C*-Correspondences and Topological Dynamical Systems Associated to Generalizations of Directed Graphs

by

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ABSTRACT

In this thesis, I investigate the C^*-algebras and related constructions that arise from combinatorial structures such as directed graphs and their generalizations. I give a complete characterization of the C^*-correspondences associated to directed graphs as well as results about obstructions to a similar characterization of these objects for generalizations of directed graphs. Viewing the higher-dimensional analogues of directed graphs through the lens of product systems, I give a rigorous proof that topological k-graphs are essentially product systems over $\mathbb{N}^k$ of topological graphs. I introduce a "compactly aligned" condition for such product systems of graphs and show that this coincides with the similarly-named conditions for topological k-graphs and for the associated product systems over $\mathbb{N}^k$ of C^*-correspondences. Finally I consider the constructions arising from topological dynamical systems consisting of a locally compact Hausdorff space and k commuting local homeomorphisms. I show that in this case, the associated topological k-graph correspondence is isomorphic to the product system over $\mathbb{N}^k$ of C^*-correspondences arising from a related Exel-Larsen system. Moreover, I show that the topological k-graph C^*-algebra has a crossed product structure in the sense of Larsen.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Chapter 1

INTRODUCTION

1.1 History

An operator algebra is an algebra of continuous linear operators on a topological vector space where the multiplication is given by composition. The operator algebras that we are concerned with are norm-closed subalgebras of $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$. For each bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, there is an associated adjoint operator $T^* : \mathcal{H} \to \mathcal{H}$. This gives rise to a natural involution map on $\mathcal{B}(\mathcal{H})$ given by $T \mapsto T^*$. The *concrete* C^*-algebras are those subalgebras of $\mathcal{B}(\mathcal{H})$ that are closed under this involution and under the operator norm. C^*-algebras also have an abstract characterization as those Banach $*$-algebras with the property that $\|a^*a\| = \|a\|^2$, for all a. Familiar examples of C^*-algebras include $M_n(\mathbb{C})$, the $n \times n$ complex matrices; $\mathcal{K}(\mathcal{H})$, the compact operators on a Hilbert space $\mathcal{H}$; and $C_0(X)$, the continuous complex-valued functions that vanish at infinity on a locally compact Hausdorff space X. In fact, the third example completely characterizes commutative C^*-algebras.

The study of C^*-algebras has its roots in the 1920's and 1930's with efforts of Stone and von Neumann to mathematically model the quantum mechanical phenomenon of non-commuting observables. Further work by Murray and von Neumann led to a description of quantum mechanics in terms of what are now called von Neumann algebras, or W^*-algebras. In this theory, quantum observables are represented by self-adjoint operators on a Hilbert space and states as functionals over those observables.

In 1943, Gelfand and Neumark [22] gave an abstract definition of C^*-algebras and characterized them as subalgebras of the algebra of bounded operators on a Hilbert space. Segal built on these results and, in his 1947 paper [48], first coined the term C^*-algebras to describe uniformly closed, self-adjoint algebras of bounded operators on a Hilbert space. The next development of interest occurred in the 1970's when the area experienced a resurgence in interest and activity.

Graph algebras

Of particular relevance to the work in this dissertation are the so-called Cuntz algebras $\mathcal{O}_n$, first described by Cuntz in 1977 [8], which are generated by isometries $s_1, \ldots, s_n$ satisfying

$$\sum_{i=1}^{n} s_i s_i^* = 1. \tag{1.1}$$

In 1980, Cuntz and Krieger [9] generalized these algebras by considering C^*-algebras associated to square matrices with entries in $\{0, 1\}$ with no rows or columns consisting of all zeros. Given such a matrix $A = (a_{ij})$, the Cuntz-Krieger C^*-algebra $\mathcal{O}_A$ is generated by partial isometries $s_1, \ldots, s_n$ satisfying

$$s_i^* s_i = \sum_{j=1}^{n} a_{ij} s_j s_j^*, \qquad \text{for } i \in \{1, \ldots, n\}. \tag{1.2}$$

It was observed by Enomoto, Fujii, and Watatani [21, 11, 51] that the Cuntz-Krieger algebras $\mathcal{O}_A$ could be viewed as C^*-algebras associated to finite graphs. Given an $n \times n$ matrix $A = (a_{ij})$ with entries in $\{0, 1\}$, we may define a graph with n vertices such that there is an edge from vertex j to vertex i if and only if $a_{ij} = 1$. In their 1997 paper, Kumjian, Pask, Raeburn, and Renault [32] described C^*-algebras associated to directed graphs in detail. Their approach involved the path groupoid of the graph and techniques introduced by Renault to construct the groupoid C^*-algebra. Kumjian, Pask, and Raeburn furthered this work in [31] and realized the C^*-algebra of the directed graph in terms of generators and relations instead of following the groupoid approach. Building on work of Pimsner in [37] in which he shows how a C^*-correspondence may be associated to an $n \times n$ matrix with entries in $\{0, 1\}$, Fowler and Raeburn describe in [19] the graph correspondence associated to a directed graph. Additionally, they show that the graph C^*-algebra can be realized as the Cuntz-Pimsner algebra of the graph correspondence.

In 2000, a generalization of a directed graph was introduced by Kumjian and Pask in [30] who built on earlier work of Robertson and Steger [45, 46]. These *higher-rank graphs* replaced the notion of the length $|\mu| \in \mathbb{N}$ of a path μ with a higher-dimensional analogue: the degree $d(\mu) \in \mathbb{N}^k$, where k is the "rank" of the graph. Every directed graph E may be viewed as a 1-graph by considering the path category E^* of the graph. In this case, the degree of a path in E^* is given by its length as a path in E. In [30], Kumjian and Pask describe the C^*-algebra of a higher-rank graph in terms of generators and relations and also as the groupoid C^*-algebra associated to the path groupoid. In [20], Fowler and Sims show that a higher-rank graph may be viewed as a product system of graphs. Raeburn, Sims, and Yeend [40, 41, 49] show that the higher-rank graph C^*-algebra may be realized as the generalized Cuntz-Pimsner algebra of the associated product system of graph correspondences.

Another generalization of a directed graph is Katsura's notion of a *topological graph* (see [26, 27, 28, 29]) in which the edge and vertex sets are locally compact Hausdorff spaces, the range map is continuous and the source map is a local homeomorphism. When the vertex space is discrete then the edge space is necessarily discrete and this definition coincides with that of directed graphs as in [39]. Katsura constructs the topological graph C^*-algebra using an associated topological graph correspondence.

In his thesis [55], Yeend developed a generalization of both higher-rank graphs and topological graphs: *topological k-graphs.* He constructs the C^*-algebra of a topological k-graph using the groupoid approach. Recently, in [7], Carlsen, Larsen, Sims, and Vittadello show that the C^*-algebra of a topological k-graph may be constructed from an associated topological k-graph correspondence.

Crossed products

The study of crossed products dates back to work of Murray and von Neumann and their group measure space construction. Generally speaking, a crossed product C^*-algebra $A \rtimes_\alpha S$ is a C^*-algebra built out of a C^*-dynamical system (i.e., a C^*-algebra A and an action of a semigroup S as endomorphisms of A). In quantum theory, one may think of this as a C^*-algebra built from an algebra of observables and operators that describe the time evolution of a quantum system. The crossed product $A \rtimes_\alpha S$ then has the same representation theory as the original system.

Given a single endomorphism $\sigma \in \mathrm{End}(A)$, there is a natural semigroup action $\theta : \mathbb{N} \to \mathrm{End}(A)$ given by $\theta_n(a) = \sigma^n(a)$. In 1977, Cuntz [8] showed that his algebra $\mathcal{O}_n$ has the structure of a crossed product of a finite simple C^*-algebra by a single endomorphism. Since that time, there have been many efforts to develop a theory of crossed products of C^*-algebras by single endomorphisms as well as by semigroups of endomorphisms.

In his 2003 paper [12], Exel introduced a new definition of crossed product, generalizing some of the earlier constructions. Exel's construction involved not only a C^*-algebra A and an endomorphism α, but also a choice of *transfer operator* that acts as a sort of left inverse for α. Exel shows that the Cuntz-Krieger algebra $\mathcal{O}_A$ may be realized as the crossed product arising from the Markov sub-shift (Ω_A, σ) and a naturally-defined transfer operator.

In 2007 with Renault [14] and in 2008 [13], Exel considered an action of a semigroup P on a commutative unital C^*-algebra A via surjective local homeomorphisms. Exel and Renault showed in [14] that if $A = C(X)$ for a compact space X and if $\theta : P \to X$ is a right action of P on X (then $\alpha : P \to \mathrm{End}(A)$ is given by $\alpha_s(f) = f \circ \theta_s$ for $s \in P$), then the crossed product $C(X) \rtimes_\alpha P$ (Exel uses the notation $C(X) \rtimes_V G$, where V is a so-called *interaction group* and G is a group that P sits naturally inside of) is isomorphic to $C^*(\mathcal{G})$ where $\mathcal{G}$ is the associated *transformation groupoid*.

Extending Exel's construction to non-unital C^*-algebras, Brownlowe, Raeburn, and Vittadello in [6] model directed graph C^* algebras as crossed products. In particular, they show that if E is a locally finite directed graph with no sources, then $C^*(E) \cong C_0(E^\infty) \rtimes_{\alpha,L} \mathbb{N}$ where E^∞ is the infinite-path space of E and α is the shift map on E^∞.

In another extension of Exel's construction, Larsen (in [34]) develops a theory of crossed products associated to Exel-Larsen dynamical systems (A, S, α, L) where A is a (not necessarily unital) C^*-algebra, S is an abelian semigroup with identity, α is an action of S by endomorphisms, L is an action of S by transfer operators, and for all $s \in S$, the maps α_s, L_s are extendible to $M(A)$ in an appropriate sense.

1.2 Overview

In Chapter 2, we provide the preliminary information necessary to understand the rest of this thesis. We begin with definitions and some examples of right-Hilbert C^*-modules and of C^*-correspondences. We then give some results about the C^*-algebras associated to these constructions in terms of their representations. In Section 2.2, we define product systems of C^*-correspondences, which are higher-dimensional generalizations of C^*-correspondences. We discuss the representations of these objects and give results about the associated C^*-algebras. Finally, in Section 2.3, we give definitions and examples of directed graphs and their generalizations and describe the constructions of the associated graph C^*-algebras.

In Chapter 3, we investigate the connection between directed graphs and C^*-correspondences. We define the C^*-correspondence of a directed graph and show in Section 3.2 that every separable nondegenerate C^*-correspondence over a commutative C^*-algebra with discrete spectrum is isomorphic to a graph correspondence. In Section 3.3, we show that this relationship is functorial. In particular, we show that there is a functor between certain categories of graphs and C^*-correspondences that is very nearly a category equivalence.

In Chapter 4, we use the language of category theory to describe product systems. Similarly to a construction in [20], we describe a tensor groupoid in which the objects are topological graphs. In Section 4.3, we show that product systems over $\mathbb{N}^k$ taking values in this tensor groupoid are equivalent to topological k-graphs up to natural isomorphism. In Section 4.4, we discuss the compactly aligned condition for topological k-graphs through the lens of product systems. We introduce a notion of what it means for a product system of graphs to be compactly aligned and show that this is equivalent to the compactly aligned condition for related constructions.

In Chapter 5, we investigate C^*-correspondences associated to generalizations of directed graphs. While for both higher-rank graph correspondences and topological graph correspondences the most naïve characterization fails, we identify obstructions in special cases that may lead to a general characterization. In Section 5.1, we describe the *skeleton* of a k-graph correspondence in terms of the structure of the k-graph and use this to identify an invariant that characterizes certain product systems. In Section 5.2, we restrict our attention to C^*-correspondences that are symmetric $C_0(V)$-imprimitivity bimodules and identify an invariant, the *Picard invariant*, that characterizes these correspondences.

In Chapter 6, we consider topological k-graphs associated to topological dynamical systems. In Section 6.1, we construct a topological k-graph from a locally compact Hausdorff space and a family of commuting local homeomorphisms and show that this graph is always compactly aligned. We discuss the product system of C^*-

correspondences arising from a related Exel-Larsen system and show that it is isomorphic to the topological k-graph correspondence. Finally, in Section 6.3, we show that the topological k-graph C^*-algebra is isomorphic to the Larsen crossed product of the Exel-Larsen system.

Chapter 2

PRELIMINARIES

In this chapter we introduce the notation and terminology that will be used throughout this thesis. The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ refer, respectively, to the sets of non-negative integers, integers, real numbers, and complex numbers. We use the convention that $0 \in \mathbb{N}$. Given a Hilbert space $\mathcal{H}$, every bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is continuous and the space of all such operators forms a C^*-algebra denoted $\mathcal{B}(\mathcal{H})$. We denote by $\mathcal{K}(\mathcal{H})$ the subset of $\mathcal{B}(\mathcal{H})$ consisting of operators T that are compact in the sense that the image under T of the unit ball is a relatively compact subset of $\mathcal{H}$. This is a subalgebra of $\mathcal{B}(\mathcal{H})$ and, in fact, is closed under taking adjoints and closed in the norm topology, hence is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

2.1 C^*-correspondences and Cuntz-Pimsner algebras

A Hilbert C^*-module is an analogue of Hilbert space in which the field of scalars is no longer $\mathbb{R}$ or $\mathbb{C}$, but is instead a C^*-algebra. In this section, we give an overview of the definitions and results about Hilbert C^*-modules and C^*-correspondences that are relevant to this dissertation. For further detail, refer to [33, 43, 3].

Definition 2.1.1. Let A be a C^*-algebra. Let X be a complex vector space with a right action of A. Let $\langle \cdot, \cdot \rangle_A : X \times X \to A$ be an A-valued inner product that is conjugate linear in the first variable and linear in the second. That is, for $\xi, \eta \in X$ and $a \in A$ we have

1. $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$.
2. $\langle \xi, \eta \cdot a \rangle_A = \langle \xi, \eta \rangle_A a$,
3. $\langle \xi, \xi \rangle_A$ is a positive element of A, and

4. $\langle \xi, \xi \rangle_A = 0$ if and only if $\xi = 0$.

The formula $\|\xi\| = \|\langle \xi, \xi \rangle_A\|^{1/2}$ defines a norm on X. If X is complete with respect to this norm, we call it a *right-Hilbert A-module*.

We say that X is *full* if we have $A = \overline{\langle X, X \rangle_A}$ where

$$\langle X, X \rangle_A = \operatorname{span}\{\langle \xi, \eta \rangle_A : \xi, \eta \in X\}.$$

Example 2.1.2. Let A be a C^*-algebra. Then A_A is an A-module with right action given by $a \cdot b = ab$ for $a, b \in A$. We may define an A-valued inner product by

$$\langle a, b \rangle_A = a^* b \qquad \text{for } a, b \in A.$$

The norm coming from this inner product is the usual one on A since

$$\|a\|_A^2 = \|\langle a, a \rangle_A\| = \|a^* a\| = \|a\|^2.$$

It follows from the existence of approximate identities in A that A_A is full.

Note that if I is any proper, closed, two-sided ideal in A, then I_A is a Hilbert A-module that is *not* full since $\overline{\langle I_A, I_A \rangle_A} = I$.

Operators on Hilbert C^*-modules

Let A be a C^*-algebra and let X, Y be Hilbert A-modules.

Definition 2.1.3. We say that a map $T : X \to Y$ is *adjointable* if there exists a map $T^* : Y \to X$ satisfying

$$\langle T\xi, \eta \rangle_A = \langle \xi, T^* \eta \rangle_A, \qquad \text{for } \xi \in X, \eta \in Y. \tag{2.1}$$

It is known that every adjointable operator on X is norm-bounded and linear, and the adjoint T^* is unique (see [43, Lemma 2.18] for details).

The collection of all adjointable operators from X to Y is denoted $\mathcal{L}(X, Y)$. We write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. If $T \in \mathcal{L}(X)$, then $T^{**} = T$ so that $T \mapsto T^*$ is an involution on $\mathcal{L}(X)$. It is straightforward to verify that $\mathcal{L}(X)$ is a C^*-algebra with respect to the operator norm (see [43, Proposition 2.21]).

Example 2.1.4. Let $\mathcal{H}$ be a Hilbert space. Then we may regard $\mathcal{H}$ as a Hilbert $\mathbb{C}$-module and $\mathcal{L}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. As we are often interested in the subalgebra $\mathcal{K}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$, we would like an analogue of the ideal $\mathcal{K}(\mathcal{H})$ inside $\mathcal{L}(X)$ for an arbitrary Hilbert A-module X.

Given two elements $\xi, \eta \in X$, we may define an adjointable operator $\Theta_{\xi,\eta}$ on X via the formula

$$\Theta_{\xi,\eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle_A. \tag{2.2}$$

Operators of this form are called *generalized rank-one operators*. Note that

$$\begin{aligned}
\langle \Theta_{\xi,\eta}(\xi'), \eta' \rangle_A &= \langle \xi \cdot \langle \eta, \xi' \rangle_A, \eta' \rangle_A \\
&= \langle \eta, \xi' \rangle_A^* \langle \xi, \eta' \rangle_A \\
&= \langle \xi', \eta \rangle_A \langle \xi, \eta' \rangle_A \\
&= \langle \xi', \eta \cdot \langle \xi, \eta' \rangle_A \rangle_A \\
&= \langle \xi', \Theta_{\eta,\xi}(\eta') \rangle_A
\end{aligned}$$

and hence $\Theta_{\xi,\eta}$ is adjointable with $\Theta_{\xi,\eta}^* = \Theta_{\eta,\xi}$.

The subspace

$$\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in X\} \tag{2.3}$$

is an essential ideal of $\mathcal{L}(X)$ whose elements are called the *generalized compact operators* on X.

Example 2.1.5. Let $X = A_A$. There is an isomorphism of A onto $\mathcal{K}(X)$ as follows. Given $a \in A$, let $L_a : X \to X$ be given by left multiplication by a. Then

L_a is adjointable with $L_a^* = L_{a^*}$. It is straightforward to show that $\|L_a\| = \|a\|$ and it follows that $L : a \mapsto L_a$ is an isomorphism of A onto a closed $*$-subalgebra of $\mathcal{L}(X)$. Since

$$\Theta_{a,b}(c) = a\langle b, c\rangle_A = ab^*c = L_{ab^*}(c),$$

we have that $\mathcal{K}(X)$ is the closure of the image under L of the linear span of products in A. It follows since A has approximate identities that these products are dense in A. Hence $\mathcal{K}(X) \cong A$.

C^*-correspondences

Definition 2.1.6. Let A be a C^*-algebra. An A-*correspondence* is a right-Hilbert A module together with a homomorphism $\phi : A \to \mathcal{L}(X)$ that we think of as implementing a left action of A on X. For convenience, we write $a \cdot \xi$ in place of $\phi(a)\xi$. We say that an A-correspondence is *nondegenerate* (also sometimes called *essential*) if the homomorphism giving the left action of A on X is nondegenerate, that is, if

$$\phi(A)X = X.$$

The A-correspondences defined above sometimes are referred to as right-Hilbert A-A bimodules in the literature. We use the term A-correspondence here in part because we feel it will make this document considerably easier to read.

Example 2.1.7. Given any C^*-algebra A, we may view A as a nondegenerate A-correspondence with bimodule actions given by multiplication in A and inner product given by $\langle a, b\rangle_A = a^*b$. Notationally, when we are considering A as an A-correspondence, we write ${}_AA_A$.

Definition 2.1.8. A *Toeplitz representation* (ψ, π) of an A-correspondence X in a C^*-algebra B is a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ satisfying

$$\psi(\xi \cdot a) = \psi(\xi)\pi(a), \tag{T1}$$

$$\psi(\xi)^*\psi(\eta) = \pi(\langle \xi, \eta \rangle_A), \text{ and} \tag{T2}$$

$$\psi(a \cdot \xi) = \pi(a)\psi(\xi) \tag{T3}$$

for $\xi, \eta \in X$ and $a \in A$.

It follows from (T2) and the automatic continuity of the homomorphism π that the linear map ψ is continuous for the norm $\| \cdot \|_A$ on X, and that ψ is isometric if π is injective.

Proposition 2.1.9. *[19, Proposition 1.3] Let X be an A-correspondence. Then there is a C^*-algebra $\mathcal{T}_X$ called the* Toeplitz algebra *and a Toeplitz representation $(i_X, i_A) : X \to \mathcal{T}_X$ such that*

- *for every Toeplitz representation (ψ, π) of X, there is a homomorphism $\psi \times \pi$ of $\mathcal{T}_X$ such that:*

$$(\psi \times \pi) \circ i_X = \psi$$

$$(\psi \times \pi) \circ i_A = \pi; \text{ and}$$

- *$\mathcal{T}_X$ is generated as a C^*-algebra by $i_X(X) \cup i_A(A)$.*

The triple $(\mathcal{T}_X, i_X, i_A)$ is unique: if (B, i'_X, i'_A) has similar properties, there is an isomorphism $\theta : \mathcal{T}_X \to B$ such that $\theta \circ i_X = i'_X$ and $\theta \circ i_A = i'_A$. Both maps i_X and i_A are injective. There is a strongly continuous action called the gauge action

$\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{T}_X$ *such that* $\gamma_z(i_A(a)) = i_A(a)$ *and* $\gamma_z(i_X(\xi)) = z i_X(\xi)$ *for* $a \in A$, $\xi \in X$, $z \in \mathbb{T}$.

Existence of the Toeplitz algebra depends on the existence of nontrivial representations of X. To establish that we have nontrivial representations of X, we describe one such representation described by Pimsner in [37]: the Fock representation.

Example 2.1.10. The Fock space of an A-correspondence X is the right-Hilbert A-module

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} X^{\otimes n}$$

where $X^{\otimes 0} = A$, $X^{\otimes 1} = X$, and for $n \geq 2$, $X^{\otimes n}$ is the n-fold tensor product (the tensor product of A-correspondences is described on page 17)

$$X^{\otimes n} = X \otimes_A \cdots \otimes_A X.$$

Note that $\mathcal{E}_+$ is also an A-correspondence with left action given by $\phi_+(a)b = ab$ for $a, b \in A = X^{\otimes 0}$ and

$$\phi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a \in A$ and $\xi_1 \otimes \cdots \otimes \xi_n \in X^{\otimes n}$.

For each $\xi \in X$, we define a *creation operator* $T_\xi \in \mathcal{L}(\mathcal{E}_+)$ by

$$T_\xi(\mu) = \xi \otimes \mu \in X^{\otimes(n+1)}$$

for $\mu \in X^{\otimes n}$. Routine calculations show that T_ξ is adjointable and

$$T_\xi^*(\mu) = \begin{cases} 0, & \text{if } \mu \in X^{\otimes 0} = A \\ \langle \xi, \mu_1 \rangle_A \cdot \mu_2 \otimes \cdots \otimes \mu_n, & \text{if } \mu = \mu_1 \otimes \cdots \otimes \mu_n \in X^{\otimes n}. \end{cases}$$

Let π_0 be a representation of A on $\mathcal{H}$. We may induce a representation $\mathcal{E}_+$-$\operatorname{Ind}_A^{\mathcal{L}(\mathcal{E}_+)} \pi_0$ of $\mathcal{L}(\mathcal{E}_+)$ on $\mathcal{E}_+ \otimes_A \mathcal{H}$ (as in [43, Proposition 2.66]) and then restrict to obtain a representation of A

$$\pi := \mathcal{E}_+ \operatorname{Ind}_A^A \pi_0|_A.$$

Let $\psi : X \to \mathcal{B}(\mathcal{E}_+ \otimes_A \mathcal{H})$ be given by

$$\psi(\xi) = \mathcal{E}_+ \operatorname{Ind}_A^{\mathcal{L}(\mathcal{E}_+)} \pi_0(T_\xi).$$

Then (ψ, π) is a Toeplitz representation of X, called the *Fock representation induced from* π_0. The representation (ψ, π) is injective whenever π_0 is.

Proposition 2.1.11. *[39, Proposition 8.9] Let X be a correspondence over a C^*-algebra A. Then*

$$\mathcal{T}_X = \overline{\operatorname{span}}\{i_X^{\otimes m}(\xi) i_X^{\otimes n}(\eta)^* : m, n \geq 0, \xi \in X^{\otimes m}, \eta \in X^{\otimes n}\}.$$

Associated to a Toeplitz representation (ψ, π) of an A-correspondence X in a C^*-algebra B is a homomorphism $\pi^{(1)} : \mathcal{K}(X) \to B$ such that

$$\pi^{(1)}(\Theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*, \qquad \text{for } \xi, \eta \in X. \tag{2.4}$$

We then have that

- $\pi^{(1)}(T)\psi(\xi) = \psi(T\xi)$ for $T \in \mathcal{K}(X)$ and $\xi \in X$;
- if ρ is a homomorphism of B into another C^*-algebra C, then $(\rho \circ \pi)^{(1)} = \rho \circ \pi^{(1)}$;
- if π is faithful, so is $\pi^{(1)}$.

Definition 2.1.12. If X is an A-correspondence, $J(X) = \phi^{-1}(\mathcal{K}(X))$ is a closed two-sided ideal in A. Given an ideal K in $J(X)$, a Toeplitz representation (ψ, π) of X is said to be *coisometric on* K if for every $a \in K$ we have

$$\pi^{(1)}(\phi(a)) = \pi(a). \tag{2.5}$$

A Toeplitz representation is said to be *Cuntz-Pimsner covariant* if (2.5) holds for every $a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^{\perp}$, where

$$(\ker \phi)^{\perp} = \{a \in A : ab = 0 \text{ for all } b \in \ker \phi\}.$$

Remark 2.1.13. Different definitions exist in the literature for Cuntz-Pimsner covariant representations. The one we use above is sometimes referred to as the "Katsura convention" and differs from that in [17], for example, in which (2.5) is required to hold for $a \in \phi^{-1}(\mathcal{K}(X))$. When the left action is injective, the two definitions coincide.

Definition 2.1.14. The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the quotient of $\mathcal{T}_X$ by the ideal generated by

$$\{i_A^{(1)}(\phi(a)) - i_A(a) : a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^{\perp}\}.$$

If $q : \mathcal{T}_X \to \mathcal{O}_X$ is the quotient map, then $(j_X, j_A) := (q \circ i_X, q \circ i_A)$ is a Cuntz-Pimsner covariant representation of X in $\mathcal{O}_X$ which is universal for Cuntz-Pimsner covariant representations of X.

Proposition 2.1.15. *[17, Proposition 1.3] Let X be an A-correspondence, and let K be an ideal in $J(X) = \phi^{-1}(\mathcal{K}(X))$. Then there is a C^*-algebra $\mathcal{O}(K, X)$ and a Toeplitz representation $(k_X, k_A) : X \to \mathcal{O}(K, X)$ called the* relative Cuntz-Pimsner algebra determined by K *which is coisometric on K and satisfies*

1. *for every Toeplitz representation (ψ, π) of X that is coisometric on K, there is a homomorphism $\psi \times_K \pi$ of $\mathcal{O}(K, X)$ such that $(\psi \times_K \pi) \circ k_X = \psi$ and $(\psi \times_K \pi) \circ k_A = \pi$; and*

2. *$\mathcal{O}(K, X)$ is generated as a C^*-algebra by $k_X(X) \cup k_A(A)$.*

The triple $(\mathcal{O}(K, X), k_X, k_A)$ is unique in the sense that if (B, k'_X, k'_A) has similar properties, there is an isomorphism $\theta : \mathcal{O}(K, X) \to B$ such that $\theta \circ k_X = k'_X$ and

$\theta \circ k_A = k'_A$. *There is a strongly continuous* gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}(K, X)$ *satisfying* $\gamma_z(k_A(a)) = k_A(a)$ *and* $\gamma_z(k_X(\xi)) = zk_X(\xi)$ *for* $a \in A$ *and* $\xi \in X$, $z \in \mathbb{T}$.

When $K = \{0\}$, the relative Cuntz-Pimsner algebra determined by K coincides with the Toeplitz algebra $\mathcal{T}_X$ and when $K = J(X) \cap (\ker \phi)^{\perp}$ it is the Cuntz-Pimsner algebra $\mathcal{O}_X$.

2.2 Product systems and generalized Cuntz-Pimsner algebras

Arveson [2] introduced continuous product systems in 1989 to develop an index theory for continuous semigroups of endomorphisms of $\mathcal{B}(\mathcal{H})$. Dinh [10] continued this work and introduced discrete product systems. Product systems over arbitrary semigroups were introduced by Fowler and Raeburn in [18] in 1998. In 2002, Fowler [16] introduced the notion of a discrete product system whose fibres are C^*-correspondences.

In this thesis, we are primarily concerned with product systems of C^*-correspondences. However, in [20], Fowler and Sims used the language of category theory to define a product system taking values in a tensor groupoid. After introducing a tensor groupoid in which the objects are directed graphs, they state without proof that product systems over $\mathbb{N}^k$ taking values in this tensor groupoid are essentially discrete k-graphs. In Chapter 4, we give a rigorous proof of this and extend the result further: using a tensor groupoid in which the objects are topological graphs, we show that product systems over $\mathbb{N}^k$ taking values in this tensor groupoid are equivalent to topological k-graphs up to natural isomorphism.

Product systems of C^*-correspondences

Given two A-correspondences X, Y, we let $X \odot Y$ denote the algebraic tensor product of X and Y (as complex vector spaces). So that the tensor product will be balanced over A, we quotient out by the subspace of $X \odot Y$ spanned by vectors of the form

$$\xi \cdot a \odot \eta - \xi \odot a \cdot \eta \qquad \text{for } \xi \in X, \eta \in Y, a \in A$$

and let $X \odot_A Y$ denote this quotient space. The equation

$$\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle_A = \langle \eta_1, \langle \xi_1, \xi_2 \rangle_A \cdot \eta_2 \rangle_A \tag{2.6}$$

gives a well-defined bounded sesquilinear form on $X \odot_A Y$. Let $N = \text{span}\{n \in X \odot_A Y : \langle n, n \rangle_A = 0_A\}$. Then $\|z + N\| = \inf\{\|\langle z + n, z + n \rangle_A\|^{1/2} : n \in N\}$ defines a norm on $(X \odot_A Y)/N$. The tensor product $X \otimes_A Y$ is the completion of $(X \odot_A Y)/N$ with respect to this norm.

If X, Y are A-correspondences and $S \in \mathcal{L}(X)$, then there is an operator $S \otimes 1_Y$ on $X \otimes_A Y$ defined by

$$(S \otimes_A 1_Y)(\xi \otimes_A \eta) = S\xi \otimes_A \eta, \qquad \text{for } \xi \in X, \eta \in Y.$$

The map $S \otimes_A 1_Y$ is adjointable, with adjoint given by $S^* \otimes_A 1_Y$.

The map $a \mapsto \phi(a) \otimes 1_Y$ is a homomorphism of A into $\mathcal{L}(X, Y)$ and the left action of A on $X \otimes_A Y$ is implemented by this homomorphism.

Definition 2.2.1. Let P be a countable semigroup with identity e, and let A be a C^*-algebra. A *product system over P of A-correspondences* is a pair (X, β) consisting of a collection $X = \{X_p : p \in P\}$ of A-correspondences and a collection $\beta = \{\beta_{p,q} : p, q \in P \setminus \{e\}\}$ of A-correspondence isomorphisms $\beta_{p,q} : X_p \otimes_A X_q \to X_{p,q}$ satisfying

$$\beta_{p,q}(\xi \otimes_A \eta) = \xi\eta, \qquad \text{for } \xi \in X_p, \eta \in X_q \tag{2.7}$$

such that

1. $X_e = A$ (viewed as an A-correspondence).

2. Multiplication in X by elements of $X_e = A$ induces maps $\beta_{p,e} : X_p \otimes_A X_e \to X_p$ and $\beta_{e,p} : X_e \otimes_A X_p \to X_p$ as in Equation 2.7. For each $p \in P$, $\beta_{p,e}$ is automatically an isomorphism (by [43, Corollary 2.7]). We do not require $\beta_{e,p}$ be an isomorphism as this is too restrictive if we want to allow for C^*-correspondences that are not essential (i.e., for which we do not necessarily have $\overline{\phi(A)X} = X$).

3. The associativity condition

$$\beta_{pq,r} \circ (\beta_{p,q} \otimes_A \mathrm{id}_r) = \beta_{p,qr} \circ (\mathrm{id}_p \otimes_A \beta_{q,r}) \tag{2.8}$$

   holds for all $p, q, r \in P$.

The collection $X = \{X_p : p \in P\}$ is in fact a semigroup with multiplication given by the maps $\beta_{p,q}$. Product systems over P of A-correspondences may equivalently be defined via a semigroup homomorphism $d : X \to P$ such that $d^{-1}(p) = X_p$ is an A-correspondence for each $p \in P$. Because of this, we will often use the semigroup X to refer to the product system (X, β).

When $P = \mathbb{N}^k$ (in fact, when P is any right-angled Artin semigroup) we may consider a product system (X, β) over P in terms of its *skeleton*. We describe the skeleton of a product system over $\mathbb{N}^k$ below. For details about more general right-angled Artin semigroups, refer to [20].

Definition 2.2.2. Let (X, β) be any product system over $\mathbb{N}^k$ of A-correspondences, and let $\{e_i \mid 1 \leq i \leq k\}$ denote the standard basis for $\mathbb{N}^k$. Then setting

$$Y_i = X_{e_i} \quad \text{and} \quad T_{i,j} = \beta_{e_j,e_i}^{-1} \circ \beta_{e_i,e_j} : Y_i \otimes_A Y_j \to Y_j \otimes_A Y_i$$

gives a collection $Y = \{Y_i \mid 1 \leq i \leq k\}$ of A-correspondences and a collection $T = \{T_{i,j} \mid 1 \leq i < j \leq k\}$ of A-correspondence isomorphisms such that the *hexagonal equation*

$$\begin{aligned}(T_{j,\ell}\otimes \mathrm{id}_i)(\mathrm{id}_j\otimes T_{i,\ell})(T_{i,j}\otimes \mathrm{id}_\ell)& \\ &= (\mathrm{id}_\ell\otimes T_{i,j})(T_{i,\ell}\otimes \mathrm{id}_j)(\mathrm{id}_i\otimes T_{j,\ell})\end{aligned} \tag{2.9}$$

holds for all $1 \le i < j < \ell \le k$, where id_i is the identity map on Y_i. (Note that both sides of Equation 2.9 are isomorphisms $Y_i \otimes_A Y_j \otimes_A Y_\ell \to Y_\ell \otimes_A Y_j \otimes_A Y_i$.) We call the pair (Y, T) the *skeleton* of (X, β).

It follows from [20, Proposition 2.11] (see also Remark 2.2.3 below) that a product system is uniquely determined, up to isomorphism, by its skeleton. More precisely, if (Z, γ) is another product system over $\mathbb{N}^k$ of A-correspondences, with skeleton (W, R), then $(X, \beta) \cong (Z, \gamma)$ if and only if $(Y, T) \cong (W, R)$ in the sense that there exist isomorphisms $\theta_i \colon Y_i \to W_i$ such that the diagram in Figure 2.1 commutes for each $1 \le i < j \le k$:

$$\begin{tikzcd} Y_i\otimes Y_j \arrow[r, "T_{i,j}"] \arrow[d, "\theta_i\otimes\theta_j"'] & Y_j\otimes Y_i \arrow[d, "\theta_j\otimes\theta_i"] \\ W_i\otimes W_j \arrow[r, "R_{i,j}"] & W_j\otimes W_i. \end{tikzcd}$$

Figure 2.1: Isomorphic product systems

Remark 2.2.3. Observe that for $k \le 2$, there are no hexagonal equations Equation 2.9, and for $k = 1$ there are no $T_{i,j}$'s. Also, defining $T_{j,i} = T_{i,j}^{-1}$ for $i < j$, we have a collection $\{T_{i,j} \mid 1 \le i \ne j \le k\}$ of correspondence isomorphisms that satisfy the hexagonal equations for all distinct i, j, l, as in [20]. But it is convenient to note that we only need the $T_{i,j}$'s for $i < j$.

Let S be a countable semigroup with identity e. Given $s, t \in S$ with $s \ne e$, we use the isomorphism $\beta_{s,t} : X_s \otimes_A X_t \to X_{st}$ to define a homomorphism $\iota_s^{st} : \mathcal{L}(X_s) \to$

$\mathcal{L}(X_{st})$ by

$$\iota_s^{st}(S)(\xi\eta) = (S\xi)\eta, \qquad \text{for } \xi \in X_s, \eta \in X_t, S \in \mathcal{L}(X_s). \tag{2.10}$$

When $s = e$, we define ι_e^t on $\mathcal{K}(X_e) \cong A$ by

$$\iota_e^t(a) = \phi_t(a), \qquad \text{for } a \in A. \tag{2.11}$$

If $s, u \in P$ with $u \neq st$ for all $t \in P$, we define $\iota_s^u : \mathcal{L}(X_s) \to \mathcal{L}(X_u)$ to be the zero map $\iota_s^u(S) = 0_{\mathcal{L}(X_u)}$ for all $S \in \mathcal{L}(X_s)$.

Throughout the rest of this section, we focus our attention on semigroups P that are part of a *quasi-lattice ordered group* (G, P). Introduced by Nica in [36], the class of such (G, P) includes all direct sums and free products of totally ordered groups.

Definition 2.2.4. A *quasi-lattice ordered group* is a pair (G, P) consisting of a discrete group G and a subsemigroup P of G such that $P \cap P^{-1} = \{e\}$ and, with respect to the partial order $p \leq q \iff p^{-1}q \in P$, any two elements $p, q \in G$ which have a common upper bound in P have a least common upper bound $p \vee q \in P$.

We write $p \vee q = \infty$ to indicate that $p, q \in G$ have no common upper bound in P, and we write $p \vee q < \infty$ otherwise.

Example 2.2.5. It is an easy exercise to see that $(\mathbb{Z}^2, \mathbb{N}^2)$ is a quasi-lattice ordered group since $\mathbb{N}^2 \cap (\mathbb{N}^2)^{-1} = \{0, 0\}$ and every (a_1, b_1), (a_2, b_2) in $\mathbb{Z}^2$ has least upper bound

$$(\max\{0, a_1, a_2\}, \max\{0, b_1, b_2\}) \in \mathbb{N}^2.$$

Example 2.2.6. For a more interesting example of a quasi-lattice ordered group, let A be a finite set. Let $\mathbb{F}_A$ be the free group on A and let $\mathbb{F}_A^+$ be the subsemigroup of $\mathbb{F}_A$ generated by A. Then $(\mathbb{F}_A, \mathbb{F}_A^+)$ is a quasi-lattice ordered group. To see this, note that if $p, q \in \mathbb{F}_A$ with $p \vee q < \infty$, then we must have $r \leq |p|$ and $s \leq |q|$ such that

$$p = \alpha_1 \cdots \alpha_r \beta_{r+1} \cdots \beta_{|p|}$$

$$q = \alpha_1 \cdots \alpha_s \gamma_{s+1} \cdots \gamma_{|q|}$$

where $\alpha_1, \ldots, \alpha_{\max\{r,s\}} \in A$ and $\beta_{r+1}, \ldots, \beta_{|p|}, \gamma_{s+1}, \ldots, \gamma_{|q|} \in A^{-1}$. Then an upper bound of p and q is

$$\alpha_1 \cdots \alpha_{\max\{r,s\}}.$$

In fact, this is the least upper bound in $\mathbb{F}_A^+$ since for any $t \in \mathbb{F}_A^+$, we have

$$\{s \in \mathbb{F}_A^+ : s \leq t\} = \{e, t_1, t_1 t_2, \ldots, t_1 \cdots t_{|t|} = t\}$$

and hence the set is totally ordered.

Representations of product systems

Definition 2.2.7. Let A be a C^*-algebra. Let S be a countable semigroup with identity and let X be a product system over S of A-correspondences. A *(Toeplitz) representation* of X in a C^*-algebra B is a map $\psi : X \to B$ such that

1. For each $s \in S$, the pair $(\psi_s, \psi_e) := (\psi|_{X_s}, \psi|_{X_e})$ is a Toeplitz representation of X_s as in Definition 2.1.8, and
2. $\psi(\xi\eta) = \psi(\xi)\psi(\eta)$, for $\xi, \eta \in X$.

For each $s \in S$, let $\psi^{(s)} : \mathcal{K}(X_s) \to B$ be the homomorphism associated to (ψ_s, ψ_e) as in (2.4). That is, $\psi^{(s)}$ satisfies

$$\psi^{(s)}(\Theta_{\xi,\eta}) = \psi_s(\xi)\psi_s(\eta)^* \qquad \text{for } \xi, \eta \in X_s.$$

Definition 2.2.8. Let A be a C^*-algebra. Let S be a countable semigroup with identity and let X be a product system over S of A-correspondences. A (Toeplitz) representation $\psi : X \to B$ is said to be *Cuntz-Pimsner covariant* if for each $s \in S$ the Toeplitz representation (ψ_s, ψ_e) is Cuntz-Pimsner covariant. That is, for each $s \in S$ we have

$$\psi^{(s)}(\phi_s(a)) = \psi_e(a) \qquad \text{for } a \in \phi_s^{-1}(\mathcal{K}(X_s)) \cap (\ker \phi_s)^{\perp}, \tag{CP-K}$$

where $\phi_s : A \to \mathcal{L}(X_s)$ is the homomorphism giving the left action of A on X_s.

As was the case with representations of C^*-correspondences, we also have a notion of what it means for a representation of a product system to be coisometric.

Definition 2.2.9. Let A be a C^*-algebra. Let S be a countable semigroup with identity and let X be a product system over S of A-correspondences. For each $s \in S$, let K_s be an ideal in $\phi_s^{-1}(\mathcal{K}(X_s))$. A (Toeplitz) representation $\psi : X \to B$ is said to be *coisometric on* $K = \{K_s\}_{s\in S}$ if each (ψ_s, ψ_0) is coisometric on K_s in the sense of Definition 2.1.12, that is, if

$$\psi^{(s)}(\phi_s(a)) = \psi_0(a), \text{ for all } a \in K_s.$$

In [16], Fowler imposed a condition called *Nica covariance* on (Toeplitz) representations that is automatic if the quasi-lattice ordered group (G, P) is totally ordered. Nica covariance was originally defined for representations of a product system on Hilbert space (see [16, Definition 5.1]) and the notion of *compactly aligned* was introduced in order to obtain a C^*-algebraic characterization of Nica covariance.

Definition 2.2.10. Given a quasi-lattice ordered group (G, P) and a product system over P of A-correspondences, we say that X is *compactly aligned* if whenever $p \vee q < \infty$ we have

$$\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q}) \qquad \text{for } S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q).$$

Definition 2.2.11. Let A be a C^*-algebra. Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of A-correspondences. A (Toeplitz) representation $\psi : X \to B$ is said to be *Nica covariant* if, for each $p, q \in P$ and for all $S \in \mathcal{K}(X_p),\ T \in \mathcal{K}(X_q)$, we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}(\iota_p^{(p\vee q)}(S)\iota_q^{(p\vee q)}(T)), & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{N}$$

Remark 2.2.12. The definition of Nica covariance above differs slightly from the one appearing in [16, Definition 5.7]. The reason for this is that Definition 2.2.11 above allows for C^*-correspondences that are not necessarily nondegenerate; that is, for which we do not necessarily have $\phi(A)X = X$.

In [49], Sims and Yeend introduced a new notion of Cuntz-Pimsner covariance for product systems (X, β) of compactly aligned A-correspondences. In order to define their notion of Cuntz-Pimsner covariance, we require some definitions. We will also give some results about when the two notions of Cuntz-Pimsner covariance coincide.

Definition 2.2.13. Let (G, P) be a quasi-lattice ordered group. We say that a statement $\mathcal{P}(s)$ (where $s \in P$) is true *for large s* if:

$$\forall q \in P,\ \exists r \in P,\ \text{such that } q \leq r \quad \text{and} \quad \forall s \geq r,\ \mathcal{P}(s) \text{ is true.}$$

Definition 2.2.14. Given a quasi-lattice ordered group (G, P) and a product system (X, β) over P of A-correspondences, define $I_e = A$ and for $p \in P \setminus \{e\}$ define $I_p = \bigcap_{e<r\leq p} \ker(\phi_r)$. Note that I_p is an ideal of A. For $q \in P$, define

$$\widetilde{X}_q = \bigoplus_{p \leq q} X_p \cdot I_{p^{-1}q}.$$

Each $\widetilde{X}_q$ is an A-correspondence with left action implemented by $\widetilde{\phi}_q : A \to \mathcal{L}(\widetilde{X}_q)$ where

$$(\widetilde{\phi}_q(a)\xi)(p) = \phi_p(a)\xi(p), \qquad \text{for } p \leq q.$$

Definition 2.2.15. Let (G, P) be a quasi-lattice ordered group and let (X, β) be a compactly aligned product system over P of A-correspondences such that $\widetilde{\phi}_q$ is injective for each $q \in P$. A (Toeplitz) representation $\psi : X \to B$ of X in a C^*-algebra B is said to be *Cuntz-Pimsner covariant* (in the sense of Sims and Yeend in [49, Definition 3.9]) if

$$\sum_{p \in F} \phi^{(p)}(T_p) = 0_B \text{ for every finite set } F \subset P \text{ and every choice of generalized}$$

$$\text{compact operators } \{T_p \in \mathcal{K}(X_p) : p \in F\} \text{ such that } \sum_{p \in F} \widetilde{\iota}_p^s(T_p) = 0 \text{ for large } s. \tag{CP-SY}$$

The notion of Cuntz-Pimsner covariance described above looks significantly different from (CP-K). The formulation of this definition was based on intuition of Sims and Yeend about the connection of product systems of C^*-correspondences to k-graph C^*-algebras. The following example should help show how the $\widetilde{X}_q$ approximate a notional "boundary" of the product system. The condition (CP-SY) encodes relations that one would expect to hold if we could make sense of the boundary of X and let the $\mathcal{K}(X_p)$ act on it.

Example 2.2.16. [49, Example 3.3] Let Λ be a finitely aligned k-graph and let $X = X(\Lambda)$ be the corresponding product system over $\mathbb{N}^k$ of $c_0(\Lambda^0)$-correspondences as in [40]. For $n \in \mathbb{N}^k$, we have $I_n = \overline{\operatorname{span}}\{\delta_v : v \in \Lambda^0, v\Lambda^m = \emptyset \text{ for all } m \leq n\}$. For $m \leq n \in \mathbb{N}^k$, we therefore have

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\delta_\mu : \mu \in \Lambda^m, s(\mu)\Lambda^{e_i} = \emptyset \text{ whenever } m_i < n_i\}.$$

In the language of [41, 42], this spanning set is familiar: $\delta_\mu \in X_m \cdot I_{n-m}$ if and only if $\mu \in \Lambda^m \cap \Lambda^{\leq n}$. That is, as a vector space, $\widetilde{X}_n = C_0(\Lambda^{\leq n})$.

In [49], Sims and Yeend describe conditions under which (CP-K) and (CP-SY) coincide.

Proposition 2.2.17. *[49, Proposition 5.1] Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of A-correspondences. Suppose that each pair in P has a least upper bound. Suppose that for each $p \in P$, the homomorphism $\phi_p : A \to \mathcal{L}(X_p)$ is injective. Let $\psi : X \to B$ be a (Toeplitz) representation of X.*

- *If ψ satisfies* (CP-SY) *then it satisfies* (CP-K)*.*
- *If $\phi_p(A) \subset \mathcal{K}(X_p)$ for each $p \in P$, and ψ satisfies* (CP-K)*, then ψ satisfies* (CP-SY)*.*

Definition 2.2.18. We call a representation that satisfies both (N) and (CP-SY) a *Cuntz-Nica-Pimsner covariant (or CNP-covariant) representation.*

Corollary 2.2.19. *[49, Corollary 5.2] Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of A-correspondences. Suppose that each pair in P has a least upper bound. Suppose that each $\phi_p : A \to \mathcal{L}(X_p)$ is injective with $\phi_p(A) \subset \mathcal{K}(X_p)$. Let $\phi : X \to B$ be a representation of X. Then ϕ is CNP-covariant if and only if $\phi^{(p)} \circ \psi_p = \phi_e$ for all $p \in P$, that is, if and only if ψ satisfies* (CP-K)*.*

Generalized Cuntz-Pimsner algebras

We now describe several C^*-algebras that may be associated to a given product system of C^*-correspondences. As was the case with C^*-algebras associated to a single C^*-correspondence, these are often defined in terms of their universal properties. Unless stated otherwise, the semigroups are countable with identity e and A is a C^*-algebra.

Proposition 2.2.20. *[16, Proposition 2.8] Let X be a product system over P of A-correspondences. Then there is a C^*-algebra $\mathcal{T}_X$, called the* Toeplitz algebra of *X, and a (Toeplitz) representation $i_X : X \to \mathcal{T}_X$, such that*

1. *for every (Toeplitz) representation ψ of X, there is a homomorphism ψ_* of $\mathcal{T}_X$ such that $\psi_* \circ i_X = \psi$; and*

2. *$\mathcal{T}_X$ is generated as a C^*-algebra by $i_X(X)$.*

The pair $(\mathcal{T}_X, i_X)$ is unique up to canonical isomorphism, and i_X is isometric.

Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of A-correspondences. Recalling the definition of Nica covariance for a (Toeplitz) representation in Definition 2.2.11, we let $\mathcal{T}_{cov}(X)$ be the quotient of $\mathcal{T}_X$ by the ideal generated by the elements

$$i_X^{(p)}(S)i_X^{(q)}(T) - i_X^{(p\vee q)}(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T))$$

where $p, q \in P$, $S \in \mathcal{K}(X_p)$, and $T \in \mathcal{K}(X_q)$. The composition of the quotient map from $\mathcal{T}_X$ onto $\mathcal{T}_{cov}(X)$ with i_X is universal for Nica covariant (Toeplitz) representations of X. Moreover, if $\psi : X \to B$ is a Nica covariant (Toeplitz) representation of X that generates B as a C^*-algebra, then

$$B = \overline{\operatorname{span}}\{\psi(\xi)\psi(\eta)^* : \xi, \eta \in X\}. \tag{2.12}$$

Definition 2.2.21. Let X be a product system over P of A-correspondences. Let $\{K_s\}_{s\in S}$ be a family of ideals K_s in $\phi_s^{-1}(\mathcal{K}(X_s))$. By an argument similar to the proof of [16, Proposition 2.9], there is a C^*-algebra $\mathcal{O}(X, \{K_s\}_{s\in S})$ called the *relative Cuntz-Pimsner algebra of X and $\{K_s\}_{s\in S}$* and a map $j_X : X \to \mathcal{O}(X, \{K_s\}_{s\in S})$ which are universal for (Toeplitz) representations of X that are coisometric on $K = \{K_s\}_{s\in S}$ in the sense of Definition 2.2.9.

Proposition 2.2.22. *[16, Proposition 2.9] Let X be a product system over P of A-correspondences. Then there is a C^*-algebra $\mathcal{O}_X$, called the* Cuntz-Pimsner algebra of X*, and a (Toeplitz) representation $j_X : X \to \mathcal{O}_X$ which is Cuntz-Pimsner covariant in the sense of CP-K, such that*

1. *for every Cuntz-Pimsner covariant representation ψ of X, there is a homomorphism ψ_* of $\mathcal{O}_X$ such that $\psi_* \circ j_X = \psi$; and*

2. *$\mathcal{O}_X$ is generated as a C^*-algebra by $j_X(X)$.*

The pair $(\mathcal{O}_X, j_X)$ is unique up to canonical isomorphism.

When $P = \mathbb{N}$, Fowler shows in [16, Proposition 2.11] that $\mathcal{T}_X$ is canonically isomorphic to the Toeplitz algebra $\mathcal{T}_{X_1}$ given by Proposition 2.1.9 of the C^*-correspondence X_1. If the left action on each fibre X_n is either isometric or by compact operators, then $\mathcal{O}_X$ is canonically isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{X_1}$ defined in Definition 2.1.14.

In [49], Sims and Yeend introduce a C^*-algebra that is universal for CNP-covariant representations of X.

Proposition 2.2.23. *[49, Proposition 3.12] Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of A-correspondences such that the homomorphisms $\widetilde{\phi}_q$ of Definition 2.2.14 are all injective. Then there exist a C^*-algebra $\mathcal{NO}_X$ and a CNP-covariant representation j_X of X in $\mathcal{NO}_X$ such that:*

1. *$\mathcal{NO}_X = \overline{\operatorname{span}}\{j_X(\xi)j_X(\eta)^* : \xi, \eta \in X\}$; and*

2. *the pair $(\mathcal{NO}_X, j_X)$ is universal in the sense that if $\psi : X \to B$ is any other CNP-covariant representation of X, then there is a unique homomorphism $\Pi_\psi : \mathcal{NO}_X \to B$ such that $\psi = \Pi_\psi \circ j_X$.*

Moreover the pair $(\mathcal{NO}_X, j_X)$ is unique up to canonical isomorphism.

More recently, in [7, Theorem 4.1], Carlsen, Larsen, Sims, and Vittadello introduce a C^*-algebra $\mathcal{NO}_X^r$ associated to a compactly aligned product system X over P

(where P is part of a quasi-lattice ordered group (G, P)) for which either the left action on each fibre is injective, or P is directed and each $\widetilde{\phi}_q$ is injective. This C^*-algebra is part of a triple $(\mathcal{NO}_X^r, j_X^r, \nu^n)$ in which j_X^r is an injective CNP-covariant representation of X whose image generates $\mathcal{NO}_X^r$ and ν^n is a normal coaction of G on $\mathcal{NO}_X^r$ such that

$$\nu^n(j_X^r(\xi)) = j_X^r(\xi) \otimes i_G(d(\xi)), \qquad \text{for } \xi \in X$$

where $d : X \to P$ is a semigroup homomorphism such that $X_p = d^{-1}(p)$. Moreover, the triple $(\mathcal{NO}_X^r, j_X^r, \nu^n)$ is co-universal in the sense that if $\psi : X \to B$ is an injective Nica covariant (Toeplitz) representation whose image generates B for which there is a coaction β of G on B satisfying

$$\beta(\psi(\xi)) = \psi(\xi) \otimes i_G(d(\xi)), \qquad \text{for } \xi \in X,$$

then there is a unique surjective $*$-homomorphism $\phi : B \to \mathcal{NO}_X^r$ such that $\phi(\psi(\xi)) = j_X^r(\xi)$ for all $\xi \in X$.

Under certain conditions $\mathcal{NO}_X^r$ is isomorphic to $\mathcal{NO}_X$. In particular, this happens when the group G is amenable. Hence if $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$ and X is a compactly aligned product system over P for which the left action on each fibre is injective, we have that $\mathcal{NO}_X^r \cong \mathcal{NO}_X$. This setting will be relevant later in Chapter 6.

2.3 Directed graphs and their generalizations

Definition 2.3.1. A *directed graph* is a system $E = (E^0, E^1, r, s)$ where E^0 and E^1 are countable sets (called the vertices and edges, respectively, of the graph) and where r and s are maps from the edges to the vertices (called the range and source maps, respectively). We view an edge e as being directed from its source $s(e)$ to its range $r(e)$.

The graph C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by a family $\{p_v\}_{v \in E^0}$ of mutually orthogonal projections (i.e., self-adjoint idempotents) in

$C^*(E)$ and a family $\{s_e\}_{e\in E^1}$ of partial isometries with mutually orthogonal range projections satisfying the *Cuntz-Krieger relations*

(CK1) $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$

(CK2) $p_v = \sum_{r(e)=v} s_e s_e^*,$ when $0 < |r^{-1}(v)| < \infty,$ and

(CK3) $s_e s_e^* \leq p_{r(e)}$ for all $e \in E^1$.

We call any such pair $\{S, P\}$, where $S = \{S_e\}_{e\in E^1}$ and $P = \{P_v\}_{v\in E^0}$ satisfy (CK1)–(CK3) above, a *Cuntz-Krieger E family.*

Example 2.3.2. For the graph E in Figure 2.2 we have $E^0 = \{u, v, w\}$, $E^1 = \{e, f, g\}$, and

$$\begin{aligned} s(e) &= v, & r(e) &= u \\ s(f) &= w, & r(f) &= u \\ s(g) &= w, & r(g) &= v. \end{aligned}$$

It follows from [39, Proposition 1.18] that $C^*(E) \cong M_4(\mathbb{C})$.

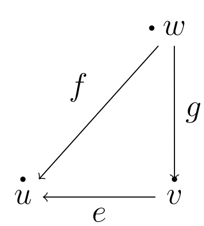

Figure 2.2: Directed graph E with $C^*(E) \cong M_4(\mathbb{C})$

Example 2.3.3. For the graph F in Figure 2.3 we have $E^0 = \{u, v\}$, $E^1 = \{e, f\}$, $r(f) = s(e) = r(e) = u$, and $s(f) = v$. It follows from the Cuntz-Krieger relations that $C^*(F)$ is generated by the single isometry $s_e + s_f$ and that $C^*(F)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$ generated by the unilateral shift.

$$\begin{tikzcd} u \arrow[loop left, "e"] & v \arrow[l, "f"'] \end{tikzcd}$$

Figure 2.3: Directed graph F with $C^*(F) \cong \mathcal{T}$

Higher-rank graphs

Higher-rank graphs, also called *k-graphs* (where $k \in \mathbb{N}$ is said to be the rank of the graph) are defined using the language of category theory. For the purposes of this thesis, a *countable category* $\mathcal{C}$ consists of two countable sets $\operatorname{Obj}\mathcal{C}$ and $\operatorname{Mor}\mathcal{C}$ (called the *objects* and *morphisms*, respectively), two functions $r, s : \operatorname{Mor}\mathcal{C} \to \operatorname{Obj}\mathcal{C}$ (called the *range* and *source* maps, respectively), and a *composition map* $\circ : \{(f, g) \in \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C} : s(f) = r(g)\} \to \operatorname{Mor}\mathcal{C}$ satisfying the following conditions:

1. For each $v \in \operatorname{Obj}\mathcal{C}$ there is a (unique) *identity morphism* $\mathrm{id}_v \in \operatorname{Mor}\mathcal{C}$ such that $r(\mathrm{id}_v) = s(\mathrm{id}_v) = v$ and

   a) for all $f \in \operatorname{Mor}\mathcal{C}$ with $r(f) = v$, we have $\mathrm{id}_v \circ f = f$

   b) for all $f \in \operatorname{Mor}\mathcal{C}$ with $s(f) = v$, we have $f \circ \mathrm{id}_v = f$.

2. For all $f, g \in \operatorname{Mor}\mathcal{C}$ with $s(f) = r(g)$, we have $r(f \circ g) = r(f)$ and $s(f \circ g) = s(g)$.

3. For all $f, g, h \in \operatorname{Mor}\mathcal{C}$ with $s(f) = r(g)$ and $s(g) = r(h)$, we have $(f \circ g) \circ h = f \circ (g \circ h)$.

Definition 2.3.4. A *k-graph* (Λ, d) is a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$. We think of the objects of Λ as *vertices* and the morphisms as *paths*. The functor d gives the *degree* or "shape" of a path and is required to satisfy the unique factorization property: given $\lambda \in \Lambda$ with $d(\lambda) = m + n$, there are unique $\mu, \nu \in \Lambda$ satisfying $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. For $m \in \mathbb{N}^k$, the set of paths of degree m is denoted $\Lambda^m := d^{-1}(\{m\})$. The set of composable paths is

$$\Lambda * \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = r(\mu)\}.$$

In [30], Kumjian and Pask assumed that their k-graphs were row-finite and had no sources. We say that a k-graph is *row-finite* if the set

$$v\Lambda^m = \{\lambda \in \Lambda^m : r(\lambda) = v\}$$

is finite for every $v \in \Lambda^0$ and $m \in \mathbb{N}^k$. A vertex $v \in \Lambda^0$ is called a *source* if there is some $m \in \mathbb{N}^k$ such that $v\Lambda^m = \emptyset$.

Example 2.3.5. Given any countable directed graph E, we have an associated 1-graph Λ_E. Let E^* denote the set of all finite paths $\mu \in E$. Set $\operatorname{Obj}\Lambda_E = E^0$, $\operatorname{Mor}\Lambda_E = E^*$, let the range and source maps be those inherited from E, and let $d(\mu)$ be the length of the path $\mu \in \Lambda_E$. Composition of paths $\mu, \nu \in \Lambda_E$ is given by $\mu\nu = \mu_1 \cdots \mu_{d(\mu)}\nu_1 \cdots \nu_{d(\nu)}$. The identity morphism on $v \in \operatorname{Obj}\Lambda_E$ is the path v of length zero.

The C^*-algebra of a k-graph may be defined as a groupoid C^*-algebra, as the C^*-algebra associated to a product system over $\mathbb{N}^k$ of C^*-correspondences, or using a direct approach generalizing the Cuntz-Krieger conditions. The first work on k-graphs was restricted to graphs that were row-finite with no sources. However, Raeburn, Sims, and Yeend [40, 41] modified the Cuntz-Krieger relations in order to work with so-called *finitely aligned* k-graphs. We define below what it means for a k-graph to be finitely aligned and then define the k-graph C^*-algebra of an arbitrary finitely aligned k-graph.

Definition 2.3.6. Let Λ be a k-graph. Given $\lambda, \mu \in \Lambda$, define the set of *minimal common extensions* of λ and μ as

$$\Lambda^{min}(\lambda, \mu) = \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\alpha)\}.$$

We say that Λ is *finitely aligned* if $|\Lambda^{min}(\lambda, \mu)| < \infty$ for all $\lambda, \mu \in \Lambda$. In particular, every row-finite k-graph is finitely aligned.

For $v \in \Lambda^0$ and $E \subset v\Lambda$, we say that E is *exhaustive* if for all $\mu \in v\Lambda$, there is $\lambda \in E$ with $\Lambda^{min}(\lambda, \mu) \neq \emptyset$.

Definition 2.3.7. Let Λ be a finitely aligned k-graph. The *k-graph C^*-algebra* $C^*(\Lambda)$ is the universal C^*-algebra generated by a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ satisfying

1. $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections
2. $s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ with $r(\mu) = s(\lambda)$
3. $s_\lambda^* s_\mu = \sum_{(\alpha,\beta)\in\Lambda^{min}(\lambda,\mu)} s_\alpha s_\beta^*$
4. $\prod_{\lambda \in F}(s_v - s_\lambda s_\lambda^*) = 0$ for $v \in \Lambda^0$ and finite exhaustive $F \subset v\Lambda$

in the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ satisfies 1-4 above, then there is a unique homomorphism $\phi : C^*(\Lambda) \to C^*(\{t_\lambda\})$ such that $\phi(s_\lambda) = t_\lambda$ for each $\lambda \in \Lambda$.

Example 2.3.8. Fix $k \in \mathbb{N}$. Let Ω_k be the countable category with $\operatorname{Obj} \Omega_k = \mathbb{N}^k$ and $\operatorname{Mor} \Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$. The range and source maps are given by $r(m, n) = m$ and $s(m, n) = n$, and composition is given by $(m, n) \circ (n, p) = (m, p)$. Let $d : \Omega_k \to \mathbb{N}^k$ be given by $d(m, n) = n - m$. Then Ω_k is a k-graph and $C^*(\Omega_k) \cong \mathcal{K}(\ell^2(\mathbb{N}^k))$.

We may visualize a k-graph by considering its *1-skeleton*, the colored directed graph $(\Lambda^0, \bigcup_{i=1}^k \Lambda^{e_i}, r, s)$ where the edges in each Λ^{e_i} are drawn in a different color so that there are k colors.

Example 2.3.9. The image in Figure 2.4 shows the 1-skeleton of the graph Ω_3. The LaTeX code for this image was based on code generously provided by Sarah Wright and used originally in [53]. The graph C^*-algebra is isomorphic to $\mathcal{K}(\ell^2(\mathbb{N}^3))$.

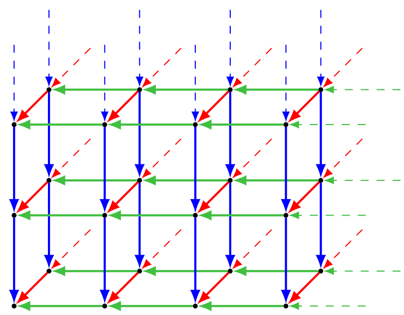

Figure 2.4: The 1-skeleton of Ω_3

Topological graphs

In [26, 27, 28, 29], Katsura introduced another generalization of a directed graph; a continuous analogue of a directed graph that he called a *topological graph*.

Definition 2.3.10. A *topological graph* is a system $E = (E^0, E^1, r, s)$ where E^0 and E^1 are locally compact Hausdorff spaces (called the vertex space and edge space, respectively, of the graph) and where $r, s : E^1 \to E^0$ are continuous maps and $s : E^1 \to E^0$ is a local homeomorphism.

Example 2.3.11. Given any directed graph E, we may construct a topological graph $E \times_{m,n} \mathbb{T}$ by replacing each vertex with a copy of the circle $\mathbb{T}$ as in [29, Section 5]. Given maps $m : E^1 \to \mathbb{Z}$ and $n : E^1 \to \mathbb{Z}_+$, we define continuous maps

$\tilde{s}, \tilde{r} : E^1 \times \mathbb{T} \to E^0 \times \mathbb{T}$ via

$$\tilde{s}(e, z) = (s(e), z^{n(e)}) \quad \text{and} \quad \tilde{r}(e, z) = (r(e), z^{m(e)}).$$

Then $E \times_{m,n} \mathbb{T} = (E^0 \times \mathbb{T}, E^1 \times \mathbb{T}, \tilde{r}, \tilde{s})$ is a topological graph.

Example 2.3.12. Let E be the directed graph from Example 2.3.2. Let $n : E^1 \to \mathbb{Z}_+$ be the function which maps each edge to 1 and let $m : E^1 \to \mathbb{Z}$ be given by

$$e \mapsto 0, \qquad f \mapsto 1, \qquad g \mapsto 1.$$

The graph $E \times_{m,n} T$ is shown in Figure 2.5. The LaTeX code for this image was based on code generously provided by Sarah Wright and used originally in [53].

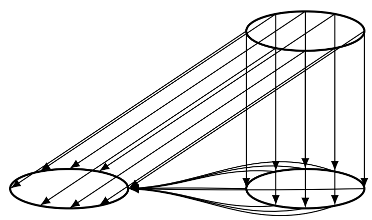

Figure 2.5: The topological graph $E \times_{m,n} \mathbb{T}$

Associated to every topological graph $E = (E^0, E^1, r, s)$ is a C^*-correspondence over the C^*-algebra $C_0(E^0)$.

Definition 2.3.13. Let $E = (E^0, E^1, r, s)$ be a topological graph. The *topological graph correspondence* X_E is the $C_0(E^0)$-correspondence

$$X_E = \{\xi \in C(E^1) : \text{ the map } v \mapsto \sum_{e \in s^{-1}(v)} |\xi(e)|^2 \text{ is in } C_0(E^0)\},$$

with $C_0(E^0)$-bimodule operations

$$(f \cdot \xi \cdot g)(e) = f(r(e))\xi(e)g(s(e))$$

and $C_0(E^0)$-valued inner product

$$\langle \xi, \eta \rangle(v) = \sum_{e \in s^{-1}(v)} \overline{\xi(e)}\eta(e).$$

The *topological graph C^*-algebra* $C^*(E)$ is defined to be $C^*(E) = \mathcal{O}_{X_E}$.

Example 2.3.14. [29, Example 5.3] Let F be the directed graph shown below

$$\overset{\bullet}{w} \xleftarrow{e} \overset{\bullet}{v}$$

Fix $\tilde{n} \in \mathbb{Z}_+, \tilde{m} \in \mathbb{Z}$, and define the functions n, m via $e \mapsto \tilde{n}$ and $e \mapsto \tilde{m}$, respectively. When $m \neq 0$, the C^*-algebra $C^*(E \times_{m,n} \mathbb{T})$ is isomorphic to $\mathbb{M}_{n+1} \otimes C(\mathbb{T})$, and is isomorphic to $(\mathbb{M}_{n+1} \otimes C(\mathbb{T})) \oplus C(\mathbb{T})$ when $m = 0$.

Topological k-graphs

In his thesis [55], Yeend developed a generalization of directed graphs in which he unified the concepts of both higher-rank graphs and topological graphs.

Definition 2.3.15. A *topological k-graph* is a higher-rank graph in which the object set and morphism set are second countable, locally compact Hausdorff spaces; the range map is continuous and the source map is a local homeomorphism; composition $\circ : \Lambda * \Lambda \to \Lambda$ is continuous and open, where $\Lambda * \Lambda$ is endowed with the relative topology inherited from the product topology on $\Lambda \times \Lambda$; d is continuous, where $\mathbb{N}^k$ has the discrete topology.

Any higher-rank graph (Λ, d) (where Λ has rank k) may be viewed as a topological k-graph where the sets $\text{Obj}(\Lambda)$ and $\text{Mor}(\Lambda)$ have the discrete topology. A more complicated topological k-graph is given by the following example.

Example 2.3.16. For $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, let $(\Omega_{k,m}, d)$ be the discrete k-graph given by

$$\text{Obj}(\Omega_{k,m}) = \{p \in \mathbb{N}^k : p \leq m\}$$

$$\text{Mor}(\Omega_{k,m}) = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}$$

$$r(p,q) = p \quad \text{and} \quad s(p,q) = q$$

$$(p,q) \circ (q,r) = (p,r)$$

$$d(p,q) = q - p.$$

As described in Example 2.3.11, we may construct a topological k-graph by replacing each vertex with a copy of the circle $\mathbb{T}$. Let m, n be the the maps defined in Example 2.3.12. The image in Figure 2.6 is the 1-skeleton of the graph $\Omega_{3,(1,1,1)} \times_{m,n} \mathbb{T}$. The LaTeX code for this image was based on code generously provided by Sarah Wright and used originally in [53].

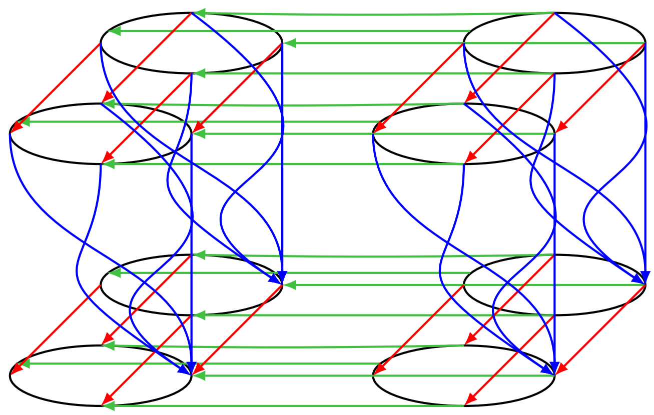

Figure 2.6: The 1-skeleton of the graph $\Omega_{3,(1,1,1)} \times_{m,n} \mathbb{T}$

Definition 2.3.17. For $m \in \mathbb{N}^k$, let $\Lambda^m = d^{-1}(\{m\})$ denote the set of paths of degree m. For $p, q \in \mathbb{N}^k$, $U \subseteq \Lambda^p$, and $V \subseteq \Lambda^q$, we let

$$U \vee V = U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q}$$

denote the set of *minimal common extensions of paths from U and V.*

The C^*-algebra $C^*(\Lambda)$ associated to a topological k-graph Λ was originally defined by Yeend in [54] as a groupoid C^*-algebra. In order to describe Yeend's construction here, we include a brief introduction to groupoids and their associated C^*-algebras. For details, see [44].

Definition 2.3.18. A groupoid G is a small category (i.e., a category G in which $\operatorname{Obj} G$ and $\operatorname{Mor} G$ are sets) in which each morphism is invertible. That is, for all $x : a \to b \in \operatorname{Mor} G$ there is $x^{-1} : b \to a \in \operatorname{Mor} G$ such that the pairs $(x, x^{-1}), (x^{-1}, x)$ are composable with

$$x \circ x^{-1} = \mathrm{id}_b \quad \text{and} \quad x^{-1} \circ x = \mathrm{id}_a.$$

We identify the objects of G with the identity morphisms and call this set the *units*, denoted $G^{(0)}$, of the groupoid. The range and source maps are given by

$$r(x) = x \circ x^{-1} \quad \text{and} \quad s(x) = x^{-1} \circ x.$$

Given a subset $U \subseteq G^{(0)}$, the *reduction of G by U* is the groupoid

$$G|_U := \{x \in G : r(x), s(x) \in U\} = r^{-1}(U) \cap s^{-1}(U) = UGU$$

with unit space $(G|_U)^{(0)} = U$ and all operations given by their restrictions of those in G. We say that U is *invariant* if $UG = GU = UGU$.

Given a groupoid G, we would like to construct an associated C^*-algebra. We may do this for a particular class of groupoids by defining a convolution and involution on $C_c(G)$, the continuous complex-valued functions on G with compact support, and completing with respect to a certain norm.

Let G be a groupoid with a second countable Hausdorff topology and give

$$G^{(2)} := \{(x, y) \in \operatorname{Mor}(G) \times \operatorname{Mor}(G) : s(x) = r(y)\}$$

the relative topology of the product topology on $G \times G$. If composition and inversion are continuous, we say that G is a *topological groupoid.* Under certain conditions (see [44, 35] for details), a topological groupoid is endowed with a *(left) Haar system on G*, that is, a family $\{\lambda^u\}_{u \in G^{(0)}}$ of non-negative Radon measures on G such that

1. $\operatorname{supp}(\lambda^u) = G^u = r^{-1}(\{u\})$ for each $u \in G^{(0)}$,

2. for $f \in C_c(G)$, the function $u \mapsto \int f d\lambda^u$ on $G^{(0)}$ is in $C_c(G^{(0)})$, and

3. *(left invariance):* for $x \in G$

$$\int f(xy)d\lambda^{s(x)}(y) = \int f(y)d\lambda^{r(x)}(y).$$

We define a convolution and involution on $C_c(G)$ as follows. For $f, g \in C_c(G)$, let

$$\begin{aligned} f * g(y) &= \int f(x)g(x^{-1}y)d\lambda^{r(y)}(x) \\ &= \int f(yx)g(x^{-1})d\lambda^{s(y)}(x), \text{ by left invariance, and} \\ f^*(y) &= \overline{f(y^{-1})}. \end{aligned}$$

For a locally compact Hausdorff groupoid for which the unit space is open and the range map is a local homeomorphism, the (left) Haar system is given by counting measures concentrated on the range-inverse sets $r^{-1}(\{u\})$ so that the integrals above are in fact sums.

Definition 2.3.19. A representation of $C_c(G)$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : C_c(G) \to \mathcal{B}(\mathcal{H})$ which is continuous with respect to the inductive-limit topology on $C_c(G)$ and the weak operator topology on $\mathcal{B}(\mathcal{H})$ and which is nondegenerate in the sense that

$$\mathcal{H} = \overline{\text{span}}\{\pi(f)\xi : f \in C_c(G), \xi \in \mathcal{H}\}.$$

It is known (see [35, Theorem 2.42]) that there is a norm on $C_c(G)$, called the I-norm, given by

$$\|f\|_I = \max\{\sup_{u \in G^{(0)}} \int |f(x)|d\lambda^u(x), \sup_{u \in G^{(0)}} \int |f(x^{-1})|d\lambda^u(x)\}$$

such that every representation π of $C_c(G)$ is bounded by the I-norm; that is, $\|\pi(f)\| \leq \|f\|_I$ for every $f \in C_c(G)$. It follows that

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a representation}\}$$

is a seminorm on $C_c(G)$. In fact $\|\cdot\|$ is a C^*-norm and the completion of $C_c(G)$ with respect to $\|\cdot\|$ is a C^*-algebra called the *full groupoid C^*-algebra of G* and denoted $C^*(G)$.

Given a topological k-graph, we now construct two associated groupoids, the path groupoid and boundary path groupoid. In order to do this, we must describe the path space of the graph Λ. We define this space as a collection of graph morphisms.

Definition 2.3.20. Given topological k-graphs (Λ_1, d_1) and (Λ_2, d_2), a *graph morphism* between Λ_1 and Λ_2 is a continuous functor $x : \Lambda_1 \to \Lambda_2$ such that $d_2(f(\lambda)) = d_1(\lambda)$ for every $\lambda \in \Lambda_1$.

Definition 2.3.21. Let (Λ, d) be a topological k-graph. The *path space* of Λ is the space

$$X_\Lambda = \bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \{x : \Omega_{k,m} \to \Lambda : x \text{ is a graph morphism}\}$$

where $\Omega_{k,m}$ is the (discrete) k-graph from Example 2.3.8. We may regard Λ as a subset of X_Λ since for every $\lambda \in \Lambda$ there is a unique graph morphism $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$ such that $x_\lambda(0, d(\lambda)) = \lambda$. We may also extend the range and degree maps to all of X_Λ via

$$r(x) = x(0) \quad \text{and} \quad d(x) = m, \qquad \text{for } x : \Omega_{k,m} \to \Lambda.$$

There is a natural action of Λ on X_Λ given by concatenation and removal of initial path segments (see [54, Lemma 3.3] for details). In particular, if $x \in X_\Lambda$, then for every $\lambda \in \Lambda$ with $s(\lambda) = r(x)$ and every $m \in \mathbb{N}^k$ with $0 \leq m \leq d(x)$ there are unique elements $\lambda x, \sigma^m x \in X_\Lambda$ such that $d(\lambda x) = d(\lambda) + d(x)$, $d(\sigma^m x) = d(x) - m$ and

$$(\lambda d)(0, m) = \begin{cases} \lambda(0, m), & 0 \leq m \leq d(\lambda) \\ \lambda x(0, m - d(\lambda)), & d(\lambda) \leq m \leq d(\lambda x), \end{cases}$$

$$(\sigma^m x)(0,n) = x(m, m+n), \qquad \text{for } 0 \le n \le d(x) - m.$$

The path groupoid G_Λ is the groupoid with objects and morphisms given by

$$\begin{aligned}
\mathrm{Obj}(G_\Lambda) &= X_\Lambda, \\
\mathrm{Mor}(G_\Lambda) &= \{(\lambda x, d(\lambda) - d(\mu), \mu x) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda : s(\mu) = s(\lambda) = r(x)\} \\
&= \{(x, m, y) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda : \exists\, p, q \in \mathbb{N}^k \text{ such that} \\
&\qquad\qquad p \le d(x),\ q \le d(y),\ p - q = m,\ \sigma^p x = \sigma^q y\}
\end{aligned}$$

and range and source maps

$$r(x, m, y) = x \quad \text{and} \quad s(x, m, y) = y.$$

Composition and inversion in G_Λ are given by

$$((x, m, y), (y, n, z)) \mapsto (x, m+n, z) \quad \text{and} \quad (x, m, y) \mapsto (y, -m, x).$$

In order to ensure that the path groupoid is a locally compact groupoid with a Haar system consisting of counting measures, we require that the topological k-graph Λ be compactly aligned in the sense of the following definition.

Definition 2.3.22. A topological k-graph (Λ, d) is *compactly aligned* if for every $p, q \in \mathbb{N}^k$ and compact $U \subseteq \Lambda^p$ and $V \subseteq \Lambda^q$, the set $U \vee V$ (as defined in Definition 2.3.17) is compact.

When Λ is a discrete k-graph, the definition of compactly aligned agrees with that of finitely aligned as in Definition 2.3.6. Yeend shows in [54, Theorem 3.16] that for every compactly aligned topological k-graph (Λ, d), the path groupoid G_Λ is a locally compact topological groupoid for which the unit space is open and there exists a Haar system consisting of counting measures. To define the topological k-graph C^*-algebra, Yeend identifies a nonempty subset $\partial\Lambda$ of $G_\Lambda^{(0)}$ that is *invariant* in the sense of Definition 2.3.18.

Definition 2.3.23. The *boundary path groupoid* is defined to be the restriction

$$\mathcal{G}_\Lambda = G_\Lambda|_{\partial\Lambda} = \{x \in G_\Lambda : r(x), s(x) \in \partial\Lambda\}$$

and is a locally compact Hausdorff topological groupoid for which the unit space is open and there exists a Haar system consisting of counting measure. The *topological k-graph C^*-algebra of* Λ (also called the *Cuntz-Krieger algebra of* Λ) is defined to be the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$.

Chapter 3

CHARACTERIZING GRAPH CORRESPONDENCES

The research described in this chapter was conducted in collaboration with Steve Kaliszewski and John Quigg. The results appear in [24].

In this chapter, we investigate the connection between directed graphs and C^*-correspondences. In Section 3.2, our main result (Theorem 3.2.1) expands and elaborates on a remark of Schweizer ([47, Section 1.6]; see also [37, Chapter 1, Example 2]) to the effect that every C^*-correspondence over C^*-algebras $c_0(X)$ and $c_0(Y)$ is unitarily equivalent to a correspondence arising from a "diagram" from X to Y. (Schweizer's diagrams can be put in our context by taking $V = X \cup Y$.) Specifically, given a (separable, nondegenerate) C^*-correspondence X over $c_0(V)$, we construct a graph E such that $X_E \cong X$ as C^*-correspondences.

In section Section 3.3, we show that the assignment $E \mapsto X_E$, where X_E is the graph correspondence of the directed graph E, can be extended to a functor Γ between certain categories of graphs and correspondences. Γ is very nearly a category equivalence: it is essentially surjective, faithful, and "essentially full" (see Proposition 3.3.5), but not full. It is also injective on objects, and reflects isomorphisms (Theorem 3.3.3).

3.1 Graph correspondences

Let $E = (E^0, E^1, r, s)$ be a directed graph with vertex set E^0, edge set E^1, and range and source maps $r, s : E^1 \to E^0$. Let X_E be the *graph correspondence* associated to E as defined in Definition 2.3.13. Since E^0 is discrete, X_E is a C^*-correspondence over $c_0(E^0)$ given by

$$X_E := \left\{ \xi \in \mathbb{C}^{E^1} \;\middle|\; \text{the map } v \mapsto \sum_{s(e)=v} |\xi(e)|^2 \text{ is in } c_0(E^0) \right\},$$

with $c_0(E^0)$-bimodule operations

$$(a\xi b)(e) = a(r(e))\xi(e)b(s(e)) \qquad \text{for } a, b \in c_0(E^0), \xi \in X_E, e \in E^1$$

and $c_0(E^0)$-valued inner product

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e).$$

We use [39] for our conventions regarding graph algebras and C^*-correspondences. It has been observed in [39, Proposition 8.8], for example, that:

- X_E is *nondegenerate* in the sense that $c_0(E^0)X_E = X_E$ (see also [19, Proposition 4.4]);
- X_E is *full* in the sense that $\overline{\operatorname{span}}\langle X_E, X_E \rangle = c_0(E^0)$ if and only if the graph E has no sinks;
- the homomorphism $\phi : c_0(E^0) \to \mathcal{L}(X_E)$ associated to the left module operation maps into the compacts $\mathcal{K}(X_E)$ if and only if no vertex receives infinitely many edges, and is faithful if and only if E has no sources.

We are concerned more with the graph correspondence as a *process* (technically, a functor) which associates correspondences to graphs, rather than with how properties of the correspondence are related to properties of the graph.

For notational convenience, we will write p_v for the characteristic function of a vertex $v \in E^0$ and χ_e for the characteristic function of an edge $e \in E^1$ respectively. Then the elements of X_E having finite support in E^1 may be regarded as formal linear combinations $\sum_{i=1}^n \alpha_i \chi_{e_i}$ with $\alpha_i \in \mathbb{C}$. Then X_E is the closed span of the linearly independent set $\{\chi_e\}_{e \in E^1}$, and we refer to the characteristic functions of the edges as *generators* of the correspondence X_E. Also, $\{p_v\}_{v \in E^0}$ is a set of mutually orthogonal minimal projections in the commutative C^*-algebra $c_0(E^0)$, and the series $\sum_{v\in E^0} p_v$ converges strictly in $M(c_0(E^0)) = \ell^\infty(E^0)$ to the identity. The $c_0(E^0)$-valued inner product is characterized on the generators by

$$\langle \chi_e, \chi_f\rangle = \begin{cases} p_v & \text{if } e = f \in s^{-1}(v) \\ 0 & \text{if } e \neq f \end{cases} \qquad \text{for } e, f \in E^1.$$

It has been observed (see [39, Example 8.13] or [26, Example 1]) that the Cuntz-Pimsner algebra $\mathcal{O}_{X_E}$ of the graph correspondence is isomorphic to the graph C^*-algebra. To see this, note that since $\phi^{-1}(\mathcal{K}(X_E))$ is spanned by $\{p_v : v \in E^0, |r^{-1}(v)| < \infty\}$ and $\phi(p_v) = 0$ if and only if v is a source ([39, Proposition 8.8]), it follows that

$$\phi^{-1}(\mathcal{K}(X_E)) \cap (\ker\phi)^\perp = \overline{\operatorname{span}}\{p_v : 0 < |r^{-1}(v)| < \infty\}.$$

Let (ψ, π) be a Toeplitz representation of X_E and set

$$P_v = \pi(p_v) \quad \text{and} \quad S_e = \psi(\chi_e), \qquad \text{for } v \in E^0, e \in E^1.$$

Then $P = \{P_v\}_{v\in E^0}$ is a family of mutually orthogonal projections and we may compute

$$\begin{aligned} S_e^* S_e &= \psi(\chi_e)^*\psi(\chi_e) \\ &= \pi(\langle \chi_e, \chi_e\rangle), \text{ by (T2)} \\ &= \pi(p_{s(e)}) \\ &= P_{s(e)} \end{aligned}$$

so that $S = \{S_e\}_{e\in E^1}$ is a family of partial isometries. The calculation above also shows that $\{S, P\}$ satisfies (CK1). Additionally, we have

$$\begin{aligned} P_{r(e)} S_e S_e^* &= \pi(p_{r(e)})\psi(\chi_e)\psi(\chi_e)^* \\ &= \psi(p_{r(e)} \cdot \chi_e)\psi(\chi_e)^*, \text{ by (T3)} \\ &= \psi(\chi_e)\psi(\chi_e)^* \end{aligned}$$

$$= S_e S_e^*$$

so that $S_e S_e^* \leq P_{r(e)}$ for all $e \in E^1$ and hence (CK3) is satisfied. Finally, we observe that (ψ, π) is Cuntz-Pimsner covariant if and only if for v with $0 < |r^{-1}(v)| < \infty$ we have

$$\begin{aligned}
\pi(p_v) &= \pi^{(1)}(\phi(p_v)) \\
&= \pi^{(1)}\left(\sum_{r(e)=v} \Theta_{\chi_e, \chi_e}\right) \\
&= \sum_{r(e)=v} \psi(\chi_e)\psi(\chi_e)^*.
\end{aligned}$$

Equivalently, (ψ, π) is Cuntz-Pimsner covariant if and only if $\{S, P\}$ is a Cuntz-Krieger E-family. By the universal properties of $C^*(E)$ and $\mathcal{O}_{X_E}$, we have that the two C^*-algebras are isomorphic.

3.2 Characterization

Let V be a countable set, and let $A = c_0(V)$. As discussed above, every directed graph E with vertex set V has associated to it a C^*-correspondence X_E over A such that $C^*(E) \cong \mathcal{O}_{X_E}$, and moreover this correspondence X_E is nondegenerate. Our purpose in this section is to prove a sort of converse:

Theorem 3.2.1. *Every nondegenerate separable correspondence over $c_0(V)$ is isomorphic to the graph correspondence of a directed graph E with vertex set V.*

Proof of Theorem 3.2.1. Let X be a nondegenerate correspondence over $A = c_0(V)$. For each $u, v \in V$ put

$$\begin{aligned}
X_v &= X p_v \\
X_{uv} &= p_u X_v = p_u X p_v.
\end{aligned}$$

Then A is the c_0-direct sum of the family $\{Ap_v\}_{v\in V}$ of one-dimensional C^*-algebras. Each X_v is a Hilbert space, with Hilbert-space inner product (conjugate-linear in the first variable)

$$\langle \xi, \eta \rangle_{X_v} = \langle \xi, \eta \rangle_A(v).$$

If $u \neq v$ then X_u is orthogonal to X_v in the sense that $\langle \xi, \eta \rangle = 0$ for all $\xi \in X_u$ and $\eta \in X_v$, because $\{p_v\}_{v\in E^0}$ is a set of mutually orthogonal projections in A and

$$\langle \xi p_u, \eta p_v \rangle = p_u \langle \xi, \eta \rangle p_v = \langle \xi, \eta \rangle p_u p_v.$$

X is the c_0-direct sum of the family $\{X_v \mid v \in V\}$ of Hilbert spaces, because finite sums of the p_v form a bounded approximate identity in A. For all u, v we have $X_{uv} \subset X_v$ because $p_u(\xi p_v) = (p_u \xi) p_v$. If $u \neq w$ then the closed subspaces X_{uv} and X_{wv} of X_v are orthogonal because

$$\langle p_u \xi, p_w \eta \rangle = \langle \xi, p_u p_w \eta \rangle.$$

Each X_v is the Hilbert space direct sum of the family $\{X_{uv} \mid u \in V\}$ of mutually orthogonal subspaces, because the left A-module action is nondegenerate.

For each $u, v \in V$, we choose an orthonormal basis E_{uv} of the Hilbert space X_{uv}, and put

$$E^1 = \bigcup_{u,v\in V} E_{uv}.$$

Since X is separable, each of the E_{uv} are countable and hence E^1 is countable. The union above is a disjoint union, so we can define $r, s : E^1 \to V$ by

$$r(e) = u \quad \text{and} \quad s(e) = v \quad \text{if} \quad e \in E_{uv}.$$

This gives a graph $E = (V, E^1, r, s)$, and we shall show that $X \cong X_E$ as A-correspondences.

We need a linear surjection $\psi : X \to X_E$ such that for all $\xi, \eta \in X$ and $a, b \in A$ we have

$$\psi(a\xi b) = a\psi(\xi)b \tag{3.1}$$

$$\langle \psi(\xi), \psi(\eta) \rangle = \langle \xi, \eta \rangle. \tag{3.2}$$

Let ψ be the map given by $e \mapsto \chi_e$ for $e \in E^1$. Then for $a = p_u$, $b = p_v$, and $\xi = e, \eta = f \in E^1$ we have

$$\begin{aligned}
\psi(p_u \cdot e p_v) &= \begin{cases} \psi(e) & e \in E_{uv} \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \chi_e & e \in E_{uv} \\ 0 & \text{else} \end{cases} \\
&= p_u \cdot \chi_e \cdot p_v \\
&= p_u \psi(e) \cdot p_v.
\end{aligned}$$

Also we may compute for $v \in V$

$$\langle e, f \rangle(v) = \begin{cases} 1 & e = f \in X_v = X \cdot p_v \\ 0 & \text{else} \end{cases}$$

and hence

$$\begin{aligned}
\langle e, f \rangle &= \begin{cases} p_v & e = f \in s^{-1}(v) \\ 0 & \text{else} \end{cases} \\
&= \langle \chi_e, \chi_f \rangle \\
&= \langle \psi(e), \psi(f) \rangle.
\end{aligned}$$

Extend ψ linearly, getting an isometric embedding of span E^1 into X_E, then appeal to continuity and density to obtain a linear map $\psi : X \to X_E$ satisfying (3.1) and (3.2). Since the image of E^1 has dense span in X_E, ψ is surjective. $\square$

Example 3.2.2. Given a countable set V, let $A = c_0(V)$ and let σ be a function of V into itself. Then we may associate to σ the directed graph $E_\sigma = (V, E^1, r, s)$ where

$$E^1_\sigma = \{(u, v) \in V \times V | u = \sigma(v)\}$$

and $r, s : E^1 \to V$ are given by

$$r(u, v) = u \quad \text{and} \quad s(u, v) = v.$$

As in [47, 1.2], let $X_\sigma = A$ be the C^*-correspondence with A-bimodule operations

$$(a\xi b) = (a \circ \sigma)\xi b \quad \text{for } a, b \in A, \xi \in X_\sigma$$

and A-valued inner product

$$\langle \xi, \eta \rangle(v) = \overline{\xi(v)}\eta(v).$$

For $u, v \in V$, let X_v, X_{uv} be as in the proof of Theorem 3.2.1. Note that these are the one-dimensional Hilbert spaces

$$X_v = \operatorname{span} \chi_{(\sigma(v),v)}, \text{ and}$$
$$X_{uv} = \begin{cases} \operatorname{span} \chi_{(u,v)}, & \sigma(v) = u \\ 0 & \text{otherwise.} \end{cases}$$

Then there is only one choice for E^1, and the graph associated to the correspondence X_σ is clearly isomorphic to E_σ as the following example illustrates.

Example 3.2.3. Let $V = \{u, v\}$ and define $\sigma : V \to V$ by $\sigma(u) = v$ and $\sigma(v) = u$. The directed graph associated to σ is shown in Figure 3.1. Define X_σ as above. Then $X_{uu}, X_{vv} = 0$ and we can choose unit vectors $e \in X_{uv} = \operatorname{span} \chi_{(u,v)}$, $f \in X_{vu} = \operatorname{span} \chi_{(v,u)}$. Setting $E^1 = \{e, f\}$ and defining r, s as before, we have that the graph associated to the correspondence X_σ is as shown in Figure 3.2.

$$\begin{tikzcd}
u \arrow[r, bend left, "{(v,u)}"] & v \arrow[l, bend left, "{(u,v)}"]
\end{tikzcd}$$

Figure 3.1: The graph associated to (V, σ)

$$\begin{tikzcd}
u \arrow[r, bend left, "f"] & v \arrow[l, bend left, "e"]
\end{tikzcd}$$

Figure 3.2: The graph associated to X_σ

3.3 Functoriality

We now turn the graph correspondence into a functor. First we define our categories: let V be a countable set, and let $\mathcal{G} = \mathcal{G}(V)$ denote the category whose objects are directed graphs with vertex set V. For $E, F \in \operatorname{Obj} \mathcal{G}$, a morphism $\phi : E \to F$ in $\mathcal{G}$ will be an injective graph morphism which is the identity on vertices. Thus, we have a commuting diagram as shown in Figure 3.3. Alternatively,

$$\begin{tikzcd}
& V & \\
E^1 \arrow[ur, "r_E"] \arrow[rr, "\phi"] \arrow[dr, "s_E"'] & & F^1 \arrow[ul, "r_F"'] \arrow[dl, "s_F"] \\
& V &
\end{tikzcd}$$

Figure 3.3: An injective graph morphism in $\mathcal{G}$

we can visualize the action of ϕ by the diagram shown in Figure 3.4.

Next let $\mathcal{C}_0 = \mathcal{C}_0(V)$ denote the category of nondegenerate C^*-correspondences over the C^*-algebra $A := c_0(V)$. We impose the nondegeneracy condition because

$$\begin{tikzcd} u & v \arrow[l, bend right, "e"'] \arrow[l, bend left, "\phi(e)"] \end{tikzcd}$$

Figure 3.4: Alternative visualization of an injective graph morphism in $\mathcal{G}$

it is automatically satisfied for graph correspondences. For $X, Y \in \operatorname{Obj} \mathcal{C}_0$, a morphism $\psi : X \to Y$ in $\mathcal{C}_0$ will be a morphism of A-correspondences, i.e., a linear map ψ satisfying

$$\psi(a\xi b) = a\psi(\xi)b \quad \text{and} \quad \langle \psi(\xi), \psi(\eta) \rangle = \langle \xi, \eta \rangle$$

for $a, b \in A$ and $\xi, \eta \in X$. In particular, each morphism will be isometric hence injective.

We want to define a functor $\Gamma : \mathcal{G} \to \mathcal{C}_0$ taking each graph to its graph correspondence, i.e., $\Gamma(E) = X_E$ in the notation of Section 3.1. Given a morphism $\phi : E \to F$ in $\mathcal{G}$, we want to define a morphism $\Gamma(\phi) : \Gamma(E) \to \Gamma(F)$ in $\mathcal{C}_0$. First, for $e \in E^1$, let $\Gamma(\phi)(\chi_e) = \chi_{\phi(e)}$. It is easily verified that this extends linearly to the span of the generators, that it preserves the operations (this requires ϕ to be injective), and thus extends by continuity to a morphism in $\mathcal{C}_0$. Then it is routine to verify that $\Gamma : \mathcal{G} \to \mathcal{C}_0$ is a functor, which we call the *graph-correspondence functor*.

We investigate the properties of the graph-correspondence functor. Our main interest is in the following property, which is the content of Theorem 3.2.1:

Property (see Theorem 3.2.1)**.** *Every object in $\mathcal{C}_0$ is isomorphic to one in the image of* Γ*.*

Recall that a functor is called *faithful* if it is injective on each hom-set. The following property is obvious from the construction:

Proposition 3.3.1. $\Gamma : \mathcal{G} \to \mathcal{C}_0$ *is faithful.*

In more detail, the above proposition says that if $\phi, \sigma : E \to F$ are two distinct morphisms in $\mathcal{G}$ with common domain and codomain, then $\Gamma(\phi) \neq \Gamma(\sigma)$.

The above two properties raise the hope that Γ might be a category equivalence, but this hope is in vain; the third property that would be necessary for Γ to be an equivalence is that it be *full*, i.e., surjective on each hom-set, and this property fails in general:

Example 3.3.2. Let E be the graph with a single vertex and a single loop edge, so that we can identify A with the complex numbers. Then $X = \Gamma(E)$ is a one-dimensional Hilbert space, and a morphism from the correspondence X to itself consists of multiplication by a number on the unit circle. However, there is only one morphism from the graph E to itself, so there are endomorphisms of X which are not of the form $\Gamma(\phi)$ for any endomorphism ϕ of E.

In spite of the above negative result, we will show that in fact Γ is surprisingly close to being an equivalence. First of all:

Theorem 3.3.3. *Let* $E, F \in \operatorname{Obj} \mathcal{G}$ *such that* $\Gamma(E) \cong \Gamma(F)$*. Then* $E \cong F$ *in* $\mathcal{G}$*. Moreover, if* $\Gamma(E) = \Gamma(F)$*, then* $E = F$ *in* $\mathcal{G}$*. That is, the graph correspondence functor is injective on objects.*

Proof. Let $X = \Gamma(E)$ and $Y = \Gamma(F)$. For $u, v \in V$, let

$$uE^1v = \{e \in E^1 | s(e) = v \text{ and } r(e) = u\}$$
$$uF^1v = \{e \in F^1 | s(e) = v \text{ and } r(e) = u\}.$$

Then for all $u, v \in V$ we have

$$|uE^1v| = \dim X_{uv} = \dim Y_{uv} = |uF^1v|,$$

so there is a bijection

$$\theta_{uv} : uE^1v \to uF^1v.$$

It follows that there is a unique graph isomorphism $\theta : E \xrightarrow{\cong} F$ such that

$$\theta|_{uE^1v} = \theta_{uv} \qquad \text{for all } u, v \in V.$$

For the other part, suppose $E \neq F$ in $\operatorname{Obj}\mathcal{G}$. If the sets E^1 and F^1 are different, then since $\Gamma(E) = \operatorname{span}\{\chi_e\}_{e\in E^1}$ and $\Gamma(F) = \operatorname{span}\{\chi_f\}_{f\in F^1}$. It immediately follows that $\Gamma(E) \neq \Gamma(F)$.

On the other hand, if $E^1 = F^1$ then we may choose $e \in E^1 = F^1$ such that either $r_E(e) \neq r_F(e)$ or $s_E(e) \neq s_F(e)$. Let $u = r_E(e)$, $v = s_E(e)$. In the first case, $p_u\chi_e = \chi_e$ in $\Gamma(E)$, but $p_u\chi_e = 0$ in $\Gamma(F)$. Similarly if $s_E(e) \neq s_F(e)$ then $\chi_e p_v = \chi_e$ in $\Gamma(E)$, but $\chi_e p_v = 0$ in $\Gamma(F)$. So the graph correspondence functor is injective on objects. □

Corollary 3.3.4. $\Gamma(\mathcal{G})$ *is a subcategory of* $\mathcal{C}_0$.

This follows immediately from injectivity of Γ on objects.

$\Gamma(\mathcal{G})$ is not a *full* subcategory of $\mathcal{C}_0$ since, as was demonstrated in Example 3.3.2, Γ is not surjective on hom-sets. On the other hand, not only is every object in $\mathcal{C}_0$ isomorphic to the image under Γ of some object in $\mathcal{G}$, but moreover, every morphism in $\mathcal{C}_0$ is “isomorphic” to the image under Γ of some morphism in $\mathcal{G}$ in the following sense:

Proposition 3.3.5. *Let* $\psi : X \to Y$ *be a morphism in* $\mathcal{C}_0$ *and let* $E \in \operatorname{Obj}\mathcal{G}$ *such that* $X \cong \Gamma(E)$. *Then there is an* $F \in \operatorname{Obj}\mathcal{G}$ *and a morphism* $\phi : E \to F$ *in* $\mathcal{G}$ *such that the diagram shown in Figure 3.5 commutes, where* Υ_E, Υ_F *are isomorphisms arising as in the proof of Theorem* 3.2.1.

$$\begin{tikzcd} X \arrow[r, "\Upsilon_E"] \arrow[d, "\psi"'] & \Gamma(E) \arrow[d, "\Gamma(\phi)"] \\ Y \arrow[r, "\Upsilon_F"'] & \Gamma(F) \end{tikzcd}$$

Figure 3.5: The functor Γ is "essentially full"

Proof. Note first that we may take E to be a directed graph constructed as in the proof of Theorem 3.2.1, since if E_X is a directed graph so constructed, then by Theorem 3.3.3 we have $E \cong E_X$.

For each $e \in E^1$, there are $u, v \in V$ such that $e \in E_{uv}$ where E_{uv} is an orthonormal basis for X_{uv}. For each $u, v \in V$, $\psi(E_{uv})$ is an orthonormal set in Y_{uv} since ψ is isometric and because

$$\psi(p_u X p_v) = p_u \psi(X) p_v.$$

So we may extend $\psi(E_{uv})$ to an orthonormal basis F_{uv} of Y_{uv} and set

$$F^1 = \bigcup_{u,v \in V} F_{uv}.$$

Similarly to the proof of Theorem 3.2.1, this is a disjoint union so we may define $r_F, s_F : F^1 \to V$ by

$$r_F(f) = u, \; s_F(f) = v, \text{ if } f \in F_{uv}.$$

Then we have a graph $F = (V, F^1, r_F, s_F) \in \operatorname{Obj} \mathcal{G}$. Define $\phi : E \to F$ by letting ϕ be ψ on E^1 and the identity on V. Then ϕ is injective since ψ is.

To see that the diagram commutes, note that it suffices to show that $\Upsilon_F \circ \psi = \Gamma(\phi) \circ \Upsilon_E$ on a subset of X whose linear span is dense, e.g., E^1 as defined in the proof of Theorem 3.2.1. So for $e \in E^1$ we have

$$\Upsilon_F \circ \psi(e) = \chi_{\psi(e)}, \text{ and}$$

$$
\begin{aligned}
\Gamma(\phi) \circ \Upsilon_E(e) &= \Gamma(\phi)(\chi_e) \\
&= \chi_{\phi(e)}, \text{ by definition of } \Gamma(\phi) \\
&= \chi_{\psi(e)}, \text{ by definition of } \phi.
\end{aligned}
$$

Thus the diagram commutes as desired. □

Remark 3.3.6. The above results, particularly Proposition 3.3.5, are what we have in mind when we say Γ is close to being a category equivalence. Since Γ is a faithful functor that is injective on objects, it is an embedding and hence gives rise to an isomorphism of categories $\mathcal{G} \to \Gamma(\mathcal{G})$. We are not experts in category theory, but it is tempting for us to speculate that functors with the above properties have been studied, and perhaps named, by category cognoscenti.

Chapter 4

PRODUCT SYSTEMS OF TOPOLOGICAL GRAPHS

In this chapter, we describe a product system of topological graphs using the language of category theory as in [20]. Similar to Fowler and Sims' construction, we describe a tensor groupoid in which the objects are topological graphs. We describe how a topological k-graph may be associated to a product system over $\mathbb{N}^k$ taking values in this tensor groupoid and how such a product system may be associated to a given topological k-graph.

In Section 4.3, we show that product systems over $\mathbb{N}^k$ taking values in this tensor groupoid are equivalent to topological k-graphs up to natural isomorphism. This result expands on a remark in [20] to the effect that (discrete) k-graphs are "essentially" product systems over $\mathbb{N}^k$ taking values in a particular tensor groupoid in which the objects are directed graphs.

In Section 4.4, we discuss the compactly aligned condition for topological k-graphs in the context of product systems of topological graphs. We introduce a notion of compactly aligned for a product system over $\mathbb{N}^k$ of topological graphs with common vertex space V and prove that this is equivalent to compactly aligned for the associated topological k-graph and product system over $\mathbb{N}^k$ of $C_0(V)$-correspondences.

4.1 Topological k-graphs and product systems

Definition 4.1.1 (Fibred product of graphs)**.** Let E, F be topological graphs with common vertex space V. The *fibred product* $E * F$ is the topological graph with vertex space $(E * F)^0 = V$, edge space

$$(E * F)^1 = \{(e, f) \in E^1 \times F^1 | s_E(e) = r_F(f)\}$$

with range and source maps

$$r_{(E*F)}(e,f) = r_E(e)$$
$$s_{(E*F)}(e,f) = s_F(f).$$

Definition 4.1.2. Let V be a (second countable) locally compact Hausdorff space. We construct a category $\mathcal{G}$ as follows. The objects in $\mathcal{G}$ are topological graphs $E = (E^0, E^1, r, s)$ with vertex space $E^0 = V$. Elements of $\hom(E, F)$ are homeomorphisms $\varphi : E^1 \to F^1$ such that

$$s_E = s_F \circ \varphi \quad \text{and} \quad r_E = r_F \circ \varphi.$$

Let $* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be the bifunctor (that is, the functor in two arguments) that takes a pair of topological graphs to their fibred product. For $\varphi_1 \in \hom(E_1, F_1)$ and $\varphi_2 \in \hom(E_2, F_2)$, define $\varphi_1 * \varphi_2 \in \hom(E_1 * E_2, F_1 * F_2)$ by

$$\varphi_1 * \varphi_2(e_1, e_2) = (\varphi_1(e_1), \varphi_2(e_2)) \qquad \text{for } e_1 \in E_1, e_2 \in E_2.$$

Define $1_{\mathcal{G}}$ to be the topological graph $V = (V, V, \mathrm{id}_V, \mathrm{id}_V)$ and let

$$\mathcal{B} = \mathcal{B}_{E_1,E_2,E_3} : E_1 * (E_2 * E_3) \to (E_1 * E_2) * E_3,$$

$$\lambda = \lambda_E : 1_{\mathcal{G}} * E \to E, \quad \text{and} \quad \rho = \rho_E : E * 1_{\mathcal{G}} \to E$$

be the natural equivalences determined by

$$\mathcal{B}_{E_1,E_2,E_3}(e_1 * (e_2 * e_3)) = (e_1 * e_2) * e_3, \qquad \text{for } e_i \in E_i,$$
$$\lambda_E(r_E(e) * e) = e, \quad \text{and}$$
$$\rho_E(e * s_E(e)) = e.$$

Then $\rho_{1_{\mathcal{G}}} = \lambda_{1_{\mathcal{G}}} : 1_{\mathcal{G}} \to 1_{\mathcal{G}}$ and the diagrams in Figure 4.1 and Figure 4.2 commute for every $E_1, E_2, E_3, E_4 \in \mathcal{G}$.

It follows that $\langle \mathcal{G}, *, 1_{\mathcal{G}}, \mathcal{B}, \lambda, \rho \rangle$ is a *tensor groupoid* in the sense of [20].

$$
\begin{tikzcd}
E_1 * (E_2 * (E_3 * E_4)) \arrow[rr, "\mathcal{B}_{E_1,E_2,E_3*E_4}"] \arrow[d, "1 * \mathcal{B}_{E_2,E_3,E_4}"'] & & (E_1 * E_2) * (E_3 * E_4) \arrow[d, "\mathcal{B}_{E_1*E_2,E_3,E_4}"] \\
E_1 * ((E_2 * E_3) * E_4) \arrow[dr, "\mathcal{B}_{E_1,E_2*E_3,E_4}"'] & & ((E_1 * E_2) * E_3) * E_4 \\
& (E_1 * (E_2 * E_3)) * E_4 \arrow[ur, "\mathcal{B}_{E_1,E_2,E_3} * 1"'] &
\end{tikzcd}
$$

Figure 4.1: Associativity in the tensor groupoid $\mathcal{G}$

$$
\begin{tikzcd}
E_1 * (1_{\mathcal{G}} * E_2) \arrow[r, "\mathcal{B}_{E_1,1_{\mathcal{G}},E_2}"] \arrow[d, "1 * \lambda_{E_2}"'] & (E_1 * 1_{\mathcal{G}}) * E_2 \arrow[d, "\rho_{E_1} * 1"] \\
E_1 * E_2 \arrow[r, "\mathrm{id}_{E_1 * E_2}"] & E_1 * E_2
\end{tikzcd}
$$

Figure 4.2: Left and right multiplication in the tensor groupoid $\mathcal{G}$

Definition 4.1.3. Given a countable semigroup S and a tensor groupoid $\mathcal{G}$, a *product system over S of topological graphs* is a pair (Y, α) in which Y is a collection $(Y_s)_{s\in S}$ of objects in $\mathcal{G}$, and α is a collection $(\alpha_{s,t})_{s,t\in S}$ of isomorphisms $\alpha_{s,t} : Y_s \otimes Y_t \to Y_{st}$ satisfying

$$\alpha_{rs,t}(\alpha_{r,s} \otimes 1_{Y_t}) = \alpha_{r,st}(1_{Y_r} \otimes \alpha_{s,t}) \qquad \text{for } r, s, t \in S.$$

If S has an identity e, we require that $Y_e = 1_{\mathcal{G}}$, and, for each $s \in S$, $\alpha_{e,s}$ and $\alpha_{s,e}$ are precisely λ_{Y_s} and ρ_{Y_s} respectively.

In the following definition and example, we use the language of Fowler and Sims to describe the product systems of A-correspondences that we defined previously in Definition 2.2.1.

Definition 4.1.4. [20, Example 1.5(3)] Let A be a C^*-algebra. Let $\mathcal{G}$ be the category in which the objects are A-correspondences. For objects $X, Y \in \mathcal{G}$, $\hom(X, Y)$ is the set of all inner product-preserving bimodule isomorphisms $X \to Y$. Let $\otimes_A$ denote the balanced tensor product over A. Define maps

$$\mathcal{B} = \mathcal{B}_{X_1,X_2,X_3} : X_1 \otimes (X_2 \otimes X_3) \to (X_1 \otimes X_2) \otimes X_3,$$

$$\lambda = \lambda_X : 1_{\mathcal{G}} \otimes X \to X, \quad \text{and} \quad \rho = \rho_X : X \otimes 1_{\mathcal{G}} \to X$$

to be the natural equivalences determined by

$$\begin{aligned} \mathcal{B}_{X_1,X_2,X_3}(\xi_1 \otimes (\xi_2 \otimes \xi_3)) &= (\xi_1 \otimes \xi_2) \otimes \xi_3, \qquad \text{for } \xi_i \in X_i, \\ \lambda_X(a \otimes \xi) &= a \cdot \xi, \quad \text{and} \\ \rho_X(\xi \otimes a) &= \xi \cdot a. \end{aligned}$$

Then $\langle \mathcal{G}, \otimes_A, A, \mathcal{B}, \lambda, \rho \rangle$ is a *tensor groupoid*, and product systems over the semigroup S that take values in this groupoid are *product systems over S of A-correspondences.*

Example 4.1.5. Let (E, α) be a product system over $\mathbb{N}^k$ of topological graphs. For each $m \in \mathbb{N}^k$, let X_m be the topological graph correspondence of E_m. For $\xi \in X_m$, $\eta \in X_n$, let $\beta(\xi \otimes \eta) = \xi\eta$ where $\xi\eta : \Gamma^{m+n} \to \mathbb{C}$ is given by

$$(\xi\eta)(\lambda) = \xi(\lambda(0, m))\eta(\lambda(m, m + n)).$$

With $\beta_{m,n} : X_m \otimes X_n \to X_{m+n}$ defined as above, (X, β) is a product system over $\mathbb{N}^k$ of $C_0(V)$-correspondences.

4.2 Constructing graphs from product systems and vice versa

To each topological k-graph (Λ, d), we may associate a product system over $\mathbb{N}^k$ of topological graphs as follows. For each $m \in \mathbb{N}^k$, define $E_m := \Lambda^m = d^{-1}(m)$.

With range and source maps inherited from Λ, E_m becomes a topological graph with vertex space V. For each $m, n \in \mathbb{N}^k$, define $\alpha_{m,n} : E_m * E_n \to E_{m+n}$ by

$$\alpha_{m,n}(e, f) := ef.$$

By [55, Lemma 3.1.9], each $\alpha_{m,n}$ is a homeomorphism and a graph isomorphism.

It is straightforward to see that $E_0 = (V, V, \mathrm{id}_V, \mathrm{id}_V)$ by identifying each $v \in V = \mathrm{Obj}(\Lambda)$ with the identity morphism. Finally, if $m \in \mathbb{N}^k$, then

$$\alpha_{0,m}(r(e), e) = r(e)e = e, \quad \text{and}$$

$$\alpha_{m,0}(e, s(e)) = es(e) = e.$$

Thus (E, α) is a product system over $\mathbb{N}^k$ of topological graphs.

Conversely, if (E, α) is a product system over $\mathbb{N}^k$ of topological graphs, we may construct an associated topological k-graph (Λ, d). Let $\mathrm{Obj}(\Lambda) = V$ and set

$$\mathrm{Mor}(\Lambda) = \bigcup_{m \in \mathbb{N}^k} \{m\} \times E_m,$$

where each $\{m\} \times E_m$ is declared to be open in $\mathrm{Mor}(\Lambda)$. That $\mathrm{Mor}(\Lambda)$ is a locally compact Hausdorff space follows since each E_m is. Define $r, s : \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ by

$$r(m, e) = r_{E_m}(e), \quad \text{and}$$

$$s(m, e) = s_{E_m}(e).$$

Define composition $\circ : \Lambda * \Lambda \to \Lambda$ by

$$(m, e)(n, f) = (m + n, \alpha_{m,n}(e, f))$$

where

$$((m, e), (n, f)) \in \Lambda * \Lambda \text{ if and only if } s(m, e) = r(n, f).$$

This is open and continuous since $\alpha_{m,n}$ is a homeomorphism.

Define $d : \Lambda \to \mathbb{N}^k$ by $d(m,e) = m$. That $d : \Lambda \to \mathbb{N}^k$ is a functor follows directly from the definitions.

Via this construction (Λ, d) is a higher-rank topological graph. To see this, first note that $\mathrm{Obj}(\Lambda)$ and $\mathrm{Mor}(\Lambda)$ are locally compact Hausdorff spaces since $\mathrm{Obj}(\Lambda) = V$ and since each E_m is a locally compact Hausdorff space. That $r, s : \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ are continuous and s is a local homeomorphism follows since each r_m, s_m are continuous and each s_m is a local homeomorphism.

Composition $\circ : \Lambda * \Lambda \to \Lambda$ is continuous and open, where $\Lambda * \Lambda$ has the relative topology inherited from the product topology on $\Lambda \times \Lambda$. This follows since each $\alpha_{m,n}$ is a homeomorphism.

The map d is continuous, where $\mathbb{N}^k$ has the discrete topology, since $d^{-1}(m) = \{(m,e) | E \in E_m\} \cong E_m$ for each $m \in \mathbb{N}^k$. Finally, (Λ, d) satisfies the unique factorization property since each of the maps $\alpha_{m,n}$ is a bijection.

4.3 The categories **k-gr** and $\mathbf{ProdSys^k}$

Definition 4.3.1. Let **k-gr**=**k-graph**(**V**) be the category whose objects are topological k-graphs $\Lambda = (\Lambda, d)$ with $\mathrm{Obj}(\Lambda) = V$. Morphisms in **k-gr** are continuous maps $\varphi : \Lambda \to \Lambda'$ satisfying

1. $r_{\Lambda'}(\varphi(\lambda)) = r_\Lambda(\lambda)$
2. $s_{\Lambda'}(\varphi(\lambda)) = s_\Lambda(\lambda)$
3. $d_{\Lambda'}(\varphi(\lambda)) = d_\Lambda(\lambda)$.

Definition 4.3.2. Let $\mathbf{ProdSys^k}$=$\mathbf{ProdSys^k}(\mathbf{V})$ be the category whose objects are product systems E over $\mathbb{N}^k$ of topological graphs with vertex space V. Morphisms in $\mathbf{ProdSys^k}$ are maps $\psi : E \to E'$ such that

1. For each $m \in \mathbb{N}^k$, $\psi_m := \psi|_{E_m} : E_m \to E'_m$ is a topological graph morphism, i.e. ψ_m is continuous and satisfies

    a) $r_{E'_m}(\psi_m(e)) = r_{E_m}(e)$

    b) $s_{E'_m}(\psi_m(e)) = s_{E_m}(e)$

    for each $e \in E_m$, and

2. ψ respects the multiplication in the semigroup E.

Define $S : \textbf{k-gr} \to \textbf{ProdSys}^{\mathbf{k}}$ by $S(\Lambda) = E^\Lambda$ for $\Lambda \in \mathrm{Obj}(\textbf{k-gr})$, where E^Λ is the product system over $\mathbb{N}^k$ of topological graphs given by the construction in Section 4.2. Given a morphism $\varphi : \Lambda \to \Lambda'$, let $S(\varphi) : E^\Lambda \to E^{\Lambda'}$ be the map given by

$$S(\varphi)(e) = \varphi(e), \qquad \text{for } e \in E^\Lambda_m.$$

Remark 4.3.3. To view $\varphi(e)$ as an edge in $E^{\Lambda'}_m$, first observe that if $e \in d_\Lambda^{-1}(m) = E^\Lambda_m$ for some $m \in \mathbb{N}^k$, then there is a path $\lambda \in \Lambda$ with $d_\Lambda(\lambda) = m$, $r_\Lambda(\lambda) = r_E(e)$, and $s_\Lambda(\lambda) = s_E(e)$. Then $\varphi(\lambda) \in \Lambda'$ satisfies $d_{\Lambda'}(\varphi(\lambda)) = d_\Lambda(\lambda) = m$, $r_{\Lambda'}(\varphi(\lambda)) = r_\Lambda(\lambda) = r_E(e)$, and $s_{\Lambda'}(\varphi(\lambda)) = s_\Lambda(\lambda) = r_E(e)$ so we may make the identification $\varphi(e) := \varphi(\lambda)$.

It is straightforward to see $S(\varphi)_m$ is a graph morphism for each $m \in \mathbb{N}^k$ since it preserves both incidence and degree. Similarly, it is straightforward to see $S(\varphi)$ preserves multiplication.

Let $m \in \mathbb{N}^k$ and let $\{e_i\} \subseteq E^\Lambda_m$, $e \in E^\Lambda_m$ with $e_i \to e$. Then

$$\begin{aligned} S(\varphi)(e_i) &= \varphi(e_i) \to \varphi(e), \text{ since } \varphi \text{ is continuous,} \\ &= S(\varphi)(e) \end{aligned}$$

and so $S(\varphi)$ is continuous.

Finally, S preserves identities and composition since

$$\begin{aligned} S(\mathrm{id}_\Lambda)(e) &= \mathrm{id}_\Lambda(e) = e \\ S(\varphi \circ \psi)(e) &= \varphi \circ \psi(e) \\ &= S(\varphi)(\psi(e)) \\ &= S(\varphi) \circ S(\psi)(e). \end{aligned}$$

Define $T : \mathbf{ProdSys^k} \to \mathbf{k\text{-}gr}$ by $T(E) = \Lambda_E$ for $E \in \mathrm{Obj}(\mathbf{ProdSys^k})$, where $\Lambda_E = (\Lambda_E, d_E)$ is the topological k-graph given by the construction in Section 4.2. Given a morphism $\psi : E \to E'$, let $T(\psi) : \Lambda_E \to \Lambda_{E'}$ be the map given by

$$T(\psi)(m, e) = (m, \psi(e)).$$

Then we have

$$\begin{aligned} r_{\Lambda_{E'}}(T(\varphi)(m, e)) &= r_{\Lambda_{E'}}(m, \varphi(e)) \\ &= r_{E'}(\varphi(e)) \\ &= r_E(e) \\ &= r_{\Lambda_E}(m, e). \end{aligned}$$

Similarly $s_{\Lambda_{E'}}(T(\varphi)(m, e)) = s_{\Lambda_E}(m, e)$, and

$$\begin{aligned} d_{\Lambda_{E'}}(T(\varphi)(m, e)) &= d_{\Lambda_{E'}}(m, \varphi(e)) \\ &= m \\ &= d_{\Lambda_E}(m, e). \end{aligned}$$

Let $\{\lambda_i\} = \{(m_i, e_i)\} \subseteq \Lambda_E$, $\lambda = (m, e) \in \Lambda_E$ with $\lambda_i \to \lambda$. Then

$$\begin{aligned} T(\varphi)(\lambda_i) &= T(\varphi)(m_i, e_i) \\ &= (m_i, \varphi(e_i)) \to (m, \varphi(e)), \text{ since } \varphi \text{ is continuous} \end{aligned}$$

$$
\begin{aligned}
&= T(\varphi)(m,e) \\
&= T(\varphi)(\lambda),
\end{aligned}
$$

so $T(\varphi)$ is continuous.

Finally, T preserves identities and composition since

$$
\begin{aligned}
T(\mathrm{id}_E)(m,e) &= (m, \mathrm{id}_E(e)) = (m,e) \\
T(\varphi \circ \psi)(m,e) &= (m, \varphi \circ \psi(e)) \\
&= T(\varphi)(m, \psi(e)) \\
&= T(\varphi) \circ T(\psi)(m,e).
\end{aligned}
$$

Define $\eta : I_{\mathbf{k\text{-}gr}} \to T \circ S$ as follows. For $\Lambda \in \mathbf{k\text{-}gr}$, let $\eta_\Lambda : \Lambda \to \Lambda_{E^\Lambda}$ be given by

$$\lambda \mapsto (d_\Lambda(\lambda), \lambda).$$

Then for $\lambda \in \Lambda$ we have

$$
\begin{aligned}
\eta_{\Lambda'} \circ \varphi(\lambda) &= \eta_{\Lambda'}(\varphi(\lambda)) \\
&= (d_{\Lambda'}(\varphi(\lambda)), \varphi(\lambda)), \text{ and} \\
(T \circ S(\varphi)) \circ \eta_\Lambda(\lambda) &= (T \circ S(\varphi))(d_\Lambda(\lambda), \lambda) \\
&= (d_{\Lambda'}(\varphi(\lambda)), \varphi(\lambda))
\end{aligned}
$$

so that the diagram in Figure 4.3 commutes:

Define $\theta : I_{\mathbf{ProdSys^k}} \to S \circ T$ as follows. For $E \in \mathbf{ProdSys^k}$, let $\theta_E : E \to E^{\Lambda_E}$ be given by

$$e \mapsto (m,e), \qquad \text{for } e \in E_m.$$

Then for $e \in E_m$ we have

$$\theta_{E'} \circ \psi(e) = \theta_{E'}(\psi(e))$$

$$\begin{tikzcd}
\Lambda \arrow[r, "\varphi"] \arrow[d, "\eta_\Lambda"'] & \Lambda' \arrow[d, "\eta_{\Lambda'}"] \\
\Lambda_{E^\Lambda} \arrow[r, "T \circ S(\varphi)"'] & \Lambda_{E^{\Lambda'}}
\end{tikzcd}.$$

Figure 4.3: The natural isomorphism η

$$\begin{aligned} &= (m, \psi(e)), \text{ since } \psi(e) \in E'_m, \text{ and} \\ (S \circ T(\psi)) \circ \theta_E(e) &= (S \circ T(\psi))(m, e) \\ &= (m, \psi(e)) \end{aligned}$$

so that the diagram in Figure 4.4 commutes:

$$\begin{tikzcd}
E \arrow[r, "\psi"] \arrow[d, "\theta_E"'] & E' \arrow[d, "\theta_{E'}"] \\
E^{\Lambda_E} \arrow[r, "S \circ T(\psi)"'] & E^{\Lambda_{E'}}
\end{tikzcd}.$$

Figure 4.4: The natural isomorphism θ

Proposition 4.3.4. *The categories* **k-*gr*** *and* ***ProdSys*$^{\mathbf{k}}$** *are equivalent.*

Proof. It is straightforward to see that η, θ are isomorphisms since for each $E \in$ **ProdSys$^{\mathbf{k}}$**$, \Lambda \in$ **k-gr** the maps η_Λ, θ_E are invertible: the maps $(\eta_\Lambda)^{-1} : \Lambda_{E^\Lambda} \to \Lambda$ and $(\theta_E)^{-1} : E^{\Lambda_E} \to E$ are given by

$$(d(\lambda), \lambda) \mapsto \lambda \quad \text{and} \quad (m, e) \mapsto e, \text{ respectively.}$$

The equivalence of categories follows since the natural transformations η and θ are isomorphisms. $\square$

4.4 Compactly aligned product systems of topological graphs

In this section, we define compactly aligned for a product system of topological graphs. We show in Proposition 4.4.5 that this notion of compactly aligned coincides with the definitions of compactly aligned for both the associated topological k-graph and topological k-graph correspondence.

Recall that if (Λ, d) is a topological k-graph, we say that (Λ, d) is *compactly aligned* if for all $p, q \in \mathbb{N}^k$ and for all compact $U \subseteq \Lambda^p, V \subseteq \Lambda^q$, the set

$$U \vee V = U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q}$$

is compact. Since we have seen that topological k-graphs are essentially product systems over $\mathbb{N}^k$ of topological graphs with common vertex space, we would like an equivalent notion of compactly aligned for these objects. Additionally, we have an associated product system over $\mathbb{N}^k$ of $C_0(V)$-correspondences.

Definition 4.4.1. Given a topological k-graph (Λ, d), let (E, α) be the associated product system over $\mathbb{N}^k$ of topological graphs. We may construct a product system over $\mathbb{N}^k$ of $C_0(V)$-correspondences as in 4.1.5. This product system is called the *topological k-graph correspondence associated to* Λ.

Definition 4.4.2. Let (E, β) be a product system over $\mathbb{N}^k$ of topological graphs with common vertex space V. We say that (E, β) is *compactly aligned* if for all $p, q \in \mathbb{N}^k$ and for all compact $U \subseteq E_p^1, V \subseteq E_q^1$, the set

$$\mathrm{MCE}(U, V) = \{\gamma \in E^1_{(p\vee q)} : \gamma(0, p) \in U, \gamma(0, q) \in V\}$$

is compact.

Remark 4.4.3. When V has the discrete topology, (E, β) is a product system over $\mathbb{N}^k$ of (directed) graphs and the definition above is equivalent to [40, Definition 5.3].

Definition 4.4.4. [16, Definition 5.7] Let (X, α) be a product system over $\mathbb{N}^k$ of A-correspondences (also called a product system over $\mathbb{N}^k$ of right-Hilbert bimodules over A). We say that (X, α) is *compactly aligned* if for all $p, q \in \mathbb{N}^k$ and for all $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$, we have

$$(S \otimes 1_{(p\vee q)-p})(1_{(p\vee q)-q} \otimes T) \in \mathcal{K}(X_{(p\vee q)}).$$

Proposition 4.4.5. *Let (Λ, d) be a topological k-graph, (E, β) the associated product system over $\mathbb{N}^k$ of topological graphs with vertex space Λ^0, and (X, α) the topological k-graph correspondence.*

The following are equivalent:

1. *(Λ, d) is compactly aligned;*
2. *(E, β) is compactly aligned;*
3. *(X, α) is compactly aligned.*

Proof. From [7, Proposition 5.15], we have that (X, α) is compactly aligned if and only if (Λ, d) is, and hence $1 \iff 3$.

To see that $1 \iff 2$, note first that for all $p \in \mathbb{N}^k$, $E_p^1 = \Lambda^p$. Then if $U \subset \Lambda^p = E_p^1, V \subseteq \Lambda^q = E_q^1$, we may consider the sets $U \vee V$ and $\mathrm{MCE}(U, V)$. These are both subsets of $\Lambda^{(p\vee q)} = E^1_{(p\vee q)}$ and

$$\begin{aligned}\gamma \in U \vee V &\iff \gamma \in U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q} \\ &\iff \gamma \in U\Lambda^{(p\vee q)-p} \quad \text{and} \quad \gamma \in V\Lambda^{(p\vee q)-q} \\ &\iff \gamma(0,p) \in U \quad \text{and} \quad \gamma(0,q) \in V\end{aligned}$$

$$\iff \gamma \in \mathrm{MCE}(U, V).$$

Thus $U \vee V$ is compact if and only if $\mathrm{MCE}(U, V)$ is, and hence (Λ, d) is compactly aligned if and only if (E, β) is. $\square$

Chapter 5

OBSTRUCTIONS TO A GENERAL CHARACTERIZATION

The research described in this chapter was conducted in collaboration with Steve Kaliszewski and John Quigg. The results appear in [25].

In this chapter, we investigate the issue of characterizing the C^*-correspondences that come from graphs in two more general contexts: that of *k-graphs*, the higher-dimensional analogues of directed graphs defined in Definition 2.3.4; and that of *topological graphs*, the continuous analogues described in Definition 2.3.10. It turns out that in each case the most naïve characterization fails: there are C^*-correspondences that are not isomorphic to any graph correspondence. However, we can in special cases identify obstructions that may lead to a general characterization.

5.1 Higher-rank graph correspondences

Throughout this section, we consider a fixed countable set V. After a brief discussion of product systems, we show how each k-graph with vertex set V gives rise to a product system over $\mathbb{N}^k$ of $c_0(V)$-correspondences, which we call a *k-graph correspondence*. Recall from Definition 2.2.2 that any product system over $\mathbb{N}^k$ of $c_0(V)$-correspondences is determined, up to isomorphism, by its *skeleton*. We describe the skeleton of a k-graph correspondence in terms of the structure of the k-graph and use this to identify an invariant that characterizes certain product systems. Furthermore, we describe the C^*-algebras associated to product systems over $\mathbb{N}^2$ of 1-dimensional $\mathbb{C}$-correspondences in terms of this invariant.

Recall from Definition 4.1.1 that if E and F are directed graphs with vertex set V, the *fibred product* $E * F$ is the graph with vertex set V, edge set

$$(E * F)^1 = \{(e, f) \in E^1 \times F^1 \mid s_E(e) = r_F(f)\},$$

and range and source maps r and s given by

$$r(e, f) = r_E(e) \quad \text{and} \quad s(e, f) = s_F(f).$$

If $\phi\colon E \to E'$ and $\psi\colon F \to F'$ are graph homomorphisms that agree on V, then the fibred product homomorphism $\phi * \psi\colon E * F \to E' * F'$ defined on edges by $\phi * \psi(e, f) = (\phi(e), \psi(f))$ and on vertices by $v \mapsto \phi(v) = \psi(v)$ is easily seen to be a graph homomorphism. (This construction makes the set of directed graphs having vertex set V into a *tensor groupoid* in the language of [20].)

As described in Chapter 4, if S is a countable semigroup with identity e, then a *product system over S of graphs on V* is a collection $E = \{E_s \mid s \in S\}$ of directed graphs E_s with vertex set V, together with a collection $\alpha = \{\alpha_{s,t} \mid s, t \in S\}$ of vertex-fixing graph isomorphisms $\alpha_{s,t}\colon E_s * E_t \to E_{st}$ such that $E_e = (V, V, \mathrm{id}_V, \mathrm{id}_V)$; $\alpha_{e,s}\colon E_e * E_s \to E_s$ and $\alpha_{s,e}\colon E_s * E_e \to E_s$ are the natural maps given on edges by $(r(e), e) \mapsto e$ and $(e, s(e)) \mapsto e$ for each $s \in S$; and the *associativity condition*

$$\alpha_{rs,t} \circ (\alpha_{r,s} * \mathrm{id}_t) = \alpha_{r,st} \circ (\mathrm{id}_r * \alpha_{s,t}) \tag{5.1}$$

holds for all $r, s, t \in S$, where id_t is the identity map on E_t. (This definition is a special case of [20, Definition 1.1].)

As observed in Chapter 4, product systems over $\mathbb{N}^k$ of graphs are essentially the same as k-graphs. In more detail, suppose (Λ, d) is a *k-graph with vertex set V*: so

Λ is a countable category, $d\colon \Lambda \to \mathbb{N}^k$ is the degree functor, and we have identified the object set $d^{-1}(0)$ with V. For each $m \in \mathbb{N}^k$, the set $d^{-1}(m)$ is the edge set of a directed graph E_m with vertex set V and range and source maps inherited from Λ. The collection $E = \{E_m \mid m \in \mathbb{N}^k\}$ together with the vertex-fixing isomorphisms $\alpha_{m,n}\colon E_m * E_n \to E_{m+n}$ given on edges by

$$\alpha_{m,n}(\mu, \nu) = \mu\nu$$

is then a product system over $\mathbb{N}^k$ of graphs on V; moreover, every such product system arises in this way from a k-graph with vertex set V.

Retaining the above notation, for each $m \in \mathbb{N}^k$, let X_m be the $c_0(V)$-correspondence associated to the graph E_m; recall from Definition 2.3.13 that

$$X_m = \Big\{\xi\colon E_m^1 \to \mathbb{C} \mid \text{the map } v \mapsto \sum_{s(e)=v} |\xi(e)|^2 \text{ is in } c_0(V)\Big\},$$

with module actions and $c_0(V)$-valued inner product given by

$$(f \cdot \xi \cdot g)(e) = f(r(e))\xi(e)g(s(e)) \quad \text{and} \quad \langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e).$$

Note that X_m is densely spanned by the set $\{\chi_e \mid e \in E_m^1\}$, where χ_e denotes the characteristic function of $\{e\}$. Further, if $p_v \in c_0(V)$ denotes the characteristic function of a vertex $v \in V$, then for any $e \in E_m^1$ and $f \in E_n^1$ we have

$$\chi_e \otimes \chi_f = \chi_e \cdot p_{s(e)} \otimes p_{r(f)} \cdot \chi_f = \chi_e \cdot p_{s(e)} p_{r(f)} \otimes \chi_f = 0$$

in $X_m \otimes_{c_0(V)} X_n$ unless $s(e) = r(f)$; thus the balanced tensor product is densely spanned by the set

$$\{\chi_e \otimes \chi_f \mid (e, f) \in (E_m * E_n)^1\}.$$

For each $m, n \in \mathbb{N}^k$ let $\beta_{m,n}\colon X_m \otimes_{c_0(V)} X_n \to X_{m+n}$ be the isomorphism determined by

$$\beta_{m,n}(\chi_e \otimes \chi_f) = \chi_{\alpha_{m,n}(e,f)} = \chi_{ef} \quad \text{for } (e, f) \in (E_m * E_n)^1. \tag{5.2}$$

Then $X = \{X_m \mid m \in \mathbb{N}^k\}$ with $\beta = \{\beta_{m,n} \mid m, n \in \mathbb{N}^k\}$ is the *k-graph correspondence* associated to (Λ, d).

Recall from Definition 2.2.2 that every product system (X, β) over $\mathbb{N}^k$ may be viewed in terms of its skeleton (Y, T). The following proposition shows how the skeleton encodes the "commuting squares" (that is, the factorizations of paths of degree $e_i + e_j$) of a k-graph.

Proposition 5.1.1. *Suppose (Λ, d) is a k-graph with vertex set V, let (E, α) be the associated product system over $\mathbb{N}^k$ of graphs, let (X, β) be the associated k-graph correspondence, and let (Y, T) be the skeleton of (X, β). Then for each $1 \leq i < j \leq k$ and each $e \in E^1_{e_i}$ and $f \in E^1_{e_j}$ we have*

$$T_{i,j}(\chi_e \otimes \chi_f) = \chi_{\tilde{f}} \otimes \chi_{\tilde{e}},$$

where $\tilde{f}$ and $\tilde{e}$ are the unique edges in E_{e_j} and E_{e_i}, respectively, such that $\tilde{f}\tilde{e} = ef$ in $E_{e_i+e_j}$.

Proof. By Equation 5.2, we have $\beta_{e_i,e_j}(\chi_e \otimes \chi_f) = \chi_{ef}$ in $X_{e_i+e_j}$, and similarly $\beta_{e_j,e_i}(\chi_{\tilde{f}} \otimes \chi_{\tilde{e}}) = \chi_{\tilde{f}\tilde{e}} = \chi_{ef}$. Since $T_{i,j} = \beta^{-1}_{e_j,e_i} \circ \beta_{e_i,e_j}$ by definition, the result follows. $\square$

One-dimensional product systems

The notion of skeleton leads to an invariant that characterizes certain product systems. Specifically, let (X, β) be a product system over $\mathbb{N}^2$ of 1-dimensional $\mathbb{C}$-correspondences, and let (Y, T) be its skeleton; thus Y_1 and Y_2 are 1-dimensional, and T consists of a single $\mathbb{C}$-correspondence isomorphism between 1-dimensional spaces:

$$T_{1,2} \colon Y_1 \otimes Y_2 \to Y_2 \otimes Y_1.$$

It follows that $T_{1,2} = \omega_X \Sigma$ for some scalar ω_X, where $\Sigma \colon Y_1 \otimes Y_2 \to Y_2 \otimes Y_1$ is the flip map; since isomorphisms of C^*-correspondences are inner-product preserving (and hence isometries), we must have $\omega_X \in \mathbb{T}$.

Theorem 5.1.2. *The assignment* $(X, \beta) \mapsto \omega_X$ *gives a complete isomorphism invariant for* 1*-dimensional* $\mathbb{C}$*-correspondences. Moreover,* (X, β) *is isomorphic to a* 2*-graph correspondence if and only if* $\omega_X = 1$.

Proof. Suppose (Z, γ) is another product system over $\mathbb{N}^2$ of 1-dimensional $\mathbb{C}$-correspondences, let (W, R) be the skeleton of (Z, γ), and let ω_Z be the associated scalar. It follows from [20] that $(X, \beta) \cong (Z, \gamma)$ if and only if there are isomorphisms $\theta_i : Y_i \to W_i$ as in Figure 2.1 such that

$$(\theta_2 \otimes \theta_1) \circ T_{1,2} = R_{1,2} \circ (\theta_1 \otimes \theta_2). \tag{5.3}$$

Further, since by construction we have

$$(\theta_2 \otimes \theta_1) \circ T_{1,2} = (\theta_2 \otimes \theta_1) \circ (\omega_X \Sigma) = \omega_X \Sigma \circ (\theta_1 \otimes \theta_2)$$

and

$$R_{1,2} \circ (\theta_1 \otimes \theta_2) = \omega_Z \Sigma \circ (\theta_1 \otimes \theta_2),$$

where in each case Σ denotes the appropriate flip map, we see that, given the existence of isomorphisms $\theta_i : Y_i \to W_i$, condition Equation 5.3 holds if and only if $\omega_X = \omega_Z$. Now, by dimensionality we can always choose $\mathbb{C}$-correspondence isomorphisms $\theta_i \colon Y_i \to W_i$, so we conclude that $(X, \beta) \cong (Z, \gamma)$ if and only if $\omega_X = \omega_Z$.

Finally, suppose (X, β) is the graph correspondence associated to a 2-graph (Λ, d) with vertex set V. Then $c_0(V) = \mathbb{C}$, so we must have $V = \{v\}$ for a single vertex v. Further, since each X_{e_i} is 1-dimensional, the associated directed graphs E_{e_i} must each consist of a single loop edge at the vertex v, say $E^1_{e_1} = \{e\}$ and

$E^1_{e_2} = \{f\}$. In particular, we must have $ef = fe$ in $E_{e_1+e_2}$, and therefore

$$T_{1,2}(\chi_e \otimes \chi_f) = \chi_f \otimes \chi_e = \Sigma(\chi_e \otimes \chi_f)$$

by Proposition 2.2.2. Thus $\omega_X = 1$ by definition. □

Corollary 5.1.3. *Let V have cardinality one. There exist uncountably many non-isomorphic product systems over $\mathbb{N}^2$ of $c_0(V)$-correspondences that are not isomorphic to 2-graph correspondences.*

Proof. In fact, we will show that the range of the invariant ω from Theorem 5.1.2 is all of $\mathbb{T}$: for any $\omega \in \mathbb{T}$, set $Y_1 = Y_2 = \mathbb{C}$ (with the canonical $\mathbb{C}$-correspondence structure), and define $T_{1,2}\colon Y_1 \otimes Y_2 \to Y_2 \otimes Y_1$ by

$$T_{1,2} = \omega\Sigma.$$

Then it is easy to check that $T_{1,2}$ is a $\mathbb{C}$-correspondence isomorphism, and the hexagonal equations as in Equation 2.9 are vacuous; it follows from [20, Theorem 2.1] that there is a product system (X, β) over $\mathbb{N}^2$ of $\mathbb{C}$-correspondences with skeleton (Y, T), and therefore such that $\omega_X = \omega$. □

Remark 5.1.4. It follows from [20] that the skeleton of a product system (X, β) over $\mathbb{N}$ of $c_0(V)$-correspondences is just a single correspondence Y; then by Theorem 3.2.1 we have $Y \cong X_E$ for some directed graph E, and thus we quickly conclude that (X, β) is isomorphic to a 1-graph correspondence. More generally, every product system (X, β) over a finitely-generated free semigroup S of $c_0(V)$-correspondences is isomorphic to one associated with a product system over S of graphs. To see this, note that, in the terminology of Fowler-Sims [20], S is a right-angled Artin semigroup with no commutation relations, so the skeleton Y of (X, β) has no $T_{i,j}$'s; again by [24] each Y_i is isomorphic to a graph correspondence X_{E_i}, and the E_i's form the skeleton of a product system over S of graphs, whose associated product system over S of correspondences can be checked to be isomorphic to (X,β) by examining the skeletons. Thus when the semigroup S is free and finitely-generated there is no obstruction to a product system over S of $c_0(V)$-correspondences coming from a product system over S of graphs.

Product systems over $\mathbb{N}^2$ of 1-dimensional $\mathbb{C}$-correspondences

Fix $\omega \in \mathbb{T}$. Let X be any representative of the isomorphism class of product systems over $\mathbb{N}^2$ of 1-dimensional $\mathbb{C}$-correspondences with invariant $\omega_X = \omega$.

Proposition 5.1.5. *$\mathcal{O}_X$ is isomorphic to the rotation algebra A_ω*

In order to simplify our proof of this, we include the following lemma.

Lemma 5.1.6. *Let $\psi : X \to B$ be a (Toeplitz) representation of X such that B is generated as a C^*-algebra by ψ. Then $B = C^*(\psi_{e_1}(1_{e_1}), \psi_{e_2}(1_{e_2}))$.*

Proof. The result above follows immediately since $X_{e_i} \cong \mathbb{C}$ for $i = 1,2$ and ψ_{e_i} is linear and ψ_0 is a unital homomorphism on the coefficient algebra $\mathbb{C}$. $\square$

Proof of Proposition 5.1.5. A_ω is the universal C^*-algebra generated by two unitaries U,V satisfying $UV = \omega VU$, so we must first show that $\mathcal{O}_X$ is generated by two universal such unitaries.

Let $j_X : X \to \mathcal{O}_X$ be the (Toeplitz) representation of X that is universal for Cuntz-Pimsner covariant (Toeplitz) representations (existence of this is given by [16, Proposition 2.9]). By Lemma 5.1.6, $\mathcal{O}_X$ is generated as a C^*-algebra by $\{j_{X_{e_1}}(1_{e_1}), j_{X_{e_2}}(1_{e_2})\}$. Set $U = j_{X_{e_1}}(1_{e_1})$ and $V = j_{X_{e_2}}(1_{e_2})$.

We may compute

$$\begin{aligned} UU^* &= j_{X_{e_1}}(1_{e_1}) j_{X_{e_1}}(1_{e_1})^* \\ &= j_{X_{\mathbf{0}}}(\langle 1_{e_1}, 1_{e_2}\rangle_{X_1}), \text{ by (T2)} \end{aligned}$$

$$
\begin{aligned}
&= j_{X_{\mathbf{0}}}(1), \text{ by the definition of the inner product on } X_1 \\
&= 1, \text{ since } j_X \text{ is a unital homomorphism on } \mathbb{C}. \\
U^*U &= j_{X_{e_1}}(1_{e_1})^* j_{X_{e_1}}(1_{e_1}) \\
&= j_X^{(e_1)}(\Theta_{1_{e_1},1_{e_1}}), \text{ by definition of } j_X^{(e_1)} \\
&= j_X^{(e_1)}(\phi_{e_1}(1)), \text{ since } \Theta_{1_{e_1},1_{e_1}} \text{ is given by multiplication by } 1 \\
&= j_{X_{\mathbf{0}}}(1), \text{ since } j_X \text{ is Cuntz-Pimsner covariant} \\
&= 1, \text{ since } j_X \text{ is a unital homomorphism on } \mathbb{C}.
\end{aligned}
$$

Similar calculations show that $VV^* = V^*V = 1$. Finally, note that

$$
\begin{aligned}
UV &= j_{X_{e_1}}(1_{e_1}) j_{X_{e_2}}(1_{e_2}) \\
&= j_{X_{(1,1)}}(1_{e_1} \otimes 1_{e_2}) \\
&= j_{X_{(1,1)}}(\omega(1_{e_2} \otimes 1_{e_1})), \text{ by definition of the isomorphism } T_{1,2} \\
&= \omega(j_X)_{(1,1)}(1_{e_2} \otimes 1_{e_1}), \text{ since } j_X \text{ is linear} \\
&= \omega j_{X_{e_2}}(1_{e_2}) j_{X_{e_1}}(1_{e_1}) \\
&= \omega VU.
\end{aligned}
$$

By the universal property of A_ω, there is a surjective homomorphism of A_ω onto $\mathcal{O}_X$. If ω is irrational, we are done since in this case A_ω is simple.

To show that the result holds for arbitrary $\omega \in \mathbb{T}$, we must show that $\mathcal{O}_X$ is universal for pairs of unitaries U, V satisfying $UV = \omega VU$. It suffices to show that if $B = C^*(\widetilde{U}, \widetilde{V})$ where $\widetilde{U}\widetilde{V} = \omega \widetilde{V}\widetilde{U}$ and $\psi : X \to B$ is a representation of X such that B is generated as a C^*-algebra by ψ, then ψ is Cuntz-Pimsner covariant in the sense of (CP-K).

In particular, it is enough to show that if $B = C^*(\psi_{e_1}(1_{e_1}), \psi_{e_2}(1_{e_2}))$ where $\psi_{e_1}(1_{e_1})\psi_{e_2}(1_{e_2}) = \omega \psi_{e_2}(1_{e_2})\psi_{e_1}(1_{e_1})$, then ψ is Cuntz-Pimsner covariant. Note that for all $m \in \mathbb{N}^2$, ϕ_m is given by multiplication so $\phi_m^{-1}(\mathcal{K}(X_m)) \cap (\ker \phi_m)^\perp$ is just $X_m \cong \mathbb{C}$. Moreover $\phi_m(a) = \Theta_{a,1}$ for all $a \in X_m$ since if $b \in X_m$ we have

$$\phi_m(a)(b) = ab = a \cdot \langle 1, b \rangle = \Theta_{a,1}(b).$$

Note that since ψ is a (Toeplitz) representation, we may write

$$\psi_m(1_m) = \omega^{r_m} \underbrace{\psi_{e_1}(1_{e_1}) \cdots \psi_{e_1}(1_{e_1})}_{m_1 \text{ factors}} \underbrace{\psi_{e_2}(1_{e_2}) \cdots \psi_{e_2}(1_{e_2})}_{m_2 \text{factors}}$$

for some $r_m \in \mathbb{N}$, and hence

$$\psi_m(1_m)^* = \overline{\omega}^{r_m} \underbrace{\psi_{e_2}(1_{e_2})^* \cdots \psi_{e_2}(1_{e_2})^*}_{m_2 \text{ factors}} \underbrace{\psi_{e_1}(1_{e_1})^* \cdots \psi_{e_1}(1_{e_1})^*}_{m_1 \text{factors}}$$

so that $\psi_m(1_m)\psi_m(1_m)^* = 1$. Then we may compute for $a \in X_m$

$$\begin{aligned} \psi^{(m)}(\phi_m(a)) &= \psi^{(m)}(\Theta_{a,1_m}) \\ &= \psi_m(a)\psi_m(1_m)^* \\ &= \psi_0(a)\psi_m(1_m)\psi_m(1_m)^* \\ &= \psi_0(a). \end{aligned}$$

Thus ψ is Cuntz-Pimsner covariant. Since $\mathcal{O}_X$ is universal for Cuntz-Pimsner covariant representations of X, it follows that $\mathcal{O}_X$ is universal for pairs U, V satisfying $UV = \omega VU$ and hence $\mathcal{O}_X \cong A_\omega$. □

5.2 Topological graph correspondences

Let V be a locally compact Hausdorff space, and let $E = (V, E^1, r, s)$ be a topological graph with vertex space V. Recall from Definition 2.3.13 that the associated *topological graph correspondence* is the space

$$X_E = \Big\{ \xi \in C(E) \mid \text{the map } v \mapsto \sum_{e \in s^{-1}(v)} |\xi(e)|^2 \text{ is in } C_0(V) \Big\},$$

with module actions and $C_0(V)$-valued inner product given by the same formulas (see Section 5.1) as in the case of a (discrete) directed graph.

We will show that in general not all $C_0(V)$-correspondences arise in this way, and this time the obstruction comes from the topology of V, namely its Picard group, or, more precisely (and equivalently, by Raeburn's version of Swan's theorem), the Picard group of $C_0(V)$.

We use the convention that $C_0(V)$-imprimitivity bimodules are complete[1], so that such an imprimitivity bimodule X is a $C_0(V)$-correspondence such that the homomorphism $\phi_{C_0(V)} : C_0(V) \to \mathcal{L}(X)$ associated to the left-module action is in fact an isomorphism onto $\mathcal{K}(X)$. By [43, Corollary 3.33], every $C_0(V)$-imprimitivity bimodule X induces a homeomorphism of V, called the *Rieffel homeomorphism*, which, by [43, Proposition 5.7], is the identity map if and only if X is *symmetric*[2] in the sense that

$$f \cdot \xi = \xi \cdot f \quad \text{for all} \quad f \in C_0(V), \xi \in X.$$

The *Picard group of* $C_0(V)$, denoted $\operatorname{Pic} C_0(V)$, is the group of isomorphism classes of $C_0(V)$-imprimitivity bimodules with group operation balanced tensor product over $C_0(V)$. Raeburn shows in [38, Proposition A.1] that $\operatorname{Pic} C_0(V)$ is isomorphic to the algebraic Picard group, comprising isomorphism classes of invertible $C_0(V)$-bimodules. The imprimitivity bimodules which are symmetric determine a normal subgroup of $\operatorname{Pic} C_0(V)$, which, by Raeburn's version of Swan's theorem (see [38, Proposition A.3]) is isomorphic to the Picard group of V, consisting of isomorphism classes of complex line bundles over V with group operation fibrewise tensor product. In fact, by [1, Theorem 1.12], $\operatorname{Pic} C_0(V)$ is a semidirect product of its symmetric part by the automorphism group of $C_0(V)$. This fact is stated for the special case of compact V in [4, Page 357].

The identity element of $\operatorname{Pic} C_0(V)$ is the class of the *trivial* $C_0(V)$-imprimitivity

[1] In some earlier literature on imprimitivity bimodules between C^*-algebras, for example [38, 4], this convention is not used.

[2] We are borrowing this terminology from [1].

bimodule, namely $C_0(V)$ itself with operations

$$f \cdot g = fg, \quad \langle f, g\rangle_{C_0(V)} = \bar{f}g, \quad \text{and} \quad {}_{C_0(V)}\langle f, g\rangle = f\bar{g}.$$

We call the element of $\operatorname{Pic} C_0(V)$ determined by a $C_0(V)$-imprimitivity bimodule X the *Picard invariant* of X, and we call the Picard invariant *trivial* if it is the identity element of $\operatorname{Pic} C_0(V)$.

In the proof of Theorem 5.2.1 below we will need the concept of *vertex-fixing isomorphism* of topological graphs: suppose E and F are topological graphs with vertex space V. By a vertex-fixing isomorphism from E to F we mean a homeomorphism $\phi : E^1 \to F^1$ such that the diagram in Figure 5.1 commutes.

$$\begin{tikzcd} V & E^1 \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "\phi"] & V \\ & F^1 \arrow[ul, "r"] \arrow[ur, "s"'] & \end{tikzcd}$$

Figure 5.1: A vertex-fixing isomorphism from E to F

Obviously, a vertex-fixing isomorphism $E \cong F$ gives rise to an isomorphism $X_E \cong X_F$ of graph correspondences.

Theorem 5.2.1. *Let V be a locally compact Hausdorff space. A symmetric $C_0(V)$-imprimitivity bimodule X is isomorphic to a topological graph correspondence if and only if its Picard invariant is trivial. Thus, if $C_0(V)$ has nontrivial Picard group, then there are $C_0(V)$-correspondences that are not isomorphic to topological graph correspondences associated to topological graphs with vertex space V.*

Proof. It is clear that the trivial $C_0(V)$-imprimitivity bimodule is isomorphic to X_E, where $E = (V, V, \mathrm{id}, \mathrm{id})$. Thus, if X has trivial Picard invariant, we have $X \cong X_E$.

Conversely, fix a symmetric $C_0(V)$-imprimitivity bimodule X, and suppose E is a topological graph with vertex space V such that $X \cong X_E$. It suffices to find a vertex-fixing isomorphism $E \cong (V, V, \mathrm{id}, \mathrm{id})$, for then X is isomorphic to the trivial $C_0(V)$-correspondence $C_0(V)$, so has trivial Picard invariant.

Furthermore, it suffices to show that the source map $s : E^1 \to V$ is a homeomorphism. To see this, note that the symmetry property of X — $f \cdot \xi = \xi \cdot f$ for $f \in C_0(V)$ and $\xi \in X$ — implies that $r = s$. Thus, if it is a homeomorphism, $s : E^1 \to V$ will be a vertex-fixing isomorphism of E onto the topological graph $(V, V, \mathrm{id}, \mathrm{id})$.

So, it remains to show that s is a homeomorphism; for this, it suffices to show that for every $v \in V$ the set $s^{-1}(v)$ has cardinality at most 1. For then s is an injective local homeomorphism, hence a homeomorphism of E^1 onto an open subset U of V corresponding to the ideal of $C_0(V)$ generated by the inner product on X. But X is full, being an imprimitivity bimodule, so we must have $U = V$.

It is obvious that the linear map

$$\xi \mapsto \xi|_{s^{-1}(v)} : X \to \ell^2(s^{-1}(v))$$

has range containing $c_c(s^{-1}(v))$ (because for $e \in s^{-1}(v)$ there exists $\xi \in c_c(E^1)$ such that $\xi(e) = 1$ and $\xi(e') = 0$ for all other $e' \in s^{-1}(v)$, since $s^{-1}(v)$ has no accumulation points in E^1) and kernel containing $X \cdot J_v$, where J_v is the maximal ideal of $C_0(V)$ comprising the functions vanishing at v. By [43, Lemma 5.12], the quotient $X/(X \cdot J_v)$ is a $C_0(V)/J_v$-imprimitivity bimodule. But $C_0(V)/J_v \cong \mathbb{C}$, and the Rieffel correspondence (for example) can be used to show that every $\mathbb{C}$-imprimitivity bimodule is 1-dimensional (because every closed subspace would automatically be a sub-bimodule, which corresponds to an ideal of $\mathbb{C}$). It follows that $c_c(s^{-1}(v))$ is has dimension at most 1, so $s^{-1}(v)$ has cardinality at most 1. $\square$

Remark 5.2.2. It follows from Theorem 5.2.1 that a (possibly non-symmetric)

$C_0(V)$-imprimitivity bimodule is isomorphic to a topological graph correspondence if and only if its Picard invariant is in $\operatorname{Aut} C_0(V)$; in fact, it follows from the above proof that in this case the topological graph can be assumed to be of the form (V, V, h, id), where $h : V \to V$ is a homeomorphism.

Remark 5.2.3. The result of Theorem 5.2.1 raises a natural question about the associated Cuntz-Pimsner algebra in the case where X is a $C_0(V)$-correspondence that does not arise from a topological graph. A result of Vasselli in [50] gives some insight. In particular, if V is a locally compact space and X is a symmetric $C(V)$-imprimitivity bimodule, then Raeburn's generalization of Swan's theorem [38, Proposition A.3] gives that X has the form $\Gamma_0(B)$ for some complex line bundle B over V. If V is in fact paracompact, then B may be given a hermitian structure (see for example [43, Hooptedoodle 4.52]). Results of Vasselli (in particular [50, Proposition 4.3]) when V is compact give that the associated Cuntz-Pimsner algebra is commutative and has as its spectrum the associated sphere bundle B_1. It seems reasonable to conjecture that Vasselli's results remain true when the compactness assumption on V is lifted.

Chapter 6

TOPOLOGICAL k-GRAPHS AND EXEL-LARSEN SYSTEMS

The research described in this chapter was conducted in collaboration with Cindy Farthing and Paulette Willis. The results appear in [15].

In this chapter, we consider a topological higher-rank graph Λ constructed from the data of a locally compact Hausdorff space and k pairwise commuting local homeomorphisms. In our main result (Theorem 6.3.5), we show that the k-graph C^*-algebra $C^*(\Lambda)$ is isomorphic to a semigroup crossed product in the sense of Larsen. To be assured that $C^*(\Lambda)$ may be constructed, we show that the graph is compactly aligned (Proposition 6.1.5). This generalizes a result of Willis in [52] in which she essentially shows that the result holds when $k = 2$, Ω is compact, and the maps T_1, T_2 $*$-commute.

In [7], the authors show that when Λ is a compactly aligned topological k-graph, the C^*-algebra $C^*(\Lambda)$ constructed from the boundary path groupoid is isomorphic to the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_{X^\Lambda}$, where X^Λ is the topological k-graph correspondence associated to Λ. We show in Proposition 6.2.5 that the product system X^{Lar} arising from an associated Exel-Larsen system is isomorphic to the topological k-graph correspondence X^Λ so that the associated Cuntz-Pimsner algebras are isomorphic. To show that $\mathcal{NO}_X$ is isomorphic to the Larsen crossed product, we show that representations of X are CNP-covariant if and only if they are coisometric on a certain family of ideals in $C_0(\Omega)$. In Theorem 6.3.5, we use the universal properties of the two C^*-algebras to conclude that $\mathcal{NO}_X$ and the Larsen crossed product are isomorphic.

6.1 Topological k-graphs and Exel-Larsen systems associated to (Ω, Θ)

Let Ω be a locally compact Hausdorff space and let $T_1, \ldots, T_k : \Omega \to \Omega$ be pairwise commuting local homeomorphisms. In order to simplify notation, we define for each $m = (m_1, \ldots, m_k) \in \mathbb{N}^k$ a local homeomorphism $\Theta_m : \Omega \to \Omega$ by

$$\Theta_m(x) = T_1^{m_1} \cdots T_k^{m_k}(x). \tag{6.1}$$

Given the above information, we construct a topological k-graph $\Lambda = (\Lambda(\Omega, \Theta), d)$ as in [54, Example 2.5(iv)]. Specifically, we have

- $\mathrm{Obj}(\Lambda) = \Omega$
- $\mathrm{Mor}(\Lambda) = \mathbb{N}^k \times \Omega$, with the product topology
- $r(n, x) = x$ and $s(n, x) = \Theta_n(x)$
- Composition is given by

$$(n, x) \circ (m, \Theta_n(x)) = (n + m, x)$$

- The degree map is defined by $d(n, x) = n$.

Remark 6.1.1. In [54, Example 7.1(iii)], Yeend describes the associated C^*-algebra in the case where the maps T_i are surjective homeomorphisms. When the maps are surjective, the topological k-graph Λ has no sources. As a result, the boundary path groupoid is amenable. When the maps are also homeomorphisms, there is an induced action α of $\mathbb{Z}^k$ on $C_0(\Omega)$ defined by

$$\alpha_m(f)(x) = f(\Theta_m(x)),$$

with universal crossed product $(C_0(\Omega) \rtimes_\alpha \mathbb{Z}^k, j_{C_0(\Omega)}, j_{\mathbb{Z}^k})$. Yeend asserts that the topological k-graph C^*-algebra is isomorphic to this crossed product. Our main result in this chapter, Theorem 6.3.5, generalizes this to the setting where the maps are local homeomorphisms that are not necessarily surjective.

It is important to verify that we may in fact construct the associated topological k-graph C^*-algebra $C^*(\Lambda)$. In order to establish this, we begin by showing that it is proper in the sense of Definition 6.1.2 below. We then prove that every proper topological k-graph is compactly aligned (see Definition 2.3.22). As discussed earlier, the compactly aligned condition ensures that the boundary path groupoid $\mathcal{G}_\Lambda$ is a locally compact r-discrete groupoid admitting a Haar system and hence that the associated C^*-algebra $C^*(\Lambda)$ may be defined.

Definition 6.1.2. A topological k-graph Λ is said to be *proper* if for all $m \in \mathbb{N}^k$, the map $r|_{\Lambda^m}$ is a proper map. That is, if for every $m \in \mathbb{N}^k$ and compact $U \subset \Lambda^0$, the set $U\Lambda^m$ is compact.

Lemma 6.1.3. *The topological k-graph $\Lambda = (\Lambda(\Omega, \Theta), d)$ is proper.*

Proof. Fix $m \in \mathbb{N}^k$ and note that $\Lambda^n = \{n\} \times \Omega$. Since

$$r : \mathrm{Mor}(\Lambda) = \mathbb{N}^k \times \Omega \to \mathrm{Obj}(\Lambda) = \Omega$$

is given by $r(n, x) = x$ for $n \in \mathbb{N}^k$ and $x \in \Omega$, it follows that $r|_{\Lambda^n} : \Lambda^n \to \Omega$ is a homeomorphism. Then we immediately have that the graph Λ is proper. $\square$

To show that $\Lambda = (\Lambda(\Omega, \Theta), d)$ is compactly aligned, we use the following result which is stated without proof in [54, Remark 6.5].

Lemma 6.1.4. *Every proper topological k-graph is compactly aligned.*

Proof. Let $p, q \in \mathbb{N}^k$ and let $U \subset \Lambda^p$ and $V \subseteq \Lambda^q$ be compact. Since U, V are compact and s is a local homeomorphism (hence continuous), it follows that $s(U)$ and $s(V)$ are both compact.

Now $\Lambda * \Lambda$ has the relative topology inherited from the product topology on $\Lambda \times \Lambda$. Then the sets $U * s(U)\Lambda^{(p \vee q)-p}$ and $V * s(V)\Lambda^{(p\vee q)-q}$ are compact. Since the composition map is continuous, it follows that the images of these sets under the composition map are compact. However, these images are precisely $U\Lambda^{(p\vee q)-p}$ and $V\Lambda^{(p\vee q)-q}$ repectively. Thus we have that

$$U \vee V = U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q}$$

is compact and hence Λ is compactly aligned. $\square$

So what we have shown is the following:

Proposition 6.1.5. *The topological k-graph $\Lambda = (\Lambda(\Omega, \Theta), d)$ is compactly aligned.*

In [52], Willis shows that when $k = 2$, Ω is compact, and the maps T_1 and T_2 $*$-commute (in the sense that whenever $T_1(x) = T_2(y)$ there is a unique $z \in \Omega$ with $T_1(z) = y$ and $T_2(z) = x$) the associated product system over $\mathbb{N}^2$ is compactly aligned. By [7, Proposition 5.15], this happens if and only if the associated 2-graph is compactly aligned. The $*$-commuting condition then ensures that the boundary path groupoid is a locally compact r-discrete topological groupoid admitting a Haar system consisting of counting measures so that the 2-graph algebra $C^*(\Lambda)$ may be constructed. Proposition 6.1.5 generalizes this result to arbitrary $k \in \mathbb{N}^k$, locally compact Ω, and additionally shows that the $*$-commuting restriction may be lifted.

We would now like to show that $C^*(\Lambda)$ has a crossed-product structure. In order to do this, we begin by restricting our attention to the setting in which the maps Θ_m have uniformly bounded cardinalities on inverse images. That is, we require that for each $m \in \mathbb{N}^k$, there is $N_m \in \mathbb{N}^k$ such that

$$\sup_{y\in\Omega} |\{x \in \Omega : \Theta_m(x) = y\}| \leq N_m. \tag{6.2}$$

Note that if Ω is compact, this is automatically satisfied. Under this assumption, we will describe how we may construct a dynamical system (in the sense of Larsen in [34]), which we will call an *Exel-Larsen system*, from the space Ω and maps Θ_m. Associated to such a system is a crossed product C^*-algebra, the *Larsen crossed product*, which we will show is isomorphic to $C^*(\Lambda)$.

Definition 6.1.6. An *Exel-Larsen system* is a quadruple (A, S, α, L) such that:

- A is a (not necessarily unital) C^*-algebra
- S is an abelian semigroup with identity element e
- $\alpha : S \to \mathrm{End}(A)$ is an action (i.e., a semigroup homomorphism) such that each α_s extends uniquely to a strictly continuous endomorphism $\overline{\alpha}_s$ of $M(A)$
- for each $s \in S$ the map $L_s : A \to A$ is continuous, linear, positive, and admits a continuous linear extension $\overline{L}_s : M(A) \to M(A)$ such that

$$L_s(\alpha_s(a)u) = a\overline{L}_s(u), \qquad \text{for } a \in A, u \in M(A). \tag{6.3}$$

Given the locally compact Hausdorff space Ω and local homeomorphisms $\Theta_m : \Omega \to \Omega$ for each $m \in \mathbb{N}^k$, we set $A = C_0(\Omega)$, $S = \mathbb{N}^k$, and define $\alpha_m \in \mathrm{End}(C_0(\Omega))$ by $\alpha_m(f) = f \circ \Theta_m$ so that α is an action of $\mathbb{N}^k$ on $C_0(\Omega)$.

Remark 6.1.7. Notice that since each Θ_m is continuous, we have that α_m is nondegenerate, i.e., $\alpha_m(C_0(\Omega))C_0(\Omega) = C_0(\Omega)$. To see this, note that it is enough to show that for $g \in C_c(\Omega)$ there is $f \in C_c(\Omega)$ such that $\alpha(f)g = g$. Since Θ is continuous and g is compactly supported, we have $\Theta(\mathrm{supp}(g))$ is compact. By Uhryson's Lemma for locally compact Hausdorff spaces, we may choose $f \in C_c(\Omega)$ such that $f|_{\Theta(\mathrm{supp}(g))} = 1$. It follows that $\alpha(f)g = g$. Since $\alpha_m(C_0(\Omega))C_0(\Omega) = C_0(\Omega)$, the unique strictly continuous extension $\overline{\alpha}_m$ defined by

$$\overline{\alpha}_m(f) = f \circ \Theta_m, \qquad \text{for } f \in C_b(\Omega)$$

is unital (see [23, Proposition 1.1.13]).

Lemma 6.1.8. *For $m \in \mathbb{N}^k$, define L_m for $f \in C_0(\Omega)$ and $x \in \Omega$ by*

$$L_m(f)(x) = \begin{cases} \sum_{\Theta_m(y)=x} f(y) & \text{if } x \in \Theta_m(\Omega) \\ 0 & \text{else.} \end{cases}$$

We use the same formula to define a map $\overline{L}_m$ for $f \in C_b(\Omega)$. Then each L_m is a continuous, linear, positive map on $C_0(\Omega)$ with continuous linear extension $\overline{L}_m$ satisfying

$$L_m(\alpha_m(f)g) = f\overline{L}_m(g).$$

Proof. Fix $m \in \mathbb{N}^k$. To see that L_m maps $C_0(\Omega)$ into $C_0(\Omega)$, let $x \in \Omega$. Then there is an open neighborhood V of x and an open neighborhood U_y for each $y \in \Theta_m^{-1}(\{x\})$ such that $\Theta_m|_{U_y} : U_y \to U_x$ is a homeomorphism. The uniform boundedness condition on inverse images ensures that there are finitely many such sets U_y. It follows then that $L_m(f)$ is the finite sum of the functions $f|_{U_y}$ and is hence is in $C_0(\Omega)$.

It is straightforward to see that, for each $m \in \mathbb{N}^k$, L_m is continuous, linear, and positive. We must show that $\overline{L}_m$ is a continuous linear extension of L_m satisfying (6.3). By an argument similar to the one above, we have that $\overline{L}_m$ maps $C_b(\Omega)$ to $C_b(\Omega)$. We have that $\overline{L}_m$ is linear since if $f, g \in C_b(\Omega)$, $a, b \in \mathbb{C}$, $x \in \Theta_m(\Omega)$ then

$$\begin{aligned}(a\overline{L}_m(f) - b\overline{L}_m(g))(x) &= a\overline{L}_m(f)(x) - b\overline{L}_m(g)(x) \\ &= \sum_{\Theta_m(y)=x} af(y) - \sum_{\Theta_m(y)=x} bg(y) \\ &= \sum_{\Theta_m(y)=x} (af - bg)(y) \\ &= \overline{L}_m(af - by)(x).\end{aligned}$$

If $x \notin \Theta_m(\Omega)$, then both $(a\overline{L}_m(f) - b\overline{L}_m(g))(x)$ and $\overline{L}_m(af - by)(x)$ are zero. Continuity of $\overline{L}_m$ follows from our bounded cardinalities hypothesis (6.2). To see this, note that for any $x \in \Theta_m(\Omega)$, we have

$$\begin{aligned}\left|\overline{L}_m(f)(x)\right| &= \left|\sum_{\Theta_m(y)=x} f(y)\right| \\ &\leq \sum_{\Theta_m(y)=x} |f(y)| \\ &\leq N_m \cdot \|f\|_\infty,\end{aligned}$$

where $N_m \in \mathbb{N}$ is the uniform bound on the cardinality of the inverse image of Θ_m given in (6.2). If $x \notin \Theta_m(\Omega)$, we have $\left|\overline{L}_m(f)(x)\right| = 0$ so the inequality

$$\left|\overline{L}_m(f)(x)\right| \leq N_m\|f\|_\infty$$

holds for all $x \in \Omega$. Taking the supremum over all $x \in \Omega$ gives that

$$\|\overline{L}_m(f)\| \leq N_m\|f\|_\infty$$

so that $\overline{L}_m$ is bounded. Since $\overline{L}_m$ is a linear map on a normed space, it follows that it is continuous.

Now if $f \in C_0(\Omega)$, $g \in C_b(\Omega)$, and $x \in \Theta_m(\Omega)$ then we have

$$\begin{aligned}L_m(\alpha(f)g)(x) &= \sum_{\Theta_m(y)=x} (\alpha_m(f)g)(y) \\ &= \sum_{\Theta_m(y)=x} f(\Theta_m(y))g(y) \\ &= f(x) \sum_{\Theta_m(y)=x} g(y) \\ &= (f\overline{L}_m(g))(x).\end{aligned}$$

Also $L_m(\alpha(f)g)(x) = 0 = (f\overline{L}_m(g))(x)$ whenever $x \notin \Theta_m(\Omega)$. Thus $L_m(\alpha_m(f)g) = f\overline{L}_m(g)$. $\square$

Thus we have that $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ is an Exel-Larsen system.

Associated to the Exel-Larsen system $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ and the topological k-graph $\Lambda = (\Lambda(\Omega, \Theta), d)$ are product systems X^{Lar} and X^Λ over $\mathbb{N}^k$ of $C_0(\Omega)$-correspondences. We show in Proposition 6.2.5 that the two product systems are in fact isomorphic.

Definition 6.2.1. Associated to the Exel-Larsen system $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ is a pair (X^{Lar}, α) where $X^{Lar} = \{X_m^{Lar}\}_{m\in\mathbb{N}^k}$ is a collection of $C_0(\Omega)$-correspondences and $\alpha = \{\alpha_{m,n}\}_{m,n\in\mathbb{N}^k}$ is a collection of $C_0(\Omega)$-correspondence isomorphisms defined as follows:

For each $m \in \mathbb{N}^k$, set $X_m^{Lar} = \{m\} \times C_0(\Omega)$ with $C_0(\Omega)$-bimodule operations

$$f \cdot (m, g) \cdot h = (m, fg\alpha_m(h)),$$

where $(fg\alpha_m(h))(x) = f(x)g(x)h(\Theta_m(x))$, and $C_0(\Omega)$-valued inner product

$$\langle (m, f), (n, g)\rangle_m(x) = L_m(f^*g)(x) = \begin{cases} \sum_{\Theta_m(y)=x} \overline{f(y)}g(y) & \text{if } x \in \Theta_m(\Omega) \\ 0 & \text{else.} \end{cases}$$

For $m, n \in \mathbb{N}^k$, define $\alpha_{m,n} : X_m^{Lar} \otimes_{C_0(\Omega)} X_n^{Lar} \to X_{m+n}^{Lar}$ by

$$\alpha_{m,n}((m, f) \otimes (n, g)) = (m + n, f\alpha_m(g)).$$

We call X^{Lar} the *product system associated to* $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$.

Definition 6.2.2. The *topological k-graph correspondence* X^Λ is a pair (X^Λ, β) consisting of a collection of $C_0(\Omega)$-correspondences $X^\Lambda = \{X_m^\Lambda\}_{m\in\mathbb{N}^k}$ and a collection of $C_0(\Omega)$-correspondence isomorphisms $\beta = \{\beta_{m,n}\}_{m,n\in\mathbb{N}^k}$ defined as follows:

For each $m \in \mathbb{N}^k$, let X_m^Λ be the topological graph correspondence (as defined in Definition 2.3.13) associated to the topological graph

$$E_m = (\Lambda^0, \Lambda^m, r|_{\Lambda^m}, s|_{\Lambda^m})$$

$$= (\Omega, \{m\} \times \Omega, r_m, s_m).$$

Note that the $C_0(\Omega)$-bimodule operations and $C_0(\Omega)$-valued inner product are given by

$$\begin{aligned}
(f \cdot \xi \cdot g)(m, x) &= f(r(m, x))\xi(m, x)g(s(m, x)) \\
&= f(x)\xi(m, x)g(\Theta_m(x)), \text{ and} \\
\langle \xi, \eta \rangle_m(x) &= \sum_{(m,y) \in s_m^{-1}(m,x)} \overline{\xi(m, y)}\eta(m, y) \\
&= \sum_{y \in \Theta_m^{-1}(x)} \overline{\xi(m, y)}\eta(m, y).
\end{aligned}$$

For $m, n \in \mathbb{N}^k$, define $\beta_{m,n} : X_m^\Lambda \otimes_{C_0(\Omega)} X_n^\Lambda \to X_{m+n}^\Lambda$ by

$$\beta_{m,n}(\xi \otimes \eta)(m + n, x) = \xi(m, x)\eta(n, \Theta_m(x)).$$

The similarities between these two product systems should be apparent. In fact, we will show in Proposition 6.2.5 below that the product systems are isomorphic in the sense that there is a map $\psi : X^{Lar} \to X^\Lambda$ satisfying

1. for each $m \in \mathbb{N}^k$, the map $\psi_m = \psi|_{X_m^{Lar}} : X_m^{Lar} \to X_m^\Lambda$ is a $C_0(\Omega)$-correspondence isomorphism that preserves inner product, and
2. ψ respects the multiplication in the semigroups X^{Lar} and X^Λ.

In order to show that the two product systems are isomorphic, the following result is helpful.

Lemma 6.2.3. *Let $E = (E^0, E^1, r, s)$ be a topological graph such that $s : E^1 \to E^0$ has uniformly bounded cardinalities on inverse images. Then $X_E = C_0(E^1)$ as an algebraic $C_0(E^0)$-bimodule.*

Proof. Since $X_E \subseteq C_0(E^1)$, it is enough to show that if $\xi \in C_0(E^1)$, then $\xi \in X_E$. Since $s : E^1 \to E^0$ has uniformly bounded cardinalities on inverse images, there is $M \in \mathbb{N}$ such that for every $v \in E^0$ we have

$$|\{e \in E^1 : s(e) = v\}| \leq M.$$

To see that the map $v \mapsto \sum_{e \in s^{-1}(v)} |\xi(e)|^2$ is in $C_0(E^0)$, note that for $v \in \Omega$ there is a neighborhood V of v and finitely many open sets U_{e_i} such that s restricts to a homeomorphism from U_{e_i} onto V. It follows that $v \mapsto \sum_{e \in s^{-1}(v)} |\xi(e)|^2$ is a finite sum of continuous functions vanishing at infinity and is therefore in $C_0(E^0)$ and hence $\xi \in X_E$. $\square$

Lemma 6.2.4. *Fix $m \in \mathbb{N}^k$. For each $f \in C_0(\Omega)$, the function $\widetilde{f} : \{m\} \times \Omega \to \mathbb{C}$ defined by*

$$\widetilde{f}(m, x) = f(x)$$

is an element of X_m^Λ.

Proof. Since $X_m^\Lambda = X_{E_m^\Lambda}$, by Lemma 6.2.3 it is sufficient to show that $\widetilde{f} \in C_0(E_m^1)$ where $E_m^1 = \{m\} \times \Omega$.

Note that $\widetilde{f}$ is the composition of f with the homeomorphism $(m, x) \mapsto x$ of $\{m\} \times \Omega$ onto Ω. Then $\widetilde{f}$ is continuous since f is. If $\varepsilon > 0$ and K is a compact set such that $|f(x)| \leq \varepsilon$ for $x \in \Omega \backslash K$, then $|\widetilde{f}(m, x)| < \varepsilon$ for $(m, x) \in E_m^1 \backslash (\{m\} \times K)$. Hence $\widetilde{f} \in X_m^\Lambda$ as desired. $\square$

By an similar argument, we see that for each $\xi \in X_m^\Lambda$, the function $\hat{\xi} = (m, \eta)$ where

$$\eta(x) = \xi(m, x)$$

is an element of X_m^{Lar}.

We define $\psi : X^{Lar} \to X^\Lambda$ by letting $\psi_m : X_m^{Lar} \to X_m^\Lambda$ by

$$\psi_m(m, f) = \widetilde{f}$$

for each $m \in \mathbb{N}^k$.

Theorem 6.2.5. *As defined above, $\psi : X^{Lar} \to X^{\Lambda}$ is an isomorphism of product systems.*

Proof. Fix $m \in \mathbb{N}^k$. That $\psi_m : X_m^{Lar} \to X_m^{\Lambda}$ is a $C_0(\Omega)$-correspondence isomorphism preserving the inner product follows directly from the fact that the map $(m, x) \mapsto x$ is a homeomorphism of $\{m\} \times \Omega$ onto Ω. To see that ψ respects the semigroup multiplication, let $(m, f) \in X_m^{Lar}, (n, g) \in X_n^{Lar}, (m+n, x) \in \{m + n\} \times \Omega$. Then

$$\begin{aligned}
(\psi_m(m, f)\psi_n(n, g))\,(m + n, x) &= \psi_m(m, f)(m, x)\psi_n(n, g)(n, \Theta_m(x)) \\
&= \widetilde{f}(m, x)\widetilde{g}(n, \Theta_m(x)) \\
&= f(x)g(\Theta_m(x)) \\
&= f\alpha_m(g)(x) \\
&= \psi_{m+n}(m + n, f\alpha_m(g))(m + n, x).
\end{aligned}$$

Hence $\psi_m(m, f)\psi_n(n, g) = \psi_{n+m}(n + m, f\alpha_m(g))$ as desired. □

6.3 The Larsen crossed product and $\mathcal{NO}_X$

Throughout this section, let S be an abelian semigroup with identity e and let A be a (not-necessarily unital) C^*-algebra. Given an Exel-Larsen system (A, S, α, L), the *Larsen crossed product* $A \rtimes_{\alpha,L} S$ is the relative Cuntz-Pimsner algebra of the product system $X(A, S, \alpha, L)$ over S of A-correspondences and the family of ideals $K = \{K_s\}_{s\in S}$, where

$$K_s = \overline{A\alpha_s(A)A} \cap \phi_s^{-1}(\mathcal{K}(X_s)). \tag{6.4}$$

Recall that, given a compactly aligned topological k-graph Λ, the *topological k-graph C^*-algebra* $C^*(\Lambda)$ is the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$ of the boundary path groupoid $\mathcal{G}_\Lambda$ defined in [56, Definition 4.1]. It is shown in [7, Theorem 5.20] that $C^*(\Lambda)$ is isomorphic to the co-universal C^*-algebra $\mathcal{NO}_X$ associated to the topological k-graph correspondence X. In this section, we show that $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k \cong \mathcal{NO}_X$ and hence that

$$C^*(\Lambda) \cong C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k.$$

Let X be the product system associated to the Exel-Larsen system $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ and let $i_X : X \to \mathcal{T}_X$ be the universal Toeplitz representation of X given by [16, Proposition 2.8]. With $\{K_m\}_{m\in\mathbb{N}^k}$ as given in (6.4), denote by $\mathcal{I}_K$ the closed two-sided ideal of $\mathcal{T}_X$ generated by

$$\{i_X(0,a) - i^{(m)}(\phi_m(a)) : m \in \mathbb{N}^k, a \in K_m\}.$$

We have from [34, Proposition 3.3] that the Larsen crossed product $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ coincides with $\mathcal{T}_X/I_K$ and the map $j_X : X \to C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ obtained by composing the quotient map $q : \mathcal{T}_X \to \mathcal{T}_X/I_K$ with i_X is universal for (Toeplitz) representations of X which are *coisometric* on K in the sense of Definition 2.2.9.

Proposition 6.3.1. *Let $X = X^{Lar} \cong X^\Lambda$ be the product system over $\mathbb{N}^k$ of $C_0(\Omega)$-correspondences described in Section 6.2. Let $K = \{K_m\}_{m\in\mathbb{N}^k}$ be the family of ideals defined by* (6.4)*. Let $\psi : X \to B$ be a (Toeplitz) representation of X in a C^*-algebra B. Then ψ is Cuntz-Pimsner covariant in the sense of* (CP-K) *if and only if it is coisometric on K.*

Proof. Recall from Remark 6.1.7 that for each $m \in \mathbb{N}^k$, the map $\phi_m : C_0(\Omega) \to \mathcal{L}(X_m)$ is given by multiplication and is therefore injective so that $(\ker \phi_m)^\perp = C_0(\Omega)$.

Then it suffices to show that each $K_m = \phi_m^{-1}(\mathcal{K}(X_m))$ so that coisometric on K is equivalent to (CP-K). Since each Θ_m is continuous, α_m is nondegenerate so that

$$\alpha_m(C_0(\Omega))C_0(\Omega) = C_0(\Omega).$$

Then we immediately have

$$\begin{aligned} K_m &= \overline{C_0(\Omega)\alpha_m(C_0(\Omega))C_0(\Omega)} \cap \phi_m^{-1}(\mathcal{K}(X_m)) \\ &= C_0(\Omega) \cap \phi_m^{-1}(\mathcal{K}(X_m)) \\ &= \phi_m^{-1}(\mathcal{K}(X_m)) \end{aligned}$$

and hence coisometric on K is equivalent to (CP-K). $\square$

We next must show that for every representation $\psi : X \to B$ of X in a C^*-algebra B, Cuntz-Pimsner covariance in the sense of (CP-K) is equivalent to CNP-covariance. Before doing so, we give some results about representations of product systems. Although we will not ultimately apply these results to obtain our main result in this section, we include them because they allow results about representations on Hilbert space to be applied to arbitrary representations on C^*-algebras, thus might be useful elsewhere.

Lemma 6.3.2. *Let S be a countable semigroup with identity e, and let X be any product system over S of A-correspondences. Let $\psi : X \to B$ be a representation of X on a C^*-algebra B and let $\pi : B \to B(\mathcal{H})$ be a faithful nondegenerate representation of B on a Hilbert space $\mathcal{H}$. Then ψ is Cuntz-Pimsner covariant in the sense of* (CP-K) *if and only if $\pi \circ \psi$ is.*

Note that for any $\xi, \eta \in X_p$, we have

$$\begin{aligned} (\pi \circ \psi)^{(p)}(\Theta_{\xi,\eta}) &= (\pi \circ \psi)(\xi)(\pi \circ \psi)(\eta)^* \\ &= \pi \circ \psi^{(p)}(\Theta_{\xi,\eta}). \end{aligned}$$

Since both $(\pi \circ \psi)^{(p)}$ and $\pi \circ \psi^{(p)}$ are linear and continuous, it follows that

$$(\pi \circ \psi)^{(p)} = \pi \circ \psi^{(p)}. \tag{6.5}$$

Proof. Fix $p \in P$ and let $a \in \phi_s^{-1}(\mathcal{K}(X_p)) \cap (\ker \phi_s)^{\perp}$. Suppose first that ψ is Cuntz-Pimsner covariant in the sense of (CP-K). Then

$$\begin{aligned}
(\pi \circ \psi)^{(p)}(\phi_p(a)) &= \pi \circ \psi^{(p)}(\phi_p(a)) \\
&= \pi \circ \psi_e(a), \text{ since } \psi \text{ is Cuntz-Pimsner covariant} \\
&= (\pi \circ \psi)_e(a), \text{ by definition of } \pi \circ \psi.
\end{aligned}$$

Hence $\pi \circ \psi$ is Cuntz-Pimsner covariant. For the converse, suppose that $\pi \circ \psi$ is Cuntz-Pimsner covariant and note that

$$(\pi \circ \psi)^{(p)}(\phi_p(a)) = \pi(\psi^{(p)}(\phi_p(a))$$

and

$$(\pi \circ \psi)_e(a) = \pi(\psi_e(a)).$$

Since π is faithful, it follows that $\psi^{(p)}(\phi_p(a)) = \psi_e(a)$ and hence ψ is Cuntz-Pimsner covariant. □

Lemma 6.3.3. *Let (G, P) be a quasi-lattice ordered group and let X be any compactly aligned product system over P of A-correspondences. Let $\psi : X \to B$ be a representation of X on a C^*-algebra B and let $\pi : B \to B(\mathcal{H})$ be a faithful nondegenerate representation of B on a Hilbert space $\mathcal{H}$. Then ψ is Nica covariant if and only if $\pi \circ \psi$ is.*

Proof. Fix $p, q \in P$ and let $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$. Suppose that ψ is Nica covariant; that is, ψ satisfies (N). We compute

$$\begin{aligned}
(\pi \circ \psi)^{(p)}(S)(\pi \circ \psi)^{(q)}(T) &= \pi \circ \psi^{(p)}(S)\pi \circ \psi^{(q)}(T) \\
&= \pi(\psi^{(p)}(S)\psi^{(q)}(T)) \\
&= \begin{cases} \pi(\psi^{(p\vee q)}(\iota_p^{(p\vee q)}(S)\iota_q^{(p\vee q)}(T))), & \text{if } p \vee q < \infty \\ \pi(0) & \text{otherwise} \end{cases}
\end{aligned}$$

$$= \begin{cases} (\pi \circ \psi)^{(p \vee q)}(\iota_p^{(p \vee q)}(S)\iota_q^{(p \vee q)}(T)), & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\pi \circ \psi$ is Nica covariant. For the converse, suppose that $\pi \circ \psi$ is Nica covariant. Again, since π is faithful, it follows that ψ is also Nica covariant. $\square$

We now show that for representations $\psi : X \to B$ of the product system described in Section 6.2, Cuntz-Pimsner covariance in the sense of (CP-K) is equivalent to CNP-covariance.

In order for CNP-covariance to make sense for a representation of X, we must have that X is compactly aligned. Recall that X is the topological k-graph correspondence associated to the topological k-graph $\Lambda = (\Lambda(\Omega, \Theta), d)$ which we showed in Proposition 6.1.5 is compactly aligned. By [7, Proposition 5.15], since Λ is compactly aligned, so is X.

We would like to apply [49, Corollary 5.2] to obtain that a representation $\psi : X \to B$ satisfies (CP-K) if and only if ψ is CNP-covariant. Since $(\mathbb{Z}^k, \mathbb{N}^k)$ is a quasi-lattice ordered group such that every pair in $\mathbb{N}^k$ has a least upper bound and X is compactly aligned, we need to establish that the left action on each fibre is injective and by compact operators. The left action is given by multiplication and is therefore injective.

Proposition 6.3.4. *The left action of $C_0(\Omega)$ on each fibre X_p is by compact operators.*

Proof. Recall that the product system X is isomorphic to the product system X_Λ of topological graph correspondences obtained from the topological k-graph Λ. Then it suffices to show that the left action of $C_0(\Omega)$ on each topological graph correspondence is by compact operators.

By [26, Proposition 1.24], we have that $\phi_p^{-1}(\mathcal{K}(X_p)) = C_0(\Omega_{fin})$ where

$$\Omega_{fin} = \{v \in \Omega : v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) \text{ is compact}\}.$$

Then it is enough to show that $\Omega_{fin} = \Omega$. This is clear, however, since $r = \mathrm{id}_\Omega$ and Ω is a locally compact space and hence every point has a compact neighborhood.

□

Thus, by [49, Corollary 5.2], if $\psi : X \to B$ is a representation of X in a C^*-algebra B, then ψ satisfies (CP-K) if and only if it is CNP-covariant. Combining this with Lemma 6.3.1, we have that ψ is coisometric on K if and only if it is CNP-covariant.

Theorem 6.3.5. *Let $C^*(\Lambda)$ and $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ be the topological k-graph C^*-algebra and Larsen crossed product associated to the locally compact Hausdorff space Ω and maps $T_1, \ldots, T_k$ as described above. Then*

$$C^*(\Lambda) \cong C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k.$$

Proof. By [7, Theorem 5.20], we have that $C^*(\Lambda) \cong \mathcal{NO}_X$ where X is the product system described in Section 6.2. Since a representation $\psi : X \to B$ is coisometric on K, where $K = \{K_m\}_{m\in\mathbb{N}^k}$ is the family of ideals defined by (6.4), if and only if it is CNP-covariant, it follows that $j_X : X \to \mathcal{NO}_X$ is coisometric on K and j_X^{Lar} is CNP-covariant. It follows from the universal properties of $\mathcal{NO}_X$ and the Larsen crossed product $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ that there are unique homomorphisms

$$\Pi_{j_X^{Lar}} : \mathcal{NO}_X \to C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$$
$$\Pi_{j_X} : C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k \to \mathcal{NO}_X$$

such that $j_X^{Lar} = \Pi_{j_X^{Lar}} \circ j_X$ and $j_X = \Pi_{j_X} \circ j_X^{Lar}$. By Definition 2.2.23, we have that

$$\mathcal{NO}_X = \overline{\mathrm{span}}\{j_X(\xi)j_X(\eta)^* : \xi, \eta \in X\}.$$

Moreover, since j_X^{Lar} is Nica covariant and generates the Larsen crossed product $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ as a C^*-algebra, we have

$$C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k = \overline{\operatorname{span}}\{j_X^{Lar}(\xi) j_X^{Lar}(\eta)^* : \xi, \eta \in X\},$$

as in (2.12). It follows that $\Pi_{j_X^{Lar}}$ and Π_{j_X} take generators to generators and hence

$$C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k \cong \mathcal{NO}_X.$$

Since, the topological k-graph C^*-algebra $C^*(\Lambda)$ is isomorphic to $\mathcal{NO}_X$ we have $C^*(\Lambda) \cong C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ as desired. $\square$

6.4 Examples

In this section we describe some examples of topological k-graphs associated to Exel-Larsen systems.

Example 6.4.1. Let A be a finite alphabet and, for $n \in \mathbb{N}$, let A^n denote the space of words of length n. We let $A^{\mathbb{N}}$ denote the one-sided infinite sequence space, which is compact by Tychonoff's Theorem. The *shift map* $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined by

$$\sigma(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$$

is a local homeomorphism of $A^{\mathbb{N}}$. Given a *block map* $d : A^n \to A$ for some $n \in \mathbb{N}$, we may define a *sliding block code* $\tau_d : A^{\mathbb{N}} \to A^{\mathbb{N}}$ via

$$\tau_d(x)_i = d(x_i \cdots x_{i+n-1}).$$

In [52, Lemma 3.3.3 and Lemma 3.3.7], Willis shows that a function $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is continuous and commutes with the shift map σ if and only if ϕ is the sliding block code associated to some block map d. Moreover, it follows from [14, Theorem 14.3] that ϕ is a local homeomorphism whenever d is *progressive* in the sense that for each $x_1 \cdots x_{n-1} \in A^{n-1}$ the function $a \mapsto d(x_1 \cdots x_{n-1} a)$ is bijective. We say that a block map d is *regressive* if for each $x_1 \cdots x_{n-1} \in A^{n-1}$ the function $a \mapsto d(ax_1 \cdots x_{n-1})$ is bijective. In [52, Proposition 3.3.32], Willis shows that if ϕ is a local homeomorphism associated to a regressive dictionary, then there is some $M \in \mathbb{N}$ such that ϕ is M-to-1 and surjective.

To consider this in our setting, it is enough to consider an alphabet A and a block map that is both progressive and regressive. Let $A = \{0, 1, 2, 3\}$ and define $d : A^2 \to A$ via $(a, b) \mapsto a + b \mod 4$. It is straightforward to see that d is both progressive and regressive and hence the associated sliding block code τ_d is a local homeomorphism that has uniformly bounded cardinalities on inverse images.

For each $(m, n) \in \mathbb{N}^2$, define $\Theta_{(m,n)} : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by $\Theta_{(m,n)}(x) = \sigma^m \tau_d^n(x)$. We may construct the topological 2-graph $\Lambda = (\Lambda(A^{\mathbb{N}}, \Theta), d)$ as in Section 6.1. With $\alpha : \mathbb{N}^2 \to C(A^{\mathbb{N}})$ given by $\alpha_{(m,n)}(f) = f \circ \Theta_{(m,n)}$ and $L_{(m,n)} : C(A^{\mathbb{N}}) \to C(A^{\mathbb{N}})$ given by

$$L_{(m,n)}(f)(x) = \sum_{\Theta_{(m,n)}(y)=x} f(y)$$

for $(m, n) \in \mathbb{N}^2$, the quadruple $(A^{\mathbb{N}}, \alpha, L, \mathbb{N}^2)$ is an Exel-Larsen system. It follows that $C^*(\Lambda) \cong C(A^{\mathbb{N}}) \rtimes_{\alpha,L} \mathbb{N}^2$.

Example 6.4.2. In [5], Brownlowe realizes the C^*-algebra of a finitely aligned discrete k-graph Λ as a crossed product. To do this, he constructs a product system X over $\mathbb{N}^k$ of C^*-correspondences called the *boundary path product system* and defines the crossed product to be the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$. He then shows that $C^*(\Lambda) \cong \mathcal{NO}_X$ (note here that X is *not* the graph correspondence of Λ). In the particular case when Λ is locally finite with no sources, Brownlowe constructs an associated Exel-Larsen system and shows that the associated product system is isomorphic to X. He then shows that $\mathcal{NO}_X$ coincides with the Larsen crossed product.

To see how our results fit in with those of Brownlowe, it is important to observe how Brownlowe's boundary path product system may be associated to a topological k-graph. In particular, given a finitely aligned discrete k-graph Λ, we let $\partial\Lambda$ denote the boundary path space of Λ, that is,

$$\partial\Lambda = \{x : \Omega_{k,m} \to \Lambda : \forall n \leq d(x), E \in x(n)\mathcal{FE}(\Lambda)\ \exists \lambda \in E, x(n, n + d(\lambda)) = \lambda\}$$

where $x(n)\mathcal{FE}(\Lambda)$ is the collection of all finite subsets E of $x(n)\Lambda$ that are *exhaustive* in the sense that for every $\mu \in x(n)\Lambda)$ there is $\lambda \in E$ with

$$\Lambda^{min}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\mu)\} = \emptyset.$$

For $m \in \mathbb{N}^k$, $\partial\Lambda^{\geq m} = \{x \in \partial\Lambda : d(x) \geq m\}$ and the shift map $\sigma_m : \partial\Lambda^{\geq m} \to \partial\Lambda$ on $\partial\Lambda^{\geq m}$ is given by $\sigma_m(x)(n) = x(m + n)$. We let $\iota : \partial\Lambda^{\geq m} \to \partial\Lambda$ denote the inclusion map. Then $E_n = (\partial\Lambda, \partial\Lambda^{\geq m}, \iota, \sigma_m)$ is a topological graph [5, Proposition 3.1] with associated topological graph correspondence X_m.

Brownlowe shows in [5, Proposition 3.2] that we also have isomorphisms

$$\pi_{m,n} : X_m \otimes X_n \to X_{m+n}$$

making X into a product system over $\mathbb{N}^k$ of $C_0(\partial\Lambda)$-correspondences.

When Λ is row-finite with no sources, $\partial\Lambda^{\geq m} = \partial\Lambda$ so that the functions α_m on $C_0(\partial\Lambda)$ given by $\alpha_m(f) = f \circ \sigma_m$ are everywhere defined and hence $\alpha : \mathbb{N}^k \to \operatorname{Aut} C_0(\partial\Lambda)$ is an action. Defining transfer operators $\mathcal{L}_m$ for each $m \in \mathbb{N}^k$, Brownlowe obtains an Exel-Larsen system $(C_0(\partial\Lambda), \alpha, \mathcal{L}, \mathbb{N}^k)$ and shows that the Larsen crossed product coincides with $\mathcal{NO}_X$.

It seems reasonable to conjecture that simple assumptions on the graph Λ will ensure that each σ_m has uniformly bounded cardinalities on inverse images. In this case, we may apply our results to the topological k-graph and Exel-Larsen system associated to the locally compact Hausdorff space $\partial\Lambda$ and the pairwise commuting local homomorphisms σ_m to obtain the content of [5, Proposition 5.7].

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