Sums of Squares of Consecutive Integers

by

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ABSTRACT

This thesis attempts to answer two questions based upon the historical observation that $1^2+2^2+\cdots+24^2 = 70^2$. The first question considers changing the starting number of the left hand side of the equation from 1 to any perfect square in the range 1 to 10000. On this question, I attempt to determine which perfect square to end the left hand side of the equation with so that the right hand side of the equation is a perfect square. Mathematically, Question #1 can be written as follows: Given a positive integer r with $1 \leq r \leq 100$, find all nontrivial solutions (N, M), if any, of $r^2 + (r+1)^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. The second question considers changing the number of terms on the left hand side of the equation to any fixed whole number in the range 1 to 100. On this question, I attempt to determine which perfect square to start the left hand side of the equation with so that the right hand side of the equation is a perfect square. Mathematically, Question #2 can be written as follows: Given a positive integer r with $1 \leq r \leq 100$, find all solutions (u, v), if any, of $u^2 + (u+1)^2 + (u+2)^2 + \cdots + (u+r-1)^2 = v^2$ with $u, v \in \mathbb{Z}^+$.

The two questions addressed by this thesis have been on the minds of many mathematicians for over 100 years. As a result of their efforts to obtain answers to these questions, a lot of mathematics has been developed. This research was done to organize that mathematics into one easily accessible place.

My findings on Question #1 can hopefully be used by future mathematicians in order to completely answer Question #1. In addition, my findings on Question #2 can hopefully be used by future mathematicians as they attempt to answer Question #2 for values of r greater than 100.

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CHAPTER ONE

Mathematically, Question #1 can be written as follows: Given a positive integer r with $1 \leq r \leq 100$, find all nontrivial solutions (N, M), if any, of $r^2+(r+1)^2+\cdots+N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. Let's begin our study of Question #1 with the case when $r = 1$. In other words, what are the nontrivial solutions (N, M), if any, of $1^2 + 2^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$? Motivation for answering this question began in 1875 when Edouard Lucas, who had been a French artillery officer in the Franco-Prussian War, challenged the readers of the Nouvelles Annales de Mathématiques to prove the following:

> A square pyramid of cannonballs contains a square number of cannonballs only when it has 24 cannonballs along its base.

Some number theorists have affectionately named this problem "The Cannonball Problem". Unfortunately, Lucas did not live to see his challenge met. The problem was not solved until 1918, when G.N. Watson gave a complicated solution based on a specially extended theory of Jacobian elliptic functions. No completely elementary solution was forthcoming until 1988, when W.S. Anglin produced one. Thus, we have the following theorem along with a synopsis of Anglin's proof:

Theorem 1.1: The only nontrivial solution of $1^2 + 2^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$ is (24, 70).

Proof Synopsis: The equation $1^2+2^2+\cdots+N^2 = M^2$ with $N \in \mathbb{Z}$ and $N > 1$ can equivalently be written as $N(N+1)(2N+1) = 6M^2$. First, suppose that N is odd. Then, since $N, N+1$, and $2N+1$ are pairwise coprime, N is either a square or a triple of a square. Hence $N \not\equiv 2 \pmod 3$. Also, $N+1$ is either double a square or six times a square. Hence, $N+1 \not\equiv 1 \pmod 3$. Thus, we may develop the following system of congruences:

$$N \equiv 1 \pmod 3$$

$$N+1 \equiv 2 \pmod 3$$

$$2N+1 \equiv 0 \pmod 3$$

Hence, there exist nonnegative integers u, v, and w such that:

$$N = u^2$$

$$N+1 = 2v^2$$

$$2N+1 = 3w^2$$

But then $6w^2 + 1 = 4N + 3 = (2u)^2 + 3$, which is a number of the form $m^2 + 3$. But $(6w^2+1)^2 - 3(4vw)^2 = 12w^2(3w^2 + 1 - 4v^2) + 1 = 1$. Since the value of x in an equation of the form $x^2 - 3c^2 = 1$ has the form $m^2 + 3$ only when $x = 7$ (see Anglin for details), we see that $6w^2 + 1 = 7$. Therefore $w = 1$ and $N = 1$.

Next, suppose that N is even. Then $N+1$ is odd and is either a square or the triple of a square. Hence, $N+1 \not\equiv 2 \pmod 3$, and $2N+1 \not\equiv 2 \pmod 3$. Thus, $N \equiv 0 \pmod 3$. Thus, there exist nonnegative integers p, q, and r such that:

$$N = 6q^2$$

$$N+1 = p^2$$

$$2N+1 = r^2$$

But then $6q^2 = (2N+1) - (N+1) = r^2 - p^2 = (r-p)(r+p)$. But p and r are both odd, which implies that q is even. Thus, q is of the form $2w$. Hence,

$(r-p)(r+p) = 6q^2 = 6(2w)^2 = 6(4w^2)$. Thus, $6w^2 = \frac{(r-p)(r+p)}{4} = \frac{r-p}{2} \cdot \frac{r+p}{2}$. Since $\frac{r-p}{2}$ and $\frac{r+p}{2}$ are coprime, we have the following two cases to consider:

Case #1: For some nonnegative integers A and B, $\frac{r-p}{2} = 3A^2$ and $\frac{r+p}{2} = 2B^2$, or vice versa. Then $p = \pm(3A^2 - 2B^2)$ and $q = 2w = 2AB$. But $6q^2+1 = N+1 = p^2$. Hence, $24A^2B^2+1 = (3A^2-2B^2)^2$. So $(3A^2-6B^2)^2 - 2(2B)^4 = 1$. Since there is no positive integer x such that $2x^4+1$ is a square (see Anglin for details), we see that $B = 0$. Therefore, $N = 6q^2 = 6(2AB)^2 = 0$.

Case #2: For some nonnegative integers A and B, $\frac{r-p}{2} = 6A^2$ and $\frac{r+p}{2} = B^2$, or vice versa. Then, $p = \pm(6A^2 - B^2)$ and $q = 2AB$. Since $6q^2+1 = p^2$, we have $24A^2B^2+1 = (6A^2-B^2)^2$. Thus, $(6A^2-3B^2)^2 - 8B^4 = 1$. Since there is only one positive integer x (namely, 1) such that $8x^4+1$ is a square (see Anglin for details), we see that $B = 0$ or 1. Therefore, $N = 6q^2 = 6(2AB)^2 = 0$ or 24. In other words, the only nontrivial solution of the Cannon-ball Problem is $(24, 70)$. □

Anglin's proof that $(24, 70)$ is the only nontrivial solution of the Cannon-ball Problem does not appear to generalize to values of r other than 1. This is because $1^2 + \cdots + (r-1)^2$ would need to be subtracted from the left-hand side of $x(x+1)(2x+1) = 6y^2$, causing us to become unable to arrive at the necessary congruences.

Moving into the twenty-first century, suppose a person was unaware of the previous quest for nontrivial solutions (N, M) of $1^2+2^2+\cdots+N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. Could technology be used to discover these solutions?

One way to do this is to begin with the formula $\frac{1}{6}N(N+1)(2N+1)$, which is equal to $1^2+2^2+\cdots+N^2$. Since we want this sum to be a square, say M^2, we need solutions of:

$$M^2 = \frac{1}{6}N(N+1)(2N+1),$$

or equivalently:

$$M^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{N}{6},$$

or equivalently:

$$81M^2 = 27N^3 + \frac{81}{2}N^2 + \frac{27}{2}N.$$

Next, letting $m = 9M$ and $n = 3N$, we obtain:

$$m^2 = n^3 + \frac{9}{2}n^2 + \frac{9}{2}n,$$

or equivalently:

$$64m^2 = 64n^3 + 9 \cdot 32n^2 + 9 \cdot 32n.$$

Next, letting $y = 8m$ and $x = 4n$, we obtain:

$$y^2 = x^3 + 18x^2 + 72x,$$

which is an elliptic curve in Weierstrass Normal Form. So, we are interested in those integer points of $y^2 = x^3 + 18x^2 + 72x$ that correspond to nontrivial solutions (N, M) of $1^2 + 2^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. Using a computer, we can determine that the integer points of $y^2 = x^3 + 18x^2 + 72x$ are:

(0, 0), (-6, 0), (-12, 0), (12, 5184), (6, 1296), (-8, 64), (-9, 81), and (288, 5040).

However, due to the substitutions that we have made, some of these points may not correspond to nontrivial solutions (N, M) of $1^2 + 2^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. Specifically, since $x = 4n = 12N$, we need 12 to divide x. Similarly, since $y = 8m = 72M$, we need 72 to divide y. This narrows our list of integer points down to:

(0, 0), (-12, 0), (12, 5184), and (288, 5040).

Now these 4 points lead to corresponding values of 0, -1, 1 and 24 for N. Since N must be positive, 0 and -1 can be disregarded. Since N must be nontrivial, 1 can be disregarded. This leaves (24, 70) as the only nontrivial solution of $1^2 + 2^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. This completes our study of the special case of Question #1 when $r = 1$.

More generally, can we answer Question #1 for positive values of r other than 1? In other words, for a fixed $r > 1$, find all nontrivial solutions (N, M), if any, of $r^2 + (r+1)^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$.

One strategy that can be used here is to note that:

$$\begin{aligned} r^2 + (r+1)^2 + \cdots + N^2 &= \left(\frac{1}{6}N(N+1)(2N+1)\right) - \left(1^2 + 2^2 + \cdots (r-1)^2\right) \\ &= \left(\frac{1}{6}N(N+1)(2N+1)\right) - \\ &\qquad \left(\frac{1}{6}(r-1)((r-1)+1)(2(r-1)+1)\right) \\ &= \frac{1}{6}N(N+1)(2N+1) - \frac{1}{6}(r-1)r(2r-1). \end{aligned}$$

Since we want this sum to be a square, say M^2, we need solutions of:

$$M^2 = \frac{1}{6}N(N+1)(2N+1) - \frac{1}{6}(r-1)r(2r-1),$$

or equivalently:

$$M^2 = \frac{1}{6}N^3 + \frac{1}{2}N^2 + \frac{N}{6} - \left(\frac{1}{3}r^3 - \frac{1}{2}r^2 + \frac{r}{6}\right),$$

or equivalently:

$$81M^2 = 27N^3 + \frac{81}{2}N^2 + \frac{27}{2}N - 81\left(\frac{1}{3}r^3 - \frac{1}{2}r^2 + \frac{r}{6}\right).$$

Next, letting $m = 9M$ and $n = 3N$, we obtain:

$$m^2 = n^3 + \frac{9}{2}n^2 + \frac{9}{2}n - 81\left(\frac{1}{3}r^3 - \frac{1}{2}r^2 + \frac{r}{6}\right),$$

or equivalently:

$$64m^2 = 64n^3 + 9 \cdot 32n^2 + 9 \cdot 32n - 5184\left(\frac{1}{3}r^3 - \frac{1}{2}r^2 + \frac{r}{6}\right),$$

or equivalently:

$$64m^2 = 64n^3 + 9 \cdot 32n^2 + 9 \cdot 32n - (1728r^3 - 2592r^2 + 864r).$$

Next, letting $y = 8m$ and $x = 4n$, we obtain:

$$y^2 = x^3 + 18x^2 + 72x - (1728r^3 - 2592r^2 + 864r),$$

which is an elliptic curve in Weierstrass Normal Form. So, we are interested in those integer points of $y^2 = x^3 + 18x^2 + 72x - (1728r^3 - 2592r^2 + 864r)$ that correspond to nontrivial solutions (N, M) of $r^2 + (r+1)^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$. There is a problem in that in order to calculate the integer points of $y^2 = x^3 + 18x^2 + 72x - (1728r^3 - 2592r^2 + 864r)$, a computer must compute generators for the Mordell-Weil group of the elliptic curve. As of 2010, there is no effective method known for doing this. Thus, for some values of r we obtain a complete list of integer points while for other values of r we obtain only a partial list of integer points. In any case, we narrow down our list of integer points of $y^2 = x^3 + 18x^2 + 72x - (1728r^3 - 2592r^2 + 864r)$ that may correspond to nontrivial solutions (N, M) of $r^2 + (r+1)^2 + \cdots + N^2 = M^2$ with $N, M \in \mathbb{Z}^+$ by considering the substitutions that we have made. Specifically, since $x = 4n = 12N$, we need 12 to divide x. Similarly, since $y = 8m = 72M$, we need 72 to divide y. So, we narrow the list accordingly. From here, we calculate corresponding values of N. Then, since N must be strictly greater than r, we discard those values of N, if any, that are less than or equal to r. This may leave one or more acceptable values for N, from which we calculate corresponding values of M using our original equation.

For example, the following computer program instructs the computer software Magma to do these computations for r from 1 to 100:

```
for r:=1 to 100 do print " "; "r=",r;
    E:=EllipticCurve([0,18,0,72,-1728*r^3+2592*r^2-864*r]);
    S:=IntegralPoints(E);
    for k:=1 to #S do
        pT:=S[k];
        x:=Z! pT[1];
        y:=Z! pT[2];
        if x mod 12 eq 0 and y mod 72 eq 0 and x gt 12*r then;
            N:=x/12;
```

```
            M:=Abs(y/72);
            print "[", N, ",", M, "]";
        end if;
    end for;
end for;
```

When this program is executed, there are some values of r for which one or more nontrivial solutions (N, M) with $N, M \in \mathbb{Z}^+$ exist. These are displayed in Table #1:

Table #1

r	N	M
1	24	70
3	4	5
	580	8075
	963	17267
7	29	92
	39	143
	56	245
	190	1518
	2215	60207
9	32	106
	552057	236818619
11	22908	2001863
13	108	652
15	111	679
	326	3406

17	39	138
	5345	225643
18	28	77
⋆ 20	21	29
	43	158
	308	3128
	1221044	778998480
21	116	724
22	80	413
	6910	331668
25	48	182
	50	195
	73	357
	578	8033
	624	9010
	3625	126035
	21624	1835940
27	59	253
	364	4017
28	77	385
	123	788
30	198	1612
32	609	8687
	4087	150878
	61281	8758575

		148856	33158210
	38	48	143
		96	531
		349	3770
		686	10384
		11918	751228
⋆	44	67	274
		93	495
	50	171	1281
		15674	1133000
	52	147	1012
		389	4433
⋆	55	3533	121268
	58	2132	56855
⋆	60	92	440
		3238	106403
	64	305	3069
	65	282	2725
		928	16332
⋆	67	116	655
		8516	453765
	73	194	1525
		22873	1997227
⋆	74	36554	4035066
	76	99	430

⋆	83	276	2619
		26003	2420957
	87	136	795
	91	332	3465

⋆ Other nontrivial solutions (N, M) with $N, M \in \mathbb{Z}^+$ may exist. This is because Magma is unsure of how successful it was in calculating the generators for the entire Mordell-Weil group.

Notice that when $r = 1$, the only solution is $(24, 70)$. This correlates with our analysis of the cannonball problem. There are other values of r in the range $1 \le r \le 100$ for which no nontrivial solutions (N, M) with $N, M \in \mathbb{Z}^+$ exist. These are when r is:

2, 4, 5, 6, 8, 10, 12, 14, 19, 23, 26, 34, 35, 36, 37, 39, 42, 43, 46,
54, 59, 61, 62, 63, 66, 69, 70, 72, 80, 82, 85, 90, 92, 97, and 98.

The reason that no nontrivial solutions (N, M) exist for these values of r may be that the rank of the corresponding elliptic curve is zero, or it may be that there are no integer points (x, y) on the corresponding elliptic curve with the properties that $12 \mid x$, $72 \mid y$, and $x > 12r$.

For all remaining values of r up to 100, Magma is unable to find any nontrivial solutions (N, M) with $N, M \in \mathbb{Z}^+$, but cannot confirm that none exist. These are when r is:

16, 24, 29, 31, 33, 40, 41, 45, 47, 48, 49, 51, 53, 56, 57, 68, 71, 75,
77, 78, 79, 81, 84, 86, 88, 89, 93, 94, 95, 96, 99, and 100.

The reason for this uncertainty is that Magma is not easily able to calculate any of the generators of these Mordell-Weil groups. However, using the features

RankBounds and TwoDescent, Magma is able to calculate the ranks of each of these corresponding Elliptic Curves. These are displayed in Table #2:

Table #2

r	Rank	
16	2	
24	2	
29	1	
31	2	
33	2	$\star\star$
40	1	
41	2	
45	2	
47	2	$\star\star$
48	3	
49	1	
51	1	
53	1	
56	2	
57	1	
68	2	
71	2	
75	2	$\star\star$
77	2	
78	2	
79	2	$\star\star$

81	2	
84	2	
86	2	
88	2	
89	2	$\star\star$
93	1	
94	2	
95	2	
96	2	
99	1	
100	2	

$\star\star$ Follows from the Selmer Conjecture, since the upper and lower bounds of the rank differ by an odd number.

This concludes our analysis of Question #1.

CHAPTER TWO

Mathematically, Question #2 can be written as follows: Given a positive integer r with $1 \leq r \leq 100$, find all solutions (u, v), if any, of $u^2 + (u+1)^2 + (u+2)^2 + \cdots + (u+r-1)^2 = v^2$ with $u, v \in \mathbb{Z}^+$. Let's begin by establishing some theory that will help us with Question #2. Suppose we are given an equation of the form $aX^2 - bY^2 = cZ^2$ with $a, b, c \in \mathbb{Z}$ and a, b, c pairwise coprime, and we are asked whether or not there is a nontrivial solution (d, e, f) with $d, e, f \in \mathbb{Z}$. Clearly, there is a non-trivial solution in integers if and only if there is a non-trivial solution in rationals. The Hasse-Minkowski Theorem tells us that such a quadratic equation over $\mathbb{Q}$ has a non-trivial solution (d, e, f) with $d, e, f \in \mathbb{Q}$ if and only if the equation has a solution in $\mathbb{R}$ and the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{p^r}$ has a nontrivial solution for all primes p and exponents $r \geq 1$. From here, Hensel's Lemma tells us when each of the congruences $aX^2 - bY^2 \equiv cZ^2 \pmod{p^r}$ has a nontrivial solution. Specifically, Hensel's Lemma tells us that for all $r \geq 1$ the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{2^r}$ has a nontrivial solution if and only if the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{8}$ has a nontrivial solution. Similarly, Hensel's Lemma tells us that for all $p \geq 3$ and for all $r \geq 1$, the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{p^r}$ has a nontrivial solution if and only if the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ has a nontrivial solution. In any case, we have the following Theorem:

Theorem 2.1: The congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ always has a nontrivial solution when p does not divide abc.

Proof. Consider that there are precisely $\frac{p-1}{2}$ distinct nonzero squares mod p.

Hence, the set $\{aX^2 - b : x \in \mathbb{Z} \mod p\}$ contains precisely $\frac{p-1}{2} + 1 = \frac{p+1}{2}$ distinct elements mod p. Similarly, the set $\{cZ^2 : Z \in \mathbb{Z} \mod p\}$ contains precisely $\frac{p+1}{2}$ distinct elements mod p. Thus, by the Pigeonhole Principle, the 2 sets share a common element mod p. Hence, there is a solution to the congruence $aX^2 - b \equiv cZ^2 \pmod{p}$. Finally, we can use that solution of $aX^2 - b \equiv cZ^2 \pmod{p}$ to create a solution of $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ by setting $Y = 1$. Notice that since $Y = 1$, this solution will be nontrivial. So, the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ always has a nontrivial solution when p does not divide abc. □

When a, b in the equation $aX^2 - bY^2 = cZ^2$ are strictly greater than zero, we see that $X = \sqrt{b}$, $Y = \sqrt{a}$, $Z = 0$ is always a real solution of the equation $aX^2 - bY^2 = cZ^2$. Thus, we shall always assume a real solution to the equation $aX^2 - bY^2 = cZ^2$. Hence, we can apply these ideas in the following way:

(1) An equation of the form $aX^2 - bY^2 = cZ^2$ with $a, b, c \in \mathbb{Z}$ and a, b, c pairwise coprime is globally solvable if and only if the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{8}$ has a nontrivial solution and the congruence(s) $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ for all p dividing abc each have a nontrivial solution.

We will be referring to (1) shortly, but for now, let's start working on Question #2.

Question #2 asks for positive integers u and v such that the ordered pair (u, v) satisfies the equation:

$$u^2 + (u+1)^2 + \cdots + (u+r-1)^2 = v^2,$$

where r is a fixed positive integer. Let's refer to this equation as "the u, v equation of r". Thus, each r has a unique u, v equation. If there exist positive integers u and v such that the ordered pair (u, v) satisfies the u, v equation, then we say that (u, v) is a solution of the equation. Otherwise, we say the u, v equation has no solutions.

At this point, let's show that each u, v equation can be written in the form $x^2 - by^2 = c$ where b and c are fixed integers. We have:

$$u^2 + (u+1)^2 + \cdots + (u+r-1)^2 = v^2,$$

or equivalently,

$$\frac{1}{6}(u+r-1)(u+r)(2u+2r-1) - \frac{1}{6}(u-1)u(2u-1) = v^2,$$

or equivalently,

$$ru^2 + (r^2-r)u + \left(\frac{1}{3}r^3 - \frac{r^2}{2} + \frac{r}{6}\right) = v^2,$$

or equivalently,

$$4rv^2 = (2ru + r^2 - r)^2 + \frac{r^4}{3} - \frac{r^2}{3},$$

or equivalently,

$$(2ru + r^2 - r)^2 - 4rv^2 = -\frac{1}{3}r^2(r^2-1),$$

or equivalently,

$$r(2u + r - 1)^2 - 4v^2 = -\frac{1}{3}r(r^2-1),$$

or equivalently,

$$4v^2 - r(2u + r - 1)^2 = \frac{1}{3}r(r^2 - 1), \qquad (*)$$

or equivalently,

$$(2v)^2 - r(2u + r - 1)^2 = \frac{1}{3}r(r^2 - 1).$$

Next, letting $x = 2v$ and $y = 2u + r - 1$, we obtain:

$$x^2 - ry^2 = \frac{1}{3}r(r^2 - 1).$$

Notice that this is an equation of the desired form (where $b = r$ and $c = \frac{1}{3}r(r^2 - 1)$). An equation of this form is known as a Pell equation. Therefore, each u, v equation has a unique Pell equation to which it corresponds. We will find $(*)$ to be a most useful formula, and will refer to $(*)$ as being the Pell equation of r corresponding to the u, v equation of r.

Since solutions of the u, v equation of r can exist only if the corresponding Pell equation for r is globally solvable, it would be helpful to have a method of determining which of these corresponding Pell equations are globally solvable and which are not.

During the remainder of this chapter, it will be important to determine, for each $1 \le r \le 100$, whether or not the Pell equation of r is globally solvable. Although not actually used in this thesis, there are two Theorems that can be used whenever they apply to show that the Pell equation of a specific r is globally unsolvable:

Theorem 2.2: If there exists a prime number p such that $p \,||\, r$ with $p \equiv \pm 5 \pmod{12}$, then the Pell equation of r is globally unsolvable.

Proof. $(*)$ has corresponding congruence $4v^2 \equiv 0 \pmod{p}$. So, $p \mid v$. Next, put $v = p\mathcal{V}$. Hence $4p^2\mathcal{V}^2 - r(2u + r - 1)^2 = \frac{1}{3}r(r^2 - 1)$. Since $p \mid r$, put $r = pR$. Thus, $4p^2V^2 - pR(2u + pR - 1)^2 = \frac{1}{3}pR(p^2R^2 - 1)$. Dividing by p, we obtain $4p\mathcal{V}^2 - R(2u + pR - 1)^2 = \frac{1}{3}R(p^2R^2 - 1)$. Next, $R(2u - 1)^2 \equiv \frac{1}{3}R \pmod{p}$. But $p \,||\, r$, so $(p, R) = 1$. Hence, $(2u-1)^2 \equiv \frac{1}{3} \pmod{p}$. Multiplying by 9, we obtain $(3(2u - 1))^2 \equiv 3 \pmod{p}$. But, since $p \equiv \pm 5 \pmod{12}$, $\left(\frac{3}{p}\right) = -1$. So, the left-hand side and the right-hand side are both congruent to $0 \pmod{p}$. So, $p \mid 3$, a contradiction. Therefore, the Pell equation of r is locally unsolvable at p, which implies that the Pell equation of r is globally unsolvable. □

Theorem 2.3: If there exists a prime number p such that $p \,||\, r+1$ with $p \equiv 3 \pmod{4}$, then the Pell equation of r is globally unsolvable.

Proof. $(*)$ has corresponding congruence $4v^2 \equiv rw^2 \pmod{p}$ where $w = 2u + r - 1$. But $4v^2 = (2v)^2$, so we have $(2v)^2 \equiv rw^2 \pmod{p}$. Next, since $p \mid r + 1, r = px - 1$, so $r \equiv -1 \pmod{p}$. Hence, $\left(\frac{r}{p}\right) = \left(\frac{-1}{p}\right) = -1$ since $p \equiv 3 \pmod{4}$. So, the left-hand side and the right-hand side are both congruent to $0 \pmod{p}$. Hence, $v \equiv w \equiv 0 \pmod{p}$. Thus, $p \mid v$ and $p \mid w$. Hence, $p^2 \mid 4v^2 - rw^2$. So $p^2 \mid \frac{1}{3}r(r^2 - 1)$, a contradiction. Therefore, the Pell equation of r is locally unsolvable at p, which implies that the Pell equation of r is globally unsolvable. □

Fortunately, the theory that we have developed provides a method for determining which Pell equations are globally solvable and which are not in the cases when we are able to write the Pell equation in the form $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$ and a, b, c pairwise coprime. This follows from the fact that we can homogenize such an equation by setting $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ yielding $aX^2 - bY^2 = cZ^2$. Notice that this is an equation to which (1) can be applied.

Since the Pell equation is globally solvable if and only if its homogenized form is globally solvable, we have the following result:

(2) Let $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$, $a, b > 0$, and a, b, c pairwise coprime. Then, if $aX^2 - bY^2 = cZ^2$ is the homogenized form of the equation, $ax^2 - by^2 = c$ is globally solvable if and only if the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{8}$ has a nontrivial solution and the congruence(s) $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ for all p dividing abc each have a nontrivial solution.

Before going any further, let's address those values of r which have corresponding Pell equations that cannot be written in the form $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$, $a, b > 0$, and a, b, c pairwise coprime. For r in the range $1 \leq r \leq 100$, this happens precisely when r is one of the following nine values:

$$9, 18, 36, 45, 63, 72, 81, 90, \text{ and } 99.$$

To help us with this, we will find the following theorem most useful:

Theorem 2.4: Given $r \in \mathbb{Z}^+$ with $9 \mid r$ and $27 \nmid r$, the corresponding u, v equation of r has no integral solutions.

Proof. It suffices to show that the corresponding Pell equation of r, $(*)$, has no integral solutions. Since $3 \mid r$, we see that 3 divides the left-hand side of $(*)$. Thus, $3 \mid 4v^2$. Hence, $3 \mid v^2$ so that $3 \mid v$. Thus, we may write $v = 3\mathcal{V}$ for some $\mathcal{V} \in \mathbb{Z}$. Hence, $(*)$ becomes $4 \cdot 9\mathcal{V}^2 - r(2u + r - 1)^2 = \frac{1}{3}r(r^2 - 1)$. Since 9 divides $4 \cdot 9\mathcal{V}^2 - r(2u + r - 1)^2$ we see that $9 \mid \frac{1}{3}r(r^2 - 1)$. Hence, $\frac{1}{3}r(r^2 - 1) \equiv 0 \pmod{9}$. Thus, $\frac{r}{9}(r^2 - 1) \equiv 0 \pmod{3}$. But, since r is a multiple of 3, we see that $3 \nmid (r+1)$ and $3 \nmid (r-1)$. Since $r^2 - 1 = (r+1)(r-1)$, it follows that $3 \nmid (r^2-1)$. Hence, $\frac{r}{9} \equiv 0 \pmod 3$. Thus, $r \equiv 0 \pmod{27}$. But then $27 \mid r$, a contradiction. Therefore, the corresponding u, v equation of r has no integral solutions. $\square$

Notice that the conditions of the previous theorem are met when r=9, 18, 36, 45, 63, 72, 90 and 99. Therefore, when r takes on any of these eight values, the corresponding u, v equation of r has no integral solutions.

Notice also that the previous Theorem cannot be extended to values of r divisible by 27. As an example, consider the case when $r = 297$. By $(*)$, the corresponding Pell equation of $r = 297$ is:

$$4v^2 - 297(2u + 297 - 1)^2 = \frac{1}{3} \cdot 297(297^2 - 1)$$
$$\Rightarrow v^2 - 297(u + 148)^2 = 2183148$$

which has (106, 4620) as a solution. Since this is also a solution of the corresponding u, v equation of $r = 297$, we see the corresponding u, v equation of $r = 297$ has an integral solution. Thus, since $27 \mid 81$, the case when $r = 81$ needs the following individual attention:

When $r = 81$, the corresponding Pell equation is:

$$4v^2 - 81(2u + 81 - 1)^2 = \frac{1}{3} \cdot 81(81^2 - 1)$$
$$\Rightarrow v^2 - 81(u + 40)^2 = 44280$$
$$\Rightarrow 3 \mid v \text{ so, put } v = 3\mathcal{V}$$
$$\Rightarrow \mathcal{V}^2 - 9(u + 40)^2 = 4920$$
$$\Rightarrow 3 \mid \mathcal{V} \text{ so, put } \mathcal{V} = 3\mathcal{W}$$

$\Rightarrow 3\mathcal{W}^2 - 3(u+40)^2 = 1640$

$\Rightarrow$3 divides the left-hand side but 3 does not divide the right-hand side

$\Rightarrow$No solution since $u, \mathcal{W} \in \mathbb{Z}^+$

So, we see that when $r = 81$, the corresponding Pell equation for r is globally unsolvable. Since the corresponding Pell equation for r must be globally solvable in order for the corresponding u, v equation for r to have solutions, this implies that the corresponding u, v equation for $r = 81$ has no solutions. With these nine cases out of the way, we can now focus our attention on the remaining 91 values of r; all of which have Pell equations that can be written in the form $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$, $a, b > 0$, and a, b, c pairwise coprime.

By statement (2), we can show that such an equation is globally unsolvable if we produce a congruence of the form $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$, with p dividing abc, that has no nontrivial solutions. Hence, it would be helpful to have a way of determining when such a congruence has no nontrivial solution. The following Theorem does just that:

Theorem 2.5: Given a congruence of the form $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ with $a, b, c \in \mathbb{Z}$ and a, b, c pairwise coprime, then any of the following conditions imply that $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ has no nontrivial solution:

(*i*) p is an odd prime exactly dividing a, and $\left(\frac{-bc}{p}\right) = -1$.

(*ii*) p is an odd prime exactly dividing b, and $\left(\frac{ac}{p}\right) = -1$.

(*iii*) p is an odd prime exactly dividing c, and $\left(\frac{ab}{p}\right) = -1$.

Proof. Suppose that p is an odd prime exactly dividing a, $\left(\frac{-bc}{p}\right) = -1$, and (d, e, f) is a nontrivial solution. It is no loss of generality to assume d, e, f

coprime. Then $a \cdot d^2 - b \cdot e^2 \equiv c \cdot f^2 \pmod{p}$. Since p divides a, $-b \cdot e^2 \equiv c \cdot f^2 \pmod{p}$. Thus, $(be)^2 \equiv -bcf^2 \pmod{p}$. Since $\left(\frac{-bc}{p}\right) = -1$, this forces $be \equiv 0 \equiv f \pmod{p}$. Since $(a,b) = 1$, we have $e \equiv 0 \equiv f \pmod{p}$. Hence, $a \cdot d^2 = b \cdot e^2 + c \cdot f^2 \equiv 0 \pmod{p^2}$. Thus, $ad^2 \equiv 0 \pmod{p^2}$. Since p exactly divides a, $d^2 \equiv 0 \pmod{p}$, so that $d \equiv 0 \pmod{p}$, contradicting d, e, f coprime. Thus, condition (i) implies that $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ has no nontrivial solution. Similarly, condition (ii) and condition (iii) each imply that $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ has no nontrivial solution. □

Of the remaining 91 values of r which have Pell equations that can be written in the form $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$, $a, b > 0$, and a, b, c pairwise coprime, precisely 69 of those equations can be shown to be globally unsolvable by statement (2) and Theorem 2.5. These are listed in the following table, which gives the value of r, the corresponding Pell equation written in the form $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$, $a, b > 0$, and a, b, c pairwise coprime, the simplified form of the congruence $aX^2 - bY^2 \equiv cZ^2 \pmod{p}$ that has no nontrivial solution, and whether p exactly divides a, b, or c.

Table #3

r	Pell equation	Congruence	$p \mid\mid$
3	$x^2 - 3y^2 = 2$	$X^2 \equiv 2Z^2 \pmod{3}$	b
5	$x^2 - 5y^2 = -2$	$X^2 \equiv -2Z^2 \pmod{5}$	b
6	$3x^2 - 2y^2 = -35$	$3X^2 \equiv 2Y^2 \pmod{7}$	c
7	$x^2 - 7y^2 = -4$	$X^2 \equiv -4Z^2 \pmod{7}$	b
8	$x^2 - 2y^2 = -21$	$X^2 \equiv 2Y^2 \pmod{3}$	c
10	$x^2 - 10y^2 = -33$	$X^2 \equiv 10Y^2 \pmod{11}$	c
12	$x^2 - 3y^2 = 143$	$X^2 \equiv 3Y^2 \pmod{13}$	c

13	$x^2 - 13y^2 = -14$	$X^2 \equiv 13Y^2 \pmod{7}$	c
14	$x^2 - 14y^2 = -65$	$X^2 \equiv -65Z^2 \pmod{7}$	b
15	$3x^2 - 5y^2 = -56$	$3X^2 \equiv -56Z^2 \pmod{5}$	b
17	$x^2 - 17y^2 = -24$	$X^2 \equiv -24Z^2 \pmod{17}$	b
19	$x^2 - 19y^2 = -30$	$X^2 \equiv -30Z^2 \pmod{19}$	b
20	$x^2 - 5y^2 = -133$	$X^2 \equiv 5Y^2 \pmod{7}$	c
21	$7x^2 - 3y^2 = 110$	$7X^2 \equiv 3Y^2 \pmod{11}$	c
22	$x^2 - 22y^2 = -161$	$X^2 \equiv 22Y^2 \pmod{23}$	c
27	$x^2 - 3y^2 = 182$	$X^2 \equiv 3Y^2 \pmod{7}$	c
28	$x^2 - 7y^2 = -261$	$X^2 \equiv -261Z^2 \pmod{7}$	b
29	$x^2 - 29y^2 = -70$	$X^2 \equiv -70Z^2 \pmod{29}$	b
30	$10x^2 - 3y^2 = 899$	$10X^2 \equiv 3Y^2 \pmod{31}$	c
31	$x^2 - 31y^2 = -80$	$X^2 \equiv -80Z^2 \pmod{31}$	b
32	$x^2 - 2y^2 = -341$	$X^2 \equiv 2Y^2 \pmod{11}$	c
34	$x^2 - 34y^2 = -385$	$X^2 \equiv -385Z^2 \pmod{17}$	b
35	$x^2 - 35y^2 = -102$	$X^2 \equiv -102Z^2 \pmod{7}$	b
37	$x^2 - 37y^2 = -114$	$X^2 \equiv 37Y^2 \pmod{19}$	c
38	$x^2 - 38y^2 = -481$	$X^2 \equiv -481Z^2 \pmod{19}$	b
39	$13x^2 - 3y^2 = 380$	$13X^2 \equiv 380Z^2 \pmod{3}$	b
40	$x^2 - 10y^2 = -533$	$X^2 \equiv -533Z^2 \pmod{5}$	b
41	$x^2 - 41y^2 = -140$	$X^2 \equiv -140Z^2 \pmod{41}$	b
42	$14x^2 - 3y^2 = 1763$	$14X^2 \equiv 3Y^2 \pmod{43}$	c
43	$x^2 - 43y^2 = -154$	$X^2 \equiv -154Z^2 \pmod{43}$	b
44	$x^2 - 11y^2 = -645$	$X^2 \equiv 11Y^2 \pmod{3}$	c
46	$x^2 - 46y^2 = -705$	$X^2 \equiv 46Y^2 \pmod{47}$	c

48	$x^2 - 3y^2 = 2303$	$X^2 \equiv 2303Z^2 \pmod{3}$	b
51	$17x^2 - 3y^2 = 650$	$3Y^2 \equiv -650Z^2 \pmod{17}$	a
53	$x^2 - 53y^2 = -234$	$X^2 \equiv -234Z^2 \pmod{53}$	b
54	$2x^2 - 3y^2 = 2915$	$2X^2 \equiv 3Y^2 \pmod{11}$	c
55	$x^2 - 55y^2 = -252$	$X^2 \equiv 55Y^2 \pmod{7}$	c
56	$x^2 - 14y^2 = -1045$	$X^2 \equiv 14Y^2 \pmod{19}$	c
57	$19x^2 - 3y^2 = 812$	$3Y^2 \equiv -812Z^2 \pmod{19}$	a
58	$x^2 - 58y^2 = -1121$	$X^2 \equiv 58Y^2 \pmod{59}$	c
60	$5x^2 - 3y^2 = 3599$	$3Y^2 \equiv -3599Z^2 \pmod{5}$	a
61	$x^2 - 61y^2 = -310$	$X^2 \equiv 61Y^2 \pmod{31}$	c
62	$x^2 - 62y^2 = -1281$	$X^2 \equiv 62Y^2 \pmod{7}$	c
65	$x^2 - 65y^2 = -352$	$X^2 \equiv 65Y^2 \pmod{11}$	c
66	$22x^2 - 3y^2 = 4355$	$22X^2 \equiv 3Y^2 \pmod{67}$	c
67	$x^2 - 67y^2 = -374$	$X^2 \equiv -374Z^2 \pmod{67}$	b
68	$x^2 - 17y^2 = -1541$	$X^2 \equiv 17Y^2 \pmod{23}$	c
69	$23x^2 - 3y^2 = 1190$	$23X^2 \equiv 3Y^2 \pmod{7}$	c
70	$x^2 - 14y^2 = -1633$	$X^2 \equiv 14Y^2 \pmod{71}$	c
71	$x^2 - 71y^2 = -420$	$X^2 \equiv 71Y^2 \pmod{3}$	c
75	$x^2 - 3y^2 = 1406$	$X^2 \equiv 3Y^2 \pmod{19}$	c
76	$x^2 - 19y^2 = -1925$	$X^2 \equiv -1925Z^2 \pmod{19}$	b
77	$x^2 - 77y^2 = -494$	$X^2 \equiv -494Z^2 \pmod{7}$	b
78	$26x^2 - 3y^2 = 6083$	$26X^2 \equiv 3Y^2 \pmod{79}$	c
79	$x^2 - 79y^2 = -520$	$X^2 \equiv -520Z^2 \pmod{79}$	b
80	$x^2 - 5y^2 = -2133$	$X^2 \equiv -2133Z^2 \pmod{5}$	b
82	$x^2 - 82y^2 = -2241$	$X^2 \equiv 82Y^2 \pmod{747}$	c

83	$x^2 - 83y^2 = -574$	$X^2 \equiv 83Y^2 \pmod{7}$	c
84	$7x^2 - 3y^2 = 7055$	$3Y^2 \equiv -7055Z^2 \pmod{7}$	a
85	$x^2 - 85y^2 = -602$	$X^2 \equiv 85Y^2 \pmod{43}$	c
86	$x^2 - 86y^2 = -2465$	$X^2 \equiv -2465Z^2 \pmod{43}$	b
87	$29x^2 - 3y^2 = 1892$	$3Y^2 \equiv -1892Z^2 \pmod{29}$	a
89	$x - 89y^2 = -660$	$X^2 \equiv -660Z^2 \pmod{89}$	b
91	$x^2 - 91y^2 = -690$	$X^2 \equiv 91Y^2 \pmod{23}$	c
92	$x^2 - 23y^2 = -2821$	$X^2 \equiv 23Y^2 \pmod{31}$	c
93	$31x^2 - 3y^2 = 2162$	$31X^2 \equiv 3Y^2 \pmod{47}$	c
94	$x^2 - 94y^2 = -2945$	$X^2 \equiv 94Y^2 \pmod{19}$	c
95	$x^2 - 95y^2 = -752$	$X^2 \equiv -752Z^2 \pmod{19}$	b
98	$x^2 - 2y^2 = -3201$	$X^2 \equiv 2Y^2 \pmod{11}$	c

So, by statement (2) and Theorem 2.5, each of these 69 values of r has a corresponding Pell equation which is globally unsolvable. Hence, the u, v equation corresponding to each of these 69 values of r has no solution.

We have narrowed our study of Question #2 down to just 100-9-69=22 remaining values of r. Some of these values of r have corresponding equations of the form $x^2 - y^2 = c$. In other words, $a = b = 1$. This happens precisely when r is one of the following 7 values:

$$1, 4, 16, 25, 49, 64, \text{ and } 100.$$

Let's analyze these values of r next. To help us with this, we will find the following theorem most useful:

Theorem 2.6: If r is of the form $4t^2$ for $t \in \mathbb{Z}^+$, then the corresponding equation of r is globally unsolvable.

Proof. First, let's show that if t is odd, then the corresponding equation of r is globally unsolvable. From $(*)$, we see that the corresponding equation of r is $4v^2-4t^2(2u+4t^2-1)^2 = \frac{1}{3}\cdot 4t^2((4t^2)^2-1)$, or equivalently, $v^2-t^2(2u+4t^2-1)^2 = \frac{1}{3}t^2(16t^4-1)$. Thus, any solution of the corresponding equation of r would also be a solution of the congruence:

$$v^2 - t^2(2u + 4t^2 - 1)^2 \equiv \frac{1}{3}t^2(16t^4 - 1) \pmod 8.$$

But since $(2u+4t^2-1)$ is odd, $(2u+4t^2-1)^2 \equiv 1 \pmod 8$. Also $16t^4-1 \equiv -1 \pmod 8$. Hence, the v in any solution of the corresponding equation of r would be a solution of the congruence $v^2 - t^2 \equiv \frac{-1}{3}t^2 \pmod 8$, or equivalently, of the congruence $v^2 \equiv \frac{2}{3}t^2 \pmod 8$. But, $\frac{2}{3}t^2 \equiv 6t^2 \pmod 8$. Thus, the v in any solution of the corresponding equation of r would be a solution of the congruence $v^2 \equiv 6t^2 \pmod 8$. From this congruence, we see that both t and v must be even. This shows that if t is odd, then the corresponding equation of r is globally unsolvable. So, assume that t (and hence v) are even. So, put $v = 2^a\mathcal{V}$ where $\mathcal{V}$ is odd and put $t = 2^b\mathcal{T}$ where $\mathcal{T}$ is odd. Then, let's show that $a \geq b$. By $(*)$, we see that the corresponding equation of r is $4(2^a\mathcal{V})^2 - 2^{2b+2}\mathcal{T}^2(2u + r - 1)^2 = \frac{1}{3}\cdot 2^{2b+2}\mathcal{T}^2(r^2-1)$, or equivalently, $2^{2a+2}\mathcal{V}^2 - 2^{2b+2}\mathcal{T}^2(2u + r - 1)^2 = \frac{1}{3}\cdot 2^{2b+2}\mathcal{T}^2(r^2-1)$. Next, since $2^{2b+2} \mid \frac{1}{3}\cdot 2^{2b+2}\mathcal{T}^2(r^2-1)$ and $2^{2b+2} \mid 2^{2b+2}\mathcal{T}^2(2u+r-1)^2$, we see that $2^{2b+2} \mid 2^{2a+1}\mathcal{V}^2$. Since $\mathcal{V}^2$ is odd, we see that $2^{2b+2} \mid 2^{2a+2}$. Hence, $2b+2 \leq 2a+2$. Thus, $a \geq b$ as desired. To complete the proof, we consider the only 2 possible cases separately; Case 1 will be when t is even and $a = b$, Case 2 will be when t is even and $a > b$.

Case 1: Since t is even, we have $t = 2^b\mathcal{T}$ with $\mathcal{T}$ odd, so that $r = 2^{2b+2}\mathcal{T}^2$.

Similarly, since v is even, we have $v = 2^a\mathcal{V}$ with $\mathcal{V}$ odd. By $(*)$, we see that the corresponding equation of r is $4(2^a\mathcal{V})^2 - 2^{2b+2}\mathcal{T}^2(2u+r-1)^2 = \frac{1}{3}\cdot 2^{2b+2}\mathcal{T}^2(r^2-1)$. But since $a = b$, this is equivalent to $2^{2a+2}\mathcal{V}^2 - 2^{2a+2}\mathcal{T}^2(2u+r-1)^2 = \frac{1}{3}\cdot 2^{2a+2}\mathcal{T}^2(r^2-1)$, or equivalently, $(\mathcal{V}+\mathcal{T}(2u+r-1))(\mathcal{V}-\mathcal{T}(2u+r-1)) = \frac{1}{3}\mathcal{T}^2(2^{2a+2}\mathcal{T}^2+1)(2^{2a+2}\mathcal{T}^2-1)$. But since $\mathcal{T},\mathcal{V}$ and $(2u+r-1)$ are all odd, the left-hand side of this equation is even while the right-hand side is odd, a contradiction. Therefore, when $a = b$, the corresponding equation of r is globally unsolvable.

Case 2: Since t is even, we have $t = 2^b\mathcal{T}$ with $\mathcal{T}$ odd, so that $r = 2^{2b+2}\mathcal{T}^2$. Similarly, since v is even, we have $v = 2^a\mathcal{V}$ with $\mathcal{V}$ odd. By $(*)$, we see that the corresponding equation of r is $4(2^a\mathcal{V})^2 - 2^{2b+2}\mathcal{T}^2(2u+r-1)^2 = \frac{1}{3}\cdot 2^{2b+2}\mathcal{T}^2(r^2-1)$. But, since $a > b$, $2a+2 > 2b+2$. Thus, by dividing by 2^{2b+2}, we obtain the equivalent equation $2^{2(a-b)}\mathcal{V}^2 - \mathcal{T}^2(2u+r-1)^2 = \frac{1}{3}\mathcal{T}^2(r^2-1)$. Thus, any solution of the corresponding equation of r would also be a solution of the congruence $2^{2(a-b)}\mathcal{V}^2 - \mathcal{T}^2(2u+r-1)^2 \equiv \frac{1}{3}\mathcal{T}^2(r^2-1) \pmod 8$. But, since $\mathcal{T},\mathcal{V}$ and $(2u+r-1)$ are odd, $\mathcal{T}^2 \equiv \mathcal{V}^2 \equiv (2u+r-1)^2 \equiv 1 \pmod 8$. In addition, $(r^2-1) \equiv -1 \pmod 8$. Hence, we obtain the equivalent equation $2^{2(a-b)} - 1 \equiv \frac{-1}{3} \pmod 8$, or equivalently, $3\cdot 2^{2(a-b)-1} \equiv 1 \pmod 4$. Since $a > b$, we have a contradiction. Therefore, when $a > b$, the corresponding equation of r is globally unsolvable. □

Notice that the conditions of the previous theorem are met when $r = 4, 16, 64$ and 100. Therefore, when r takes on any of these four values, the corresponding u, v equation of r has no integral solutions. Next, let's analyze the cases when $r = 1, 25$, and 49:

$r = 1$: The corresponding equation is:

$$4v^2 - 1(2u + 1 - 1)^2 = \frac{1}{3} \cdot 1(1^2 - 1)$$
$$\Rightarrow v^2 - u^2 = 0$$
$$\Rightarrow u = v \text{ since } u, v \text{ are positive}$$

So, this corresponding equation has infinitely many solutions; all of which are trivial of the form (n, n) for $n \in \mathbb{Z}^+$. Thus, the corresponding u, v equation has no nontrivial solutions.

$r = 25$: The corresponding equation is:

$$4v^2 - 25(2u + 25 - 1)^2 = \frac{1}{3} \cdot 25(25^2 - 1)$$
$$\Rightarrow v^2 - 25(u + 12)^2 = 1300$$
$$\Rightarrow 5 \mid v \text{ so, put } v = 5\mathcal{V}$$
$$\Rightarrow \mathcal{V}^2 - (u + 12)^2 = 52$$
$$\Rightarrow (\mathcal{V} + (u + 12))(\mathcal{V} - (u + 12)) = 52$$

Since the only factorizations of 52 are:

1(52), 2(26), 4(13), (-1)(-52), (-2)(-26), and (-4)(-13),

and $\mathcal{V} + (u + 12)$ cannot be negative, the only possibilities are:

$$\left.\begin{array}{l} \mathcal{V} + (u + 12) = 1 \\ \mathcal{V} - (u + 12) = 52 \end{array}\right\} \Longrightarrow \quad (u, v) = (\tfrac{-75}{2}, \tfrac{265}{2})$$

$$\left.\begin{array}{l} \mathcal{V} + (u + 12) = 52 \\ \mathcal{V} - (u + 12) = 1 \end{array}\right\} \Longrightarrow \quad (u, v) = (\tfrac{27}{2}, \tfrac{265}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+12)=2\\ \mathcal{V}-(u+12)=26\end{array}\right\} \Longrightarrow \quad (u,v)=(-24,70)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+12)=26\\ \mathcal{V}-(u+12)=2\end{array}\right\} \Longrightarrow \quad (u,v)=(0,70)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+12)=4\\ \mathcal{V}-(u+12)=13\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-33}{2},\tfrac{85}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+12)=13\\ \mathcal{V}-(u+12)=4\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-15}{2},\tfrac{85}{2})$$

Since we require that $u \in \mathbb{Z}^+$, the corresponding u, v equation has no solution.

$r = 49$: The corresponding equation is:

$$4v^2 - 49(2u + 49 - 1)^2 = \frac{1}{3} \cdot 49(49^2 - 1)$$

$$\Rightarrow v^2 - 49(u + 24)^2 = 9800$$

$$\Rightarrow 7 \mid v \text{ so, put } v = 7\mathcal{V}$$

$$\Rightarrow \mathcal{V}^2 - (u + 24)^2 = 200$$

$$\Rightarrow (\mathcal{V} + (u + 24))(\mathcal{V} - (u + 24)) = 200$$

Since the only factorizations of 200 are:

1(200), 2(100), 4(50), 8(25), 5(40), 10(20), (-1)(-200), (-2)(-100), (-4)(-50), (-8)(-25), (-5)(-40), and (-10)(-20),

and $\mathcal{V} + (u + 24)$ cannot be negative, the only possibilities are:

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=1\\ \mathcal{V}-(u+24)=200\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-247}{2},\tfrac{1407}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=200\\ \mathcal{V}-(u+24)=1\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{151}{2},\tfrac{1407}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=2\\ \mathcal{V}-(u+24)=100\end{array}\right\} \Longrightarrow \quad (u,v)=(-73,357)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=100\\ \mathcal{V}-(u+24)=2\end{array}\right\} \Longrightarrow \quad (u,v)=(25,357)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=4\\ \mathcal{V}-(u+24)=50\end{array}\right\} \Longrightarrow \quad (u,v)=(-47,189)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=50\\ \mathcal{V}-(u+24)=4\end{array}\right\} \Longrightarrow \quad (u,v)=(-1,189)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=8\\ \mathcal{V}-(u+24)=25\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-65}{2},\tfrac{231}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=25\\ \mathcal{V}-(u+24)=8\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-31}{2},\tfrac{231}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=5\\ \mathcal{V}-(u+24)=40\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-83}{2},\tfrac{315}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=40\\ \mathcal{V}-(u+24)=5\end{array}\right\} \Longrightarrow \quad (u,v)=(\tfrac{-13}{2},\tfrac{315}{2})$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=10\\ \mathcal{V}-(u+24)=20\end{array}\right\} \Longrightarrow \quad (u,v)=(-29,105)$$

$$\left.\begin{array}{l}\mathcal{V}+(u+24)=20\\ \mathcal{V}-(u+24)=10\end{array}\right\} \Longrightarrow \quad (u,v)=(-19,105)$$

Since we require that $u \in \mathbb{Z}^+$, the corresponding u, v equation has precisely 1 solution: (25, 357).

Before going on, let's summarize our progress on Question #2 thus far: For $r = 49$, the u, v equation has 1 unique solution: (25, 357). For all remaining r except:

$$2, 11, 23, 24, 26, 33, 47, 50, 52, 59, 73, 74, 88, 96, \text{ and } 97,$$

the corresponding u, v equation has no nontrivial solution.

Now, each of these remaining 15 values of r has a corresponding Pell equation of the form $ax^2 - by^2 = c$ with $a, b, c \in \mathbb{Z}$, $a, b > 0$, and a, b, c pairwise coprime. By statement (2), these Pell equations are globally solvable (have nontrivial solutions of the form (d, e) with $d, e \in \mathbb{Q}$) if and only if the congruences $aX^2 - bY^2 \equiv cZ^2 \pmod 8$ and $aX^2 - bY^2 \equiv cZ^2 \pmod p$ for p an odd prime dividing abc all have nontrivial solutions. So the "bad" primes (those needing individual attention) are 2 and the odd primes dividing abc. But we can do even better. The product formula for the Hilbert Norm Residue Symbol guarantees that congruences with only trivial solutions come in pairs. Hence, to determine the global solvability of these Pell equations, it suffices to check the solvability of the congruences for all but one of the "bad" primes. In practice, it is easiest not to bother with $p = 2$. Since it turns out that each of these 15 Pell equations passes all such tests, we can deduce that each of these 15 Pell equations is globally solvable. Therefore, it makes sense to start looking for integral solutions to these equations. If an integral solution is found for one of these Pell equations, that Pell equation will have infinitely many integral solutions. To see this, and an algorithm for producing these solutions, we may refer to the following two theorems which are presented by Charles Vanden Eynden in his book "Elementary Number Theory" (a synopsis of his proof of each is included):

Theorem 2.7: Suppose $d > 0$ is not a perfect square. If x, y is any positive solution to $x^2 - dy^2 = 1$, and if $n > 0$, then the integers x_n, y_n uniquely defined by $x_n + y_n\sqrt{d} = (x + y\sqrt{d})^n$ are also a positive solution to $x^2 - dy^2 = 1$, and these are distinct for distinct values of n.

Proof Synopsis. First, let's show that x_n and y_n are uniquely defined. Expanding $(x+y\sqrt{d})^n$ gives an expression of the form $A+B\sqrt{d}$ with $A, B \in \mathbb{Z}^+$. So, suppose that $A + B\sqrt{d} = C + D\sqrt{d}$ with $A, B, C, D \in \mathbb{Z}^+$. Then, $(B - D)\sqrt{d} = C - A$. If $B - D \neq 0$, then $(B - D)\sqrt{d} = C - A$ implies that $\sqrt{d}$ is rational, a contradiction. Hence, $D = B$ and so $C = A$. This shows that x_n and y_n are uniquely defined. Next, let's show that x_n, y_n is a solution of $x^2 - dy^2 = 1$ for each n. Notice if we expand $(x - y\sqrt{d})^n$, all of the odd powers of $\sqrt{d}$ enter with a minus sign, and we get $A - B\sqrt{d} = x_n - y_n\sqrt{d}$. Hence, $x_n^2 - dy_n^2 = (x_n - y_n\sqrt{d})(x_n + y_n\sqrt{d}) = (x - y\sqrt{d})^n(x + y\sqrt{d})^n = (x^2 - dy^2)^n = 1^n = 1$. This shows that x_n, y_n is a solution of $x^2 - dy^2 = 1$ for each n. Finally, let's show that $x_n, y_n \in \mathbb{Z}^+$. Since $x + y\sqrt{d} > 1$, the numbers $x_n + y_n\sqrt{d} = (x + y\sqrt{d})^n$ form a strictly increasing sequence. Since $x_n^2 - dy_n^2 = 1$, both sequences x_n and y_n increase. Therefore, $x_n, y_n \in \mathbb{Z}^+$. $\square$

Theorem 2.8: Suppose $d > 0$ is not a perfect square. If a, b is any positive solution to $a^2 - db^2 = c$, and if x, y is any of the infinitely many positive solutions of $x^2 - dy^2 = 1$, then the pair $\mathcal{U}, \mathcal{V}$ defined by $\mathcal{U} = ax + byd, \mathcal{V} = ay + bx$ is also a positive solution of $a^2 - db^2 = c$, and these are distinct for distinct positive solutions x, y.

Proof Synopsis. From the equations $x^2 - dy^2 = 1$ and $a^2 - db^2 = c$, we have $\mathcal{U}^2 - d\mathcal{V}^2 = a^2x^2 + 2abxyd + by^2d^2 - d(a^2y^2 + 2abxy + b^2x^2) = a^2(x^2 - dy^2) - db^2(x^2 - dy^2) = a^2 - db^2 = c$. This shows that $\mathcal{U}, \mathcal{V}$ is a solution to $\mathcal{U}^2 - d\mathcal{V}^2 = c$. Further, if x_2, y_2 and x_1, y_1 are two solutions to $x^2 - dy^2 = 1$ with $x_2 > x_1$, then $y_2 > y_1$. Hence, $\mathcal{U}_2 > \mathcal{U}_1$ and $\mathcal{V}_2 > \mathcal{V}_1$. $\square$

So, from the previous two theorems, we see that if an integral solution is found for one of these Pell equations, that Pell equation will have infinitely many solutions. However, these may or may not pull back to solutions of the corresponding u, v equation. So, let's individually address each of these 15 values of r:

$r = 2$: The corresponding Pell equation is:

$$\begin{aligned} &4v^2 - 2(2u + 2 - 1)^2 = \frac{1}{3} \cdot 2(2^2 - 1) \\ \Rightarrow &(2u + 1)^2 - 2v^2 = -1 \end{aligned}$$

So, let's try to find a solution of $a^2 - 2b^2 = -1$. Through trial and error, we obtain the solution $a = b = 1$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: Since $\sqrt{2} = [1; 2, 2]$, and $1 + \frac{1}{2} = \frac{3}{2}$, and $3^2 - 2 \cdot 2^2 = 1$, let $x = 3$ and $y = 2$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} = &\quad ax + byd & \qquad \mathcal{V} = &\quad ay + bx \\ = &\quad 1 \cdot 3 + 1 \cdot 2 \cdot 2 \quad \text{and} & = &\quad 1 \cdot 2 + 1 \cdot 3 \\ = &\quad 7 & = &\quad 5 \end{aligned}$$

Checking: $7^2 - 2 \cdot 5^2 = -1$.

Now, applying to our Pell equation:

$$\begin{aligned} &2u + 1 = \mathcal{U} = 7 & \qquad & v = \mathcal{V} = 5 \\ &2u = 6 & \text{and} & \\ &u = 3 & & \end{aligned}$$

So, our desired solution is (3,5). This means that $3^2 + 4^2 = 5^2$. Finally, since $2u + 1$ is odd, all that is required in order for a particular solution (a_1, b_1) of

$a^2 - 2b^2 = -1$ to pull back to a solution of the corresponding u, v equation is for a_1 to be odd. But, since $-2b_1$ is even and -1 is odd, ${a_1}^2$ must be odd. Thus, a_1 must be odd. Hence, all of the infinitely many integral solutions of $a^2 - 2b^2 = -1$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 11$: The corresponding Pell equation is:

$$\begin{aligned} 4v^2 - 11(2u + 11 - 1)^2 &= \frac{1}{3} \cdot 11(11^2 - 1) \\ \Rightarrow & v^2 - 11(u+5)^2 = 110 \\ \Rightarrow & 11 \mid v \text{ so, put } v = 11R \\ \Rightarrow & (u+5)^2 - 11R^2 = -10 \end{aligned}$$

So, let's try to find a solution of $a^2 - 11b^2 = -10$. Through trial and error, we obtain the solution $a = b = 1$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{11} = [3; 3, 6]$, and $3 + \frac{1}{3} = \frac{10}{3}$, and $10^2 - 11 \cdot 3^2 = 1$, let $x = 10$ and $y = 3$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd & \qquad \mathcal{V} &= ay + bx \\ &= 1 \cdot 10 + 1 \cdot 3 \cdot 11 \quad \text{and} & &= 1 \cdot 3 + 1 \cdot 10 \\ &= 43 & &= 13 \end{aligned}$$

Checking: $43^2 - 11 \cdot 13^2 = -10$.

Now, applying to our Pell equation:

$$\begin{aligned} u + 5 &= \mathcal{U} = 43 \quad \text{and} \quad v = 11R = 11\mathcal{V} = 11 \cdot 13 = 143 \\ u &= 38 \end{aligned}$$

So, our desired solution is (38,143). This means that $38^2 + 39^2 + \cdots + 48^2 = 143^2$. Finally, since $u + 5$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 11b^2 = -10$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 11b^2 = -10$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 23$: The corresponding Pell equation is:

$$\begin{aligned} 4v^2 - 23(2u + 23 - 1)^2 &= \frac{1}{3} \cdot 23(23^2 - 1) \\ \Rightarrow & v^2 - 23(u + 11)^2 = 1012 \\ \Rightarrow & 23 \mid v \text{ so, put } v = 23R \\ \Rightarrow & (u + 11)^2 - 23R^2 = -44 \end{aligned}$$

So, let's try to find a solution of $a^2 - 23b^2 = -44$. Through trial and error, we obtain the solution $a = 18$ and $b = 4$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{23} = [4; 1, 3, 1, 8]$, and $4 + \frac{1}{1+\frac{1}{3+\frac{1}{1}}} = \frac{24}{5}$, and $24^2 - 23 \cdot 5^2 = 1$, let $x = 24$ and $y = 5$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} = & \; ax + byd & \qquad \mathcal{V} = & \; ay + bx \\ = & \; 18 \cdot 24 + 4 \cdot 5 \cdot 23 \quad \text{and} & = & \; 18 \cdot 5 + 4 \cdot 24 \\ = & \; 892 & = & \; 186 \end{aligned}$$

Checking: $892^2 - 23 \cdot 186^2 = -44$.

Now, applying to our Pell equation:

$$\begin{aligned} & u + 11 = \mathcal{U} = 892 \quad \text{and} \quad v = 23R = 23\mathcal{V} = 23 \cdot 186 = 4278 \\ & u = 881 \end{aligned}$$

So, our desired solution is (881, 4278). This means that $881^2 + 882^2 + \cdots + 903^2 = 4278^2$. Finally, since $u + 11$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 23b^2 = -44$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 23b^2 = -44$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 24$: The corresponding Pell equation is:

$$\begin{aligned} & 4v^2 - 24(2u + 24 - 1)^2 = \frac{1}{3} \cdot 24(24^2 - 1) \\ \Rightarrow & v^2 - 6(2u + 23)^2 = 1150 \\ \Rightarrow & 2 \mid v \text{ so, put } v = 2R \\ \Rightarrow & (2R)^2 - 6(2u + 23)^2 = 1150 \end{aligned}$$

So, let's try to find a solution of $a^2 - 6b^2 = 1150$. Through trial and error, we obtain the solution $a = 34$ and $b = 1$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{6} = [2; 2, 4]$, and $2 + \frac{1}{2} = \frac{5}{2}$, and $5^2 - 6 \cdot 2^2 = 1$, let $x = 5$ and $y = 2$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd & \qquad \mathcal{V} &= ay + bx \\ &= 34 \cdot 5 + 1 \cdot 2 \cdot 6 \quad \text{and} & &= 34 \cdot 2 + 1 \cdot 5 \\ &= 182 & &= 73 \end{aligned}$$

Checking: $182^2 - 6 \cdot 73^2 = 1150$.

Now, applying to our Pell equation:

$$\begin{aligned} & 2u + 23 = \mathcal{V} = 73 & \qquad & v = 2R = \mathcal{U} = 182 \\ & 2u = 50 & \text{and} & \\ & u = 25 & & \end{aligned}$$

So, our desired solution is (25, 182). This means that $25^2 + 26^2 + \cdots + 48^2 = 182^2$. Finally, since $2R$ is even and $2u + 23$ is odd, all that is required in order for a particular solution (a_1, b_1) of $a^2 - 6b^2 = 1150$ to pull back to a solution of the corresponding u, v equation is for a_1 to be even and b_1 to be odd. But, since $-6b_1^2$ is even and 1150 is even, a_1^2 must be even. Thus, a_1 must be even. Then, b_1 must be odd since otherwise 4 would divide 1150, which it doesn't. Hence, all of the infinitely many integral solutions of $a^2 - 6b^2 = 1150$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 26$: The corresponding Pell equation is:

$$
\begin{aligned}
&4v^2 - 26(2u + 26 - 1)^2 = \frac{1}{3} \cdot 26(26^2 - 1) \\
\Rightarrow &2v^2 - 13(2u + 25)^2 = 2925 \\
\Rightarrow &13 \mid v \text{ so, put } v = 13R \\
\Rightarrow &(2u + 25)^2 - 26R^2 = -225
\end{aligned}
$$

So, let's try to find a solution of $a^2 - 26b^2 = -225$. Through trial and error, we obtain the solution $a = b = 3$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{26} = [5; 5, 10]$, and $5 + \frac{1}{10} = \frac{51}{10}$, and $51^2 - 26 \cdot 10^2 = 1$, let $x = 51$ and $y = 10$. So, our desired solution is:

$$
\begin{aligned}
\mathcal{U} = &\quad ax + byd & \qquad \mathcal{V} = &\quad ay + bx \\
= &\quad 3 \cdot 51 + 3 \cdot 10 \cdot 26 \quad \text{and} & = &\quad 3 \cdot 10 + 3 \cdot 51 \\
= &\quad 933 & = &\quad 183
\end{aligned}
$$

Checking: $933^2 - 26 \cdot 183^2 = -225$.

Now, applying to our Pell equation:

$$2u + 25 = \mathcal{U} = 933 \qquad v = 13R = 13\mathcal{V} = 13 \cdot 183 = 2379$$

$$2u = 908 \qquad \text{and}$$

$$u = 454$$

So, our desired solution is (454, 2379). This means that $454^2 + 455^2 + \cdots + 479^2 = 2379^2$. Finally, since $2u + 25$ is odd, all that is required in order for a particular solution (a_1, b_1) of $a^2 - 26b^2 = -225$ to pull back to a solution of the corresponding u, v equation is for a_1 to be odd. But, since $-26b_1^2$ is even and -225 is odd, a_1^2 must be odd. Therefore, a_1 must be odd. Hence, all of the infinitely many integral solutions of $a^2 - 26b^2 = -225$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 33$: The corresponding Pell equation is:

$$4v^2 - 33(2u + 33 - 1)^2 = \frac{1}{3} \cdot 33(33^2 - 1)$$

$$\Rightarrow v^2 - 33(u + 16)^2 = 2992$$

$$\Rightarrow 11 \mid v \text{ so, put } v = 11R$$

$$\Rightarrow (11R)^2 - 33(u + 16)^2 = 2992$$

So, let's try to find a solution of $a^2 - 33b^2 = 2992$. Through trial and error, we obtain the solution $a = 55$ and $b = 1$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{33} = [5; 1, 2, 1, 10]$, and $5 + \frac{1}{1+\frac{1}{2+\frac{1}{1}}} = \frac{23}{4}$, and $23^2 - 33 \cdot 4^2 = 1$, let $x = 23$ and $y = 4$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} = &\quad ax + byd & \mathcal{V} = &\quad ay + bx \\ = &\quad 55 \cdot 23 + 1 \cdot 4 \cdot 33 \quad \text{and} & = &\quad 55 \cdot 4 + 1 \cdot 23 \\ = &\quad 1397 & = &\quad 243 \end{aligned}$$

Checking: $1397^2 - 33 \cdot 243^2 = 2992$.

Now, applying to our Pell equation:

$$\begin{aligned} &u + 16 = \mathcal{V} = 243 \quad \text{and} \quad v = 11R = \mathcal{U} = 1397 \\ &u = 227 \end{aligned}$$

So, our desired solution is (227, 1397). This means that $227^2 + 228^2 + \cdots + 259^2 = 1397^2$. Finally, since $11R$ can be even or odd and $u + 16$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 33b^2 = 2992$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 33b^2 = 2992$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 47$: The corresponding Pell equation is:

$$\begin{aligned} 4v^2 - 47(2u + 47 - 1)^2 &= \frac{1}{3} \cdot 47(47^2 - 1) \\ \Rightarrow v^2 - 47(u + 23)^2 &= 8648 \\ \Rightarrow 47 \mid v \text{ so, put } v &= 47R \\ \Rightarrow (u + 23)^2 - 47R^2 &= -184 \end{aligned}$$

So, let's try to find a solution of $a^2 - 47b^2 = -184$. Through trial and error, we obtain the solution $a = b = 2$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{47} = [6; 1, 5, 1, 12]$, and $6 + \frac{1}{1+\frac{1}{5+\frac{1}{1}}} = \frac{48}{7}$, and

$48^2 - 47 \cdot 7^2 = 1$, let $x = 48$ and $y = 7$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} = &\ ax + byd \\ = &\ 2 \cdot 48 + 2 \cdot 7 \cdot 47 \\ = &\ 754 \end{aligned} \quad \text{and} \quad \begin{aligned} \mathcal{V} = &\ ay + bx \\ = &\ 2 \cdot 7 + 2 \cdot 48 \\ = &\ 110 \end{aligned}$$

Checking: $754^2 - 47 \cdot 110^2 = -184$.

Now, applying to our Pell equation:

$$u + 23 = \mathcal{U} = 754 \quad \text{and} \quad v = 47R = 47\mathcal{V} = 47 \cdot 110 = 5170$$
$$u = 731$$

So, our desired solution is (731, 5170). This means that $731^2 + 732^2 + \cdots + 777^2 = 5170^2$. Finally, since $u + 23$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 47b^2 = -184$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 47b^2 = -184$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 50$: The corresponding Pell equation is:

$$\begin{aligned} &4v^2 - 50(2u + 50 - 1)^2 = \frac{1}{3} \cdot 50(50^2 - 1) \\ \Rightarrow &2v^2 - 25(2u + 49)^2 = 20825 \\ \Rightarrow &5 \mid v \text{ so, put } v = 5R \\ \Rightarrow &(2u + 49)^2 - 2R^2 = -833 \end{aligned}$$

So, let's try to find a solution of $a^2 - 2b^2 = -833$. Through trial and error, we obtain the solution $a = 7$ and $b = 21$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{2} = [1; 2]$, and $1 + \frac{1}{2} = \frac{3}{2}$, and $3^2 - 2 \cdot 2^2 = 1$, let $x = 3$ and $y = 2$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd & \qquad \mathcal{V} &= ay + bx \\ &= 7 \cdot 3 + 21 \cdot 2 \cdot 2 \quad \text{and} & &= 7 \cdot 2 + 21 \cdot 3 \\ &= 105 & &= 77 \end{aligned}$$

Checking: $105^2 - 2 \cdot 77^2 = -833$.

Now, applying to our Pell equation:

$$\begin{aligned} &2u + 49 = \mathcal{U} = 105 & \qquad & v = 5R = 5\mathcal{V} = 5 \cdot 77 = 385 \\ &2u = 56 & \text{and} & \\ &u = 28 & & \end{aligned}$$

So, our desired solution is (28, 385). This means that $28^2 + 29^2 + \cdots + 77^2 = 385^2$. Finally, since $2u + 49$ is odd, all that is required in order for a particular solution (a_1, b_1) of $a^2 - 2b^2 = -833$ to pull back to a solution of the corresponding u, v equation is for a_1 to be odd. But, since $-2b^2$ is even and -833 is odd, a_1^2 must be odd. Thus, a_1 must be odd. Hence, all of the infinitely many integral solutions of $a^2 - 2b^2 = -833$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 52$: The corresponding Pell equation is:

$$\begin{aligned} &4v^2 - 52(2u + 52 - 1)^2 = \frac{1}{3} \cdot 52(52^2 - 1) \\ \Rightarrow &v^2 - 13(2u + 51)^2 = 11713 \\ \Rightarrow &13 \mid v \text{ so, put } v = 13R \\ \Rightarrow &(2u + 51)^2 - 13R^2 = -901 \end{aligned}$$

So, let's try to find a solution of $a^2 - 13b^2 = -901$. Through trial and error, we obtain the solution $a = 36$ and $b = 13$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{13} = [3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6]$, and

$$3 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}}}}}} = \frac{649}{180},$$

and $649^2 - 13 \cdot 180^2 = 1$, let $x = 649$ and $y = 180$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} = &\ ax + byd & \mathcal{V} = &\ ay + bx \\ = &\ 36 \cdot 649 + 13 \cdot 180 \cdot 13 \quad \text{and} & = &\ 36 \cdot 180 + 13 \cdot 649 \\ = &\ 53784 & = &\ 14917 \end{aligned}$$

Checking: $53784^2 - 13 \cdot 14917^2 = -901$.

Now, applying to our Pell equation:

$$2u + 51 = \mathcal{U} = 53784$$

$$2u = 53733$$

$$\Rightarrow \text{no solution in } \mathbb{Z}$$

Yes, we can find infinitely many more integral solutions of the Pell equation $a^2 - 13b^2 = -901$, but none of them will pull back to solutions of the u, v equation. To see this, consider that we want solutions of $a^2 - 13b^2 = -901$ with a odd. But the odd numbers squared are congruent to 1 (mod 8). So, we want:

$$1 + 3b^2 \equiv 3 \pmod 8$$

$$\Rightarrow 3b^2 \equiv 2 \pmod 8$$

$$\Rightarrow b^2 \equiv 6 \pmod 8 \text{ multiplying by 3}$$

$$\Rightarrow \text{ no solution}$$

Therefore, the u, v equation has no solution.

$r = 59$: The corresponding Pell equation is:

$$4v^2 - 59(2u + 59 - 1)^2 = \frac{1}{3} \cdot 59(59^2 - 1)$$

$$\Rightarrow v^2 - 59(u + 29)^2 = 17110$$

$$\Rightarrow 59 \mid v \text{ so, put } v = 59R$$

$$\Rightarrow (u + 29)^2 - 59R^2 = -290$$

So, let's try to find a solution of $a^2 - 59b^2 = -290$. Through trial and error, we obtain the solution $a = 51$ and $b = 7$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{59} = [7; 1, 2, 7, 2, 1, 14]$, and $7 + \frac{1}{1+\frac{1}{2+\frac{1}{7+\frac{1}{2+\frac{1}{1}}}}} = \frac{530}{69}$, and $530^2 - 59 \cdot 69^2 = 1$, let $x = 530$ and $y = 69$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd & \qquad \mathcal{V} &= ay + bx \\ &= 51 \cdot 530 + 7 \cdot 69 \cdot 59 \quad \text{and} & &= 51 \cdot 69 + 7 \cdot 530 \\ &= 55527 & &= 7229 \end{aligned}$$

Checking: $55527^2 - 59 \cdot 7229^2 = -290$.

Now, applying to our Pell equation:

$$u + 29 = \mathcal{U} = 55527 \quad \text{and} \quad v = 59R = 59\mathcal{V} = 59 \cdot 7229 = 426511$$

$$u = 55498$$

So, our desired solution is (55498, 426511). This means that $55498^2 + 55499^2 + \cdots + 55556^2 = 426511^2$. Finally, since $u + 29$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 59b^2 = -290$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 59b^2 = -290$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 73$: The corresponding Pell equation is:

$$\begin{aligned} &4v^2 - 73(2u + 73 - 1)^2 = \frac{1}{3} \cdot 73(73^2 - 1) \\ \Rightarrow &v^2 - 73(u + 36)^2 = 32412 \\ \Rightarrow &73 \mid v \text{ so, put } v = 73R \\ \Rightarrow &(u + 36)^2 - 73R^2 = -444 \end{aligned}$$

So, let's try to find a solution of $a^2 - 73b^2 = -444$. Through trial and error, we obtain the solution $a = 478$ and $b = 56$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{73} = [8; 1, 1, 5, 5, 1, 1, 16, 1, 1, 5, 5, 1, 1, 16]$, and

$$8 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{16 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{1 + \frac{1}{1}}}}}}}}}}}}} = \frac{2281249}{267000},$$

and $2281249^2 - 73 \cdot 267000^2 = 1$, let $x = 2281249$ and $y = 267000$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd & \mathcal{V} &= ay + bx \\ &= 478 \cdot 2281249 + 56 \cdot 267000 \cdot 73 \quad \text{and} & &= 478 \cdot 267000+ \\ &= 2181933022 & &\quad 56 \cdot 2281249 \\ & & &= 255375944 \end{aligned}$$

Checking: $2181933022^2 - 73 \cdot 255375944^2 = -444$.

Now, applying to our Pell equation:

$$\begin{aligned} u + 36 &= \mathcal{U} = 2181933022 \quad \text{and} \quad v &&= 73R = 73\mathcal{V} = 73 \cdot 255375944 \\ u &= 2181932986 && = 18642443910 \end{aligned}$$

So, our desired solution is (2181932986, 18642443910). This means that $2181932986^2 + 2181932987^2 + \cdots + 2181933058^2 = 18642443910^2$. Finally, since $u + 36$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 73b^2 = -444$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 73b^2 = -444$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 74$: The corresponding Pell equation is:

$$\begin{aligned} &4v^2 - 74(2u + 74 - 1)^2 = \frac{1}{3} \cdot 74(74^2 - 1) \\ \Rightarrow &2v^2 - 37(2u + 73)^2 = 67525 \\ \Rightarrow &37 \mid v \text{ so, put } v = 37R \\ \Rightarrow &(2u + 73)^2 - 74R^2 = -1825 \end{aligned}$$

So, let's try to find a solution of $a^2 - 74b^2 = -1825$. Through trial and error, we obtain the solution $a = b = 5$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{74} = [8; 1, 1, 1, 1, 16, 1, 1, 1, 1, 16]$, and

$$8 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{16+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}}}}}} = \frac{3699}{430},$$

and $3699^2 - 74 \cdot 430^2 = 1$, let $x = 3699$ and $y = 430$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd & \qquad \mathcal{V} &= ay + bx \\ &= 5 \cdot 3699 + 5 \cdot 430 \cdot 74 \quad \text{and} & &= 5 \cdot 430 + 5 \cdot 3699 \\ &= 177595 & &= 20645 \end{aligned}$$

Checking: $177595^2 - 74 \cdot 20645^2 = -1825$.

Now, applying to our Pell equation:

$$\begin{aligned} 2u + 73 &= \mathcal{U} = 177595 & \qquad v &= 37R = 37\mathcal{V} = 37 \cdot 20645 \\ 2u &= 177522 \qquad \text{and} & &= 763865 \\ u &= 88761 & & \end{aligned}$$

So, our desired solution is (88761, 763865). This means that $88761^2 + 88762^2 + \cdots + 88834^2 = 763865^2$. Finally, since $2u + 73$ is odd, all that is required in order for a particular solution (a_1, b_1) of $a^2 - 74b^2 = -1825$ to pull back to a solution of the corresponding u, v equation is for a_1 to be odd. But, since $-74b^2$ is even and -1825 is odd, a_1^2 must be odd. Thus, a_1 must be odd. Hence, all the infinitely many integral solutions of $a^2 - 74b^2 = -1825$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 88$: The corresponding Pell equation is:

$$4v^2 - 88(2u + 88 - 1)^2 = \frac{1}{3} \cdot 88(88^2 - 1)$$

$$\Rightarrow v^2 - 22(2u+87)^2 = 56782$$

$$\Rightarrow 11 \mid v \text{ so, put } v = 11R$$

$$\Rightarrow 11R^2 - 2(2u+87)^2 = 5162$$

$$\Rightarrow 2 \mid R \text{ so, put } R = 2T$$

$$\Rightarrow (2u+87)^2 - 22T^2 = -2581$$

So, let's try to find a solution of $a^2 - 22b^2 = -2581$. Through trial and error, we obtain the solution $a = 9$ and $b = 11$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{22} = [4; 1, 2, 4, 2, 1, 8]$, and $4 + \frac{1}{1+\frac{1}{2+\frac{1}{4+\frac{1}{2+\frac{1}{1}}}}} = \frac{197}{42}$ and $197^2 - 22 \cdot 42^2 = 1$, let $x = 197$ and $y = 42$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} &= ax + byd \\ &= 9 \cdot 197 + 11 \cdot 42 \cdot 22 \\ &= 11937 \end{aligned} \quad \text{and} \quad \begin{aligned} \mathcal{V} &= ay + bx \\ &= 9 \cdot 42 + 11 \cdot 197 \\ &= 2545 \end{aligned}$$

Checking: $11937^2 - 22 \cdot 2545^2 = -2581$.

Now, applying to our Pell equation:

$$\begin{aligned} 2u + 87 &= \mathcal{U} = 11937 \\ 2u &= 11850 \\ u &= 5925 \end{aligned} \quad \text{and} \quad \begin{aligned} v &= 11R = 11 \cdot 2T = 11 \cdot 2 \cdot \mathcal{V} \\ &= 11 \cdot 2 \cdot 2545 \\ &= 55990 \end{aligned}$$

So, our desired solution is (5925, 55990). This means that $5925^2 + 5926^2 + \cdots + 6012^2 = 55990^2$. Finally, since $2u + 87$ is odd, all that is required in order for a particular solution (a_1, b_1) of $a^2 - 22b^2 = -2581$ to pull back to a solution of the corresponding u, v equation is for a_1 to be odd. But, since $-22b^2$ is even and -2581 is odd, a_1^2 must be odd. Thus, a_1 must be odd. Hence, all of the infinitely many integral solutions of $a^2 - 22b^2 = -2581$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 96$: The corresponding Pell equation is:

$$
\begin{aligned}
4v^2 - 96(2u + 96 - 1)^2 &= \frac{1}{3} \cdot 96(96^2 - 1) \\
\Rightarrow & v^2 - 24(2u + 95)^2 = 73720 \\
\Rightarrow & 2 \mid v \text{ so, put } v = 2R \\
\Rightarrow & R^2 - 6(2u + 95)^2 = 18430 \\
\Rightarrow & 2 \mid R \text{ so, put } R = 2T \\
\Rightarrow & (2T)^2 - 6(2u + 95)^2 = 18430
\end{aligned}
$$

So, let's try to find a solution of $a^2 - 6b^2 = 18430$. Through trial and error, we obtain the solution $a = 142$ and $b = 17$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{6} = [2; 2, 4]$, and $2 + \frac{1}{2} = \frac{5}{2}$ and $5^2 - 6 \cdot 2^2 = 1$, let $x = 5$ and $y = 2$. So, our desired solution is:

$$
\begin{aligned}
\mathcal{U} = &\ ax + byd & \qquad \mathcal{V} = &\ ay + bx \\
= &\ 142 \cdot 5 + 17 \cdot 2 \cdot 6 \quad \text{and} & = &\ 142 \cdot 2 + 17 \cdot 5 \\
= &\ 914 & = &\ 369
\end{aligned}
$$

Checking: $914^2 - 6 \cdot 369^2 = 18430$.

Now, applying to our Pell equation:

$$
\begin{aligned}
2u + 95 &= \mathcal{V} = 369 & \qquad v &= 2R = 2 \cdot 2T = 2 \cdot \mathcal{U} \\
2u &= 274 \qquad \text{and} & &= 2 \cdot 914 \\
u &= 137 & &= 1828
\end{aligned}
$$

So, our desired solution is (137, 1828). This means that $137^2 + 138^2 + \cdots + 232^2 = 1828^2$. Finally, since $2T$ is even and $2u+95$ is odd, all that is required in order for a particular solution (a_1, b_1) of $a^2 - 6b^2 = 18430$ to pull back to a solution of the corresponding u, v equation is for a_1 to be even and b_1 to be odd. But, since $-6b_1^2$ is even and 18430 is even, a_1^2 must be even. Thus, a_1 must be even. Then, b_1 must be odd since otherwise 4 would divide 18430, which it doesn't. Hence, all of the infinitely many integral solutions of $a^2 - 6b^2 = 18430$ pull back to solutions of the corresponding u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

$r = 97$: The corresponding Pell equation is:

$$\begin{aligned} 4v^2 - 97(2u + 97 - 1)^2 &= \frac{1}{3} \cdot 97(97^2 - 1) \\ \Rightarrow & v^2 - 97(u + 48)^2 = 76048 \\ \Rightarrow & 97 \mid v \text{ so, put } v = 97R \\ \Rightarrow & (u + 48)^2 - 97R^2 = -784 \end{aligned}$$

So, let's try to find a solution of $a^2 - 97b^2 = -784$. Through trial and error, we obtain the solution $a = 63$ and $b = 7$. Since this is a Pell equation, infinitely many solutions will exist.

Let's find another: since $\sqrt{97} = [9;$ 1,5,1,1,1,1,1,1,5,1,18,1,5,1,1,1,1,1,1,5, 1,18], and

$$9+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{18+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\frac{1}{1}}}}}}}}}}}}}}}}}}}}} = \frac{62809633}{6377352},$$

and $62809633^2 - 97 \cdot 6377352^2 = 1$, let $x = 62809633$ and $y = 6377352$. So, our desired solution is:

$$\begin{aligned} \mathcal{U} = &\quad ax + byd & \qquad \mathcal{V} = &\quad ay + bx \\ = &\quad 63 \cdot 62809633 + 7 \cdot 6377352 \cdot 97 \quad \text{and} & = &\quad 63 \cdot 6377352 + \\ = &\quad 8287228887 & &\qquad 7 \cdot 62809633 \\ & & = &\quad 841440607 \end{aligned}$$

Checking: $8287228887^2 - 97 \cdot 841440607^2 = -784$.

Now, applying to our Pell equation:

$$\begin{aligned} u + 48 = \mathcal{U} &= 8287228887 \quad \text{and} \quad v &= 97R = 97\mathcal{V} \\ u &= 8287228839 & = 97 \cdot 841440607 \\ & & = 81619738880 \end{aligned}$$

So, our desired solution is (8287228839, 81619738880). This means that $8287228839^2 + 8287228840^2 + \cdots + 8287228935^2 = 81619738880^2$. Finally, since $u + 48$ can be even or odd, we see that there are no restrictions in order for a particular solution (a_1, b_1) of $a^2 - 97b^2 = -784$ to pull back to a solution of the corresponding u, v equation. Hence, all of the infinitely many integral solutions of $a^2 - 97b^2 = -784$ pull back to solutions of the corresponding

u, v equation. Therefore, the corresponding u, v equation has infinitely many solutions.

It has been a long journey, but we can now summarize the situation for all values of r from 1 to 100 on Question #2:

For $r = 49$, the u, v equation has 1 unique solution: (25, 357). For r=2, 11, 23, 24, 26, 33, 47, 50, 59, 73, 74, 88, 96, and 97, the u, v equations each have an infinite number of solutions. We list the two smallest solutions for these values of r in Table #4. For all other values of r not listed above, the u, v equations have no nontrivial solutions. This concludes our analysis of Question #2.

Table #4

r	(u, v)
2	(3,5)
	(20, 29)
11	(38, 143)
	(854, 2849)
23	(7, 92)
	(881, 4278)
24	(25, 182)
	(353, 1786)
26	(454, 2379)
	(47569, 242619)
33	(227, 1397)
	(11161, 64207)
47	(731, 5170)
	(72359, 496226)

50	(28, 385)
	(287, 2205)
59	(22, 413)
	(55498, 426511)
73	(442, 4088)
	(2181932986, 18642443910)
74	(88761, 763865)
	(656923866, 5651073085)
88	(5925, 55990)
	(2351541, 11090112)
96	(137, 1828)
	(1789, 17996)
97	(15, 679)
	(8287228839, 81619738880)

REFERENCES

[1] Anglin, W.S., *The Queen of Mathematics*, AH Dordrecht, The Netherlands: Kluwer Academic Publishers, 1995.

[2] Rosen, K.H., *Elementary Number Theory*, AT&T Laboratories, 2005.

[3] Silverman, J.H. and John Tate, *Rational Points on Elliptic Curves*, New York, NY: Springer Science + Business Media, LLC, 1992.

[4] Vanden Eynden, C., *Elementary Number Theory*, New York, NY: Random House, Inc., 1987.