

Monotonicity and Manipulability of Ordinal and Cardinal Social Choice Functions

by

Andrew Jennings

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

ARIZONA STATE UNIVERSITY

December 2010

Monotonicity and Manipulability of Ordinal and Cardinal Social Choice Functions

by

Andrew Jennings

has been approved

July 2010

Graduate Supervisory Committee:

Glenn Hurlbert, Co-Chair

Hélène Barcelo, Co-Chair

Michel Balinski

Rida Laraki

Don Jones

ACCEPTED BY THE GRADUATE COLLEGE

ABSTRACT

Borda's social choice method and Condorcet's social choice method are shown to satisfy different monotonicities and it is shown that it is impossible for any social choice method to satisfy them both. Results of a Monte Carlo simulation are presented which estimate the probability of each of the following social choice methods being manipulable: plurality (first past the post), Borda count, instant runoff, Kemeny-Young, Schulze, and majority Borda. The Kemeny-Young and Schulze methods exhibit the strongest resistance to random manipulability. Two variations of the majority judgment method, with different tie-breaking rules, are compared for continuity. A new variation is proposed which minimizes discontinuity.

A framework for social choice methods based on grades is presented. It is based on the Balinski-Laraki framework, but doesn't require aggregation functions to be strictly monotone. By relaxing this restriction, strategy-proof aggregation functions can better handle a polarized electorate, can give a societal grade closer to the input grades, and can partially avoid certain voting paradoxes. A new cardinal voting method, called the linear median is presented, and is shown to have several very valuable properties. Range voting, the majority judgment, and the linear median are also simulated to compare their manipulability against that of the ordinal methods.

ACKNOWLEDGEMENTS

I thank my family, especially my wife, Rebekah, for seemingly unending support of my academic activities.

I thank Glenn Hurlbert for his flexibility in working with me on pursuing a degree in this unique subject, for taking the time to delve into social choice and mentor my research despite the intense demands on his time.

I thank H el ene Barcelo for an amazing ability to orchestrate and facilitate my studies while on leave at the Mathematical Sciences Research Institute in Berkeley, California.

I thank Michel Balinski and Rida Laraki for inviting me to be involved with their research and proofread their forthcoming book which should greatly advance the science of social choice, also for making possible a trip to France to study under their tutelage. I thank Ecole Polytechnique and G.I.S. Sciences de la D ecision for providing financing for that trip.

TABLE OF CONTENTS

	Page
TABLE OF CONTENTS	v
LIST OF TABLES	viii
LIST OF FIGURES	ix
1 INTRODUCTION	1
1.1 History	1
1.2 Arrow's Impossibility Theorem	2
1.3 Organization of this dissertation	4
1.4 Notations	5
I MONOTONICITY AND MANIPULABILITY OF EXISTING SOCIAL CHOICE	
METHODS	7
2 MONOTONICITY IN TRADITIONAL SOCIAL CHOICE	8
2.1 Monotonocities	8
Winner-monotonicity	8
Choice-monotonicity	9
Rank-monotonicity	10
Rank-order-monotonicity	11
2.2 Incompatibility	12
2.3 Conclusion	14
3 MANIPULATION IN TRADITIONAL SOCIAL CHOICE	15
3.1 Random manipulability	16
3.2 Voter manipulability	17
3.3 The Borda Majority method	18
3.4 Simulations	19
3.5 Conclusion	19
4 SOCIAL CHOICE BASED ON GRADING	22
4.1 Existing methods	22

Chapter	Page
4.2 Arrow's theorem and cardinal voting	23
4.3 Majority judgment tie-breaking rules	24
4.4 Conclusion	30
II WEAK MONOTONICITY AND THE LINEAR MEDIAN	32
5 AGGREGATION FUNCTIONS AND STRATEGIC MEDIANS	33
5.1 Aggregation functions	33
5.2 The linear median	34
5.3 Strategic medians	34
5.4 Characterizing strategy-proof aggregation-functions	35
5.5 Conclusion	40
6 EVALUATING GRADING CURVES WITH THE EUCLIDEAN NORM	42
6.1 Notes on the grading language	42
6.2 Minimizing input-output distance	43
6.3 Minimizing distance in the strategy-proof space	46
6.4 Interpretation of G_p	46
6.5 Proof of theorem 18	47
6.6 Example	49
6.7 Uniform Distributions	50
6.8 Non-monotone G_p functions	50
6.9 Distribution of distributions	54
6.10 Conclusions	54
7 EVALUATING WITH OTHER NORMS	56
7.1 Minimizing the input-output distance for various norms	56
7.2 Manipulability	59
7.3 Conclusion	65
8 CARDINAL SYSTEMS IN A COMPETITIVE CONTEXT	67
8.1 The no-show paradox	68
8.2 Random and voter manipulability	69

Chapter	Page
8.3 Results	70
8.4 Conclusion	72
BIBLIOGRAPHY	73
A MANIPULABILITY SIMULATION DATA	76
A.1 Random manipulability - Ordinal Systems	76
A.2 Voter manipulability - Ordinal Systems	77
A.3 Random manipulability - Cardinal Systems	78
A.4 Voter manipulability - Cardinal Systems	79

LIST OF TABLES

Table	Page
1.1 Social decision method criteria chart	3
4.1 Discontinuity example for standard tie-breaking rule	26
4.2 Discontinuity example for Gale's tie-breaking rule	27

LIST OF FIGURES

Figure	Page
3.1 Manipulability of different social choice methods	20
4.1 Standard tie-breaking rule for majority judgment	25
4.2 Gale's tie-breaking rule for majority judgment	26
4.3 Suggested tie-breaking rule for majority judgment	27
4.4 Discontinuity of Gale's rule at grade transitions	28
4.5 Continuity of the standard rule at the grade transitions	29
4.6 Continuity of the suggested rule in the interior and at the transitions	30
6.1 A sample probability function and its grading function	50
6.2 Five piecewise uniform distributions and their corresponding G_p functions . . .	51
6.3 Error functions for a piecewise uniform distribution for different numbers of voters	53
8.1 Manipulability of cardinal voting systems.	71

Chapter 1

INTRODUCTION

1.1 History

Decision theory, or social choice, is the study of all mechanisms by which a group of people can come to a collective decision from individual preferences. In traditional social choice, the only input to the decision-making mechanism is each individual's rank ordering of the possible alternatives, and the output of any decision making mechanism is a "societal" rank ordering meant to represent the aggregated wishes of the group.

An example of a well-known and straightforward social decision method is the *Borda Count* [5], where the decision among n alternatives is made by assigning n points to each individual's favorite alternative, $n - 1$ points to each person's second favorite, etc., down to 1 point assigned to each person's least favorite. The points for each alternative are summed and the alternatives are put in descending order of the total number of points received. This method was popularized in the late eighteenth century, by Jean-Charles, chevalier de Borda.

A contemporary of Borda, Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet, was of the opinion that the societal rank ordering should match the majority rule as much as possible. That is, for every pair of alternatives, whichever alternative was preferred by the most people should be preferred in the societal rank ordering. Condorcet was critical of Borda's method for going against the majority rule in many circumstances, but he also noted that this criterion is not always able to be satisfied; there are cases where alternative A is preferred to alternative B by a majority, B is preferred to C by a majority and C to A by a majority. This situation is called Condorcet's paradox. The Condorcet criterion for social decision methods says that if there is one alternative that is preferred by a majority to every other alternative, then it should be selected as the best alternative in the societal ranking. Condorcet suggested a social decision method¹ consistent with this criterion²: Points

¹See [7] and [16]

²Also known as the Kemeny-Young method [10]

are awarded to all possible societal rankings in accordance with the number of people who agree with each of the pairwise outcomes in the ranking. These points are summed and the ranking with the highest score is declared the societal ranking.

Since the framework of social choice allows any social decision mechanism imaginable, the space of possible mechanisms is rather large³. There are some notably degenerate social decision functions. A constant social decision function would always yield the same societal ranking regardless of what the individual preferences were. A *dictatorship* is a social decision function that prefers one individual and always chooses his ranking of the alternatives as the societal ranking.

1.2 Arrow's Impossibility Theorem

Here are some standard criteria for evaluating social decision methods:

- *Impartiality to individuals* - If the preferences of two individuals are exchanged, the societal ranking should not change.
- *Impartiality to alternatives* - If two alternatives exchanged position in every individual's preference order, the only change to the societal ranking should be those two alternatives exchanging positions.
- *Pairwise unanimity* - For each pair of alternatives, if one alternative is preferred to the other by every individual, then that must be true in the societal ranking as well.
- *Independence of irrelevant alternatives* - Which alternative of any pair is preferred in the societal ranking must not change when any individual moves a third alternative (not in the pair) to a different place in his ranking (leaving the rest of the ranking unchanged).

The impartiality criteria are natural requirements for any system that is used in practice, but many times are omitted in theory because they are overly restrictive. Pairwise una-

³Define \mathcal{N} as the set of all possible rankings of the alternatives. There are $n!$ when only strict rank orderings are considered, more if ties are allowed. The possible social decision mechanisms are all functions from \mathcal{N}^m to \mathcal{N} .

nimity is a very weak condition which ensures the societal ranking has some relationship to the individuals' preferences. Any sensible social decision method will satisfy pairwise unanimity. Independence of irrelevant alternatives (or IIA) is a very desirable criterion for a social decision method; the societal decision between two alternatives should not change merely when opinions shift about other alternatives.

Table 1.1 shows how well a selection of social decision methods satisfy these criteria.

Method	Impartial to individuals	Impartial to alternatives	Pairwise unanimous	IIA
Borda	Y	Y	Y	N
Condorcet	Y	Y	Y	N
Constant fn	Y	N	N	Y
Dictatorship	N	Y	Y	Y

Table 1.1: Social decision method criteria chart

Despite being extremely desirable, independence of irrelevant alternatives is extremely difficult to satisfy: any social decision method that is pairwise unanimous and IIA must be a dictatorship. This is Kenneth Arrow's renowned impossibility theorem [1].

Theorem 1. (Arrow) *If a social choice method for three or more alternatives returns a societal rank-ordering for every possible combination of input rank-orderings, and if it is pairwise unanimous and independent of irrelevant alternatives, then it is a dictatorship.*

Another important and well-studied criterion that is impossible to satisfy is that a social decision method should not allow *manipulation* or *strategic voting*—that is, an individual should not be able to cause the top-ranked alternative to switch to an alternative he prefers by changing his ranking away from his honest one to something else. This is known as the Gibbard-Satterthwaite theorem [9] [13].

Theorem 2. (Gibbard-Satterthwaite) *If a social choice method for three or more alternatives is not a dictatorship and for every alternative there is some profile of rankings that could cause that alternative to win, then there exists some profile where some individual*

could, by submitting a vote that does not indicate his true preferences, alter the top-ranked alternative to one he prefers more.

1.3 Organization of this dissertation

Part I of this work examines the concepts of monotonicity and manipulability in well-known social choice methods from several perspectives. Chapters 2 and 3 take place within the traditional social choice framework, where each individual submits a ranking of the alternatives. In chapter 2, we explore the concept of monotonicity in this framework, which generally means that no negative effects should come to a candidate when one or more individuals move that candidate up in their rankings. Another type of monotonicity, for rankings, is introduced. It is proved that Borda's method and Condorcet's method satisfy different monotonicities and that it is impossible for any method to satisfy them both. In chapter 3, we examine the manipulability of several social choice functions, including a relatively new method, the majority Borda method[3]. This method is expected to have some strong manipulability-resistance properties, but it has not been widely studied. The manipulability of the different methods is analyzed from a probability standpoint, with the main original result being some Monte Carlo simulations which give concrete manipulability scores for these methods. The majority Borda method does indeed fare well in some circumstances, though Condorcet's method fares the best in general. Chapter 4 examines an alternative social choice system, based on individuals submitting grades or evaluations of each alternative instead of ranking them. Some existing social choice methods within this framework are approval voting, score voting, and the majority judgment. Several variations of the majority judgment, including a proposed new variant, are analyzed for continuity.

In part II, we introduce a new framework for the analysis of social choice methods based on grades. It is a variation of the Balinski-Laraki[4] framework, but relaxes one key requirement and thus admits many more possible ways of aggregating individual grades into a societal grade. In chapter 5, we provide a new characterization of all aggregation functions that are resistant to strategic voting. A new method, called the linear median, is introduced which has several very valuable aggregation properties. Chapter 6 explores

how to choose the aggregation function that will minimize the distance between the voter input grades and the aggregated output grade. The linear median is shown to be the optimal aggregation function for this purpose in certain circumstances. Chapter 7 extends this analysis to other norms. It also examines which aggregation functions will minimize the influence that a single voter can have on the election outcome. The linear median is shown to be the unique aggregation function in its class that can minimize the influence of a single voter, as measured with the uniform norm (L^∞). In chapter 8, we examine the no-show paradox and how it affects the majority judgment and the linear median. Additionally, three of the social choice functions that are based on grading, including the linear median, are examined and simulated to give concrete manipulability scores which are compared to the ordinal functions computed in chapter 3.

In summary, part II aims to provide a comprehensive analysis of a new social choice framework. That analysis shows that the linear median and the majority judgment stand out as the most useful, sensible, and viable of all cardinal social choice methods. We promote further study of these methods and encourage their implementation in electoral situations large and small.

1.4 Notations

For alternatives in the abstract, we will generally use upper-case letters A, B, C , etc. .

To indicate that one or more individuals prefer A to B to C , we will use the notation

$$A \succ B \succ C.$$

To indicate individuals who are indifferent to A and B , we will indicate $A \approx B$. Two individuals who prefer A to B to C will be notated as

$$2 : A \succ B \succ C.$$

To indicate that a social choice method gives a societal outcome preferring A to B to C , we will use to subscripted relation \succ_S , as in $A \succ_S B \succ_S C$.

When we refer to “Condorcet’s method” we are referring to the social choice method generally known as Kemeny-Young, in accordance with the deduction in [16] that this was probably the method Condorcet had in mind in his writings.

In some places, where it seems more natural, we will refer specifically to the context of elections. That is, we will refer to alternatives as candidates and to individuals as voters. All results, however, apply to social choice situations in general, not merely to elections.

Part I

MONOTONICITY AND MANIPULABILITY OF EXISTING SOCIAL CHOICE
METHODS

Chapter 2

MONOTONICITY IN TRADITIONAL SOCIAL CHOICE

One of the most basic criteria used to analyze social choice mechanisms is monotonicity. It seems to be a basic tenet of fairness that when some voters move a candidate up in their rankings, he should not be negatively affected. When a social choice method allows a candidate to become worse off when they are ranked higher by some voters, the election outcomes are never without doubt. Additionally, such methods are susceptible to some of the most blatant forms of tactical voting, for there will be situations where voters are tempted to rank candidates lower in order to help them.

In this chapter we examine several different monotonicity criteria. We then turn to an idea of Peyton Young, that choosing a “best” societal ranking and choosing one “best” alternative are not fully compatible. We expand on Young’s results with some principles of monotonicity. The results in this section were published as a joint work with Michel Balinski and Rida Laraki in November 2009[2].

2.1 Monotoncities

Winner-monotonicity

The most common monotonicity considered in the literature is *winner-monotonicity*, which means that the winning candidate should still win if they were to be ranked higher by some of the voters (with no other changes to the voter preference profile). Borda’s method clearly satisfies winner-monotonicity, because moving the winner up in some rankings will increase his Borda score and will decrease or leave unchanged the scores of the other candidates, meaning that no candidate can overtake the winning candidate when he is moved up in ranking by some voters. Similarly, Condorcet’s method is winner-monotone because moving the winner up in one voter’s ranking will add exactly one point to every ranking which ranks that candidate over the candidate who was moved down and will subtract one point from every other ranking. In particular, every ranking which ranks that candidate first

will have its points increased by one, and no ranking will increase in points by more than one, so no ranking can overtake the winning ranking when the winner is moved up by one or more voters.

An example of a method that can fail winner-monotonicity is instant runoff voting. In a three winner race, for example, there are times when the winner being moved up in some voters' rankings will change which candidate is eliminated first and allow the former winner to be defeated by the other candidate. Here is a specific example:

$$8 : A \succ B \succ C \quad 2 : B \succ A \succ C \quad 5 : B \succ C \succ A \quad 6 : C \succ A \succ B.$$

C is eliminated in the first round and A defeats B by a score of 14 to 7 in the final round. If the two $B \succ A \succ C$ voters move A up so they become $A \succ B \succ C$ voters, then we have the following profile:

$$10 : A \succ B \succ C \quad 5 : B \succ C \succ A \quad 6 : C \succ A \succ B$$

where B is eliminated in the first round and then C defeats A by a score of 11 to 10 in the final round. Thus, with instant runoff voting, a candidate can lose the election by being ranked higher by some voters.

Choice-monotonicity

A more thorough notion of monotonicity is that moving any candidate up in ranking should never cause that candidate to lose to someone they defeated before they were moved up. Formally, we define *choice-monotonicity* to mean that if a profile of individual preferences yields a societal ranking with alternative A preferred to alternative B or A tied with B and then alternative A is moved to a higher rank or B is moved to a lower rank by one person, the societal ranking should (strictly) prefer A to B . Choice-monotonicity can be seen as a generalization of winner-monotonicity in that winner-monotonicity only requires this condition to hold for the winning candidate in the original profile. Choice-monotonicity requires that it be true for every candidate.

Borda's method also satisfies choice-monotonicity. Again, this is because moving a candidate up in ranking will strictly increase his Borda points, while decreasing or leaving

unaltered the points of every other candidate. Condorcet's method, on the other hand, is not choice-monotone. The following profile, \mathcal{P}_0 , shows the failure of Condorcet's method to satisfy choice-monotonicity:

$$4 : A \succ B \succ C \succ D \succ E$$

$$4 : B \succ E \succ C \succ A \succ D$$

$$2 : D \succ E \succ C \succ A \succ B$$

$$1 : D \succ C \succ E \succ A \succ B$$

$$1 : D \succ E \succ A \succ C \succ B.$$

Condorcet's method will give a two-way tie for first between the societal rankings

$$A \succ_S B \succ_S C \succ_S D \succ_S E \quad \text{and} \quad B \succ_S E \succ_S C \succ_S A \succ_S D,$$

so we can create profile \mathcal{P}_1 from \mathcal{P}_0 by having one of the voters who ranked C immediately above A swap them. This causes Condorcet's method to give a unique best societal ranking of $A \succ_S B \succ_S C \succ_S D \succ_S E$. We can also create profile \mathcal{P}_2 from \mathcal{P}_0 by having the one voter who ranked A immediately above C swap them, which yields a unique best societal ranking of $B \succ_S E \succ_S C \succ_S A \succ_S D$. Thus, moving from \mathcal{P}_1 to \mathcal{P}_2 is accomplished by having two voters move C up one rank, which causes C to become defeated by E in the societal ranking where C had defeated E before.

Rank-monotonicity

A different way to generalize winner-monotonicity is to require that when the winner is moved up in rank by some voters, the entire societal ranking should stay the same, not just the winner. This is the definition of *rank-monotonicity*. Formally, a social choice method is rank-monotone if whenever \mathcal{P}_0 is a profile that causes candidate A to win and \mathcal{P}_1 is identical to \mathcal{P}_0 except that A is moved up in some voters' rankings, then \mathcal{P}_1 should give the same societal ranking as \mathcal{P}_0 .

Unlike choice-monotonicity, it will be fairly rare for a social choice method to satisfy rank-monotonicity. Borda's method fails it, because raising the winner in some voters' rankings will decrease the score of any candidates who are moved down in the process, and

this will commonly cause a re-ordering of the non-winners in the societal ranking. In the following profile

$$3 : A \succ B \succ C \quad 2 : C \succ B \succ A,$$

the candidates A , B , and C receive 11, 10, and 9 Borda points respectively, giving the societal ranking $A \succ_S B \succ_S C$. If the two voters who rank A last were to raise him one position, this would reward A with two Borda points at the expense of B , giving a societal outcome of $A \succ_S C \succ_S B$

Condorcet's method, however, meets this criterion for the same reason that it satisfies winner-monotonicity: it assigns points to the various societal rankings, and moving the winner up in one voter's rankings will add one point to the winning societal ranking and can not add more than one points to any other ranking, so the winning societal ranking must remain the same.

Rank-order-monotonicity

Yet, another form of monotonicity is *rank-order-monotonicity*, which means that no candidate will move down in the societal ranking when he is moved up in the rankings of one or more voters. This is another criterion, like choice-monotonicity, that is identical to winner-monotonicity when it is applied only to the winner. In fact, rank-monotonicity is between choice-monotonicity and winner monotonicity. Any method that is choice-monotone is necessarily rank-order-monotone, and any method that is rank-order-monotone is necessarily winner-monotone. Although it appears to have more to do with ranking than pairwise comparison, rank-order-monotonicity is more closely related to choice-monotonicity than rank-monotonicity.

Borda's method satisfies rank-order-monotonicity, as it is choice-monotone, but Condorcet's method does not. Here is a counterexample which is very similar to the counterex-

ample for choice-monotonicity. It uses six candidates instead of five:

$$\begin{aligned}
4 &: A \succ B \succ C \succ D \succ E \succ F \\
4 &: B \succ E \succ F \succ C \succ A \succ D \\
1 &: F \succ D \succ E \succ C \succ A \succ B \\
1 &: F \succ D \succ A \succ C \succ E \succ B \\
1 &: D \succ C \succ E \succ A \succ F \succ B \\
1 &: D \succ E \succ F \succ C \succ A \succ B.
\end{aligned}$$

Condorcet's method will give a two-way tie for first between

$$A \succ_S B \succ_S C \succ_S D \succ_S E \succ_S F \quad \text{and} \quad B \succ_S E \succ_S F \succ_S C \succ_S A \succ_S D,$$

so we construct a pair of profiles \mathcal{P}_0 by having one voter move A above C and \mathcal{P}_1 by having one voter move C above A . Then this pair illustrates a failure of rank-order-monotonicity failure because two voters moving C above A will cause C to move from third to fourth in the societal ranking.

In [2], we prove that for Condorcet's method, it is impossible to find a counterexample for rank-order-monotonicity with five or fewer candidates. We also prove that finding a counterexample for choice-monotonicity in Condorcet's method is impossible with four or fewer candidates.

2.2 Incompatibility

It was observed by Peyton Young [15] that choosing a "best" societal ranking of all alternatives and choosing one "best" alternative are distinct, and not necessarily compatible, goals. For instance, an algorithm to produce a ranking of sports teams might reasonably choose the one that minimizes the number of upsets in the past season. Such an algorithm would not necessarily rank first the team that had the highest probability of defeating all other teams. In a probabilistic sense the "best" alternative that comes from the preferences of the individuals may be different than the top-ranked alternative in the "best" ranking that comes from those same preferences. Young showed that Condorcet's method was the opti-

mal method for choosing a societal ranking and Borda's method was optimal for choosing one alternative.

From the definitions above, we can examine Young's observation from a monotonicity perspective. If a social decision method is good for selecting a societal ranking, then it should be rank-monotone, because it seems that the societal ranking should be stable if some voters decide to move the winner up. Choice-monotonicity is desirable for a choosing a societal winner because it ensures relatively stable pairwise outcomes. As noted above Borda's method is choice- but not rank-monotone and that Condorcet's method is rank- but not choice-monotone.

Theorem 3. (*Balinski, Jennings, Laraki*) *There is no social choice method that is both rank- and choice-monotone that is also impartial to individuals and alternatives and respects unanimity (for at least three alternatives and at least two individuals).*

Proof. Let $2k + i$ equal the number of voters, with i either 0 or 1, and \mathcal{P} be the profile

$$\begin{aligned} k : A \succ B \succ C \succ A_1 \succ \dots \succ A_n & \quad k : B \succ C \succ A \succ A_1 \succ \dots \succ A_n \\ i : A \approx B \approx C \succ A_1 \succ \dots \succ A_n. \end{aligned}$$

By impartiality, the profile

$$\begin{aligned} k : B \succ A \succ C \succ A_1 \succ \dots \succ A_n & \quad k : B \succ C \succ A \succ A_1 \succ \dots \succ A_n \\ i : A \approx B \approx C \succ A_1 \succ \dots \succ A_n \end{aligned}$$

implies $A \approx_S C$. The profile \mathcal{P} is obtained when the first k voters move A above B . By choice-monotonicity, profile \mathcal{P} must imply $A \succ_S C$.

Similarly, the profile

$$\begin{aligned} k : A \succ B \succ C \succ A_1 \succ \dots \succ A_n & \quad k : B \succ A \succ C \succ A_1 \succ \dots \succ A_n \\ i : A \approx B \approx C \succ A_1 \succ \dots \succ A_n \end{aligned}$$

implies $A \approx_S B$ and changes into profile \mathcal{P} when the second group of voters move C above A . Thus the profile \mathcal{P} must imply $B \succ_S A$. Unanimity now determines the complete outcome for \mathcal{P} to be $B \succ_S A \succ_S C \succ_S A_1 \succ_S \dots \succ_S A_n$.

By rank-monotonicity, the profile

$$k : B \succ A \succ C \succ A_1 \succ \dots \succ A_n \quad k : B \succ C \succ A \succ A_1 \succ \dots \succ A_n$$

$$i : A \approx B \approx C \succ A_1 \succ \dots \succ A_n,$$

must imply the same outcome as \mathcal{P} , including $A \succ_S C$, which contradicts the earlier impartiality result ($A \approx_S C$) for this profile. \square

In [2], we give examples of social choice methods that are rank- and choice-monotone and respect unanimity if one is willing to give up either impartiality towards voters or impartiality towards candidates.

2.3 Conclusion

Monotonicity is one of the most basic properties of social choice systems that can be studied. Methods that fail the most basic type of monotonicity, which is winner-monotonicity, are manipulable and can be difficult to study mathematically. There are many possible ways to extend winner-monotonicity into different types of monotonicity that can be analyzed. Borda's method and Condorcet's method satisfy different monotonicities. That Condorcet's method satisfies rank-monotonicity and Borda's method fails it, while Borda's method but not Condorcet's satisfies choice-monotonicity, gives us further insight into Peyton Young's result that Condorcet's method is more appropriate for choosing a societal ranking and Borda's method is more appropriate for choosing one winner.

Chapter 3

MANIPULATION IN TRADITIONAL SOCIAL CHOICE

Significant complication is added to the analysis of social choice methods by the consideration of strategy. Many social choice methods which work well with voters that are honest can fail miserably if participants try to manipulate the results. For situations as consequential as political elections, we must assume that some voters will do everything within their power to affect the outcome. The tools of game theory can be of some help in studying strategy in social choice methods, but their application is limited where there is collaboration and elections tend to be highly collaborative environments. In addition, voters often have incomplete and outdated information, and it is unclear that elections in real-life situations ever reach equilibrium.

In an ideal social choice method, no voter would ever be able to benefit from submitting a rank-ordering of the candidates that was not honest. Then, without any incentive to be dishonest, voters could be instructed to vote honestly. After the election, all participants could trust that no group of voters took unfair advantage of the election. The voter opinions expressed could be assumed to be honest, allowing further analysis to be performed on the election results as well as providing sample data for understanding the statistical properties of an honest voter profile. Unfortunately, no such system exists. As indicated in chapter 1, the Gibbard-Satterthwaite theorem proved that there will always be cases where a voter can benefit by submitting a dishonest vote. Additionally, a social choice method which fails any of the monotonicities in the previous chapter, but especially if it fails winner-monotonicity, will be susceptible in some way to rewarding voters who vote dishonestly.

Even if the ideal social choice method that perfectly incentivized honest voting existed, there is another form of manipulation that would need to be considered: strategic nomination. Since Arrow's impossibility theorem proves that the outcome of the election can depend on which candidates are running, even those who have very little chance of winning, there will always be an incentive for political parties and other powerful groups to influence

whether or not certain candidates enter the race or drop out in an attempt to maximize their advantage, but a good voting system would minimize this effect to the maximum degree possible.

3.1 Random manipulability

Though it is an extremely important topic to study, manipulability has proven very difficult to actually quantify. Usually it has been studied using criteria, and social choice methods are evaluated as either satisfying a given criterion or not. Because of the variety of impossibility theorems that indicate the futility of trying to satisfy all theoretical criteria at once, it seems that it would be more useful to have a probabilistic model that indicated how often each social decision method was susceptible to different types of manipulability. The primary difficulty with a probabilistic model, however, is finding the right distribution over which to sum. Even when selecting the *honest* preferences of many voters, it is not clear which probability distribution should be used to select the preferences. A common way to solve this problem is to model one, two, or n -dimensional issue spaces with Euclidean geometries, randomly placing the candidates and the voters within the space and choosing a suitable norm to compute the rank-ordering for each voter from the distances to the candidates. As useful as these models are, they still seem unrealistic in many ways, and leave questions about the applicability of the results to real-life elections.

Once the honest profiles are selected, they should be adjusted to account for voter strategy. Especially when trying to quantify manipulability, one should be careful to simulate voters who are making an attempt to act strategically, but this brings with it a whole new set of complications. Voters have imperfect information. Some voters will choose not to strategize, and others will use suboptimal strategies. Groups of voters will collude even when it may be against their best interest, and they can consider both the past and the future when making their decisions about the current election. As a result of these difficulties, it becomes almost impossible to truly simulate strategic voters in a way that will be widely accepted as neutral and reasonable.

A recent paper by Friedgut, Kalai, and Nisan [8] takes a completely different approach.

It ignores all of these difficulties, not even attempting to simulate realistic honest preferences, and considers only the uniform distribution. Their measure of manipulability, which we call here *random manipulability*, calculates how likely it is that a voter in a random voter profile (chosen uniformly over all possible voter profiles), by changing his preference to a random dishonest ranking, will effect a profitable manipulation—that is, will change the winner of the election to someone he prefers more (in terms of his original ranking, which is assumed to be his honest ranking). It is unclear whether the authors intended their manipulability measure to be used in practice or only theoretically, but we use it later in this chapter to make actual empirical measurements of several different methods. That this manipulability measure makes no attempt to divine a proper probability distribution for the voting profiles may be seen as a strength instead of a weakness; if it is not realistic, at least it is not contrived. It won't tell us everything about the various social choice methods' manipulability, but surely it tells us something.

The main result of [8] is that elections can be manipulated “often”. Regardless of the social choice method used, there will be a voter who can manipulate the election with probability at least on the order of $1/n$.

3.2 Voter manipulability

We propose a companion measure to random manipulability called *voter manipulability* which calculates how likely it is for a voter in a random profile (again chosen uniformly over all possible voter profiles) to have any available rank-ordering to which he could change his vote that would change the winner to one he prefers more. In other words, once an honest voter profile is chosen, random manipulability is concerned with the probability that a selected voter might profit by changing his vote randomly, and voter manipulability measures the probability that he might profit by carefully choosing to which rank-ordering he should change his vote. For a specific number of voters and candidates the voter manipulability score of a method will always be greater than or equal to its random manipulability score. The ratio between the two manipulability scores can tell us something about how careful a voter must be when trying to manipulate, whether there are many available dishonest

votes that will benefit the voter or whether he needs to be very selective in choosing his manipulation.

There are several ways that these manipulability measures are imperfect. As mentioned above, they assume that the voters' honest preferences come from a uniform distribution, which is not the case for most real-life situations, and they assume that all voters except one maintain their honesty while only one voter changes his rank-order to a dishonest one. In addition, voter manipulability assumes that the manipulative voter has perfect knowledge of the rest of the voters' preferences and is allowed to act on that information. In spite of these shortcomings, we still feel that these two manipulability metrics have value, especially for their simplicity and transparency.

3.3 The Borda Majority method

In a forthcoming book, Balinski and Laraki suggest a social decision method called the *Borda majority method* [3]. The alternatives are assigned collections of points according to their positions in each individual's rank-ordering, as they are with the traditional Borda Count, but instead of summing those points, they are analyzed using the majority judgment method, where the sets are ordered by their median grades¹. The tie-breaking process, if there are ties, is discussed in chapter 4.

The median function is an example of a function that is *strategy-proof*, meaning that any voter who submitted a grade higher than the societal output grade would not be able to raise the societal grade if he could alter his submitted grade. Nor could any voter who submitted a grade lower than the societal output grade lower the societal grade by altering his submitted grade. This property is the basis for several valuable strategy-resistance properties of the majority judgment, though it does not imply that the majority judgment is perfectly resistant to strategy. Further, any strategy-resistance that the majority judgment does have will not necessarily transfer directly to the Borda majority method. Since the Borda majority method is based on rank orderings, it is not possible to raise one candidate's point value

¹If the number of grades is even, however, the lower middlemost value is used.

without lowering another. Still, it is theorized[3] that the Borda majority method will likely inherit some resistance to strategic voting. We desired to test this hypothesis.

3.4 Simulations

In order to collect data about the manipulability of various methods, a Monte Carlo simulation was performed. For each trial, each voter's rank ordering was chosen at random from the set of possible rank orders. The winner was computed for each social choice method. Any ties were broken lexicographically, so each method produced one unique, strict rank ordering.

Then, one voter was chosen and his vote was changed to another rank order at random. For each method, it was recorded whether or not this changed the winner to someone the voter liked better, in terms of his original rank order, for the random manipulability measurement.

Also, for the same voter profile and the same voter, all the other possible rank orders were tried. For each method, it was recorded whether or not there was any rank order that changed the winner to someone the voter preferred in his original ranking, for the voter manipulability measurement.

Simulations were run for 3, 4, and 6 candidates and 10, 32, 100, 320, and 1000 voters. Between sixty thousand and three million trials were run for each combination of candidates and voters. The data is provided in appendix A, and charts are shown in figure 3.1.

3.5 Conclusion

The most salient trend in this manipulability data is that the Condorcet methods, Schulze and Kemeny-Young, consistently have the best scores for both random manipulability and voter manipulability. It is also interesting to note that Borda's method performs well with respect to the random manipulability measure but fares poorly in voter manipulability. This indicates that the susceptibility of Borda's method to strategic voting depends heavily on the amount of information that the voters have. Borda's method should be considerably more strategy-resistant in situations where voters truly have no idea which candidates are

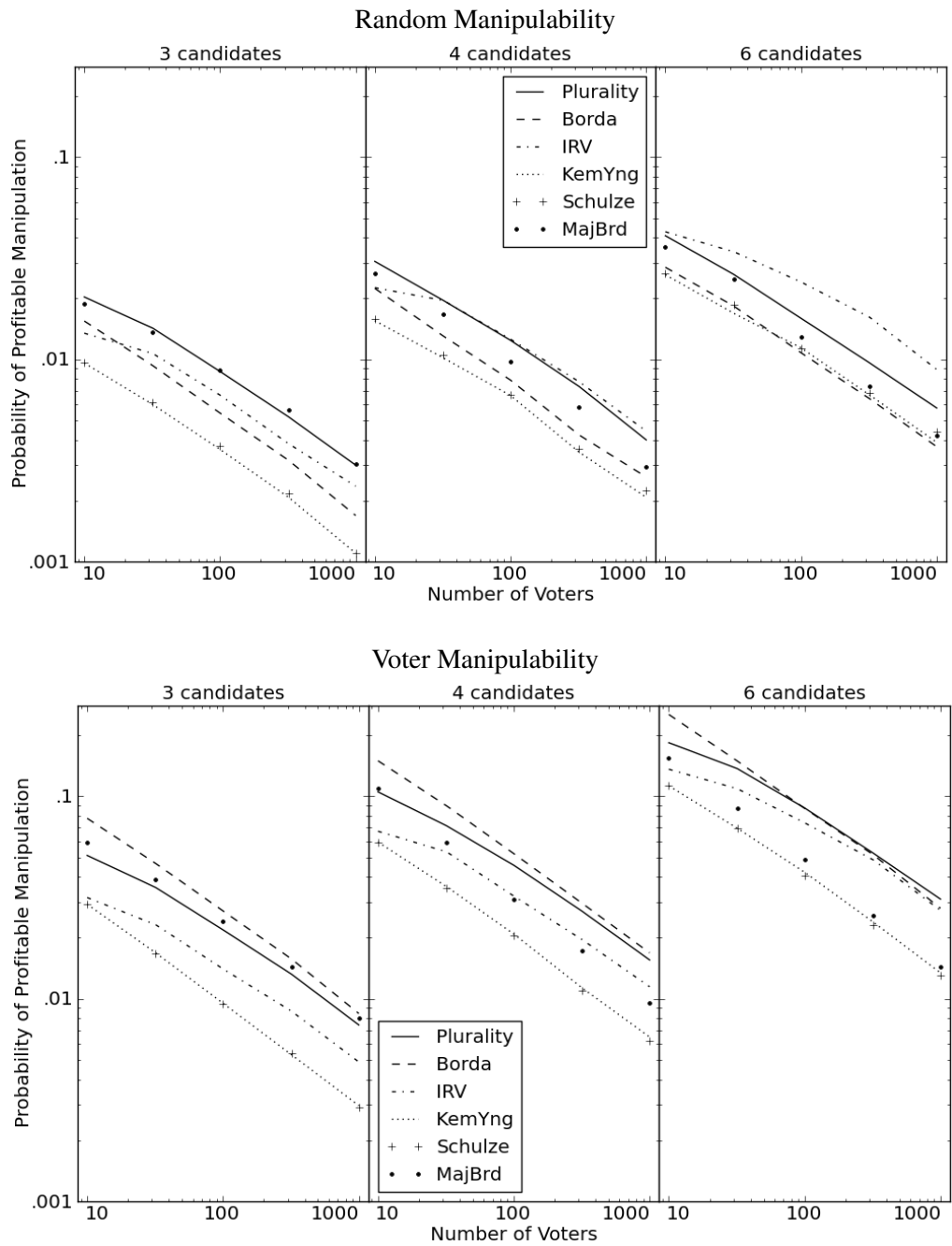


Figure 3.1: Manipulability of different social choice methods

popular and how others are voting than it is when voters are somewhat informed of who the front-runners are.

Majority Borda, the new method invented by Balinski and Laraki, seems to improve in strategy-resistance relative to other methods as the number of candidates increases. Indeed, it is the only one of the methods benchmarked which is competitive with the Condorcet methods in voter manipulability for 6 candidates and 1000 voters. It appears that more than four candidates are needed for the strategy-resistance properties of the median to have a significant effect.

Chapter 4

SOCIAL CHOICE BASED ON GRADING

Since 1978, social choice theorists have also studied mechanisms that are outside of the traditional framework. They don't limit the individual inputs to a rank ordering of the alternatives. Instead, they allow each individual to submit a grade for each alternative. Each alternative's set of grades is then aggregated separately, so it is impossible for the grades given to one alternative to affect another alternative's final grade.

4.1 Existing methods

- Approval voting [6] - Each individual submits a binary grade for each alternative, "approve" or "disapprove". Each alternative's aggregate grade is the fraction of individuals who approve it.
- Range voting [14] - Each individual submits a number from some pre-determined real interval as a grade for each alternative. Each alternative's aggregate grade is the arithmetic mean of its grades.
- Majority-judgment [4] - Each individual submits a grade from a pre-determined fully-ordered set (need not be numeric). Each alternative's aggregate grade is the median of its grades when the number of grades submitted is odd, and when the number of grades submitted is even, it is the lower of the two middlemost values. This number is called the *majority-grade*.

Each of these methods respects unanimity: if one alternative is graded strictly higher than another by all individuals, then it ends up with a higher aggregate grade. They are also independent of irrelevant alternatives in the sense that one or more people changing the grade of one alternative will affect neither the final grades nor the order of finish among the other alternatives. None of them is a dictatorship. Thus, Arrow's impossibility theorem is avoided in each case.

However, it is not entirely accurate to consider approval voting and range voting as pure cardinal methods. Much of the existing literature on approval voting assumes a context of ordinal voting, imagining the voter to have a preferred rank ordering of the candidates and needing to choose how many candidates he should approve to maximize his expected influence on the election outcome. If voters indeed behave this way, then it is possible for the addition or deletion of candidates (and for candidates fading in and out of legitimacy) to affect the voters' evaluations of the other candidates, and thus for approval voting to fail the IIA criterion. A similar complication occurs with range voting, where it is practically universal to assume that every voter should focus mostly on electable candidates and should re-scale their grades to utilize the full grading spectrum in order to avoid wasting voting power. This also ruins independence of irrelevant alternatives because the evaluations of all candidates may change when any candidate is added, deleted, or achieves perceived legitimacy.

In contrast, the inventors of the majority judgment system devote considerable attention to the importance of convincing the voters to evaluate the candidates independently.[3] While this ideal will probably never be fully achievable¹, it is the goal that we must aim for if we desire a social choice method that is truly independent of irrelevant alternatives.

4.2 Arrow's theorem and cardinal voting

It is instructive here to elaborate how these cardinal voting systems fit into the traditional ordinal framework. Technically, if we consider a version of Arrow's theorem that allows ties in the inputs and outputs of the social decision functions, then approval voting falls under the definition of a social decision function with a restricted domain. Each voter's approved set of candidates can be converted into a rank-order with two ranks: the approved candidates are all tied, and are preferred to the non-approved candidates, who are also all tied. By failing to meet the unrestricted domain criterion, approval voting can be simultaneously pairwise unanimous, non-dictatorial, and independent of irrelevant alternatives.

¹For each voter, there is probably someone in the world regarded so highly (or poorly) that were he to declare candidacy, the voter would adjust his other grades.

Range voting and the majority judgment, on the other hand, can not be forced to satisfy the traditional definition of a social decision function. Any voter preference profile from a range voting or majority judgment ballot can be converted into a rank-ordered voter profile if ties are allowed, but this mapping is not one-to-one, so a given rank-ordered voter profile usually maps back to multiple different cardinal voter profiles. Since each one of these cardinal profiles may have a different societal ranking outcome when resolved with a cardinal voting method (range voting or the majority judgment), it is impossible to classify and analyze these functions within the traditional social choice framework where each ordinal voter profile must result in at most one societal ranking.

4.3 Majority judgment tie-breaking rules

The remainder of this chapter will be devoted to an analysis of the tie-breaking rule of the majority-judgment. Balinski and Laraki suggest a tie-breaking rule to go with the majority-judgment [3]. In the case of ties, the majority-grade is removed from the set of grades (only one instance of it is removed if there are multiple), and the majority-grade of the new set is calculated. This is repeated until all ties are resolved.

This rule is sensible whether there are few individuals or many, but if there are many individuals then it can be characterized more efficiently. For grades $\bar{a} = (a_1, \dots, a_n)$, let $\alpha(\bar{a})$ be the fraction of grades strictly greater than the majority-grade and $\beta(\bar{a})$ be the fraction of grades strictly less than the majority-grade. Define

$$\gamma_0(\bar{a}) = \begin{cases} \alpha(\bar{a}), & \text{if } \alpha(\bar{a}) > \beta(\bar{a}) \\ -\beta(\bar{a}), & \text{if } \alpha(\bar{a}) \leq \beta(\bar{a}). \end{cases}$$

For two alternatives with the same majority-grade, ties are broken in favor of the one with the greater γ_0 value. (If the γ_0 values were the same, further tie breaking rounds would be necessary, but with many individuals, such ties are very rare.) γ_0 is shown in figure 4.1.

David Gale suggested an alternative tie-breaking rule [11],

$$\gamma_1(\bar{a}) = \alpha(\bar{a}) - \beta(\bar{a}),$$

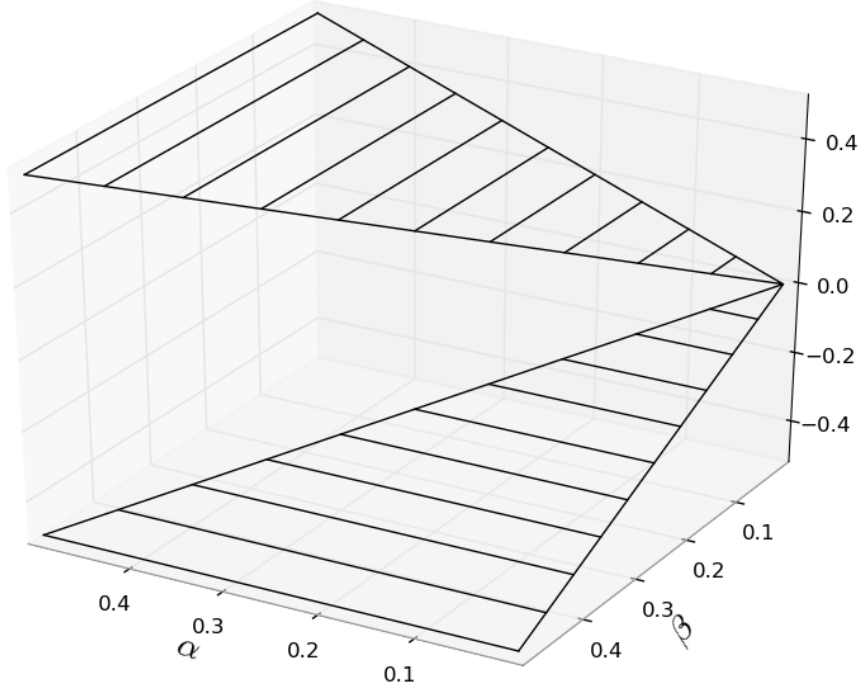


Figure 4.1: Standard tie-breaking rule for majority judgment

and we suggest a third possible tie-breaking rule,

$$\gamma_2(\bar{a}) = \frac{\alpha(\bar{a}) - \beta(\bar{a})}{2(1 - \beta(\bar{a}) - \alpha(\bar{a}))}.$$

γ_1 is shown in 4.2 and γ_2 is shown in 4.3.

γ_0 seems the least susceptible to strategic manipulation, since almost everywhere it is locally constant with respect to either α or β . Its major shortcoming seems to be its discontinuity at $\alpha = \beta$. Table 4.1 shows an example of five candidates with the same median grade where the societal ranking is $A \succ_S B \succ_S C \succ_S D \succ_S E$, but if one voter changes his grade for A from above the median grade to below it, then A will fall past the other four candidates with their widely varying γ_0 scores to the bottom of the societal ranking.

If we desire a continuous tie-breaking rule, then at first, γ_1 seems better in this regard, but γ_1 exhibits discontinuity where there is a transition to a higher or lower majority-grade (see figure 4.4). Table 4.2 shows an example where a candidate can rise or fall past several

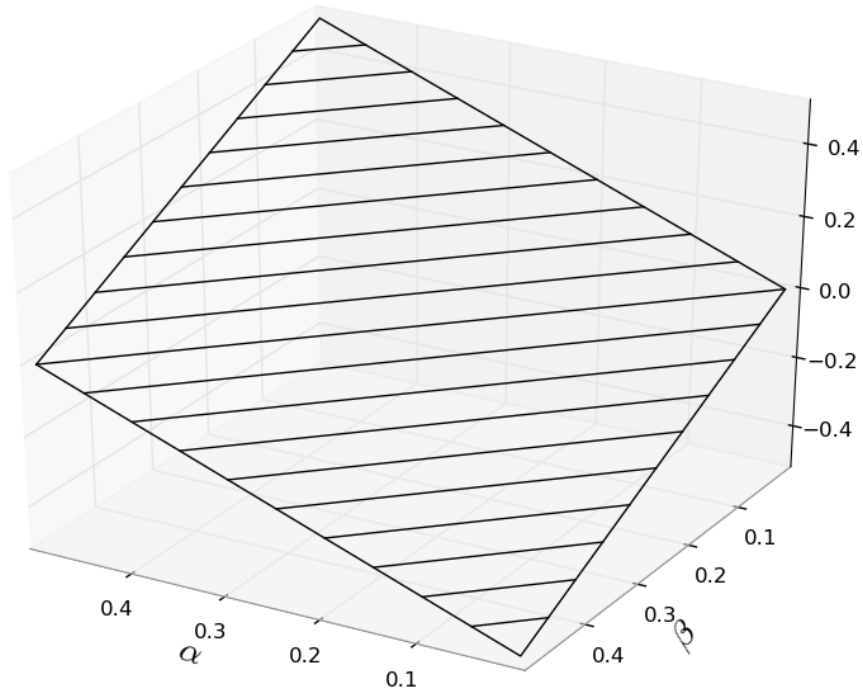


Figure 4.2: Gale's tie-breaking rule for majority judgment

candidates at once with the change of one voter's grade (moving between profile A and A'). In order to be continuous at the interface between grades, a tie-breaking rule must achieve 0.5 when α equals 0.5 and must approach -0.5 as β approaches 0.5. γ_0 satisfies this requirement, and it is shown in figure 4.5 that despite the continuity at $\alpha = \beta$, γ_0 is continuous at the transition between grades (the transition between the solid and dashed lines).

Candidate	Votes above median	Votes below median	γ_0	γ_2
A	41	40	0.41	0.026
B	30	20	0.30	0.100
C	12	7	0.12	0.031
D	7	12	-0.12	-0.031
E	20	30	-0.30	-0.100
A'	40	41	-0.41	-0.026

Table 4.1: Discontinuity of γ_0 for five candidates with the same median grade (100 voters)

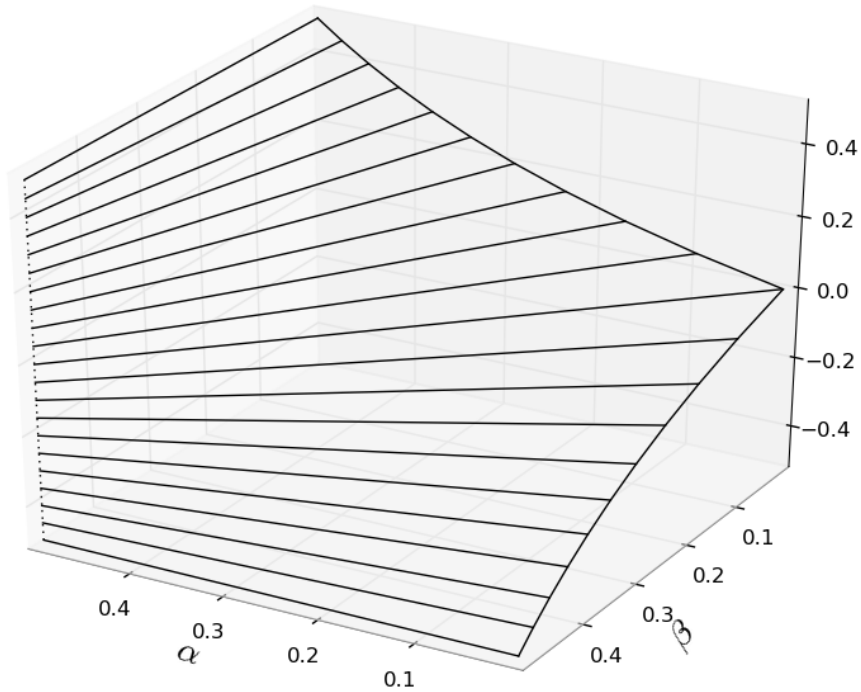


Figure 4.3: Suggested tie-breaking rule for majority judgment

Candidate	Grade distribution				Maj. grade	Tie-breaking rule	
	Excellent	Good	Fair	Poor		γ_1	γ_2
<i>A</i>	40	11	9	40	Good	-0.09	-0.409
<i>B</i>	35	20	11	34	Good	-0.10	-0.250
<i>C</i>	29	25	13	33	Good	-0.17	-0.340
<i>D</i>	35	12	26	27	Fair	0.20	0.385
<i>E</i>	24	21	26	29	Fair	0.16	0.308
<i>A'</i>	40	10	10	40	Fair	0.10	0.500

Table 4.2: Discontinuity of γ_1 for five candidates (100 voters)

γ_2 , on the other hand, exhibits continuity on the entire interior of the domain as well as at the transitions between grades (see figure 4.6). The only point of discontinuity for γ_2 is where $1 - \alpha - \beta$ is zero. But $\alpha \leq 0.5$ and $\beta < 0.5$, so this situation is outside the set of achievable α and β values, though it is on the boundary. Indeed, $1 - \alpha - \beta$ indicates the fraction of grades that are identical to the majority-grade, which can never be zero because,

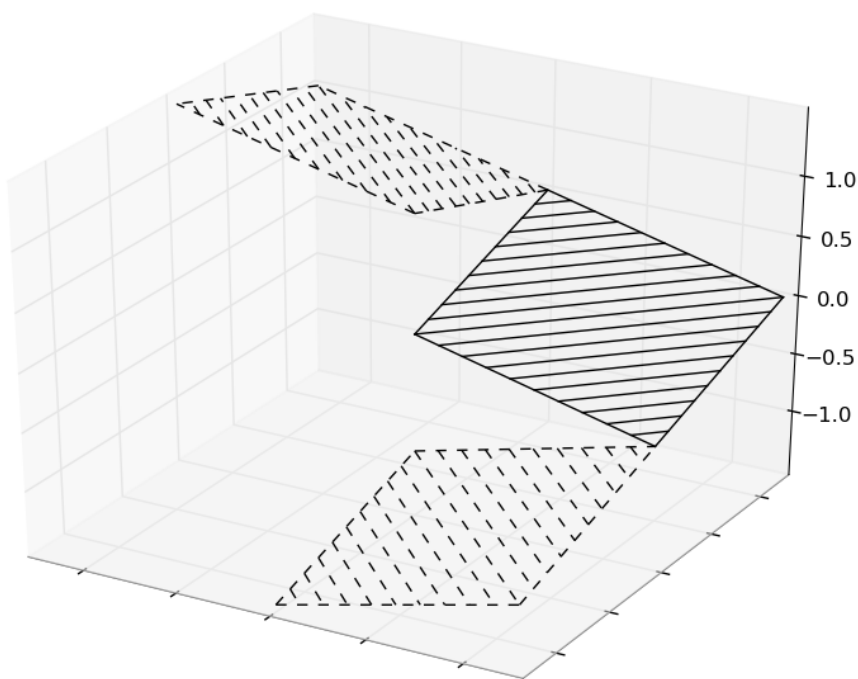


Figure 4.4: Discontinuity of Gale's rule at the transition to different grades (the transition between the solid-line area and the dashed-line areas)

by construction, the majority-grade (the median) comes from the set of input grades. And since, $\alpha \leq 1 - \alpha$ and $\beta \leq 1 - \beta$, it follows that $|\alpha - \beta| \leq 1 - \alpha - \beta$, so γ_2 is indeed bounded between -0.5 and 0.5 on the entire domain. We record these observations as Theorem 4, below.

Theorem 4. (Jennings) γ_0 is continuous at the transition between grades, but not on the line $\alpha = \beta$.

γ_1 is continuous on the line $\alpha = \beta$, but not at the transition between grades.

γ_2 is continuous on the interior of the domain and at the transition between grades, but has a discontinuity at $\alpha = \beta = 0.5$.

In fact, since tie-breaking functions are really only relevant when the number of voters is large and the set of grades is small and discrete, the probability that any of the grades will

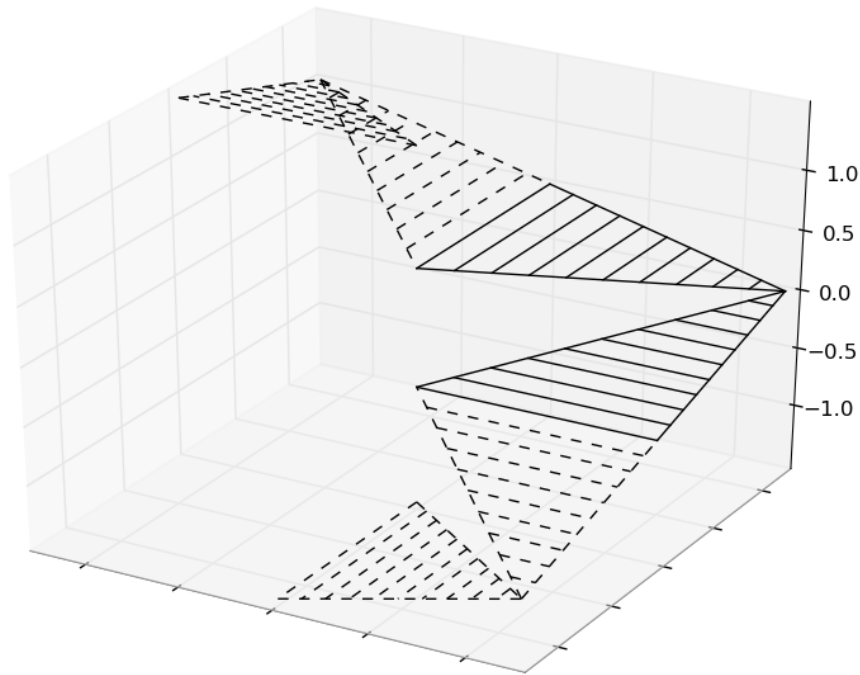


Figure 4.5: Continuity of the standard rule at the grade transitions (the interface between the solid-line area and dashed-line areas). The discontinuity at $\alpha = \beta$ remains, however.

be awarded zero times is negligible, so it does no great harm to bound the domain away from this point of discontinuity. Tables 4.1 and 4.2 each have a column indicating the γ_2 tie-breaking value for two problematic voter profiles introduced above. In both of these specific examples, the continuous tie-breaking rule would re-order the existing winners so that profiles A and A' are adjacent in the societal ordering. Thus, with the continuous tie-breaking rule, neither of these one-voter changes would change the societal candidate ordering at all.

Both of the examples above involved a highly polarizing candidate who had lots of extreme grades and relatively few middling grades leapfrogging multiple other candidates who had fewer extreme grades and lots of middling grades. It seems that this is the most problematic scenario. Although γ_2 is continuous, it is still very steep near $\alpha = \beta = 0.5$ (the most polarized distribution of grades). By making the polarization even more extreme than

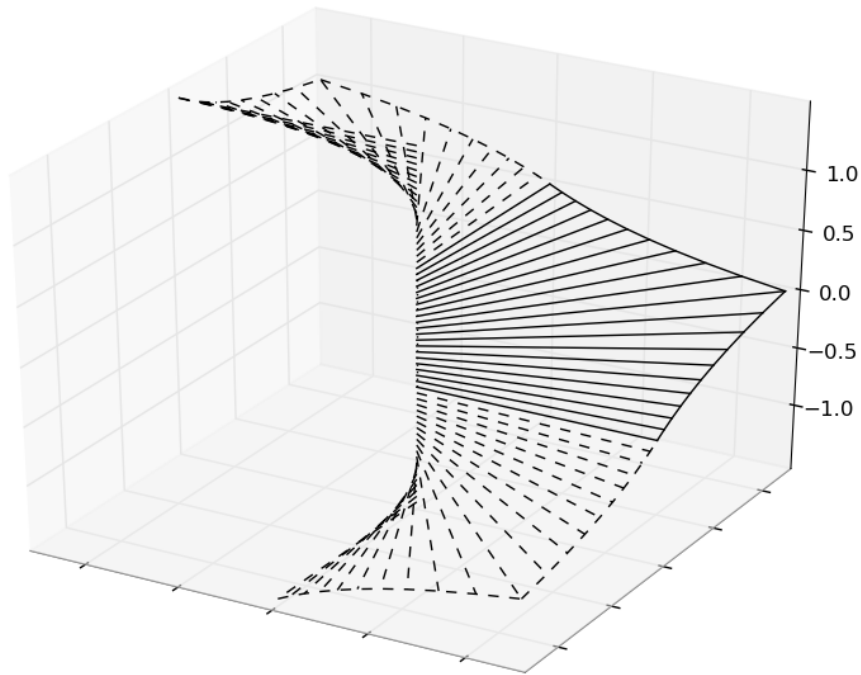


Figure 4.6: Continuity of the suggested rule in the interior and at the transitions

the two examples above, it is possible to concoct scenarios with γ_2 where one polarizing candidate jumps over multiple non-polarizing candidates with a very small change in his grade distribution.

4.4 Conclusion

The study of social choice functions based on voter grades instead of voter rank-orderings is a promising field of research. These functions allow social decisions that are pairwise unanimous, non-dictatorial, and independent of irrelevant alternatives, which is not possible in the traditional social choice framework. Three promising such methods are approval voting, range voting, and the majority judgment.

An analysis of the majority judgment tie-breaking rule shows that it is not continuous. While this, of itself, is not of great concern since we are always dealing with a finite number of voters and a discretization of the tie-breaking function anyways, an example was given

where a one-voter change in the profile can cause a candidate to fall several rankings in the societal output, past candidates that had tie-breaking scores that were some distance apart. An alternative rule, suggested by Gale, has a different discontinuity and a different example of a one-voter change to a profile which changes the societal output significantly was given for this rule.

A third rule was presented which is continuous everywhere and behaves better in the two example cases above. It is not possible, however, to fully eliminate the problem of large changes in the societal output ranking from small changes in the grade distribution. As a result, it is unclear how significant is the benefit to using this continuous tie-breaking rule, especially since it requires sacrificing one of the advantages of the original rule: that it is locally constant almost everywhere with respect to either α or β , which definitely decreases its manipulability.

Part II

WEAK MONOTONICITY AND THE LINEAR MEDIAN

Chapter 5

AGGREGATION FUNCTIONS AND STRATEGIC MEDIANS

We desire to explore cardinal voting methods in general. Any cardinal voting method that seeks independence of irrelevant alternatives should be based on a process that generates a societal grade for a given alternative based only on the grades given to that alternative, not on the grades given to any of the other alternatives. In this chapter, we formalize this concept in the form of aggregation functions. We also introduce the concept of strategy-proofness, whereby a cardinal voting method can avoid rewarding voter dishonesty. We characterize all possible strategy-proof aggregation functions. An aggregation function called the linear median is presented as an example of strategy-proofness. In later chapters, the linear median will be shown to have several important qualities and ultimately it emerges as a very valuable cardinal voting method.

5.1 Aggregation functions

A function, f from \mathbb{R}^m to \mathbb{R} is an *aggregation function* if it satisfies the following three properties:

- Unanimity - For all r , $f(r, \dots, r) = r$.
- Anonymity - Permuting the entries of the input vector preserves the value of f .
- Monotonicity - If $s_i \geq r_i$ for all i , then $f(\bar{s}) \geq (\bar{r})$.

Balinski and Laraki define an aggregation function similarly, but include a condition of *strict monotonicity* which requires that when all components of the input vector are increased, the output of f must strictly increase. [4] In this work, we explore the space of social choice functions that are based on aggregation functions that are not strictly monotone.

5.2 The linear median

Define the *linear median*, $\mathcal{M} : [0, 1]^n \rightarrow [0, 1]$, by

$$\mathcal{M}(a_1, \dots, a_n) = \sup \left\{ y \in [0, 1] \mid \frac{\#(a_i \geq y)}{n} \geq y \right\}.$$

The function \mathcal{M} gives the largest value y such that the proportion of the input arguments that are at least y is greater than or equal to y . This function is motivated by considering a game where n actors submit approval votes with each actor attempting to make the output equal to some personal target grade. The outcome indicated by $\mathcal{M}(a_1, \dots, a_n)$, where a_1, \dots, a_n are the personal target grades of the actors, is a Nash equilibrium for this game.

In fact, the function \mathcal{M} is *strategy-proof*: no actor can benefit (bring the output closer to his target grade) by lying about his target grade. If the actors know that the grades given will be aggregated with \mathcal{M} , then all actors responding honestly (revealing their target grade) is a Nash equilibrium of pure strategies.

5.3 Strategic medians

Balinski and Laraki proved [3] that the only strategy-proof strictly-monotone aggregation functions are the order statistics (the functions that return, respectively, the maximum argument, the second-highest argument, the third-highest argument, etc., down to the minimum argument). If the strict monotonicity condition is omitted, then there is a larger class of strategy-proof aggregation functions, including the linear median, \mathcal{M} , that are available.

A function, $f : [0, R]^n \rightarrow [0, R]$, is a *strategic median* if there exists an increasing function $g : [0, R] \rightarrow [0, 1]$ with $g(x) > 0$ for $x > 0$ and

$$f(x) = \sup \left\{ y \in [0, R] \mid \frac{\#(x_i \geq y)}{n} \geq g(y) \right\}.$$

We call g the *grading curve* (or *grading function*) of f . A fixed grading curve will generate a family of strategic medians, one for each n .

A strategic median based on grading function g will give the largest value of y such that the proportion of the input arguments that are at least y is greater than or equal to $g(y)$. The order statistics are strategic medians with constant grading curves.

5.4 Characterizing strategy-proof aggregation-functions

Theorem 5. (Jennings) *All strategic medians are aggregation functions.*

Proof. Let f be a strategic median.

(anonymity) f is anonymous because the formulation

$$f(x) = \sup \left\{ y \in [0, R] \mid \frac{\#(x_i > y)}{n} \geq g(y) \right\}$$

is a function of $\#(x_i > y)$, which is anonymous.

(unanimity) Let z in $[0, R]$, define $\mathbf{z} = (z, z, \dots, z)$. and consider $f(\mathbf{z})$. Since for every y with $y > z$, $\frac{\#(\mathbf{z}_i \geq y)}{n} = 0 < g(y)$ and for all $y \leq z$, $\frac{\#(\mathbf{z}_i \geq y)}{n} = 1 \geq g(y)$, it follows that we have $f(\mathbf{z}) = \sup([0, z]) = z$.

(monotonicity) Let $r = (r_1, \dots, r_j, \dots, r_n)$ and $r' = (r_1, \dots, s, \dots, r_n)$, and suppose that s is strictly greater than r_j . Then,

$$\begin{aligned} \#(r_i \geq y) &\leq \#(r'_i \geq y), \quad (\text{for all } y \in [0, R]) \\ \left\{ y \in [0, R] \mid \frac{\#(r_i \geq y)}{n} \geq g(y) \right\} &\subseteq \left\{ y \in [0, R] \mid \frac{\#(r'_i \geq y)}{n} \geq g(y) \right\} \\ \sup \left\{ y \in [0, R] \mid \frac{\#(r_i \geq y)}{n} \geq g(y) \right\} &\leq \sup \left\{ y \in [0, R] \mid \frac{\#(r'_i \geq y)}{n} \geq g(y) \right\} \\ f(r) &\leq f(r'). \end{aligned} \quad \square$$

Lemma 6. (Jennings) *Let f be a strategic median. Let $r, s \in [0, R]^n$ differ only in dimension i . If $f(r)$ is outside of the interval between r_i and s_i , then $f(r) = f(s)$.*

Proof. For x in $[0, R]^n$, define $h_x : [0, R] \rightarrow [0, 1]$ by $h_x(y) = \frac{\#(x_i \geq y)}{n}$. Then

$$f(x) = \sup\{h_x \geq g\}.$$

We note that h_r and h_s differ only on the interval between r_i and s_i (closed on the right). For convenience, we name this interval $(m, M]$.

If $f(r) < m$, then $m > \sup\{h_r \geq g\}$, so $h_r(m) < g(m)$. $h_s(m) = h_r(m) < g(m)$, so $[m, R]$ is disjoint from $\{h_r \geq g\}$ and $\{h_s \geq g\}$. Since h_r and h_s are identical on $[0, m)$, it follows that $\{h_r \geq g\} = \{h_s \geq g\}$ and $f(r) = f(s)$.

If $f(r) > M$, then choose $M < x < f(r)$. $x < \sup\{h_r \geq g\}$, so $h_r(x) \geq g(x)$. We have $h_s(x) = h_r(x) \geq g(x)$, so $[0, M]$ is a subset of both $\{h_r \geq g\}$ and $\{h_s \geq g\}$. Since h_r and h_s are identical on $(M, R]$, it follows that $\{h_r \geq g\} = \{h_s \geq g\}$ and $f(r) = f(s)$. \square

Theorem 7. (Jennings) *The strategic medians are strategy-proof.*

Proof. Let f be a strategic median. Let r in $[0, R]^n$ and j in $1, \dots, n$. Suppose $r_j > f(r)$. Let

$$s = (r_1, \dots, r_{j-1}, s_j, r_{j+1}, \dots, r_n).$$

If $s_j \leq r_j$ then monotonicity of f gives $f(s) \leq f(r)$. If $s_j > r_j > f(r)$, then lemma 6 gives $f(s) = f(r)$. \square

Lemma 8. (Jennings) *Let f be a strategy-proof aggregation function. Let $r, s \in [0, R]^n$ differ only in dimension i . If $f(r)$ is outside of the interval between r_i and s_i , then it is true that $f(r) = f(s)$.*

Proof. Suppose $f(r)$ is outside of the interval between r_i and s_i .

Case (i): ($r_i < s_i$) By monotonicity, $f(r) \leq f(s)$. If $f(r) < r_i$, then by the definition of strategy-proof, $f(s) \leq f(r)$. If $f(r) > s_i$, then $s_i < f(s)$ and by the definition of strategy-proof, $f(r) \geq f(s)$.

Case (ii): ($r_i > s_i$) By monotonicity, $f(r) \geq f(s)$. If $f(r) < s_i$, then $f(s) < s_i$ and by the definition of strategy-proof, $f(r) \leq f(s)$. If $f(r) > r_i$, then by the definition of strategy-proof, $f(s) \geq f(r)$.

We conclude that $f(r) = f(s)$. \square

Lemma 9. (Jennings) *Let f be a strategic median or a strategy-proof aggregation function. If $r, s \in [0, R]^n$ differ only in dimension i , then $|f(s) - f(r)| \leq |s_i - r_i|$.*

Proof. By lemmas 6 and 8, if $f(r)$ or $f(s)$ is outside of the interval between r_i and s_i , then $f(r) = f(s)$, so $f(s)$ and $f(r)$ can only differ if they are both within this interval. We conclude that $|f(s) - f(r)| \leq |s_i - r_i|$. \square

Theorem 10. (Jennings) *If f is a strategic median or a strategy-proof aggregation function, then f is continuous.*

Proof. Let $\varepsilon > 0$. Choose r, s in $[0, R]^n$ such that $|r_i - s_i| < \frac{\varepsilon}{n}$.

$$\begin{aligned} |f(r) - f(s)| &\leq \sum_{i=1}^n |f(s_1, \dots, s_{i-1}, r_i, \dots, r_n) - f(s_1, \dots, s_i, r_{i+1}, \dots, r_n)| \\ &\leq \sum_{i=1}^n |r_i - s_i| \quad (\text{by lemma 9}) \\ &< \frac{\varepsilon}{n} \cdot n = \varepsilon. \quad \square \end{aligned}$$

Lemma 11. (Jennings) *Let f be a strategy-proof aggregation function. Let r in $[0, R]^n$ and j in $1, \dots, n$. Suppose $r_j > f(r)$. For s_j in $[f(r), R]$,*

$$f(r_1, \dots, r_{j-1}, s_j, r_{j+1}, \dots, r_n) = f(r).$$

Proof. For s_j in $(f(r), R]$, this is immediate from lemma 8. The conclusion for $s_j = f(r)$ follows from continuity of strategy-proof aggregation functions. \square

For any grading curve g and $n > 0$, we define the *grading values* of g to be the $n - 1$ real numbers $\alpha_i = \sup g^{-1}([0, \frac{i}{n}])$ for $i = 1, \dots, n - 1$. Then we can prove that the output of f_g is the same as that given by computing the median of the n voters' grades combined with these $n - 1$ grading values. For notational convenience, we define $\alpha_0 = -\infty$ and $\alpha_n = +\infty$.

Lemma 12. (Jennings) *Let $n > 0$. Let g be a grading curve and f_g be the strategic median based on g . Let $\alpha_0, \dots, \alpha_n$, be the grading values of g as defined above. Then for every input vector (x_1, \dots, x_n) , the output of f_g is governed by one of the following two rules:*

(I) *If there is i in $1, \dots, n - 1$ such that $\#(x_k < \alpha_i) \leq n - i$ and $\#(x_k > \alpha_i) \leq i$, then $f_g(x) = \alpha_i$.*

(II) *If there is i in $1, \dots, n$ and j in $0, \dots, n - 1$ such that x_i is strictly between α_j and α_{j+1} , $\#(x_k < x_i) \leq n - 1 - j$, and $\#(x_k > x_i) \leq j$, then $f_g(x) = x_i$.*

Proof. (I) Suppose there is i in $1, \dots, n-1$ with $\#(x_k < \alpha_i) \leq n-i$ and $\#(x_k > \alpha_i) \leq i$. For any $s < \alpha_i$,

$$\frac{\#(x_k \geq s)}{n} \geq \frac{\#(x_k \geq \alpha_i)}{n} \geq \frac{i}{n} \geq g(s),$$

so $f_g(x) \geq \alpha_i$. For any $t > \alpha_i$,

$$\frac{\#(x_k \geq t)}{n} \leq \frac{\#(x_k > \alpha_i)}{n} \leq \frac{i}{n} < g(t),$$

so $f_g(x) \leq \alpha_i$.

(II) Suppose there is i in $1, \dots, n$ and j in $0, \dots, n-1$ such that $\alpha_j < x_i < \alpha_{j+1}$, and $\#(x_k < x_i) \leq n-1-j$, and $\#(x_k > x_i) \leq j$. Since $x_i < \alpha_{j+1}$, we have $g(x_i) \leq \frac{j+1}{n}$ and

$$\frac{\#(x_k \geq x_i)}{n} \geq \frac{j+1}{n} \geq g(x_i),$$

so $f_g(x) \geq x_i$. For any $t > x_i$,

$$\frac{\#(x_k \geq t)}{n} \leq \frac{\#(x_k > x_i)}{n} \leq \frac{j}{n} < g(t),$$

so $f_g(x) \leq x_i$.

(III) It remains to be shown that these two cases are exhaustive. Let x be given. For $i = 0, \dots, n-1$, we define $C(i) = \#(x_k \leq \alpha_{i+1}) + i + 1$. Let

$$j = \min\{i \in 0, \dots, n-1 \mid C(i) \geq n\}.$$

(The set is non-empty because $C(n-1) = \#(x_k \leq \alpha_n) + n = n + n = 2n$.)

If $\#(x_k < \alpha_{j+1}) + j < n$, then $\#(x_k < \alpha_{j+1}) < n-j$ so $\#(x_k < \alpha_{j+1}) \leq n-j-1$. Also, by the definition of j , $\#(x_k \leq \alpha_{j+1}) + j + 1 \geq n$ so $\#(x_k > \alpha_{j+1}) \leq j+1$. This satisfies the conditions for case (I) above (with $i = j+1$).

On the other hand, if $\#(x_k < \alpha_{j+1}) + j \geq n$, then $\#(x_k \geq \alpha_{j+1}) \leq j$. We choose i so that when x_1, \dots, x_n are put in ascending order, x_i is in position $n-j$. Then $\#(x_k \leq x_i) \geq n-j$ and $\#(x_k \geq x_i) \geq j+1$, which is equivalent to $\#(x_k > x_i) \leq j$ and $\#(x_k < x_i) \leq n-j-1$. Since j is the smallest number with $C(j) \geq n$, it follows that $n > C(j-1) = \#(x_k \leq \alpha_j) + j$. Equivalently, $\#(x_k \leq \alpha_j) < n-j$. This means that $\#(x_k \leq \alpha_j) < \#(x_k \leq x_i)$ and also that $\#(x_k \geq \alpha_{j+1}) < \#(x_k \geq x_i)$, from which it follows that $\alpha_j < x_i < \alpha_{j+1}$. This satisfies the conditions for case (II) above. \square

In [12], Hervé Moulin proved that any strategy-proof aggregation function is equivalent to calculating the median of the n input arguments together with $n - 1$ constants. The above lemma confirms that our characterization, the strategic medians, agrees with his. Although the other half of our characterization is redundant with Moulin's result, it is re-proved here (in our notation) because of its significance.

Theorem 13. (Moulin) *Any strategy-proof aggregation function is a strategic median.*

Proof. Let f be a strategy-proof aggregation function.

Define $g : [0, R] \rightarrow [0, 1]$ by

$$g(y) = \begin{cases} 0, & y = 0 \\ \min \left\{ \frac{\#(x_i \geq y)}{n} : x \in f^{-1}(y) \right\}, & \text{otherwise.} \end{cases}$$

(Because of unanimity, $f^{-1}(y)$ contains at least one element, (y, y, \dots, y) .)

First, we show that g is increasing. Let $0 < y \leq z \leq R$.

Let $p = n \cdot g(z) = \min \{ \#(x_i \geq z) : x \in f^{-1}(z) \}$. Choose x in $[0, R]^n$ with $f(x) = z$ and $\#(x_i \geq z) = p$. By applying lemmas 8 and 11, we can turn this into

$$f(\underbrace{z, \dots, z}_p, \underbrace{0, \dots, 0}_{n-p}) = z$$

By monotonicity and unanimity, we know that $f(\underbrace{y, \dots, y}_p, \underbrace{0, \dots, 0}_{n-p}) \leq y$. If it were true that $f(\underbrace{y, \dots, y}_p, \underbrace{0, \dots, 0}_{n-p})$ was less than y , repeated application of lemma 8 would give

$$f(\underbrace{z, \dots, z}_p, \underbrace{0, \dots, 0}_{n-p}) = f(\underbrace{y, \dots, y}_p, \underbrace{0, \dots, 0}_{n-p})$$

and we would have

$$f(\underbrace{z, \dots, z}_p, \underbrace{0, \dots, 0}_{n-p}) < y \leq z,$$

a contradiction. Thus, $f(\underbrace{y, \dots, y}_p, \underbrace{0, \dots, 0}_{n-p}) = y$.

This implies that p is in $\{ \#(x_i \geq y) : x \in f^{-1}(y) \}$. And we have that

$$g(y) = \min \left\{ \frac{\#(x_i \geq y)}{n} : x \in f^{-1}(y) \right\} \leq \frac{p}{n} = g(z),$$

which proves that g is increasing.

It remains to show that

$$f(x) = \sup \left\{ y \in [0, R] : \frac{\#(x_i \geq y)}{n} \geq g(y) \right\}.$$

Fix r in $[0, R]^n$. Then $r \in f^{-1}(f(r))$, so $g(f(r)) \leq \frac{\#(r_i \geq f(r))}{n}$. Thus,

$$f(r) \leq \sup \left\{ y \in [0, R] : \frac{\#(r_i \geq y)}{n} \geq g(y) \right\}$$

Let $\phi > f(r)$, and let s be an element of $f^{-1}(\phi)$. If $\#(r_i \geq \phi)$ were greater than or equal to $\#(s_i \geq \phi)$, applying lemmas 8 and 11 repeatedly (in each dimension) would give

$$f(r) = f(s) = \phi > f(r),$$

a contradiction.

Thus, $\#(r_i \geq \phi) < \#(s_i \geq \phi)$. Since s was arbitrary, we have

$$\#(r_i \geq \phi) < \min \{ \#(x_i \geq \phi) : x \in f^{-1}(\phi) \}.$$

Equivalently,

$$\frac{\#(r_i \geq \phi)}{n} < g(\phi).$$

So ϕ is not in $\left\{ y \in [0, R] : \frac{\#(r_i \geq y)}{n} \geq g(y) \right\}$. Since $\phi > f(r)$ was arbitrary,

$$f(r) \geq \sup \left\{ y \in [0, R] : \frac{\#(r_i \geq y)}{n} \geq g(y) \right\}.$$

We conclude that

$$f(r) = \sup \left\{ y \in [0, R] : \frac{\#(r_i \geq y)}{n} \geq g(y) \right\}. \quad \square$$

5.5 Conclusion

Aggregation functions, as introduced in this chapter, provide a broad framework to analyze all possible cardinal voting systems. Any function from \mathbb{R}^n to \mathbb{R} which is anonymous, unanimous, and monotone can generate a cardinal voting system that is potentially independent of irrelevant alternatives. Our framework is similar to one presented by Balinski and Laraki in [4], except we use a weaker monotonicity requirement.

We focus on aggregation functions that are strategy-proof, which means a voter can never, by submitting a dishonest grade, force the societal grade to move towards his honest grade. We have characterized all such strategy-proof aggregation functions. Each one can be characterized as the function that finds the intersection of the cumulative grade distribution with a specific grading curve. We relate our characterization to one provided by Moulin[12] which characterizes the strategy-proof aggregation functions as the functions that compute the median of the n input grades with $n - 1$ specific fixed values. This extension of Moulin's characterization into a characterization in terms of grading functions is significant because it allows us to do three things. First, we can generate a family of aggregation functions that each apply to a different number of voters. Second, we can better understand the role of the $n - 1$ fixed constants in the aggregation process. Third, it provides a basis for us to find the optimal aggregation function with respect to different criteria, as we will do in later chapters.

The linear median was a specific strategy-proof aggregation function introduced in this process. It is significant because it arises naturally from a simple continuous approval voting model, yet it is disallowed from the Balinski-Laraki framework because it is not strictly monotone. In the next few chapters it is shown to be quite an important aggregation function which generates a valuable cardinal voting system.

Chapter 6

EVALUATING GRADING CURVES WITH THE EUCLIDEAN NORM

One great advantage to dropping the strict monotonicity requirement and considering the broader class of strategic medians, instead of only the order statistics, is that they can better handle polarized situations. If all individuals submit maximal or minimal grades, then any order statistic also yields an extreme grade. In these polarized cases, it may be more useful to know how many individuals submitted high grades and how many submitted low grades than it is to know the median of the grades submitted. Aggregation functions that mediate polarity serve this function. \mathcal{M} is an example of a strategy-proof aggregation function that mediates polarity. When all voters give extreme grades, it yields the arithmetic mean of those input grades.

On the other hand, eliminating the strict monotonicity requirement admits all possible strategic medians as acceptable, and requires accepting the responsibility of distinguishing which ones are best for any given situation. One way to evaluate the suitability of an aggregation function, f , is to determine the distance between the input values and the aggregated output value, with respect to some norm. Since f takes an n -length vector as input and yields a real number, we refer to $\|(x_1, \dots, x_n) - f(x) \cdot (1, \dots, 1)\|$ as the distance between the aggregation function's inputs and outputs (with respect to any norm).

This chapter will briefly examine the aggregation functions that minimize the l^q -norms pointwise for $q \geq 1$, but the bulk of the chapter will consider how to choose the strategy-proof aggregation function that will minimize the l^2 distance between the inputs and the output. It will be shown that if the input grades come from a uniform distribution, then the ideal aggregation function is the linear median. Otherwise, we give a formula for generating the optimal grading curve from the input grade distribution.

6.1 Notes on the grading language

At this point, it is evident that care should be taken in choosing the grading language to use with a strategic median. With Moulin's characterization of the strategic medians (as

the median of the n votes along with $n - 1$ fixed values), it could make sense to use any fully-ordered set as the grading language, as long as we had a sensible way to choose the $n - 1$ fixed values. Our characterization of the strategic medians as the intersection of the input grade distribution with a grading curve indicates that the strategic median is closely related to the distribution and meaning of the input grades. And now, as we discuss distances between grades, it becomes even clearer that the grading language should be an *interval scale*, a numerical scale where the difference between two values is meaningful in a consistent manner along the entire scale. Which scale should be used will be explored later, but for now, we proceed with the general idea that we are working with an interval scale.

6.2 Minimizing input-output distance

Let $x = (x_1, \dots, x_m)$ be a vector of input values.

For an aggregation function f , the l^1 -norm distance between the inputs and outputs at x is minimized when $\sum_{i=1}^m |x_i - x_{out}|$ is minimized, which happens when it is true that $\#(x_i < x_{out}) = \#(x_i > x_{out})$. If m is odd, then x_{out} must be the median of x_1, \dots, x_m . If m is even then x_{out} can be any value in the closed interval between the two middlemost input values. So we note that the majority-grade is one aggregation function that everywhere minimizes the l^1 -norm between its inputs and outputs. Strategy-proofness basically comes for free in this case. As will be shown below, there is no other l^q norm where minimizing the distance between the inputs and output on a pointwise basis will produce a strategy-proof aggregation function!

The l^2 -norm distance between the inputs and outputs of f at x will be minimized when $\sum_{i=1}^m (x_i - x_{out})^2$ is minimized, which happens if and only if x_{out} is the arithmetic mean of x_1, \dots, x_m . Thus the arithmetic mean is the unique aggregation function that everywhere minimizes the l^2 -norm between its inputs and outputs. Since the arithmetic mean is not strategy-proof, there is no strategy-proof aggregation function that everywhere minimizes the l^2 -norm between its inputs and outputs.

In fact, for any q greater than 1, the following theorem and corollary show that we can

minimize the l^q -norm distance between the inputs of f and its output on a pointwise basis to form an aggregation function.

Theorem 14. (Jennings) *If $d : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly convex function with a minimum at 0, then the function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ that minimizes $\sum_{i=1}^m d(x_i - y)$ on a pointwise basis is an aggregation function.*

Proof. Fix $x = x_1, \dots, x_m$. Then $u(y) = \sum_{i=1}^m d(x_i - y)$ is strictly convex in y with a unique minimum, so f is well-defined.

The function f is anonymous by construction. It is unanimous because if all of the x_i values are equal, then $u(y)$ will be minimized at the common value.

Let y_{min} be the y value that minimizes $u(y)$. Let \hat{x} be identical to x except strictly larger in the i^{th} component. We define $\hat{u}(y) = \sum_{i=1}^m d(\hat{x}_i - y)$. For any $y < y_{min}$, because of the convexity of d ,

$$\hat{u}(y) - u(y) = d(\hat{x}_i - y) - d(x_i - y) > d(\hat{x}_i - y_{min}) - d(x_i - y_{min}) = \hat{u}(y_{min}) - u(y_{min}),$$

which means that it is impossible for the minimum to move to the left. This establishes the monotonicity of f . □

Corollary 15. *For $q > 1$, the function f that minimizes $\sum_{i=1}^m |x_i - f(x_1, \dots, x_m)|^q$ on a pointwise basis is an aggregation function.*

Theorem 16. (Jennings) *For $q > 1$, the function f that minimizes $\sum_{i=1}^m |x_i - f(x_1, \dots, x_m)|^q$ on a pointwise basis is not strategy-proof.*

Proof. Fix $x = x_1, \dots, x_m$. Let $d(y) = |y|^q$ and $u(y) = \sum_{i=1}^m d(x_i - y)$. Let y_{min} be the y value that minimizes u . Let \hat{x} be identical to x except strictly larger in the i^{th} component, and let $\hat{u}(y) = \sum_{i=1}^m d(\hat{x}_i - y)$. Since $q > 1$, d is differentiable everywhere, with increasing derivative, hence $d'(\hat{x}_i - y) > d'(x_i - y)$.

$\hat{u}'(y_{min}) = u'(y_{min}) - d'(\hat{x}_i - y) + d'(x_i - y) < 0$. Since \hat{u} has a unique minimum, this minimum must be to the right of y_{min} .

This proves that f is strictly monotone in each of the input components, which means it cannot be strategy-proof. \square

Theorem 17. (Jennings) For $0 < q < 1$ and $m > 3$, the function that minimizes the expression $\sum_{i=1}^m |x_i - f(x_1, \dots, x_m)|^q$ pointwise (restricted to the domain where the unique minimum exists) is not monotone, hence doesn't qualify as an aggregation function.

Proof. Let $u(y) = \sum_{i=1}^m |x_i - y|^q$. First, we note that for any interval $a < b$, if none of x_1, \dots, x_m falls between a and b , then the minimum of u on $[a, b]$ occurs at one of the endpoints. This results from the fact that u is concave on $(-\infty, 0)$ and $(0, \infty)$, so when there are no x values between a and b , then the function u on $[a, b]$ is the sum of m concave functions, hence is concave itself. From this, it follows that to find the minimum of u we need only check the x_i values. For convenience, we again define $d(y) = |y|^q$.

Case (i): If m is even, let $x_1 = \dots = x_{\frac{m}{2}} = 0$, $x_{\frac{m}{2}+1} = \dots = x_{m-1} = 1$, and $x_m = 1 + \varepsilon$.

$$\begin{aligned} u(0) &= \frac{m}{2}d(0) + \left(\frac{m}{2} - 1\right)d(1) + d(1 + \varepsilon) = \left(\frac{m}{2} - 1\right)d(1) + d(1 + \varepsilon) \\ u(1) &= \frac{m}{2}d(1) + \left(\frac{m}{2} - 1\right)d(0) + d(\varepsilon) = \frac{m}{2}d(1) + d(\varepsilon) \\ u(1 + \varepsilon) &= \frac{m}{2}d(1 + \varepsilon) + \left(\frac{m}{2} - 1\right)d(\varepsilon) + d(0) = \frac{m}{2}d(1 + \varepsilon) + \left(\frac{m}{2} - 1\right)d(\varepsilon) \end{aligned}$$

In this case, $u(0)$ is less than $u(1)$ because $d(1 + \varepsilon) < d(1) + d(\varepsilon)$. And

$$u(0) < \frac{m}{2}d(1 + \varepsilon) < u(1 + \varepsilon),$$

so $y = 0$ is the unique global minimum.

If we create \hat{x} by moving the first component of x down from 0 to $-\varepsilon$, and the last component down from $1 + \varepsilon$ to 1, we have the reflection of the above situation, and $y = 1$ is the unique global minimum for \hat{x} .

Case (ii): If m is odd, we assign the first $m - 1$ input arguments as in the even case above. The last input argument, x_m , we assign to a large positive number N . Since the derivative of d goes to 0 as y goes to infinity, we can choose N large enough so that $d(N) - d(N - (1 + \varepsilon))$ is as small as we like. In particular, we choose N large enough that $y = 0$ is still the global minimum for x and $y = 1$ is still the global minimum for \hat{x} . \square

6.3 Minimizing distance in the strategy-proof space

In general, we desire to find the strategy-proof aggregation function that, over the domain of input values, minimizes the l^q -norm distance between its inputs and output. This requires choosing a separate norm to apply as we integrate the l^q -norm over the domain of input values. The L^q -norm, weighted by the probability distribution from which the input values are drawn, is the natural choice. The analysis below proceeds using this norm.

Because of the ubiquity of the l^2 -norm, we examine the l^2L^2 case first. As mentioned above, the aggregation function that minimizes the l^2 -norm is the arithmetic mean, which is by far the most common way to aggregate a set of data.

Let p be a probability distribution with compact support. Define the lower endpoint $m = \inf\{x \mid \int_{-\infty}^x p(y)dy > 0\}$ and the upper endpoint $M = \sup\{x \mid \int_x^{\infty} p(y)dy > 0\}$. Define E_p for a, b in (m, M) as

$$E_p(a, b) = \frac{\int_a^b t p(t) dt}{\int_a^b p(t) dt}.$$

$E_p(a, b)$ is the expected value of a number drawn from distribution p given that it is between a and b . Technically E_p is undefined where $\int_a^b p(t)dt = 0$, but we will overlook this fact since we are only concerned with $E_p(m, \cdot)$ and $E_p(\cdot, M)$, where E_p is always well-defined.

Define G_p on the interval (m, M) as

$$G_p(x) = \frac{x - E_p(m, x)}{E_p(x, M) - E_p(m, x)}.$$

We note that $\lim_{x \rightarrow m^+} G_p(x) = 0$ and $\lim_{x \rightarrow M^-} G_p(x) = 1$, and that G_p is continuous.

Theorem 18. (*Jennings*) *If G_p is monotone, then the strategic median that uses G_p as its grading function is the unique strategy-proof aggregation function that minimizes the l^2 -norm distance from the input arguments over the probability distribution p .*

6.4 Interpretation of G_p

For any value of x between m and M , notice that

$$x = (1 - G_p(x))E_p(m, x) + G_p(x)E_p(x, M).$$

So $G_p(x)$ indicates the relative placement of x between $E_p(m, x)$ and $E_p(x, M)$. In fact, x is the expected value of the mean of a data set where $G_p(x)$ is the proportion of the data values greater than x .

Thus, when the grading curve can be chosen to equal G_p (which is whenever G_p is monotone), the corresponding strategic median will always give the expected value of the mean of the input arguments, as it will return x whenever the proportion of the input arguments greater than x is $G_p(x)$. As mentioned above, the l^2 -norm prefers the arithmetic mean, so it is natural that this is the best strategic median according to the l^2 -norm.

6.5 Proof of theorem 18

Proof. Fix n . Our goal is to choose f from the family of strategy-proof aggregation functions that minimizes:

$$E(f, p) = \int_m^M \cdots \int_m^M \sum_{i=1}^n (f(x) - x_i)^2 \prod_{i=1}^n p(x_i) dx_n \cdots dx_1$$

Let f be a strategy-proof aggregation function with grading curve g . Let $\alpha_0, \dots, \alpha_n$ be the grading values of g as in lemma 12. We use lemma 12 and symmetry to rewrite E as:

$$\begin{aligned} E((\alpha_1, \dots, \alpha_{n-1}), p) = & \\ & \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!} \underbrace{\int_m^{\alpha_j} \cdots \int_m^{\alpha_j}}_{n-j} \underbrace{\int_{\alpha_j}^M \cdots \int_{\alpha_j}^M}_j \sum_{i=1}^n (\alpha_j - x_i)^2 \prod_{i=1}^n p(x_i) dx_n \cdots dx_1 \\ & + \sum_{j=0}^{n-1} \frac{n!}{j!(n-j-1)!} \int_{\alpha_j}^{\alpha_{j+1}} \underbrace{\int_m^{x_1} \cdots \int_m^{x_1}}_{n-j-1} \underbrace{\int_{x_1}^M \cdots \int_{x_1}^M}_j \sum_{i=1}^n (x_1 - x_i)^2 \prod_{i=1}^n p(x_i) dx_n \cdots dx_1 \end{aligned}$$

Note that the only form in which f or g shows up in this expression is by way of the real numbers $\alpha_1, \dots, \alpha_{n-1}$. Let us consider this set of variables to be our primary parameters. We fix k in $1, \dots, n-1$, and differentiate with respect to α_k :

$$E_{\alpha_k}((\alpha_1, \dots, \alpha_{n-1}), p) =$$

$$\frac{n!}{k!(n-k)!} \underbrace{\int_m^{\alpha_k} \cdots \int_m^{\alpha_k}}_{n-k} \underbrace{\int_{\alpha_k}^M \cdots \int_{\alpha_k}^M}_{k} \sum_{i=1}^n 2(\alpha_k - x_i) \prod_{i=1}^n p(x_i) dx_n \cdots dx_1$$

We note that E_{α_k} does not depend on any of the other α_i 's, so minimizing E is simply a matter of optimizing each α_i independently. First we scale E_{α_k} to remove constants.

$$\hat{E}_{\alpha_k}((\alpha_1, \dots, \alpha_{n-1}), p) = \underbrace{\int_m^{\alpha_k} \cdots \int_m^{\alpha_k}}_{n-k} \underbrace{\int_{\alpha_k}^M \cdots \int_{\alpha_k}^M}_{k} \sum_{i=1}^n (\alpha_k - x_i) \prod_{i=1}^n p(x_i) dx_n \cdots dx_1$$

Now we change variables. Define $P(x) = \int_m^x p(a) da$. Then let $\chi_i = P(x_i)$ and we have $d\chi_i = p(x_i) dx_i$. For notational ease, we define $\phi = P(\alpha_k)$.

$$\begin{aligned} \hat{E}_{\alpha_k}((\alpha_1, \dots, \alpha_{n-1}), p) &= \underbrace{\int_0^{\phi} \cdots \int_0^{\phi}}_{n-k} \underbrace{\int_{\phi}^1 \cdots \int_{\phi}^1}_{k} \sum_{i=1}^n (\alpha_k - P^{-1}(\chi_i)) d\chi_n \cdots d\chi_1 \\ &= n \underbrace{\int_0^{\phi} \cdots \int_0^{\phi}}_{n-k} \underbrace{\int_{\phi}^1 \cdots \int_{\phi}^1}_{k} \alpha_k d\chi_n \cdots d\chi_1 \\ &\quad - \sum_{i=1}^n \underbrace{\int_0^{\phi} \cdots \int_0^{\phi}}_{n-k} \underbrace{\int_{\phi}^1 \cdots \int_{\phi}^1}_{k} P^{-1}(\chi_i) d\chi_n \cdots d\chi_1 \\ &= n\phi^{n-k} (1-\phi)^k \alpha_k \\ &\quad - \sum_{i=1}^{n-k} \phi^{n-k-1} (1-\phi)^k \int_0^{\phi} P^{-1}(\chi_i) d\chi_i \\ &\quad - \sum_{i=n-k+1}^n \phi^{n-k} (1-\phi)^{k-1} \int_{\phi}^1 P^{-1}(\chi_i) d\chi_i \\ &= n\phi^{n-k} (1-\phi)^k \alpha_k \\ &\quad - (n-k)\phi^{n-k-1} (1-\phi)^k \int_m^{\alpha_k} xp(x) dx \\ &\quad - k\phi^{n-k} (1-\phi)^{k-1} \int_{\alpha_k}^M xp(x) dx \\ &= \phi^{n-k-1} (1-\phi)^{k-1} n \left(\phi(1-\phi)\alpha_k - \left(1 - \frac{k}{n}\right)(1-\phi) \int_m^{\alpha_k} xp(x) dx - \frac{k}{n}\phi \int_{\alpha_k}^M xp(x) dx \right) \end{aligned}$$

Since $\phi = P(\alpha_k) = \int_m^{\alpha_k} p(x)dx$ is strictly positive for $\alpha_k > m$ and $1 - \phi$ is strictly positive for $\alpha_k < M$, we consider E as a function of α_k on (m, M) . This function is increasing when

$$\phi(1 - \phi)\alpha_k - \left(1 - \frac{k}{n}\right)(1 - \phi) \int_m^{\alpha_k} xp(x)dx - \frac{k}{n} \phi \int_{\alpha_k}^M xp(x)dx > 0.$$

Equivalently, (since $\phi = \int_m^{\alpha_k} p(a)da$ and $1 - \phi = \int_{\alpha_k}^M p(a)da$)

$$\left(\frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx} - \frac{\int_{\alpha_k}^M xp(x)dx}{\int_{\alpha_k}^M p(x)dx} \right) \frac{k}{n} > \frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx} - \alpha_k.$$

Here we note that

$$m \leq \frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx} \leq \alpha_k$$

and

$$\alpha_k \leq \frac{\int_{\alpha_k}^M xp(x)dx}{\int_{\alpha_k}^M p(x)dx} \leq M,$$

so our condition for E to be increasing becomes:

$$\frac{k}{n} < \frac{\alpha_k - \frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx}}{\frac{\int_{\alpha_k}^M xp(x)dx}{\int_{\alpha_k}^M p(x)dx} - \frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx}} = G_p(\alpha_k).$$

Similarly, E as a function of α_k on (m, M) will be decreasing when

$$\frac{k}{n} > \frac{\alpha_k - \frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx}}{\frac{\int_{\alpha_k}^M xp(x)dx}{\int_{\alpha_k}^M p(x)dx} - \frac{\int_m^{\alpha_k} xp(x)dx}{\int_m^{\alpha_k} p(x)dx}} = G_p(\alpha_k).$$

If G_p is one-to-one, E will be decreasing when $\alpha_k < G_p^{-1}(\frac{k}{n})$ and increasing whenever $\alpha_k > G_p^{-1}(\frac{k}{n})$, so E will have one minimum, at $\alpha_k = G_p^{-1}(\frac{k}{n})$. This must hold for all k from 1 to $n - 1$, and since n was arbitrary, it must be true for all n .

Since α_k is defined as $\sup g^{-1}([0, \frac{k}{n}])$, it follows that $g = G_p$ is the unique grading function that will minimize the l^2 norm between the inputs and the output. \square

6.6 Example

$$p(x) = 3x^2 \text{ on } [0, 1] \text{ with } m = 0 \text{ and } M = 1.$$

$$\begin{aligned}
 E_p(0,x) &= \frac{3}{4}x \\
 E_p(x,1) &= \frac{3}{4} \left(\frac{1+x+x^2+x^3}{1+x+x^2} \right) \\
 G_p(x) &= \frac{x}{3}(1+x+x^2)
 \end{aligned}$$

See figure 6.1.

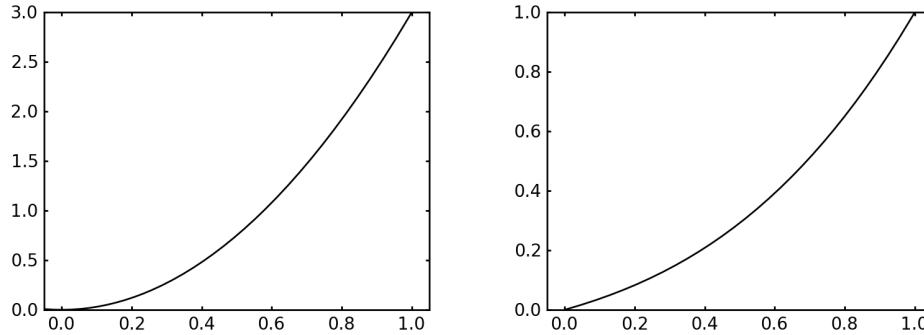


Figure 6.1: $p(x) = 3x^2$ (left) and its grading function G_p (right)

6.7 Uniform Distributions

If the input distribution is uniform on a certain interval $[m, M]$, then the G_p will be the line that goes through points $(m, 0)$ and $(M, 1)$. So if the input distribution is uniform on $[0, 1]$, then $G_p(x) = x$, and if the input distribution is a uniform distribution on $[0.5, 1]$, then $G_p(x) = 2(x - 0.5)$. It is instructive to observe how G_p changes as we move between these two distributions. Consider the family of split uniform distributions

$$p_s(x) = \begin{cases} s & , 0 \leq x < \frac{1}{2} \\ 2-s & , \frac{1}{2} \leq x \leq 1 \end{cases}$$

as we vary s from 0 to 1. The grading functions for $s = 0, \frac{1}{100}, \frac{1}{10}, \frac{1}{2}, 1$ are shown in figure 6.2.

6.8 Non-monotone G_p functions

The distributions in figure 6.2 with $s = 0.1$ and $s = 0.01$ are examples of distributions that result in non-monotone G_p functions. Consider the case when $s = 0.1$:

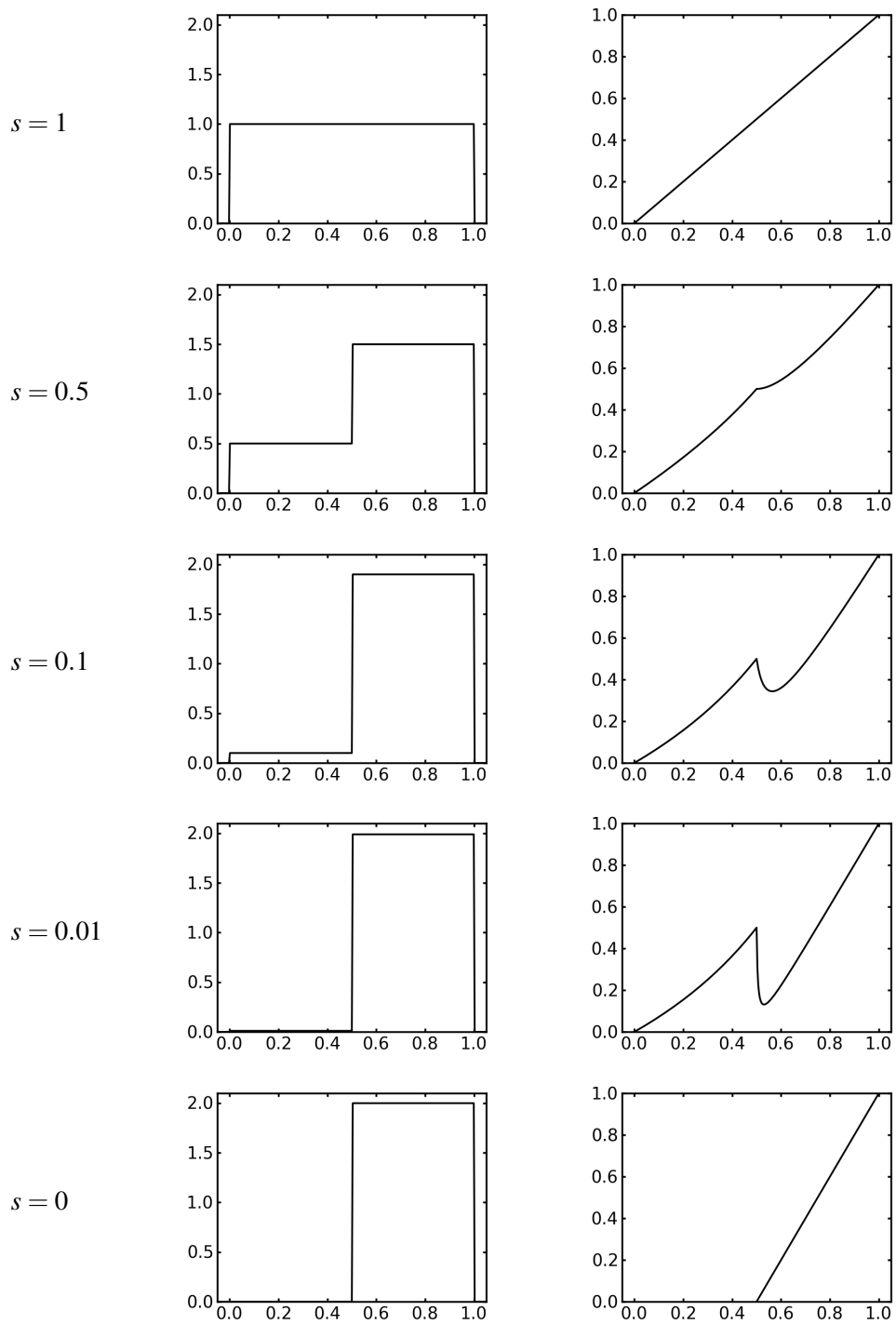


Figure 6.2: Five piecewise uniform distributions (left) and their corresponding G_p functions (right)

$$p(x) = \begin{cases} \frac{1}{10} & , 0 \leq x < \frac{1}{2} \\ \frac{19}{10} & , \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{with } m = 0 \text{ and } M = 1.$$

$$E_p(0, x) = \begin{cases} \frac{x}{2} & , 0 \leq x < \frac{1}{2} \\ \frac{38x^2 - 9}{76x - 36} & , \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$E_p(x, 1) = \begin{cases} \frac{29 - 2x^2}{40 - 4x} & , 0 \leq x < \frac{1}{2} \\ \frac{1+x}{2} & , \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$G_p(x) = \begin{cases} \frac{20x - 2x^2}{29 - 20x} & , 0 \leq x < \frac{1}{2} \\ \frac{38x^2 - 36x + 9}{20x - 9} & , \frac{1}{2} \leq x \leq 1 \end{cases}$$

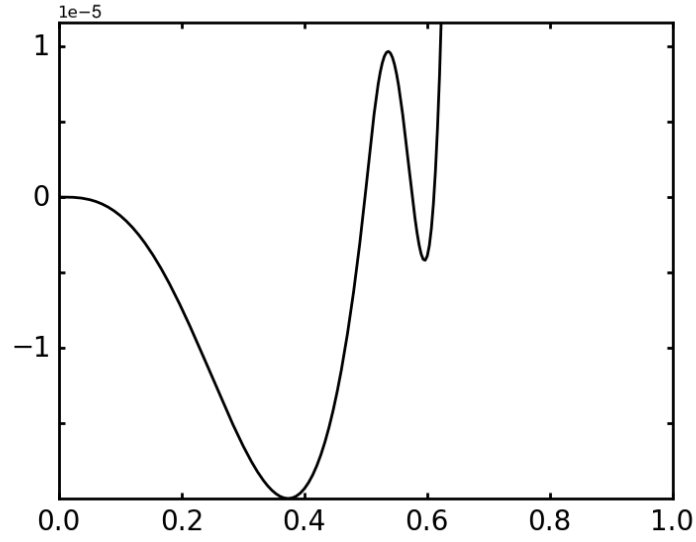
Non-monotonic G_p functions do not qualify as grading functions. A non-monotonic G_p function indicates that for certain values of y , the set $G_p^{-1}(y)$ has more than one value, in which case there is more than one local minimum in attempting to minimize the l^2 -norm. Therefore, the appropriate way to convert G_p into a one-to-one function is to go back to the construction of G_p in the proof of theorem 18 and choose the global minimum from among the local minima. Specifically, one must compute the following integral

$$\int (\phi(t)^{1-y}(1-\phi(t))^y)^n \left(t - (1-y) \frac{\int_m^t xp(x)dx}{\int_m^t p(x)dx} - y \frac{\int_t^M xp(x)dx}{\int_t^M p(x)dx} \right) dt,$$

between the two local minima. If the integral is positive, then the leftmost minimum is the global minimum. Otherwise the rightmost minimum is the global one.

Notice that this integral depends on n , so for a given probability function, it is possible for \hat{G}_p to depend on n , the number of voters. In fact, for the split uniform distribution with $s = 0.08$, the solutions to $G_p(x) = \frac{1}{3}$ are $x = 0.373$, $x = 0.536$, and $x = 0.596$. The middle one represents a local maximum. 0.373 is the global minimum when $n = 3$ and $k = 1$ and 0.596 is the global minimum when $n = 6$ and $k = 2$. The error functions that we seek to minimize in these cases are shown in figure 6.3.

$k = 1, n = 3$



$k = 2, n = 6$

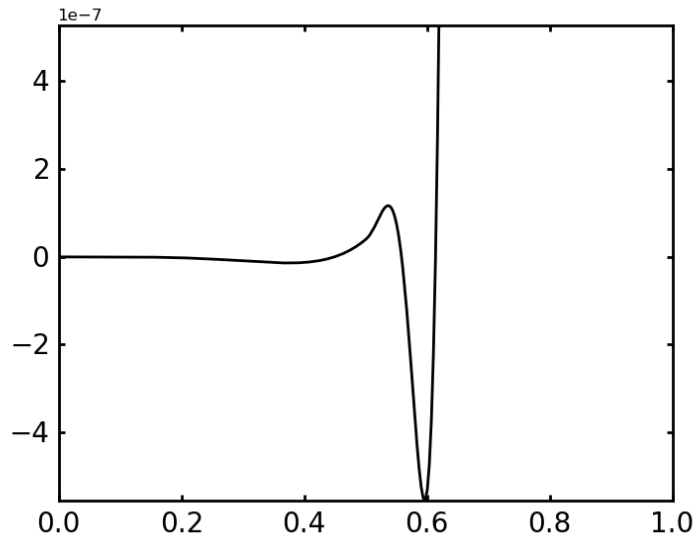


Figure 6.3: Error functions for a piecewise uniform distribution for 3 and 6 voters. Changing the number of voters can change the relative vertical positions of the local minima, thus altering the global minimum.

6.9 Distribution of distributions

Suppose, instead of knowing that the input grades are drawn from a certain distribution, we have a family of probability distributions $\{p_d\}_{d \in \mathcal{D}}$ from which one distribution will be chosen, and then all the input grades will be drawn from that distribution. Ideally, we would be able to collapse the family of distributions into one master distribution and find the grading curve with the G_p formulation given above. Unfortunately, this approach is unsuccessful. The grading function must be found by adding the error terms for all the probability distributions and re-solving for $\frac{n}{k}$ as in the proof of theorem 18. For generality, we suppose that P_d is the probability with which p_d will be chosen as the grading input distribution. Then, the total error term is

$$\begin{aligned} \hat{E}_{\alpha_k} &= \int_{d \in \mathcal{D}} P_d \underbrace{\int_m^{\alpha_k} \cdots \int_m^{\alpha_k}}_{n-k} \underbrace{\int_{\alpha_k}^M \cdots \int_{\alpha_k}^M}_k \sum_{i=1}^n (\alpha_k - x_i) \prod_{i=1}^n p_d(x_i) dx_n \cdots dx_1 dd \\ &= \int_{d \in \mathcal{D}} P_d \phi_d^{n-k} (1 - \phi_d)^k n \left(\alpha_k - \left(1 - \frac{k}{n}\right) \frac{\int_m^{\alpha_k} x p_d(x) dx}{\phi_d} - \frac{k}{n} \frac{\int_{\alpha_k}^M x p_d(x) dx}{1 - \phi_d} \right) dd. \end{aligned}$$

Thus, the formula for the master grading function is

$$G_{\mathcal{D}} = \frac{\int_{d \in \mathcal{D}} P_d \phi_d^{n-k} (1 - \phi_d)^k \left(\alpha_k - \frac{\int_m^{\alpha_k} x p_d(x) dx}{\phi_d} \right) dd}{\int_{d \in \mathcal{D}} P_d \phi_d^{n-k} (1 - \phi_d)^k \left(\frac{\int_{\alpha_k}^M x p_d(x) dx}{1 - \phi_d} - \frac{\int_m^{\alpha_k} x p_d(x) dx}{\phi_d} \right) dd}.$$

We note that the grading function in this situation, varies with n , the number of voters.

6.10 Conclusions

It is natural to seek an aggregation function where the output value is representative of the input data. Any aggregation function must be unanimous, anonymous, and monotone, which ensures at least a minimal level of correspondence between its inputs and output. Using the l^q -norms to measure the distance between the inputs and output of an aggregation function allows us to examine more deeply the quality of the data aggregation.

For $q \geq 1$, we can generate an aggregation function pointwise by minimizing the appropriate norm for each vector of input values in the domain. For $q > 1$, this generates a unique

aggregation function that is not strategy-proof. For $q = 1$ this generates a family of satisfactory aggregation functions, including the majority judgment, which is strategy-proof.

It is also possible consider just the strategy-proof aggregation functions and find the one that minimizes the l^q distance between the inputs and the output. In this chapter, we have done so for the l^2 -norm and found a formula that gives the optimal grading curve for a given input probability distribution. The formula will generally yield monotone grading curves that are independent of the number of voters, but for some grading distributions, it will yield non-monotone grading functions which must be monotonized. The monotonization procedure is outlined, and in this case the optimal grading curve may change depending on the number of participating voters. The next chapter will cover other l^q -norms.

For the specific case when the input grades come from a uniform distribution, the optimal grading curve for the l^2 -norm is the diagonal grading curve, which corresponds to the linear median. This Euclidean norm is one of the most commonly used norms, which gives additional weight to the linear median. Additionally, our determination that the strategic medians require an interval scale for the grading language indicates that the distance between grades has an important meaning that should be consistent through the entire grading scale. As such, we should be biased towards treating the input distribution as uniform and towards using the linear median if at all possible.

EVALUATING WITH OTHER NORMS

While the l^1 and l^2 norms are the most interesting because they are the most common, in this chapter we continue the analysis of the previous chapter in minimizing the l^q distance between the inputs and the output of strategy-proof aggregation functions. We also examine how much influence one voter is able to have in swaying the final grade, seeking to minimize this quantity with respect to different L^q norms.

7.1 Minimizing the input-output distance for various norms

As in theorem 18, when we are dealing with probability distribution p below, we define $P(x) = \int_m^x p(a)da$.

Theorem 19. (Jennings) *For $q \geq 1$, the grading curve that minimizes the $l^q L^q$ -norm within the space of strategy-proof aggregation functions, for input coming from probability distribution p is*

$$G_p(x) = \left(1 + \frac{\int_x^M (t-x)^{q-1} p(t) dt}{\int_x^M p(t) dt} \frac{\int_m^x (x-t)^{q-1} p(t) dt}{\int_m^x p(t) dt} \right)^{-1}$$

if it is monotone.

Proof. We desire to minimize $E(f) = \int_m^M \dots \int_m^M \sum_{i=1}^n |x_i - f(x)|^q \prod_{i=1}^n p(x_i) dx_m \dots dx_1$.

Fix n a natural number. Let f be a strategy-proof aggregation function with grading values $\alpha_1, \dots, \alpha_{n-1}$ as in lemma 12. We use this lemma and symmetry to rewrite $E(f)$ as:

$$\begin{aligned} E(f) &= \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!} \underbrace{\int_m^{a_j} \dots \int_m^{a_j}}_{n-j} \underbrace{\int_{a_j}^M \dots \int_{a_j}^M}_{j} \sum_{i=1}^n |x_i - a_j|^q \prod_{i=1}^n p(x_i) dx_n \dots dx_1 \\ &+ \sum_{j=0}^{n-1} \frac{n!}{j!(n-j-1)!} \int_{a_j}^{a_{j+1}} \underbrace{\int_m^{x_1} \dots \int_m^{x_1}}_{n-j-1} \underbrace{\int_{x_1}^M \dots \int_{x_1}^M}_{j} \sum_{i=1}^n |x_i - x_1|^q \prod_{i=1}^n p(x_i) dx_n \dots dx_1. \end{aligned}$$

We differentiate with respect to a_k :

$$E_{a_k}(f) = \frac{q \cdot n!}{k!(n-k)!} \underbrace{\int_m^{a_k} \dots \int_m^{a_k}}_{n-k} \underbrace{\int_{a_k}^M \dots \int_{a_k}^M}_{k} \sum_{i=1}^{n-k} (a_k - x_i)^{q-1} \prod_{i=1}^n p(x_i) dx_n \dots dx_1$$

$$-\frac{q \cdot n!}{k!(n-k)!} \underbrace{\int_m^{a_k} \cdots \int_m^{a_k}}_{n-k} \underbrace{\int_{a_k}^M \cdots \int_{a_k}^M}_k \sum_{i=n-k+1}^n (x_i - a_k)^{q-1} \prod_{i=1}^n p(x_i) dx_n \cdots dx_1$$

and set that equal to zero:

$$0 = (n-k)P(a_k)^{n-k-1}(1-P(a_k))^k \int_m^{a_k} p(x)(a_k-x)^{q-1} dx$$

$$-kP(a_k)^{n-k}(1-P(a_k))^{k-1} \int_{a_k}^M p(x)(x-a_k)^{q-1} dx$$

$$(n-k)(1-P(a_k)) \int_m^{a_k} p(x)(a_k-x)^{q-1} dx = kP(a_k) \int_{a_k}^M p(x)(x-a_k)^{q-1} dx$$

$$\frac{n-k}{k} = \frac{P(a_k) \int_{a_k}^M p(x)(x-a_k)^{q-1} dx}{(1-P(a_k)) \int_m^{a_k} p(x)(a_k-x)^{q-1} dx}$$

$$\frac{k}{n} = \left(1 + \frac{\frac{\int_{a_k}^M p(t)(t-a_k)^{q-1} dt}{\int_{a_k}^M p(t) dt}}{\frac{\int_m^{a_k} p(t)(a_k-t)^{q-1} dt}{\int_m^{a_k} p(t) dt}} \right)^{-1}$$

As in theorem 18, if the right-hand side is monotone, then it generates the unique grading curve that minimizes the l^q norm:

$$G_p(x) = \left(1 + \frac{\frac{\int_x^M (t-x)^{q-1} p(t) dt}{\int_x^M p(t) dt}}{\frac{\int_m^x (x-t)^{q-1} p(t) dt}{\int_m^x p(t) dt}} \right)^{-1}.$$

□

This result generalizes, and agrees with, the l^2 -norm result proven in theorem 18. As in that theorem, if G_p generates a non-monotone candidate grading function, it can be monotonized with a similar process.

In the case where we are dealing with the uniform distribution, G_p becomes

$$G_p(x) = \frac{1}{1 + \left(\frac{M-x}{x-m}\right)^{q-1}},$$

which is consistent with our earlier results that $G_p(x) = \frac{1}{2}$ when $q = 1$ and $G_p(x) = \frac{x-m}{M-m}$ when $q = 2$. As q goes to ∞ , the grading curve will approach the step function:

$$\begin{cases} 0 & , \text{ if } x < \frac{m+M}{2} \\ 1 & , \text{ if } x > \frac{m+M}{2} \end{cases} .$$

An actual step function is not a true grading curve, as it fails to satisfy $g(y) > 0$ for all $y > 0$. Even if we replace the 0 with some very small positive number, such a grading curve is not very useful, because it will yield a strategic median that almost always gives the same output grade. If the step is at $x = s$, for instance, then the strategic median will output the grade s whenever at least one of the input arguments is less than s and at least one of the arguments is greater than s . For large n , this is practically all of the time. When all of the arguments are less than s it will return the maximum of the arguments and when they are all greater than s it will return the minimum argument.

One could argue that a grading curve that is close to a step function, but still continuous, generates an aggregation function that is reminiscent of approval voting. In the case of the function above, any input grade above $\frac{m+M}{2}$ is considered approval and any grade below that point is considered disapproval and the output grade is an increasing function of the number of approvals received. It would be silly to actually run an approval election this way, asking voters to submit grades on a scale between m and M and choosing the winner to be the one who received the most grades above $\frac{m+M}{2}$, because it asks the voters to submit so much more information than is actually used. It does, however, have the advantage of being able to break ties if more than one candidate is approved by everyone (choose the one with the highest minimum grade) or if all candidates are disapproved by everyone (choose the one with the highest maximum grade). In any case, this relationship between strategic medians in the limit and approval voting may indicate that there is some sense in which approval voting is the system that minimizes the $l^\infty L^\infty$ distance between the input grades and the output grade. In truth, this step function is being chosen as the ideal grading curve for the $l^\infty L^\infty$ norm not because of any relationship to approval voting, but because of its degenerate behavior of almost always assigning a societal grade of $\frac{m+M}{2}$.

In summary, the ideal strategic median for keeping the output grade close to the input grades depends on the norms used when integrating over the possible profiles. To use the l^1L^1 -norm is to minimize the expected value of the absolute difference between the output grade and the input grades, which is accomplished best with the majority judgment voting method (a horizontal grading curve). As we minimize the distance between inputs and outputs for the general l^qL^q -norm, the optimal grading curve gets steeper with increasing q . We note that if $q = 2$ and we are dealing with a uniform distribution for the input grades, then the ideal grading curve is the diagonal one (corresponding to the linear median). At $q = \infty$, the ideal grading curve is a step function, with the step at the midpoint between the extents of the grading language.

7.2 Manipulability

For this section, we restrict the the range of grades is 0 to 1. Consider the cumulative grade distribution function $d(x_1, \dots, x_n; y) = \frac{\#(x_i \geq y)}{n}$. This comes from the definition of a strategic median, and it is what gets compared against the grading curve in order to determine the output grade.

If one voter drops out of the electorate and his grade was r , this cumulative distribution function will shift downwards to the left of r and upwards to the right of r by a magnitude of no more than $\frac{1}{n-1}$. If a voter is added to the electorate, the cumulative distribution function will shift upwards to the left of his vote and downwards to the right of it, by no more than $\frac{1}{n+1}$. If a voter changes his vote, the cumulative distribution function will shift upwards or downwards on the interval between his old vote and his new vote, by no more than $\frac{1}{n}$.

In any of these cases, if the grading curve is horizontal, then a small vertical shift in the distribution function can cause a large shift in the output grade. However, if we choose the diagonal grading curve (the linear median), then adding a voter, removing a voter, or allowing one voter to change his vote can only change the output grade by $\frac{1}{n-1}$, $\frac{1}{n+1}$, or $\frac{1}{n}$ respectively.

In general, we define $\Delta(g, \varepsilon)$ to be the maximum horizontal change in a grading curve, g , caused by moving up or down by ε . Then, for the strategic median generated by grading

curve g , adding a voter, removing a voter, and allowing a voter to change his vote can only change the output grade by $\Delta(g, \frac{1}{n-1})$, $\Delta(g, \frac{1}{n+1})$, and $\Delta(g, \frac{1}{n})$ respectively.

There are many possible definitions of manipulability. If manipulability is defined to be the maximum effect on the final grade that one voter can have by changing his grade, over all possible voter input profiles, then the linear median is the strategy-proof aggregation function that minimizes manipulability.

Theorem 20. (*Jennings*) *The linear median is the unique strategy-proof aggregation function that minimizes $\max_{x_1, \dots, x_{n-1} \in [0,1]} f(x_1, \dots, x_{n-1}, 1) - f(x_1, \dots, x_{n-1}, 0)$ for any dimension n .*

Proof. Fix the dimension n . First we will show that every aggregation function (including non-strategy-proof ones) has $\max_{x_1, \dots, x_{n-1} \in [0,1]} f(x_1, \dots, x_{n-1}, 1) - f(x_1, \dots, x_{n-1}, 0) \geq \frac{1}{n}$. Let f be an aggregation function.

$$\begin{aligned} 1 &= f(1, \dots, 1) - f(0, \dots, 0) \\ &= (f(1, \dots, 1) - f(1, \dots, 1, 0)) + \dots + (f(0, \dots, 0, 1) - f(0, \dots, 0)) \\ &\leq n \cdot \max_{x_1, \dots, x_{n-1} \in [0,1]} f(x_1, \dots, x_{n-1}, 1) - f(x_1, \dots, x_{n-1}, 0) \end{aligned}$$

(The arithmetic mean is an example of an aggregation function that is not strategy proof that achieves this minimal manipulability.)

Now if f is a strategy-proof aggregation function, we obtain its grading curve g and its grading values $\alpha_1, \dots, \alpha_{n-1}$. Again, for convenience, we define $\alpha_0 = 0$ and $\alpha_n = 1$. For each i in $1, \dots, n$, there is an input profile where one voter can cause the output grade to move from α_{i-1} to α_i . Namely, if $n - i$ voters give grades of 0 and $i - 1$ voters give grades of 1 then the last voter will be able to swing the output grade between α_{i-1} and α_i . Also, since the aggregation function can be formulated as the median of the n submitted grades along with $\alpha_1, \dots, \alpha_{n-1}$, it will be impossible for any one voter to unilaterally move the output grade across any of the α_j values, so

$$\max_{x_1, \dots, x_{n-1} \in [0,1]} f(x_1, \dots, x_{n-1}, 1) - f(x_1, \dots, x_{n-1}, 0) = \max_{i \in 1, \dots, n} \alpha_i - \alpha_{i-1}.$$

This indicates that the α_j values should be spaced evenly within to open interval from 0 to 1. The diagonal function is the unique grading curve that accomplishes this for all dimensions n . \square

The fact that the linear median minimizes this measure of manipulability is independent of the distribution of the input grades. Minimizing this “maximum” manipulability is equivalent to minimizing the L^∞ -norm (uniform-norm) distance between $f(x_1, \dots, x_{n-1}, 1)$ and $f(x_1, \dots, x_{n-1}, 0)$ on $[0, 1]^{n-1}$. We can also attempt to minimize a different L^q -norm.

Theorem 21. (Jennings) Fix the dimension n , and fix $q > 1$. For input coming from probability distribution p , if there is a unique strategic median f that minimizes the L^q -norm distance between $f(x_1, \dots, x_{n-1}, 1)$ and $f(x_1, \dots, x_{n-1}, 0)$ on $[0, 1]^{n-1}$, then the corresponding grading values $\alpha_1, \dots, \alpha_{n-1}$ satisfy:

$$\frac{j}{n-j} = \left(\frac{1-P(\alpha_j)}{P(\alpha_j)} \right) \frac{\int_{\alpha_j}^{\alpha_{j+1}} \left(\frac{1-P(x)}{1-P(\alpha_j)} \right)^j (x-\alpha_j)^{q-2} dx}{\int_{\alpha_{j-1}}^{\alpha_j} \left(\frac{P(x)}{P(\alpha_j)} \right)^{n-j} (\alpha_j-x)^{q-2} dx}, \quad \text{for } j = 1, \dots, n-1.$$

Proof. We desire to minimize

$$E = \int_0^1 \cdots \int_0^1 (f(x_1, \dots, x_{n-1}, 1) - f(x_1, \dots, x_{n-1}, 0))^q \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1.$$

We divide the integral into four pieces depending on whether $f(\dots, 1)$ and $f(\dots, 0)$ are equal to one of the x_i values or one of the α_i values, again using lemma 12 and symmetry to simplify:

$$\begin{aligned} E &= \sum_{k=1}^n \frac{n!}{(n-k)!(k-1)!} \underbrace{\int_0^{\alpha_{k-1}} \cdots \int_0^{\alpha_{k-1}}}_{n-k} \underbrace{\int_{\alpha_k}^1 \cdots \int_{\alpha_k}^1}_{k-1} (\alpha_k - \alpha_{k-1})^q \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \\ &+ \sum_{k=1}^n \frac{n!}{(n-k)!(k-2)!} \int_{\alpha_{k-1}}^{\alpha_k} \underbrace{\int_0^{\alpha_{k-1}} \cdots \int_0^{\alpha_{k-1}}}_{n-k} \underbrace{\int_{x_1}^1 \cdots \int_{x_1}^1}_{k-2} (x_1 - \alpha_{k-1})^q \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \\ &+ \sum_{k=1}^n \frac{n!}{(n-k-1)!(k-1)!} \int_{\alpha_{k-1}}^{\alpha_k} \underbrace{\int_0^{x_1} \cdots \int_0^{x_1}}_{n-k-1} \underbrace{\int_{\alpha_k}^1 \cdots \int_{\alpha_k}^1}_{k-1} (\alpha_k - x_1)^q \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \end{aligned}$$

$$+ \sum_{k=1}^n \frac{n!}{(n-k-1)!(k-2)!} \int_{\alpha_{k-1}}^{\alpha_k} \int_{\alpha_{k-1}}^{x_1} \underbrace{\int_0^{x_2} \cdots \int_0^{x_2}}_{n-k-1} \underbrace{\int_{x_1}^1 \cdots \int_{x_1}^1}_{k-2} (x_1 - x_2)^q \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1$$

For $j = 1, \dots, n-1$, we can take the partial derivative with respect to α_j . (Knowing that the original integrand, $f(\dots, 1) - f(\dots, 0)$ was continuous, we can avoid worrying about the boundaries of these integrals and just differentiate the integrands.)

$$\begin{aligned} E\alpha_j &= q \frac{n!}{(n-j)!(j-1)!} \underbrace{\int_0^{\alpha_{j-1}} \cdots \int_0^{\alpha_{j-1}}}_{n-j} \underbrace{\int_{\alpha_j}^1 \cdots \int_{\alpha_j}^1}_{j-1} (\alpha_j - \alpha_{j-1})^{q-1} \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \\ &- q \frac{n!}{(n-j-1)!j!} \underbrace{\int_0^{\alpha_j} \cdots \int_0^{\alpha_j}}_{n-j-1} \underbrace{\int_{\alpha_{j+1}}^1 \cdots \int_{\alpha_{j+1}}^1}_j (\alpha_{j+1} - \alpha_j)^{q-1} \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \\ &- q \frac{n!}{(n-j-1)!(j-1)!} \int_{\alpha_j}^{\alpha_{j+1}} \underbrace{\int_0^{\alpha_j} \cdots \int_0^{\alpha_j}}_{n-j-1} \underbrace{\int_{x_1}^1 \cdots \int_{x_1}^1}_{j-1} (x_1 - \alpha_j)^{q-1} \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \\ &+ q \frac{n!}{(n-j-1)!(j-1)!} \int_{\alpha_{j-1}}^{\alpha_j} \underbrace{\int_0^{x_1} \cdots \int_0^{x_1}}_{n-j-1} \underbrace{\int_{\alpha_j}^1 \cdots \int_{\alpha_j}^1}_{j-1} (\alpha_j - x_1)^{q-1} \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \cdots dx_1 \\ &= q \frac{n!}{(n-j)!(j-1)!} P(\alpha_{j-1})^{n-j} (1 - P(\alpha_j))^{j-1} (\alpha_j - \alpha_{j-1})^{q-1} \\ &- q \frac{n!}{(n-j-1)!j!} P(\alpha_j)^{n-j-1} (1 - P(\alpha_{j+1}))^j (\alpha_{j+1} - \alpha_j)^{q-1} \\ &- q \frac{n!}{(n-j-1)!(j-1)!} P(\alpha_j)^{n-j-1} \int_{\alpha_j}^{\alpha_{j+1}} (1 - P(x))^{j-1} (x - \alpha_j)^{q-1} p(x) dx \\ &+ q \frac{n!}{(n-j-1)!(j-1)!} (1 - P(\alpha_j))^{j-1} \int_{\alpha_{j-1}}^{\alpha_j} P(x)^{n-j-1} (\alpha_j - x)^{q-1} p(x) dx \end{aligned}$$

We integrate these two integrals by parts with $dv = (1 - P(x))^{j-1} p(x) dx$ for the first integral and $dv = P(x)^{n-j-1} p(x) dx$ for the second integral, which eliminates the first two terms and leaves

$$\begin{aligned} E\alpha_j &= -q(q-1) \frac{n!}{(n-j-1)!j!} P(\alpha_j)^{n-j-1} \int_{\alpha_j}^{\alpha_{j+1}} (1 - P(x))^j (x - \alpha_j)^{q-2} dx \\ &+ q(q-1) \frac{n!}{(n-j)!(j-1)!} (1 - P(\alpha_j))^{j-1} \int_{\alpha_{j-1}}^{\alpha_j} P(x)^{n-j} (\alpha_j - x)^{q-2} dx \end{aligned}$$

We set this to zero:

$$\frac{P(\alpha_j)^{n-j-1}}{j} \int_{\alpha_j}^{\alpha_{j+1}} (1 - P(x))^j (x - \alpha_j)^{q-2} dx = \frac{(1 - P(\alpha_j))^{j-1}}{n-j} \int_{\alpha_{j-1}}^{\alpha_j} P(x)^{n-j} (\alpha_j - x)^{q-2} dx$$

$$\frac{j}{n-j} = \left(\frac{1-P(\alpha_j)}{P(\alpha_j)} \right) \frac{\int_{\alpha_j}^{\alpha_{j+1}} \left(\frac{1-P(x)}{1-P(\alpha_j)} \right)^j (x-\alpha_j)^{q-2} dx}{\int_{\alpha_{j-1}}^{\alpha_j} \left(\frac{P(x)}{P(\alpha_j)} \right)^{n-j} (\alpha_j-x)^{q-2} dx}$$

□

This gives us $n-1$ equations to solve for $\alpha_1, \dots, \alpha_{n-1}$ (since $\alpha_0 = 0$ and $\alpha_n = 1$). Each equation relates α_{j-1}, α_j , and α_{j+1} , and the equations are generally well behaved. For fixed α_{j-1} and α_{j+1} , the right hand side of the equation is continuous for α_j in between those values. It approaches infinity as α_j approaches α_{j-1} from above and zero as α_j approaches α_{j+1} from below, so there will be a unique α_j that satisfies the equation. It is probable, though unproven, that there is always a unique solution to this system of equations.

Theorem 22. (Jennings) For any dimension n , and for probability distribution p , the strategic median f that minimizes the L^1 -norm distance between $f(x_1, \dots, x_{n-1}, 1)$ and $f(x_1, \dots, x_{n-1}, 0)$ on $[0, 1]^{n-1}$ is characterized by grading values $\alpha_1, \dots, \alpha_n$ which are all equal, with the common value that satisfies $\int_0^\alpha p(t) dt = \frac{1}{2}$.

Proof. For $q = 1$, the E_{α_j} equation from the proof of theorem 21 would be:

$$\begin{aligned} E_{\alpha_j} &= q \frac{n!}{(n-j)!(j-1)!} P(\alpha_{j-1})^{n-j} (1-P(\alpha_j))^{j-1} \\ &\quad - q \frac{n!}{(n-j-1)!j!} P(\alpha_j)^{n-j-1} (1-P(\alpha_{j+1}))^j \\ &\quad - q \frac{n!}{(n-j-1)!(j-1)!} P(\alpha_j)^{n-j-1} \int_{\alpha_j}^{\alpha_{j+1}} (1-P(x))^{j-1} p(x) dx \\ &\quad + q \frac{n!}{(n-j-1)!(j-1)!} (1-P(\alpha_j))^{j-1} \int_{\alpha_{j-1}}^{\alpha_j} P(x)^{n-j-1} p(x) dx \\ &= q \frac{n!}{(n-j)!(j-1)!} P(\alpha_{j-1})^{n-j} (1-P(\alpha_j))^{j-1} \\ &\quad - q \frac{n!}{(n-j-1)!j!} P(\alpha_j)^{n-j-1} (1-P(\alpha_{j+1}))^j \\ &\quad - q \frac{n!}{(n-j-1)!j!} P(\alpha_j)^{n-j-1} ((1-P(\alpha_{j+1}))^j - (1-P(\alpha_j))^j) \\ &\quad + q \frac{n!}{(n-j)!(j-1)!} (1-P(\alpha_j))^{j-1} (P(\alpha_j)^{n-j} - P(\alpha_{j-1})^{n-j}) \end{aligned}$$

Setting this to zero, we get:

$$0 = -(n-j)P(\alpha_j)^{n-j-1}(1-P(\alpha_j))^j + jP(\alpha_j)^{n-j}(1-P(\alpha_j))^{j-1}$$

$$\frac{j}{n} = 1 - P(\alpha_j)$$

This indicates a decreasing grading function, $y = 1 - P(x)$. Of course, this is not a valid grading curve, and it indicates that all of the α_i values coalesce to the same real number and the relevant function is a step function. So we proceed with the assumption that all of the α_i values equal a common value, α .

We go back to the definition of E :

$$\begin{aligned} E &= \int_0^1 \cdots \int_0^1 (f(x_1, \dots, x_{n-1}, 1) - f(x_1, \dots, x_{n-1}, 0)) \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \dots dx_1 \\ &= \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_{n-1}, 1) \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \dots dx_1 \\ &\quad - \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_{n-1}, 0) \prod_{i=1}^{n-1} p(x_i) dx_{n-1} \dots dx_1 \end{aligned}$$

f will return the common α value when at least one of the input arguments is less than α and at least one is greater than α , so

$$\begin{aligned} E &= \left(\alpha(1 - (1 - P(\alpha))^{n-1}) + (n-1) \int_{\alpha}^1 x(1 - P(x))^{n-2} dx \right) \\ &\quad - \left(\alpha(1 - P(\alpha)^{n-1}) + (n-1) \int_0^{\alpha} xP(x)^{n-2} dx \right) \\ &= \alpha(P(\alpha)^{n-1} - (1 - P(\alpha))^{n-1}) + (n-1) \left(\int_{\alpha}^1 x(1 - P(x))^{n-2} dx - \int_0^{\alpha} xP(x)^{n-2} dx \right) \end{aligned}$$

Setting the derivative equal to zero:

$$\begin{aligned} 0 = E_{\alpha} &= (P(\alpha)^{n-1} - (1 - P(\alpha))^{n-1}) + \alpha p(\alpha)(n-1)(P(\alpha)^{n-2} + (1 - P(\alpha))^{n-2}) \\ &\quad + (n-1)\alpha p(\alpha)(-(1 - P(\alpha))^{n-2} - P(\alpha)^{n-2}) \\ &= (P(\alpha)^{n-1} - (1 - P(\alpha))^{n-1}) \\ &\quad P(\alpha) = \frac{1}{2} \end{aligned}$$

□

As was the case earlier, a step function emerges as the ideal grading curve not because of any coincidental relationship to approval voting, but because it causes the aggregation function to return the same grade practically all of the time. This does technically minimize the expected value of a lone voter's ability to influence the output grade, but only by completely ruining the expressiveness of the aggregation process.

These three theorems indicate the grading curves that will minimize the effect one voter can have on the output grade as different L^q norms are used to measure the effect. Using the L^1 -norm is equivalent to minimizing the expected value of one voter's influence, which is done by using a step function as the grading curve, with the step occurring at the point where an input grade is equally likely to be above as below. As q is increased, the optimal grading curve gets less and less steep until it becomes diagonal at $q = \infty$. Using the L^∞ -norm is equivalent to minimizing the maximum possible influence that one voter may have, which is accomplished by the linear median (a diagonal grading curve) independent of the distribution of the input grades.

7.3 Conclusion

In this chapter, we were able to determine the grading curves that will minimize the distance between aggregation function inputs and outputs for norms other than the Euclidean norm. Additionally, we determined the grading curves that minimize the influence of single voters according to different L^q norms. The linear median is the strategy-proof aggregation function that minimizes single voter influence according to the L^∞ norm.

In our exploration of aggregation functions, especially in focusing on strategy-proof ones, we have discovered that the linear median is quite a valuable aggregation function. The specific advantages discovered up to this point are:

- The linear median is strategy-proof.
- The linear median handles polarized situations well, returning the arithmetic mean of the input grades.

- If the input grades are assumed to come from a uniform distribution, then among all strategy-proof aggregation functions, the linear median minimizes the Euclidean norm distance between inputs and output.
- Regardless of the distribution of the input grades, the linear median is the unique strategy-proof aggregation function that minimizes the worst-case (L^∞) influence that one voter may have on the outcome. It achieves the theoretical minimum possible for any aggregation function (even non-strategy-proof ones).¹

A further advantage of the linear median with a grading language of 0-100 is that it is probably the easiest strategy-proof aggregation function to understand outside of the order statistics. (In this case, the linear median will be the largest number, x , where x percent of the voters gave a grade of x or higher.)

The main disadvantage to the linear median is that it requires an interval scale. This means we need to choose a numeric grading scale, but it also means that we need to choose one that the voters are likely to use linearly. This is a psychological question as much as a mathematical one. We propose that a scale of 0 to 100 be used, and that the voters are instructed to indicate “what approval rating they give each candidate”. The 100-point scale is used, in large part, because using any other grading scale with the linear median will be far more difficult to explain. This scale, however, does carry the risk that some people in the United States will interpret as it is used in the educational system, where 75 means “acceptable” and 50 means “failing”. We hope that instructing voters to give their approval rating for each candidate will help them avoid this trap. We also hope that time and experience with the linear median will help people become familiar with the 100-point scale in an election context and use it more linearly over time.

¹Although there is at least one non-strategy-proof aggregation function, namely the arithmetic mean, which achieves this same theoretical minimum.

Chapter 8

CARDINAL SYSTEMS IN A COMPETITIVE CONTEXT

In chapter 5, we examined the criterion of strategy-proofness and found all aggregation functions where no voter can, by submitting a dishonest grade, move the societal grade closer to his honest grade. This implies that honesty is a dominant strategy for voters dealing with a strategy-proof aggregation function. In chapter 7, we analyzed the manipulability of aggregation functions in another way. We measured the influence one voter could have on the election outcome if all the other votes remained unchanged. Both of these methods of analyzing manipulability apply to the process of aggregating n grades into one societal grade. We desire here to examine strategies and incentives in an election context, where grades are being aggregated for multiple candidates simultaneously.

In a competitive environment such as an election, a voter's utility does not depend only on the final scores of the candidates and how close they are to the voter's personal scores. It also depends on the election outcome. This means that even when we use a strategy-proof aggregation function, which incentivizes honesty in the single output case, there are profiles in the multi-candidate case where a voter can gain an advantage by voting dishonestly. Indeed the criterion we have called "strategy-proofness" is, in [4], more accurately called "strategy-proofness-in-grading". It is distinguished there from "strategy-proofness-in-ranking", which would be a voting system where it is never possible for any voter to change the winner to one he likes better by submitting a dishonest vote. In [3], Balinski and Laraki prove that there is no strategy-proof-in-ranking aggregation function.

As an introduction to competitive context, we will examine the no-show paradox. Then we present random-manipulability and voter-manipulability simulation data for range voting, majority judgment, and the linear median in the same vein as that presented for ordinal systems in chapter 3.

8.1 The no-show paradox

One criticism that has been raised against the majority judgment method is that it is susceptible to the no-show paradox, where the addition of a voter who prefers candidate A to candidate B can cause A to lose and B to win. Here is an example:

$$6: A - 90 \quad B - 40$$

$$5: A - 10 \quad B - 40.$$

Here A has a majority-grade of 90 and B has a majority grade of 40, but if another voter is added who grades them both very poorly but slightly prefers A :

$$1: A - 10 \quad B - 0,$$

then A 's majority-grade falls to 10 and B 's remains at 40. So the addition of a voter who prefers A to B indeed causes the winner to change from A to B . Range voting is not susceptible to this no-show paradox.

One of the contributors to the no-show paradox in majority judgment is that there are profiles where the majority-grade changes drastically with the addition of just one voter. As shown in chapter 7, we can choose a strategy-proof aggregation function where a lone voter has considerably less influence on the election outcome, thereby decreasing the likelihood of the no-show paradox. In the above example, the linear median of the scores of candidates A and B before the addition of the twelfth voter are 54.5 and 40, respectively, and after the addition of the last voter they are 50 and 40, so the no-show paradox is avoided by the linear median in this case.

It is not possible, however, to eliminate the no-show paradox entirely. Here is an example of a profile where the no-show paradox will occur with the linear median:

$$7: A - 100 \quad B - 69$$

$$3: A - 0 \quad B - 69.$$

With this profile A 's score is 70 and B 's score is 69, so A wins. If another voter shows up who prefers A to B , but grades them both relatively poorly:

$$1: A - 60 \quad B - 50,$$

then B 's score remains at 69 but A 's score decreases to 63.6, so now B wins. Again, the addition of a voter who preferred A to B has caused the winner to change from A to B .

In both of these cases, if the additional voter had given A a high enough grade, then A would've remained the winner, so in addition to illustrating the no-show paradox for their respective voting systems, these profiles show a case where a voter would have the incentive to vote dishonestly.

8.2 Random and voter manipulability

As a simple measure of how manipulable are these cardinal voting systems, we use the same measures used in chapter 3, namely random manipulability and voter manipulability. For cardinal systems, of course, instead of their preferences being drawn from the set of possible candidate orderings as they were for ordinal voting systems, the voters' evaluations of the candidates are chosen uniformly from the set of grade tuples. It is then determined how likely it is for a random manipulation or a deliberate manipulation by one voter to change the winner to a candidate more preferred by that voter. In the three systems examined, range voting, majority judgment, and linear median, the deliberate manipulation is devised by giving a maximal score to anyone the voter prefers to the candidate who would win if he voted honestly and a minimal score to the rest.

For each system, there is some ambiguity about whether a continuous or discrete interval should be chosen for the set of possible grades, and if a discrete interval is chosen, whether the voter is allowed to have preferences among candidates to whom he gives identical grades. We chose to use the grades for each system which we feel are most likely to be used in real elections. For range voting, we used the integers from 0 to 10. For majority judgment, we used six different grades. And for the linear median, we used the integers from 0 to 100. For range voting and the linear median, whenever a voter gave identical grades to two or more candidates, we simulated him as being indifferent between them. That is, a random or deliberate manipulation that switches the winner to a candidate with the same grade as the original winner is not considered a profitable manipulation. For ma-

majority judgment, the grading language seemed too coarse for this approach¹, so instead we simulated each voter's opinion of each candidate on an integer scale from 0 to 11. These opinions were then converted into the six-term grading language for purposes of simulating elections, but for creating the voter's deliberate manipulation and determining whether a manipulation was profitable, the voter's original opinion (on the scale from 0 to 11) was used.

The results are shown in figure 8.1, and the corresponding data is found in appendix A. The manipulability scores of the ordinal systems from chapter 3 are shown in a shaded grey region for comparison.

8.3 Results

For random manipulability, range voting performs best and the linear median worst, with majority judgment in between. The cardinal systems seem to become more competitive with the ordinal ones as the number of candidates increases. For three candidates, they are competitive only with the most manipulable ordinal methods, but in six-candidate situations, they are competitive with the least manipulable ordinal methods.

For voter manipulability, majority judgment performs best, and the linear median is close behind. Again the cardinal methods become more competitive with the ordinal ones as more candidates are introduced. For three candidates, the best cardinal methods are more manipulable than the worst ordinal methods, but when there are six candidates, the best cardinal methods are almost as good as the best ordinal methods.

When performing the manipulation simulations on these cardinal systems, there are several parameters that can be adjusted, mostly having to do with the grading language used, as detailed above. We tried to choose configurations that would represent the likely dynamics of real elections. It does seem like the linear median might be disadvantaged by having such a large grading language, but the best way to correct this bias is not clear and

¹that is, it would've been unfairly advantageous to the majority judgment by disallowing too many profitable manipulations

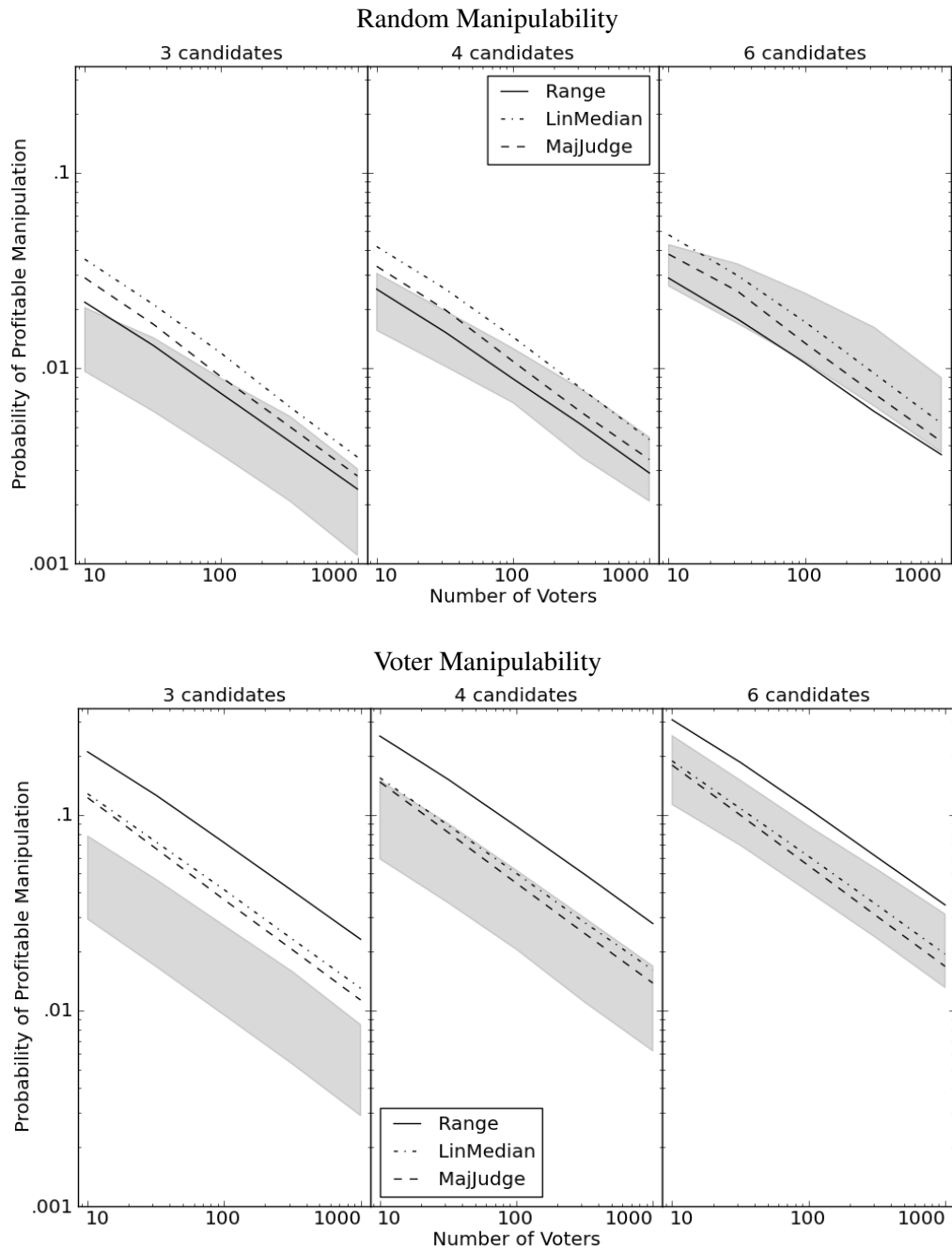


Figure 8.1: Manipulability of cardinal voting systems. Shaded areas indicate the manipulability range of ordinal systems examined in chapter 3 (see figure 3.1).

should be the topic of further research. Still, these simulations give us a rough idea of the manipulability of these cardinal systems and how they compare to the ordinal ones.

8.4 Conclusion

In previous chapters, it was shown that the linear median has many nice aggregation and strategy-resistance properties when it is aggregating the grades for one issue or candidate into a societal grade. In this chapter we explored the dynamics that were introduced when grades are being aggregated for multiple candidates simultaneously. Although the linear median is susceptible to the same no-show paradox that afflicts the majority judgment in competitive situations, the linear median is more effective at limiting the effect that can be had by one voter, so it should be able to decrease the frequency of the no-show paradox.

The majority judgment and the linear median, as well as range voting, were simulated in multi-candidate elections to determine how they fare in terms of random manipulability and voter manipulability, the two manipulability measures introduced in chapter 3. The results showed that with six candidates, the majority judgment and the linear median are competitive with the best ordinal voting methods in terms of minimizing manipulability. Since they satisfy so many theoretical and practical criteria and they offer the hope of elections where each candidate is truly evaluated independently on his own merits, it is clear that these two methods should be included in the canon of acceptable social choice mechanisms and should be attempted in actual elections so we can determine how well they achieve the improvements to social choice and public governance that they promise in theory.

BIBLIOGRAPHY

- [1] K. Arrow. A Difficulty in the Concept of Social Welfare. *The Journal of Political Economy*, 58(4):328–346, August 1950.
- [2] M. Balinski, A. Jennings, and R. Laraki. Monotonic incompatibility between electing and ranking. *Economics Letters*, 105(2):145–147, November 2009.
- [3] M. Balinski and R. Laraki. *Majority Judgement: Measuring, Ranking and Electing*. MIT Press. (forthcoming).
- [4] M. Balinski and R. Laraki. A theory of measuring, electing, and ranking. *Proc. Natl. Acad. Sci. U.S.A.*, 104:8720–8725, May 2007.
- [5] Le Chevalier Jean-Charles de Borda. Mémoire sur les élections au scrutin. In *Histoire de l'Académie Royale des Sciences*, pages 657–665. 1784.
- [6] S. Brams and P. Fishburn. Approval Voting. *American Political Science Review*, 72(3):831–847, 1978.
- [7] Nicolas de Caritat marquis de Condorcet. Essai sur l'application de l'analyse à la probabilité des dcisions rendues à la pluralité des voix, 1785.
- [8] E. Friedgut, G. Kalai, and N. Nisan. Elections Can be Manipulated Often. In *Proc. 49th FOCS, 2008*, pages 243–249. IEEE Computer Society, 2008.
- [9] A. Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, 41(4):587–601, 1973.
- [10] J. Kemeny. *Mathematics without numbers*, volume 88, pages 571–591. Daedalus, 1959.
- [11] R. Laraki. Personal Communication, May 2008.
- [12] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, (35):437–455, 1980.
- [13] M. Satterthwaite. Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions. *Journal of Economic Theory*, 10:187–217, April 1975.

- [14] W. Smith. Range voting. <http://math.temple.edu/~wds/homepage/rangevote.pdf>, December 2000. (retrieved on 7 Oct 2009).
- [15] H. P. Young. Optimal ranking and choice from pairwise comparisons. In B. Groffman and G. Owen, editors, *Information Pooling and Group Decision Making*, pages 113–122. JAI Press, Greenwich, CT, 1986.
- [16] H. P. Young. Condorcets theory of voting. *American Political Science Review*, 82:1231–1244, 1988.

Appendix A

MANIPULABILITY SIMULATION DATA

A.1 Random manipulability - Ordinal Systems

3 candidates					
Number of voters					
	10	32	100	320	1000
Plurality	0.0204	0.0143	0.0087	0.0052	0.0030
Borda	0.0155	0.0093	0.0054	0.0032	0.0017
IRV	0.0135	0.0107	0.0067	0.0038	0.0024
Kemeny-Young	0.0096	0.0060	0.0036	0.0021	0.0011
Schulze	0.0097	0.0061	0.0037	0.0021	0.0011
Majority Borda	0.0188	0.0136	0.0088	0.0057	0.0031
Elections Simulated	2,721,270	2,456,231	988,385	65,978	286,190

4 candidates					
Number of voters					
	10	32	100	320	1000
Plurality	0.0305	0.0196	0.0124	0.0074	0.0040
Borda	0.0225	0.0130	0.0079	0.0042	0.0026
IRV	0.0226	0.0196	0.0126	0.0078	0.0044
Kemeny-Young	0.0156	0.0102	0.0066	0.0035	0.0021
Schulze	0.0158	0.0105	0.0067	0.0036	0.0023
Majority Borda	0.0266	0.0167	0.0098	0.0058	0.0030
Elections Simulated	62,894	155,325	71,989	124,683	115,376

6 candidates					
Number of voters					
	10	32	100	320	1000
Plurality	0.0409	0.0263	0.0160	0.0096	0.0058
Borda	0.0286	0.0183	0.0108	0.0064	0.0037
IRV	0.0428	0.0341	0.0241	0.0162	0.0089
Kemeny-Young	0.0262	0.0169	0.0114	0.0067	0.0039
Schulze	0.0265	0.0186	0.0114	0.0068	0.0044
Majority borda	0.0361	0.0249	0.0129	0.0074	0.0042
Elections Simulated	60,939	60,901	61,062	60,862	61,910

A.2 Voter manipulability - Ordinal Systems

3 candidates

	Number of voters				
	10	32	100	320	1000
Plurality	0.0511	0.0354	0.0219	0.0132	0.0074
Borda	0.0779	0.0466	0.0272	0.0157	0.0085
IRV	0.0316	0.0232	0.0141	0.0087	0.0049
Kemeny-Young	0.0293	0.0169	0.0095	0.0053	0.0030
Schulze	0.0292	0.0167	0.0095	0.0054	0.0029
Majority Borda	0.0594	0.0389	0.0242	0.0144	0.0080
Elections Simulated	2,721,270	2,456,231	988,385	65,978	286,190

4 candidates

	Number of voters				
	10	32	100	320	1000
Plurality	0.1052	0.0715	0.0457	0.0270	0.0156
Borda	0.1498	0.0897	0.0521	0.0296	0.0169
IRV	0.0672	0.0532	0.0322	0.0196	0.0115
Kemeny-Young	0.0595	0.0358	0.0209	0.0113	0.0065
Schulze	0.0594	0.0353	0.0205	0.0109	0.0062
Majority Borda	0.1097	0.0589	0.0310	0.0172	0.0096
Elections Simulated	62,894	155,325	71,989	124,683	115,376

6 candidates

	Number of voters				
	10	32	100	320	1000
Plurality	0.1839	0.1369	0.0876	0.0524	0.0311
Borda	0.2536	0.1492	0.0872	0.0515	0.0280
IRV	0.1361	0.1090	0.0740	0.0485	0.0274
Kemeny-Young	0.1127	0.0695	0.0422	0.0239	0.0134
Schulze	0.1130	0.0693	0.0404	0.0232	0.0131
Majority Borda	0.1541	0.0873	0.0488	0.0258	0.0144
Elections Simulated	60,939	60,901	61,062	60,862	61,910

A.3 Random manipulability - Cardinal Systems

3 candidates

	Number of voters				
	10	32	100	320	1000
Range voting	0.0217	0.0130	0.0074	0.0042	0.0024
Majority judgment	0.0289	0.0167	0.0090	0.0050	0.0028
Linear median	0.0360	0.0211	0.0119	0.0063	0.0035

4 candidates

	Number of voters				
	10	32	100	320	1000
Range voting	0.0253	0.0152	0.0088	0.0051	0.0029
Majority judgment	0.0331	0.0198	0.0108	0.0059	0.0034
Linear median	0.0418	0.0253	0.0144	0.0078	0.0043

6 candidates

	Number of voters				
	10	32	100	320	1000
Range voting	0.0288	0.0178	0.0106	0.0060	0.0036
Majority judgment	0.0382	0.0247	0.0134	0.0074	0.0042
Linear median	0.0480	0.0297	0.0172	0.0094	0.0052

Each probability represents a simulation of at least one million elections.

A.4 Voter manipulability - Cardinal Systems

3 candidates

	Number of voters				
	10	32	100	320	1000
Range voting	0.2095	0.1253	0.0720	0.0406	0.0231
Majority judgment	0.1222	0.0668	0.0368	0.0203	0.0113
Linear median	0.1280	0.0723	0.0414	0.0230	0.0130

4 candidates

	Number of voters				
	10	32	100	320	1000
Range voting	0.2523	0.1508	0.0872	0.0493	0.0278
Majority judgment	0.1471	0.0807	0.0447	0.0246	0.0138
Linear median	0.1543	0.0878	0.0502	0.0280	0.0161

6 candidates

	Number of voters				
	10	32	100	320	1000
Range voting	0.3056	0.1841	0.1070	0.0604	0.0346
Majority judgment	0.1791	0.0989	0.0549	0.0301	0.0168
Linear median	0.1882	0.1070	0.0613	0.0344	0.0194

Each probability represents a simulation of at least one million elections.

BIOGRAPHICAL SKETCH

Andrew Jennings was born in Mesa, Arizona in 1979 to Gordon and Melinda Jennings. He graduated from high school in 1997, served a religious mission from 1998 to 2000, and obtained a B.S. Mathematics degree from Arizona State University in 2003. He was married in 2002 to Rebekah Hoku Lewis, and they now have four wonderful children, Warren, Abigail, Avery and Madeleine, ages 7, 5, 3, and 1.