Reachability in K-Colored Tournaments

by

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ABSTRACT

Let T be a tournament with edges colored with any number of colors. A rainbow triangle is a 3-colored 3-cycle. A monochromatic sink of T is a vertex which can be reached along a monochromatic path by every other vertex of T. In 1982, Sands, Sauer, and Woodrow asked if T has no rainbow triangles, then does T have a monochromatic sink? I answer yes in the following five scenarios: when all 4-cycles are monochromatic, all 4-semi-cycles are near-monochromatic, all 5-semi-cycles are near-monochromatic, all back-paths of an ordering of the vertices are vertex disjoint, and for any vertex in an ordering of the vertices, its back edges are all colored the same. I provide conjectures related to these results that ask if the result is also true for larger cycles and semi-cycles.

A ruling class is a set of vertices in T so that every other vertex of T can reach a vertex of the ruling class along a monochromatic path. Every tournament contains a ruling class, although the ruling class may have a trivial size of the order of T. Sands, Sauer, and Woodrow asked (again in 1982) about the minimum size of ruling classes in T. In particular, in a 3-colored tournament, must there be a ruling class of size 3? I answer yes when it is required that all 2-colored cycles have an edge xy so that y has a monochromatic path to x. I conjecture that there is a ruling class of size 3 if there are no rainbow triangles in T.

Finally, I present the new topic of α-step-chromatic sinks along with related results. I show that for certain values of α, a tournament is not guaranteed to have an α-step-chromatic sink. In fact, similar to the previous results in this thesis, α-step-chromatic sinks can only be demonstrated when additional restrictions are put on the coloring of the tournament's edges, such as excluding rainbow triangles. However, when proving the existence of α-step-chromatic sinks, it is only necessary to exclude special types of rainbow triangles.

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INTRODUCTION

Let $D = (V, E)$ be a **directed graph** (also called a **digraph**) with vertex set $V(D)$ and edge set $E(D)$. The **order** of D is the cardinality of $V(D)$, which is represented by $|V(D)|$, $|D|$, and n in this thesis. The cardinality of $E(D)$ is represented by $|E(D)|$. An edge $uv \in E(D)$ is considered to be directed from u to v and we write $u \to v$. A variation of a digraph is the **oriented graph**, where for any two vertices u and v of D, it is not the case that both $uv \in E(D)$ and $vu \in E(D)$. We define digraphs to not contain **loops**, where a loop is an edge of the form $vv \in E(D)$, for some vertex $v \in V(D)$.

Definition 1.1. *A complete oriented graph is called a* ***tournament****.*

Note that since a tournament T is a complete oriented graph, for any pair of distinct vertices u and v in T, there is exactly one edge between u and v. In the following sections, we introduce definitions and notation from three different sources: Bang-Jensen and Gutin in [2], Diestel in [4], and West in [15].

1.1 Paths and Cycles

We begin with some common definitions about digraphs.

Definition 1.2. *A* ***path*** $P = (V, E)$ *is a digraph with at least one vertex with vertex set* $V = \{v_0, v_1, \ldots, v_{t-1}\}$ *and edge set* $E(P) = \{v_0v_1, v_1v_2, \ldots, v_{t-2}v_{t-1}\}$*, where all* v_i *are distinct.*

If P is a path on t vertices as described in the definition above, we may write $v_0 \to v_1 \to \cdots \to v_{t-1}$ to represent the path or $v_0v_1 \ldots v_{t-1}$. A path on t vertices has a **length** of $t-1$ (the number of edges on the path). We have the following definition for what it means for a path to be **Hamiltonian**.

Definition 1.3. *A path P in a digraph D is **Hamiltonian** if $V(P) = V(D)$.*

It is an easy exercise to show the following. We will give one possible proof.

Theorem 1.4. *Every tournament has a Hamiltonian path.*

Proof. Let T be a tournament on n vertices. We argue by induction on n that T has a Hamiltonian path. For the base case of $n = 1$, this is clearly true. So we assume $n > 1$ and the theorem to be true for all tournaments of order less than n. Let $v \in V(T)$. By the induction hypothesis, the tournament $T - \{v\}$ has a Hamiltonian path $P = v_0v_1 \dots v_{n-2}$. If $v_{n-2} \to v$, then Pv is a Hamiltonian path. If $v \to v_0$, then vP is a Hamiltonian path. So we assume otherwise. Then there must exist some $i \in \{0, 1, \dots, n-1\}$ so that $v_i \to v$ and $v \to v_{i+1}$. Replacing the edge v_iv_{i+1} with the path v_ivv_{i+1} along P yields a Hamiltonian path in T. $\square$

Definition 1.5. *If $P = v_0v_1 \dots v_{s-1}$ is a path on $s \geq 3$ vertices, then the graph $C := P + v_{s-1}v_0$ is called a **cycle**.*

As with paths, we have two methods to write a cycle. If C is a cycle as described in the definition above, then we may write $v_0 \to v_1 \to \cdots v_{s-1} \to v_0$ or $v_0v_1 \dots v_{s-1}v_0$. We also may refer to a cycle on s vertices as an **s-cycle**. A digraph without any cycles is called **acyclic**. A 3-cycle is commonly referred to as a **triangle**.

Definition 1.6. *Let $C = v_0v_1 \dots v_{s-1}v_0$ be a cycle on s vertices. An edge of the type v_iv_{i+2}, for some $i \in \{0, 1, \dots s-1\}$, is called a **square** edge.*

We now define a Hamiltonian cycle similarly to how we defined a Hamiltonian path.

Definition 1.7. *A cycle C in a digraph D is **Hamiltonian** if $V(C) = V(D)$.*

We have the following definition for a strong digraph.

Definition 1.8. *A digraph is **strong** if for each ordered pair u, v of vertices, there is a path from u to v.*

The following is a well-known theorem attributed to Moon in [12].

Theorem 1.9 (Moon, 1966, [12])**.** *A tournament is strong if and only if there exists a Hamiltonian cycle.*

Definition 1.10. *A digraph D is **transitive** if $uv \in E(D)$ and $vz \in E(D)$ imply that $uz \in E(D)$.*

Note that a tournament is acyclic if and only if it is transitive. We provide a short proof.

Theorem 1.11. *A tournament is acyclic if and only if it is transitive.*

Proof. $(\Rightarrow)$ Assume that T is an acyclic tournament. Consider two edges uv and vz of T. If $z \to u$, then $uvzu$ is a cycle, which would contradict that T is acyclic. Hence $uz \in E(T)$ and so T is transitive.

$(\Leftarrow)$ Assume that T is a transitive tournament. For contradiction, assume T has a cycle and let $C = v_0 v_1 \ldots v_{s-1} v_0$ be a minimum cycle of T. A cycle in a tournament has at least 3 vertices. Suppose that $|C| > 3$. Then either $v_1 \to v_{s-1}$ or $v_{s-1} \to v_1$. In the former case, $v_0 v_1 v_{s-1} v_0$ is a smaller cycle than C, contradicting that C is minimum. In the latter case, $v_{s-1} v_1 C v_{s-1}$ is a smaller cycle than C, again contradicting that C is miniumum. Thus C is a 3-cycle and the edges of C contradict that T is transitive. Thus T is acyclic. $\square$

1.2 Colorings

Definition 1.12. *A function $f : E(D) \to \{0,1,\ldots,k-1\}$ is a **k-coloring** (or **coloring**) of the edges of a digraph D. A digraph D is k-colored if the edges of D have been colored by some fixed coloring f.*

A coloring is not required to be proper. Thus, there are $k^{|E(D)|}$ distinct k-colorings of a digraph D. If the edge directed from a vertex u to a vertex v is colored with c, we write $u \xrightarrow{c} v$.

Definition 1.13. *A digraph D is **monochromatic** if every edge of D is colored with the same color.*

If there exists a monochromatic path from u to v, we write $u \mapsto v$. A monochromatic path colored with some color c from u to v is notated by $u \overset{c}{\mapsto} v$.

Definition 1.14. *A digraph D is a **rainbow** digraph if no two edges are colored with the same color.*

A common tournament referred to throughout this thesis is a **rainbow triangle**.

1.3 Score Sequences

For a vertex v of a digraph D, let $N^+(v) = \{x \in V(D) : v \to x\}$ be its **out-neighborhood** and $N^-(v) = \{x \in V(D) : x \to v\}$ be its **in-neighborhood**. Let let $N^+[v] = N^+(v) \cup \{v\}$ be its **closed out-neighborhood** and $N^-[v] = N^-(v) \cup \{v\}$ be its **closed in-neighborhood**.

Definition 1.15. *The* ***out-degree*** *of a vertex v in a digraph D is equal to $|N^+(v)|$ and is denoted by $d^+(v)$. The* ***in-degree*** *of a vertex v in a digraph D is equal to $|N^-(v)|$ and is denoted by $d^-(v)$.*

It is common to call the out-degree of a vertex its **score**.

Definition 1.16. *A* ***score sequence*** $(s_0, s_1, \ldots, s_{n-1})$ *is a listing of the scores for each vertex of the tournament.*

We choose to list the scores of a score sequence in ascending order.

Definition 1.17. *Let D be a digraph on n vertices. A vertex of D with a score of 0 is called a* ***dominated*** *vertex. A vertex of D with a score of $n-1$ is called a* ***dominating*** *vertex.*

Combining the definitions of a transitive tournament and dominated/dominating vertices, we get the following fact.

Fact 1.18. *If T is a transitive tournament, then T has exactly one dominating vertex and exactly one dominated vertex.*

In fact, we see that the score sequence of a transitive tournament on n vertices is $(0, 1, 2, \ldots, n-1)$.

1.4 Tournaments on 4 Vertices

There are four different non-isomorphic tournaments on 4 vertices. A tournament on 4 vertices is isomorphic to a **transitive** tournament, a **Hamiltonian** tournament, a **dominating triangle** tournament, or a **dominated triangle** tournament. Examples of these tournaments can be seen in Figures 1.1, 1.2, 1.3, and 1.4.

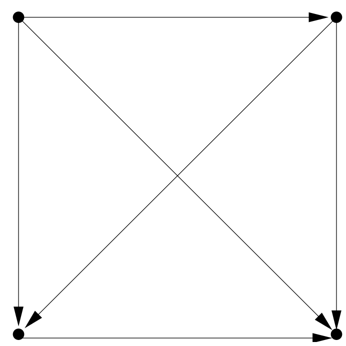

Figure 1.1: Transitive Tournament

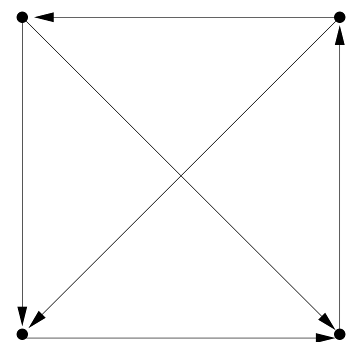

Figure 1.2: Hamiltonian Tournament

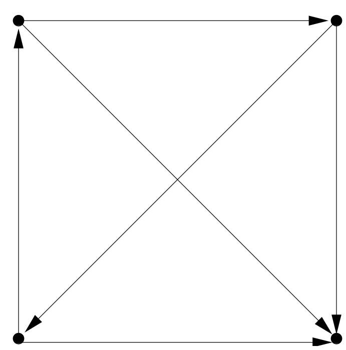

Figure 1.3: Dominating Triangle

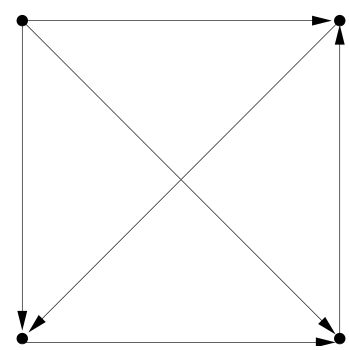

Figure 1.4: Dominated Triangle

These will be useful to refer to later in the proofs of Theorems 2.19 and 2.23. It is worth noting that in a transitive tournament and a dominating triangle tournament, there exists a dominated vertex. Similary, in a transitive tournament and a dominated triangle, there exists a dominating vertex.

1.5 Serfs and Sinks

Not every digraph contains a dominated (or dominating) vertex, so we look for vertices with similar properties within a digraph.

Definition 1.19. *A vertex v in a digraph D is a* ***serf*** *if every other vertex can reach v along a path of length at most two.*

A serf in a digraph D is a **king** in the converse of D. That is, a king is a vertex in a digraph which can reach any other vertex in the digraph along a path of length at most 2. Many useful results have been proven true for kings; we will rephrase them for our use in terms of serfs.

Theorem 1.20 (Chvátal and Lovász, 1972, [3])**.** *Every tournament has a king.*

The proof of the following theorem is similar to that given in [3] to prove Theorem 1.20.

Theorem 1.21. *Every tournament has a serf.*

Proof. Let T be a tournament. We argue by induction on $|T|$ that T has a serf. The base case of $|T| = 1$ is clearly true. So assume $|T| > 1$ and assume the theorem is true for all tournaments of order less than $|T|$. Let $v \in V(T)$ so that $d^-(v) < n-2$. Let $T' = T - \{N^-[v]\}$. Note that for all $v' \in V(T')$, $v \to v'$ in T. By the induction hypothesis, T' has a serf, v_m. Then $v \to v_m$ in T and for all $v' \in N^-(v)$, $v' \to v \to v_m$. Thus v_m is a serf of T. □

The following is a result attributed to Jacob and Meyniel. Havet and Thomassé provided an alternate proof in [8].

Theorem 1.22 (Jacob and Meyniel, 1996, [9])**.** *Every tournament has a king. Further, if a tournament has no dominating vertex, then it has at least three kings.*

We prove the following theorem using the same method Chvátal and Lovász used in the proof of Theorem 1.20.

Theorem 1.23. *Every tournament has a serf. Further, if a tournament has no dominated vertex, then it has at least three serfs.*

Proof. Let T be a tournament on n vertices. We argue by induction on n that T has a serf. Further, if T has no dominated vertex, then it has at least three serfs. The base case of $n = 2$ is trivially true. So assume $n > 2$ and assume the result is true for all tournaments with order less than n. Let v be some vertex in T. If v is dominated, then v is a serf of T and we are done. So assume T does not have a dominated vertex and let $T' = T - N^-[v]$. By the Induction Hypothesis, T' has a serf, v_0. Since $v_0 \notin N^-[v]$, $v \to v_0$. And further, for all $u \in N^-(v)$, $u \to v \to v_0$. Thus v_0 is a serf of T. Now let $T'' = T - N^-[v_0]$. By the Induction Hypothesis, T'' has a serf, v_1. Similar reasoning yields that v_1 is a serf of T. Since $v \notin V(T'')$, $v \neq v_1$. Thus reasoning similarly on the tournament $T''' = T - N^-[v_1]$ yields a serf, v_2, in T that is different from both v_0 and v_1. Thus T has at least three serfs. □

This result is best possible as we give an example of a tournament without a dominating vertex which has exactly three serfs.

Fact 1.24. *For any $n \geq 4$, there exists a tournament on n vertices without a dominated vertex that has exactly three serfs.*

Proof. Let T be a tournament isomorphic to the Hamiltonian tournament on 4 vertices (as previously seen in Figure 1.2). This tournament has exactly 3 serfs. To create a tournament on $n > 4$ vertices with exactly 3 serfs, add $n-4$ vertices which dominate all vertices of the T. Thus, for any $n \geq 4$, we can find a tournament on n vertices which has exactly 3 serfs. □

By weakening the requirement that a serf must be reached by every vertex along a path of at most 2, we obtain the following definition.

Definition 1.25. *A vertex v is a **sink** of a digraph D if every vertex can reach v along a path (with no restriction on the length of the path).*

A dominated vertex is both a sink and a serf of a digraph. We write the relationships between sinks, serfs, and dominated vertices as the following three facts, but will use these facts without reference throughout the thesis.

Fact 1.26. *If v is a dominated vertex of a digraph D, then v is a serf of D.*

Fact 1.27. *If v is a serf of a digraph D, then v is a sink of D.*

The third fact follows easily from the first two facts.

Fact 1.28. *If v is a dominated vertex of a digraph D, then v is a sink of D.*

A digraph is not guaranteed to contain a dominated or dominating vertex. A digraph is guaranteed to contain a dominating and dominated set of vertices, though the sets may be of a trivial size.

Definition 1.29. *A set R is a **dominated set** of vertices in a digraph D if for all $v \notin R$, there exists a vertex $u \in R$, so that $v \to u$. We can similarly define a **dominating set** in a digraph D to be a set of vertices R' so that for all $v \notin R'$, there exists a vertex $u \in R'$ so that $u \to v$.*

We use these two definitions in the next section.

1.6 Linear Orders

It is sometimes helpful to view digraphs and tournaments as partially ordered sets.

Definition 1.30. *A **partially ordered set** (or **poset**) is a pair $(P,<)$, where P is a set and $<$ is a relation on P satisfying the following three conditions:*

- *Reflexivity: for all $x \in P$, $x < x$.*
- *Antisymmetry: for all $x, y \in P$, if $x < y$ and $y < x$, then $x = y$.*
- *Transitivity: for all $x, y, z \in P$, if $x < y$ and $y < z$, then $x < z$.*

Two elements u and v of a set are said to be **comparable** if either $u < v$ or $v < u$. From a poset $(P,<)$, we can create a digraph by taking P as the set of vertices and directing an edge from u to v if and only if $u < v$. Note that this graph is acyclic and transitive. The converse is also true and so we write this as the following fact.

Fact 1.31. *A transitive, acyclic digraph D can be formed from a poset $(P,<)$. Conversely, a poset $(P,<)$ can be formed from a transitive, acyclic digraph D.*

It is useful to be able to find orders on tournaments, so we introduce linear orders.

Definition 1.32. *A **linear order** is a partial order with the property that every pair of elements are comparable.*

Reasoning similarly as above, we can obtain a linear order from a transitive tournament and obtain a transitive tournament from a linear order. We write this as the following fact.

Fact 1.33. *A transitive tournament T can be formed from a linear order $(P,<)$ so that $uv \in E(T)$ when $u < v$. Conversely, a linear order $(P,<)$ can be formed from a transitive tournament T so that $u < v$ when $uv \in E(T)$.*

As we will see later in the thesis, it is sometimes necessary to create a transitive tournament from an acyclic digraph. The following theorem called the Order Extension Principle allows us to do this. First, we must establish one more definition.

Definition 1.34. *If $(P,<)$ is a poset, a* ***linear extension*** *of P is a relation $<^*$ on P so that $(P,<^*)$ is a linear order and $u < v$ implies $u <^* v$.*

We now give the Order Extension Principle, which was first published by Marczewski in 1930.

Theorem 1.35 (Marczewski, 1930, [10])**.** *Every finite poset $(P,<)$ has a linear extension.*

We then combine Fact 1.33 and Theorem 1.35 to obtain the following fact.

Fact 1.36. *If D is an acyclic, transitive digraph, then there exists a transitive tournament T so that if $uv \in E(D)$, then $uv \in E(T)$.*

In [1], Alon, Brightwell, Kierstead, Kostochka, and Winkler introduced the definition of a p-majority tournament.

Definition 1.37. *Let $<_1, <_2, \ldots, <_{2p-1}$ be $2p-1$ linear orders on a set of n vertices. A* ***p-majority tournament*** *is one which an edge uv exists if $v < u$ in at least p of the linear orders.*

A p-majority tournament is said to be realized by these $2p-1$ orders. The following theorem appeared in [1].

Theorem 1.38 (Alon, Brightwell, Kierstead, Kostochka, and Winkler, 2006, [1])**.** *Every 2-majority tournament has a dominating set of size at most three. Moreover, if T does not have a dominating set of size one, then it has a dominating set of size three that induces a directed cycle.*

It is easy then to find a dominated set of size at most three in a 2-majority tournament. Consider a 2-majority tournament T. Reverse the orientation of every edge in T to obtain T'. The new tournament T' is still a 2-majority tournament. By Theorem 1.38, T' has a dominating set of size at most 3. This dominating set of vertices is then a dominated set of vertices in T. We pose this result as the following theorem.

Theorem 1.39. *Every 2-majority tournament has a dominated set of size at most three.*

This theorem will be used in Chapter 3.

1.7 Modular Counting

It is assumed when performing addition or subtraction on colors from the set $\{0,1,\ldots,k-1\}$, it is done modulo k. It is assumed when performing addition or subtraction on the indices of the vertices of a cycle $\{0,1,\ldots,s-1\}$, it is done modulo s. Calculations will be made without reference to these facts throughout the thesis.

1.8 Organization

In Chapter 2, we introduce the definition of a monochromatic sink. We also introduce the first of two questions posed by Sands, Sauer, and Woodrow in [13], which asks about the existence of monochromatic sinks in k-colored tournaments without rainbow triangles. We present work from Shen (in [14]), Galeana-Sánchez (in [5] and [6]), and Rojas-Monroy (in [6]) which frames the progress made on answering the question from Sands, Sauer, and Woodrow. We use their results as inspiration and direction towards our new results. Additionally, we introduce the notion of nearly transitive tournaments and continue to prove results related to the the question posed by Sands, Sauer, and Woodrow. We will find the existence of monochromatic sinks in k-colored nearly transitive tournaments. Melcher and Reid introduced the concept of upset tournaments (a type of nearly transitive tournament) in [11] and we prove a slightly stronger result than their main result. We additionally provide a result about the existence of monochromatic sinks in a k-colored nearly transitive tournament with some restrictions on the coloring of its back-edges.

In Chapter 3, we introduce the notion of ruling classes and introduce the second question asked by Sands, Sauer, and Woodrow in [13]. We begin by presenting a small result by Galeana-Sánchez and Rojas-Monroy from [7]. With this being the extent of work published towards answering this problem, we then give results (and a corresponding proof method) as to the existence of ruling classes in k-colored tournaments. We end with a conjecture that would allow us to make conclusions about the size of ruling classes in tournaments similar to those in Chapter 2.

The primary focus of this thesis is that of monochromatic paths and monochromatic sinks. In Chapter 4, we introduce the notion of α-step-chromatic sinks and provide initial results and proofs pertaining to the question introduced in Chapter 2.

Chapter 2

MONOCHROMATIC SINKS

In Chapter 1, we presented some results about the existence of serfs in tournaments and the various relations between serfs, dominated vertices, and sinks. In this chapter, we look for vertices in colored digraphs that essentially serve as sinks of the digraph. That is, we look for a vertex in a digraph that can be reached by any other vertex of the digraph along a monochromatic path.

Definition 2.1. *A vertex in a digraph D is a **monochromatic sink** if every other vertex in D can reach it along a monochromatic path.*

The purpose of this chapter is to find monochromatic sinks in k-colored tournaments with various color restrictions. The work done in this chapter is inspired by a question posed by Sands, Sauer, and Woodrow (Question 2.6) and subsequent work done by Shen as well as Galeana-Sánchez and Rojas-Monroy.

Definition 2.2. *Let T be a colored tournament on n vertices. A **dominated rainbow triangle** on $3 \leq i \leq n$ vertices is a subtournament of T containing a rainbow triangle and $i-3$ vertices, all of which dominate the vertices of the rainbow triangle. We denote a dominated rainbow triangle on i vertices by T_i^*. A dominated rainbow triangle on all n vertices will be denoted T^*.*

Figure 2.1 is an example of a 3-colored T_5^*. Note that the T_5^* in Figure 2.1 does not have a monochromatic sink. In fact, there does not exist a monochromatic sink in any k-colored tournament that is a T^*.

Fact 2.3. *A colored tournament that is a T^* does not have a monochromatic sink.*

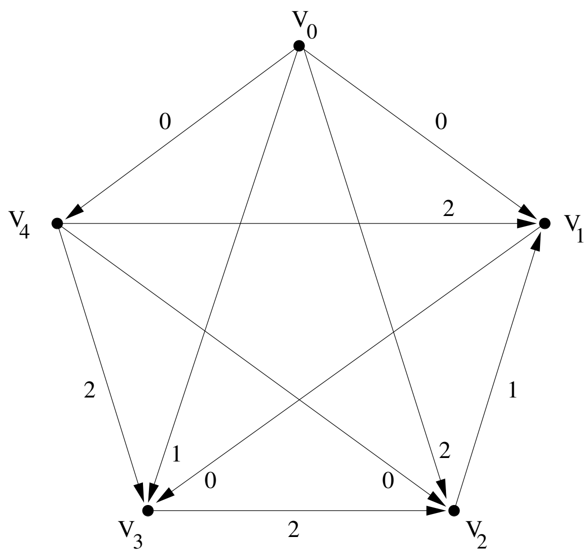

Figure 2.1: An example of a T_5^*.

Clearly, then, if a colored tournament has a monochromatic sink, it is not a T^*. However, must a colored tournament that is not a T^* have a monochromatic sink? The answer is no, and many tournaments can be created to show this. For example, the tournament given later in Figure 2.2 has no monochromatic sink, yet is not a T^*. In the next section, we present a question posed by Sands, Sauer, and Woodrow with stronger conditions.

2.1 Monochromatic Reachability Question

We can view a path as a finite sequence of distinct vertices $v_0v_1v_2\ldots$ of a digraph D so that there is a directed edge from v_i to v_{i+1}, for each i. If this sequence is infinite, then the path is said to be an **infinite outward path**. Throughout this thesis, we always view paths as finite sequences, but the definition of an infinite outward path is useful for the following theorem proven by Sands, Sauer, and Woodrow in [13].

Theorem 2.4 (Sands, Sauer, and Woodrow, 1982, [13]). *Let D be a digraph whose edges are colored with two colors such that D contains no monochromatic infinite* *outward path. Then there is an independent set S of vertices of D such that, for every vertex x not in S, there is a monochromatic path from x to a vertex of S.*

In a 2-colored tournament, this independent set is of size one and thus this single vertex is a monochromatic sink.

Theorem 2.5 (Sands, Sauer, and Woodrow, 1982, [13])**.** *If T is a 2-colored tournament, then T has a monochromatic sink.*

We will give a proof of Theorem 2.5 in the next section. In a 3-colored tournament, we may no longer use Theorem 2.5 to find a monochromatic sink and in fact, we can find a very basic tournament which has no monochromatic sink. Consider a rainbow triangle. This is a tournament on 3 vertices which has no monochromatic sink. In general, for any $n,k \geq 3$, the k-colored tournament T^* is a tournament on n vertices without a monochromatic sink. Naturally, then, Sands, Sauer, and Woodrow posed the following question.

Question 2.6 (Sands, Sauer, and Woodrow, 1982, [13])**.** *If T is a k-colored tournament without rainbow triangles, does T contain a monochromatic sink?*

Shen showed in [14] that the answer is no when $k \geq 5$. Figure 2.2 gives a tournament on five vertices without rainbow cycles and without a monochromatic sink. A tournament on $n \geq 6$ vertices without rainbow triangles and without a monochromatic sink can be created by adding $n-5$ vertices one at a time and directing all edges to all prior vertices (with edges colored any color).

Similar to Shen's example when $k = 4$, Galeana-Sánchez and Rojas-Monroy provided in [6] a tournament on six vertices without rainbow cycles and without a monochromatic sink (see Figure 2.3). A tournament on $n \geq 7$ vertices can be created from Figure 2.3 similarly to how Shen's tournament was extended.

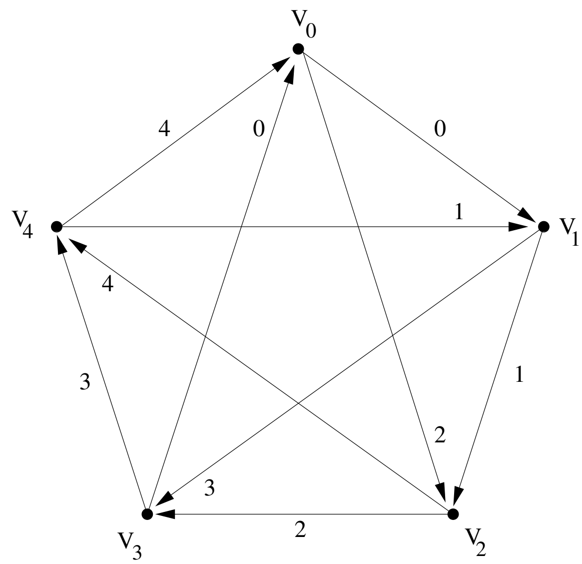

Figure 2.2: k-Colored Tournament With No Monochromatic Sink When $k \geq 5$.

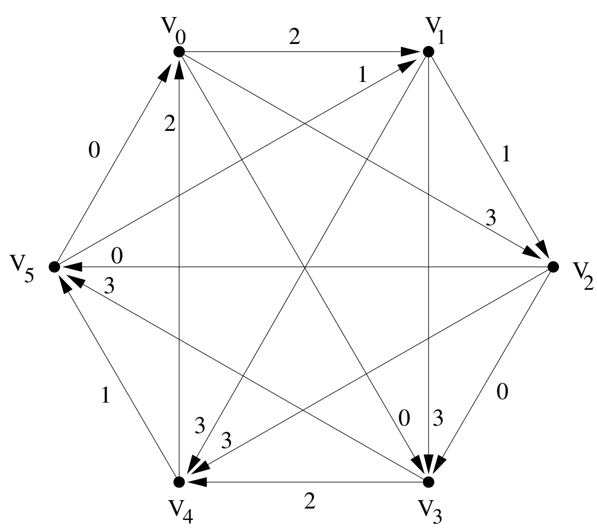

Figure 2.3: k-Colored Tournament With No Monochromatic Sink When $k \geq 4$.

We then rephrase the question to reflect the work done since the original posing of the question.

Question 2.7. *If T is a 3-colored tournament without rainbow triangles, does T contain a monochromatic sink?*

Question 2.7 still remains open. Question 2.6 can be answered with additional restrictions put on the coloring of the edges and we will present these results in this chapter.

2.2 Minimum Counter Example

Within a proof of Shen's in [14], a structure for a minimum counter example to Question 2.6 is obtained. We provide a lemma, with proof, so we may use this substructure throughout the thesis. First, we must introduce some terminology.

Definition 2.8. *A cycle $C = v_0v_1 \ldots v_{s-1}v_0$ is a* ***dominating cycle*** *in a digraph D if for all $i \in \{0, 1, \ldots, s-1\}$, v_i is a monochromatic sink in $D - v_{i+1}$, but there is no monochromatic path from v_{i+1} to v_i in D.*

Throughout the thesis, we consider Hamiltonian dominating cycles.

Definition 2.9. *A property $\mathscr{P}$ is a* ***hereditary property*** *of a colored digraph D if the digraph $D - S$ has the property $\mathscr{P}$ for all $S \subset V(D)$.*

It is worth noting that a particular hereditary property is dependant not only on the number of vertices in the digraph, but also on the coloring of the digraph. A common hereditary property that is considered in this thesis is that a k-colored digraph has no rainbow triangles. Below is the proof based on the proof given by Shen in [14], however it has been generalized to consider any hereditary property $\mathscr{P}$, and not just the property that a tournament does not have rainbow triangles.

Lemma 2.10. *Let T be a minimum k-colored tournament with hereditary property $\mathscr{P}$ so that there is no monochromatic sink. Then T has a Hamiltonian dominating cycle.*

Proof. Let T be a minimum tournament with hereditary property $\mathscr{P}$ so that there is no monochromatic sink. Then for each $v \in V(T)$, there exists some vertex $f(v) \in V(T)$ so that $x \mapsto f(v)$ for all $x \in V(T) \backslash \{v\}$. Since T has no monochromatic sink, $v \not\mapsto f(v)$, and therefore $f(v) \to v$. Note also that for any two distinct vertices $u, v \in V(T)$, $f(u) \neq f(v)$, for otherwise $f(u)$ is a monochromatic sinks in T, a contradiction. So we can assume that f is a bijection and that $v \not\mapsto f(v)$ for all $v \in V(T)$. By the relabeling $f(v_i) = v_{i+1}$, $V(T)$ is partitioned into cycles

$$v_1 v_2 \dots v_{s_1} v_1, \;\; v_{s_1+1} v_{s_1+2} \dots v_{s_2} v_{s_1+1}, \;\; \dots \; .$$

If there is more than one cycle, then consider the tournament with vertex set $\{v_1, v_2, \dots, v_{s_1}\}$. Call this tournament T'. This is a smaller tournament than T, so there exists a monochromatic sink v_i in T'. In particular, since $v_{i+1} \in V(T')$, this implies that $v_{i+1} \mapsto v_i$, which is a contradiction. Thus there is only one cycle and this cycle is a dominating cycle. $\square$

Observe the following proof of Theorem 2.5 which utilizes Lemma 2.10.

Proof. Let T be a 2-colored tournament. We consider the 2-coloring of the tournament to be its hereditary property. We will argue on $|T|$ that T has a monochromatic sink. For the base case when $|T| = 2$, this is trivially true. So assume $|T| > 2$ and assume T has no monochromatic sink. By the Induction Hypothesis, this tournament must be a minimum counter example and so by Lemma 2.10, T has a Hamiltonian dominating cycle, $C = v_0 v_1 \dots v_{n-1} v_0$. If C is monochromatic, then every vertex of C is a monochromatic sink of T. So assume there exist consecutive edges along C that are colored differently. Without loss of generality, say $v_0 \overset{0}{\to} v_1 \overset{1}{\to} v_2$. Since C is a dominating cycle, either $v_2 \overset{0}{\mapsto} v_0$ or $v_2 \overset{1}{\mapsto} v_0$. In the former case, $v_2 \overset{0}{\mapsto} v_1$, a contradiction. In the latter case, $v_1 \overset{1}{\mapsto} v_0$, a contradiction. Thus both cases lead to a contradiction and we conclude that T must have a monochromatic sink. $\square$

The property that a tournament has no rainbow triangles (or dominated rainbow triangles) is a hereditary property and as a result, we will be able to use Lemma 2.10 for many of the proofs throughout this thesis in the same fashion that it was used in the previous proof.

2.3 Near-Monochromatic Cycles

We begin this section by defining a slight variation of a monochromatic digraph.

Definition 2.11. *A digraph D is* ***near-monochromatic*** *if all edges of D are colored the same with the possible exception of one edge.*

Certainly, a monochromatic digraph is also a near-monochromatic digraph. The converse statement is not necessarily true. If a digraph is not near-monochromatic (or not monochromatic), we use the term **non-near-monochromatic** (or **non-monochromatic**) to describe this property.

Definition 2.12. *For some color c, a digraph with ℓ edges is said to be* ***near-monochromatic with c*** *if at least $\ell - 1$ of the edges are colored with c.*

Galeana-Sánchez proved the following theorem.

Theorem 2.13 (Galeana-Sánchez, 1996, [5])**.** *Let T be a k-colored tournament. If every 3-cycle is monochromatic, then T has a monochromatic sink.*

Note that the requirement that a 3-cycle is monochromatic prevents the graph from containing a rainbow triangle. Galeana-Sánchez proved Theorem 2.13 by showing that in a k-colored tournament T where all 3-cycles are monochromatic, it is the case that all 4-cycles are near-monochromatic. Then, using the following theorem, T must have a monochromatic sink.

Theorem 2.14 (Galeana-Sánchez, 1996, [5])**.** *Let T be a k-colored tournament so that every 3- and 4-cycle is near-monochromatic. Then T has a monochromatic sink.*

Similar to the coloring restriction on 3-cycles in Theorem 2.13, the near-monochromatic coloring of 3-cycles in Theorem 2.14 prevents the tournament from containing a rainbow triangle. In Theorem 2.19, we prove a result similar to that of Theorem 2.13. We will show that a k-colored tournament that is not a T^* has a monochromatic sink when all 4-cycles are monochromatic. We must first, however, prove some initial results. We begin with the following fact.

Fact 2.15. *If D is a digraph with a near-monochromatic Hamiltonian cycle, then D has a monochromatic sink.*

The previous fact is easy to see when considering a monochromatic Hamiltonian path (if the cycle is not monochromatic, there is exactly one monochromatic Hamiltonian path) along the near-monochromatic Hamiltonian cycle. The last vertex of the path is the monochromatic sink of the digraph. We next establish two lemmas, Lemma 2.16 and Lemma 2.17, to be used in the proof of Theorem 2.18. Lemma 2.16 will also be used repeatedly in the proof of Theorem 2.19. Theorem 2.18 will be used to prove Theorem 2.19 in a similar fashion to how Galeana-Sánchez used Theorem 2.14 to prove Theorem 2.13.

Lemma 2.16. *Let T be a k-colored tournament on n vertices where all 4- and 5-cycles are near-monochromatic. Suppose for two edges xy and yz, there exists a* *monochromatic path P of length at least two from z to x. Then either $y \mapsto x$ or $z \mapsto y$.*

Proof. Let T be a k-colored tournament as described in the hypothesis. For colors $\alpha, \beta \in \{0, 1, \ldots, k-1\}$, let $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$. Note that this allows α to equal β. If P is colored with either α or β, then either $z \overset{\alpha}{\mapsto} y$ or $y \overset{\beta}{\mapsto} x$, respectively. In either case, we are done, so we assume that P is colored with a color different than α and β, say with γ. Note that the path P has length at least 3, since otherwise it would create a non-near-monochromatic 4-cycle, a contradiction. In fact, we can reason similarly that P has length at least 4. Also, we can assume $y \notin V(P)$, as otherwise $z \mapsto y$ and we are done. Let $P = u_0 u_1 \ldots u_t$, where $u_0 = z, u_t = x$, and $t > 3$. If $u_2 \rightarrow y$, then $y z u_1 u_2 y$ is a 4-cycle with $z \xrightarrow{\gamma} u_1$ and $u_1 \xrightarrow{\gamma} u_2$, so then $u_2 \xrightarrow{\gamma} y$. Then $z \mapsto y$. Assume that $y \rightarrow u_2$. If $u_3 \rightarrow y$, then similar reasoning finds that $z \mapsto y$. Then we may assume that $y \rightarrow u_3$. If for all $i \in \{2, \ldots, t-2\}$, $y \rightarrow u_i$, then $y u_{t-2} u_{t-1} x y$ is a 4-cycle and therefore $y \xrightarrow{\gamma} u_{t-2}$. But then $y \overset{\gamma}{\mapsto} x$. So assume otherwise. Let $i \in \{2, \ldots, t-2\}$ be minimum so that $u_i \rightarrow y$. Then $u_i y u_{i-2} u_{i-1} u_i$ is a 4-colored cycle with $u_{i-2} \xrightarrow{\gamma} u_{i-1}$ and $u_{i-1} \xrightarrow{\gamma} u_i$. Thus either $u_i \xrightarrow{\gamma} y$ or $y \xrightarrow{\gamma} u_{i-2}$ (or both). If $u_i \xrightarrow{\gamma} y$, then since P is colored with γ, $z P u_i y$ is a monochromatic path from z to y. Hence $z \overset{\gamma}{\mapsto} y$. If $y \xrightarrow{\gamma} u_{i-2}$, then $y u_{i-2} P x$ is a monochromatic path from y to x. So in all cases, we have that either $y \mapsto x$ or $z \mapsto y$, as desired. □

The following lemma will be used as the base case for Theorem 2.18.

Lemma 2.17. *Let T be a k-colored tournament on 5 vertices that is not a T^* so that all 4- and 5-cycles are near-monochromatic. Then, T has a monochromatic sink.*

Proof. Let T be a k-colored tournament on 5 vertices. By Theorem 1.4, T has a Hamiltonian path, P. Let $P = v_0 v_1 v_2 v_3 v_4$. If $v_4 \rightarrow v_0$, then $v_0 v_1 v_2 v_3 v_4 v_0$ is a 5-cycle and by Fact 2.15, T has a monochromatic sink. So assume $v_0 \to v_4$. If $v_1v_2v_3v_4$ is monochromatic, then v_4 is a monochromatic sink of T. So assume otherwise.

First assume $v_4 \to v_1$. Since $v_1v_2v_3v_4$ is not monochromatic, three edges of the 4-cycle $v_1v_2v_3v_4v_1$ must be colored the same and the other must be colored differently. Without loss of generality, let three of the edges be colored with 0 and the other colored with 1. We have already established that it cannot be that v_4v_1 is the edge colored with 1, so we have three cases.

Case 1: v_3v_4 is colored with 1. If $v_0 \to v_3$, then v_3 is a monochromatic sink of T. So assume otherwise. That is, assume $v_3 \to v_0$. If the 4-cycle $v_0v_1v_2v_3v_0$ is monochromatic, then v_0 is a monochromatic sink of T. So assume otherwise. Then $v_0v_1v_2v_3v_0$ is near-monochromatic, with three of its edges colored 0 and one other edge colored 1. If $v_3 \xrightarrow{0} v_0$, then v_0 is a monochromatic sink of T and if $v_0 \xrightarrow{0} v_1$, then v_3 is a monochromatic sink of T. In either case, we find a monochromatic sink of T.

Case 2: v_2v_3 is colored with 1. If $v_0 \to v_2$, then v_2 is a monochromatic sink of T. So assume otherwise, that $v_2 \to v_0$. Then $v_2v_0v_4v_1v_2$ is a 4-cycle and so at least one of v_2v_0 or v_0v_4 is colored with 0. If $v_2 \xrightarrow{0} v_0$, then v_0 is a monochromatic sink of T. If $v_0 \xrightarrow{0} v_4$, then v_2 is a monochromatic sink of T.

Case 3: v_1v_2 is colored with 1. Then v_1 is a monochromatic sink of T.

In all three cases, we found a monochromatic sink of T. So we then assume that $v_1 \to v_4$. If $v_2 \mapsto v_4$, the v_4 is a monochromatic sink of T. So assume otherwise, that $v_4 \to v_2$ and the edges v_2v_3 and v_3v_4 are colored differently. Without loss of generality, let $v_2 \xrightarrow{0} v_3$ and $v_3 \xrightarrow{1} v_4$. If $v_3 \to v_0$, then $v_3v_0v_1v_4v_2v_3$ is a 5-cycle. But then this cycle is near-monochromatic and therefore all vertices along the cycle can reach at least one vertex on the cycle along a monochromatic path, and this vertex is a monochromatic sink. So assume $v_0 \to v_3$. If $v_3 \to v_1$, then $v_3v_1v_4v_2v_3$

is a 4-cycle. If it is monochromatic, then v_4 is a monochromatic sink. So exactly three edges of the 4-cycle are colored the same and one is colored differently. If the color 0 is shared on three edges, then in all three cases ($v_1 \xrightarrow{0} v_4 \xrightarrow{0} v_2 \xrightarrow{0} v_3$, $v_4 \xrightarrow{0} v_2 \xrightarrow{0} v_3 \xrightarrow{0} v_1$, or $v_2 \xrightarrow{0} v_3 \xrightarrow{0} v_1 \xrightarrow{0} v_4$) there is a monochromatic sink (v_3, v_1 or v_4, respectively). Then the edges v_3v_1, v_1v_4, and v_4v_2 are colored the same (not with 0). If $v_2 \to v_0$, then $v_2v_0v_3v_1v_4v_2$ is a 5-cycle and by Fact 2.15, T has a monochromatic sink. So we may assume $v_0 \to v_2$. Since v_3v_1, v_1v_4, and v_4v_2 are all colored with the same color, v_2 is a monochromatic sink of T. So assume that $v_1 \to v_3$. Assume $v_2 \to v_0$. Then $v_2v_0v_1v_3v_4v_2$ is a 5-cycle and by Fact 2.15, T has a monochromatic sink. So assume $v_0 \to v_2$. If $v_4 \xrightarrow{c} v_2$, where $c \in \{2,3,\dots,k-1\}$, then T is a T^*, a contradiction. If $v_4 \xrightarrow{0} v_2$, then v_3 is a monochromatic sink of T. So it must be that $v_4 \xrightarrow{1} v_2$ and therefore v_2 is a monochromatic sink of T. $\square$

Theorem 2.18. *Let T be a k-colored tournament on $n \geq 5$ vertices that is not a T^* so that all 4- and 5-cycles are near-monochromatic. Then, T has a monochromatic sink.*

Proof. Let T be a k-colored tournament on $n \geq 5$ vertices as described in the hypothesis. We argue by induction on $|T|$ that T has a monochromatic sink. For the base case when $|T| = 5$, Lemma 2.17 gives that T has a monochromatic sink. So let $|T| > 4$ and assume T does not have a monochromatic sink. Then by the Induction Hypothesis, T is a minimum counter example to the result. By Lemma 2.10, there is a Hamiltonian dominating cycle, C. Let $C = v_0v_1\dots v_{n-1}v_0$. If C is monochromatic, then every vertex is a monochromatic sink of T. So assume otherwise. Then there exists $i \in \{0,1,\dots,n-1\}$ so that v_iv_{i+1} is colored differently than $v_{i+1}v_{i+2}$. Without loss of generality, let $v_0 \xrightarrow{0} v_1 \xrightarrow{1} v_2$. Since C is a dominating cycle, $v_3 \mapsto v_0$ along some path $P = u_0u_1\dots u_t$, where $v_3 = u_0$ and $v_0 = u_t$. We then have three cases: either $v_2 \xrightarrow{0} v_3$, $v_2 \xrightarrow{1} v_3$, or $v_2 \xrightarrow{c} v_3$, where $c \in \{2,3,\ldots,k-1\}$.

Case 1: Assume $v_2 \xrightarrow{0} v_3$. We then have three subcases: either $v_3 \stackrel{0}{\mapsto} v_0$, $v_3 \stackrel{1}{\mapsto} v_0$, or $v_3 \stackrel{c}{\mapsto} v_0$, where $c \in \{2,3,\ldots,k-1\}$, along P.

Case 1a: Assume $v_3 \stackrel{0}{\mapsto} v_0$ along P. Then $v_2v_3Pv_0v_1$ is a monochromatic path from v_2 to v_1, a contradiction.

Case 1b: Assume $v_3 \stackrel{1}{\mapsto} v_0$ along P. Since all 4- and 5-cycles are near-monochromatic, we have that $t > 2$. If $u_1 \to v_1$, then $u_1v_1v_2v_3u_1$ is a 4-cycle. Therefore $u_1 \xrightarrow{1} v_1$ and thus $v_3 \stackrel{1}{\mapsto} v_2$, a contradiction. So $v_1 \to u_1$. Similarly, we reason that $v_1 \to u_2$. Suppose for all $i \in \{1,2,\ldots,t-2\}$, $v_1 \to u_i$. Then $v_1u_{t-2}u_{t-1}v_0v_1$ is a 4-cycle. Therefore $v_1 \xrightarrow{1} u_{t-2}$ and thus $v_1 \stackrel{1}{\mapsto} v_0$, a contradiction. So choose $i \in \{3,4,\ldots,t-2\}$ to be minimum so that $u_i \to v_1$. Then $v_1u_{i-2}u_{i-1}u_iv_1$ is a 4-cycle where $u_{i-2} \xrightarrow{1} u_{i-1}$ and $u_{i-1} \xrightarrow{1} u_i$. Then either $v_1 \xrightarrow{1} u_{i-2}$ or $u_i \xrightarrow{1} v_1$. If $v_1 \xrightarrow{1} u_{i-2}$, then $v_1 \stackrel{1}{\mapsto} v_0$, a contradiction. If $u_i \xrightarrow{1} v_1$, then $v_3 \stackrel{1}{\mapsto} v_2$, a contradiction.

Case 1c: Assume $v_3 \stackrel{c}{\mapsto} v_0$, where $c \in \{2,3,\ldots,k-1\}$, along P. Without loss of generality, let $v_3 \stackrel{2}{\mapsto} v_0$. We reason similarly as in Case 1b to find that $t > 2$ and both $v_1 \to u_1$ and $v_1 \to u_2$. Suppose for all $i \in \{1,2,\ldots,t-2\}$, $v_1 \to u_i$. Then $v_1u_{t-2}u_{t-1}v_0v_1$ is a 4-cycle. Therefore $v_1 \xrightarrow{2} u_{t-2}$ and thus $v_1 \stackrel{2}{\mapsto} v_0$, a contradiction. So choose $i \in \{3,4,\ldots,t-2\}$ to be minimum so that $u_i \to v_1$. Then $v_1u_{i-2}u_{i-1}u_iv_1$ is a 4-cycle where $u_{i-2} \xrightarrow{2} u_{i-1}$ and $u_{i-1} \xrightarrow{2} u_i$. Then either $v_1 \xrightarrow{2} u_{i-2}$ or $u_i \xrightarrow{2} v_1$. If $v_1 \xrightarrow{2} u_{i-2}$, then $v_1 \stackrel{2}{\mapsto} v_0$, a contradiction. So then $u_i \xrightarrow{2} v_1$. But then by Lemma 2.16, we arrive at a contradiction.

In all three subcases, we arrive at a contradiction. So assume Case 1 not to be true.

Case 2: Assume $v_2 \xrightarrow{1} v_3$. We again have three subcases: $v_3 \stackrel{0}{\mapsto} v_0$, $v_3 \stackrel{1}{\mapsto} v_0$, or $v_3 \stackrel{c}{\mapsto} v_0$, where $c \in \{2,3,\ldots,k-1\}$, along P.

Case 2a: Assume $v_3 \overset{0}{\mapsto} v_0$ along P. By Lemma 2.16, we have that either $v_3 \mapsto v_2$ or $v_2 \mapsto v_1$. In either case, we have a contradiction.

Case 2b: Assume $v_3 \overset{1}{\mapsto} v_0$ along P. Then $v_1 v_2 v_3 P v_0$ is a monochromatic path from v_1 to v_0, a contradiction.

Case 2c: Assume $v_3 \overset{c}{\mapsto} v_0$, where $c \in \{2,3,\ldots,k-1\}$, along P. If $u_2 \to v_2$, then $u_2 v_2 v_3 u_1 u_2$ is a 4-cycle which is either non-near-monochromatic or gives a monochromatic path from v_3 to v_2. In either case, we arrive at a contradiction. So $v_2 \to u_2$. If $u_3 \to v_2$, we arrive at a similar contradiction, so assume $v_2 \to u_3$. If for all $i \in \{4,5,\ldots,t-1\}$, $v_2 \to u_i$, then $v_2 u_{t-1} v_0 v_1 v_2$ is a 4-cycle that is colored with at least 3 colors, a contradiction. So choose $i \in \{4,5,\ldots,t-1\}$ to be minimum so that $u_i \to v_2$. Then $v_2 u_{i-2} u_{i-1} u_i v_2$ is a 4-cycle with $u_{i-2} \xrightarrow{c} u_{i-1}$ and $u_{i-1} \xrightarrow{c} u_i$. Therefore $v_2 \xrightarrow{c} u_{i-2}$ or $u_i \xrightarrow{c} v_2$ (or both). If $u_i \xrightarrow{c} v_2$, then $v_3 \overset{c}{\mapsto} v_2$, which is a contradiction. So then assume $v_2 \xrightarrow{c} u_{i-2}$. Then by Lemma 2.16, either $v_2 \mapsto v_1$ or $v_1 \mapsto v_0$. In either case, we have a contradiction.

In all three subcases, we arrive at contradictions. So assume Case 2 not to be true.

Case 3: Assume $v_2 \xrightarrow{c} v_3$, where $c \in \{2,3,\ldots,k-1\}$. Without loss of generality, let $v_2 \xrightarrow{2} v_3$. We then have four cases: $v_3 \overset{0}{\mapsto} v_0$, $v_3 \overset{1}{\mapsto} v_0$, $v_3 \xrightarrow{2} v_0$, or $v_3 \xrightarrow{c'} v_2$, where $c' \in \{3,4,\ldots,k-1\}$.

Case 3a: Assume $v_3 \overset{0}{\mapsto} v_0$ along P. Then by Lemma 2.16, either $v_3 \mapsto v_2$ or $v_2 \mapsto v_1$. In either case, we have a contradiction.

Case 3b: Assume $v_3 \overset{1}{\mapsto} v_0$ along P. If $v_2 \to u_{t-1}$, then $v_0 v_1 v_2 u_{t-1} v_0$ is a 4-cycle and so $v_2 \xrightarrow{1} u_{t-1}$. But then $v_1 \overset{1}{\mapsto} v_0$, a contradiction. So $u_{t-1} \to v_2$. Similarly, we reason that $u_{t-2} \to v_2$. If for all $i \in \{1,2,\ldots,t-3\}$, $u_i \to v_2$, then $v_3 u_1 u_2 v_2 v_3$ is a 4-cycle with $v_3 \xrightarrow{1} u_1 \xrightarrow{1} u_2$. Since $v_2 \xrightarrow{2} v_3$ and all 4-cycles are near-monochromatic,

$u_2 \xrightarrow{1} v_2$. Then $v_3u_1u_2v_2$ is a monochromatic path from v_3 to v_2, a contradiction. Then choose a maximum $i \in \{1,2,\ldots,t-3\}$ so that $v_2 \to u_i$. Then $v_2u_iu_{i+1}u_{i+2}v_2$ is a 4-cycle with $u_i \xrightarrow{1} u_{i+1} \xrightarrow{1} u_{i+2}$. Then either $u_{i+2} \xrightarrow{1} v_2$ or $v_2 \xrightarrow{1} u_i$ (or both). If $v_2 \xrightarrow{1} u_i$, then $v_1v_2u_iPv_0$ is a monochromatic path from v_1 to v_0. If $u_{i+2} \xrightarrow{1} v_2$, then $v_3Pu_{i+1}v_2$ is a monochromatic path from v_3 to v_2. In either case, we have a contradiction.

Case 3c: Assume $v_3 \overset{2}{\mapsto} v_0$ along P. Then by Lemma 2.16, either $v_2 \mapsto v_1$ or $v_1 \mapsto v_0$, both contradictions.

Case 3d: Assume $v_3 \xrightarrow{c'} v_2$, where $c' \in \{3,4,\ldots,k-1\}$. Without loss of generality, let P be colored with 3. Then both $u_{t-1} \to v_2$ and $u_{t-2} \to v_2$, as otherwise, there would be a 3-colored 4- or 5-cycle. If for all $i \in \{1,2,3,\ldots,t-1\}$, $u_i \to v_2$, then $v_2v_3u_1u_2v_2$ is a 4-cycle with $v_3 \xrightarrow{3} u_1 \xrightarrow{3} u_2$. Since $v_2 \xrightarrow{2} v_3$ and the 4-cycle is near-monochromatic, $u_2 \xrightarrow{3} v_2$. But then $v_3 \overset{3}{\mapsto} v_2$, a contradiction. Then choose a maximum $i \in \{1,2,\ldots,t-3\}$ so that $v_2 \to u_i$. Then $v_2u_iu_{i+1}u_{i+2}v_2$ is a 4-cycle with $u_i \xrightarrow{3} u_{i+1} \xrightarrow{3} u_{i+2}$. Then either $u_{i+2} \xrightarrow{3} v_2$ or $v_2 \xrightarrow{3} u_i$ (or both). If $v_2 \xrightarrow{3} u_i$, then by Lemma 2.16, either $v_2 \mapsto v_1$ or $v_1 \mapsto v_0$ (or both). In either case, we have a contradiction. If $u_{i+2} \xrightarrow{3} v_2$, then $v_3Pu_{i+2}v_2$ is a monochromatic path from v_3 to v_2, a contradiction.

In all 4 subcases we arrive at contradictions. Thus in all 3 cases, we arrive at contradictions and so we must conclude that T has a monochromatic sink. $\square$

Theorems 2.14 and 2.18 are independent of each other. One does not imply the other. In Figure 2.4 , we see that Theorem 2.14 does not imply Theorem 2.18. In particular, all 3- and 4-cycles of the tournament in Figure 2.4 are near-monochromatic, yet the 5-cycle $v_0v_4v_3v_2v_1v_0$ is not near-monochromatic. In Figure 2.5, we see that Theorem 2.18 does not imply Theorem 2.14. In particular, all 4- and

5-cycles of the tournament in Figure 2.5 are near-monochromatic, yet the 3-cycle $v_1v_4v_3$ is not near-monochromatic.

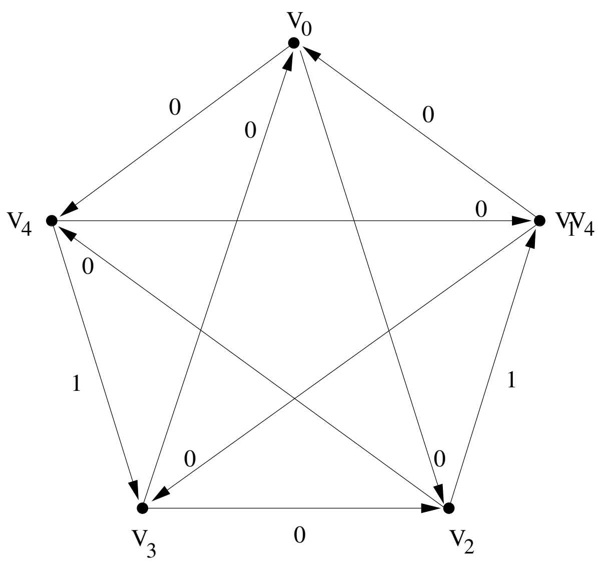

Figure 2.4: Thm. 2.14 $\not\Rightarrow$ Thm. 2.18

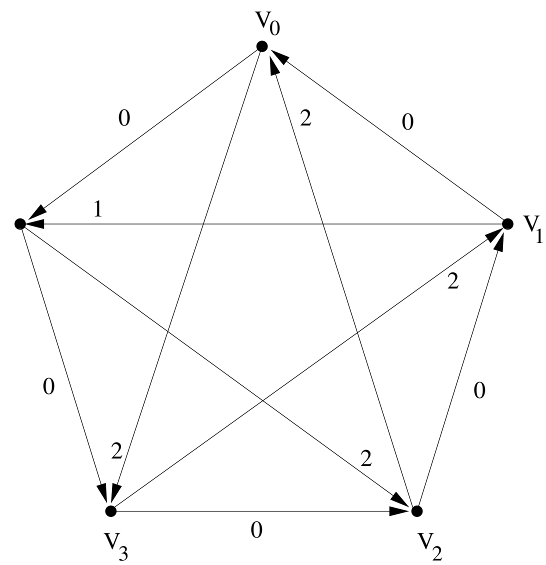

Figure 2.5: Thm. 2.18 $\not\Rightarrow$ Thm. 2.14

We now give Theorem 2.19.

Theorem 2.19. *Let T be a k-colored tournament on $n \geq 5$ vertices that is not a T^* so that all 4-cycles of T are monochromatic. Then T has a monochromatic sink.*

Proof. Let T be a k-colored tournament as described in the hypothesis. We argue by induction on the number of non-near-monochromatic 5-cycles in T that T has a monochromatic sink. For the base case, consider when there are zero non-near-monochromatic 5-cycles. Thus every 5-cycle of T is near-monochromatic and so by Result 2.18, T has a monochromatic sink. So assume there are $\ell > 0$ non-near-monochromatic 5-cycles and assume a tournament with less than ℓ non-near-monochromatic 5-cycles has a monochromatic sink. Let $C = v_0v_1v_2v_3v_4v_0$ be a 5-cycle of T that is non-near-monochromatic. First assume there are no square edges of C. Then for each $i \in \{0,1,2,3,4\}$, let $C_i = v_iv_{i-2}v_{i-1}v_{i+2}v_i$ be a 4-cycle of T. Each C_i is monochromatic. For each $i \in \{0,1,2,3,4\}$, $E(C_i) \cap E(C_{i+1}) = \{v_i v_{i-2}\}$, where addition is done modulo 4. Thus each C_i is colored the same and since $E(C) \subset \bigcup_{i=0}^{4} E(C_i)$, C is monochromatic, contradicting that C was non-near-monochromatic.

Assume, then, that C has at least two square edges. Without loss of generality, we let $v_0v_2 \in E(T)$ be a square edge of C. Additionally, let $i \in \{1,2,3,4\}$ so that $v_iv_{i+2} \in E(T)$. Then $C' = v_0v_2v_3v_4v_0$ and $C'' = v_iv_{i+2}v_{i+3}v_{i+4}v_i$ are 4-cycles. They are both monochromatic. Without loss of generality, C' is colored with 0 and v_0v_1 is colored with 1. Since $|E(C') \cap E(C'')| \geq 1$, C'' is also colored with 0. Since v_0v_1 or v_1v_2 (or both) is an edge of C'', we then have that C is near-monochromatic, a contradiction.

Finally, then, assume v_0v_2 is the only square edge. Again, let $C' = v_0v_2v_3v_4v_0$ be colored with 0 and v_0v_1 be colored with 1. The remaining edges of $T[C]$ are then v_3v_1, v_4v_2, v_0v_3, and v_1v_4. We will show that there is a monochromatic path colored 0 from v_0 to v_1. We will then recolor the edge v_0v_1 with 0. Then, we will show that the recoloring did not create any non-monochromatic 4-cycles or any non-near-monochromatic 5-cycles. Additionally, we will show that the recoloring did not create a T^*. By recoloring any 5-cycle that is not near-monochromatic, we may then apply Theorem 2.18 to the recolored tournament to obtain a monochromatic sink in T. Any monochromatic path that follows a recolored edge xy in the recolored tournament will instead follow the monochromatic path colored from x to y in T.

The 4-cycles $v_1v_4v_2v_3v_1$ and $v_0v_3v_1v_4v_0$ each have an edge colored 0. Therefore, all edges of the two 4-cycles are colored 0. Then $v_0 \xrightarrow{0} v_3 \xrightarrow{0} v_1$ is a monochromatic path colored 0 from v_0 to v_1. Obtain the tournament T' from T by recoloring v_0v_1 with 0. It is necessary to show that the recoloring did not create a T^*, any non-monochromatic 4-cycles, or any non-near-monochromatic 5-cycles.

First assume that the recoloring of v_0v_1 has created a non-monochromatic 4-cycle. This cycle must be colored 1 since v_0v_1 was originally colored 1. So there are vertices x and y so that $v_0v_1xyv_0$ is a 4-cycle where $v_0 \xrightarrow{0} v_1$ and $v_1 \xrightarrow{1} x \xrightarrow{1} y \xrightarrow{1} v_0$. First assume $x = v_2$. Consider the edge between y and v_1. If $y \to v_1$, then $yv_1v_4v_2y$ is a 4-cycle in T colored with more than one color, a contradiction. So assume $v_1 \to y$. But then $v_1yv_0v_3v_1$ is a 4-cycle colored with more than 1 color, a contradiction. So assume $x \neq v_2$. If $y \to v_3$, then yv_3v_1xy is a 4-cycle colored with more than 1 color, a contradiction. So then $v_3 \to y$. But then $v_3yv_0v_2v_3$ is a 4-cycle colored with more than 1 color, a contradiction. So, in any case, we arrive at a contradiction. The recoloring of v_0v_1 does not create a non-monochromatic 4-cycle.

Next, we show that T is not a T^*. It is enough to show a rainbow triangle was not created. Assume for contradiction that there exists $x \in V(T)$ so that $v_0v_1xv_1$ is a rainbow triangle in the recolored tournament. Let $\alpha, \beta \in \{0, 1, \ldots, k-1\}$ so that $v_1 \xrightarrow{\alpha} x \xrightarrow{\beta} v_0$. Note that $\alpha \neq \beta$ and $\alpha, \beta \neq 0$. Then $v_0v_3v_1xv_0$ is a 4-cycle colored with more than 1 color, a contradiction. Thus a rainbow triangle was not created by the recoloring of v_0v_1 and therefore T is not a T^*.

Finally, we show that no additional non-near-monochromatic 5-cycles were created. Assume there exist $x, y, z \in V(T)$ so that the recoloring of v_0v_1 from 1 to 0 creates a non-near-monochromatic 5-cycle $C' = v_0v_1xyzv_0$ that was originally near-monochromatic when v_0v_1 was colored with 1. We have two cases, $v_1v_2 \in E(C')$ or $v_1v_2 \notin E(C')$. First assume $v_1v_2 \in E(C')$. That is, $x = v_2$. Since $v_0 \xrightarrow{0} v_2$, the 4-cycle $v_0v_2yzv_0$ is colored with 0. But then C' is a near-monochromatic 5-cycle, which we assumed was not the case. So assume $v_1v_2 \notin E(C')$. If both $v_1 \stackrel{1}{\not\to} x$ and $x \stackrel{1}{\not\to} y$, then, since C' was near-monochromatic when $v_0 \xrightarrow{1} v_1$, we have that v_1x, xy, yz, and zv_0 are all colored the same, which contradicts that we assumed C' to be non-near-monochromatic after recoloring v_0v_1 with 0. We arrive at a similar contradiction if we assume that both $y \overset{1}{\not\to} z$ and $z \overset{1}{\not\to} v_0$. Thus we may say that the following is a fact: at least one of v_1x or xy is colored with 1 and at least one of yz or zv_0 is colored with 1. Consider the edge between y and v_3. If $y \to v_3$, then yv_3v_1xy is a non-monochromatic 4-cycle, a contradiction. So $v_3 \to y$. But then $v_3yzv_0v_3$ is a non-monochromatic 4-cycle, a contradiction .

There are now less than ℓ non-near-monochromatic 5-cycles after the recoloring of the edge v_0v_1. By the Induction Hypothesis, the recolored tournament has a monochromatic sink. If the edge v_0v_1 is used along a monochromatic path to the monochromatic sink in the recolored tournament, then the monochromatic path will instead use the path $v_0v_3v_1$ in T. Therefore, T has a monochromatic sink. $\square$

It is then natural to make the following conjecture.

Conjecture 2.20. *Let T be a k-colored tournament that is not a T^*. If there exists $\ell \in \{3,4,\ldots,n\}$ so that all ℓ-cycles are monochromatic, then T has a monochromatic sink.*

One of the difficulties, it seems, with answering this conjecture in the same manner that Theorem 2.19 was proven is that the proof of a base case, if we are to argue on the number of vertices, becomes quite complicated once the order of the tournament gets large. Further, in the proof of Theorem 2.19, it was necessary to use Theorem 2.18. If the same proof method were to be used for higher values of ℓ in the previous conjecture, then it would be necessary to prove results similar to Theorem 2.18 where the color restrictions were put on cycles larger than 4- and 5-cycles.

2.4 Semi-Cycles

Shen showed the following theorem in [14].

Theorem 2.21 (Shen, 1986, [14]). *If T is a k-colored tournament so that any subtournament of order 3 is near-monochromatic, then T has a monochromatic sink.*

We generalize the definition of the subtournaments of order 3 from Shen's result to include larger subgraphs.

Definition 2.22. *If C is a digraph that is either a cycle or has an edge xy so that $C - xy + yx$ is a cycle, then C is called a **semi-cycle**.*

A cycle is then a semi-cycle. A semi-cycle on s vertices is sometimes called an **s-semi-cycle**. The method for denoting semi-cycle varies. If an s-semi-cycle is just an s-cycle, then we denote the semi-cycle the same way we would a cycle. However, for an s-semi-cycle on vertices $\{v_0, v_1, \dots, v_{s-1}\}$ consisting of the path $v_0 v_1 \dots v_{s-1}$ and the edge $v_0 v_{s-1}$, we write $\mathrm{SC}(v_0 v_1 \dots v_{s-2}; v_{s-1})$. For example, in Figure 2.4, $\mathrm{SC}(v_3 v_0; v_2)$ is a 3-semi-cycle.

Theorems 2.23 and 2.24 are similar to Theorem 2.21, but instead of requiring all 3-semi-cycles to be near-monochromatic (as Shen did in Theorem 2.21), we require all 4-semi-cycles to be near-monochromatic (in Theorem 2.23) and all 5-semi-cycles to be near-monochromatic (in Theorem 2.24).

It is important to note that Theorem 2.19 is not proven by the following theorem. In Theorem 2.23, the fact that the coloring on semi-cycles - specifically, the semi-cycles which are not also cycles - is restricted puts additional requirements on the coloring that is not covered by the hypothesis of Theorem 2.19.

Theorem 2.23. *Let T be a k-colored tournament on $n \geq 4$ vertices so that all 4-semi-cycles are near-monochromatic. Then T has a monochromatic sink.*

Proof. Let T be a k-colored tournament as described in the hypothesis. We argue by induction on $|T|$ that T has a monochromatic sink. For the base case, let $|T| = 4$.

Recall that T is isomorphic to one of the four tournaments listed in Section 1.4. If T is transitive or a dominating triangle, then T is has a dominated vertex and this vertex is a monochromatic sink. If T is Hamiltonian, then by Fact 2.15, T has a monochromatic sink. So assume T is a dominated triangle. Let $v_1v_2v_3v_1$ be the triangle and let v_0 dominate every vertex of the triangle. Assume first that the triangle is a rainbow triangle. Any two edges from v_0 to the triangle must be colored the same as they create a 4-semi-cycle with two (differently colored) edges of the triangle. Thus all edges from v_0 are colored the same. Since the triangle is rainbow, there exist two edges not colored the same as the edges from v_0, say v_1v_2 and v_2v_3. Then $\mathrm{SC}(v_0v_1v_2;v_3)$ is 3-colored, a contradiction. Then we assume the triangle is not rainbow and so at least two edges of the triangle are colored the same, say again v_1v_2 and v_2v_3. Then v_3 is a monochromatic sink of T.

So assume $|T| > 4$ and assume that T has no monochromatic sink. By the Induction Hypothesis, T is a minimum counter example to the theorem. By Lemma 2.10, T has a dominating Hamiltonian cycle. Let $C = v_0v_1\dots v_{n-1}v_0$ be the Hamiltonian dominating cycle. If C is monochromatic, then every vertex along C is a monochromatic sink of T, a contradiction. So there must exist $i \in \{0,1,\dots,n-1\}$ so that v_iv_{i+1} and $v_{i+1}v_{i+2}$ are colored differently. Without loss of generality, say $v_0 \overset{0}{\to} v_1$ and $v_1 \overset{1}{\to} v_2$. Since C is a dominating cycle, there is a monochromatic path, $P = u_0u_1\dots u_t$, from $u_0 = v_2$ to $u_t = v_0$. If P is colored with 0, then $v_2Pv_0v_1$ is a monochromatic path from v_2 to v_1, a contradiction. Similarly, if P is colored with 1, then $v_1v_2Pv_0$ is a monochromatic path from v_1 to v_0, a contradiction. So assume, without loss of generality, that P is colored with 2. First consider when $|P| > 1$. Then $v_2u_1u_2v_1v_2$ is a cycle on 4 vertices or $\mathrm{SC}(v_2u_1u_2;v_1)$ is a semi-cycle on 4 vertices. In both cases, either $u_2 \overset{2}{\to} v_1$ or $v_1 \overset{2}{\to} u_2$. If $u_2 \overset{2}{\to} v_1$, then $v_2u_1u_2v_1$ is a monochromatic path from v_2 to v_1, a contradiction. So $v_1 \overset{2}{\to} u_2$. But then $v_1u_2Puv_0$

is a monochromatic path from v_1 to v_0, a contradiction. Now consider when $|P| = 1$. That is, $v_2 \xrightarrow{2} v_0$. If $v_3 \to v_0$, then $v_0v_1v_2v_3v_0$ is a 4-cycle and so either $v_1 \overset{1}{\mapsto} v_0$ or $v_2 \overset{0}{\mapsto} v_1$, a contradiction. So assume $v_0 \to v_3$. Note that since $\mathrm{SC}(v_0v_1v_2;v_3)$ is a 4-semi-cycle, and is therefore near-monochromatic, either $v_2 \xrightarrow{0} v_3$ or $v_2 \xrightarrow{1} v_3$. First assume $v_3 \to v_1$. Then $v_1v_2v_0v_3v_1$ is a 4-cycle with edges $v_1 \xrightarrow{1} v_2$ and $v_2 \xrightarrow{2} v_0$. Then either $v_0 \xrightarrow{2} v_3 \xrightarrow{2} v_1$ (in which case, $v_2 \overset{2}{\mapsto} v_1$) or $v_0 \xrightarrow{1} v_3 \xrightarrow{1} v_1$ (in which case, $v_3 \overset{1}{\mapsto} v_2$). In both cases, we arrive at a contradiction. So then assume $v_1 \to v_3$. Since $v_2 \xrightarrow{2} v_0, v_0 \xrightarrow{0} v_1$, either $v_2 \xrightarrow{0} v_3$ or $v_2 \xrightarrow{1} v_3$, and $\mathrm{SC}(v_2v_0v_1;v_3)$ is a 4-semi-cycle, then we conclude that $v_1 \xrightarrow{0} v_3$. But then $\mathrm{SC}(v_1v_2v_0;v_3)$ is non-near-monochromatic since $v_1 \xrightarrow{1} v_2, v_2 \xrightarrow{2} v_0$, and $v_1 \xrightarrow{0} v_3$, a contradiction. Thus there is no tournament whose 4-semi-cycles are near-monochromatic without a monochromatic sink. □

This result can also be shown when all 5-semi-cycles are near-monochromatic.

Theorem 2.24. *Let T be a k-colored tournament on $n \geq 5$ vertices so that all 5-semi-cycles are near-monochromatic. Then T has a monochromatic sink.*

Proof. Let T be a k-colored tournament on $n \geq 5$ vertices so that all 5-semi-cycles are near-monochromatic. We will argue by induction on $|T|$ that T has a monochromatic sink. For the base case, consider when $|T| = 5$. We have three cases: either T has a dominating vertex, a dominated vertex, or neither a dominating nor a dominated vertex.

Case 1: Assume T has a dominated vertex. Then this vertex is a monochromatic sink of T.

Case 2: Assume T has a dominating vertex, v_0, and no dominated vertex. The remaining four vertices induce a tournament of one of the four forms in Section 1.4. Let T_4 be the tournament on 4 vertices with vertices where $V(T_4) = \{v_1, v_2, v_3, v_4\}$. Since T_4 is assumed not to have a dominated vertex, it must be of the Hamiltonian form or a dominated triangle form. First assume T_4 is Hamiltonian. Let $C := v_1v_2v_3v_4v_1$ be a 4-cycle in T_4. If there exists $i \in \{1,2,3,4\}$ so that $v_iv_{i+1}, v_{i+1}v_{i+2}$, and $v_{i+2}v_{i+3}$ are all colored the same, then v_{i+3} is a monochromatic sink of T. So assume that is not the case. Since $\mathrm{SC}(v_0v_1v_2v_3;v_4)$ must be near-monochromatic, let at least 4 of its edges be colored, without loss of generality, with 0. Then exactly one of the edges v_1v_2, v_2v_3, or v_3v_4 is not colored with 0. Say, without loss of generality, it is colored with 1. Let v_iv_{i+1}, where $i \in \{1,2,3\}$ be this edge colored 1. Then the edges of $\mathrm{SC}(v_0v_iv_{i+1}v_{i+2};v_{i+3})$ are colored 0 with the exception of v_iv_{i+1}, which is colored with 1. Thus $v_{i+1}v_{i+2}v_{i+3}v_i$ is a monochromatic path (colored with 0), which contradicts our earlier assumption.

So then assume T_4 is a dominated triangle. Without loss of generality, let $v_0 \to v_1$ and let $v_2 \to v_3 \to v_4 \to v_2$. Additionally, let both v_0 and v_1 dominate v_2, v_3, and v_4. If there exists $i \in \{2,3,4\}$ so that $v_iv_{i+1}v_{i+2}$ is a monochromatic path, then v_{i+2} is a monochromatic sink of T. So assume the triangle $v_2v_3v_4v_2$ is rainbow. Say $v_2 \xrightarrow{0} v_3, v_3 \xrightarrow{1} v_4$, and $v_4 \xrightarrow{2} v_2$. We then look at three different 5-semi-cycles: $\mathrm{SC}(v_0v_1v_2v_3;v_4)$, $\mathrm{SC}(v_0v_1v_3v_4;v_2)$, and $\mathrm{SC}(v_0v_1v_4v_2;v_3)$. Exactly one edge of $\mathrm{SC}(v_0v_1v_2v_3;v_4)$ is colored differently than the rest with the two colors on $\mathrm{SC}(v_0v_1v_2v_3;v_4)$ being 0 and 1. Exactly one edge of $\mathrm{SC}(v_0v_1v_3v_4;v_2)$ is colored differently than the rest with the two colors on $\mathrm{SC}(v_0v_1v_3v_4;v_2)$ being 1 and 2. Exactly one edge of $\mathrm{SC}(v_0v_1v_4v_2;v_3)$ is colored differently than the rest with the two colors on $\mathrm{SC}(v_0v_1v_4v_3;v_3)$ being 0 and 2. Also note that two edges are shared between any pair of the 5-semi-cycles. Therefore, if 4 edges of $\mathrm{SC}(v_0v_1v_2v_3;v_4)$ are colored with 0, then this contradicts that no edges of $\mathrm{SC}(v_0v_1v_3v_4;v_2)$ are colored with 0. Also, if 4 edges of $\mathrm{SC}(v_0v_1v_2v_3;v_4)$ are colored with 1, then this contradicts that no edges of $\mathrm{SC}(v_0v_1v_4v_2;v_3)$ are colored with 1. In either case, we have arrived at a contradiction.

Case 3: Assume T does not have either a dominating nor a dominated vertex. Let T_4 be obtained by removing any vertex of T, say v_0. If T_4 is transitive, then let v_1 be the dominating vertex in T_4 and v_4 be the dominated vertex in T_4. Without loss of generality, let $v_2 \to v_3$. Since T has neither a dominating vertex nor a dominated vertex, we have that $v_4 \to v_0$ and $v_0 \to v_1$. Then $v_0v_1v_2v_3v_4v_0$ is a 5-cycle and by Fact 2.15, T has a monochromatic sink. So assume T_4 is Hamiltonian. Let $v_1v_2v_3v_4v_1$ be a 4-cycle of T. There then exists $i \in \{1,2,3,4\}$ so that $v_i \to v_0$ and $v_0 \to v_{i+1}$. Then $v_0v_{i+1}v_{i+2}v_{i+3}v_iv_0$ is a 5-cycle and by Fact 2.15, T has a monochromatic sink. So assume T_4 is a dominating triangle. Let $v_1 \to v_2 \to v_3 \to v_1$ where $v_i \to v_4$, for all $i \in \{1,2,3\}$. Since v_4 is not dominated in T, $v_4 \to v_0$. Since v_0 is not dominated in T, there exists $i \in \{1,2,3\}$ so that $v_0 \to v_i$. Then $v_0v_iv_{i+1}v_{i+2}v_4v_0$ is a 5-cycle and by Fact 2.15, T has a monochromatic sink. Finally assume that T_4 is a dominated triangle. Let $v_2 \to v_3 \to v_4 \to v_2$ where $v_1 \to v_i$, for all $i \in \{2,3,4\}$. Since v_1 is not dominating in T, $v_0 \to v_1$. Since v_0 is not dominating in T, there exists $i \in \{2,3,4\}$ so that $v_i \to v_0$. Then $v_1v_0v_iv_{i+1}v_{i+2}v_1$ is a 5-cycle of T and by Fact 2.15, T has a monochromatic sink.

Next assume $|T| > 5$ and assume that T does not have a monochromatic sink. By the Induction Hypothesis, T is a minimum counter example to the theorem. By Lemma 2.10, T has a Hamiltonian dominating cycle, $C = v_0v_1\dots v_{n-1}v_0$. If C is monochromatic, then every vertex of T is a monochromatic sink of T and we are done. So assume there are consecutive edges along C colored differently. Without loss of generality, say v_0v_1 and v_1v_2 are these edges and $v_0 \xrightarrow{0} v_1 \xrightarrow{1} v_2$. If $v_4 \to v_0$, then either $v_2 \xmapsto{0} v_1$ or $v_1 \xmapsto{1} v_0$, contradicting that C is a dominating cycle. So $v_0 \to v_4$. Let $P = u_0u_1\dots u_t$ be the monochromatic path from $u_0 = v_4$ to $u_t = v_0$. Note that $t > 1$ since $v_0 \to v_4$. Now $\mathrm{SC}(v_0v_1v_2v_3;v_4)$ has edges $v_0 \xrightarrow{0} v_1$ and $v_1 \xrightarrow{1} v_2$. So then either $v_2 \xrightarrow{1} v_3 \xrightarrow{1} v_4$ or $v_2 \xrightarrow{0} v_3 \xrightarrow{0} v_4$. First assume $v_2 \xrightarrow{1} v_3 \xrightarrow{1} v_4$. Then

SC$(v_2v_3v_4u_1;u_2)$ has two edges of P as well as $v_2 \xrightarrow{1} v_3 \xrightarrow{1} v_4$. Thus since both edges of P are colored the same, both are colored with 1. But then $v_1v_2v_3v_4Pv_0$ is a monochromatic path from v_1 to v_0, a contradiction. So instead assume $v_2 \xrightarrow{0} v_3 \xrightarrow{0} v_4$. Then similar reasoning yields that $v_2v_3v_4Pv_0v_1$ is a monochromatic path colored 0 from v_2 to v_1, a contradiction. Thus we conclude that T must have a monochromatic sink. □

It is worth noting that the Theorems 2.21 and 2.23 do not imply one another as well as Theorems 2.23 and 2.24 do not imply one another. Figure 2.6 is a tournament on 4 vertices where all 3-semi-cycles are near-monochromatic, however the same can not be said for all its 4-semi-cycles. In particular, the 4-cycle $v_0v_3v_2v_1v_0$ is 3-colored. Figure 2.7 is a tournament on 4 vertices where all 4-semi-cycles are near-monochromatic, but the same is not true for all its 3-semi-cycles. In particular, the 3-cycle $v_0v_3v_2v_0$ is 3-colored. Let **NM** mean near-monochromatic.

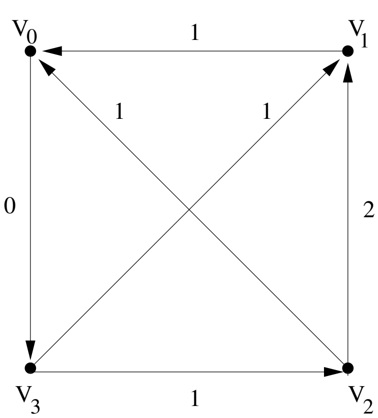

Figure 2.6: NM 3-semi-cycles.

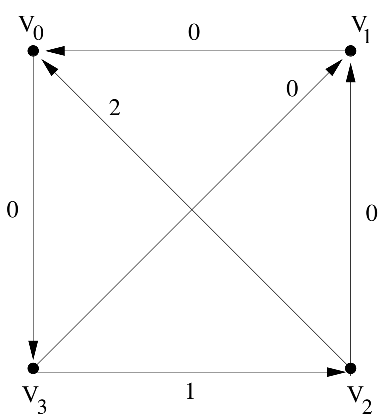

Figure 2.7: NM 4-semi-cycles.

We see similar results for 4- and 5-semi-cycles in Figures 2.8 and 2.9. Figure 2.8 is a tournament on 5 vertices where all 4-semi-cycles are near-monochromatic, however the same can not be said for all its 5-semi cycles. In particular, the 5-cycle $v_0v_1v_2v_3v_4v_0$ is non-near-monochromatic. Figure 2.9 is a tournament on 5 vertices where all 5-semi-cycles are near-monochromatic, but the same is not true for all its 4-semi-cycles. In particular, SC($v_1 v_2 v_3; v_4$) is non-near-monochromatic.

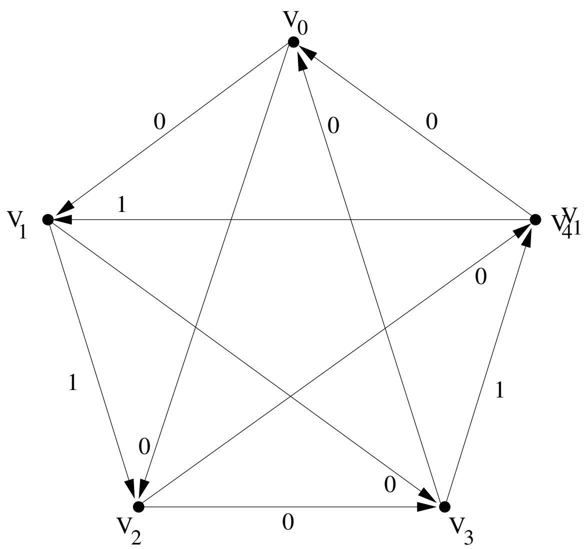

Figure 2.8: NM 4-semi-cycles.

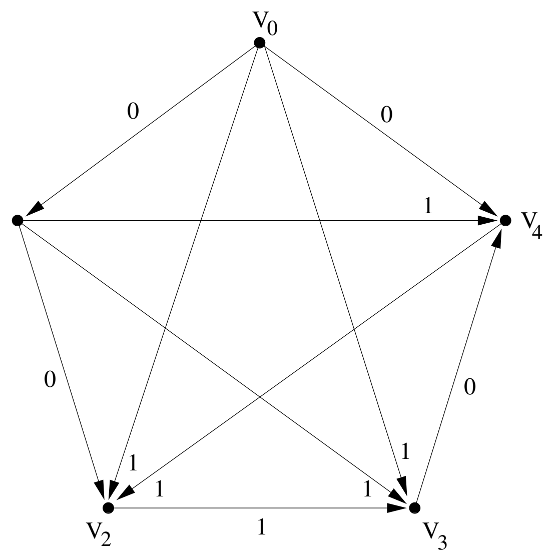

Figure 2.9: NM 5-semi-cycles.

Shen's result, along with Theorems 2.23 and 2.24, gives us the following theorem.

Theorem 2.25. *Let T be a k-colored tournament of order n. If there exists $\ell \in \{3,4,5\}$ so that every ℓ-semi-cycle of T is near-monochromatic, then T has a monochromatic sink.*

A natural conjecture to pose then is the following.

Conjecture 2.26. *Let T be a k-colored tournament of order n. If there exists $\ell \in \{3,4,\ldots,n\}$ so that every ℓ-semi-cycle of T is near-monochromatic, then T has a monochromatic sink.*

Similar to Conjecture 2.20, the main difficulty with answering this conjecture seems to be proving the base case for tournaments with large orders, assuming of course we are arguing on the number of vertices in the tournament, as we did in the proofs of Theorem 2.23 and 2.24. The induction step proves to be a simple argument, as it follows similar to the proof of Theorem 2.24. We make the following conjecture to be used in a proof for Conjecture 2.26.

Conjecture 2.27. *Let T be a k-colored tournament of order n. If all n-semi-cycles of T are near-monochromatic, then T has a monochromatic sink.*

If this conjecture is true, then we can answer Conjecture 2.26.

Theorem 2.28. *Let T be a k-colored tournament of order n. If there exists $\ell \in \{3,4,\dots,n\}$ so that every ℓ-semi-cycle of T is near-monochromatic and Conjecture 2.27 is true for $n=\ell$, then T has a monochromatic sink.*

Proof. Fix $\ell \geq 3$. Let T be a k-colored tournament of order n. Assume all ℓ-semi-cycles of T are near-monochromatic. We will argue by induction on n that T has a monochromatic sink. For the base case of $n=\ell$, T has a monochromatic sink (since Conjecture 2.27 is assumed to be true). So we assume $n>\ell$ and assume that T has no monochromatic sink. By the induction hypothesis, T is a minimum counter example to the theorem. By Lemma 2.10, T has a Hamiltonian dominating cycle, $C = v_0v_1\dots v_{n-1}v_0$. If C is monochromatic, then every vertex of T is a monochromatic sink of T and we are done. So assume there are consecutive edges along C colored differently. Without loss of generality, say v_0v_1 and v_1v_2 are these edges and $v_0 \xrightarrow{0} v_1 \xrightarrow{1} v_2$. If $v_{\ell-1} \to v_0$, then either $v_2 \overset{0}{\mapsto} v_1$ or $v_1 \overset{1}{\mapsto} v_0$, contradicting that C is a dominating cycle. So $v_0 \to v_{\ell-1}$. Let $P = u_0u_1\dots u_t$ be the monochromatic path from $u_0 = v_{\ell-1}$ to $u_t = v_0$. Note that $t>1$ since $v_0 \to v_{\ell-1}$. Now $\mathrm{SC}(v_0v_1\dots v_{\ell-2};v_{\ell-1})$ has edges $v_0 \xrightarrow{0} v_1$ and $v_1 \xrightarrow{1} v_2$. So then either $v_2 \xrightarrow{1} v_3 \xrightarrow{1} \cdots \xrightarrow{1} v_{\ell-1}$ or $v_2 \xrightarrow{0} v_3 \xrightarrow{0} \cdots \xrightarrow{0} v_{\ell-1}$. First assume $v_2 \xrightarrow{1} v_3 \xrightarrow{1} \cdots \xrightarrow{1} v_{\ell-1}$.

Then $\mathrm{SC}(v_2v_3\ldots v_{\ell-1}u_1;u_2)$ has two edges of P as well as $v_2 \xrightarrow{1} v_3 \xrightarrow{1} \cdots \xrightarrow{1} v_{\ell-1}$. Thus since both edges of P are colored the same, both are colored with 1. But then $v_1v_2\ldots v_{\ell-1}Pv_0$ is a monochromatic path from v_1 to v_0, a contradiction. So instead assume $v_2 \xrightarrow{0} v_3 \xrightarrow{0} \cdots \xrightarrow{0} v_{\ell-1}$. Then similar reasoning yields that $v_2v_3\ldots v_{\ell-1}Pv_0v_1$ is a monochromatic path colored 0 from v_2 to v_1, a contradiction. Thus we conclude that T must have a monochromatic sink. $\square$

At this point, it is necessary only to show that Conjecture 2.27 is true.

2.5 Nearly Transitive Tournaments

The term **nearly transitive tournament** was used by Melcher and Reid in [11] to describe a tournament which is transitive after changing the orientation of a *small* number of edges. We will prove two results regarding these nearly transitive tournaments.

Definition 2.29. *Let the vertices of T be ordered $(v_{n-1}, v_{n-2}, \ldots, v_1, v_0)$. We call T an* ***upset tournament*** *if the reversal of the edges along a path from v_0 to v_{n-1} results in a transitive tournament where v_{n-1} is the dominating vertex and v_0 is the dominated vertex.*

Melcher and Reid showed the following result regarding upset tournaments.

Theorem 2.30 (Melcher and Reid, 2010, [11])**.** *Let T be a k-colored tournament without rainbow triangles so that every strong component is either a single vertex or an upset tournament. Then T has a monochromatic sink.*

We will prove a slightly stronger statement in Theorem 2.33, but must first establish two definitions.

Definition 2.31. *If* $(v_{n-1}, v_{n-2}, \ldots, v_1, v_0)$ *is an ordering of vertices of a digraph D, call an edge* $v_i v_j$ *a* ***back-edge*** *if* $i < j$*.*

Definition 2.32. *If* $(v_{n-1}, v_{n-2}, \ldots, v_1, v_0)$ *is an ordering of vertices of a digraph D, a* ***back-path*** *is a path of D composed entirely of back-edges.*

We can now prove a slightly stronger result than that of Theorem 2.30. In addition, this proof is slightly smaller than that of Theorem 2.30 and is self-contained (that is, no additional lemmas are required for its proof).

Theorem 2.33. *Let* T *be a* k*-colored tournament without rainbow triangles with an ordering* $(v_{n-1}, v_{n-2}, \ldots, v_1, v_0)$ *of its vertices so that any two maximal back-paths are vertex disjoint. Then* T *has a monochromatic sink.*

Proof. Let T be a k-colored tournament as described in the hypothesis. We argue by induction on $|T|$ that T has a monochromatic sink. If $|T| = 1$, then the result is trivially true. So assume the result is true when the order of the tournament is less than $|T|$ and assume $|T| > 1$. If P and Q are any two maximal back-paths, $V(P) \cap V(Q) = \emptyset$. Thus for any $v \in V(T)$, there is at most one back-edge directed away from v and at most one back-edge directed into v. If $d^+(v_0) = 0$, then v_0 is a monochromatic sink. So assume there exists $i > 0$ so that $v_0 \to v_i$. Without loss of generality, $v_0 \xrightarrow{0} v_i$. If $v_i \mapsto v_0$, then v_0 is a monochromatic sink of T and we are done. So assume $v_i \not\mapsto v_0$. If there exists $j > 0$ so that $v_j \overset{0}{\mapsto} v_0$, then apply the induction hypothesis to $T - \{v_j\}$ to get a monochromatic sink v_m. If $m = 0$, then $v_j \mapsto v_0$ and therefore v_0 is a monochromatic sink of T. If $m > 0$, then $v_0 \overset{0}{\mapsto} v_m$ and thus $v_j \overset{0}{\mapsto} v_m$. Therefore v_m is a monochromatic sink of T. So assume for all $j > 0$, $v_j \overset{0}{\not\mapsto} v_0$. If there exists $i' < i$ so that $v_i \xrightarrow{1} v_{i'}$, then either $v_i \xrightarrow{1} v_{i'} \xrightarrow{1} v_0$, in which case v_0 is a monochromatic sink of T, or $v_i \xrightarrow{1} v_{i'} \xrightarrow{\ell} v_0 \xrightarrow{0} v_i$ (for $\ell \geq 2$), which contradicts that T has no rainbow triangles. If there exists $i' < i$ so that $v_i \xrightarrow{\ell} v_{i'}$, for some $\ell \geq 2$, then either $v_i \xrightarrow{\ell} v_{i'} \xrightarrow{\ell} v_0$, in which case v_0 is a monochromatic sink of T, or $v_i \xrightarrow{\ell} v_{i'} \xrightarrow{\ell'} v_0 \xrightarrow{0} v_i$, where $\ell' \notin \{0,2\}$, which contradicts that T has no rainbow triangles. Thus, we assume $v_i \xrightarrow{0} v_{i'}$ for all $0 < i' < i$. Apply the induction hypothesis to $T - \{v_0\}$ to get a monochromatic sink $v_{m'}$. If $m' \leq i$, then $v_0 \xrightarrow{0} v_i \xrightarrow{0} v_{m'}$ and so $v_{m'}$ is a monochromatic sink of T. So assume $m' > i$. If $v_i \overset{0}{\mapsto} v_{m'}$, then $v_0 \overset{0}{\mapsto} v_{m'}$ and so $v_{m'}$ is a monochromatic sink of T. So assume that $v_i \overset{0}{\not\mapsto} v_{m'}$. Without loss of generality, say $v_i \overset{1}{\mapsto} v_{m'}$. Since $v_i \xrightarrow{0} v_i'$ for all $0 < i' < i$, this implies that there exists $i'' > i$ so that $v_i \xrightarrow{1} v_{i''}$. Since $v_i \not\mapsto v_0$, this implies that $v_{i''} \overset{1}{\not\to} v_0$. To avoid a rainbow triangle, $v_{i''} \overset{\ell}{\not\to} v_0$, where $\ell > 1$. Therefore, $v_{i''} \xrightarrow{0} v_0$. But this has already been assumed to not be the case. Thus $m' \leq i$ and therefore $v_{m'}$ is a monochromatic sink of T. $\square$

We additionally prove the following result which restricts the colors on the back-edges.

Theorem 2.34. *Let T be a k-colored tournament without rainbow triangles with an ordering $(v_{n-1}, v_{n-2}, \ldots, v_1, v_0)$ of its vertices so that any back-edge of a vertex is colored the same as the other back-edges at that vertex. Then T has a monochromatic sink.*

Proof. Let T be a k-colored tournament as described in the hypothesis. We argue by induction on $|T|$ that T has a monochromatic sink. The base case of $|T| = 1$ is trivially true. So assume the result is true when the order of the tournament is less than $|T|$ and assume $|T| > 1$. If $d^+(v_0) = 0$, then v_0 is a monochromatic sink. So assume $d^+(v_0) > 0$. Without loss of generality, let 0 be the only color appearing on the back-edges of v_0. If there exists $i \in \{1, \ldots, n-1\}$ so that $v_i \overset{0}{\mapsto} v_0$, then we apply the induction hypothesis to $T-\{v_i\}$ to get a monochromatic sink, v_m. But then $v_i \overset{0}{\mapsto} v_0 \overset{0}{\mapsto} v_m$ and so v_m is a monochromatic sink of T. Assume then for all $i \in \{1,\ldots,n-1\}$, $v_i \overset{0}{\not\mapsto} v_0$. Let j be minimum so that $v_j \not\mapsto v_0$. If there is no such j, then v_0 is a monochromatic sink of T. So assume j exists. Then $v_0 \overset{0}{\to} v_j$. Apply the induction hypothesis to $T-\{v_0\}$ to get a monochromatic sink $v_{m'}$. If $m' = j$, then $v_0 \to v_{m'}$ and so we are done. So there exists a monochromatic path $P = u_0u_1\ldots u_s$ from $v_j = u_0$ to $v_{m'} = u_s$. If $v_j \overset{0}{\mapsto} v_{m'}$, we are done as then $v_0 \overset{0}{\to} v_j \overset{0}{\mapsto} v_{m'}$. So assume without loss of generality that the path is colored 1. We claim that for any $i \in \{0,1,\ldots,s\}$, $v_0 \overset{0}{\to} u_i$. We prove this claim with induction on $|P|$. For the base case when $|P| = 1$, if $v_{m'} \overset{\ell}{\to} v_0$, where $\ell > 1$, then $v_0v_jv_{m'}v_0$ is a rainbow triangle, and therefore $v_0 \overset{0}{\to} v_{m'}, v_{m'} \overset{0}{\to} v_0$, or $v_{m'} \overset{1}{\to} v_0$. It has been assumed that $v_{m'} \overset{0}{\not\to} v_0$. Also, if $v_{m'} \overset{1}{\to} v_0$, then $v_j \overset{1}{\mapsto} v_0$, which was also assumed not to be the case. Thus we conclude that $v_0 \overset{0}{\to} v_{m'}$. Now assume the claim is true for paths of length less than $|P|$ and assume $|P| > 1$. Consider $u_s \in V(P)$. If $u_s \overset{\ell}{\to} v_0$, for some $\ell > 1$, then $v_0u_{s-1}u_sv_0$ is a rainbow triangle and therefore $v_0 \overset{0}{\to} u_s, u_s \overset{0}{\to} v_0$, or $u_s \overset{1}{\to} v_0$. As seen earlier, it has been assumed that $u_s \overset{0}{\not\to} v_0$. Also, if $u_s \overset{1}{\to} v_0$, then $v_j \overset{1}{\mapsto} v_0$, which was also assumed not to be the case. Thus we conclude that $v_0 \overset{0}{\to} u_s$. Since $u_s = v_{m'}$, we conclude that $v_0 \to v_{m'}$ and so $v_{m'}$ is a monochromatic sink of T. $\square$

Chapter 3

RULING CLASSES

In Chapter 2, the goal was to find a monochromatic sink in a k-colored tournament with various restrictions on the coloring of its edges. In this chapter, we look at tournaments that may not necessarily have a monochromatic sink, but instead relax the definition of a monochromatic sink. We will look for a set of vertices in a tournament so that for every vertex in the tournament, it is either in the set or has a monochromatic path to a vertex in the set. We call a set of vertices that possess this property a ruling class.

Definition 3.1. *Let R be a set of vertices in a digraph D. The set R is called a* ***ruling class*** *of D if for every vertex v in D, either $v \in R$ or v has a monochromatic path to some vertex of R.*

Very little work has been done towards finding ruling classes in k-colored tournaments. We will present a question posed by Sands, Sauer, and Woodrow in [13] about the existence of ruling classes in k-colored tournament and a small result of Galeana-Sánchez's. After which, we will provide a theorem that partially answers the question asked by Sands, Sauer, and Woodrow and a conjecture that would give the size of a ruling class in a 3-colored tournament without rainbow triangles. In the next chapter, we will again relax the definition of a monochromatic sink by allowing color changes along the path to a vertex of the tournament.

3.1 Ruling Class Question

Consider the 3-colored tournament $T = (V,E)$ given by Sands, Sauer, and Woodrow in [13] where $V(T) = \{v_0, v_1, \ldots, v_8\}$ and $E(T)$ contains the following colored edges (as seen in Figure 3.1):

$$v_0v_1, v_3v_4, \text{ and } v_6v_7 \text{ colored with } 0,$$

$$v_1v_2, v_4v_5 \text{ and } v_7v_8 \text{ colored with } 1,$$

$$v_2v_0, v_5v_3 \text{ and } v_8v_6 \text{ colored with } 2,$$

$$v_iv_j \text{ colored with 0 for all } i \in \{0,1,2\} \text{ and } j \in \{3,4,5\},$$

$$v_iv_j \text{ colored with 1 for all } i \in \{3,4,5\} \text{ and } j \in \{6,7,8\}, \text{ and}$$

$$v_iv_j \text{ colored with 2 for all } i \in \{6,7,8\} \text{ and } j \in \{0,1,2\}.$$

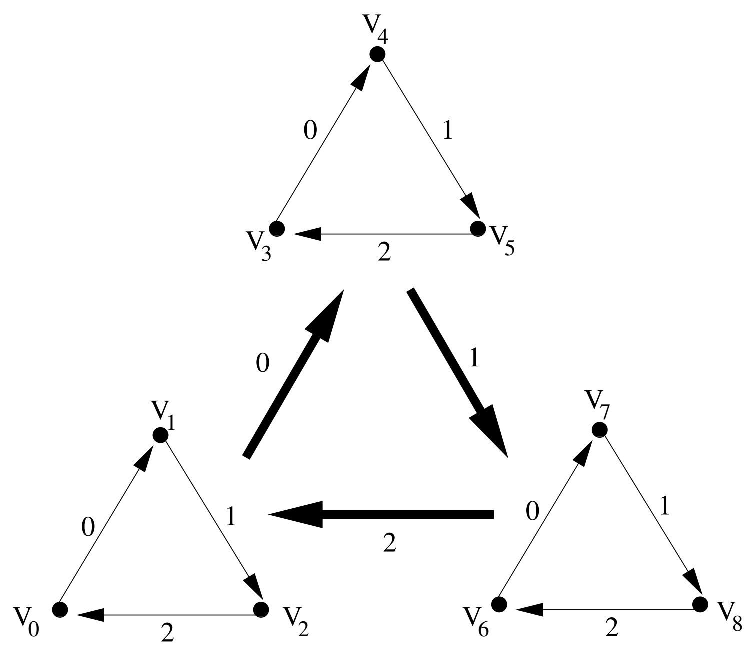

Figure 3.1: 3-colored tournament with ruling class of size 3.

This is a tournament on 9 vertices and has a minimum ruling class of size 3. A similar tournament could be created on $n > 9$ vertices by adding $n-9$ vertices to T that dominate the original 9 vertices of T. This tournament led Sands, Sauer, and Woodrow to ask the following question:

Question 3.2 (Sands, Sauer, and Woodrow, 1982, [13]). *For any integer k, is there a (least) positive integer $f(k)$ so that every k-colored tournament has a ruling class of size $f(k)$? In particular, is $f(3) = 3$?*

In Figure 3.1, we see that $f(3) \geq 3$. It remains to show $f(3) \leq 3$.

3.2 Previous Work

Other than the tournament given in Figure 3.1 that shows $f(3) \geq 3$, only one other result has been published which attempts to answer Question 3.2. In [7], Galeana-Sánchez and Rojas-Monroy restrict each vertex to be incident with edges of at most 2 different colors. They showed that in a k-colored tournament, where $k \geq 4$, this color restriction ensures that the tournament has a monochromatic sink.

Theorem 3.3 (Galeana-Sánchez and Rojas-Monroy, 2005, [7]). *If T is a k-colored tournament, where $k \geq 4$, so that every vertex is incident with edges of at most 2 colors, then T has a monochromatic sink.*

In a 3-colored tournament, this color restriction yields the following theorem.

Theorem 3.4 (Galeana-Sánchez and Rojas-Monroy, 2005, [7]). *Let T be a 3-colored tournament so that every vertex is incident with edges of at most 2 colors. Then T has a ruling class of size 3.*

We have now seen the extent of work published which attempts to answer Question 3.2.

3.3 Ruling Classes in 3-Colored Tournaments

In Theorem 3.7, we give a result that comes close to answering Question 3.2. Additionally, the proof provides a method that differs from the main method used in Chapter 2, which was to look at a minimum counter example and work within the

Hamiltonian dominating cycle that is guaranteed to exist. We also give a conjecture which would give a partial answer to Question 2.6. Many terms and theorems used in this section are from Section 1.6. We first need to give one more definition and a fact to be used in the proof of Theorem 3.7.

Definition 3.5. *Let D be an acyclic digraph. The* ***transitive closure*** *of D is the digraph $D' = D \cup \{uv : uv \notin E(D)$ and u has a path to v in D$\}$.*

The following is a fact similar to the 2-colored cycle restriction in the hypothesis of Theorem 3.7, but instead handles monochromatic cycles.

Fact 3.6. *If C is a monochromatic cycle, then for every edge $uv \in E(C)$, $v \mapsto u$.*

This is obvious from the fact that the cycle is monochromatic and therefore every vertex along the cycle can reach every other vertex by following the monochromatic path along C. We now give the main result of this section.

Theorem 3.7. *Let T be a 3-colored tournament so that if C is a 2-colored cycle, then there exists an edge $uv \in E(C)$ so that $v \mapsto u$. Then T has a ruling class of size 3.*

Proof. Let T be a 3-colored tournament as described in the hypothesis. We first give a quick outline to this proof. We will create three transitive tournaments from the edges of T. If an edge uv of T does not appear in at least 2 of the 3 transitive tournaments, this is because $v \mapsto u$ in T. From these three transitive tournaments, we can obtain 3 linear orders of the vertices, which will yield a 2-majority tournament T''. We can then conclude that if $uv \in E(T'')$, then $uv \in E(T)$ or $v \mapsto u$ in T. Thus the dominated set of size 3 in T'' obtained by Theorem 1.39 is then a ruling class of size 3 in T. We now begin the proof.

Let $T' = T - \{uv : v \mapsto u\}$. Note that any cycle of T' is 3-colored. For each $i \in \{0,1,2\}$, let $T_i = T' - \{uv : u \xrightarrow{i} v\}$. Note that each T_i is acyclic and is colored with at most 2 colors. Let T_i' be the transitive tournament guaranteed by Fact 1.36 after taking the transitive closure of each T_i. For each T_i, Fact 1.33 guarantees the existence of a linear order $<_i$ so that if $uv \in E(T_i)$, then $u <_i v$. Let T'' be the 2-majority tournament realized by these three linear orders. Each edge of T'' is an edge of T or represents a monochromatic path in T. By Theorem 1.39, T'' has a dominated set R of size at most 3. Thus R is a ruling class in T. □

Not only does Theorem 3.7 make some progress in answering if $f(3) \leq 3$ in Question 3.2, but it provides an opportunity to show if there exists a ruling class of size 3 in a 3-colored tournament without rainbow triangles (a tournament of the type described in Sands, Sauer, and Woodrow's first question - Question 2.6). It is with great confidence that we make the following conjecture.

Conjecture 3.8. *Let T be a 3-colored tournament without rainbow triangles so that T has a 2-colored Hamiltonian cycle C. Then there exists $uv \in E(C)$ so that $v \mapsto u$.*

Namely, this says that every 2-colored cycle C in a colored tournament without rainbow cycles has an edge uv so that $v \mapsto u$. This is particularly useful in that if Conjecture 3.8 is true, then in a 3-colored tournament without rainbow triangles, the hypothesis for Theorem 3.7 is satisfied. Therefore, the tournament would have a ruling class of size 3.

Theorem 3.9. *Let T be a k-colored tournament with no rainbow cycles. If Conjecture 3.8 is true, then T has a ruling class of size 3.*

Chapter 4

STEP-CHROMATIC SINKS

Up to this point in the thesis, we have concerned ourselves entirely with reachability along monochromatic paths. What if we allow for color changes along the path? Specifically, since we color the edges of a digraph with numbers, what if we allowed for color changes between consecutive edges along a path, but not by *too much*? As we did in Chapter 3, we relax the definition of a monochromatic sink. We will instead look for a vertex that can be reached by any other vertex in the tournament along a path with a special coloring: for any two consecutive edges along the path, the colors on the edges differ by at most some constant. The definition below allows us to work with paths of this type.

Definition 4.1. *For $\alpha \in \mathbb{N}$, a path P in a k-colored digraph is an **α-step-chromatic path** if for any two consecutive edges along P, $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$, if $v_i \xrightarrow{c} v_{i+1}$ for some $c \in \{0, 1, \ldots, k-1\}$, then $v_{i+1} \xrightarrow{c'} v_{i+2}$ for some $c' \in \{c, c+1, \ldots, c+\alpha\}$. (Recall that arithmetic done within a set of colors is done modulo k.)*

We can quickly establish the following fact about α-step-chromatic paths.

Fact 4.2. *If P is an α-step-chromatic path in a digraph D, then P is also a β-step-chromatic path in D, for any integer $\beta \geq \alpha$.*

In this chapter, we will seek the existence of α-step-chromatic sinks, which will be defined in the next section. Additionally, in the next section, we give some theorems that give a guide as to what needs to be proven regarding α-step-chromatic sinks. In Section 4.2, we will give a lemma similar to Lemma 2.10, which was used to prove all of the major theorems in Chapter 2. In the final section, we provide results regarding the existence of α-step-chromatic sinks in k-colored tournaments.

4.1 Step-Chromatic Sinks

We extend the notion of an α-step-chromatic path to the reachability definitions from earlier.

Definition 4.3. *For $\alpha \in \mathbb{N}$, an **α-step-chromatic sink** is a vertex in a digraph D so that every other vertex in D can reach it along an α-step-chromatic path.*

Note that a monochromatic sink is a 0-step-chromatic sink. Call a 1-step-chromatic sink a **step-chromatic sink**. We initially establish that some type of subtournament must be restricted to find a step-chromatic sink.

Theorem 4.4. *For any $n, k \geq 3$, there is a k-colored tournament T so that T does not have a step-chromatic sink.*

Proof. Let $T = (V, E)$ be a tournament with $V(T) = \{x, y, z\}$. Let $x \xrightarrow{1} y \xrightarrow{3} z \xrightarrow{2} x$. There is no step-chromatic sink in this tournament. If we wish to create a tournament on more than three vertices without a step-chromatic sink, add vertices one at a time while directing all edges to all existing vertices. (The added edges may be any color.) □

Specifically, the tournament in the last result is a T^*, as we have seen earlier in the thesis. The existence of a rainbow triangle in a T^* prevents the existence of any type of monochromatic sink and so, similarly, the existence of rainbow triangles in a colored tournament could possibly prevent the existence of an α-step-chromatic sink. If we consider k-colored tournaments which do not have rainbow cycles, we find quickly that there is not necessarily a step-chromatic sink.

Theorem 4.5. *For any $n, k \geq 5$, there is a k-colored tournament T without rainbow triangles that does not have a step-chromatic sink.*

Proof. The 5-colored tournament on five vertices given in Figure 4.5, which was obtained from Shen's tournament (Figure 2.2) by replacing color i with color $5 - i$, has no rainbow triangles and no step-chromatic sinks. If we wish to create a tournament on $n \geq 6$ vertices, then add vertices one at a time, directing all new edges to all existing vertices. Color these new edges with any color. This tournament still has no step-chromatic sink. $\square$

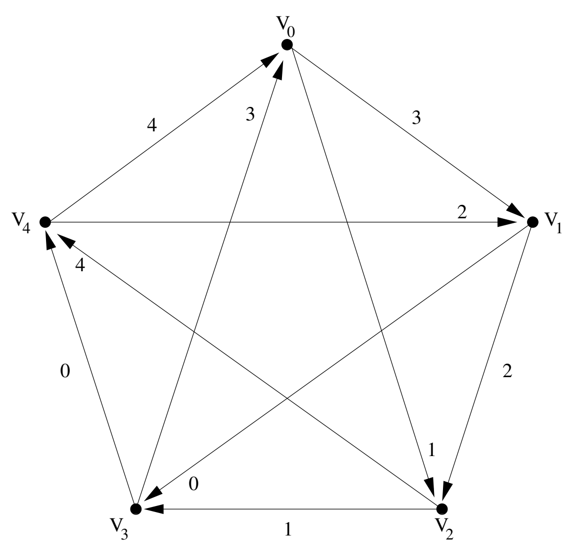

Figure 4.1: Tournament for Theorem 4.5.

By Theorem 2.4, a 2-colored tournament has a step-chromatic sink (since it has a monochromatic sink). We are left to answer whether a 3- or 4-colored tournament without rainbow triangles has a step-chromatic sink.

Question 4.6. *If T is a k-colored tournament without rainbow triangles, where $k \in \{3, 4\}$, does T have a step-chromatic sink?*

Because we are allowing color changes along our α-step-chromatic paths to find α-step-chromatic sinks, it makes sense then to not exclude all rainbow triangles, as we do in Chapter 2. In both Theorems 4.4 and 4.5, it isn't merely the existence of rainbow triangles that prevent an α-step-chromatic sink from existing. The existence a rainbow triangle with a particularly *bad* coloring prevents the existence of an α-step-chromatic sink. We give a definition for a triangle (not necessarily rainbow) with a particularly *good* coloring.

Definition 4.7. *Let $\ell \geq 0$. Let $C = xyzx$ be a triangle in a digraph with $x \xrightarrow{c_0} y \xrightarrow{c_1} z \xrightarrow{c_2} x$. If there exist distinct $i, j \in \{0,1,2\}$ so that $(c_i - c_{i-1}), (c_j - c_{j-1}) \leq \ell$, then C is a **Type-ℓ triangle**.*

That is, there exist two 2-paths along C, say xyz and yzx so that the colors increase by at most ℓ along each path. For example, in a k-colored tournament, the rainbow triangle $x \xrightarrow{0} y \xrightarrow{1} z \xrightarrow{2} x$ is a Type-1 triangle, but the the rainbow triangle $x \xrightarrow{2} y \xrightarrow{1} z \xrightarrow{0} x$ is not a Type-1 triangle. A monochromatic triangle is a Type-0 triangle. We see then in Theorems 4.4 and 4.5 that the tournaments contain triangles that are not Type-1. So we have two goals in this chapter. One, find the existence of α-step-chromatic sinks in k-colored tournaments without rainbow triangles and two, find the existence of ℓ-step-chromatic sinks in k-colored tournaments with only triangles of Type-ℓ.

4.2 Minimum Counter Example

In Chapter 2, the Hamiltonian dominating cycle that existed in many of the counter examples to the theorems provided the information necessary to prove the theorems. This chapter is similar in that a form of a dominating cycle exists in counter examples to the theorems and this cycle gives us the information needed for the proof.

So we must establish a similar definition to a dominating cycle when considering α-step-chromatic paths and sinks.

Definition 4.8. *For some integer $\alpha \geq 0$, a cycle $C = v_0 v_1 \ldots v_{s-1} v_0$ in a digraph D is an **α-step-dominating cycle** if for all $i \in \{0, 1, \ldots, s-1\}$, v_i is an α-step-chromatic sink in $D - v_{i-1}$, but there is no α-step-chromatic path from v_{i-1} to v_i in D.*

Similar to Lemma 2.2, we give the structure of a minimum k-colored tournament without an α-step-chromatic sink.

Lemma 4.9. *Let $\alpha \geq 0$. Let T be a minimum tournament with hereditary property $\mathscr{P}$ so that there is no α-step-chromatic sink. Then T has a Hamiltonian α-step-dominating cycle.*

Proof. Let $\alpha \geq 1$. Let T be a minimum tournament with hereditary property $\mathscr{P}$ so that there is no α-step-chromatic sink. Then for each $v \in V(T)$, there exists some vertex $f(v) \in V(T)$ so that x has an α-step-chromatic path to $f(v)$ for all $x \in V(T) \setminus \{v\}$. Since T has no α-step-chromatic sink, v has no α-step-chromatic path to $f(v)$. Thus $f(v) \to v$. Note also that for any two distinct vertices $u, v \in V(T)$, $f(u) \neq f(v)$, for otherwise $f(u)$ and $f(v)$ are both α-step-chromatic sinks in T, a contradiction. So we can assume that f is a bijection and that v does not have an α-step-chromatic path to $f(v)$ for all $v \in V(T)$. By the relabeling $f(v_i) = v_{i+1}$, $V(T)$ is partitioned into cycles

$$v_1 v_2 \ldots v_{s_1} v_1, \;\; v_{s_1+1} v_{s_1+2} \ldots v_{s_2} v_{s_1+1}, \;\; \ldots \; .$$

If there is more than one cycle, then consider the tournament with vertex set $\{v_1, v_2, \ldots, v_{s_1}\}$. Call this tournament T'. This is a smaller tournament than T, so there exists an α-step-chromatic sink v_i in T'. In particular, since $v_{i+1} \in V(T')$, this implies that v_{i+1}

has an α-step-chromatic path to v_i, which is a contradiction. Thus there is only one cycle and this cycle is an α-step-dominating cycle. $\square$

4.3 Results

We can now give the results regarding the existence of α-step-chromatic sinks in k-colored tournaments. The first theorem, Theorem 4.10, shows the existence of α-step-chromatic sinks in k-colored tournaments whose triangles are all of Type-α.

Theorem 4.10. *Let $\alpha \geq 0$. If T is a k-colored tournament on n vertices so that every triangle is a Type-α triangle, then T has an α-step-chromatic sink.*

Proof. Let $\alpha \in \mathbb{N}$. Let T be a k-colored tournament on n vertices so that any triangle is a Type-α triangle. We will argue by induction on n that T has an α-step-chromatic sink. The base case of $n = 1$ is trivial. Assume then that $n > 1$ and assume that T does not have an α-step-chromatic sink. By the induction hypothesis, T is then a minimum counter example to the Theorem. By Lemma 4.9, T has a Hamiltonian α-step-chromatic dominating cycle $C = v_0 v_1 \dots v_{n-1}$. Let $P = u_0 u_1 \dots u_t$ be a minimum α-step-chromatic path from $v_2 = u_0$ to $v_0 = u_t$. Let $i \in \{1, 2, \dots, t\}$ be minimum so that $u_i \to v_1$. Then $v_1 \to u_{i-1}$. There exist colors $a, b, c \in \{0, 1, \dots, k-1\}$ so that $v_1 \xrightarrow{a} u_{i-1}, u_{i-1} \xrightarrow{b} u_i$, and $u_i \xrightarrow{c} v_1$. Since every triangle is a Type-α triangle, either $b - a \leq \alpha$ or $c - b \leq \alpha$. In the former case, we then have that $v_1 u_{i-1} P v_0$ is an α-step-chromatic path from v_1 to v_0, a contradiction. In the latter case, we have that $v_2 P u_i v_1$ is an α-step-chromatic path from v_2 to v_1, a contradiction. So in either case we arrive at a contradiction and can conclude that T must have an α-step-chromatic sink. $\square$

Recall, from Chapter 2, Galeana-Sánchez's result (Theorem 2.13) that a k-colored tournament with monochromatic triangles has a monochromatic sink. Also recall that a Type-0 triangle is a monochromatic triangle and a 0-step-chromatic path is a monochromatic path. Using Theorem 4.10, we find the same result, which we will pose as the following corollary.

Corollary 4.11. *Let T be a k-colored tournament so that any triangle is a Type-0 triangle. Then T has a monochromatic sink.*

Theorem 4.10 requires every triangle in the tournament to be colored in a particular manner. If, much like in Chapter 2, we don't allow rainbow triangles at all in the tournament (but put no restriction on the coloring of the other triangles), then we can find an α-step-chromatic sink when α is big enough.

Theorem 4.12. *If T is a k-colored tournament without rainbow triangles, then there exists a $\lfloor k/2 \rfloor$-step-chromatic sink.*

Proof. Let T be a k-colored tournament on n vertices without rainbow cycles. We argue by induction on n that T has a $\lfloor k/2 \rfloor$-step-chromatic sink. The base case when $n = 1$ is trivial. So assume $n > 1$ and that T does not have a $\lfloor k/2 \rfloor$-step-chromatic sink. By the induction hypothesis, T is then a minimum counter example to the theorem. By Lemma 4.9, T has a $\lfloor k/2 \rfloor$-step-dominating Hamiltonian cycle, $C = v_0 v_1 \dots v_{n-1} v_0$. There exists $i \in \{0, 1, \dots, n-1\}$ so that $v_i \xrightarrow{j} v_{i+1}$ and $v_{i+1} \xrightarrow{j'} v_{i+2}$, for some $j' \in \{j + \lfloor k/2 \rfloor + 1, j + \lfloor k/2 \rfloor + 2, \dots, j-1\}$. Otherwise, every vertex can reach every other vertex along a $\lfloor k/2 \rfloor$-step-chromatic path, namely the Hamiltonian cycle, and this was assumed not to be the case. Then, without loss of generality, say $v_0 \xrightarrow{0} v_1$ and $v_1 \xrightarrow{j} v_2$, where $j \in \{\lfloor k/2 \rfloor + 1, \lfloor k/2 \rfloor + 2, \dots, k-1\}$. Let $P = u_0 u_1 \dots u_t$ be a minimum $\lfloor k/2 \rfloor$-step-chromatic path from $v_2 = u_0$ to $v_0 = u_t$. Let $i \in \{1, 2, \dots, t\}$ be minimum so that $u_i \to v_1$. Then $v_1 \to u_{i-1}$. There exist colors $a, b, c \in \{0, 1, \dots, k-1\}$ so that $v_1 \xrightarrow{a} u_{i-1}, u_{i-1} \xrightarrow{b} u_i$, and $u_i \xrightarrow{c} v_1$. Since T

has no rainbow triangles, at least one of the following cases is true: $a = b, b = c$, or $a = c$. If $a = b$, then $v_1u_{i-1}Pv_0$ is a $\lfloor k/2 \rfloor$-step-chromatic path from v_1 to v_0, a contradiction. If $b = c$, then $v_2Pu_iv_1$ is a $\lfloor k/2 \rfloor$-step-chromatic path from v_2 to v_1, a contradiction. Therefore, $a = c$. If $a \in \{b, b+1, \ldots, b+\lfloor k/2 \rfloor\}$, then $v_2Pu_iv_1$ is a $\lfloor k/2 \rfloor$-step-chromatic path from v_2 to v_1, a contradiction. So then $a \in \{b + \lfloor k/2 \rfloor + 1, b + \lfloor k/2 \rfloor + 2, \ldots, b-1\}$. But then $v_1u_{i-1}Pv_0$ is a $\lfloor k/2 \rfloor$-step-chromatic path from v_1 to v_0, a contradiction. In all cases, we arrive at a contradiction, thus it must be the case that there is no minimum k-colored tournament on n vertices without rainbow cycles that does not have a $\lfloor k/2 \rfloor$-step-chromatic sink. Thus the result is true. □

Thus, we see that when $k = 3$, there is a step-chromatic sink, which answers the question if a 3-colored tournament without rainbow triangles has a step-chromatic sink (which was part of Question 4.6).

Theorem 4.13. *If T is a 3-colored tournament without rainbow triangles, then T has a step-chromatic sink.*

We find that Theorem 4.12 guarantees the existence of a 2-step-chromatic sink in a 4-colored tournament without rainbow triangles, but we are still left with the following question (the remaining half of Question 4.6 not answered by Theorem 4.11).

Question 4.14. *If T is a 4-colored tournament without rainbow triangles, must T have a step-chromatic sink?*

Chapter 5

CONCLUSION

In this thesis, we provided two questions posed by Sands, Sauer, and Woodrow in [13] and the subsequent work done towards answering the questions. We provided our work towards answering the questions, which primarily built off of previous work done by Shen (in [14]), Galeana-Sánchez (in [5], [6], and [7]), Rojas-Monroy (in [6] and [7]), and Melcher and Reid (in [11]). Additionally, we introduced a proof method towards answering the second question (Question 3.2) asked by Sands, Sauer, and Woodrow. Finally, we introduced the concept of step-chromatic sinks and provided initial results about their existence in colored tournaments.

In Chapter 2, we presented three different types of results pertaining to the first question posed by Sands, Sauer, and Woodrow. First, we built off the results of Galeana-Sánchez and Rojas-Monroy. They had shown that that if all 3- and 4-cycles are near-monochromatic in a k-colored tournament, then there exists a monochromatic sink. We showed that if all 4- and 5-cycles are near-monochromatic in a k-colored tournament (that is not a T^*), then there exists a monochromatic sink. We showed that our result is independent of Galeana-Sánchez and Rojas-Monroy's result. Galeana-Sánchez also showed that if all 3-cycles are monochromatic in a k-colored tournament, then there exists a monochromatic sink. We showed that if all 4-cycles are monochromatic in a k-colored tournament that is not a T^*, there exists a monochromatic sink. This last result led to Conjecture 2.20, but the Conjecture seems difficult to prove given the rather lengthy arguments given in Theorem 2.13 and Theorem 2.19 when the cycles were of small size (3- and 4- cycles respectively).

Next in Chapter 2, we presented Shen's result that k-colored tournaments with near-monochromatic 3-semi-cycles have monochromatic sinks. We were able to show that the result is true if all 4-semi-cycles are near-monochromatic as well the result is true if all 5-semi-cycles are near-monochromatic. We showed that none of these results imply the other, so it is worth exploring whether it is true in tournaments whose ℓ-semi-cycles are near-monochromatic, for values of $\ell > 5$. Naturally, then, we conjecture that in a k-colored tournament whose ℓ-semi-cycles are near-monochromatic, for some $\ell \in \{3,4,\ldots,n\}$, there exists a monochromatic sink (Conjecture 2.26). Much like our concerns with proving Conjecture 2.20 to be true, proving the base case true is difficult when arguing by induction on the number of vertices. However, if the base case were to be true, then Conjecture 2.26 can be proven, and we showed this in Theorem 2.28. We are then left to prove that the base case is true and this is stated in Conjecture 2.27.

We also presented a result from Melcher and Reid that monochromatic sinks in nearly transitive tournaments. We slightly improved their result in Theorem 2.33 and presented a new result in Theorem 2.34.

In Chapter 3, we presented results towards answering the second question asked by Sands, Sauer, and Woodrow. Very little work has been done towards answering this question and we present this work (attributed to Galeana-Sánchez and Rojas-Monroy). We then show that there is a ruling class of size 3 in a 3-colored tournament with a small requirement on the 2-colored cycles in the tournament. There are two important things we can take from this result. The proof method we use, which includes the use of p-majority tournaments, is a new method which finally makes some progress on Sands, Sauer, and Woodrow's second question. In Chapter 2, we were able to use a dominating Hamiltonian cycle as a basis for our proofs. We don't have this luxury in Chapter 3. Thus, this new method that we have introduced finally gives us some structure to work with and gives hope of answering the question. Additionally, if we now consider k-colored tournaments without rainbow triangles, we conjecture that we can find a ruling class of size 3. The result is certainly less desireable as what is being sought after in all of Chapter 2 (which is just a single monochromatic sink), but it would be a result on the reachability of the tournament which doesn't put any additional restrictions on the coloring of the tournament other than "no rainbow triangles". In order to show this, it is only necessary to prove that if a 3-colored tournament without rainbow cycles has a 2-colored Hamiltonian cycle C, then there exists $uv \in E(C)$ so that $v \mapsto u$. We pose this as Conjecture 3.8.

Finally, we end the thesis with a new topic, α-step-chromatic sinks. We provide initial results towards the existence of α-step-chromatic sinks in a k-colored tournament. The results are proven using the main strategy from Chapter 2, but in this chapter, we instead use dominating step-chromatic Hamiltonian cycles within the proofs.

Throughout this thesis, two main proof methods are used to prove the results, dominating cycles (in Chapters 2 and 4) and p-majority tournaments (in Chapter 3). This is quite representative of the work already done towards answering these two questions. It would be useful to find another method to prove these results as this new method, much like our p-majority tournament method, could open up possibilities to prove stronger statements than what we have presented and proven in this thesis.

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