Erdős–Ko–Rado Theorems: New Generalizations, Stability Analysis and Chvátal's Conjecture

by

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ABSTRACT

One of the seminal results in extremal combinatorics, due to Erdős, Ko and Rado, states that if $\mathscr{F}$ is an *intersecting* family of r-subsets of an n-element set, i.e. for any $A,B \in \mathscr{F}$, $A \cap B \neq \emptyset$, then $|\mathscr{F}| \leq \binom{n-1}{r-1}$ if $r \leq n/2$. Furthermore, when $r < n/2$, the only structure which attains this *extremal* number is that of a *star*. A major part of this dissertation considers extensions of the Erdős–Ko–Rado theorem motivated by a graph-theoretic generalization due to Holroyd, Spencer and Talbot. A conjecture of Holroyd and Talbot is proved for a large class of graphs, namely *chordal* graphs which contain at least one isolated vertex. A stronger result is also shown to exist for a special class of chordal graphs obtained by blowing up edges of a path into complete graphs.

Next, a well-known generalization of the EKR theorem due to Frankl is considered. For some $k \geq 2$, let $\mathscr{F}$ be a k-wise intersecting family of r-subsets of an n-element set, i.e. for any $F_1,\ldots,F_k \in \mathscr{F}$, $\cap_{i=1}^{k} F_i \neq \emptyset$. If $r \leq \dfrac{(k-1)n}{k}$, then $|\mathscr{F}| \leq \binom{n-1}{r-1}$. A *stability* version of this theorem is proved using an analog of Katona's *circle* method. A graph-theoretic generalization of Frankl's theorem analogous to a theorem of Bollobás and Leader is also formulated and proved.

A collection of families $\mathscr{A}_1,\ldots,\mathscr{A}_k$ is called *cross-intersecting* if for any $i,j \in [k]$ with $i \neq j$, $A \in \mathscr{A}_i$ and $B \in \mathscr{A}_j$ implies $A \cap B \neq \emptyset$. Hilton proved a best possible upper bound on the sum of the cardinalities of *uniform* cross-intersecting subfamilies. In this thesis, extensions of Hilton's theorem are formulated and proved for chordal graphs and cycles.

One of the motivations in formulating these graph-theoretic generalizations for EKR theorems is a long-standing conjecture of Chvátal for *hereditary* set systems. A set system $\mathscr{F}$ is said to be *hereditary* if for any $F \in \mathscr{F}$, if $G \subseteq F$, then $G \in \mathscr{F}$. Chvátal's conjecture states that the set of maximum-sized intersecting subfamilies of a hereditary set system contains a star. It can be observed that the family of

all independent sets in a graph is hereditary. A different class of hereditary vertex families in a graph is studied, namely the family of all *cycle-free* vertex subsets of a graph. Finally, a powerful tool of Erdős and Rado is used to prove Chvátal's conjecture for hereditary families with small rank.

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To my parents, Amar, Amit and Neha

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Chapter 1

INTRODUCTION

The primary focus of this dissertation lies in the area of extremal combinatorics, in particular intersection theorems in extremal set theory. A starting point in this line of research is the following question. Consider a collection of subsets of an n-element set X, such that no pair of subsets in the collection is disjoint. Call it an *intersecting* family. How large can such an intersecting family be? As it turns out, this question is surprisingly easy to answer. An intersecting family of subsets can have size at most 2^{n-1}, because for any subset A, at most one out of the pair $(A, X \setminus A)$ can be in the family. Furthermore, one of the structures which attains this *extremal* number is the family of all subsets which contain a specific element, called a *maximum star*. In general, a family $\mathscr{F}$ with $\cap_{F \in \mathscr{F}} F \neq \emptyset$ is called a *star*.

A related question is the following: For a set $X = [n] = \{1, \ldots, n\}$, and $r \geq 1$, let $\binom{X}{r}$ be the family of all subsets of X of size r, also called the complete r-uniform hypergraph on n vertices, and let $\mathscr{A} \subseteq \binom{X}{r}$ be intersecting. How large can $\mathscr{A}$ be? If $r > n/2$, any pair of r-subsets have a non-empty intersection, but the case $r \leq n/2$ is non-trivial. In the paper that initiated the study of intersecting set systems, Erdős, Ko and Rado [21] proved the following seminal result.

1.1 The Erdős–Ko–Rado Theorem

Theorem 1.1.1 (Erdős–Ko–Rado)**.** *For a set $X = [n]$ and $2 \leq r \leq n/2$, if $\mathscr{A} \subseteq \binom{X}{r}$ is intersecting, then $|\mathscr{A}| \leq \binom{n-1}{r-1}$.*

Moreover, Hilton and Milner [32], as part of a stronger result, proved that if $r < n/2$, then equality holds iff $\mathscr{A} = \binom{X}{r}_x = \{A : A \in \binom{X}{r}, x \in A\}$; in other words, $\mathscr{A}$ is a maximum r-uniform star centered at x.

The original inductive proof in [21] used the so-called shifting (or compression) method, a widely used technique in extremal set theory. Frankl [26] gives an excellent survey of the use of this technique. There have been other interesting proofs too. Daykin [14] proved the theorem using the Kruskal-Katona theorem, while Katona [39] gave what was probably the simplest proof, an elegant argument using double counting. Later in the chapter we briefly review shifting and Katona's method, the two main techniques we extensively use in our arguments.

The Erdős–Ko–Rado theorem is one of the fundamental theorems in combinatorics, and has inspired a large number of beautiful results, many of which have found applications not only within combinatorics, but also in the fields of information theory and probability. A particularly elegant application to probability was by Liggett [43], who proved a result on sums of independent Bernoulli random variables using the bound in Theorem 1.1.1.

The broader area of extremal combinatorics has applications to the theory of computing. For instance, the fundamental *lower bounds problem*, which is to prove that a given function cannot be computed within a certain amount of time or space, is an extremal problem, and techniques from extremal set theory have been extensively used to prove results of this type. A striking example was due to Razborov [54], who used the *Sunflower Lemma* of Erdős and Rado [23] to prove a lower bounds theorem for monotone circuits.

Many of the outstanding achievements in the field also have connections with other areas of mathematics; for instance, Szemerédi's regularity lemma [59] was born out of a conjecture in number theory, while the Kruskal-Katona theorem, particularly the version due to Lovasz, led to seminal work of Bollobás and Thomason [7] which proved the existence of thresholds for monotone properties.

A very fine survey of the the avenues of research, pursued as extensions of the

Erdős–Ko–Rado theorem, in the 1960's, 70's and 80's, is presented by Deza and Frankl [15]. In this chapter, we will present some of the most important directions, and ones most relevant to the focus of this thesis.

Generalization for t-intersecting families

The most natural extension of the theorem is to t-intersecting r-uniform families, i.e. r-uniform families in which every pair of subsets intersect in at least t elements for some $t \geq 1$. As with the case $t = 1$, the problem is interesting only when $n > 2r - t$, since otherwise, $\binom{[n]}{r}$ is t-intersecting. This problem was first considered by Erdős et al. in their seminal paper, who proved the t-intersecting version of the EKR theorem for sufficiently large n (in terms of t and r). The following theorem appears in Bollobás [5], with a slightly better bound on n than the one obtained by Erdős et al.

Theorem 1.1.2. *Let $2 \leq t < r$, $n \geq 2tr^3$, and suppose $\mathscr{F} \subseteq \binom{[n]}{r}$ is t-intersecting. Then, $|\mathscr{F}| \leq \binom{n-t}{r-t}$, and equality holds if and only if $\mathscr{F}$ is a t-star, i.e. $\mathscr{F} = \{A \in \binom{[n]}{r} : [t] \subseteq A\}$.*

However, the t-star structure is not the only candidate for creating a large t-intersecting family. For $0 \leq k \leq r-t$, let $L_k = [t+2k]$. Now, consider the following families. Let $\mathscr{F}_k = \{A \in \binom{[n]}{r} : |A \cap L_k| \geq t+k\}$. It is not hard to see that for each $0 \leq k \leq r-t$, $\mathscr{F}_k$ is a t-intersecting family.

The following proposition, about the sizes of the F_i's can be easily verified.

Proposition 1.1.3.

1. *If $n > (t+1)(r-t+1)$, then $|\mathscr{F}_0| > \max_{1 \leq k \leq r-t} |\mathscr{F}_k|$.*
2. *If $n = (t+1)(r-t+1)$, then $|\mathscr{F}_0| = |\mathscr{F}_1|$.*
3. *If $n < (t+1)(r-t+1)$, then $|\mathscr{F}_0| < |\mathscr{F}_1|$.*

Frankl [25] conjectured that one of the families $\mathscr{F}_k$ $(0 \leq i \leq r-l)$ has maximum cardinality among all t-intersecting families. In particular, by Proposition 1.1.3, he conjectured that if $n \geq (t+1)(r-t+1)$, then for any t-intersecting family on $[n]$, $|\mathscr{F}| \leq \binom{n-t}{r-t}$. Frankl [25] proved this conjecture for all $t \geq 15$ and Wilson [61] later proved this for all t. Finally, Ahlswede and Khatchatrian [1] gave an outstanding proof of what is now called the "complete intersection theorem", by finding the size and structure of the extremal families for all values of n, including when $n < (t+1)(r-t+1)$. One of the many remarkable achievements of this theorem was to highlight a deep connection between intersection theorems in finite set theory and computational complexity theory. Indeed, the $t = 2$ case of the theorem was a crucial component in the work of Dinur and Safra [18] which proved that approximating the *Minimum Vertex Cover* problem to within a factor of 1.3606 is NP-hard.

1.2 Chvátal's Conjecture

We've seen that the maximum size of an intersecting subset of the power set of a set $[n]$, denoted henceforth by $2^{[n]}$, is at most 2^{n-1} and that the maximum star is one of the structures which achieves this size. It turns out that the star is not the only extremal structure in this case. We can construct another extremal family when n is odd and $n \geq 3$. Consider the family $\mathscr{G} = \{G \subseteq [n] : |G| > n/2\}$. Clearly the family is intersecting, since every set has size more than half the size of the ground set and from every $(A, X \setminus A)$ pair in $2^{[n]}$, we have picked exactly one set, more precisely the larger set so the size of the family is 2^{n-1}. It is also trivial to note that $\mathscr{G}$ is not a star of size 2^{n-1} since it does not contain any singletons. We say that the set system $2^{[n]}$ is *EKR* since the set of maximum-sized intersecting subfamilies of $2^{[n]}$ contains a star. Similarly, we say that the family of r-subsets of $[n]$, denoted by

$\binom{[n]}{r}$, is *strictly* EKR for $n > 2r$, since every member in the set of maximum-sized intersecting subfamilies is a star. Note that by our preceding observations, $2^{[n]}$ is not strictly EKR. We also point out that when $n = 2r$, $\binom{[n]}{r}$ is EKR, but not strictly EKR. The following simple construction demonstrates this. Let $\mathscr{H} = \{H \in \binom{[n]}{r} : 1 \notin H\}$. Thus, $\mathscr{H}$ consists of all r-subsets of $[2r] \setminus \{1\}$, and is intersecting. It also has size $\binom{2r-1}{r} = \binom{2r-1}{r-1}$.

We turn our attention back to the power set $2^{[n]}$. $2^{[n]}$ is a special example of a *hereditary* family, also referred to in the literature as an *ideal* or a *downset*. A family $\mathscr{F}$ is said to be hereditary if $A \in \mathscr{F}$ and $A' \subseteq A$ implies that $A' \in \mathscr{F}$. Chvátal conjectured that with regards to maximum intersecting subsets, all hereditary set systems exhibit behavior similar to $2^{[n]}$. More precisely, he conjectured the following.

Conjecture 1.2.1 (Chvátal)**.** *If $\mathscr{F}$ is a hereditary family, then $\mathscr{F}$ is EKR.*

There are a few results which verify the conjecture for specific hereditary families. Among the most important ones is a result of Chvátal himself [12]. Let $\mathscr{F}$ be a hereditary family on a set X, which has a total ordering of its elements induced by a relation $\preceq$. Chvátal proved the conjecture when $\mathscr{F}$ is *compressed*. Snevily [57] further extended Chvátal's theorem and proved the conjecture when the family is compressed with respect to a specific element x. In Chapter 5, we will discuss these, and other related results in greater detail, and consider this conjecture for hereditary families with small rank.

1.3 Erdős–Ko–Rado For Graphs

Partially motivated by Chvátal's conjecture, and earlier results of Berge [4] and Bollobás-Leader [6], one of the recent generalizations of Theorem 1.1.2 considers hereditary families of vertex sets of a graph G. It is not hard to observe that the family of all *independent* vertex sets (subsets of vertices containing no edges) of a graph G is hereditary. In particular, Holroyd, Spencer and Talbot [34] consider *uniform* subfamilies of this family. For a graph G, vertex $v \in V(G)$ and some integer $r \geq 1$, denote the family of independent sets of size r of $V(G)$ by $\mathscr{I}^{(r)}(G)$ and the star subfamily $\{A \in \mathscr{I}^{(r)}(G) : v \in A\}$ by $\mathscr{I}_v^{(r)}(G)$. Call G (strictly) r-EKR if $\mathscr{I}^{(r)}(G)$ is (strictly) EKR.

Earlier results by Berge [4], Deza and Frankl [15], and Bollobas and Leader [6], while not explicitly stated in graph-theoretic terms, hint in this direction. The following interesting conjecture was posed by Holroyd and Talbot [35]. For graph G, let $\mu(G)$ be the minimum size of a maximal independent set.

Conjecture 1.3.1. *Let G be any graph and let $1 \leq r \leq \frac{1}{2}\mu$; then G is r-EKR(and is strictly so if $2 < r < \frac{1}{2}\mu$).*

One of the main contributions of this dissertation involves verifying this conjecture for a large class of graphs, in particular encompassing earlier results by Borg-Holroyd [10] and Holroyd et al. [34]. Call a graph G *chordal* if every cycle in G, of length at least 4, has a chord, i.e. an edge between non-adjacent vertices of the cycle.

Theorem 1.3.2 (Hurlbert, Kamat)**.** *If G is a disjoint union of chordal graphs, including at least one isolated vertex, and if $r \leq \frac{1}{2}\mu(G)$, then G is r-EKR.*

The isolated vertex condition, in the hypothesis of the theorem, allows us to determine the center of a maximum star in the graph (in a graph with an isolated vertex, it is not hard to show that one of the maximum stars is centered at the isolated vertex). More importantly, it makes it easy to extend Theorem 1.3.2 in the direction of Chvátal's conjecture in the form of the following corollary for a class of hereditary families. Let $\mathscr{I}^{(\leq r)}(G)$ be the hereditary family of all independent vertex sets of size at most r.

Corollary 1.3.3. *If G is a disjoint union of chordal graphs, including at least one isolated vertex, and if $r \leq \frac{1}{2}\mu(G)$, then $\mathscr{I}^{(\leq r)}(G)$ satisfies Conjecture 1.2.1.*

In Chapter 2, we will give a proof of Theorem 1.3.2 and also consider similar problems for trees and other classes of chordal graphs without isolated vertices.

1.4 k-wise Intersection Theorems

A natural extension of the notion of intersection is *k-wise* intersection, for $k \geq 2$. Call $\mathscr{F} \subset \binom{[n]}{r}$ k-wise intersecting if for any $F_1, \ldots, F_k \in \mathscr{F}$, $\bigcap_{i=1}^{k} F_i \neq \emptyset$. One of the main results for k-wise intersecting families is the following generalization of the EKR theorem, due to Frankl [28].

Theorem 1.4.1 (Frankl)**.** *Let $\mathscr{F} \subset \binom{[n]}{r}$ be k-wise intersecting. If $r \leq \dfrac{(k-1)n}{k}$, then $|\mathscr{F}| \leq \binom{n-1}{r-1}$.*

It is trivial to note that the $k = 2$ case of Theorem 1.4.1 is the Erdős–Ko–Rado theorem. This theorem of Frankl led to the following problem of Katona's, for the case $k = 3$. Suppose, for some $s \geq 0$, we require the condition $F_1 \cap F_2 \cap F_3 \neq \emptyset$, only for those triples which satisfy $|F_1 \cup F_2 \cup F_3| \leq s$. For which values of s does this condition give the Erdős–Ko–Rado bound, i.e. for which s values is $|\mathscr{F}| \leq \binom{n-1}{r-1}$. Frankl and Furedi [29] proved, for large n, that for the range $2r \leq s \leq 3r$, the extremal number, as well as the extremal structures remain unchanged. More precisely, they proved the following theorem.

Theorem 1.4.2. *Let $\mathscr{F} \subseteq \binom{[n]}{r}$ be such that for any $F_1, F_2, F_3 \in \mathscr{F}$ satisfying $|F_1 \cup F_2 \cup F_3| \leq 2r$, $F_1 \cap F_2 \cap F_3 \neq \emptyset$ holds. Then, $|\mathscr{F}| \leq \binom{n-1}{r-1}$, with equality holding if and only if $\mathscr{F}$ is a star.*

In this thesis, we will be mostly interested in Theorem 1.4.1, and its generalizations along the lines of the Erdős–Ko–Rado property for graphs defined in the previous section. More particularly, for a graph G and $r \geq 1$, let $\mathscr{M}^r(G)$ denote the family of all vertex sets of size r containing a maximum independent set, and let $\mathscr{H}^r(G) = \mathscr{I}^r(G) \cup \mathscr{M}^r(G)$, where, as before, $\mathscr{I}^r(G)$ denotes the set of all independent vertex sets of size r in G. For a vertex $x \in V(G)$, let $\mathscr{H}_x^r(G) = \{A \in \mathscr{H}^r(G) : x \in A\}$. Define $\mathscr{I}_x^r(G)$ and $\mathscr{M}_x^r(G)$ in a similar manner. We restrict our attention to the case when $G = M_n$, the perfect matching graph on $2n$ vertices (and n edges). Note that $|\mathscr{H}_x^r(M_n)| = 2^{r-1}\binom{n-1}{r-1}$ when $r \leq n$ and $|\mathscr{H}_x^r(M_n)| = 2^{2n-r}\binom{n-1}{r-n-1} + 2^{2n-r-1}\binom{n-1}{r-n}$, when $r > n$. We will consider k-wise intersecting families in $\mathscr{H}^r(M_n)$, and prove the following analog of Frankl's theorem.

Theorem 1.4.3. *For $k \geq 2$, let $r \leq \dfrac{(k-1)(2n)}{k}$, and let $\mathscr{F} \subseteq \mathscr{H}^r(M_n)$ be k-wise intersecting. Then,*

$$|\mathscr{F}| \leq \begin{cases} 2^{r-1}\binom{n-1}{r-1} & \text{if } r \leq n, \text{ and} \\ 2^{2n-r}\binom{n-1}{r-n-1} + 2^{2n-r-1}\binom{n-1}{r-n} & \text{otherwise.} \end{cases}$$

If $r < \dfrac{(k-1)(2n)}{k}$, then equality holds if and only if $\mathscr{F} = \mathscr{H}_x^r(M_n)$ for some $x \in V(M_n)$.

It can be seen that the $k = 2$ case of Theorem 1.4.3 is the theorem of Bollobás and Leader [6] we referred to in the earlier section. Note that when $r \leq n$, $\mathscr{H}^r(M_n) = \mathscr{I}^r(M_n)$.

Theorem 1.4.4 (Bollobás-Leader)**.** *Let $1 \leq r \leq n$, and let $\mathscr{F} \subseteq \mathscr{I}^r(M_n)$. Then, $|\mathscr{F}| \leq 2^{r-1}\binom{n-1}{r-1}$. If $r < n$, equality holds if and only if $\mathscr{F} = \mathscr{I}_x^r(M_n)$ for some $x \in V(M_n)$.*

1.5 Stability for Erdős–Ko–Rado theorems

One of the other interesting questions that is a natural extension in this line of inquiry, is to ask what happens when the extremal family is excluded, i.e if we suppose that there is no element contained in all sets of the family, i.e. $\cap\mathscr{F} = \emptyset$. This problem was solved by the result of Hilton and Milner [32], who gave an upper bound on the size of non-star intersecting families, and discovered the structure of these "second best" families.

Theorem 1.5.1. *Suppose $r < n/2$, and $\mathscr{F} \subseteq \binom{[n]}{r}$ is an intersecting family such that $\cap_{F\in\mathscr{F}} = \cap\mathscr{F} = \emptyset$. Then, $|\mathscr{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. Equality holds if and only if $\mathscr{F} \simeq \{F \in \binom{[n]}{r} : 1 \in F, \{2,\ldots,r+1\} \cap F \neq \emptyset\} \cup \{2,\ldots,r+1\}$.*

While the original proof of this theorem is complicated, simpler proofs exist, one of them due to Frankl and Furedi, which uses shifting. In our brief introduction to the shifting technique, we will present this proof.

Note that a different way of stating Theorem 1.5.1 is the following: For $r < n/2$, if $\mathscr{F} \subseteq \binom{[n]}{r}$ is intersecting and $|\mathscr{F}| > \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$, then $\cap\mathscr{F} \neq \emptyset$, in other words, $\mathscr{F}$ has the structure of a star, although possibly not of the maximum size.

The above form of the Hilton-Milner theorem leads nicely to the problem of stability analysis of extremal theorems, an area that has attracted a lot of attention in recent years. We will briefly present this line of investigation here, and discuss it in more detail while discussing our main result in this area in Chapter 3.

The classical extremal problem is to determine the maximum size, and possibly structure, of a family on a given ground set of size n, which avoids a given forbidden configuration $\mathscr{F}$. For example, the Erdős–Ko–Rado theorem finds the maximum size and structure of a set system on the set $[n]$, which does not have a pair of disjoint subsets. Often, only a few trivial structures attain this extremal number. In case of the EKR theorem, the only extremal structure, when $r < \frac{n}{2}$, is that of a star in $\binom{[n]}{r}$. A natural further step is to ask whether non-extremal families, which have size close to the extremal number, also have structure similar to any of the extremal structures. This approach was first pioneered by Simonovits [56], to answer a question in extremal graph theory, and a similar notion for set systems was recently formulated by Mubayi [50]. The Hilton-Milner [32] theorem, as we observed earlier, by giving an upper bound on the maximum size of non-star intersecting families, proves a stability result for the Erdős-Ko-Rado theorem. Other stability results for the Erdős–Ko–Rado theorem have been recently proved by Dinur-Friedgut [17], Keevash [40], Keevash-Mubayi [41] and others.

In this thesis, we will be interested in stability analysis for k-wise Erdős–Ko–Rado theorems, in particular Theorem 1.4.1. Our main result will be the following.

Theorem 1.5.2. *For some $k \geq 2$, let $1 \leq r < \frac{(k-1)n}{k}$, and let $\mathscr{F} \subseteq \binom{[n]}{r}$ be a k-wise intersecting family. Then for any $0 \leq \varepsilon < 1$, there exists a $0 \leq \delta < 1$ such that if $|\mathscr{F}| \geq (1-\delta)\binom{n-1}{r-1}$, then there is an element $v \in [n]$ such that $|\mathscr{F}(v)| \geq (1-\varepsilon)\binom{n-1}{r-1}$.*

We note that if $\mathscr{F}$ is k-wise intersecting, for some $k \geq 2$, then it is intersecting. Hence, if $r < n/2$, the results obtained in the papers mentioned above suffice, as stability results for Theorem 1.4.1. Consequently, the main interest of our theorem is in the structural information that it provides when $n/2 \leq r < (k-1)n/k$.

1.6 Cross-intersection Theorems for Graphs

Consider a collection of k subfamilies of $2^{[n]}$, say $\mathscr{A}_1, \ldots, \mathscr{A}_k$. Call this collection *cross-intersecting* if for any $i, j \in [k]$ with $i \neq j$, $A \in \mathscr{A}_i$ and $B \in \mathscr{A}_j$ implies $A \cap B \neq \emptyset$. Note that the individual families themselves do not need to be either non-empty or intersecting, and a subset can lie in more than one family in the collection.

We will be interested in *uniform* cross-intersecting families, i.e. cross-intersecting subfamilies of $\binom{[n]}{r}$ for suitable values of r. There are two main kinds of problems concerning uniform cross-intersecting families that have been investigated, the *maximum product* problem and the *maximum sum* problem. One of the main results for the maximum product problem due to Matsumoto and Tokushige [45] states that for $r \leq n/2$ and $k \geq 2$, the product of the cardinalities of k cross-intersecting subfamilies $\{\mathscr{A}_1, \ldots, \mathscr{A}_k\}$ of $\binom{[n]}{r}$ is maximum if $\mathscr{A}_1 = \cdots = \mathscr{A}_k = \{A \subseteq \binom{[n]}{r} : x \in A\}$ for some $x \in [n]$.

We will be more interested in the maximum sum problem, particularly the following theorem of Hilton [31], which establishes a best possible upper bound on the sum of cardinalities of cross-intersecting families and also characterizes the extremal structures.

Theorem 1.6.1 (Hilton)**.** *Let $r \leq n/2$ and $k \geq 2$. Let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be cross-intersecting subfamilies of $\binom{[n]}{r}$, with $\mathscr{A}_1 \neq \emptyset$. Then,*

$$\sum_{i=1}^{k} |\mathscr{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq n/r, \text{ and} \\ k\binom{n-1}{r-1} & \text{if } k \geq n/r. \end{cases}$$

If equality holds, then

1. *$\mathscr{A}_1 = \binom{[n]}{r}$ and $\mathscr{A}_i = \emptyset$, for each $2 \leq i \leq k$, if $k < \dfrac{n}{r}$,*
2. *$|\mathscr{A}_i| = \binom{n-1}{r-1}$ for each $i \in [k]$ if $k > \dfrac{n}{r}$, and*
3. *$\mathscr{A}_1, \ldots, \mathscr{A}_k$ are as in case 1 or 2 if $k = \dfrac{n}{r} > 2$.*

It can be observed that Theorem 1.6.1 is a generalization of the Erdős-Ko-Rado theorem [21] in the following manner: put $k > n/r$, let $\mathscr{A}_1 = \cdots = \mathscr{A}_k$, and we obtain the EKR theorem.

There have been a few generalizations of Hilton's cross-intersection theorem, most recently for permutations by Borg ([8] and [9]) and for uniform cross-intersecting subfamilies of independent sets in graph M_n which is the perfect matching on $2n$ vertices, by Borg and Leader [11]. Borg and Leader proved an extension of Hilton's theorem for *signed* sets, which we will state in the language of graphs as follows.

Theorem 1.6.2 (Borg-Leader [11])**.** *Let $r \leq n$ and $k \geq 2$. Let $\mathscr{A}_1, \ldots, \mathscr{A}_k \subseteq \mathscr{J}^r(M_n)$ be cross-intersecting. Then*

$$\sum_{i=1}^{k} |\mathscr{A}_i| \leq \begin{cases} \binom{n}{r} 2^r & \text{if } k \leq 2n/r, \text{ and} \\ k\binom{n-1}{r-1} 2^{r-1} & \text{if } k \geq 2n/r. \end{cases}$$

Suppose equality holds and $\mathscr{A}_1 \neq \emptyset$. Then,

1. *If $k \leq 2n/r$, then $\mathscr{A}_1 = \mathscr{J}^r(M_n)$ and $\mathscr{A}_2 = \cdots = \mathscr{A}_k = \emptyset$,*

2. *If $k \geq 2n/r$, then for some $x \in V(M_n)$, $\mathscr{A}_1 = \cdots = \mathscr{A}_k = \mathscr{J}_x^r(M_n)$, and*

3. *If $k = 2n/r > 2$, then $\mathscr{A}_1, \ldots, \mathscr{A}_k$ are as in either of the first two cases.*

In Chapter 4, we will consider extensions of this result to any disjoint union of complete graphs and further investigate this problems for other classes of graphs. In particular, we restrict our attention to *cross-intersecting pairs*, i.e. we fix $k = 2$ and prove cross-intersection theorems for larger classes of graphs, namely chordal graphs and cycles.

1.7 Proof Techniques

Shifting

Set systems typically have little structure, so shifting is a technique that frequently makes them easier to work with. More importantly, it preserves many of the properties of the set system, such as size and intersecting nature, so it proves useful when proving intersection theorems. The technique was first employed by Erdős et al. in the original proof of the EKR theorem. In this section, we will first define the operation, present some simple, yet useful properties of the operation, and then present an inductive proof of the bound in the EKR theorem. We will also present a proof of the Hilton-Milner theorem, due to Frankl and Furedi. In both cases, our approach will be similar to the one of Frankl [26].

Let $[n]$ be the ground set, and let $\mathscr{F} \subseteq \binom{[n]}{r}$ be an intersecting family. For $1 \leq i < j \leq n$, define a shifting operation for a set $F \in \mathscr{F}$ as follows:

$$S_{ij}(F) = \begin{cases} (F - \{j\}) \cup \{i\} & \text{if } i \notin F, j \in F, (F - \{j\} \cup \{i\} \notin \mathscr{F}, \text{ and} \\ F & \text{otherwise.} \end{cases}$$

Based on this definition, we can define the corresponding shifting operation for the family $\mathscr{F}$ as follows.

$$S_{ij}(\mathscr{F}) = \{S_{ij}(F) : F \in \mathscr{F}\}.$$

We will now present the following well-known lemma about shifting.

Lemma 1.7.1. *Suppose $\mathscr{F} \subseteq \binom{[n]}{r}$ is intersecting. Then,*

1. $|S_{ij}(\mathscr{F})| = |\mathscr{F}|$.

2. $S_{ij}(\mathscr{F})$ *is r-uniform.*

3. $S_{ij}(\mathscr{F})$ *is intersecting.*

Proof. It is clear, from the definition of shifting itself, that the first two parts of the lemma are trivial. So, we will only prove the final part of the lemma. Let $F_1, F_2 \in S_{ij}(\mathscr{F})$, and suppose $F_1 \cap F_2 = \emptyset$. Now, let G_1, G_2 be such that $S_{ij}(G_k) = F_i$, for $k \in \{1,2\}$. Since $G_1 \cap G_2 \neq \emptyset$ but $F_1 \cap F_2 = \emptyset$, we have $j \in G_1 \cap G_2$. Moreover, we also have $j \notin F_1 \cap F_2$ and $i \notin F_1 \cap F_2$. Without loss of generality, suppose $j \notin F_1$.

Then, $F_1 = G_1 \setminus \{j\} \cup \{i\}$. However, we then have $F_2 = G_2$. Now, since $j \in G_2$ but $i \notin G_2$, and G_2 was not changed by the shifting operation S_{ij}, this means $G_2' = G_2 \setminus \{j\} \cup \{i\} \in \mathscr{F}$. This gives $|F_1 \cap F_2| = |(G_1 \setminus \{j\} \cup \{i\}) \cap (G_2' \setminus \{i\} \cup \{j\})| = |G_1 \cap G_2'| > 0$, which completes the proof of the lemma. □

It is not hard to see that by carrying out the shifting operation on $\mathscr{F}$ with all ij pairs, with $i < j$, we end up with a *shifted* family $\mathscr{G}$, i.e. a family with the following nice structure. For all $1 \leq i < j \leq n$, we have $S_{ij}(\mathscr{G}) = \mathscr{G}$. Before we proceed to a proof of the bound in Theorem 1.1.1, we will prove another lemma, about shifted families, which will prove that for a family that is shifted and intersecting, any two elements in it will intersect on the first $2k-1$ elements of $[n]$. More precisely, we prove the following.

Lemma 1.7.2. *Suppose $\mathscr{F}$ is r-uniform, intersecting and shifted. Then, for all $A, B \in \mathscr{F}$, $A \cap B \cap [2r-1] \neq \emptyset$.*

Proof. We will give a proof by contradiction. Pick a counterexample which maximizes $A \cap [2r-1]$. Let $j > 2r-1$ such that $j \in A \cap B$. Since $j > 2r-1$, $A \cup B \not\subseteq [2r-1]$. Now, pick an $i \notin A \cup B$, such that $i \leq 2r-1$, and replace A by $A' = A \setminus \{j\} \cup \{i\}$. Since $\mathscr{F}$ is shifted, $A' \in \mathscr{F}$, and $A' \cap B \cap [2r-1] = \emptyset$, and we obtain a counterexample, where $A' \cap [2r-1] > A \cap [2r-1]$. This contradicts the maximality of original counterexample, completing the proof. □

A Proof of the EKR theorem using shifting

We now proceed to prove the bound in Theorem 1.1.1, using shifting.

Proof of Theorem 1.1.1. Let $n \geq 2r$, and suppose $\mathscr{F}$ is an r-uniform, intersecting family on $[n]$. We do induction on r. The statement is trivial when $r = 1$, so suppose $r \geq 2$. We now do induction on n. Suppose $n = 2r$. Now, for any $F \in \mathscr{F}$, its complement, which also has cardinality r, $[n]\setminus F \notin \mathscr{F}$. This gives us the bound $|\mathscr{F}| \le \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1}$. So suppose $n > 2r$. Using Lemma 1.7.1, we can assume $\mathscr{F}$ to be shifted. We define the following families. For $0 \le i \le r$, let $\mathscr{F}_i = \{F \cap [2r] : F \in \mathscr{F}, F \cap [2r] = i\}$. By Lemma 1.7.2, $\mathscr{F}_0$ is empty and each $\mathscr{F}_i$ is intersecting. By the induction hypothesis for r, for $1 \le i \le r-1$, we get $|\mathscr{F}_i| \le \binom{2r-1}{i-1}$. When $i = r$, we get $|\mathscr{F}_r| \le \binom{2r-1}{r-1}$ by the induction hypothesis for n. Now, given $F \in F_i$, at most $\binom{n-2r}{r-i}$ sets $G \in \mathscr{F}$ have $G \cap [2r] = F$. This gives us the required bound, as follows.

$$|\mathscr{F}| \le \sum_{1 \le i \le r} |\mathscr{F}_i| \binom{n-2r}{r-i} \le \sum_{1 \le i \le r} \binom{2k-1}{i-1}\binom{n-2r}{r-i} \le \binom{n-1}{r-1}.$$

□

Proof of the Hilton-Milner theorem

Proof of Theorem 1.5.1. Let $\mathscr{F}$ be a family that satisfies the hypothesis of the theorem, i.e. $\mathscr{F} \subseteq \binom{[n]}{r}$, with $r < n/2$, and suppose $\cap_{F \in \mathscr{F}} F = \cap \mathscr{F} = \emptyset$. We can assume $\mathscr{F}$ is of maximal size. Consider the effect of an arbitrary shift operation S_{ij}, for some $i < j$. By the properties of the shifting operation, $S_{ij}(\mathscr{F})$ is intersecting, but there are two possibilities: either $\cap S_{ij}(\mathscr{F}) = \emptyset$ or $S_{ij}(\mathscr{F})$ has a star structure. If the former is true, keep applying shift operations till be obtain a shifted family which satisfies the hypothesis of the theorem. So, suppose the latter is true. To simplify, we can assume (relabeling, if necessary) that the operation S_{12} results in the family attaining the star structure. It is not hard to observe that $1 \in \cap S_{12}(\mathscr{F})$. This also implies that $\{1,2\} \cap F \neq \emptyset$ for all $F \in \mathscr{F}$. By maximality of $\mathscr{F}$, we can assume that $\mathscr{H} = \{G \in \binom{[n]}{r} : \{1,2\} \subset G\} \subset \mathscr{F}$. Also, note that $\cap \mathscr{F} = \emptyset$.

Now, apply all S_{ij} operations for $3 \le i < j \le n$. $\mathscr{H} \subseteq \mathscr{F}$ implies that $\cap S_{ij}(\mathscr{F}) = \emptyset$. Eventually, after all the above S_{ij} operations, we get a family, which we denote by $\mathscr{F}$, satisfying $S_{ij}(\mathscr{F}) = \mathscr{F}$ for $3 \le i < j \le n$. This shifted property of $\mathscr{F}$ implies that $\{i,3,4,\ldots,r+1\} \in \mathscr{F}$ for $i \in \{1,2\}$. This gives us $\binom{[r+1]}{r} \subseteq \mathscr{F}$, because $\mathscr{H} \subseteq \mathscr{F}$. Now, applying any S_{ij} operation, even with $i = 1,2$ will leave $\binom{[r+1]}{r}$ unchanged, and the property $\cap\mathscr{F} = \emptyset$ will be maintained. So, we have shown that in proving Theorem 1.5.1, we can assume that the family is shifted. Now, $\mathscr{F}$ being shifted implies that $\binom{[r+1]}{r} \subseteq \mathscr{F}$.

We now proceed by induction on n. Define the families $\mathscr{F}_i$ as before, i.e. $\mathscr{F}_i = \{F \cap [2r] : F \in \mathscr{F}, |F \cap [2r]| = i\}$. From Lemma 1.7.2, each $\mathscr{F}_i$ is intersecting. In particular, $\mathscr{F}_0 = \emptyset$, and using the fact that $\binom{[r+1]}{r} \subseteq \mathscr{F}$, we also get $\mathscr{F}_1 = \emptyset$. We will now prove the following proposition.

Proposition 1.7.3. *1.* $|\mathscr{F}_i| \leq \binom{2r-1}{i-1}$*, if* $2 \leq i < r$.

2. $|\mathscr{F}_r| \leq \binom{2r-1}{r-1} - \binom{r-1}{r-1} + 1$.

Proof. Assume first that $\cap\mathscr{F} \neq \emptyset$. Now, since $\binom{[r+1]}{r} \subseteq \mathscr{F}$, no set of the form $\{l \cup \{A\} : A \in \binom{\{r+2,r+3,\ldots,2r\}}{i-1}\}$ can be in $\mathscr{F}$, since it would have empty intersection with some set from $\binom{[r+1]}{r}$. This implies $|\mathscr{F}| \leq \binom{2r-1}{i-1} - \binom{r-1}{i-1}$, giving the required bound. So suppose $\cap\mathscr{F} = \emptyset$. Now, $\mathscr{F}_i$ satisfies the assumptions of the main theorem, so by the induction hypothesis (of the proof of the main theorem), we get $|\mathscr{F}_i| \leq \binom{2r-1}{i-1} - \binom{2r-i-1}{i-1} + 1$, which leads to the required bound. The case $i = r$ is trivial, since, as seen before, $\mathscr{F} \leq \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1}$, which gives us the bound. ◇

Now, for any $S \subseteq [2r]$, there are at most $\binom{n-2r}{r-|S|}$ sets $F \in \mathscr{F}$ with $F \cap [2r] = S$. This observation, with the above proposition gives us the required upper bound as follows.

$$
\begin{aligned}
|\mathscr{F}| &\leq \sum_{i=1}^{r} \binom{n-2r}{r-i} |\mathscr{F}_i| \\
&\leq 1 + \sum_{i=2}^{r} \binom{n-2r}{r-i} \left(\binom{2r-1}{i-1} - \binom{r-1}{i-1} \right) \\
&= 1 + \binom{n-1}{r-1} - \binom{n-r-1}{r-1}. \qquad (1.1)
\end{aligned}
$$

□

Katona's circle method

Arguably the most elegant, and certainly the simplest proof of the Erdős–Ko–Rado theorem was given by Katona [39], who devised what is now referred to as the *circle method*. Moreover, for a number of suitably structured generalizations (see Bollobas-Leader [6], for one example), this method can often be extended to give short, simple proofs. In this small section, we will reproduce this proof, which includes a proof of the bound in Theorem 1.1.1, and the characterization of the extremal families. We will also discuss some applications of this method to other settings.

Katona's proof of the EKR theorem

Proof. We begin by defining a cyclic order on $[n]$ to be a bijection from the set $[n]$ to itself. We say that a set F of size r is an interval in a cyclic order f if there exists an $i \in [n]$ such that $F = \{f(i), f(i+1), \ldots, f(i+r-1)\}$. Note that in the proof, addition will be carried out mod n, more precisely, $i = i - n$ if $i > n$. We say that F begins in i and ends in $i+r-1$, and contains the points $i, i+1, \ldots, i+r-1$. As before, let $\mathscr{F}$ be a r-uniform, intersecting family on $[n]$, with $r < n/2$. The main part of the proof will be the following lemma.

Lemma 1.7.4. *Let f be a cyclic order on $[n]$. Then, there are at most r elements in $\mathscr{F}$ which are intervals in f.*

Proof. Suppose $A \in \mathscr{F}$ is an interval in f. Let $A = \{f(i), \ldots, f(i+r-1)\}$, for some $i \in [n]$. Now, for $i \leq j \leq i+r-2$, let A_i be the set that ends in i and let B_i be the set that begins at $i+1$. Since $r \leq n/2$, for each i, $A_i \cap B_i = \emptyset$, so at most one out of A_i and B_i can be an interval in f. Since there are $r-1$ (A_i, B_i) pairs, this gives us the required bound. $\diamond$

Now, regarding two cyclic orders as identical if one can be obtained from the other by rotation, there are $(n-1)!$ cyclic orders, and each $F \in \mathscr{F}$ is an interval in $r!(n-r)!$ cyclic orders, we get the following inequality, using Lemma 1.7.4. $|\mathscr{F}|r!(n-r)! \leq r(n-1)!$, which simplifies to the required bound $|\mathscr{F}| \leq \binom{n-1}{r-1}$. Next, let $r < n/2$ and suppose equality holds. Then, for each cyclic order f, there are exactly r elements in $\mathscr{F}$ that are intervals in f. It is not hard to see that in this case, each of the r intervals will contain a common point. Now, let $r < n/2$ and suppose there are exactly r elements in $\mathscr{F}$ that are intervals in a cyclic order f. Without loss of generality, suppose the common point is r. Let $A_1 = \{f(1), \ldots, f(r)\}$ be the set that ends in r and let $A_2 = \{f(r), \ldots, f(2r-1)\}$ be the set that begins in r. Let $b \in [n]$ such that $b \cap (A_1 \cup A_2) = \emptyset$. Now, $A' = A_1 \setminus \{f(i)\} \cup \{b\} \notin \mathscr{F}$, since $A' \cap A_2 = \emptyset$. Consider a cyclic order g where $g(n) = b$, $g(k) = f(k)$, when $1 \leq k \leq r$, and the rest of the bijection is defined arbitrarily. By Lemma 1.7.4, there are r sets in $\mathscr{F}$ that are intervals in g. Now, since $A' \notin \mathscr{F}$, but $A_1 \in \mathscr{F}$, this means that the r sets in $\mathscr{F}$ that are intervals in g also contain the same common point, i.e. r. This clearly shows that $\{F : f(r) \in F, |F| = r\} \subseteq \mathscr{F}$, completing the proof. $\square$

Cayley graphs and application to Stability analysis

In our proof of Theorem 1.5.2, we will use a special generalization of Katona's circle method, first formulated by Keevash [40]. We will briefly describe the method here, while a more detailed exposition will be presented while giving the actual proof of the theorem in Chapter 3.

Keevash's method considers the Cayley graph of the symmetric group S_n, with the adjacent transpositions as the generating set. Using a result of Bacher [3], it was shown by Keevash that this Cayley graph has desirable expansion properties. Katona's lemma states that for an intersecting family $\mathscr{F} \subseteq \binom{[n]}{r}$, there are exactly r sets which are intervals in a cyclic order (which can now be regarded as a permutation on the set $\{1,2,\ldots,n-1\}$), then they all contain a common point. Call such cyclic orders *v-complete*, with respect to $\mathscr{F}$, where v is the point common to all intervals. If we take an intersecting family which has size close to the extremal number, which is $\binom{n-1}{r-1}$, we can show that there are many complete cyclic orders. The rest of this strategy involves consideration of the subgraph of the above-mentioned Cayley graph (on S_{n-1}) containing all the complete cyclic orders, and finding a large component in this subgraph, using the expansion properties of this graph. An argument similar to the one used to characterize the extremal structures in Katona's proof can then be used to conclude that this large component contains complete cyclic orders which are v-complete, for some vertex v.

Application to Erdős–Ko–Rado graphs

Graphs generally have complicated structure, and a generalization of Katona's method to any graph seems extremely hard to formulate. However, for certain vertex-transitive graphs, in particular those where all the independent sets are "identical", it seems possible for Katona's method to be generalized, by finding a suitable class of structures to average over. One of the classic examples of this is Theorem 1.4.4, the result of Bollobás and Leader [6] which proves that if $G = M_n$, where M_n is a perfect matching on $2n$ vertices (and n edges), G is r-EKR for all $r \leq n$. We present the short proof of the bound in that theorem, which is very similar to Katona's method.

Proof of Theorem 1.4.4. Let $V(M_n) = \{1, \ldots, 2n\}$, $E(M_n) = \{i\, i+n : i \in [n]\}$, $1 \leq r \leq n$ and suppose $\mathscr{F} \subseteq \mathscr{J}^r(M_n)$ is intersecting. We will consider cyclic orders on this vertex set, but since we are interested in only those r sets which are independent sets, we consider only certain cyclic orders, which we call *good* cyclic orders. Consider those cyclic orders on $V(M_n)$ where for every $i \in [n]$, its neighbor is exactly n vertices apart. To put more precisely, if we denote a cyclic order by a function $f : V(M_n) \to [2n]$, then for every $i \in [n]$, $f(i+n) = f(i)+n$, where addition is carried out mod n, i.e. $f(i)+n$ is equal to $f(i)+n$ or $f(i)-n$ depending on which lies in $[2n]$. It is not hard to observe that, considering two cyclic orders equivalent under rotation, there are $2^{n-1}(n-1)!$ good cyclic orderings on $V(M_n)$. Consider a set $M \in \mathscr{F}$. In how many good cyclic orders is M an interval? The answer is $r!(n-r)!2^{n-r}$. Since $r \leq n = 1/2(2n)$, we can use Lemma 1.7.4 to conclude that for any fixed good cyclic order f, at most r sets from $\mathscr{F}$ can be intervals in f. This gives us the following inequality, $r!(n-r)!2^{n-r}|\mathscr{F}| \leq r(n-1)!2^{n-1}$, which simplifies to $|\mathscr{F}| \leq 2^{r-1}\binom{n-1}{r-1}$, as required. □

Note that a few reasons why this method could be generalized was because there exists at least one good cyclic ordering, i.e. one ordering where all intervals are independent sets in M_n. Secondly, not only is M_n vertex-transitive, it has an even stronger property: every independent set in M_n is identical, in the sense that every independent set in M_n lies in the same number of good cyclic orderings. It seems possible that this strategy can be generalized in other settings with similarly desirable properties, for example the family of all perfect matchings in a complete graph of even order. In this thesis, we will employ a generalization of this method to prove an extension of Theorem 1.4.4 for k-wise intersecting families.

Chapter 2

GRAPHS WITH ERDŐS–KO–RADO PROPERTY

The Erdős–Ko–Rado property for graphs is defined in the following manner.

For a graph G, vertex $v \in V(G)$ and some integer $r \geq 1$, denote the family of independent r-sets of $V(G)$ by $\mathscr{I}^{(r)}(G)$ and the subfamily $\{A \in \mathscr{I}^{(r)}(G) : v \in A\}$ by $\mathscr{I}_v^{(r)}(G)$, called a star. Then, G is said to be r-EKR if no intersecting subfamily of $\mathscr{I}^{(r)}(G)$ is larger than the largest star in $\mathscr{I}^{(r)}(G)$. If every maximum sized intersecting subfamily of $\mathscr{I}^{(r)}(G)$ is a star, then G is said to be strictly r-EKR. This can be viewed as the Erdős–Ko–Rado property on a ground set, but with additional structure on this ground set. In fact, the Erdős–Ko–Rado theorem can be restated in these terms as follows.

Theorem 1.1.1 (Erdős–Ko–Rado)**.** *The graph on n vertices with no edges is r-EKR if $n \geq 2r$ and strictly r-EKR if $n > 2r$.*

There are some results giving EKR-type theorems for different types of graphs. The following theorem was originally proved by Berge [4], with Livingston [44] characterizing the extremal case.

Theorem 2.0.5 (Berge [4], Livingston [44])**.** *If $r \geq 1$, $t \geq 2$ and G is the disjoint union of r copies of K_t, then G is r-EKR and strictly so unless $t = 2$.*

Other proofs of this result were given by Gronau [30] and Moon [49]. Berge [4] proved a stronger result.

Theorem 2.0.6 (Berge [4])**.** *If G is the disjoint union of r complete graphs each of order at least 2, then G is r-EKR.*

A generalization of Theorem 2.0.5 was first stated by Meyer [47] and proved by Deza and Frankl [15].

Theorem 2.0.7 (Meyer [47], Deza and Frankl [15])**.** *If* $r \geq 1$*,* $t \geq 2$ *and* G *is a disjoint union of* $n \geq r$ *copies of* K_t*, then* G *is* r*-EKR and strictly so unless* $t = 2$ *and* $r = n$*.*

In the paper which introduced the notion of the r-EKR property for graphs, Holroyd, Spencer and Talbot [34] prove a generalization of Theorems 2.0.6 and 2.0.7.

Theorem 2.0.8 (Holroyd et al. [34])**.** *If* G *is a disjoint union of* $n \geq r$ *complete graphs each of order at least* 2*, then* G *is* r*-EKR.*

The compression technique used in [34], which is equivalent to contracting an edge in a graph, was employed by Talbot[60] to prove a theorem for the k^{th} power of a cycle.

Definition 2.0.9. *The* k^{th} *power of a cycle* C_n^k *is a graph with vertex set* $[n]$ *and edges between* $a, b \in [n]$ *iff* $1 \leq |a - b \bmod n| \leq k$*.*

Theorem 2.0.10 (Talbot [60])**.** *If* $r, k, n \geq 1$*, then* C_n^k *is* r*-EKR and strictly so unless* $n = 2r + 2$ *and* $k = 1$*.*

An analogous theorem for the k^{th} power of a path is also proved in [34].

Definition 2.0.11. *The* k^{th} *power of a path* P_n^k *is a graph with vertex set* $[n]$ *and edges between* $a, b \in [n]$ *iff* $1 \leq |a - b| \leq k$*.*

Theorem 2.0.12 (Holroyd et al. [34])**.** *If* $r, k, n \geq 1$*, then* P_n^k *is* r*-EKR.*

It can be observed here that the condition $r \leq n/2$ is not required for the graphs C_n^k and P_n^k because for each of the two graphs, there is no independent set of size greater than $n/2$, so the r-EKR property holds vacuously if $r > n/2$.

The compression proof technique is also employed to prove a result for a larger class of graphs.

Theorem 2.0.13 (Holroyd et al. [34])**.** *If G is a disjoint union of $n \geq 2r$ complete graphs, cycles and paths, including an isolated singleton, then G is r-EKR.*

The problem of finding if a graph G is 2-EKR is addressed by Holroyd and Talbot in [35].

Theorem 2.0.14 (Holroyd–Talbot [35])**.** *Let G be a non-complete graph of order n with minimum degree δ and independence number α.*

1. *If $\alpha = 2$, then G is strictly 2-EKR.*
2. *If $\alpha \geq 3$, then G is 2-EKR if and only if $\delta \leq n-4$ and strictly so if and only if $\delta \leq n-5$, the star centers being the vertices of minimum degree.*

Holroyd and Talbot also present an interesting conjecture in [35], which we first stated in Chapter 1 and recall here.

Definition 2.0.15. *The minimum size of a maximal independent vertex set of a graph G is the minimax independent number, denoted by $\mu(G)$.*

It can be noted here that $\mu(G) = i(G)$, where $i(G)$ is the independent domination number of graph G. We now restate the conjecture of Holroyd and Talbot.

Conjecture 1.3.1. *Let G be any graph and let $1 \leq r \leq \frac{1}{2}\mu$; then G is r-EKR (and is strictly so if $2 < r < \frac{1}{2}\mu$).*

This conjecture seems hard to prove or disprove; however, restricting attention to certain classes of graphs makes the problem easier to tackle. Borg and Holroyd [10] prove the conjecture for a large class of graphs, which contain a singleton as a component.

Definition 2.0.16 (Borg, Holroyd [10])**.** *For a monotonic non-decreasing (*mnd*) sequence $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ of non-negative integers, let $M = M(\mathbf{d})$ be the graph such that $V(M) = \{x_i : i \in \mathbb{N}\}$ and for $x_a, x_b \in V(M)$ with $a < b$, $x_a x_b \in E(M)$ iff $b \leq a + d_a$. Let $M_n = M_n(\mathbf{d})$ be the subgraph of M induced by the subset $\{x_i : i \in [n]\}$ of $V(M)$. Call M_n an* mnd *graph.*

Definition 2.0.17 (Borg, Holroyd [10])**.** *For $n > 2$, $1 \leq k < n-1$, $0 \leq q < n$, let $C^{k,k+1}_{q,n}$ be the graph with vertex set $\{v_i : i \in [n]\}$ and edge set $E(C^k_n) \cup \{v_i v_{i+k+1 \text{ mod } n} : 1 \leq i \leq q\}$. If $q > 0$, call $C^{k,k+1}_{q,n}$ a modified k^{th} power of a cycle.*

Borg and Holroyd [10] prove the following theorem.

Theorem 2.0.18. *Conjecture 1.3.1 is true if G is a disjoint union of complete multipartite graphs, copies of* mnd *graphs, powers of cycles, modified powers of cycles, trees, and at least one singleton.*

One of the main results in this dissertation extends the class of graphs which satisfy Conjecture 1.3.1 by proving the conjecture for all chordal graphs which contain a singleton. It can be noted that the mnd graphs in Theorem 2.0.18 are chordal.

We also define a special class of chordal graphs, and prove a stronger EKR result for these graphs. Finally, we consider similar problems for two classes of bipartite graphs, trees and ladder graphs.

2.1 Chordal Graphs

Definition 2.1.1. *A graph G is a chordal graph if every cycle of length at least* 4 *has a chord.*

It is easy to observe that if G is chordal, then every induced subgraph of G is also chordal.

Definition 2.1.2. *A vertex v is called simplicial in a graph G if its neighborhood is a clique in G.*

Consider a graph G on n vertices, and let $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices of G. Let the graph G_i be the subgraph obtained by removing the vertex set $\{v_1, \ldots, v_{i-1}\}$ from G. Then σ is called a *simplicial elimination ordering* if v_i is simplicial in the graph G_i, for each $1 \leq i \leq n$. We state a well known characterization of chordal graphs, due to Dirac [19].

Theorem 2.1.3. *A graph G is a chordal graph if and only if it has a simplicial elimination ordering.*

It is easy to see, using this characterization of chordal graphs, that the mnd graphs of Definition 2.0.16 are chordal.

Proposition 2.1.4. *If M_n is an* mnd *graph on n vertices, M_n is chordal.*

Proof. It can be seen that ordering the vertices of M_n, according to the corresponding degree sequence **d**, as stated in Definition 2.0.16, gives a simplicial elimination ordering. □

Note that, with or without the non-decreasing condition on the sequence **d**, the resulting graph is an interval graph — use the interval $[a, a+d_a]$ for vertex x_a — which is chordal regardless.

We prove Theorem 1.3.2, which is the non-strict part of Conjecture 1.3.1 for disjoint unions of chordal graphs, containing at least one singleton.

Theorem 1.3.2. *If G is a disjoint union of chordal graphs, including at least one singleton, and if $r \leq \frac{1}{2}\mu(G)$, then G is r-EKR.*

We also consider graphs which do not have singletons. Consider a class of chordal graphs constructed as follows.

Let P_{n+1} be a path on n edges with $V(P_{n+1}) = \{v_1, \ldots, v_{n+1}\}$. Label the edge $v_i v_{i+1}$ as i, for each $1 \leq i \leq n$. A *chain* of complete graphs, of length n, is obtained from P_{n+1} by replacing each edge of P_{n+1} by a complete graph of order at least 2 in the following manner: to convert edge i of P_{n+1} into K_s, introduce a complete graph K_{s-2} and connect v_i and v_{i+1} to each of the $s-2$ vertices of the complete graph. Call the resulting complete graph G_i, and call each G_i a link of the chain. We call v_i and v_{i+1} the *connecting vertices* of this complete graph, with the exception of G_1 and G_n, which have only one connecting vertex each (the ones shared with G_2 and G_{n-1} respectively). In general, for each $2 \leq i \leq n$, call v_i the $(i-1)^{th}$ connecting vertex of G. Unless otherwise specified, we will refer to a chain of complete graphs as just a chain. We will call an isolated vertex a *trivial chain* (of length 0), while a complete graph is simply a chain of length 1. Call a chain of length n *special* if $n \in \{0, 1\}$ or if $n \geq 2$ and the following conditions hold:

1. $|G_i| \geq |G_{i-1}| + 1$ for each $2 \leq i \leq n-1$, and

2. $|G_n| \geq |G_{n-1}|$.

We prove the following results for special chains.

Theorem 2.1.5. *If G is a special chain, then G is r-EKR for all $r \geq 1$.*

Theorem 2.1.6. *If G is a disjoint union of 2 special chains, then G is r-EKR for all $r \geq 1$.*

We will also consider similar problems for bipartite graphs. A basic observation about complete bipartite graphs, along with an obvious generalization for complete multipartite graphs, is mentioned below.

- If $G = K_{m,n}$ and $m \leq n$, then G is r-EKR for all $r \leq \frac{m}{2}$.
- If $G = K_{m_1,\ldots,m_k}$, with $m_1 \leq m_2 \leq \ldots \leq m_k$, then G is r-EKR for all $r \leq \frac{m_1}{2}$.

It is easy to see why these hold. If $\mathscr{B} \subseteq \mathscr{J}^r(G)$ is intersecting, then each $A \in \mathscr{B}$ lies in the same partite set. Clearly, if $2r \leq m \leq n$, then G is r-EKR by Theorem 1.1.2. A similar argument works for complete multipartite graphs as well.

Holroyd and Talbot [35] proved Conjecture 1.3.1 for a disjoint union of two complete multipartite graphs.

If we consider non-complete bipartite graphs with high minimum degree, it seems that they usually have low μ (always at most $\min\{n-\delta, n/2\}$). Instead, in this paper, we consider bipartite graphs with low maximum degree in order to have higher values of μ (always at least $\frac{n}{\Delta+1}$). In particular, we look at trees and ladder graphs, two such classes of sparse bipartite graphs.

One of the difficult problems in dealing with graphs without singletons is that of finding centers of maximum stars. We consider this problem for trees, and conjecture that there is a maximum star in a tree that is centered at a leaf.

Conjecture 2.1.7. *For any tree T on n vertices, there exists a leaf x such that for any $v \in V(T)$, $|\mathscr{J}^r_v(T)| \leq |\mathscr{J}^r_v(T)|$.*

We prove this conjecture for $r \leq 4$.

Theorem 2.1.8. *Let $1 \leq r \leq 4$. Then, a maximum sized star of r-independent vertex sets of T is centered at a leaf.*

We will also prove that the ladder graph is 3-EKR.

Definition 2.1.9. *The ladder graph L_n with n rungs can be defined as the cartesian product of K_2 and P_n.*

It is not hard to see that, for L_n, $\mu(L_n) \leq \lceil \frac{n+1}{2} \rceil$. In fact, we show that equality holds.

Proposition 2.1.10.

$$\mu(L_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. The result is trivial if $n \leq 2$, so let $n \geq 3$. Suppose $\mu(L_n) < \lceil \frac{n+1}{2} \rceil$ and let A be a maximal independent set of size $\mu(L_n)$. Then, there exist two consecutive rungs, say the i^{th} and $(i+1)^{st}$ in L_n, with endpoints $\{x_i, y_i\}$ and $\{x_{i+1}, y_{i+1}\}$ respectively, such that $\{x_i, y_i\} \cap A = \emptyset$ and $\{x_{i+1}, y_{i+1}\} \cap A = \emptyset$. Let $u = x_i$, $v = x_{i-1}$ and $w = y_i$ if $i > 1$, otherwise, let $u = x_{i+1}$, $v = x_{i+2}$ and $w = y_{i+1}$. $A \cup \{u\}$ is not independent, since A is maximal. Then, $v \in A$ and $A \cup \{w\}$ is independent, a contradiction. □

Theorem 2.1.11. *The graph L_n is 3-EKR for all $n \geq 1$.*

An Erdős–Ko–Rado theorem for chordal graphs

We begin by fixing some notation. For a graph G and a vertex $v \in V(G)$, let $G - v$ be the graph obtained from G by removing vertex v. Also, let $G \downarrow v$ denote the graph obtained by removing v and its set of neighbors from G. We note that if G is a disjoint union of chordal graphs and if $v \in G$, the graphs $G - v$ and $G \downarrow v$ are also disjoint unions of chordal graphs.

We state and prove a series of lemmas, which we will use in the proof of Theorem 1.3.2.

Lemma 2.1.12. *Let G be a graph containing an isolated vertex x. Then, for any vertex $v \in V(G)$, $|\mathscr{I}_v^r(G)| \leq |\mathscr{I}_x^r(G)|$.*

Proof. Let $v \in V(G)$, $v \neq x$. We define a function $f : \mathscr{I}_v^r(G) \to \mathscr{I}_x^r(G)$ as follows.

$$f(A) = \begin{cases} A & \text{if } x \in A, \text{ and} \\ A \setminus \{v\} \cup \{x\} & \text{otherwise.} \end{cases}$$

It is easy to see that the function is injective, and this completes the proof. □

Lemma 2.1.13. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then, the following inequalities hold:*

1. *$\mu(G - v_2) \geq \mu(G)$;*

2. *$\mu(G \downarrow v_2) + 1 \geq \mu(G)$.*

Proof. We begin by noting that the condition $N[v_1] \subseteq N[v_2]$ implies that $v_1 v_2 \in E(G)$.

1. We will show that if I is a maximal independent set in $G - v_2$, then I is maximally independent in G. Suppose I is not a maximal independent set in G. Then, $I \cup \{v_2\}$ is an independent set in G. Thus, for any $u \in N[v_2]$, $u \notin I$. In particular, for any $u \in N[v_1]$, $u \notin I$. Thus, $I \cup \{v_1\}$ is an independent set in $G - v_2$. This is a contradiction. Thus, I is a maximal independent set in G.

   Taking I to be the smallest maximal independent set in $G - v_2$, we get $\mu(G - v_2) = |I| \geq \mu(G)$.

2. We will show that if I is a maximal independent set in $G \downarrow v_2$, then $I \cup \{v_2\}$ is a maximal independent set in G. Of course, $I \cup \{v_2\}$ is independent, so suppose it is not maximal. Then, for some vertex $u \in G \downarrow v_2$ and $u \notin I \cup \{v_2\}$, $I \cup \{u, v_2\}$ is an independent set. Thus, $I \cup \{u\}$ is an independent set in $G \downarrow v_2$, a contradiction.

   Taking I to be the smallest maximal independent set in $G \downarrow v_2$, we get $\mu(G \downarrow v_2) + 1 = |I| + 1 \geq \mu(G)$.

□

Corollary 2.1.14. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then, the following statements hold:*

1. *If $r \leq \frac{1}{2}\mu(G)$, then $r \leq \frac{1}{2}\mu(G - v_2)$;*
2. *If $r \leq \frac{1}{2}\mu(G)$, then $r - 1 \leq \frac{1}{2}\mu(G \downarrow v_2)$.*

Proof. 1. This follows trivially from the first part of Lemma 2.1.13.

2. To prove this part, we use the second part of Lemma 2.1.13 to show

$$r - 1 \leq \frac{1}{2}\mu(G) - 1 = \frac{\mu(G) - 2}{2} \leq \frac{\mu(G \downarrow v_2)}{2} - \frac{1}{2}.$$

□

Let H be a component of G, so H is a chordal graph on m vertices, $m \geq 2$. Let $\{v_1, \ldots, v_m\}$ be a simplicial elimination ordering of H and let $v_1 v_i \in E(H)$ for some $i \geq 2$. Let $\mathscr{A} \subseteq \mathscr{I}^r(G)$ be an intersecting family. We define a compression operation $f_{1,i}$ for the family $\mathscr{A}$. Before we give the definition, we note that if A is an independent set and if $v_i \in A$, then $A \setminus \{v_i\} \cup \{v_1\}$ is also independent.

$$f_{1,i}(A) = \begin{cases} A \setminus \{v_i\} \cup \{v_1\} & \text{if } v_i \in A, v_1 \notin A, A \setminus \{v_i\} \cup \{v_1\} \notin \mathscr{A}, \text{ and} \\ A & \text{otherwise.} \end{cases}$$

Then, we define the family $\mathscr{A}'$ by

$$\mathscr{A}' = f_{1,i}(\mathscr{A}) = \{f_{1,i}(A) : A \in \mathscr{A}\}.$$

It is not hard to see that $|\mathscr{A}'| = |\mathscr{A}|$. Next, we define the families

$$\mathscr{A}_i' = \{A \in \mathscr{A}' : v_i \in A\},$$

$$\bar{\mathscr{A}}_i' = \mathscr{A}' \setminus \mathscr{A}_i', \text{ and}$$

$$\mathscr{B}' = \{A \setminus \{v_i\} : A \in \mathscr{A}_i'\}.$$

Then we have

$$\begin{aligned} |\mathscr{A}| &= |\mathscr{A}'| \\ &= |\mathscr{A}_i'| + |\bar{\mathscr{A}}_i'| \\ &= |\mathscr{B}'| + |\bar{\mathscr{A}}_i'|. \end{aligned} \tag{2.1}$$

We prove the following lemma about these families.

Lemma 2.1.15. *1.* $\bar{\mathscr{A}}_i' \subseteq \mathscr{J}^r(G - v_i)$.

2. $\mathscr{B}' \subseteq \mathscr{J}^{(r-1)}(G \downarrow v_i)$.

3. $\bar{\mathscr{A}}_i'$ *is intersecting.*

4. $\mathscr{B}'$ *is intersecting.*

Proof. It follows from the definitions of the families that $\bar{\mathscr{A}}_i' \subseteq \mathscr{J}^r(G - v_i)$ and $\mathscr{B}' \subseteq \mathscr{J}^{(r-1)}(G \downarrow v_i)$. So, we only prove that the two families are intersecting. Consider $A, B \in \bar{\mathscr{A}}_i'$. If $v_1 \in A$ and $v_1 \in B$, we are done. If $v_1 \notin A$ and $v_1 \notin B$, then $A, B \in \mathscr{A}$ and hence $A \cap B \neq \emptyset$. So, suppose $v_1 \notin A$ and $v_1 \in B$. Then, $A \in \mathscr{A}$.

Also, either $B \in \mathscr{A}$, in which case we are done or $B_1 = B \setminus \{v_1\} \cup \{v_i\} \in \mathscr{A}$. Then, $|A \cap B| = |A \cap B \setminus \{v_1\} \cup \{v_i\}| = |A \cap B_1| > 0$.

Finally, consider $A, B \in \mathscr{B}'$. Since $A \cup \{v_i\} \in \mathscr{A}'_{v_i}$, $A \cup \{v_1\} \in \mathscr{A}$ and $A \cup \{v_i\} \in \mathscr{A}$. A similar argument works for B. Thus, $|(A \cup \{v_1\}) \cap (B \cup \{v_i\})| > 0$ and hence, $|A \cap B| > 0$. □

The final lemma we prove is regarding the star family $\mathscr{I}_x^r(G)$, where x is an isolated vertex.

Lemma 2.1.16. *Let G be a graph containing an isolated vertex x and let $v \in V(G)$, $v \neq x$. Then, we have*

$$|\mathscr{I}_x^r(G)| = |\mathscr{I}_x^r(G - v)| + |\mathscr{I}_x^{(r-1)}(G \downarrow v)|.$$

Proof. Partition the family $\mathscr{I}_x^r(G)$ into two parts. Let the first part contain all sets containing v, say $\mathscr{F}_v$, and let the second part contain all sets which do not contain v, say $\bar{\mathscr{F}}_v$. Then

$\mathscr{F}_v = \mathscr{I}_x^{(r-1)}(G \downarrow v)$ and $\bar{\mathscr{F}}_v = \mathscr{I}_x^r(G - v)$. □

We proceed to a proof of Theorem 1.3.2.

Proof. The theorem trivially holds for $r = 1$, so suppose $r \geq 2$. Let G be a disjoint union of chordal graphs, including at least one singleton, and let $\mu(G) \geq 2r$. We do induction on $|G|$. If $|G| = \mu(G)$, then $G = E_{|G|}$, and we are done by the Erdős–Ko–Rado theorem. So, suppose $|G| > \mu(G)$, and there is one component, say H, which is a chordal graph having m vertices, $m \geq 2$. Let $\{v_1, \ldots, v_m\}$ be a simplicial ordering of H and suppose $v_1 v_i \in E(H)$ for some $i \geq 2$. Since the neighborhood of v_1 is a clique, we have $N[v_1] \subseteq N[v_i]$. Also, let x be an isolated vertex in G. Let $\mathscr{A} \subseteq \mathscr{I}^r(G)$ be intersecting.

Define the compression operation $f_{1,i}$ and the families $\bar{\mathscr{A}}_i'$ and $\mathscr{B}'$ as before. Using Equation 2.1, Lemmas 2.1.12, 2.1.13, 2.1.15, 2.1.16, Corollary 2.1.14 and the induction hypothesis, we have

$$\begin{aligned} |\mathscr{A}| &= |\bar{\mathscr{A}}_i'| + |\mathscr{B}'| \\ &\leq |\mathscr{I}_x^r(G - v_i)| + |\mathscr{I}_x^{(r-1)}(G \downarrow v_i)| \\ &= |\mathscr{I}_x^r(G)|. \end{aligned} \tag{2.2}$$

□

2.2 Graphs without isolated vertices

The main technique we use to prove Theorem 2.1.5 is a compression operation that is equivalent to compressing a clique to a single vertex. In a sense, it is a more general version of the technique used in [34]. We begin by stating and proving a technical lemma, similar to the one proved in [34]. We will then use it to prove Theorem 2.1.5 by induction.

Generalized compression techniques

Let $H \subseteq G$ with $V(H) = \{v_1, \ldots, v_s\}$. Let G/H be the graph obtained by contracting the subgraph H to a single vertex. The contraction function c is defined as follows.

$$c(x) = \begin{cases} v_1 & : x \in H, \text{ and} \\ x & : x \notin H. \end{cases}$$

When we contract H to v_1, the edges which have both endpoints in H are lost and if there is an edge $xv_i \in E(G)$ such that $x \in V(G) \setminus V(H)$, then there is an edge $xv_1 \in E(G/H)$. Duplicate edges are disregarded.

Also, let $G - H$ be the (possibly disconnected) graph obtained from G by removing all vertices in H.

Lemma 2.2.1. *Let $G=(V,E)$ be a graph and let $\mathscr{A} \subseteq \mathscr{J}^r(G)$ be an intersecting family of maximum size. If H is a subgraph of G with vertex set $\{v_1,\dots,v_s\}$, and if H is isomorphic to K_s, then there exist families $\mathscr{B}$, $\{\mathscr{C}_i\}_{i=2}^s$, $\{\mathscr{D}_i\}_{i=2}^s$, $\{\mathscr{E}_i\}_{i=2}^s$ satisfying:*

1. $|\mathscr{A}| = |\mathscr{B}| + \sum_{i=2}^s |\mathscr{C}_i| + |\bigcup_{i=2}^s \mathscr{D}_i| + \sum_{i=2}^s |\mathscr{E}_i|$;
2. $\mathscr{B} \subseteq \mathscr{J}^r(G/H)$ *is intersecting; and*
3. *for each* $2 \le i \le s$,
   a) $\mathscr{C}_i \subseteq \mathscr{J}^{r-1}(G-H)$ *is intersecting,*
   b) $\mathscr{D}_i = \{A \in \mathscr{A} : v_1 \in A$ *and* $N(v_i) \cap (A \setminus \{v_1\}) \neq \emptyset\}$, *and*
   c) $\mathscr{E}_i = \{A \in \mathscr{A} : v_i \in A$ *and* $N(v_1) \cap (A \setminus \{v_i\}) \neq \emptyset\}$.

To prove Lemma 2.2.1, we will need a claim, which we state and prove below.

Claim 2.2.2. *Let $H \subseteq G$ be isomorphic to K_s, $s \ge 3$. Let $\mathscr{A} \subseteq \mathscr{J}^r(G)$ be an intersecting family of maximum size. Suppose $A \cup \{v_i\}, A \cup \{v_j\} \in \mathscr{A}$ for some $i, j \neq 1$ and $c(A \cup \{v_i\}) = A \cup \{v_1\} \in \mathscr{J}^r(G/H)$. Then $A \cup \{v_1\} \in \mathscr{A}$.*

Proof. Since we have $c(A \cup \{v_i\}) \in \mathscr{J}^r(G/H)$, $B = A \cup \{v_1\} \in \mathscr{J}^r(G)$. Suppose $B \notin \mathscr{A}$. Since $\mathscr{A}$ is an intersecting family of maximum size, $\mathscr{A} \cup \{B\}$ is not an intersecting family. So, there exists a $C \in \mathscr{A}$ such that $B \cap C = \emptyset$. So, we have $C \cap (A \cup \{v_i\}) = v_i$ and $C \cap (A \cup \{v_j\}) = v_j$. Thus, $v_i, v_j \in C$. This is a contradiction since v_i and v_j are adjacent to each other. □

Proof. (Proof of Lemma 2.2.1) Define the following families:

1. $\mathscr{B} = \{c(A) : A \in \mathscr{A}$ and $c(A) \in \mathscr{J}^r(G/H)\}$; and
2. for each $2 \le i \le s$:

a) $\mathscr{C}_i = \{A \setminus \{v_1\} : v_1 \in A \text{ and } A \setminus \{v_1\} \cup \{v_i\} \in \mathscr{A}\}$,

b) $\mathscr{D}_i = \{A \in \mathscr{A} : v_1 \in A \text{ and } N(v_i) \cap (A \setminus \{v_1\}) \neq \emptyset\}$, and

c) $\mathscr{E}_i = \{A \in \mathscr{A} : v_i \in A \text{ and } N(v_1) \cap (A \setminus \{v_i\}) \neq \emptyset\}$.

If $A, B \in \mathscr{A}$ and $A \neq B$, then $c(A) = c(B)$ iff $A \triangle B = \{v_i, v_j\}$ for some $1 \leq i, j \leq s$. Using this and Claim 2.2.2 (if $s \geq 3$), we have

$$|\{A \in \mathscr{A} : c(A) \in \mathscr{J}^r(G/H)\}| = |\mathscr{B}| + \sum_{i=2}^{s} |\mathscr{C}_i|.$$

Also, if $A \in \mathscr{A}$, then $c(A) \notin \mathscr{J}^r(G/H)$ iff $A \in \bigcup_{i=2}^{s} \mathscr{D}_i \cup \bigcup_{i=2}^{s} \mathscr{E}_i$. Thus, we have $|\mathscr{A}| = |\mathscr{B}| + \sum_{i=2}^{s} |\mathscr{C}_i| + |\bigcup_{i=2}^{s} \mathscr{D}_i| + |\bigcup_{i=2}^{s} \mathscr{E}_i|$. By the definition of the $\mathscr{E}_i$'s, $\bigcup_{i=2}^{s} \mathscr{E}_i$ is a disjoint union, so we have

$$|\mathscr{A}| = |\mathscr{B}| + \sum_{i=2}^{s} |\mathscr{C}_i| + |\bigcup_{i=2}^{s} \mathscr{D}_i| + \sum_{i=2}^{s} |\mathscr{E}_i|$$

It is obvious to show that $\mathscr{B}$ is intersecting since $\mathscr{A}$ is.

Let $2 \leq i \leq s$. To see that $\mathscr{C}_i$ is intersecting, suppose $C, D \in \mathscr{C}_i$ and $C \cap D = \emptyset$. But $C \cup \{v_1\}$ and $D \cup \{v_i\}$ are in $\mathscr{A}$ and hence, are intersecting. This is a contradiction. □

Before we move to the proof of Theorem 2.1.5, we will prove one final claim regarding maximum sized star families in G.

Claim 2.2.3. *If G is special chain of length n, then a maximum sized star is centered at an internal vertex of G_1.*

Proof. First note that for any i, there is a trivial injection from a star centered at a connecting vertex of G_i to a star centered at an internal vertex of G_i, which replaces the star center by that internal vertex in every set of the family. So suppose $\mathscr{Q}$ is a star centered at a internal vertex u of any of the graphs G_i, $i \neq 1$. Let $G_1 = K_m$. Consider the following cases.

1. Suppose u is in G_2. In this case, define an arbitrary bijection between the $m-1$ internal vertices of G_1 and any $m-1$ internal vertices of G_2 containing u, such that u corresponds to an internal vertex of G_1, say v (note that this can always be done, since if $n=2$, then $|G_2| \geq m$, with one connecting vertex, while if $n \geq 3$, then $|G_2| \geq m+1$, with two connecting vertices).

2. Suppose u is in some G_i such that $i \geq 3$. Then, define an arbitrary bijection between the m vertices of G_1 and any m internal vertices of G_i including u such that u corresponds to an internal vertex of G_1, say v.

Next, consider any set in $\mathscr{Q}$. If it contains a vertex w in G_1, replace that vertex by b and replace u by the vertex in G_i corresponding to w. If it does not contain a vertex in G_1, replace u by v. This defines the injection from $\mathscr{Q}$ to a star centered at v. $\square$

We now give a proof of Theorem 2.1.5.

Proof. Let $\mathscr{I}_1^r(G)$ be a maximum sized star family in G, where 1 is an internal vertex of G_1.

We do induction on r. The result is trivial for $r=1$. Let $r \geq 2$. We do induction on n (n is the number of links). For $n=1$, result is vacuously true. If $n=2$, then for $r=2$, we use Theorem 2.0.14 to conclude that G is 2-EKR while the result is vacuously true for $r \geq 3$. So, let $n \geq 3$. Let $\mathscr{A} \subseteq \mathscr{I}^r(G)$ be an intersecting family of maximum cardinality. Let the vertices of $G_n = K_s$ be labeled from n_1 to n_s (let n_1 be the connecting vertex which also belongs to G_{n-1}). Define the compression operation c on G and the clique K_s as before. Let the families $\mathscr{B}$, $\{\mathscr{C}_i\}_{i=2}^s$, $\{\mathscr{D}_i\}_{i=2}^s$, $\{\mathscr{E}_i\}_{i=2}^s$ be defined as in Lemma 2.2.1.

Clearly, for G, $\mathscr{D}_i = \emptyset$ for each $2 \leq i \leq s$. So, by Lemma 2.2.1,

$$\mathscr{A} = \mathscr{B} + \sum_{i=2}^{s} |\mathscr{C}_i| + \sum_{i=2}^{s} |\mathscr{E}_i|.$$

Let $G_{n-1} = K_t$. Let the vertices of G_{n-1} be labeled from m_1 to $m_t (t \leq s)$, with $m_t = n_1$. For every $1 \leq i \leq t-1$ and $2 \leq j \leq s$ define a set $\mathscr{H}_{ij}$ of families by

$$\mathscr{H}_{ij} = \{A \in \mathscr{A} : m_i \in A, n_j \in A\}.$$

We note that $\bigcup_{i=1}^{t-1} H_{ij} = \mathscr{E}_j$ for each $2 \leq j \leq s$, and since each of the $\mathscr{H}_{ij}$'s are also disjoint, we have

$$\sum_{i=2}^{s} |\mathscr{E}_i| = \sum_{1 \leq i \leq t-1, 2 \leq j \leq s} |\mathscr{H}_{ij}|.$$

Now, consider a complete bipartite graph $K_{t-1,s-1}$. Label the vertices in part 1 from m_1 to m_{t-1} and vertices in part 2 from n_2 to n_s.

Partition the edges of the bipartite graph $K_{t-1,s-1}$ into $s-1$ matchings, each of size $t-1$. For each matching M_k $(1 \leq k \leq s-1)$, define the family

$$\mathscr{F}_{M_k} = \bigcup_{i,j,m_i n_j \in M_k} (\mathscr{H}_{ij} - \{n_j\}),$$

where a family $\mathscr{H} - \{a\}$ is obtained from $\mathscr{H}$ by removing a from all its sets. Then of course

$$\sum_{1 \leq i \leq t-1, 2 \leq j \leq s} |\mathscr{H}_{ij}| = \sum_{1 \leq i \leq s-1} |\mathscr{F}_{M_i}|.$$

For each $1 \leq k \leq s-1$, F_{M_k} is a disjoint union and is intersecting. The intersecting property is obvious if both sets are in the same $\mathscr{H}_{ij} - \{n_j\}$ since they contain m_i. If in different such sets, adding distinct elements which were removed (during the above operation) gives sets in the original family which are intersecting.

Finally, if we consider families $C_{n_i} \cup F_{M_{i-1}} \subseteq \mathscr{J}^{(r-1)}(G - G_n)$ for $2 \leq i \leq s$, each such family is a disjoint union. It is also intersecting since for $C \in C_{n_i}$ and $F \in F_{M_{i-1}}$, $C \cup \{n_1\}$ and $F \cup \{n_j\}$ for some $j \neq 1$ gives us sets in $\mathscr{A}$. So, we get

$$
\begin{aligned}
|\mathscr{A}| &= |\mathscr{B}| + \sum_{i=2}^{s} |\mathscr{C}_{n_i}| + \sum_{1\le i\le s-1, 2\le j\le s} |\mathscr{H}_{ij}| \\
&= |\mathscr{B}| + \sum_{i=2}^{s} |\mathscr{C}_{n_i}| + \sum_{1\le i\le s-1} |\mathscr{F}_{M_i}| \\
&= |\mathscr{B}| + \sum_{i=2}^{s} |(C_{n_i} \cup \mathscr{F}_{M_{i-1}})| \\
&\le \mathscr{I}_1^r(G/G_n) + (s-1)\mathscr{I}_1^{(r-1)}(G-G_n) \\
&= \mathscr{I}_1^r(G).
\end{aligned}
$$

The last inequality is obtained by partitioning the star based on whether or not it contains one of $\{n_2, \ldots, n_s\}$. □

We now proceed to a proof of Theorem 2.1.6.

Proof. We do induction on r. Since the case $r = 1$ is trivial, let $r \ge 2$. Let G be a disjoint union of 2 *special* chains G' and G'', with lengths n_1 and n_2 respectively. We will do induction on $n = n_1 + n_2$. If $n = 0$, the result holds trivially if $r = 2$ and vacuously if $r \ge 3$. So, let $n \ge 1$. If $n = 1$ or if $n_1 = n_2 = 1$, then $\alpha(G) = 2$. In this case, G is vacuously r-EKR for $r \ge 3$. Also, if $r = 2$, then we are done by Theorem 2.0.14. So, without loss of generality, we assume that G_1 has length at least 2. We can now proceed as in the proof of Theorem 2.1.5. □

2.3 Bipartite graphs

Trees

In this section, we give a proof of Theorem 2.1.8, which states that for a given tree T and $r \le 4$, there is a maximum star family centered at a leaf of T.

Proof. The statement is trivial for $r = 1$. If $r = 2$, we use the fact that for any vertex v, $|\mathscr{I}_v^2(T)| = n - 1 - d(v)$, where $d(v)$ is the degree of vertex v, and thus it will be maximum when v is a leaf.

Let $3 \leq r \leq 4$. Let v be an internal vertex ($d(v) \geq 2$) and let $\mathscr{A} = \mathscr{I}_v^r(T)$ be the star centered at v. Consider T as a tree rooted at v. We find an injection f from $\mathscr{A}$ to a star centered at some leaf. Let v_1 and v_2 be any two neighbors of v and let u be a leaf with neighbor w. Let $A \in \mathscr{A}$.

1. If $u \in A$, then let $f(A) = A$.

2. If $u \notin A$, then we consider two cases.

   a) If $w \notin A$, let $f(A) = A \setminus \{v\} \cup \{u\}$.

   b) If $w \in A$, then $B = A \setminus \{w\} \cup \{u\} \in \mathscr{A}$. We consider the following two cases separately.

      - $r = 3$

        Let $A = \{v, w, x\}$. We know that x cannot be connected to both v_1 and v_2 since that would result in a cycle. Without loss of generality, suppose that $xv_1 \notin E(T)$. Then, let $f(A) = A \setminus \{v, w\} \cup \{u, v_1\}$.

      - $r = 4$

        Let $A = \{v, w, w_1, w_2\}$. We first note that if there is a leaf at distance two from v, then by using 1 and $2(a)$ above, we can show that the size of the star at this leaf is at least as much as the given star. We again consider two cases.

        - Suppose that $\{v_1, v_2\} \nsubseteq N(w_1) \cup N(w_2)$. By symmetry, suppose $v_1 \notin N(w_1) \cup N(w_2)$. In this case, let $f(A) = A \setminus \{w, v\} \cup \{u, v_1\}$.

        - Suppose that $\{v_1, v_2\} \subseteq N(w_1) \cup N(w_2)$. Label so that $v_i \in N(w_i)$ for $1 \leq i \leq 2$ (in particular, v_i is the parent of w_i). Since neither w_1 nor w_2 is a leaf, they have at least one child, say

x_1 and x_2, respectively. In this case, let $f(A) = \{u, x_1, x_2, v_1\}$. For this case, injection is less obvious. We show it by contradiction as follows. Let $f(\{v, w, w_1, w_2\}) = f(\{v, w, y_1, y_2\}) = \{u, x_1, x_2, v_1\}$. We may assume that $y_1 \neq w_1$ and let y_i be the child of v_i and x_i be the child of y_i; then certainly $v_1 w_1 x_1 y_1 v_1$ gives a cycle in T, a contradiction.

□

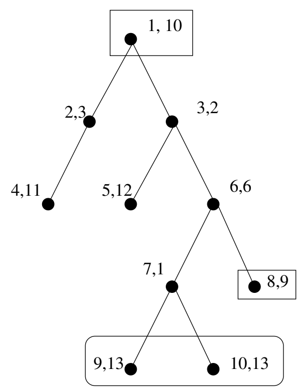

Figure 2.1: Tree T on 10 vertices, $r = 5$.

We believe that Conjecture 2.1.7 holds true for all r. However, it is harder to prove because it is not true that every leaf centered star is bigger than every non-leaf centered star; an example is illustrated in Figure 2.1.

For each vertex, the first number denotes the label, while the second number denotes the size of the star centered at that vertex. We note that $\mathscr{I}_8^5(T) = 9$, while $\mathscr{I}_1^5(T) = 10$. However, we note that the maximum sized stars are still centered at leaves 9 and 10.

We also point out that this example satisfies an interesting property, first observed by Colbourn [13].

Property 2.3.1. *Let G be a bipartite graph with bipartition $V = \{V_1, V_2\}$ and let $r \geq 1$. We say that G has the* bipartite degree sort *property if for all $x, y \in V_i$ with $d(x) \leq d(y)$, $\mathscr{I}_x^r(T) \geq \mathscr{I}_y^r(T)$.*

Not all bipartite graphs satisfy this property. Neiman [53] constructed the following counterexample, with $r = 3$.

Fix positive integers t and k with $t \geq 2k \geq 4$. Let $G = G_{t,k}$ be the graph obtained from the complete bipartite graph $K_{2,t}$ and P_{2k} by identifying one endpoint of P_{2k} to be a vertex in $K_{2,t}$ lying in the bipartition of size 2. Let x be the other endpoint of the path, and let y be a vertex in $K_{2,t}$ lying in the bipartition of size t, of degree 2. An example is shown in Figure 2.2.

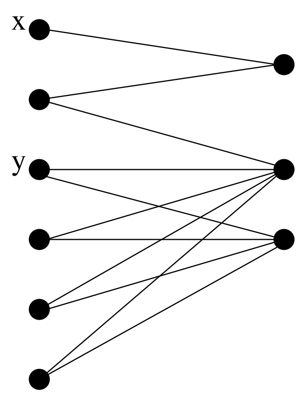

Figure 2.2: $G_{4,2}$

Let $Y = \mathscr{I}_y^3(G)$ and let $X = \mathscr{I}_x^3(G)$. We have, for $t \geq 2k$,

$$\begin{aligned}
Y - X &= \mathscr{I}^2(G \downarrow y) - \mathscr{I}^2(G \downarrow x) \\
&= \binom{t+2k-2}{2} - |E(G \downarrow y)| - \binom{t+2k-1}{2} + |E(G \downarrow x)| \\
&= \binom{t+2k-2}{2} - \binom{t+2k-1}{2} + 2t - 1 \\
&= (t+2k-2)(-1) + 2t - 1 \\
&= t - 2k + 1 \\
&> 0. \qquad (2.3)
\end{aligned}$$

We show that a similar construction acts as a counterexample for all $r > 3$. Given $r > 3$, consider the graph $G = G_{t,2}$, $t > r$. Let x and y be as defined before, with $d(x) = 1$ and $d(y) = 2$. Let $Y = \mathscr{I}_y^r(G)$ and $X = \mathscr{I}_x^r(G)$. We have $X = \binom{t+1}{r-1}$ and $Y = \binom{t+1}{r-1} + \binom{t-1}{r-2}$. It follows that, for $t > r$, $Y > X$.

If we consider trees, it can be seen that the tree in Figure 2.1 satisfies this property. It is also not hard to show that the path P_n satisfies this property, since for all $r \geq 1$, $\mathscr{I}_{v_1}^r(P_n) = \mathscr{I}_{v_n}^r(P_n) \geq \mathscr{I}_{v_i}^r(P_n)$ holds for each $2 \leq i \leq n-1$.

Another infinite family of trees that satisfy the property are the depth-two stars shown in Figure 2.3 below.

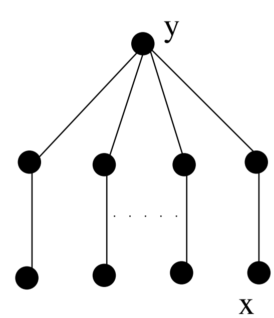

Figure 2.3: Tree T on $2n+1$ vertices which satisfies Conjecture 2.3.1.

Let $Y = \mathscr{I}_y^r(T)$ and let $X = \mathscr{I}_x^r(T)$. Then, we have $Y = \mathscr{I}^{r-1}(T \downarrow y) = \binom{n}{r-1}$ and $X = \binom{n-1}{r-2} + 2^{r-1}\binom{n-1}{r-1}$. It is then easy to note that when $r \geq 1$, $X - Y \geq 0$.

However, it turns out that not all trees satisfy this property. A counterexample, for $n = 10$ and $r = 5$, is shown in Figure 2.4. Observe that the vertex labeled 8, with

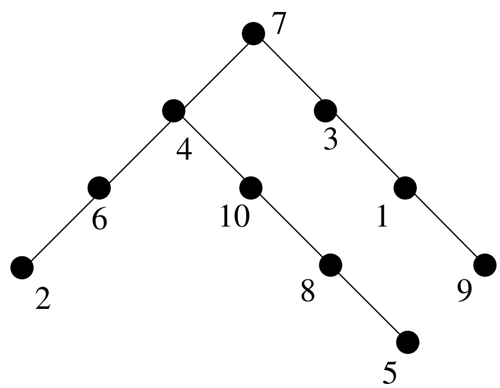

Figure 2.4: Tree T_1 which does not satisfy Property 2.3.1

degree 2, and the vertex labeled 4, with degree 3, lie in the same partite set, but we have $\mathscr{I}_4^5(T_1) = \{\{2,3,4,8,9\},\{2,3,4,5,9\}\}$ and $\mathscr{I}_8^5(T_1) = \{\{2,3,4,8,9\}\}$. Note that, in this example, $r = \frac{n}{2}$. Another counterexample, with $n = 12$ and $r = 5$, is shown in Figure 2.5.

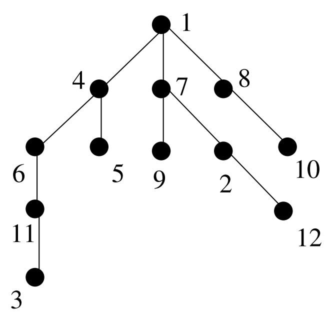

Figure 2.5: Tree T_2 which does not satisfy Property 2.3.1

We see that the vertices labeled 1 and 2, with degrees 3 and 2 respectively, lie in the same partite set. It can be checked that $|\mathscr{I}_1^5(T_2)| = 32$ and $|\mathscr{I}_2^5(T_2)| = 28$.

In this section, we give a proof of Theorem 2.1.11, which states that the ladder graph L_n is 3-EKR for all $n \geq 1$. First, we state and prove a claim about maximum star families in L_n.

Let $G = L_n$ be a ladder with n rungs. Let the rung edges be $x_i y_i (1 \leq i \leq n)$. First, we show that $\mathscr{I}_x^r(G)$ is a maximum sized star for $x \in \{x_1, y_1, x_n, y_n\}$.

Claim 2.3.2. *If G is a ladder with n rungs, $\mathscr{I}_x^r(G)$ is a maximum sized star for $x \in \{x_1, y_1, x_n, y_n\}$.*

Proof. We prove the claim for $x = x_n$. The claim is obvious if $n \leq 2$, so suppose $n \geq 3$. Let $\mathscr{A}$ be a star centered at some $x \in V(G)$. Without loss of generality, we assume that $x = x_k$ for some $1 < k < n$. We now construct an injection from $\mathscr{A}$ to $\mathscr{I}_{x_n}^r(G)$. Define functions f and g as follows.

$$f(x) = \begin{cases} x_{i \text{ mod n } +1} & \text{if } x = x_i, \text{ and} \\ y_{i \text{ mod n } +1} & \text{if } x = y_i. \end{cases}$$

$$g(x) = \begin{cases} y_i & \text{if } x = x_i, \text{ and} \\ x_i & \text{if } x = y_i. \end{cases}$$

Consider the function f^{n-k}. For every $A \in \mathscr{A}$, define $f^{n-k}(A) = \{f^{n-k}(x) : x \in A\}$ and similarly for g. We define a function $h : \mathscr{A} \to \mathscr{I}_{x_n}^r(G)$ as follows.

$$h(A) = \begin{cases} A & \text{if } \{x_1, x_n\} \subseteq A, \\ g(A) & \text{if } \{y_1, y_n\} \subseteq A, \text{ and} \\ f^{n-k}(A) & \text{otherwise.} \end{cases}$$

Clearly, $x_n \in h(A)$ for every $A \in \mathscr{A}$. We will show that h is an injection. Suppose $A, B \in \mathscr{A}$ and $A \neq B$. We show that $h(A) \neq h(B)$. If both A and B are in the same category(out of the three mentioned in the definition of h), then it is obvious. So, suppose not. If $\{x_1, x_n\} \subseteq A$ and $\{y_1, y_n\} \subseteq B$, then $x_k \in h(A)$, but $x_k \notin h(B)$. Then, let A be in either of the first two categories, and let B be in the third category. Then, $\{x_1, x_n\} \subseteq h(A)$, but $\{x_1, x_n\} \nsubseteq h(B)$. This holds because otherwise, we would have $\{x_k, x_{k+1}\} \subseteq B$, a contradiction. □

We give a proof of Theorem 2.1.11.

Proof. We do induction on the number of rungs. If $n = 1$, we have $G = P_2$, which is trivially r-EKR for $r = 1$ and vacuously true for $r = 2$ and $r = 3$. Similarly, for $n = 2$, $G = C_4$, so it is trivially r-EKR for each $1 \leq r \leq 2$ and vacuously true for $r = 3$. So, let $n \geq 3$. The case $r = 1$ is trivial. If $r = 2$, since $\delta(G) = 2$ and $|G| \geq 6$, we can use Theorem 2.0.14 to conclude that G is 2-EKR. So consider G such that $n \geq 3$ and $r = 3$. If $n = 3$, the maximum size of an intersecting family of independent sets of size 3 is 1, so 3-EKR again holds trivially. So, suppose $n \geq 4$. Let $G' = L_{n-1}$, $G'' = L_{n-2}$. Also, let $Z = \{x_{n-2}, y_{n-2}, x_{n-1}, y_{n-1}, x_n, y_n\}$. Define a function c as follows.

$$c(x) = \begin{cases} x_{n-1} & \text{if } x = x_n, \\ y_{n-1} & \text{if } x = y_n, \text{ and} \\ x & \text{otherwise.} \end{cases}$$

Let $\mathscr{A} \subseteq \mathscr{J}^r(G)$ be intersecting.

Define the following families.

$$\mathscr{B} = \{c(A) : A \in \mathscr{A} \text{ and } c(A) \in \mathscr{J}^r(G')\}$$

$$\mathscr{C}_1 = \{A \setminus \{x_n\} : x_n \in A \in \mathscr{A} \text{ and } A \setminus \{x_n\} \cup \{x_{n-1}\} \in \mathscr{A}\}$$

$$\mathscr{C}_2 = \{A \setminus \{y_n\} : y_n \in A \in \mathscr{A} \text{ and } A \setminus \{y_n\} \cup \{y_{n-1}\} \in \mathscr{A}\}$$

$$\mathscr{D}_1 = \{A \in \mathscr{A} : A \cap Z = \{x_{n-2}, x_n\}\}$$

$$\mathscr{D}_2 = \{A \in \mathscr{A} : A \cap Z = \{y_{n-2}, y_n\}\}$$

$$\mathscr{D}_3 = \{A \in \mathscr{A} : A \cap Z = \{x_{n-1}, y_n\}\}$$

$$\mathscr{D}_4 = \{A \in \mathscr{A} : A \cap Z = \{y_{n-1}, x_n\}\}$$

$$\mathscr{D}_5 = \{\{x_{n-2}, y_{n-1}, x_n\}\}$$

$$\mathscr{D}_6 = \{\{y_{n-2}, x_{n-1}, y_n\}\}$$

Define the families $\mathscr{E} = \mathscr{C}_1 \cup (\mathscr{D}_1 - \{x_n\})$ and $\mathscr{F} = \mathscr{C}_2 \cup (\mathscr{D}_2 - \{y_n\})$. Then both $\mathscr{E} \subseteq \mathscr{J}^{r-1}(G'')$ and $\mathscr{F} \subseteq \mathscr{J}^{r-1}(G'')$.

Proposition 2.3.3. *The family $\mathscr{E}(\mathscr{F})$ is a disjoint union of $\mathscr{C}_1$ and $\mathscr{D}_1 - \{x_n\}$($\mathscr{C}_2$ and $\mathscr{D}_2 - \{y_n\}$) and is intersecting.*

Proof. We prove the proposition for $\mathscr{E}$. The proof for $\mathscr{F}$ follows similarly. Each $D \in \mathscr{D}_1 - \{x_n\}$ contains x_{n-2}. However, no member in $\mathscr{C}_1$ contains x_{n-2}. Thus, $\mathscr{E}$ is a disjoint union. To show that it is intersecting, observe that $\mathscr{C}_1$ is intersecting since for any $C_1, C_2 \in \mathscr{C}_1$, $C_1 \cup \{x_{n-1}\}$ and $C_2 \cup \{x_n\}$ are intersecting. Also, $\mathscr{D}_1 - \{x_n\}$ is intersecting since each member of the family contains x_{n-2}. So, suppose $C \in \mathscr{C}_1$ and $D \in \mathscr{D}_1 - \{x_n\}$. Then, $C \cup \{x_{n-1}\}$ and $D \cup \{x_n\}$ are intersecting. □

Proposition 2.3.4. *If $G = L_n$, where $n \geq 4$, then we have*

$$|\mathscr{J}^3_{x_1}(G)| \geq |\mathscr{J}^3_{x_1}(G')| + 2|\mathscr{J}^2_{x_1}(G'')| + 2.$$

Proof. Each $A \in \mathscr{J}^3_{x_1}(G')$ is also a member of $\mathscr{J}^3_{x_1}(G)$, containing neither x_n nor y_n. Each $A \in \mathscr{J}^3_{x_1}(G'')$ contributes two members to $\mathscr{J}^3_{x_1}(G)$, $A \cup \{x_n\}$ and $A \cup \{y_n\}$. Also, $\{x_1, x_{n-1}, y_n\}, \{x_1, y_{n-1}, x_n\} \in \mathscr{J}^3_{x_1}(G)$. This completes the argument. □

We have

$$
\begin{aligned}
|\mathscr{A}| &= |\mathscr{B}| + \sum_{i=1}^{2} |\mathscr{C}_i| + \sum_{i=1}^{6} |\mathscr{D}_i| \\
&= |\mathscr{B}| + |\mathscr{E}| + |\mathscr{F}| + \sum_{i=3}^{6} |\mathscr{D}_i|. \qquad (2.4)
\end{aligned}
$$

We consider two cases.

- $\mathscr{D}_3 \neq \emptyset$ and $\mathscr{D}_4 \neq \emptyset$.

  In this case, we must have $\mathscr{D}_3 = \{\{a, x_{n-1}, y_n\}\}$ and $\mathscr{D}_4 = \{\{a, y_{n-1}, x_n\}\}$ for some $a \notin \{y_{n-2}, x_{n-2}\}$ and hence, $|\mathscr{D}_3| = |\mathscr{D}_4| = 1$. Also $\mathscr{D}_5 = \mathscr{D}_6 = \emptyset$. So, using Equation 2.4, Propositions 2.3.3 and 2.3.4 and the induction hypothesis, we have

$$
\begin{aligned}
|\mathscr{A}| &= |\mathscr{B}| + |\mathscr{E}| + |\mathscr{F}| + \sum_{i=3}^{6} |\mathscr{D}_i| \\
&\leq |\mathscr{J}_{x_1}^{r}(G')| + 2|\mathscr{J}_{x_1}^{r-1}(G'')| + 2 \\
&\leq |\mathscr{J}_{x_1}^{r}(G)|.
\end{aligned}
$$

- Without loss of generality, we suppose that $\mathscr{D}_4 = \emptyset$. If $\mathscr{D}_3 = \emptyset$, then $\sum_{i=3}^{6} |\mathscr{D}_i| \leq 1$, so we are done by Proposition 2.3.4. So, suppose $|\mathscr{D}_4| > 0$. We again consider two cases.

  1. Suppose $\mathscr{C}_1 = \emptyset$ and $\mathscr{D}_1 = \emptyset$.

     We note that at most one out of $\mathscr{D}_5$ and $\mathscr{D}_6$ can be nonempty. We also note that $|\mathscr{D}_3| \leq 2(n-3)$ and $\mathscr{J}_{x_1}^{2}(G'') = 2(n-3) - 1$. So, using Proposition 2.3.4

$$
\begin{aligned}
|\mathscr{A}| &= |\mathscr{B}| + |\mathscr{F}| + |\mathscr{D}_3| + 1 \\
&\leq |\mathscr{J}_{x_1}^{r}(G')| + |\mathscr{J}_{x_1}^{r-1}(G'')| + 2(n-3) + 1 \\
&\leq |\mathscr{J}_{x_1}^{r}(G)|.
\end{aligned}
$$

2. Suppose that either $\mathscr{C}_1 \neq \emptyset$ or $\mathscr{D}_1 \neq \emptyset$. Let $C = \{a,b\} \in \mathscr{C}_1$ and $D \in \mathscr{D}_3$. We have $C \cup \{x_n\} \cap D \neq \emptyset$. So, we have $D \setminus \{y_n, x_{n-1}\} = \{a\}$ or $D \setminus \{y_n, x_{n-1}\} = \{b\}$. So, $|\mathscr{D}_3| \leq 2$. If $|\mathscr{D}_3| = 2$, then $y_{n-2} \notin \{a,b\}$, so $\mathscr{D}_6 = \emptyset$. Also, $\mathscr{D}_5 = \emptyset$ since $\mathscr{D}_3$ is nonempty. If $|\mathscr{D}_3| \leq 1$, then $|\mathscr{D}_6| \leq 1$. Thus, in either case, $\sum_{i=3}^{6} |\mathscr{D}_i| \leq 2$. Thus, using Equation 2.4 and Proposition 2.3.4, we are done. A similar argument works if $\mathscr{D}_1$ is nonempty.

$\square$

Chapter 3

k-WISE INTERSECTION THEOREMS

A family $\mathscr{F} \subseteq \binom{[n]}{r}$ is called *intersecting* if for any $A, B \in \mathscr{F}$, $A \cap B \neq \emptyset$. Similarly, call $\mathscr{F} \subseteq \binom{[n]}{r}$ k-wise intersecting if for any $F_1, \ldots, F_k \in \mathscr{F}$, $\bigcap_{i=1}^{k} F_i \neq \emptyset$. Frankl [28] proved the following theorem for k-wise intersecting families.

Theorem 1.4.1. *Let $\mathscr{F} \subseteq \binom{[n]}{r}$ be k-wise intersecting. If $r \leq \dfrac{(k-1)n}{k}$, then $|\mathscr{F}| \leq \binom{n-1}{r-1}$.*

We generalize Frankl's theorem in two directions. First, we formulate and prove a stability version, which shows that every k-wise intersecting family contains an arbitrarily large star, provided that its size is sufficiently close to the extremal number $\binom{n-1}{r-1}$. Next, we will formulate a graph-theoretic generalization of Frankl's theorem and prove an analog of the theorem for k-wise intersecting families of vertex sets of a perfect matching graph which are either independent or contain a maximum-sized independent set.

3.1 Structure and Stability of k-wise Intersecting Families

The classical extremal problem is to determine the maximum size and structure of a family on a given ground set of size n which avoids a given forbidden configuration $\mathscr{F}$. For example, the Erdős-Ko-Rado theorem finds the maximum size of a set system on the set $[n]$, which does not have a pair of disjoint subsets. Often only a few trivial structures attain this extremal number. In case of the EKR theorem, the only extremal structure when $r < \frac{n}{2}$ is that of a star in $\binom{[n]}{r}$. A natural further step is to ask whether non-extremal families which have size close to the extremal number also have structure similar to any of the extremal structures. This approach was first pioneered by Simonovits [56] to answer a question in extremal graph theory and a similar notion for set systems was recently formulated by Mubayi [50]. We will adopt the definition of stability from Mubayi [50] to formulate the notions of *weak stability* and *strong stability* for the properties of *intersection* and *k-wise intersection* for set systems. We state the definitions for the intersection property below.

Definition 3.1.1 (Weak Stability)**.** *Let $r \geq 2$. Then for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon, r)$ and $\delta > 0$ such that the following holds for all $n > n_0$: if $\mathscr{F} \subseteq \binom{[n]}{r}$ is intersecting and $|\mathscr{F}| > (1-\delta)\binom{n-1}{r-1}$ then there exists a $v \in [n]$ such that $|\mathscr{F}_v| > (1-\varepsilon)\binom{n-1}{r-1}$.*

Definition 3.1.2 (Strong Stability)**.** *Let $1 < r < n/2$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathscr{F} \subseteq \binom{[n]}{r}$ has the intersecting property and $|\mathscr{F}| \geq (1-\delta)\binom{n-1}{r-1}$, then there exists a $v \in [n]$ such that $|\mathscr{F}_v| \geq (1-\varepsilon)\binom{n-1}{r-1}$.*

Typically, we ask questions of this nature for *monotone* properties of set systems. Informally, a property $\mathscr{P}$ is said to be monotone if for any $\mathscr{A}$, if $\mathscr{A}$ has property $\mathscr{P}$ and $\mathscr{B} \subseteq \mathscr{A}$, then $\mathscr{B}$ has property $\mathscr{P}$ as well. Weak stability is true for a monotone property of set systems if it can be proved for set systems where the uniformity is a constant and the number of vertices is comparatively much larger. Strong stability on the other hand holds for a property if it can be proved when the uniformity of a set system/hypergraph is comparable to the size of ground set, and results of this type are harder to prove.

For the k-wise intersection property, the definition of weak stability remains unchanged. In fact, it is sufficient to prove weak stability for intersecting families, as k-wise intersecting implies intersecting. For strong stability, the range of values r can take is bigger for k-wise intersection, in particular we have $1 < r < (k-1)n/k$. It turns out that proving weak stability for intersecting families (and as a result,

k-intersecting families) is fairly easy. One possible approach involves invoking the Hilton-Milner theorem, but we present an even simpler proof below.

Theorem 3.1.3. *Let $r \geq 2$. Then there exists a $n_0 = n_0(r)$ and $\delta > 0$ such that the following holds for all $n > n_0$: if $\mathscr{F} \subseteq \binom{[n]}{r}$ is intersecting and $|\mathscr{F}| > (1-\delta)\binom{n-1}{r-1}$ then $\mathscr{F}$ is a star.*

Proof. Let $r \geq 2$. Choose n_0 and $\delta > 0$ such that $\frac{1}{r^3} >> \frac{1}{n_0} >> \delta$. Now suppose $\mathscr{F}$ is a non-star intersecting family. We will show that $|\mathscr{F}| \leq (1-\delta)\binom{n-1}{r-1}$. Since there is no element which lies in all sets of $\mathscr{F}$, there exist three sets $A, B, C \in \mathscr{F}$ which are pairwise intersecting but with $A \cap B \cap C = \emptyset$. This gives $|A \cup B \cup C| \leq 3r-3$. Now any other set in $\mathscr{F}$ intersects $A \cup B \cup C$ in at least 2 elements, since there is no element which lies in all 3 sets. This gives us the following upper bound on the size of $\mathscr{F}$: $|\mathscr{F}| \leq 3 + \binom{3r-3}{2}\binom{n-2}{r-2}$. It can be verified that since $\frac{1}{r^3} >> \frac{1}{n_0} >> \delta$, we get $3 + \binom{3r-3}{2}\binom{n-2}{r-2} \leq (1-\delta)\binom{n-1}{r-1}$, which completes the proof of the theorem. □

Strong stability results for the Erdős-Ko-Rado theorem are considerably harder to prove, but results of this nature for intersecting families have recently appeared, due to Dinur-Friedgut [17], Keevash [40], Keevash-Mubayi [41] and others. We will prove the following strong stability result for k-wise intersecting families.

Theorem 1.5.2 *For some $k \geq 2$, let $1 \leq r < \frac{(k-1)n}{k}$, and let $\mathscr{F} \subseteq \binom{[n]}{r}$ be a k-wise intersecting family. Then for any $0 \leq \varepsilon < 1$, there exists a $0 \leq \delta < 1$ such that if $|\mathscr{F}| \geq (1-\delta)\binom{n-1}{r-1}$, then there is an element $v \in [n]$ such that $|\mathscr{F}(v)| \geq (1-\varepsilon)\binom{n-1}{r-1}$.*

We note that for $k \geq 2$, $\mathscr{F}$ is k-wise intersecting implies that it is intersecting. Hence if $r < n/2$, the results obtained in the papers mentioned above suffice as stability results for Theorem 1.4.1. Consequently, the main interest of our theorem is in the structural information about large k-wise intersecting families that it provides when r is much closer to n, more specifically when $n/2 \leq r < (k-1)n/k$.

Proof of Stability

Suppose $\mathscr{F} \subseteq \binom{[n]}{r}$ is a k-wise intersecting family, with $r < \frac{(k-1)n}{k}$. For any $0 \leq \varepsilon < 1$, let $\delta = \frac{\varepsilon}{2rn(n^3+1)}$ and suppose $|\mathscr{F}| \geq (1-\delta)\binom{n-1}{r-1}$. We will show that $\mathscr{F}$ contains a large star.

Katona-type Lemmas for k-wise Intersecting Families

In this section, we will prove some Katona-type lemmas which we will employ later in the proof of the main theorem. We introduce some notation first. Consider a permutation $\sigma \in S_n$ as a sequence $(\sigma(1), \ldots, \sigma(n))$. We say that two permutations μ and π are *equivalent* if there is some $i \in [n]$ such that $\pi(x) = \mu(x+i)$ for all $x \in [n]$. Note that addition is carried out modulo n; more precisely, $x+i$ is either $x+i$ or $x+i-n$, depending on which lies in $[n]$. Let P_n be the set of equivalence classes, called *cyclic* orders on $[n]$. For a cyclic order σ and some $x \in [n]$, call the set $\{\sigma(x), \ldots, \sigma(x+r-1)\}$ a σ-interval of length r *starting* at x, *ending* in $x+r-1$, and *containing* the points $(x, x+1, \ldots, x+r-1)$ (addition again mod n). Denote this interval by $I_{\sigma,r}(x)$. The following lemma is due to Frankl [26]. We include the short proof below as we will build on these ideas in the proofs of the other lemmas.

Lemma 3.1.4 (Frankl). *Let $\sigma \in P_n$ be a cyclic order on $[n]$, and $\mathscr{F}$ be a k-wise intersecting family of σ-intervals of length $r \leq (k-1)n/k$. Then, $|\mathscr{F}| \leq r$.*

Proof. Let $\mathscr{F}^c = \{[n] \setminus F : F \in \mathscr{F}\}$. Let $|\mathscr{F}| = |\mathscr{F}^c| = m$. We will prove that $m \leq r$. Since $r \leq (k-1)n/k$, we have $n \leq k(n-r)$. Suppose $G_1, \ldots, G_k \in \mathscr{F}^c$. Clearly $\cup_{i=1}^k G_i \neq [n]$; otherwise $\cap_{i=1}^k ([n] \setminus G_i) = \emptyset$, which is a contradiction. Let $G \in \mathscr{F}^c$. Without loss of generality, suppose G ends in n. We now assign indices from $[1,k(n-r)]$ to sets in $\mathscr{F}^c$. For every set $G' \in \mathscr{F}^c \setminus \{G\}$, assign the index x to G' if G' ends in x. Assign all indices in $[n,k(n-r)]$ for G. Consider the set of indices $[k(n-r)]$ and partition them into equivalence classes mod $n-r$. Suppose there is an equivalence class such that all k indices in that class are assigned. Let $\{H_i\}_{i\in[k]}$ be the k sets in $\mathscr{F}^c$ which end at the k indices in the equivalence class. It is easy to note that $\cup_{i=1}^{k} H_i = [n]$, which is a contradiction. So for every equivalence class, there exists an index which has not been assigned to any set in $\mathscr{F}^c$. This implies that there are at least $n-r$ indices in $[k(n-r)]$ which are unassigned. Each set in $\mathscr{F}^c \setminus \{G\}$ has one index assigned to it, and G has $k(n-r)-n+1$ indices assigned to it. This gives us $m-1+k(n-r)-n+1+n-r \leq k(n-r)$, which simplifies to $m \leq r$, completing the proof.

◇

We will now characterize the case when $|\mathscr{F}| = r$, in the following lemma.

Lemma 3.1.5. *Let $\sigma \in P_n$ be a cyclic order on $[n]$, and let $\mathscr{F}$ be a k-wise intersecting family of intervals of length $r < (k-1)n/k$. If $|\mathscr{F}| = r$, then $\mathscr{F}$ consists of all intervals which contain a point x.*

Proof. As in the proof of Lemma 3.1.4, we consider $\mathscr{F}^c$ and assume (without loss of generality) that there exists $F \in \mathscr{F}^c$ which ends in n. It is clear from the proof of Lemma 3.1.4 that if $|\mathscr{F}| = r$, there are exactly $n-r$ indices in $[k(n-r)]$, one from each equivalence class, which are not assigned to any set in $\mathscr{F}^c$. Since F ends in n, all indices in $[n,k(n-r)]$ (and there will be at least 2) will be assigned. It will be sufficient to show that the set of unassigned indices is $I_{\sigma,n-r}(x)$ for some $x \in [r]$, as this would imply that every set in $\mathscr{F}$ contains x.

Let x be the smallest unassigned index in $[n-1]$. Clearly $x \leq r$. Let $x \equiv j \bmod n-r$. We will show that $x+i$ is unassigned for each $0 \leq i \leq n-r-1$.

We argue by induction on i, with the base case being $i = 0$. Let $y = x + i$ for some $1 \leq i \leq n - r - 1$. Suppose y is assigned and Y is the set in $\mathscr{F}^c$ that ends in y. We know by the induction hypothesis that $y - 1$ is unassigned, so every other index in the same equivalence class as $y - 1$ is assigned. Call this equivalence class E_{y-1}. Consider all indices in E_{y-1} which lie in $(y - 1, n]$ and let I_1 be the set of these indices, all assigned. Similarly, consider all indices in E_{y-1} which lie in $[1, y - 1)$ and call this set I_2. Let $I_2' = \{j + 1 : j \in I_2\}$. I_2' contains indices in the same equivalence class as y, and are assigned (as they are all less than x and x is the smallest unassigned index). Let $J = I_1 \cup I_2'$, and since J contains only assigned indices, let $\mathscr{H}$ be the subfamily of $\mathscr{F}^c$ to which indices in J are assigned. Let p be the largest index in I_1 and let q be the smallest index in I_2'. Since $n < k(n - r)$, the set which ends in q contains $p + 1$. The family $\mathscr{H} \cup \{Y\}$ has at most k sets, and the union of all sets in this family is $[n]$. This is a contradiction. Thus y is unassigned.

$\diamond$

Now let $\mathscr{F} \subseteq \binom{[n]}{r}$ be a k-wise intersecting family for some $r < \dfrac{(k-1)n}{k}$. For each cyclic order $\sigma \in P_n$, let $\mathscr{F}_\sigma$ be the subfamily of sets in $\mathscr{F}$ that are intervals in σ. We say that σ is *saturated* if $|\mathscr{F}_\sigma| = r$; otherwise call it unsaturated. By Lemma 3.1.5, if σ is saturated, all sets in $\mathscr{F}_\sigma$ contain a common point, say v, so call σ v-saturated to identify the common point.

For $i \leq n$, define an *adjacent transposition* A_i on a cyclic order σ as an operation that swaps the elements in positions i and $i + 1$ ($i + 1 = 1$ if $i = n$) of σ. We are now ready to prove our next lemma.

Lemma 3.1.6. *For a k-wise intersecting family $\mathscr{F} \subseteq \binom{[n]}{r}$ with $r < \dfrac{(k-1)n}{k}$, let $\sigma \in P_n$ be a v-saturated cyclic order. Let μ be the cyclic order obtained from σ by* *an adjacent transposition A_i, $i \in [n] \setminus \{v, v-1\}$ ($v-1 = n$ if $v = 1$). If μ is saturated, then it is v-saturated.*

Proof. Without loss of generality (relabeling if necessary), assume σ is n-saturated, so $1 \leq i \leq n-2$. Let μ be saturated. As before, we consider the family of complements $\mathscr{F}^c$ and observe that the interval $I_{\sigma,n-r}(n)$ contains all the $n-r$ unassigned indices.

- Suppose $i \in (n-r-1, n-1)$. Let $A = I_{\mu,n-r}(i+1)$ and let A end in index $j = i+n-r$. Clearly, $j \neq n-r-1$. Suppose first that $j \in (n-r-1, n)$. Then all indices in the interval $I_{\mu,n-r}(n)$ are still unassigned, so μ is n-saturated. Next we argue that if $j \in [1, n-r-1)$, j cannot be an assigned index. This is because all the indices in the set $\{n\} \cup [1, j) \cup (j, n-r-1]$ are unassigned in μ, and by Lemma 3.1.5, all the unassigned indices in a saturated order occur in an interval of length $n-r$. So assume $j = n$ and suppose j is assigned. By Lemma 3.1.5, the index $n-r$ will be unassigned, which is only possible if $i = n-r$ (otherwise $I_{\mu,n-r}(1) = I_{\sigma,n-r}(1)$). This implies that $n = 2(n-r)$ and hence $k \geq 3$. Now consider the following intervals, all of which are sets in $\mathscr{F}^c$: $I_{\sigma,n-r}(1)$, $I_{\sigma,n-r}(n-r)$ and $I_{\mu,n-r}(n-r+1)$. The union of these three sets is $[n]$, a contradiction.

- Suppose $i = n-r-1$. As before, the only possibilities to consider are when either n or $n-r-1$ are assigned indices in μ. Suppose n is assigned in μ. This means that $i+1 = n-r$ is unassigned in μ, by Lemma 3.1.5. However this is not possible since $I_{\mu,n-r}(1) = I_{\sigma,n-r}(1)$. So suppose $n-r-1$ is assigned in μ. By Lemma 3.1.5, $n-1$ is unassigned in μ. This is only possible if the interval ending in $n-1$ starts at $i+1$. This means $n = 2(n-r)$ and an argument identical to Case 1 suffices.

- Suppose $i \in [1, n-r-1)$. Now $I_{\sigma,n-r}(n) = I_{\mu,n-r}(n)$, so $n-r-1$ is unassigned in μ. Hence assume n is assigned in μ. Then the interval $I_{\mu,n-r}(i+1)$ ends in n and is a set in $\mathscr{F}^c$. Clearly, the union of this set with $I_{\sigma,n-r}(1)$, which is also a set in $\mathscr{F}^c$, is $[n]$, a contradiction.

$\diamond$

Cayley Graphs

In this small section, we gather some facts about expansion properties of a specific Cayley graph of the symmetric group. We will consider the Cayley graph G on S_{n-1} generated by the set of adjacent transpositions $A = \{(12), \ldots, (n-2\; n-1)\}$. In particular, the vertex set of G is S_{n-1} and two permutations σ and μ are adjacent if $\mu = \sigma \circ a$, for some $a \in A$. We note that the transposition operates by exchanging adjacent positions (as opposed to consecutive values). G is an $n-2$-regular graph. It was shown by Keevash [40], using a result of Bacher [3], that G is an α-expander for some $\alpha > \frac{1}{n^3}$, i.e. for any $H \subseteq V(G)$ with $|H| \leq \frac{|V(G)|}{2}$, we have $N(H) \geq \alpha|H| > \frac{|H|}{n^3}$, where $N(H)$ is the set of all vertices in $V(G) \setminus H$ which are adjacent to some vertex in H.

Proof of Main Theorem

Proof of Theorem 1.5.2. We will finish the proof of Theorem 1.5.2 in this section. We can identify every cyclic order in P_n with a permutation $\sigma \in S_n$ having $\sigma(n) = n$. Restricting σ to $[n-1]$ gives a bijection between P_n and S_{n-1}. Let U be the set of unsaturated cyclic orders in P_n. We have

$$\begin{aligned} r!(n-r)!|\mathscr{F}| &= \sum_{\sigma \in P_n} |\mathscr{F}_\sigma| \\ &\leq \sum_{\sigma \in P_n} r - |U| \\ &= r(n-1)! - |U|. \end{aligned}$$

This gives us $|U| \le r(n-1)! - r!(n-r)!(1-\delta)\binom{n-1}{r-1} = r\delta(n-1)!$, implying that there are at least $(1-r\delta)(n-1)!$ saturated orders in P_n.

We now consider the Cayley graph G defined above, with the vertex set being P_n and the generating set being the set of adjacent transpositions $A = \{(12), \ldots, (n-2\ n-1)\}$. Suppose S is a subset of saturated cyclic orders. We can use the expansion property of G to conclude that if $n^3 r\delta \le \frac{|S|}{(n-1)!} \le \frac{1}{2}$, we get $N(S) > |S|/n^3 \ge r\delta(n-1)!$. This means that there is a saturated cyclic order in $N(S)$. We will use this observation to show that the subgraph of G induced by the set of all saturated cyclic orders, say H, has a large component. Consider the set of all components in H. Now a component in H can be either *small*, i.e. have size at most $n^3 r\delta(n-1)!$ or be *large*, i.e. have size bigger than $(n-1)!/2$. Clearly there can be at most one large component. We argue that the total size of all small components is at most $n^3 r\delta(n-1)!$. Suppose not. Let S' be the union of (at least 2) small components such that $n^3 r\delta(n-1)! \le |S'| \le 2n^3 r\delta(n-1)! \le (n-1)!/2$. Now using the above observation, $N_H(S')$ is non-empty, a contradiction. Thus there is a large component of size at least $(1-n^3 r\delta)(n-1)!$. Call this component H'. Suppose σ is a v-saturated cyclic order in H'. By Lemma 3.1.6, every cyclic order in H' is v-saturated. Thus, $r!(n-r)!|\mathscr{F}(v)| \ge \sum_{\sigma \in H'} |\mathscr{F}_\sigma| \ge r(1 - r\delta - n^3 r\delta)(n-1)!$, which gives $|\mathscr{F}(v)| \ge (1-\frac{\varepsilon}{2n})\binom{n-1}{r-1}$, since $\delta = \frac{\varepsilon}{2rn(n^3+1)}$. □

Remark: The proof of Theorem 1.5.2 also contains a proof of the structural uniqueness of the extremal configurations for Theorem 1.4.1 when $r < (k-1)n/k$. This can be easily observed by putting $\varepsilon = 0$ in the statement of the theorem, or by just using Lemmas 3.1.4, 3.1.5 and 3.1.6. We note that the original proof by Frankl in [28] did not include this structural information. However in [26], Frankl gave another proof of Theorem 1.4.1 using the Kruskal-Katona theorem, which includes the characterization of the extremal structures for $r \leq (k-1)n/k$ when $k \geq 3$ and $r < (k-1)n/k$ when $k = 2$. An alternate proof of this characterization is also given by Mubayi and Verstraete [52].

3.2 k-wise Intersecting Vertex Families in Graphs

Next, we consider a graph-theoretic generalization of Theorem 1.4.1. For a graph G (with vertex set and edge set denoted by $V(G)$ and $E(G)$ respectively) and $r \geq 1$, let $\mathscr{J}^r(G)$ denote the set of all independent vertex sets of size r. Let $\mathscr{M}^r(G)$ denote the family of all vertex sets of size r containing a maximum independent set and let $\mathscr{H}^r(G) = \mathscr{J}^r(G) \cup \mathscr{M}^r(G)$. For a vertex $x \in V(G)$, let $\mathscr{H}^r_x(G) = \{A \in \mathscr{H}^r(G) : x \in A\}$. Define $\mathscr{J}^r_x(G)$ and $\mathscr{M}^r_x(G)$ in a similar manner. Henceforth we will consider the perfect matching graph on $2n$ vertices (and n edges), and denote it by M_n. Note that $|\mathscr{H}^r_x(M_n)| = 2^{r-1}\binom{n-1}{r-1}$ when $r \leq n$ and $|\mathscr{H}^r_x(M_n)| = 2^{2n-r}\binom{n-1}{r-n-1} + 2^{2n-r-1}\binom{n-1}{r-n}$, when $r > n$. We will consider k-wise intersecting families in $\mathscr{H}^r(M_n)$, and prove the following analog of Frankl's theorem.

Theorem 1.4.3 *For $k \geq 2$, let $r \leq \dfrac{(k-1)(2n)}{k}$, and let $\mathscr{F} \subseteq \mathscr{H}^r(M_n)$ be k-wise intersecting. Then,*

$$|\mathscr{F}| \leq \begin{cases} 2^{r-1}\binom{n-1}{r-1} & \text{if } r \leq n, \text{ and} \\ 2^{2n-r}\binom{n-1}{r-n-1} + 2^{2n-r-1}\binom{n-1}{r-n} & \text{otherwise.} \end{cases}$$

If $r < \dfrac{(k-1)(2n)}{k}$, then equality holds if and only if $\mathscr{F} = \mathscr{H}^r_x(M_n)$ for some $x \in V(M_n)$.

It is not hard to observe that the $k = 2$ case of Theorem 1.4.3 is Theorem 1.4.4 of Bollobás and Leader [6].

Theorem 1.4.4 *Let $1 \leq r \leq n$, and let $\mathscr{F} \subseteq \mathscr{J}^r(M_n)$ be an intersecting family.*

Then, $|\mathscr{F}| \leq 2^{r-1}\binom{n-1}{r-1}$*. If* $r < n$*, equality holds if and only if* $\mathscr{F} = \mathscr{J}_x^r(M_n)$ *for some* $x \in V(M_n)$*.*

Note that if $r < n$, then $\mathscr{H}^r(M_n) = \mathscr{J}^r(M_n)$ and $\mathscr{M}^r(M_n) = \emptyset$. Similarly if $r > n$, $\mathscr{H}^r(M_n) = \mathscr{M}^r(M_n)$ and $\mathscr{J}^r(M_n) = \emptyset$. In the case $r = n$, we have $\mathscr{H}^r(M_n) = \mathscr{J}^r(M_n) = \mathscr{M}^r(M_n)$. We also observe that the main interest of our theorem is in the case $r > n$ for the bound and $r \geq n$ for the characterization of the extremal structures. This is because of the previously fact that if a family $\mathscr{F}$ is k-wise intersecting ($k \geq 2$), it is also intersecting.

A k-wise Intersection Theorem for Perfect Matchings

Let $V(M_n) = \{1,2,\ldots,2n\}$, and let $E(M_n) = \{\{1,n+1\},\{2,n+2\},\ldots,\{n,2n\}\}$. Call two vertices which share an edge as *partners*. We consider cyclic orderings of the set $V(G)$, i.e. a bijection between $V(G)$ and $[2n]$ with certain properties. In particular, call a cyclic ordering of $V(G)$ *good* if all partners are exactly n apart in the cyclic order. More formally, if c is a bijection from $V(G)$ to $[2n]$, c is a good cyclic ordering if for any $i \in [n]$, $c(i+n) = c(i)+n$ (modulo $2n$, so $c(i+n) = c(i)-n$ if $c(i) > n$). It is fairly simple to note that the total number of good cyclic orderings, regarding cyclically equivalent orderings as identical, is $2^{n-1}(n-1)!$. Every interval in a good cyclic ordering will be either an independent set in M_n (if $r \leq n$) or contain a maximum independent set (if $r > n$). Now let $\mathscr{F} \subseteq \mathscr{H}^r(G)$ be k-wise intersecting for $r \leq \frac{(k-1)(2n)}{k}$. Using an argument identical to the proof of Lemma 3.1.4, we can conclude that for any good cyclic ordering c, there can be at most r sets in $\mathscr{F}$ that are intervals in c. For a given set $F \in \mathscr{F}$, in how many good cyclic orderings is it an interval? The answer depends on the value of r. Suppose $r \leq n$. In this case, F is an interval in $r!(n-r)!2^{n-r}$ good cyclic orderings. Thus we have $|\mathscr{F}|r!(n-r)!2^{n-r} \leq r(n-1)!2^{n-1}$, giving $|\mathscr{F}| \leq \binom{n-1}{r-1}2^{r-1}$. Note that this bound also follows directly from Theorem 1.4.4, since $r \leq n$ implies that $\mathscr{H}^r(M_n) = \mathscr{J}^r(G)$. Now suppose $r > n$. Then $\mathscr{J}^r(G) = \emptyset$ and $\mathscr{H}^r(G) = \mathscr{M}^r(G)$. We can think of each set in $\mathscr{F}$ as containing both vertices from $r-n$ edges, and exactly 1 vertex each from the remaining $2n-r$ edges. Hence the number of good cyclic orders in which a set $F \in \mathscr{F}$ is contained is $(2n-r)!(r-n)!2^{r-n}$. This gives us the following inequality.

$$
\begin{aligned}
|\mathscr{F}| &\leq \frac{r(n-1)!2^{n-1}}{(2n-r)!(r-n)!2^{r-n}} \\
&= \frac{n(n-1)!2^{2n-r-1}}{(2n-r)!(r-n)!} + \frac{(r-n)(n-1)!2^{2n-r-1}}{(2n-r)!(r-n)!} \\
&= \binom{n}{r-n}2^{2n-r-1} + \binom{n-1}{r-n-1}2^{2n-r-1} \\
&= \binom{n-1}{r-n-1}2^{2n-r-1} + \binom{n-1}{r-n}2^{2n-r-1} + \binom{n-1}{r-n-1}2^{2n-r-1} \\
&= 2^{2n-r}\binom{n-1}{r-n-1} + 2^{2n-r-1}\binom{n-1}{r-n}.
\end{aligned}
$$

This completes the proof of the bound. We will now prove that the extremal families are essentially unique. Suppose that $r < \frac{(k-1)(2n)}{k}$ and $|\mathscr{F}| = 2^{2n-r}\binom{n-1}{r-n-1} + 2^{2n-r-1}\binom{n-1}{r-n}$. Then for each good cyclic ordering c, there are exactly r sets from $\mathscr{F}$ that are intervals in c. Using Lemma 3.1.5, we can conclude that each good cyclic ordering is saturated. To simplify the argument, and because Theorem 1.4.4 suffices when $r < n$, we henceforth assume $r \geq n$ so $k \geq 3$ and $2n - r \leq n$.

Consider the good cyclic ordering π defined by $\pi(i) = i$ for $1 \leq i \leq 2n$ and assume without loss of generality that it is $2n$-saturated. Since the number of good cyclic orderings are $2^{n-1}(n-1)!$, we will identify all good cyclic orderings with bijections σ from $[2n]$ to itself that satisfy $\sigma(n) = n$ and $\sigma(2n) = 2n$.

For each permutation $p \in S_{n-1}$, define the following good cyclic ordering σ on $[2n]$: for $1 \leq i \leq n-1$, let $\sigma(i) = p(i)$ and for $n+1 \leq i \leq 2n-1$, let $\sigma(i) = p(i-n)+n$. Also let $\sigma(i) = i$ if $i \in \{n, 2n\}$. Denote the set of good cyclic orders obtained from permutations in S_{n-1} in this manner by C_{n-1}. Now for $1 \le i \le n-2$, define an analogous adjacent transposition T_i for any good cyclic ordering σ as an operation that swaps the elements in positions i and $i+1$ and also the elements in positions $i+n$ and $i+n+1$ of σ, so the resulting cyclic ordering, say μ, is also a good cyclic ordering. Note also that if $\sigma \in C_{n-1}$, then $\mu \in C_{n-1}$. We now prove a lemma that is similar to Lemma 3.1.6. The proof will be very similar to that of Lemma 3.1.6, so we will omit many of the details. As before, for $x, l \in [2n]$, let $I_{\sigma,l}(x)$ be the interval of length l in the good cyclic ordering σ that begins in x, ends in $x+l-1$ and contains the elements $\sigma(x), \ldots, \sigma(x+l-1)$.

Lemma 3.2.1. *For a k-wise intersecting family $\mathscr{F} \subseteq \mathscr{H}^r(M_n)$, with $r < \dfrac{(k-1)(2n)}{k}$, let σ be a $2n$-saturated good cyclic ordering. Let μ be the good cyclic order obtained from σ by an adjacent transposition T_i, $i \in [n-2]$. If μ is saturated, then it is $2n$-saturated.*

Proof. As in Lemmas 3.1.4, 3.1.5 and 3.1.6, we again consider the family of compliments of sets in $\mathscr{F}$, denoted by $\mathscr{F}^c$, that are intervals in σ. By Lemma 3.1.5, we know that $I_{\sigma,2n-r}(2n)$ (which ends in $2n-r-1$) contains all of the $2n-r$ unassigned indices. Now let T_i be an adjacent transposition for $1 \le i \le n-2$. Recall that T_i swaps elements in position i and $i+1$, and also the elements in positions $i+n$ and $i+n+1$. Suppose μ, obtained from σ by T_i is saturated, but not $2n$-saturated. We consider the following cases.

- Suppose $i = 2n-r-1$. In this case, $2n$ cannot be an assigned index in μ since that would mean $2n-r$ is unassigned in μ. This would be a contradiction because $I_{\sigma,2n-r}(1) = I_{\mu,2n-r}(1)$. So suppose $2n-r-1$ is assigned, implying that $2n-1$ is unassigned in μ. This means that the interval $I_{\sigma,2n-r}(3n-r)$ ends in $2n-1$, giving $3n = 2r$ (and hence, $2n-r = n/2$). This yields $k \ge$

5. Now consider the following sets: $I_{\mu,n/2}(2n)$, $I_{\sigma,n/2}(1)$, $I_{\sigma,n/2}(n/2+1)$, $I_{\sigma,n/2}(n+1)$ and $I_{\sigma,n/2}(3n/2)$. All of these are sets in $\mathscr{F}^c$ and their union is $[2n]$, a contradiction.

- Suppose $i \in [1, 2n-r-1)$. In this case we have $I_{\sigma,2n-r}(2n) = I_{\mu,2n-r}(2n)$ and $I_{\sigma,2n-r}(1) = I_{\mu,2n-r}(1)$, so $2n-r-1$ is an unassigned index in μ and $2n-r$ is assigned. This implies by Lemma 3.1.5 that the interval of unassigned indices remains unchanged in μ, as required.

- Suppose $i \in (2n-r-1, n-1)$. Here $I_{\sigma,2n-r}(2n) = I_{\mu,2n-r}(2n)$, so suppose $2n$ is assigned in μ. This means that $2n-r$ is unassigned in μ, implying $i = 2n-r$. Since $2n$ is assigned in μ, we have $(i+n+1)+(2n-r-1) = 2n$, which yields $i = r-n$. Hence $3n = 2r$ and $k \geq 5$. Now consider the following 5 intervals, all sets in $\mathscr{F}^c$: $I_{\mu,n/2}(3n/2+1)$, $I_{\sigma,n/2}(1)$, $I_{\sigma,n/2}(n/2+1)$, $I_{\sigma,n/2}(n+1)$ and $I_{\sigma,n/2}(3n/2)$. The union of the 5 sets is $[2n]$, a contradiction.

$\diamond$

Now for $1 \leq i \leq n$, define a *swap* operation W_i on a good cyclic ordering σ as an operation that exchanges the elements in positions i and $n+i$ of σ, so the resulting cyclic order is also good. We will now prove the following lemma about the swap operation.

Lemma 3.2.2. *For a k-wise intersecting family $\mathscr{F} \subseteq \mathscr{H}^r(M_n)$ with $n < r < \dfrac{(k-1)(2n)}{k}$, let σ be a $2n$-saturated good cyclic ordering. Let μ be the good cyclic order obtained from σ by the swap W_{n-1}. If μ is saturated, then it is $2n$-saturated.*

Proof. We first observe that $n < r$ implies $k \geq 3$. We consider two cases for the proof. As before, $I_{\sigma,2n-r}(2n)$ contains all the unassigned indices.

- Suppose $r = n+1$, so $2n-r = n-1$. Now $I_{\sigma,2n-r}(2n) = I_{\mu,2n-r}(2n)$, so $2n-r-1$ is still unassigned in μ. This implies that $2n$ is assigned in μ and $2n-r = n-1$ is unassigned. Now consider the following three intervals: $I_{\sigma,n-1}(1)$, $I_{\mu,n-1}(3)$ and $I_{\mu,n-1}(n+2)$. All 3 sets lie in $\mathscr{F}^c$, and their union is $[2n]$, a contradiction. We note here that the argument assumes $n \geq 4$. If $n \leq 3$ and $r = n+1$, we have $k \geq 4$ and a trivial ad hoc argument suffices.

- Suppose $n-1 > 2n-r$. Now the intervals of length $2n-r$ ending at the points in the interval $[2n-r-1, n-1)$ (which has length at least 2) are the same in both σ and μ. In other words, $2n-r-1$ is unassigned in μ and all the other indices in the interval are assigned. This means that the set of unassigned indices remains unchanged in μ, as required.

$\diamond$

We are now ready to finish the proof of Theorem 1.4.3. We consider two cases, $r = n$ and $r > n$, since the proofs are slightly different. Suppose first that $r > n$. Since every good cyclic ordering is saturated (and since we have assumed that π is $2n$-saturated), we can use Lemmas 3.2.1 and 3.2.2 to infer that every good cyclic ordering is $2n$-saturated. To finish the proof of this case, we will show that each set in $\mathscr{H}^r_{2n}(M_n)$ is an interval in some such good cyclic ordering. Let $A \in \mathscr{H}^r_{2n}(M_n)$. Then A contains $r-n$ edges (i.e. both vertices in $r-n$ edges) and $2n-r$ other vertices, one each from the other $2n-r$ edges. Suppose first that $n \in A$, so A contains the edge $\{n, 2n\}$. Let the other $r-n-1$ edges be $\{\{x_1, y_1\}, \ldots, \{x_{r-n-1}, y_{r-n-1}\}\}$, with each $x_i \in [n-1]$ and each $y_i \in [n+1, 2n-1]$. Let $L = \{l_1, \ldots, l_{2n-r}\}$ be the set of the remaining $2n-r$ vertices in A. We now construct a good cyclic ordering σ in which A is an interval. To define σ, it clearly suffices to define values of $\sigma(i)$ for $1 \leq i \leq n-1$. So for $1 \leq i \leq r-n-1$, let $\sigma(i) = x_i$, and for $1 \leq i \leq 2n-r$, let

$\sigma(i+r-n-1) = l_i$. Here the σ-interval of length r, ending at index $r-1$, is precisely A. Now suppose that $n \notin A$. Let the $r-n$ edges be $\{\{x_1, y_1\}, \ldots, \{x_{r-n}, y_{r-n}\}\}$ and let $L = \{l_1, \ldots, l_{2n-r-1}\}$ be the other $2n-r-1$ vertices (excluding $2n$). A good cyclic ordering σ in which A is an interval can be constructed as follows: for $1 \leq i \leq 2n-r-1$, let $\sigma(i) = l_i$ and for $2n-r \leq i \leq n-1$, let $\sigma(i) = x_{i-(2n-r-1)}$. In this case, the σ-interval of length r ending at index $n-1$, is A.

For $r = n$, we observe by Lemma 3.2.1 that every good cyclic ordering in C_{n-1} is $2n$-saturated. Again, we will show that every set in $\mathscr{H}^r_{2n}(M_n)$ is an interval in some $\sigma \in C_{n-1}$. Let $A \in \mathscr{H}^r_{2n}(M_n)$. Note that A is a maximum independent set in M_n and contains no edges. Let $V = A \cap [n-1]$, $|V| = s$, for some $s \leq r$ and let $W = A \setminus \{V \cup \{2n\}\}$. Let $V = \{v_1, \ldots, v_s\}$ and $W = \{w_1, \ldots, w_{r-1-s}\}$. Construct a good cyclic ordering $\sigma \in C_{n-1}$ as follows: for $1 \leq i \leq s$, define $\sigma(i) = v_i$, and for $s+1 \leq i \leq r-1$, set $\sigma(i) = w_{i-s} - n$. Then the σ-interval of length r, ending at s, is A. This completes the proof of the theorem. □

Chapter 4

CROSS-INTERSECTION THEOREMS FOR GRAPHS

Consider a collection of k subfamilies of $2^{[n]}$, say $\mathscr{A}_1, \ldots, \mathscr{A}_k$. Call this collection *cross-intersecting* if for any $i, j \in [k]$ with $i \neq j$, $A \in \mathscr{A}_i$ and $B \in \mathscr{A}_j$ implies $A \cap B \neq \emptyset$. Note that the individual families themselves do not need to be either non-empty or intersecting, and a subset can lie in more than one family in the collection. We will be interested in *uniform* cross-intersecting families, i.e. cross-intersecting subfamilies of $\binom{[n]}{r}$ for suitable values of r. There are two main kinds of problems concerning uniform cross-intersecting families that have been investigated, the *maximum product* problem and the *maximum sum* problem. One of the main results for the maximum product problem due to Matsumoto and Tokushige [45] states that for $r \leq n/2$ and $k \geq 2$, the product of the cardinalities of k cross-intersecting subfamilies $\{\mathscr{A}_1, \ldots, \mathscr{A}_k\}$ of $\binom{[n]}{r}$ is maximum if $\mathscr{A}_1 = \cdots = \mathscr{A}_k = \{A \subseteq \binom{[n]}{r} : x \in A\}$ for some $x \in [n]$.

We will be more interested in the maximum sum problem, particularly the following theorem of Hilton [31], which establishes a best possible upper bound on the sum of cardinalities of cross-intersecting families and also characterizes the extremal structures.

Theorem 1.6.1. *Let $r \leq n/2$ and $k \geq 2$. Let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be cross-intersecting subfamilies of $\binom{[n]}{r}$, with $\mathscr{A}_1 \neq \emptyset$. Then,*

$$\sum_{i=1}^{k} |\mathscr{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq n/r, \text{ and} \\ k\binom{n-1}{r-1} & \text{if } k \geq n/r. \end{cases}$$

If equality holds, then

1. $\mathscr{A}_1 = \binom{[n]}{r}$ *and* $\mathscr{A}_i = \emptyset$, *for each* $2 \leq i \leq k$, *if* $k < \dfrac{n}{r}$,

2. $|\mathscr{A}_i| = \binom{n-1}{r-1}$ *for each* $i \in [k]$ *if* $k > \dfrac{n}{r}$, *and*

3. $\mathscr{A}_1, \ldots, \mathscr{A}_k$ *are as in case* 1 *or* 2 *if* $k = \dfrac{n}{r} > 2$.

It is simple to observe that Theorem 1.6.1 is a generalization of the Erdős-Ko-Rado theorem [21] in the following manner: put $k > n/r$ and let $\mathscr{A}_1 = \cdots = \mathscr{A}_k$

There have been a few generalizations of Hilton's cross-intersection theorem, most recently for permutations by Borg ([8] and [9]) and for uniform cross-intersecting subfamilies of independent sets in graph M_n which is the perfect matching on $2n$ vertices, by Borg and Leader [11]. Borg and Leader proved an extension of Hilton's theorem for *signed* sets, which we will state in the language of graphs as we are interested in formulating a graph-theoretic analogue of Theorem 1.6.1 similar to the one developed in [34] for Theorem 1.1.1. For graph G, let $\mathscr{J}^{(r)}(G)$ be the family of all independent sets of size r in G. Also for any vertex $x \in V(G)$, let $\mathscr{J}_x^r(G) = \{A \in \mathscr{J}^r(G) : x \in A\}$.

Theorem 1.6.2 *Let* $r \le n$ *and* $k \ge 2$. *Let* $\mathscr{A}_1, \ldots, \mathscr{A}_k \subseteq \mathscr{J}^r(M_n)$ *be cross-intersecting. Then*

$$\sum_{i=1}^{k} |\mathscr{A}_i| \le \begin{cases} \binom{n}{r} 2^r & \text{if } k \le 2n/r, \text{ and} \\ k\binom{n-1}{r-1} 2^{r-1} & \text{if } k \ge 2n/r. \end{cases}$$

Suppose equality holds and $\mathscr{A}_1 \ne \emptyset$. *Then,*

1. *If* $k \le 2n/r$, *then* $\mathscr{A}_1 = \mathscr{J}^r(M_n)$ *and* $\mathscr{A}_2 = \cdots = \mathscr{A}_k = \emptyset$,

2. *If* $k \ge 2n/r$, *then for some* $x \in V(M_n)$, $\mathscr{A}_1 = \cdots = \mathscr{A}_k = \mathscr{J}_x^r(M_n)$, *and*

3. *If* $k = 2n/r > 2$, *then* $\mathscr{A}_1, \ldots, \mathscr{A}_k$ *are as in either of the first two cases.*

In fact, Borg and Leader proved a slightly more general result with the same argument, for a disjoint union of complete graphs, all having the same number of vertices s, for some $s \ge 2$. We consider extensions of this result to any disjoint union of complete graphs. Let G be a disjoint union of complete graphs, with each component containing at least 2 vertices. We first prove a theorem which bounds the sum of the cardinalities of cross-intersecting subfamilies $\mathscr{A}_1,\dots,\mathscr{A}_k$ of $\mathscr{J}^r(G)$ when k is sufficiently small.

Theorem 4.0.3. *Let $G_1,\dots,G_n$ be n complete graphs with $|G_i| \geq 2$ for each $1 \leq i \leq n$. Let G be the disjoint union of these n graphs and let $r \leq n$. For some $2 \leq k \leq \min_{i=1}^{n}\{|G_i|\}$, let $\mathscr{A}_1,\dots,\mathscr{A}_k \subseteq \mathscr{J}^r(G)$ be cross-intersecting families. Then,*

$$\sum_{i=1}^{k} |\mathscr{A}_i| \leq |\mathscr{J}^{(r)}(G)|.$$

This bound is best possible, and can be obtained by letting $\mathscr{A}_1 = \mathscr{J}^r(G)$ and $\mathscr{A}_2 = \cdots = \mathscr{A}_k = \emptyset$.

Cross-intersecting pairs

We now restrict our attention to *cross-intersecting pairs* in $\mathscr{J}^r(G)$, i.e. we fix $k=2$. The following Corollary of Theorem 1.6.2 is immediately apparent.

Corollary 4.0.4. *Let $r \leq n$. Let $(\mathscr{A},\mathscr{B})$ be a cross-intersecting pair in $\mathscr{J}^r(M_n)$. Then,*

$$|\mathscr{A}| + |\mathscr{B}| \leq 2^r \binom{n}{r}.$$

If $r < n$, then equality holds if and only if $\mathscr{A} = \mathscr{J}^r(M_n)$ and $\mathscr{B} = \emptyset$ (or vice-versa).

We give an alternate proof of Corollary 4.0.4. The bound in the statement of Corollary 4.0.4 will follow immediately from Theorem 4.0.3, while the Theorem 1.4.4 of Bollobás and Leader [6] is used to characterize the extremal structures. The following corollary can also be directly obtained from Theorem 4.0.3.

Corollary 4.0.5. *Let $r \leq n$ and suppose $(\mathscr{A},\mathscr{B})$ is a cross-intersecting pair in $\mathscr{J}^r(G)$, where G is a disjoint union of n complete graphs, each having at least*

2 *vertices. Then* $|\mathscr{A}|+|\mathscr{B}| \leq |\mathscr{J}^r(G)|$. *This bound is best possible, and can be attained by letting* $\mathscr{A} = \mathscr{J}^r(G)$ *and* $\mathscr{B} = \emptyset$.

We now consider this problem for a larger class of graphs, namely chordal graphs, but with a slightly stronger restriction on r. As defined before, let $\mu = \mu(G)$ be the minimum size of a maximal independent set in G. We prove the following theorem for chordal graphs.

Theorem 4.0.6. *Let G be a chordal graph and let $r \leq \mu(G)/2$. Then for any cross-intersecting pair* $(\mathscr{A},\mathscr{B})$ *in* $\mathscr{J}^r(G)$, $|\mathscr{A}|+|\mathscr{B}| \leq |\mathscr{J}^r(G)|$.

We conjecture that the statement of Theorem 4.0.6 should hold for all graphs.

Conjecture 4.0.7. *Let G be a graph and $r \leq \mu(G)/2$. If* $(\mathscr{A},\mathscr{B})$ *is a cross-intersecting pair in* $\mathscr{J}^r(G)$, *then* $|\mathscr{A}|+|\mathscr{B}| \leq |\mathscr{J}^r(G)|$.

We end by proving Conjecture 4.0.7 when $G = C_n$, the cycle on $n \geq 2$ vertices (defining C_n to be a solitary edge when $n = 2$), which is non-chordal when $n \geq 4$. In fact we prove the following stronger statement.

Theorem 4.0.8. *For $r \geq 1$, $n \geq 2$, and any cross-intersecting pair* $(\mathscr{A},\mathscr{B})$ *in* $\mathscr{J}^r(C_n)$, $|\mathscr{A}|+|\mathscr{B}| \leq |\mathscr{J}^r(G)|$.

The main tool we use to prove Theorems 4.0.6 and 4.0.8 is the shifting technique, appropriately modified for the respective graphs.

4.1 Disjoint union of complete graphs

We start by giving a proof of Theorem 4.0.3. We will use a strategy of Borg [8] in conjunction with Theorem 2.0.8. The strategy is to construct an intersecting family from a collection of cross-intersecting families and obtain the cross-intersection result by invoking Theorem 2.0.8, the full statement of which we recall below. We require Theorem 2.0.8 by Holroyd, Spencer and Talbot [34], the full statement of which we recall below.

Theorem 2.0.8. *Let G be a disjoint union of $n \geq r$ complete graphs, each on at least 2 vertices. If $\mathscr{A} \subseteq \mathscr{J}^r(G)$ is intersecting, then $|\mathscr{A}| \leq \max_{x \in V(G)} |\mathscr{J}_x^r(G)|$.*

Proof of Theorem 4.0.3. Let G be a disjoint union of n complete graphs $G_1, \ldots, G_n$ with $|G_i| \geq 2$ for each $i \in [n]$. Let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be cross-intersecting subfamilies of $\mathscr{J}^r(G)$, with $r \leq n$ and $2 \leq k \leq \min_{i=1}^n \{|G_i|\}$.

We create an auxiliary graph $G' = G \cup G_{n+1}$ where $G_{n+1} = K_k$, the complete graph on k vertices and $V(G_{n+1}) = \{v_1, \ldots, v_k\}$. Let $V(G') = V(G) \cup V(G_{n+1})$ and $E(G') = E(G) \cup E(G_{n+1})$. For each $1 \leq i \leq k$, let $\mathscr{A}_i' = \{A \cup \{v_i\} : A \in \mathscr{A}_i\}$. Let $\mathscr{A}' = \bigcup_{i=1}^k \mathscr{A}_i'$. Clearly, $|\mathscr{A}'| = \sum_{i=1}^k |\mathscr{A}_i'| = \sum_{i=1}^k |\mathscr{A}_i|$ and $\mathscr{A}' \subseteq \mathscr{J}^{r+1}(G')$. We now prove that $\mathscr{A}'$ is intersecting.

Claim 4.1.1. *$\mathscr{A}'$ is intersecting.*

Proof. Let $A, B \in \mathscr{A}'$. If $A, B \in \mathscr{A}_i'$ for some $i \in [k]$, then $v_i \in A \cap B$, so assume $A \in \mathscr{A}_i'$ and $B \in \mathscr{A}_j'$ for some $i \neq j$. For $A' = A \setminus \{v_i\}$ and $B' = B \setminus \{v_j\}$, we have $A' \in \mathscr{A}_i$ and $B' \in \mathscr{A}_j$, which implies $A' \cap B' \neq \emptyset$. This gives $A \cap B \neq \emptyset$ as required.

$\diamond$

Using Theorem 2.0.8 and Claim 4.1.1, we get $|\mathscr{A}'| \leq |\mathscr{J}_x^{r+1}(G')|$, where x is any vertex in a component with the smallest number of vertices. In particular we can let $x \in V(G_{n+1})$, since $k \leq \min_{i=1}^n \{|G_i|\}$. This gives us $|\mathscr{J}_x^{r+1}(G')| = |\mathscr{J}^r(G)|$, completing the proof of the theorem. $\square$

We can now use Theorem 4.0.3 to give the following short alternate proof of Corollary 4.0.4. As mentioned before we require Theorem 1.4.4 to characterize the extremal structures.

Theorem 1.4.4 *Let $r \le n$ and suppose $\mathscr{A} \subseteq \mathscr{J}^r(M_n)$ is intersecting. Then $|\mathscr{A}| \le 2^{r-1}\binom{n-1}{r-1}$. If $r < n$, then equality holds if and only if $\mathscr{A} = \mathscr{J}_x^r(M_n)$ for some $x \in V(M_n)$.*

Proof of Corollary 4.0.4. It is clear that when $k = 2$, the bound in Corollary 4.0.4 follows immediately from Theorem 4.0.3. So suppose that $r < n$ and $|\mathscr{A}| + |\mathscr{B}| = 2^r\binom{n}{r}$. Assume $\mathscr{A}'$ is defined as in the proof of Theorem 4.0.3, so $\mathscr{A}' \subseteq \mathscr{J}^{r+1}(M_{n+1})$ is intersecting. Let v_1v_2 be the edge added to M_n to obtain M_{n+1}. Now $|\mathscr{A}'| = |\mathscr{A}| + |\mathscr{B}| = 2^r\binom{n}{r}$. By using the characterization of equality in Theorem 1.4.4, we get $\mathscr{A}' = \mathscr{J}_x^{r+1}(M_{n+1})$ for some $x \in V(M_{n+1})$. But by the construction of $\mathscr{A}'$, every set in $\mathscr{A}'$ contains either v_1 or v_2, so $x \in \{v_1, v_2\}$. Without loss of generality, let $x = v_1$. This implies that no set in $\mathscr{A}'$ contains v_2. Thus we get $\mathscr{A} = \mathscr{J}^r(M_n)$ and $\mathscr{B} = \emptyset$. $\diamond$

4.2 Chordal graphs

In this section, we prove Theorem 4.0.6. We begin by fixing some notation. For a graph G and a vertex $v \in V(G)$, let $G - v$ be the graph obtained from G by removing vertex v. Also let $G \downarrow v$ denote the graph obtained by removing v and its set of neighbors from G. We now recall the characterization of chordal graphs, due to Dirac [19].

Definition 4.2.1. *A vertex v is called simplicial in a graph G if its neighborhood is a clique in G.*

Consider a graph G on n vertices, and let $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices of G. Let the graph G_i be the subgraph obtained by removing the vertex set $\{v_1, \ldots, v_{i-1}\}$ from G. Then σ is called a *simplicial elimination ordering* if v_i is simplicial in the graph G_i, for each $1 \leq i \leq n$.

Theorem 2.1.3. *A graph G is a chordal graph if and only if it has a simplicial elimination ordering.*

We state a lemma regarding the graph parameter μ. Note that the proof of this fact appears in Chapter 2, so we present it here without proof.

Lemma 2.1.13. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then the following inequalities hold:*

1. $\mu(G - v_2) \geq \mu(G)$;

2. $\mu(G \downarrow v_2) + 1 \geq \mu(G)$.

The following corollary is an easy consequence of Lemma 2.1.13.

Corollary 2.1.14 *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then the following statements hold:*

1. *If $r \leq \frac{1}{2}\mu(G)$, then $r \leq \frac{1}{2}\mu(G - v_2)$;*

2. *If $r \leq \frac{1}{2}\mu(G)$, then $r - 1 \leq \frac{1}{2}\mu(G \downarrow v_2)$.*

We now proceed with the proof of Theorem 4.0.6. We do induction on r, the base case being $r = 1$. Since $\mu(G) \geq 2$, G has at least two vertices so the bound follows trivially. Let $r \geq 2$ and let G be a chordal graph with $\mu(G) \geq 2r$. We now do induction on $|V(G)|$. If $|V(G)| = \mu(G)$, G is the empty graph on $|V(G)|$ vertices, and we are done by Theorem 1.6.1. So let $|V(G)| > \mu(G) \geq 2r$. This implies that there is a component of G, say H on at least 2 vertices. We know from the definition of chordal graphs that any induced subgraph of a chordal graph is also chordal. So by using Theorem 2.1.3 for H, we can find a simplicial elimination ordering in H. Let this ordering be $[v_1, \ldots, v_m]$ where $m = |V(H)|$ and let $v_1 v_i \in E(H)$ for some $2 \leq i \leq m$. Let $\mathscr{A}$ and $\mathscr{B}$ be a cross-intersecting pair in $\mathscr{J}^r(G)$.

We define two compression operations $f_{1,i}$ and $g_{1,i}$ for sets in the families $\mathscr{A}$ and $\mathscr{B}$ respectively. Before we give the definitions, we note that $N[v_1] \subseteq N[v_i]$ and that if A is an independent set with $v_i \in A$, then $A \setminus \{v_i\} \cup \{v_1\}$ is also independent.

$$f_{1,i}(A) = \begin{cases} A \setminus \{v_i\} \cup \{v_1\} & \text{if } v_i \in A, A \setminus \{v_i\} \cup \{v_1\} \notin \mathscr{A}, \text{ and} \\ A & \text{otherwise.} \end{cases}$$

$$g_{1,i}(B) = \begin{cases} B \setminus \{v_i\} \cup \{v_1\} & \text{if } v_i \in B, B \setminus \{v_i\} \cup \{v_1\} \notin \mathscr{B}, \text{ and} \\ B & \text{otherwise.} \end{cases}$$

We define $\mathscr{A}' = f_{1,i}(\mathscr{A}) = \{f_{1,i}(A) : A \in \mathscr{A}\}$. Also define $\mathscr{B}'$ in an analogous manner. Next, we define the following families for $\mathscr{A}'$ (the families for $\mathscr{B}'$ are also defined in an identical manner).

$$\mathscr{A}'_i = \{A \in \mathscr{A}' : v_i \in A\},$$

$$\bar{\mathscr{A}}'_i = \mathscr{A}' \setminus \mathscr{A}'_i, \text{ and}$$

$$\mathscr{A}''_i = \{A \setminus \{v_i\} : A \in \mathscr{A}'_i\}.$$

It is not hard to observe that $|\mathscr{A}| = |\mathscr{A}'| = |\mathscr{A}''_i| + |\bar{\mathscr{A}}'_i|$ and $|\mathscr{B}| = |\mathscr{B}'| = |\mathscr{B}''_i| + |\bar{\mathscr{B}}'_i|$. Consider the pair $(\mathscr{A}''_i, \mathscr{B}''_i)$ and the pair $(\bar{\mathscr{A}}'_i, \bar{\mathscr{B}}'_i)$. We will prove the following lemma about these pairs.

Lemma 4.2.2. *1. $(\mathscr{A}''_i, \mathscr{B}''_i)$ is a cross-intersecting pair in $\mathscr{J}^{r-1}(G \downarrow v_i)$.*

2. $(\bar{\mathscr{A}}'_i, \bar{\mathscr{B}}'_i)$ is a cross-intersecting pair in $\mathscr{J}^{r}(G - v_i)$.

Proof. 1. Let $A \in \mathscr{A}''_i$ and $B \in \mathscr{B}''_i$. Then $A_1 = A \cup \{v_i\} \in \mathscr{A}$ and $B_1 = B \cup \{v_i\} \in \mathscr{B}$. Also, $A_2 = A \cup \{v_1\} \in \mathscr{A}$, otherwise A_1 could have been shifted to A_2 by $f_{1,i}$. Since $B_1 \cap A_2 \neq \emptyset$, we get $A \cap B \neq \emptyset$ as required.

2. Let $A \in \bar{\mathscr{A}}_i'$ and $B \in \bar{\mathscr{B}}_i'$. If $A \in \mathscr{A}$ and $B \in \mathscr{B}$, we are done, so suppose $A \notin \mathscr{A}$. Then we must have $v_1 \in \mathscr{A}$. Assuming $v_1 \notin B$, we get $B \in \mathscr{B}$. Since $(A \setminus \{v_1\} \cup \{v_i\}) \in \mathscr{A}$, we have $(A \setminus \{v_1\} \cup \{v_i\}) \cap B \neq \emptyset$, implying $A \cap B \neq \emptyset$ as required.

$\diamond$

We are now in a position to complete the proof of Theorem 4.0.6 as follows, using Lemma 4.2.2. We can use Corollary 2.1.14 to infer that $G - v_i$ satisfies the induction hypothesis for r and $G \downarrow v_i$ satisfies the induction hypothesis for $r-1$.

$$\begin{aligned} |\mathscr{A}| + |\mathscr{B}| &= (|\bar{\mathscr{A}}_i'| + |\bar{\mathscr{B}}_i'|) + (|\mathscr{A}_i''| + |\mathscr{B}_i''|) \\ &\leq |\mathscr{J}^r(G - v_i)| + |\mathscr{J}^{r-1}(G \downarrow v_i)| \\ &= |\mathscr{J}^r(G)|. \end{aligned}$$

The last equality can be explained by a simple partitioning of the family $\mathscr{J}^r(G)$ based on whether or not a set in the family contains v_i. There are exactly $|\mathscr{J}^{r-1}(G \downarrow v_i)|$ sets which contain v_i and $|\mathscr{J}^r(G - v_i)|$ sets which do not contain v_i. $\square$

4.3 Cycles

Proof of Theorem 4.0.8. As mentioned earlier, the main tool we use to prove Theorem 4.0.8 is a shifting operation first employed by Talbot [60] to prove an EKR theorem for the cycle. Proceeding by induction on r as before with $r = 1$ being the trivial base case, we suppose $r \geq 2$ and do induction on n. The statement is vacuously true when $n \in \{2, 3\}$, so suppose $n \geq 4$. Let $V(C_n) = \{1, \ldots, n\}$ and $E(C_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$. Suppose $(\mathscr{A}, \mathscr{B})$ is a cross-intersecting pair in $\mathscr{J}^r(C_n)$. Consider the graph obtained by contracting the edge $e_1 = \{n-1, n\}$ in C_n. We will identify this contraction by the function $c : [n] \to [n-1]$ defined by $c(n) = n-1$ (and $c(x) = x$ elsewhere), so the resulting graph is C_{n-1}. Similarly identify the graph obtained from C_{n-1} by contracting the edge $e_2 = \{n-2, n-1\}$ as C_{n-2}. We define the following two subfamilies for $\mathscr{A}$. Let $\mathscr{A}_1 = \{A - \{n\} : n-2, n \in A \in \mathscr{A}\}$ and $\mathscr{A}_2 = \{A - \{n-1\} : n-1, 1 \in A \in \mathscr{A}\}$. Define $\mathscr{B}_1$ and $\mathscr{B}_2$ similarly. Now no set in either $\mathscr{A}_1$ or $\mathscr{B}_1$ contains 1. Similarly no set in either $\mathscr{A}_2$ or $\mathscr{B}_2$ contains $n-2$. Moreover, no set in any of the families $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2$ contains either n or $n-1$. This implies that $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2 \subseteq \mathscr{J}^{r-1}(C_{n-2})$. Let $\mathscr{A}_1' = \{A \in \mathscr{A} : n-2, n \in A\}$ and $\mathscr{A}_2' = \{A \in \mathscr{A} : 1, n-1 \in A\}$, with $\mathscr{B}_1'$ and $\mathscr{B}_2'$ defined similarly. We consider the families $\mathscr{A}^* = \mathscr{A} \setminus (\mathscr{A}_1' \cup \mathscr{A}_2')$ and $\mathscr{B}^* = \mathscr{B} \setminus (\mathscr{B}_1' \cup \mathscr{B}_2')$. Note that $(\mathscr{A}^*, \mathscr{B}^*)$ is a cross-intersecting pair in $\mathscr{J}^r(C_n)$. We will now define two shifting operations, one for $\mathscr{A}^*$ and one for $\mathscr{B}^*$ with respect to the vertices n and $n-1$.

$$f(A) = \begin{cases} A \setminus \{n\} \cup \{n-1\} & \text{if } n \in A, A \setminus \{n\} \cup \{n-1\} \notin \mathscr{A}^*, \text{ and} \\ A & \text{otherwise.} \end{cases}$$

$$g(B) = \begin{cases} B \setminus \{n\} \cup \{n-1\} & \text{if } n \in B, B \setminus \{n\} \cup \{n-1\} \notin \mathscr{B}^*, \text{ and} \\ B & \text{otherwise.} \end{cases}$$

Let $f(\mathscr{A}^*) = \{f(A) : A \in \mathscr{A}^*\}$ and $f(\mathscr{B}^*) = \{f(B) : B \in \mathscr{B}^*\}$. As before, we partition $f(\mathscr{A}^*)$ (and similarly, $f(\mathscr{B}^*)$) into two parts as follows. Let $\mathscr{A}' = \{A \in f(\mathscr{A}^*) : n \notin A\}$ and let $\mathscr{A}_3 = \{A - \{n\} : A \in f(\mathscr{A}^*) \setminus \mathscr{A}'\}$. We have $\mathscr{A}', \mathscr{B}' \subseteq \mathscr{J}^r(C_{n-1})$. Also $\mathscr{A}_3, \mathscr{B}_3 \subseteq \mathscr{J}^{r-1}(C_{n-2})$ because for any set $S \in \mathscr{A}_3 \cup \mathscr{B}_3$, $S \cap \{1, n-1, n\} = \emptyset$. Let $\tilde{\mathscr{A}} = \bigcup_{i \in [3]} \mathscr{A}_i$ and $\tilde{\mathscr{B}} = \bigcup_{i \in [3]} \mathscr{B}_i$. We consider the pair $(\mathscr{A}', \mathscr{B}')$ in $\mathscr{J}^r(C_{n-1})$ and the pair $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$ in $\mathscr{J}^{r-1}(C_{n-2})$. We first state and prove some claims about these families.

Claim 4.3.1. *1. Let $A \in \mathscr{A}_3$. Then $A \cup \{n-1\} \in \mathscr{A}^*$.*

2. Let $B \in \mathscr{B}_3$. Then $B \cup \{n-1\} \in \mathscr{B}^*$.

Proof. It suffices to prove the claim for $\mathscr{A}_3$. We know that $A \cup \{n\} \in f(\mathscr{A}^*)$. This means that $A \cup \{n\} \in \mathscr{A}^*$ and $A \cup \{n\}$ was not shifted to $A \cup \{n-1\}$ by f, implying $A \cup \{n-1\} \in \mathscr{A}^*$. $\diamond$

The next claim will show that $\tilde{\mathscr{A}} = \bigcup_{i \in [3]} \mathscr{A}_i$ and $\tilde{\mathscr{B}} = \bigcup_{i \in [3]} \mathscr{B}_i$ are disjoint unions.

Claim 4.3.2. *1. For any $i, j \in [3]$ with $i \neq j$, $\mathscr{A}_i \cap \mathscr{A}_j = \emptyset$.*

2. For any $i, j \in [3]$ with $i \neq j$, $\mathscr{B}_i \cap \mathscr{B}_j = \emptyset$.

Proof. As before, it suffices to prove the claim for the $\mathscr{A}_i$'s. It is clear from the definitions of $\mathscr{A}_1$ and $\mathscr{A}_2$ that $\mathscr{A}_1 \cap \mathscr{A}_2 = \emptyset$. Since every $(r-1)$-set in $\mathscr{A}_3$ is obtained by removing n from an r-set, no set in $\mathscr{A}_3$ contains 1. So it remains to prove that no set in $\mathscr{A}_3$ contains $n-2$. By the previous claim we know that for any $A \in \mathscr{A}_3$, $A \cup \{n-1\} \in \mathscr{A}^*$. This gives $n-2 \notin A$ as required. $\diamond$

Claim 4.3.3. *1. $(\mathscr{A}', \mathscr{B}')$ is a cross-intersecting pair in $\mathscr{J}^r(C_{n-1})$.*

2. $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$ is a cross-intersecting pair in $\mathscr{J}^{r-1}(C_{n-2})$.

Proof. 1. Suppose $A \in \mathscr{A}'$ and $B \in \mathscr{B}'$. If $A \in \mathscr{A}^*$ and $B \in \mathscr{B}^*$, then $A \cap B \neq \emptyset$ so suppose $A \notin \mathscr{A}^*$. This gives $n-1 \in A$. Assume $n-1 \notin B$ so $B \in \mathscr{B}^*$. Since $A_1 = (A \setminus \{n-1\} \cup \{n\}) \in \mathscr{A}^*$, we have $A_1 \cap B \neq \emptyset$, which gives $A \cap B \neq \emptyset$.

2. Let $A \in \tilde{\mathscr{A}}$ and $B \in \tilde{\mathscr{B}}$. So $A \in \mathscr{A}_i$ and $B \in \mathscr{B}_j$ for some $i, j \in [3]$. First consider the case when $i = j$. Each set in $\mathscr{A}_1$ and $\mathscr{B}_1$ has $n-2$, while each set in $\mathscr{A}_2$ and $\mathscr{B}_2$ has 1, so let $A \in \mathscr{A}_3$ and $B \in \mathscr{B}_3$. We have $A \cup \{n\} \in \mathscr{A}^*$. Also, $B \cup \{n-1\} \in \mathscr{B}^*$ by Claim 4.3.1, so $(A \cup \{n\}) \cap (B \cup \{n-1\}) \neq \emptyset$, giving $A \cap B \neq \emptyset$ as required. Next, let $i \neq j$. We only consider cases when $i < j$, since the other cases follow identically. Suppose $i = 1$ and $j = 2$. In

this case we have $(A \cup \{n\}) \in \mathscr{A}$, $(B \cup \{n-1\}) \in \mathscr{B}$, which gives $A \cap B \neq \emptyset$. If $i = 1$ and $j = 3$, we again have $A \cup \{n\} \in \mathscr{A}$ while Claim 4.3.1 implies $B \cup \{n-1\} \in \mathscr{B}$, giving $A \cap B \neq \emptyset$. Similarly for $i = 2$ and $j = 3$ we have $A \cup \{n-1\} \in \mathscr{A}$ and $B \cup \{n\} \in \mathscr{B}$.

$\diamond$

The final claim we prove is regarding the size of $\mathscr{J}^r(C_n)$.

Claim 4.3.4. $|\mathscr{J}^r(C_n)| = |\mathscr{J}^r(C_{n-1})| + |\mathscr{J}^{r-1}(C_{n-2})|$.

Proof. Consider all sets in $\mathscr{J}^r(C_n)$ which contain neither n nor both $n-1$ and 1. The number of these sets is clearly $\mathscr{J}^r(C_{n-1})$. Now consider the subfamily containing the remaining sets, i.e. those which either have n or both 1 and $n-1$. Call it $\mathscr{F}$. We define the following correspondence between $\mathscr{F}$ and $\mathscr{J}^{r-1}(C_{n-2})$. For $A \in \mathscr{F}$, define $f(A) = A - \{n\}$ if $n \in A$ and $f(A) = A - \{n-1\}$ if $1, n-1 \in A$. Clearly $f(A) \in \mathscr{J}^{r-1}(C_{n-2})$ and f is bijective, giving $|\mathscr{F}| = |\mathscr{J}^{r-1}(C_{n-2})|$ as required. $\diamond$

We can now finish the proof of Theorem 4.0.8 as follows, using Claim 4.3.3 and the inductive hypothesis. The final equality follows from Claim 4.3.4.

$$\begin{aligned} |\mathscr{A}| + |\mathscr{B}| &= |\mathscr{A}^*| + |\mathscr{B}^*| + \sum_{i=1}^{2} (|\mathscr{A}_i| + |\mathscr{B}_i|) \\ &= (|\mathscr{A}'| + |\mathscr{B}'|) + \sum_{i=1}^{3} (|\mathscr{A}_i| + |\mathscr{B}_i|) \\ &= (|\mathscr{A}'| + |\mathscr{B}'|) + (|\tilde{\mathscr{A}}| + |\tilde{\mathscr{B}}|) \\ &\leq |\mathscr{J}^r(C_{n-1})| + |\mathscr{J}^{r-1}(C_{n-2})| \\ &= |\mathscr{J}^r(C_n)|. \end{aligned} \tag{4.1}$$

$\square$

Chapter 5

NEW DIRECTIONS AND GENERALIZATIONS

The Erdős–Ko–Rado theorem and its many generalizations continue to inspire a tremendous amount of research in extremal set theory. In addition to the conjectures and scope for strengthening the main theorems mentioned in the preceding sections, there are other avenues for future research. A few of them are outlined in this section.

5.1 Chvátal's conjecture for hereditary families of small rank

As we mentioned in the introduction, Chvátal's conjecture is one of the long-standing open problems in the field. We recollect some definitions, before recalling the conjecture itself. A family $\mathscr{F} \subseteq 2^{[n]}$ is called *hereditary* if $A \in \mathscr{F}$ and $B \subseteq A$ imply that $B \in \mathscr{F}$. We say that a family $\mathscr{F} \subseteq 2^{[n]}$ is *EKR* is the set of all maximum-sized intersecting subfamilies of $\mathscr{F}$ contains a *star*, i.e. a subfamily $\mathscr{S}$ with $\cap_{S \in \mathscr{S}} S \neq \emptyset$. We've seen that $2^{[n]}$ is EKR, as is $\binom{[n]}{r}$, if $r \leq n/2$. Chvátal's conjecture can now be stated as follows.

Conjecture 1.2.1 *If $\mathscr{F} \subseteq 2^{[n]}$ is hereditary, then $\mathscr{F}$ is EKR.*

There are a few results which verify the conjecture for specific hereditary families. Among the most important ones is a result of Chvátal himself [12]. Let $\mathscr{F}$ be a hereditary family on a set X, which has a total ordering of its elements defined by a relation $\preceq$. Chvátal proved the conjecture when $\mathscr{F}$ is *compressed*, i.e. if $\{x_1, \ldots, x_r\} \in \mathscr{F}$ and $y_i \preceq x_i$ for each $1 \leq i \leq r$, then $\{y_1, \ldots, y_r\} \in \mathscr{F}$. Snevily [57] further extended Chvátal's theorem and proved the conjecture when the family is compressed with respect to a specific element x, i.e. if $F \in \mathscr{F}$ such that $y \in F$ but $x \notin F$, then $F \setminus \{y\} \cup \{x\} \in \mathscr{F}$. There have been other results, which have involved the maximal members of a hereditary family $\mathscr{F}$.

We begin this section by mentioning an important lemma of Kleitman [42], on hereditary families. We call $\mathscr{G}$ a *reverse hereditary* family if $A \in \mathscr{G}$ and $A \subseteq B$ implies that $B \in \mathscr{G}$. Kleitman's lemma can be now stated as follows:

Lemma 5.1.1 (Kleitman [42])**.** *If $\mathscr{F}$ is a hereditary family, and $\mathscr{G}$ is a reverse hereditary family on $[n]$, then $|\mathscr{G}||\mathscr{F}| \geq 2^n|\mathscr{G} \cap \mathscr{F}|$.*

An important consequence of Kleitman's lemma is the following corollary. The corollary appears as an exercise in Anderson [2] and can be proved as follows. Let $I(\mathscr{F})$ denote the size of the maximum intersecting family in $\mathscr{F}$.

Corollary 5.1.2. *If $\mathscr{F}$ is a hereditary family on $[n]$, then $I(\mathscr{F}) \leq \frac{1}{2}|\mathscr{F}|$.*

Proof. For a hereditary family $\mathscr{F}$, suppose $\mathscr{I}$ is a maximum intersecting family in $\mathscr{F}$, with $|\mathscr{I}| = I(\mathscr{F})$. Suppose $A \in \mathscr{I}$, and let $A \subseteq B \in \mathscr{F}$. We must have $B \in \mathscr{I}$, otherwise by maximality, there exists some $C \in \mathscr{I}$ with $B \cap C = \emptyset$, giving $A \cap C = \emptyset$, a contradiction. Now, define a family $\mathscr{G}$ as follows: let $\mathscr{G} = \mathscr{I} \cup \{A : B \subseteq A \text{ for some } B \in \mathscr{I}\}$. It is easy to note that $\mathscr{G}$ is a reverse hereditary family, and $\mathscr{I} = \mathscr{F} \cap \mathscr{G}$. Moreover, $\mathscr{G}$ is intersecting, giving $|\mathscr{G}| \leq 2^{n-1}$. Using Kleitman's lemma, we get the required bound as follows:

$$|\mathscr{I}| = |\mathscr{G} \cap \mathscr{F}| \leq \frac{|\mathscr{G}||\mathscr{F}|}{2^n} \leq \frac{|\mathscr{F}|}{2}.$$

□

Corollary 5.1.2 is an important observation that leads to further partial results regarding Chvátal's conjecture. Many of the partial results concerning the conjecture have involved the maximum elements of a given hereditary family. We define the maximal elements of a hereditary family in the obvious manner as follows.

Definition 5.1.3. *For any hereditary family $\mathscr{F} \subseteq 2^{[n]}$, $F \in \mathscr{F}$ is a maximal element if $F \subseteq G$ and $G \in \mathscr{F}$ implies that $F = G$.*

Schönheim [55] proved that if the maximal elements of a hereditary family satisfy additional conditions, then equality holds in the conclusion of Corollary 5.1.2.

Theorem 5.1.4 (Schönheim [55])**.** *If the maximal elements of a hereditary family $\mathscr{F}$ have non-empty intersection, then Chvátal's conjecture is true for $\mathscr{F}$, and $I(\mathscr{F}) = \frac{1}{2}|\mathscr{F}|$.*

Further results have also been obtained in this direction by Stein [58] and Miklós [48], among others. We will be interested in proving results for hereditary families where the maximal elements have small cardinality, and all maximum intersecting families are sufficiently large. Let $\binom{[n]}{\leq k} = \{A \subseteq [n] : |A| \leq k\}$. In our proof, we will make use of the *Sunflower Lemma* of Erdős and Rado [23]. We first define *sunflowers*, then state the lemma, before proceeding to a statement and proof of the theorem.

Definition 5.1.5. *A sunflower, with k petals and a core X is a family of k sets $S_1, \ldots, S_k$ such that for any $i, j \in [k]$ with $i \neq j$, we have $S_i \cap S_j = X$. Also, the sets $S_i \setminus X$, called the petals, should be nonempty.*

We can now state the Sunflower Lemma as follows.

Lemma 5.1.6 (Erdős–Rado [23])**.** *Let $\mathscr{F}$ be a family of sets, each of cardinality s. If $|\mathscr{F}| > s!(k-1)^s$, then $\mathscr{F}$ contains a sunflower with k petals.*

We now state and prove the following theorem for hereditary families in $\binom{[n]}{\leq 3}$.

Theorem 5.1.7 (Hurlbert–Kamat–Mubayi [38]). *Let $\mathscr{F} \subseteq \binom{[n]}{\leq 3}$ be a hereditary family, and let $\mathscr{I} \subseteq \mathscr{F}$ be a maximum intersecting family. If $|\mathscr{I}| \geq 166$, then $\mathscr{I}$ is a star.*

Proof. Let $\mathscr{I}_i = \mathscr{I} \cap \binom{[n]}{i}$, for $i = 1,2,3$. We can assume $\mathscr{I}_1 = \emptyset$, since otherwise, $\mathscr{I}$ is a star. Similarly, we can assume $|\mathscr{I}_2| \leq 3$. Thus, we have $|\mathscr{I}_3| \geq 163$. Since $163 = 3!(4-1)^3 + 1$, we can use the Sunflower Lemma to conclude that $\mathscr{I}$ contains a sunflower with at least 4 petals. Let S be a sunflower with the maximum number of petals, and let C be the core of S. If $|C| = 1$, we can conclude that I is a star, since I is intersecting, so suppose $|C| = 2$. Let $C = \{a,b\}$. Let $\mathscr{A} = \{A \in \mathscr{I}_3 : A \cap C = \{a\}\}$, and let $\mathscr{B} = \{B \in \mathscr{I}_3 : B \cap C = \{b\}\}$. We have $|\mathscr{I}_3| = |S| + |\mathscr{A}| + |\mathscr{B}|$. Let $\mathscr{A}' = \{A - \{a\} : A \in \mathscr{A}\}$, and $\mathscr{B}' = \{B - \{b\} : B \in \mathscr{B}\}$. If $\mathscr{A}' = \emptyset$ or $\mathscr{B}' = \emptyset$, we can conclude that $\mathscr{I}_3$, and hence, $\mathscr{I}$ is a star (centered at either a or b), so suppose both are non-empty. Since $\mathscr{I}$ is intersecting, $\mathscr{A}'$ and $\mathscr{B}'$ are cross-intersecting families, i.e. for any $A \in \mathscr{A}'$ and $B \in \mathscr{B}'$, $A \cap B \neq \emptyset$. Let $V(\mathscr{A}')$ and $V(\mathscr{B}')$ be the vertex sets of $\mathscr{A}'$ and $\mathscr{B}'$ respectively, and let $n(\mathscr{X}) = |V(\mathscr{X})|$ for $\mathscr{X} \in \{\mathscr{A}', \mathscr{B}'\}$. We first prove the following claims.

Claim 5.1.8. *If both $\mathscr{A}'$ and $\mathscr{B}'$ are intersecting, or $|\mathscr{A}'| \geq 2$ and $|\mathscr{B}'| \geq 2$, then, $|\mathscr{X}| \leq 2 + n(\mathscr{X})$ for $\mathscr{X} \in \{\mathscr{A}', \mathscr{B}'\}$.*

Proof. If $\mathscr{A}'$ is intersecting, it is either a triangle, or a star. In either case, the bound follows trivially. A similar argument works for $\mathscr{B}'$, so suppose, without loss of generality that $\mathscr{A}'$ has two disjoint edges, say $\{xy, x'y'\}$. $\mathscr{B}' \subseteq \{xy', y'y, yx', x'x\}$, giving the required bound for $\mathscr{B}'$. Now, if $\mathscr{B}'$ has two disjoint edges, we can use a similar argument for $\mathscr{A}'$, so suppose $\mathscr{B}'$ is intersecting. Without loss of generality, suppose $\mathscr{B}' = \{xy', y'y\}$. Then $\mathscr{A}' \subseteq \{xy, x'y'\} \cup \{A \in \binom{[n]}{2} : y' \in A\}$, giving the bound $|n(\mathscr{A}')| \geq |\mathscr{A}'|$. This completes the proof of the claim. ◇

Claim 5.1.9. *If $\mathscr{A}'$ has a pair of disjoint edges, and $|\mathscr{B}'| = 1$, then $n(\mathscr{A}') - |\mathscr{A}'| \geq -(|S|+1)$.*

Proof. Let $\{xy, x'y'\}$ be a pair of disjoint edges in $\mathscr{A}'$, and, wlog, let $\mathscr{B}' = \{xx'\}$. Let $\mathscr{A}'_x = \{A \in \mathscr{A}' : x \in A\}$, and let $\mathscr{A}'_{x'} = \{A \in \mathscr{A}' : x' \in A\}$. Let $X = \{v \in [n] : v \neq x', xv \in \mathscr{A}_x\}$, $X' = \{v \in [n] : v \neq x, x'v \in \mathscr{A}_{x'}\}$ and $R = X \cap X'$. Now, $|\mathscr{A}'| \leq 2|R| + |X \setminus R| + |X' \setminus R| + 1$, and $n(\mathscr{A}') = 2 + |R| + |X \setminus R| + |X' \setminus R|$. So, $n(\mathscr{A}') - |\mathscr{A}'| \geq -(|R|+1)$. Since $|R| \leq |S|$ (otherwise, R would be a bigger sunflower with core $\{a,x\}$ (or $\{a,x'\}$), contradicting the choice of S), we have $n(\mathscr{A}') - |\mathscr{A}'| \geq -(|S|+1)$. ◇

In the next claim, we give lower bounds on the sizes of $\mathscr{F}_a$ and $\mathscr{F}_b$.

Claim 5.1.10.

- $|\mathscr{F}_a| \geq 1 + (|S| + n(\mathscr{A}') + 1) + (|S| + |\mathscr{A}'|)$.
- $|\mathscr{F}_b| \geq 1 + (|S| + n(\mathscr{B}') + 1) + (|S| + |\mathscr{B}'|)$.

Proof. We will only give the proof for $\mathscr{F}_a$, as the proof for $\mathscr{F}_b$ follows identically. We know that $|\mathscr{F}_a| = \sum_{i=1}^{3} |\mathscr{F}_a^i|$, where $\mathscr{F}_a^i = \mathscr{F}_a \cap \binom{[n]}{i}$ for $i \in \{1,2,3\}$. It is trivial to note that $|\mathscr{F}_a^1| = 1$. Now, consider $\mathscr{F}_a^2$. First, $\{a,b\} \in \mathscr{F}_a^2$. Also, for every $\{a,b,s\} \in S$, $\{a,s\} \in \mathscr{F}_a^2$, as $\mathscr{F}$ is a downset. Similarly, for every $s \in V(\mathscr{A}')$, there exists a $t \in V(\mathscr{A}')$ such that $\{a,s,t\} \in \mathscr{I}_3$, and hence, $\{a,s\} \in \mathscr{F}_a^2$. Thus, $|\mathscr{F}_a^2| \geq |S| + n(\mathscr{A}') + 1$. Also, it is not hard to see that $|\mathscr{F}_a^3| \geq |S| + |\mathscr{A}'|$. This completes the proof of the claim. ◇

We will now prove that either $\mathscr{F}_a$ or $\mathscr{F}_b$ is bigger than $\mathscr{I}$, which will complete the proof of the theorem. It will be sufficient to prove the following claim.

Claim 5.1.11. $|\mathscr{F}_a| + |\mathscr{F}_b| > 2(|\mathscr{I}_3| + 3)$.

Proof. We will consider two cases, depending on whether or not the hypothesis of Claim 5.1.8 is true. Suppose the hypothesis of Claim 5.1.8 holds, so we have $n(\mathscr{X}) - |\mathscr{X}| \geq -2$, for $\mathscr{X} \in \{\mathscr{A}', \mathscr{B}'\}$. Thus, since $|S| > 3$, we have

$$\begin{aligned} |\mathscr{F}_a| + |\mathscr{F}_b| &\geq 4 + 4|S| + |\mathscr{A}'| + |\mathscr{B}'| + n(\mathscr{A}') + n(\mathscr{B}') \\ &= (2|S| + 2|\mathscr{A}'| + 2|\mathscr{B}'| + 6) + (n(\mathscr{A}') - |\mathscr{A}'|) + (n(\mathscr{B}') - |\mathscr{B}'|) + 2|S| - 2 \\ &\geq 2(|\mathscr{I}_3| + 3) + (2|S| - 6) \\ &> 2(|\mathscr{I}_3| + 3). \end{aligned}$$

Now, assume the hypothesis of Claim 5.1.8 is false, so, without loss of generality, suppose $\mathscr{A}'$ has a pair of disjoint edges, and $|\mathscr{B}'| = 1$. Clearly, $n(\mathscr{B}') - |\mathscr{B}'| = 1$ and we can use Claim 5.1.9 to conclude that $n(\mathscr{A}') - |\mathscr{A}'| \geq -(|S| + 1)$. Thus, we have

$$\begin{aligned} |\mathscr{F}_a| + |\mathscr{F}_b| &\geq 4 + 4|S| + |\mathscr{A}'| + |\mathscr{B}'| + n(\mathscr{A}') + n(\mathscr{B}') \\ &\geq (2|S| + 2|\mathscr{A}'| + 2|\mathscr{B}'| + 6) - (|S| + 1) + 1 + 2|S| - 2 \\ &\geq 2(|\mathscr{I}_3| + 3) + |S| - 2 \\ &> 2(|\mathscr{I}_3| + 3). \end{aligned}$$

$\diamond$

$\square$

It would be nice to be able to prove a more general version of this theorem, for any k, with sufficiently large n, along the following lines.

Conjecture 5.1.12 (Hurlbert–Kamat–Mubayi)**.** *Let $\mathscr{F} \subseteq \binom{[n]}{\leq k}$ be a hereditary family, and let $\mathscr{I} \subseteq \mathscr{F}$ be a maximum intersecting family. Then, there exists an $c_0(k)$ such that if $|\mathscr{I}| > c_0(k)$, then $\mathscr{I}$ is a star.*

It is clear, though, that the mostly ad-hoc methods used in the proof of Theorem 5.1.7 would most likely not work for the general case, and some new ideas would be needed.

5.2 Families of Cycle-Free Subsets of Graphs

Next, we aim to continue our investigation of the Erdős–Ko–Rado properties of hereditary families of vertex sets of certain graphs, along the lines of Chapter 2. In Chapter 2, we considered the families of independent vertex sets of graphs, and proved EKR-type theorems for certain classes. In this section, we consider a similar problem, but for different types of vertex sets of a graph G, namely sets which induce a forest in G; in other words, cycle-free vertex subsets of graph G. We begin with some notation and definitions.

Let $\mathscr{A}_G = \{A \subseteq V(G) : A$ induces a forest $\}$, $\mathscr{A}_G^r = \{A \in \mathscr{A}_G : |A| = r\}$, and for any $x \in V(G)$, let $\mathscr{A}_G^r(x) = \{A \in \mathscr{A}_G^r : x \in A\}$, called a star of $\mathscr{A}_G^r$, centered at vertex x. We say that $\mathscr{A}_G^r$ is EKR if no intersecting subfamily of $\mathscr{A}_G^r$ is larger than the largest star of $\mathscr{A}_G^r$.

In this preliminary work, we will prove a theorem which states that $\mathscr{A}_G^r$ is EKR when G is disjoint union of complete graphs satisfying some additional conditions. Call a complete graph K_s trivial if $s \in \{1,2\}$. If H is subgraph of G, we let $G-H$ denote the graph obtained by removing the vertex set $V(H)$ from $V(G)$. Finally, if $\mathscr{A}$ is a family of sets such that there is a set S which is a subset of every set in $\mathscr{A}$, let $\mathscr{A} - S = \{A \setminus S : A \in \mathscr{A}\}$. We are now ready to state and prove our result.

Theorem 5.2.1. *Let G be a disjoint union of n complete graphs such that there is at least one trivial component and each non-trivial component has order at least* 5. *If $r \leq n/2$, then $\mathscr{A}_G^r$ is EKR.*

Before we proceed further, we note for the graphs G in Theorem 5.2.1, each

$A \in \mathscr{A}_G^r$ contains at most 2 vertices from each component of G; in other words, A can be represented as a union of an independent set and a matching. This is a key property that we will exploit in the proof.

Proof. We do induction on r and $|V(G)|$. The statement is trivial when $r = 1$, so assume $r \geq 2$. Suppose that $r = 2$. If G is a disjoint union of n complete graphs, with $n \geq 4$, then $|V(G)| \geq 4 = 2r$, and every vertex subset of size 2 in G will induce a forest, so we are done by the Erdős–Ko–Rado theorem. Consequently, we'll assume $r \geq 3$, and suppose G is a disjoint union of at least $n \geq 2r \geq 6$ complete graphs. If G is the disjoint union of n trivial graphs, every vertex subset of size r will be a member of $\mathscr{A}_G^r$, so by the assumption $n \geq 2r$, we are again done by the Erdős–Ko–Rado theorem. So suppose there exists some non-trivial component K_s, with $s \geq 5$, with $V(K_s) = [s]$. Let $\mathscr{A} \subseteq \mathscr{A}_G^r$ be an intersecting family. Finally, choose a vertex $x \in V(G)$ from any trivial component of G. Note that $|\mathscr{A}_G^r(x)|$ will be the same for any choice of x, since every trivial component is acyclic.

For $1 \leq i < j \leq n$, we define the shift operator $f_{i,j}$ as follows.

$$f_{i,j}(A) = \begin{cases} A \setminus \{j\} \cup \{i\} & \text{if } j \in A, i \notin A, A \setminus \{j\} \cup \{i\} \notin \mathscr{A}, \text{ and} \\ A & \text{otherwise.} \end{cases}$$

Let $f_{i,j}(\mathscr{A}) = \{f_{i,j}(A) : A \in \mathscr{A}\}$. Note that $|f_{i,j}(\mathscr{A})| = |\mathscr{A}|$. We also know from the properties of shifting that if $\mathscr{A}$ is r-uniform and intersecting, then $f_{i,j}(\mathscr{A})$ is also r-uniform and intersecting. Moreover, since we do shifting only on the vertices of K_s, $f_{i,j}(\mathscr{A}) \subseteq \mathscr{A}_G^r$. Hence we may assume that $\mathscr{A}$ is shifted, i.e. for any $i < j$, $f_{i,j}(\mathscr{A}) = \mathscr{A}$. We now partition $\mathscr{A}$ as follows. Let $\mathscr{A}_0 = \{A \in \mathscr{A} : A \cap [s] = \emptyset\}$, and for each $1 \leq i \leq s$, $\mathscr{A}_i = \{A \in \mathscr{A} : A \cap [s] = \{i\}\}$. Also, for each $1 \leq i < j \leq s$, let $\mathscr{A}_{i,j} = \{A \in \mathscr{A} : A \cap [s] = \{i, j\}\}$. It is clear that the disjoint union $\mathscr{A}_0 \cup \mathscr{A}_1 \subseteq \mathscr{A}_{G-[2,s]}^r$, and is intersecting.

Now, we consider the families $\mathscr{A}_{1,4}$ and $\mathscr{A}_{2,3}$ (recall that $s \geq 5$), and define the following *double shifting* operation, with respect to these two families.

$$g(A) = \begin{cases} A \setminus \{2,3\} \cup \{1,4\} & \text{if } A \in \mathscr{A}_{2,3}, A \setminus \{2,3\} \cup \{1,4\} \notin \mathscr{A}, \text{ and} \\ A & \text{otherwise.} \end{cases}$$

Let $g(\mathscr{A}_{2,3}) = \{g(A) : A \in \mathscr{A}_{2,3}\}$, and let $\mathscr{F} = \mathscr{A}_{1,4} \cup g(\mathscr{A}_{2,3})$. We now partition $\mathscr{F}$ into $\mathscr{F}_{1,4} = \{A \in \mathscr{F} : \{1,4\} \in A\}$ and $\mathscr{F}_{2,3} = \mathscr{F} \setminus \mathscr{F}_{1,4}$. Clearly, $\mathscr{F}_{2,3} \subseteq \mathscr{A}_{2,3}$ and is an intersecting family. Consider $\mathscr{D}_{2,3} = \mathscr{F}_{2,3} - \{2,3\}$. $\mathscr{D}_{2,3}$ is also an intersecting family, since for $D_1, D_2 \in \mathscr{D}_{2,3}$, $(D_1 \cup \{1,4\}) \cap (D_2 \cup \{2,3\}) \neq \emptyset$. Hence $D_{2,3} \subseteq \mathscr{A}^{r-2}_{G-[s]}$.

For each $2 \leq j \leq s-1$, $j \neq 4$, let $\mathscr{C}_j = (\mathscr{A}_{1,j} - \{j\}) \cup (\mathscr{A}_{j+1} - \{j+1\})$. If $j = s$, let $\mathscr{C}_s = (\mathscr{A}_{1,s} - \{s\}) \cup (\mathscr{A}_2 - \{2\})$. If $j = 4$, we let $\mathscr{C}_4 = (\mathscr{F}_{1,4} - \{4\}) \cup (\mathscr{A}_5 - \{5\})$. It is clear that for each i, $\mathscr{C}_i$ is a union of two disjoint families, and $\mathscr{C}_i \subseteq \mathscr{A}^{r-1}_{G-[2,s]}$. We also claim that each $\mathscr{C}_i$ is intersecting.

Claim 5.2.2. *For each $2 \leq i \leq s$, $\mathscr{C}_i$ is an intersecting family.*

Proof. First consider the case when $i \neq 4$. The family $\mathscr{A}_{1,i} - \{i\}$ is intersecting, since 1 lies in every set of the family. Consider the family $\mathscr{A}_{i+1} - \{i+1\}$ (Replace $i+1$ by 2 if $i = s$). Since $\mathscr{A}$ is a shifted family, for any $A \in \mathscr{A}_{i+1} - \{i+1\}$, $A \cup \{1\} \in \mathscr{A}$. Thus, for $A, B \in \mathscr{A}_{i+1} - \{i+1\}$, $A \cup \{i+1\}, B \cup \{1\} \in \mathscr{A}$, so $A \cap B \neq \emptyset$, as $i+1 \neq 1$. So suppose $A \in \mathscr{A}_{1,i} - \{i\}$ and $B \in \mathscr{A}_{i+1} - \{i+1\}$. Then $A \cup \{i\}, B \cup \{i+1\} \in \mathscr{A}$, giving $A \cap B \neq \emptyset$.

We now turn our attention to the case $i = 4$. As before, the family $\mathscr{F}_{1,4} - \{4\}$ is intersecting, so consider $\mathscr{A}_5 - \{5\}$. Since $\mathscr{A}$ is shifted, for any $A \in \mathscr{A}_5 - \{5\}$, $A \cup \{1\} \in \mathscr{A}$. Consequently, for $A, B \in \mathscr{A}_5 - \{5\}$, $A \cup \{5\}, B \cup \{1\} \in \mathscr{A}$, giving $A \cap B \neq \emptyset$. So assume $A \in \mathscr{F}_{14} - \{4\}$, and $B \in \mathscr{A}_5 - \{5\}$. If $A \in \mathscr{A}_{1,4} - \{4\}$, then $A \cup \{4\}, B \cup \{5\} \in \mathscr{A}$, implying $A \cap B \neq \emptyset$. So suppose not. Then we have

$A' = A \setminus \{1\} \cup \{2,3\} \in \mathscr{A}_{2,3}$. Thus $A' \cap (B \cup \{5\}) \neq \emptyset$, giving $A' \cap B \neq \emptyset$, since $5 \notin A'$. This implies $A \cap B \neq \emptyset$, as $B \cap [s] = \emptyset$. $\diamond$

We now consider the remaining $\mathscr{A}_{i,j}$'s, for $i \neq 1$. For any $i < j$, $i \neq 1$, and $\{i,j\} \neq \{2,3\}$, consider the families $\mathscr{D}_{i,j} = \mathscr{A}_{i,j} - \{i,j\}$. Clearly, $\mathscr{D}_{i,j} \subseteq \mathscr{A}^{r-2}_{G-[s]}$, for every i,j pair. We now show that all these families are also intersecting.

Claim 5.2.3. *$\mathscr{D}_{i,j}$ is intersecting, for all $i < j$, $i \neq 1$ and $\{i,j\} \neq \{2,3\}$.*

Proof. Suppose $A,B \in \mathscr{D}_{i,j}$. Since $\{i,j\} \neq \{2,3\}$ and $i \neq 1$, there is some $i' < i$ and $j' < j$ and $\{i,j\} \cap \{i',j'\} \neq \emptyset = \emptyset$. Since $\mathscr{A}$ is a shifted family, $A \cup \{i',j'\} \in \mathscr{A}$. Thus, since $(A \cup \{i',j'\}), (B \cup \{i,j\}) \in \mathscr{A}$, $A \cap B \neq \emptyset$. $\diamond$

Using the two claims, and the two induction hypotheses, we can complete the argument as follows. Recall that we have fixed $x \in V(G)$ to be some vertex in a trivial component of G.

$$
\begin{aligned}
|\mathscr{A}| &= \sum_{i=0}^{s} |\mathscr{A}_i| + \sum_{i,j \in [s], i<j} |\mathscr{A}_{i,j}| \\
&= |\mathscr{A}_0 \cup \mathscr{A}_1| + \sum_{i=2}^{s} |\mathscr{C}_i| + \sum_{i,j \in [2,s], i<j} |\mathscr{D}_{i,j}| \\
&\leq |\mathscr{A}^r_{G-[2,s]}(x)| + (s-1)|\mathscr{A}^{r-1}_{G-[2,s]}(x)| + \binom{s-1}{2} |\mathscr{A}^{r-2}_{G-[s]}(x)| \\
&= |\mathscr{A}^r_G(x)|.
\end{aligned}
$$

The last equality can be explained by partitioning $\mathscr{A}^r_G(x)$ into three parts: all sets containing no elements from $[2,s]$, all sets containing exactly one element from $[2,s]$, and all sets containing exactly 2 elements from $[2,s]$. $\square$

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