Unlike the linear space $\mathcal{H}$, $P(\mathcal{H})$ is a (nonlinear) manifold, so that, as in classical mechanics, the flow generated by an observable consists of nonlinear symplectic transformations. The geometric view of quantum mechanics has this feature in common with the standard symplectic formulation of classical mechanics. In addition to the symplectic structure $\Omega_{P(\mathcal{H})}$, the quantum phase space $P(\mathcal{H})$ has the Riemannian metric $g_{P(\mathcal{H})}$ arising from the real part of the Fubini-Study Kähler structure, which accounts for the uncertainty structure found in quantum mechanics. The metric $g_{P(\mathcal{H})}$ also allows a description of transitions between states in the phase space $P(\mathcal{H})$.

2.3 The Uncertainty Principle in Terms of the Kähler Structure

The possible outcomes of measurement of a self-adjoint quantum operator $\hat{A}$ on the state $\psi \in \mathcal{H}$ have a probability distribution with mean equal to the expectation value

$$\langle \hat{A} \rangle_\psi := \langle \psi, \hat{A}\psi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian metric on $\mathcal{H}$. The proof of the generalized Heisenberg uncertainty principle given by Shankar[86] is suitable for adaptation to the setting of the quantum phase space $P(\mathcal{H})$, as shown below. Define the *dispersion* or uncertainty of the function values as

$$(\Delta \hat{A})_\psi := [\langle \psi, (\hat{A} - \langle \hat{A} \rangle_\psi)^2 \psi \rangle]^{1/2} . \tag{2.30}$$

Observe that $(\Delta \hat{A})_\psi = 0$ for eigenvectors of $\hat{A}$. In the theory of geometric quantum mechanics, it is emphasized that the dispersion $(\Delta \hat{A})_\psi$ is directly related to the Riemannian metric at $[\psi] \in P(\mathcal{H})$.[5, 8, 28, 78] For two self-adjoint operators $\hat{A}$ and $\hat{B}$ on $\mathcal{H}$, define the *covariance* or correlation function as

$$C(\hat{A}, \hat{B})_\psi := \frac{1}{2} \langle \psi, (\hat{A}\hat{B} + \hat{B}\hat{A})\psi \rangle - \langle \hat{A} \rangle_\psi \langle \hat{B} \rangle_\psi . \tag{2.31}$$

In (2.31) and elsewhere throughout this thesis, as necessary, we assume that the operators $\hat{A}$ and $\hat{B}$ have a common core, and that ψ is taken from the common core. In practice, the operators commonly considered in quantum mechanics are differential operators and multiplication (Töplitz) operators acting on $L^2(\mathcal{M}, \mathbb{C})$, such as the momentum and position operators. In these cases, the set of C^∞ functions with compact support constitutes a common core.

Lemma 3 *The component of the vector $\hat{A}\psi \in \mathcal{H}$ that is Hermitian orthogonal to ψ is*

$$(\hat{A} - \langle \hat{A} \rangle_\psi)\psi . \tag{2.32}$$

If $\hat{A}$ is self-adjoint, then $\hat{A}\psi$ decomposes as

$$\hat{A}\psi = \langle \hat{A} \rangle_\psi \psi + (\Delta \hat{A})_\psi \chi , \tag{2.33}$$

where $\langle \chi, \psi \rangle = 0$ and $\langle \chi, \chi \rangle = 1$. (Note: we have assumed that $\langle \psi, \psi \rangle = 1$; if we do not assume so, we must normalize.)

Proof: We can assume that $\hat{A}\psi = \alpha\psi + \beta\chi$, for some α, $\beta \in \mathbb{C}$ with $\langle \chi, \psi \rangle = 0$ and $\langle \chi, \chi \rangle = 1$. Taking the scalar product of both sides of this equation with ψ gives $\langle \psi, \hat{A}\psi \rangle = \alpha\langle \psi, \psi \rangle = \alpha$. To determine β, observe that $\beta\chi = \hat{A}\psi - \langle \psi, \hat{A}\psi \rangle\psi$. Then $|\beta|^2 = \langle \beta\chi, \beta\chi \rangle = \langle \hat{A}\psi - \langle \psi, \hat{A}\psi \rangle\psi, \hat{A}\psi - \langle \psi, \hat{A}\psi \rangle\psi \rangle$. If $\hat{A}$ is self-adjoint, then $|\beta|^2 = (\Delta\hat{A})^2_\psi$. □

Proposition 4 *The Riemannian metric $g : T(P(\mathcal{H})) \times T(P(\mathcal{H})) \to \mathbb{R}$ which is the real part of the Fubini-Study Kähler metric on $P(\mathcal{H})$ is*

$$g_\psi(X_A, X_A) = \frac{2}{\hbar}\langle (\hat{A} - \langle \hat{A} \rangle_\psi)\psi, (\hat{A} - \langle \hat{A} \rangle_\psi)\psi \rangle = \frac{2}{\hbar}(\Delta\hat{A})^2_\psi \qquad (2.34)$$

$$g_\psi(X_A, X_B) = \frac{2}{\hbar}C(\hat{A}, \hat{B})_\psi. \qquad (2.35)$$

The uncertainty principle gives a lower bound on the product $(\Delta\hat{A})_\psi\,(\Delta\hat{B})_\psi$. In general, the lower bound depends on the operator and on the state, and in some cases, the lower bound is independent of the state.

Theorem 5 *(Uncertainty Principle) Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators on $\mathcal{H}$ and $\psi \in \mathcal{H}$. Then*

$$(\Delta\hat{A})^2_\psi\,(\Delta\hat{B})^2_\psi - (C(\hat{A}, \hat{B})_\psi)^2 \geq \frac{1}{4\hbar^2}\Omega(\hat{A}\psi, \hat{B}\psi)^2\,, \qquad (2.36)$$

where Ω is the symplectic form on $P(\mathcal{H})$.

Proof: By using the self-adjoint property, observe that the product of the dispersions takes the form

$$(\Delta\hat{A})^2_\psi\,(\Delta\hat{B})^2_\psi = |(\hat{A} - \langle \hat{A} \rangle_\psi)\psi|^2\,|(\hat{B} - \langle \hat{B} \rangle_\psi)\psi|^2\,. \qquad (2.37)$$

By applying the Schwartz inequality,

$$(\Delta\hat{A})^2_\psi\,(\Delta\hat{B})^2_\psi \;\geq\; |\langle (\hat{A} - \langle \hat{A} \rangle_\psi)\psi, (\hat{B} - \langle \hat{B} \rangle_\psi)\psi \rangle|^2\,. \qquad (2.38)$$

Now use the fact that the Hermitian orthogonal part of $\hat{A}\psi$ is $(\hat{A} - \langle\hat{A}\rangle_\psi)\psi$.

$$(\Delta\hat{A})^2_\psi\,(\Delta\hat{B})^2_\psi \;\geq\; \frac{1}{4\hbar^2}|\langle\hat{A}\psi,\hat{B}\psi\rangle_{P(\mathcal{H})}|^2 \tag{2.39}$$
$$= \frac{1}{4\hbar^2}\left(g(\hat{A}\psi,\hat{B}\psi)^2 + \Omega(\hat{A}\psi,\hat{B}\psi)^2\right), \tag{2.40}$$

where g and Ω are the Riemannian metric and symplectic form on $P(\mathcal{H})$. Finally, since g is symmetric, we have $g(\hat{A}\psi,\hat{B}\psi) = C(\hat{A},\hat{B})_\psi$, so the result follows. □

Proposition 6 *Suppose that $\psi : [0,\tau] \to \mathcal{H}$ is an integral curve of the vector field X_A on $\mathcal{H}$ corresponding to the operator A. Let C be the projected curve $\pi(\psi) \subset P(\mathcal{H})$. Then the distance along the curve is*

$$s = \int_C (\Delta\hat{A})_\psi(t)\,dt\,. \tag{2.41}$$

Proof: If we use the condition $\langle\psi(t),\frac{d\psi}{dt}\rangle = 0$ to define the horizontal subspace of $T_{\psi(t)}(S(\mathcal{H}))$ (see section 2.4), then we can decompose

$$\frac{d}{dt}\psi(t) = i\hat{A}(t)\psi(t) \tag{2.42}$$

into horizontal and vertical parts.

$$\frac{d\psi}{dt} = (\frac{d\psi}{dt})_{vert} + (\frac{d\psi}{dt})_{horiz} \tag{2.43}$$

where

$$\frac{d\psi}{dt}_{vert} = \langle\psi,\frac{d\psi}{dt}\rangle\psi$$

corresponds to the covariant derivative in the associated line bundle. Using the result of the lemma, we have

$$(\frac{d\psi}{dt})_{vert} = i\langle\hat{A}\rangle_\psi\psi \;\text{ and }\; (\frac{d\psi}{dt})_{horiz} = i(\Delta\hat{A})_\psi\chi\,.$$

Hence, the distance along the curve C is

$$s = \frac{\hbar}{2}\int_C (g_{\psi(t)}(X_A,X_A))^{1/2}dt = \int_C \langle(\frac{d\psi}{dt})_{horiz},(\frac{d\psi}{dt})_{horiz}\rangle^{1/2}dt = \int_C (\Delta\hat{A})_\psi(t)\,dt$$

□

2.4 The Role of the Principal Connection

The focus of this section is the connection on the principal bundle $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ that is defined through the Hermitian scalar product on $calH$. The curvature of this connection pulls back to a 2-form on $P(\mathcal{H})$ that is identical to the symplectic form $\Omega_{P(\mathcal{H})}$ on $P(\mathcal{H})$. Thus, the holonomy of a curve in $P(\mathcal{H})$ can be described as an integral of the symplectic form $\Omega_{P(\mathcal{H})}$.

The Hermitian inner product on $\mathcal{H}$ determines a natural principal connection on the principal bundle $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$. To see this, first note that, since $S(\mathcal{H}) \subset \mathcal{H}$, an element z in $S(\mathcal{H})$ is an element of $\mathcal{H}$. Also, since $\mathcal{H}$ is a linear space, a tangent vector $v \in T_z(S(\mathcal{H}))$, also may be viewed as an element of $\mathcal{H}$ and thus the inner product (z, v) is defined. We can use the requirement of hermitian orthogonality, that is, $(z, v) = 0$, to define horizontal vectors as follows.[6] For $z \in S(\mathcal{H})$, let $x = \pi(z)$. The vector z is a euclidean normal vector to the sphere $S(\mathcal{H})$ at z and the fiber $F_x = \pi^{-1}(x) = \{z \mid \pi(z) = x\}$ is the complex line determined by z. The idea is to decompose $v \in T_z(S(\mathcal{H}))$ into horizontal and vertical components: one in F_x and the other in the hermitian-orthogonal direction. The component of v in F_x is the vertical part and is determined by the requirement that $\pi_* v = 0$. Note that hermitian-orthogonal to the vector z means euclidean-orthogonal to the vectors z and iz. The vector iz is a vector tangent to the circle in which the sphere intersects the complex line passing through z, that is, iz is in $T_z(F_x)$. Thus the component of the vector v which is hermitian-orthogonal to z is tangent to the sphere $S(\mathcal{H})$ and euclidean-orthogonal to the circle in which the sphere intersects the line F_x. With this understanding, for each $z \in S(\mathcal{H})$, we define the horizontal subspace of $T_z(S(\mathcal{H}))$ as the set of vectors tangent to $S(\mathcal{H})$ at z that are hermitian-orthogonal to z:

$$H_z(S(\mathcal{H})) = \{v \in T_z(S(\mathcal{H})) \mid \langle z, v \rangle = 0\}. \tag{2.44}$$

Thus, a curve $\psi : [0, 1] \to S(\mathcal{H})$ is horizontal if $\langle \psi(t), \psi'(t) \rangle = 0$, for all $t \in [0, 1]$. Since $\langle \psi, \psi \rangle = 1$ for all $\psi \in S(\mathcal{H})$, it is always true that Re $\langle \psi(t), \psi'(t) \rangle = 0$. Hence, the horizontal condition amounts to Im $\langle \psi(t), \psi'(t) \rangle = 0$. So, define the connection 1-form ω

on $S(\mathcal{H})$ as

$$\omega_z(v) := i\,\mathrm{Im}\,\langle z, v\rangle \text{ for } v \in T_z(S(\mathcal{H}))\,. \tag{2.45}$$

Indeed, since the Lie algebra of $U(1)$ is $u(1) = i\mathbb{R}$, ω is a Lie-algebra valued 1-form, and it can be shown that ω satisfies the conditions of a connection form.

In quantum mechanics, symmetry transformations of the system are determined by unitary operators in order to guarantee invariance of the Hermitian form. The natural connection on $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ determined by the Hermitian inner product as described above is the unique connection that is invariant under the unitary group $U(\mathcal{H})$.[20] Invariance of the connection under $U(\mathcal{H})$ requires the horizontal subspace of $T_z(\mathcal{H})$ to be invariant under $U(\mathcal{H}')$, where $\mathcal{H}' = \{z\}^{\perp}$ is the orthogonal complement to $[z]$ in $\mathcal{H}$, with the result that this invariant subspace is orthogonal to the fiber.

If V is a trivializing neighborhood of $P(\mathcal{H})$ and $s : V \to S(\mathcal{H})$ is a local section, then, for $x \in V$, the pull-back $\mathcal{A} := s^*\omega$ acts on a vector $u \in T_x(P(\mathcal{H}))$ as

$$\mathcal{A}\,u = \omega_{s(x)}(s_{*x}(u)) = i\,\mathrm{Im}\langle s(x), s_{*x}(u)\rangle\,. \tag{2.46}$$

Here $s_*(u) = u(ds)$, is the directional derivative of the $\mathcal{H}$ valued function s in the direction u, as in (6.30)

Recall that a connection 1-form on a principal bundle gives rise to the holonomy of a closed path in the base of the bundle. For the case of the connection form ω (2.45) on the principal bundle $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ the result takes the following form.

Proposition 7 *For each $z \in S(\mathcal{H})$, let the horizontal subspace of $T_z(S(\mathcal{H}))$ be given as $H_z(S(\mathcal{H})) = \{v \in T_z(S(\mathcal{H})) \mid \langle z, v\rangle = 0\}$. If $\psi : [0,1] \to S(\mathcal{H})$ is any smooth closed curve in $S(\mathcal{H})$ which projects to a closed curve α in $P(\mathcal{H})$, then*

$$\exp\left(-\int_0^1 \langle \psi(t), \psi'(t)\rangle\, dt\right) \tag{2.47}$$

is the holonomy of the path α.

Proof: Let $\alpha : [0,1] \to P(\mathcal{H})$. Let $\psi : [0,1] \to S(\mathcal{H})$ be any smooth lift of α, that is, ψ has the property $\pi(\psi(t)) = \alpha(t)$ for all $t \in [0,1]$. A horizontal lift of α must be of the form

$\alpha^{\uparrow}(t) = \psi(t)g(t)$ for some curve $g : [0,1] \to G$. Since $G = U(1)$, then $g(t) = e^{i\theta(t)}$ with $\theta : [0,1] \to \mathbb{R}$. Hence, $\alpha^{\uparrow}(t) = \psi(t)e^{i\theta(t)}$ and the (horizontal) vector tangent to the curve at $\alpha^{\uparrow}(t)$ is

$$(\alpha^{\uparrow})'(t) = i\theta'(t)\alpha^{\uparrow}(t) \,+\, e^{i\theta(t)}\psi'(t)\,. \tag{2.48}$$

Next, take the inner product of both sides of Equation (2.48) on the left with $\alpha^{\uparrow}(t)$. Since $\alpha^{\uparrow}$ is assumed to be horizontal, $\langle \alpha^{\uparrow}(t), (\alpha^{\uparrow})'(t)\rangle = 0$. Since $\alpha^{\uparrow} \subset S(\mathcal{H})$, $\langle \alpha^{\uparrow}(t), \alpha^{\uparrow}(t)\rangle = 1$. Thus,

$$\begin{aligned} 0 &= i\theta'(t) \,+\, \langle \alpha^{\uparrow}(t), e^{i\theta(t)}\psi'(t)\rangle \\ -i\theta'(t) &= \langle \psi(t), \psi'(t)\rangle\,, \end{aligned} \tag{2.49}$$

so that, by integrating along the path, we have

$$\begin{aligned} \theta(1) - \theta(0) &= i\int_0^1 \langle \psi(t), \psi'(t)\rangle\, dt && (2.50) \\ &= i\int_0^1 \omega_{\psi(t)}(\psi'(t))\, dt\,. && (2.51) \end{aligned}$$

where ω is the connection 1-form (2.45).

Now suppose that α is a closed curve in $P(\mathcal{H})$ so that $\alpha(0) = \alpha(1)$. As in the proof of Theorem 3, without loss of generality we may suppose that $\alpha([0,1])$ is contained in a trivializing neighborhood $V \subset P(\mathcal{H})$ and that s is a local section defined on V. Then a natural choice for a lift of the path α is $\psi = s \circ \alpha$. In this case, $\psi(1) = \psi(0)$. Thus $\alpha^{\uparrow} = \psi e^{i\theta}$ implies that $\alpha^{\uparrow}(1) = \alpha^{\uparrow}(0)e^{i(\theta(1)-\theta(0))}$. Thus, (2.51) says that the difference in phase between initial and final points on the horizontal lift of the loop α is equal to the line integral of the connection 1-form along the closed lift ψ. Equation (2.51) may be written as

$$\theta(1) - \theta(0) = i\int_0^1 \mathcal{A}(\alpha'(t))\, dt\,, \tag{2.52}$$

where $\mathcal{A} := s^*\omega$ is the local 1-form on $P(\mathcal{H})$ whose action on tangent vectors is given by (2.46). By exponentiating both sides of (2.52), it may be seen that this statement is the content of Corollary 32 (section 6.3) for the case of the connection on $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ induced by the Hermitian scalar product on $\mathcal{H}$. $\square$

Now turn to the topic of time-dependent quantum mechanical systems. Michael Berry[14] showed that, in a system with a parameter dependent Hamiltonian operator, the cyclic adiabatic evolution of energy eigenfunctions contains a phase factor that depends on the geometrical structure of the parameter space, and does not depend on the duration of the evolution. Berry's phase is discussed in Chapter 4 in the context of the adiabatic approximation.

In the aftermath of Berry's discovery, Aharonov and Anandan[3] found a geometric phase that can be viewed as a generalization of Berry's phase, since it does not require any adiabatic approximation. A very interesting result is the equivalence of the geometric phase factor and the holonomy (2.47). The relation between the geometric phase and the holonomy of a connection was first pointed out by Simon[87] in the adiabatic context, to be discussed in the next chapter.

Theorem 8 *Let $\mathcal{H}$ be a complex Hilbert space. Let ω be the connection form on $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ induced by the Hermitian scalar product given on $\mathcal{H}$. Let $\alpha : [0, \tau] \to P(\mathcal{H})$ be a closed path, and let $\beta(\tau)$ be given by*

$$\beta(\tau) = i \int_0^\tau \langle \tilde{\psi}(t), \tilde{\psi}'(t) \rangle dt \,. \tag{2.53}$$

with $\tilde{\psi}$ a smooth closed lift of α (need a local trivialization here?). The Aharonov-Anandan phase factor $exp(i\beta(\tau))$ defined for cyclic evolution of a quantum system is exactly the holonomy (2.47) of the path α.

Proof: Following the original work[3], suppose that the curve $\psi : [0, \tau] \to S(\mathcal{H})$ satisfies Schrödinger's equation (2.15)with Hamiltonian $\hat{H}$. Define $\phi \in \mathbb{R}$ to be the phase difference between initial and final state vectors: $\psi(\tau) = e^{i\phi}\psi(0)$. Now define $\tilde{\psi} : [0, \tau] \to S(\mathcal{H})$ by $\tilde{\psi}(t) = e^{-if(t)}\psi(t)$ where $f : [0, \tau] \to \mathbb{R}$ so that $f(\tau) - f(0) = \phi$. Then $\tilde{\psi}(\tau) = \tilde{\psi}(0)$. From (2.15), we have

$$\frac{df}{dt} = \frac{-1}{\hbar} \langle \psi(t), \hat{H}\psi(t) \rangle + i \langle \tilde{\psi}(t), \tilde{\psi}'(t) \rangle \,. \tag{2.54}$$

Thus, integrating along the path ψ, we find that the total phase difference ϕ is decomposed into two parts:

$$
\begin{aligned}
\phi &= f(\tau) - f(0) = \chi(\tau) + \beta(\tau) && (2.55)\\
\chi(\tau) &= \frac{-1}{\hbar}\int_0^\tau \langle \psi(t), H\psi(t)\rangle dt\,, && (2.56)
\end{aligned}
$$

and $\beta(\tau)$ is given by (2.53). The component $\chi(\tau)$ is known as the *dynamical phase* and $\beta(\tau)$ is known as the *geometric phase*.

Now observe that $\tilde{\psi}$ may be written in terms of a section $s : P(\mathcal{H}) \to S(\mathcal{H})$, that is, $\tilde{\psi}(t) = (s\circ\pi\circ\psi)(t)$, and that the same $\tilde{\psi}(t)$ can be chosen for every curve $\gamma : [0,\tau] \to S(\mathcal{H})$ with the property $\pi \circ \psi = \pi \circ \gamma$ by appropriate choice of $f(t)$. Hence β is independent of ϕ and H for a given closed curve $\alpha = \pi \circ \psi$ in $P(\mathcal{H})$. □

As observed by Page[77], the curvature 2-form Ω of the connection form ω on $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ is proportional to the Ricci curvature tensor for the Fubini-Study metric, which (being Kähler-Einstein) is proportional to the Kähler form. Thus, the expression (6.16) for the holonomy in terms of the curvature Ω is equivalent to an integral of the symplectic 2-form. That is, the geometric phase is equal to the symplectic area of the surface spanned by the closed path in $P(\mathcal{H})$. The symplectic structure of the Aharonov-Anandan geometric phase was pointed out in [5, 22, 39].

Corollary 9 *In the finite-dimensional case, we can choose local coordinates so that, by using Stoke's theorem, the geometric phase is*

$$
\beta = \frac{1}{\hbar}\oint_\alpha \sum_k Q_k dP_k = \frac{1}{\hbar}\int_S \sum_k dP_k \wedge dQ_k\,, \tag{2.57}
$$

where S is any two dimensional manifold whose boundary is α.

Proof: By Darboux's theorem, locally there exist symplectic coordinates, that is, Q_k and P_k satisfy $JQ_k = -P_k$ and $JP_k = Q_k$. As in [5], we can choose, for example, $Q_j = w_j(1 + \sum \bar{w}_k w_k)^{-1/2}$, and $P_j = \sqrt{-1}Q_j$, where the w_k are the "homogeneous" local coordinates introduced in Example 9. □

The integral (2.57), known as the integral invariant of Poincaré-Cartan[6], is a symplectic invariant. That is, for any symplectomorphism S of $\mathcal{H}$, the geometric phase $\beta(S(C)) = \beta(C)$ for any closed curve $C \in P(\mathcal{H})$. The invariant integral is related to $c_{HZ}(U)$, the Hofer-Zehnder symplectic capacity[50]. Suppose that U is a bounded, connected, open set in $\mathcal{H}$, and the function $A : \mathcal{H} \to \mathbb{R}$ defined by $A(\psi) = \langle \hat{A} \rangle_\psi$ has a regular value 1 such that $U = A^{-1}([0,1])$ and $\partial U = A^{-1}(1)$. Let $X_A = J\nabla A$ be the corresponding Hamiltonian vector field defined by $i_{X_A}\Omega = -dA$. The images of the periodic solutions of Hamilton's equation, $z' = X_A(z)$, are called the "closed characteristics" of ∂U. The capacity $c_{HZ}(U)$ is defined as the minimum symplectic area β of all closed characteristics. We know the value of $c_{HZ}(P(\mathcal{H}))$! It has been calculated to be the number π by Hofer and Viterbo[49]. In the present context, we expect a factor of $\hbar$ to appear.

2.5 Quantum Mechanics as an Integrable System

Quantum mechanics as an infinite dimensional integrable system is discussed in [29]). Here, we discuss only the finite-dimensional case. Let $(\mathcal{M}, \Omega, X_H)$ be a Hamiltonian system with dim $\mathcal{M} = 2n$. Then $(\mathcal{M}, \Omega, X_H)$ is *integrable* on $\mathcal{N} \subset \mathcal{M}$ if there exist n independent Poisson commuting functions $F_i : \mathcal{M} \to \mathbb{R}$, $i = 1, ..., n$ that also commute with the Hamiltonian energy function H. That is, the differentials $dF_i(z), ..., dF_n(z)$ are linearly independent in some open dense subset of $\mathcal{M}$, and $\{F_i, F_j\} = \{F_i, E\} = 0$, for each i, j. Because the functions F_i commute with H, they are constant along the flow of X_H, and are thus called *constants of the motion*. The level sets $T_c = \{z \in \mathcal{N} | F_i(z) = c_i\}$ form n-dimensional submanifolds invariant under the Hamiltonian flow of H and under the Hamiltonian flows of the F_i. Observe that, at each point $z \in \mathcal{M}$, the hamiltonian vector fields X_{F_i} generate a *isotropic* subspace of $T_z(\mathcal{M})$: $\Omega(X_{F_i}, X_{F_j}) = \{F_i, F_j\} = 0$. A finite-dimensional Hamiltonian system $(\mathcal{M}, \Omega, X_H)$ is *integrable* (or completely integrable) if it admits $n = \frac{1}{2}$ dim($\mathcal{M}$) independent constants of the motion which pairwise Poisson commute.

Let $(\mathcal{M}, \Omega, X_H)$ be an integrable system of dimension $2n$ with integrals of the motion $F_1 = H, ..., F_n$. Let $c \in \mathbb{R}^n$ be a regular value of $F := (F_1, ..., F_n)$. The corresponding level set, $F^{-1}(c)$, is a *lagrangian submanifold*. Generally, a submanifold $\mathcal{N}$ of a $2n$-dimensional symplectic manifold $(\mathcal{M}, \Omega)$ is lagrangian if it is n-dimensional and if $i^*\Omega = 0$ where $i : \mathcal{N} \hookrightarrow \mathcal{M}$ is the inclusion map. One can show that, if the level set $F^{-1}(c)$ is compact and connected, then it is diffeomorphic to the n-torus, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Moreover, in a neighborhood of every such invariant torus, one can find *action-angle coordinates* $z = \Phi(\xi, \eta)$, where $\xi \in \mathbb{T}^n$ and $\eta \in \mathbb{R}^n$ such that the Hamiltonian flow in the coordinates (ξ, η) is given by $\dot{\xi} = \frac{\partial K}{\partial \eta}$, $\dot{\eta} = 0$. Thus, the Hamiltonian function $K = H \circ \Phi$ depends only on η and the space is foliated by invariant tori on which the Hamiltonian flow is given by straight lines. The map F is a *lagrangian fibration* (or foliation); it is locally trivial and its fibers are lagrangian submanifolds. The coordinates along the fibers are the angle coordinates, ξ. The action coordinates, η, Poisson commute among themselves and satisfy $\{\xi_i, \eta_j\} = \delta_{ij}$.

The authors of [13] have shown that geometric quantum mechanics naturally describes an integrable system, as follows. Let $\mathcal{H}$ be a complex separable Hilbert space of dimension $n+1$, and view $(P(\mathcal{H}), \Omega, X_H)$ as a Hamiltonian dynamical system on the phase space $P(\mathcal{H})$, equipped with symplectic form Ω arising from the Fubini-Study Hermitian metric. Let $\hat{H}$ be the self-adjoint Hamiltonian operator for the system, and assume that each eigenspace of $\hat{H}$ is one-dimensional. Choose an orthonormal basis $\{e_0, ..., e_n\}$ for $\mathcal{H}$ consisting of eigenvectors of $\hat{H}$. Define the projection operators $\{\hat{P}_0, \cdots, \hat{P}_n\}$ by

$$\hat{P}_j : \mathcal{H} \to \mathcal{H}, \quad v \mapsto \hat{P}_j(v) = \langle e_j, v\rangle e_j \,. \tag{2.58}$$

Without loss of generality, set the lowest eigenvalue to be 0, so that

$$\hat{H} = \sum_{j=1}^{n} \lambda_j \hat{P}_j \,. \tag{2.59}$$

Observe that the projectors $\{\hat{P}_j\}$ form a mutually commuting set of n operators on $\mathcal{H}$, and we can define the set of Hamiltonian functions $\{P_j\}$ by

$$P_j : P(\mathcal{H}) \to \mathbb{R}, \quad P_j([v]) = \frac{\langle v, \hat{P}_j v\rangle}{\langle v, v\rangle} \,. \tag{2.60}$$

The n functions $\{\hat{P}_1, \cdots, \hat{P}_n\}$ are independent, Poisson commute among themselves, and each are constants of the motion, since $\{H, P_j\} = 0$ for each j. The torus $\mathbb{T}^{n+1}$ acts on

$e_j \in \mathcal{H}$ by $e_j \mapsto e^{i\beta} e_j$, $\beta \in [0, 2\pi)$, and this action descends to an effective action of $G := \mathbb{T}^n$ on $P(\mathcal{H})$. The orbits $G \cdot [v]$ are n-dimensional Lagrangian tori. In summary, the toral action given by the projection operators is a Hamiltonian action, foliating $P(\mathcal{H})$ into Lagrangian tori.

Theorem 10 *[13] Under the above assumptions, the Hamiltonian dynamical system $(P(\mathcal{H}), \Omega, X_H)$ is integrable. The projection operators $\{\hat{P}_j\}$ are the generators of the torus action, and the action variables I_j coincide with the transition probabilities, i.e., for $v = \sum_{k=0}^{n} \alpha_k e_k \in \mathcal{H}$, the action variable I_j is $|\alpha_j|^2 = P_j([v])$, $j = 1, \cdots, n$.*

The authors of [13] further extend the analogy between quantum dynamics and the dynamics of a classical mechanical system by interpreting a cyclic adiabatic perturbation of the Hamiltonian as a migration of the Lagrangian tori, and observing that the Berry phases corresponds to the Hannay's angles that characterize a classical integrable system.

Chapter 3

J-HOLOMORPHIC MAPS AND THE UNCERTAINTY PRINCIPLE

3.1 Introduction

This chapter revisits the quantum uncertainty principle discussed in section 2.3. As was shown in (2.36), the uncertainty principle may be written in the form (with change of notation $\psi \to z$ to avoid later notational confusions)

$$(\Delta\hat{A})^2_z\,(\Delta\hat{B})^2_z - (C(\hat{A},\hat{B})_z)^2 \geq \frac{1}{2\hbar}\Omega(\hat{A}z,\hat{B}z)^2\,, \tag{3.1}$$

a relation between the dispersion, covariance and the symplectic form for the operators $\hat{A}$, $\hat{B}$ acting on $z \in \mathcal{H}$. The central result of the chapter is that the quantum mechanical uncertainty principle is a special case of an inequality from J-holomorphic map theory; this result is used to show how the condition of minimum uncertainty is equivalent to the requirement that a map $u : \Sigma \to \mathcal{H}$ be J-holomorphic. We begin by writing the inequality in terms of the symplectic area $\Omega(\hat{A}z,\hat{B}z)$ and the determinant of the quantum covariance matrix, $D(\hat{A},\hat{B},z)$. Geometrically, $D(\hat{A},\hat{B},z)$ represents the squared metric area of a parallelogram. Then it becomes clear that the uncertainty principle can be cast in a form that compares a metric area to a symplectic area. It is shown that the quantum covariance matrix determinant is equal to the *harmonic energy* of the map u. Moreover, requiring u to be holomorphic minimizes the difference between the quantum covariance matrix determinant $D(\hat{A},\hat{B},z)$ and the square of the symplectic area $\Omega(\hat{A}z,\hat{B}z)$. When equality is achieved, the off-diagonal part $C(\hat{A},\hat{B})_z)$ of the covariance matrix vanishes and the product of the variances $(\Delta\hat{A})_z\,(\Delta\hat{B})_z$ is a topological invariant, within a fixed homology class of the curve.

The inspiration for our idea to study the uncertainty principle from the standpoint of symplectic topology comes from Oh's lecture notes[74], wherein he suggests that Gromov's non-squeezing theorem is a classical analogue of the quantum mechanical uncertainty principle. The literature on the uncertainty principle is extensive. (add references) The work of Narcowich[66] on uncertainty and the covariance matrix for a Wigner distribution function is relevant to the present study. Especially notable is the work of de Gosson and Luef[34, 33], who have related the quantum covariance matrix and the uncertainty principle to symplectic capacitance and Gromov's non-squeezing theorem. Here, we focus on developing the method of holomorphic curves as a tool for studying dispersion, minimality, and stability of a quantum system under deformation.

3.2 The Covariance Matrix Determinant

The left hand side of (3.1) is equal to the square root of the determinant of the covariance matrix. We show here that this determinant is also equal to the determinant of the Jacobian matrix of an immersion map of a Riemann surface Σ into $\mathcal{H}$. Immersions into $P(\mathcal{H})$ or into submanifolds of $\mathcal{H}$ or $P(\mathcal{H})$ are expected to have a similar property.

Let z vary over $\mathcal{H}$, define the vector fields

$$\begin{aligned} v, w &: \mathcal{H} \to T(\mathcal{H}) \\ v(z) &= (\hat{A} - \langle \hat{A} \rangle_z) z \\ w(z) &= (\hat{B} - \langle \hat{B} \rangle_z) z \,. \end{aligned} \tag{3.2}$$

Lemma 11 *Let $\hat{A}$ and $\hat{B}$ be two self-adjoint, linear operators on $\mathcal{H}$ and let $z \in \mathcal{H}$. Define*

$$D(\hat{A}, \hat{B}, z) := (\Delta \hat{A})_z^2 \, (\Delta \hat{B})_z^2 - (C(\hat{A}, \hat{B})_z)^2 \,. \tag{3.3}$$

$D(\hat{A}, \hat{B}, z)$ represents the squared area of the parallelogram defined by the vectors $v(z), w(z)$ defined in (3.2), measured in the metric g on $P(\mathcal{H})$.

$$D(\hat{A}, \hat{B}, z) = g(\hat{A}z, \hat{A}z) \, g(\hat{B}z, \hat{B}z) - g(\hat{A}z, \hat{B}z) \, g(\hat{B}z, \hat{A}z) \tag{3.4}$$

Proof: Recalling the definitions of the dispersion (2.30) and the (2.31), observe that $(\Delta \hat{A})_z^2 = cg(\hat{A}z, \hat{A}z)$, $(\Delta \hat{B}_z^2) = cg(\hat{B}z, \hat{B}z)$, and $C(\hat{A}, \hat{B})_z = cg(\hat{A}z, \hat{B}z) = cg(\hat{B}z, \hat{A}z)$, where $c = \frac{1}{4\hbar^2}$. Recall a formula for the area of a parallelogram defined by two vectors v and w, which are elements of a Hermitian vector space with scalar product $\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + i\omega(\cdot, \cdot)$. The metric area of the parallelogram is $A = |v| \, |w| \, |\sin \theta|$, where θ is the angle between v

and w. So, we may write the square of that area as

$$\begin{aligned} A^2 &= |v|^2|w|^2(1-\cos^2\theta) \\ &= g(v,v)g(w,w) - g(v,w)g(v,w)\,. \end{aligned} \tag{3.5}$$

It is now clear that (3.4)represents the squared area of a parallelogram defined by the vectors $\hat{A}z$ and $\hat{B}z$ in $\mathbb{C}^{n+1}$. Note, however, that g here is the metric on $\mathbb{C}P^n$, not the metric on $\mathbb{C}^{n+1}$. □

Proposition 12 *The uncertainty principle (3.1) relates the metric and symplectic differential areas defined by the vectors (3.2).*

$$\sqrt{D(\hat{A},\hat{B},z)} \geq \Omega(\hat{A}z,\hat{B}z)\,. \tag{3.6}$$

Proof: The symplectic form Ω also measures an area in $\mathbb{C}P^n$, since it is a volume form in two dimensions. So, by the lemma, the uncertainty principle in the form (3.6) displays a relation between the two 2-dimensional differential areas. □

3.3 J-Holomorphic Maps

Consider a map between almost complex manifolds,

$$u : (\Sigma, j) \to (\mathcal{M}, J)\,. \tag{3.7}$$

where the target $\mathcal{M}, J$ is an almost complex manifold we wish to study, and (Σ, j) is another almost complex manifold with dim $\Sigma = 2$. Usually, $\Sigma = S^2$ or D^2. The condition for u to be *J-holomorphic* is

$$J \circ du = du \circ j\,, \tag{3.8}$$

where du is regarded as a vector-valued one-form with values in $u^*T(\mathcal{M})$.

Decomposing into J-holomorphic and J-antiholomorphic parts,[63, 75]

$$du = \partial_J u + \bar{\partial}_J u\,, \tag{3.9}$$

so that

$$\begin{aligned}\partial_J u &:= \frac{1}{2}(du - J\circ du\circ j) && (3.10)\\ \bar{\partial}_J u &:= \frac{1}{2}(du + J\circ du\circ j)\,. && (3.11)\end{aligned}$$

Thus, u is J-holomorphic if and only if $\bar{\partial}_J u = 0$. This is exactly the (multivariable) Cauchy-Riemann equation when J is integrable.

To be explicit, following Oh[75], fix a Riemannian metric h on Σ. Then, for any unit vector $e_1 \in T(\Sigma)$, $\{e_1, e_2\}$ is an orthonormal frame with $e_2 = je_1$. With the metric g on $\mathcal{M}$, the norm $|du|$ of the map $du : T(\Sigma) \to T(\mathcal{M})$ is defined by

$$|du|_g^2 := \sum_{i=1}^{2} |du(e_i)|_g^2\,. \tag{3.12}$$

In coordinates, (x^1, x^2) on Σ and $(y_1, \cdots, y_{2n})$ on $\mathcal{M}$, write $g = \sum g_{\alpha\beta} dy_\alpha dy_\beta$ and $h = \sum h_{ij} dx_i dx_j$, and $(h^{ij}) = (h_{ij})^{-1}$. Then

$$|du|_g^2 = \sum_{i,j,\alpha,\beta} g_{\alpha\beta}(u(x)) h^{ij}(x) \frac{\partial u_\alpha}{\partial x^i}\frac{\partial u_\beta}{\partial x^j}\,. \tag{3.13}$$

The *harmonic energy* of a smooth map $u : \Sigma \to \mathcal{M}$ is defined[63] as the L^2-norm of the one-form $du \in \Lambda^1(\Sigma, u^*T(\mathcal{M}))$:

$$E(u) := \frac{1}{2}\int_\Sigma |du|_g^2\, dvol_\Sigma\,. \tag{3.14}$$

If u is a J-holomorphic curve in a symplectic manifold, the harmonic energy of u is a topological invariant that depends only on the homology class of the curve modulo its boundary.

Lemma 13 *[63, 75] (Calibrated property of J-holomorphic curves) Let $(\mathcal{M}, \omega)$ be a symplectic manifold with compatible almost complex structure J. Every smooth map $u : \Sigma \to \mathcal{M}$ satisfies*

$$\begin{aligned}E(u) &= \int_\Sigma |\bar{\partial}_J(u)|_g^2\, dvol_\Sigma + \int_\Sigma u^*\omega. && (3.15)\\ &\geq \int_\Sigma u^*\omega. && (3.16)\end{aligned}$$

and equality holds precisely when u is J-holomorphic. When Σ is a closed surface without boundary (resp. with boundary fixed or with free boundary on a fixed lagrangian submanifold), then $\int u^*\omega$ is constant in a fixed homology class (resp. in a relative homology class). In this case, if u is J-holomorphic map, then the metric area Area$_{g_J}(u) := E(u)$ depends only on $[u]$, the homology class represented by u:

$$\mathrm{Area}_{g_J}(u) = [\omega]([u]) \tag{3.17}$$

Corollary 14 *[75] If $u : \Sigma \to \mathcal{M}$ is a J-holomorphic map, then near each regular point of u on Σ, the image of u is a minimal surface with respect to the metric $g_J = \omega(\cdot, J\cdot)$.*

3.4 Minimal Uncertainty and the Holomorphic Condition

To apply the method of J-holomorphic curves to the quantum uncertainty principle, we make the choice of Σ as an open subset of $\mathbb{C}$ immersed in $\mathcal{H} \cong \mathbb{C}^m$, the Hilbert space of quantum state vectors, with the standard complex structures.

$$\begin{aligned} u &: ((0,1)\times(0,1), j) \to (\mathbb{C}^m, J) \\ (s,t) &\mapsto u(s,t) = sv + tw\,, \end{aligned} \tag{3.18}$$

where $v, w \in \mathbb{C}^m \simeq \mathbb{R}^{2m}$ are the vectors defined by (3.2). In coordinates

$$\begin{aligned} v &= (v_1, \cdots, v_{2m})^T \\ w &= (w_1, \cdots, w_{2m})^T. \end{aligned} \tag{3.19}$$

Then

$$\begin{aligned} u(s,t) &= (y_1(s,t), \cdots, y_{2m}(s,t))^T \\ &= (sv_1 + tw_1, \cdots, sv_{2m} + tw_{2m})^T \end{aligned} \tag{3.20}$$

$$u^*(dy_\alpha) = v_\alpha ds + w_\alpha dt, \quad \alpha = 1, \cdots, 2m\,. \tag{3.21}$$

Then du is the matrix of partial derivatives, with columns v and w:

$$du = \begin{pmatrix} \frac{\partial y_1}{\partial s} & \frac{\partial y_1}{\partial t} \\ \vdots & \vdots \\ \frac{\partial y_{2m}}{\partial s} & \frac{\partial y_{2m}}{\partial t} \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ \vdots & \vdots \\ v_{2m} & w_{2m} \end{pmatrix}. \tag{3.22}$$

Denote by G the flat metric on $\mathbb{C}^m$; $G_{\alpha\beta} = \delta_{\alpha\beta}$. Pulling back the metric G to get h on Σ,

$$\begin{aligned} h &= u^*G = u^* \left(\sum_\alpha (dy_\alpha)^2 \right) = \sum_\alpha (u^*(dy_\alpha))^2 \\ &= |v|^2 ds^2 + |w|^2 dt^2 + 2\langle v, w\rangle ds\, dt\,. \end{aligned} \tag{3.23}$$

Comparing to $h = \sum h_{ij} ds_i dt_j$ gives $h_{11} = |v|^2$, $h_{22} = |w|^2$, and $h_{12} = h_{21} = \langle v, w\rangle$. Inverting (h_{ij}) and using (3.13) we have

$$\begin{aligned} |du|^2 &= \sum_{\alpha,i,j} h^{ij}(s,t)(\frac{\partial y_\alpha}{\partial s}\frac{\partial y_\alpha}{\partial t})_{ij} \\ &= \frac{1}{\det(h_{ij})}(2|v|^2|w|^2 - 2\langle v, w\rangle^2) = 2\,. \end{aligned} \tag{3.24}$$

Hence, evaluating the harmonic energy (3.14), we find that

$$\begin{aligned} E(u) &= \int_\Sigma dvol_\Sigma = \int_\Sigma \sqrt{det(h_{ij})} ds \wedge dt \\ &= \sqrt{|v|^2|w|^2 - \langle v, w\rangle^2}\,. \end{aligned} \tag{3.25}$$

Thus, looking back at (3.4), we see that $E(u)$ is equal to the area of the parallelogram defined by the vectors (3.2), as well as to the square root of the determinant of the covariance matrix on the left hand side of the quantum uncertainty principle. We have shown

Lemma 15 *The harmonic energy $E(u)$ of the immersion u which maps the open unit square into the parallelogram defined by the vectors (3.2) is equal to the area of the parallelogram $\sqrt{D(\hat{A}, \hat{B}, z)}$.*

$$E(u) = \sqrt{(\Delta\hat{A})_z^2\,(\Delta\hat{B})_z^2 - (C(\hat{A},\hat{B})_z)^2}\,. \tag{3.26}$$

Now let us specify the J-holomorphic condition for the map (3.18). Let

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and let J be the $2m \times 2m$ block diagonal matrix with each block equal to the 2×2 matrix j. The J-holomorphic condition

$$\bar{\partial}_J u = \frac{1}{2} \begin{pmatrix} v_1 - w_2 & w_1 + v_2 \\ v_2 + w_1 & w_2 - v_1 \\ v_3 - w_4 & w_3 + v_4 \\ v_4 + w_3 & w_4 - v_3 \\ \vdots & \vdots \end{pmatrix} = 0 \tag{3.27}$$

is equivalent to the component-wise Cauchy-Riemann equations:

$$\begin{aligned} v_\alpha &= \frac{\partial y_\alpha}{\partial s} = \frac{\partial y_{\alpha+1}}{\partial t} = w_{\alpha+1} \\ v_{\alpha+1} &= \frac{\partial y_{\alpha+1}}{\partial s} = -\frac{\partial y_\alpha}{\partial t} = -w_\alpha \, , \end{aligned} \tag{3.28}$$

Thus, we have the result:

Lemma 16 *The map u defined in (3.18) is J-holomorphic if and only if $v_\alpha + iv_{\alpha+1} = -i(w_\alpha + iw_{\alpha+1})$ if and only if $v = -Jw$.*

Now observe that the off-diagonal part $C(\hat{A}, \hat{B})_z = g(\hat{A}z, \hat{B}z)$ of the covariance matrix is real and hence vanishes in the case that $v = -Jw$. Therefore, we have the following:

Corollary 17 *When the map u defined in (3.18) is J-holomorphic, the covariance matrix has vanishing off-diagonal components and minimum determinant. In this case, the unit square is mapped to a square.*

Now we can compare the uncertainty principle inequality and the minimal surface inequality term by term. By the lemma, the left sides of (2.36) and (3.15) are equal, and the right sides are each equal to the symplectic area of the parallelogram determined by the vectors (3.2). This makes it possible to compare the additional terms that occur when equality is not achieved, that is, the antiholomorphic part of the image of the map u on the one hand, and orthogonal part lost when v is projected onto w on the other hand. It could be illuminating to exhibit this comparison explicitly. Observe the topological invariant for QT.

To compare with the other quantities in the equality (3.15), we compute

$$
\begin{aligned}
u^*\omega &= u^*\left(\sum_\alpha dy_\alpha \wedge dy_{\alpha+1}\right) = \sum_\alpha u^*(dy_\alpha) \wedge u^*(dy_{\alpha+1}) \\
&= \sum_\alpha (v_\alpha ds + w_\alpha dt) \wedge (v_{\alpha+1} ds + w_{\alpha+1} dt) \\
&= \sum_\alpha (v_\alpha w_{\alpha+1} - v_{\alpha+1} w_\alpha) ds \wedge dt \\
&= \sum_\alpha \det \begin{pmatrix} v_\alpha & w_\alpha \\ v_{\alpha+1} & w_{\alpha+1} \end{pmatrix} ds \wedge dt \qquad (3.29) \\
&= -\langle du, Jdu \rangle \, ds \wedge dt \qquad (3.30)
\end{aligned}
$$

In summary,

Theorem 18 *The square root of the determinant $\sqrt{(\Delta\hat{A})_z^2\,(\Delta\hat{B})_z^2 - (C(\hat{A},\hat{B})_z)^2}$ of the covariance matrix is given by the harmonic energy $E(u)$ of the map*

$$
\begin{aligned}
&u : (0,1) \times (0,1) \to (\mathbb{C}^m, J) \\
&(s,t) \mapsto u(s,t) = sv + tw\,,
\end{aligned}
$$

with $v, w : \mathcal{H} \to T(\mathcal{H})$ defined by

$$
\begin{aligned}
v(z) &= (\hat{A} - \langle \hat{A} \rangle_z) z \\
w(z) &= (\hat{B} - \langle \hat{B} \rangle_z) z\,,
\end{aligned}
$$

$E(u)$ represents the area of the image of the open unit square under the map u. The quantum uncertainty principle (3.1) in integral form is

$$
E(u) = \frac{1}{2}\int_\Sigma |du|_g^2 ds \wedge dt \geq \int_\Sigma u^*\Omega\,, \qquad (3.31)
$$

with equality holding if and only if $u : \Sigma \to \mathcal{H}$ is a J-holomorphic map. When equality is achieved, the off-diagonal part $C(\hat{A},\hat{B})_z$ of the covariance matrix vanishes and the product of the variances $(\Delta\hat{A})_z\,(\Delta\hat{B})_z$ is equal to $\int_\Sigma u^*\Omega$, a topological invariant within a fixed homology class of the curve.

3.5 Examples

By choosing the operators $\hat{A}$ and $\hat{B}$ judiciously, we can use the understanding of the uncertainty principle gained from the holomorphic map approach by specific applications to quantum systems. An obvious choice is $\hat{X}$ and $\hat{P}$, the position and momentum operators, since the uncertainty principle is most commonly couched in these terms. Moreover, in analogy to the cotangent bundle phase space of classical mechanics, we can view (x, p) as canonical coordinates on the quantum phase space $P(\mathcal{H})$. Recall that taking the horizontal component of $\hat{A}z$, that is $(\hat{A} - \langle\hat{A}\rangle)z$, is equivalent to projection onto $P(\mathcal{H})$. Thus, by choosing $\hat{A} = \hat{X}$ and $\hat{B} = \hat{P}$, the image of the surface Σ under the map u is a two-dimensional region in $P(\mathcal{H})$ and $\Omega(\hat{X}z, \hat{P}z)$ represents its symplectic area. With these coordinates, $\Omega = dx \wedge dp$, so that $\Omega(\hat{X}z, \hat{P}z) = \mathrm{I}m\langle z, [\hat{X}, \hat{P}]z\rangle = \hbar$. We may interpret the result of the theorem to mean that when the map u is J-holomorphic, the harmonic energy (a metric area) has the value $\hbar$. de Gosson and Leuf[34] have analyzed this example in terms of the symplectic capacitance of the quantum phase space, and pointed out its meaning in terms of Gromov's nonsqueezing theorem.

If we allow the operators $\hat{A}$ and $\hat{B}$ to be related through time-dependence, as in $\hat{B}(t) = \hat{U}(t)\hat{B}\hat{U}(t)^{-1}$ for a family of unitaries $\hat{U}(t)$, we can study the linear response of a quantum system to a perturbation by using a Kubo Formula,[38]

$$\delta\langle\hat{A}\rangle = \langle\hat{A}\rangle - \langle\hat{A}\rangle_0 = \frac{i}{\hbar}\int dt\langle\psi, [\hat{B}(t), \hat{A}]\psi\rangle \tag{3.32}$$

We return to this example in Chapter 5, where it is shown that the dielectric response function in the form (3.32) is equivalent to the curvature of a connection on a principal bundle.

A closely related quantity derived from a cumulant generating function[88] is the second cumulant average

$$\langle X_i X_j\rangle_c = \langle X_i X_j\rangle - \langle X_i\rangle\langle X_j\rangle\,. \tag{3.33}$$

For a finite system, $\hat{X} := \sum_{i=1}^{N} \hat{x}_i$, so that $\hat{X}/N$ is the position operator for the center of mass of the N electrons in the finite volume. For an extended many-body system, similar quantities are defined in [88] by using special "twisted" boundary conditions. These authors find the second moment (3.33)

$$\begin{aligned} \langle X_i X_j \rangle_c &= \frac{V}{2\pi}^3 \int dk T_{ij}(k) \\ T_{ij}(k) &= \langle \partial_{k_i} \Phi_k, \partial_{k_j} \Phi_k \rangle - \langle \partial_{k_i} \Phi_k, \Phi_k \rangle \langle \Phi_k, \partial_{k_j} \Phi_k \rangle, \end{aligned} \tag{3.34}$$

where Φ_k is a many-body cell-periodic wave function, and $T_{ij}(k)$ is a gauge-invariant quantity called the *quantum geometric tensor*.[15] Analysis of these results using holomorphic maps could provide interesting and useful insights.

An important set of examples takes one of the operators in (3.1) to be the Hamiltonian energy operator, $\hat{H}$. Viewing the quantum system as an integral dynamical system, the system dynamics follows the flow of the associated vector field, as described in section 2.5. This flow is tangent to the lagrangian submanifolds defined as the intersection of the level sets of the constants of the motion. If we choose a second vector field to be normal to a lagrangian submanifold, the associated flow will take the system away from the one determined by $\hat{H}$, producing a variation or deformation of the dynamical flow. The mean curvature vector field determines such a deforming flow, since it is normal to the immersed submanifold. The choice of $\hat{A}$ as the Hamiltonian operator $\hat{H}$, and $\hat{B}$ as a variation or deformation of $\hat{H}$ is an instance of the energy-time uncertainty principle of quantum mechanics. By choosing a family of immersion maps $u_t : \Sigma \to \mathcal{M}$ so that the boundary of the image of u_0 lies in a lagrangian submanifold of $P(\mathcal{H})$ (or of $\mathcal{H}$), we can study these deformations and gain information about topological invariants. In the next chapter, we study the mean curvature flow and associated Maslov cohomology class, with particular interest in its relationship to the geometric phase of an evolving quantum system.

As a specific example, consider the following J-map that could provide insight into the problem of the intersection of Lagrangian tori, as in a family of integrable systems. We can specify that the boundary of Σ maps into a torus which is a Lagrangian submanifold of

M, as follows.

$$\begin{aligned}
\tilde{u} \quad &: \quad ((0,2\pi)\times(0,2\pi), j) \to (\mathbb{C}, J) \\
&(s,t) \mapsto \tilde{u}(s,t) = vR_s + wR_t \\
&R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\end{aligned} \tag{3.35}$$

with $v = (v_1, \cdots, v_{2m})^T$, and $w = (w_1, \cdots, w_{2m})^T$ defined by (3.2) as before. Then

$$\tilde{u}(s,t) = \begin{pmatrix} v_1\cos s - v_2 \sin s + w_1 \cos t - w_2 \sin t \\ v_1 \sin s + v_2 \cos s + w_1 \sin t + w_2 \cos t \\ \vdots \end{pmatrix}. \tag{3.36}$$

Using the matrices j and J as before to compute $\bar{\partial}_J u = \frac{1}{2}(d\tilde{u} + J \circ d\tilde{u} \circ j)$, we find that the condition $\bar{\partial}_J \tilde{u} = 0$ for $\tilde{u}$ to be J-holomorphic is

$$\begin{pmatrix} -v_1 \sin s - v_2 \cos s - w_1 \cos s + w_2 \sin s & -w_1 \sin t - w_2 \cos t + v_1 \cos t + v_2 \sin t \\ v_1 \cos s - v_2 \sin s - w_1 \sin s - w_2 \cos s & w_1 \cos t - w_2 \sin t - v_1 \sin t + v_2 \cos t \\ \vdots & \vdots \end{pmatrix} = 0\,,$$

which holds for all $s, t \in (0, 2\pi)$ if and only if $v_1 = w_2, v_2 = -w_1$ if and only if $v = -Jw$. This is precisely the result found with the linear map (3.18). In the present context, we would like to interpret the result as an equivalence between the holomorphic map condition and the condition for transversality of two lagrangian tori corresponding to initial and deformed integrable systems.

Pulling back the metric $h = \tilde{u}^* g$ yields (after some algebra) $h = \sum_{i,j=1}^{2} h_{ij} ds_i dt_j$ with

$$\begin{aligned}
h_{11} &= |v|^2 = g(v,v), \\
h_{22} &= |w|^2 = g(w,w), \\
h_{12} &= h_{21} = g(v,w)\cos(s-t) + g(v,Jw)\sin(s-t)\,.
\end{aligned} \tag{3.37}$$

Chapter 4

DEFORMATIONS OF A QUANTUM SYSTEM

Let us now consider deformations of an integrable quantum system. We would like to compare two Lagrangian submanifolds L and $L' = \phi_T(L)$ of a symplectic manifold $\mathcal{M}$, where $\{\phi_t\}_{t=0}^T$ is a family of (Hamiltonian) symplectomorphisms parameterized by t. As we study this question, keep in mind two perspectives:

1) Let $u_0 : \Sigma \to \mathcal{M}$ be a J-map that maps the boundary of Σ into L and similarly, let $u_T : \Sigma \to \mathcal{M}$ map the boundary of Σ into L'. The pullback bundle $u^*(T\mathcal{M})$ is a way to get the Maslov classes for comparison[72]. Transversality is the key feature.

2) Consider the single map (3.35), where $v := \hat{A}z$, $w := \hat{U}(T)\hat{A}\hat{U}(T)^{-1}z$, and $\hat{U}(t)$ is the unitary operator corresponding to ϕ_t.

4.1 Symplectic and Hamiltonian Deformations

Let Diff($\mathcal{M}$) denote the group of diffeomorphisms of a smooth manifold $\mathcal{M}$. An *action* of a Lie group G on $\mathcal{M}$ is a group homomorphism $\psi : G \to \mathrm{D}iff(\mathcal{M})$ defined by $g \mapsto \psi_g$. Let $(\mathcal{M}, \omega)$ be a symplectic manifold and let Symp($\mathcal{M}, \omega$) denote the (infinite dimensional) group of symplectomorphisms of $\mathcal{M}$, that is, the group of diffeomorphisms of $\mathcal{M}$ that preserve ω: Symp$(\mathcal{M}, \omega) = \{\psi \in \mathrm{D}iff(\mathcal{M}) \,|\, \psi^*\omega = \omega\}$. The action ψ is a *symplectic action* if

$$\psi : G \to \mathrm{S}ymp(\mathcal{M}, \omega) \subset \mathrm{D}iff(\mathcal{M})\,.$$

Let G be a Lie group with action $\psi : G \to$ Diff($\mathcal{M}$), and $\mathfrak{g}$ the Lie algebra of G with dual $\mathfrak{g}^*$. Then the action ψ is a *hamiltonian action* if there exists a map

$$\mu : \mathcal{M} \to \mathfrak{g}^*$$

that satisfies the following two conditions:

1. For each $\xi \in \mathfrak{g}$, let $\mu^\xi : \mathcal{M} \to \mathbb{R},\ \mu^\xi(p) := \langle \mu(p), \xi \rangle$ be the component of μ along ξ, and let $\xi^\sharp$ be the fundamental vector field on $\mathcal{M}$ generated by the one-parameter

subgroup $\{\exp t\xi \,|\, t \in \mathbb{R}\} \subset G$. Then $d\mu^{\xi} = \iota_{\xi^{\sharp}}\omega$. That is, the function μ^{ξ} is a hamiltonian function for the vector field $\xi^{\sharp}$.

2. The map μ is equivariant with respect to (or intertwines) the action ψ of G on $\mathcal{M}$ and the coadjoint action Ad* of G on $\mathfrak{g}^*$:

$$\mu \circ \psi_g = Ad_g^* \circ \mu\,, \quad \text{f}or\ all\, g \in G.$$

In this case, μ is called a *moment map*.

Observe that the set of complete vector fields on $\mathcal{M}$ are in one-to-one correspondence with actions of $\mathbb{R}$ on $\mathcal{M}$. The diffeomorphism $\psi_t : \mathcal{M} \to \mathcal{M}$ associated to $t \in \mathbb{R}$ is the time-t map $\exp tX$ defined by the flow of the vector field X. The set of complete symplectic (resp. hamiltonian) vector fields on $(\mathcal{M}, \omega)$ are in one-to-one correspondence with symplectic (resp. hamiltonian) actions of $\mathbb{R}$ on $\mathcal{M}$.

Consider $(\mathbb{C}, \omega_0)$ with symplectic S^1-action given by complex multiplication $z \to e^{it}z$, $t \in S^1$. The S^1-action is generated by the vector field $X = -ydx + xdy$. The moment map is $\mu(z) = -\frac{1}{2}|z|^2$, since $\iota_X\omega = -ydy - xdx$.

Consider again the torus action discussed at the end of Chapter 2 in the context of a finite-dimensional quantum Hamiltonian system $(P(\mathcal{H}), \Omega, X_H)$ viewed as an integrable system. The Lie group $G = \mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$ with Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ has trivial coadjoint action. $\mathbb{T}^n \cong S^1 \times \cdots \times S^1$ acts on $\mathbb{C}^n$ by

$$(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n})(z_1, \cdots, z_n) = (e^{ik_1\theta_1}z_1, e^{ik_2\theta_2}z_2, \cdots, e^{ik_n\theta_n}z_n)\,. \tag{4.1}$$

where $k_1, k_2, \cdots, k_n$ are fixed, and $\theta_i \in \mathbb{R}$ for each i. This action is hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^* \cong \mathbb{R}$

$$\mu(z_1, \cdots, z_n) = -\frac{1}{2}(k_1|z_1|^2, \cdots, k_n|z_n|^2)\,. \tag{4.2}$$

Choose a basis $\{w_0, \cdots, w_n\}$ of eigenvectors of the Hamiltonian energy operator $\hat{H}$. Then, Hamilton's (Schrödinger's) equation for an eigenvector is

$$i\hbar\frac{dw_k}{dt} = \hat{H}w_k \tag{4.3}$$

$$w_k(t) = e^{i\hbar\lambda_k} w_k \tag{4.4}$$

Now consider parameterized families of $\mathbb{R}$ actions. A *symplectic isotopy* of $\mathcal{M}$ is a smooth map $[0,1] \times \mathcal{M} \to \mathcal{M} : (t,q) \mapsto \psi_t(q)$ such that $\psi_t \in \mathrm{S}ymp(\mathcal{M},\omega)$ for each $t \in [0,1]$ and $\psi_0 = \mathrm{i}d$. Any such isotopy is generated by a unique family of vector fields $X_t : \mathcal{M} \to T(\mathcal{M})$ such that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t. \tag{4.5}$$

Since ψ_t is a symplectomorphism for every t, the vector fields X_t are symplectic and so, each associated one-form is closed:

$$d\iota_{X_t}\omega = 0.$$

If all of these one-forms are exact, then there exists a smooth family of Hamiltonian functions $H_t : \mathcal{M} \to \mathbb{R}$ such that

$$\iota_{X_t}\omega = dH_t\,. \tag{4.6}$$

In this case, the family $\{\psi_t\}$ is a *hamiltonian isotopy*, and the family H_t is a time-dependent Hamiltonian. If $\mathcal{M}$ is simply connected, then every symplectic isotopy is hamiltonian. Oh studied isotopies of lagrangian submanifolds.[70, 71, 72, 73]

A symplectomorphism $\psi \in \mathrm{S}ymp(\mathcal{M},\omega)$ is called *hamiltonian* if there exists a hamiltonian isotopy $\psi_t \in \mathrm{S}ymp(\mathcal{M},\omega)$ from $\psi_0 = \mathrm{i}d$ to $\psi_1 = \psi$. The space of hamiltonian symplectomorphisms, denoted by Ham$(\mathcal{M},\omega)$, is a normal subgroup of Symp$(\mathcal{M},\omega)$, and its Lie algebra is the algebra of all hamiltonian vector fields.[62]

The flux homomorphism[62] Flux : $\widetilde{\mathrm{S}ymp}_0(\mathcal{M},\omega) \to H^1(\mathcal{M},\mathbb{R})$ is defined by

$$Flux(\{\psi_t\}) = \int_0^1 [\iota(X_t)\omega]dt \in H^1(\mathcal{M};\mathbb{R}) \tag{4.7}$$

where X_t is determined by (4.5).

Now consider the application of the concepts of symplectic and hamiltonian isotopies to the quantum dynamics of time-dependent Hamiltonians. Giavarini and Onofri[42] studied Hamiltonians of the form

$$H(t) = U(t)H(0)U(t)\dagger\,, \tag{4.8}$$

where $U(t)$ takes values in a unitary irreducible representation of a Lie group, and $H(0)$ is a generator in its Lie algebra $\mathfrak{g}$. Applications include the geometric phase and coherent states. It would be interesting to study these as examples of hamiltonian isotopies. Consider the following conjectures for future study:

Conjecture 1: Lagrangian deformations correspond to coadjoint orbits of the symplectic group.

Conjecture 2: Hamiltonian deformations correspond to coadjoint orbits of the unitary group.

The geometric phase is the holonomy of a curve (class) in the base $\mathcal{M}$ of a principle $U(1)$ bundle. In the adiabatic case, there is a correspondence between each point in $\mathcal{M}$ and a lagrangian submanifold, so that a path in $\mathcal{M}$ corresponds to a family of lagrangian submanifolds.

4.2 Mean Curvature and the Maslov Class

Let us now consider the time evolution of quantum states, viewed as a Hamiltonian dynamical system. For the initial system, take the example $(P(\mathcal{H}), \Omega_0, X_H)$ with Hilbert space $\mathcal{H} \cong \mathbb{C}^{n+1}$ from section 2.5, and recall that the toral action of the projection operators $\{\hat{P}_j\}$ foliates $\mathcal{H}$ into lagrangian tori of dimension $n+1$, and foliates the phase space $P(\mathcal{H})$ into lagrangian tori of dimension n. The lagrangian condition means that means, in particular, that $\Omega(u, v) = 0$ for any $u, v \in T(\mathcal{L})$.

There are many possibilities to consider: How does a system relax after an initial perturbation? How do states of a perturbed system evolve to eigenstates of a future Hamiltonian? Will an unperturbed system initially in $\mathcal{L}$ evolve into another lagrangian submanifold $\mathcal{L}'$ of the same leaf and foliation? For now, we will not commit to any of these questions, but instead ask: How do states initially in $\mathcal{L}$ evolve along the flow of the mean curvature vector field determined by $\mathcal{L}$? A nice property of the case of a lagrangian submanifold embedded in a symplectic manifold is that the submanifold evolves under the mean curvature flow into a lagrangian submanifold, so that the system remains integrable.

Furthermore, mean curvature plays an important role in the symplectic topology of Lagrangian submanifolds.[73]. An enlightening discussion of the Maslov class may be found in [12]. See also the paper by Cieliebak and Goldstein [26], in which they prove the following relation between the symplectic area, Maslov class, and the mean curvature one-form of a Lagrangian immersion in a Kähler-Einstein manifold.

$$\lambda\omega(F) = \pi\mu(F) + \sigma_L(\partial F)\,. \tag{4.9}$$

Morvan[64] proved that the mean curvature vector H_i of the lagrangian immersion $i : \mathcal{L} \to \mathcal{C}^n$ represents the Maslov cohomology class of i. More precisely, the one form $\frac{1}{\pi}\alpha_{H_i}$ on $\mathcal{L}$ defined by

$$\begin{aligned} &\alpha_{H_i} : T(\mathcal{L}) \to \mathbb{R}, \\ &\alpha_{H_i}(v) = \Omega(H_i, v)|_{T(\mathcal{L})} \end{aligned} \tag{4.10}$$

represents the Maslov class $\mu \in H^1(\mathcal{L}, \mathbb{Z})$.

Next we show a calculation of the mean curvature vector H_i for the Lagrangian immersion $i : \mathcal{L} = \mathbb{T}^m \to \mathbb{C}^m$.

Let G and Ω be the real and imaginary parts of the standard Hermitian inner product on $\mathbb{C}^m$. By the definition of a Lagrangian immersion, $i^*(\Omega) = 0$ on all points of $\mathcal{L}$. Putting the metric $i^*(G)$ on $\mathcal{L}$ makes i an isometric immersion. The normal bundle $N(\mathbb{T}^m)$ is the orthogonal complement to $T(\mathbb{T}^m)$ in $T(\mathbb{C}^m)$.

Choose an orthonormal basis $\{\psi_0, ..., \psi_n\}$ for $\mathcal{H}$ consisting of eigenvectors of $\hat{H}$. Label the one-dimensional (circle) components of $\mathbb{T}^m$ as S_k, $k = 1, \cdots, m$, where $S_k = \{z_k = x_k + iy_k \,:\, x_k, y_k \in \mathbb{R} x_k^2 + y_k^2 = |r_k|^2\}$.

The mean curvature vector H_i of the immersion i at $z \in \mathcal{L}$ is the normal vector

$$H_i(z) := \mathrm{Tr}\, II = \sum_{k=1}^{m} II(v_k, v_k)\,, \tag{4.11}$$

where $\{v_1, \cdots, v_m\}$ is an orthonormal basis of $T_z(\mathcal{C}^m)$, and II is the *second fundamental* *form* associated to i, defined as

$$\begin{aligned} II &: T(\mathbb{T}^m) \otimes T(\mathbb{T}^m) \to N(\mathbb{T}^m) \\ II(U,V) &= \hat{\nabla}_U V - \nabla_U V = p_N(\nabla_U V)\,, \end{aligned} \tag{4.12}$$

where $\hat{\nabla}$ is the trivial metric on $\mathbb{C}^m$, ∇ is the Levi-Civita connection on $T(\mathbb{T}^m)$, and p_N is the orthogonal projection onto $N(\mathbb{T}^m)$.

At a point $z = (z_1, \cdots, z_m) \in \mathbb{T}^m$, in coordinates $z_k = x_k + iy_k$, the unit vector tangent to the circle S_k with radius r_k is

$$v_k = \frac{1}{r_k}\left(y_k \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial y^k}\right) . \tag{4.13}$$

The connection ∇ corresponds to the flat metric on $\mathbb{C}^m$. Then

$$\nabla_{v_k}(v_k) = \nabla_{\left(\frac{y_k}{r_k}\frac{\partial}{\partial x^k} - \frac{x_k}{r_k}\frac{\partial}{\partial y^k}\right)} \left(\frac{y_k}{r_k}\frac{\partial}{\partial x^k} - \frac{x_k}{r_k}\frac{\partial}{\partial y^k}\right) \tag{4.14}$$

(more here)

$$\nabla_{v_k}(v_k) = -\frac{1}{r_k^2}\left(x_k \frac{\partial}{\partial x^k} + y_k \frac{\partial}{\partial y^k}\right) . \tag{4.15}$$

This vector is already along the normal direction (along the radial vector $x_k \frac{\partial}{\partial x^k} + y_k \frac{\partial}{\partial y_k}$), so

$$p_N \nabla_{v_k}(v_k) = -\frac{1}{r_k^2}\left(x_k \frac{\partial}{\partial x^k} + y_k \frac{\partial}{\partial y^k}\right) , \tag{4.16}$$

so that the mean curvature vector is

$$H_i = -\sum_{k=1}^{m} \frac{1}{r_k^2}\left(x_k \frac{\partial}{\partial x^k} + y_k \frac{\partial}{\partial y^k}\right) . \tag{4.17}$$

In our coordinates, $\Omega = \sum_{k=1}^m dx_k \wedge dy_k$, so that the mean curvature one-form is

$$\alpha_{H_i} := \frac{1}{\pi} i_{H_i}\Omega = \sum_{k=1}^{m} \frac{1}{\pi r_k^2}\left(y_k dx_k - x_k dy_k\right) . \tag{4.18}$$

A theorem of Dazord[31] states that, on Kähler manifolds, the one-form α_{H_i} satisfies

$$d\alpha_{H_i} = i^*\rho\,, \tag{4.19}$$

where $i : \mathcal{L} \to (\mathcal{M}, \Omega)$ and ρ is the Ricci form of the Kähler metric G. In, particular, if $(\mathcal{M}, \Omega, J)$ is Einstein-Kähler, that is, if $\rho = c\Omega$ for $c \in \mathbb{R}$, then the one-form α_{H_i} on $\mathcal{L}$ is closed and so defines a real cohomology class on $\mathcal{L}$.

Here we calculate the Maslov class for a typical generator of the homology basis $\{\gamma_k\}$. For $(x_k, y_k) = (r_k \cos(2\pi t), r_k \sin(2\pi t))$, $t \in [0,1]$, define

$$\gamma_k(t) = (0, 0, \cdots, r_k \cos(2\pi t), r_k \sin(2\pi t), 0, \cdots)\,. \tag{4.20}$$

Then

$$\int_{\gamma_k} \alpha_{H_i} = \frac{1}{\pi} \int_0^1 \frac{-2\pi}{r_k^2} (r_k^2 \sin^2(2\pi t) + r_k^2 \cos^2(2\pi t)) dt = -2\,. \tag{4.21}$$

The Maslov class takes each of the generators to the number 2.

Taking the exterior derivative of the result (4.18) for the one-form α_{H_i} corresponding to the immersion $i : \mathcal{L} = \mathbb{T}^N \to \mathbb{C}^N$ gives

$$d\alpha_{H_i} = \sum_{k=1}^{m} \frac{1}{\pi r_k^2} dx_k \wedge dy_k\,. \tag{4.22}$$

In summary, we have calculated the mean curvature vector field H_i of an immersion $i : \mathcal{L} \to \mathcal{M}$ of a lagrangian submanifold $\mathcal{L}$ in $\mathcal{M}$; H_i is the trace of the second fundamental form of $\mathcal{L}$. As such, it is a section of the normal bundle to $\mathcal{L}$ in $\mathcal{M}$, and generates a local flow on $\mathcal{L}$. A family of submanifolds evolves under mean curvature flow if the velocity at each point of a submanifold is given by the mean curvature vector at that point. In addition, the mean curvature one-form $\alpha_{H_i} := \frac{1}{\pi} \iota_{H_i} \Omega$ represents the Maslov class $\mu \in H^1(\mathcal{L}, \mathbb{Z})$.[64] Since the quantum phase space $P(\mathcal{H})$ is Einstein-Kähler, the one form α_{H_i} on $\mathcal{L} = \mathbb{T}^n \subset P(\mathcal{H})$ is closed[31], and so defines a real cohomology class in $H^1(\mathcal{L}, \mathbb{R})$. In particular, this means that lagrangian submanifolds stay lagrangian under the mean curvature flow on $P(\mathcal{H})$. Moreover, in this case, under a Hamiltonian isotopy of the Lagrangian immersion $i : \mathcal{L} \to \mathcal{M}$, the one forms α_{H_t} represent the same cohomology class.

4.3 Berry's Phase and Mean Curvature Flow

The original descriptions of the quantum geometric phase by Berry[14] and Simon[87] were not in terms of the principal bundle $U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ used to describe the geometric phase in Chapter 2. Instead, Berry[14] considered a quantum system that can be described in terms of a parameterized family of Hamiltonian operators $\{\hat{H}(r)\}$; the time evolution is determined by smoothly varying the parameter. In this spirit, let the parameter space be an oriented, smooth, compact manifold, denoted by $\mathcal{M}$. Suppose that, for each $r \in \mathcal{M}$, $\hat{H}(r)$ acts on elements of a complex Hilbert space $\mathcal{H}$ with inner product $\langle , \rangle$. Berry assumed that, for each $r \in \mathcal{M}$, $\hat{H}(r)$ has a purely discrete spectrum, and that the n^{th} eigenvalue of $\hat{H}(r)$ is nondegenerate (has multiplicity one). On a local chart domain $V \subset \mathcal{M}$, define a map $s_n : V \to \mathcal{H}$ so that, for each $r \in V$, $s_n(r)$ is an element of the n^{th} eigenspace of $H(r)$ with $\langle s_n(r), s_n(r) \rangle = 1$. The eigenvectors $s_n(r)$ of $\hat{H}(r)$ satisfy

$$\hat{H}(r)s_n(r) = E_n(r)s_n(r) \tag{4.23}$$

with energies $E_n(r)$.

Let $\{\hat{H}(r)\}_{r\in\mathcal{M}}$ be a family of Hamiltonian operators parameterized by elements of $\mathcal{M}$ and let $C : [0,T] \to \mathcal{M}$, $t \mapsto r(t)$ be a path in $\mathcal{M}$. Suppose that a quantum system is prepared with state vector $\psi_0 = s_n(r(0))$. The state vector $\psi(t)$ of the system evolves according to Schrödinger's equation

$$i\hbar\psi'(t) = \hat{H}(r(t))\psi(t)\,. \tag{4.24}$$

If $\hat{H}$ is slowly altered by varying r, it follows from the quantum adiabatic theorem[53, 67, 11], that the system will evolve with $\hat{H}$ and will remain in the n^{th} eigenspace of $\hat{H}(r(t))$ for all $t \in [0,T]$. Thus, the state vector $\psi(t)$ equals $s_n(r(t))$ modulo a phase factor and may be written as

$$\psi(t) = \exp(i\chi_n(t))\exp(i\beta_n(t))s_n(r(t))\,, \tag{4.25}$$

where $\chi_n(t) = \frac{-1}{\hbar}\int_0^t E_n(r(t'))dt'$ is the dynamical phase. The phase function $\beta_n(t)$ is determined by requiring that $\psi(t)$ satisfy Schrödinger's equation. Applying (4.24) to (4.25) and using (4.23) yields

$$\begin{aligned} \beta_n'(t)s_n(r(t)) &= i(s_n \circ r)'(t) = ir'(t)(s_n(r(t))) = ids(r'(t)) && (4.26)\\ \beta_n'(t) &= i\langle s_n(r(t)), ds(r'(t))\rangle\,. && (4.27) \end{aligned}$$

Here, $ds(r'(t))$ is the derivative of s in the direction $r'(t)$, as in (6.30). Now suppose that the path C is closed, that is, $r(0) = r(T)$. In this case, $\psi(T)$ may differ from $\psi(0)$ only by a phase factor. The total phase factor change of ψ after traversing C is given by

$$\psi(T) = \psi(0)\exp(i\chi_n(T))\exp(i\beta_n(C))\,, \tag{4.28}$$

where the geometrical phase change, known as *Berry's phase*,

$$\beta_n(C) = i \oint \langle s_n(r(t)), ds(r'(t))\rangle dt \,, \tag{4.29}$$

depends on the geometry of $\mathcal{M}$ and on the path C.

Note the similarity between the expressions (4.29) for Berry's phase and (2.53) or (2.52) for the AA phase. In fact, Aharonov and Anandan[3] showed that the AA phase reduces to Berry's phase in the adiabatic limit. Simon[87] observed that Berry's phase could be interpreted geometrically in terms of a holonomy of a connection in a complex line bundle over the parameter space $\mathcal{M}$. Wilczek and Zee[94] found a nonabelian generalization of Berry's phase for the case of a parameterized family of Hamiltonian operators, each with a spectrum containing an eigenvalue of fixed multiplicity greater than one.

A goal for future work is to establish the relationship between Berry's phase and the mean curvature flow of a quantum system discussed in the prvious section. An outline of the plan follows. Consider a time-dependent family of immersions,

$$\begin{aligned} f \quad &: \quad \mathcal{L} \times [0,1] \to \mathcal{H} \\ \frac{\partial}{\partial t} f(z,t) \quad &= \quad H(z,t) \quad z \in \mathcal{H},\, t \geq 0 \qquad (4.30) \\ f(\cdot, 0) \quad &= \quad f_0 \,, \qquad (4.31) \end{aligned}$$

where $H(z,t)$ is the mean curvature vector at $z \in \mathcal{H}$, $t \geq 0$. Since $H(z,t)$ is an element of the normal bundle $N(L)$, the mean curvature flow is normal to the level surfaces of the integrable system. We want to compare the initial and time-evolved states of the system.

A vector $\psi \in \mathcal{H}$ evolves according to $\psi_t = \sum_{k=0}^{n} \lambda_k P_k(t)$, while the elements of the phase space $P(\mathcal{H})$ may be written as $p(z,t)$. The following lemma makes it possible to relate a flow direction (tangent vector) in $\mathcal{H}$ to the evolution of a point $p(z,t) = |v(t)\rangle\langle v(t)| \in P(\mathcal{H})$.

Lemma 19 *[89] Let* $v, w \in \mathcal{H}$, $|v|^2 = 1$, $w := w_1 + w_2$, *with* $w_1 \in [v]$, $w_2 \in \{v\}^\perp$. *Set*

$v(t) := v + tw$. *Then*

$$\frac{d}{dt} \quad \left(\langle v(t)|v(t)\rangle^{-1} |v(t)\rangle\langle v(t)| \right)|_{t=0} \tag{4.32}$$

$$= \quad |w_2\rangle\langle v| + |v\rangle\langle w_2| \,. \tag{4.33}$$

To compare the flows of a projector $\hat{P}$ and its variation $\dot{\hat{P}}$, use the lemma to calculate

$$[\dot{\hat{P}}, \hat{P}] = [(|w_2\rangle\langle v| + |v\rangle\langle w_2|), |v\rangle\langle v|] = |w_2\rangle\langle v| - |v\rangle\langle w_2| \,, \tag{4.34}$$

similar to multiplication by J. Moreover, $\hat{H}_{ex} := [\dot{\hat{P}}, \hat{P}]$ plays the role of a perturbing Hamiltonian operator, since

$$[\hat{H}_{ex}, \hat{P}] = \dot{\hat{P}} \,. \tag{4.35}$$

The plan is to relate $\hat{H}_{ex}$ to the deformation of a quantum integrable system, using the geometric phase as a watermark.

In his paper[93], Weinstein considered the cyclic evolution of a lagrangian submanifold of a symplectic manifold $(\mathcal{M}, \omega)$, and presented a way to define a connection over a foliation on the space $\Lambda(\mathcal{M})$ of closed, embedded lagrangian submanifolds of $\mathcal{M}$. In order to compare to the case of adiabatic motion studied by Berry[14], the evolution is restricted to *isodrastic* deformations, which requires the (reduced) action integral

$$A = \int_S \omega$$

to be constant. Denote by Γ_ω the *period group* of the symplectic form ω on $\mathcal{M}$, that is, the subset of $\mathbb{R}$ consisting of the integrals of ω over all integer 2-cycles in $\mathcal{M}$. If Γ_ω is trivial or cyclic, so that $G_\omega := \mathbb{R}/\Gamma_\omega$ is $\mathbb{R}$ or a circle, then there is a principal G_ω bundle $\pi : \mathcal{Q} \to \mathcal{M}$ with a connection form α whose curvature form is the symplectic structure, ω.[93] By Stokes theorem, the action integral may be written as

$$A = \int_\gamma \alpha \tag{4.36}$$

where γ is a loop in $\mathcal{M}$ enclosing the surface S. The action integral is thus equal to the holonomy of the connection α around γ. The holonomy around a loop is unchanged when the loop undergoes an isodrastic deformation.

Isodrastic deformations are also called *exact* deformations, or *hamiltonian* deformations, since they can be obtained by flowing along globally hamiltonian vector fields. In particular, they are realized by the trajectories of adiabatically varying completely integrable systems.

4.4 The Vector Bundle Classification Theorem

The applications to physics that we wish to describe rely heavily on the geometrical structure of a complex line bundle over a parameter space and the generalization to higher dimensional spectral bundles, which are *not* principal bundles, but rather vector bundles. Hence, this section begins by defining the relevant fiber bundles and the relationships between them. It is shown that the quantum adiabatic theorem can be viewed as a statement relating isomorphisms of certain vector bundles to the adiabatic time evolution of a quantum system.

Here we begin from more general definitions.[51] A *bundle* is a triple (E, π, B) where $\pi : E \to B$ is a map. The space B is called the base space, the space E is called the total space, and the map π is called the projection of the bundle. For each $b \in B$, the space $\pi^{-1}(b)$ is called the fiber of the bundle over b.

A *bundle map* (or bundle morphism) between a pair of bundles (E, π, B) and (E', π', B') is a pair of maps (h, f) where $h : E \to E'$ and $f : B \to B'$ such that $\pi' \circ h = f \circ \pi$. Note that the last equality implies that, for all $b \in B,\ h(\pi^{-1}(b)) \subset \pi'^{-1}(f(b))$, that is, the pair of maps (h, f) maps fibers into fibers. When π is surjective, the map f is uniquely determined by the map h. If (E, π, B) and (E', π', B) are two bundles over B, a bundle morphism over B (or B-morphism) $u : (E, \pi, B) \to (E', \pi', B)$ is a map $u : E \to E'$ such that $\pi = \pi' \circ u$.

If $\lambda = (E, \pi, B)$ is a bundle and f is a map from another manifold B' into the base B, then the *induced bundle* or *pull-back* of λ is the bundle $f^*(\lambda)$ over B' with total space $E' := \{(b', p) \in B' \times E \mid f(b') = \pi(p)\}$ and with the projection map $\pi' : E' \to B'$ defined by $\pi'(b', p) = b'$. There is a natural bundle map from the pull-back bundle $f^*(\lambda)$ to the original bundle λ, defined as the pair of maps (f_λ, f), where $f_\lambda : E' \to E$ and $f_\lambda(b', p) := p$.

An n-dimensional complex *vector bundle* $F \hookrightarrow E \xrightarrow{\pi} B$ is a bundle together with the structure of an n-dimensional complex vector space on each fiber $\pi^{-1}(b) \cong F$ such that the following local triviality condition is satisfied. For each $b \in B$, there exists a neighborhood $U \subset B$ of b and a U-isomorphism $h : U \times \mathbb{C}^n \to \pi^{-1}(U)$ such that, for all $y \in U$, the restriction $\{y\} \times \mathbb{C}^n \to \pi^{-1}(y)$ is a vector space isomorphism.

Definition: Canonical vector bundle. The *complex Grassmann manifold* (or *Grassmannian*), $G_N^n(\mathbb{C})$, is the set of n-dimensional complex subspaces in $\mathbb{C}^N$, provided with the structure of a complex manifold.[57], p.133. In particular, the Grassmannian $G_N^1(\mathbb{C})$ is N-dimensional complex projective space $\mathbb{C}P_N$ defined in Example 9 of the appendix and used to describe physical states in quantum mechanics. There is a canonical vector bundle $\mathcal{E}^n$ over G_N^n: any point $x \in G_N^n$ is an n-dimensional subspace in $\mathbb{C}^N$, and the fiber F_x^n over x is this subspace itself. That is,

$$\begin{aligned} &\mathcal{E}^n = \{(v, w) \mid v \in w\} \subset \mathbb{C}^N \times G_N^n \\ &\pi = \mathcal{E}^n \to G_N^n\,, \quad \pi(v, w) := w\,. \end{aligned} \tag{4.37}$$

Example: Spectral line bundle. Let $\mathcal{H}$ be a separable Hilbert space. Suppose that $\{H(x)\}_{x\in\mathcal{M}}$ is a family of self-adjoint operators on $\mathcal{H}$, and that the family is parameterized by points x of a smooth manifold $\mathcal{M}$. Assume that for each $x \in \mathcal{M}$, $H(x)$ has a multiplicity one eigenvalue $\lambda(x)$ that is separated from the rest of the spectrum by a gap, that is, there exist two real-valued functions f and g on $\mathcal{M}$ such that $dist[(g(x), h(x); \lambda(x)] > 0$. Also assume that the function $\lambda : \mathcal{M} \to \mathbb{R}$ is differentiable. Define a map $P : \mathcal{M} \to \mathbb{C}P_N$ such that Ran $P(x) \cong V_x$ where V_x is the one-dimensional eigenspace associated to $\lambda(x)$. The spectral line bundle $F \hookrightarrow E \xrightarrow{\pi} \mathcal{M}$ is the one-dimensional vector bundle, with fibers $F_x = \pi^{-1}(x) \cong V_x$ and total space $F = \bigsqcup_{x\in\mathcal{M}} = \{(x, v) \mid v \in V_x\}$, isomorphic to the pullback bundle $P^*(\mathcal{E}^1)$ of the canonical line bundle over $\mathbb{C}P_N$. The generalization to n-dimensional vector spectral bundles, given in the next section, enables us to study systems with Hamiltonians with more varied spectra.

Next we wish to discuss an important result that we will use to relate the theory of vector bundles to the adiabatic theorem. That is, any n-dimensional complex vector bundle is isomorphic to a pullback bundle of the canonical vector bundle over the Grassmannian $G_N^n(\mathbb{C})$[37]. Moreover, the n-dimensional vector bundles are sorted into isomorphism classes by the homotopy classes of maps into $G_N^n(\mathbb{C})$. In order to explain this construction, we first recall some facts about projection operators.

Let $\mathcal{N}$ be a closed subspace of a Hilbert space $\mathcal{H}$. The *orthogonal complement* of $\mathcal{N}$ in $\mathcal{H}$, denoted $\mathcal{N}^{\perp}$, is the set of vectors in $\mathcal{H}$ that are orthogonal to $\mathcal{N}$, that is, $\mathcal{N}^{\perp} = \{x \in \mathcal{H} \mid (x,n) = 0 \; for \, all \, n \in \mathcal{N}\}$. The projection theorem states that every $x \in \mathcal{H}$ can be written uniquely as $x = z + w$, where $z \in \mathcal{N}$ and $w \in \mathcal{N}^{\perp}$. Let P be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. If $P^2 = P$, then P is called a *projector*. If $P = P^*$, then P is called an *orthogonal projector*. The range of a projector P, denoted Ran P, is always a closed subspace on which P acts as the identity. If P is orthogonal, then P acts as the zero operator on (Ran $P)^{\perp}$. In this case, for any $x \in \mathcal{H}$, the projection theorem guarantees the decomposition $x = y + z$, with $y \in$ Ran P and $z \in$ (Ran $P)^{\perp}$. Then $Px = y$ and P is called the orthogonal projector onto Ran P. Thus, the projection theorem sets up a one to one correspondence between orthogonal projectors and closed subspaces.[79]

Since the Grassmannian $G_N^n(\mathbb{C})$ is the manifold of n-dimensional complex subspaces in $\mathbb{C}^N$, the projection theorem provides a correspondence between the points of $G_N^n(\mathbb{C})$ and orthogonal projectors, as follows. Suppose that $\{u_1, \cdots, u_n\}$ is a basis for the closed subspace $U \subset \mathbb{C}^N$. In the case that N is a finite number, let $M^*(N, n; \mathbb{C})$ be the space of $N \times n$ complex matrices of rank n. Each element of $M^*(N, n; \mathbb{C})$ may be considered as a set of linearly independent vectors in $\mathbb{C}^N$, which, in turn, determines an n-dimensional complex subspace in $\mathbb{C}^N$. Thus, we have the natural projection $M^*(N, n; \mathbb{C}) \rightarrow G_N^n(\mathbb{C})$.[57] If A is the matrix with the vectors $\{u_1, \cdots, u_n\}$ as columns, then the projection is the $n \times N$ matrix $P_A = A(A^{\dagger}A)^{-1}A^{\dagger}$. Since the matrix P can be bordered by zeros, the dimension N can be made arbitrarily large.

Two useful identities hold for any differentiable projector function $P : [0,1] \rightarrow \mathcal{L}(\mathcal{H})$,

$t \mapsto P(t)$. By differentiating both sides of the statement $P^2 = P$, we have

$$\dot{P} = P\dot{P} + \dot{P}P\,. \tag{4.38}$$

Multiplying both sides of (4.38) by P on the right and using $P^2 = P$ again gives

$$P\dot{P}P = 0\,. \tag{4.39}$$

Now we can show that an n-dimensional vector bundle over a manifold $\mathcal{M}$ is isomorphic to a pullback bundle by an orthogonal projector valued function on $\mathcal{M}$.[37]. Let $P(x)$ be such a function on $\mathcal{M}$, its values being $N \times N$ matrices and suppose that rank($P(x)) = n$, for all $x \in \mathcal{M}$. Then Ran $P(x) \in G_N^n$, $P^2(x) = P(x)$, and $\mathrm{tr}\, P(x) = n$. The function $P(x)$ defines a vector bundle, $\mathcal{E}$, over $\mathcal{M}$ as follows. The bundle $\mathcal{E}$ over $\mathcal{M}$ induced by the map

$$f = \mathrm{R}an\, P : \, \mathcal{M} \to G_N^n\,. \tag{4.40}$$

is defined as the pull-back of the canonical bundle $\mathcal{E}^n$ by the map f. The fiber F_x of the bundle $\mathcal{E}$ over $\mathcal{M}$ is isomorphic to the range of $P(x)$, a subspace in $\mathbb{C}^N$.

The Grassmannians G_N^n are called *classifying spaces* for n-dimensional vector bundles. The map (4.40) is called a *characteristic map*, or classifying map, and is used to state and prove the classifying theorem of vector bundles. Recall that a *homotopy* is a continuous one-parameter family of maps: $f_t : X \to Y$, $t \in [0,1]$ such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. Two maps f and $g : X \to Y$ are *homotopic* provided there is a homotopy f_t with $f = f_0$ and $g = f_1$.

Theorem 20 *[37, 51] (Classifying Theorem) Any n-dimensional vector bundle over a compact base is isomorphic to a bundle, induced by (4.40) for N sufficiently large. Two such bundles are isomorphic if and only if their characteristic maps are homotopic for a sufficiently large N.*

Fedosov[37] uses the following lemma to prove one implication of the second statement of the classifying theorem, that is, that n-dimensional vector bundles with homotopic characteristic maps are isomorphic.

Lemma 21 *[37] (Intertwining Lemma) Let $P(x,t)$ be a smooth projector-valued function on $\mathcal{M} \times [0,1]$. Then there exists an invertible matrix function $U(x,t)$ such that*

$$P(x,t) = U(x,t)P(x,0)U^{-1}(x,t)\,. \tag{4.41}$$

Proof: We fix $x = x_0 \in \mathcal{M}$ and suppress the x-dependence of P and U. Define $U(t)$ as the solution of

$$\dot{U}U^{-1} = [\dot{P}, P] \tag{4.42}$$

with the initial condition $U(0) = 1$. (The solution exists since $[\dot{P}, P]$ is bounded (Theorem X.69 [80]).) Define $\tilde{P}(t) = U(t)P(0)U^{-1}(t)$. Then $\tilde{P}(t)$ and the given function $P(t)$ satisfy the same differential equation

$$\dot{P}(t) = [\dot{U}U^{-1}, P]\,. \tag{4.43}$$

Indeed, from (4.42), we have

$$[\dot{U}U^{-1}, P] = [[\dot{P}, P], P] = \dot{P}P - P\dot{P}P - P\dot{P}P + P\dot{P} = \dot{P} \tag{4.44}$$

by using (4.38) and (4.39). On the other hand,

$$[\dot{U}U^{-1}, \tilde{P}] = \dot{U}P(0)U^{-1} + UP(0)\frac{dU^{-1}}{dt} = \dot{\tilde{P}}\,. \tag{4.45}$$

Moreover, since $U(0) = 1$, $\tilde{P}$ and P have the same initial value. Hence, P and $\tilde{P}$ coincide everywhere. Thus, we have shown that $P(x_0, t) = \tilde{P}(x_0, t)$ for all t. The argument holds for each $x \in \mathcal{M}$. We need to use joint smoothness in x and t here. $\square$

Given a homotopy $P_t \equiv P(\cdot, t)$ between the characteristic maps P_0 and P_1 of two vector bundles over $\mathcal{M}$, the result (4.41) means that the subspaces Ran P_0 and Ran P_1 are isomorphic (*and* that they are isomorphic to all intermediate subspaces Ran P_t for all $t \in [0,1]$). Hence the pullback bundles $P_0^*(\mathcal{E}^n)$ and $P_1^*(\mathcal{E}^n)$ are isomorphic to each other (*and* to all of the intervening pullback bundles $P_t^*(\mathcal{E}^n)$ for $t \in [0,1]$).

4.5 The Adiabatic Theorem

At first sight, it is perhaps remarkable that a version of the Intertwining Lemma is also used in proofs[53, 67, 11] of the adiabatic theorem for quantum mechanics. The adiabatic theorem is a statement about the validity of an approximation to the dynamics generated by Hamiltonians that vary slowly in time. Observe that, in general, the Schrödinger equation, $i\hbar\dot{\psi} = H(t)\psi$, with time-dependent Hamiltonian operator H has no stationary states. The theorem states that, if $H(t)$ varies with t slowly enough, then a system initially in one part of the spectrum of $H(0)$ will pass through the corresponding part of the spectrum of $H(t)$ for all t.

To be more precise, let $s = t/T$, where T is the intrinsic time scale of the system. In rescaled variables, the Schrödinger equation is

$$i\hbar\frac{d\psi(s)}{ds} = TH(s)\psi(s)\,, \tag{4.46}$$

where $s \in [0,1]$. Under general conditions on H,[80, Thm. X.71] there exists a unitary propagator U_T on $L^2(\mathbb{R}^n)$ which satisfies

$$i\hbar\frac{\partial U_T(s;s')}{\partial s} = TH(s)U_T(s;s')\,,\ \ s \geq s'\,,\ \ U_T(s',s') = 1\,. \tag{4.47}$$

so that $\psi(s) = U_T(s;s')\psi(s')$ satisfies (4.46). The adiabatic limit corresponds to the limit $T \to \infty$. Kato's version[53] of the theorem assumes $H(0)$ has an isolated eigenvalue, whose multiplicity may be greater than one. Avron, Seiler, and Yaffe[11] and Nenciu[67] give proofs under less restrictive assumptions about the spectrum of H, that is, H is only required to satisfy the gap condition: a part of the spectrum of H is separated by an energy gap from the other parts. Avron and Elgart[4] have proved a version of the theorem without assuming the gap condition. All three proofs[53, 67, 11] of the theorem with the gap assumption consist of two parts. In the first part, the lemma is used to obtain the unitary evolution operator U_a representing an adiabatic transformation of the system. In the second part, one shows that the true dynamic evolution operator determined by equation (4.47) approaches U_a as $T \to \infty$. It is only for the second part that different proofs are required for the different assumptions made about the spectra of the Hamiltonians $H(t)$ in [53, 67, 11].

The adiabatic theorem can be understood in terms of a spectral vector bundle over parameter space, which is a generalization of the spectral line bundle.

Example: Spectral vector bundle. As for the spectral line bundle, suppose that $\{H(x)\}_{x\in\mathcal{M}}$

is a family of self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and that the family is parameterized by points x of a smooth manifold $\mathcal{M}$. For each $x \in \mathcal{M}$, the spectral theorem for self-adjoint operators associates with $H(x)$ a set of projections that completely determines $H(x)$. Let $P(x)$ be a map from $\mathcal{M}$ to rank n projectors onto a part of the spectrum of $H(x)$ that is separated from the rest by a gap. Let $\mathcal{H} \hookrightarrow \mathcal{H} \times \mathcal{M} \to \mathcal{M}$ be the trivial bundle over $\mathcal{M}$. Construct a sub-bundle whose fibers F_x are the vector subspaces $V_x =$ Ran $P(x)$, that is, the fiber at x is $F_x :=$ Ran $P(x) \subset \mathcal{H}$. We call the vector bundle $F \hookrightarrow \mathcal{H}_P \to \mathcal{M}$ with total space $\mathcal{H}_P := \bigcup_{x \in \mathcal{M}} F_x$ a *spectral bundle*. Note that the spectral bundle $F \hookrightarrow \mathcal{H}_P \to \mathcal{M}$ is isomorphic to the pullback bundle $P^*(\mathcal{E}^n)$ and that

$$\mathcal{H}_P = \{(x,p) \in \mathcal{M} \times \mathcal{E}^n \mid \mathrm{R}an\, P(x) = \pi(p)\} \tag{4.48}$$
$$\pi = \mathcal{E}^n \to G_N^n\,, \quad \pi(v,w) := w\,.$$

Now consider how the intertwining lemma serves to establish the existence of a unitary propagator that determines adiabatic time-evolution of state vectors in $\mathcal{H}_P$. A path $[0,1] \to \mathcal{M}$ in the space of parameters $\mathcal{M}$ gives rise to a path in the family of parameterized Hamiltonians, that is, to a time-dependent Hamiltonian $H(t) = H(x(t))$. In turn, for each $t \in [0,1]$, $P(t)$ is defined as a projector onto one part of the spectrum $\sigma(H(t))$, separated from the rest of the spectrum by a gap. If we begin the time evolution with a state vector $\psi(0) \in$ Ran $P(0) \subset \mathcal{H}$, then $P(0)\psi(0) = \psi(0)$. In the adiabatic approximation, at any later time t of order T, the state vector $\psi(t)$ is required to be an element of Ran $P(t)$, that is, $P(t)\psi(t) = \psi(t)$. This requirement is satisfied if there exists a unitary operator U_a that time evolves ψ:

$$\psi(t) = U_a(t)\psi(0) \tag{4.49}$$

where U_a has the intertwining property

$$P(t) = U_a(t)P(0)U_a^{-1}(t) \tag{4.50}$$

which is (4.41) of the lemma statement. Indeed, if (4.50) holds, then

$$\begin{aligned} P(t)U_a(t)\psi(0) &= U_a(t)P(0)\psi(0) \\ &= U_a(t)\psi(0)\,. \end{aligned} \tag{4.51}$$

Then, by (4.49),

$$P(t)\psi(t) = \psi(t)\,. \tag{4.52}$$

Observe that adiabatic time evolution can be viewed as parallel transport by lifting a path in $\mathcal{M}$ to a path in a Hilbert bundle. If $x \in \mathcal{M}$ is fixed, then $H_x := H(x)$, regarded as time-independent Hamiltonian, generates a one-parameter group $\{U_x(t) := \exp(-itH_x) \mid t \in \mathbb{R}\}$. Take a sequence of partitions of a given interval $[s, t]$

$$s \equiv t_0 < t_1 < \cdots t_{n-1} < t_n \equiv t,\;\; \lim_{n\to\infty} \sup(t_{k-1} - t_k) = 0\,, \tag{4.53}$$

we can represent a unitary propagator as a limit

$$U(t,s)\psi = \lim_{n\to\infty} U_{x(t_n)}(t_n - t_{n-1})U_{x(t_{n-1})}(t_{n-1} - t_{n-2}) \cdots U_{x(t_1)}(t_1 - t_0)\psi \tag{4.54}$$

see [2, p.59]

From the proof of the lemma, it is clear that U_a satisfies (4.50) if U_a satisfies

$$i\hbar \frac{dU_a(s)}{ds} = TH_K(s)U_a(s) \tag{4.55}$$

$$H_K(s) := \frac{i\hbar}{T}[\dot{P}, P]\,. \tag{4.56}$$

The solution U_a determined from (4.55) with the "Kato Hamiltonian" (4.56) is not unique. Avron *e*t al.[11] showed that the requirement that the adiabatic evolution approximates the true physical evolution "as best as possible" determines it uniquely. This is the evolution generated by

$$H_a(s, P) := H(s) + i/\tau[P'(s), P(s)]\,. \tag{4.57}$$

The crux of the adiabatic theorem is that the true time evolution operator U_T, which solves (4.47), is approximated by U_a in the adiabatic limit:

Theorem 22 *[53, 67, 11](Adiabatic theorem) Let $\{H(s)\}$ be a smooth family of Hamiltonians, $s = t/T \in [0,1]$, and let $U_T(s)$ be the true time evolution operator (4.47). Let $\{P(s)\}$ be the family of finite rank projectors onto a band of the spectrum $\sigma(H(s))$ separated by a gap. Then*

$$\lim_{T\to\infty} \|U_T(s)P(0) - P(s)U_T(s)\| = 0\,.$$

The line bundle over a parameter space, $\mathcal{M}$, described by Simon[87] is a spectral bundle that is constructed as follows. As in the discussion of Berry's work, assume $H(x)$ is a Hermitian operator depending smoothly on the parameters $x \in \mathcal{M}$, with an isolated nondegenerate eigenvalue $E_n(x)$ depending continuously on x. Then $\{(x, \psi) \mid H(x)\psi = E_n(x)\psi\}$ defines a spectral line bundle over parameter space. The fiber at $x \in \mathcal{M}$ is the one-dimensional eigenspace L_x of $H(x)$ corresponding to the eigenvalue $E_n(x)$ and associated eigenvector $\psi_n(x)$: $L_x = \{\psi \in S(\mathcal{H}) \mid \psi = c\psi_n(x),\, c \in \mathbb{C}\}$.

The spectral line bundle over parameter space can be related to the principal bundle $S(\mathcal{H})$ over $P(\mathcal{H})$ as follows[21]. The principal bundle $\eta : U(1) \hookrightarrow S(\mathcal{H}) \xrightarrow{\pi} P(\mathcal{H})$ used to describe the Aharonov and Anandan phase is the universal classifying bundle for $U(1)$ principle fiber bundles and the classifying theorem states that any $U(1)$ principle bundle over $\mathcal{M}$ is isomorphic to the pull-back bundle $f^*(\eta)$ for some continuous function $f : \mathcal{M} \to \mathbb{C}P_\infty$. The spectral line bundle is the associated bundle of this $U(1)$ principal bundle.

In gauge field theory[18, 32, 65], there is a background field, such as the electromagnetic field, that is given by a connection A defined on a principal bundle over spacetime, $\mathcal{M}$. The structure group of this bundle represents the internal symmetries of the background field. The possible interactions of the background field with matter are determined by the representations of the group. Each particle field ψ is a section of the vector bundle over $\mathcal{M}$ associated to the principle bundle through a group representation. The Hilbert space $\mathcal{H}$ in quantum mechanics is the space of sections of this vector bundle. For example, if the structure group is $U(1)$, then such a section is a function $\psi : \mathcal{M} \to \mathbb{C}$.

Simon[87] observed that the adiabatic theorem defines a connection on such a vector bundle, which is the same as the Hermitian connection inherited from the embedding of the total space of this bundle into the Hilbert space. The adiabatic evolution transports state vectors from the range of the projector $P(x)$ to the range of $P(x')$. The adiabatic connection can be taken to be the operator-valued one-form[10]

$$A(P)(x) = -[(dP)(x), P(x)]\,, \tag{4.58}$$

and the covariant derivative corresponding to the adiabatic connection is

$$\nabla := d + A = Pd\,. \tag{4.59}$$

In summary, we have given an interpretation of the adiabatic theorem in terms of the intertwining lemma used to prove the vector bundle classifying theorem. To link the discussion with Weinstein's[93] interpretation of the adiabatic theorem in terms of cyclic evolution of a lagrangian submanifold of a symplectic manifold, the lemma can be used to define an isomorphism of lagrangian subbundles defined on the lagrangian tori corresponding to an evolving integrable Hamiltonian system. For each lagrangian immersion $i : \mathcal{L} \to P(\mathcal{H})$ along a path in $\Lambda(\mathcal{L})$, define a line bundle over $\mathcal{L}$, determined by a one dimensional eigenspace of the Hamiltonian operator $\hat{H}(t)$, and corresponding projection operator, $P(z,t)$, $z \in \mathcal{L}$. The time evolution problem may be viewed as a homotopy of classifying maps

$$P(z,t) = U(z,t)P(z,0)U(z,t)^{-1} \tag{4.60}$$

$$\frac{dU}{dt}U^{-1} = \left[\frac{dP}{dt}, P\right] \tag{4.61}$$

By the classifying theorem of vector bundles, the homotopy $P(z,t)$ gives an isomorphism of the line bundles. This is a plan for future work.

Chapter 5

ADIABATIC CURVATURE AND THE RESPONSE OF A CRYSTALLINE SOLID

This chapter discusses the curvature of the connection on a fiber bundle associated with the energy spectrum of a periodic Schrödinger operator, and shows the relationship between this curvature and an electromagnetic linear response function. The curvature of the connection of the spectral bundle is the pullback of the symplectic form on $P(\mathcal{H})$. The first Brillouin zone is a parameter space and a moduli space of connections on the space of lattice periodic wave functions.

5.1 Symplectic Structure of the Periodic Solid

Let $a_1, \cdots, a_N \in \mathbb{R}^N$ be N linearly independent vectors. The set

$$\Lambda = \{a \in \mathbb{R}^N \mid a = \sum_{i=1}^{N} m_i a_i,\ m_i \in \mathbb{Z}\} \tag{5.1}$$

is called the *lattice* generated by $\{a_i\}_{i=1}^N$.

The set Λ is a normal subgroup of $\mathbb{R}^N$, viewed as a group under addition. The set of equivalence classes under the equivalence relation $x \sim y$ if and only if $x - y \in \Lambda$ is itself a group, denoted by $\mathbb{R}^n/\Lambda$. That is, the equivalence class of $x \in \mathbb{R}^N$ is $[x] = \{y \in \mathbb{R}^N | \, x \sim y\}$ and $\mathbb{R}^N/\Lambda = \{[x] \,|\, x \in \mathbb{R}^N\}$. In addition to being a group, $\mathbb{R}^N/\Lambda$ is a smooth manifold, which is an N-dimensional torus, $\mathbb{T}^N$. Indeed, let Q denote the half open parallelepiped spanned by the $\{a_i\}$, that is,

$$Q := \{y \in \mathbb{R}^N \mid y = x_1 a_1 + \cdots x_N a_N,\ 0 \le x_i < 1\ \forall i\}\,. \tag{5.2}$$

The map $Q \to \mathbb{R}^N/\Lambda$, $x \mapsto [x]$ is injective; its inverse is a coordinate map for $\mathbb{R}^N/\Lambda$. The translates of Q cover $\mathbb{R}^N$ in a space-filling, one-to-one manner: $\bigcup_{a\in\Lambda} aQ = \mathbb{R}^N$. Hence, Q is called a *primitive cell* of the lattice.

In this chapter, we are interested in functions that are periodic with respect to Λ. A function $f : \mathbb{R}^N \to \mathbb{R}$ is periodic with period a, if $f(x + a) = f(x)$ whenever $a \in \Lambda$. For example, the function $f(x) = e^{ig\cdot x}$ has period a if and only if $e^{ig\cdot(x+a)} = e^{ig\cdot a}$. This is true if and only if $e^{ig\cdot a} = 1$, that is, if and only if $g \cdot a = 2\pi n$, for some $n \in \mathbb{Z}$. Thus, we are interested in the set $\{g \in \mathbb{R}^N | \, g \cdot a \in 2\pi\mathbb{Z}$ for all $a \in \Lambda\}$.

Let Λ be a lattice in $\mathbb{R}^N$. Define the dual basis $\{g_j\}_1^N$ by $a_i g_j = 2\pi\delta_{ij}$. The dual lattice $\Lambda^* = \{g \in \mathbb{R}^N \mid g = \sum_{j=1}^N m_j g_j, \;\; m_j \in \mathbb{Z}\}$ is called the *reciprocal lattice* and

$$B := \{y \in \mathbb{R}^N \mid y = x_1 g_1 + \cdots x_N g_N, \; 0 \le x_i < 1 \; \forall i\} \cong \mathbb{R}^N/\Lambda^* \tag{5.3}$$

is called the *Brillouin zone*.

Let us restrict our attention to the case $N = 3$. We will show that the manifold $Q \times B$ is a symplectic manifold. Let $\{a_1, a_2, a_3\}$ be a basis of primitive translation vectors of the lattice, Λ. The volume of the 3d parallelepiped spanned by $\{a_1, a_2, a_3\}$ may be written in terms of the triple scalar product,

$$V_{cell} = a_1 \cdot (a_2 \times a_3) \tag{5.4}$$

The standard definition of the dual basis is

$$b_1 = \frac{2\pi}{V_{cell}}(a_2 \times a_3)\,, \;\; b_2 = \frac{2\pi}{V_{cell}}(a_3 \times a_1)\,, \;\; b_3 = \frac{2\pi}{V_{cell}}(a_1 \times a_2)\,. \tag{5.5}$$

Viewing the triple scalar product as a volume form on the unit cell,

$$\begin{aligned} &vol : T_Q \times T_Q \times T_Q \to \mathbb{R} \\ &vol(x_1, x_2, x_3) = x_1 \cdot (x_2 \times x_3) = V_{cell} \end{aligned} \tag{5.6}$$

Then the dual vector b_1 is a map $T_Q \to \mathbb{R}$,

$$b_1(\cdot) = \frac{2\pi}{V_{cell}} vol(\cdot, a_2, a_3) \tag{5.7}$$

that is, b_1 is a one-form on Q, and similarly, so are b_2 and b_3. Define a two-form on $Q \times B$ by

$$\omega((s_1, n_1), (s_2, n_2)) = n_2(s_1) - n_1(s_2)\,. \tag{5.8}$$

Compare to the canonical symplectic form on the cotangent bundle of a torus: $\omega = \sum_{i=1}^3 dr_i \wedge dk_i$. Here, $b_1 = k_i dr^i$.

5.2 The Spectral Bundle and the One-Band Subbundle

Consider a single electron in $\mathbb{R}^N$ subject to a periodic potential, V. The Hamiltonian for the electron in the lattice is

$$H = \frac{p^2}{2m} + V\,, \tag{5.9}$$

where $p = -i\hbar\nabla_x$. We assume here that V satisfies $\int_Q |V(x)|^2 d^n x < \infty$. The translation operators $T_j = \exp(ia_j \cdot p/\hbar)$, defined by their operation on wave functions $\psi \in L^2(\mathbb{R}^N)$: $T_j\psi(x) = \psi(x + a_j)$, then commute with the Hamiltonian and with each other. Hence, H has eigenfunctions which are simultaneously eigenfunctions of the translation operators. The eigenvalues of T_j may be parameterized as $\lambda_j = \exp(ik \cdot a_j)$ by the *Bloch wave vector* k, which is an element of the reciprocal lattice. The Bloch wave functions $\psi(x + a_j) = \exp(ik \cdot a_j)\psi(x)$ are invariant if k is translated by a reciprocal lattice vector. Thus, it is sufficient to restrict k to the first Brillouin zone.

Now consider the eigenvalue problem

$$H\psi = E\psi\,, \tag{5.10}$$

for H given by (5.9). Since the Hamiltonian commutes with translations by lattice vectors, we may choose the eigenvectors (weak eigensolutions) of H to be Bloch vectors ψ_k^n with eigenvalues E_k^n:

$$H\psi_k^n = E_k^n \psi_k^n \tag{5.11}$$

$$\psi_k^n(x) = e^{ikx} u_k^n(x) \tag{5.12}$$

Here n labels a discrete set of finite-dimensional eigenspaces, and k is a continuous parameter. The u_k^n are smooth functions with the same periodicity as the lattice. To see why the result (5.11) with (5.12) is true, substitute (5.12) into (5.10) and find that, for any real vector k, u_k^n solves

$$H_k u_k^n = E_k u_k^n \tag{5.13}$$

where $H_k = e^{-ikx} H e^{ikx}$.

For the Hamiltonian (5.9),

$$H_k = \frac{1}{2m}(p + \hbar k)^2 + V\,. \tag{5.14}$$

Now solve the eigenvalue equation (5.13), imposing periodic boundary conditions on u_k^n. As discussed by Odeh and Keller in [69], H_k defines a symmetric operator in the space of continuously differentiable functions whose first partial derivatives are absolutely continuous. The smoothness conditions on the potential are sufficient to assure that there is a unique self-adjoint extension of H_k in $L^2(Q)$, also denoted by H_k. This operator, being a regular uniformly elliptic self-adjoint operator defined in a bounded domain, possesses a discrete set of eigenvalues $\{E_k^n\}_{n=1}^{\infty}$, each of finite multiplicity, and corresponding eigenfunctions u_k^n. By extending each u_k^n to the whole space by periodicity, we obtain solutions of the form (5.12).[69] The authors[69] showed that the set of Bloch waves $\{\psi_k^n\}$, where k ranges over $\mathbb{B}$ and n ranges over the integers, is complete in $L^2(\mathbb{R}^3)$ that is, complete in the sense of eigenfunction expansions. Reed and Simon[81] attribute the following theorem to Gel'fand[41].

Theorem 23 *(Gelfand, Odeh-Keller) Let the map $U_B : L^2(\mathbb{R}^N) \to L^2(\mathbb{N} \times \mathcal{B})$ be given by*

$$(U_B f)(n,k) = \int_{\mathbb{R}^N} \psi_{nk}(x) f(x) d^N x\,. \tag{5.15}$$

a) U_B is an isometric isomorphism of $L^2(\mathbb{R}^N)$ onto $L^2(\mathbb{N} \times \mathcal{B})$ with inverse given by

$$(U_B^{-1} F)(x) = \int_{\mathbb{B}} \sum_{n=1}^{\infty} F_n(k) \overline{\psi_{nk}(x)} d^N k\,. \tag{5.16}$$

b) U_B diagonalizes the Hamiltonian (5.9), that is, for all $F \in L^2(\mathbb{N} \times \mathbb{B})$,

$$(U_B H U_B^{-1} F)(n,k) = E_k^n F_n(k)\,. \tag{5.17}$$

Moreover, in the notation E_k^n for the n^{th} isolated eigenvalue of the auxiliary problem (5.13), Odeh and Keller[69] showed that E^n is an analytic, though not necessarily single-valued, function of k. As k varies continuously over the Brillouin zone, the eigenvalues trace out a set of energy bands.

Next construct a bundle of Hilbert spaces over the Brillouin zone.

Hilbert space bundle Definition[17, 40, 16]: Let $(X, \mathcal{A}, \mu)$ be a measure space with a non-negative σ-finite measure μ, and let $\mathcal{R}$ be a nonempty family of separable Hilbert spaces. Consider the case in which all members of the family $\mathcal{R}$ have the same dimension. To each $x \in X$ assign $\mathcal{H}(x) \in \mathcal{R}$, with norm $| \cdot |_x$ and inner product $(\, , \,)_x$. This assignment is a *Hilbert space bundle* with total space $\mathcal{P} = \sqcup_{x \in X} \mathcal{H}(x)$ and base space X. The fiber of the bundle at $x \in X$ is the Hilbert space $\mathcal{H}(x)$. A section of the bundle is then a function $F : X \to \mathcal{P}$ so that $F(x) \in \mathcal{H}(x)$. Accordingly, $\mathcal{V} := \times_{x \in X} \mathcal{H}(x)$ is the set of sections of the bundle. The elements of $\mathcal{V}$ are called *vector-valued functions* and $\mathcal{V}$ becomes a vector space when equipped with pointwise defined algebraic operations. A set $\{E_n\} \subset \mathcal{V}$ is called a *measurability basis* if $\{E_n(x)\}$ is an orthonormal basis in $\mathcal{H}(x)$ for any $x \in X$. Given a measurability basis $\{E_n\}$, the set of measurable vector-valued functions in $\mathcal{V}$ is defined as $\mathcal{V}_\mu(\{E_n\}) := \{F \in \mathcal{V} \mid x \mapsto (E_n(x), F(x))_x$ is measurable for all $n\}$. The map $x \mapsto \mathcal{H}(x)$ together with the subspace $\mathcal{V}_\mu \subset \mathcal{V}$ is called a *measurable Hilbert space field* on $(X, \mathcal{A}, \mu)$. Now define the *direct integral of the measurable field* $(x \mapsto \mathcal{H}(x), \mathcal{V}_\mu)$ on $(X, \mathcal{A}, \mu)$ as

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x) d\mu(x) := \mathcal{L}^2 / \mathcal{L}_0 \, , \tag{5.18}$$

where $\mathcal{L}^2 := \{F \in \mathcal{V}_\mu \mid \int_X |F(x)|_x^2 d\mu(x) < \infty\}$ is the set of square-integrable sections and $\mathcal{L}_0 := \{F \in \mathcal{V}_\mu \mid F(x) = 0 \, \mu a.e.\}$ is the set of null-sections. The inner product on $\mathcal{H}$ is defined as

$$(F, G)_{\mathcal{H}} := \int_X (F(x), G(x))_x d\mu(x) \, . \tag{5.19}$$

It can be shown that $\mathcal{H}$ defined in (5.18) and (5.19) is a separable Hilbert space[16], p.30. For the case of identical summands considered here, $\mathcal{H}(x) = \mathcal{H}'$ for all $x \in X$. In this case, $\mathcal{H} = L^2(X, d\mu; \mathcal{H}')$ and $\mathcal{H}$ is called a *constant fiber direct integral*.

This Hilbert bundle construction can be applied to the periodic Schrödinger operator problem, as follows. The Brillouin zone with Lebesgue measure $(\mathcal{B}, d^N k)$ plays the role of the base space of the bundle. The elements k of $\mathcal{B}$ parameterize a family of self-adjoint operators $\{H_k\}_{k \in \mathcal{B}}$ as defined in (5.14); for each k, H_k is densely defined on $L^2(Q)$. Hence,

$\mathcal{R} = \{\mathcal{H}(k)\}$ is a family of Hilbert spaces with $\mathcal{H}(k) = L^2(Q)$ for each k. The sections $F : \mathcal{B} \to \mathcal{P} := \sqcup_{k\in\mathcal{B}} L^2(Q)$ comprise the set $\mathcal{V}$. For each k, order the eigenvalues of H_k as $\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^n, \cdots$. Then, the eigenfunctions $\{u_k^{n,\alpha(n)}\}$ of H_k may be chosen to form an orthonormal basis for $\mathcal{H}(k)$. Here, n indexes the eigenvalues and $\alpha(n)$ distinguishes the eigenvectors corresponding to λ_k^n. Then $u^{n,\alpha(n)} : \mathcal{B} \to \mathcal{P}$ defined by $u^{n,\alpha(n)}(k) = u_k^{n,\alpha(n)}$ is a section for each n, and $\{u^{n,\alpha(n)}\}$ is a measurability basis for the measurable sections: $F(k) = \sum_{n,\alpha(n)} F_n(k) u_{nk}$ where $F_n(k) := (u_{nk}, F(k))_{L^2(Q)}$. This gives the constant fiber direct integral

$$\mathcal{H} := \int_{\mathcal{B}}^{\oplus} L^2(Q) d^N k = L^2(\mathcal{B}, d^N k; L^2(Q))\,. \tag{5.20}$$

Also, it is straightforward to show that $\mathcal{H} \cong L^2(\mathbb{N} \times \mathcal{B})$. A comparable approach[81, 9] uses a constant fiber direct integral of the l^2 Hilbert spaces for each $k \in Q^*$.

Gel'fand's theorem may be used to expand elements of $\mathcal{H}$ in terms of Bloch waves (Fourier-Bloch transform):

Corollary 24 *Let $F \in \mathcal{H}$. Then*

$$\begin{aligned} F_n(k) = F_n(k) &= \int_{\mathcal{B}} \sum_{n=1}^{\infty} \eta(kn; k'n') F_n(k') d^N k'\,, && (5.21)\\ \eta(kn; k'n') &= \int_{\mathbb{R}^N} \psi_{nk'}(x) \overline{\psi_{nk}(x)} d^N x\,. && (5.22) \end{aligned}$$

Proof: By (5.15) and (5.16) of the theorem,

$$\begin{aligned} F_n(k) &= (U_B U_B^{-1} F)(n, k) && (5.23)\\ &= \int_{\mathbb{R}^N} \int_{\mathcal{B}} \sum_{n'=1}^{\infty} F_{n'}(k') \psi_{k'n'}(x) \overline{\psi_{kn}(x)} d^N k' d^N x\,, && (5.24) \end{aligned}$$

□

In the case of the periodic Schrödinger problem, we have seen that, for each real vector $k \in \mathcal{B}$, the Hilbert space $\mathcal{H}_k$ is isomorphic to a direct sum of finite-dimensional eigenspaces of H_k, and the eigenvalues E_k^n are continuous in k. Now suppose that the n^{th} eigenvalue of H_k is isolated for each k. This means that there exists a band of the spectrum of H that is isolated. Let $\sigma(H)$ denote the spectrum of H. A bounded set $\sigma_0 \subset \mathbb{R}$

is an *isolated band of* H if $\sigma(H) = \sigma_0 \bigcup \sigma_1$, with $dist(\sigma_0, \sigma_1) > 0$. Define $P(k)$ to be the spectral projection of H_k corresponding to the isolated n^{th} band. More precisely, for each $k \in \mathcal{B}$, select the n^{th} eigenvalue E_k^n of H_k, and let $P(k)$ be the associated spectral projection operator:

$$P(k) = \frac{1}{2\pi i} \oint_\Gamma \frac{dz}{z - H_k}\,, \tag{5.25}$$

where the contour Γ circles E_k^n in the complex z-plane.

The spectral projection operator P^n corresponding to the n^{th} isolated band is constructed from the $\{P(k)\}_{k\in\mathcal{B}}$. This is a direct integral decomposition. Define a Hilbert space $\mathcal{H}^n$ of sections of the spectral bundle for the n^{th} band.

In the adiabatic approximation the gaps between eigenvalues $E_k^{n+1} - E_k^n$ are assumed to be large compared to the quasiparticle excitation energies. For now, I assume that each eigenspace is one-dimensional. Let $\mathcal{E}_k^n = \{u_k^n e^{i\alpha} \mid \alpha \in \mathbb{R}\}$. Hereafter, we focus exclusively on the nth eigenspace by fixing n and dropping the index n. We then have that the u_k are eigenfunctions of a family of Hamiltonians $\{H_k\}$ parameterized by k. Let $\mathcal{E} = \bigsqcup \mathcal{E}_k$, where $\bigsqcup$ denotes a disjoint union and let $\pi : \mathcal{E} \to \mathcal{B}$ be defined by $u_k \to k$. Then $U(1) \hookrightarrow \mathcal{E} \to \mathcal{B}$ is a principal fiber bundle. The base space $\mathcal{B}$ is isomorphic to a torus and is isomorphic to a subset of projective Hilbert space. The total space $\mathcal{E}$ of the bundle is a subset of the full Hilbert space, and $\mathcal{H}^n = \overline{span\{u_k\}}$. The inner product on $\mathcal{H}^n$ is given by

$$\langle u_k, u_{k'} \rangle = \frac{1}{v} \int_Q u_k^*(x) u_{k'}(x)\, dx\,, \tag{5.26}$$

and v denotes the volume of Q.

5.3 The Current-Current Correlation Function

The connection 1-form on the principal bundle $U(1) \hookrightarrow \mathcal{E} \to \mathcal{B}$ is[96]

$$\mathcal{A}_{u_k} = -\mathrm{Im}\langle u_k, \frac{\partial u_k}{\partial k_\mu}\rangle dk_\mu\,. \tag{5.27}$$

Since $U(1)$ is an abelian group, the curvature of the connection is given by $\Omega_{\mu\nu} = -\mathrm{Im}(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu)$, or

$$\Omega_{u_k} = -\mathrm{Im} \sum_{\mu,\nu} \left(\langle \frac{\partial u_k}{\partial k_\mu}, \frac{\partial u_k}{\partial k_\nu} \rangle - \langle \frac{\partial u_k}{\partial k_\nu}, \frac{\partial u_k}{\partial k_\mu} \rangle \right) dk_\mu \wedge dk_\nu\,. \tag{5.28}$$

This section relates the curvature Ω to the electromagnetic response function for the periodic solid, which may be written in terms of a correlation function. Thouless and others have established the relation between Ω and the interband conductivity, which involves inner products between vectors of different eigenspaces of the Hamiltonian H. In contrast, for the intraband scattering problem considered here, one needs to consider a correlation function involving inner products between different k values for a fixed eigenspace.

Using the corollary, insert

$$1_{\mathcal{H}^n} = \int_{Q^*} d\mu_{k'}\, u_{k'}^* \otimes u_{k'} \,, \tag{5.29}$$

into (5.28). Then, the curvature takes the form

$$\Omega_{u_k} = -\mathrm{I}m \int d\mu_{k'} \, \left(\langle \frac{\partial u_k}{\partial k_\mu}, u_{k'} \rangle \langle u_{k'}, \frac{\partial u_k}{\partial k_\nu} \rangle - \langle \frac{\partial u_k}{\partial k_\nu}, u_{k'} \rangle \langle u_{k'}, \frac{\partial u_k}{\partial k_\mu} \rangle \right) dk_\mu \wedge dk_\nu \,. \tag{5.30}$$

Differentiating the eigenvalue equation (5.13) with respect to k_μ and taking the inner product with $u_{k'}$ on the left yields

$$\langle u_{k'}, \frac{\partial H_k}{\partial k_\mu} u_k \rangle + \langle u_{k'}, H_k \frac{\partial u_k}{\partial k_\mu} \rangle = \frac{\partial E_k}{\partial k_\mu} \langle u_{k'}, u_k \rangle + E_k \langle u_{k'}, \frac{\partial u_k}{\partial k_\mu} \rangle \,. \tag{5.31}$$

The calculation uses equation (5.31) to eliminate the factors of the form $\langle u_{k'}, \frac{\partial u_k}{\partial k_i} \rangle$ and their conjugates in equation (5.30) in favor of terms involving $\langle u_{k'}, \frac{\partial H_k}{\partial k_\mu} u_k \rangle$. Dealing with the second term on the left hand side of (5.31), a straightforward calculation shows that

$$H_k u_{k'} = E_{k'} u_{k'} + e^{-ikx} [H, e^{i(k-k')x}] e^{ik'x} u_{k'} \,. \tag{5.32}$$

For the remainder of this work, we consider only the Hamiltonian (5.9) without spin-orbit coupling. In this case,

$$H_k u_{k'} = E_{k'} u_{k'} + \frac{\hbar^2}{2m} (k^2 - k'^2) u_{k'} + \frac{\hbar}{m} (k - k') \cdot p \, u_{k'} \,. \tag{5.33}$$

Now define

$$E_{kk'} \equiv \langle (\frac{\hbar^2}{2m} (k^2 - k'^2) u_{k'} + \frac{\hbar}{m} (k - k') \cdot p \, u_{k'}), \frac{\partial u_k}{\partial k_\mu} \rangle / \langle u_{k'}, \frac{\partial u_k}{\partial k_\mu} \rangle \,. \tag{5.34}$$

Then, for $k \neq k'$, we have

$$\langle u_{k'}, \frac{\partial u_k}{\partial k_\mu} \rangle = \frac{\langle u_{k'}, \left(\frac{\partial H_k}{\partial k_\mu} - \frac{\partial E_k}{\partial k_\mu} \right) u_k \rangle}{E_k - E_{k'} - E_{kk'}} \,. \tag{5.35}$$

By equation (5.14), we have

$$\frac{\partial H_k}{\partial k_\mu} = \frac{\hbar}{m}(p + \hbar k)_\mu \,. \tag{5.36}$$

Hence,

$$\langle u_{k'}, \frac{\partial u_k}{\partial k_\mu}\rangle = \frac{\langle u_{k'}, (\frac{p_\mu}{m} + \frac{\hbar k_\mu}{m} - \frac{1}{\hbar}\frac{\partial E_k}{\partial k_\mu})u_k\rangle}{\omega_{kk'}}\,, \tag{5.37}$$

where $\omega_{kk'} \equiv (E_k - E_{k'} - E_{kk'})/\hbar$. It may be important to keep in mind that $\omega_{kk'}$ is, in general, complex.

We can write the matrix element in (5.37) in terms of the Bloch functions, as follows. (Here I only show the one-dimensional argument.) We have $u_k(x) = u_k(x + ma)$ and $\frac{\partial u_k}{\partial x_\mu}(x) = \frac{\partial u_k}{\partial x_\mu}(x + ma)$ for any translation $x \to x + ma$ by integer multiple of the lattice spacing a. Hence, assuming a finite lattice of $2N$ sites,

$$\langle u_{k'}, (\frac{p_\mu}{m} + \frac{\hbar k_\mu}{m} - \frac{1}{\hbar}\frac{\partial E_k}{\partial k_\mu})u_k\rangle = \frac{1}{V}\int_{-Na}^{Na} u^*_{k'}(x)(\frac{p_\mu}{m} + \frac{\hbar k_\mu}{m} - \frac{1}{\hbar}\frac{\partial E_k}{\partial k_\mu})u_k(x)\,dx\,, \tag{5.38}$$

where $V = 2Nv$ is the volume of the lattice. Using $u_k(x) = e^{-ikx}\psi_k(x)$ yields

$$(p_\nu + \hbar k_\nu)u_k(x) = e^{-ikx}p_\nu\psi_k(x)\,. \tag{5.39}$$

Thus,

$$\langle u_{k'}, (\frac{p_\mu}{m} + \frac{\hbar k_\mu}{m} - \frac{1}{\hbar}\frac{\partial E_k}{\partial k_\mu})u_k\rangle = \frac{1}{V}\int_{-Na}^{Na} e^{i(k-k')x}\psi^*_{k'}(x)\left(\frac{p_\mu}{m} - \frac{1}{\hbar}\frac{\partial E_k}{\partial k_\mu}\right)\psi_k(x)\,dx\,. \tag{5.40}$$

Finally, defining

$$j_\mu(k) \equiv \frac{p_\mu}{m} - \frac{1}{\hbar}\frac{\partial E_k}{\partial k_\mu}(k)\,, \tag{5.41}$$

$$\langle j_\mu\rangle_{k'k} \equiv \frac{1}{V}\int_{Solid} e^{i(k-k')x}\psi^*_{k'}(x)j_\mu(k)\psi_k(x)\,dx\,, \tag{5.42}$$

the expression for the curvature Ω in equation (5.30) takes the form

$$\Omega_{u_k} = -\mathrm{Im}\int d\mu_{k'}\,\frac{\overline{\langle j_\mu\rangle_{k'k}}\langle j_\nu\rangle_{k'k} - \overline{\langle j_\nu\rangle_{k'k}}\langle j_\mu\rangle_{k'k}}{\mid \omega_{kk'}\mid^2}dk_\mu \wedge dk_\nu\,. \tag{5.43}$$

The frequency and wave-vector electrical conductivity tensor $\sigma_{\mu\nu}$ describes the linear response of a solid to an applied electromagnetic field. The Kubo formula for the intraband conductivity is given in terms of a current-current correlation function. Mahan[59] writes

$$\sigma_{\mu\nu}(q,\omega) = \frac{1}{\omega V}\int_{-\infty}^{t} e^{i\omega(t-t')}\langle\psi \mid [j_\mu^\dagger(q,t), j_\nu(q,t')] \mid \psi\rangle + \frac{n_0 e^2}{m\omega} i\delta_{\mu\nu}\,. \tag{5.44}$$

The first term on the right hand side of (5.44) may be compared to the expression (5.43). A way to clarify the time-dependence is to use the identity

$$\frac{i}{\omega + i\alpha} = \int_0^\infty e^{i(\omega+i\alpha)t}\,dt \tag{5.45}$$

(ω, α real) in (5.43) with $\omega_{kk'} = \omega + i\alpha$. Then, write $e^{i\hbar\omega_{kk'}t}$ as a product of exponentials and move a factor like $e^{i(E_k - E_{k'})t}$into the matrix element so that

$$j_\mu \to j_\mu(t) = e^{-iH_{k'}t} j_\mu e^{iH_k t}\,. \tag{5.46}$$

The dielectric tensor and the conductivity tensor are closely related. The conductivity σ is the current response to an EM field and the dielectric tensor ϵ^{-1} is the density response to such a field. Hence, from the continuity equation, ϵ^{-1} corresponds to a time-derivative of σ, giving an additional factor of frequency in the denominator.

A plan for future work is to make a generalization of an earlier paper[85] that used the longitudinal ϵ^{-1} for the electron-phonon gas to derive the inelastic scattering term in the Boltzmann equation. Instead, the new work would use the response function tensor for the crystal and elucidate its geometric meaning in terms of the curvature of the connection on the spectral bundle for the lattice. With this approach, I hope to study the anomalous Hall effect which involves a transverse electrical response in the presence of perpendicular electric and magnetic fields. I will also want to solve the case where spin-orbit coupling is included. In that case, the Hamiltonian is

$$H_{so} = \frac{p^2}{2m} + \frac{e^2}{4m^2c^2} p\cdot\sigma\times\Delta V + V\,, \tag{5.47}$$

where σ denotes the Pauli matrices.By including spin in the calculation, I hope to deal with the spin Hall effect, which has generated much interest, both for theoretical and technological reasons.

Chapter 6

FUNDAMENTALS AND EXAMPLES

6.1 Vector Fields, Lie Groups, and Homogeneous Spaces

Let $\mathcal{M}$ be a differentiable (smooth) manifold and let $p \in \mathcal{M}$. Let $\mathcal{F}(p)$ be the algebra of differentiable real-valued functions defined in a neighborhood of p, and let $\mathcal{F}(\mathcal{U})$ be the algebra of such functions defined on $U \subseteq \mathcal{M}$. If $\alpha : \mathbb{R} \to \mathcal{M}$ is a differentiable curve such that $\alpha(t_0) = p$, then the *tangent vector to the curve* α at p is a mapping $\alpha'(t_0) : \mathcal{F}(p) \to \mathbb{R}$ defined by

$$\alpha'(t_0)f := \frac{df(\alpha(t))}{dt} \,|_{t_0} \ . \tag{6.1}$$

That is, $\alpha'(t_0)f$ is the derivative of f in the direction of the curve α at $t = t_0$. A *tangent vector at* p may be defined as an equivalence class of curves in $\mathcal{M}$: curves $\sigma_1, \sigma_2 : \mathbb{R} \to \mathcal{M}$ are equivalent if $\sigma_1(t_0) = \sigma_2(t_0) = p$, and, in some local coordinate system $(x^1, \cdots, x^m)$ around p, $\frac{d}{dt}(x^i \circ \sigma_1)|_{t_0} = \frac{d}{dt}(x^i \circ \sigma_2)|_{t_0}$ for $i = 1, \cdots, m$. The set of tangent vectors at a point $p \in \mathcal{M}$ is a vector space over $\mathbb{R}$ called the *tangent space* at p and denoted $T_p(\mathcal{M})$. The vector space dual to $T_p(\mathcal{M})$ is called the *cotangent space* at p and is denoted $T_p^*(\mathcal{M})$. The collection of all tangent vectors to $\mathcal{M}$ itself forms a differentiable manifold $T(\mathcal{M}) := \bigsqcup_{p\in\mathcal{M}} T_p(\mathcal{M})$; similarly $T^*(\mathcal{M}) := \bigsqcup_{p\in\mathcal{M}} T_p^*(\mathcal{M})$ forms a differentiable manifold.

Given a mapping f of a manifold $\mathcal{M}$ into another manifold $\mathcal{M}'$, the *push forward* or derivative of f at p is the linear mapping f_* of $T_p(\mathcal{M})$ into $T_{f(p)}(\mathcal{M}')$ defined as follows. For $v \in T_p(\mathcal{M})$, choose a curve $\alpha : \mathbb{R} \to \mathcal{M}$ such that v is the tangent vector to α at $p = \alpha(t_0)$. Then $f_*(v)$ is the tangent vector tangent to the curve $f \circ \alpha$ at $f(p) = f(\alpha(t_0))$.

A *vector field* X on an open set $U \subset \mathcal{M}$ is an assignment of a tangent vector $X_p \in T_p(\mathcal{M})$ to each point $p \in U$, that is, $X : U \to T(M)$, $p \mapsto X_p$. A *1-form* α on U is an assignment of a covector $\alpha_p \in T_p^*(\mathcal{M})$ to each point $p \in U$. The dual pairing of vector fields and 1-forms gives a real valued function on $\mathcal{M} : (\alpha(X))_p = \alpha_p(X_p) \in \mathbb{R}$. An important example of a 1-form is the differential df of $f \in \mathcal{F}(U)$, which may be defined by $df(X) = X(f)$, for $X \in \mathfrak{X}$.

The set $\mathfrak{X}(\mathcal{M})$ of differentiable vector fields on $\mathcal{M}$ forms a vector space over $\mathbb{R}$. A vector field $X \in \mathfrak{X}(\mathcal{M})$ operates on functions in $\mathcal{F}(\mathcal{M})$ as follows. To each $f \in \mathcal{F}(\mathcal{M})$, X assigns the real-valued function Xf on $\mathcal{M}$ defined by $Xf(p) = X_p f$ for all $p \in \mathcal{M}$. The map $X : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$ defined by $f \mapsto Xf$ has the properties of a *derivation*: for $f, g \in \mathcal{F}(\mathcal{M})$ and $a, b \in \mathbb{R}$,

1. $X(af + bg) = aXf + bXg$,
2. $X(fg) = X(f)g + fX(g)$.

Conversely, any derivation on $\mathcal{F}(\mathcal{M})$ arises from a smooth vector field. The view of vector fields as derivations yields the following bilinear operation on $\mathfrak{X}(\mathcal{M})$. Let $X, Y \in \mathfrak{X}(\mathcal{M})$. Define the *Lie bracket* $[X, Y]$ of X and Y by $[X, Y] : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$, $[X, Y]f = X(Yf) - Y(Xf)$. The bracket $[X, Y]$ is a derivation on $\mathcal{F}(\mathcal{M})$ that assigns to each $p \in \mathcal{M}$ the tangent vector $[X, Y]_p$ such that

$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)\,. \tag{6.2}$$

The bracket operation satisfies

$$[X, Y] = -[Y, X] \tag{6.3}$$

and the Jacobi identity,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\,, \tag{6.4}$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

A *Lie algebra* is a vector space with a bilinear operation $[\cdot, \cdot]$ satisfying (6.3) and (6.4). Thus, $\mathfrak{X}(\mathcal{M})$ is a Lie algebra (of infinite dimension).

A curve $\alpha : [a, b] \to \mathcal{M}$ for $a, b \in \mathbb{R}$ is called an *integral curve* of $X \in \mathfrak{X}(\mathcal{M})$ if, for every parameter value $t \in [a, b]$, $X_{\alpha(t)}$ is the tangent vector to the curve α at t, that is, $X_{\alpha(t)} = \alpha'(t)$.

A *one-parameter group of diffeomorphisms* of $\mathcal{M}$ is a mapping of $\mathbb{R} \times \mathcal{M}$ into $\mathcal{M}$, $(t, p) \mapsto \phi_t(p) \in \mathcal{M}$ such that for each $t \in \mathbb{R}$, $\phi_t : p \mapsto \phi_t(p)$ is a diffeomorphism of $\mathcal{M}$, and

$$\phi_{t+s}(p) = \phi_t(\phi_s(p)) \text{ } for \, all \, t, s \in \mathbb{R} \, and \, p \in \mathcal{M} \, . \tag{6.5}$$

Thus, a one-parameter group of diffeomorphisms may be regarded as a family of diffeomorphisms $\{\phi_t\}$ of $\mathcal{M}$ satisfying (6.5). Each one-parameter group $\{\phi_t\}$ induces a vector field X as follows. For every point $p \in \mathcal{M}$, define the *orbit of* p as the curve $x_p : \mathbb{R} \to \mathcal{M}$ given by $x_p(t) = \phi_t(p)$ with $x_p(0) = p$. Define X_p to be the vector tangent to x_p at $t = 0$, that is, $X_p = x_p'(0)$. Then $X : \mathcal{M} \to T(\mathcal{M})$, $p \mapsto X_p$ is a vector field on $\mathcal{M}$ and the orbit x_p is an integral curve of X starting at p.

A *local one-parameter group of local diffeomorphisms* (or, more briefly, a *local one-parameter group*) is defined similarly except that $\phi_t(p)$ is defined only for $t \in (-\epsilon, \epsilon)$ for some $\epsilon < 0$, and for p in an open set U of $\mathcal{M}$; the group homomorphism property (6.5) then becomes

$$if \, t, s, t+s \in (-\epsilon, \epsilon), and \, if \, p, \phi_s(p) \in U, \, then \, \phi_{t+s}(p) = \phi_t(\phi_s(p)) \, . \tag{6.6}$$

A local one-parameter group induces a vector field just as the global version does, except defined only on U. The converse holds true as well [56, p.13]:

Proposition 25 *Let $X \in \mathfrak{X}(\mathcal{M})$. For each point p of $\mathcal{M}$, there exists $\epsilon > 0$, a neighborhood U of p, and a local one-parameter group of local diffeomorphisms $\phi_t^X : U \to \mathcal{M}$, $t \in (-\epsilon, \epsilon)$, which induces the given X.*

In this case, we say that X *generates* the local one-parameter group $\{\phi_t^X\}$, and $\{\phi_t^X\}$ is called the *local flow* of X. The proof uses the fundamental theorem for systems of linear ordinary differential equations.

If there exists a global one-parameter group of diffeomorphisms of $\mathcal{M}$ which induces a vector field X, then X is called *complete*. It can be shown that every vector field on a compact manifold is complete.

Now consider how the local flow of a vector field on $\mathcal{M}$ is affected by a diffeomorphism of $\mathcal{M}$.

Proposition 26 *[56, p.14] Let $\psi : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism. If a vector field $X \in \mathfrak{X}(\mathcal{M})$ generates a local flow $t \mapsto \phi_t^X$, then the vector field $\psi_* X$ generates the local flow $t \mapsto \psi \circ \phi_t^X \circ \psi^{-1}$.*

Proof: Let $p \in \mathcal{M}$ and $q = \psi^{-1}(p)$. Proposition 1 guarantees that the vector field $\psi_* X$ generates a local flow. Since ϕ_t^X induces X, the tangent vector $X_q \in T_q(\mathcal{M})$ is tangent to the curve $x_q : \mathbb{R} \to \mathcal{M}$ defined by $x_q(t) = \phi_t^X(q)$ with $x_q(0) = q$. Hence, $(\psi_* X)_p = \psi_*(X_q) \in T_p(\mathcal{M})$ is tangent to the curve $\psi \circ \phi_t^X(q) = \psi \circ \phi_t^X \circ \psi^{-1}(p)$. $\square$

We say that a vector field X is *invariant under the diffeomorphism* ψ if $\psi_* X = X$, that is, if $\psi_*(X_m) = X_{\psi(m)}$ for all $m \in \mathcal{M}$. Proposition 2 provides the following simple criterion for invariance of vector fields:

Corollary 27 *The vector field X induced by the local one-parameter group $t \mapsto \phi_t$ is invariant under a diffeomorphism ψ if and only if $\psi \circ \phi_t = \phi_t \circ \psi$.*

A *Lie group* G is a group which is at the same time a differentiable manifold such that the group operations $G \times G \to G$, $(a, b) \mapsto ab$ and $G \to G$, $a \mapsto a^{-1}$ are differentiable mappings.

Example 1 *Let $M_n\mathbb{C}$ be the set of all $n \times n$ complex matrices, and associate the matrix $A = (a_{ij}) \in M_n\mathbb{C}$ to the point in $\mathbb{C}^{n^2}$ whose coordinates are $a_{11}, a_{12}, \cdots, a_{nn}$. Then $M_n\mathbb{C}$ is topologically equivalent to the Euclidean space $\mathbb{C}^{n^2}$.*

a) *Define the complex general linear group to be $GL_n(\mathbb{C}) = \{A \in M_n\mathbb{C} \mid \mathrm{d}etA \neq 0\}$ under the usual matrix multiplication. Since $GL_n(\mathbb{C})$ is an open subset of a Euclidean space, it is an n^2-dimensional manifold, and the group operations $(a, b) \mapsto ab$ and $a \mapsto a^{-1}$ are differentiable. Hence, $GL_n(\mathbb{C})$ is a Lie group.*

b) *The unitary group is $U(n) = \{A \in GL_n(\mathbb{C}) \mid A\overline{A}^t = I\}$. Since $U(n)$ is a closed subgroup of $GL_n(\mathbb{C})$, it is a submanifold of $GL_n(\mathbb{C})$, and hence a Lie subgroup of $GL_n(\mathbb{C})$.*

c) *Similarly, the special unitary group* $SU(n) = \{A \in U(n) \mid \mathrm{det} A = 1\}$ *is a Lie subgroup of* $U(n)$.

c) *Define* Diff$(\mathcal{M})$*, an infinite dimensional Lie group.*

If G is a Lie group, we denote by L_a the left translation of G by an element $a \in G$, that is, $L_a b = ab$ for every $b \in G$. For each $a \in G$, L_a is a diffeomorphism of G with inverse $L_{a^{-1}}$. A vector field X on G is called *left invariant* if it is invariant by all left translations L_a, $a \in G$, that is, if $(L_a)_*(X_b) = X_{ab}$ for all $a, b \in G$. We define the Lie algebra $\mathfrak{g}$ of G to be the set of all left invariant vector fields on G with the usual addition, scalar multiplication and the Lie bracket operation (6.2). Then, $\mathfrak{g}$ is a Lie subalgebra of the Lie algebra of differentiable vector fields $\mathfrak{X}(G)$. As a vector space, $\mathfrak{g}$ is isomorphic to the tangent space $T_e(G)$ at the identity, $e \in G$; the isomorphism is given by the mapping which sends $A \in \mathfrak{g}$ into $A_e \in T_e(G)$. The inverse of this mapping sends $v \in T_e(G)$ into $A \in \mathfrak{g}$ defined by $A_a = (L_a)_{*e}(v)$. Hence, the dimension of $\mathfrak{g}$ is equal to dim $T_e(G) =$ dim G.

A *one-parameter subgroup* of a Lie group G is a smooth homomorphism γ mapping the group $\mathbb{R}$ under addition into G. This means that $\gamma : \mathbb{R} \to G$ is a curve with the properties $\gamma(s+t) = \gamma(s)\gamma(t)$, $\gamma(0) = e$, and $\gamma(-t) = [\gamma(t)]^{-1}$.

A key result in Lie group theory is that every left invariant vector field on a Lie group G is complete, even if G is noncompact. That is, every $A \in \mathfrak{g}$ generates a global 1-parameter group of diffeomorphisms $\{\phi_t^A\}$ of G. This result may be proved as follows. By Proposition 1, the vector field $A \in \mathfrak{X}(G)$ generates a local one-parameter group $t \mapsto \phi_t^A$ in a neighborhood U of $e \in G$, $|t| < \epsilon$ for some $\epsilon > 0$. Since A is left-invariant, ϕ_t^A commutes with L_g for each $g \in G$ (by Corollary 3). Hence, $\phi_t^A(g)$ is defined for all $g \in G$ and $|t| < \epsilon$ by $\phi_t^A(g) = \phi_t^A(L_g(e)) = L_g(\phi_t^A(e))$. Since $\phi_t^A(g)$ is defined for $|t| < \epsilon$ for every $g \in G$, $\phi_t^A(g)$ for $|t| < \infty$ for every $g \in G$.

Thus there exists a unique integral curve of the vector field $A \in \mathfrak{g}$ starting at $e \in G$ defined by $a_e : \mathbb{R} \to G$, $t \mapsto a_e(t) = \phi_t^A(e)$ with $a_e'(0) = A_e$ and $a_e(0) = e$, defined for all $t \in \mathbb{R}$, with the property $a_e(s+t) = a_e(s)a_e(t)$ for all $s, t \in \mathbb{R}$. Hence, a_e is a one-parameter subgroup of G; we call a_e the one-parameter subgroup generated by A. Denote $a_e(1)$ by $\exp(A_e)$. It follows that $\exp tA_e = a_e(t)$ for all t. The map $\exp : T_e(G) \to G$ given by $A_e \mapsto \exp A_e := \exp tA_e \mid_{t=1}$ is called the *exponential map*. Conversely, all smooth one-parameter subgroups of G are of the form $\exp tA_e$ for some $A \in \mathfrak{g}$ (see [61] p. 273). In other words, there is a one-to-one correspondence between one-parameter subgroups of G and elements of $\mathfrak{g}$.

Example 2 *Let $\mathcal{M}$ be a differentiable manifold, let $X \in \mathfrak{X}(\mathcal{M})$ be complete, and let $\{\phi_t^X\}$ be the flow of X on $\mathcal{M}$. The map $t \mapsto \phi_t^X$ which defines a one-parameter group of diffeomorphisms of $\mathcal{M}$ in (6.5) is a smooth isomorphism from the additive group of $\mathbb{R}$ into the Lie group* Diff*($\mathcal{M}$). Thus, the collection $\{\phi_t^X\}_{t\in\mathbb{R}}$ is a one-parameter subgroup of* Diff*($\mathcal{M}$), and hence corresponds to an element of its Lie algebra. Thus, a complete vector field X on $\mathcal{M}$ may be viewed as an element of Lie(*Diff*($\mathcal{M}$)).*

The adjoint map $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is basic to our discussions; it is defined as follows. For $g \in G$, *conjugation* by g is the inner automorphism $a_g : G \to G$ defined by $a_g(h) = ghg^{-1}$. That is, $a_g = L_g \circ R_{g^{-1}}$. The map a_g is a diffeomorphism. Moreover, $a_g(e) = e$, so that (under the identification $\mathfrak{g} = G_e$) the derivative $(a_g)_{*e}$ maps $\mathfrak{g}$ isomorphically onto $\mathfrak{g}$; thus, $(a_g)_{*e}$ belongs to $Aut(\mathfrak{g})$. $(a_g)_{*e}$ is denoted Ad_g and the assignment $g \mapsto Ad_g$ is called the adjoint representation of G. If G is a matrix Lie Group, then $Ad_g(A) = gAg^{-1}$, for $g \in G$ and $A \in \mathfrak{g}$ Remark: Notice the equivalence of Ad-invariance and $(R_g)_*$-invariance: for every $g \in G$ and $A \in \mathfrak{g}$, we have $Ad_g(A) = (R_{g^{-1}})_*A$, because $Ad_g(A) = (a_g)_*A = (L_g)_*(R_{g^{-1}})_*A$ and $(R_{g^{-1}})_*A$ is left invariant.

Group Action Definition: Consider a Lie group G, a C^∞ manifold $\mathcal{M}$, and a C^∞ map $\sigma : \mathcal{M} \times G \to \mathcal{M}$ defined by $\sigma(x, g) = x \cdot g$. We say that G *acts on* $\mathcal{M}$ *on the right* via this map if

1. the map $R_g : \mathcal{M} \to \mathcal{M}$ defined by $R_g(x) = x \cdot g$ is a diffeomorphism for all $g \in G$,

2. $x \cdot (gh) = (x \cdot g) \cdot h$ for all $x \in \mathcal{M}$ and $g, h \in G$.

In this case, $\mathcal{M}$ is called a *right G-space*. We say that G *acts effectively* if e is the only element g with R_g the identity map of $\mathcal{M}$, and we say that G *acts freely* (or without fixed point) if the following stronger condition holds: if $x \cdot g = x$ for some $x \in \mathcal{M}$, then $g = e$. An action of G is called *transitive* if, for any $x, y \in \mathcal{M}$, there exists a $g \in G$ such that $g \cdot x = y$. If $x \in \mathcal{M}$, the set $G_x = \{g \in G \mid g \cdot x = x\}$ is called the *isotropy subgroup* at x. The *orbit* of a point $x \in \mathcal{M}$ is the set $G \cdot x = \{g \cdot x \mid g \in G\}$.

A *homogeneous space* is a manifold $\mathcal{M}$ with a transitive action of a Lie group G on $\mathcal{M}$. Equivalently, as the following proposition shows[7, p.66], it is a coset manifold of the form $G/K = \{gK \mid g \in G\}$, where K is a closed subgroup of G.

Proposition 28 *Let $G \times \mathcal{M} \to \mathcal{M}$ be a transitive right action of a Lie group G on a manifold $\mathcal{M}$, and let $K = G_m$ be the isotropy subgroup of a point $m \in \mathcal{M}$. Then:*

a) *The subgroup K is a closed subgroup of G.*

b) *The natural map $j : G/K \to \mathcal{M}$ defined by $j(gK) = g \cdot m$ is a diffeomorphism. That is, the orbit $G \cdot m$ is diffeomorphic to G/K.*

c) *(If G is finite-dimensional) The dimension of G/K is* $\dim G -$ $\dim K$.

Example 3 *($P_n(\mathbb{C})$ is a homogeneous space diffeomorphic to $SU(n+1)/U(n)$) Complex projective space $P_n(\mathbb{C})$ is the manifold of all complex lines in $\mathbb{C}^{n+1}$ that pass through $0 \in \mathbb{C}^{n+1}$. More precisely, $P_n(\mathbb{C})$ is the quotient space of $\mathbb{C}^{n+1} - \{0\}$ under the equivalence relation $(z_1, \cdots, z_{n+1}) \sim (cz_1, \cdots, cz_{n+1})$, $c \in \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of complex numbers. Show transitive action: The unitary group $U(n+1)$ acts on $\mathbb{C}^{n+1}$ (how?) and transforms complex subspaces into complex subspaces, in particular, lines into lines. Hence, $U(n+1)$ also acts on $P_n(\mathbb{C})$.*

Example 4 *(The n-dimensional torus is diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$)*

The action of a group G on a manifold $\mathcal{M}$ results in a homomorphism $\sharp : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$, as follows. Suppose G acts on $\mathcal{M}$ on the right and let $A \in \mathfrak{g}$. The 1-parameter subgroup of G generated by A acts on $\mathcal{M}$ and thus induces a vector field, $A^\sharp$, on $\mathcal{M}$ as follows. Let $x \in \mathcal{M}$. Since the curve $t \mapsto \exp tA_e$ is in G, the curve $c_x(t) := x \cdot (\exp tA_e) = R_{\exp tA_e}(x)$ is in $\mathcal{M}$. Define $A_x^\sharp = c_x'(0)$. Then $A^\sharp$ is a complete vector field on $\mathcal{M}$. Note that the map $\sharp : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ defined by $A \mapsto A^\sharp$ may also be described as follows. For $x \in \mathcal{M}$, let $\sigma_x : G \to \mathcal{M}$ be defined by $\sigma_x(g) = x \cdot g$. Then $A_x^\sharp = (\sigma_x)_*(A_e)$. The map given by $A \mapsto A^\sharp$ is a Lie algebra homomorphism [56, p.42]. If G acts effectively on $\mathcal{M}$, the map is actually an isomorphism onto its image. We will use this construction of vector fields on $\mathcal{M}$ by the map $\sharp : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ in the context of principal bundles.

6.2 Principal Bundles and Connections

Principal Fiber Bundle Definition: Let $\mathcal{B}$ be a manifold and G be a Lie group. A differentiable *principal fiber bundle* $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ consists of a manifold $\mathcal{P}$ and an action of G on $\mathcal{P}$ satisfying

1. G acts freely on $\mathcal{P}$ on the right: $\sigma : \mathcal{P} \times G \to \mathcal{P},\;\; \sigma(p, g) = p \cdot g$.
2. $\mathcal{B}$ is the quotient space of $\mathcal{P}$ by the equivalence relation induced by G, $\mathcal{B} = \mathcal{P}/G$, and the canonical projection $\pi : \mathcal{P} \to \mathcal{B}$ is differentiable.
3. (Local triviality) For each $x \in \mathcal{B}$ there exists an open neighborhood U and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times G$ such that $\Psi(p) = (\pi(p), \phi(p))$ where $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(p \cdot g) = \phi(p)g$ for all $p \in \pi^{-1}(U)$ and $g \in G$.

We call $\mathcal{P}$ the *total space*, $\mathcal{B}$ the *base space*, G the *structure group*, and π the *projection*. One sometimes also says that $\mathcal{P}$ is a principal fiber bundle over $\mathcal{B}$ with group G, or that $\pi : \mathcal{P} \to \mathcal{B}$ is a principal G-bundle. For each $x \in \mathcal{B}$, $\pi^{-1}(x)$ is a closed submanifold of $\mathcal{P}$, called the *fiber* over x. If p is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points $p \cdot g,\; g \in G$, and hence is also called the fiber through p. Every fiber is diffeomorphic to G.

Example 5 *(The trivial G-bundle over $\mathcal{B}$) Let P be the product manifold, $P = \mathcal{B} \times G$, let π be the projection onto the first factor $\pi : \mathcal{B} \times G \to \mathcal{B}$, and define the action to be*

$\sigma((x,h),g) = (x,h)\cdot g = (x,hg)$. *In this case, we may take the trivializing neighborhood* U *to be all of* $\mathcal{B}$ *and* Ψ *to be the identity map on* $\pi^{-1}(U) = \pi^{-1}(\mathcal{B}) = \mathcal{B}\times G$.

Example 6 *(the tangent bundle and the cotangent bundle)*

Example 7 *(Homogeneous spaces:* $H \hookrightarrow G \xrightarrow{\pi} G/H$*)*

Example 8 *(The complex Hopf bundle) Let* $G = U(1)$ *and let* $P = S^3$*, the unit* 3*-sphere in* $\mathbb{C}^2$ *defined by* $\{(z_1,z_2) : |z_1|^2 + |z_2|^2 = 1\}$. *Define the action* $\sigma : S^3 \times U(1) \to S^3$ *by* $\sigma(p,g) = (z^1,z^2)\cdot g = (z^1 g, z^2 g)$. *The orbit space is* $\mathcal{B} = S^2$ *and we let* $\pi : S^3 \to S^2$ *be the quotient map. (but see p. 30, NaberII on two inequivalent bundles defined in this way).*

Example 9 *(*$\mathbb{C}^* \hookrightarrow (\mathbb{C}^{n+1}-0) \xrightarrow{\pi} P_n(\mathbb{C})$ *)The multiplicative group* $\mathbb{C}^*$ *of non-zero complex numbers acts freely on* $\mathbb{C}^{n+1}-0$ *by* $\sigma : (\mathbb{C}^{n+1}-0)\times\mathbb{C}^* \to (\mathbb{C}^{n+1}-0)$, $(z,c)\mapsto zc = cz$. *Define* complex projective space, $P_n(\mathbb{C}) := (\mathbb{C}^{n+1}-0)/\mathbb{C}^*$, *that is,* $P_n(\mathbb{C})$ *is the manifold of all complex lines passing through the origin in* $\mathbb{C}^{n+1}$. *Let* $\pi : (\mathbb{C}^{n+1}-0) \to P_n(\mathbb{C})$ *be the projection which takes a point* $z \neq 0$ *in* $\mathbb{C}^{n+1}$ *to the complex line through* 0 *and* z. *Then* $\mathbb{C}^{n+1}-0$ *is a principal fiber bundle over* $P_n(\mathbb{C})$ *with group* $\mathbb{C}^*$. *To show local triviality, let* $z^0, z^1, \cdots, z^n$ *be the natural coordinate system in* $\mathbb{C}^{n+1}$. *For* $j \in \{0,1,\cdots,n\}$, *let* V_j^* *be the set of points of* $\mathbb{C}^{n+1}-0$ *where* $z^j \neq 0$ *and let* $V_j = \pi(V_j^*)$. *Then* $P_n(\mathbb{C}) \subset \cup_{j=0}^n V_j$ *and, for each* j*, the functions* $z^0/z^j, \cdots, z^{j-1}/z^j, z^{j+1}/z^j, \cdots, z^n/z^j$ *provide a coordinate system on* V_j. *Local triviality* $\Psi_j : \pi^{-1}(V_j) \xrightarrow{\simeq} V_j \times \mathbb{C}^*$ *is given by* $\Psi_j(z) = (\pi(z), z^j)$ *for* $z = (z^0,\cdots,z^n) \in \mathbb{C}^{n+1}-0$.

Example 10 *(*$U(1) \hookrightarrow S^{2n+1} \xrightarrow{\pi} P_n(\mathbb{C})$*) Let* S^{2n+1} *be the unit sphere (of real dimension* $2n+1$*) in* $\mathbb{C}^{n+1}$ *defined by* $|z^0|^2 + \cdots + |z^n|^2 = 1$, *and* $U(1)$ *is the multiplicative group of complex numbers of unit modulus. Let* $\pi : S^{2n+1} \to P_n(\mathbb{C})$ *be the bundle projection map. Then* S^{2n+1} *is a principal fiber bundle over* $P_n(\mathbb{C})$ *with group* $U(1)$. *Local triviality* $\psi_j :$ $\pi^{-1}(u_j) \cong u_j \times U(1)$ *is given by* $\psi(j) = (\pi(z), z/|z^j|) \in u_j \times U(1)$ *for* $z = (z^1,\cdots,z^n) \in S^{2n+1}$. *Note that when* $n = 1$, $P_1(\mathbb{C})$ *can be identified with* S^2, *and this case is the* $U(1)$ *bundle over* S^2 *described in Example 8.*

If p is a point of $\pi^{-1}(x)$ and $\imath : \pi^{-1}(x) \to \mathcal{P}$ is the inclusion map, then the image of the tangent space $\imath_*(T_p(\pi^{-1}(x)))$ is a subspace of $T_p(\mathcal{P})$ called the *vertical subspace at* p and denoted $V_p(\mathcal{P})$. Tangent vectors in $V_p(\mathcal{P})$ are tangent to the fiber through p and are called *vertical tangent vectors* at p. It follows that $v \in T_p(\mathcal{P})$ is vertical if and only if $\pi_*(v) = 0$.

If $\mathcal{P}$ is a principal fiber bundle with group G, then the action of G on $\mathcal{P}$ induces the homomorphism $\sharp$ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{X}(\mathcal{P})$ of vector fields on $\mathcal{P}$ (see Example 4). For each $A \in \mathfrak{g}$, $A^\sharp$ is called the *fundamental vector field corresponding to* A. Since the action of G sends each fiber into itself, $A^\sharp_p$ is tangent to the fiber at each $p \in \mathcal{P}$. Since the action of G on $\mathcal{P}$ is free, and the dimension of each fiber is equal to dim $\mathfrak{g}$, the mapping given by $A \mapsto A^\sharp_p$ is a linear isomorphism of $\mathfrak{g}$ into the tangent space at p of the fiber through p, that is, $\mathfrak{g}$ is isomorphic to $V_p(\mathcal{P})$, the set of vertical vectors at p.

The following result shows how the group action induced on $V_p(\mathcal{P})$ appears in terms of the adjoint map.[56, p.51]

Proposition 29 *Let $\mathcal{P}$ be a principal fiber bundle with group G and let $A^\sharp$ be the fundamental vector field corresponding to $A \in \mathfrak{g}$. For each $g \in G$, $(R_g)_* A^\sharp$ is the fundamental vector field corresponding to $Ad_{g^{-1}}(A) \in \mathfrak{g}$.*

Proof: Observe that $A^\sharp$ generates the 1-parameter group of diffeomorphisms $\{R_{\exp tA_e}\}$. Hence, by Proposition 2, the vector field $(R_g)_* A^\sharp$ generates the 1-parameter group of diffeomorphisms $R_g R_{\exp tA_e} R_{g^{-1}} = R_{g^{-1}(\exp tA_e)g}$. Moreover, $g^{-1}(\exp tA_e)g$ is the 1-parameter group generated by $Ad_{g^{-1}}(A) \in \mathfrak{g}$. $\square$

We shall need to compare points of neighboring fibers of the total space $\mathcal{P}$ and hence we seek vector fields with direction vectors that are *not* tangent to the fibers. Thus we need the elements of $T_p(\mathcal{P})$ that complement the vertical vectors in $V_p(\mathcal{P})$. This motivates the following definition of a connection on $\mathcal{P}$.

Principal Connection Definition: Let $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ be a principal fiber bundle over a manifold

$\mathcal{B}$ with group G. A *principal connection* Γ in $\mathcal{P}$ is an assignment of a subspace $H_p(\mathcal{P})$ of $T_p(\mathcal{P})$ to each $p \in \mathcal{P}$ such that

1. $T_p(\mathcal{P}) = V_p(\mathcal{P}) \oplus H_p(\mathcal{P})$ (direct sum)

2. $H_{pg}(\mathcal{P}) = (R_g)_* H_p(\mathcal{P})$ for each $p \in \mathcal{P}$ and $g \in G$, where $R_g : \mathcal{P} \to \mathcal{P}$ is defined by $R_g p = p \cdot g$.

3. H_p depends differentiably on p.

A vector $v \in T_p(\mathcal{P})$ is called *horizontal* if it lies in $H_p(\mathcal{P})$. Condition 1 means that any tangent vector $v \in T_p(\mathcal{P})$ can be decomposed uniquely into a sum of vertical and horizontal components lying in $V_p(\mathcal{P})$ and $H_p(\mathcal{P})$, respectively, and denoted by $vert(v)$ and $hor(v)$, respectively. Similarly, a vector field X on $\mathcal{P}$ can be decomposed uniquely into a sum of vector fields $vert(X)$ and $hor(X)$ on $\mathcal{P}$ with the property that for all $p \in \mathcal{P}$, $vert(X)_p \in V_p(\mathcal{P})$ and $hor(X)_p \in H_p(\mathcal{P})$. Condition 2 means that the assignment $p \mapsto H_p(\mathcal{P})$ is invariant by G. Equivalently, condition 2 means that, for every vector field X on $\mathcal{P}$ and for every $g \in G$, $hor(X)$ is invariant by $(R_g)_*$. Using left-invariance, condition 2 is equivalent to $hor(X) = (L_{g^{-1}})_*(R_g)_* hor(X) = Ad_{g^{-1}}(hor(X))$. In other words, $hor(X)$ is $Ad_{g^{-1}}$ invariant.

The device of a principal connection may also be viewed in terms of a 1-form on $\mathcal{P}$ with values in $\mathfrak{g}$, as follows.

Connection Form Definition: Let Γ be a connection in the principal fiber bundle $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ and let $H_p(\mathcal{P})$ denote the assigned horizontal subspace of $T_p(\mathcal{P})$, for each p. For each $v \in T_p(\mathcal{P})$, define $\omega_p(v)$ to be the unique $A \in \mathfrak{g}$ such that $A_p^\sharp = vert(v)$. Since the map $A \mapsto A_p^\sharp$ is an isomorphism from $\mathfrak{g}$ onto $V_p(\mathcal{P})$, $\omega_p(T_p(\mathcal{P}))$ is isomorphic to $V_p(\mathcal{P})$. Thus, it follows that $v \in H_p(\mathcal{P})$ if and only if $\omega_p(v) = 0$. The $\mathfrak{g}$-valued 1-form ω is called the *connection form* of Γ, and satisfies the following properties[56, p.64]:

1. $\omega_p(A_p^\sharp) = A$ for all $p \in \mathcal{P},\ A \in \mathfrak{g}$

2. $(R_g)^*\omega = Ad_{g^{-1}}\omega$

   $(that\, is,\ \omega((R_g)_*X) = Ad_{g^{-1}}\omega(X)$ for all $g \in G,\ X \in \mathfrak{X}(\mathcal{P}))$.

Property 1 follows directly from the definition of ω. To verify Property 2, observe that every $X \in \mathfrak{X}(\mathcal{P}))$ can be decomposed uniquely into a sum of $vert(X)$ and $hor(X)$. Let $g \in G$. Since $hor(X)$ is horizontal, $(R_g)_*(hor(X))$ is horizontal, by condition 2 of the connection definition. Hence, $\omega((R_g)_*(hor(X))) = 0$ and $Ad_{g^{-1}}\omega(hor(X)) = 0$. Since $vert(X)$ is vertical, we may assume that $vert(X)$ is a fundamental vector field $A^\sharp$ corresponding to $A \in \mathfrak{g}$. Then, by Proposition 4, $(R_g)_*(vert(X))$ is the fundamental vector field corresponding to $Ad_{g^{-1}}(A)$. Hence, $\omega((R_g)_*(vert(X))) = Ad_{g^{-1}}(A) = Ad_{g^{-1}}(\omega(vert(X)))$. Hence, Property 2 holds for all $X \in \mathfrak{X}(\mathcal{P}))$.

Since the connection 1-form ω is a $\mathfrak{g}$-valued one-form on $\mathcal{P}$, the exterior derivative $d\omega$ is a $\mathfrak{g}$-valued 2-form on $\mathcal{P}$, that is, $d\omega$ operates on pairs of tangent vectors to $\mathcal{P}$ and produces elements of $\mathfrak{g}$.

Curvature Form Definition: The *curvature* Ω of a connection form ω on $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ is its covariant exterior derivative, defined by having $d\omega$ operate only on horizontal parts, that is, for each $p \in \mathcal{P}$ and for all $v, w \in T_p(\mathcal{P})$, $\Omega_p(v,w) = (d\omega)_p(hor(v), hor(w))$. If $\mathcal{P}$ is a smooth principal bundle over $\mathcal{B}$ with group G, connection form ω, and curvature form Ω, then the Cartan Structure Equation holds[56]: $\Omega = d\omega + \frac{1}{2}\omega \wedge \omega$. Because ω is $\mathfrak{g}$-valued, the wedge product is the one determined by the Lie bracket pairing in $\mathfrak{g}$, that is, $(\omega \wedge \omega)_p(v,w) = [\omega_p(v), \omega_p(w)] - [\omega_p(w), \omega_p(v)] = 2[\omega_p(v), \omega_p(w)]$. For abelian groups, $\omega \wedge \omega = 0$.

Example 11 *Yang-Mills gauge theory of electrodynamics consists of a principal U(1)-bundle over the manifold of space-time. A connection on this bundle corresponds to the electromagnetic vector potential, and the curvature of this connection corresponds to the electromagnetic field strength 2-form.*

A homomorphism of a principal fiber bundle $G' \hookrightarrow \mathcal{P}' \xrightarrow{\pi'} \mathcal{B}'$ into another principal fiber bundle $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ consists of a map $h : \mathcal{P}' \to \mathcal{P}$ and a homomorphism $f : G' \to G$

such that $h(p'g') = h(p')f(g')$ for all $p' \in \mathcal{P}'$ and $g' \in G'$. (more: p.53 and/or p.79ff of KN I) An automorphism f of the bundle $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ is called an *automorphism of a connection* Γ in $\mathcal{P}$ if it maps Γ into Γ. In this case, we say that the connection Γ is *invariant by* f.

6.3 Horizontal Lifts and Holonomy

By definition, a connection on $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ provides a decomposition $T_p(\mathcal{P}) = V_p(\mathcal{P}) \oplus H_p(\mathcal{P})$ at each point $p \in \mathcal{P}$. From this decomposition, and the fact that $V_p(\mathcal{P})$ is the kernel of $\pi_* : T_p(\mathcal{P}) \to T_{\pi(p)}(\mathcal{B})$, it follows that π_* is an isomorphism of $H_p(\mathcal{P})$ onto $T_{\pi(p)}(\mathcal{B})$ for each $p \in \mathcal{P}$. Hence, to each vector field X on $\mathcal{B}$ there exists a unique vector field, $X^{\uparrow}$, on $\mathcal{P}$ such that for all $p \in \mathcal{P}$, $\pi_*(X_p^{\uparrow}) = X_{\pi(p)}$ and $vert(X_p^{\uparrow}) = 0$. This vector field $X^{\uparrow}$ is called the *horizontal lift of the vector field* X. A *horizontal lift of a smooth curve* $\alpha : [a, b] \to \mathcal{B}$ is a curve $\alpha^{\uparrow} : [a, b] \to \mathcal{P}$ such that $\pi(\alpha^{\uparrow}(t)) = \alpha(t)$ and $vert(\alpha^{\uparrow\prime}(t)) = 0$ for all $t \in [a, b]$. The theorem of this section asserts that a connection on $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ guarantees the existence of a unique horizontal lift for a given curve in $\mathcal{B}$. Horizontal lifting, in turn, well-defines the parallel translation map between fibers $\pi^{-1}(\alpha(0)) \to \pi^{-1}(\alpha(1))$, as well as the holonomy group of the connection.

For the theorem proof, we shall need the notion of a section of a principal bundle. A *local section* (or cross-section) of $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ defined on an open set $V \subseteq \mathcal{B}$ is a smooth map $s : V \to \pi^{-1}(V)$ that satisfies $\pi \circ s = id_V$, i.e., it is a smooth selection of an element from each fiber above V. A section s on V gives rise to a local trivialization (V, Ψ), where $\Psi : \pi^{-1}(V) \to V \times G$ is given by $\Psi(s(x) \cdot g) = (x, g)$. Conversely, a trivialization (V, Ψ) gives rise to a section called the *canonical section* $s : V \to \pi^{-1}(V)$ defined by $s(x) = \Psi^{-1}(x, e)$ and this correspondence between trivializations and sections is bijective. Now, let $\{U_i\}$ be an open covering of $\mathcal{M}$ and, for each i, let s_i be a local section defined on U_i. If ω is a connection form on $\mathcal{P}$, then the pull-back $s_i^*\omega$ is a $\mathfrak{g}$-valued one-form on U_i.

For the theorem proof, we shall also need the following result for differentials on product spaces[56, p.11].

Lemma 30 *(Leibniz's formula) Let ϕ be a mapping of the product manifold $M \times N$ into* *another manifold W. The differential ϕ_* at $(p,q) \in M \times N$ can be expressed as follows. If $v \in T_{(p,q)}(M \times N)$ corresponds to $(v_x, v_y) \in T_p(M) \times T_q(N)$, then*

$$\phi_*(v) = \phi_{1*}(v_x) + \phi_{2*}(v_y)\,,$$

where $\phi_1 : M \to W$ and $\phi_2 : N \to W$ are defined by

$$\phi_1(p') = \phi(p', q)\,\mathrm{for}\, p' \in M \,\mathrm{and}\, \phi_2(q') = \phi(p, q')\,\mathrm{for}\, q' \in N\,.$$

Now we are ready to state and prove the horizontal lift theorem for principal bundles:

Theorem 31 *Let $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ be a differentiable principal bundle and ω a connection form on $\mathcal{P}$. Let $\alpha : [0,1] \to \mathcal{B}$ be a piecewise C^1 curve in $\mathcal{B}$ and let $p_0 \in \pi^{-1}(\alpha(0))$. Then there exists a unique horizontal lift $\alpha^\uparrow$ in $\mathcal{P}$ such that $\alpha^\uparrow(0) = p_0$.*

Proof: We may suppose without loss of generality that α is contained in a trivializing neighborhood V and take a section s over V. Then $\gamma : [0,1] \to \mathcal{P}$ defined by $\gamma(t) = (s \circ \alpha)(t)$ is a curve in $\mathcal{P}$ such that $\pi \circ \gamma = \alpha$ and we may suppose that $\gamma(0) = p_0$. If it exists, the lift $\alpha^\uparrow$ of α must be of the form $\alpha^\uparrow(t) = \gamma(t) \cdot g(t)$ for some curve $g : [0,1] \to G$ with $g(0) = e$. Recalling that the action $\sigma : \mathcal{P} \times G \to \mathcal{P}$ is defined by $\sigma(p,g) = p \cdot g$, we have

$$\begin{aligned} \alpha^\uparrow(t) &= \sigma(\gamma(t), g(t)) = \sigma \circ (\gamma, g)(t)\,, \\ (\alpha^\uparrow)'(\tau) &= (\sigma \circ (\gamma, g))'(\tau) = \sigma_{*\,(\gamma(\tau), g(\tau))}(\gamma'(\tau), g'(\tau))\,. \end{aligned}$$

Since σ maps the product manifold $\mathcal{P} \times G$ into $\mathcal{P}$, we may apply the lemma (Leibniz's formula) with $\phi = \sigma$ by defining $\phi_1 := R_{g(\tau)} : \mathcal{P} \to \mathcal{P},\, p \mapsto p \cdot g(\tau)$, defining $\phi_2 := \sigma_{\gamma(\tau)} : G \to \mathcal{P},\, g \mapsto \gamma(\tau) \cdot g$, and $(v_x, v_y) := (\gamma'(\tau), g'(\tau))$. Thus,

$$(\alpha^\uparrow)'(\tau) = (R_{g(\tau)})_{*\,\gamma(\tau)}(\gamma'(\tau)) + (\sigma_{\gamma(\tau)})_{*\,g(\tau)}(g'(\tau))\,. \tag{6.7}$$

Using the chain rule and recalling that $\gamma(\tau) \cdot g(\tau) = \alpha^\uparrow(\tau)$, the second term on the right side is

$$\begin{aligned}(\sigma_{\gamma(\tau)})_{*\,g(\tau)}(g'(\tau)) &= (\gamma(\tau)\cdot g(t))'(\tau) \\ &= (\alpha^{\uparrow}(\tau)\,[g(\tau)^{-1}\,g(t)])'(\tau) \\ &= (\sigma_{\alpha^{\uparrow}(\tau)})_{*}\,(g(\tau)^{-1}g(t))'(\tau) \\ &= (\sigma_{\alpha^{\uparrow}(\tau)})_{*}(A_e) \\ &= A^{\sharp}_{\alpha^{\uparrow}(\tau)}\,,\end{aligned}$$

where we have defined

$$A_e := (g(\tau)^{-1}g(t))'(\tau) \in T_e(G)\,, \tag{6.8}$$

and, in the last step, we used $(\sigma_p)_* A_e = A^{\sharp}_p$ for any $p \in \mathcal{P}$ and $A \in \mathfrak{g}$. Next, apply the connection form ω to both sides of Equation(6.7), using the fact that $\alpha^{\uparrow}$ is horizontal on the left side and properties 1) and 2) of the connection form on the right side to find

$$\begin{aligned}\omega_{\alpha^{\uparrow}(\tau)}((\alpha^{\uparrow})'(\tau)) &= \omega_{\alpha^{\uparrow}(\tau)}((R_{g(\tau)})_{*\,\gamma(\tau)}(\gamma'(\tau)) + \omega_{\alpha^{\uparrow}(\tau)}(A^{\sharp}_{\alpha^{\uparrow}(\tau)}) \\ 0 &= Ad_{g(\tau)^{-1}}\omega_{\alpha^{\uparrow}(\tau)}(\gamma'(\tau)) + A\,.\end{aligned} \tag{6.9}$$

Evaluating the vector fields at $e \in G$ and assuming that G is a matrix group to rewrite Ad yields,

$$A_e = -g(\tau)^{-1}(\omega_{\alpha^{\uparrow}(\tau)}(\gamma'(\tau)))_e g(\tau)\,. \tag{6.10}$$

Observe that, by (6.8), $(L_{g(\tau)^{-1}})_*\,g'(\tau) = A_e$. Hence, with (6.10) we have

$$(L_{g(\tau)^{-1}})_*\,g'(\tau) = -g(\tau)^{-1}(\omega_{\alpha^{\uparrow}(\tau)}(\gamma'(\tau)))_e g(\tau)\,. \tag{6.11}$$

Since $L_{g(\tau)}$ is a linear operator, $(L_{g(\tau)})_* = L_{g(\tau)}$, so that it may be applied on the left to both sides of the equation. Since the resulting equation holds for arbitrary $\tau \in [0,1]$, we obtain an ordinary differential equation for g.

$$g'(t) = -(\omega_{\alpha^{\uparrow}(t)}(\gamma'(t)))_e g(t)\,. \tag{6.12}$$

Recalling that $\gamma := s \circ \alpha$, so that

$$\omega_{\alpha^{\uparrow}(t)}(\gamma'(t)) = \omega_{\alpha^{\uparrow}(t)}(s_*(\alpha'(t))) = (s^*\omega)_{\alpha(t)}(\alpha'(t))\,,$$

we have

$$g'(t) = -((s^*\omega)_{\alpha(t)}(\alpha'(t)))_e g(t) = -(\mathcal{A}_{\alpha(t)}(\alpha'(t)))_e\, g(t)\,, \tag{6.13}$$

where we have defined $\mathcal{A} := s^*\omega$. The fundamental theorem of ordinary differential equations guarantees the existence and uniqueness of the solution. (Show global: use compactness to get it over [0,1].) □

The existence of horizontal lifts described in the theorem provides a way to relate the fibers above any two points in $\mathcal{B}$ that can be joined by a smooth curve, as follows. Suppose that $x_0,\, x_1 \in \mathcal{B}$ and $\alpha : [0,1] \to \mathcal{B}$ is a smooth curve with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Then, by the theorem, for any $p_0 \in \pi^{-1}(x_0)$ there exists a unique curve $\alpha^{\uparrow}_{p_0} : [0,1] \to \mathcal{P}$ with initial point p_0 that lifts α and has horizontal tangent vector at each point. Since $\alpha^{\uparrow}_{p_0}(1) \in \pi^{-1}(x_1)$, this defines a map $\tau_\alpha : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$, called *parallel translation along* α determined by the connection form ω, given by $\tau_\alpha(p_0) = \alpha^{\uparrow}_{p_0}(1)$.

If α is a smooth loop in $\mathcal{B}$ with $\alpha(0) = \alpha(1) = x_0$, then $\tau_\alpha : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$. In this case, using the section s as in the proof, we have

$$\begin{aligned} \tau_\alpha(p_0) = \alpha^{\uparrow}_{p_0}(1) &= s(\alpha(1)) \cdot g(1) \\ &= s(\alpha(0)) \cdot g(1) \\ &= \alpha^{\uparrow}_{p_0}(0) \cdot g(1) = p_0 \cdot g(1)\,. \end{aligned} \tag{6.14}$$

Thus, the horizontal lift of a closed path α in $\mathcal{B}$ is not, in general, closed. Instead, $\alpha^{\uparrow}_{p_0}(1)$ differs from $\alpha^{\uparrow}_{p_0}(0)$ by the action of an element $g(1)$ of G, called the *holonomy of* α, *measured from* p_0. If G is abelian, one can solve Equation(6.13).[60, p.41] If α is a loop, one obtains

$$g(1) = \exp\left(-\int_0^1 ((s^*\omega)_{\alpha(t)}(\alpha'(t)))_e\, dt\right) = \exp\left(-\iint\, s^*\, d\omega\right)\,, \tag{6.15}$$

where the last equality is obtained by Stoke's Theorem (using $ds^*\omega = s^*d\omega$) and the double integral is taken over a two dimensional submanifold of $\mathcal{B}$ whose boundary is the loop α. Such a surface may not always exist, in which case, the last equality is dropped. Recalling that for abelian Lie groups, $d\omega = \Omega$, where Ω is the curvature of the connection, we have

Corollary 32 *If $\pi : \mathcal{P} \to \mathcal{B}$ is a principal G-bundle with G abelian, ω is a principal connection 1-form on $\mathcal{P}$, and s is a local section over a trivializing neighborhood $U \subset \mathcal{B}$, then the holonomy of a closed path α in U is given by the group element*

$$holonomy = \exp\left(-\oint_{\alpha} s^*\omega\right) = \exp\left(-\iint s^*\Omega\right), \tag{6.16}$$

where the double integral in the last expression is taken over any two-dimensional submanifold in $\mathcal{B}$ whose boundary is α.

Since G acts transitively on the fibers of $\mathcal{P}$, for each $p_0 \in \pi^{-1}(x_0)$ there exists a unique $g \in G$ such that $\tau_\alpha(p_0) = p_0 \cdot g$, if α is closed. Holding p_0 fixed, but letting α vary over all smooth loops at $x_0 = \pi(p_0)$ in $\mathcal{B}$ we obtain a subset $H(p_0)$ of G consisting of all those g such that p_0 is parallel transported to $p_0 \cdot g$ over some smooth loop at $\pi(p_0)$ in $\mathcal{B}$. $H(p_0)$ is a subgroup of G called the *holonomy group* of the connection form ω at p_0.

6.4 Associated Bundles and the Covariant Derivative

Consider the following way to construct a fiber bundle with base $\mathcal{B}$ that is associated in a precise way with a specified principal bundle $\lambda = G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ having the same base space.

Associated bundle Definition: Let F be a smooth manifold on which G acts smoothly on the left: $(g, f) \in G \times F \mapsto gf \in F$. G acts on the product manifold $\mathcal{P} \times F$ on the right as follows: $(p, f) \cdot g \mapsto (p \cdot g, g^{-1} \cdot f) \in \mathcal{P} \times F$. An equivalence relation $\sim$ is then defined as: $(p_1, f_1) \sim (p_2, f_2)$ if and only if there exists a $g \in G$ such that $(p_2, f_2) = (p_1, f_1) \cdot g$. The equivalence class containing (p, f) is $[p, f] = \{(p \cdot g, g^{-1} \cdot f) \mid g \in G\}$. The quotient space of $\mathcal{P} \times F$ by this group action is denoted by $E = \mathcal{P} \times_G F$. Now define $\pi_F : \mathcal{P} \times_G F \to \mathcal{B}$ by $\pi_F([p, f]) = \pi(p)$. (differentiable structure) Then $\lambda(F) = F \hookrightarrow \mathcal{P} \times_G F \xrightarrow{\pi_F} \mathcal{B}$ is a fiber bundle over $\mathcal{B}$ with fiber F that is called the *fiber bundle associated with* $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ by the given left action of G on F.

A vector bundle may be viewed as an example of an associated bundle, as follows. Let $F = V$ be a finite-dimensional vector space and $\rho : G \to Gl(V)$ a smooth representation of G on V. Then ρ gives rise to a smooth left action of G on V, that is, $g \cdot v = (\rho(g))(v)$.

The fiber bundle associated with the principal bundle $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ by this action is denoted $V \hookrightarrow \mathcal{P} \times_\rho V \xrightarrow{\pi_\rho} \mathcal{B}$ and is called *the vector bundle associated with* $G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ by the representation ρ. In this case, each fiber $\pi_\rho^{-1}(x) = \{[p, v] \mid v \in V\}$, $p \in \pi^{-1}(x)$, is a copy of V.

Thus, a complex line bundle is the associated vector bundle of a principle $U(1)$ bundle. Let $U(1) \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ be a principal $U(1)$-bundle and let $V = \mathbb{C}$. If $\rho : U(1) \to Gl(\mathbb{C})$ is a representation of $U(1)$ on $\mathbb{C}$, then the associated vector bundle $\mathbb{C} \hookrightarrow \mathcal{P} \times_\rho \mathbb{C} \xrightarrow{\pi_\rho} \mathcal{B}$ has fibers that are copies of $\mathbb{C}$. The usual choice for the representation ρ is defined by $(\rho(g))(z) = gz$. Then, for each $\theta \in [0, 2\pi)$, $g = e^{i\theta} \in U(1)$, and $\rho(g)$ is a rotation by θ.

The concepts of connections and parallel transport on principal bundles may be extended to associated fiber bundles. Let ω be a connection in the principal bundle $\lambda = G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ and let $\lambda(F) = F \hookrightarrow \mathcal{P} \times_G F \xrightarrow{\pi_F} \mathcal{B}$ be the bundle associated to λ via the left action of G on F. The *vertical subspace* of the tangent space $T_y(\mathcal{P} \times_G F)$, $y \in P \times_G F$, is defined as $V_y(\mathcal{P} \times_G F) := \{w \in T_y(\mathcal{P} \times_G F) \mid \pi_F(w) = 0\}$. Let $k_v : \mathcal{P} \to \mathcal{P} \times_G F$, $v \in F$, be defined by $k_v(p) := [p, v]$. Then the *horizontal subspace* of the tangent space $T_{[p,v]}(\mathcal{P} \times_G F)$ is defined as $H_{[p,v]}(\mathcal{P} \times_G F) := k_{v*}(H_p(\mathcal{P}))$. Let $\alpha : [a, b] \to \mathcal{B}$ and let $[p, v]$ be any point in $\pi_F^{-1}(\alpha(a))$. Let $\alpha^\uparrow$ be the unique horizontal lift of α to $\mathcal{P}$ such that $\alpha^\uparrow(a) = p$. Then the path $\alpha_F^\uparrow(t) := k_v(\alpha^\uparrow(t)) = [\alpha^\uparrow(t), v]$ is the horizontal lift of α to $\mathcal{P} \times_G F$ that passes through $[p, v]$ at $t = a$. Thus, the parallel translation map in the associated bundle is $\tau_F : \pi_F^{-1}(\alpha(a)) \to \pi_F^{-1}(\alpha(b))$, $y \mapsto \alpha_F^\uparrow(b)$, where $t \mapsto \alpha_F^\uparrow(t)$ is the horizontal lift of α to $\mathcal{P} \times_G F$ that passes through y.

In order to define a derivative of a cross-section $\psi : \mathcal{B} \to \mathcal{P} \times_\rho V$ of a vector bundle, one needs to compare values of ψ at neighboring points in two different fibers of $\mathcal{P} \times_\rho V$. When the bundle is equipped with a parallel translation τ_V the composition $\tau_V \circ \psi$ provides a way to make the necessary comparison. Equivalently, a connection on a principal bundle induces a unique covariant derivative in each associated bundle.

Covariant Derivative Definition: Let $\lambda = G \hookrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{B}$ be a principal G-bundle and let V be a vector space that carries a linear representation ρ of G. Let $\alpha : [0, \epsilon] \to \mathcal{B}$, $\epsilon > 0$, be a path in $\mathcal{B}$ with $\alpha(0) = x_0 \in \mathcal{B}$, and let $\psi : \mathcal{B} \mapsto \mathcal{P} \times_\rho V$ be a cross-section of the associated vector bundle, $E = \lambda(V)$. Then the *covariant derivative* of ψ in the direction α at x_0 is

$$\nabla_{\alpha'(0)}\psi := \lim_{t\to 0}\left(\frac{\tau_V^t \psi(\alpha(t)) - \psi(x_0)}{t}\right) \in \pi_\rho^{-1}(x_0) \tag{6.17}$$

where τ_V^t is the (linear) parallel transport map from the vector space $\pi_\rho^{-1}(\alpha(t))$ to the vector space $\pi_\rho^{-1}(x_0)$.

For $v \in T_x(\mathcal{M})$, the covariant derivative $\nabla_v\psi$ of ψ in the direction of v is defined by letting $\alpha : [-\epsilon, \epsilon] \to \mathcal{B}$, $\epsilon > 0$, be a path in $\mathcal{B}$ with $v = \alpha'(0)$. Then

$$\nabla_v\psi = \nabla_{\alpha'(0)}\psi\,. \tag{6.18}$$

A section ψ of E defined on an open subset U of $\mathcal{B}$ is parallel if and only if $\nabla_v\psi = 0$ for all $v \in T_x(U),\ x \in U.$

If X is a vector field on $\mathcal{B}$, then the covariant derivative $\nabla_X\psi$ of ψ in the direction of X is defined by

$$(\nabla_X\psi)(x) = \nabla_{X_x}\psi\,.$$

Thus, a covariant derivative is a map that associates to each vector field X on $\mathcal{B}$ a linear map $\nabla_X : \Gamma(E) \to \Gamma(E)$. In a local trivialization, the section ψ is represented by a function $\psi_i : U_i \to V$ and $(\nabla_X(\psi))_i = d\psi_x(X_x) + A_i(x)\psi(x)$, where $A_i : U_i \to \mathrm{End}(V)$.

6.5 Metrics, Symplectic Manifolds, and Hamiltonian Vector Fields

A classical Hamiltonian mechanical system is described in terms of a differentiable manifold $\mathcal{M}$ called the phase space, or the space of states. This manifold is equipped with a symplectic form, Ω, which plays a special role in determining the time-evolution of the states of the system. This section defines symplectic manifolds and explains how Hamiltonian dynamics is determined by a special class of vector fields that are related by Ω to functions $f : \mathcal{M} \to \mathbb{R}$ called observables.

Recall that an inner product on a real vector space is a symmetric and positive definite bilinear form. A *Riemannian metric* on a differentiable manifold $\mathcal{M}$ is an assignment to each $p \in \mathcal{M}$ an inner product $g_p =< \cdot, \cdot >_p$ on the tangent space $T_p(\mathcal{M})$, which depends smoothly on the base point p. More precisely, the smoothness condition means that, for every pair $X, Y \in \mathfrak{X}(\mathcal{M})$ in a neighborhood of $p \in \mathcal{M}$, the map $p \mapsto < X_p, Y_p >_p$ is smooth. A *Riemannian manifold* is a differentiable manifold, equipped with a Riemannian metric. An *isometry* is a diffeomorphism $h : \mathcal{M} \to \mathcal{N}$ between Riemannian manifolds that preserves the Riemannian metric. That is, for $p \in \mathcal{M}$ and $u, v \in T_p(\mathcal{M})$,

$$< u, v >_{\mathcal{M}} =< h_*(u), h_*(v) >_{\mathcal{N}} , \tag{6.19}$$

where $< \cdot, \cdot >_{\mathcal{M}}$ and $< \cdot, \cdot >_{\mathcal{N}}$ denote the inner products on $T_p(\mathcal{M})$ and $T_{h(p)}(\mathcal{N})$ respectively. A Riemannian manifold is called *symmetric* if for each $p \in \mathcal{M}$ there exists an isometry $\sigma_p : \mathcal{M} \to \mathcal{M}$ such that $\sigma_p(p) = p$ and $(\sigma_p)_*(v) = -v$. An isometry with these properties is also called an *involution*.

Let V be a real Banach space, possibly infinite-dimensional. A continuous bilinear form $B : V \times V \to \mathbb{R}$ is called *nondegenerate* (or *weakly nondegenerate*) if it has the following property: if $B(u, v) = 0$ for all $v \in V$, then $u = 0$. Define the associated linear map $B^\flat : V \to V^*$ by $B^\flat(u)(v) := B(u, v)$. Nondegeneracy of B is equivalent to injectivity of $B^\flat$, that is, to the property: if $B^\flat(u) = 0$, then $u = 0$. The form B is called *strongly nondegenerate* if $B^\flat$ is an isomorphism. If V is finite-dimensional, then weak degeneracy and strong degeneracy are equivalent, since then $B^\flat$ is injective if and only if it is onto.

A *symplectic linear structure* on V is a skew-symmetric and nondegenerate bilinear form B on V. The pair (V, B) is called a *symplectic vector space*. If B is strongly nondegenerate, (V, B) is called a *strong symplectic vector space.*

Example 12 *(Standard symplectic linear structure on $\mathbb{R}^{2n}$) By an appropriate choice of basis $\{e_1, e_2, \cdots, e_{2n}\}$ with $\{e_1^*, e_2^*, \cdots, e_{2n}^*\}$ as its dual, the standard symplectic form ω_0 appears as*

$$\omega_0 = \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^* , \tag{6.20}$$

$$\omega_0(v, v') = \sum_{i=1}^{n} (x_i y_i' - x_i' y_i) \tag{6.21}$$

when $v \in V$ has components $v = \sum_{i=1}^{n}(x_i e_i + y_i e_{i+n})$. Defining the matrix

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{6.22}$$

and the column vectors $v,\ v'$ with $v^T = (x_1, \cdots, x_n, y_1, \cdots, y_n)$, the standard (or canonical) symplectic form (6.21) is

$$\omega_0(v, v') = v^T J_0 v', . \tag{6.23}$$

Example 13 *(Inner product on $\mathcal{C}^N$) Let $z = (z_1, \cdots, z_N)$ and $w = (w_1, \cdots, w_N) \in V = \mathcal{C}^N$. Define the inner product of z and w in the standard way:*

$$\langle z, w\rangle = \sum_{j=1}^{N} z_j \bar{w}_j = \sum_{j=1}^{N}(x_j u_j + y_j v_j) + i\sum_{j=1}^{N}(u_j y_j - v_j x_j)\,, \tag{6.24}$$

where $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$. Re $\langle z, w\rangle$ is a euclidean inner product and -Im $\langle z, w\rangle$ is a symplectic linear structure when $\mathcal{C}^N$ is identified with $\mathbb{R}^N \times \mathbb{R}^N$ by mapping $(x_1 + iy_1, \cdots, x_N + iy_N) \in \mathbb{C}^{\mathbb{N}}$ into $(x_1, \cdots, x_N, y_1, \cdots, y_N) \in \mathbb{R}^N \times \mathbb{R}^N$.

A linear endomorphism J of a real vector space V is called a *complex structure* if $J^2 = -\mathrm{id}_V$. If (V, B) is symplectic and J is a complex structure on V, then we say that J is *compatible* with B if

$$B(Jv, Jw) = B(v, w)\, for\, all\, u, v \in V\,. \tag{6.25}$$

If J is a complex structure compatible with the symplectic form B, then we may define

$$g(v, w) := B(v, Jw)\, for\, v, w \in V\,. \tag{6.26}$$

Compatibility of J and B implies that

$$g(Jv, w) = B(v, w)\,. \tag{6.27}$$

From $J^2 = -1$ and the skew-symmetry of B, we also have

$$g(v, w) = g(w, v)\,, \tag{6.28}$$

and

$$g(Jv, Jw) = g(v, w)\,. \tag{6.29}$$

That is, g is a symmetric, bilinear form compatible with J. As B, g is also nondegenerate. When $g(v, v) \geq 0$ for all $v \in V$, g is a Riemannian metric and we call the triple (V, B, J) a *Kähler vector space*.

Example 14 *The matrix J_0 defined in (6.22) has the property: $J_0^2 = -id_{\mathbb{R}^{2n}}$, so J_0 is a complex structure on $\mathbb{R}^{2n}$. t is straightforward to verify that J_0 is compatible with the standard symplectic form ω_0, defined in (6.21). Then g_0 defined by $g_0(v, v') = \omega_0(v, J_0 v') = -v^\dagger v$ is the corresponding Riemannian metric.*

Example 15 *If $\mathcal{H}$ is a complex Hilbert space, then $J : \mathcal{H} \to \mathcal{H}$ defined by $J(v) = iv$ (where $i = \sqrt{-1}$) is a complex structure on $\mathcal{H}$. Moreover, this complex structure is compatible with the symplectic linear structure defined in Example 13 so that, in particular, Re $\langle iz, w\rangle = -$Im $\langle z, w\rangle$. That is, $\mathbb{C}^N$ with the standard inner product and this J is a Kähler vector space. Similarly, any complex Hilbert space can be given a Kahler structure.*

Let $\mathcal{M}$ be a real smooth manifold, and denote by $\Lambda^j(\mathcal{M})$ the set of j-forms on $\mathcal{M}$ ($\alpha \in \Lambda^k(\mathcal{M})$ if and only if, for each $x \in \mathcal{M}$, $\alpha_x : T_x(\mathcal{M}) \times \cdots \times T_x(\mathcal{M}) \to \mathbb{R}$, such that $\alpha_x(X_1, \cdots, X_k)$ is antisymmetric in $X_1, \cdots, X_k \in T_x(\mathcal{M})$). Let $\Omega \in \Lambda^2(\mathcal{M})$. Then, for each $m \in \mathcal{M}$, Ω_m maps $T_m(\mathcal{M}) \times T_m(\mathcal{M})$ into $\mathbb{R}$. We say that Ω is *nondegenerate* if, for each $m \in \mathcal{M}$, Ω_m is nondegenerate on $T_m(\mathcal{M})$. Thus, $\Omega \in \Lambda^2(\mathcal{M})$ is nondegenerate if, for each $m \in \mathcal{M}$, Ω_m is a symplectic linear structure on $T_m(\mathcal{M})$. If $d\Omega = 0$, we say that Ω is *closed*. A *symplectic form* Ω on $\mathcal{M}$ is a closed (weakly) nondegenerate 2-form on $\mathcal{M}$. The pair $(\mathcal{M}, \Omega)$ is called a *symplectic manifold*. If Ω is strongly nondegenerate, then $(\mathcal{M}, \Omega)$ is called a *strong symplectic manifold*. If $\mathcal{M}$ is a finite-dimensional symplectic manifold, then dim $\mathcal{M}$ is even. Infinite dimensional symplectic manifolds are discussed in [24, 61, 55].

Example 16 *Every cotangent bundle is a symplectic manifold. (Each has an invariant 1-form.)*

Example 17 *The electromagnetic field strength tensor is a symplectic form on space-time. (Note that it is closed, since $\mathcal{F}_{\mu\nu} = d\mathcal{A}$, where $\mathcal{A}$ is the electromagnetic vector potential.)*

As a manifold, a vector space, V, has a trivial tangent bundle. Thus, as in the following example, one can identify the tangent space at any point of V with V itself. This perspective enables us to define a symplectic form on a complex Hilbert space $\mathcal{H}$ by giving $\mathcal{H}$ a symplectic linear structure.

Example 18 *(Symplectic form on a complex Hilbert space) Let V be a real or complex vector space. For each $p \in V$, the tangent space $T_p(V)$ can be identified with V, as follows. For each $v \in V$, define $v_p \in T_p(V)$ by $v_p := \alpha'(0)$, where $\alpha : \mathbb{R} \to V$ is given by $\alpha(t) = p + tv$. Then $v \mapsto v_p$ is the* canonical isomorphism *of V onto $T_p(V)$. Equivalently, by the definition (6.1),*

$$v_p f(p) := \frac{d}{dt} f(p + tv) \mid_{t=0} \tag{6.30}$$

for any $f \in C^\infty(V)$. Now, if V is a complex Hilbert space $\mathcal{H}$, we may define the symplectic 2-form $\Omega_{\mathcal{H}}$ on $\mathcal{H}$ by using the symplectic structure in Example 6:

$$(\Omega_{\mathcal{H}})_p(v_p, w_p) := 2 \, \mathrm{Im}\langle v, w\rangle \, , \tag{6.31}$$

for each $p \in \mathcal{H}$ and $v_p, w_p \in T_p(\mathcal{H})$, with $\langle \, , \, \rangle$ denoting the Hermitian inner product on $\mathcal{H}$. The form (6.31) is used in Chapter 2 to show the symplectic character of quantum dynamics.

An *almost complex structure* on a real differentiable manifold $\mathcal{M}$ is a tensor field J which is, for each $m \in \mathcal{M}$, an endomorphism of the tangent space $T_m(\mathcal{M})$ such that $J^2 = -1$, where 1 denotes the identity transformation of $T_m(\mathcal{M})$. A manifold with a fixed almost complex structure is called an *almost complex manifold*.

Let $\mathcal{M}$ be an almost complex manifold with almost complex structure J and suppose that $\mathcal{M}$ is equipped with a symplectic form Ω. $(\mathcal{M}, \Omega, J)$ is called a *Kahler manifold* if, for every point $m \in \mathcal{M}$, the triple $(T_m(\mathcal{M}), \Omega_m, J_m)$ is a Kahler vector space.

Example 19 *(Projective Hilbert space as a Kahler manifold) Let $\mathcal{H}$ be a complex Hilbert space. We have remarked that the Hermitian inner product on $\mathcal{H}$ gives $\mathcal{H}$ a euclidean structure and a symplectic structure (as in Example 15) that are related by the complex structure. Thus, by the canonical isomorphism of $\mathcal{H}$ onto $T_p(\mathcal{H})$ for each $p \in \mathcal{H}$ (Example 18), $\mathcal{H}$ is itself a (strong) Kahler manifold. Suppose that $\mathcal{H}$ is isomorphic to $\mathbb{C}^{n+1}$.* Projective Hilbert space*, $P(\mathcal{H})$, is isomorphic to $P_n(\mathbb{C})$ as defined in Example 9, with the added ingredient that $P(\mathcal{H})$ arises from $\mathcal{H}$, which has a Hermitian inner product. It is possible to generalize to infinite dimensional Hilbert spaces[24, 61]. Now observe that $P(\mathcal{H})$ is endowed with a Hermitian structure induced by the one on $\mathcal{H}$, making $P(\mathcal{H})$ a strong Kahler manifold as well.[61] Indeed, let $\pi : (\mathcal{H} - \{0\}) \to P(\mathcal{H})$ denote the projection map that sends a vector $\psi \in (\mathcal{H} - \{0\})$ to the complex line $[\psi]$. The tangent space $T_{[\psi]}(P(\mathcal{H}))$ is isomorphic to $(\mathbb{C}\psi)^{\perp} = \{\phi \in \mathcal{H} | \langle \phi, \psi \rangle = 0\}$ and $(\pi_*)_{\psi}|_{(\mathbb{C}\psi)^{\perp}}$ is a complex linear isomorphism onto $T_{[\psi]}(P(\mathcal{H}))$. Note that $(\pi_*)_{\psi}|_{(\mathbb{C}\psi)^{\perp}}$ depends on the chosen representative ψ in $[\psi]$ since $\lambda(\pi_*)_{\lambda\psi}|_{(\mathbb{C}\psi)^{\perp}} = (\pi_*)_{\psi}|_{(\mathbb{C}\psi)^{\perp}}$. However, if we restrict to vectors of unit length, then we may define an inner product on $P(\mathcal{H})$ that does not depend on the choice of representative. That is, if $[\psi] \in P(\mathcal{H})$, $|\psi| = 1$, and $\phi_1, \phi_2 \in (\mathbb{C}\psi)^{\perp}$, then the formula*

$$\langle \pi_*(\phi_1), \pi_*(\phi_2) \rangle_{P(\mathcal{H})} := 2\hbar \langle \phi_1, \phi_2 \rangle_{\mathcal{H}} \tag{6.32}$$

gives a well-defined Hermitian inner product on $T_{[\psi]}(P(\mathcal{H}))$. Moreover,

$$\Omega_{[\psi]}(\pi_*(\phi_1), \pi_*(\phi_2)) := -2\hbar \mathrm{Im} \langle \phi_1, \phi_2 \rangle \tag{6.33}$$

defines a strong symplectic form on $P(\mathcal{H})$, and

$$g_{[\psi]}(\pi_*(\phi_1), \pi_*(\phi_2)) := -2\hbar \mathrm{Re} \langle \phi_1, \phi_2 \rangle \tag{6.34}$$

defines a strong Riemannian metric on $P(\mathcal{H})$ called the Fubini-Study metric*. Both Ω and g are invariant under all transformations $[U]$, for all unitary operators U on $\mathcal{H}$.*

Let $(\mathcal{M}, \Omega)$ and $(\mathcal{M}', \Omega')$ be symplectic manifolds. A C^{∞} mapping $\phi : \mathcal{M} \to \mathcal{M}'$ is called *symplectic* or *canonical* or a *symplectomorphism* if $\phi^*\Omega' = \Omega$, that is, if $\Omega_z(v, w) = \Omega'_{\phi(z)}((\phi_*)_z(v), (\phi_*)_z(w))$ for all $z \in \mathcal{M}$ and $v, w \in T_z(\mathcal{M})$. In particular, if $\phi : \mathcal{M} \to \mathcal{M}$

is an automorphism with this property, we say that Ω is *invariant by* ϕ. This property yields Liouiville's theorem, since Ω^N is a volume form.

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