# Multivariate Charts for Multivariate

# Poisson-Distributed Data

by

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ABSTRACT

There has been much research involving simultaneous monitoring of several correlated quality characteristics that rely on the assumptions of multivariate normality and independence. In real world applications, these assumptions are not always met, particularly when small counts are of interest. In general, the use of normal approximation to the Poisson distribution seems to be justified when the Poisson means are large enough. A new two-sided Multivariate Poisson Exponentially Weighted Moving Average (MPEWMA) control chart is proposed, and the control limits are directly derived from the multivariate Poisson distribution. The MPEWMA and the conventional Multivariate Exponentially Weighted Moving Average (MEWMA) charts are evaluated by using the multivariate Poisson framework. The MPEWMA chart outperforms the MEWMA with the normal-theory limits in terms of the in-control average run lengths.

An extension study of the two-sided MPEWMA to a one-sided version is performed; this is useful for detecting an increase in the count means. The results of comparison with the one-sided MEWMA chart are quite similar to the two-sided case. The implementation of the MPEWMA scheme for multiple count data is illustrated, with step by step guidelines and several examples. In addition, the method is compared to other model-based control charts that are used to monitor the residual values such as the regression adjustment. The MPEWMA scheme shows better performance on detecting the mean shift in count data when positive correlation exists among all variables.

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# TABLE OF CONTENTS

Page

LIST OF TABLES.......... vi

LIST OF FIGURES.......... x

CHAPTER

1 INTRODUCTION.......... 1

1.1 Overview.......... 1

1.2 Statement of the Problem.......... 2

1.3 Potential Contributions.......... 3

2 BACKGROUND LITERATURE.......... 7

2.1 Background.......... 7

2.2 Statistical Performance of the MEWMA chart.......... 7

2.3 Robustness to Non-normal Data.......... 8

2.4 Multivariate Poisson Distribuiton.......... 11

2.4.1 Multivariate Poisson Distribuiton.......... 11

2.4.2 Theta Parameter Estimation Methods.......... 12

2.5 Multivariate Charts for the Multivariate Poisson Distribuiton. 13

2.6 MEWMA Chart and Its Extension to the One-sided Version.. 15

2.7 Other Control Chart Techniques.......... 19

3 TWO-SIDED MEWMA CONTROL CHART.......... 24

3.1 Introduction.......... 24

3.2 Normality test on the Poisson distribution.......... 26

CHAPTER Page

3.3 The Multivariate Poisson Exponentially Wegihted Moving Average (MPEWMA Control Chart ............................................ 27

3.4 Data Simulation ........................................................................ 29

3.5 Results ........................................................................ 32

3.6 General Equation of the Average Run Length ........................ 39

3.7 Effect of Changing the Common Variable ............................. 43

3.8 Comparison of the MPEWMA and MEWMA Control Chart . 47

3.8.1 Out-of-Control ARL Comparison ............................... 47

3.8.2 In-Control ARL Comparison ...................................... 48

3.8.3 SDRL Comparison ...................................................... 49

3.9 Examples ........................................................................ 59

3.9.1 True Parameter Value ................................................. 59

3.9.2 Parameter Estimation .................................................. 63

4 ONE-SIDED MEWMA CONTROL CHART ................................... 65

4.1 Introduction ........................................................................ 65

4.2 One-sided MPEWMA chart ..................................................... 66

4.3 Simulation Conditions ............................................................ 68

4.4 Results ........................................................................ 69

4.5 The one-sided MPEWMA and MEWMA Chart Comparisons 71

4.6 Individuals and a Row of Out-of-Control Signal ..................... 78

4.7 Examples ........................................................................ 83

CHAPTER Page

5 THE MODEL-BASED CONTROL CHARTS.................................. 87

5.1 Introduction.................................................................. 87

5.2 Methodology.................................................................. 88

5.2.1 Regression Adjustment .............................................. 88

5.2.2 Exponentially Weighted Moving Average Chart......... 89

5.3 Simulation Results.......................................................... 90

5.4 Comparison of Model-Based Control Charts ........................... 91

5.5 Example .......................................................................... 92

6 CONCLUSION AND RECOMMENDATIONS ............................. 104

6.1 Conclusion ...................................................................... 104

6.2 Future Works ................................................................... 106

References .................................................................................... 109

APPENDIX

A MATLAB CODE FOR OBTAINING THE CONTROL LIMITS OF THE MPEWMA CHART ......................................................... 117

LIST OF TABLES

Table Page

1. Summary of the $p$-values from the Anderson-Darling test ($n$ = 100-200) ..... 26

2. Shift Matrix for four-variable and six-variable cases ..... 30

3. Examples of the shift size calculation on four-variable case ..... 32

4. The chosen control limit to achieve an in-control ARL of 200 ..... 33

5. ARLs for the MPEWMA control chart when $\theta = 0.5$ ..... 34

6. ARLs for the MPEWMA control chart when $\theta = 1$ ..... 35

7. SDRLs for the MPEWMA control chart when $\theta = 0.5$ ..... 36

8. SDRLs for the MPEWMA control chart when $\theta = 1$ ..... 37

9. The SAS output of fitting the GLM with the exponential distribution to the ARLs obtained from simulation ..... 40

10. The SAS output after dropping the fix common mean variable from the model ..... 41

11. The ARL performance of the MPEWMA chart for detecting the mean shift in one variable when varying $\theta$ ..... 44

12. The ARL performance of the MPEWMA chart for detecting the mean shift in two variables when varying $\theta$ ..... 45

13. The ARL performance of the MPEWMA chart for detecting the mean shift in all variables when varying $\theta$ ..... 46

14. The Steady State ARL for the MEWMA chart proposed by Prabhu and Runger (1997) ..... 48

LIST OF TABLES

Table Page

15. Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.05$ .................................... 51

16. Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.1$ ...................................... 52

17. Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 1$ and $\lambda = 0.05$ ....................................... 53

18. Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 1$ and $\lambda = 0.1$ ......................................... 54

19. Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.05$ .................................... 55

20. Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.1$ ...................................... 56

21. Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 1$ ............................................................ 57

22. Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 1$ ............................................................ 58

23. Example of calculations the T-square statistics of the MPEWMA chart for the first 10 observations ............................................................ 61

24. Summary of the ARL for the one-sided MPEWMA control chart when $\theta = 0.5$ ........................................................................ 72

LIST OF TABLES

Table Page

25. Summary of the ARL for the one-sided MPEWMA control chart when $\theta = 1$ .......... 73

26. Summary of the SDRL for the one-sided MPEWMA control chart when thetafix = 0.5 .......... 74

27. Summary of the SDRL for the one-sided MPEWMA control chart when thetafix = 1 .......... 75

28. Summary of the one-sided MEWMA chart's performance proposed by Joner *et al*. (2008) for 10 variables .......... 76

29. Comparison of the performance between the one-sided MEWMA and MPEWMA charts for in-control ARL of 100 when $\theta = 5$ .......... 76

30. The performance of the one-sided MEWMA applying to the multivariate Poisson distribution with $\theta = 5$ (The advertised in-control ARL of 100 .......... 78

31. The percentage of cases that detected an out-of-control signal for all four scenarios .......... 81

32. Summary of the first out-of-control signal period detected by the one-sided MPEWMA chart for all four scenarios .......... 83

33. Example of the calculation of the first 10 samples of the one-sided MPEWMA statistics .......... 86

34. The Average Run Lengths performance of the regression adjustment with the multiple EWMA charts for 4 variables case .......... 93

LIST OF TABLES

Table Page

35. Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 0.5 and 4 variables case.. .................... 94

36. Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 0.5 and 6 variables case.. .................... 95

37. Comparison of the Average Run Lengths between four multivariate control charts for thetafix = 0.5 and 10 variables case. .................... 96

38. Comparison of the Average Run Lengths between four multivariate control charts for thetafix = 0.5 and 15 variables case. .................... 97

39. Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 1 and 4 variables case.. ....................... 98

40. Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 1 and 6 variables case.. ....................... 99

41. Comparison of the Average Run Lengths between four multivariate control charts for thetafix = 1 and 10 variables case. ..................... 100

42. Comparison of the Average Run Lengths between four multivariate control charts for thetafix = 1 and 15 variables case. ..................... 101

LIST OF FIGURES

Figure Page

1. Comparison of the simulated and calculated ARL plots separated by the number of variables .................................................... 42

2. Comparison of the MPEWMA and MEWMA charts on monitoring the number of defects..................................................................... 61

3. Comparison of the MPEWMA and MEWMA charts based on the in-control condition ..................................................................... 62

4. Comparison of the two MPEWMA charts using the true and estimated mean and variance-covariance matrix.............................................. 64

5. The one-sided MPEWMA chart of the hourly CO concentration...... 86

6. Plot of EWMA charts of the regression adjustment on both Y and Z scales ......................................................................................... 102

7. Plot of MPEWMA chart with $H = 11.49$ ......................................... 103

# Chapter 1

# INTRODUCTION

## 1.1 Overview

The multivariate control charts are widely used to simultaneously monitor several quality characteristics for detecting the mean changes in manufacturing industries (i.e. the measurements in production lines or during the inspection). Various types of the multivariate charts have been explored and discussed extensively, including the Hotelling $T^2$, Multivariate Cumulative Sum (MCUSUM), and Multivariate Exponentially Weighted Moving Average (MEWMA) charts. Typically, the MEWMA scheme is used to detect a shift in the process means, especially for the small shift. The application of the MEWMA chart is not only limited to the manufacturing and service business, but has also been extended to public health and biosurveillance problems. For example, control charting has become more widespread for monitoring disease data and activity during recent years.

Two important assumptions (a multivariate normal distribution and the independence of observations) are made before applying the MEWMA scheme. In practice, the data are most likely observed as counts or number of events of interest, but sometimes the normal assumption is violated. The departure from normality can affect the statistical performance of the MEWMA chart. Rather than ignoring the normality violation, there is a need to investigate the MEWMA chart's performance and develop an appropriate way of monitoring multiple count data.

## 1.2 Statement of the Problem

The objective of this research is applying the multivariate EWMA control chart to a specific problem in industry and syndromic surveillance, particularly for modeling counts or count rates from multiple sources. The situation can be found, for example, in monitoring several types of defects on a layer of wafer (e.g. particles, scratches, and pattern defects) during the fabrication process in the semiconductor industry. Indeed, the defects are considered to be count data and being monitored at very low level. Such data tend to follow the Poisson distribution and depart significantly from the assumption of normality.

The effect of violation of multivariate normality involving the MEWMA chart has not been intensively investigated, and therefore employing the traditional MEWMA scheme to monitor the changes in those defects becomes questionable. It could possibly result in a high early false alarm rate or a poor performance of detecting a shift in the means. In addition, the study of the robustness of the normal approximation to the Poisson distribution is too small, and it could be problematic for determining the appropriate mean value of the Poisson variable to properly approximate by the normal. Thus, an improvement of the traditional MEWMA chart is necessary to increase the accuracy of the detection performance by assuming a proper structure to those counts. A new MEWMA chart for monitoring the multiple correlated count data is proposed as an alternative method to the traditional one.

### 1.3 Potential Contributions

This dissertation consists of three topics related to monitoring multivariate Poisson count data. Firstly, there has been some suspicion on the adequacy of using the MEWMA chart to monitor correlated counts from multiple sources. Since a Poisson distribution is commonly assumed in monitoring count data, the new type of the MEWMA chart that relies on the multivariate Poisson distributed-data is introduced to tackle this problem. The multivariate Poisson model is composed as a sum of two Poisson variables (one to represent the positive correlation among all variables). This new method is referred to as the Multivariate Poisson Exponentially Weighted Moving Average (or MPEWMA) control chart. The control limits are straightforwardly derived from Monte Carlo simulation results based on the multivariate Poisson distribution, instead of the usual the normality assumption.

A general framework for the construction and use of MPEWMA scheme to detect the mean changes in both upward and downward directions is presented. The statistical performance of the MPEWMA chart is examined through the run length distributions in terms of both the average and standard deviation. A comparison of the efficacy is made between the MPEWMA and traditional MEWMA charts at several combinations of the factors, such as the number of variables and the mean values. Besides understanding the effects on the MEWMA performance against the Poisson data, the result can help to clarify whether the use of the normal approximation to the Poisson distribution is appropriate or not and under what circumstances.

Secondly, an extension of the two-sided MPEWMA chart to the one-sided version is discussed for detecting any shift in a specified direction, an upward trend. In many situations, it is not necessary to monitor the mean count changes in both increasing and decreasing directions. For example in public health surveillance, one monitors the number of patients with respiratory disease visiting hospital emergency departments or the incidence rates of disease in various regions. It is desirable to detect only an increase in those counts because the downward shift indicates a better condition, i.e. people tend to become healthier or the spread of disease is not apparent. Hence, applying the two-sided MPEWMA scheme does not seem appropriate and useful since it should not be signaled by a decrease in the count (number of patients) or count rate (the incidence rates of disease).

The Poisson limits of the one-sided MPEWMA chart are again obtained through the same simulation procedure. The one-sided MPEWMA chart's performance is investigated and the results reported are the average and standard deviation of the run lengths. The performance comparison of the one-sided MPEWMA and one-sided MEWMA is examined under a variety of parameter conditions. The results could help to understand the robustness of the one-sided MEWMA chart to the multivariate Poisson distribution and determine when it is appropriate to use the normal approximation to the Poisson data.

For syndromic surveillance application, interpreting an out-of-control signal beyond the control limit as an out-of-control condition is considered to be uncertain. A claim that disease has dramatically increased is sometimes overreacting if the out-of-control signal is truly a false alarm. However, waiting too long before making the claim can cause delay in the prevention of the disease when the disease rate has already increased. This is a trade-off between the detection time and the confidence in an increased rate of disease. In general, the time for detecting the mean shifts tends to be longer while waiting for more signals to occur to ensure an increase in the disease rate. To understand an effect of detection time delay on making the claim, the one-sided MPEWMA scheme's performance is examined not only in an individual out-of-control signal, but also a run of consecutive out-of-control signal (2, 3, 4, or 5 points in a row). The results are reported in terms of the detection time and the percentage of correct detection of the out-of-control state under each of the out-of-control condition described above. This would help determine whether the risks of making decisions affect the detection time or not and how big is the effect; therefore the user should be able to make a better decision on detecting a positive shift in the disease rate.

Thirdly, another type of control scheme called the model-based control chart has been utilized to monitor several correlated count data. The model-based control approaches embrace the process knowledge concept into the use of conventional control charts to improve their sensitivity and efficiency. There are several ways to implement process knowledge, but one implementation discussed in this study is fitting a model to gain more insights into the relationships of quality characteristics being monitored. The residuals after fitting the model will be plotted on the conventional control charts, and therefore it is sometimes referred to as the residual-based control chart. The regression adjustment technique is chosen and used in conjunction with the univariate EWMA scheme. The EWMA scheme is selected for study because it is known to be an effective method of detecting a small mean shift.

The performances of those model-based control charts are investigated for several combinations of the parameters including mean values, number of variables, and various sizes of shift. The average run length (ARL) performances are reported and then compared with the two-sided MPEWMA chart. The results are discussed in more detail to explain how well the regression analysis works with multiple correlated counts, i.e. the performance in removing the correlation and the ability to transform data into an approximately normal distribution. Moreover, a comparison of the ARL results can assist in determining whether the proposed MPEWMA scheme is more useful for early detection of the count changes than those model-based control methods.

Chapter 2

# BACKGROUND LITERATURE

## 2.1 Background

A Multivariate Exponentially Weighted Moving Average (MEWMA) chart is one type of multivariate control charts involving a simultaneous monitoring of several correlated quality characteristics. The MEWMA scheme was firstly introduced by Lowry *et al*. (1992) as a multivariate version of the univariate EWMA chart for detecting a shift in the mean vectors. In general, the MEWMA scheme is applied to monitor the process changes in the manufacturing industries. Testik and Borror (2004) have recommended the use of MEWMA to detect small and moderate shifts in the process means. Typically, a smaller smoothing weight ($\lambda$) is used in favor of detection of a smaller size of shift. Bersimis *et al.* (2006) suggested that the MEWMA scheme outperforms the multivariate Shewhart chart, and for many practitioners it is easier to implement than a multivariate cumulative sum control chart (Fricker, Knitt, and Hu, 2008).

## 2.2 Statistical Performance of the MEWMA chart

The statistical performance of the MEWMA chart is computed and reported in terms of the run length properties. There are three different methods of calculation. The first method is the simulation technique. Both Average Run Length (ARL) and the standard errors of the ARL are derived from simulation over 6,000 times. If all variables have equal interest in monitoring the changes, the ARL performance will be based on a function of the noncentrality parameter ($\delta$) or the shift size (Lowry *et al*., 1992). If the quality characteristics being monitored are not of equal interest (assuming unequal smoothing weight), the ARL will depend on the direction of the shift and can be obtained through the regression adjustment method (Hawkins, 1991).

The second method is using an integral equation. The integral and double-integral equations are developed to approximate the ARL values. The ARL for the in-control case can be estimated by solving a single integral equation whereas the ARL for the out-of-control case is computed by solving a double integral equation (Rigdon, 1995a; and Rigdon, 1995b). The third method involves the Markov Chain approach. The Markov chain model has been extended to estimate the ARL. The MEWMA chart's performance is presented in two conditions: the 'zero-state' and 'steady-state' ARL. The 'zero-state' ARL is obtained as the process starts at the normal condition. The 'steady-state' ARL is calculated by assuming a shift has been introduced after the normal operating process runs for a certain period of time. The ARLs are also reported in terms of a quantity of shift size for several parameter combinations (Prabhu and Runger, 1997; Runger and Prabhu, 1996).

### 2.3 Robustness to non-normal data

The basic assumptions of independence and multivariate normality significantly affect the adequacy of the MEWMA method. Few articles have appeared concerning the MEWMA scheme and its performance when it is applied to non-normal data. The performances of the MEWMA chart were investigated using the multivariate t and gamma distributions with various values of skewness and kurtosis up to ten variables (Stoumbos and Sullivan, 2002) and up to twenty variables (Testik, Runger, and Borror, 2003). The MEWMA chart's performance is found to be better than Chi-square ($\chi^2$) chart in terms of both larger in-control and smaller out-of-control ARL values. In both works the MEWMA scheme relies on the asymptotic covariance. Stoumbos and Sullivan (2002) mentioned that the use of an exact covariance matrix in calculating the MEWMA statistics can actually decrease the robustness again the non-normal data. The MEWMA scheme with a large number of data points and a range of the smoothing weight (between 0.02 and 0.05) is sufficient to ensure a central limit theorem and hold for robustness. For the high dimensional case, a smaller value of the smoothing weight ($\lambda$) is recommended for increasing robustness to a non-normal distribution. However, a significant chance of having early false alarm leads to a departure from the multivariate normality, and therefore the robustness becomes an issue.

Testik, Runger, and Borror (2003) state that the in-control performance may be decreased in monitoring non-normal data, that is, the false alarm rate is likely to increase. Generally, the MEWMA chart with the weight constant of 0.05 is recommended due to its good performance in detecting the changes and robustness under non-normal conditions, similarly to Testik and Borror (2004). Testik and Borror (2004) noted that the smaller $\lambda$ value can provide greater robustness, but it also delays the detection time when the MEWMA vectors go in the opposite direction relative to the occurrence of a shift. It is referred to as the inertia problem. For more details of this problem and solutions, see Lowry *et al*., 1992; Niaki and Abbasi, 2005; Woodall and Mahmoud, 2005. Unfortunately, there was no further study on the robustness again multivariate Poisson data for the MEWMA scheme.

In fact, the MEWMA chart is also employed to monitor Poisson counts by assuming normality. Typically, the normal approximation for Poisson data will suffice for the large mean counts. For the univariate and multivariate control charts, several authors have suggested that a good approximation to the Poisson distribution with a normal distribution can be obtained if the Poisson mean is 5 or more (Xie, Goh, and Kuralmani, 2002), the Poisson mean is greater than 10 (Joner *et al*., 2008), the Poisson mean exceeds 12 (Box, Luceño, and Paniagua-Quiñones, 2009) and the Poisson mean is at least 15 (Montgomery, 2009). Moreover, Testik, Runger, and Borror (2003) advised that the central limit theorem can be applied to the MEWMA if the number of samples is large enough. However, there has been no clear cut-off values for the mean and appropriate sample sizes to provide more accurate approximation to the Poisson.

Since those earlier reviews do not provide much information about the efficiency of the traditional MEWMA scheme to Poisson-distributed data, the use of the MEWMA chart with the normal-theory limits in such a scenario remains in doubt due to the accuracy of the normal approximation. In our study, we introduce a new MEWMA scheme based on the multivariate Poisson distribution. The simulation method is used to calculate the appropriate control limits and estimate the performance of the proposed chart. Details on the multivariate Poisson model and parameter estimation are described in Section 2.4. In Section 2.5, a review of the multivariate control charts that rely on the multivariate

Poisson assumption are discussed. In Section 2.6, an extension of the MEWMA scheme to the one-sided test is presented. A discussion of other control chart techniques for dealing with multivariate data is provided in Section 2.7.

## 2.4 Multivariate Poisson Distribution

### 2.4.1 Multivariate Poisson Random Variables

The multivariate Poisson distribution was introduced in two different forms. Kawamura (1979) presented the multivariate Poisson model in terms of the sum of $p$ independent random Poisson variates. Johnson, Kotz, and Balakrishnan (1997) proposed a structure of a multivariate Poisson distribution involving the correlated Poisson variates. For the control chart application in this research, the multivariate Poisson distribution is based on the work of Johnson, Kotz, and Balakrishnan (1997). The $p$ multivariate Poisson random variables are defined as

$$X_i = Y_i + Y \text{, for } i = 1, 2, \ldots, p \tag{1}$$

where $Y$ and $Y_i$ are independent Poisson random variables with means $\theta$ and $\theta_i$, respectively and $X_i$ are Poisson random variables with means $\theta + \theta_i$ for $i$ = 1, 2, ..., $p$. The variance-covariance matrix of $X_1,...,X_p$ has diagonal elements, $Var(X_i) = \theta + \theta_i$ and off-diagonal elements, $Cov(X_i, X_j) = \theta$. Elements of the variance-covariance matrix are

$$Var(X_i) = \theta + \theta_i \qquad , i = 1, 2, \ldots, p \tag{2}$$

$$Cov(X_i, X_j) = \theta \qquad , j = 1, 2, \ldots, p \text{ and } i \neq j \tag{3}$$

The fixed parameter, $\theta$, corresponds to an event or mean common to all $p$ random variables. Let's use the previous scenario where monitoring three types of defects (particles, scratches, and pattern defects) as an example. The particle defects on the layer of the wafer ($X_1$) are the combination of the effect of particle defects on the layer ($Y_1$) (such as etching process) and the effect of original wafer quality ($Y$). The quality of the original wafer can also affect other types of defects; in other words, the scratch defects on the layer of the wafer ($X_2$) are the combination of the effect of scratch defects on the layer ($Y_2$) (such as polishing process and handling equipment) and, again, the effect of original wafer quality ($Y$). Thus, the effect of original wafer quality is considered as the common relationships among all types of defects. Skinners, Runger, and Montgomery (2006) have recommended using this model for monitoring several types of defects per unit of product (such as defects in assembly automobiles) or defects per area of product (such as defects in paper or cloth products).

The estimation of all parameters, especially the fixed parameter, is an important issue, if it is not assumed to be known. We provide a brief description of the various methods for obtaining $\theta_i$ and $\theta$ as follows.

### 2.4.2 Theta Parameter Estimation Methods

Holgate (1964) compared two ways of estimating the parameter $\theta$ for the bivariate Poisson distribution: 1) the maximum-likelihood estimation and 2) the method of moments. The method of moments is considered efficient with two uncorrelated variables. If the correlation increases, the efficiency of the method of moments tends to decrease whereas the maximum-likelihood method provide more precise due to the reduction in variance of maximum-likelihood estimator. Karlis (2003) used an EM algorithm to approximate the parameters $\theta_i$

of multivariate Poisson distributions. The E-step is used to calculate the estimates (or pseudo-values). The estimates are then updated by the M-step. One restriction is to pick the initial values for $\theta_i$ in the feasible range such as $\theta_i > 0$, otherwise the final values will not be in the admissible range.

Jost *et al*. (2006) proposed a new approach based on the composite likelihood concept of Lindsay (1988) to estimate parameters. The optimal composite likelihood estimator can be derived by using an iterative approach to solve the equation relating a pairwise log-likelihood function below

$$\sum_{u=1}^{m-1} \sum_{v=u+1}^{m} w_{uv} \frac{\partial l_{uv}(\theta)}{\partial \theta} \Big|_{\theta=\hat{\theta}} = 0 \tag{4}$$

where $m$ is the number of variables and $w_{uv}$ is the weight where in general $w_{uv}=1$ for $1 \leq u < v \leq m$, and $l_{uv}(\theta)$ is the bivariate marginal log-likelihood function between two variables $X_u$ and $X_v$.

This new method is more effective than the method of moments, and requires less computational effort than the maximum-likelihood method. He also mentioned the disadvantage of Karlis (2003) that the computation becomes more complicated as the multivariate Poisson distribution involves a large number of variables (eight or more).

### 2.5 Multivariate charts for the multivariate Poisson distribution

There have been many articles involved in introducing the new types of multivariate charts that relied on the multivariate Poisson distribution. The first control chart for the multivariate Poisson distribution was presented by Patel (1973). The ‘G-statistic’, similar to the Hotelling T-square statistic, is calculated and plotted on the chi-square control chart. The control scheme discussed in Patel (1973) has not been used in practice because of the complexity of obtaining the 'G-statistic'. Skinners, Runger, and Montgomery (2006) proposed two types of schemes to detect the change in the means of multiple Poisson counts. Firstly, the Deleted-Y chart based on the moment estimator is recommended for only one or two variables shifted when the all mean counts are assumed equal. Secondly, the $\overline{Y}$ chart computed from the sample mean is proposed to detect a change in all variables. Since both Deleted-Y and $\overline{Y}$ statistics are plotted on $p$ individual Shewhart charts, they may not be easy to use for the higher-dimensional problems.

Chiu and Kuo (2008) studied two new types of control charts for monitoring multivariate Poisson counts with correlated variables: 1) the multivariate Poisson (MP) chart and 2) a Shewhart-type chart. The control limits of the MP chart can be obtained by either the exact distribution based on the sum of all Poisson variates or a multiple linear regression method (Kuo and Chiu, 2008). The control limits of the Shewhart-type scheme are derived from the normal approximation to the Poisson distribution. The result shows that using the normal approximation to the Poisson distribution is good for a mean count of five or larger. It can be seen that the MP chart performs better than the Shewhart-type in terms of the in-control ARL, but the out-of-control ARL performance is sensitive to an increase in the coefficient of correlation. One limitation of the result is that the authors only examined the run length performance for two and three variable problems, not in the higher-dimensional case. In addition, it is restricted to the case of positive correlation among the variables being monitored because the multivariate Poisson model used in this work is expanded from the bivariate Poisson model proposed by Holgate (1964).

### 2.6 MEWMA chart and its extension to the one-sided version

The MEWMA chart is generally applied to monitor both positive and negative changes in the process means. In addition to the industrial and business applications, the quality control method also has great potential for use in the area of public health-care and bioterrorism surveillance. An increasing number of papers have studied the outcome from applying the control chart to detect and monitor diseases in public health surveillance. The implementation of the MEWMA chart for public-care and bioterrorism monitoring has been recently discussed by many authors (Burkom *et al*., 2005; Yan, Chen, and Zeng, 2008; and Woodall, 2006). Rolka *et al*. (2007) addressed the MEWMA chart as one of several techniques for detecting events of bioterrorism-related outbreak. However, it is necessary to improve the outcome and avoid false alarm triggered by unrelated events. Fricker, Knitt and Hu (2008) found a similar performance between the directional MCUSUM and MEWMA charts in biosurveillance application. However, the MEWMA scheme is preferred based on practical reasons for selecting parameters.

A review of statistical methods in modern biosurveillance, describing a variety of control charts including the MEWMA chart, is given by Shmueli (2009). The author outlines some concerns with applying traditional multivariate charts to syndromic data. One concern is the data most often do not follow a multivariate normal distribution nor is the independence assumption satisfied. It is difficult to justify that bio-surveillance data follow a multivariate normal distribution since the variety of data sources come from widely diverse environments (Shmueli and Fienberg, 2006). Fricker (2009) also noted that the natural occurrence of autocorrelated data cannot be well monitored by standard SPC techniques used in manufacturing. Another concern is related to the covariance structure for standard multivariate SPC techniques. The covariance structure is often assumed to be constant across time. Empirical evidence has shown that when the data is syndromic, the covariance structure changes over time. Therefore, applying the standard multivariate charts in these situations should be done with caution since the covariance structure departs from its intended application and original setting. Finally, it is more reasonable to detect only when an increase in syndromic data has occurred. Consequently, the standard control charts must be modified so that they are more sensitive to certain directional shifts. For example, one-sided monitoring techniques modified for surveillance of syndromic data will often result in better detection performance than two-sided monitoring methods (Lotze and Shmueli, 2008).

Discussions of the design of MEWMA control chart are extended to a one-sided MEWMA for detecting only an increase in the mean shift. The one-sided MEWMA scheme has been studied and appears in many literatures. FassÒ (1999) modified the multivariate EWMA chart for the bivariate case by using a restricted maximum likelihood estimator (MLE) to the MEWMA statistics. The resulting one-sided MEWMA control chart is designed to monitor an upward shift in at least one quality characteristic when no variables have a decreased rate. Unfortunately, this approach has not been extensively used because of the method's complexity and the restrictive assumptions.

Testik and Runger (2006) extended the one-sided MEWMA proposed by FassÒ (1999) for use in a higher dimensional problem. Another control method is proposed for the case where at least one variable shifts either upward or downward (one-sided test for some variables) and others move in any direction (two-sided test for the remaining variables). The new approach is referred to as the partial one-sided control chart. FassÒ and Locatelli (2007) also developed an asymmetric MEWMA chart that is similar to the partial one-sided chart by Testik and Runger (2006) which allows the remaining quality characteristics to change in both upward and downward direction. Testik and Runger (2006) and FassÒ and Locatelli (2007) obtained the MEWMA statistics by quadratic programming. The slight difference between the two methods is that the control chart statistics of the asymmetric MEWMA is computed using the asymptotic covariance matrix, but is not necessary for the partial one-sided chart.

Sonesson and Frisén (2005) recommended applying an individual upper CUSUM limit to the MEWMA chart introduced by Lowry *et al*. (1992). The proposed method can detect an upward shift in some quality characteristics without being affected by the downward shifts of other quality characteristics. Stoto *et al*. (2006) modified the multivariate CUSUM (MCUSUM) chart to detect positive shifts by limiting the MCUSUM statistics to be positive values only.

Note that each of the methods given above is based on the multivariate normality assumption.

Joner *et al*. (2005) and Joner *et al*. (2008) presented a new one-sided MEWMA chart to detect a small upward shift in the incidence rates of disease. This one-sided MEWMA scheme is built up from two works - Sonesson and Frisén (2005) and Stoto *et al*. (2006). One good feature of the new control method is that it should not take too long to detect an abrupt increase when there is evidence of a continuing decrease in the incidence rates before. This is a result of placing a 'barrier' (or zero) within the equation of MEWMA statistics calculation to prevent the negative results. Consequently, the decrease does not greatly affect the next computation for detecting the upcoming increase in the incidence rates. This approach relies on the assumption that the normal approximation to the true underlying distribution (such as the Poisson) is appropriate (means greater of 10 or more). There are, however, some situations where the normal approximation to the Poisson distribution is not necessarily true. In particular, when the process mean is quite small. In these situations, an adequate mean for using the normal approximation is still an issue in the multivariate case and, therefore monitoring techniques based directly on the Poisson distribution are recommended.

Recently, there has been a review of the robustness of the one-sided MEWMA chart to multivariate Poisson data. Yahav and Shmueli (2010) investigated the performance of the Hotelling T-square and two types of one-sided MEWMA charts (modified Follmann (1996) and Testik and Runger (2006)'s work) under a simulated multivariate Poisson distribution. The multivariate Poisson model is generated by the work of Yahav and Shmueli (2009). The mean rates (from 1, 5, 10, and 20) are tested and the variance-covariance matrix is assumed to be known. Two extended one-sided MEWMA charts show superior performance to the Hotelling T-square based on the in-control ARL. This finding is similar to the result that the two-sided MEWMA chart is more robust to the multivariate t and gamma distributions than the Hotelling T-square, as previously discussed by Stoumbos and Sullivan (2002).

### 2.7 Other control chart techniques

Control chart methods are normally employed on the raw data. A new method has been developed by combining other modeling techniques with the quality control monitoring of multivariate data. In the other words, this new method consists of two steps in data monitoring: 1) a pre-process step and 2) a control step. The first step is transforming the multivariate data to gain more insights into a diagnosis such as applying regression analysis. Once the model has been found, the residuals are calculated and used in the next step. The second step is monitoring these residuals on control chart for detecting the mean changes. Thus, the method is called the model-based control chart, or sometimes it is referred to as the residual-based control chart.

There have been many papers recently that developed the model-based control charts (see Hawkins, 1991; 1993; Healy, 1987; Mandel, 1969; Skinner, Montgomery, and Runger, 2003; and Zhang, 1984). The model-based control technique was firstly introduced by Mandel (1969). The Regression control chart was aimed to monitor the varying mean by using the conventional control chart in conjunction with the regression method. The idea of Mandel (1969) has been extended to a cause-selecting chart (Zhang, 1984) for monitoring two process steps. The outgoing variable is monitored by applying regression to adjust for the effect of an incoming variable. One good feature is that it can help determine which subprocess goes out of control. Healy (1987) expanded a CUSUM control method to detect the mean shift in the multivariate case. The proposed CUSUM chart based on a linear combination of the variables is recommended if shifts in a known direction are expected. If the shifts are expected in more than one direction, the CUSUM of orthogonal linear combinations is needed to assure independence.

Two types of regression adjustment are proposed by Hawkins (1991; 1993). The first method involves the problem of correlated variables and expecting a shift in the mean of a single variable does not affect the remaining variables. Hawkins (1991) recommended applying the Z transformation to the data rescaled to zero mean and unit variance for further improving the Hotelling T-squared chart. The control chart based on the Z scaled residuals are obtained from regressing each variable on all others (e.g. regression $X_j$ on $X_1$, $X_2$,…, $X_{j-1}$, $X_{j+1}$,…, $X_p$) and plotting them in multiple univariate control charts, such as CUSUM charts. The second method relates to a process having a natural ordering, and therefore a shift can affect some or all subsequent variables, not the prior variables. It is referred to as a cascade process. Hawkins (1993) introduced other ways to transform the X scale to the vector Y and W scales by firstly standardizing variables to zero mean and unit standard deviation. The Y scaled residuals are computed from regression each variable on all preceding variables (e.g. regression $X_j$ on $X_1$, $X_2$,…, $X_{j-1}$). The vector W is defined by scaling vector X using principal components. The W scale will work reasonably if a shift has occurred in the direction of one of the principal components. It is considered less useful than other scales due to a restriction that shifts of the mean should be in a direction along one of the principal components axes. However, it is not obvious that Y or Z scales give a better performance. Those decomposition approaches can be extended for use with other schemes, including the univariate and multivariate EWMA charts.

Besides the regression adjustment, another regression technique is proposed for situations where the data are obtained from a biosurveillance system. Burkom *et al*. (2004) discussed the concept of sliding buffers under the baseline period for aggregated data. He suggested applying the control chart method to normalized data (i.e. the residuals of linear or Poisson regression) if the raw data show systematic behaviors. The comparison results indicate that using a multiple EWMA chart with the baseline length obtained from the empirical test provides better performance than the Hotelling T-square chart with the residuals of Provider-count regression. Fricker, Knitt, and Hu (2008) applied the "adaptive regression model with a sliding baseline" presented by Burkom *et al*. (2004) and Burkom, Murphy, and Shmueli (2007) to remove the systematic components in the biosurveillance data. The residuals are plotted on the directional multivariate CUSUM and EWMA charts.

To effectively eliminate the systematic components, there is a need to determine the appropriate values of parameters used in the adaptive regression such as forms of the regression model (linear and quadratic models) and the length of the sliding baseline. Pre-processing is also suggested to remove deviation from the normality assumption and autocorrelation (Lotze, Murphy, and Shmueli, 2008; and Yahav and Shmueli, 2010). Those residuals that go through the pre-process will satisfy the control chart requirements, and then they are applicable for the quality control methods.

Lotze, Murphy, and Shmueli (2008) pointed out that the preconditioning (e.g. linear regression, log regression, and differencing) can reduce the seasonality impact in the syndromic data. Since there may be many explainable patterns in the data, a failure to remove all those patterns could have a remarkable effect on the results of the control charting methods. In particular, biosurveillance data with extremely low counts significantly departs from the normality assumption. Hence, using the control chart on the unprocessed data may lead to failure of detection of the presence of an outbreak or an increasing numbers of false alarm rates. The preprocessing methods used before applying the CUSUM chart to the actual data have shown improvements by removing variation from other irrelevant sources.

The ordinary least square regression technique above is limited to the normally distributed data. For non-normal data, the generalized linear model-based control charts were initiated to monitor counts (Skinner, Montgomery, and Runger, 2003) and over-dispersed counts (Skinner, Montgomery, and Runger, 2004) from multiple sources. The deviance residuals are calculated by using the predicted value obtained from fitting the generalized linear model with an appropriate link. The deviance residuals used in conjunction with the *C* chart show superior performance to the *C* chart itself in both univariate and bivariate cases. The link for the model should be selected with care, since it could result in bad predictions. Lewis, Montgomery, and Myers (2001) investigated the confidence interval coverage of the mean response when the incorrect link is assumed. The result demonstrated that a misspecified link has an impact on the model performance, especially for Poisson data. Precision is reduced and the confidence interval coverage is degraded by the misspecified link. In addition, the normal probability plot of the deviance residuals also showed the possible insufficiency of the fit model.

Chapter 3

# TWO-SIDED MEWMA CONTROL CHART

## 3.1 Introduction

A Multivariate Exponentially Weighted Moving Average (MEWMA) control chart is generally used to simultaneously monitor several correlated quality characteristics for many applications in manufacturing and business. The implementation of the MEWMA chart requires an assumption of a multivariate normal distribution. In real world situations, there has been interest in monitoring a small change in the count or count rate of occurrence of an event. A few simple examples of quantities that are monitored are the number of defects found at inspection stations, the number of car accidents that occurred at major junctions during peak traffic periods, and the number of customer complaints about service quality to service providers. These sample counts are usually assumed to follow a Poisson distribution. Since no extension of the MEWMA chart is developed for the multivariate Poisson distribution, the normal approximation to the Poisson can be used for applying the MEWMA chart.

There has been no extensive assessment of the MEWMA control scheme performance for monitoring multiple Poisson-distributed variables when the assumption of the normal approximation to the Poisson distribution is not necessarily valid. The adequacy of the normal-distribution model for traditionally Poisson-distributed data is an issue of concern, particularly if the process means are small (say 5 or less). In addition, control chart performance, which corresponds to the normal approximation assumption, is often evaluated assuming the covariance structure does not change along with a shift or change in the process mean. This is not true for the Poisson distribution because an upward shift in the mean also results in an increase in the variance. If the covariance matrix remains constant after a shift in the mean has occurred, then it could affect the shift size calculation and probably lead to an incorrect summary of the run length distribution.

A new type of multivariate EWMA chart that relies on the multivariate Poisson distribution has been studied and proposed to properly handle this problem. It can be referred to as the multivariate Poisson Exponentially Weighted Moving Average (MPEWMA) control chart. Monte Carlo simulation is utilized to obtain the appropriate control limits which correspond to an in-control Average Run Length (ARL) of 200. The statistical performance of the MPEWMA chart is reported in the form of ARL and Standard Deviation of the Run Length (SDRL). In addition, comparison of the proposed MPEWMA and the traditional MEWMA chart's performance is made in terms of the ARLs.

In Section 3.2, we assess the normality of the multivariate Poisson distribution. Section 3.3 describes the MEWMA chart. Section 3.4 discusses the details of simulation method. Section 3.5 presents and summarizes the ARL and SDRL results. Section 3.6 develops the general equation to estimate the control chart's performance. Section 3.7 compares the performance of the traditional MEWMA and MPEWMA charts. Section 3.8 illustrates an example of using the MPEWMA scheme.

### 3.2 Normality test on the Poisson distribution

The adequacy of the normal approximation to multivariate Poisson-distributed data is examined by performing the Anderson-Darling normality test. We illustrate an example of testing the normality on a four-variate Poisson distribution, X = [X1, X2, X3, X4]. Suppose all four means ($\theta_1 + \theta, \theta_2 + \theta, \theta_3 + \theta, \theta_4 + \theta$) are assumed equal. We consider five multivariate Poisson distributions with mean 5, 15, 25, 30, and 35, respectively. Each sample data ($X_1$, $X_2$, $X_3$, and $X_4$) is randomly generated from each of these five distributions with various $\theta$ values (0, 0.0005, 0.05, and 1) for a minimum sample size ($n$) of 100 to 200 observations. The Normal probability plots are constructed and the resulting $p$-values from the Anderson-Darling test were calculated. The $p$-values of the first two variables (X1 and X2) are reported in Table 1 below.

**Table 1** Summary of the $p$-values from the Anderson-Darling Test ($n$ = 100-200)

| Mean | $\theta = 0$ | | $\theta = 0.0005$ | | $\theta = 0.05$ | | $\theta = 1$ | |
|---|---|---|---|---|---|---|---|---|
| | P-value of $X_1$ | P-value of $X_2$ | P-value of $X_1$ | P-value of $X_2$ | P-value of $X_1$ | P-value of $X_2$ | P-value of $X_1$ | P-value of $X_2$ |
| 5 | < 0.005 | < 0.005 | < 0.005 | < 0.005 | < 0.005 | < 0.005 | < 0.005 | 0.006 |
| 15 | 0.024 | 0.028 | < 0.005 | < 0.005 | < 0.005 | 0.11 | 0.015 | 0.067 |
| 25 | 0.226 | 0.018 | 0.308 | 0.16 | 0.005 | 0.109 | 0.065 | 0.085 |
| 30 | 0.193 | 0.017 | 0.053 | 0.048 | 0.466 | 0.021 | 0.098 | 0.229 |
| 35 | 0.021 | 0.384 | 0.131 | 0.191 | 0.389 | 0.115 | 0.057 | 0.332 |

It can be seen that the normal approximation is not always valid, particularly when the mean of the Poisson process is small (e.g., means of 5 and

15). As a result, control charts based on the assumption of normal-theory limits may not be appropriate when monitoring Poisson data. Thus, there is a need for monitoring techniques based on the true underlying distribution of the data.

### 3.3 The Multivariate Poisson Exponentially Weighted Moving Average (MPEWMA) Control Chart

The new type of the MEWMA scheme is developed based on the traditional multivariate EWMA chart. Lowry *et al*. (1992) proposed the MEWMA as an extension to the univariate EWMA chart. The MEWMA scheme takes into account recent past data which often results in quicker detection of the shifts in the process mean. Let's say that $p$ quality characteristics are being monitored simultaneously. The MEWMA statistic is given by

$$\mathbf{Z}_t = \mathbf{R}\mathbf{X}_t + (\mathbf{I} - \mathbf{R})\mathbf{Z}_{t-1} \tag{5}$$

where $\boldsymbol{Z}_t$ is the $t^{\text{th}}$ MEWMA statistics vector, $\boldsymbol{X}_t$ is the $t^{\text{th}}$ observation vector for $t$ = 1, 2, ..., $n$ and $\boldsymbol{Z}_0 = \mathbf{0}$. The vector $\boldsymbol{R}$ consists of weights assigned to past observations in each of the $p$ quality characteristics being monitored and $\boldsymbol{I}$ is the $p \times p$ identity matrix. Specifically, let $r_j$, represent the weight assigned to the $j^{\text{th}}$ quality characteristic, then $\boldsymbol{R} = \text{diag}(r_1, r_2, \ldots, r_p)$, where $0 < r_j \leq 1$ and $j$ = 1, 2,…, $p$. If equal weight is assigned to each random variable so that $r_1 = r_2 = \ldots = r_p = \lambda$, then

$$\mathbf{Z}_t = \lambda\mathbf{X}_t + (1 - \lambda)\mathbf{Z}_{t-1} \tag{6}$$

The covariance matrix for the random variable $\mathbf{Z}_t$ is

$$\Sigma_{\mathbf{Z}_t} = \left\{ \frac{\lambda\left[1-(1-\lambda)^{2t}\right]}{2-\lambda} \right\} \Sigma \tag{7}$$

where $\sum$ is the covariance matrix for the $p$ random variables and is assumed to be known. (Assuming a known covariance matrix is common when evaluating monitoring techniques). If the covariance matrix is unknown, then it can be estimated using a number of possible methods (see, e.g., Sullivan and Woodall, 1995; and Williams *et al*., 2006). As $t \to \infty$, the asymptotic covariance matrix can be written as

$$\Sigma_{\mathbf{Z}_t \to \infty} = \left\{ \frac{\lambda}{2-\lambda} \right\} \Sigma \tag{8}$$

The MEWMA control chart statistic is given by

$$T_t^2 = \mathbf{Z}_t' \Sigma_{\mathbf{Z}_t}^{-1} \mathbf{Z}_t \tag{9}$$

An out-of-control signal will occur if $T_t^2 > H$, where $H > 0$ is a threshold limit selected in order to achieve a desired in-control ARL. The choices of the parameters $H$ and $\lambda$ can have significant effects on the performance of the MEWMA chart and should be selected with care.

Since all $p$ random variables being monitored truly follow a multivariate Poisson distribution, the data are generated from the multivariate Poisson model, as earlier discussed in Section 2.4.1, using Monte Carlo simulation. The MPEWMA statistic is obtained simply through the same steps for calculating the MEWMA statistics (from Equation (5) – Equation (9)). The asymptotic covariance matrix as shown in Equation (8) is used as the covariance matrix of the MPEWMA chart. We also consider two additional factors (the mean value and the thetafix parameter) in determining the control limits of the MPEWPA chart. We will present results for various combinations of these parameters in order to obtain in-control ARL of interest.

### 3.4 Data Simulation

The proposed control chart is called the multivariate Poisson exponentially weighted moving average (or MPEWMA) chart. In this simulation study, the means $\theta + \theta_1, \ldots, \theta + \theta_k$ generated from the Poisson distribution to be investigated are 3, 5, 8, 10, and 15. Two smoothing weights ($\lambda$ = 0.05, and 0.1) are selected for $p$ = 4, 6, 8, and 10 variables. To simplify the study, the means of all variables are assumed equal. As previously mentioned, the chosen values of $\lambda$ have been shown to be effective in detecting small shifts in the process mean. Values of $\theta$ were arbitrarily chosen to be 0.5 and 1. The MPEWMA control chart is studied under the "steady-state" condition. A "steady-state" control chart is defined as a control chart that operates in statistical control for some period of time. To simulate the steady-state condition and then a shift in the process mean, we allow the control chart to run under normal conditions for one-hundred time periods before a shift in the process is introduced at time period 101. The simulation continues until either the first out-of-control signal is found or the simulation routine reaches 100,000 iterations. Each simulation is replicated 50,000 times to provide more accurate results.

We are interested in the capability of the monitoring scheme to detect the increase in the mean (shifts) for one or more of the variables. The scenario of interest is limited to a permanent upward shift, or a long-lasting increase in the means. The performance of the MPEWMA scheme is evaluated using various sizes of the mean shifts such as increases of one up to four units in one or more variables. Table 2 displays a list of all shifts that we applied to the four-variable and then the six-variable cases. To illustrate how to interpret the notation in Table 2, suppose we have four Poisson processes with equal means that are being monitored simultaneously and a shift of 2 units has occurred in only one of the processes, say process 3. This can be represented by the notation [0, 0, 2, 0], which can be interpreted as no shift in the mean for the first two processes, a two unit shift in mean for the third process, and no shift in the mean for the fourth process.

**Table 2** Shift matrix for four-variable and six-variable cases

| No. of Shift matrix | Variable shift $X_1, X_2, X_3, X_4$ | Variable shift $X_1, X_2, X_3, X_4 X_5, X_6$ |
|---|---|---|
| 1 | 0,0,0,0 | 0,0,0,0,0,0 |
| 2 | 1,0,0,0 | 1,0,0,0,0,0 |
| 3 | 0,1,0,0 | 0,1,0,0,0,0, |
| 4 | 2,0,0,0 | 2,0,0,0,0,0 |
| 5 | 0,2,0,0 | 0,2,0,0,0,0 |
| 6 | 1,1,0,0 | 1,1,0,0,0,0 |
| 7 | 2,2,0,0 | 2,2,0,0,0,0 |
| 8 | 1,0,1,0 | 1,0,1,0,0,0 |
| 9 | 0,0,2,0 | 0,0,2,0,0,0 |
| 10 | 1,0,0,1 | 1,0,0,1,0,0 |
| 11 | 0,0,0,2 | 0,0,0,2,0,0 |
| 12 | 1,1,1,1 | 1,0,0,0,1,0 |
| 13 | 2,2,2,2 | 0,0,0,0,2,0 |
| 14 | 3,3,3,3 | 1,0,0,0,0,1 |
| 15 | 4,4,4,4 | 0,0,0,0,0,2 |
| 16 | | 1,1,1,1,1,1 |
| 17 | | 2,2,2,2,2,2 |
| 18 | | 3,3,3,3,3,3 |
| 19 | | 4,4,4,4,4,4 |

The shift size or 'noncentrality parameter' ($\delta$) is based on Lowry *et al.*'s work (1992) and defined as

$$\delta = [(\boldsymbol{\mu} - \boldsymbol{\mu}_0)' \Sigma^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)]^{1/2} \quad (10)$$

where $\boldsymbol{\mu}_0$ represents the mean vector for an in-control process, $\boldsymbol{\mu}$ represents the mean vector after a shift has occurred, and $\sum$ is the variance-covariance matrix. It can be noted that equation (10) is also referred to as the Mahalanobis' distance. As shown in equation (10), the shift size is related to changes in both the mean and covariance matrix.

For the multivariate Poisson model, an increase in any element of the variance-covariance matrix, $\sum$, corresponds to a shift in one or more means, $\theta + \theta_i$, which are diagonal elements of the covariance matrix. In other words, we take into account the effect of the mean shifts on the variance-covariance matrix to obtain a better estimate of the MPEWMA statistics in equation (9) and the shift size calculation in equation (10). Table 3 displays the shift size calculation assuming the means of all four variables are 3 with two values of $\theta$ ($\theta = 0.5$, and 1). For example, the shift size using equation (10) for the case of [0, 0, 2, 0] would be $\delta = 0.912$ (for $\theta = 0.5$) and $\delta = 0.953$ (for $\theta = 1$). It is important to note that shifts in the process means will not always result in the same overall shift size ($\delta$). For example, [0, 0, 2, 0] and [1, 1, 0, 0] both represent a total two-unit shift in the process. However, $\delta = 0.912$ for [0, 0, 2, 0] and $\delta = 0.689$ for [1, 1, 0, 0] when $\theta = 0.5$. Therefore, there can be slightly different resulting shift sizes for any two processes that may have the same total *unit* shift.

**Table 3** Examples of the shift size calculation on four-variable case

| No. of Shift matrix | Variable shift $X_1, X_2, X_3, X_4$ | shift size (δ) | |
|---|---|---|---|
| | | Mean = 3 $\theta$ = 0.5 | Mean = 3 $\theta$ = 1 |
| 1 | 0,0,0,0 | 0 | 0 |
| 2 | 1,0,0,0 | 0.512 | 0.542 |
| 3 | 0,1,0,0 | 0.512 | 0.542 |
| 4 | 2,0,0,0 | 0.912 | 0.953 |
| 5 | 0,2,0,0 | 0.912 | 0.953 |
| 6 | 1,1,0,0 | 0.689 | 0.707 |
| 7 | 2,2,0,0 | 1.239 | 1.265 |
| 8 | 1,0,1,0 | 0.689 | 0.707 |
| 9 | 0,0,2,0 | 0.912 | 0.953 |
| 10 | 1,0,0,1 | 0.689 | 0.707 |
| 11 | 0,0,0,2 | 0.912 | 0.953 |
| 12 | 1,1,1,1 | 0.853 | 0.756 |
| 13 | 2,2,2,2 | 1.569 | 1.414 |
| 14 | 3,3,3,3 | 2.191 | 2.000 |
| 15 | 4,4,4,4 | 2.744 | 2.309 |

### 3.5 Results

The statistical performance of the proposed MPEWMA chart is investigated by assessing the run length distribution, including the average run length (ARL) and standard deviation of the run length (SDRL). The control limit ($H$) was chosen to provide the in-control ARL of 200. The appropriate control limits to achieve the steady-state in-control ARL of 200 are summarized in Table 4. The ARL performance for different smoothing weights ($\lambda$) and various number of quality characteristics ($p$) are presented in Tables 5 and 6 for $\theta$ = 0.5 and $\theta$ = 1, respectively.

**Table 4** The chosen control limit obtained by simulation to achieve an in-control ARL of 200

| Mean | $\lambda$ | 4 variables | | 6 variables | | 10 variables | | 15 variables | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\theta = 1$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 0.5$ |
| 3 | 0.1 | 13.02 | 13.01 | 16.68 | 16.65 | 23.20 | 23.17 | 30.65 | 30.62 |
| | 0.05 | 11.49 | 11.49 | 14.95 | 14.93 | 21.19 | 21.17 | 28.41 | 28.39 |
| 5 | 0.1 | 12.95 | 12.95 | 16.56 | 16.54 | 23.03 | 23.04 | 30.49 | 30.49 |
| | 0.05 | 11.48 | 11.48 | 14.92 | 14.90 | 21.14 | 21.15 | 28.34 | 28.34 |
| 8 | 0.1 | 12.92 | 12.91 | 16.50 | 16.50 | 22.98 | 22.98 | 30.41 | 30.40 |
| | 0.05 | 11.47 | 11.47 | 14.91 | 14.89 | 21.13 | 21.14 | 28.32 | 28.31 |
| 10 | 0.1 | 12.90 | 12.90 | 16.50 | 16.48 | 22.94 | 22.95 | 30.38 | 30.36 |
| | 0.05 | 11.46 | 11.46 | 14.91 | 14.89 | 21.13 | 21.13 | 28.31 | 28.30 |
| 15 | 0.1 | 12.89 | 12.89 | 16.48 | 16.49 | 22.94 | 22.91 | 30.35 | 30.33 |
| | 0.05 | 11.46 | 11.46 | 14.90 | 14.90 | 21.12 | 21.12 | 28.30 | 28.30 |

**Table 5** ARLs for the MPEWMA Control Chart when $\theta = 0.5$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ |
| 4 | δ / H | 11.49 | 13.01 | δ / H | 11.48 | 12.95 | δ / H | 11.47 | 12.91 | δ / H | 11.46 | 12.90 | δ / H | 11.46 | 12.89 |
| | 0.00 | 200.124 | 200.012 | 0.00 | 200.114 | 199.771 | 0.00 | 199.806 | 200.234 | 0.00 | 199.200 | 198.817 | 0.00 | 199.549 | 199.811 |
| | 0.51 | 28.843 | 32.322 | 0.56 | 25.287 | 27.1677 | 0.46 | 33.0807 | 37.142 | 0.57 | 24.971 | 25.576 | 0.49 | 30.681 | 34.584 |
| | 0.91 | 13.115 | 12.491 | 1.05 | 10.964 | 10.033 | 0.88 | 13.801 | 13.017 | 1.09 | 10.505 | 9.378 | 0.93 | 12.826 | 11.886 |
| | 1.57 | 6.858 | 5.708 | 1.37 | 7.951 | 6.775 | 1.70 | 6.062 | 4.995 | 1.58 | 6.618 | 5.518 | 1.36 | 7.912 | 6.753 |
| | 2.19 | 4.569 | 3.607 | 1.95 | 5.196 | 4.172 | 2.18 | 4.486 | 3.532 | 2.03 | 4.847 | 3.877 | 1.77 | 5.713 | 4.656 |
| | 2.74 | 3.478 | 2.657 | 2.95 | 3.143 | 2.378 | 2.63 | 3.573 | 2.749 | 2.46 | 3.845 | 2.977 | 2.16 | 4.488 | 3.551 |
| 6 | δ / H | 14.93 | 16.65 | δ / H | 14.90 | 16.54 | δ / H | 14.89 | 16.50 | δ / H | 14.89 | 16.48 | δ / H | 14.9 | 16.49 |
| | 0.00 | 200.023 | 199.591 | 0.00 | 199.306 | 200.009 | 0.00 | 199.670 | 199.801 | 0.00 | 199.214 | 199.801 | 0.00 | 200.150 | 200.066 |
| | 0.52 | 31.746 | 36.831 | 0.41 | 43.168 | 51.347 | 0.46 | 36.643 | 42.586 | 0.42 | 42.204 | 49.655 | 0.49 | 34.287 | 39.700 |
| | 0.92 | 14.015 | 13.896 | 0.77 | 18.188 | 18.535 | 0.88 | 15.045 | 14.597 | 0.81 | 16.973 | 16.805 | 0.68 | 21.478 | 22.381 |
| | 1.26 | 9.588 | 8.559 | 1.05 | 11.943 | 11.002 | 1.39 | 8.590 | 7.388 | 1.29 | 9.366 | 8.191 | 1.11 | 11.274 | 10.152 |
| | 1.79 | 6.509 | 5.326 | 1.59 | 7.330 | 6.147 | 2.00 | 5.515 | 4.43 | 1.87 | 5.951 | 4.830 | 1.62 | 7.006 | 5.829 |
| | 2.52 | 4.328 | 3.350 | 2.27 | 4.802 | 3.775 | 2.57 | 4.058 | 3.142 | 2.41 | 4.372 | 3.411 | 2.11 | 5.078 | 4.051 |
| 10 | δ / H | 21.17 | 23.17 | δ / H | 21.15 | 23.04 | δ / H | 21.14 | 22.98 | δ / H | 21.13 | 22.95 | δ / H | 21.12 | 22.91 |
| | 0.00 | 200.104 | 199.268 | 0.00 | 199.475 | 200.122 | 0.00 | 199.927 | 199.828 | 0.00 | 199.332 | 200.06 | 0.00 | 199.441 | 199.469 |
| | 0.52 | 36.169 | 43.709 | 0.42 | 49.796 | 61.486 | 0.47 | 66.215 | 50.685 | 0.42 | 48.818 | 59.331 | 0.49 | 39.676 | 47.551 |
| | 1.08 | 13.747 | 12.783 | 0.98 | 15.491 | 14.796 | 0.89 | 17.002 | 16.873 | 0.81 | 19.333 | 19.708 | 0.70 | 23.809 | 25.45 |
| | 1.28 | 10.628 | 9.517 | 1.86 | 6.966 | 5.667 | 1.66 | 7.837 | 6.559 | 1.56 | 8.433 | 7.136 | 1.36 | 9.845 | 8.592 |
| | 2.05 | 6.374 | 5.1 | 2.68 | 4.534 | 3.483 | 2.41 | 5.031 | 3.933 | 2.27 | 5.355 | 4.253 | 2.00 | 6.174 | 4.98 |
| | 2.93 | 4.226 | 3.201 | 3.44 | 3.407 | 2.522 | 3.11 | 3.735 | 2.812 | 2.94 | 3.947 | 2.996 | 2.61 | 4.476 | 3.477 |
| 15 | δ / H | 28.39 | 30.62 | δ / H | 28.34 | 30.49 | δ / H | 28.31 | 30.40 | δ / H | 28.30 | 30.36 | δ / H | 28.30 | 30.33 |
| | 0.00 | 199.862 | 199.034 | 0.00 | 200.025 | 200.182 | 0.00 | 199.650 | 199.478 | 0.00 | 199.677 | 199.244 | 0.00 | 200.098 | 200.112 |
| | 0.52 | 40.647 | 50.393 | 0.42 | 56.139 | 70.966 | 0.47 | 47.175 | 58.461 | 0.42 | 54.607 | 68.097 | 0.49 | 44.790 | 55.192 |
| | 0.93 | 17.875 | 18.304 | 0.77 | 23.031 | 25.069 | 0.97 | 17.252 | 16.940 | 0.91 | 18.531 | 18.507 | 0.81 | 21.820 | 22.693 |
| | 1.29 | 11.514 | 10.561 | 1.07 | 15.233 | 14.138 | 1.88 | 7.549 | 6.192 | 1.78 | 8.003 | 6.655 | 1.58 | 9.105 | 7.757 |
| | 2.24 | 6.515 | 5.138 | 2.07 | 6.869 | 5.542 | 2.74 | 4.854 | 3.745 | 2.60 | 5.090 | 3.961 | 2.32 | 5.724 | 4.541 |
| | 3.22 | 4.307 | 3.217 | 3.00 | 4.486 | 3.403 | 3.55 | 3.599 | 2.660 | 3.38 | 3.763 | 2.807 | 3.04 | 4.167 | 3.171 |

**Table 6** ARLs for the MPEWMA Control Chart when $\theta = 1$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 |
| 4 | δ / H | 11.49 | 13.02 | δ / H | 11.48 | 12.95 | δ / H | 11.47 | 12.92 | δ / H | 11.46 | 12.90 | δ / H | 11.46 | 12.89 |
| | 0.00 | 199.031 | 199.730 | 0.00 | 199.142 | 200.252 | 0.00 | 199.309 | 200.236 | 0.00 | 200.141 | 200.059 | 0.00 | 198.954 | 199.299 |
| | 0.54 | 26.558 | 29.416 | 0.56 | 25.254 | 27.102 | 0.46 | 33.638 | 37.845 | 0.53 | 27.063 | 29.124 | 0.49 | 30.577 | 34.326 |
| | 0.95 | 12.349 | 11.641 | 1.04 | 11.029 | 10.024 | 1.11 | 10.352 | 9.155 | 1.03 | 11.288 | 10.205 | 0.89 | 13.512 | 12.652 |
| | 1.41 | 7.833 | 6.653 | 1.26 | 8.857 | 7.672 | 1.60 | 6.529 | 5.431 | 1.50 | 7.048 | 5.921 | 1.31 | 8.289 | 7.138 |
| | 2.00 | 5.141 | 4.133 | 1.81 | 5.691 | 4.646 | 2.07 | 4.789 | 3.812 | 1.94 | 5.165 | 4.138 | 1.71 | 5.99 | 4.913 |
| | 2.53 | 3.885 | 2.988 | 2.31 | 4.252 | 3.319 | 2.50 | 3.800 | 2.948 | 2.36 | 4.059 | 3.166 | 2.09 | 4.672 | 3.716 |
| 6 | δ / H | 14.95 | 16.68 | δ / H | 14.92 | 16.56 | δ / H | 14.91 | 16.50 | δ / H | 14.91 | 16.50 | δ / H | 14.90 | 16.48 |
| | 0.00 | 199.747 | 200.020 | 0.00 | 199.905 | 199.622 | 0.00 | 199.439 | 199.093 | 0.00 | 200.111 | 198.872 | 0.00 | 199.983 | 200.237 |
| | 0.55 | 29.068 | 33.037 | 0.43 | 41.300 | 49.471 | 0.46 | 36.722 | 42.578 | 0.42 | 42.481 | 50.155 | 0.49 | 34.002 | 29.296 |
| | 0.97 | 13.404 | 12.852 | 1.07 | 11.740 | 10.76 | 0.88 | 15.061 | 14.517 | 0.80 | 17.153 | 16.935 | 0.68 | 21.573 | 22.593 |
| | 1.55 | 7.836 | 6.538 | 1.41 | 8.579 | 7.291 | 1.26 | 9.67 | 8.459 | 1.18 | 10.418 | 9.216 | 1.04 | 12.188 | 11.157 |
| | 2.22 | 5.111 | 4.014 | 2.04 | 5.508 | 4.403 | 1.84 | 6.143 | 4.973 | 1.73 | 6.545 | 5.402 | 1.53 | 7.545 | 6.339 |
| | 2.83 | 3.836 | 2.916 | 2.62 | 4.093 | 3.146 | 2.38 | 4.513 | 3.513 | 2.25 | 4.79 | 3.765 | 2.00 | 5.438 | 4.366 |
| 10 | δ / H | 21.19 | 23.20 | δ / H | 21.14 | 23.03 | δ / H | 21.13 | 22.98 | δ / H | 21.13 | 22.94 | δ / H | 21.12 | 22.94 |
| | 0.00 | 199.484 | 199.341 | 0.00 | 199.739 | 199.64 | 0.00 | 200.081 | 199.334 | 0.00 | 200.194 | 198.783 | 0.00 | 199.188 | 198.847 |
| | 0.56 | 32.922 | 38.808 | 0.43 | 47.399 | 58.248 | 0.47 | 41.389 | 49.981 | 0.43 | 48.300 | 58.690 | 0.49 | 39.267 | 47.107 |
| | 0.98 | 14.984 | 14.719 | 1.10 | 12.889 | 12.056 | 0.89 | 16.741 | 16.680 | 0.81 | 19.058 | 19.381 | 0.68 | 24.465 | 25.556 |
| | 1.69 | 8.168 | 6.754 | 1.58 | 8.581 | 7.224 | 1.45 | 9.322 | 8.010 | 1.38 | 9.852 | 8.514 | 1.24 | 11.153 | 9.933 |
| | 2.45 | 5.317 | 4.101 | 2.30 | 5.513 | 4.318 | 2.12 | 5.914 | 4.710 | 2.02 | 6.196 | 4.995 | 1.83 | 6.913 | 5.692 |
| | 3.16 | 3.965 | 2.945 | 2.98 | 4.095 | 3.074 | 2.76 | 4.352 | 3.327 | 2.64 | 4.525 | 3.502 | 2.39 | 4.998 | 3.94 |
| 15 | δ / H | 28.41 | 30.65 | δ / H | 28.34 | 30.49 | δ / H | 28.32 | 30.41 | δ / H | 28.31 | 30.38 | δ / H | 28.30 | 30.35 |
| | 0.00 | 200.047 | 199.953 | 0.00 | 199.949 | 199.448 | 0.00 | 199.995 | 200.083 | 0.00 | 199.587 | 199.686 | 0.00 | 199.23 | 199.436 |
| | 0.57 | 36.455 | 44.795 | 0.44 | 52.785 | 67.044 | 0.48 | 46.022 | 57.209 | 0.43 | 53.747 | 67.435 | 0.49 | 44.103 | 54.575 |
| | 0.98 | 16.549 | 16.602 | 0.87 | 20.435 | 20.823 | 0.91 | 18.326 | 18.567 | 0.82 | 21.075 | 22.098 | 0.71 | 26.549 | 28.853 |
| | 1.37 | 10.796 | 9.721 | 1.11 | 14.017 | 13.400 | 1.58 | 9.357 | 7.971 | 1.52 | 9.756 | 8.400 | 1.39 | 10.715 | 9.393 |
| | 1.78 | 8.703 | 7.138 | 2.48 | 5.714 | 4.431 | 2.32 | 5.946 | 4.701 | 2.24 | 6.142 | 4.893 | 2.05 | 6.682 | 5.415 |
| | 2.60 | 5.641 | 4.321 | 3.23 | 4.227 | 3.158 | 3.04 | 4.383 | 3.304 | 2.93 | 4.502 | 3.439 | 2.70 | 4.826 | 3.744 |

**Table 7** SDRLs for the MPEWMA control chart when $\theta = 0.5$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| p | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ | | $\lambda = 0.05$ | $\lambda = 0.1$ |
| 4 | δ / H | 11.49 | 13.01 | δ / H | 11.48 | 12.95 | δ / H | 11.47 | 12.91 | δ / H | 11.46 | 12.90 | δ / H | 11.46 | 12.89 |
| | 0.00 | 201.330 | 201.809 | 0.00 | 200.489 | 200.438 | 0.00 | 201.513 | 201.147 | 0.00 | 200.473 | 200.082 | 0.00 | 201.750 | 201.745 |
| | 0.51 | 19.132 | 25.756 | 0.56 | 16.089 | 20.775 | 0.46 | 23.128 | 30.656 | 0.57 | 15.324 | 20.253 | 0.49 | 20.918 | 28.135 |
| | 0.91 | 6.608 | 7.461 | 1.05 | 5.267 | 5.568 | 0.88 | 7.231 | 7.971 | 1.09 | 5.056 | 5.147 | 0.93 | 6.524 | 7.079 |
| | 1.57 | 2.762 | 2.594 | 1.37 | 3.465 | 3.328 | 1.70 | 2.554 | 2.287 | 1.58 | 2.841 | 2.618 | 1.36 | 3.569 | 3.401 |
| | 2.19 | 1.693 | 1.471 | 1.95 | 2.059 | 1.811 | 2.18 | 1.793 | 1.518 | 2.03 | 1.981 | 1.707 | 1.77 | 2.443 | 2.158 |
| | 2.74 | 1.220 | 1.041 | 2.95 | 1.140 | 0.950 | 2.63 | 1.375 | 1.149 | 2.46 | 1.516 | 1.281 | 2.16 | 1.858 | 1.576 |
| 6 | δ / H | 14.93 | 16.65 | δ / H | 14.90 | 16.54 | δ / H | 14.89 | 16.50 | δ / H | 14.89 | 16.48 | δ / H | 14.9 | 16.49 |
| | 0.00 | 202.741 | 200.140 | 0.00 | 200.880 | 200.923 | 0.00 | 200.194 | 201.853 | 0.00 | 200.153 | 200.417 | 0.00 | 202.306 | 201.493 |
| | 0.52 | 21.313 | 29.153 | 0.41 | 32.147 | 44.941 | 0.46 | 25.956 | 35.852 | 0.42 | 31.258 | 43.152 | 0.49 | 23.723 | 32.888 |
| | 0.93 | 6.750 | 8.234 | 0.77 | 10.049 | 12.450 | 0.88 | 7.764 | 9.101 | 0.81 | 9.148 | 10.969 | 0.68 | 12.583 | 15.975 |
| | 1.26 | 4.221 | 4.308 | 1.05 | 5.744 | 6.164 | 1.39 | 3.734 | 3.589 | 1.29 | 4.181 | 4.153 | 1.11 | 5.348 | 5.573 |
| | 1.79 | 2.454 | 2.252 | 1.59 | 2.980 | 2.801 | 2.00 | 2.153 | 1.868 | 1.87 | 2.407 | 2.138 | 1.62 | 2.963 | 2.729 |
| | 2.52 | 1.487 | 1.274 | 2.27 | 1.768 | 1.528 | 2.57 | 1.520 | 1.273 | 2.41 | 1.683 | 1.413 | 2.11 | 2.039 | 1.761 |
| 10 | δ / H | 21.17 | 23.17 | δ / H | 21.15 | 23.04 | δ / H | 21.14 | 22.98 | δ / H | 21.13 | 22.95 | δ / H | 21.12 | 22.91 |
| | 0.00 | 202.254 | 200.787 | 0.00 | 200.668 | 202.111 | 0.00 | 200.902 | 201.510 | 0.00 | 200.966 | 201.003 | 0.00 | 203.114 | 202.550 |
| | 0.52 | 24.965 | 36.515 | 0.42 | 38.094 | 54.956 | 0.47 | 30.552 | 43.745 | 0.42 | 37.096 | 52.683 | 0.49 | 28.210 | 40.633 |
| | 1.08 | 6.135 | 6.998 | 0.98 | 7.502 | 8.722 | 0.89 | 8.817 | 10.639 | 0.81 | 10.555 | 13.171 | 0.70 | 13.996 | 18.423 |
| | 1.28 | 4.623 | 4.802 | 1.86 | 2.581 | 2.366 | 1.66 | 3.126 | 2.918 | 1.56 | 3.485 | 3.274 | 1.36 | 4.278 | 4.278 |
| | 2.05 | 2.197 | 1.972 | 2.68 | 1.527 | 1.287 | 2.41 | 1.822 | 1.554 | 2.27 | 1.990 | 1.739 | 2.00 | 2.422 | 2.113 |
| | 2.93 | 1.309 | 1.104 | 3.44 | 1.095 | 0.900 | 3.11 | 1.282 | 1.056 | 2.94 | 1.403 | 1.152 | 2.61 | 1.672 | 1.404 |
| 15 | δ / H | 28.39 | 30.62 | δ / H | 28.34 | 30.49 | δ / H | 28.31 | 30.40 | δ / H | 28.30 | 30.36 | δ / H | 28.30 | 30.33 |
| | 0.00 | 201.321 | 200.938 | 0.00 | 201.462 | 201.443 | 0.00 | 200.514 | 200.623 | 0.00 | 202.975 | 201.435 | 0.00 | 203.567 | 203.247 |
| | 0.52 | 28.583 | 43.190 | 0.42 | 44.097 | 64.634 | 0.47 | 35.018 | 51.605 | 0.42 | 42.622 | 61.689 | 0.49 | 32.725 | 48.306 |
| | 0.93 | 9.120 | 11.565 | 0.77 | 13.029 | 17.766 | 0.97 | 8.561 | 10.305 | 0.91 | 9.527 | 11.835 | 0.81 | 12.032 | 15.595 |
| | 1.29 | 5.044 | 5.398 | 1.07 | 6.971 | 8.121 | 1.88 | 2.813 | 2.594 | 1.78 | 3.089 | 2.871 | 1.58 | 3.765 | 3.606 |
| | 2.24 | 2.071 | 1.849 | 2.07 | 2.384 | 2.156 | 2.74 | 1.638 | 1.368 | 2.60 | 1.774 | 1.498 | 2.32 | 2.107 | 1.821 |
| | 3.22 | 1.230 | 1.031 | 3.00 | 1.395 | 1.174 | 3.55 | 1.150 | 0.935 | 3.38 | 1.244 | 1.016 | 3.04 | 1.470 | 1.206 |

**Table 8** SDRLs for the MPEWMA control chart when $\theta = 1$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| p | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 |
| 4 | δ / H | 11.49 | 13.02 | δ / H | 11.48 | 12.95 | δ / H | 11.47 | 12.92 | δ / H | 11.46 | 12.90 | δ / H | 11.46 | 12.89 |
| | 0.00 | 200.534 | 202.437 | 0.00 | 200.347 | 202.086 | 0.00 | 200.577 | 201.463 | 0.00 | 201.088 | 202.059 | 0.00 | 199.868 | 201.212 |
| | 0.54 | 17.088 | 22.958 | 0.56 | 25.254 | 20.765 | 0.46 | 23.548 | 31.484 | 0.53 | 17.600 | 22.721 | 0.49 | 20.877 | 27.846 |
| | 0.95 | 6.068 | 6.752 | 1.04 | 11.029 | 5.572 | 1.11 | 4.900 | 4.941 | 1.03 | 5.501 | 5.685 | 0.89 | 6.989 | 7.138 |
| | 1.41 | 3.249 | 3.128 | 1.26 | 8.857 | 3.846 | 1.60 | 2.759 | 2.521 | 1.50 | 3.067 | 2.842 | 1.31 | 3.796 | 3.622 |
| | 2.00 | 1.932 | 1.706 | 1.81 | 5.691 | 2.039 | 2.07 | 1.929 | 1.658 | 1.94 | 2.120 | 1.847 | 1.71 | 2.569 | 2.285 |
| | 2.53 | 1.381 | 1.170 | 2.31 | 4.252 | 1.375 | 2.50 | 1.486 | 1.246 | 2.36 | 1.621 | 1.358 | 2.09 | 1.942 | 1.657 |
| 6 | δ / H | 14.95 | 16.68 | δ / H | 14.92 | 16.56 | δ / H | 14.91 | 16.5 | δ / H | 14.91 | 16.50 | δ / H | 14.90 | 16.48 |
| | 0.00 | 200.636 | 200.901 | 0.00 | 201.829 | 202.104 | 0.00 | 202.657 | 200.512 | 0.00 | 201.069 | 199.387 | 0.00 | 201.725 | 204.029 |
| | 0.55 | 18.914 | 26.116 | 0.43 | 30.244 | 42.729 | 0.46 | 26.022 | 35.892 | 0.42 | 31.583 | 43.567 | 0.49 | 23.524 | 32.678 |
| | 0.97 | 6.552 | 7.483 | 1.07 | 5.565 | 5.913 | 0.88 | 7.815 | 8.956 | 0.80 | 9.285 | 11.035 | 0.68 | 12.789 | 16.110 |
| | 1.55 | 3.016 | 2.884 | 1.41 | 3.574 | 3.449 | 1.26 | 4.297 | 4.291 | 1.18 | 4.731 | 4.850 | 1.04 | 5.878 | 6.296 |
| | 2.22 | 1.786 | 1.546 | 2.04 | 2.061 | 1.814 | 1.84 | 2.465 | 2.169 | 1.73 | 2.672 | 2.410 | 1.53 | 3.246 | 3.000 |
| | 2.83 | 1.270 | 1.070 | 2.62 | 1.446 | 1.217 | 2.38 | 1.693 | 1.426 | 2.25 | 1.850 | 1.577 | 2.00 | 2.207 | 1.900 |
| 10 | δ / H | 21.19 | 23.2 | δ / H | 21.14 | 23.03 | δ / H | 21.13 | 22.98 | δ / H | 21.13 | 22.94 | δ / H | 21.12 | 22.94 |
| | 0.00 | 200.799 | 201.361 | 0.00 | 201.164 | 200.237 | 0.00 | 201.381 | 200.540 | 0.00 | 202.238 | 200.292 | 0.00 | 200.625 | 200.826 |
| | 0.56 | 21.760 | 31.4404 | 0.43 | 35.670 | 51.263 | 0.47 | 29.843 | 43.042 | 0.43 | 36.632 | 52.183 | 0.49 | 27.814 | 40.026 |
| | 0.98 | 7.334 | 4.32852 | 1.10 | 6.053 | 6.683 | 0.89 | 8.698 | 10.500 | 0.81 | 10.376 | 12.891 | 0.68 | 14.588 | 19.435 |
| | 1.69 | 2.894 | 2.77813 | 1.58 | 3.290 | 3.168 | 1.45 | 3.854 | 3.771 | 1.38 | 4.181 | 4.154 | 1.24 | 5.014 | 5.138 |
| | 2.45 | 1.671 | 1.4507 | 2.30 | 1.876 | 1.635 | 2.12 | 2.165 | 1.897 | 2.02 | 2.343 | 2.060 | 1.83 | 2.754 | 2.478 |
| | 3.16 | 1.181 | 0.98221 | 2.98 | 1.314 | 1.089 | 2.76 | 1.500 | 1.248 | 2.64 | 1.602 | 1.352 | 2.39 | 1.870 | 1.578 |
| 15 | δ / H | 28.41 | 30.65 | δ / H | 28.34 | 30.49 | δ / H | 28.32 | 30.41 | δ / H | 28.31 | 30.38 | δ / H | 28.30 | 30.35 |
| | 0.00 | 200.324 | 202.701 | 0.00 | 200.396 | 201.100 | 0.00 | 201.743 | 202.246 | 0.00 | 199.869 | 200.416 | 0.00 | 199.234 | 200.128 |
| | 0.57 | 24.491 | 37.468 | 0.44 | 40.432 | 60.442 | 0.48 | 33.894 | 50.238 | 0.43 | 41.730 | 60.927 | 0.49 | 32.055 | 47.477 |
| | 0.98 | 8.175 | 10.076 | 0.87 | 10.324 | 13.399 | 0.91 | 9.548 | 11.839 | 0.82 | 11.517 | 15.070 | 0.71 | 15.648 | 21.470 |
| | 1.37 | 4.546 | 4.734 | 1.11 | 6.586 | 7.578 | 1.58 | 3.653 | 3.545 | 1.52 | 3.916 | 3.853 | 1.39 | 4.557 | 4.615 |
| | 1.78 | 2.880 | 2.772 | 2.48 | 1.805 | 1.563 | 2.32 | 2.023 | 1.763 | 2.24 | 2.183 | 1.897 | 2.05 | 2.504 | 2.222 |
| | 2.60 | 1.635 | 1.416 | 3.23 | 1.253 | 1.035 | 3.04 | 1.398 | 1.165 | 2.93 | 1.489 | 1.245 | 2.70 | 1.706 | 1.431 |

The results show that the out-of-control ARLs can be quite different for processes that have the same number of unit shifts. For example, a shift of two units in a single mean such as [2, 0, 0, 0] (or $\delta = 0.912$) with $\lambda = 0.1$ and $\theta = 0.5$ has an ARL of 12.491 (see Table 5), while a two-unit shift in two variables such as [1, 1, 0, 0] (or $\delta = 0.689$) has an approximate ARL of 19.648 (again with $\lambda = 0.1$ and $\theta = 0.5$). Both are a "shift" of 2 units, but the ARLs and the shift-size of the mean are fairly different. However, the out-of-control ARLs are similar when the same number of variables shifts by the same amount. For example, the ARL for a shifted process such as [1, 1, 0, 0] is roughly the same as the ARL for the resulting process of [1, 0, 1, 0] or the process of [1, 0, 0, 1] for the same value of $\lambda$.

For a complete investigation of the performance of the proposed MPEWMA chart, the standard deviation of each scenario is calculated and summarized in Tables 7 and 8. The pooled standard deviations are applied within the same shift size. Moreover, the accuracy of the true mean of the population can be evaluated by calculating the standard error of the ARL ($SE_{ARL}$). The formula of the $SE_{ARL}$ is given by

$$SE_{ARL} = \frac{S}{\sqrt{n}} \tag{11}$$

where $S$ is the standard deviation, and $n$ is the number of replicates. A total of 50,000 replications is used for each scenario. Thus, the standard error of the mean is comparatively small due to the large sample size. The maximum value of $SE_{ARL}$ is around 0.91 whereas the minimum value is approximately 0.005. A smaller value of $SE_{ARL}$ implies a more accurate estimate of the true mean of the run length, and therefore the sample mean is close to the population mean of the run length. The results clearly demonstrate that the $SE_{ARL}$ decreases monotonically with an increase of the shift size. The $SE_{ARL}$ computed from a large shift size tends to be small and one obtains a better estimate of the true mean than for a small size of shift.

### 3.6 General Equation of the Average Run Length

Since the simulation study is limited to the certain values of the parameter combination as discussed in Section 3.4, the average run length performance of the proposed MPEWMA control chart could be further extended to include other values of those parameters by performing a multiple regression. We fit both the multiple linear regression and Generalized Linear Models (GLM) to the ARL values. The five possible variables considered affecting the out-of-control ARL values are the shift size ($\delta$), the fixed common mean ($\theta$), the smoothing weight ($\lambda$), the number of variables being monitored ($p$), and the process mean of interest which is assumed to be equal among all process means ($\mu$). Cases of no shift ($\delta$ = 0) related to an in-control ARL are removed to provide a more accurate model. The preliminary results show that the GLM with the exponential distribution provides a better fit to the ARL performance of the MPEWMA scheme as the ARL values tend to decrease exponentially with increasing the size of shift. The SAS output indicates that only four parameters ($p$, $\delta$, $\lambda$, and $\mu$) are statistically significant (p-value < alpha level of 0.05) as shown in Table 9. The fixed common mean, $\theta$, is not significant and dropped from the fitted model. Table 10 reports the output of fitting the model without the fixed common mean variable ($\theta$) in SAS. Thus, the fitted model is

$$\hat{y} = \frac{-1}{0.007 + 0.0008\,p - 0.0004\,\mu + 0.0547\,\lambda - 0.0840\,\delta} \tag{12}$$

**Table 9** The SAS output of fitting the GLM with the exponential distribution to the ARL values obtained from simulation.

The GENMOD Procedure

Model Information

| | | |
|---|---|---|
| Data Set | WORK.ALLDATA | |
| Distribution | Gamma | |
| Link Function | Power(-1) | |
| Dependent Variable | ARL | ARL |

| | |
|---|---|
| Number of Observations Read | 1900 |
| Number of Observations Used | 1900 |

Criteria For Assessing Goodness Of Fit

| Criterion | DF | Value | Value/DF |
|---|---|---|---|
| Deviance | 1894 | 37.8351 | 0.0200 |
| Scaled Deviance | 1894 | 1906.2848 | 1.0065 |
| Pearson Chi-Square | 1894 | 32.5436 | 0.0172 |
| Scaled Pearson X2 | 1894 | 1639.6761 | 0.8657 |
| Log Likelihood | | -5132.6491 | |
| Full Log Likelihood | | -5132.6491 | |
| AIC (smaller is better) | | 10279.2981 | |
| AICC (smaller is better) | | 10279.3573 | |
| BIC (smaller is better) | | 10318.1454 | |

Algorithm converged.

Analysis Of Maximum Likelihood Parameter Estimates

| Parameter | DF | Estimate | Standard Error | Wald 95% Confidence Limits | | Wald Chi-Square | Pr > ChiSq |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | -0.0068 | 0.0006 | -0.0079 | -0.0056 | 129.39 | <.0001 |
| Var | 1 | -0.0008 | 0.0000 | -0.0008 | -0.0007 | 1576.43 | <.0001 |
| Mean | 1 | 0.0004 | 0.0000 | 0.0003 | 0.0004 | 225.06 | <.0001 |
| Lambda | 1 | -0.0547 | 0.0031 | -0.0608 | -0.0486 | 308.19 | <.0001 |
| Shiftsize | 1 | 0.0840 | 0.0005 | 0.0830 | 0.0851 | 24258.5 | <.0001 |
| Thetafix | 1 | -0.0003 | 0.0003 | -0.0009 | 0.0003 | 1.08 | 0.2996 |
| Scale | 1 | 50.3840 | 1.6293 | 47.2897 | 53.6807 | | |

NOTE: The scale parameter was estimated by maximum likelihood.

**Table 10** The SAS output after dropping the fix common mean variable from the model

```
                              The GENMOD Procedure
                               Model Information
               Data Set            WORK.ALLDATA
               Distribution               Gamma
               Link Function          Power(-1)
               Dependent Variable           ARL    ARL
                       Number of Observations Read        1900
                       Number of Observations Used        1900

                         Criteria For Assessing Goodness Of Fit
               Criterion                    DF           Value        Value/DF
               Deviance                   1895         37.8565          0.0200
               Scaled Deviance            1895       1906.2884          1.0060
               Pearson Chi-Square         1895         32.5559          0.0172
               Scaled Pearson X2          1895       1639.3756          0.8651
               Log Likelihood                       -5133.1867
               Full Log Likelihood                  -5133.1867
               AIC (smaller is better)              10278.3734
               AICC (smaller is better)             10278.4178
               BIC (smaller is better)              10311.6711
         Algorithm converged.
```

Analysis Of Maximum Likelihood Parameter Estimates

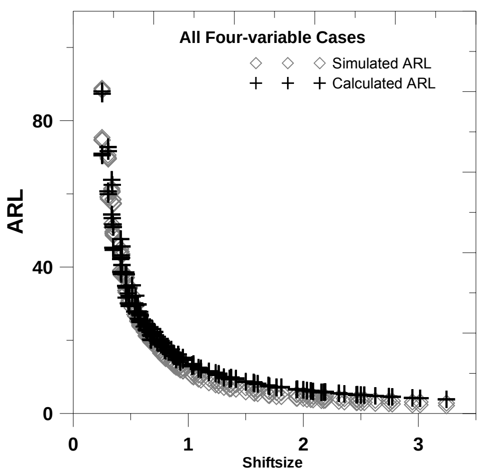
| Parameter | DF | Estimate | Standard Error | Wald 95% Confidence Limits | | Wald Chi-Square | Pr > ChiSq |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | -0.0070 | 0.0006 | -0.0081 | -0.0059 | 157.61 | <.0001 |
| Var | 1 | -0.0008 | 0.0000 | -0.0008 | -0.0007 | 1574.78 | <.0001 |
| Mean | 1 | 0.0004 | 0.0000 | 0.0003 | 0.0004 | 224.29 | <.0001 |
| Lambda | 1 | -0.0547 | 0.0031 | -0.0608 | -0.0485 | 307.95 | <.0001 |
| Shiftsize | 1 | 0.0840 | 0.0005 | 0.0830 | 0.0851 | 24266.2 | <.0001 |
| Scale | 1 | 50.3557 | 1.6284 | 47.2631 | 53.6505 | | |

NOTE: The scale parameter was estimated by maximum likelihood.

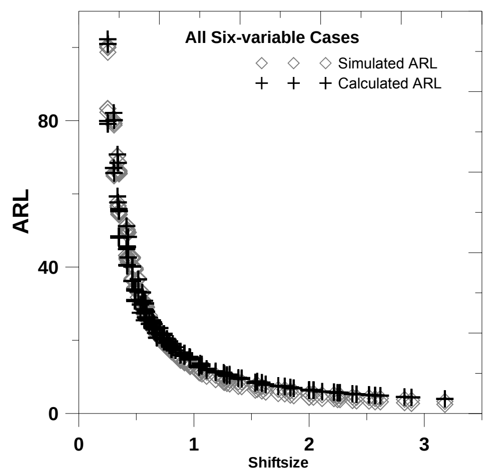
A comparison between the simulated and fitted ARL from the above equation is provided in Figure 1. Figure 1 displays the plot of the out-of-control ARL values of the simulation and those from the fitted model for all cases as a function of δ. However, using the general equation gives a slightly larger out-of-control ARL than the simulation method. It can be noted that the general equation may not provide good approximations for the small shift size and large number of variables. We can see from Figure 1.d, when the size of shift is around 0.25 with fifteen-variables, the out-of-control ARL based on equation (12) is approximately

380, with the simulated value being close to 125. Thus, it can be noted that one should use the general equation with caution to approximate the out-of-control ARL value for the small shift size (say less than 0.3) and large number of variables (ten or more).

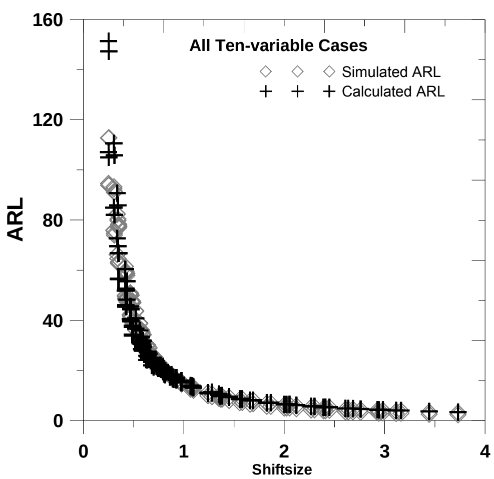
**Figure 1** The comparison of the simulated and calculated ARL plots separated by the number of variables.



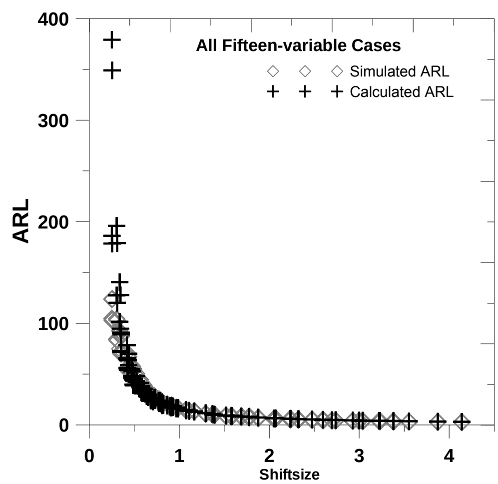

a. All Four-variable Cases




b. All Six-variable Cases




c. All Ten-variable Cases




d. All Fifteen-variable Cases

### 3.7 Effect of Changing the Common Variable

In Section 3.5, we discussed the statistical performances of the MPEWMA chart with two thetafix values ($\theta$ = 0.5, and 1). The ARL values reported in Table 5 – 6 are calculated by assuming the mean of the common variable ($\theta$) remains the same. However, sometimes both the mean of the common variable and other variables (i.e. $\theta_i$) increase simultaneously. Thus, we extend the investigation of the proposed MPEWMA scheme into the case of monitoring the mean shift in the common variable ($X$). Five mean values of the common variable are tested ($\theta$ = 0.5, 1, 1.5, 2, and 2.5). The smoothing weight of 0.05 is selected in this study. Table 11 shows the ARL performances of the MPEWMA chart when the common variable and one of the other variables shift together. For two variables shifted and all variables shifted, please see Table 12 and 13, respectively.

It can be seen that an increase in the common variable has an effect on the in-control ARL for the mean of 3, but there is a little or none for the mean of 5 or larger. The out-of-control ARL values show a significant decrease in case of one or two variables shifted (Table 11 - 12), particularly for a small unit of shift matrix. It is worthy to note that the combination of increase in both common variable and all variable means could cause a dramatic increase in the out-of-control ARL values (see Table 13). This corresponds to the small shift size ($\delta$) computed from Equation (10). An increase in the mean of the common variable can result in a large covariance matrix, and produce the small shift size.

**Table 11** The ARL performance of the MPEWMA chart for detecting the mean shift in one variable when varying $\theta$

| | | Number of variable (p) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | θ | 4 (H = 11.49) | | | 6 (H = 14.93) | | | 10 (H = 21.17) | | | 15 (H = 28.39) | | |
| Mean | Shift matrix | (0,0,0,0) | (1,0,0,0) | (2,0,0,0) | (0,0,0,0) | (1,0,0,0) | (2,0,0,0) | (0,0,0,0) | (1,0,0,0) | (2,0,0,0) | (0,0,0,0) | (1,0,0,0) | (2,0,0,0) |
| 3 | 0.5 | 200.899 | 29.082 | 13.2 | 200.303 | 31.984 | 14.41 | 201.338 | 36.47 | 16.223 | 199.065 | 40.688 | 17.967 |
| | 1 | 199.975 | 26.672 | 12.392 | 199.957 | 28.987 | 13.404 | 199.19 | 32.722 | 15.014 | 199.155 | 36.719 | 16.565 |
| | 1.5 | 201.616 | 23.733 | 11.374 | 199.355 | 25.659 | 12.305 | 196.607 | 28.811 | 13.759 | 197.186 | 31.953 | 15.164 |
| | 2 | 198.211 | 20.396 | 10.276 | 197.622 | 21.852 | 11.073 | 194.4 | 24.491 | 12.349 | 194.223 | 27.097 | 13.644 |
| | 2.5 | 192.625 | 16.421 | 8.954 | 189.496 | 17.638 | 9.725 | 184.353 | 19.763 | 10.916 | 184.593 | 21.982 | 12.03 |
| | θ | 4 (H = 11.48) | | | 6 (H = 14.9) | | | 10 (H = 21.15) | | | 15 (H = 28.39) | | |
| 5 | 0.5 | 203.365 | 39.066 | 16.616 | 200.739 | 43.543 | 18.281 | 200.868 | 50.151 | 20.799 | 199.901 | 56.135 | 23.038 |
| | 1 | 201.421 | 37.545 | 16.103 | 199.229 | 41.576 | 17.592 | 201.805 | 47.368 | 19.849 | 199.312 | 53.626 | 21.978 |
| | 1.5 | 200.763 | 35.745 | 15.474 | 200.49 | 38.849 | 16.659 | 201.581 | 44.286 | 18.763 | 200.611 | 49.101 | 20.709 |
| | 2 | 201.773 | 33.223 | 14.532 | 199.294 | 36.007 | 15.775 | 199.91 | 40.448 | 17.543 | 197.88 | 44.811 | 19.373 |
| | 2.5 | 198.748 | 30.548 | 13.692 | 198.194 | 32.852 | 14.653 | 197.967 | 36.886 | 16.345 | 199.281 | 41.03 | 18.03 |
| | θ | 4 (H = 11.47) | | | 6 (H = 14.89) | | | 10 (H = 21.14) | | | 15 (H = 28.31) | | |
| 8 | 0.5 | 201.058 | 51.889 | 21.304 | 199.038 | 57.542 | 23.544 | 199.441 | 67.08 | 27.079 | 199.901 | 74.77 | 30.091 |
| | 1 | 201.337 | 50.981 | 20.957 | 199.146 | 56.781 | 23.079 | 200.657 | 65.22 | 26.18 | 199.599 | 72.481 | 29.221 |
| | 1.5 | 200.735 | 50.029 | 20.543 | 200.124 | 55.053 | 22.441 | 200.394 | 62.689 | 25.456 | 201.379 | 69.929 | 28.295 |
| | 2 | 202.275 | 48.248 | 20.042 | 199.421 | 52.925 | 21.689 | 199.258 | 60.062 | 24.394 | 200.633 | 67.131 | 27.086 |
| | 2.5 | 200.318 | 46.437 | 19.188 | 198.62 | 50.895 | 20.82 | 200.982 | 57.329 | 23.284 | 200.836 | 64.091 | 25.845 |
| | θ | 4 (H = 11.46) | | | 6 (H = 14.89) | | | 10 (H = 21.13) | | | 15 (H = 28.3) | | |
| 10 | 0.5 | 200.526 | 59.933 | 24.217 | 200.501 | 66.497 | 26.918 | 201.528 | 76.53 | 30.751 | 199.439 | 84.578 | 34.564 |
| | 1 | 200.119 | 58.798 | 23.895 | 201.651 | 65.197 | 26.429 | 201.571 | 74.724 | 30.246 | 200.96 | 83.475 | 33.82 |
| | 1.5 | 200.409 | 58.061 | 23.632 | 198.849 | 63.842 | 25.881 | 201.211 | 73.064 | 29.413 | 199.343 | 81.214 | 32.869 |
| | 2 | 199.177 | 56.667 | 22.99 | 200.793 | 62.369 | 25.263 | 200.28 | 70.894 | 28.737 | 200.137 | 78.585 | 31.783 |
| | 2.5 | 199.768 | 55.344 | 22.512 | 200.472 | 60.506 | 24.44 | 199.337 | 68.725 | 27.549 | 201.232 | 75.538 | 30.437 |
| | θ | 4 (H = 11.46) | | | 6 (H = 14.9) | | | 10 (H = 21.12) | | | 15 (H = 28.3) | | |
| 15 | 0.5 | 199.361 | 74.765 | 30.755 | 200.394 | 83.528 | 34.659 | 202.492 | 94.707 | 40.03 | 200.783 | 104.669 | 45.072 |
| | 1 | 200.78 | 74.995 | 30.538 | 199.645 | 82.837 | 34.103 | 201.015 | 94.489 | 39.423 | 200.466 | 104.326 | 44.396 |
| | 1.5 | 199.96 | 74.662 | 30.466 | 200.647 | 82.247 | 33.986 | 199.99 | 92.543 | 38.785 | 200.009 | 102.403 | 43.607 |
| | 2 | 199.545 | 73.663 | 29.986 | 201.459 | 81.225 | 33.18 | 201.115 | 91.802 | 38.058 | 200.4 | 100.586 | 42.759 |
| | 2.5 | 200.377 | 72.989 | 29.761 | 199.867 | 79.958 | 32.746 | 200.54 | 90.295 | 37.251 | 199.495 | 99.43 | 41.852 |

**Table 12** The ARL performance of the MPEWMA chart for detecting the mean shift in two variables when varying $\theta$

| | θ | Number of variable (p) 4 (H = 11.49) | | | 6 (H = 14.93) | | | 10 (H = 21.17) | | | 15 (H = 28.39) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean | Shift matrix | (0,0,0,0) | (1,1,0,0) | (2,2,0,0) | (0,0,0,0) | (1,1,0,0) | (2,2,0,0) | (0,0,0,0) | (1,1,0,0) | (2,2,0,0) | (0,0,0,0) | (1,1,0,0) | (2,2,0,0) |
| 3 | 0.5 | 200.899 | 19.291 | 8.997 | 200.303 | 20.834 | 9.611 | 201.338 | 23.062 | 10.643 | 199.065 | 25.422 | 11.568 |
| | 1 | 199.975 | 18.573 | 8.693 | 199.957 | 19.339 | 9.153 | 199.19 | 21.074 | 9.922 | 199.155 | 22.901 | 10.829 |
| | 1.5 | 201.616 | 16.972 | 8.136 | 199.355 | 17.376 | 8.43 | 196.607 | 18.521 | 9.126 | 197.186 | 20.125 | 9.887 |
| | 2 | 198.211 | 14.688 | 7.352 | 197.622 | 14.767 | 7.558 | 194.4 | 15.851 | 8.242 | 194.223 | 17.128 | 8.945 |
| | 2.5 | 192.625 | 11.691 | 6.308 | 189.496 | 11.906 | 6.593 | 184.353 | 12.851 | 7.227 | 184.593 | 14.038 | 7.911 |
| | θ | 4 (H = 11.48) | | | 6 (H = 14.9) | | | 10 (H = 21.15) | | | 15 (H = 28.39) | | |
| 5 | 0.5 | 203.365 | 25.364 | 11.05 | 200.739 | 27.632 | 11.982 | 200.868 | 31.505 | 13.445 | 199.901 | 34.836 | 14.721 |
| | 1 | 201.421 | 25.444 | 11.068 | 199.229 | 27.147 | 11.797 | 201.805 | 29.934 | 12.985 | 199.312 | 33.054 | 14.098 |
| | 1.5 | 200.763 | 24.725 | 10.883 | 200.49 | 25.726 | 11.35 | 201.581 | 28.135 | 12.311 | 200.611 | 30.577 | 13.346 |
| | 2 | 201.773 | 23.47 | 10.492 | 199.294 | 24.117 | 10.748 | 199.91 | 26.084 | 11.623 | 197.88 | 28.24 | 12.516 |
| | 2.5 | 198.748 | 22.041 | 9.935 | 198.194 | 22.219 | 10.097 | 197.967 | 23.687 | 10.853 | 199.281 | 25.565 | 11.706 |
| | θ | 4 (H = 11.47) | | | 6 (H = 14.89) | | | 10 (H = 21.14) | | | 15 (H = 28.31) | | |
| 8 | 0.5 | 201.058 | 33.17 | 13.856 | 199.038 | 36.731 | 15.13 | 199.441 | 42.53 | 17.058 | 199.901 | 47.496 | 18.857 |
| | 1 | 201.337 | 33.744 | 13.919 | 199.146 | 36.705 | 15.145 | 200.657 | 41.862 | 16.82 | 199.599 | 46.155 | 18.441 |
| | 1.5 | 200.735 | 33.678 | 13.921 | 200.124 | 36.229 | 14.911 | 200.394 | 40.363 | 16.361 | 201.379 | 44.243 | 17.78 |
| | 2 | 202.275 | 33.446 | 13.881 | 199.421 | 35.164 | 14.47 | 199.258 | 38.785 | 15.871 | 200.633 | 42.627 | 17.247 |
| | 2.5 | 200.318 | 32.583 | 13.618 | 198.62 | 33.969 | 14.086 | 200.982 | 36.971 | 15.264 | 200.836 | 40.37 | 16.475 |
| | θ | 4 (H = 11.46) | | | 6 (H = 14.89) | | | 10 (H = 21.13) | | | 15 (H = 28.3) | | |
| 10 | 0.5 | 200.526 | 37.926 | 15.523 | 200.501 | 42.361 | 17.122 | 201.528 | 48.657 | 19.295 | 199.439 | 54.924 | 21.404 |
| | 1 | 200.119 | 38.845 | 15.717 | 201.651 | 42.712 | 17.088 | 201.571 | 48.478 | 19.204 | 200.96 | 54.215 | 21.102 |
| | 1.5 | 200.409 | 38.886 | 15.793 | 198.849 | 42.135 | 17.006 | 201.211 | 47.404 | 18.921 | 199.343 | 52.575 | 20.6 |
| | 2 | 199.177 | 38.574 | 15.742 | 200.793 | 41.426 | 16.803 | 200.28 | 46.339 | 18.415 | 200.137 | 50.897 | 20.023 |
| | 2.5 | 199.768 | 38.227 | 15.538 | 200.472 | 40.63 | 16.422 | 199.337 | 44.585 | 17.879 | 201.232 | 49.065 | 19.42 |
| | θ | 4 (H = 11.46) | | | 6 (H = 14.9) | | | 10 (H = 21.12) | | | 15 (H = 28.3) | | |
| 15 | 0.5 | 199.361 | 49.029 | 19.483 | 200.394 | 54.725 | 21.545 | 202.492 | 63.201 | 24.45 | 200.783 | 71.011 | 27.651 |
| | 1 | 200.78 | 49.802 | 19.78 | 199.645 | 54.979 | 21.83 | 201.015 | 63.152 | 24.555 | 200.466 | 70.635 | 27.381 |
| | 1.5 | 199.96 | 50.062 | 19.891 | 200.647 | 55.602 | 21.735 | 199.99 | 62.764 | 24.426 | 200.009 | 70.303 | 27.001 |
| | 2 | 199.545 | 50.273 | 19.916 | 201.459 | 54.948 | 21.674 | 201.115 | 62.095 | 24.094 | 200.4 | 69.062 | 26.556 |
| | 2.5 | 200.377 | 50.35 | 19.851 | 199.867 | 54.785 | 21.402 | 200.54 | 61.221 | 23.702 | 199.495 | 67.45 | 26.085 |

**Table 13** The ARL performance of the MPEWMA chart for detecting the mean shift in all variables when varying $\theta$

| | | Number of variable (p) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | θ | 4 (H = 11.49) | | | 6 (H = 14.93) | | | 10 (H = 21.17) | | | 15 (H = 28.39) | | |
| Mean | Shift matrix | (1,1,1,1) | (2,2,2,2) | (3,3,3,3) | (1,1,1,1) | (2,2,2,2) | (3,3,3,3) | (1,1,1,1) | (2,2,2,2) | (3,3,3,3) | (1,1,1,1) | (2,2,2,2) | (3,3,3,3) |
| 3 | 0.5 | 14.712 | 6.85 | 4.569 | 13.994 | 6.467 | 4.327 | 13.801 | 6.373 | 4.227 | 14.054 | 6.511 | 4.309 |
| | 1 | 17.436 | 7.859 | 5.149 | 17.463 | 7.859 | 5.116 | 18.353 | 8.166 | 5.301 | 19.668 | 8.71 | 5.647 |
| | 1.5 | 19.817 | 8.791 | 5.707 | 20.716 | 9.078 | 5.856 | 22.622 | 9.833 | 6.309 | 24.83 | 10.66 | 6.85 |
| | 2 | 22.408 | 9.695 | 6.263 | 23.864 | 10.275 | 6.561 | 26.885 | 11.383 | 7.227 | 29.894 | 12.573 | 7.981 |
| | 2.5 | 24.898 | 10.617 | 6.781 | 27.145 | 11.45 | 7.26 | 31.125 | 12.928 | 8.161 | 35.06 | 14.44 | 9.079 |
| | θ | 4 (H = 11.48) | | | 6 (H = 14.9) | | | 10 (H = 21.15) | | | 15 (H = 28.34) | | |
| 5 | 0.5 | 17.891 | 7.984 | 5.175 | 16.448 | 7.371 | 4.803 | 15.485 | 6.971 | 4.552 | 15.254 | 6.867 | 4.5 |
| | 1 | 20.308 | 8.842 | 5.704 | 19.686 | 8.569 | 5.507 | 19.787 | 8.554 | 5.51 | 20.519 | 8.885 | 5.711 |
| | 1.5 | 22.539 | 9.744 | 6.235 | 22.61 | 9.682 | 6.166 | 23.714 | 10.063 | 6.404 | 25.326 | 10.69 | 6.803 |
| | 2 | 24.892 | 10.518 | 6.7 | 25.498 | 10.788 | 6.806 | 27.521 | 11.42 | 7.261 | 30.035 | 12.409 | 7.821 |
| | 2.5 | 27.071 | 11.346 | 7.149 | 28.299 | 11.77 | 7.414 | 31.339 | 12.85 | 8.069 | 34.422 | 14.042 | 8.791 |
| | θ | 4 (H = 11.47) | | | 6 (H = 14.89) | | | 10 (H = 21.14) | | | 15 (H = 28.31) | | |
| 8 | 0.5 | 22.511 | 9.522 | 6.069 | 20.028 | 8.631 | 5.513 | 17.995 | 7.849 | 5.05 | 17.258 | 7.54 | 4.851 |
| | 1 | 24.592 | 10.345 | 6.544 | 22.942 | 9.699 | 6.133 | 22.166 | 9.35 | 5.911 | 22.184 | 9.36 | 5.944 |
| | 1.5 | 26.613 | 11.129 | 6.966 | 25.64 | 10.731 | 6.725 | 25.904 | 10.727 | 6.733 | 26.757 | 11.035 | 6.946 |
| | 2 | 28.677 | 11.819 | 7.399 | 28.326 | 11.679 | 7.307 | 29.398 | 12.012 | 7.463 | 31.079 | 12.631 | 7.87 |
| | 2.5 | 30.531 | 12.577 | 7.839 | 31.043 | 12.648 | 7.85 | 32.902 | 13.301 | 8.222 | 35.286 | 14.085 | 8.768 |
| | θ | 4 (H = 11.46) | | | 6 (H = 14.89) | | | 10 (H = 21.13) | | | 15 (H = 28.3) | | |
| 10 | 0.5 | 25.088 | 10.552 | 6.63 | 22.173 | 9.414 | 5.953 | 19.742 | 8.45 | 5.368 | 18.531 | 7.988 | 5.106 |
| | 1 | 27.269 | 11.298 | 7.054 | 25.077 | 10.468 | 6.548 | 23.613 | 9.854 | 6.191 | 23.407 | 9.752 | 6.144 |
| | 1.5 | 29.238 | 12.05 | 7.449 | 27.809 | 11.452 | 7.098 | 27.276 | 11.155 | 6.984 | 27.806 | 11.349 | 7.113 |
| | 2 | 31.254 | 12.739 | 7.919 | 30.311 | 12.344 | 7.658 | 30.771 | 12.405 | 7.71 | 32.084 | 12.869 | 7.99 |
| | 2.5 | 33.185 | 13.402 | 8.3 | 32.971 | 13.256 | 8.181 | 34.137 | 13.622 | 8.41 | 36.424 | 14.322 | 8.808 |
| | θ | 4 (H = 11.46) | | | 6 (H = 14.9) | | | 10 (H = 21.12) | | | 15 (H = 28.3) | | |
| 15 | 0.5 | 31.598 | 12.886 | 7.924 | 27.526 | 11.348 | 7.027 | 23.981 | 9.883 | 6.187 | 21.912 | 9.129 | 5.74 |
| | 1 | 33.276 | 13.494 | 8.335 | 30.13 | 12.23 | 7.525 | 27.542 | 11.183 | 6.914 | 26.383 | 10.738 | 6.681 |
| | 1.5 | 35.437 | 14.139 | 8.689 | 32.66 | 13.157 | 8.037 | 30.982 | 12.377 | 7.601 | 30.784 | 12.261 | 7.579 |
| | 2 | 37.172 | 14.87 | 9.066 | 35.227 | 13.95 | 8.522 | 34.268 | 13.553 | 8.306 | 34.885 | 13.687 | 8.377 |
| | 2.5 | 38.95 | 15.47 | 9.412 | 37.73 | 14.871 | 9.032 | 37.787 | 14.695 | 8.94 | 38.852 | 15.064 | 9.17 |

## 3.8 Comparison of the MPEWMA and MEWMA Control Chart

To gain more insight into the performance of the MEWMA chart under the multivariate Poisson distribution, a comparison is made between the MEWMA and MPEWMA scheme's performance. The control limits of the steady-state MEWMA chart based on the work of Prabhu and Runger (1997) are established assuming the multivariate normal distribution. The ARL performance of the MEWMA chart is reported in Table 14 in terms of a quantity of the shift size. For the proposed MEWMA monitoring scheme, the multivariate Poisson data are generated using the simulation method and conditions similar to that discussed in Section 3.4. The normal-theory limits are placed on the MPEWMA chart to obtain the true performance of the MEWMA based on the multivariate Poisson. The ARL performance comparisons are shown in Tables 15 – 16 for $\theta$ = 0.5 and 1, respectively. The in-control and out-of-control ARLs are both investigated to evaluate the robustness of the MEWMA chart

### 3.8.1 Out of-Control ARL Comparison

Comparing two control charts that have shifted by some amount, the chart with a smaller out-of-control ARL is preferred. For a fair comparison, the in-control ARLs for the two methods must be approximately equal. The traditional MEWMA chart and the proposed MPEWMA chart perform similarly in detecting the same shift in the process means as reported in Tables 15 and 16. It can be noticed that if the shift size, $\delta$, is less than 0.6, the out-of-control ARLs obtained from the MPEWMA chart are slightly but not much worse than the out-of-control ARLs for the MEWMA chart.

**Table 14** The Steady State ARL for the MEWMA chart proposed by Prabhu and Runger (1997)

| p | δ (Shift size) | λ 0.05 | λ 0.1 |
|---|---|---|---|
| 4 | H | 11.22 | 12.73 |
| | 0 | 199.98 | 200.05 |
| | 0.5 | 29.52 | 33.12 |
| | 1 | 12.27 | 11.38 |
| | 1.5 | 7.75 | 6.7 |
| | 2 | 5.71 | 4.8 |
| | 3 | 3.82 | 3.14 |
| 6 | H | 14.6 | 16.27 |
| | 0 | 199.88 | 200 |
| | 0.5 | 31.91 | 37.08 |
| | 1 | 13.31 | 12.41 |
| | 1.5 | 8.43 | 7.25 |
| | 2 | 6.23 | 5.18 |
| | 3 | 4.17 | 3.38 |
| 10 | H | 20.72 | 22.67 |
| | 0 | 200.06 | 200.06 |
| | 0.5 | 36.87 | 44.19 |
| | 1 | 15.23 | 14.32 |
| | 1.5 | 9.64 | 8.23 |
| | 2 | 7.11 | 5.83 |
| | 3 | 4.76 | 3.79 |
| 15 | H | 27.82 | 30.03 |
| | 0 | 200.05 | 199.95 |
| | 0.5 | 41.78 | 51.23 |
| | 1 | 17.13 | 16.3 |
| | 1.5 | 10.8 | 9.21 |
| | 2 | 7.97 | 6.48 |
| | 3 | 5.32 | 4.18 |

### 3.8.2 In-Control ARL Comparison

The most important result from this study concerns the in-control ARLs. As alluded to previously, out-of-control ARLs of two or more monitoring schemes can only be compared fairly if the corresponding in-control ARLs are approximately the same value. Notice the in-control ARLs reported in Tables 15 and 16. The in-control ARLs for the MPEWMA for data that comes from a multivariate Poisson distribution are close to 200. However, when the MEWMA using the normal-theory limits from Prahbu and Runger (1997) is applied to multivariate Poisson distributed data, the in-control ARLs are not the advertised value of 200. In fact, when the control limit for the MEWMA based on the normality assumption is applied to the multivariate Poisson-distributed data, a substantial reduction in the in-control ARL occurs. For example, instead of an in-control ARL near 200 as expected, the true in-control ARL ranges from 170 to 190 (see Tables 15 - 18). That is, using the MEWMA chart assuming multivariate normality when the underlying distribution is truly multivariate Poisson, results in a 5-15% reduction the in-control ARL and thus, an increase in false alarms. This reduction is quite obvious for small process means (5 or less) and a large number of variables (10 or more). A large in-control ARL for the MPEWMA indicates that it will result in fewer false alarms than the MEWMA scheme for normally distributed data.

The results have significant implications in practice. Specifically, we have shown that if one simply assumes the normality assumption applies (and uses published normal-theory limits) when in fact the underlying distribution is something such as the multivariate Poisson, expect an increase in out-of-control signals when the process is truly in control. The practitioner may be stopping the process when a signal occurs, when in fact the process is still in control.

### 3.8.3 SRDL Comparison

We also examined the standard deviation of the run length (SDRL). Tables 19, 20, 21 and 22 give the SDRLs for the MEWMA and MPEWMA control charts for two values of $\theta$ ($\theta$ = 0.5 and $\theta$ = 1, respectively). The SDRL results are quite similar to those ARL results for both in-control and out-of-control processes. The MEWMA relied on the normal theory approximation provides considerably lower SDRL values than the MPEWMA for the in-control case. On the other hand, there is no difference in the out-of-control SDRL performance, particularly when the size of shift becomes large.

**Table 15** Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.05$

| p | | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | |
| | | λ = 0.05 | | | | | | | | | | | | | | |
| 4 | δ / H | 11.49 | 11.22 | δ / H | 11.48 | 11.22 | δ / H | 11.47 | 11.22 | δ / H | 11.46 | 11.22 | δ / H | 11.46 | 11.22 | |
| | 0.00 | 200.124 | 183.885 | 0.00 | 200.114 | 184.081 | 0.00 | 199.806 | 184.378 | 0.00 | 199.200 | 184.579 | 0.00 | 199.549 | 183.509 | |
| | 0.51 | 28.843 | 27.944 | 0.56 | 25.287 | 24.657 | 0.46 | 33.08067 | 32.244 | 0.57 | 24.971 | 24.460 | 0.49 | 30.68125 | 29.962 | |
| | 0.91 | 13.115 | 12.849 | 1.05 | 10.964 | 10.814 | 0.88 | 13.801 | 13.530 | 1.09 | 10.505 | 10.309 | 0.93 | 12.826 | 12.573 | |
| | 1.57 | 6.858 | 6.715 | 1.37 | 7.951 | 7.854 | 1.70 | 6.062 | 5.978 | 1.58 | 6.618 | 6.539 | 1.36 | 7.912 | 7.778 | |
| | 2.19 | 4.569 | 4.489 | 1.95 | 5.196 | 5.107 | 2.18 | 4.486 | 4.421 | 2.03 | 4.847 | 4.774 | 1.77 | 5.713 | 5.635 | |
| | 2.74 | 3.478 | 3.427 | 2.95 | 3.143 | 3.089 | 2.63 | 3.573 | 3.527 | 2.46 | 3.845 | 3.769 | 2.16 | 4.488 | 4.408 | |
| 6 | δ / H | 14.93 | 14.60 | δ / H | 14.90 | 14.6 | δ / H | 14.89 | 14.60 | δ / H | 14.89 | 14.6 | δ / H | 14.9 | 14.6 | |
| | 0.00 | 200.023 | 180.555 | 0.00 | 199.306 | 182.533 | 0.00 | 199.670 | 181.292 | 0.00 | 199.214 | 180.262 | 0.00 | 200.150 | 181.355 | |
| | 0.52 | 31.746 | 30.773 | 0.41 | 43.168 | 41.592 | 0.46 | 36.643 | 35.3168 | 0.42 | 42.204 | 40.455 | 0.49 | 34.287 | 33.070 | |
| | 0.92 | 14.341 | 14.016 | 0.77 | 18.188 | 17.796 | 0.88 | 15.045 | 14.709 | 0.81 | 16.973 | 16.562 | 0.68 | 21.478 | 20.766 | |
| | 1.26 | 9.588 | 9.398 | 1.05 | 11.943 | 11.745 | 1.39 | 8.590 | 8.428 | 1.29 | 9.366 | 9.217 | 1.11 | 11.274 | 10.998 | |
| | 1.79 | 6.509 | 6.390 | 1.59 | 7.330 | 7.233 | 2.00 | 5.515 | 5.402 | 1.87 | 5.951 | 5.850 | 1.62 | 7.006 | 6.886 | |
| | 2.52 | 4.328 | 4.248 | 2.27 | 4.802 | 4.726 | 2.57 | 4.058 | 3.991 | 2.41 | 4.372 | 4.290 | 2.11 | 5.078 | 4.983 | |
| 10 | δ / H | 21.17 | 20.72 | δ / H | 21.15 | 20.72 | δ / H | 21.14 | 20.72 | δ / H | 21.13 | 20.72 | δ / H | 21.12 | 20.72 | |
| | 0.00 | 200.104 | 176.191 | 0.00 | 199.475 | 177.746 | 0.00 | 199.927 | 179.165 | 0.00 | 199.332 | 179.396 | 0.00 | 199.441 | 180.616 | |
| | 0.52 | 36.169 | 34.635 | 0.42 | 49.796 | 47.262 | 0.47 | 42.214 | 40.368 | 0.42 | 48.818 | 46.410 | 0.49 | 39.676 | 38.183 | |
| | 1.08 | 13.747 | 13.478 | 0.98 | 15.491 | 15.066 | 0.89 | 17.002 | 16.597 | 0.81 | 19.333 | 18.851 | 0.70 | 23.809 | 23.041 | |
| | 1.28 | 10.628 | 10.338 | 1.86 | 6.966 | 6.835 | 1.66 | 7.837 | 7.712 | 1.56 | 8.433 | 8.268 | 1.36 | 9.845 | 9.655 | |
| | 2.05 | 6.374 | 6.262 | 2.68 | 4.534 | 4.458 | 2.41 | 5.031 | 4.927 | 2.27 | 5.355 | 5.272 | 2.00 | 6.174 | 6.057 | |
| | 2.93 | 4.226 | 4.157 | 3.44 | 3.407 | 3.343 | 3.11 | 3.735 | 3.663 | 2.94 | 3.947 | 3.889 | 2.61 | 4.476 | 4.407 | |
| 15 | δ / H | 28.39 | 27.82 | δ / H | 28.34 | 27.82 | δ / H | 28.31 | 27.82 | δ / H | 28.30 | 27.82 | δ / H | 28.30 | 27.82 | |
| | 0.00 | 199.862 | 174.170 | 0.00 | 200.025 | 177.180 | 0.00 | 199.65 | 178.284 | 0.00 | 199.677 | 178.350 | 0.00 | 200.098 | 179.302 | |
| | 0.52 | 40.647 | 38.745 | 0.42 | 56.139 | 52.923 | 0.47 | 47.175 | 44.857 | 0.42 | 54.607 | 51.791 | 0.49 | 44.790 | 42.781 | |
| | 0.93 | 17.875 | 17.333 | 0.77 | 23.031 | 22.292 | 0.97 | 17.252 | 16.860 | 0.91 | 18.531 | 18.076 | 0.81 | 21.820 | 21.242 | |
| | 1.29 | 11.514 | 11.243 | 1.07 | 15.233 | 14.869 | 1.88 | 7.549 | 7.386 | 1.78 | 8.003 | 7.832 | 1.58 | 9.105 | 8.953 | |
| | 2.24 | 6.515 | 6.385 | 2.07 | 6.869 | 6.762 | 2.74 | 4.854 | 4.766 | 2.60 | 5.090 | 5.012 | 2.32 | 5.724 | 5.622 | |
| | 3.22 | 4.307 | 4.237 | 3.00 | 4.486 | 4.417 | 3.55 | 3.599 | 3.532 | 3.38 | 3.763 | 3.687 | 3.04 | 4.167 | 4.094 | |

**Table 16** Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.1$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | $\lambda = 0.1$ | | | | | | | | | | | | | | |
| 4 | δ / H | 13.01 | 12.73 | δ / H | 12.95 | 12.73 | δ / H | 12.91 | 12.73 | δ / H | 12.90 | 12.73 | δ / H | 12.89 | 12.73 |
| | 0.00 | 200.012 | 182.348 | 0.00 | 199.771 | 184.738 | 0.00 | 200.234 | 186.29 | 0.00 | 198.817 | 187.280 | 0.00 | 199.811 | 189.834 |
| | 0.51 | 32.322 | 30.968 | 0.56 | 27.168 | 26.320 | 0.46 | 37.142 | 36.123 | 0.57 | 25.576 | 25.301 | 0.48 | 34.584 | 33.689 |
| | 0.91 | 12.491 | 12.168 | 1.05 | 10.033 | 9.867 | 0.88 | 13.017 | 12.806 | 1.09 | 9.378 | 9.276 | 0.93 | 11.886 | 11.740 |
| | 1.57 | 5.708 | 5.585 | 1.37 | 6.775 | 6.692 | 1.70 | 4.995 | 4.953 | 1.58 | 5.518 | 5.434 | 1.36 | 6.753 | 6.698 |
| | 2.19 | 3.607 | 3.537 | 1.95 | 4.172 | 4.110 | 2.18 | 3.532 | 3.501 | 2.03 | 3.877 | 3.837 | 1.77 | 4.656 | 4.627 |
| | 2.74 | 2.657 | 2.616 | 2.95 | 2.378 | 2.328 | 2.63 | 2.749 | 2.703 | 2.46 | 2.977 | 2.942 | 2.16 | 3.551 | 3.506 |
| 6 | δ / H | 16.65 | 16.27 | δ / H | 16.54 | 16.27 | δ / H | 16.50 | 16.27 | δ / H | 16.48 | 16.27 | δ / H | 16.49 | 16.27 |
| | 0.00 | 199.591 | 179.161 | 0.00 | 200.009 | 182.997 | 0.00 | 199.801 | 186.126 | 0.00 | 199.801 | 185.212 | 0.00 | 200.066 | 187.169 |
| | 0.52 | 36.831 | 34.849 | 0.41 | 51.347 | 49.196 | 0.46 | 42.5856 | 41.1436 | 0.42 | 49.655 | 47.697 | 0.49 | 39.700 | 38.219 |
| | 0.92 | 13.789 | 13.467 | 0.77 | 18.535 | 18.0427 | 0.88 | 14.597 | 14.278 | 0.81 | 16.805 | 16.473 | 0.68 | 22.381 | 21.747 |
| | 1.26 | 8.559 | 8.304 | 1.05 | 11.002 | 10.758 | 1.39 | 7.388 | 7.301 | 1.29 | 8.191 | 8.033 | 1.11 | 10.152 | 10.042 |
| | 1.79 | 5.326 | 5.205 | 1.59 | 6.147 | 6.044 | 2.00 | 4.43 | 4.36 | 1.87 | 4.830 | 4.773 | 1.62 | 5.829 | 5.744 |
| | 2.52 | 3.350 | 2.410 | 2.27 | 3.775 | 3.711 | 2.57 | 3.142 | 3.099 | 2.41 | 3.411 | 3.365 | 2.11 | 4.051 | 3.993 |
| 10 | δ / H | 23.17 | 22.67 | δ / H | 23.04 | 22.67 | δ / H | 22.98 | 22.67 | δ / H | 22.95 | 22.67 | δ / H | 22.91 | 22.67 |
| | 0.00 | 199.268 | 175.325 | 0.00 | 200.122 | 182.320 | 0.00 | 199.828 | 183.217 | 0.00 | 200.06 | 184.810 | 0.00 | 199.469 | 184.569 |
| | 0.52 | 43.709 | 40.707 | 0.42 | 61.486 | 57.736 | 0.47 | 50.685 | 48.499 | 0.42 | 59.331 | 56.619 | 0.49 | 47.551 | 45.914 |
| | 1.08 | 12.783 | 12.353 | 0.98 | 14.796 | 14.379 | 0.89 | 16.873 | 16.535 | 0.81 | 19.708 | 19.280 | 0.70 | 25.45 | 24.436 |
| | 1.28 | 9.517 | 9.29 | 1.86 | 5.667 | 5.568 | 1.66 | 6.559 | 6.489 | 1.56 | 7.136 | 7.068 | 1.36 | 8.592 | 8.484 |
| | 2.05 | 5.1 | 4.998 | 2.68 | 3.483 | 3.448 | 2.41 | 3.933 | 3.901 | 2.27 | 4.253 | 4.196 | 2.00 | 4.98 | 4.943 |
| | 2.93 | 3.201 | 3.129 | 3.44 | 2.522 | 2.488 | 3.11 | 2.812 | 2.767 | 2.94 | 2.996 | 2.960 | 2.61 | 3.477 | 3.436 |
| 15 | δ / H | 30.62 | 30.03 | δ / H | 30.49 | 30.03 | δ / H | 30.40 | 30.03 | δ / H | 30.36 | 30.03 | δ / H | 30.33 | 30.03 |
| | 0.00 | 199.034 | 173.298 | 0.00 | 200.182 | 178.251 | 0.00 | 199.478 | 184.199 | 0.00 | 199.244 | 184.422 | 0.00 | 200.112 | 184.677 |
| | 0.52 | 50.393 | 46.438 | 0.42 | 70.966 | 65.627 | 0.47 | 58.461 | 55.399 | 0.42 | 68.097 | 64.615 | 0.49 | 55.192 | 52.887 |
| | 0.93 | 18.304 | 17.521 | 0.77 | 25.069 | 24.029 | 0.97 | 16.940 | 16.443 | 0.91 | 18.507 | 18.093 | 0.81 | 22.693 | 22.21 |
| | 1.29 | 10.561 | 10.295 | 1.07 | 14.138 | 14.059 | 1.88 | 6.192 | 6.139 | 1.78 | 6.655 | 6.577 | 1.58 | 7.757 | 7.668 |
| | 2.24 | 5.138 | 5.02 | 2.07 | 5.542 | 5.468 | 2.74 | 3.745 | 3.683 | 2.60 | 3.961 | 3.916 | 2.32 | 4.541 | 4.492 |
| | 3.22 | 3.217 | 3.144 | 3.00 | 3.403 | 1.165 | 3.55 | 2.660 | 2.621 | 3.38 | 2.807 | 2.772 | 3.04 | 3.171 | 3.142 |

**Table 17** Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 1.0$ and $\lambda = 0.05$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| p | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | $\lambda = 0.05$ | | | | | | | | | | | | | | |
| 4 | δ / H | 11.49 | 11.22 | δ / H | 11.48 | 11.22 | δ / H | 11.47 | 11.22 | δ / H | 11.46 | 11.22 | δ / H | 11.46 | 11.22 |
| | 0.00 | 199.031 | 182.379 | 0.00 | 199.142 | 183.332 | 0.00 | 199.309 | 183.735 | 0.00 | 200.141 | 183.361 | 0.00 | 198.954 | 184.046 |
| | 0.54 | 26.558 | 25.863 | 0.56 | 25.254 | 24.599 | 0.46 | 33.638 | 32.645 | 0.53 | 27.063 | 26.531 | 0.49 | 30.577 | 29.815 |
| | 0.95 | 12.349 | 8.536 | 1.04 | 11.029 | 10.853 | 1.11 | 10.352 | 10.127 | 1.03 | 11.288 | 11.031 | 0.89 | 13.512 | 13.233 |
| | 1.41 | 7.833 | 7.722 | 1.26 | 8.857 | 8.702 | 1.60 | 6.529 | 6.413 | 1.50 | 7.048 | 6.942 | 1.31 | 8.289 | 8.186 |
| | 2.00 | 5.141 | 5.075 | 1.81 | 5.691 | 5.618 | 2.07 | 4.789 | 4.723 | 1.94 | 5.165 | 5.064 | 1.71 | 5.99 | 5.886 |
| | 2.53 | 3.885 | 3.808 | 2.31 | 4.252 | 4.174 | 2.50 | 3.800 | 3.752 | 2.36 | 4.059 | 3.991 | 2.09 | 4.672 | 4.610 |
| 6 | δ / H | 14.95 | 14.6 | δ / H | 14.92 | 14.6 | δ / H | 14.91 | 14.60 | δ / H | 14.91 | 14.60 | δ / H | 14.90 | 14.6 |
| | 0.00 | 199.747 | 179.675 | 0.00 | 199.905 | 182.015 | 0.00 | 199.439 | 182.435 | 0.00 | 200.111 | 182.727 | 0.00 | 199.983 | 181.930 |
| | 0.55 | 29.068 | 28.056 | 0.43 | 41.300 | 39.8285 | 0.46 | 36.7216 | 35.318 | 0.42 | 42.481 | 40.609 | 0.49 | 34.002 | 32.829 |
| | 0.97 | 13.404 | 13.120 | 1.07 | 11.740 | 11.524 | 0.88 | 15.061 | 14.759 | 0.80 | 17.153 | 16.615 | 0.68 | 21.573 | 20.990 |
| | 1.55 | 7.836 | 7.679 | 1.41 | 8.579 | 8.391 | 1.26 | 9.67 | 9.458 | 1.18 | 10.418 | 10.191 | 1.04 | 12.188 | 11.893 |
| | 2.22 | 5.111 | 5.020 | 2.04 | 5.508 | 5.427 | 1.84 | 6.143 | 6.034 | 1.73 | 6.545 | 6.417 | 1.53 | 7.545 | 7.401 |
| | 2.83 | 3.836 | 3.771 | 2.62 | 4.093 | 4.036 | 2.38 | 4.513 | 4.425 | 2.25 | 4.79 | 4.700 | 2.00 | 5.438 | 5.358 |
| 10 | δ / H | 21.19 | 20.72 | δ / H | 21.14 | 20.72 | δ / H | 21.13 | 20.72 | δ / H | 21.13 | 20.72 | δ / H | 21.12 | 20.72 |
| | 0.00 | 199.484 | 176.648 | 0.00 | 199.739 | 178.941 | 0.00 | 200.081 | 180.012 | 0.00 | 200.194 | 180.788 | 0.00 | 199.188 | 180.138 |
| | 0.56 | 32.922 | 31.503 | 0.43 | 47.399 | 44.908 | 0.47 | 41.389 | 39.589 | 0.43 | 48.300 | 46.012 | 0.49 | 39.267 | 37.7468 |
| | 0.98 | 14.984 | 14.615 | 1.10 | 12.889 | 12.593 | 0.89 | 16.741 | 16.308 | 0.81 | 19.058 | 18.599 | 0.68 | 24.465 | 23.703 |
| | 1.69 | 8.168 | 7.988 | 1.58 | 8.581 | 8.408 | 1.45 | 9.322 | 9.173 | 1.38 | 9.852 | 9.666 | 1.24 | 11.153 | 10.937 |
| | 2.45 | 5.317 | 5.214 | 2.30 | 5.513 | 5.42 | 2.12 | 5.914 | 5.809 | 2.02 | 6.196 | 6.077 | 1.83 | 6.913 | 6.807 |
| | 3.16 | 3.965 | 3.883 | 2.98 | 4.095 | 4.012 | 2.76 | 4.352 | 4.264 | 2.64 | 4.525 | 4.455 | 2.39 | 4.998 | 4.916 |
| 15 | δ / H | 28.41 | 27.82 | δ / H | 28.34 | 27.82 | δ / H | 28.32 | 27.82 | δ / H | 28.31 | 27.82 | δ / H | 28.30 | 27.82 |
| | 0.00 | 200.047 | 174.974 | 0.00 | 199.949 | 176.419 | 0.00 | 199.995 | 178.083 | 0.00 | 199.587 | 178.383 | 0.00 | 199.23 | 178.563 |
| | 0.57 | 36.455 | 34.602 | 0.44 | 52.785 | 49.834 | 0.48 | 46.022 | 43.848 | 0.43 | 53.747 | 50.911 | 0.49 | 44.10313 | 42.174 |
| | 0.98 | 16.549 | 16.062 | 0.87 | 20.435 | 19.861 | 0.91 | 18.326 | 17.912 | 0.82 | 21.075 | 20.454 | 0.71 | 26.549 | 25.470 |
| | 1.37 | 10.796 | 10.512 | 1.11 | 14.017 | 13.679 | 1.58 | 9.357 | 9.156 | 1.52 | 9.756 | 9.553 | 1.39 | 10.715 | 10.518 |
| | 1.78 | 8.703 | 8.523 | 2.48 | 5.714 | 5.613 | 2.32 | 5.946 | 5.848 | 2.24 | 6.142 | 6.030 | 2.05 | 6.682 | 6.557 |
| | 2.60 | 5.641 | 5.548 | 3.23 | 4.227 | 4.163 | 3.04 | 4.383 | 4.308 | 2.93 | 4.502 | 4.411 | 2.70 | 4.826 | 4.749 |

**Table 18** Comparison of the ARL performance between the MPEWMA and MEWMA charts when $\theta = 1.0$ and $\lambda = 0.1$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | λ = 0.1 | | | | | | | | | | | | | | |
| 4 | δ / H | 13.02 | 12.73 | δ / H | 12.95 | 12.73 | δ / H | 12.92 | 12.73 | δ / H | 12.90 | 12.73 | δ / H | 12.89 | 12.73 |
| | 0.00 | 199.730 | 179.263 | 0.00 | 200.252 | 185.260 | 0.00 | 200.236 | 187.762 | 0.00 | 200.059 | 186.609 | 0.00 | 199.299 | 187.992 |
| | 0.54 | 29.416 | 28.3795 | 0.56 | 27.102 | 26.413 | 0.46 | 37.845 | 36.714 | 0.53 | 29.124 | 28.472 | 0.49 | 34.326 | 33.352 |
| | 0.95 | 11.641 | 11.341 | 1.04 | 10.024 | 9.821 | 1.11 | 9.155 | 9.113 | 1.03 | 10.205 | 10.096 | 0.89 | 12.652 | 12.460 |
| | 1.41 | 6.653 | 6.523 | 1.26 | 7.672 | 7.553 | 1.60 | 5.431 | 5.342 | 1.50 | 5.921 | 5.84 | 1.31 | 7.138 | 7.055 |
| | 2.00 | 4.133 | 4.035 | 1.81 | 4.646 | 4.567 | 2.07 | 3.812 | 3.772 | 1.94 | 4.138 | 4.096 | 1.71 | 4.913 | 4.849 |
| | 2.53 | 2.988 | 2.925 | 2.31 | 3.319 | 3.274 | 2.50 | 2.948 | 2.898 | 2.36 | 3.166 | 3.125 | 2.09 | 3.716 | 3.670 |
| 6 | δ / H | 16.68 | 16.27 | δ / H | 16.56 | 16.27 | δ / H | 16.50 | 16.27 | δ / H | 16.50 | 16.27 | δ / H | 16.48 | 16.27 |
| | 0.00 | 200.020 | 176.851 | 0.00 | 199.622 | 183.283 | 0.00 | 199.093 | 184.287 | 0.00 | 198.872 | 186.141 | 0.00 | 200.237 | 187.447 |
| | 0.55 | 33.037 | 31.254 | 0.43 | 49.4705 | 47.0775 | 0.46 | 42.5778 | 40.982 | 0.42 | 50.155 | 48.0322 | 0.49 | 29.296 | 37.9475 |
| | 0.97 | 12.852 | 12.4578 | 1.07 | 10.76 | 10.574 | 0.88 | 14.517 | 14.143 | 0.80 | 16.935 | 16.543 | 0.68 | 22.593 | 21.997 |
| | 1.55 | 6.538 | 6.407 | 1.41 | 7.291 | 7.16 | 1.26 | 8.459 | 8.331 | 1.18 | 9.216 | 9.109 | 1.04 | 11.157 | 10.953 |
| | 2.22 | 4.014 | 3.94 | 2.04 | 4.403 | 4.321 | 1.84 | 4.973 | 4.907 | 1.73 | 5.402 | 5.316 | 1.53 | 6.339 | 6.24 |
| | 2.83 | 2.916 | 2.851 | 2.62 | 3.146 | 3.085 | 2.38 | 3.513 | 3.461 | 2.25 | 3.765 | 3.712 | 2.00 | 4.366 | 4.31 |
| 10 | δ / H | 23.20 | 22.67 | δ / H | 23.03 | 22.67 | δ / H | 22.98 | 22.67 | δ / H | 22.94 | 22.67 | δ / H | 22.94 | 22.67 |
| | 0.00 | 199.341 | 172.773 | 0.00 | 199.64 | 181.505 | 0.00 | 199.334 | 184.767 | 0.00 | 198.783 | 183.536 | 0.00 | 198.847 | 185.910 |
| | 0.56 | 38.808 | 36.2755 | 0.43 | 58.248 | 54.906 | 0.47 | 49.981 | 47.704 | 0.43 | 58.690 | 56.2496 | 0.49 | 47.1071 | 45.463 |
| | 0.98 | 14.7194 | 14.1605 | 1.10 | 12.056 | 11.763 | 0.89 | 16.680 | 16.145 | 0.81 | 19.381 | 18.949 | 0.68 | 25.556 | 26.001 |
| | 1.69 | 6.754 | 6.604 | 1.58 | 7.224 | 7.080 | 1.45 | 8.010 | 7.892 | 1.38 | 8.514 | 8.426 | 1.24 | 9.933 | 9.786 |
| | 2.45 | 4.101 | 4.016 | 2.30 | 4.318 | 4.251 | 2.12 | 4.710 | 4.654 | 2.02 | 4.995 | 4.912 | 1.83 | 5.692 | 5.620 |
| | 3.16 | 2.945 | 2.883 | 2.98 | 3.074 | 3.026 | 2.76 | 3.327 | 3.283 | 2.64 | 3.502 | 3.45 | 2.39 | 3.94 | 3.881 |
| 15 | δ / H | 30.65 | 30.03 | δ / H | 30.49 | 30.03 | δ / H | 30.41 | 30.03 | δ / H | 30.38 | 30.03 | δ / H | 30.35 | 30.03 |
| | 0.00 | 199.953 | 171.475 | 0.00 | 199.448 | 179.022 | 0.00 | 200.083 | 182.487 | 0.00 | 199.686 | 182.693 | 0.00 | 199.436 | 184.636 |
| | 0.57 | 44.795 | 41.439 | 0.44 | 67.044 | 62.124 | 0.48 | 57.209 | 54.1469 | 0.43 | 67.435 | 63.7484 | 0.49 | 54.575 | 52.145 |
| | 0.98 | 16.602 | 15.9079 | 0.87 | 20.823 | 20.098 | 0.91 | 18.567 | 18.149 | 0.82 | 22.098 | 21.455 | 0.71 | 28.853 | 28.004 |
| | 1.37 | 9.721 | 9.399 | 1.11 | 13.400 | 12.940 | 1.58 | 7.971 | 7.84 | 1.52 | 8.400 | 8.256 | 1.39 | 9.393 | 9.285 |
| | 1.78 | 7.138 | 6.953 | 2.48 | 4.431 | 4.375 | 2.32 | 4.701 | 4.633 | 2.24 | 4.893 | 4.827 | 2.05 | 5.415 | 5.346 |
| | 2.60 | 4.321 | 4.237 | 3.23 | 3.158 | 1.033 | 3.04 | 3.304 | 3.265 | 2.93 | 3.439 | 3.382 | 2.70 | 3.744 | 3.708 |

**Table 19** Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.05$

| p | | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | | | | | | | | λ = 0.05 | | | | | | | |
| 4 | δ / H | 11.49 | 11.22 | δ / H | 11.48 | 11.22 | δ / H | 11.47 | 11.22 | δ / H | 11.47 | 11.22 | δ / H | 11.46 | 11.22 |
| | 0.00 | 201.330 | 185.840 | 0.00 | 200.489 | 183.939 | 0.00 | 201.513 | 184.764 | 0.00 | 200.473 | 185.678 | 0.00 | 201.750 | 183.501 |
| | 0.51 | 19.132 | 18.479 | 0.56 | 16.089 | 15.611 | 0.46 | 23.128 | 22.446 | 0.57 | 15.324 | 15.346 | 0.49 | 20.918 | 21.096 |
| | 0.91 | 6.608 | 6.487 | 1.05 | 5.267 | 5.241 | 0.88 | 7.231 | 6.990 | 1.09 | 5.056 | 4.955 | 0.93 | 6.524 | 6.410 |
| | 1.57 | 2.762 | 2.752 | 1.37 | 3.465 | 3.453 | 1.70 | 2.554 | 2.520 | 1.58 | 2.841 | 2.812 | 1.36 | 3.569 | 3.523 |
| | 2.19 | 1.693 | 1.669 | 1.95 | 2.059 | 2.032 | 2.18 | 1.793 | 1.777 | 2.03 | 1.981 | 1.956 | 1.77 | 2.443 | 2.414 |
| | 2.74 | 1.220 | 1.225 | 2.95 | 1.140 | 1.143 | 2.63 | 1.375 | 1.376 | 2.46 | 1.516 | 1.511 | 2.16 | 1.858 | 1.832 |
| 6 | δ / H | 14.93 | 14.60 | δ / H | 14.90 | 14.6 | δ / H | 14.89 | 14.60 | δ / H | 14.89 | 14.6 | δ / H | 14.9 | 14.6 |
| | 0.00 | 202.741 | 180.962 | 0.00 | 200.880 | 183.318 | 0.00 | 200.194 | 183.088 | 0.00 | 200.153 | 182.516 | 0.00 | 202.306 | 184.962 |
| | 0.52 | 21.313 | 20.611 | 0.41 | 32.147 | 30.828 | 0.46 | 25.956 | 25.025 | 0.42 | 31.258 | 30.007 | 0.49 | 23.723 | 22.976 |
| | 0.92 | 6.750 | 7.016 | 0.77 | 10.049 | 9.798 | 0.88 | 7.764 | 7.706 | 0.81 | 9.148 | 9.067 | 0.68 | 12.583 | 12.306 |
| | 1.26 | 4.221 | 4.155 | 1.05 | 5.744 | 5.596 | 1.39 | 3.734 | 3.699 | 1.29 | 4.181 | 4.159 | 1.11 | 5.348 | 5.296 |
| | 1.79 | 2.454 | 2.443 | 1.59 | 2.980 | 2.946 | 2.00 | 2.153 | 2.153 | 1.87 | 2.407 | 2.384 | 1.62 | 2.963 | 2.943 |
| | 2.52 | 1.487 | 1.473 | 2.27 | 1.768 | 1.767 | 2.57 | 1.520 | 1.514 | 2.41 | 1.683 | 1.678 | 2.11 | 2.039 | 2.026 |
| 10 | δ / H | 21.17 | 20.72 | δ / H | 21.15 | 20.72 | δ / H | 21.14 | 20.72 | δ / H | 21.13 | 20.72 | δ / H | 21.12 | 20.72 |
| | 0.00 | 202.254 | 176.506 | 0.00 | 200.668 | 178.685 | 0.00 | 200.902 | 180.116 | 0.00 | 200.966 | 179.352 | 0.00 | 203.114 | 180.620 |
| | 0.52 | 24.965 | 23.623 | 0.42 | 38.094 | 47.262 | 0.47 | 30.552 | 29.142 | 0.42 | 37.096 | 35.125 | 0.49 | 28.210 | 27.117 |
| | 1.08 | 6.135 | 6.060 | 0.98 | 7.502 | 15.066 | 0.89 | 8.817 | 8.656 | 0.81 | 10.555 | 10.278 | 0.70 | 13.996 | 13.517 |
| | 1.28 | 4.623 | 4.538 | 1.86 | 2.581 | 6.835 | 1.66 | 3.126 | 3.104 | 1.56 | 3.485 | 3.435 | 1.36 | 4.278 | 4.241 |
| | 2.05 | 2.197 | 2.176 | 2.68 | 1.527 | 4.458 | 2.41 | 1.822 | 1.788 | 2.27 | 1.990 | 1.977 | 2.00 | 2.422 | 2.385 |
| | 2.93 | 1.309 | 1.308 | 3.44 | 1.095 | 3.343 | 3.11 | 1.282 | 1.275 | 2.94 | 1.403 | 1.391 | 2.61 | 1.672 | 1.667 |
| 15 | δ / H | 28.39 | 27.82 | δ / H | 28.34 | 27.82 | δ / H | 28.31 | 27.82 | δ / H | 28.30 | 27.82 | δ / H | 28.30 | 27.82 |
| | 0.00 | 201.321 | 174.435 | 0.00 | 201.462 | 177.958 | 0.00 | 200.514 | 180.049 | 0.00 | 202.975 | 178.751 | 0.00 | 203.567 | 180.638 |
| | 0.52 | 28.583 | 26.987 | 0.42 | 44.097 | 41.010 | 0.47 | 35.018 | 33.097 | 0.42 | 42.622 | 40.060 | 0.49 | 32.725 | 31.051 |
| | 0.93 | 9.120 | 8.848 | 0.77 | 13.029 | 12.595 | 0.97 | 8.561 | 8.375 | 0.91 | 9.527 | 9.329 | 0.81 | 12.032 | 11.701 |
| | 1.29 | 5.044 | 4.906 | 1.07 | 6.971 | 6.874 | 1.88 | 2.813 | 2.797 | 1.78 | 3.089 | 3.034 | 1.58 | 3.765 | 3.672 |
| | 2.24 | 2.071 | 2.053 | 2.07 | 2.384 | 2.366 | 2.74 | 1.638 | 1.632 | 2.60 | 1.774 | 1.768 | 2.32 | 2.107 | 2.093 |
| | 3.22 | 1.230 | 1.224 | 3.00 | 1.395 | 1.396 | 3.55 | 1.150 | 1.149 | 3.38 | 1.244 | 1.244 | 3.04 | 1.470 | 1.449 |

**Table 20** Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 0.5$ and $\lambda = 0.1$

| p | | Mean 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | | | | | | | | $\lambda = 0.1$ | | | | | | | |
| 4 | δ / H | 13.01 | 12.73 | δ / H | 12.95 | 12.73 | δ / H | 12.91 | 12.73 | δ / H | 12.90 | 12.73 | δ / H | 12.89 | 12.73 |
| | 0.00 | 202.314 | 183.725 | 0.00 | 200.438 | 187.493 | 0.00 | 201.147 | 189.018 | 0.00 | 200.082 | 188.435 | 0.00 | 201.745 | 190.196 |
| | 0.51 | 25.675 | 24.481 | 0.56 | 20.775 | 20.118 | 0.46 | 30.656 | 29.775 | 0.57 | 20.253 | 19.163 | 0.48 | 28.135 | 27.300 |
| | 0.91 | 7.444 | 7.231 | 1.05 | 5.568 | 5.512 | 0.88 | 7.971 | 7.841 | 1.09 | 5.147 | 5.140 | 0.93 | 7.079 | 6.941 |
| | 1.57 | 2.583 | 2.568 | 1.37 | 3.328 | 3.278 | 1.70 | 2.287 | 2.286 | 1.58 | 2.618 | 2.585 | 1.36 | 3.401 | 3.342 |
| | 2.19 | 1.477 | 1.459 | 1.95 | 1.811 | 1.787 | 2.18 | 1.518 | 1.522 | 2.03 | 1.707 | 1.707 | 1.77 | 2.158 | 2.159 |
| | 2.74 | 1.038 | 1.025 | 2.95 | 0.950 | 0.965 | 2.63 | 1.149 | 1.146 | 2.46 | 1.281 | 1.264 | 2.16 | 1.576 | 1.574 |
| 6 | δ / H | 16.65 | 16.27 | δ / H | 16.54 | 16.27 | δ / H | 16.50 | 16.27 | δ / H | 16.48 | 16.27 | δ / H | 16.49 | 16.27 |
| | 0.00 | 200.140 | 180.710 | 0.00 | 200.923 | 183.194 | 0.00 | 201.853 | 185.680 | 0.00 | 200.417 | 187.714 | 0.00 | 201.493 | 190.234 |
| | 0.52 | 29.153 | 28.209 | 0.41 | 44.941 | 42.839 | 0.46 | 35.852 | 34.384 | 0.42 | 43.152 | 41.402 | 0.49 | 32.888 | 31.624 |
| | 0.92 | 8.234 | 8.059 | 0.77 | 12.450 | 11.981 | 0.88 | 9.101 | 8.738 | 0.81 | 10.969 | 10.706 | 0.68 | 15.975 | 15.597 |
| | 1.26 | 4.308 | 4.190 | 1.05 | 6.164 | 5.951 | 1.39 | 3.589 | 3.554 | 1.29 | 4.153 | 4.100 | 1.11 | 5.573 | 5.531 |
| | 1.79 | 2.252 | 2.211 | 1.59 | 2.801 | 2.752 | 2.00 | 1.868 | 1.875 | 1.87 | 2.138 | 2.132 | 1.62 | 2.729 | 2.696 |
| | 2.52 | 1.274 | 1.257 | 2.27 | 1.528 | 1.515 | 2.57 | 1.273 | 1.263 | 2.41 | 1.413 | 1.413 | 2.11 | 1.761 | 1.737 |
| 10 | δ / H | 23.17 | 22.67 | δ / H | 23.04 | 22.67 | δ / H | 22.98 | 22.67 | δ / H | 22.95 | 22.67 | δ / H | 22.91 | 22.67 |
| | 0.00 | 200.787 | 174.891 | 0.00 | 202.111 | 182.105 | 0.00 | 201.510 | 184.177 | 0.00 | 201.003 | 185.787 | 0.00 | 202.550 | 185.687 |
| | 0.52 | 36.515 | 33.516 | 0.42 | 54.956 | 51.041 | 0.47 | 43.745 | 41.437 | 0.42 | 52.683 | 49.893 | 0.49 | 40.633 | 38.871 |
| | 1.08 | 6.998 | 6.780 | 0.98 | 8.722 | 8.470 | 0.89 | 10.639 | 10.370 | 0.81 | 13.171 | 12.740 | 0.70 | 18.423 | 17.486 |
| | 1.28 | 4.802 | 4.694 | 1.86 | 2.366 | 2.321 | 1.66 | 2.918 | 2.895 | 1.56 | 3.274 | 3.281 | 1.36 | 4.278 | 4.183 |
| | 2.05 | 1.972 | 1.946 | 2.68 | 1.287 | 1.283 | 2.41 | 1.554 | 1.535 | 2.27 | 1.739 | 1.703 | 2.00 | 2.113 | 2.102 |
| | 2.93 | 1.104 | 1.096 | 3.44 | 0.900 | 0.896 | 3.11 | 1.056 | 1.052 | 2.94 | 1.152 | 1.152 | 2.61 | 1.404 | 1.382 |
| 15 | δ / H | 30.62 | 30.03 | δ / H | 30.49 | 30.03 | δ / H | 30.40 | 30.03 | δ / H | 30.36 | 30.03 | δ / H | 30.33 | 30.03 |
| | 0.00 | 202.938 | 175.244 | 0.00 | 201.443 | 178.855 | 0.00 | 200.623 | 184.936 | 0.00 | 201.435 | 186.337 | 0.00 | 203.247 | 185.506 |
| | 0.52 | 43.190 | 39.496 | 0.42 | 64.634 | 59.346 | 0.47 | 51.605 | 48.502 | 0.42 | 61.689 | 58.197 | 0.49 | 48.306 | 45.765 |
| | 0.93 | 11.565 | 10.961 | 0.77 | 17.766 | 16.890 | 0.97 | 10.305 | 9.989 | 0.91 | 11.835 | 11.393 | 0.81 | 15.595 | 15.126 |
| | 1.29 | 5.398 | 5.213 | 1.07 | 8.121 | 7.945 | 1.88 | 2.594 | 2.568 | 1.78 | 2.871 | 2.854 | 1.58 | 3.606 | 3.525 |
| | 2.24 | 1.849 | 1.820 | 2.07 | 2.156 | 2.142 | 2.74 | 1.368 | 1.367 | 2.60 | 1.498 | 1.495 | 2.32 | 1.821 | 1.807 |
| | 3.22 | 1.031 | 1.025 | 3.00 | 1.174 | 1.165 | 3.55 | 0.935 | 0.934 | 3.38 | 1.016 | 1.011 | 3.04 | 1.206 | 1.195 |

**Table 21** Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 1.0$ and $\lambda = 0.05$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | $\lambda = 0.05$ | | | | | | | | | | | | | | |
| 4 | δ / H | 11.49 | 11.22 | δ / H | 11.48 | 11.22 | δ / H | 11.47 | 11.22 | δ / H | 11.46 | 11.22 | δ / H | 11.46 | 11.22 |
| | 0.00 | 200.534 | 183.040 | 0.00 | 200.347 | 183.983 | 0.00 | 200.577 | 183.385 | 0.00 | 201.088 | 185.203 | 0.00 | 199.868 | 184.600 |
| | 0.54 | 17.088 | 16.729 | 0.56 | 25.254 | 15.536 | 0.46 | 23.548 | 22.832 | 0.53 | 17.600 | 17.232 | 0.49 | 20.877 | 20.261 |
| | 0.95 | 6.068 | 5.953 | 1.04 | 11.029 | 5.220 | 1.11 | 4.900 | 4.790 | 1.03 | 5.501 | 5.361 | 0.89 | 6.989 | 6.872 |
| | 1.41 | 3.249 | 3.230 | 1.26 | 8.857 | 3.867 | 1.60 | 2.759 | 2.731 | 1.50 | 3.067 | 3.019 | 1.31 | 3.796 | 3.743 |
| | 2.00 | 1.932 | 1.912 | 1.81 | 5.691 | 2.267 | 2.07 | 1.929 | 1.910 | 1.94 | 2.120 | 2.093 | 1.71 | 2.569 | 2.533 |
| | 2.53 | 1.381 | 1.361 | 2.31 | 4.252 | 1.599 | 2.50 | 1.486 | 1.475 | 2.36 | 1.621 | 1.599 | 2.09 | 1.942 | 1.919 |
| 6 | δ / H | 14.95 | 14.6 | δ / H | 14.92 | 14.6 | δ / H | 14.91 | 14.60 | δ / H | 14.91 | 14.60 | δ / H | 14.90 | 14.6 |
| | 0.00 | 200.636 | 180.352 | 0.00 | 201.829 | 183.272 | 0.00 | 202.657 | 181.310 | 0.00 | 201.069 | 185.575 | 0.00 | 201.725 | 182.679 |
| | 0.55 | 18.914 | 18.133 | 0.43 | 30.244 | 29.034 | 0.46 | 26.022 | 25.037 | 0.42 | 31.583 | 30.182 | 0.49 | 23.524 | 22.735 |
| | 0.97 | 6.552 | 6.418 | 1.07 | 5.565 | 5.479 | 0.88 | 7.815 | 7.589 | 0.80 | 9.285 | 9.128 | 0.68 | 12.789 | 12.479 |
| | 1.55 | 3.016 | 3.003 | 1.41 | 3.574 | 3.524 | 1.26 | 4.297 | 4.257 | 1.18 | 4.731 | 4.704 | 1.04 | 5.878 | 5.793 |
| | 2.22 | 1.786 | 1.772 | 2.04 | 2.061 | 2.047 | 1.84 | 2.465 | 2.426 | 1.73 | 2.672 | 2.660 | 1.53 | 3.246 | 3.211 |
| | 2.83 | 1.270 | 1.265 | 2.62 | 1.446 | 1.434 | 2.38 | 1.693 | 1.689 | 2.25 | 1.850 | 1.844 | 2.00 | 2.207 | 2.190 |
| 10 | δ / H | 21.19 | 20.72 | δ / H | 21.14 | 20.72 | δ / H | 21.13 | 20.72 | δ / H | 21.13 | 20.72 | δ / H | 21.12 | 20.72 |
| | 0.00 | 200.799 | 179.518 | 0.00 | 201.164 | 180.636 | 0.00 | 201.381 | 181.373 | 0.00 | 202.238 | 180.771 | 0.00 | 200.625 | 179.907 |
| | 0.56 | 21.760 | 20.724 | 0.43 | 35.670 | 33.455 | 0.47 | 29.843 | 28.323 | 0.43 | 36.632 | 34.644 | 0.49 | 27.814 | 26.556 |
| | 0.98 | 7.334 | 7.134 | 1.10 | 6.053 | 5.941 | 0.89 | 8.698 | 8.455 | 0.81 | 10.376 | 10.121 | 0.68 | 14.588 | 14.059 |
| | 1.69 | 2.894 | 2.851 | 1.58 | 3.290 | 3.245 | 1.45 | 3.854 | 3.793 | 1.38 | 4.181 | 4.113 | 1.24 | 5.014 | 4.938 |
| | 2.45 | 1.671 | 1.670 | 2.30 | 1.876 | 1.870 | 2.12 | 2.165 | 2.141 | 2.02 | 2.343 | 2.324 | 1.83 | 2.754 | 2.725 |
| | 3.16 | 1.181 | 1.182 | 2.98 | 1.314 | 1.308 | 2.76 | 1.500 | 1.488 | 2.64 | 1.602 | 1.600 | 2.39 | 1.870 | 1.861 |
| 15 | δ / H | 28.41 | 27.82 | δ / H | 28.34 | 27.82 | δ / H | 28.32 | 27.82 | δ / H | 28.31 | 27.82 | δ / H | 28.30 | 27.82 |
| | 0.00 | 200.324 | 175.553 | 0.00 | 200.396 | 175.814 | 0.00 | 201.743 | 178.416 | 0.00 | 199.869 | 179.730 | 0.00 | 199.234 | 178.772 |
| | 0.57 | 24.491 | 23.118 | 0.44 | 40.432 | 38.147 | 0.48 | 33.894 | 32.106 | 0.43 | 41.730 | 39.225 | 0.49 | 32.055 | 30.478 |
| | 0.98 | 8.175 | 7.921 | 0.87 | 10.324 | 10.112 | 0.91 | 9.548 | 9.326 | 0.82 | 11.517 | 11.174 | 0.71 | 15.648 | 15.103 |
| | 1.37 | 4.546 | 4.488 | 1.11 | 6.586 | 6.435 | 1.58 | 3.653 | 3.616 | 1.52 | 3.916 | 3.884 | 1.39 | 4.557 | 4.528 |
| | 1.78 | 2.880 | 2.850 | 2.48 | 1.805 | 1.794 | 2.32 | 2.023 | 2.026 | 2.24 | 2.183 | 2.164 | 2.05 | 2.504 | 2.483 |
| | 2.60 | 1.635 | 1.614 | 3.23 | 1.253 | 1.252 | 3.04 | 1.398 | 1.394 | 2.93 | 1.489 | 1.493 | 2.70 | 1.706 | 1.698 |

**Table 22** Comparison of the SDRL performance between the MPEWMA and MEWMA charts when $\theta = 1.0$ and $\lambda = 0.1$

| | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA | | MPEWMA | MEWMA |
| | λ = 0.1 | | | | | | | | | | | | | | |
| 4 | δ / H | 13.02 | 12.73 | δ / H | 12.95 | 12.73 | δ / H | 12.92 | 12.73 | δ / H | 12.90 | 12.73 | δ / H | 12.89 | 12.73 |
| | 0.00 | 202.437 | 178.674 | 0.00 | 202.086 | 184.831 | 0.00 | 201.463 | 189.096 | 0.00 | 202.059 | 187.053 | 0.00 | 201.212 | 187.930 |
| | 0.54 | 22.958 | 21.944 | 0.56 | 20.765 | 20.203 | 0.46 | 31.484 | 30.149 | 0.53 | 22.721 | 22.194 | 0.49 | 27.846 | 26.951 |
| | 0.95 | 6.752 | 6.567 | 1.04 | 5.572 | 5.487 | 1.11 | 4.941 | 4.921 | 1.03 | 5.685 | 5.697 | 0.89 | 7.138 | 7.520 |
| | 1.41 | 3.128 | 3.085 | 1.26 | 3.846 | 3.809 | 1.60 | 2.521 | 2.514 | 1.50 | 2.842 | 2.814 | 1.31 | 3.622 | 3.599 |
| | 2.00 | 1.706 | 1.686 | 1.81 | 2.039 | 2.031 | 2.07 | 1.658 | 1.652 | 1.94 | 1.847 | 1.827 | 1.71 | 2.285 | 2.286 |
| | 2.53 | 1.170 | 1.168 | 2.31 | 1.375 | 1.374 | 2.50 | 1.246 | 1.231 | 2.36 | 1.358 | 1.354 | 2.09 | 1.657 | 1.652 |
| 6 | δ / H | 16.68 | 16.27 | δ / H | 16.56 | 16.27 | δ / H | 16.5 | 16.27 | δ / H | 16.50 | 16.27 | δ / H | 16.48 | 16.27 |
| | 0.00 | 200.901 | 176.880 | 0.00 | 202.104 | 183.693 | 0.00 | 200.512 | 184.501 | 0.00 | 199.387 | 188.195 | 0.00 | 204.029 | 190.544 |
| | 0.55 | 26.116 | 24.535 | 0.43 | 42.729 | 40.519 | 0.46 | 35.892 | 34.530 | 0.42 | 43.567 | 41.579 | 0.49 | 32.678 | 31.352 |
| | 0.97 | 7.483 | 7.210 | 1.07 | 5.913 | 5.835 | 0.88 | 8.956 | 8.724 | 0.80 | 11.035 | 10.832 | 0.68 | 16.110 | 15.790 |
| | 1.55 | 2.884 | 2.847 | 1.41 | 3.449 | 3.401 | 1.26 | 4.291 | 4.220 | 1.18 | 4.850 | 4.827 | 1.04 | 6.296 | 6.171 |
| | 2.22 | 1.546 | 1.538 | 2.04 | 1.814 | 1.789 | 1.84 | 2.169 | 2.147 | 1.73 | 2.410 | 2.397 | 1.53 | 3.000 | 2.986 |
| | 2.83 | 1.070 | 1.063 | 2.62 | 1.217 | 1.206 | 2.38 | 1.426 | 1.425 | 2.25 | 1.577 | 1.565 | 2.00 | 1.900 | 1.895 |
| 10 | δ / H | 23.2 | 22.67 | δ / H | 23.03 | 22.67 | δ / H | 22.98 | 22.67 | δ / H | 22.94 | 22.67 | δ / H | 22.94 | 22.67 |
| | 0.00 | 201.3614 | 173.263 | 0.00 | 200.237 | 182.603 | 0.00 | 200.540 | 185.776 | 0.00 | 200.292 | 182.879 | 0.00 | 200.826 | 185.468 |
| | 0.56 | 31.44039 | 29.060 | 0.43 | 51.263 | 48.110 | 0.47 | 43.042 | 40.660 | 0.43 | 52.183 | 49.555 | 0.49 | 40.026 | 38.423 |
| | 0.98 | 4.328523 | 8.332 | 1.10 | 6.683 | 6.469 | 0.89 | 10.500 | 10.009 | 0.81 | 12.891 | 12.426 | 0.68 | 19.435 | 19.025 |
| | 1.69 | 2.778131 | 2.717 | 1.58 | 3.168 | 3.152 | 1.45 | 3.771 | 3.728 | 1.38 | 4.154 | 4.100 | 1.24 | 5.138 | 5.074 |
| | 2.45 | 1.450696 | 1.435 | 2.30 | 1.635 | 1.628 | 2.12 | 1.897 | 1.887 | 2.02 | 2.060 | 2.048 | 1.83 | 2.478 | 2.451 |
| | 3.16 | 0.982212 | 0.976 | 2.98 | 1.089 | 1.086 | 2.76 | 1.248 | 1.252 | 2.64 | 1.352 | 1.337 | 2.39 | 1.578 | 1.575 |
| 15 | δ / H | 30.65 | 30.03 | δ / H | 30.49 | 30.03 | δ / H | 30.41 | 30.03 | δ / H | 30.38 | 30.03 | δ / H | 30.35 | 30.03 |
| | 0.00 | 202.701 | 172.504 | 0.00 | 201.100 | 178.522 | 0.00 | 202.246 | 182.345 | 0.00 | 200.416 | 182.840 | 0.00 | 200.128 | 184.179 |
| | 0.57 | 37.468 | 34.277 | 0.44 | 60.442 | 55.375 | 0.48 | 50.238 | 47.159 | 0.43 | 60.927 | 56.984 | 0.49 | 47.477 | 45.037 |
| | 0.98 | 10.076 | 9.571 | 0.87 | 13.399 | 12.922 | 0.91 | 11.839 | 11.490 | 0.82 | 15.070 | 14.541 | 0.71 | 21.470 | 20.590 |
| | 1.37 | 4.734 | 4.620 | 1.11 | 7.578 | 7.225 | 1.58 | 3.545 | 3.533 | 1.52 | 3.853 | 3.829 | 1.39 | 4.615 | 4.548 |
| | 1.78 | 2.772 | 2.723 | 2.48 | 1.563 | 1.560 | 2.32 | 1.763 | 1.768 | 2.24 | 1.897 | 1.899 | 2.05 | 2.222 | 2.200 |
| | 2.60 | 1.416 | 1.401 | 3.23 | 1.035 | 1.033 | 3.04 | 1.165 | 1.161 | 2.93 | 1.245 | 1.236 | 2.70 | 1.431 | 1.428 |

### 3.9 Examples

We illustrate how to apply the proposed MPEWMA control chart to a situation where we monitor four different types of GaN-epitaxial layer defects (particles, micropits, microcracks, crescents) that may occur after polishing the sapphire substrates in the light emitting diode (LED) manufacturing process. The count for each defect type follows a Poisson distribution with a mean of 3. A total of three-hundred observations are collected at inspection points over two months. It is appropriate to apply the MPEWMA scheme since the numbers of defects tends to follow a multivariate Poisson rather than the normality assumption. The values of all theta parameters ($\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\theta$) are needed to be determined before using the MPEWMA control chart. Two ways of obtaining these parameter values are: using the true mean value (if they are known) and the estimated value of the means (if they are unknown).

#### 3.9.1 True Parameter Value

Suppose we know that the true value of $\theta$ is 1. The sample mean and the variance-covariance matrix of the four-variate Poisson data are given by

$$\overline{\mathbf{X}}^* = \begin{bmatrix} \theta_1+\theta \\ \theta_2+\theta \\ \theta_3+\theta \\ \theta_4+\theta \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{S}^* = \begin{bmatrix} \theta_1+\theta & \theta & \theta & \theta \\ \theta & \theta_2+\theta & \theta & \theta \\ \theta & \theta & \theta_3+\theta & \theta \\ \theta & \theta & \theta & \theta_4+\theta \end{bmatrix} = \begin{bmatrix} 3 & 1.0 & 1.0 & 1.0 \\ 1.0 & 3 & 1.0 & 1.0 \\ 1.0 & 1.0 & 3 & 1.0 \\ 1.0 & 1.0 & 1.0 & 3 \end{bmatrix}$$

Thus, all $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ are equal to 2. The MPEWMA chart is constructed using a smoothing weight of 0.05 ($\lambda$ = 0.05). To demonstrate the $T_t^2$ computations, consider the first period, $\mathbf{X}_1 = \begin{bmatrix} 6 \\ 1 \\ 3 \\ 5 \end{bmatrix}$, $\lambda = 0.05$, and $\mathbf{Z}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

The calculation of $\mathbf{Z}_1$ is $\mathbf{Z}_1 = \lambda(\mathbf{X}_1 - \overline{\mathbf{X}}^*) + (1-\lambda)\mathbf{Z}_0 = \begin{bmatrix} 0.15 \\ -0.10 \\ 0 \\ 0.10 \end{bmatrix}$. Since

$\mathbf{Z}_1 = \begin{bmatrix} 0.15 \\ -0.10 \\ 0 \\ 0.10 \end{bmatrix}$, and $\Sigma_{\mathbf{Z}_1} = \left\{ \frac{\lambda}{2-\lambda} \right\} \Sigma = \left\{ \frac{0.05}{2-0.05} \right\} \mathbf{S}^*$, we obtain

$$T_1^2 = \mathbf{Z}_1' \Sigma_{\mathbf{Z}_1}^{-1} \mathbf{Z}_1 = 0.7556$$

Table 23 presents the sample calculations of $Z_t$ and $T_t^2$ for the first ten observations. The control limit of the MPEWMA chart can be read directly from Table 4 ($H$ = 11.49). If the MEWMA scheme is employed instead of the MPEWMA, the normal-theory limit is obtained from Table 11 ($H$ = 11.22). It is noticed that the T-square statistics of the MPEWMA and MEWMA control charts are identical, but the control limit of the proposed MPEWMA chart ($H$ = 11.49 for Poisson limit) is slightly wider than the traditional MEWMA chart ($H$ = 11.22 for Normal limit).

**Table 23** Example of calculations the T-square statistics of the MPEWMA chart for the first 10 observations.

| | | | | | MPEWMA (λ = 0.05) | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Obs | x1 | x2 | x3 | x4 | $Z_t$ | | | | $T^2_t$ |
| 1 | 6 | 1 | 3 | 5 | 0.1500 | -0.1000 | 0.0000 | 0.1000 | 0.7556 |
| 2 | 7 | 7 | 6 | 4 | 0.3425 | 0.1050 | 0.1500 | 0.1450 | 1.5595 |
| 3 | 1 | 3 | 1 | 3 | 0.2254 | 0.0998 | 0.0425 | 0.1378 | 0.7597 |
| 4 | 3 | 3 | 4 | 5 | 0.2141 | 0.0948 | 0.0904 | 0.2309 | 0.9772 |
| 5 | 4 | 1 | 1 | 2 | 0.2534 | -0.0100 | -0.0141 | 0.1693 | 1.3007 |
| 6 | 4 | 5 | 5 | 7 | 0.2907 | 0.0905 | 0.0866 | 0.3609 | 2.2616 |
| 7 | 2 | 2 | 1 | 0 | 0.2262 | 0.0360 | -0.0178 | 0.1928 | 1.1327 |
| 8 | 3 | 3 | 2 | 4 | 0.2149 | 0.0342 | -0.0669 | 0.2332 | 1.5099 |
| 9 | 4 | 2 | 1 | 4 | 0.2149 | 0.0342 | -0.0669 | 0.2332 | 2.8385 |
| 10 | 2 | 1 | 1 | 2 | 0.1914 | -0.1166 | -0.2554 | 0.2079 | 3.0921 |

**Figure 2** Comparison of the MPEWMA and MEWMA charts on monitoring the number of defects

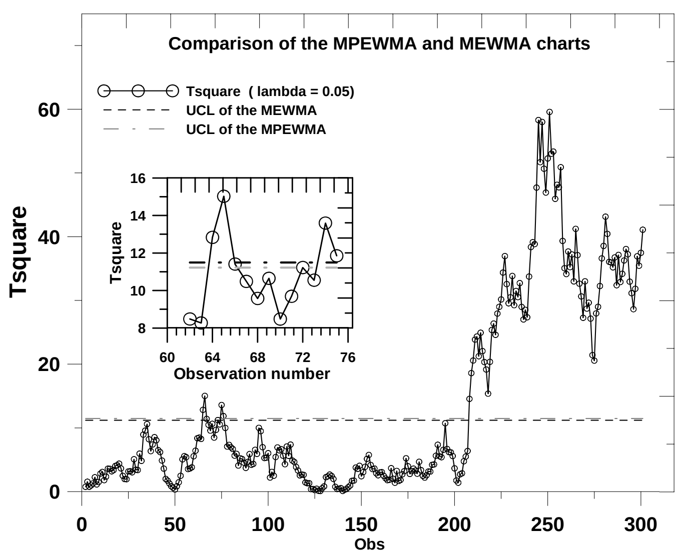
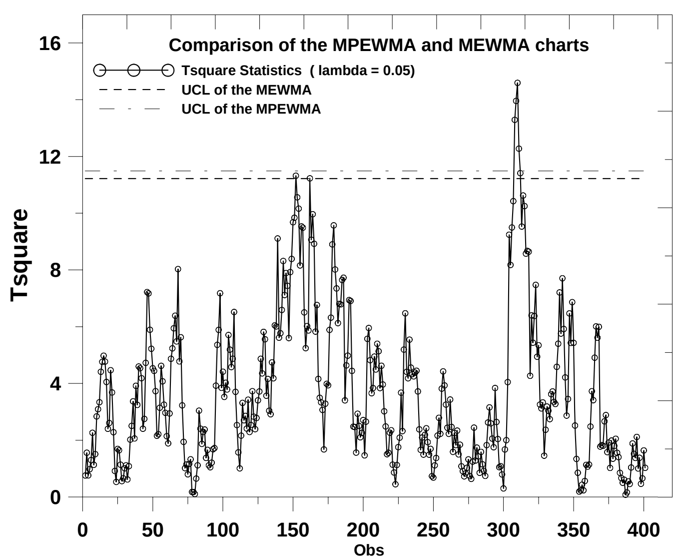



Figure 2 displays the comparison of applying the normal and Poisson control limits to the T-square statistics when the data truly comes from the multivariate Poisson distribution. These two limits perform similarly when the process shifts to an out of control state as the first out-of-control signal occurs at the same period 207. This may result from the similar out-of-control ARL performance. However, it can be seen that the normal-theory limit gives 5 false alarms whereas only 3 false alarms occur under the Poisson limit. These false alarms arise between period 62 and period 74 as shown in Figure 2 inset. The MPEWMA scheme reduces the number of false alarms that indicate the chance of misinterpreting the in-control process to be out-of-control status due to the better in-control ARL performance.

**Figure 3** Comparison of the MPEWMA and MEWMA charts based on the in-control condition




We provide another example to amplify the importance of a larger in-control ARL value. Let's continue with the previous example by considering the new scenario of an in-control process over a long period (a total of 400 observations). Figure 3 shows a comparison between the MPEWMA and MEWMA charts on monitoring an in-control process. The plot shows that no alarm is given in the first two hundred periods by using the Poisson limit on the MPEWMA scheme whereas the MEWMA relied on the normal limit signals 2 false alarms at period 151 and 161, respectively. After that both schemes simultaneously detect the out-of-control signals at period 307 to 310. The result demonstrates a difference in false alarm rate as the wider control limit produces fewer false alarms. Thus, the MEWMA chart tends to have more false alarms than the MPEWMA, resulting in more stops in production pace to investigate and fix a problem when one does not occur.

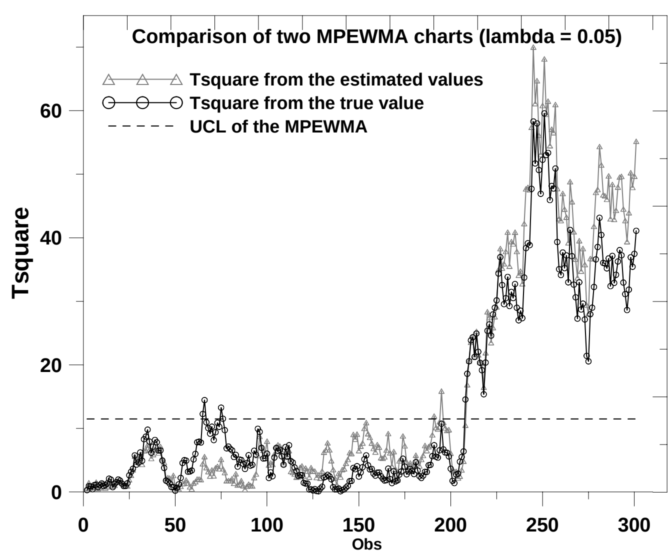
### 3.9.2 Parameter Estimation

We also look at the scenario where the estimated values of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\theta$ are used in place of the true values. If the values of all parameters are unknown, we would estimate these values from historical data if available. (See Section 3.2.1 for details on various methods of estimating these parameters.). The purpose of using estimates is that in practice, the mean values will not necessarily be known, but historical data from an in-control process may be available. Going back to the previous example, all the theta parameter values ($\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\theta$) are obtained by applying the composite likelihood concept (Jost *et al*. (2006)) to the historical data set with 100 observations. The estimation of the theta parameters are $\theta_1 = 2.3791$, $\theta_2 = 2.0652$, $\theta_3 = 2.1085$, $\theta_4 = 2.4543$, and $\theta = 0.7607$. Therefore, the sample means and variance-covariance matrix based on the estimated theta parameters are

$$\overline{\mathbf{X}}^* = \begin{bmatrix} 2.3791 + 0.7607 \\ 2.0652 + 0.7607 \\ 2.1085 + 0.7607 \\ 2.4543 + 0.7607 \end{bmatrix} = \begin{bmatrix} 3.1398 \\ 2.8259 \\ 2.8692 \\ 3.2150 \end{bmatrix} \quad \text{and} \quad \mathbf{S}^* = \begin{bmatrix} 3.1398 & 0.7607 & 0.7607 & 0.7607 \\ 0.7607 & 2.8259 & 0.7607 & 0.7607 \\ 0.7607 & 0.7607 & 2.8692 & 0.7607 \\ 0.7607 & 0.7607 & 0.7607 & 3.2150 \end{bmatrix}$$

It can be seen that the mean values of all four variables are roughly three and the thetafix parameter is about 1. Thus, the same control limit ($H$ = 11.49) is chosen for the smoothing weights of 0.05. Again, we calculate the T-square statistics by following the steps described above. Two T-square statistics based on the true and estimated value of all theta parameters are plotted in Figure 4. These two MPEWMA schemes show the same pattern, but they have the magnitude differences of the T-square statistics. The effect of the estimation of the theta parameters ($\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\theta$) could make substantial differences in magnitude for both directions toward increasing or decreasing the T-square statistics.

**Figure 4** Comparison of the two MPEWMA charts using the true and estimated mean and variance-covariance matrix.



Chapter 4

# ONE-SIDED MEWMA CONTROL CHART

## 4.1 Introduction

The multivariate exponentially weighted moving average (MEWMA) control chart is frequently used to monitor both decreasing and increasing mean shifts in several processes concurrently. The MEWMA scheme is commonly employed in industry and manufacturing, but is finding increased popularity in monitoring public health and bioterrorism surveillance data. Regardless of application, the MEWMA control chart is most often constructed assuming that the underlying distribution of the data is multivariate normal. In other words, the common assumption of the central limit theorem or the normal approximation will apply when the true underlying distribution of the data is not normal (or multivariate normal) and in some cases not even continuous. Examples would include monitoring the increase in the rate of occurrences such as the number of cracks in road pavement surfaces, number of misprints and errors found on manuscript pages, or number of failures observed during testing processes. In each of these situations, the data collected most commonly follows a Poisson or multivariate Poisson distribution. However, the control charts applied are based on normal theory assuming the central limit theorem will apply.

There is an interest in detecting a positive shift in count since the upward trend is evidence for abnormal conditions in the manufacturing process or public health surveillance. For instance, a large number of defects observed during the inspection periods or an increasing number of daily visit to clinic and health care counts. These can be considered as a signal to stop and fix an existing problem whereas the downward direction shows a good performance (i.e. less number of defects found or the process has improved product quality). Hence, applying the one-sided MEWMA scheme is more appropriate than the two-sided because it will not signal if the mean counts decrease.

We propose a one-sided multivariate Poisson EWMA (MPEWMA) control chart to detect small and medium upward shifts in the process when the process consists of count data. For our method, we do not assume the normal approximation is appropriate and instead construct the control charts using the multivariate Poisson distribution. The average run length (ARL) and standard deviation of the run length (SDRL) are examined for both steady-state in-control and out-of-control processes. We then compare the MPEWMA with the MEWMA control schemes. In addition, we examine the performance of the MPEWMA chart when a signal is defined as two or more points in a row beyond the control limits. There are several applications where a single point beyond the control limits is not of concern, but rather a run of say 2, 3, 4, or 5 points is of concern. This is often the case in monitoring public health data.

### 4.2 One-sided MPEWMA chart

The one-sided MPEWMA chart has been established by the works of Joner *et al*. (2005) and Joner *et al*. (2008). The one-sided MEWMA statistic is

$$\mathbf{Z}_t = \max\{\lambda(\mathbf{X}_t - \mu_0) + (\mathbf{I} - \lambda)\mathbf{Z}_{t-1},\ \mathbf{0}\} \qquad (13)$$

where $\boldsymbol{Z}_0 = \mathbf{0}$, and $\lambda$ is the smoothing weight. The maximum operator is defined as a comparison of the two element-wise vectors. Thus, $\boldsymbol{Z}_t$ will be equal to or greater than 0, and the one-sided MPEWMA chart shows only a signal for an increase in the means. Suppose we are monitoring $p$ random variables simultaneously. Assuming all $p$ variables are given equal weight ($\lambda > 0$), the covariance matrix of $\mathbf{Z}_t$ is given by

$$\Sigma_{Z_t} = \left\{ \frac{\lambda\left[1-(1-\lambda)^{2t}\right]}{2-\lambda} \right\} \Sigma \tag{14}$$

where $\sum$ is assumed to be the known covariance matrix of the $p$ random variables. The asymptotic covariance matrix ($t \rightarrow \infty$) can be shown to be

$$\Sigma_{Z_{t\rightarrow\infty}} = \left\{ \frac{\lambda}{2-\lambda} \right\} \Sigma \tag{15}$$

An out-of-control signal is generated if

$$MEW_t = Z_t^{'} \textstyle\sum_{Z_t}^{-1} Z_t > H \tag{16}$$

where $H$ is the control limit chosen to achieve a specific in-control ARL. The asymptotic covariance matrix is again used to calculate the statistics $MEW_t$ given in Equation (16). To monitor the multivariate Poisson data, the one-sided MPEWMA statistics are computed using the above Equations (13) – (16). The covariance matrix of the one-sided MPEWMA chart is obtained from the asymptotic covariance matrix given in Equation (15). The performance of the MEWMA control chart depends on several parameters including the number of variables ($p$), the variable mean values, and the smoothing weight ($\lambda$). Similar to the two-sided MPEWMA chart, the mean and the thetafix values are additionally considered in establishing the control limits of the one-sided MPEWMA when the multivariate Poisson data are simulated following the method of the two-sided case. We obtain appropriate control limits that will result in the desired in-control ARL for various practical combinations of these parameters.

### 4.3 Simulation Conditions

Monte-Carlo simulation is used to generate the multivariate Poisson distribution as the sum of two independent Poisson random variables. The Poisson data is produced under the same conditions applied to the two-sided version. The threshold or control limit (*H*) was selected to achieve a desired in-control average run length (ARL) of 200. To evaluate the statistical performance of the proposed one-sided scheme, we consider both average run length (ARL) and standard deviation of run length (SDRL) when testing various combinations of parameters discussed previously for various shift sizes. Several shift sizes are added to the mean of one or more variables simultaneously by one, two, and three units. Joner *et al*. (2008) noted that the shift size should be calculated in terms of percentage change, not the units of the standard deviation as previously proposed by Lowry *et al*. (1992). Since a Poisson distribution has the property that the variance is equal to its mean, it may not be easy to interpret the shift size in situations where the out-of-control condition is due to a standard deviation increase. The percentage of change developed based on Lucas (1985) is given by

$$\%\, of\ shift = \frac{u}{\mu_a} \times 100 \tag{17}$$

where $u$ is the unit of the shift size, and $\mu_a$ is the mean of the data before the shift has occurred. To illustrate, suppose we simultaneously monitor four Poisson process means when all means are equal to 3. In addition, we are interested in a two unit shift in the second process variable, and there are no shifts in the other variables (variables 1, 3, and 4). The shift size percentage in the second process using Equation (17) is $\frac{2}{3} \times 100 = 67\%$ whereas the percentages of shift size in the other processes (variable 1, 3, or 4) are zero. Thus, it can be seen that a two unit shift in any of these four variables (variable 1-4) would result in the same shift size (67%). Consider a unit of two shifts of one unit in the first and second variables that also represents a two-unit shift in the process means. The percentages of shift size in the first and second process means using equation (17) are $\frac{1}{3} \times 100$ = 33%. It shows that the percentage of two-unit shift size calculated from a two-unit shift in one variable is not equivalent to one-unit shift in each of two variables. Therefore, the same unit shift in the mean will not always have the same values for the percentage of the shift.

### 4.4 Results

We investigate the one-sided MPEWMA control chart under the "steady-state" condition. That is, we assume that the control chart operates under normal conditions for two-hundred time periods before a shift occurrs at period 201. The out-of-control average run length (ARL) is calculated as the average number of samples taken before detecting an upward shift when the process actually goes out of control. It helps to determine how quickly the proposed chart detects this upward shift. Summaries of the ARL performances for the control limits chosen to achieve the steady-state ARL of 200 are shown in Table 24 (for $\theta$ = 0.5) and

Table 25 (for $\theta = 1$). Examining these tables, it can be seen that the out-of-control ARL drops significantly when a large shift is added to any single mean. We illustrate this reduction using the previous example of monitoring four process means. From Table 20 with $\lambda = 0.05$, and $\theta = 0.5$, the ARL value decreases from 28.574 (33% shift in one variable) to 12.277 (67% shift in one variable). In addition, the out-of-control ARL value in Table 20 is reduced from 28.574 (33% shift in any one variable shifted) to 16.907 (33% shift in any two variables shifted), and to 10.263 (33% shift in all four variables shifted).

Since the means of all variables are equal, we found that a number of units shifted in the mean and not the variable shifted is directly related to the chart's ability in detecting the upward shift. Again, we use the previous example of monitoring four variables to demonstrate this relationship. Suppose now we are interested in a two-unit shift in the process mean. A shift of two units can be denoted by either two units in a single mean (variable 1 [2, 0, 0, 0], variable 2 [0, 2, 0, 0], variable 3 [0, 0, 2, 0], and variable 4 [0, 0, 0, 2]) or one unit in any two means (variable 1 and 2 [1, 1, 0, 0], variable 1 and 3 [1, 0, 1, 0], and variable 1 and 4 [1, 0, 0, 1]). The out-of-control ARL of 12.277 for two units in a single mean shift are approximately the same among these four variables ([2, 0, 0, 0], [0, 2, 0, 0], [0, 0, 2, 0], and [0, 0, 0, 2]). However, the out-of-control ARL of 16.907 for a unit shift in two means [1, 1, 0, 0] is similar to those of [1, 0, 1, 0] or [1, 0, 0, 1].

In addition to the ARL performance of the one-sided MPEWMA chart, we examine the standard deviation of the run length (SDRL) for all scenarios. The

SDRL is calculated by pooling the standard deviations among the same shift size. Table 26 and 27 display the summarized SDRL values of the one-sided MPEWMA scheme for $\theta = 0.5$ and $\theta = 1$, respectively. The SDRLs have the same behavior as the ARLs, but are substantially lower than the ARL.

### 4.5 The one-sided MPEWMA and MEWMA Chart Comparisons

We next compare the proposed one-sided MPEWMA with the one-sided MEWMA chart introduced by Joner *et al*. (2008). The one-sided MEWMA scheme has been developed by using normal approximations to Poisson distributions. The means of each count data are large enough (let's say 10 or larger) to appropriately assume the normal approximation. Thus, the data are simulated from a multivariate normal distribution and the in-control ARL values are calculated based upon 10,000 replicates. The control limits are chosen to provide a specific in-control ARL of 100 with certain correlation value ($\rho = 0.2$, 0.5, and 0.7). The summary of the one-sided MEWMA chart's performance is presented in Table 28.

In this study, we investigate three different scenarios of a 20% shift in 10 variables with all means $\mu = 10$: 1) a shift in variable 1 only; 2) a shift in variables 1, 2, and 4; and, 3) a shift in variables 1, 6, and 10. The control limits of each shift case (1) $H = 12.325$; 2) $H = 14.695$; and 3) $H = 14.430$ for different selected values of $\lambda$) are presented on the left side of Table 28. For the purpose of comparison, the control limits of the proposed one-sided MPEWMA chart are developed based on the same correlation value ($\rho = 0.5$). The multivariate Poisson correlation structure is given by

**Table 24** Summary of the ARL for the one-sided MPEWMA control chart when thetafix = 0.5

| | | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| p | Actual Region Shifted | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 |
| 4 | | %shift / H | 10.29 | 12.11 | %shift / H | 10.44 | 12.09 | %shift / H | 10.56 | 12.07 | %shift / H | 10.59 | 12.07 | %shift / H | 10.66 | 12.04 |
| | None | 0 | 200.132 | 199.393 | 0 | 199.979 | 200.044 | 0 | 199.210 | 199.244 | 0 | 198.993 | 199.965 | 0 | 200.094 | 198.770 |
| | 1 or 2 | 33 | 28.574 | 35.739 | 20 | 37.294 | 47.189 | 12.5 | 48.236 | 59.875 | 10 | 54.341 | 67.115 | 6.67 | 66.881 | 80.618 |
| | 1 and 2 | 33 | 16.907 | 19.263 | 20 | 21.500 | 25.153 | 12.5 | 27.462 | 32.873 | 10 | 30.992 | 37.347 | 6.67 | 39.054 | 47.021 |
| | All | 33 | 10.263 | 10.713 | 20 | 12.327 | 13.204 | 12.5 | 15.169 | 16.563 | 10 | 16.968 | 18.950 | 6.67 | 21.234 | 24.010 |
| | 1 or 2 | 67 | 12.277 | 12.799 | 40 | 15.152 | 16.447 | 25 | 19.140 | 21.559 | 20 | 21.614 | 24.935 | 13.33 | 27.415 | 32.552 |
| | 1 and 2 | 67 | 7.551 | 7.246 | 40 | 8.978 | 8.903 | 25 | 11.103 | 11.366 | 20 | 12.340 | 12.969 | 13.33 | 15.374 | 16.641 |
| | All | 67 | 4.748 | 4.299 | 40 | 5.427 | 5.004 | 25 | 6.436 | 6.053 | 20 | 7.045 | 6.797 | 13.33 | 8.580 | 8.454 |
| | All | 100 | 3.162 | 2.685 | 60 | 3.512 | 3.052 | 37.5 | 4.058 | 3.547 | 30 | 4.410 | 3.901 | 20 | 5.212 | 4.738 |
| | All | 133 | 2.422 | 1.981 | 80 | 2.620 | 2.171 | 50 | 2.960 | 2.488 | 40 | 3.177 | 2.721 | 26.67 | 3.710 | 3.231 |
| 6 | | %shift / H | 12.64 | 14.70 | %shift / H | 12.95 | 14.76 | %shift / H | 13.21 | 14.87 | %shift / H | 13.31 | 14.93 | %shift / H | 13.50 | 14.98 |
| | None | 0 | 199.654 | 199.052 | 0 | 199.631 | 199.630 | 0 | 199.883 | 199.726 | 0 | 199.889 | 199.750 | 0 | 199.793 | 199.770 |
| | 1 or 2 | 33 | 32.891 | 42.462 | 20 | 43.440 | 55.798 | 12.5 | 55.911 | 70.976 | 10 | 62.441 | 78.533 | 6.67 | 77.254 | 92.574 |
| | 1 and 2 | 33 | 19.471 | 23.056 | 20 | 25.154 | 30.357 | 12.5 | 32.325 | 39.652 | 10 | 36.451 | 45.102 | 6.67 | 45.730 | 56.105 |
| | All | 33 | 9.159 | 9.273 | 20 | 10.553 | 10.951 | 12.5 | 12.701 | 13.499 | 10 | 13.954 | 15.099 | 6.67 | 17.175 | 18.920 |
| | 1 or 2 | 67 | 13.843 | 14.788 | 40 | 17.234 | 19.148 | 25 | 21.879 | 25.369 | 20 | 24.697 | 29.445 | 13.33 | 31.486 | 38.432 |
| | 1 and 2 | 67 | 8.547 | 8.288 | 40 | 10.314 | 10.363 | 25 | 12.741 | 13.332 | 20 | 14.165 | 15.259 | 13.33 | 17.774 | 19.753 |
| | All | 67 | 4.250 | 3.746 | 40 | 4.717 | 4.254 | 25 | 5.434 | 4.958 | 20 | 5.875 | 5.494 | 13.33 | 7.036 | 6.725 |
| | All | 100 | 2.825 | 2.342 | 60 | 3.060 | 2.571 | 37.5 | 3.434 | 2.936 | 30 | 3.668 | 3.195 | 20 | 4.309 | 3.807 |
| 10 | | %shift / H | 16.54 | 19.04 | %shift / H | 17.04 | 19.22 | %shift / H | 17.59 | 19.51 | %shift / H | 17.84 | 19.66 | %shift / H | 18.25 | 19.93 |
| | None | 0 | 200.061 | 199.575 | 0 | 199.373 | 199.910 | 0 | 199.826 | 199.163 | 0 | 199.004 | 199.536 | 0 | 200.185 | 199.704 |
| | 1 or 2 | 33 | 39.484 | 53.100 | 20 | 52.459 | 70.076 | 12.5 | 67.814 | 86.424 | 10 | 75.015 | 94.463 | 6.67 | 89.894 | 108.846 |
| | 1 and 2 | 33 | 23.383 | 29.029 | 20 | 30.646 | 39.289 | 12.5 | 39.780 | 51.051 | 10 | 45.049 | 57.308 | 6.67 | 56.124 | 69.982 |
| | All | 33 | 8.255 | 8.132 | 20 | 9.031 | 9.112 | 12.5 | 10.365 | 10.717 | 10 | 11.260 | 11.780 | 6.67 | 13.423 | 14.459 |
| | 1 or 2 | 67 | 16.110 | 17.866 | 40 | 20.273 | 23.696 | 25 | 25.987 | 31.713 | 20 | 29.565 | 36.633 | 13.33 | 37.660 | 48.016 |
| | 1 and 2 | 67 | 9.927 | 9.868 | 40 | 12.227 | 12.723 | 25 | 15.135 | 16.543 | 20 | 17.077 | 19.018 | 13.33 | 21.299 | 24.949 |
| | All | 67 | 3.587 | 3.332 | 40 | 4.058 | 3.559 | 25 | 4.517 | 4.008 | 20 | 4.819 | 4.337 | 13.33 | 5.604 | 5.137 |
| | All | 100 | 2.556 | 2.069 | 60 | 2.648 | 2.165 | 37.5 | 2.867 | 2.373 | 30 | 3.027 | 2.546 | 20 | 3.451 | 2.945 |
| 15 | | %shift / H | 20.77 | 23.74 | %shift / H | 21.39 | 23.94 | %shift / H | 22.17 | 24.38 | %shift / H | 22.58 | 24.67 | %shift / H | 23.33 | 25.15 |
| | None | 0 | 200.176 | 199.289 | 0 | 199.987 | 199.835 | 0 | 199.189 | 199.020 | 0 | 199.574 | 199.640 | 0 | 200.187 | 199.189 |
| | 1 or 2 | 33 | 45.706 | 63.655 | 20 | 61.135 | 82.896 | 12.5 | 78.218 | 100.325 | 10 | 86.847 | 108.851 | 6.67 | 102.719 | 122.000 |
| | 1 and 2 | 33 | 27.073 | 35.126 | 20 | 36.240 | 48.275 | 12.5 | 47.378 | 62.236 | 10 | 53.446 | 69.523 | 6.67 | 66.092 | .82.68557 |
| | All | 33 | 7.841 | 7.520 | 20 | 8.274 | 8.046 | 12.5 | 9.143 | 9.143 | 10 | 9.717 | 9.864 | 6.67 | 11.290 | 11.671 |
| | 1 or 2 | 67 | 18.236 | 21.037 | 40 | 23.271 | 28.372 | 25 | 30.095 | 38.220 | 20 | 34.245 | 44.364 | 13.33 | 44.070 | 57.366 |
| | 1 and 2 | 67 | 11.211 | 11.443 | 40 | 13.906 | 14.973 | 25 | 17.580 | 19.881 | 20 | 19.807 | 23.008 | 13.33 | 24.999 | 30.269 |
| | All | 67 | 3.728 | 3.141 | 40 | 3.770 | 3.233 | 25 | 4.019 | 3.496 | 20 | 4.222 | 3.711 | 13.33 | 4.775 | 4.225 |
| | All | 100 | 2.464 | 1.957 | 60 | 2.459 | 1.964 | 37.5 | 2.565 | 2.080 | 30 | 2.668 | 2.181 | 20 | 2.951 | 2.446 |

**Table 25 Summary of the ARL for the one-sided MPEWMA control chart when thetafix = 1**

| | | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| p | Actual Region Shifted | % | λ = 0.05 | λ = 0.1 | % | λ = 0.05 | λ = 0.1 | % | λ = 0.05 | λ = 0.1 | % | λ = 0.05 | λ = 0.1 | % | λ = 0.05 | λ = 0.1 |
| 4 | | %shift / H | 9.80 | 11.68 | %shift / H | 9.98 | 11.68 | %shift / H | 10.18 | 11.73 | %shift / H | 10.27 | 11.76 | %shift / H | 10.40 | 11.83 |
| | None | 0 | 200.163 | 199.483 | 0 | 200.196 | 199.975 | 0 | 199.399 | 199.056 | 0 | 199.435 | 199.644 | 0 | 200.147 | 199.948 |
| | 1 or 2 | 33 | 28.114 | 35.087 | 20 | 37.854 | 47.618 | 12.5 | 49.152 | 61.373 | 10 | 55.515 | 68.614 | 6.67 | 68.032 | 82.524 |
| | 1 and 2 | 33 | 17.476 | 20.014 | 20 | 22.618 | 26.707 | 12.5 | 28.783 | 34.510 | 10 | 32.353 | 39.033 | 6.67 | 40.130 | 48.786 |
| | All | 33 | 11.993 | 12.765 | 20 | 13.827 | 15.082 | 12.5 | 16.494 | 18.382 | 10 | 18.256 | 20.450 | 6.67 | 22.155 | 25.375 |
| | 1 or 2 | 67 | 12.017 | 12.500 | 40 | 15.175 | 16.531 | 25 | 19.361 | 21.916 | 20 | 21.886 | 25.339 | 13.33 | 27.692 | 33.180 |
| | 1 and 2 | 67 | 7.733 | 7.426 | 40 | 9.349 | 9.351 | 25 | 11.501 | 11.915 | 20 | 12.769 | 13.408 | 13.33 | 15.756 | 17.375 |
| | All | 67 | 5.434 | 5.014 | 40 | 5.985 | 5.624 | 25 | 6.908 | 6.629 | 20 | 7.551 | 7.304 | 13.33 | 8.973 | 8.948 |
| | All | 100 | 3.554 | 3.068 | 60 | 3.846 | 3.362 | 37.5 | 4.329 | 3.855 | 30 | 4.649 | 4.189 | 20 | 5.458 | 5.013 |
| | All | 133 | 2.664 | 2.218 | 80 | 2.843 | 2.393 | 50 | 3.163 | 2.677 | 40 | 3.362 | 2.892 | 26.67 | 3.895 | 3.410 |
| 6 | | %shift / H | 12.02 | 14.17 | %shift / H | 12.23 | 14.16 | %shift / H | 12.56 | 14.30 | %shift / H | 12.71 | 14.39 | %shift / H | 13.00 | 14.56 |
| | None | 0 | 199.222 | 199.563 | 0 | 199.904 | 199.405 | 0 | 199.747 | 200.181 | 0 | 199.941 | 199.756 | 0 | 199.383 | 199.314 |
| | 1 or 2 | 33 | 32.074 | 41.381 | 20 | 43.698 | 56.772 | 12.5 | 57.354 | 73.158 | 10 | 64.214 | 80.675 | 6.67 | 78.204 | 95.394 |
| | 1 and 2 | 33 | 19.724 | 23.288 | 20 | 26.182 | 32.160 | 12.5 | 33.882 | 42.148 | 10 | 38.134 | 47.493 | 6.67 | 47.470 | 58.624 |
| | All | 33 | 11.388 | 11.910 | 20 | 12.436 | 13.363 | 12.5 | 14.278 | 15.592 | 10 | 15.565 | 17.138 | 6.67 | 18.416 | 20.709 |
| | 1 or 2 | 67 | 13.462 | 14.288 | 40 | 17.175 | 19.174 | 25 | 22.066 | 25.945 | 20 | 25.047 | 30.005 | 13.33 | 32.041 | 39.487 |
| | 1 and 2 | 67 | 8.581 | 8.306 | 40 | 10.619 | 10.738 | 25 | 13.183 | 13.949 | 20 | 14.746 | 15.947 | 13.33 | 18.310 | 20.709 |
| | All | 67 | 5.160 | 4.667 | 40 | 5.491 | 5.000 | 25 | 6.088 | 5.701 | 20 | 6.522 | 6.128 | 13.33 | 7.551 | 7.308 |
| | All | 100 | 3.371 | 2.858 | 60 | 3.495 | 2.998 | 37.5 | 3.819 | 3.308 | 30 | 4.052 | 3.548 | 20 | 4.607 | 4.130 |
| 10 | | %shift / H | 15.87 | 18.48 | %shift / H | 16.03 | 18.34 | %shift / H | 16.47 | 18.53 | %shift / H | 16.73 | 18.69 | %shift / H | 17.25 | 19.04 |
| | None | 0 | 199.992 | 199.530 | 0 | 199.679 | 199.917 | 0 | 200.280 | 199.965 | 0 | 199.352 | 199.363 | 0 | 199.772 | 200.170 |
| | 1 or 2 | 33 | 38.145 | 50.879 | 20 | 52.721 | 70.580 | 12.5 | 68.982 | 88.991 | 10 | 77.147 | 97.368 | 6.67 | 93.048 | 112.051 |
| | 1 and 2 | 33 | 11.017 | 11.271 | 20 | 31.705 | 40.840 | 12.5 | 41.741 | 54.006 | 10 | 47.263 | 60.880 | 6.67 | 58.430 | 73.642 |
| | All | 33 | 23.098 | 28.319 | 20 | 11.397 | 11.804 | 12.5 | 12.428 | 13.183 | 10 | 13.273 | 14.128 | 6.67 | 15.192 | 16.514 |
| | 1 or 2 | 67 | 15.551 | 17.103 | 40 | 20.139 | 23.543 | 25 | 26.225 | 32.322 | 20 | 29.944 | 37.740 | 13.33 | 38.509 | 49.457 |
| | 1 and 2 | 67 | 9.833 | 9.673 | 40 | 12.377 | 12.885 | 25 | 15.677 | 17.162 | 20 | 17.629 | 19.943 | 13.33 | 22.180 | 26.199 |
| | All | 67 | 5.098 | 4.470 | 40 | 5.101 | 4.528 | 25 | 5.382 | 4.869 | 20 | 5.629 | 5.147 | 13.33 | 6.271 | 5.843 |
| | All | 100 | 3.321 | 2.745 | 60 | 3.279 | 2.735 | 37.5 | 3.399 | 2.875 | 30 | 3.524 | 3.000 | 20 | 3.863 | 3.348 |
| 15 | | %shift / H | 20.30 | 23.41 | %shift / H | 20.22 | 22.96 | %shift / H | 20.69 | 23.10 | %shift / H | 21.00 | 23.29 | %shift / H | 21.74 | 23.74 |
| | None | 0 | 199.107 | 199.935 | 0 | 199.582 | 199.955 | 0 | 200.203 | 199.338 | 0 | 199.629 | 199.450 | 0 | 199.009 | 199.874 |
| | 1 or 2 | 33 | 44.384 | 60.342 | 20 | 61.529 | 82.747 | 12.5 | 80.577 | 103.312 | 10 | 89.354 | 112.559 | 6.67 | 105.641 | 124.554 |
| | 1 and 2 | 33 | 26.676 | 33.861 | 20 | 37.187 | 49.306 | 12.5 | 49.646 | 65.901 | 10 | 56.204 | 73.554 | 6.67 | 69.393 | 86.792 |
| | All | 33 | 10.996 | 10.944 | 20 | 10.881 | 11.012 | 12.5 | 11.572 | 11.916 | 10 | 11.994 | 12.526 | 6.67 | 13.344 | 14.163 |
| | 1 or 2 | 67 | 17.725 | 20.155 | 40 | 23.026 | 28.041 | 25 | 30.386 | 39.051 | 20 | 34.805 | 45.518 | 13.33 | 44.944 | 59.080 |
| | 1 and 2 | 67 | 11.107 | 4.000 | 40 | 14.091 | 15.128 | 25 | 18.092 | 20.574 | 20 | 20.466 | 24.113 | 13.33 | 26.073 | 31.844 |
| | All | 67 | 5.226 | 11.227 | 40 | 4.997 | 4.366 | 25 | 5.087 | 4.496 | 20 | 5.197 | 4.642 | 13.33 | 5.655 | 5.102 |
| | All | 100 | 3.441 | 2.774 | 60 | 3.240 | 2.647 | 37.5 | 3.217 | 2.669 | 30 | 3.279 | 2.726 | 20 | 3.486 | 2.938 |

**Table 26 Summary of the SDRL for the one-sided MPEWMA control chart when thetafix = 0.5**

| | | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | | |
| p | Actual Region Shifted | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 |
| 4 | | %shift / H | 10.29 | 12.11 | %shift / H | 10.42 | 12.09 | %shift / H | 10.54 | 12.07 | %shift / H | 10.59 | 12.07 | %shift / H | 10.66 | 12.04 |
| | None | 0 | 201.017 | 199.308 | 0 | 199.984 | 199.010 | 0 | 199.952 | 199.333 | 0 | 198.858 | 200.251 | 0 | 201.911 | 200.091 |
| | 1 or 2 | 33 | 20.642 | 30.527 | 20 | 29.552 | 42.781 | 12.5 | 40.952 | 56.111 | 10 | 47.670 | 63.695 | 6.67 | 60.784 | 78.320 |
| | 1 and 2 | 33 | 10.789 | 14.802 | 20 | 15.203 | 20.925 | 12.5 | 27.462 | 28.889 | 10 | 24.703 | 33.243 | 6.67 | 32.813 | 43.502 |
| | All | 33 | 5.944 | 7.301 | 20 | 7.803 | 9.760 | 12.5 | 15.169 | 13.161 | 10 | 12.108 | 15.559 | 6.67 | 16.171 | 20.702 |
| | 1 or 2 | 67 | 6.561 | 8.185 | 40 | 8.909 | 11.634 | 25 | 19.140 | 16.771 | 20 | 14.685 | 19.987 | 13.33 | 20.308 | 27.854 |
| | 1 and 2 | 67 | 3.621 | 4.000 | 40 | 4.733 | 5.450 | 25 | 11.103 | 7.707 | 20 | 7.358 | 9.128 | 13.33 | 9.941 | 12.667 |
| | All | 67 | 2.189 | 2.195 | 40 | 2.672 | 2.784 | 25 | 6.436 | 3.668 | 20 | 3.902 | 4.276 | 13.33 | 5.044 | 5.717 |
| | All | 100 | 1.329 | 1.242 | 60 | 1.591 | 1.510 | 37.5 | 4.058 | 1.880 | 30 | 2.188 | 2.132 | 20 | 2.731 | 2.755 |
| | All | 133 | 0.966 | 0.875 | 80 | 1.130 | 1.026 | 50 | 2.960 | 1.251 | 40 | 1.509 | 1.404 | 26.67 | 1.847 | 1.758 |
| 6 | | %shift / H | 12.64 | 14.70 | %shift / H | 12.95 | 14.76 | %shift / H | 13.21 | 14.87 | %shift / H | 13.31 | 14.93 | %shift / H | 13.50 | 14.98 |
| | None | 0 | 199.059 | 200.657 | 0 | 199.991 | 199.044 | 0 | 200.956 | 200.998 | 0 | 200.115 | 198.565 | 0 | 201.259 | 199.951 |
| | 1 or 2 | 33 | 24.177 | 36.947 | 20 | 34.999 | 51.207 | 12.5 | 48.248 | 66.971 | 10 | 55.448 | 75.086 | 6.67 | 71.160 | 89.443 |
| | 1 and 2 | 33 | 12.433 | 18.035 | 20 | 17.858 | 25.568 | 12.5 | 25.135 | 35.434 | 10 | 29.301 | 41.056 | 6.67 | 39.025 | 52.345 |
| | All | 33 | 5.040 | 6.010 | 20 | 6.293 | 7.668 | 12.5 | 8.177 | 10.140 | 10 | 9.357 | 11.784 | 6.67 | 12.233 | 15.479 |
| | 1 or 2 | 67 | 7.351 | 9.591 | 40 | 10.158 | 13.781 | 25 | 14.284 | 19.971 | 20 | 16.863 | 24.115 | 13.33 | 23.494 | 33.341 |
| | 1 and 2 | 67 | 4.023 | 4.579 | 40 | 5.348 | 6.368 | 25 | 7.421 | 9.071 | 20 | 8.472 | 10.833 | 13.33 | 11.531 | 15.325 |
| | All | 67 | 1.866 | 1.830 | 40 | 2.247 | 2.238 | 25 | 2.758 | 2.809 | 20 | 3.098 | 3.242 | 13.33 | 3.915 | 4.253 |
| | All | 100 | 1.146 | 1.042 | 60 | 1.333 | 1.228 | 37.5 | 1.606 | 1.497 | 30 | 1.762 | 1.675 | 20 | 2.171 | 2.092 |
| 10 | | %shift / H | 16.57 | 19.04 | %shift / H | 17.04 | 19.22 | %shift / H | 17.59 | 19.51 | %shift / H | 17.84 | 19.66 | %shift / H | 18.25 | 19.93 |
| | None | 0 | 200.588 | 198.185 | 0 | 198.970 | 201.254 | 0 | 200.225 | 201.240 | 0 | 198.589 | 199.881 | 0 | 201.142 | 198.063 |
| | 1 or 2 | 33 | 29.815 | 47.553 | 20 | 43.250 | 65.526 | 12.5 | 59.798 | 83.118 | 10 | 67.795 | 90.760 | 6.67 | 84.038 | 106.631 |
| | 1 and 2 | 33 | 15.168 | 23.324 | 20 | 22.207 | 34.149 | 12.5 | 31.677 | 46.479 | 10 | 37.231 | 53.106 | 6.67 | 49.047 | 66.394 |
| | All | 33 | 4.261 | 4.887 | 20 | 5.043 | 5.980 | 12.5 | 6.195 | 7.506 | 10 | 7.039 | 8.549 | 6.67 | 8.870 | 11.163 |
| | 1 or 2 | 67 | 8.585 | 11.942 | 40 | 12.006 | 17.605 | 25 | 17.196 | 25.707 | 20 | 20.569 | 30.813 | 13.33 | 28.649 | 42.648 |
| | 1 and 2 | 67 | 4.632 | 5.502 | 40 | 6.275 | 7.995 | 25 | 8.576 | 11.579 | 20 | 10.213 | 13.938 | 13.33 | 13.938 | 19.914 |
| | All | 67 | 1.602 | 1.514 | 40 | 1.840 | 1.762 | 25 | 2.189 | 2.133 | 20 | 2.405 | 2.376 | 13.33 | 2.951 | 3.033 |
| | All | 100 | 0.988 | 0.876 | 60 | 1.215 | 1.000 | 37.5 | 1.295 | 1.168 | 30 | 1.416 | 1.278 | 20 | 1.678 | 1.564 |
| 15 | | %shift / H | 20.77 | 23.74 | %shift / H | 21.39 | 23.94 | %shift / H | 22.17 | 24.38 | %shift / H | 22.58 | 24.67 | %shift / H | 23.33 | 25.15 |
| | None | 0 | 200.206 | 201.687 | 0 | 200.613 | 201.856 | 0 | 198.282 | 199.909 | 0 | 200.066 | 199.655 | 0 | 199.341 | 199.404 |
| | 1 or 2 | 33 | 35.338 | 57.871 | 20 | 51.625 | 78.815 | 12.5 | 70.168 | 97.199 | 10 | 79.426 | 105.943 | 6.67 | 97.200 | 120.284 |
| | 1 and 2 | 33 | 17.875 | 29.028 | 20 | 26.937 | 42.868 | 12.5 | 38.612 | 57.857 | 10 | 45.228 | 65.548 | 6.67 | 58.759 | 79.362 |
| | All | 33 | 3.846 | 4.305 | 20 | 4.386 | 4.990 | 12.5 | 5.197 | 6.100 | 10 | 5.704 | 6.751 | 6.67 | 7.023 | 8.564 |
| | 1 or 2 | 67 | 9.855 | 14.532 | 40 | 14.061 | 21.708 | 25 | 20.344 | 31.895 | 20 | 24.331 | 38.286 | 13.33 | 34.267 | 52.081 |
| | 1 and 2 | 67 | 5.190 | 6.471 | 40 | 7.140 | 9.545 | 25 | 10.081 | 14.173 | 20 | 11.996 | 17.275 | 13.33 | 16.709 | 24.723 |
| | All | 67 | 1.479 | 1.354 | 40 | 1.639 | 1.528 | 25 | 1.876 | 1.784 | 20 | 2.028 | 1.950 | 13.33 | 2.404 | 2.371 |
| | All | 100 | 0.914 | 0.793 | 60 | 1.007 | 0.878 | 37.5 | 1.136 | 0.993 | 30 | 1.219 | 1.071 | 20 | 1.412 | 1.266 |

**Table 27** Summary of the SDRL for the one-sided MPEWMA control chart when thetafix = 1

| | | Mean | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 3 | | | 5 | | | 8 | | | 10 | | | 15 | |
| p | Actual Region Shifted | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 | | λ = 0.05 | λ = 0.1 |
| 4 | | %shift / H | 9.80 | 11.68 | %shift / H | 9.98 | 11.68 | %shift / H | 10.18 | 11.73 | %shift / H | 10.27 | 11.76 | %shift / H | 10.40 | 11.83 |
| | None | 0 | 200.167 | 199.051 | 0 | 199.131 | 201.920 | 0 | 198.409 | 199.168 | 0 | 200.182 | 200.321 | 0 | 200.190 | 199.759 |
| | 1 or 2 | 33 | 20.176 | 30.022 | 20 | 30.209 | 43.317 | 12.5 | 41.905 | 57.499 | 10 | 48.417 | 65.484 | 6.67 | 62.032 | 79.560 |
| | 1 and 2 | 33 | 11.232 | 15.367 | 20 | 16.144 | 22.324 | 12.5 | 22.282 | 30.283 | 10 | 25.948 | 35.238 | 6.67 | 33.902 | 45.356 |
| | All | 33 | 7.239 | 9.142 | 20 | 8.971 | 11.428 | 12.5 | 11.412 | 14.762 | 10 | 13.279 | 16.884 | 6.67 | 16.908 | 21.878 |
| | 1 or 2 | 67 | 6.344 | 7.931 | 40 | 8.863 | 11.705 | 25 | 12.545 | 17.051 | 20 | 14.901 | 20.413 | 13.33 | 20.458 | 28.493 |
| | 1 and 2 | 67 | 3.675 | 4.159 | 40 | 4.970 | 5.729 | 25 | 6.652 | 8.089 | 20 | 7.685 | 9.526 | 13.33 | 10.308 | 13.343 |
| | All | 67 | 2.544 | 2.637 | 40 | 3.025 | 3.191 | 25 | 3.745 | 4.065 | 20 | 4.227 | 4.663 | 13.33 | 5.350 | 6.105 |
| | All | 100 | 1.500 | 1.429 | 60 | 1.760 | 1.679 | 37.5 | 2.115 | 2.063 | 30 | 2.342 | 2.306 | 20 | 2.866 | 2.942 |
| | All | 133 | 1.078 | 0.975 | 80 | 1.240 | 1.134 | 50 | 1.462 | 1.353 | 40 | 1.596 | 1.500 | 26.67 | 1.937 | 1.857 |
| 6 | | %shift / H | 12.02 | 14.17 | %shift / H | 12.23 | 14.16 | %shift / H | 12.56 | 14.30 | %shift / H | 12.71 | 14.39 | %shift / H | 13.00 | 14.56 |
| | None | 0 | 200.393 | 199.906 | 0 | 199.915 | 199.723 | 0 | 200.391 | 202.496 | 0 | 198.936 | 200.606 | 0 | 199.895 | 200.656 |
| | 1 or 2 | 33 | 23.282 | 35.771 | 20 | 35.276 | 52.146 | 12.5 | 49.634 | 69.444 | 10 | 57.356 | 77.038 | 6.67 | 72.045 | 93.233 |
| | 1 and 2 | 33 | 12.626 | 18.136 | 20 | 18.772 | 27.326 | 12.5 | 26.517 | 37.702 | 10 | 31.080 | 43.446 | 6.67 | 40.689 | 55.182 |
| | All | 33 | 6.518 | 8.166 | 20 | 7.722 | 9.769 | 12.5 | 9.423 | 12.052 | 10 | 10.689 | 13.591 | 6.67 | 13.442 | 17.243 |
| | 1 or 2 | 67 | 7.033 | 9.157 | 40 | 10.069 | 13.785 | 25 | 14.381 | 20.488 | 20 | 17.142 | 24.621 | 13.33 | 23.921 | 34.397 |
| | 1 and 2 | 67 | 3.994 | 4.562 | 40 | 5.495 | 6.632 | 25 | 7.520 | 9.579 | 20 | 8.824 | 11.494 | 13.33 | 11.988 | 16.095 |
| | All | 67 | 2.308 | 2.332 | 40 | 2.653 | 2.691 | 25 | 3.140 | 3.318 | 20 | 3.479 | 3.696 | 13.33 | 4.247 | 4.698 |
| | All | 100 | 1.365 | 1.261 | 60 | 1.542 | 1.442 | 37.5 | 1.792 | 1.691 | 30 | 1.959 | 1.872 | 20 | 2.345 | 2.314 |
| 10 | | %shift / H | 15.87 | 18.48 | %shift / H | 16.03 | 18.34 | %shift / H | 16.47 | 18.53 | %shift / H | 16.73 | 18.69 | %shift / H | 17.25 | 19.04 |
| | None | 0 | 201.214 | 199.656 | 0 | 201.007 | 199.539 | 0 | 201.433 | 198.822 | 0 | 198.083 | 200.703 | 0 | 199.960 | 201.283 |
| | 1 or 2 | 33 | 28.468 | 45.478 | 20 | 43.603 | 65.939 | 12.5 | 61.046 | 85.496 | 10 | 69.332 | 93.708 | 6.67 | 87.358 | 109.570 |
| | 1 and 2 | 33 | 6.013 | 7.336 | 20 | 23.212 | 35.524 | 12.5 | 33.544 | 49.574 | 10 | 39.421 | 56.651 | 6.67 | 51.438 | 70.250 |
| | All | 33 | 14.950 | 21.197 | 20 | 6.700 | 8.131 | 12.5 | 7.751 | 9.682 | 10 | 8.518 | 10.711 | 6.67 | 10.354 | 13.088 |
| | 1 or 2 | 67 | 8.193 | 11.271 | 40 | 11.900 | 17.442 | 25 | 17.317 | 26.335 | 20 | 20.913 | 31.866 | 13.33 | 29.314 | 44.166 |
| | 1 and 2 | 67 | 4.548 | 5.390 | 40 | 6.346 | 8.080 | 25 | 8.911 | 11.993 | 20 | 10.605 | 14.773 | 13.33 | 14.470 | 20.945 |
| | All | 67 | 2.119 | 2.078 | 40 | 2.332 | 2.315 | 25 | 2.638 | 2.673 | 20 | 2.839 | 2.911 | 13.33 | 3.339 | 3.522 |
| | All | 100 | 1.256 | 1.138 | 60 | 1.376 | 1.235 | 37.5 | 1.538 | 1.406 | 30 | 1.645 | 1.507 | 20 | 1.882 | 1.773 |
| 15 | | %shift / H | 20.30 | 23.41 | %shift / H | 20.22 | 22.96 | %shift / H | 20.69 | 23.10 | %shift / H | 21.00 | 23.29 | %shift / H | 21.74 | 23.74 |
| | None | 0 | 198.975 | 199.843 | 0 | 201.491 | 201.107 | 0 | 201.665 | 199.258 | 0 | 199.413 | 201.215 | 0 | 199.651 | 200.306 |
| | 1 or 2 | 33 | 33.893 | 54.671 | 20 | 52.006 | 78.552 | 12.5 | 72.659 | 100.328 | 10 | 81.790 | 109.949 | 6.67 | 100.185 | 122.947 |
| | 1 and 2 | 33 | 17.692 | 27.876 | 20 | 27.805 | 43.989 | 12.5 | 40.781 | 61.365 | 10 | 47.782 | 69.572 | 6.67 | 62.088 | 83.690 |
| | All | 33 | 5.657 | 6.853 | 20 | 6.082 | 7.336 | 12.5 | 6.913 | 8.380 | 10 | 7.349 | 9.110 | 6.67 | 8.749 | 10.791 |
| | 1 or 2 | 67 | 9.520 | 13.782 | 40 | 13.876 | 21.407 | 25 | 20.470 | 32.696 | 20 | 24.798 | 39.441 | 13.33 | 35.086 | 53.870 |
| | 1 and 2 | 67 | 5.124 | 2.052 | 40 | 7.232 | 9.656 | 25 | 10.325 | 14.943 | 20 | 12.393 | 18.547 | 13.33 | 17.437 | 26.176 |
| | All | 67 | 2.046 | 6.389 | 40 | 2.184 | 2.116 | 25 | 2.403 | 2.345 | 20 | 2.532 | 2.505 | 13.33 | 2.886 | 2.918 |
| | All | 100 | 1.212 | 1.084 | 60 | 1.285 | 1.149 | 37.5 | 1.395 | 1.250 | 30 | 1.486 | 1.331 | 20 | 1.656 | 1.507 |

$$\rho_{ij} = \frac{\theta}{\sqrt{(\theta+\theta_i)(\theta+\theta_j)}}, \; i \neq j \qquad (18)$$

where $\theta$, $\theta_i$, and $\theta_j$ are the Poisson means of $Y$, $Y_i$, and $Y_j$, respectively. Since we know that the process means of the first two variables ($Y_1$ and $Y_2$) are 10 ($\theta + \theta_1$ and $\theta + \theta_2$), then the value of $\theta$ is determined to be 5 (obtained using Equation (18)) and, consequently, both $\theta_1$ and $\theta_2$ are equal to 5.

**Table 28** Summary of the one-sided MEWMA chart's performance proposed by Joner *et al*. (2008) for 10 variables with $\rho = 0.5$.

| Variable Shifted | % shift | $\mu_0$ | λ | One-sided MEWMA chart<br>Control limit (H) | out-of-control ARL | $SE_{ARL}$ |
|---|---|---|---|---|---|---|
| 1 | 20 | 10 | 0.05 | 12.325 | 15.01 | 0.031 |
| 1 | 20 | 50 | 0.19 | 15.960 | 5.10 | 0.009 |
| 1 | 10 | 100 | 0.11 | 14.695 | 8.34 | 0.015 |
| 1, 2, and 4 | 20 | 10 | 0.11 | 14.695 | 9.09 | 0.019 |
| 1, 2, and 4 | 20 | 50 | 0.37 | 16.970 | 3.00 | 0.005 |
| 1, 2, and 4 | 10 | 100 | 0.22 | 16.255 | 4.97 | 0.009 |
| 1, 6, and 10 | 20 | 10 | 0.10 | 14.430 | 7.08 | 0.013 |
| 1, 6, and 10 | 20 | 50 | 0.44 | 17.120 | 2.28 | 0.004 |
| 1, 6, and 10 | 10 | 100 | 0.26 | 16.510 | 3.80 | 0.007 |
| All | 10 | 100 | 0.34 | 16.890 | 3.15 | 0.006 |

**Table 29** Comparison of the performance between the one-sided MEWMA and MPEWMA charts for in-control ARL of 100 when θ = 5.

| | | | | Normal theory limits with u = 10 | | | Poisson limits with u = 10 | | |
|---|---|---|---|---|---|---|---|---|---|
| p | Variable Shifted | % shift | λ | Control limit (H) | out-of-control ARL | $SE_{ARL}$ | Control limit (H) | out-of-control ARL | $SE_{ARL}$ |
| 10 | 1 | 20 | 0.05 | 12.325 | 15.010 | 0.031 | 12.68 | 20.144 | 0.063 |
| | 1, 2, and 4 | 20 | 0.11 | 14.695 | 9.090 | 0.019 | 15.420 | 10.903 | 0.035 |
| | 1, 6, and 10 | 20 | 0.10 | 14.430 | 7.080 | 0.013 | 15.140 | 10.737 | 0.034 |

Table 29 shows a comparison of two one-sided schemes for an in-control ARL of 100. The Poisson-limits obtained for the chosen three cases are 12.68, 15.420, and 15.14 respectively. The out-of-control ARLs are 20.144, 10.903, and 10.737. The results indicate that the control limits calculated from the normal approximation are narrower than the Poisson distribution themselves. The differences in detecting the same shift in the mean vectors of both one-sided control charts are quite small.

We investigate another issue of applying the normal-theory limits to the multivariate Poisson distribution. Let us use the above conditions as an example. Suppose we ignore the Poisson assumption and use the normal approximation for the Poisson distribution. In the other word, the control limits from the normal-theory ($H$ = 12.325, 14.695, and 14.430) are applied to the data generated from the multivariate Poisson distribution. The important result here is that the one-sided MEWMA charts of all three scenarios have in-control ARLs much lower than the stated level of 100 as shown in Table 30. The in-control ARLs are sufficiently dropped to 91.392, 81.773, and 83.565, respectively. In order to achieve the in-control ARL of 100, the Poisson limits ($H$ = 12.68, 15.42, and 15.14) should be applied instead of those normal-theory limits. Moreover, the out-of-control ARLs are 19.428, 10.208, and 10.114 and they are quite similar to the expected values (20.144, 10.903, and 10.737) on the right hand side of Table 29. In practice, one can also expect an earlier false alarm when the normal approximation is applied to multivariate Poisson data.

**Table 30** The performance of the one-sided MEWMA applied to the multivariate Poisson distribution with θ = 5 (The advertised in-control ARL of 100).

| | | | | Normal theory limits with u = 10 | | | |
|---|---|---|---|---|---|---|---|
| p | Variable Shifted | % shift | λ | Control limit (H) | In-control ARL | out-of-controlARL | $SE_{ARL}$ |
| 10 | 1 | 20 | 0.05 | 12.325 | 91.392 | 19.428 | 0.061 |
| | 1, 2, and 4 | 20 | 0.11 | 14.695 | 81.773 | 10.208 | 0.033 |
| | 1, 6, and 10 | 20 | 0.10 | 14.430 | 83.565 | 10.114 | 0.045 |

### 4.6 Individual and a Row of Out-of-Control Signal

Commonly, an out-of-control signal in the one-sided MPEWMA control chart occurs if a single point is out of control. However, there are some situations where we are interested in consecutive points plotting beyond the control limit for a signal. For example, suppose we are monitoring the incidence rates of asthma from several locations over 120 weeks. Suppose one out-of-control signal is given at period 100 and no other signals are detected. It is quite difficult to conclude that the asthma rate has increased and that there is evidence for spread of asthma disease in those areas since only one signal has occurred. Thus, it would be better to wait for several out-of-control signals in a row rather than an individual out-of-control signal. In this section, we study the detection performance on four test cases in which data are generated by the multivariate Poisson model. The details of each case are described below.

Case 1: Four variables each with $\mu = 3$ and $\theta = 1$.

Case 2: Four variables each with $\mu = 3$ and $\theta = 0.5$.

Case 3: Six variables each with $\mu = 5$ and $\theta = 1$.

Case 4: Six variables each with $\mu = 5$ and $\theta = 0.5$.

To assure the steady-state condition, the process runs in control for the first 200 time periods. After that, the process shifts to an out-of-control state by one of three different shift sizes randomly applied to demonstrate the proposed chart performance – one unit shift in the first variable, one unit shift in the first two variables, and one unit shift in all variables. The shifts [1, 0, 0, 0], [1, 1, 0, 0], and [1, 1, 1, 1] are used for the case of four variables. Two smoothing weights used are $\lambda = 0.05$ and $\lambda = 0.1$. The control limits are obtained from Tables 24-25. For instance, the control limits of case 1 are 9.8 ($\lambda = 0.05$) and 11.68 ($\lambda = 0.1$) as shown in Table 25.

We examine two different approaches for signaling an out-of-control state using the same control limits - an individual signal and a run of signals. A run of signals is defined by any two or more consecutive out-of-control signals. Four scenarios of the runs of out-of-control signals are tested, including two, three, four, and five points in a row, respectively. To demonstrate how to declare an out-of-control signal in these scenarios, let's assume that we are interested in a run of three out-of-control signals. If an individual signal or two consecutive out-of-control points signal are found, the process is not considered out-of-control. When we detect three out-of-control points in a row, it means that we also detect an individual and two out-of-control points in a row in the earlier period. Suppose three consecutive out-of-control points are found at period 205, 206 and 207. We treat these three out-of-control signals as one of an individual out-of-control point at the time period 205, one of the two out-of-control points in a row at the time period 206, and one of the three out-of-control points in a row at the time period 207.

We evaluate the performance of the one-sided MPEWMA chart for monitoring the 500 simulated data based on two criteria – the percentage of cases where an out-of-control signal is detected and the time period of the first out-of-control signal detection after the shift has occurred at period 201. Table 31 presents the percentage of cases where an out-of-control signal has been detected using the one-sided MPEWMA chart. The one-sided MPEWMA scheme is able to detect the mean shift when at least one out-of-control signal has occurred. The results indicate that the percentage of cases decreases as the number of consecutive out-of-control points increases, particularly for mean shifts of one or two variables. The reason for the reduction in the percentage of cases corresponds to the lower chance of having consecutive out-of-control points, particularly for a small shift size. A significant decrease in the percentages of detected cases occurred for larger smoothing weights. For example, consider a unit shift in one out of six variables from Case 4 (i.e. the shift matrix [1, 0, 0, 0, 0, 0]). When $\lambda = 0.05$, there is no obvious difference between using either an individual signal (99.98%) or two to five out-of-control points in a row (98.63% - 99.94%). However, for $\lambda = 0.1$, the differences between two the approaches are large - 99.67% for detecting an individual signal and as low as 71.64% for detecting a run of two to five out-of-control signals.

**Table 31** The percentage of cases that detected an out-of-control signal for all four scenarios

| Case | λ | shift matrix | The percentage of detecting an out-of-control | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1 point | 2 points | 3 points | 4 points | 5 points |
| 1 | 0.05 | [1,0,0,0] | 100.00 | 100.00 | 100.00 | 100.00 | 99.99 |
| | | [1,1,0,0] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | | [1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | 0.1 | [1,0,0,0] | 100.00 | 99.95 | 99.56 | 98.47 | 96.30 |
| | | [1,1,0,0] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | | [1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 99.99 |
| 2 | 0.05 | [1,0,0,0] | 100.00 | 100.00 | 99.99 | 99.99 | 99.99 |
| | | [1,1,0,0] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | | [1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | 0.1 | [1,0,0,0] | 100.00 | 99.86 | 99.22 | 97.66 | 94.55 |
| | | [1,1,0,0] | 100.00 | 100.00 | 100.00 | 100.00 | 99.99 |
| | | [1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| 3 | 0.05 | [1,0,0,0,0,0] | 99.98 | 99.87 | 99.69 | 99.27 | 98.63 |
| | | [1,1,0,0,0,0] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | | [1,1,1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | 0.1 | [1,0,0,0,0,0] | 99.68 | 96.99 | 90.88 | 82.07 | 71.52 |
| | | [1,1,0,0,0,0] | 100.00 | 99.99 | 99.66 | 98.71 | 96.60 |
| | | [1,1,1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| 4 | 0.05 | [1,0,0,0,0,0] | 99.98 | 99.94 | 99.62 | 99.2 | 98.63 |
| | | [1,1,0,0,0,0] | 100.00 | 99.99 | 99.99 | 99.99 | 99.99 |
| | | [1,1,1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| | 0.1 | [1,0,0,0,0,0] | 99.67 | 97.38 | 91.71 | 82.41 | 71.64 |
| | | [1,1,0,0,0,0] | 100.00 | 100.00 | 99.86 | 99.04 | 97.42 |
| | | [1,1,1,1,1,1] | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |

Table 32 reports the average period of time to detect the first out-of-control signal for all four scenarios. The average of the times is calculated from the first out-of-control signal detected by the one-sided MPEWMA scheme within each replication. For example, the first case study with $\lambda = 0.05$ shows that the first individual out-of-control signal is detected with average time of 29.0907 (period = 230) after the shift [1, 0, 0, 0] has occurred at period 200. The results demonstrate that the shift in one and two variables can be quickly detected by a single out-of-control point as compared to a run of consecutive out-of-control points for all four cases. A run of 5 consecutive out-of-control points is the slowest out of control condition to be detected since we have to wait until all five consecutive points exceed the control limit. For example, consider Case 4 with $\lambda$ = 0.1. The time until the first out-of-control signal occurs increases from 58.284 (an individual) to 82.7172, 101.8237, 117.0436, and 127.6594 for two, three, four, and five points in a row, respectively. Thus, it will take an average of 69 additional periods to detect an out-of-control situation when applying a run of five out-of-control points instead of an individual.

The time-delay for detection tends to become shorter with larger numbers of variables shifted. Consider the previous example with individual out-of-control signal. The time to detect the first out-of-control signal reduces from 58.284 to 31.711, and 11.9623 when the number of variables shifted increases to two and six variables, respectively. The use of an individual approach is recommended over a run of 2 or more occurs for detecting a shift in one or two variables. The consecutive out-of-control points method improves if shifts occur in two or more variables. There is no considerable increase in detection time. For instance, if all six variables shifted (the shift [1, 1, 1, 1, 1, 1]) such as in Case 4, the detection time increases from 12.7318 to 14.2853, 15.6401, 16.914, and 18.1616.

**Table 32** Summary of the first out-of-control signal period detected by the one-sided MPEWMA chart for all four scenarios

| Case | λ | shift matrix | The first period to detect an out-of-control signal | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1 point | 2 points | 3 points | 4 points | 5 points |
| 1 | 0.05 | [1,0,0,0] | 29.0907 | 34.2840 | 38.9742 | 43.0704 | 46.9790 |
| | | [1,1,0,0] | 18.5769 | 21.3951 | 23.8374 | 26.0050 | 28.2012 |
| | | [1,1,1,1] | 13.9113 | 15.7539 | 17.3141 | 18.7701 | 20.1483 |
| | 0.1 | [1,0,0,0] | 36.6166 | 51.0260 | 65.4608 | 78.1542 | 90.3734 |
| | | [1,1,0,0] | 20.8885 | 27.3465 | 33.4023 | 39.5808 | 45.7962 |
| | | [1,1,1,1] | 13.9071 | 17.1756 | 21.0479 | 24.4276 | 27.6739 |
| 2 | 0.05 | [1,0,0,0] | 29.5135 | 35.0677 | 39.9803 | 44.3125 | 48.7262 |
| | | [1,1,0,0] | 18.1188 | 20.9985 | 23.4479 | 25.6407 | 27.7250 |
| | | [1,1,1,1] | 12.6268 | 13.9811 | 15.2246 | 16.4003 | 17.5337 |
| | 0.1 | [1,0,0,0] | 37.3561 | 52.6916 | 67.6304 | 82.7736 | 95.1820 |
| | | [1,1,0,0] | 20.1862 | 26.5004 | 32.5465 | 38.7679 | 45.035 |
| | | [1,1,1,1] | 11.7540 | 14.7180 | 17.3741 | 19.8542 | 22.3003 |
| 3 | 0.05 | [1,0,0,0,0,0] | 45.1418 | 55.6943 | 64.5467 | 72.4852 | 79.9108 |
| | | [1,1,0,0,0,0] | 27.3583 | 32.4666 | 37.0147 | 40.9685 | 44.9837 |
| | | [1,1,1,1,1,1] | 13.9702 | 16.1032 | 17.8595 | 19.4734 | 20.9719 |
| | 0.1 | [1,0,0,0,0,0] | 58.31 | 82.0819 | 101.2547 | 115.6195 | 126.9829 |
| | | [1,1,0,0,0,0] | 33.9501 | 47.3816 | 61.2324 | 74.6788 | 87.5655 |
| | | [1,1,1,1,1,1] | 14.1226 | 18.3333 | 22.1978 | 26.0304 | 29.8291 |
| 4 | 0.05 | [1,0,0,0,0,0] | 45.2154 | 55.7593 | 64.6804 | 73.2164 | 80.8801 |
| | | [1,1,0,0,0,0] | 26.4281 | 31.411 | 35.5505 | 39.4522 | 43.3637 |
| | | [1,1,1,1,1,1] | 12.7318 | 14.2853 | 15.6401 | 16.914 | 18.1616 |
| | 0.1 | [1,0,0,0,0,0] | 58.284 | 82.7172 | 101.8237 | 117.0436 | 127.6594 |
| | | [1,1,0,0,0,0] | 31.711 | 44.8519 | 57.8931 | 70.8977 | 84.1669 |
| | | [1,1,1,1,1,1] | 11.9623 | 15.2091 | 18.1433 | 20.9336 | 23.6991 |

### 4.7 Examples

We illustrate an example of using the one-sided MPEWMA chart to monitor public-health data. Let's consider the monitoring of one of six common air pollutants, Carbon Monoxide (or CO). The hourly CO concentration (in parts per million (ppm)) at 4 different stations being monitored are denoted by $X_i$ where

$i$ = 1, 2, 3, and 4. The hourly average CO concentration measured from each station, $X_i$, is the combination of the overall and the area effects. The effect on overall CO in the atmosphere can be represented by $Y$ and the effect of CO emissions at each area is represented by $Y_i$ for $i$ = 1, 2, 3, and 4. Thus, it is reasonable to assume the data can be sufficiently modeled by a multivariate Poisson distribution assuming a correlation exists between variables. Suppose the mean hourly CO concentration for each of the 4 stations is 3 ppm and the common effect is 0.5 ($\theta$ = 0.5). Given this information, the sample mean and the covariance matrix are given by

$$\boldsymbol{\mu_0} = [3, 3, 3, 3] \qquad \text{and} \qquad \Sigma = \begin{bmatrix} 3 & 0.5 & 0.5 & 0.5 \\ 0.5 & 3 & 0.5 & 0.5 \\ 0.5 & 0.5 & 3 & 0.5 \\ 0.5 & 0.5 & 0.5 & 3 \end{bmatrix}$$

Data are collected on day $t$ for $t$ = 1, 2,…, 200 over a 6 month period. In this example, we use the known means and covariance matrix to compute the one-sided MPEWMA (or $MEW_t$) statistics. A smoothing weight of 0.05 is selected, and the control limit obtained from Table 24 is 10.29. For From Equation (13),

$$\mathbf{Z}_1 = \max\{\lambda(\mathbf{X}_1 - \mu_0) + (1-\lambda)\mathbf{Z}_0,\ \mathbf{0}\}$$

To illustrate the calculations of the $MEW_t$ statistics considered

$$\mathbf{Z}_1 = \max\left\{ 0.05(\begin{bmatrix} 3 \\ 3 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}) + (1-0.05)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}$$

Using Equation (15) we obtain:

$$\Sigma_{Z_1} = \frac{\lambda}{2-\lambda}\Sigma = \frac{0.05}{2-0.05}\Sigma .$$

Day 1: the $MEW_t$ statistics using Equation (16) is

$$MEW_1 = Z_1^{'}\sum\nolimits_{Z_1}^{-1} Z_1 = 0.5547$$

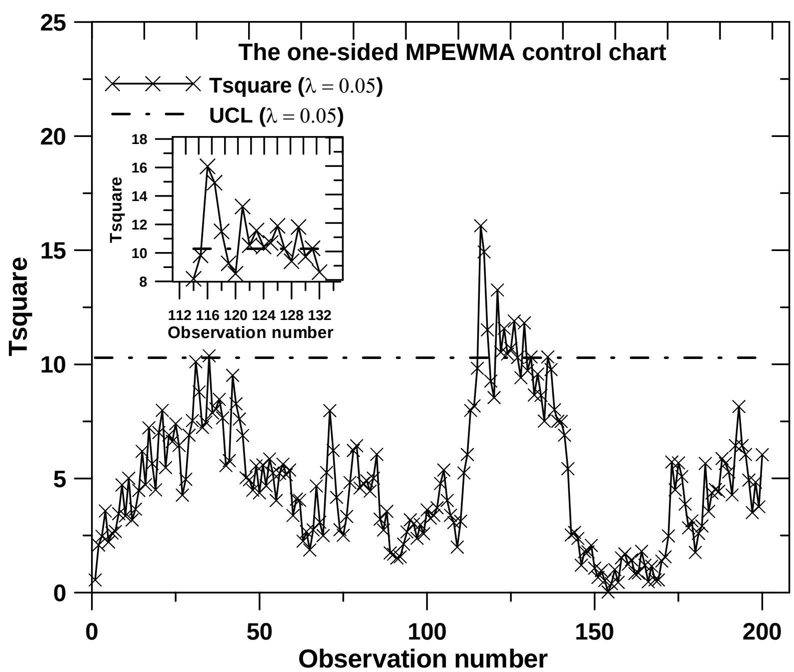
The calculations of the $MEW_t$ statistics for the first 10 samples are presented in Table 33. If the values of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\theta$ are unknown, we can estimate all these parameters from historical data by using several methods (for more details, see Section 2.4.2).

Figure 5 displays the plot of the one-sided MPEWMA chart for the hourly CO concentration. The first out-of-control signal is given at period 116 since the $MEW_t$ statistics of 16.0734 exceeds the control limits ($H$ = 10.29). If we consider a run of out-of-control signals instead of an individual, the first out-of-control signal is still found at the same period 116 for the cases of two and three signals in a row. However, the first out-of-control signal is detected at period 121 while waiting for four and five out-of-control signals to occur. The time-delay of detection with 5 periods could be problematic, particularly if the mean hourly CO concentration exceeds the air quality standard. Hence, the individual signal method can be implemented if the mean hourly CO concentration lies near the level of the air quality standard. Implementation of the long-run (n = 4, and 5) method can be applied when the mean hourly CO concentration is far beyond the standard level of the air quality.

**Table 33** Example of the calculation of the first 10 samples of the one-sided MPEWMA statistics.

| | | | | | λ = 0.05 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Obs | x1 | x2 | x3 | x4 | $Z_t$ | | | | $MEW_t$ |
| | | | | | 0 | 0 | 0 | 0 | |
| 1 | 3 | 3 | 3 | 7 | 0.0000 | 0.0000 | 0.0000 | 0.2000 | 0.5547 |
| 2 | 6 | 8 | 8 | 5 | 0.1500 | 0.2500 | 0.2500 | 0.2900 | 2.0814 |
| 3 | 3 | 5 | 4 | 2 | 0.1425 | 0.3375 | 0.2875 | 0.2255 | 2.4673 |
| 4 | 1 | 3 | 3 | 7 | 0.0354 | 0.3206 | 0.2731 | 0.4142 | 3.5767 |
| 5 | 2 | 0 | 4 | 1 | 0.0000 | 0.1546 | 0.3095 | 0.2935 | 2.2160 |
| 6 | 5 | 3 | 3 | 5 | 0.1000 | 0.1469 | 0.2940 | 0.3788 | 2.6136 |
| 7 | 3 | 5 | 1 | 4 | 0.0950 | 0.2395 | 0.1793 | 0.4099 | 2.6793 |
| 8 | 3 | 2 | 4 | 5 | 0.0902 | 0.1775 | 0.2203 | 0.4894 | 3.4562 |
| 9 | 2 | 0 | 4 | 5 | 0.0357 | 0.0187 | 0.2593 | 0.5649 | 4.7149 |
| 10 | 6 | 3 | 4 | 1 | 0.1840 | 0.0177 | 0.2963 | 0.4367 | 3.3632 |

**Figure 5** The one-sided MPEWMA chart of the hourly CO concentration



Chapter 5

# COMPARISON OF THE MULTIVARIATE CHARTS FOR MULTIVARIATE POISSON DATA

## 5.1 Introduction

There have been many studies on the quality control methods for monitoring multivariate data. In general, the control charts are applied to the raw or unprocessed data. The adequacy of the normality and independence assumptions must be assessed before applying any control scheme. It is not always appropriate to assume normality in the situation where the variables follow the Poisson distribution, particularly for small mean counts. If the data depart from the normality assumption, then methods based on other distributions should be employed. The process knowledge has been utilized to improve the sensitivity of the control chart by fitting a regression model to the data. The coefficients of the model are estimated by the regression technique. The residuals from the model are plotted on the conventional control chart. Thus, this method is referred to as the model-based or the residual-based control charting. The model-based control method has relied on the normally distributed data because the control statistics are based on residuals.

There are a few studies of the model-based control approach on monitoring multivariate Poisson data. It is interesting to investigate the model-based control chart's performance in detecting a shift in the mean count. Two regression analyses are chosen to demonstrate the ability for modeling the Poisson counts. The Poisson counts are generated through Monte Carlo simulation. The residuals are plotted on the multiple Exponentially Weighted Moving Average (EWMA) charts because of the good performance for detecting a small mean shift. The Average Run Length (ARL) performances are reported and evaluated by several combinations of the parameters including mean values, number of variables, and various sizes of shift. In addition, we make a comparison between those model-based schemes and the multivariate Poisson EWMA chart, for which the control limits are directly obtained from the Poisson distribution. The results can help clarify a better method for the early detection of a mean shift.

## 5.2 Methodology

Typically, the Ordinary Least Squares (OLS) regression is performed to estimate the coefficients of the model. The model computed from the OLS is limited to normal data. In this study, two regression techniques are selected to model the Poisson distributed data - the regression adjustment and generalized linear regression. The details of each regression method are discussed below.

### 5.2.1 Regression Adjustment

Hawkins (1993) introduced the regression-adjustment based on Y and Z scales. The standardization of the original scale is recommended before transformation into the Y and Z scales. All X, Y, and Z scales correspond to the Hotelling $T^2$ statistics. The Hotelling $T^2$ statistic is

$$T_i^2 = (\mathbf{X}_i - \mu)' \Sigma^{-1} (\mathbf{X}_i - \mu) \tag{19}$$

The $T^2$ can be expressed as

$$T_i^2 = \mathbf{Y}'\mathbf{Y} = \sum_{j=1}^{p} Y_j^2 \tag{20}$$

where $Y_j$ is the decomposition of $T^2$. $\boldsymbol{Y}$ can be rewrite in terms of the linear transformation of **X** as

$$Y = C(\mathbf{X} - \mu) \tag{21}$$

where $C$ is the Cholesky decomposition over the lower triangular root of the inverse of the covariance matrix ($CC' = \Sigma^{-1}$). Another decomposition of $T^2$ is

$$T_i^2 = (\mathbf{X}_i - \mu)'\mathbf{Z} \tag{22}$$

where $\mathbf{Z} = \Sigma^{-1}(\mathbf{X}_i - \mu)$ and $\boldsymbol{Z}$ is a $p \times 1$ vector.

The residuals obtained from both regression techniques above are plotted on the multiple Exponentially Weighted Moving Average (EWMA) charts to monitor the shift in means separately since the residuals are considered as independent and approximately normally distributed. The discussion of the EWMA chart is provided below.

### 5.2.2 Exponentially Weighted Moving Average Chart

Roberts (1959) proposed the EWMA chart by defining the control statistics as previously shown in Equation (6) where $\mathbf{Z}_0 = \boldsymbol{\mu}_0$. The value of the smoothing weight ($\lambda$) ranges from 0 to 1. The control limits of the steady-state EWMA are given by

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \tag{27}$$

$$CL = \mu_0 \tag{28}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \quad (29)$$

where *L* is the multiple of standard deviation used in the control limits.

### 5.3 Simulation Results

The multivariate Poisson data is simulated by the Monte Carlo simulation as the sum of two independent Poisson random variables. The Poisson counts are generated under the same conditions as previously discussed in Sections 3.4 and 4.3. Two different regression methods are applied to the simulated multivariate Poisson data. The residuals are computed and plotted on multiple EWMA charts. In this study, we consider using the EWMA chart with the smoothing weight ($\lambda$) of 0.05 due to its good performance in detection of a small shift. For each regression technique, the control limits (*L*) of the EWMA chart are independently chosen to achieve an in-control ARL of 200 (performed with 10000 repetitions).

The out-of-control ARL performances of these two residual-based control charts are tested against a wide variety of conditions. We shift the mean of one or more variables, at the same time, by adding one, two, three and four unit sizes. The shift has occurred at the period of 200 to ensure the steady-state condition. The results appear in the relation between the region shifted and percentage of change. The percentage of change is calculated using Equation (17). For example, the shift of one unit in the mean of 3 is $\frac{1}{3}\times 100 = 33\%$. The performance of all two residual-based control methods on monitoring the multivariate Poisson-distributed data are reported in terms of the ARL values below.

The appropriate multiple of sigma employed in the control limits for the regression adjustment are $L$ = 2.15, 2.2, 2.25, and 2.3 (for $\theta$ = 0.5) and $L$ = 2.35, 2.4, 2.45, and 2.55 (for $\theta$ = 1) to obtain the desired in-control ARL of 200 for the case of four, six, ten, and fifteen variables, respectively. Table 30 presents the ARL performance of four EWMA charts (four-variable case) obtained from both Y and Z scales. The ARL values reported in all the tables below are chosen from the lowest ARL among all multiple EWMA charts. The regression adjustment on the Y scale performs as well as the Z scale for both thetafix values ($\theta$ = 0.5, and 1) due to the similar in-control and out-of-control ARLs. It is noticed that the in-control ARLs of the Z scale are slightly less than the Y scale, but the difference tends to be larger for a mean of 5 or smaller. However, it is unclear whether the Y scale has actually outperformed the Z scale or not.

### 5.4 Comparison of model-based control charts

We compare the proposed two-sided multivariate Poisson EWMA schemes to the other two model-based control methods for monitoring the multiple counts. As we stated earlier, there is no difference in applying the regression adjustment on the Y and Z scales. For the purpose of comparison, the regression adjustment on the Y scale is selected based on the larger in-of-control ARL performance. The comparisons of all control charts are summarized in Table 33 – 40. The performances of the two model-based control methods are quite comparable due to the similar out-of-control ARLs. It is clearly shown that the two-sided MPEWMA chart provides the smallest out-of-control ARL values among all three methods. Hence, the two-sided MPEWMA method has superior performance than those two residual-based control charts for all scenarios.

### 5.5 Examples

Reconsider the problem of monitoring four types of defects in the LED manufacturing process as we early mentioned in Section 3.9. We simulate the number of defects by setting the mean of each defect type to 3 with a common mean of 0.5. A shift in the first defect type to the new mean of 4 is generated to determine the out-of-control performance. Three different control methods are used to monitor the defects: the regression adjustment with the EWMA chart, the generalized linear model with the EWMA chart, and the multivariate Poisson EWMA chart. Both the EWMA and MPEWMA charts are constructed using the smoothing weight of 0.05 ($\lambda = 0.05$).

A comparison of the EWMA charts for the regression adjustment on both Y and Z scales is shown in Figure 6. The EWMA statistics and the control limits for the Y scale are little different from the Z scale. The out-of-control signals are given by the EWMA chart of Y1 and Z1 at the same time during period 276 to period 293. In addition, the EWMA scheme of Y2 also detects one additional out-of-control signal during period 270 to 272. In other words, there is an indication that the process has changed. The MPEWMA chart is plotted in Figure 7. It signals an out-of-control condition at period 266 because the T-square statistics exceed the control limit. Thus, it will require 10 samples less than the regression adjustment technique to detect an increase in the mean number of defects.

**Table 34** The Average Run Lengths performance of the regression adjustment with the multiple EWMA charts for 4 variables case.

| Mean | Actual Region Shifted | % shift | I. Regression adjustment Y scale + EWMA Thetafix = 0.5, L = 2.15 and λ = 0.05 | I. Regression adjustment Z scale + EWMA Thetafix = 0.5, L = 2.15 and λ = 0.05 | I. Regression adjustment Y scale + EWMA Thetafix = 0.5, L = 2.35 and λ = 0.05 | I. Regression adjustment Z scale + EWMA Thetafix = 0.5, L = 2.35 and λ = 0.05 |
|---|---|---|---|---|---|---|
| | | | ARL* | ARL* | ARL* | ARL* |
| 3 | None | 0 | 199.779 | 201.066 | 204.186 | 205.795 |
| | 1 | 33 | 30.486 | 29.748 | 31.753 | 28.791 |
| | 2 | 33 | 31.362 | 29.460 | 34.370 | 28.786 |
| | (1,2) | 33 | 30.067 | 29.805 | 31.497 | 28.170 |
| | All | 33 | 29.335 | 29.047 | 30.771 | 27.684 |
| | 1 | 67 | 14.943 | 14.584 | 16.459 | 14.689 |
| | 2 | 67 | 15.317 | 14.413 | 17.484 | 14.837 |
| | (1,2) | 67 | 14.094 | 14.042 | 14.506 | 14.500 |
| | All | 67 | 12.579 | 12.940 | 13.240 | 13.492 |
| | All | 100 | 5.041 | 5.255 | 5.472 | 5.744 |
| | All | 133 | 2.057 | 2.124 | 2.185 | 2.682 |
| 5 | None | 0 | 209.706 | 207.846 | 228.679 | 227.277 |
| | 1 | 20 | 39.606 | 39.378 | 44.504 | 42.768 |
| | 2 | 20 | 40.446 | 39.349 | 46.206 | 42.553 |
| | (1,2) | 20 | 39.245 | 39.155 | 43.553 | 41.850 |
| | All | 20 | 38.630 | 38.528 | 43.105 | 41.368 |
| | 1 | 40 | 20.155 | 20.027 | 23.356 | 22.335 |
| | 2 | 40 | 20.595 | 20.158 | 23.861 | 22.210 |
| | (1,2) | 40 | 19.432 | 19.959 | 21.982 | 21.609 |
| | All | 40 | 18.389 | 18.870 | 20.753 | 20.584 |
| | All | 60 | 9.647 | 9.998 | 11.013 | 11.062 |
| | All | 80 | 4.418 | 4.593 | 5.340 | 5.816 |
| 8 | None | 0 | 211.364 | 212.532 | 239.254 | 237.014 |
| | 1 | 12.5 | 48.887 | 48.899 | 55.520 | 54.901 |
| | 2 | 12.5 | 48.697 | 47.983 | 57.197 | 55.284 |
| | (1,2) | 12.5 | 48.269 | 48.389 | 55.433 | 54.388 |
| | All | 12.5 | 47.461 | 47.558 | 55.005 | 54.360 |
| | 1 | 25 | 26.103 | 26.016 | 29.807 | 29.458 |
| | 2 | 25 | 25.994 | 25.731 | 30.647 | 29.604 |
| | (1,2) | 25 | 26.243 | 26.510 | 29.135 | 28.948 |
| | All | 25 | 24.777 | 25.330 | 28.328 | 28.320 |
| | All | 37.5 | 14.489 | 14.864 | 17.231 | 17.556 |
| | All | 50 | 8.406 | 8.692 | 10.466 | 10.482 |
| 10 | None | 0 | 213.104 | 211.939 | 240.106 | 237.920 |
| | 1 | 10 | 52.220 | 52.308 | 60.219 | 59.850 |
| | 2 | 10 | 52.406 | 51.910 | 61.246 | 59.852 |
| | (1,2) | 10 | 51.620 | 52.794 | 59.943 | 58.780 |
| | All | 10 | 51.291 | 51.414 | 59.139 | 58.520 |
| | 1 | 20 | 28.679 | 28.873 | 33.925 | 33.557 |
| | 2 | 20 | 29.157 | 28.858 | 34.482 | 33.709 |
| | (1,2) | 20 | 28.514 | 28.185 | 33.738 | 33.178 |
| | All | 20 | 28.170 | 28.809 | 32.537 | 32.458 |
| | All | 30 | 17.274 | 17.724 | 20.165 | 20.023 |
| | All | 40 | 10.644 | 10.987 | 12.941 | 12.741 |
| 15 | None | 0 | 210.976 | 210.613 | 241.091 | 240.296 |
| | 1 | 6.67 | 58.348 | 58.322 | 67.401 | 67.355 |
| | 2 | 6.67 | 59.274 | 58.726 | 68.229 | 67.650 |
| | (1,2) | 6.67 | 58.701 | 58.278 | 67.194 | 67.202 |
| | All | 6.67 | 58.554 | 58.155 | 67.178 | 67.197 |
| | 1 | 13.33 | 35.133 | 35.190 | 41.751 | 41.693 |
| | 2 | 13.33 | 35.237 | 34.922 | 42.177 | 41.573 |
| | (1,2) | 13.33 | 35.397 | 35.181 | 41.640 | 41.226 |
| | All | 13.33 | 34.586 | 34.386 | 40.691 | 40.410 |
| | All | 20 | 22.213 | 22.716 | 26.309 | 26.558 |
| | All | 26.67 | 15.465 | 15.877 | 18.262 | 18.557 |

Note: ARL* represents the lowest ARL obtained from those EWMA control charts

**Table 35** Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 0.5 and 4 variables case.

| Mean | Actual Region Shifted | % shift | I. Regression Adjustment of Y scale + EWMA<br>L = 2.15 and λ = 0.05 | II MPEWMA<br>λ = 0.05 |
|---|---|---|---|---|
| 3 | | | ARL* | H = 11.49 |
| | None | 0 | 199.779 | 210.573 |
| | 1 | 33 | 30.486 | 23.033 |
| | 2 | 33 | 31.362 | 23.139 |
| | (1,2) | 33 | 30.067 | 15.693 |
| | All | 33 | 29.335 | 12.170 |
| | 1 | 67 | 14.943 | 10.053 |
| | 2 | 67 | 15.317 | 10.014 |
| | (1,2) | 67 | 14.094 | 7.202 |
| | All | 67 | 12.579 | 5.724 |
| | All | 100 | 5.041 | 3.839 |
| | All | 133 | 2.057 | 2.945 |
| 5 | | | ARL* | H = 11.48 |
| | None | 0 | 209.706 | 211.002 |
| | 1 | 20 | 39.606 | 31.196 |
| | 2 | 20 | 40.446 | 31.417 |
| | (1,2) | 20 | 39.245 | 21.916 |
| | All | 20 | 38.630 | 15.469 |
| | 1 | 40 | 20.155 | 13.682 |
| | 2 | 40 | 20.595 | 13.725 |
| | (1,2) | 40 | 19.432 | 9.496 |
| | All | 40 | 18.389 | 7.087 |
| | All | 60 | 9.647 | 4.661 |
| | All | 80 | 4.418 | 3.541 |
| 8 | | | ARL* | H = 11.47 |
| | None | 0 | 211.364 | 209.804 |
| | 1 | 12.5 | 48.887 | 38.307 |
| | 2 | 12.5 | 48.697 | 38.359 |
| | (1,2) | 12.5 | 48.269 | 28.732 |
| | All | 12.5 | 47.461 | 20.030 |
| | 1 | 25 | 26.103 | 18.503 |
| | 2 | 25 | 25.994 | 18.484 |
| | (1,2) | 25 | 26.243 | 12.445 |
| | All | 25 | 24.777 | 8.833 |
| | All | 37.5 | 14.489 | 5.650 |
| | All | 50 | 8.406 | 4.268 |
| 10 | | | ARL* | H = 11.46 |
| | None | 0 | 213.104 | 210.250 |
| | 1 | 10 | 52.220 | 40.090 |
| | 2 | 10 | 52.406 | 40.379 |
| | (1,2) | 10 | 51.620 | 32.180 |
| | All | 10 | 51.291 | 22.557 |
| | 1 | 20 | 28.679 | 21.395 |
| | 2 | 20 | 29.157 | 21.345 |
| | (1,2) | 20 | 28.514 | 14.056 |
| | All | 20 | 28.170 | 9.755 |
| | All | 30 | 17.274 | 6.298 |
| | All | 40 | 10.644 | 4.719 |
| 15 | | | ARL* | H = 11.46 |
| | None | 0 | 210.976 | 211.350 |
| | 1 | 6.67 | 58.348 | 43.318 |
| | 2 | 6.67 | 59.274 | 43.390 |
| | (1,2) | 6.67 | 58.701 | 37.281 |
| | All | 6.67 | 58.554 | 28.040 |
| | 1 | 13.33 | 35.133 | 27.658 |
| | 2 | 13.33 | 35.237 | 27.209 |
| | (1,2) | 13.33 | 35.397 | 18.160 |
| | All | 13.33 | 34.586 | 12.045 |
| | All | 20 | 22.213 | 7.677 |
| | All | 26.67 | 15.465 | 5.700 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Table 36** Comparison of the Average Run Lengths between six multiple EWMA charts for thetafix = 0.5 and 6 variables case.

| | | | I. Regression Adjustment of Y scale + EWMA | II MPEWMA |
|---|---|---|---|---|
| Mean | Actual Region Shifted | % shift | L = 2.2 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 14.93 |
| | None | 0 | 203.872 | 210.021 |
| | 1 | 33 | 30.944 | 24.692 |
| | 2 | 33 | 32.276 | 25.041 |
| | (1,2) | 33 | 30.344 | 16.836 |
| | All | 33 | 29.855 | 11.546 |
| | 1 | 67 | 15.350 | 10.708 |
| | 2 | 67 | 15.676 | 10.775 |
| | (1,2) | 67 | 14.402 | 7.565 |
| | All | 67 | 12.488 | 5.429 |
| | All | 100 | 4.795 | 3.613 |
| | All | 133 | 1.871 | 2.806 |
| 5 | | | ARL* | H = 14.9 |
| | None | 0 | 212.519 | 210.145 |
| | 1 | 20 | 40.244 | 33.167 |
| | 2 | 20 | 41.369 | 33.810 |
| | (1,2) | 20 | 40.128 | 23.745 |
| | All | 20 | 39.677 | 14.179 |
| | 1 | 40 | 20.844 | 14.994 |
| | 2 | 40 | 21.380 | 14.956 |
| | (1,2) | 40 | 19.997 | 10.191 |
| | All | 40 | 18.651 | 6.526 |
| | All | 60 | 9.409 | 4.340 |
| | All | 80 | 4.405 | 3.302 |
| 8 | | | ARL* | H = 14.89 |
| | None | 0 | 216.016 | 209.733 |
| | 1 | 12.5 | 49.459 | 39.937 |
| | 2 | 12.5 | 50.370 | 40.162 |
| | (1,2) | 12.5 | 49.531 | 30.922 |
| | All | 12.5 | 49.134 | 17.841 |
| | 1 | 25 | 26.927 | 20.249 |
| | 2 | 25 | 26.815 | 20.105 |
| | (1,2) | 25 | 26.197 | 13.422 |
| | All | 25 | 25.408 | 7.926 |
| | All | 37.5 | 14.814 | 5.193 |
| | All | 50 | 8.384 | 3.913 |
| 10 | | | ARL* | H = 14.89 |
| | None | 0 | 216.923 | 209.776 |
| | 1 | 10 | 53.915 | 42.413 |
| | 2 | 10 | 54.153 | 42.072 |
| | (1,2) | 10 | 53.222 | 34.616 |
| | All | 10 | 53.477 | 20.322 |
| | 1 | 20 | 29.879 | 23.477 |
| | 2 | 20 | 30.146 | 23.382 |
| | (1,2) | 20 | 29.411 | 15.354 |
| | All | 20 | 28.775 | 8.804 |
| | All | 30 | 17.345 | 5.687 |
| | All | 40 | 10.932 | 4.226 |
| 15 | | | ARL* | H = 14.9 |
| | None | 0 | 220.222 | 211.031 |
| | 1 | 6.67 | 60.392 | 44.087 |
| | 2 | 6.67 | 60.793 | 44.251 |
| | (1,2) | 6.67 | 60.3692 | 39.832 |
| | All | 6.67 | 59.676 | 25.218 |
| | 1 | 13.33 | 36.041 | 30.108 |
| | 2 | 13.33 | 36.487 | 29.906 |
| | (1,2) | 13.33 | 36.558 | 19.734 |
| | All | 13.33 | 35.546 | 10.772 |
| | All | 20 | 22.579 | 6.844 |
| | All | 26.67 | 15.369 | 5.077 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Table 37** Comparison of the Average Run Lengths between ten multiple EWMA charts for thetafix = 0.5 and 10 variables case.

| Mean | Actual Region Shifted | % shift | I. Regression Adjustment of Y scale + EWMA (parshift) | IV MPEWMA |
|---|---|---|---|---|
| | | | L = 2.25 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 21.17 |
| | None | 0 | 205.650 | 211.413 |
| | 1 | 33 | 31.829 | 27.844 |
| | 2 | 33 | 31.874 | 27.484 |
| | (1,2) | 33 | 31.273 | 18.429 |
| | All | 33 | 29.756 | 10.817 |
| | 1 | 67 | 15.555 | 11.864 |
| | 2 | 67 | 15.804 | 11.947 |
| | (1,2) | 67 | 14.370 | 8.194 |
| | All | 67 | 12.269 | 5.138 |
| | All | 100 | 4.917 | 3.490 |
| | All | 133 | 1.813 | 2.683 |
| 5 | | | ARL* | H = 21.15 |
| | None | 0 | 216.317 | 211.050 |
| | 1 | 20 | 41.380 | 36.894 |
| | 2 | 20 | 41.635 | 36.646 |
| | (1,2) | 20 | 41.451 | 26.348 |
| | All | 20 | 40.070 | 13.183 |
| | 1 | 40 | 21.224 | 16.634 |
| | 2 | 40 | 21.592 | 16.645 |
| | (1,2) | 40 | 20.671 | 11.122 |
| | All | 40 | 18.588 | 6.134 |
| | All | 60 | 9.400 | 4.055 |
| | All | 80 | 4.183 | 3.129 |
| 8 | | | ARL* | H = 21.14 |
| | None | 0 | 222.257 | 211.074 |
| | 1 | 12.5 | 50.793 | 42.561 |
| | 2 | 12.5 | 50.869 | 41.737 |
| | (1,2) | 12.5 | 50.351 | 34.729 |
| | All | 12.5 | 49.767 | 16.073 |
| | 1 | 25 | 27.551 | 22.901 |
| | 2 | 25 | 27.271 | 22.945 |
| | (1,2) | 25 | 26.903 | 15.084 |
| | All | 25 | 25.235 | 7.221 |
| | All | 37.5 | 14.683 | 4.768 |
| | All | 50 | 8.421 | 3.593 |
| 10 | | | ARL* | H = 21.13 |
| | None | 0 | 222.673 | 210.791 |
| | 1 | 10 | 54.563 | 44.087 |
| | 2 | 10 | 55.142 | 43.807 |
| | (1,2) | 10 | 54.506 | 37.626 |
| | All | 10 | 54.382 | 17.910 |
| | 1 | 20 | 30.709 | 26.519 |
| | 2 | 20 | 31.236 | 26.900 |
| | (1,2) | 20 | 30.345 | 17.321 |
| | All | 20 | 29.378 | 7.910 |
| | All | 30 | 17.628 | 5.184 |
| | All | 40 | 10.567 | 3.874 |
| 15 | | | ARL* | H = 21.12 |
| | None | 0 | 223.709 | 210.355 |
| | 1 | 6.67 | 62.139 | 45.261 |
| | 2 | 6.67 | 61.988 | 44.668 |
| | (1,2) | 6.67 | 62.373 | 42.046 |
| | All | 6.67 | 61.821 | 21.944 |
| | 1 | 13.33 | 37.378 | 33.580 |
| | 2 | 13.33 | 38.140 | 33.371 |
| | (1,2) | 13.33 | 37.641 | 22.388 |
| | All | 13.33 | 37.154 | 9.415 |
| | All | 20 | 22.978 | 6.048 |
| | All | 26.67 | 15.284 | 4.525 |

Note: Al Note: ARL* represents the lowest ARL obtained from those individual EWMA control cha

**Table 38** Comparison of the Average Run Lengths between fifteen multiple EWMA charts for thetafix = 0.5 and 15 variables case.

| Mean | Actual Region Shifted | % shift | I. Regression Adjustment of Y scale + EWMA | II MPEWMA |
|---|---|---|---|---|
| | | | L = 2.3 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 28.39 |
| | None | 0 | 205.740 | 212.430 |
| | 1 | 33 | 31.672 | 30.162 |
| | 2 | 33 | 32.463 | 30.304 |
| | (1,2) | 33 | 31.251 | 20.124 |
| | All | 33 | 30.273 | 10.565 |
| | 1 | 67 | 16.030 | 12.937 |
| | 2 | 67 | 16.176 | 12.811 |
| | (1,2) | 67 | 14.781 | 8.785 |
| | All | 67 | 12.330 | 5.019 |
| | All | 100 | 4.798 | 3.405 |
| | All | 133 | 1.858 | 2.620 |
| 5 | | | ARL* | H = 28.34 |
| | None | 0 | 219.482 | 210.961 |
| | 1 | 20 | 42.388 | 39.355 |
| | 2 | 20 | 42.255 | 38.929 |
| | (1,2) | 20 | 41.857 | 28.677 |
| | All | 20 | 40.785 | 12.826 |
| | 1 | 40 | 21.743 | 18.390 |
| | 2 | 40 | 22.007 | 18.343 |
| | (1,2) | 40 | 20.841 | 12.093 |
| | All | 40 | 18.679 | 5.891 |
| | All | 60 | 9.459 | 3.952 |
| | All | 80 | 4.272 | 3.032 |
| 8 | | | ARL* | H = 28.31 |
| | None | 0 | 224.872 | 211.828 |
| | 1 | 12.5 | 52.618 | 44.036 |
| | 2 | 12.5 | 52.194 | 43.131 |
| | (1,2) | 12.5 | 51.708 | 37.589 |
| | All | 12.5 | 51.556 | 15.397 |
| | 1 | 25 | 28.099 | 25.128 |
| | 2 | 25 | 28.215 | 25.454 |
| | (1,2) | 25 | 27.550 | 16.411 |
| | All | 25 | 25.760 | 6.839 |
| | All | 37.5 | 14.868 | 4.520 |
| | All | 50 | 8.234 | 3.445 |
| 10 | | | ARL* | H = 28.30 |
| | None | 0 | 227.023 | 210.381 |
| | 1 | 10 | 56.338 | 44.687 |
| | 2 | 10 | 56.652 | 44.460 |
| | (1,2) | 10 | 56.398 | 39.743 |
| | All | 10 | 55.517 | 16.573 |
| | 1 | 20 | 31.323 | 29.444 |
| | 2 | 20 | 31.613 | 29.497 |
| | (1,2) | 20 | 31.127 | 18.847 |
| | All | 20 | 29.791 | 7.423 |
| | All | 30 | 17.806 | 4.856 |
| | All | 40 | 10.566 | 3.674 |
| 15 | | | ARL* | H = 28.30 |
| | None | 0 | 228.241 | 211.097 |
| | 1 | 6.67 | 62.679 | 45.247 |
| | 2 | 6.67 | 63.514 | 45.592 |
| | (1,2) | 6.67 | 63.251 | 43.665 |
| | All | 6.67 | 62.422 | 19.954 |
| | 1 | 13.33 | 38.284 | 36.140 |
| | 2 | 13.33 | 38.806 | 36.230 |
| | (1,2) | 13.33 | 38.494 | 24.969 |
| | All | 13.33 | 37.236 | 8.740 |
| | All | 20 | 23.182 | 5.645 |
| | All | 26.67 | 15.644 | 4.220 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Table 39** Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 1 and 4 variables case.

| | | | I. Regression Adjustment of Y scale + EWMA | II MPEWMA |
|---|---|---|---|---|
| Mean | Actual Region Shifted | % shift | L = 2.35 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 11.49 |
| | None | 0 | 204.186 | 209.945 |
| | 1 | 33 | 31.753 | 20.736 |
| | 2 | 33 | 34.370 | 20.817 |
| | (1,2) | 33 | 31.497 | 14.953 |
| | All | 33 | 30.771 | 13.964 |
| | 1 | 67 | 16.459 | 9.149 |
| | 2 | 67 | 17.484 | 9.207 |
| | (1,2) | 67 | 14.506 | 6.882 |
| | All | 67 | 13.240 | 6.408 |
| | All | 100 | 5.472 | 4.268 |
| | All | 133 | 2.185 | 3.270 |
| 5 | | | ARL* | H = 11.48 |
| | None | 0 | 228.679 | 211.177 |
| | 1 | 20 | 44.504 | 30.391 |
| | 2 | 20 | 46.206 | 30.196 |
| | (1,2) | 20 | 43.553 | 21.806 |
| | All | 20 | 43.105 | 17.223 |
| | 1 | 40 | 23.356 | 13.242 |
| | 2 | 40 | 23.861 | 13.126 |
| | (1,2) | 40 | 21.982 | 9.591 |
| | All | 40 | 20.753 | 7.746 |
| | All | 60 | 11.013 | 5.099 |
| | All | 80 | 5.340 | 3.875 |
| 8 | | | ARL* | H = 11.47 |
| | None | 0 | 239.254 | 210.074 |
| | 1 | 12.5 | 55.520 | 37.854 |
| | 2 | 12.5 | 57.197 | 37.638 |
| | (1,2) | 12.5 | 55.433 | 28.970 |
| | All | 12.5 | 55.005 | 21.736 |
| | 1 | 25 | 29.807 | 18.116 |
| | 2 | 25 | 30.647 | 18.225 |
| | (1,2) | 25 | 29.135 | 12.614 |
| | All | 25 | 28.328 | 9.508 |
| | All | 37.5 | 17.231 | 6.068 |
| | All | 50 | 10.466 | 4.570 |
| 10 | | | ARL* | H = 11.46 |
| | None | 0 | 240.106 | 209.354 |
| | 1 | 10 | 60.219 | 39.709 |
| | 2 | 10 | 61.246 | 40.308 |
| | (1,2) | 10 | 59.943 | 32.740 |
| | All | 10 | 60.139 | 24.204 |
| | 1 | 20 | 33.925 | 21.140 |
| | 2 | 20 | 34.482 | 21.131 |
| | (1,2) | 20 | 41.227 | 14.247 |
| | All | 20 | 32.537 | 10.489 |
| | All | 30 | 20.165 | 6.681 |
| | All | 40 | 12.941 | 4.974 |
| 15 | | | ARL* | H = 11.46 |
| | None | 0 | 241.091 | 209.510 |
| | 1 | 6.67 | 67.401 | 42.515 |
| | 2 | 6.67 | 68.229 | 42.743 |
| | (1,2) | 6.67 | 67.194 | 37.760 |
| | All | 6.67 | 67.178 | 29.715 |
| | 1 | 13.33 | 41.751 | 27.403 |
| | 2 | 13.33 | 42.177 | 27.163 |
| | (1,2) | 13.33 | 41.640 | 18.112 |
| | All | 13.33 | 40.691 | 12.778 |
| | All | 20 | 26.309 | 8.038 |
| | All | 26.67 | 18.262 | 5.918 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Table 40** Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 1 and 6 variables case.

| Mean | Actual Region Shifted | % shift | I. Regression Adjustment of Y scale + EWMA | II MPEWMA |
|---|---|---|---|---|
| | | | L = 2.4 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 14.95 |
| | None | 0 | 200.876 | 208.494 |
| | 1 | 33 | 31.785 | 22.197 |
| | 2 | 33 | 33.611 | 22.134 |
| | (1,2) | 33 | 30.815 | 15.431 |
| | All | 33 | 30.243 | 13.465 |
| | 1 | 67 | 16.433 | 9.808 |
| | 2 | 67 | 17.326 | 9.734 |
| | (1,2) | 67 | 14.716 | 7.068 |
| | All | 67 | 12.736 | 6.255 |
| | All | 100 | 5.226 | 4.141 |
| | All | 133 | 1.993 | 3.178 |
| 5 | | | ARL* | H = 14.92 |
| | None | 0 | 227.943 | 209.71 |
| | 1 | 20 | 44.665 | 32.773 |
| | 2 | 20 | 46.177 | 32.322 |
| | (1,2) | 20 | 44.064 | 23.033 |
| | All | 20 | 43.544 | 16.392 |
| | 1 | 40 | 23.334 | 14.274 |
| | 2 | 40 | 24.001 | 14.368 |
| | (1,2) | 40 | 22.108 | 10.003 |
| | All | 40 | 20.676 | 7.453 |
| | All | 60 | 10.632 | 4.883 |
| | All | 80 | 5.036 | 3.706 |
| 8 | | | ARL* | H = 14.91 |
| | None | 0 | 240.600 | 210.096 |
| | 1 | 12.5 | 56.772 | 39.621 |
| | 2 | 12.5 | 57.988 | 39.804 |
| | (1,2) | 12.5 | 56.733 | 31.109 |
| | All | 12.5 | 55.291 | 20.175 |
| | 1 | 25 | 30.547 | 19.880 |
| | 2 | 25 | 30.956 | 19.747 |
| | (1,2) | 25 | 30.068 | 13.406 |
| | All | 25 | 28.798 | 8.900 |
| | All | 37.5 | 16.806 | 5.757 |
| | All | 50 | 9.930 | 4.284 |
| 10 | | | ARL* | H = 14.91 |
| | None | 0 | 240.673 | 210.320 |
| | 1 | 10 | 62.300 | 41.720 |
| | 2 | 10 | 62.523 | 42.455 |
| | (1,2) | 10 | 61.038 | 35.049 |
| | All | 10 | 60.527 | 22.494 |
| | 1 | 20 | 34.627 | 23.239 |
| | 2 | 20 | 34.970 | 22.995 |
| | (1,2) | 20 | 34.269 | 15.485 |
| | All | 20 | 33.082 | 9.718 |
| | All | 30 | 20.394 | 6.218 |
| | All | 40 | 13.038 | 4.648 |
| 15 | | | ARL* | H = 14.90 |
| | None | 0 | 244.579 | 209.452 |
| | 1 | 6.67 | 69.809 | 43.628 |
| | 2 | 6.67 | 69.694 | 43.870 |
| | (1,2) | 6.67 | 68.722 | 39.950 |
| | All | 6.67 | 68.576 | 27.270 |
| | 1 | 13.33 | 43.161 | 30.238 |
| | 2 | 13.33 | 43.677 | 29.860 |
| | (1,2) | 13.33 | 43.185 | 19.963 |
| | All | 13.33 | 42.221 | 11.535 |
| | All | 20 | 26.667 | 7.308 |
| | All | 26.67 | 18.113 | 5.398 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Table 41** Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 1 and 10 variables case.

| | | | I. Regression Adjustment of Y scale + EWMA | II MPEWMA |
|---|---|---|---|---|
| Mean | Actual Region Shifted | % shift | L = 2.45 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 21.19 |
| | None | 0 | 197.275 | 212.060 |
| | 1 | 33 | 32.077 | 24.349 |
| | 2 | 33 | 32.365 | 24.588 |
| | (1,2) | 33 | 31.020 | 16.435 |
| | All | 33 | 29.859 | 13.139 |
| | 1 | 67 | 16.505 | 10.540 |
| | 2 | 67 | 17.134 | 10.564 |
| | (1,2) | 67 | 14.974 | 7.416 |
| | All | 67 | 12.428 | 6.050 |
| | All | 100 | 4.932 | 4.054 |
| | All | 133 | 1.935 | 3.084 |
| 5 | | | ARL* | H = 21.14 |
| | None | 0 | 228.754 | 211.92 |
| | 1 | 20 | 44.765 | 35.644 |
| | 2 | 20 | 46.138 | 35.442 |
| | (1,2) | 20 | 44.175 | 25.038 |
| | All | 20 | 43.858 | 16.014 |
| | 1 | 40 | 23.215 | 15.755 |
| | 2 | 40 | 23.726 | 15.704 |
| | (1,2) | 40 | 22.255 | 10.711 |
| | All | 40 | 20.034 | 7.254 |
| | All | 60 | 10.483 | 4.773 |
| | All | 80 | 4.796 | 3.606 |
| 8 | | | ARL* | H = 21.13 |
| | None | 0 | 242.032 | 212.212 |
| | 1 | 12.5 | 57.075 | 41.982 |
| | 2 | 12.5 | 58.389 | 42.035 |
| | (1,2) | 12.5 | 57.266 | 34.383 |
| | All | 12.5 | 56.624 | 19.318 |
| | 1 | 25 | 30.948 | 22.011 |
| | 2 | 25 | 31.413 | 22.269 |
| | (1,2) | 25 | 30.376 | 14.767 |
| | All | 25 | 28.893 | 8.502 |
| | All | 37.5 | 16.691 | 5.466 |
| | All | 50 | 9.634 | 4.102 |
| 10 | | | ARL* | H = 21.13 |
| | None | 0 | 243.229 | 209.904 |
| | 1 | 10 | 61.899 | 43.342 |
| | 2 | 10 | 63.042 | 43.475 |
| | (1,2) | 10 | 62.262 | 37.833 |
| | All | 10 | 62.045 | 21.214 |
| | 1 | 20 | 35.300 | 26.039 |
| | 2 | 20 | 35.549 | 25.893 |
| | (1,2) | 20 | 35.086 | 17.165 |
| | All | 20 | 33.288 | 9.069 |
| | All | 30 | 20.053 | 5.859 |
| | All | 40 | 12.468 | 4.387 |
| 15 | | | ARL* | H = 21.12 |
| | None | 0 | 247.037 | 211.668 |
| | 1 | 6.67 | 69.722 | 44.877 |
| | 2 | 6.67 | 70.594 | 45.454 |
| | (1,2) | 6.67 | 70.066 | 42.3493 |
| | All | 6.67 | 69.559 | 24.629 |
| | 1 | 13.33 | 43.729 | 33.124 |
| | 2 | 13.33 | 44.513 | 33.064 |
| | (1,2) | 13.33 | 43.757 | 22.402 |
| | All | 13.33 | 42.201 | 10.523 |
| | All | 20 | 26.792 | 6.738 |
| | All | 26.67 | 17.866 | 4.998 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Table 42** Comparison of the Average Run Lengths between four multiple EWMA charts for thetafix = 1 and 15 variables case.

| Mean | Actual Region Shifted | % shift | I. Regression Adjustment of Y scale + EWMA | II MPEWMA |
|---|---|---|---|---|
| | | | L = 2.55 and λ = 0.05 | λ = 0.05 |
| 3 | | | ARL* | H = 28.41 |
| | None | 0 | 201.580 | 209.948 |
| | 1 | 33 | 32.816 | 26.690 |
| | 2 | 33 | 33.627 | 26.711 |
| | (1,2) | 33 | 32.145 | 17.530 |
| | All | 33 | 30.419 | 12.831 |
| | 1 | 67 | 17.629 | 11.360 |
| | 2 | 67 | 17.898 | 11.326 |
| | (1,2) | 67 | 15.920 | 7.863 |
| | All | 67 | 12.801 | 5.956 |
| | All | 100 | 5.156 | 3.912 |
| | All | 133 | 2.012 | 3.015 |
| 5 | | | ARL* | H = 28.34 |
| | None | 0 | 236.314 | 210.28 |
| | 1 | 20 | 46.573 | 37.585 |
| | 2 | 20 | 47.508 | 37.684 |
| | (1,2) | 20 | 46.552 | 27.600 |
| | All | 20 | 45.137 | 15.965 |
| | 1 | 40 | 24.260 | 17.299 |
| | 2 | 40 | 24.845 | 17.218 |
| | (1,2) | 40 | 23.929 | 11.493 |
| | All | 40 | 20.944 | 7.162 |
| | All | 60 | 11.135 | 4.700 |
| | All | 80 | 5.065 | 3.563 |
| 8 | | | ARL* | H = 28.32 |
| | None | 0 | 247.979 | 212.357 |
| | 1 | 12.5 | 59.843 | 43.275 |
| | 2 | 12.5 | 60.348 | 43.340 |
| | (1,2) | 12.5 | 59.056 | 36.480 |
| | All | 12.5 | 58.423 | 18.810 |
| | 1 | 25 | 32.939 | 24.787 |
| | 2 | 25 | 33.154 | 24.597 |
| | (1,2) | 25 | 31.877 | 16.130 |
| | All | 25 | 30.136 | 8.248 |
| | All | 37.5 | 17.611 | 5.371 |
| | All | 50 | 10.208 | 4.005 |
| 10 | | | ARL* | H = 28.31 |
| | None | 0 | 250.813 | 211.739 |
| | 1 | 10 | 64.449 | 44.388 |
| | 2 | 10 | 65.815 | 44.621 |
| | (1,2) | 10 | 64.435 | 39.922 |
| | All | 10 | 63.555 | 20.420 |
| | 1 | 20 | 36.650 | 28.375 |
| | 2 | 20 | 37.369 | 28.833 |
| | (1,2) | 20 | 36.436 | 18.800 |
| | All | 20 | 34.622 | 8.776 |
| | All | 30 | 21.103 | 5.724 |
| | All | 40 | 13.093 | 4.284 |
| 15 | | | ARL* | H = 28.3 |
| | None | 0 | 255.313 | 211.468 |
| | 1 | 6.67 | 72.635 | 45.452 |
| | 2 | 6.67 | 73.671 | 46.359 |
| | (1,2) | 6.67 | 72.568 | 43.198 |
| | All | 6.67 | 71.895 | 23.741 |
| | 1 | 13.33 | 46.328 | 35.940 |
| | 2 | 13.33 | 46.750 | 36.070 |
| | (1,2) | 13.33 | 46.248 | 24.901 |
| | All | 13.33 | 44.604 | 10.014 |
| | All | 20 | 27.925 | 6.431 |
| | All | 26.67 | 18.832 | 4.808 |

Note: ARL* represents the lowest ARL obtained from those individual EWMA control charts

**Figure 6** Plots of EWMA charts of the regression adjustment on both Y and Z scales

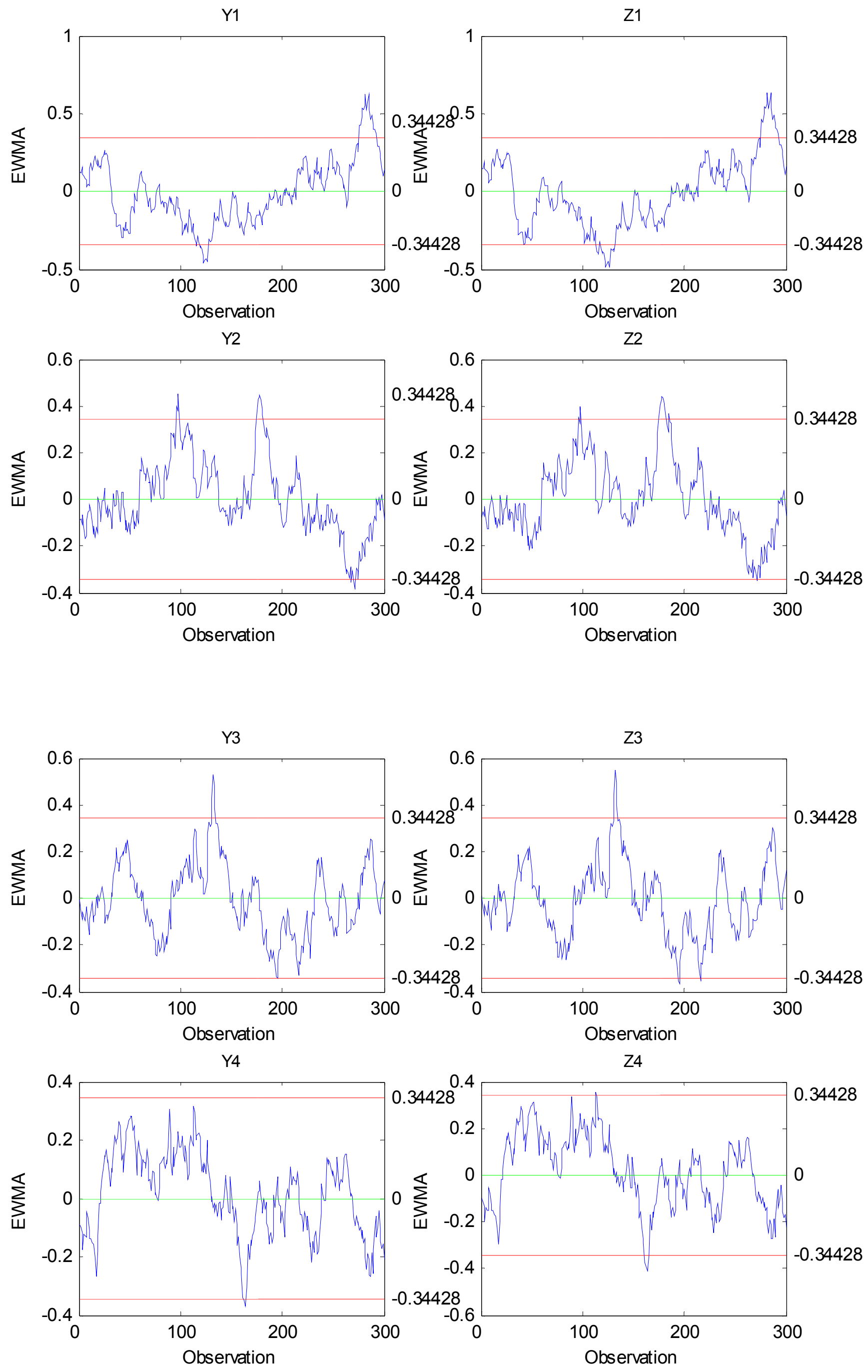
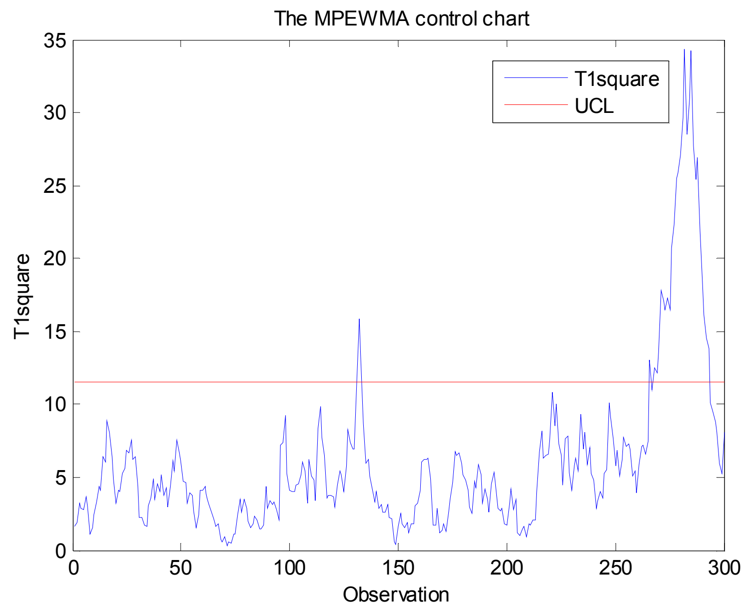


**Figure 7** Plots of MPEWMA chart with $H = 11.49$



Chapter 6

# CONCLUSION AND RECOMMENDATIONS

## 6.1 Conclusion

We presented a new type of the multivariate Exponentially Weighted Moving Average control chart for monitoring multiple related count data. This kind of data can usually be found when monitoring several types of defects per unit of product or defects per area of product in the manufacturing process. In fact, often the number of defects is small and tends to depart from a normal distribution. There is also some common relationship among all variables and consequently it can be assumed that the multivariate Poisson distribution holds. The multivariate Poisson EWMA (or MPEWMA) chart has been proposed to detect small and medium changes in the mean counts. The Poisson limits are directly derived from the multivariate Poisson distribution, instead of the normality.

We have demonstrated that control chart performance in monitoring multivariate Poisson-distributed data is slightly different between a scheme based on normal-theory limits and a scheme based directly on multivariate Poisson-distribution limits. ARL tables are presented to show the general performance of the MPEWMA scheme. The control limits of the proposed method are slightly wider than those that relied on the normality assumption. Based on the ARL results, we find that the proposed control chart produces out-of-control ARL values similar to the standard normal-theory MEWMA. However, the MPEWMA control chart is superior to the traditional MEWMA in terms of the in-control ARL. The use of the normal-theory limits can lead to substantially smaller in-control ARL values than what is stated when the data follow a multivariate Poisson distribution. Thus, the result shows the potential of using the MPEWMA with Poisson-distributed data in reducing the false alarm rate. The standard deviation run length is provided, and therefore the standard error of the mean can be obtained if desired. Furthermore, we illustrate some examples of implementation the MPEWMA chart in practice.

We extended the two-sided multivariate Poisson EWMA to the one-sided control chart based on the multivariate Poisson assumption as a method for detecting only upward shifts in the mean of multiple count data. The control limits are again established using the multivariate Poisson distribution instead of the normal approximation limits. The results indicate that the multivariate Poisson-distribution limits are wider than the normal-theory limits. The statistical performances of the one-sided MPEWMA scheme are presented by both average and standard deviation of the run length. The results indicate that applying the one-sided MEWMA with the normal-theory limits to the multivariate Poisson distribution can result in a smaller in-control ARL than the advertised value. MEWMA causes a high false alarm rate when the process is actually in control.

Four case studies are illustrated to investigate the one-sided MPEWMA performance for detecting a single and a run of out-of-control signals (2 to 5 consecutive points). The time-delay in detection tends to increase with the amount of out-of-control points waiting to signal, particularly when there is a shift in a few variables. The use of the consecutive points method is similar to the single point method when monitoring a shift in all variables because it takes a slightly longer time to detect the first out-of-control signal. The single point approach is preferred to detect a shift in one or possible two variables since it reduces detection times compared to a long run of out-of-control signals.

The proposed MPEWMA chart is also compared with other model-based control charts for monitoring count data from multiple sources. Two techniques for model building are investigated: 1) the regression adjustment and 2) the generalized linear model. We consider the regression adjustment based on two ways of decomposing the Hotelling $T^2$ statistics, and they are called the Y and Z scales. For the generalized linear model, Poisson regression is selected for modeling the Poisson distribution. The residuals (computed from the regression adjustment) and deviance residuals (calculated from the Poisson regression) are plotted on the multiple EWMA charts as those residuals are approximately normally distributed. The comparison results show that the MPEWMA scheme outperforms two residual-based control charts for all scenarios due to the small out-of-control ARL values. Hence, the MPEWMA chart can detect changes in the mean of a Poisson count earlier than those model-based control methods. Fewer samples will be taken to indicate that the process mean has increased.

### 6.2 Future work

Several concerns of the multivariate Poisson EWMA chart still require further exploration. Firstly, this paper uses a multivariate Poisson model that allowed only positive correlation. It is also interesting to develop the multivariate

Poisson distribution with the general correlation structure, and therefore a new control scheme can be extended to allow negative correlation among variables. A theoretical framework of the multivariate Poisson model and an effective method for generating data are desirable to examine the statistical performance of the new scheme.

Secondly, one disadvantage of using the multivariate control scheme is in interpretation of the out-of-control signal. It is not easy to determine which quality characteristic is associated with the mean shift signal, particularly for the high dimensional case. Moreover, the change in either local ($\theta_i$) or common ($\theta$) variables can result in an increased mean. An advanced method is needed to identify the variables that correspond to an increase in the mean. Consequently, a proper action can be taken to correct the problem.

Thirdly, it is necessary to investigate additional conditions of the parameters. For example, a study on the effect of the thetafix parameter and its role in the average run length performance. In this research, we have only explored two values of $\theta$, 0.5 and 1, and the ARL values of $\theta = 0.5$ appear to be slightly larger than for the $\theta = 1$. The parameter should vary over a wide range of values in oder to investigate the MPEWMA chart's performance. The results are needed to gain more insight into the detection of the mean change in the common variable. In addition, we simplified the process monitoring problem by assuming that all quality characteristics have equal means. This is too restrictive an assumption in many real word applications. The mean of one variable can be different than the others. Thus, it is more appealing to estimate the control chart performance under this circumstance (i.e. unequal means of the variables). One more thing, the change in the means is limited to a permanent upward shift, in other words, the count means increase and hold on to the new level after the shift has occurred. However, the shift is sometimes happen during a certain period of time (e.g. the spike of the mean shift). It is also a good idea to find some way to detect this spike shift as well as the permanent shift in the multiple count data.

Lastly, an existing method of estimation the theta parameters in the multivariate Poisson model is not guaranteed to have a good performance in the high dimensional problems. The advanced method is needed to provide more accurate the theta estimates. Thus, phase I of the proposed MPEWMA chart can be established based on this method. The statistical performance of the MPEWMA scheme in phase I will be evaluated on the basis of the run length distribution.

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APPENDIX A

APPENDIX A

MATLAB CODING FOR OBTAINING THE CONTROL LIMITS

OF THE MPEWMA CHART

```
close all;
clear all;

% Set the variables
n = 4;                  % number of variables of interest
lamda = 0.05;           % define value of lambda
M = 3;                  % define value of the mean vector
thetafix = 0.5;         % define value of the common mean
t = 0;                  % define the trail value of the control limit
shift_position = 201;   % set the occurrence of the mean shift
cycle = 50000;          % set the number of maximum cycles
lp_max = 100000;        % set the number of maximum loops to prevent infinity run

% Set the location to safe result file
dir = 'H:\Research\Result\';

% Normal Theory limits
nlamdaH =[4,0.05,11.22;4,0.1,12.73;6,0.05,14.60;6,0.1,16.27;10,0.05,20.72;
10,0.1,22.67;15,0.05,27.82;15,0.1,30.03];

% Assign shift matrix
shiftmatrix =[0,0,0,0;1,0,0,0;0,1,0,0;2,0,0,0;0,2,0,0;1,1,0,0;2,2,0,0;1,0,1,0;0,0,2,0;
1,0,0,1;0,0,0,2;1,1,1,1;2,2,2,2;3,3,3,3;4,4,4,4];
countshift = max(size(shiftmatrix));

% Fix variables
Me = zeros(n,1);
Cov = zeros(n,n);
Yi = zeros(n,1);
Xi0 = zeros(n+1,1);
theta_tmp = zeros(n,1);
countall = ones(1,cycle);

% Using trial and error based on the normal limits to obtain the Poisson limits
countnlH = max(size(nlamdaH(:,1)));
for i=1:countnlH
   if (nlamdaH(i,1)== n) & (nlamdaH(i,2)== lamda)
      H = nlamdaH(i,3);
```

```
    end
end
H = H + t;

% Find mean vector assuming all means are equal
for i = 1:n
    Me(i) = M;
end
% Find covariance matrix
for i = 1:n
    for j = 1:n
        if i == j
            Cov(i,j) = Me(i);
        else
            Cov(i,j) = thetafix;
        end
    end
end

lamda001 = 1/(lamda/(2-lamda));
thetafix_N = thetafix*ones(n,1);

% Calculate the control limits for each shift matrix
for s = 1:countshift
    shift = shiftmatrix(s,:)';
    cyc_cnt = 1;

    % To satisfy the steady-state condition, each cycle will loop for at least 200
    % periods. If fail before reaching 200 loops, we re-do simulation. If not fail
    % after 200 loops pass, the simulation continues until fail or reach lp_max
    while (cyc_cnt<=cycle)
      Me_shift = Me;
      theta_tmp = Me_shift - thetafix_N;
      z = zeros(n,1);
      lp_cnt = 1;
      while ( lp_cnt < lp_max )
          % Generate Xi and Yi
          Xi0 = poissrnd([thetafix;theta_tmp]);
          for i=1:n
              Yi(i) = Xi0(i+1) + Xi0(1);
          end

          % Calculate T-square based on asymptotic assumption
          z = (lamda*(Xi-Me)) + ((1-lamda)* z);
          invCov = inv(Cov);
```

```
        Covarinv = lamda001*invCov;
        T1square = z' * Covarinv * z;


        % Check if the T-square is in or out of control
        if (T1square > H)
          if (lp_cnt<shift_position)
            lp_cnt = 1;
            z = zeros(n,1);
            Me_shift = Me;
            theta_tmp = Me_shift - thetafix_N;
          else
            break;
          end
        else
          if (lp_cnt==shift_position)
            %Change in the mean vector after the shift period
            Me_shift = Me + shift;
            theta_tmp = Me_shift - thetafix_N;
            %Change in the variance-covaraince matrix after the shift period
            for i = 1:n
               for j = 1:n
                  if i == j
                     Cov(i,j) = Me_shift(i);
                  else
                  end
               end
            end
          end
          lp_cnt = lp_cnt + 1;
        end
    end
    countall(cyc_cnt) = lp_cnt - shift_position - 1;
    cyc_cnt = cyc_cnt+1;
end
countall_s(s,:) = countall(:);

% Create the result file
str = strcat('Total_ARL','M=',num2str(M),'thetafix=',num2str(thetafix),'H=',
num2str(H),'VAR=',num2str(n),'S=',num2str(s),'Lamda=',num2str(lamda),
'Poi=',num2str(shift_position),'.dat');
filename = [dir,str];
fid1 = fopen(filename,'w');
% Print the average run length from all cycles and run length of each cycle
fprintf(fid1,'\n%12s','ARL');
```

```
    MeanARL = mean(countall_s(s,:));
    fprintf(fid1,' %12.3f',MeanARL);
    fprintf(fid1,'\n%12s','Loop');
    fprintf(fid1,' %10s%2d','No,s=',s);
    fprintf(fid1,'\n');
    for cyc_cnt = 1:cycle;
        fprintf(fid1,'%12d',cyc_cnt);
        fprintf(fid1,' %12d',countall_s(s,cyc_cnt));
        fprintf(fid1,'\n');
    end
    fclose(fid1);
end
```