The First-Fit Algorithm Uses Many Colors on Some Interval Graphs

by

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ABSTRACT

Graph coloring is about allocating resources that can be shared except where there are certain pairwise conflicts between recipients. The simplest coloring algorithm that attempts to conserve resources is called first fit. Interval graphs are used in models for scheduling (in computer science and operations research) and in biochemistry for one-dimensional molecules such as genetic material.

It is not known precisely how much waste in the worst case is due to the first-fit algorithm for coloring interval graphs. However, after decades of research the range is narrow. Kierstead proved that the performance ratio R is at most 40. Pemmaraju, Raman, and Varadarajan proved that R is at most 10. This can be improved to 8.

Witsenhausen, and independently Chrobak and Slusarek, proved that R is at least 4. Slusarek improved this to 4.45. Kierstead and Trotter extended the method of Chrobak and Slusarek to one good for a lower bound of 4.99999 or so.

The method relies on number sequences with a certain property of order. It is shown here that each sequence considered in the construction satisfies a linear recurrence; that R is at least 5; that the Fibonacci sequence is in some sense minimally useless for the construction; and that the Fibonacci sequence is a point of accumulation in some space for the useful sequences of the construction. Limitations of all earlier constructions are revealed.

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Chapter 1

INTRODUCTION

The main result presented here is a lower bound of 5 for the performance ratio of the first-fit algorithm for coloring interval graphs. First appears a brief introduction to graphs and coloring.

1.1 Notation

The sets of integers, nonnegative integers, and positive integers are denoted $\mathbb{Z}, \mathbb{N}$, and $\mathbb{P}$. When $n \in \mathbb{P}$, let $[n]$ denote $\{1,2,3,\ldots,n\}$, and let $[0]$ be the empty set $\emptyset$. When S is a finite set, $|S|$ denotes its size. Subset T of set S is a *k-subset* of S if $|T| = k$. The collection of k-subsets of S is denoted $\binom{S}{k}$, and $\binom{n}{k}$ denotes the binomial coefficient $\left|\binom{[n]}{k}\right|$. When f is a function, the image $\{f(x) \mid x \in A\}$ of set A is denoted $f(A)$, and $f^{-1}(B)$ denotes $\{x \mid f(x) \in B\}$, even when f has no inverse (with respect to composition). The set of real numbers is denoted $\mathbb{R}$. Real intervals are denoted as in the example $(a,b]$ for $\{x \in \mathbb{R} \mid a < x \leq b\}$. When pair (a,b) is intended as an open interval, the word interval precedes it. The set of real intervals that are each nonempty, closed, and bounded is denoted $\mathscr{I}$. When $x \in \mathbb{R}$, let $\lfloor x \rfloor = \max\ \mathbb{Z} \cap (-\infty, x]$ and $\lceil x \rceil = \min\ \mathbb{Z} \cap [x, \infty)$. When z is a complex number, its real part, imaginary part, and complex conjugate are denoted $\mathrm{Re}[z], \mathrm{Im}[z]$, and z^*.

1.2 Graphs

Pair (V,E) is a *graph* if $E \subseteq \binom{V}{2}$. Elements of V are *vertices*, and elements of E are *edges*. So an edge is an unordered pair of vertices. Often omitted from the notation for such a pair are comma and braces, so the expression $uv \in E$ means $\{u,v\} \in E$. Vertex u is a *neighbor* of vertex v if $uv \in E$. The set of neighbors of v is denoted $N(v)$, and $N[v]$ denotes $N(v) \cup \{v\}$. For the present purpose, V is always finite. Graph $G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If $E' = E \cap \binom{V'}{2}$ for some $V' \subseteq V$, then G' is *induced* by V', and in this case G' is denoted $G[V']$. When $G[V']$ has no edge, V' is *stable* or *independent*.

The *clique* or *complete graph* on vertex set V is $\left(V, \binom{V}{2}\right)$. Each induced subgraph of a clique is also a clique. As another example, when $n \in \mathbb{N}$, the *n-vertex path* is the graph $([n], \{\{v, v+1\} \mid v \in [n-1]\})$. When $V = \mathbb{Z}/n\mathbb{Z}$ is the (additive) cyclic group on $n > 2$ elements, the *n-cycle* is the graph $(V, \{\{v, v+1\} \mid v \in V\})$.

When (V, E) and (V', E') are graphs and $\varphi\colon V \to V'$ is a bijection, φ is an *isomorphism* if $E' = \{\varphi(u)\varphi(v) \mid uv \in E\}$. Graph theory is the study of relationships among graph invariants (i.e., properties preserved by isomorphism). Some examples of invariants are number of vertices, number of edges, size of a largest complete subgraph, and smallest number of colors in a proper coloring.[1] Isomorphic graphs may be taken to be equal, although this custom is not observed consistently. While there is only one clique on n vertices, denoted K_n, it may be said, for example, that K_4 has 4 subgraphs K_3. Generally, K_n has $\binom{n}{k}$ subgraphs K_k. Case $k = 2$ means simply that there is one subgraph K_2 of K_n for each edge.

1.3 Coloring

When $G = (V, E)$ is a graph, an assignment f on V is a *proper coloring*, or simply a *coloring*, if $f(u) \neq f(v)$ for every edge $uv \in E$. Put another way, each color class is stable. That is, for each color k, graph $G\left[f^{-1}(k)\right]$ has no edge.

[1] Coloring is defined below.

A coloring[2] is a solution to a certain kind of problem. Suppose, for example, that a zookeeper is responsible for a collection V of animals, some incompatible. Each incompatible pair is an edge. Then a coloring of (V,E) compatibly assigns enclosures to animals. Colorings of many colors are easy to find, while colorings of few colors may be desired.

The number $\chi(G)$ of colors in a minimum coloring of G is its *chromatic number*. An obvious lower bound for $\chi(G)$ is the number $\omega(G)$ of vertices in a largest complete subgraph of G. An obvious upper bound for $\chi(G)$ is $|V|$. Of course the upper bound holds with equality (i.e., $\chi(G) = |V|$) only when G is a clique, and in that case the lower bound does too (i.e., $\omega(G) = \chi(G)$).

Colorings are easy to find. Each vertex could have its own color, say. Good colorings use few colors, so consider the following obvious procedure for improving a coloring. Let colors be numbered $1, 2, \ldots$.

> While some vertex v has no neighbor of some color smaller than its own, reassign v that color; if this creates a gap in the set of colors used, renumber colors to eliminate it.

The procedure does no harm; each step produces a (proper) coloring, and the total number of colors used can only decrease. The procedure halts; the sum (over all vertices) of colors is nonnegative and can only decrease. The result is a coloring

[2] The name *coloring* for such an assignment apparently comes from the nineteenth-century question whether 4 colors suffice to color a planar map of territories (abstractly, a planar graph). Each territory is a vertex, and each pair of territories that share a border is an edge, so a proper coloring gives them distinct colors. Mathematically, the identities of the colors aren't important, so perhaps what is sought is not a coloring, but a *graph partition*, a partition of the vertex set that respects graph structure.

$f\colon V \to \mathbb{P}$ so that $f(N[v]) \supseteq [f(v)]$ for each $v \in V$. Such a coloring is *Grundy*. One wonders whether the number of colors in a Grundy coloring is optimal. How large can it be?

It can be quite large. Some graphs that can be colored with 2 colors have Grundy colorings with arbitrarily many colors.[3] That is, coloring haphazardly can yield bad colorings, even when conservation is a deliberate goal. The question arises whether some slightly more clever coloring scheme could produce consistently good colorings. As it turns out, the problem of determining $\chi(G)$ for a general graph G is NP-complete (cf. Garey and Johnson [10]), which is to say challenging.

People have been studying Grundy colorings for over 70 years. Some results that have appeared during that time are discussed in Erdős, Hedetniemi, Laskar, and Prins [8].

1.4 Online coloring

Suppose the n vertices of graph G are $v_1, \ldots, v_n$. A coloring algorithm is *online* if it assigns for each $k \in [n]$ a color to v_k that depends only on $G[v_1, \ldots, v_k]$. In other words, the algorithm assigns irrevocable colors to vertices in prescribed order $v_1, \ldots, v_n$ without awareness of their future neighbors. A trivial online algorithm might assign color k to v_k; obviously the result would be a (bad) coloring with n colors.

[3] The graph with vertex set $\{\pm 1, \ldots, \pm n\}$ and edge set $\left\{ kk' \;\middle|\; \{k, -k'\} \in \binom{[n]}{2} \right\}$ is 2-chromatic, and the coloring $k \mapsto |k|$ is a Grundy coloring of n colors. There are more such examples even among graphs without cycle subgraphs. Every cycle-free graph is 2-chromatic. One can construct for each $n \in \mathbb{N}$ a cycle-free graph G_n and a Grundy coloring of G_n with n colors.

Toward the goal of using few colors, the simple online coloring algorithm known as *first-fit* assigns the least positive integer not used already on a neighbor. Naturally the result is a Grundy coloring. The number of colors used by the first-fit algorithm on a given graph depends on the vertex order, and it can vary greatly.

When vertices are ordered favorably, first-fit uses only $\chi(G)$ colors. At the other extreme, the largest number of colors used by first-fit over all possible vertex orders is denoted $\chi_{FF}(G)$. A bad order can cause first-fit to use strictly more than $\chi(G)$ colors; a small example is a path on 4 vertices with ends presented first. The *performance ratio* of first-fit on graph G is $\chi_{FF}(G)/\chi(G)$. It compares the number of colors used by first-fit on a worst order to the optimum number of colors.

1.5 Perfect graphs

Every graph $G = (V,E)$ satisfies $\omega(G) \leq \chi(G) \leq |V|$. Obviously $|V|$ can be much bigger than $\chi(G)$. And $\chi(G)$ can be much bigger than $\omega(G)$. In fact, $\chi(G)$ can be arbitrarily large even when $\omega(G)$ is just 2. Such graphs were constructed by Zykov [40] and[4] Mycielski [29].

Graphs with $\chi(G) = \omega(G)$ are of special interest, particularly those that satisfy a stronger condition:[5] G is *perfect* when $\chi(H) = \omega(H)$ for every induced subgraph H of G. Not every graph is perfect; consider for example a 5-cycle. But cliques are perfect, as are many other classes. Grötschel, Lovász, and Schrijver [14] found

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[4] Friend of the trees A. Gyárfás [17] used only 9 lines of text to define and explain his triangle-free infinite-chromatic graph. Of course, the vertex set of this graph is not finite. It is countable, though.

[5] If (V,E) is a graph and (V',E') a clique with $|V| \leq |V'|$ and $V \cap V' = \emptyset$, then $G = (V \cup V', E \cup E')$ is a graph with $\chi(G) = \omega(G)$ that is not inherently interesting (cf. Seymour [33]).

that when G is perfect, $\chi(G)$ can be determined in polynomial time (which is to say quickly), although the procedure is not simple.

When graphs are not perfect, how far from perfect are they? A measure discussed by Gyárfás [16] is a function f so that for every graph G of some class, $\chi(G) \leq f(\omega(G))$. Then f is a χ-*binding function*, and the class is described in terms of f. For example, when f is a degree-1 polynomial, the class is said to be linearly χ-bound. In particular, perfect graphs are χ-bound by the identity function $x \mapsto x$. The existence of graphs with χ arbitrarily large and $\omega = 2$ means that graphs are not generally χ-bound. Erdős [7] showed something more, that for all $g > 0$ there are graphs with χ arbitrarily large and no cycle subgraph on fewer than g vertices.

A *tree* is a graph T with each pair of its vertices linked by precisely 1 path subgraph of T. A tree therefore has no cycle subgraph. Gyárfás [15] and Sumner [36] conjectured independently that for every tree T there is a function f so that if G has no induced subgraph T, then $\chi(G) \leq f(\omega(G))$. The conjecture is unresolved, although there has been progress. The *radius* of tree $T = (V,E)$ is $\min_{v \in V} \max_{u \in V} \operatorname{dist}(u,v)$, where $\operatorname{dist}(u,v)$ is the number of edges in the path that links u and v in T. The conjecture was confirmed for trees of radius at most 2 by Gyárfás, Szemerédi, and Tuza [20] when $\omega(G) \leq 2$, and by Kierstead and Penrice [24] with $\omega(G)$ unrestricted.

Binding functions may be used also to characterize the first-fit chromatic number. When $\chi_{FF}(G) \leq f(\omega(G))$ for each graph G of some class, the class is *first-fit* χ*-bound* by f.

A graph which for all $m > 3$ has no induced m-cycle subgraph is *chordal* or *triangulated*. Chordal graphs are perfect. When G is chordal, $\chi(G)$ is computed easily in polynomial time (cf. Golumbic [11]). Irani [21] showed that there are constants b and c so that when G is a chordal graph G on n vertices, $\chi_{FF}(G) \leq b\omega(G)\log n + c$.

Interval graphs

Graph $G = (V,E)$ is an *interval graph* if there is a function $\iota\colon V \to \mathscr{I}$ so that

$$E = \left\{ uv \in \binom{V}{2} \;\middle|\; \iota(u) \cap \iota(v) \neq \emptyset \right\}.$$

In this case ι is an *interval representation*, or simply a *representation*, of G.

In certain scheduling problems, some resource is reserved for a bounded duration. Suppose for example that only one conference at a time can be held in a conference room. If V is a set of conferences and $\iota(v)$ is the time interval associated with conference v, where ι is a representation of $G = (V,E)$, then $\chi(G)$ is the smallest number of conference rooms needed to accommodate all conferences.

The definition of graph is abstract in the sense that vertices need not be points in space or numbers or any other particular type of object. Vertices could be real intervals, as in the previous paragraph. Usually in this case the interval representation is the identity map (so is redundant). For the construction of Kierstead and Trotter that appears in this paper, it is desired to associate more than one vertex with a given interval. So interval representations below are often nontrivial and in particular noninjective.

Every interval graph G is perfect. Here is a sketch of a proof:

- fix some interval representation ι of G;
- assume vertices $v_1, \ldots, v_n$ are ordered by interval left endpoint;
- color G by first fit.

For all $k \in [n]$ and $v \in N(v_k) \cap \{v_1, \ldots, v_{k-1}\}$, one has $\min \iota(v_k) \in \iota(v)$. So $G[N[v_k] \cap \{v_1, \ldots, v_k\}]$ is a clique, and $f(v_k) \leq |N[v_k]| \leq \omega(G)$.

Interval graphs are also chordal. When G is an interval graph, $\chi(G)$ is trivially determined in polynomial time, say by modifying the procedure above.

Many graphs can be regarded as generalized interval graphs. In a circular arc graph, each vertex is an arc in the unit circle, and the edges are the pairs of arcs that meet. (The stereographic projection takes intervals in the real line to arcs in a circle.) These and other intersection graphs are discussed by Golumbic [11]. In a box graph, each vertex is a t-dimensional cartesian box, and the edges are the pairs of boxes that meet. So interval graphs are the box graphs with $t = 1$. See *boxicity* in Trotter [37]. Tolerance graphs are like interval graphs, but some overlap between intervals of non-neighboring vertices may be allowed. Golumbic and Trotter [13] proved that tolerance graphs are perfect. Tolerance graphs are treated at length by Golumbic and Trenk [12].

First-fit algorithms are relatively easy to analyze and implement, so are much studied and often used. Both the performance ratio of first-fit coloring of interval graphs and the methods used to find it are therefore interesting.

1.6 The origin of the question

The question of the performance ratio of first-fit for coloring interval graphs was raised by Woodall [39] in 1973. There was, however, some earlier discussion in an unpublished technical memorandum on dynamic storage allocation by M. D. McIlroy in 1968, and in subsequent work by R. L. Graham and others, according to Coffman [5].

McDiarmid [28] showed $\chi_{FF}(G) \leq (2+\varepsilon)\chi(G)$ for all $\varepsilon > 0$ for almost all graphs G. Gyárfás and Lehel [19] found for $\chi_{FF}(G)$ upper bounds $\omega(G)+1$ when G is a split graph; $1.5\omega(G)$ when G is the complement[6] of a 2-chromatic graph; and $2\omega(G)-1$ when G is the complement of a chordal graph.

Gyárfás and Lehel [18] defined a *type-k wall* (or simply k-wall) to be essentially an interval graph with a Grundy coloring so that each vertex has among its greater-colored neighbors no stable set of more than k vertices. The *height* of such a wall is the number of colors in the Grundy coloring, and its *density* is the size of a largest clique in its graph. They proved that the height of a 1-wall is no more than 24 times its density, giving an upper bound on the performance ratio of first-fit on a special class of interval graphs. They pointed out that known high walls at the time were 1-walls. The constructions of Kierstead and Trotter in the present paper turn out to be 1-walls also. Gyárfás and Lehel also showed that for all k, when a k-wall exists, so does a 3-wall of the same height and density. The best known upper bounds currently do not exploit this. It could still yield some benefit. Kierstead and Qin [25] improved the upper bound of 24 for 1-walls to 8.

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[6] The *complement* of graph (V,E) is $\left(V, \binom{V}{2} - E\right)$.

Chapter 2

FIRST-FIT COLORING OF INTERVAL GRAPHS

The *performance ratio* of the first-fit algorithm for coloring interval graphs is

$$R = \sup\left\{ \frac{\chi_{FF}(G)}{\chi(G)} \;\middle|\; G \text{ is an interval graph} \right\}.$$

2.1 Results

Theorem 1 (Kierstead and Trotter [27]). *Some online algorithm uses at most* $3\omega(G)-2$ *colors on each interval graph G, and no online algorithm is better.*

So $R \geq 3$ because first-fit is online and interval graphs are perfect. Originally, the result was expressed in terms not of coloring interval graphs, but of partitioning interval posets (a *poset* is a partially ordered set) into chains (a *chain* is a set of pairwise comparable elements). When $G = (V,E)$ is an interval graph with representation $\iota : V \to \mathscr{I}$, there is a partial order $<_I$ on V, called *interval order*, defined by $u <_I v$ if $x < y$ for each $x \in \iota(u)$ and $y \in \iota(v)$. The order associated with a given poset is an *interval order* if it is realized in this way by some interval graph. The condition of adjacency in an interval graph is complementary with that of comparability in its interval order. A color class of a coloring of an interval graph is a chain in the interval order. So the problem of coloring the vertices of an interval graph is the same as that of partitioning the elements of a poset into chains, and the first-fit algorithm can be applied to the latter. A clique in an interval graph is a set of pairwise incomparable elements in the interval order. Such a subset of a poset is an *antichain*. The size of a largest antichain of a poset is its *width*. The width of the interval order realized by interval graph G is $\omega(G)$.

Theorem 2 (Kierstead [23]). $R \leq 40$.

This established for the first time that interval graphs are linearly first-fit χ-bound. Kierstead and Qin [25] later proved $R \leq 25.72$.

Theorem 3 (Witsenhausen [38])**.** $R \geq 4$.

Independently, Chrobak and Ślusarek [3] found the following result implying $R \geq 4$: for every natural number k there is an interval graph G such that $\omega(G) = k$ and $\chi_{FF}(G) \geq 4k-9$. Ślusarek [34] improved this to $R \geq 4.45$.

Theorem 4 (Pemmaraju, Raman, and Varadarajan [31])**.** $R \leq 10$.

This relied on a counting argument applied to what they called columns in a wall. They nearly proved $R \leq 8$. Brightwell, Kierstead, and Trotter [2] and Narayanaswamy and Subhash Babu [30] finished the job.

Kierstead and Saoub [26] generalized this upper bound on the performance ratio of first-fit to a class of tolerance graphs called p-tolerance graphs. The bound is $8\left\lceil \frac{1}{1-p} \right\rceil$, where $0 \leq p < 1$. A graph is a 0-tolerance graph if and only if it is an interval graph. At the other extreme, they showed that 1-tolerance graphs (also known as bounded tolerance graphs) are not linearly first-fit χ-bound.

Two chains X and Y in a poset are *incomparable* if there are no $x \in X$ and $y \in Y$ so that x and y are comparable. Interval orders have been characterized as those with no two incomparable chains each with 2 elements (cf. Fishburn [9]). For a poset of width w with no two incomparable chains each with s elements, Bosek, Krawczyk, and Szczypka [1] found an upper bound on χ_{FF} of $(3s-2)(w-1)w+w$. Joret and Milans [22] extended the column method of Pemmaraju, Raman, and Varadarajan to generalize the upper bound of 8 on χ_{FF} to $8(s-1)^2w$ for posets of width w

without two incomparable chains each with s elements. While the bound on χ_{FF} of Bosek, Krawczyk, and Szczypka was quadratic in w, this one is linear.

Theorem 5. $R \geq 5$.

Theorem 5 confirms a conjecture of Kierstead and Trotter, who obtained by an unpublished method a lower bound of 4.99999 or so. The graphs of their construction are described by a linear recurrence relation of order 3. Key to the proof is understanding the behavior of linear recurring sequences, in particular, whether like the Fibonacci sequence they increase strictly after a few initial terms. A proof of Theorem 5 appears in Chapter 5, after all the needed results.

2.2 Walls and caps

A coloring is Grundy if each vertex has a neighbor of every lesser color.

Definition. Coloring $f\colon V \to \mathbb{P}$ of graph (V,E) is *Grundy* if $f(N[v]) \supseteq [f(v)]$ for each $v \in V$.

Let $G = (V,E)$ be a graph. By definition, a witness to a lower bound for $\chi_{FF}(G)$ is a linear order of the vertices of G. There may be many vertex orders that witness the same lower bound. In fact, there may be many vertex orders that result in the same Grundy coloring. Given a Grundy coloring of G, it is easy to find such a vertex order. Vertices colored 1 could be first, then those colored 2, and so on. So a Grundy coloring serves as a witness.

Proposition 6. *Every graph $G = (V,E)$ has a Grundy coloring f with* $\max f(V) = \chi_{FF}(G)$.

Proof. The first-fit algorithm produces a Grundy coloring. □

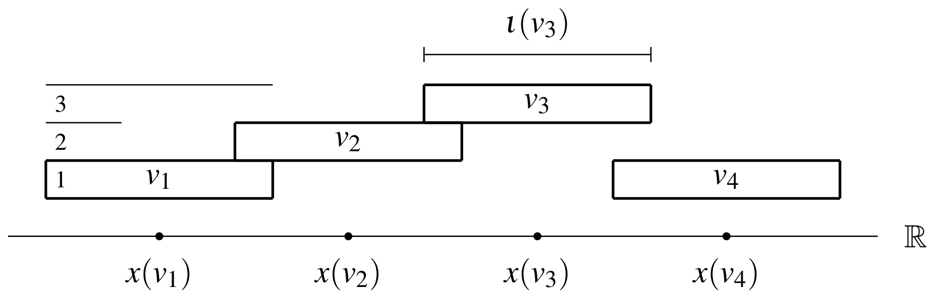

Figure 2.1: First-fit uses 3 colors on a 4-vertex path

Proposition 7. *If f is a Grundy coloring of graph $G = (V,E)$, then* $\max f(V) \leq \chi_{FF}(G)$.

Proof. Let V be ordered $v_1, \ldots, v_n$ where $f(v_j) \leq f(v_k)$ whenever $j \leq k$. □

Definition. Pair (G, f) is a *wall* when $G = (V,E)$ is an interval graph and $f\colon V \to \mathbb{P}$ is a Grundy coloring.

Example 8. Figure 2.1 depicts a wall on 4-vertex path $P_4 = v_1v_2v_3v_4$ with a Grundy coloring of 3 colors (and interval representation ι). So $\chi_{FF}(P_4) \geq 3$. Of course $\chi_{FF}(P_4) \leq 1 + \Delta(P_4) = 3$, where $\Delta(G)$ denotes $\max\{|N(v)| : v \text{ is a vertex of } G\}$.

When intervals are plumped up into unit-height boxes as in Figure 2.1, a wall somewhat resembles a real brick wall. It has discrete levels, notably a bottom level (i.e., level 1). Bricks, or boxes as they are called, are not permitted to overlap. Each box must be supported from below, although of course the Grundy condition is not a real condition for support of bricks.

Example 8 illustrates the essential obstacle to optimality of a Grundy coloring. Given a linear order $<_C$ on the vertices of graph $G = (V,E)$, the result of first-fit coloring in this order is a Grundy coloring f. Chvátal [4] observed that if $\max f(V) > \chi(G)$, then G has an induced 4-vertex-path subgraph *abcd* so that $a <_C b$ and $d <_C c$ (each end of the path precedes its neighbor).

Proposition 9. *Let* (G, f) *be a wall with* $G = (V, E)$. *When* $0 \le m \le \max f(V)$, *there is a wall* (G', f') *with* $G' = (V', E')$ *so that*

$$\begin{aligned} \max f'(V') &= m \\ \omega(G') &\le \omega(G). \end{aligned}$$

Proof. Remove f-maximum (or f-minimum) vertices from G (in whole levels). $\square$

A common way to obtain new graphs from old is to attach all vertices of some graph by all possible edges to some set of vertices in another graph.

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be graphs with $V \cap V' = \emptyset$. Let $U \subseteq V$. Then the graph formed by *attachment* of G' to G at U is the graph with vertex set $V \cup V'$ and edge set $E \cup E' \cup \{uv' \mid u \in U \text{ and } v' \in V'\}$.

Note 10. If G is the graph formed by attachment of $G_1 = (V_1, E_1)$ to $G_0 = (V_0, E_0)$ at U_0, then $G[V_0] = G_0$ and $G[V_1] = G_1$. Each largest clique in G is either

- a largest clique in G_0 or
- induced by U_0 and the vertices of a largest clique in G_1.

In a wall, each vertex must have neighbors in a range of levels. And a wall of height h contains vertices of levels $1, \ldots, h$. So it seems useful for a vertex of a wall to be adjacent to an entire other wall below. However, when large cliques are to be avoided, such a lower wall should not be adjacent to extraneous vertices. This motivates the use of attachment at selected vertices to build walls.

Definition. When $x \in \mathbb{R}$ and $G = (V, E)$ is an interval graph with representation ι, let $V(x) = V_\iota(x) = \{v \in V \mid x \in \iota(v)\}$.

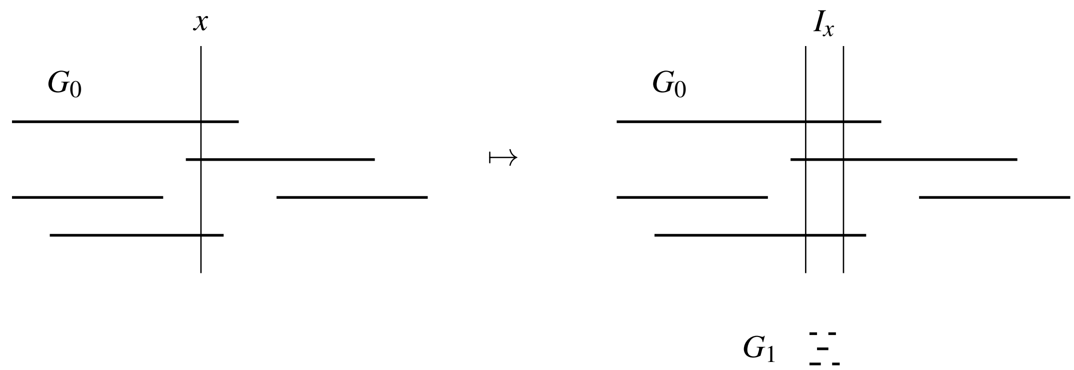

Figure 2.2: Stretching an interval and attaching a graph

Note 11. Let $G = (V,E)$ be an interval graph with representation $\iota\colon V \to \mathscr{I}$. If $x \in \mathbb{R}$, then $G[V(x)]$ is a clique. On the other hand, when $U \subseteq V$ induces a clique in G, there exists $x \in \mathbb{R}$ so that $x \in \iota(v)$ for all $v \in U$.

Proposition 12. *If G is formed by attachment of an interval graph $G_1 = (V_1, E_1)$ (with representation ι_1) to an interval graph $G_0 = (V_0, E_0)$ (with representation ι_0) at $V_0(x)$ for some $x \in \mathbb{R}$, then G is an interval graph.*

Proof. Attach G_1 to G_0 by stretching x (modifying ι_0 as in fig. 2.2) to an interval I_x that contains $\cup\iota_1(V_1)$. □

The operation of Proposition 12 could be called attachment of G_1 to G_0 at x. In every known proof of a lower bound for R, a new wall is produced from a special wall-like structure, here called a cap, by attaching many older walls, resulting in a sequence of walls with large performance ratio eventually. After caps are defined, some noteworthy instances appear in Examples 13 and 14. Then the definition of cap is explained. Procedures for attaching older walls and constructing wall sequences are described in the proof of Theorem 15.

Definition. Here *r-cap* is defined. Let $G = (V,E)$ be an interval graph with $|V| > 0$ and interval representation ι. Let $r > 1$ be rational and $f\colon V \to \mathbb{P}$ a coloring. Let v_I be the vertex represented by interval I; if there are many[1] such vertices, choose the one greatest in f. That is, $f(v_I) = \max f\left(\iota^{-1}(I)\right)$. Then (G,ι,f) is an *r-cap* if there exist a function $t\colon \iota(V) \to \mathbb{P}$ (t for cone top) and an injection[2] $x\colon \iota(V) \to \mathbb{R}$ so that for each $I \in \iota(V)$,

$$x(I) \in I \tag{2.1}$$

$$f(V(x(I))) \subseteq [t(I),\infty) \tag{2.2}$$

$$\max f(V) - t(I) + 1 \geq r\,|V(x(I))| \tag{2.3}$$

$$f(N[v_I]) \supseteq [t(I), f(v_I)] \cap \mathbb{P}. \tag{2.4}$$

Example 13. Figure 2.1, already considered as a wall, depicts also a 3-cap. Table 2.1 contains a precise definition of a 3-cap like it. (In this paper, caps are rarely walls. The coincidence has no use here.)

Example 14. Figure 2.3 depicts a 4-cap of Chrobak and Ślusarek [3]. It is symmetric except for a slight difference in vertical position of the twin unit-height boxes at the top. Only the right part of the cap is shown. Tables 2.2 and 2.3 together define a cap like this one.

[1] In the construction of Kierstead and Trotter, often many vertices are represented by the same interval.

[2] $x(I)$ is the horizontal location of an older wall for attachment as in Proposition 12, and $t(I)$ is the level just above the top level of that older wall.

v	$f(v)$	$\iota(v) = I$	$t(I)$	$x(I)$
v_1	1	$[0,2]$	1	1
v_2	2	$[2,4]$	1	3
v_3	3	$[4,6]$	1	5
v_4	1	$[6,8]$	1	7

Table 2.1: A 3-cap like the one of Figure 2.1

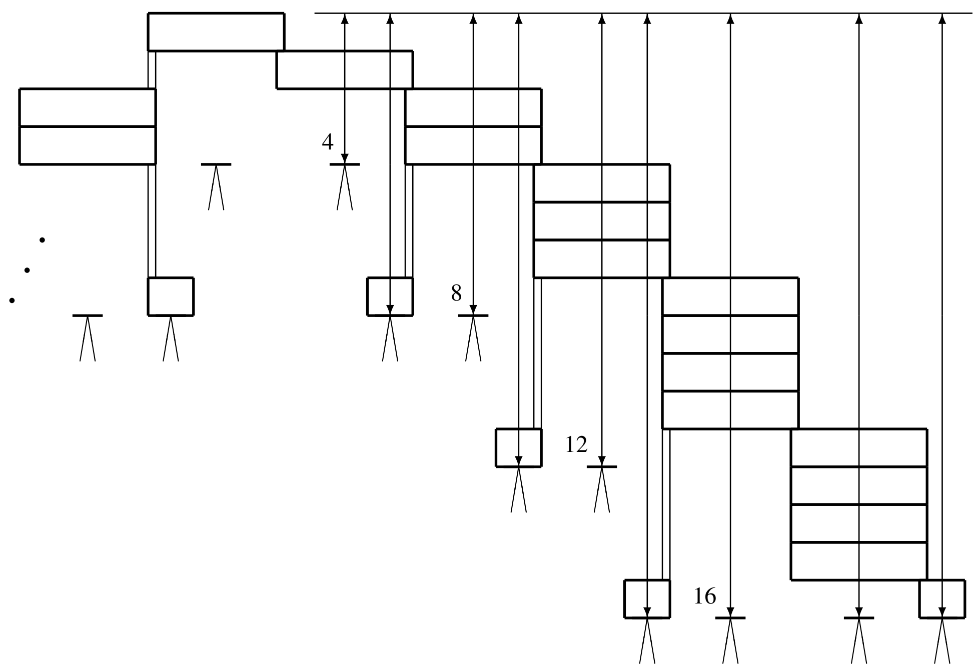

Figure 2.3: A 4-cap of Chrobak and Ślusarek

v	$f(v)$	$\iota(v) = I$	$t(I)$	$x(I)$
1	15	$[0,3]$	13	1.5
2	14	$[3,6]$	9	4.5
3	13	$[3,6]$		
4	9	$[2,3]$	9	2.5
5	12	$[6,9]$	5	7.5
6	11	$[6,9]$		
7	10	$[6,9]$		
8	5	$[5,6]$	5	5.5
9	9	$[9,12]$	1	10.5
10	8	$[9,12]$		
11	7	$[9,12]$		
12	6	$[9,12]$		
13	1	$[8,9]$	1	8.5
14	5	$[12,15]$	1	13.5
15	4	$[12,15]$		
16	3	$[12,15]$		
17	2	$[12,15]$		
18	1	$[15,16]$	1	15.5

Table 2.2: A 4-cap of Chrobak and Ślusarek, right half

It is easy to verify that each of Example 13 and Example 14 define a cap. Observe in Figure 2.3 that

- each interval has a cone beneath it marking where an older wall is to be attached;
- each interval meets some other in each level between its own and the cone top;
- above each cone, at most 1 level in 4 is occupied by an interval.

These are true also of Figure 2.1 with the number 3 in place of 4, but no cones are drawn there. The idea is to place a cap atop many copies of previously constructed

v	$f(v)$	$\iota(v) = I$	$t(I)$	$x(I)$
-1	16	$[-3,0]$	13	-1.5
-2	14	$[-6,-3]$	9	-4.5
-3	13	$[-6,-3]$		
-4	9	$[-3,-2]$	9	-2.5
-5	12	$[-9,-6]$	5	-7.5
-6	11	$[-9,-6]$		
-7	10	$[-9,-6]$		
-8	5	$[-6,-5]$	5	-5.5
-9	9	$[-12,-9]$	1	-10.5
-10	8	$[-12,-9]$		
-11	7	$[-12,-9]$		
-12	6	$[-12,-9]$		
-13	1	$[-9,-8]$	1	-8.5
-14	5	$[-15,-12]$	1	-13.5
-15	4	$[-15,-12]$		
-16	3	$[-15,-12]$		
-17	2	$[-15,-12]$		
-18	1	$[-16,-15]$	1	-15.5

Table 2.3: A 4-cap of Chrobak and Ślusarek, left half

walls by attachment. Observe in Figure 2.3 the sequence $(2,3,4,4)$ of bundles of right-side supporting boxes. Each bundle of k unit-height boxes in consecutive levels will be in future representations fused into a single box of height k. The top level of the bundle corresponding to a given interval I is occupied by vertex $v(I)$. Observe for example (see Figure 2.3 and Table 2.2) that

$$\begin{array}{lcrcrcr} f(v_{[3,6]}) & = & 14 & & & & \\ f(v_{[6,9]}) & = & 12 & = & 14 & - & 2 \\ f(v_{[9,12]}) & = & 9 & = & 12 & - & 3 \\ f(v_{[12,15]}) & = & 5 & = & 9 & - & 4 \\ f(v_{[15,16]}) & = & 1 & = & 5 & - & 4 \end{array}$$

etc.

It is just for simplicity that x gives each interval its own attachment point for an older wall. Also technical is condition (2.2), which keeps attached older walls from sharing levels with adjacent cap vertices. The last two conditions are important. Condition (2.3) means that between the top of each attached older wall and the top of the cap, at most 1 level in r is occupied by a cap vertex. So the sequence of walls tends to r in the ratio of height to largest clique size. Condition (2.4) ensures that each cap vertex has a neighbor in each level down to the top of an attached older wall. The older wall is assumed to provide low-level neighbors to the cap vertex. High-level neighbors appear in the cap itself. So f could be considered as a pseudo-Grundy coloring.

Theorem 15. *When (G,ι,f) is an r-cap with $r>1$, there exist $b\in\mathbb{R}$ and a wall (G_k,f_k) for each $k\geq 0$ so that*

$$\begin{aligned}\omega(G_k) &\leq k\\ \chi_{FF}(G_k) &\geq rk-b.\end{aligned}$$

Proof. By induction on k. Let $G=(V,E)$. Let $\mu=\max f(V)$ and $b=r\mu$. For the base case, when either

$$0\leq k\leq\mu \tag{2.5}$$

or

$$\lceil rk-b\rceil<\mu<k, \tag{2.6}$$

let $G_k=K_k$ and f_k be onto $[k]$. Clearly $\omega(G_k)\leq k$ and f_k is Grundy. In case (2.5),

$$k\geq 0\geq r(k-\mu)=rk-b,$$

and in both cases (2.5) and (2.6), Proposition 7 provides that

$$\chi_{FF}(G_k)\geq k\geq rk-b.$$

For the inductive step, construct (G_k, f_k), where $(G_{k'}, f_{k'})$ has been constructed for $0 \leq k' < k$. Note that $\mu < k$ and

$$\mu \leq \lceil rk - b \rceil. \tag{2.7}$$

Let $I \in \iota(V)$. Then $\iota^{-1}(I) \neq \emptyset$, so $0 < |V(x(I))| \leq \mu < k$, and $G_{k-|V(x(I))|}$ has been constructed already, where

$$\begin{aligned} \chi_{FF}(G_{k-|V(x(I))|}) &\geq \lceil r(k - |V(x(I))|) - b \rceil \\ &\geq \lceil rk - b - (\mu - t(I) + 1) \rceil \text{ (by (2.3))} \\ &= \lceil rk - b \rceil - \mu + t(I) - 1. \end{aligned}$$

By Proposition 9 applied to $G_{k-|V(x(I))|}$, there is an interval graph H_I on vertex set U_I with

$$\omega(H_I) \leq k - |V(x(I))| \tag{2.8}$$

and a Grundy coloring g_I of H_I with

$$g_I(U_I) = [\lceil rk - b \rceil - \mu + t(I) - 1]. \tag{2.9}$$

Build G_k by attaching (as in Proposition 12) to G for each $I \in \iota(V)$ an instance of graph H_I at $x(I)$. Proposition 12 provides that G_k is an interval graph, and Note 10 shows how to assess $\omega(G_k)$. The largest cliques in G have no more than $\mu < k$ vertices. By (2.8), the largest cliques resulting from attachment have at most $|V(x(I))| + (k - |V(x(I))|) = k$ vertices. So $\omega(G_k) \leq k$. Let

$$f_k(v) = \left\{ \begin{array}{ll} \lceil rk - b \rceil - \mu + f(v) & \text{if } v \in V \\ g_I(v) & \text{if } v \in U_I \end{array} \right\}.$$

That f_k is a coloring is shown first. There are three kinds of edge in G_k: those in G, those in some H_I, and those formed by attachment between G and some H_I. If

$vv' \in E$, then $f_k(v) \neq f_k(v')$ because f is a coloring. If vv' is an edge of H_I, then $f_k(v) \neq f_k(v')$ because g_I is a coloring. If $vv' \in V(x(I)) \times U_I$, then

$$\begin{aligned} f_k(v) &= \lceil rk - b \rceil - \mu + f(v) \\ &\geq \lceil rk - b \rceil - \mu + t(I) \text{ (by (2.2))} \\ &> g_I(v') \text{ (by (2.9))} \\ &= f_k(v'). \end{aligned}$$

So f_k is a coloring.

It remains to show that f_k is Grundy. Let $v \in V$. From (2.7) follows $f_k(v) \geq f(v) > 0$, so $f_k(v) \in \mathbb{P}$. In G_k, by (2.4),

$$\begin{aligned} f_k(N[v]) &= f_k(N[v_I]) \\ &\supseteq [\lceil rk - b \rceil - \mu + t(I), f_k(v_I)] \cap \mathbb{P} \\ &\supseteq [\lceil rk - b \rceil - \mu + t(I), f_k(v)] \cap \mathbb{P}, \end{aligned}$$

and by (2.9),

$$f_k(N[v]) \supseteq [\lceil rk - b \rceil - \mu + t(I) - 1].$$

□

Chapter 3

A CONSTRUCTION

The goal of this chapter is to produce r-caps for Theorem 15. First comes a comment on notation. Many of the following definitions depend on r. Because the value of r remains constant throughout the construction, the dependency is not emphasized. That is, in this chapter, $r > 2$ is some fixed rational number.

The goal (i.e., to produce caps) will not be met entirely in this chapter. Only a scheme for producing caps is described. The plan is to relax the definition of cap. The scheme begins with a structure that is cap-like, trivial, and deficient. The deficiency is corrected in many little positive steps, resulting in something large and essentially like half a cap. A minor modification (including doubling the half) yields a cap. By the end of the chapter, the problem of finding caps will have been replaced by the problem of finding number sequences with desired properties. In particular, the sequences will

- satisfy a linear recurrence relation of order 3 and
- avoid the behavior of some sequences, like the Fibonacci sequence, that increase strictly after the first few entries.

In the special caps constructed here, the f-lesser neighbors of v_I of condition (2.4) are associated with at most three intervals, including I itself, as depicted in Figure 3.1. Each box in that figure represents possibly many vertices of a common interval, where the vertices are associated with a range of levels. In the 4-cap of Chrobak and Ślusarek of Figure 2.3, each interior box has support only from an attached older wall (i.e., its cone). But in the present construction, each box may have support from another box on the left or the right (or both or neither). The twin

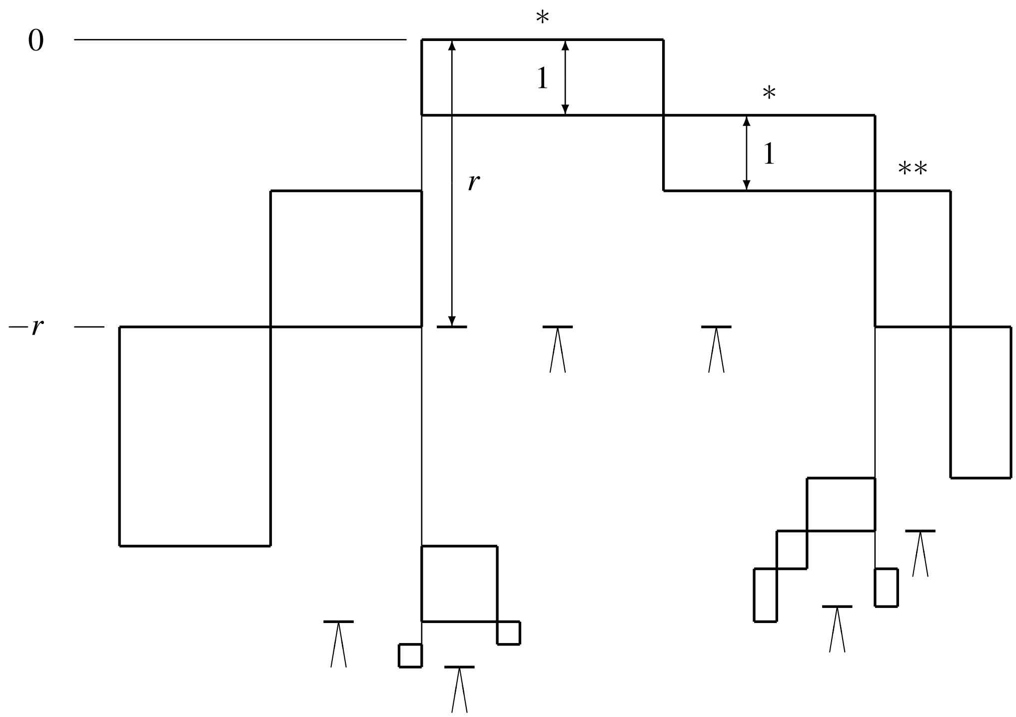

Figure 3.1: A cap of Kierstead and Trotter, in small part

unit-height boxes ($*$) at the top of Figure 3.1 are arbitrary exceptions to this rule, having only left or right support.

In Figure 3.1 there is a vertical extent labeled r. This might seem odd, as one expects the distance to be integral, while r is taken to be rational. It will be shown that these diagrams can be scaled vertically, so rational distances can be made integral. It is important in the present scheme to allow rational vertical distances. The aforementioned relaxation of the definition of cap is that the top of a certain box ($**$) in Figure 3.1 might not be high enough to meet the bottom of the lower of the twin unit-height boxes ($*$). That box ($**$) itself will have its height increased in small (rational) amounts (i.e., to correct its deficiency). Its height begins at only 1 (i.e., is deficient, assuming $r > 3$). Its height is great enough to meet the bottom of the lower of the twin unit-height boxes when it reaches $r-2$.

These elements of the construction (i.e., the 3-interval restriction and fractional levels) appear in the definition of an auxiliary object called a box stack (describing what is depicted in Figure 3.1). Before the rather technical definition, here are some notes. A box stack has n boxes. Horizontally, each box k is associated with an interval I_k and a distinguished point $x_k \in I_k$. The remaining 4 components $\tau(k), \beta(k), \dot{y}(k), y(k)$ give the levels of vertical features of box k. On the 3-interval restriction, each box k is supported (see Figure 3.2) in the cap by a high box whose top is at its own bottom $\beta(k) = \ddot{y}(k)$ and whose bottom is at $\dot{y}(k)$; or if box k has no high supporter, $\ddot{y}(k) = \dot{y}(k)$. And box k is supported in the cap by a low box whose top is at $\dot{y}(k)$ and whose bottom is at $y(k)$; or if box k has no low supporter, $\dot{y}(k) = y(k)$. The top of the cone under box k is at level $y(k)$. Condition (2.2) corresponds to a condition for box stacks called clearance.

Chrobak and Ślusarek construct caps from the top down. So do Kierstead and Trotter. At the start, there are twin unit-height boxes at the top, and boxes are added until the cap is complete. So initially the values of the pseudo-Grundy function at the top of the cap are unknown. It is convenient then to set momentarily to 0 the level of the cap top, and make an adjustment (by positive translation) at the end. Accordingly, until adjustment, levels of box features are usually negative.

Definition. For the present purpose, a *box* is a cartesian product $I \times L$ of real intervals I and L, where L represents a range of levels. A *box stack* is a 6-tuple

$$\big(\{I_k\}_{k\in[n]}, \{x_k\}_{k\in[n]}, \tau, \beta, \dot{y}, y\big)$$

featuring

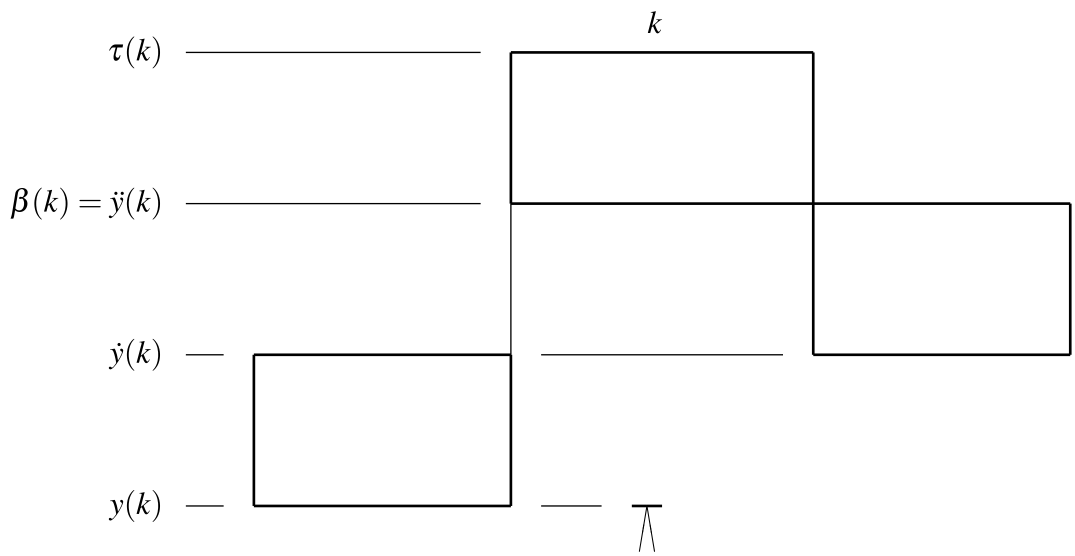

Figure 3.2: Support of a box

- n (distinct) intervals $I_1, \ldots, I_n \in \mathscr{I}$,
- n (distinct) real numbers $x_1, \ldots, x_n$ with $x_k \in I_k$ for all $k \in [n]$,
- n (distinct) boxes $B_k = I_k \times (\beta(k), \tau(k)]$ where $\beta, \tau \colon [n] \to \mathbb{Q}$ with $\beta(k) < \tau(k)$ for all $k \in [n]$ ($\beta(k)$ and $\tau(k)$ are bottom and top of box B_k)

so that

$$B_k \cap B_{k'} = \emptyset \tag{3.1}$$

whenever $kk' \in \binom{[n]}{2}$, and for each $k \in [n]$ there are $y(k) \leq \dot{y}(k) \leq \ddot{y}(k) = \beta(k)$ with

- (*support*) When $y(k) < \dot{y}(k)$, there is some $k' \in [n]$ with $I_k \cap I_{k'} \neq \emptyset$ and

$$\begin{aligned} \beta(k') &= y(k) \\ \tau(k') &= \dot{y}(k); \end{aligned}$$

and similarly, when $\dot{y}(k) < \ddot{y}(k)$, there is some $k' \in [n]$ with $I_k \cap I_{k'} \neq \emptyset$ and

$$\begin{aligned} \beta(k') &= \dot{y}(k) \\ \tau(k') &= \ddot{y}(k). \end{aligned}$$

- (*clearance*) $y(k) \leq \beta(k')$ for all $k' \in [n]$ with $x_k \in I_{k'}$.

Something like condition (2.3) is needed. A box might be considered good if between its cone top and the cap top (i.e., level 0), at most 1 level in r is occupied by an interval. Then the goodness of box k is measured by

$$g(k) = -y(k) - r \sum_{k' \in [n]} \{\tau(k') - \beta(k') \mid x_k \in I_{k'}\}.$$

Precise conditions on the value of $g(k)$ are discussed below, but here one should imagine that box k is good when $g(k) \geq 0$. When two box stacks σ' and σ are being considered, such as when new box stack σ' is built from old box stack σ, these sums are denoted g' and g accordingly.

Note 16. If a fixed continuous injection $\mathbb{R} \to \mathbb{R}$ is applied to each interval $I_1, \ldots, I_n$ and point $x_1, \ldots, x_n$ of a box stack σ, the result is a box stack σ' with $g' = g$. In particular, if a box stack and $a, b \in \mathbb{R}$ with $a < b$ are given, then under some such injection there is a box stack with all its intervals inside interval (a, b).

Definition. A box stack is a *wholestack* if $\max \tau([n]) = 0$ and $g(k) \geq 0$ for each $k \in [n]$.

Lemma 17. *When a wholestack exists, one exists with $\beta([n]) \cup \tau([n]) \subseteq \mathbb{Z}$.*

Proof. Scale by some fixed positive integer each of $\beta, \tau, y, \dot{y}$. □

Theorem 18. *When a wholestack exists, a cap exists.*

Proof. By Lemma 17 assume $\beta([n]) \cup \tau([n]) \subseteq \mathbb{Z}$. A box in a wholestack corresponds to a bundle of intervals in a cap. Let

$$V = \bigcup_{k \in [n]} \left(\{k\} \times [\tau(k) - \beta(k)]\right).$$

Next is a positive translation of levels (by some amount M). For each $k \in [n]$,

$$\begin{aligned} x(I_k) &= x_k \\ t(I_k) &= y(k) + M; \end{aligned}$$

and for each $j \in [\tau(k) - \beta(k)]$,

$$\begin{aligned} \iota((k, j)) &= I_k \\ f((k, j)) &= \tau(k) - j + M; \end{aligned}$$

where M is the smallest integer so that $f(V) \subseteq \mathbb{P}$. An interval graph $G = (V, E)$ with representation ι has been defined. Because $\max \tau([n]) = 0$, there is some $k \in [n]$ with $\tau(k) = 0$, so $(k, 1) \in V \neq \emptyset$, and $\max f(V) = f((k, 1)) = \tau(k) - 1 + M = M - 1$.

Because boxes in a box stack do not overlap, it is no surprise that f is a coloring. Suppose $(k, j)(k', j') \in E$. So $k \neq k'$ or $j \neq j'$. If $k \neq k'$, then by (3.1) and the fact that the vertices are adjacent, $(\beta(k), \tau(k)] \cap (\beta(k'), \tau(k')] = \emptyset$, and one can assume

$\tau(k) \leq \beta(k')$, so

$$\begin{aligned} f((k,j)) &= \tau(k) - j + M \\ &< \tau(k) + M \\ &\leq \beta(k') + M \\ &\leq \tau(k') - j' + M \\ &= f((k',j')). \end{aligned}$$

If $k = k'$, then $f((k,j)) = \tau(k) - j + M \neq \tau(k) - j' + M = f((k',j'))$.

Verification of the remaining conditions is mostly a reconciliation of definitions of cap and box stack. Let $I \in \iota(V)$. Then $I = I_k$ for some $k \in [n]$. For (2.2), suppose $(k',j') \in V(x(I)) = V(x(I_k)) = V(x_k)$. Then $x_k \in \iota((k',j')) = I_{k'}$, so $y(k) \leq \beta(k')$, and

$$\begin{aligned} f((k',j')) &= \tau(k') - j' + M \\ &\geq \beta(k') + M \\ &\geq y(k) + M \\ &= t(I_k) \\ &= t(I). \end{aligned}$$

For (2.3),

$$\begin{aligned} 0 &\leq -y(k) - r \sum_{k' \in [n]} \{\tau(k') - \beta(k') \mid x_k \in I_{k'}\} \\ &= \max f(V) - t(I_k) + 1 - r|V(x(I_k))|. \end{aligned}$$

For (2.4),

$$
\begin{aligned}
f(N[v_{I_k}]) &\supseteq f(\{(k,j) \mid j \in [\tau(k)-\beta(k)]\}) \\
&= \{\tau(k)-j+M \mid j \in [\tau(k)-\beta(k)]\} \\
&= (\beta(k)+M-1, f(v_{I_k})] \cap \mathbb{P};
\end{aligned}
$$

and for $z \in \{y(k), \dot{y}(k)\}$, when $\dot{z} > z$,

$$
\begin{aligned}
f(N[v_{I_k}]) &\supseteq f(\{(k',j) \mid j \in [\dot{z}-z]\}) \\
&= \{\tau(k')-j+M \mid j \in [\dot{z}-z]\} \\
&= \{\dot{z}-j+M \mid j \in [\dot{z}-z]\} \\
&= (z+M-1, \dot{z}+M-1] \cap \mathbb{P};
\end{aligned}
$$

so

$$
\begin{aligned}
f(N[v_{I_k}]) &\supseteq (y(k)+M-1, f(v_{I_k})] \cap \mathbb{P} \\
&= [t(I_k), f(v_{I_k})] \cap \mathbb{P}.
\end{aligned}
$$

□

The new goal is to produce a wholestack. As discussed in the introduction to the chapter, this goal is approached by correcting a deficiency, or perhaps advancing toward completion, incrementally. The deficient object is called a halfstack, depicted in Figure 3.3. It is like half a cap, as in Figure 2.3, but does not include either of the twin unit-height top boxes. For this reason, boxes destined to go under one of the twin unit-height top boxes must have goodness in excess, that is, goodness r instead of 0. A single parameter θ, which stands for the height of the top[1] box of

[1] The top box of a halfstack is always indexed $k = 1$ in this construction.

the halfstack, expresses the degree of completion of the halfstack. A θ-halfstack is nascent when $\theta = 1$, and complete when $\theta = r-2$.

Definition. A box stack is a θ-*halfstack* when

$$\begin{aligned} \theta &= \tau(1) - \beta(1) \\ -r &= \beta(1) \\ 0 &\in I_1 \subseteq [0,\infty) \end{aligned}$$

and for each $k \in [n]$

$$\begin{aligned} \tau(k) &\leq \tau(1) \\ x_k &\neq 0 \\ g(k) &\geq \left\{ \begin{array}{ll} 0 & \text{if } x_k > 0 \\ r & \text{if } x_k < 0 \end{array} \right\}. \end{aligned}$$

A trivial halfstack begins the construction.

Proposition 19. *A* 1*-halfstack exists.*

Proof. One is $(\{[0,1]\},\{1\},\tau,\beta,\dot{y},y)$ where $\tau(1) = 1-r$ and $\beta(1) = \dot{y}(1) = y(1) = -r$. □

An $(r-2)$-halfstack is final.

Lemma 20. *When an* $(r-2)$*-halfstack exists, a wholestack exists.*

Proof of Lemma 20 is deferred while a theorem with a similar proof is stated. Here is how to improve halfstacks.

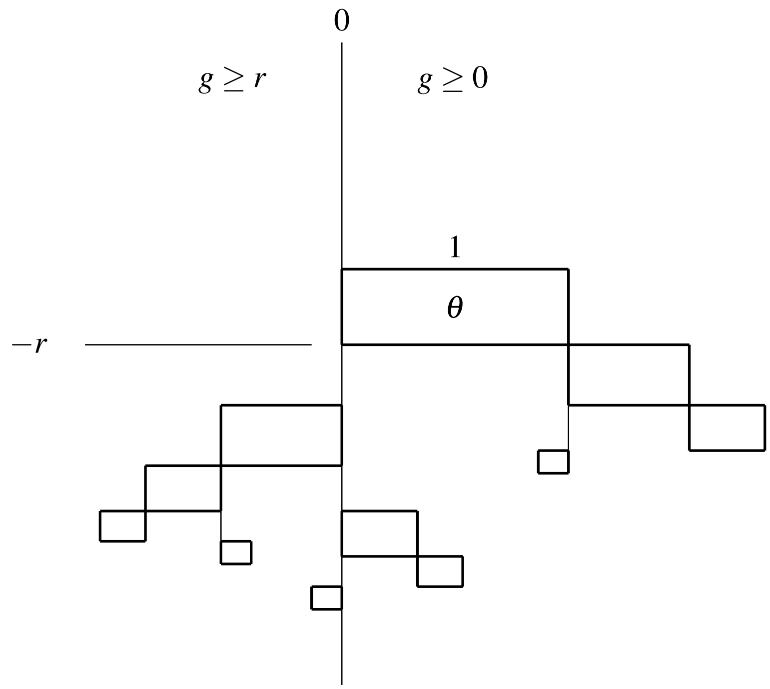

Figure 3.3: A halfstack

Definition. When r, θ, and δ are real numbers, linear recurring sequence (u_n) is:

$$\begin{aligned} u_0 = u_1 &= 1 \\ u_2 &= \theta + \delta \\ u_n &= (r-\theta)u_{n-1} - (r-2\theta)u_{n-2} - \theta u_{n-3} \end{aligned}$$

for $n \geq 3$.

Theorem 21. *When there exist a θ-halfstack, $1 \leq \theta < r$, and rational $\delta > 0$ with*

$$u_1 < u_2 < \cdots < u_N \geq u_{N+1}$$

for some $N \in \mathbb{P}$, a $(\theta + \delta)$-halfstack exists.

The rest of Chapter 3 proves Lemma 20 and Theorem 21. First is described generally how box stacks are assembled. The box-stack assembly procedure is used in both proofs.

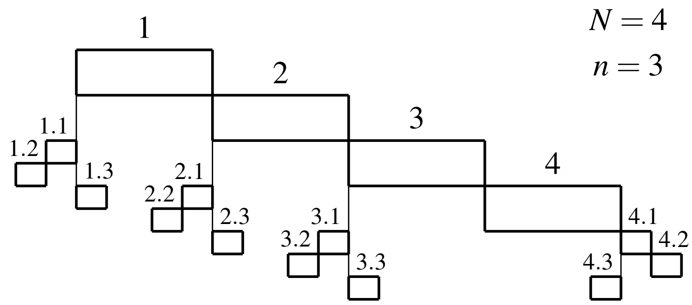

Figure 3.4: A new box stack

3.1 Box-stack assembly

Figure 3.4 depicts an assembled box stack. A box stack is made from new boxes $1, \ldots, N$ plus instances $1, \ldots, N$ of a single θ-halfstack of n boxes, where the instances are independently scaled (vertically and horizontally) and translated (vertically and horizontally) as needed. All instances but one (i.e., N, the last) are reversed horizontally. Before a formal account of the details comes a sketch of a proof of Theorem 21.

Improvement of halfstacks

Here is an informal account of how halfstacks are improved (as in Theorem 21). Scaling and translating instances horizontally is not really interesting, as Figure 3.4 is adequate to show where they belong. Nor is it difficult, as the instances just need to be so narrow as to avoid meeting.

Scaling and translating instances vertically is the challenge, as each box must be good. An important guiding principle is that with few exceptions, no box of the new halfstack is better (i.e., more good) than required (i.e., r-good or 0-good). The exceptions are new box N and the boxes of instance N. Those may be extra good, like the right-most box[2] of Figure 2.3.

A new halfstack is built from the top down, as was the 4-cap of Chrobak and Ślusarek. Recall the sequence $(2,3,4,4)$ of right-side supporting bundles in Figure 2.3. The N new boxes in a new halfstack are much like those bundles. A halfstack does not include the twin unit-height boxes at the top, so the first box (i.e., new box 1) to be introduced is something like the 2-bundle of Figure 2.3. By the method of Chrobak and Ślusarek, that box would be of height $r-2$. However, in the construction of Kierstead and Trotter, only an incremental advance of $\delta > 0$ in the height of box 1 is made (in the previous halfstack it is θ, so in the new one it is $\theta + \delta$). The choice of δ should be regarded as arbitrary. Due to more rules below, this is the only choice in the construction. It is an important choice, though. When it is too large, the construction fails. Choosing δ is the subject of Chapter 4.

The new halfstack consists so far only of new box 1, and box 1 must be supported. The level of the cone under this box should be determined now. The next addition to the new halfstack is instance 1. It is to be positioned so that its own box 1 (labeled 1.1 in Figure 3.4) is the low supporter of new box 1. In particular, the instance must be scaled and translated vertically so that the old box 1 rests at the level of the top of the cone of new box 1. There is a unique solution to this problem, subject to the guiding principle (that boxes of instance 1 are good but not extra good in the new halfstack). Caution: instance 1 is special in that some of its boxes may need to be r-good, and others 0-good. This is because the left edge of new box 1 is at $x = 0$ (see Figure 3.3), which might divide instance 1. After instance 1 is in

[2] Note that its cone is 16 levels from the cap top, only 1 of which is occupied by an interval.

place, if there is a gap between the bottom of new box 1 and the top of box 1.1, then add new box 2 with just enough height to cover the gap (i.e., to complete the support of new box 1).

Now the recently added new box 2 must be supported. As before, the level of the cone of new box 2 is determined. Instance 2 is scaled and translated so that its own box 1 (labeled 2.1 in Figure 3.4) is the low supporter of box 2, subject to the guiding principle for boxes of instance 2. If needed, add new box 3.

And so on. It is unclear how this procedure ends. It should be supposed that the value of N is not foreordained, but instead some terminal condition arises in the course of the construction. By the preceding paragraphs, that condition would seem to be that some instance provides enough support to its corresponding new box that a subsequent new box is unnecessary. But another condition suffices. In the sequence of new boxes, if some new box N is no taller than its predecessor, then instance N can be just a (horizontally reversed) copy of instance $N-1$, where instance N gives right-side support to new box N rather than left (see Figure 3.4). The procedure ends with all boxes good.

This terminal condition explains the hypothesis of Theorem 21, as sequence (u_n) is defined to implement[3] the procedure above. Note that (perhaps unexpectedly) u_{n+1} is the height of new box n for $n \in [N]$ (see Figure 3.6). The terminal condition also explains why most instances are reversed horizontally. Halfstacks have excess goodness on the left. This property is preserved by vertical scaling, although the amount of excess varies. All boxes of a given instance are scaled by the same amount. But the sequence of new boxes increases in height from left to right,

[3] Does the procedure halt for some given r, θ, and δ? It does if $u_{n+1} \leq u_n$ for some $n > 0$.

so a given instance needs more goodness for its right boxes than its left boxes. Horizontal reversal sends excess goodness where it is needed (i.e., where the burden above is greater, the right).

A definite account

Given a halfstack σ of n boxes, here is how a box stack σ' of N new boxes and N instances of σ is obtained. While features of box stack σ are denoted I, x, τ, g, etc., those of σ' are denoted I', x', τ', g', etc. For example, $n' = N + Nn$. In Lemma 20, the resulting box stack σ' is a wholestack. In Theorem 21, σ' is a halfstack. Because wholestacks and halfstacks have different conditions for goodness, Note 24 shows only how to assess g'. The actual assessments appear separately, in the proofs of Lemma 20 and Theorem 21. Until Note 24, the aim is to show that σ' is a box stack. Scaling and translating vertically the N instances of σ is a major concern. The affine transformation that implements these operations for instance K, where $K \in [N]$, is denoted ψ_K in the following.

Given

- integer $N \geq 2$
- halfstack $\sigma = \left(\{I_k\}_{k\in[n]}, \{x_k\}_{k\in[n]}, \tau, \beta, \dot{y}, y\right)$

- rational numbers[4]

$$\begin{array}{rlcl} & \tau'(1) & > & \beta'(1) \\ = & \tau'(2) & > & \beta'(2) \\ = & \tau'(3) & > & \cdots \\ & \cdots & > & \beta'(N-1) \\ = & \tau'(N) & > & \beta'(N) \end{array}$$

- an increasing[5] affine function $\psi_K\colon \mathbb{Q} \to \mathbb{Q}$ for each $K \in [N]$ with

$$\psi_N \circ \tau(1) = \beta'(N) \tag{3.2}$$

$$\psi_K \circ \tau(1) = \beta'(K+1) \tag{3.3}$$

for each $K \in [N-1]$,

it will be shown that some 6-tuple

$$\sigma' = \left(\{I'_k\}_{k\in[n']}, \{x'_k\}_{k\in[n']}, \tau', \beta', \dot{y}', y'\right)$$

is a box stack where $n' = N(n+1)$. Then g' will be expressed in terms of g. Already σ' is partly defined by τ', β' as given above.

Condition (3.3) ensures that for each $K \in [N-1]$, instance $K+1$ is positioned to support new box K as in Figure 3.4. Condition (3.2) deals with case $K = N$. Accordingly, let $y'(K) = \psi_K \circ \beta(1)$ for each $K \in [N]$; let $\dot{y}'(K) = \beta'(K+1)$ for all $K \in [N-1]$ and $\dot{y}'(N) = \beta'(N)$.

[4] In practice, only $\tau'(1), \tau'(2), \ldots, \tau'(N), \beta'(N)$ of these are defined, and the equations are implied.

[5] ψ_K is increasing simply because box stacks are never reversed vertically.

The horizontal placement of new boxes should not present any difficulty. But to avoid any misunderstanding, let

$$\begin{array}{rcccl} & & x'_0 & < & 0 \\ = l'_1 & < & x'_1 & < & r'_1 \\ = l'_2 & < & x'_2 & < & r'_2 \\ = l'_3 & < & \cdots & & \\ & & \cdots & < & r'_{N-1} \\ = l'_N & < & x'_N & < & r'_N \\ & < & x'_{N+1} & & \end{array}$$

be real numbers and $I'_K = [l'_K, r'_K]$ for $K \in [N]$. This completes the definition of σ' for the first N elements of $[n']$, i.e., the N new boxes.

In order to concisely address the remaining Nn elements of $[n']$, i.e., boxes of the N instances, the notation

$$K.k = N + n(K-1) + k$$

is defined for each $K \in [N]$ and $k \in [n]$. (This notation is used to label boxes in Figure 3.4.) The instances are scaled and translated horizontally, and all but the last is reversed horizontally. This is accomplished by a transformation φ_K for each $K \in [N]$. For each $K \in [N-1]$ let $\varphi_K : \mathbb{R} \to \mathbb{R}$ be a continuous (strictly) decreasing[6] function with

$$\begin{array}{rcl} \varphi_K(0) & = & l'_K \\ \varphi_K(I_k) & \subseteq & \text{interval } (x'_{K-1}, x'_K) \end{array}$$

[6] φ_K is decreasing for $K \in [N-1]$ to effect horizontal reversal of instance K.

for each $k \in [n]$. Let $\varphi_N \colon \mathbb{R} \to \mathbb{R}$ be a continuous increasing function with

$$\begin{aligned}
\varphi_N(0) &= r'_N \\
\varphi_N(I_k) &\subseteq \text{interval } (x'_N, x'_{N+1}).
\end{aligned}$$

For each $K \in [N]$ and $k \in [n]$ let

$$\begin{aligned}
I'_{K.k} &= \varphi_K(I_k) \\
x'_{K.k} &= \varphi_K(x_k) \\
\tau'(K.k) &= \psi_K \circ \tau(k) \\
\beta'(K.k) &= \psi_K \circ \beta(k) \\
\dot{y}'(K.k) &= \psi_K \circ \dot{y}(k) \\
y'(K.k) &= \psi_K \circ y(k).
\end{aligned}$$

That is, every feature of instance K undergoes the same horizontal transformation φ_K and vertical transformation ψ_K. This completes the definition of σ' for the N instances. Indeed, the definition of σ' is complete.

Many conditions of box stacks are to be verified for σ', and many of these are addressed in several cases. Certain conditions are so obvious in σ' that they are mentioned, but not explained.

- intervals $I'_1, \ldots, I'_{n'}$ are distinct
- points $x'_1, \ldots, x'_{n'}$ are distinct
- $x'_K \in I'_K$ for each $K \in [N]$
- $x'_{K.k} \in I'_{K.k}$ for each $K \in [N]$ and $k \in [n]$.

Also

- $\beta'(K) < \tau'(K)$ for each $K \in [N]$
- $\beta'(K.k) < \tau'(K.k)$ for each $K \in [N]$ and $k \in [n]$
- $y'(K) \le \dot{y}'(K) \le \beta'(K)$ for each $K \in [N]$
- $y'(K.k) \le \dot{y}'(K.k) \le \beta'(K.k)$ for each $K \in [N]$ and $k \in [n]$.

Further

- if $KK' \in \binom{[N]}{2}$, then $B_K \cap B_{K'} = \emptyset$
- if $K, K' \in [N]$ and $K \notin \{K'-1, K'\}$, then $B_K \cap B_{K'.k} = \emptyset$ for each $k \in [n]$ because $I_K \cap I_{K'.k} = \emptyset$
- if $\{K.k, K.k'\} \in \binom{[n']}{2}$, then $B_{K.k} \cap B_{K.k'} = \emptyset$
- if $\{K.k, K'.k'\} \in \binom{[n']}{2}$ with $K \neq K'$, then $B_{K.k} \cap B_{K'.k'} = \emptyset$
- $B_{N-1} \cap B_{N.k} = \emptyset$ for each $k \in [n]$.

Easy conditions (for σ' being a box stack) have been verified. Others remain.

Proposition 22. *$B_K \cap B_{K.k} = \emptyset$ for each $K \in [N]$ and $k \in [n]$.*

Proof.

$$\begin{aligned}
\tau'(K.k) &= \psi_K \circ \tau(k) \\
&\le \psi_K \circ \tau(1) \\
&= \dot{y}'(K) \\
&\le \beta'(K).
\end{aligned}$$

□

Proposition 23. *$B_K \cap B_{(K+1).k} = \emptyset$ for each $K \in [N-2]$ and $k \in [n]$.*

Proof.

$$\begin{aligned}
\tau'((K+1).k) &= \psi_{K+1} \circ \tau(k) \\
&\leq \psi_{K+1} \circ \tau(1) \\
&= \dot{y}'(K+1) \\
&\leq \beta'(K+1) \\
&< \beta'(K).
\end{aligned}$$

□

So $B_k \cap B_{k'} = \emptyset$ for all $kk' \in \binom{[n']}{2}$. Here support is addressed:

- $\dot{y}'(K) < \ddot{y}'(K)$ for each $K \in [N-1]$ with support from box $K+1$;
- $y'(K) < \dot{y}'(K)$ for each $K \in [N]$ with support from box $K.1$.

Support is provided in the remaining cases by definition of φ_K and ψ_K. The issue of clearance is straightforward. Note that $x'_K > 0$ for all $K \in [N]$. When $K \in [N]$ and $k \in [n]$, one has $x'_{K.k} < 0$ if and only if $K = 1$ and $x_k > 0$. So σ' is a box stack (given $N, \sigma, \tau', \beta', \psi$ as described at the beginning of the section).

Generic properties of box stacks have been verified. Now consider those special to wholestacks and halfstacks. Here g' is expressed in terms convenient for use below.

Note 24. Assume $\psi_K = c_K \mathrm{id} + d_K$ where c_K and d_K are rational, $c_K > 0$, and id is the identity function $x \mapsto x$. For $K \in [N]$,

$$\begin{aligned} g'(K) &= -\psi_K \circ \beta(1) - r(\tau'(K) - \beta'(K)) \\ &= rc_K - d_K - r(\tau'(K) - \beta'(K)). \end{aligned}$$

For $K \in \{1, N\}$ and $k \in [n]$, when $x_k > 0$,

$$\begin{aligned} g'(K.k) &= -\psi_K \circ y(k) - r \sum_{k' \in [n]} \{\psi_K \circ \tau(k') - \psi_K \circ \beta(k') \mid x_k \in I_{k'}\} \\ &= -c_K \circ y(k) - d_K - rc_K \sum_{k' \in [n]} \{\tau(k') - \beta(k') \mid x_k \in I_{k'}\} \\ &= c_K g(k) - d_K, \end{aligned}$$

and when $x_k < 0$,

$$\begin{aligned} g'(K.k) &= -\psi_K \circ y(k) - r(\tau'(K) - \beta'(K)) \\ &\quad -r \sum_{k' \in [n]} \{\psi_K \circ \tau(k') - \psi_K \circ \beta(k') \mid x_k \in I_{k'}\} \\ &= c_K g(k) - d_K - r(\tau'(K) - \beta'(K)). \end{aligned}$$

For $1 < K < N$ and $k \in [n]$, when $x_k > 0$,

$$\begin{aligned} g'(K.k) &= -\psi_K \circ y(k) - r(\tau'(K-1) - \beta'(K-1)) \\ &\quad -r \sum_{k' \in [n]} \{\psi_K \circ \tau(k') - \psi_K \circ \beta(k') \mid x_k \in I_{k'}\} \\ &= c_K g(k) - d_K - r(\tau'(K-1) - \beta'(K-1)), \end{aligned}$$

and when $x_k < 0$,

$$\begin{aligned} g'(K.k) &= -\psi_K \circ y(k) - r(\tau'(K) - \beta'(K)) \\ &\quad -r \sum_{k' \in [n]} \{\psi_K \circ \tau(k') - \psi_K \circ \beta(k') \mid x_k \in I_{k'}\} \\ &= c_K g(k) - d_K - r(\tau'(K) - \beta'(K)). \end{aligned}$$

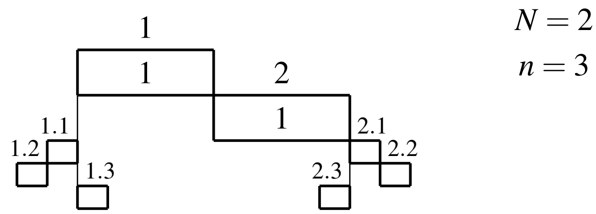

Figure 3.5: A new wholestack as in Lemma 20

3.2 Proofs pending

It remains to verify that box stacks constructed by the procedure of Section 3.1 have the desired properties of wholestacks or halfstacks. A wholestack resulting from Lemma 20 is depicted in Figure 3.5.

Proof of Lemma 20. In Section 3.1, let

$$\begin{aligned} N &= 2 \\ \tau'(1) &= 0 \\ \tau'(2) &= -1 \\ \beta'(2) &= -2 \\ \psi_1 = \psi_2 &= \text{id.} \end{aligned}$$

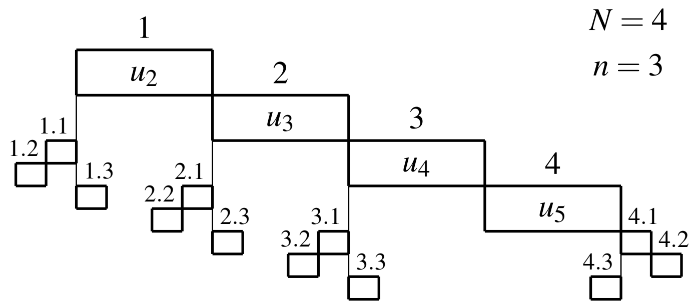

Figure 3.6: A new halfstack as in Theorem 21

Equations (3.3) and (3.2) hold:

$$\begin{aligned}
\psi_1 \circ \tau(1) &= \tau(1) \\
&= \theta + \beta(1) \\
&= (r-2) - r \\
&= \beta'(2) \\
\psi_2 \circ \tau(1) &= \beta'(2),
\end{aligned}$$

so σ' is a box stack. For $K \in [2]$, (by Note 24)

$$g'(K) = r - r(1) = 0,$$

and for $k \in [n]$, when $x_k > 0$,

$$g'(K.k) = g(k) \geq 0,$$

and when $x_k < 0$,

$$g'(K.k) = g(k) - r \geq 0,$$

so σ' is a wholestack. $\square$

A halfstack resulting from Theorem 21 is depicted in Figure 3.6.

Proof of Theorem 21. Note that $u_2 = \theta + \delta > 1 = u_1$, so $N \geq 2$. In Section 3.1, let

$$\begin{aligned}
\tau'(1) &= -r + u_2 \\
\tau'(K+1) &= \tau'(K) - u_{K+1} \\
\beta'(N) &= \tau'(N) - u_{N+1} \\
\psi_K &= (u_{K+1} - u_K)\mathrm{id} - ru_K \\
\psi_N &= \psi_{N-1}
\end{aligned}$$

for $K \in [N-1]$. It should be verified that each new box has positive height. Clearly

$$\tau'(K) - \beta'(K) = \tau'(K) - \tau'(K+1) = u_{K+1} > u_0 > 0$$

for $K \in [N-1]$, and for the remaining case $K = N$, observe that

$$\begin{aligned}
\tau'(N) - \beta'(N) &= u_{N+1} \\
&= (r-\theta)u_N - (r-2\theta)u_{N-1} - \theta u_{N-2} \\
&= (r-\theta)(u_N - u_{N-1}) + \theta(u_{N-1} - u_{N-2}) > 0.
\end{aligned}$$

Of course ψ_K is increasing for $K \in [N]$. By induction on K is shown (3.3). For base case $K = 1$,

$$\begin{aligned}
\beta'(2) &= -r - u_3 \\
&= -r - (r-\theta)u_2 + (r-2\theta) + \theta \\
&= \theta(u_2 - 1) - ru_2 \\
&= (u_2 - u_1)(\theta - r) - ru_1 \\
&= \psi_1 \circ \tau(1).
\end{aligned}$$

For the inductive step where $1 < K < N$,

$$\begin{aligned}
\beta'(K+1) &= \beta'(K) - u_{K+2} \\
&= \psi_{K-1} \circ \tau(1) - u_{K+2} \text{ (by inductive assumption)} \\
&= (u_K - u_{K-1})(\theta - r) - ru_{K-1} - (r-\theta)u_{K+1} + (r-2\theta)u_K + \theta u_{K-1} \\
&= u_K(\theta - r) - (r-\theta)u_{K+1} + (r-2\theta)u_K \\
&= (u_{K+1} - u_K)(\theta - r) - ru_K \\
&= \psi_K \circ \tau(1).
\end{aligned}$$

The last case $K = N-1$ of (3.3) is identical to (3.2). So σ' is a box stack.

It remains to show that each box is good (again using Note 24). In most cases, boxes need be only 0-good. But in one half of instance 1, boxes must be r-good.

$$g'(N) = r(u_N - u_{N-1}) + ru_{N-1} - ru_N = 0,$$

and for $K \in [N-1]$,

$$g'(K) = r(u_{K+1} - u_K) + ru_K - ru_{K+1} = 0.$$

For $k \in [n]$, when $x_k > 0$,

$$\begin{aligned}
g'(1.k) &= c_1 g(k) - d_1 \geq ru_1 = r \\
g'(N.k) &= c_N g(k) - d_N \geq ru_{N-1} \geq 0,
\end{aligned}$$

and when $x_k < 0$,

$$\begin{aligned}
g'(1.k) &= c_1 g(k) - d_1 - r(\tau'(1) - \beta'(1)) \\
&\geq (u_2 - u_1)r + ru_1 - ru_2 = 0 \\
g'(N.k) &= c_N g(k) - d_N - r(\tau'(N) - \beta'(N)) \\
&\geq (u_N - u_{N-1})r + ru_{N-1} - ru_N = 0.
\end{aligned}$$

For $1 < K < N$ and $k \in [n]$, when $x_k > 0$,

$$\begin{aligned} g'(K.k) &= c_K g(k) - d_K - r(\tau'(K-1) - \beta'(K-1)) \\ &\geq ru_K - ru_K = 0, \end{aligned}$$

and when $x_k < 0$,

$$\begin{aligned} g'(K.k) &= c_K g(k) - d_K - r(\tau'(K) - \beta'(K)) \\ &\geq (u_{K+1} - u_K)r + ru_K - ru_{K+1} = 0. \end{aligned}$$

Finally,

$$\begin{aligned} \tau'(1) - \beta'(1) &= u_2 = \theta + \delta \\ \beta'(1) &= -r \\ 0 &\in I'_1 = [0, r'_1] \subseteq [0, \infty) \\ x'_k &\neq 0 \\ \tau'(k) &\leq \tau'(1) \end{aligned}$$

for all $k \in [n']$, so σ' is a $(\theta + \delta)$-halfstack. $\square$

Chapter 4

A SPECIAL SEQUENCE

The previous chapter suggests that caps may be gained if the behavior of a particular sequence (u_n) is understood. Here again is its definition.

Definition. When r, θ, and δ are real numbers, sequence (u_n) is:

$$
\begin{aligned}
u_0 = u_1 &= 1 \\
u_2 &= \theta + \delta \\
u_n &= (r-\theta)u_{n-1} - (r-2\theta)u_{n-2} - \theta u_{n-3}
\end{aligned}
$$

for $n \geq 3$.

Sequence (u_n) and its difference sequence $(u_{n+1} - u_n)$ are linear homogeneous recurring sequences. The ordinary power series generating function

$$f(x) = \sum_{n \geq 0} (u_{n+1} - u_n) x^n$$

of the difference sequence $(u_{n+1}-u_n)$ is[1] (cf. chapter 4 of Stanley [35])

$$f(x)=x[(\theta+\delta)(1-x)-1]/q(x) \tag{4.1}$$

where

$$\begin{aligned} q(x) &= 1-(r-\theta)x+(r-2\theta)x^2+\theta x^3 && (4.2)\\ &= 1+rx(x-1)+\theta x(x-1)^2 && (4.3)\\ &= (1-x/\alpha)(1-x/\beta)(1-x/\gamma). && (4.4)\end{aligned}$$

Note that q depends on r and θ, but not δ. Equation (4.4) gives names α,β, and γ to the (complex) roots of q. The roots partly determine the asymptotic behavior of the sequence. When they are distinct,

$$f(x)=\frac{A}{1-x/\alpha}+\frac{B}{1-x/\beta}+\frac{C}{1-x/\gamma}$$

1

$$\begin{aligned}
f_0(x) &= \sum_{n\geq 0} u_n x^n\\
0 &= \sum_{n\geq 3} u_n x^n-(r-\theta)\sum_{n\geq 3} u_{n-1}x^n+(r-2\theta)\sum_{n\geq 3} u_{n-2}x^n+\theta\sum_{n\geq 3} u_{n-3}x^n\\
&= [f_0(x)-1-x-(\theta+\delta)x^2]-(r-\theta)x[f_0(x)-1-x]+\\
&\quad (r-2\theta)x^2[f_0(x)-1]+\theta x^3[f_0(x)]\\
f_0(x) &= p_0(x)/q(x)\\
p_0(x) &= 1+(1+\theta-r)x+\delta x^2\\
xf(x) &= \sum_{n\geq 0} u_{n+1}x^{n+1}-x\sum_{n\geq 0} u_n x^n\\
&= (1-x)f_0(x)-1\\
xq(x)f(x) &= (1-x)p_0(x)-q(x)\\
&= 1+(1+\theta-r)x+\delta x^2-x-(1+\theta-r)x^2-\delta x^3\\
&\quad -1+(r-\theta)x-(r-2\theta)x^2-\theta x^3\\
&= (\theta+\delta-1)x^2-(\theta+\delta)x^3.
\end{aligned}$$

and

$$u_{n+1}-u_n = A\alpha^{-n}+B\beta^{-n}+C\gamma^{-n} \tag{4.5}$$

for some complex numbers A,B,C. By (4.4),

$$q(x)f(x) = A(1-x/\beta)(1-x/\gamma)+B(1-x/\alpha)(1-x/\gamma)+C(1-x/\alpha)(1-x/\beta). \tag{4.6}$$

The next observation holds in all cases that matter: some root, say α, is real and unimportant. (It is not yet clear whether β and γ are real.)

Proposition 25. *When $1 \le \theta \le 0.5r \le 3\theta$, it can be assumed that $\alpha < -1$ and $0 < \beta\gamma < 1$.*

Proof. The first conclusion follows from the intermediate value theorem and (4.3):

$$\begin{aligned} q(-1) &= 1+2r-4\theta > 0 \\ q(-10) &= 1+110r-1210\theta < 0. \end{aligned}$$

The second conclusion follows from Viète's laws:

$$-\alpha\beta\gamma = \theta^{-1} \le 1.$$

□

For the rest of the chapter, r and θ are restricted in order to facilitate computation:

$$4.999 \le r \le 5$$

$$1 \le \theta \le 2.2.$$

So Proposition 25 applies.

Proposition 26. *When $\theta \leq 2.13$, it can be assumed that*

$$0 < \gamma < 0.56 < \beta < 1.$$

Proof. By the intermediate value theorem. Using (4.3),

$$q(0) = q(1) = 1 > 0$$

and

$$q(0.56) \leq 1 - (4.999)(0.56)(0.44) + (2.13)(0.56)(0.44)^2 < 0.$$

$\square$

Theorem 27. *If $\alpha < -1 < 0 < \gamma < \beta < 1$, then $u_{n+1} - u_n \to -\infty$ when $\theta + \delta < \dfrac{1}{1-\gamma}$.*

Proof. In this case,

$$u_{n+1} - u_n \sim C\gamma^{-n}$$

(by (4.5)), and the desired conclusion follows when $C < 0$. By equations (4.1) and (4.6),

$$\begin{aligned} x[(\theta+\delta)(1-x)-1] &= A(1-x/\beta)(1-x/\gamma)+ \\ &\quad B(1-x/\alpha)(1-x/\gamma)+ \\ &\quad C(1-x/\alpha)(1-x/\beta). \end{aligned} \tag{4.7}$$

Substituting γ for x in (4.7),

$$\gamma[(\theta+\delta)(1-\gamma)-1] = C(1-\gamma/\alpha)(1-\gamma/\beta).$$

Because

$$0 < (1-\gamma/\alpha)(1-\gamma/\beta),$$

one has $C < 0$ when

$$(\theta+\delta)(1-\gamma)-1<0,$$

or

$$\theta+\delta<\frac{1}{1-\gamma}.$$

$\square$

Proposition 28. *If $\theta \leq 2.13$, then γ decreases (strictly) in r, and so does $\dfrac{1}{1-\gamma}-\theta$.*

Proof. Suppose

$$4.999 \leq r_0 < r_1 \leq 5.$$

By (4.3), for $j \in \{0,1\}$,

$$q_j(x) = 1 + r_j x(x-1) + \theta x(x-1)^2$$

has roots

$$0 < \gamma_j < \beta_j < 1$$

by Proposition 26. When $\gamma_0 \leq x \leq \beta_0$,

$$0 \geq q_0(x) = 1 + r_0 x(x-1) + \theta x(x-1)^2 \tag{4.8}$$

and

$$0 > (r_1 - r_0)x(x-1). \tag{4.9}$$

Adding (4.9) to (4.8),

$$0 > 1 + r_1 x(x-1) + \theta x(x-1)^2 = q_1(x),$$

and in particular, $q_1(\gamma_0) < 0$. Because $q_1(x) \geq 0$ for $x \in [0,\gamma_1] \cup [\beta_1, 1]$, it follows that $\gamma_1 < \gamma_0$. That is, γ decreases in r. The second conclusion follows from the first. $\square$

Proposition 29. *If* $\theta \leq 2.13$ *and* $r = 5$, *then* $\theta \leq \dfrac{1}{1-\gamma}$.

Proof. By (4.3),

$$q\left(1-\theta^{-1}\right) = 1 + 5\left(1-\theta^{-1}\right)\left(-\theta^{-1}\right) + \theta\left(1-\theta^{-1}\right)\theta^{-2} = \left(1-2\theta^{-1}\right)^2 \geq 0,$$

so $\gamma \geq 1-\theta^{-1}$ and $\theta \leq \dfrac{1}{1-\gamma}$. □

Proposition 30. *If* $\theta = 2.13$, *then* $\dfrac{1}{1-\gamma} - \theta > 0.04$.

Proof. Proposition 28 implies that γ is least when $r = 5$. Evaluate $q(0.54)$ there (using (4.3)) to obtain a lower bound for γ:

$$q(0.54) = 1 - (5)(0.54)(0.46) + (2.13)(0.54)(0.46)^2 > 0,$$

So $\gamma > 0.54$ and $\dfrac{1}{1-\gamma} - \theta > 0.04$. □

The discriminant D of q is (cf. pp. 95-102 of Rotman [32])

$$D = -27\theta^2 - 4\theta^3 + 6\theta^2 r + 6\theta r^2 + \theta^2 r^2 - 4r^3 - 2\theta r^3 + r^4,$$

and $D < 0$ if and only if $\mathrm{Im}(\gamma)\mathrm{Im}(\beta) \neq 0$.

Proposition 31. *If* $\theta = 2.15$, *then* $D < 0$.

Proof. If $\theta \geq 2.1$, then D increases in r:

$$\begin{aligned}
\frac{\partial D}{\partial r} &= 6\theta^2 + 12\theta r + 2\theta^2 r - 12r^2 - 6\theta r^2 + 4r^3 \\
&\geq 6(2.1)^2 + 12(2.1)(4.9) + 2(2.1)^2(4.9) - 12(5)^2 - 6(2.2)(5)^2 + 4(4.9)^3 \\
&> 0.
\end{aligned}$$

So D is greatest when $r = 5$.

$$\begin{aligned} D &= -27(2.15)^2 - 4(2.15)^3 + 6(2.15)^2(5) + 6(2.15)(5)^2 + \\ &\quad (2.15)^2(5)^2 - 4(5)^3 - 2(2.15)(5)^3 + (5)^4 < 0. \end{aligned}$$

□

Proposition 32. *If $D < 0$ and $|\alpha| > 1 > |\beta| = |\gamma| > 0$, then $u_n \geq u_{n+1}$ for some $n > 0$.*

Proof. In this case, (4.5) is equivalent to

$$u_{n+1} - u_n = A\alpha^{-n} + 2\mathrm{Re}[C\gamma^{-n}],$$

because[2] formula (4.7) for A, B, and C is symmetric, and complex conjugation is a field automorphism.

Clearly $A\alpha^{-n} \to 0$ as $n \to \infty$. The behavior of the other term $2\mathrm{Re}[C\gamma^{-n}]$ follows from a well-known characterization of complex multiplication. The *modulus* $|z|$ of complex number $z = x + iy$ is the distance $\sqrt{x^2 + y^2}$ in the complex plane of z from 0. By a formula due to Euler, when z is a nonzero complex number,

$$z = |z|\exp(i\zeta),$$

where ζ (the *argument* of z) is an angle in the complex plane from the positive real axis to the ray emanating from 0 to z. Therefore complex multiplication is

[2] Note that

$$\begin{aligned} B &= \frac{\beta[(\theta+\delta)(1-\beta)-1]}{(1-\beta/\alpha)(1-\beta/\gamma)} = \frac{\gamma^*[(\theta+\delta)(1-\gamma^*)-1]}{(1-\gamma^*/\alpha^*)(1-\gamma^*/\beta^*)} = \frac{\gamma^*[(\theta+\delta)(1-\gamma)^*-1]}{(1-(\gamma/\alpha)^*)(1-(\gamma/\beta)^*)} \\ &= \frac{\gamma^*[((\theta+\delta)(1-\gamma))^*-1]}{(1-\gamma/\alpha)^*(1-\gamma/\beta)^*} = \frac{\gamma^*[(\theta+\delta)(1-\gamma)-1]^*}{[(1-\gamma/\alpha)(1-\gamma/\beta)]^*} = \frac{(\gamma[(\theta+\delta)(1-\gamma)-1])^*}{[(1-\gamma/\alpha)(1-\gamma/\beta)]^*} = C^*. \end{aligned}$$

multiplicative in modulus and additive in argument. Because $|C\gamma^{-n}| \to \infty$ as $n \to \infty$, the desired conclusion holds if the ray emanating from 0 to $C\gamma^{-n}$ is near the negative real axis for some n large enough that the second term $2\mathrm{Re}[C\gamma^{-n}]$ of the sum is dominant.

Indeed, $2\mathrm{Re}[C\gamma^{-n}]$ oscillates in sign. Because $|\alpha| > 1$, eventually

$$|A\alpha^{-n}| < 0.1,$$

say, while

$$\{n \in \mathbb{P} \mid 2\mathrm{Re}[C\gamma^{-n}] < -0.1\}$$

is infinite (cf. exercise II.1.11 of Conway [6]).[3] $\square$

---

[3] Consider the group $[0, 2\pi)$ with addition modulo 2π. When $\zeta \in [0, 2\pi) - \{2\pi q \mid q \in \mathbb{Q}\}$, the subset $D_\zeta = \{\zeta n \mid n \in \mathbb{P}\}$ is infinite. So for all $\varepsilon > 0$, some two elements ζm_1 and ζm_2 are less than ε apart. Thus every interval of length at least ε meets $\{\zeta jn \mid n \in \mathbb{P}\}$ where $j = |m_2 - m_1|$. It follows that every tail of D_ζ meets every neighborhood of $[0, 2\pi)$ infinitely often.

If $\frac{\zeta}{2\pi} \in \mathbb{Q}$, then D_ζ contains $\{2\pi n/N \mid n \in \mathbb{P}\}$ for some integer N. That is, $\{\exp(id) \mid d \in D_\zeta\}$ contains the complex Nth roots of 1. Because γ is imaginary, $N \geq 3$. Finally, note that $\arccos(-0.1) < 2\pi/3$.

Chapter 5

CONCLUSION

This chapter

- proves Theorem 5 by collecting results of the last three chapters,
- reveals a limitation of the method of Chrobak and Ślusarek, and
- shows why the present method does no better than $R \geq 5$.

Kierstead and Trotter knew already that their construction could not yield a 5-cap. However, there is new insight here for anyone who has attempted their method, say with computer assistance, only to find progress difficult for r near 5 and especially so for (r,θ) near $(5,2)$.

Proof of Theorem 5. Fix some rational r with $4.999 < r < 5$. (Here r, θ, and γ mean what they did in preceding chapters.) Let Θ be the set of θ for which a θ-halfstack exists. Then $1 \in \Theta$ by Proposition 19. Recall that γ depends on r and θ. Though r has been fixed, θ is free, so γ is a function of θ. Let

$$F = \left\{ \frac{1}{1-\gamma} - \theta \;\middle|\; 1 \leq \theta \leq 2.13 \right\}.$$

Then F is closed because it is the continuous image of a compact set. Propositions 28 and 29 imply $\inf F > 0$. Fix some rational δ with $0 < \delta < \inf F$ so that $(2.13 - 1)/\delta$ is an integer. Theorem 27 and Propositions 25 and 26 imply $u_{n+1} - u_n \to -\infty$ for all $\theta \in [1, 2.13]$. So invoking Theorem 21 many (i.e., $1.13/\delta$) times yields $2.13 \in \Theta$. Proposition 30 and Theorem 21 imply $2.15 \in \Theta$ because $2.15 - 2.13 < 0.04$. Theorem 21 and Propositions 31 and 32 imply $r - 2 \in \Theta$. By Lemma 20 a wholestack exists, and by Theorem 18 an r-cap exists. So $R \geq 5$ by Theorem 15. □

Sequence (u_n) is *useless*[1] for Theorem 21 if $u_n < u_{n+1}$ for all $n > 0$. If $(r, \theta, \delta) = (5, 2, 0)$, then (u_n) is the Fibonacci sequence $(1, 1, 2, 3, 5, 8, \ldots)$. Of course the Fibonacci sequence is useless, and it turns out to be r-minimal among useless sequences with $\delta = 0$. Figure 5.1 depicts the 0-level set of $\frac{1}{1-\gamma} - \theta$. Observe[2] that the minimum value of r on the curve occurs when $dr/d\theta = 0, \theta = 2$, and $r = 5$.

If $(r, \theta, \delta) = (4, 1, 1)$, then (u_n) begins $(1, 1, 2, 3, 4, 4)$. This subsequence appears in the 4-cap of Chrobak and Ślusarek of Figure 2.3. Finally, consider the optimal value of r in case the first application of Theorem 21 to advance from Proposition 19 is also the last (as in the construction of Ślusarek [34]). Let

$$r_+ = 1.5 + 0.5\sqrt{13 + 16\sqrt{2}} \approx 4.48.$$

Proposition 33. *If r is rational,* $4 < r < r_+$, $\theta = 1$, *and* $\delta = r - 3$, *then* $u_n \geq u_{n+1}$ *for some* $n > 0$.

---

[1] Such a sequence might be called (strictly) increasing, but it can't, as $u_0 = u_1$.

[2] Note that

$$\begin{aligned}
\theta &= \frac{1}{1-\gamma} \\
1 &= \theta(1-\gamma) \\
0 &= q(\gamma) \\
&= 1 - (r-\theta)\gamma + (r-2\theta)\gamma^2 + \theta\gamma^3 \\
&= 1 - r(\gamma - \gamma^2) + \theta(\gamma - 2\gamma^2 + \gamma^3) \\
r &= \frac{1 + \theta\gamma(1-\gamma)^2}{\gamma(1-\gamma)} \\
&= 1 + \theta/\gamma \\
&= 1 + \theta(1-\theta^{-1})^{-1} \\
\frac{dr}{d\theta} &= (1-\theta^{-1})^{-1} - \theta^{-1}(1-\theta^{-1})^{-2}.
\end{aligned}$$

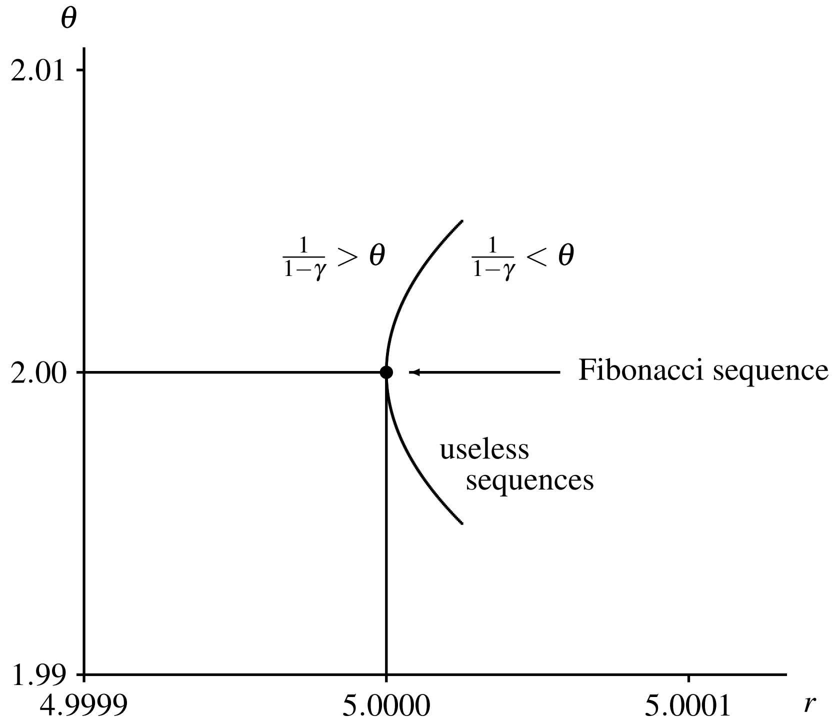

Figure 5.1: The Fibonacci sequence is minimally useless

Proof. Proposition 25 applies, and the sole positive real root of the discriminant

$$D = -31 + 6r + 7r^2 - 6r^3 + r^4$$

of q is r_+ (cf. pp. 44-49 of Rotman [32]). So

- $D < 0$
- $\beta = \gamma^*$
- $0 < |\gamma| < 1$ by Proposition 25

and $u_n \geq u_{n+1}$ for some $n > 0$ by Proposition 32. $\square$

The stronger conclusion ($4.48 > 4.45$) would not surprise Ślusarek, who conjectured a lower bound of 4.5. But 4.5 cannot be reached in this way. When $r > r_+$,

$$|\alpha| > 1 > |\beta| > \gamma > 0,$$

so $u_n < u_{n+1}$ for all $n > 0$ unless $C \leq 0$ or the difference sequence takes a nonpositive value early on. Neither of these happens.

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