Estimating Low Generalized Coloring Numbers of Planar Graphs

by

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ABSTRACT

The chromatic number $\chi(G)$ of a graph $G = (V, E)$ is the minimum number of colors needed to color $V(G)$ such that no adjacent vertices receive the same color. The coloring number $\mathrm{col}(G)$ of a graph G is the minimum number k such that there exists a linear ordering of $V(G)$ for which each vertex has at most $k-1$ backward neighbors. It is well known that the coloring number is an upper bound for the chromatic number. The weak r-coloring number $\mathrm{wcol}_r(G)$ is a generalization of the coloring number, and it was first introduced by Kierstead and Yang [23]. The weak r-coloring number $\mathrm{wcol}_r(G)$ is the minimum integer k such that for some linear ordering L of $V(G)$ each vertex v can reach at most $k-1$ other smaller vertices u (with respect to L) with a path of length at most r and u is the smallest vertex in the path. This dissertation proves that $\mathrm{wcol}_2(G) \leq 23$ for every planar graph G.

The exact distance-3 graph $G^{[\natural 3]}$ of a graph $G = (V, E)$ is a graph with V as its set of vertices, and $xy \in E(G^{[\natural 3]})$ if and only if the distance between x and y in G is 3. This dissertation improves the best known upper bound of the chromatic number of the exact distance-3 graphs $G^{[\natural 3]}$ of planar graphs G, which is 105, to 95. It also improves the best known lower bound, which is 7, to 9.

A class of graphs is nowhere dense if for every $r \geq 1$ there exists $t \geq 1$ such that no graph in the class contains a topological minor of the complete graph K_t where every edge is subdivided at most r times. This dissertation gives a new characterization of nowhere dense classes using generalized notions of the domination number.

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INTRODUCTION

In 1852, Francis Guthrie asked wether every map can be colored with four colors so that no two adjacent regions have the same color. The answer is affirmative and this very famous fact is known as the Four Color Theorem. The first accepted proof was given by Appel and Haken in 1977. Thirty-seven years ago, Steven J. Brams invented the Map-Coloring Game hoping to find a game-theoretic proof of the Four Color Theorem. Even though his approach was not successful, we are left with an interesting map-coloring game.

The Map-Coloring Problem is illustrated as follows. Suppose that Alice and Bob, with Alice playing first, are coloring the regions of a map alternatively, using four colors, so that no adjacent regions receive the same color. Alice wins the game if every region receives a color, Bob wins the game otherwise, i.e., Bob wins the game if at some point during the game there is an uncolored region and all the four colors are used to color its neighbors. Steven J. Brams was trying to find a strategy for Alice to win the game no matter how Bob plays. Unfortunately, such a strategy does not exist. Scientists came up with several maps and strategies for Bob to win the game no matter how Alice plays. It was then natural to ask what is the smallest number of colors for which Alice has a winning strategy. In the context of general graphs, this number is called the *game chromatic number* $\chi_g(G)$ of a graph G, as every map can be viewed as a graph with the regions as its set of vertices and each edge corresponds to adjacent regions. Since then the game chromatic number has received special attention and several approaches have been developed to find this number.

During the journey of attacking this problem a very useful tool came from an unexpected source. Chan and Schelp [5] proved that the class of planar graphs has a linear Ramsey number by bounding, for any vertex v, the number of smaller vertices, with respect to a fixed linear ordering L, that can be reached from v by a path of length two whose internal vertex is greater than v. This was the birth of one of the generalized coloring numbers which was later called the *strong* 2-*coloring number* $\mathrm{scol}_2(G)$. Kierstead and Trotter [22] realized that this was the missing tool. They proved that $\chi_g(G) \leq \chi(G)(1 + \mathrm{scol}_2(G))$, and as the class of planar graphs has bounded generalized coloring numbers, $\chi_g(G)$ is also bounded. Kierstead and Trotter [21] gave the rise of the *strong* 4-*coloring numbers* $\mathrm{scol}_4(G)$ and *weak* 4-*coloring numbers* $\mathrm{wcol}_4(G)$ by considering paths of length four with large internal vertices in their study of oriented game chromatic numbers. The general notions of *strong* r-*coloring numbers* $\mathrm{scol}_r(G)$ and *weak* r-*coloring numbers* $\mathrm{wcol}_r(G)$ were first introduced by Kierstead and Yang [23]. Since then several applications have been found. Zhu [37] proved that these invariants are strongly related to low tree depth decompositions and they can be used to characterize both bounded expansion and nowhere dense classes of graphs. The generalized coloring numbers can also be used to give a linear-time constant-factor algorithm for approximation of the k-domination number in classes of graphs with bounded expansion (Dvořák [8]).

In this dissertation, we focus on the weak 2-coloring number of planar graphs since this number is an upper bound for the *star list chromatic number* $ch_s(G)$. The best known upper bound for $ch_s(G)$ was 28 where G is any planar graph; in this dissertation, we prove that the weak 2-coloring number for any planar graph is at most 23. We also focus on the chromatic numbers of exact distance-3 graphs. We prove that for any planar graph G, the chromatic number of the *exact distance*-3 *graph* $G^{[\natural 3]}$ is at most 95.

This dissertation is organized as follows. This chapter contains the needed background and a brief history of each notion. In Chapter 2, we prove that for every planar graph G, $\mathrm{wcol}_2(G) \leq 23$. In Chapter 3, we prove that for every planar graph G, $\chi(G^{[\natural 3]}) \leq 95$. We also give an example of a planar graph G with $\chi(G^{[\natural 3]}) \geq 9$. In Chapter 4, we investigate a linear ordering L defined by Van den Heuvel et al. [14]; we show that $\mathrm{wcol}_2[G, L] \leq 26$ and we show that, by giving an example, this result is tight. In Chapter 5, we give a new characterization of nowhere dense class of graphs.

1.1 Generalized Coloring Numbers

All graphs in this paper are finite and simple. For a graph $G = (V, E)$ and vertices $x, y \in V$, the *distance* between x and y in G is the number of edges in the shortest path between x and y and it is denoted by $\mathrm{dist}_G(x, y)$. For $v \in G$, we denote the set of neighbors of v by $N^G(v)$, or briefly by $N(v)$, and for $r \in \mathbb{N}$, we denote the set of vertices within distance at most r from v by $N_r[v]$. Note that $v \in N_r[v]$. We call $N_r[v]$ the *closed r-neighborhood of v*.

Let $\mathit{\Pi}(G)$ be the set of all linear orders of the vertices $V(G)$ of a graph G. For $L \in \mathit{\Pi}(G)$, we write $u <_L v$ if u is smaller than v with respect to L, and $u \leq_L v$ if $u <_L v$ or $u = v$. Let X and Y be two disjoint sets of vertices of G. We write $X <_L Y$ if every $x \in X$ and $y \in Y$ satisfy $x <_L y$. Let r be a positive integer and $u, v \in V(G)$. We say that u is *strongly r-reachable* from v with respect to L if $u \leq_L v$ and there is a path $P = u \dots v$ in G with $\|P\| \leq r$ and $v <_L w$ for all internal vertices w of

P. If we allow paths of any length we say that u is *strongly reachable* from v. We denote the set of all vertices u such that u is strongly r-reachable (strongly reachable) from v with respect to L by $\mathrm{Scol}_r[G, L, v]$ ($\mathrm{Scol}_\infty[G, L, v]$). Let $\mathrm{scol}_r[G, L, v]$ be the cardinality of $\mathrm{Scol}_r[G, L, v]$. *Let* $\mathrm{scol}_r[G, L] := \max_{v \in V(G)} \mathrm{scol}_r[G, L, v]$. The *strong r-coloring number* $\mathrm{scol}_r(G)$ *of* G is defined by

$$\mathrm{scol}_r(G) := \min_{L \in \Pi(G)} \mathrm{scol}_r[G, L].$$

We write $\mathrm{scol}_r[L, u]$ and $\mathrm{Scol}_r[L, u]$ instead of $\mathrm{scol}_r[G, L, u]$ and $\mathrm{Scol}_r[G, L, u]$ respectively if G is known from the context.

We say that u is *weakly r-reachable* from v with respect to L if there is a path $P = u \ldots v$ in G with $\|P\| \leq r$ and $u \leq_L w$ for all vertices w of P. If we allow paths of any length we say that u is *weakly reachable* from v. We denote the set of all vertices u such that u is weakly r-reachable (weakly reachable) from v with respect to L by $\mathrm{Wcol}_r[G, L, v]$ ($\mathrm{Wcol}_\infty[G, L, v]$). Let $\mathrm{wcol}_r[G, L, v]$ be the cardinality of $\mathrm{Wcol}_r[G, L, v]$. Let $\mathrm{wcol}_r[G, L] := \max_{v \in V(G)} \mathrm{wcol}_r[G, L, v]$. The *weak r-coloring number* $\mathrm{wcol}_r(G)$ *of* G is defined by

$$\mathrm{wcol}_r(G) := \min_{L \in \Pi(G)} \mathrm{wcol}_r[G, L].$$

Note that when $r = 1$, then $\mathrm{scol}_r(G) = \mathrm{wcol}_r(G) = \mathrm{col}(G)$. We write $\mathrm{wcol}_r[L, u]$ and $\mathrm{Wcol}_r[L, u]$ instead of $\mathrm{wcol}_r[G, L, u]$ and $\mathrm{Wcol}_r[G, L, u]$ respectively if G is known from the context.

As noticed in [23], the generalized coloring numbers are related; for every graph G we have the following

$$\mathrm{scol}_r(G) \leq \mathrm{wcol}_r(G) \leq (\mathrm{scol}_r(G))^r.$$

Thus if the weak r-coloring number is bounded for a particular class of graphs, then the strong r-coloring number is also bounded and vice versa.

Let L be a linear ordering of the vertices of a graph G. The *fill-in* of G with respect to L is the graph G_L obtained inductively by adding for every vertex $v \in G$ (starting with the greatest vertex in the ordering L) an edge uw for all $u, w \in N(v), u \neq w$ satisfying that $u <_L v$ and $w <_L v$. The *elimination-width* of the ordering L is $\omega(G_L) - 1$ where $\omega(G_L)$ is the size of the largest clique in G_L.

Observe that $\omega(G_L)$ equals the maximum over all vertices $v \in G$ of the number of vertices u such that $u \leq_L v$ and u can be reached from v in G by a path whose internal vertices are greater than v with respect to L. So $\omega(G_L) = \max_{v \in V(G)} \mathrm{scol}_\infty[G, L, v]$; thus $\min_{L \in \Pi(G)} \omega(G_L) = \mathrm{scol}_\infty(G)$.

The generalized coloring numbers are gradations between the coloring number $\mathrm{col}(G)$ and the tree-width $\mathrm{tw}(G)$ and the tree-depth $\mathrm{td}(G)$.

Proposition 1.1.1. *Let G be a graph. Then*

1. $\mathrm{col}(G) \leq \mathrm{scol}_2(G) \leq \ldots \leq \mathrm{scol}_\infty(G) = \mathrm{tw}(G) + 1$, *and*
2. $\mathrm{col}(G) \leq \mathrm{wcol}_2(G) \leq \ldots \leq \mathrm{wcol}_\infty(G) = \mathrm{td}(G)$.

The equality $\mathrm{scol}_\infty(G) = \mathrm{tw}(G) + 1$ is proved in [11] and the equality $\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$ is proved in (Lemma 6.5, [31]).

There are several ways to define the *tree-width* $\mathrm{tw}(G)$ of G, we take this equality as its definition, i.e., $\mathrm{tw}(G) = \mathrm{scol}_\infty(G) - 1$.

Determining the weak r-coloring numbers is NP-complete when $r \geq 3$ [11], while the problem remains open for $r = 2$ and for the strong r-coloring numbers when $r \geq 2$.

It is natural to ask about the values of the generalized coloring numbers for a particular class of graphs. Grohe et al. [11] proved that for every graph G, the weak r-coloring number is bounded above by $\binom{r+\mathrm{tw}(G)}{\mathrm{tw}(G)}$. In the same paper, they gave for every positive integers r and k, a graph G such that $\mathrm{tw}(G) = k$ and $\mathrm{wcol}_r(G) \geq k + 1$.

Theorem 1.1.2. *Let r be a positive integer.*

1. *If G is a graph with* $\mathrm{tw}(G) \leq k$ *then* $\mathrm{wcol}_r(G) \leq \binom{r+k}{k}$.
2. *For every positive integer k, there exists a graph G with* $\mathrm{tw}(G) = k$ *and* $\mathrm{wcol}_r(G) \geq k + 1$.

Van den Heuvel et al. [14] gave a creative proof that dramatically improved the bounds of the generalized coloring numbers for the class of graphs that exclude K_t as a minor.

Theorem 1.1.3. *Let G be a graph and r be a positive integer. If G excludes K_t as a minor then*

1. $\mathrm{scol}_r(G) \leq \binom{t-1}{2} \cdot (2r + 1)$, *and*
2. $\mathrm{wcol}_r(G) \leq \binom{r+t-2}{t-2} \cdot (t - 3)(2r + 1)$.

Those results were huge improvements from the exponential bounds of Grohe et al. [11] to linear bounds in r for the strong r-coloring numbers and to polynomial bounds for the weak r-coloring numbers.

For graphs with bounded genus g, Van den Heuvel et al. [14] proved the following.

Theorem 1.1.4. *Let G be a graph with genus g. Then*

1. $\mathrm{scol}_r(G) \leq (4g + 5)r + 2g + 1$, *and*
2. $\mathrm{wcol}_r(G) \leq (2g + \binom{r+2}{2}) \cdot (2r + 1)$.

It is well known that every planar graph excludes K_5 as a minor, so Theorem 1.1.3 bounds the generalized coloring numbers for planar graphs too. However, Theorem 1.1.4 gives better bounds for planar graphs ($g = 0$).

Theorem 1.1.5. *(Van den Heuvel et al. [14]) Let G be a planar graph and r be a positive integer. Then*

1. $\mathrm{scol}_r(G) \leq 5r + 1$, *and*
2. $\mathrm{wcol}_r(G) \leq \binom{r+2}{2}(2r + 1)$.

For every $r \in \mathbb{N}$, there exists a planar graph G such that $\mathrm{scol}_r(G) \geq \frac{r}{2}$ [14], so the above upper bound for $\mathrm{scol}_r(G)$ gives a constant-factor approximation. If $r = 1$, the bound on $\mathrm{col}(G) = \mathrm{wcol}_1(G) = \mathrm{scol}_1(G)$ is tight where G is a planar graph. If G excludes K_t as a minor for $t = 2$, 3, Van den Heuvel et al. [14] showed the following.

Theorem 1.1.6.

1. *If G is a graph that excludes K_2 as a minor,* $\mathrm{scol}_r(G) = \mathrm{wcol}_r(G) = 1$.
2. *If G is a graph that excludes K_3 as a minor,* $\mathrm{scol}_r(G) \leq 2$ *and* $\mathrm{wcol}_r(G) \leq r + 1$.

In Chapter 4, we will study the linear ordering L that Van den Heuvel et al. [14] constructed to witness $\mathrm{scol}_r(G) \leq 5r + 1$.

The generalized coloring numbers when $r = 2$ have attracted special attention since they play an important role on bounding other types of graph chromatic numbers, namely, *game coloring number* $\mathrm{gcol}(G)$, *star chromatic number* $\chi_s(G)$ and *degenerate coloring number* $ch_d(G)$. We state the best known bounds for $\mathrm{scol}_2(G)$ and $\mathrm{wcol}_2(G)$ where G is any planar graph, then we provide their relations with the other types of graph colorings.

Researchers have succeeded in their mission of finding the strong 2-coloring number $\mathrm{scol}_2(G)$ for planar graphs. Chan and Schelp [5] proved that the strong 2-coloring number for planar graphs is bounded above by 761. Kierstead and Trotter [22] improved the bound to 10, then to 9 (Kierstead et al. [24]). On the other hand, several planar graphs with strong 2-coloring number equal to 8 have been found [22, 24]. Eventually, Dvořák et al. [9] proved that the strong 2-coloring number of planar graphs is at most 8.

Theorem 1.1.7.

1. *For every planar graph G,* $\mathrm{scol}_2(G) \leq 8$*.*
2. *There exists a planar graph G such that* $\mathrm{scol}_2(G) = 8$*.*

Unlike the strong 2-coloring number, researchers are still on the hunt for the weak 2-coloring number. Theorem 1.1.5 gives 30 as a bound for $\mathrm{wcol}_2(G)$ for any planar graph G. Kierstead and Yang [25] improved this bound to 28. On the other hand, Albertson et al. [2] constructed a planar graph G with star chromatic number at least 10, as the star chromatic number is a lower bound for the weak 2-coloring number, $\mathrm{wcol}_2(G) \geq 10$.

In this dissertation, we prove that the weak 2-coloring number of any planar graph is at most 23.

1.1.1 Game Coloring Numbers

Let G be a graph. Assume that two players, Alice and Bob, with Alice playing first, play a game by taking turns on choosing an unchosen vertex from $V(G)$. The game ends when all $V(G)$ are chosen. For a vertex v of G, define $b(v)$ to be the number of neighbors of v chosen before v during the game. The *score* of the game is $s = 1 + max_{v \in V(G)} b(v)$.

Alice goal is to minimize s, while Bob goal is to maximize s. The *game coloring number* $\mathrm{gcol}(G)$ is the smallest s such that Alice has a strategy to make the score at most s no matter how Bob plays.

Bartnicki et al. [3] showed the following.

Theorem 1.1.8. *Every graph G satisfies that* $\operatorname{gcol}(G) \leq 3\operatorname{scol}_2(G) - 1$.

Since $\operatorname{scol}_2(G) \leq 8$ for every planar graph G, we immediately get $\operatorname{gcol}(G) \leq 23$. However, this bound is not the best bound; Kierstead [19] proved that 18 colors are always sufficient when G is a planar graph. Later, Zhu [38] improved the bound to 17. On the other hand, Wu and Zhu [36] constructed a planar graph G and a strategy for Bob to make the score at least 11 no matter how Alice plays.

1.1.2 Star chromatic Numbers

A *star coloring* of a graph G is a proper coloring of G such that any subgraph of G that is induced by two colors has no P_4, i.e. every path of length three in G is colored with at least three colors. The least number r such that G has a star coloring using r colors is called the *star chromatic number* of G, and it is denoted by $\chi_s(G)$. If we assign for every $v \in G$ a list of colors $L(v)$ and we ask to color each vertex using a color from its list such that we get a star coloring, the coloring is called *star list coloring.* The smallest integer k such that for any set of lists $\{L(v) : v \in G\}$ satisfying $|L(v)| = k$ there exists a star list coloring of G is called the *star list chromatic number* of G, and it is denoted by $ch_s(G)$. Clearly, $\chi_s(G) \leq ch_s(G)$ for every graph G.

The problem of finding the star chromatic numbers was first suggested in 1973 by Grünbaum [13], and has been investigated recently by other authors.

For planar graphs, Albertson et al. [2] proved the following.

Theorem 1.1.9. *If G is a planar graph then* $\chi_s(G) \leq 20$.

In the same paper, Albertson et al. constructed a planar graph G with $\chi_s(G) \geq 10$. Unfortunately, the proof of Theorem 1.1.9 did not improve the star list chromatic number $ch_s(G)$. The following theorem is a well known relation between the star list chromatic number and the weak 2-coloring number.

Theorem 1.1.10. *For every graph G, $ch_s(G) \leq \mathrm{wcol}_2(G)$.*

As $\mathrm{wcol}_2(G) \leq 23$ for planar graphs, this number is the new upper bound for $ch_s(G)$ too.

For planar bipartite graph, a better bound is found.

Theorem 1.1.11. *(Kirestead et al. [20]) If G is planar bipartite graph then $ch_s(G) \leq 14$.*

In the same paper, Kierstead et al. constructed a planar bipartite graph G such that $\chi_s(G) \geq 8$.

1.1.3 Degenerate Coloring Numbers

Assume that a graph G is vertex colored. Let S be a set of color classes of G. We denote the subgraph of G that is induced by the set of all vertices v such that v is in a color class in S by $G[S]$. A proper coloring of a graph G is *a degenerate coloring* if for every set S of color classes, every subgraph H of $G[S]$ has a vertex w such that $d_H(w) < |S|$. In other words, a proper coloring of a graph G is a degenerate coloring if for every subgraph H of G, there is a vertex w in H such that the degree of w in H is less than the number of colors appearing in H. The least number r such that G has a degenerate coloring using r colors is called the *degenerate number* of G, and it is denoted by $\chi_d(G)$. The degenerate coloring was first introduced by Borodin [4]. The degenerate coloring is a strengthening of the *acyclic coloring number* $\chi_\alpha(G)$ which is the least number of colors needed to properly color the graph G such that the union of any two color classes induces a forest.

There is another version of coloring called the *list degenerate coloring.* In this coloring, for each vertex $v \in G$, a list $L(v)$ of colors is assigned, and we are asked to color each vertex v of G from its list $L(v)$ such that the coloring is degenerate. The least k such that for any set of color lists $\{L(v) : v \in G\}$ with $|L(v)| = k$ the graph G has a list degenerate coloring is called the *list degenerate number,* and it is denoted by $ch_d(G)$. Clearly, for every graph G, we have $\chi_d(G) \leq ch_d(G)$.

The strong 2-coloring number is related to the list degenerate coloring number as the following theorem shows.

Theorem 1.1.12. *(Kierstead et al. [24]) For every graph G, $ch_d(G) \leq \mathrm{scol}_2(G)$.*

As $\mathrm{scol}_2(G) \leq 8$ for every planar graph G, $ch_d(G) \leq 8$ too. On the other hand, there are infinity many planar graphs where each vertex is either of degree 5 or 6. Clearly, those graphs cannot have degenerate coloring with less than 6 colors.

1.2 Path Decomposition for Planar Graphs

Let H_1 and H_2 be vertex-disjoint subgraphs of G. We say that H_1 is *adjacent* to H_2 if there is an edge $uv \in E(G)$ such that $v \in H_1$ and $u \in H_2$.

Definition 1.2.1. A sequence $\mathcal{H} = (H_1, \ldots, H_s)$ of none-empty subgraphs of a graph G is a *decomposition* of G if the sets $V(H_1), \ldots, V(H_s)$ partition $V(G)$. The decomposition is *connected* if each H_i is connected.

Assume that a decomposition $\mathcal{H} = (H_1, \ldots, H_s)$ of a graph G is given. We denote the subgraph of G induced by $\bigcup_{i \le j \le s} V(H_j)$ by $G[H_{\ge i}]$.

Definition 1.2.2. Let $\mathcal{H} = (H_1, \ldots, H_s)$ be a decomposition of a graph G. Let C be a component of $G[H_{\ge i}]$ where $1 < i \le s$. The *separating number* of C is the number t of graphs $H' \in \{H_1, \ldots, H_{i-1}\}$ such that H' is adjacent to C. The *width* of $\mathcal{H}$ is the maximum separating number of a component C of $G[H_{\ge i}]$, maximized over all $i, 1 < i \le s$.

Lemma 1.2.3. *(Van den Heuvel et al. [14]) Let $\mathcal{H} = (H_1, \ldots, H_s)$ be a connected decomposition of a graph G of width at most k. By contracting each connected subgraph H_i to a single vertex h_i, we obtain a graph $H = G/\mathcal{H}$ with s vertices and tree-width at most k. More precisely, the elimination-width of the ordering $L = h_1, \ldots, h_s$ is at most k.*

There is a relation between the elimination-width and weak reachability as next theorem states.

Theorem 1.2.4. *(Grohe et al. [11]) Let G be a graph and let L be a linear order of $V(G)$ with elimination-width at most k. For all $r \in \mathbb{N}$ and all $v \in V(G)$,*

$$\mathrm{wcol}_r[G, L, v] \le \binom{r+k}{k}.$$

From Theorem 1.2.4 and Lemma 1.2.3 we have the following.

Lemma 1.2.5. *Let $\mathcal{H} = (H_1, \ldots, H_s)$ be a connected decomposition of a graph G of width at most k. Let H be the graph obtained from G by contracting each $H_i \in \mathcal{H}$ to a single vertex h_i where $1 \le i \le s$. For all $r \in \mathbb{N}$ and all $h_i \in V(H)$,*

$$\mathrm{wcol}_r[H, L, h_i] \le \binom{r+k}{k}$$

where L is the ordering of $V(H)$ that arises naturally from $\mathcal{H}$, i.e., $L = h_1, \ldots, h_s$.

A path in a graph G is *isometric* if it is the shortest path in G between its endpoints. Observe that if a path is isometric then all its subpaths are also isometric.

Lemma 1.2.6. *Let $P = x_1 \dots x_n$ be an isometric path in G, and let $v \in G$. Then $|N_r[v] \cap P| \leq 2r + 1$.*

Proof. Suppose $N_r[v] \cap P \neq \emptyset$, and let $x, y \in N_r[v] \cap P$ such that $\mathrm{dist}_P(x, y)$ is maximum. Since $x, y \in N_r[v]$, $\mathrm{dist}_G(x, y) \leq 2r$. As P is isometric, $\mathrm{dist}_P(x, y) = \mathrm{dist}_G(x, y)$. Thus $|N_r[v] \cap P| \leq 2r + 1$. $\square$

A decomposition $\mathcal{H} = (H_1, \dots, H_s)$ of a graph G is called an *isometric-path decomposition* if for every $i, 1 \leq i \leq s$, H_i is an isometric path in $G[H_{\geq i}]$.

Van den Heuvel et al. [14] showed that every maximal planar graph G has an isometric-path decomposition $\mathscr{P} = (P_1, \dots, P_s)$ of width at most 2. This decomposition is a very useful tool and it is the base of our work. For completeness, we include their proof.

Lemma 1.2.7. *(Van den Heuvel et al. [14]) Every maximal planar graph G has an isometric-path decomposition of width at most* 2.

Proof. Fix a plane drawing $\widetilde{G}$ of G. For simplicity, we write G for $\widetilde{G}$. The lemma is trivial if $|G| = 3$, so assume that $|G| \geq 4$. We inductively construct an isometric-path decomposition $P_1, \dots, P_k, k \geq 2$ such that

($*$) for any component C of $G[P_{\geq k+1}]$, the boundary of the region of $R^2 \setminus G[P_1 \cup \dots \cup P_k]$ containing C is a cycle in G that has its vertices in exactly two paths from $P_1, \dots, P_k$.

Let P_1 be any edge of the outer face and let P_2 be the vertex of the outer face that is not contained in P_1. Then $G[P_{\geq 3}]$ has only one component and the boundary of the

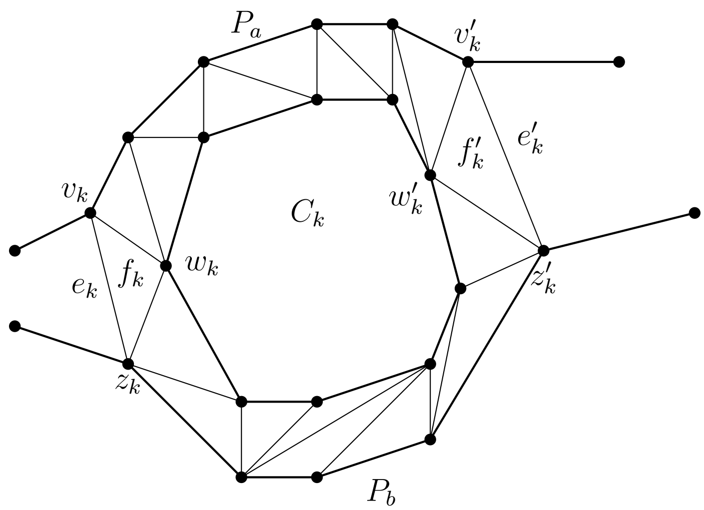

Figure 1.1. Paths Decomposition of Width at Most 2

region of $R^2 \setminus G[P_1 \cup P_2]$ containing this component is the cycle forming the outer face.

Now assume that $P_1, \ldots, P_{k-1}$ are constructed as described above. Let C_k be a component of $G[P_{\geq k}]$. Let D_k be the cycle forming the boundary of the region of $R^2 \setminus G[P_1 \cup \ldots \cup P_{k-1}]$ in which C_k lies. Let P_a and P_b where $1 \leq a < b \leq k-1$ be the paths such that $V(D_k) \subseteq V(P_a \cup P_b)$. Since any cycle contains at least three vertices, one of the paths, say P_a, has more than one vertex.

Since P_a is an isometric, any edge $vu \in D_k$ where $u, v \in V(P_a)$ is also an edge in P_a. The same is true for P_b. Since P_a and P_b are vertex-disjoint, D_k has exactly two edges e_k and e'_k such that $e_k, e'_k \notin P_a \cup P_b$. Say $e_k = v_k z_k, e'_k = v'_k z'_k, v_k, v'_k \in P_a, z_k, z'_k \in P_b$. It is possible that $v_k = v'_k$ or $z_k = z'_k$ but not both as G does not have multiple edges. Note that there could be lots of edges between P_a and P_b but only two of them are in D_k. Since every face in G is triangulated, each one of those two edges belongs to the boundary of a triangle face contained in the interior of D_k, see Figure 1.1 on page 14. Let f_k and f'_k be those triangle faces, say $e_k \in G[f_k]$ and $e'_k \in G[f'_k]$ where $G[f]$ denotes the boundary of the face f. Let $w_k \in G[f_k] - \{v_k, z_k\}$ and $w'_k \in G[f'_k] - \{v'_k, z'_k\}$ (it is possible that $w_k = w'_k$). From the definition of D_k, both w_k and w'_k are in C_k. Choose P_k to be the shortest path between w_k and w'_k in C_k.

Let C' be a component of $G[P_{\geq k+1}]$ such that $C' \subset C_k$. From the way P_k is defined, C' is adjacent to at most two paths among P_a,P_b and P_k, and no such component is adjacent to both P_a and P_b. Now we just need to show that C' is adjacent to exactly two paths among P_a, P_b and P_k. Let D'' be the cycle forming the boundary of the region of $R^2 \setminus G[P_1 \cup \ldots \cup P_k]$ containing C'. Assume that $V(D'') \subseteq V(P_i)$ for some $i \in \{a, b, k\}$. Then there exists an edge $e = uv \in D''$ such that $u, v \in V(P_i)$ but $e \notin P_i$; this is in contradiction to the fact that P_i is an isometric path. $\square$

Let $\mathscr{P} = (P_1, \ldots, P_s)$ be the resulting decomposition. Let L be any ordering of $V(G)$ satisfying that $V(P_i) <_L V(P_j)$ if $i < j$ (i.e., if $x \in P_i$ and $y \in P_j$ where $i < j$ then $x <_L y$). Contract each P_i to a single vertex p_i, and call the resulting graph H, i.e., $H = G/\mathscr{P}$. Let $v \in G$ and let P_i be the path containing v. From Lemma 1.2.5, there are at most $\binom{r+2}{2}$ vertices in H that are weakly r-reachable from p_i with respect to the ordering $L' = p_1, \ldots, p_s$. Those vertices correspond to the only paths in $\mathscr{P}$ that may contain vertices that are weakly r-reachable from v in G. Since each one of those paths P_j is isometric in $G[P_{\geq j}]$, Lemma 1.2.6 tells us that $\mathrm{wcol}_r[G, L, v] \leq \binom{r+2}{2} \cdot (2r + 1)$.

Theorem 1.2.8. *(Van den Heuvel et al. [14]) Let G be a planar graph and $r \in \mathbb{N}$. Then* $\mathrm{wcol}_r(G) \leq \binom{r+2}{2} \cdot (2r + 1)$*.*

For $r = 2$, we immediately get $\mathrm{wcol}_2(G) \leq 30$. Note that the restriction of L on $V(P_i), i \in [s]$ was arbitrary. If we fix any endpoint of P_i and order the vertices toward this end then we easily get $\mathrm{wcol}_r(G) \leq \binom{r+2}{2} \cdot (2r + 1) - r$. This gives $\mathrm{wcol}_2(G) \leq 28$.

In Chapter 2, we will modify the technique used in the proof of Lemma 1.2.7 to prove that $\mathrm{wcol}_2(G) \leq 23$ for every planar graph G.

1.3 Distance-r-coloring Numbers

Van den Heuvel et al. [15] defined another variant of the generalized coloring numbers called *distance-r-coloring number* $\mathrm{dcol}_r(G)$. Let $L \in \Pi(G)$. For every $v \in V(G)$, let $\mathrm{Dcol}_r[G, L, v]$ be the set of all vertices u such that there is a path $P = x_0 \dots x_s$ in G with $x_0 = u, x_s = v$ satisfying:

- $\|P\| = s \leq r$;
- u is the minimum vertex in P with respect to L;
- $v \leq_L x_i$ for $\lfloor \frac{r}{2} \rfloor + 1 \leq i \leq s$.

We denote the cardinality of $\mathrm{Dcol}_r[G, L, v]$ by $\mathrm{dcol}_r[G, L, v]$. Let

$$\mathrm{dcol}_r[G, L] := \max_{v \in V(G)} \mathrm{dcol}_r[G, L, v].$$

The *distance-r-coloring number* $\mathrm{dcol}_r(G)$ is defined as follows.

$$\mathrm{dcol}_r(G) := \min_{L \in \Pi(G)} \mathrm{dcol}_r[G, L].$$

We write $\mathrm{Dcol}_r[L, v]$ and $\mathrm{dcol}_r[L, v]$ instead of $\mathrm{Dcol}_r[G, L, v]$ and $\mathrm{dcol}_r[G, L, v]$ respectively if G is known from the context.

It is not hard to see that $\mathrm{Scol}_r[L, v] \subseteq \mathrm{Dcol}_r[L, v] \subseteq \mathrm{Wcol}_r[L, v]$, so

$$\mathrm{scol}_r(G) \leq \mathrm{dcol}_r(G) \leq \mathrm{wcol}_r(G). \tag{1.3.1}$$

Also as $\mathrm{Wcol}_{\lfloor \frac{r}{2} \rfloor + 1}[L, v] \subseteq \mathrm{Dcol}_r[L, v]$, $\mathrm{wcol}_{\lfloor \frac{r}{2} \rfloor + 1}(G) \leq \mathrm{dcol}_r(G)$.

Van den Heuvel et al. [15] gave explicit upper bounds of $\mathrm{dcol}_r(G)$ for some classes of graphs. They proved the following.

Theorem 1.3.1. *Let $r \in \mathbb{N}$.*

1. *If G is a planar graph then* $\mathrm{dcol}_r(G) \leq \binom{\lfloor \frac{r}{2} \rfloor + 3}{2} \cdot (2r+1) - r$.
2. *Let $t \in \mathbb{N}$ and G a graph with tree-width at most t. Then* $\mathrm{dcol}_r(G) \leq \binom{t + \lfloor \frac{r}{2} \rfloor + 1}{t}$.

The motivation behind defining the distance-r-coloring number $\mathrm{dcol}_r(G)$ is to bound the chromatic number of the exact distance graphs as we see in the following section.

1.4 Exact Distance Graphs

Let $p \in \mathbb{N}$ and $G = (V, E)$ is a graph. The *p-th power graph* G^p of G is a graph with V as its set of vertices, and $xy \in E(G^p)$ if and only if $\mathrm{dist}_G(x, y) \leq p$. Problems related to the chromatic number of G^p were first considered by Kramer and Kramer [27, 26]. If G is a star then G^2 is a clique, so $\chi(G^2) = |V(G)|$. Thus there are not many graphs G for which $\chi(G^p)$ is bounded by a constant. It is not hard to show that for a graph G with maximum degree $\Delta(G) \geq 3$,

$$\chi(G^p) \leq 1 + \Delta(G^p) \leq 1 + \Delta(G) \cdot \sum_{i=0}^{p-1} (\Delta(G) - 1)^i \in \mathcal{O}(\Delta(G)^p).$$

There are some classes of graphs for which the upper bound is better. Recall that a graph G is *k-degenerate* if for every subgraph $H \subseteq G$, H contains a vertex of degree at most k.

Theorem 1.4.1. *(Agnarsson and Halldórsson [1]) Let $k, p \in \mathbb{N}$. Let G be a k-degenerate graph. Then there exists a constant $c = c(k, p)$ such that $\chi(G^p) \leq c \cdot \Delta(G)^{\lfloor p/2 \rfloor}$.*

The exponent $\lfloor \frac{p}{2} \rfloor$ is best possible. In particular, for planar graphs G, $\chi(G^2)$ has a linear bound in $\Delta(G)$. In 1977, Wegner [35] conjectured that if G is a planar graph and

$\Delta(G) \geq 8$ then $\chi(G^2) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + 1$, and he gave examples showing that this bound would be best possible. Since then this conjecture has attracted special attention.

Nešetřil and Ossona de Mendez (Section 11.9, [31]) introduced the notions of *exact power* and *exact distance graphs*. Let $p \in \mathbb{N}$ and $G = (V, E)$ a graph. The *exact p-power graph* $G^{\natural p}$ of G is a graph with V as its set of vertices, and $xy \in E(G^{\natural p})$ if and only if there is a path of length p between x and y in G (the path need not be a shortest path). While the *exact distance-p graph* $G^{[\natural p]}$ is a graph with V as its set of vertices, and $xy \in E(G^{[\natural p]})$ if and only if $\mathrm{dist}_G(x, y) = p$. Clearly, $\chi(G^{[\natural p]}) \leq \chi(G^{\natural p}) \leq \chi(G^p)$ as $E(G^{[\natural p]}) \subseteq E(G^{\natural p}) \subseteq E(G^p)$.

Every planar graph is 5-degenerate, so Theorem 1.4.1 gives that $\chi(G^{\natural p}) \in \mathcal{O}(\Delta(G)^{\lfloor p/2 \rfloor})$ for every planar graph G. This bound is optimal even for outerplanar graphs [15]. The situation is different for exact distance-p graphs. For every planar graph G and an odd integer p, $\chi(G^{[\natural p]}) \in O(1)$, and when p is even we have $\chi(G^{[\natural p]}) \in O(\Delta(G))$. Indeed, those results are special cases from more general results.

Let $\mathcal{K}$ be a class of graphs with bounded expansion (the definition will be given next section). Nešetřil and Ossona de Mendez (Theorem 11.8, [31]) proved that for $G \in \mathcal{K}$ and an odd integer p we have $\chi(G^{[\natural p]}) \in \mathcal{O}(1)$. Van den Heuvel et al. [15] proved that for $G \in \mathcal{K}$ and an even integer p we have $\chi(G^{[\natural p]}) \in \mathcal{O}(\Delta(G))$. Since every class of planar graphs is a class with bounded expansion, their results hold for planar graphs too.

Theorem 1.4.2. *Let $\mathcal{K}$ be a class of graphs with bounded expansion.*

1. *Let p be an odd positive integer. Then there exists a constant $C = C(\mathcal{K}, p)$ such that for every $G \in \mathcal{K}$ we have $\chi(G^{[\natural p]}) \leq C$.*
2. *Let p be an even positive integer. Then there exists a constant $C' = C'(\mathcal{K}, p)$ such that for every $G \in \mathcal{K}$ we have $\chi(G^{[\natural p]}) \leq C'.\Delta(G)$.*

We have seen that in general we cannot bound $\chi(G^{\natural p})$ without involving $\Delta(G)$ even if p is an odd integer and G is a planar graph. However, if we require that G has large enough *odd girth* then we can bound $\chi(G^{\natural p})$ without involving $\Delta(G)$ where p is odd.

Theorem 1.4.3. *(Nešetřil and Ossona de Mendez (Theorem 11.7, [31])) Let $\mathcal{K}$ be a class of graphs with bounded expansion, and p an odd integer. If $G \in \mathcal{K}$ with odd girth at least $p+1$ then there exists a constant $C = C(\mathcal{K}, p)$ such that $\chi(G^{\natural p}) \leq C$.*

A class of graphs with bounded expansion can be characterized in many different ways, one of them is in terms of generalized coloring numbers.

Theorem 1.4.4. *(Zhu [37]) A class of graphs $\mathcal{K}$ has* bounded expansion *if and only if for every $r \in \mathbb{N}$ there exists a constant c_r such that* $\mathrm{wcol}_r(G) \leq c_r$ *for all $G \in \mathcal{K}$.*

Van den Heuvel et al. [15] proved Theorem 1.4.2(2) and gave two different proofs with better bounds for Theorem 1.4.2(1) using this characterization. They proved the following.

Theorem 1.4.5. *Let G be a graph.*

1. *If p is an odd positive integer then $\chi(G^{[\natural p]}) \leq \mathrm{dcol}_{2p-1}(G)$.*
2. *If p is an even positive integer then $\chi(G^{[\natural p]}) \leq \mathrm{dcol}_{2p}(G) \cdot \Delta(G)$.*

This theorem together with (1.3.1) and Theorem 1.4.4 give Theorem 1.4.2. They also proved the following.

Theorem 1.4.6. *Let G be a graph and p an odd integer. Set $q = \mathrm{wcol}_p(G)$.*

1. *We have $\chi(G^{[\natural p]}) \leq (\lfloor \frac{p}{2} \rfloor + 2)^q$.*
2. *If G has odd girth at least $p+1$ then $\chi(G^{\natural p}) \leq (\lfloor \frac{p}{2} \rfloor + 2)^q$.*

Clearly, Theorem 1.4.6(1) leads to Theorem 1.4.2(1).

For the class of planar graphs, the class of bounded tree-width graphs and other classes, Van den Heuvel et al. [15] give explicit upper bounds of $\chi(G^{[\natural p]})$ by finding upper bounds of $\mathrm{dcol}_p(G)$.

Theorem 1.4.7. *Let $p \in \mathbb{N}$.*

1. *If G is a planar graph then* $\mathrm{dcol}_p(G) \leq \binom{\lfloor \frac{p}{2} \rfloor + 3}{2} \cdot (2p+1) - p$.
2. *If G is a graph with genus g then* $\mathrm{dcol}_p(G) \leq (2g + \binom{\lfloor \frac{p}{2} \rfloor + 3}{2}) \cdot (2p+1) - p$.
3. *Let $t \in \mathbb{N}$ and G is a graph with tree-width at most t. Then* $\mathrm{dcol}_p(G) \leq \binom{t + \lfloor \frac{p}{2} \rfloor + 1}{t}$.
4. *If G is a graph that excludes K_t as a minor then* $\mathrm{dcol}_p(G) \leq \binom{t + \lfloor \frac{p}{2} \rfloor - 1}{t-2} \cdot (t-3)(2p+1)$.

Since every outerplanar graph G has tree-width at most 2, Theorem 1.4.7(3) and Theorem 1.4.5 together give $\chi(G^{[\natural 3]}) \leq 10$.

From the proof of Theorem 11.8 [31], it follows that for any planar graph G, $\chi(G^{[\natural 3]}) \leq 5.2^{20,971,522}$. On the other hand, Exercise 11.4 [31] gives an example of planar graph G with $\chi(G^{[\natural 3]}) = 6$. From Theorem 1.4.5 together with Theorem 1.4.7(1), Van den Heuvel et al. [14] improved the upper bound to 105. In the same paper, they gave an example for planar graph G with $\chi(G^{[\natural 3]}) = 7$.

In this dissertation, we tighten the range of $\chi(G^{[\natural 3]})$; we prove that $\chi(G^{[\natural 3]}) \leq 95$ for every planar graph G and we give an example for planar graph G such that $\chi(G^{[\natural 3]}) \geq 9$.

1.5 Nowhere Dense Classes and Classes with Bounded Expansion

The class of nowhere dense graphs was first introduced by Nešetřil and Ossona de Mendez [30, 31]. It generalizes and includes other types of sparse graph classes such as classes that exclude minors and classes with bounded expansion. Nowhere dense graph classes have useful algorithmic properties [6, 31, 12]; some algorithmic hard problems can be solved efficiently when they are restricted to nowhere dense classes. Nowhere dense classes are a limit for the efficient solvability of a wide class of problems [10, 28]. Grohe et al. [12] proved that for every graph in a fixed nowhere dense class, every first-order property can be decided in almost linear time. Nowhere dense classes can be characterized in several seemingly different ways. They can be characterized in terms of shallow minor densities [30], consequently in terms of generalized coloring numbers (by a result from [37]), sparse neighborhood covers [12, 11], just to name a few.

Definition 1.5.1. Let $\mathcal{C}$ be a class of graphs. We say that $\mathcal{C}$ is *nowhere dense* if for every $r \geq 1$ there exists $t \geq 1$ such that no graph in the class contains a topological minor of the complete graph K_t where every edge is subdivided at most r times. A class of graphs $\mathcal{C}$ is *somewhere dense* if it is not nowhere dense.

Nowhere dense classes can also be defined in terms of ordinary minors. Denote the average degree of G by $d(G)$. A class of graphs $\mathcal{C}$ is nowhere dense if for every positive integer r and every $\epsilon > 0$, there exists a positive integer n_0 such that if H is a graph with $|V(H)| \geq n_0$ and H is depth-r minor of some $G \in \mathcal{C}$, then $d(H) \leq |V(H)|^{\epsilon}$.

Theorem 1.5.2. *(by a result from [30]) A class of graphs $\mathcal{C}$ is nowhere dense if and only if for every $r \in \mathbb{N}$ and $\epsilon > 0$ there exists an integer n_0 such that for every graph*

$G \in \mathcal{C}$ with $|V(G)| \geq n_0$ we have

$$\mathrm{wcol}_r(G) \leq |V(G)|^{\epsilon}.$$

There are more restrictive sparse classes which are also characterized in terms of the generalized coloring numbers.

Definition 1.5.3. Let $\mathcal{C}$ be a class of graphs. We say that $\mathcal{C}$ has *bounded expansion* if for every $r \geq 1$ there exists $t \geq 1$ such that no graph in the class contains a topological minor of a graph H where every edge is subdivided at most r times and $d(H) \geq t$.

Theorem 1.5.4. *(Zhu [37]) A class of graphs $\mathcal{C}$ is a class with bounded expansion if and only if for every positive integer r there exists a constant number c such that* $\mathrm{wcol}_r(G) \leq c$ *for every $G \in \mathcal{C}$.*

Next we consider other characterizations of nowhere dense classes; the generalized coloring numbers were used to prove the forward implication of Theorem 1.5.10 and Lemma 1.5.13.

1.5.1 Poset Dimension

Nowhere dense classes can also be characterized in terms of poset dimension. Joret et al. [16] showed that the property of being nowhere dense class can be captured by looking at the dimension of posets whose order diagrams are in the class when seen as graphs.

Let P be a poset. The *dimension* $\dim(P)$ is the least number of total orders such that the intersection of those orders gives rise to P. The standard way of representing a poset is by drawing it's *diagram:* if $v <_P u$ we draw v below u and we draw a curve from v up to u if the relation $v <_P u$ is not implied by transitivity. Any relation

$v <_P u$ not implied by transitivity is called *cover relation.* The *height* of a poset is the maximum number of vertices in a chain in the poset. The diagram corresponds in a natural way to undirected graph G, where $V(G)$ are the elements of the poset and $E(G)$ correspond to the pairs of elements in a cover relation. The graph G is called the *cover graph* of P.

Recall that a *monotone* class means a class closed under taking subgraphs.

Theorem 1.5.5. *(Joret et al. [16]) Let $\mathcal{C}$ be a monotone class of graphs. Then $\mathcal{C}$ is nowhere dense if and only if for every $h \in \mathbb{N}$ and $\epsilon > 1$, n-element posets of height at most h whose cover graphs are in $\mathcal{C}$ have dimension $\mathcal{O}(n^\epsilon)$.*

It is conjectured that classes with bounded expansion can be characterized in terms of poset dimension, where the dimension is bounded by a function of the height.

Conjecture 1.5.6. (Joret et al. [17]) A monotone class of graphs $\mathcal{C}$ has bounded expansion if and only if for every fixed $h \geq 1$, posets of height at most h whose cover graphs are in $\mathcal{C}$ have bounded dimension.

The forward implication of the conjecture is shown in the same paper, but the backward direction is still open.

1.5.2 Sparse Neighborhood Covers

Another characterization of nowhere dense graphs is in terms of sparse neighborhood covers. Neighborhood covers with small size and radius play an important role in the design of many data structures for distributed systems [33, 32].

For a positive integer r, an *r-neighborhood cover* $\mathcal{X}$ of a graph G is a set of connected subgraphs of G called *clusters* such that for every vertex v in G there is a cluster $X \in \mathcal{X}$ with $N_r(v) \subseteq X$. The *radius* of the cover $\mathcal{X}$ is $\max\{\text{rad}(X) : X \in \mathcal{X}\}$ and it is denoted by $\text{rad}(\mathcal{X})$. The *degree* $d^{\mathcal{X}}(v)$ of v in $\mathcal{X}$ is the number of clusters containing v. The *degree* of $\mathcal{X}$ is $\max\{d^{\mathcal{X}}(v) : v \in G\}$.

Definition 1.5.7. A class $\mathcal{C}$ *admits sparse neighborhood covers* if for every positive integer r, there exists a positive integer c such that for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $G \in \mathcal{C}$ with $|G| \geq n_0$, there exists an r-neighborhood cover $\mathcal{X}$ with $\text{rad}(\mathcal{X}) \leq cr$ at degree at most $|G|^{\epsilon}$.

The generalized coloring numbers are useful tools in finding neighborhood covers with small radius.

Theorem 1.5.8. *(Grohe et al. [12]) Let G be a graph. If* $\text{wcol}_{2r}(G) = d$ *then G admits an r-neighborhood cover $\mathcal{X}$ of radius at most $2r$ and degree at most d. Therefore, if $\mathcal{C}$ is nowhere dense class then $\mathcal{C}$ admits sparse neighborhood covers.*

Theorem 1.5.9. *(Grohe et al. [11]) Let $\mathcal{C}$ be a monotone class of graphs. If $\mathcal{C}$ admits sparse neighborhood covers then $\mathcal{C}$ is nowhere dense.*

From Theorem 1.5.8 and Theorem 1.5.9, we have the following.

Theorem 1.5.10. *A monotone class of graphs $\mathcal{C}$ is nowhere dense if and only if it admits sparse neighborhood covers.*

1.5.3 The k-domination Numbers

A set $D \subseteq V(G)$ is a *dominating set* if for every $v \in V(G)$, either $v \in D$ or $N(v) \cap D \neq \phi$. The minimum size of a dominating set is called the *domination number* of G, and it is denoted by $\text{dom}(G)$. Determining the domination number is a famous problem in algorithmic graph theory and it is known to be NP-complete in general (Karp [18]). Even approximating $\mathrm{dom}(G)$ within a factor better than $O(\log(|V(G)|)$ is NP-complete (Raz and Safra [34]). However, when restricted to some sparse classes, the problem becomes more manageable. For example, for every G with degeneracy at most c, there exists a linear-time algorithm approximating $\mathrm{dom}(G)$ within a factor $O(c^2)$ (Lenzen and Wattenhofer [29]).

The k-domination number generalizes the notion of the domination number. The *k-domination number* of a graph G is the minimum size of a set $D \subseteq V(G)$ such that for every $v \in G$, $\mathrm{dist}_G(v, u) \leq k$ for some $u \in D$. The k-domination number of a graph G is denoted by $\mathrm{dom}_k(G)$.

The generalized coloring numbers can be used to give a linear-time constant-factor algorithm approximating the k-domination number in classes of graphs with bounded expansion.

Definition 1.5.11. Let G be a graph and $A \subseteq V(G)$. The subset A is *d-independent* if for every two vertices $x, y \in A$, $\mathrm{dist}_G(x, y) > d$. The maximum size of a d-independent set in G is denoted by $\alpha_d(G)$.

It is easy to see that for any graph G, $\alpha_{2k}(G) \leq \mathrm{dom}_k(G)$. When we restrict the problem to the classes with bounded expansion, then $\mathrm{dom}_k(G) = \mathcal{O}(\alpha_{2k}(G))$.

Theorem 1.5.12. *(Dvorak [8]) If $k \geq 1$ and $1 \leq m \leq 2k + 1$ are integers and G is a graph satisfying that $\mathrm{wcol}_m(G) \leq c$, then $\mathrm{dom}_k(G) \leq c^2 \alpha_m(G)$.*

From Theorem 1.5.12 and Theorem 1.5.2 we have the following.

Lemma 1.5.13. *Let $\mathcal{C}$ be a nowhere dense class of graphs. Then for every $r \in \mathbb{N}$ and $\epsilon > 0$ there exists an integer n_0 such that for every graph $G \in \mathcal{C}$ with $|V(G)| \geq n_0$ we* *have*

$$\mathrm{dom}_r(G) \leq |V(G)|^{2\epsilon} \alpha_{2r}(G).$$

In Chapter 5, we prove that the converse of Lemma 1.5.13 is also true for every monotone class of graphs $\mathcal{C}$.

Chapter 2

ON THE WEAK 2-COLORING NUMBER OF PLANAR GRAPHS

In this chapter, we prove the following.

Theorem 2.0.1. *For every planar graph G,* $\mathrm{wcol}_2(G) \leq 23$.

We first give an isometric-path decomposition $\mathscr{P} = (P_1, \ldots, P_s)$ for G. After that we use $\mathscr{P}$ to define a linear ordering L of $V(G)$. Then we prove that $\mathrm{wcol}_2[G, L] \leq 23$. We also give an example of a planar graph G such that $\mathrm{wcol}_2[G, L] = 23$. This shows that the upper bound 23 is optimal in terms of our method.

2.1 Isometric-path Decomposition $\mathscr{P}$

We may assume without loss of generality that G is a maximal planar graph since removing edges does not increase the weak coloring numbers. We refine the isometric-path decomposition that is defined by Van den Heuvel et al. [14]. We gave the construction of their decomposition in the proof of Lemma 1.2.7. Here we add an extra condition when we choose the shortest path P_k between w_k and w'_k. If there are more than one isometric paths then break ties as follows.

(∗∗) Choose P_k so that the interior of the cycle $D' := v_k P_a v'_k w'_k P_k w_k v_k$ has as few vertices as possible.

Let $\mathscr{P} = (P_1, \ldots, P_s)$ be the resulting isometric-path decomposition. We give now some notations. For each $i \in [s]$, denote the endpoints of P_i by w_i and w'_i. Let $i \in [s] \smallsetminus \{1, 2\}$, then P_i is adjacent to exactly two paths $P_a, P_b \in \{P_1, \ldots, P_{i-1}\}, a < b$.

We call P_a *the strong parent* and P_b *the weak parent of* P_i. We say that $P_j, j \in [s]$ is a *parent* of P_i if P_j is either the strong or the weak parent of P_i. For each $i \in [s] \setminus \{1, 2\}$, denote the component of $G[P_{\geq i}]$ containing P_i by C_i, the cycle forming the boundary of the region of $R^2 \setminus G[P_1 \cup \ldots \cup P_{i-1}]$ in which C_i lies by D_i, and the two edges $e \in D_i$ such that $e \notin P_a \cup P_b$ by e_i and e'_i. Denote the unique vertex in $V(P_a) \cap e_i$ by v_i, and the unique vertex in $V(P_a) \cap e'_i$ by v'_i. Denote the unique vertex in $V(P_b) \cap e_i$ by z_i, and the unique vertex in $V(P_b) \cap e'_i$ by z'_i. The edges e_i, e'_i each belong to the boundary of a triangle face contained in the interior of D_i. Let f_i and f'_i be those triangle faces with $e_i \in G[f_i]$ and $e'_i \in G[f'_i]$. Denote the unique vertex in $G[f_i] - \{v_i, z_i\}$ by w_i, and the unique vertex in $G[f'_i] - \{v'_i, z'_i\}$ by w'_i. Denote the cycle $v_i P_a v'_i w'_i P_i w_i v_i$ by $D_{i,1}$ and the cycle $w_i P_i w'_i z'_i P_b z_i w_i$ by $D_{i,2}$. Denote the interior of $D_{i,1}$ by $O_{i,1}$, and the interior of $D_{i,2}$ by $O_{i,2}$.

The following four lemmas follow directly from the construction of $\mathscr{P}$.

Lemma 2.1.1. *Let $P_i \in \mathscr{P}, i \geq 3$. If v is a vertex in the interior of D_i, then $v \in P_j \in \mathscr{P}$ for some $j \geq i$.*

Lemma 2.1.2. *Let $v \in P_i, i \geq 3$. Let P_a and P_b be the strong and weak parents of P_i respectively. If P is a path between v and a vertex $u \in V(P_1 \cup \ldots \cup P_{i-1})$ then P intersects with $P_a \cup P_b$, more precisely, P intersects with either $v_i P_a v'_i$ or $z_i P_b z'_i$.*

Lemma 2.1.3. *Let $P_i \in \mathscr{P}, i \geq 3$. Let P_a be the strong parent, and P_b the weak parent of P_i. Then P_a is a parent of P_b, i.e., if $b \geq 3$, P_a is either the strong parent or the weak parent of P_b.*

Lemma 2.1.4. *Let $P_i, P_j \in \mathscr{P}, i \geq 3$. If P_i is a parent of P_j then $C_j \subset C_i$.*

We will use the next lemma often in the remainder of this chapter.

Lemma 2.1.5. *Let $P_i \in \mathscr{P}, i \in [s] \smallsetminus \{1,2\}$ and let $x, y \in P_i$. Assume that there exists a path Q between x and y in $O_{i,1} \cup P_i$ such that Q contains at least one vertex in $O_{i,1}$ (i.e., Q is not a subpath of P_i). Then $\|xP_iy\| \leq \|Q\| - 1$.*

Proof. We may assume without loss of generality that $\mathrm{dist}_{P_i}(x, w_i) \leq \mathrm{dist}_{P_i}(y, w_i)$. The path P_i is isometric in C_i, so xP_iy is also isometric in C_i. Both xP_iy and Q are paths between x and y in C_i. Thus $\|xP_iy\| \leq \|Q\|$. Assume that $\|xP_iy\| = \|Q\|$. Starting from the x-end of Q find the last vertex $x' \in Q$ such that $x' \in w_iP_ix$. Starting from the y-end of $x'Qy$ find the last vertex $y' \in x'Qy$ such that $y' \in yP_iw_i'$. The vertices x' and y' exist because x and y respectively are candidates. Assume that $x' = x$ and $y' = y$. As $Q \subseteq O_{i,1} \cup P_i$ and Q contains a vertex in $O_{i,1}$, the number of vertices in the interior of the cycle $v_iP_av_i'w_i'P_iyQxP_iw_iv_i$ is less than the number of vertices in $O_{i,1}$, a contradiction to $(**)$. Thus $\|xP_iy\| < \|Q\|$ as desired. Assume now that either $x' \neq x$ or $y' \neq y$. Clearly, $\|xP_iy\| < \|x'P_iy'\|$. The path $x'P_iy'$ is isometric in C_i, and both $x'P_iy'$ and $x'Qy'$ are paths between x' and y' in C_i. Then $\|x'P_iy'\| \leq \|x'Qy'\|$. Since either $x' \neq x$ or $y' \neq y$, $\|x'Qy'\| < \|Q\|$. Hence $\|xP_iy\| \leq \|Q\| - 2$. $\square$

Now we are ready to define the ordering L that witnesses $\mathrm{wcol}_2[G, L] \leq 23$.

2.2 The ordering L

Let $x \in P_i, y \in P_j, i \neq j$. Then $x <_L y$ if and only if $i < j$. We inductively give the restriction of L on $V(P_i), i \in [s]$. Fix any endpoint r_1 of P_1 and make it the first vertex in P_1 (the first vertex in L too), then order $V(P_1)$ toward r_1. The path P_2 is a single vertex r_2, so r_2 is the third vertex in L.

Let $P_i \in \mathscr{P}, i \in [s] \smallsetminus \{1,2\}$. If $v_i = v_i'$ (recall that v_i and v_i' are vertices in the strong parent of P_i) then denote arbitrary the endpoints of P_i by w_i and w_i' and set

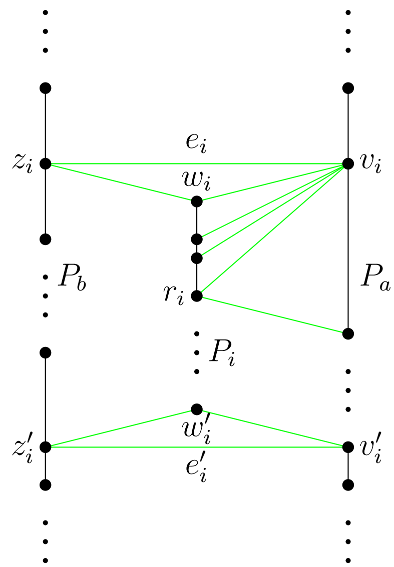

Figure 2.1. P_a the Strong Parent of P_i and P_b the Weak Parent of P_i

$r_i = w'_i$. If $v_i \neq v'_i$ then choose notations so that $v_i <_L v'_i$. Let $r_i \in P_i \cap N(v_i)$ such that $\text{dist}_{P_i}(w_i, r_i)$ is maximum over all the vertices in $P_i \cap N(v_i)$, see Figure 2.1 on page 30. Let r_i be the first vertex in P_i with respect to L, then order $V(P_i)$ toward r_i, i.e., choose any linear ordering of $V(P_i)$ satisfying that for any $x, y \in V(P_i)$ with $\text{dist}_{P_i}(r_i, x) < \text{dist}_{P_i}(r_i, y)$ then $x <_L y$. Note that if $\text{dist}_{P_i}(r_i, x) = \text{dist}_{P_i}(r_i, y)$, we arbitrary choose the vertex that comes first. The ordering L is now completed.

Let $v \in P_i$ such that $v \neq r_i$. If they exist, let $v^-, v^+ \in P_i \cap N(v)$ such that $v^- <_L v <_L v^+$. If the vertices v^- and v^+ exist then they are unique.

Each vertex $y \in P_i$ gives a rise to two subpaths of P_i, they are $w_i P_i y$ and $y P_i w'_i$. The next lemma follows from the definition of L.

Lemma 2.2.1. *Let $x, y \in P_i, i \in [s]$ with $x <_L y$. Let $Q \in \{w_i P_i y, y P_i w'_i\}$ such that $x \notin Q$, and let $z \in Q - y$. Then $y <_L z$.*

Proof. As $x <_L y$, $\text{dist}_{P_i}(r_i, x) \leq \text{dist}_{P_i}(r_i, y)$. Thus $r_i \in P_i - Q$. So $\text{dist}_{P_i}(r_i, y) < \text{dist}_{P_i}(r_i, z)$. $\square$

Before we prove $\mathrm{wcol}_2[G, L] \le 23$ for every planar graph G, we illustrate the construction of $\mathscr{P}$ and the ordering L by giving an example of a maximal planar graph G such that $\mathrm{wcol}_2[G, L] = 23$.

2.3 Example

The graph G is given in Figure 2.2 on page 32. It is easy to see that G is a planar graph and every face is triangulated. We illustrate the decomposition $\mathscr{P}$ together with the ordering L. The first path P_1 is the edge u_1u_2 which is an edge incident to the outer face. Let u_1, u_2 be the first and second vertices with respect to L respectively. The ordering L is represented by purple numerals next to the vertices. The second path P_2 is the vertex u_3 which is the vertex of the outer face that is not in P_1. The vertex u_3 is the third vertex in L.

Now there is only one component C_3 in $G[P_{\ge 3}]$. The edge u_1u_3 belongs to the boundary of a triangle face incident to a unique vertex (which is u_4) in C_3. Similar is true for u_2u_3, and the vertex associated with u_2u_3 is u_9. Now we pick an isometric path between u_4 and u_9 in C_3 satisfying $(**)$, so this path is $P_3 := u_4u_5u_6u_7u_8u_9$. The path P_3 is adjacent to P_1 and P_2, and as $1 < 2$, P_1 is the strong parent of P_3. Since $u_1 <_L u_2$, $v_3 = u_1, v'_3 = u_2, w_3 = u_4, w'_3 = u_9$. As $N(v_3) \cap P_3 = \{u_4, u_5, u_6, u_7\}$ and u_7 is the farthest of these from w_3, $r_3 = u_7$. There are more than one way to order $V(P_3)$, for example the second vertex could be u_6 or u_8. Order $V(P_3)$ as $u_7 <_L u_6 <_L u_8 <_L u_5 <_L u_9 <_L u_4$.

There is only one component C_4 in $G[P_{\ge 4}]$. The next path in $\mathscr{P}$ has to be an isometric path between u_{10} and u_{14} in C_4 satisfying $(**)$. This path is $P_4 := u_{10}u_{11}u_{12}u_{13}u_{14}$. The strong parent of P_4 is P_2, and as P_2 is the single vertex u_3

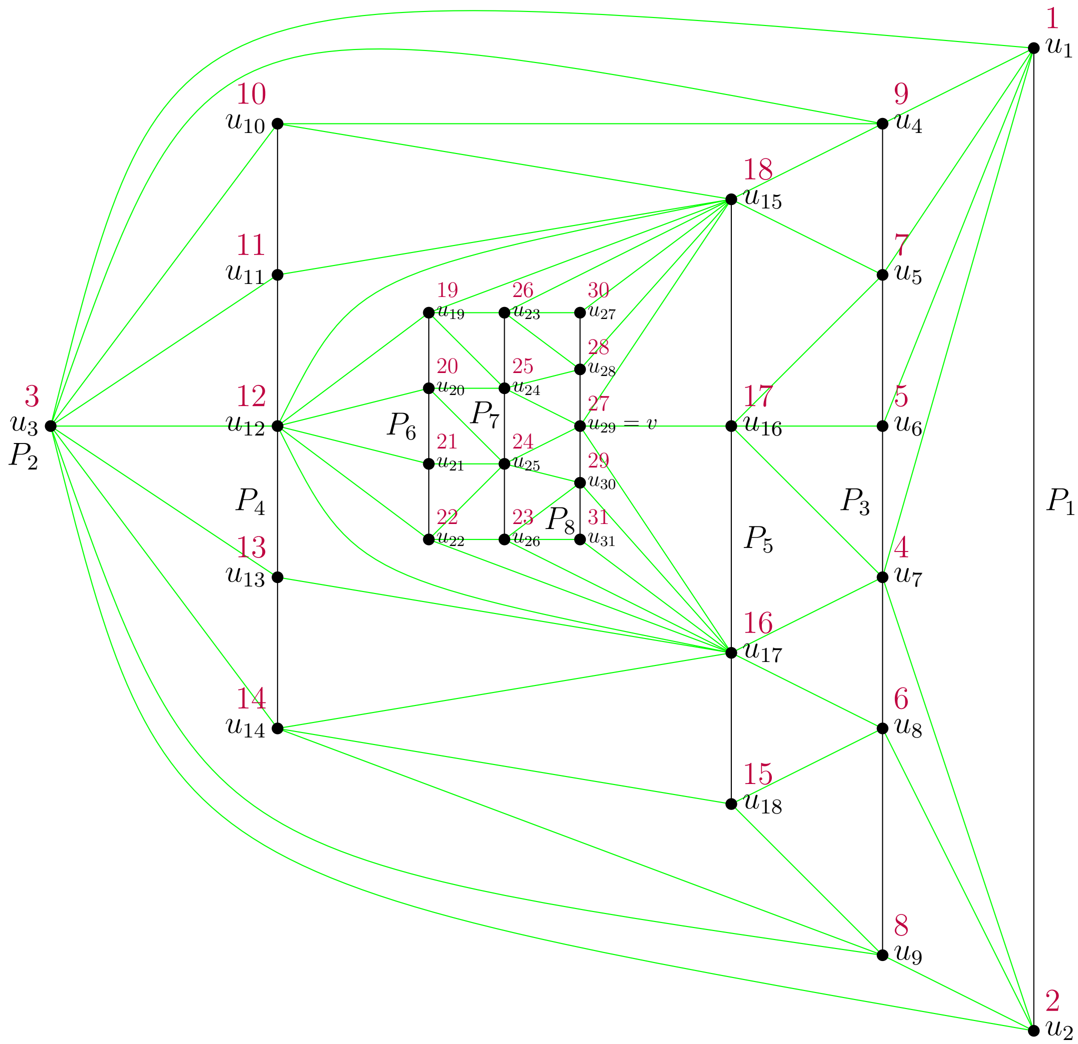

Figure 2.2. A Planar Graph G and an Ordering L Satisfying $\mathrm{wcol}_2[G, L] = 23$. The Ordering L is Represented by the Purple Numbers

then $v_4 = v_{4'} = u_3$. Any endpoint of P_4 is a candidate for w_4. Let $w_4 = u_{14}$, so $r_4 = w_4' = u_{10}$ and the order of $V(P_4)$ is $u_{10} <_L u_{11} <_L u_{12} <_L u_{13} <_L u_{14}$.

Denote the unique component in $G[P_{\geq 5}]$ by C_5. The next path in $\mathscr{P}$ has to be an isometric path between u_{15} and u_{18} in C_5 satisfying $(**)$. This path is $P_5 := u_{15}u_{16}u_{17}u_{18}$. The strong parent of P_5 is P_3, as $u_9 <_L u_4$, $v_5 = u_9$ and $v_5' = u_4$. As $N(v_5) \cap P_5 = \{u_{18}\}$, $r_5 = u_{18}$ and the order of $V(P_5)$ is $u_{18} <_L u_{17} <_L u_{16} <_L u_{15}$.

Denote the unique component in $G[P_{\geq 6}]$ by C_6. The next path in $\mathscr{P}$ is an isometric path between u_{19} and u_{22} in C_6 satisfying $(**)$. This path is $P_6 := u_{19}u_{20}u_{21}u_{22}$. The strong parent of P_6 is P_4 and $v_6 = v_6' = u_{12}$. So any endpoint of P_6 is a candidate for w_6. Choose u_{22} to be w_6, so $r_6 = w_6' = u_{19}$ and the order of $V(P_6)$ is $u_{19} <_L u_{20} <_L u_{21} <_L u_{22}$.

Denote the unique component in $G[P_{\geq 7}]$ by C_7. The next path is $P_7 := u_{23}u_{24}u_{25}u_{26}$ since it is an isometric path between u_{23} and u_{26} satisfying $(**)$. The strong parent of P_7 is P_5 and as $u_{17} <_L u_{15}$, $v_7 = u_{17}$ and $v_7' = u_{15}$. As $N(v_7) \cap P_7 = \{u_{26}\}$, $r_7 = u_{26}$ and so the order of $V(P_7)$ is $u_{26} <_L u_{25} <_L u_{24} <_L u_{23}$. The next path P_8 in $\mathscr{P}$ has to be a path between u_{27} and u_{31} in $G[P_{\geq 8}]$; since there is only one such path, $P_8 := u_{27}u_{28}u_{29}u_{30}u_{31}$. The strong parent of P_8 is P_5, as $u_{17} <_L u_{15}$, $v_8 = u_{17}$ and $v_8' = u_{15}$. So $w_8 = u_{31}$ and $w_8' = u_{27}$. As $N(v_8) \cap P_8 = \{u_{29}, u_{30}, u_{31}\}$ and u_{29} is the farthest of these from w_8, $r_8 = u_{29}$. There are more than way to order the $V(P_8)$ toward r_8, we choose the order $u_{29} <_L u_{28} <_L u_{30} <_L u_{27} <_L u_{31}$. The decomposition $\mathscr{P}$ and the ordering L are now completed.

Claim 2.3.1. $\mathrm{wcol}_2[L, u_{29}] = 23$.

For simplicity we write v for u_{29}, and we denote the set of vertices $\mathrm{Wcol}_2[G, L, v]$ and the number $\mathrm{wcol}_2[G, L, v]$ by $\mathrm{W}(v)$ and $\mathrm{w}(v)$ respectively. Let $A := \{P_3, P_4, P_5, P_6, P_7, P_8\}$. Observe that $\mathrm{W}(v) \subseteq \bigcup\limits_{P \in A} P$ as v is with distance greater than two from any vertex in $P_1 \cup P_2$. The vertex v is the smallest in P_8, so $|\,\mathrm{W}(v) \cap P_8| = 1$. Now we find $|\,\mathrm{W}(v) \cap P_7|$. Both $u_{24}, u_{25} \in N(v)$ and the paths $vu_{28}u_{23}, vu_{30}u_{26}$ are witnessing $u_{23}, u_{26} \in \mathrm{W}(v)$ respectively. Thus $|\,\mathrm{W}(v) \cap P_7| = 4$. The four paths $vu_{24}u_{19}, vu_{25}u_{20}, vu_{25}u_{21}, vu_{25}u_{22}$ are witnessing $u_{19}, u_{20}, u_{21}, u_{22} \in \mathrm{W}(v)$ respectively. Thus $|\,\mathrm{W}(v) \cap P_6| = 4$. The vertices u_{15}, u_{16}, u_{17} are neighbors of v and the path $vu_{17}u_{18}$ is witnessing $u_{18} \in \mathrm{W}(v)$. Thus $|\,\mathrm{W}(v) \cap P_5| = 4$. The vertex u_{15} is a common neighbor of $v, u_{10}, u_{11}, u_{12}$, so $\{u_{10}, u_{11}, u_{12}\} \subseteq \mathrm{W}(v)$. Similarly, the vertex u_{17} is a common neighbor of v, u_{13}, u_{14}, so $\{u_{13}, u_{14}\} \subseteq \mathrm{W}(v)$. Thus $|\mathrm{W}(v) \cap P_4| = 5$. The five paths $vu_{15}u_4, vu_{15}u_5, vu_{16}u_6, vu_{16}u_7, vu_{17}u_8$ are witnessing $u_4, u_5, u_6, u_7, u_8 \in \mathrm{W}(v)$ respectively. Observe that $\mathrm{dist}(v, u_9) > 2$, so $u_9 \notin \mathrm{W}(v)$, thus $|\mathrm{W}(v) \cap P_3| = 5$. Hence all in all $\mathrm{w}(v) = 23$.

2.4 Proof of Theorem 2.0.1

Theorem 2.4.1. $\mathrm{wcol}_2[G, L] \leq 23$.

Proof. Let $v \in G$; then $v \in P_k$ for some $k \in [s]$. Recall that $\mathrm{W}(v)$ denotes the set of vertices $\mathrm{Wcol}_2[G, L, v]$, and $\mathrm{w}(v)$ denotes the number $\mathrm{wcol}_2[G, L, v]$.

Claim 2.4.2. $|\mathrm{W}(v) \cap P_k| \leq 3$.

Proof. Follows from Lemma 2.1.5 and Lemma 2.2.1. $\square$

If $k \leq 2$ then $\mathrm{W}(v) \subseteq P_1 \cup P_2$. So $\mathrm{w}(v) \leq 3$ and we are done. So assume $k \geq 3$. Then P_k is adjacent to exactly two paths $P_h, P_j \in \{P_1, \ldots, P_{k-1}\}, h < j$. So P_h is the strong parent of P_k, and P_j is the weak parent of P_k. Observe that P_h is a parent of P_j too. If $j \leq 2$ then $h = 1, j = 2$ and $\mathrm{W}(v) \subseteq P_1 \cup P_2 \cup P_k$; from Lemma 1.2.6, $\mathrm{w}(v) \leq 2 * 5 + 3 = 13$. So assume $j \geq 3$, then P_j is adjacent to exactly two paths in $\{P_1, \ldots, P_{j-1}\}$, one of them is P_h and the other one is P_i, say. Note that P_i is adjacent to P_h, so either P_h is a parent of P_i or P_i is a parent of P_h. If $h \leq 2$ then $\mathrm{W}(v) \subseteq P_1 \cup P_2 \cup P_i \cup P_j \cup P_k$; so $\mathrm{w}(v) \leq 4 * 5 + 3 = 23$. So assume $h \geq 3$; then P_h is adjacent to exactly two paths $P_f, P_g \in \{P_1, \ldots, P_{h-1}\}$. If $i < h$ then $i \in \{f, g\}$, so $\mathrm{W}(v) \subseteq P_f \cup P_g \cup P_h \cup P_j \cup P_k$, and thus $\mathrm{w}(v) \leq 4 * 5 + 3 = 23$. So assume $h < i$. See Figure 2.3 on page 35.

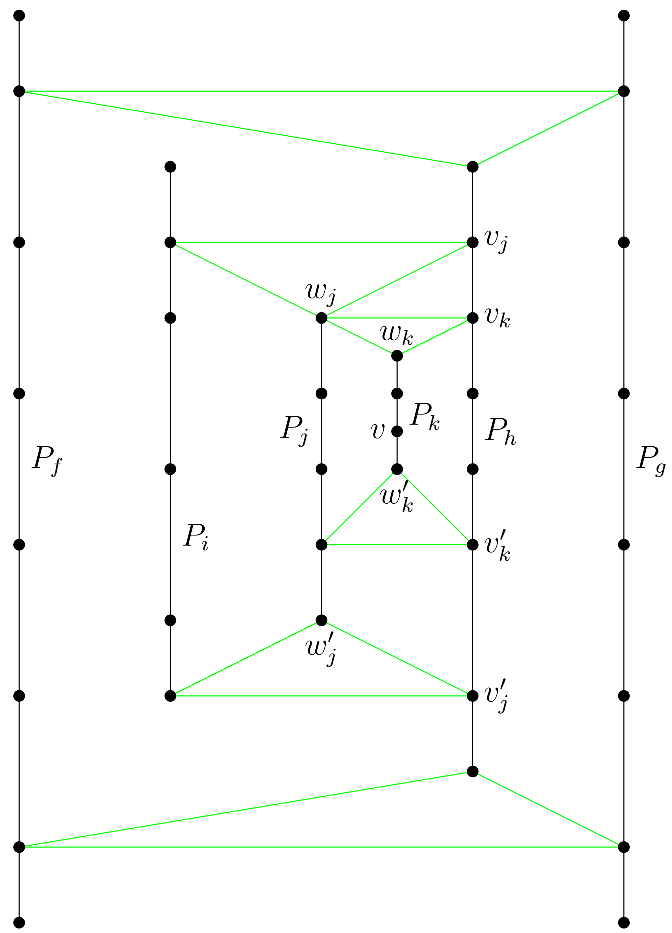

Figure 2.3. $V(D) \subseteq P_f \cup P_h$

Let $A := \{P_f, P_g, P_h, P_i, P_j, P_k\}$. Let $u \in \mathrm{W}(v)$. Then $u \in P$ for some $P \in A$. Therefore $\mathrm{w}(v) = \sum_{P \in A} |\mathrm{W}(v) \cap P|$. In the remainder of this proof we estimate $|\mathrm{W}(v) \cap P|$ for each $P \in A$.

Claim 2.4.3. $|\mathrm{W}(v) \cap P_j| \leq 4$.

Proof. Recall that $V(D_j) \subseteq P_h \cup P_i$ as P_h and P_i are the strong and weak parents of P_j. Let $u, u' \in \mathrm{W}(v) \cap P_j$ such that $\mathrm{dist}_{P_j}(u, u')$ is maximum. There exist two paths each of length at most two in $O_{j,1} \cup P_j$ (as $v \in O_{j,1}$) witnessing that $u \in \mathrm{W}(v)$ and $u' \in \mathrm{W}(v)$. Those two paths combined contain a path Q of length at most four between u and u' in $O_{j,1} \cup P_j$. If $v \notin Q$ then $\|uP_ju'\| \leq \|Q\| \leq 2$, if $v \in Q$

then Q contains a vertex in $O_{j,1}$; from Lemma 2.1.5, $\|uP_ju'\| \leq \|Q\| - 1 \leq 3$. Thus $|\,\mathrm{W}(v) \cap P_j| \leq 4$. $\square$

Claim 2.4.4. If $x, y \in N(v) \cap P_j, x \neq y$ then $xy \in P_j$.

Proof. The path xvy is a path in $O_{j,1} \cup P_j$ and $v \in O_{j,1}$. From Lemma 2.1.5, $\|xP_jy\| \leq \|xvy\| - 1 = 1$. Thus $xy \in P_j$. $\square$

Claim 2.4.5. $|\,\mathrm{W}(v) \cap P_i| \leq 4$.

Proof. Let $y_1, y_2 \in \mathrm{W}(v) \cap P_i$ such that $\mathrm{dist}_{P_i}(y_1, y_2)$ is maximum. Let Q_1 be a path witnessing $y_1 \in \mathrm{W}(v)$ and Q_2 a path witnessing $y_2 \in \mathrm{W}(v)$. As P_k is not adjacent to P_i, both Q_1 and Q_2 are of length two, say $Q_1 = vy'y_1, Q_2 = vy''y_2$. Note that $Q_1, Q_2 \subseteq C_i$ and $y', y'' \in P_j \cup P_h$. As $h < i$, every vertex in P_h is smaller with respect to L than every vertex in P_i. From the definition of weak reachability, $y', y'' \in P_j$. If $y' = y''$ then $\|y_1P_iy_2\| \leq \|y_1y'y_2\| \leq 2$. Assume that $y' \neq y''$, then $y', y'' \in N(v) \cap P_j$. From Claim 2.4.4, $y'y'' \in P_j$, so $\|y_1P_iy_2\| \leq \|y_1y'y''y_2\| = 3$. Thus $|\,\mathrm{W}(v) \cap P_i| \leq 4$. $\square$

Let C be the component of $G[P_{\geq h+1}]$ containing v. Let D be the cycle forming the boundary of the region of $R^2 \setminus G[P_1 \cup \ldots \cup P_h]$ in which C lies. Then either $V(D) \subseteq P_f \cup P_h$ or $V(D) \subseteq P_g \cup P_h$. Assume without loss of generality that $V(D) \subseteq P_f \cup P_h$, see the example in Figure 2.3 on page 35. Set

$$R := |\,\mathrm{W}(v) \cap P_k| + |\,\mathrm{W}(v) \cap P_h| + |\,\mathrm{W}(v) \cap P_f|.$$

Claim 2.4.6. *If $v = r_k$ then $R \leq 10$.*

Proof. Clearly, $|\,\mathrm{W}(v) \cap P_k| = 1$. Note that $v \in N(v_k)$.

Case 1: $v \notin N(v_k')$ or $v_k = v_k'$.

Note that $v_k, v_k' \in P_h$ as P_h is the strong parent of P_k. Let $z \in \mathrm{W}(v) \cap P_f$, and let Q be a path witnessing $z \in \mathrm{W}(v)$. As P_k is not adjacent to P_f, Q has to be of length exactly two. So $Q = vz'z$ where $z' \in P_h \cup P_j$. The path P_j is not adjacent to P_f, so $z' \in P_h$. Thus either $z' = v_k$ or $z' = v'_k$. Since $v \notin N(v'_k)$ or $v_k = v'_k$, $z' = v_k$. The vertex v_k has at most three neighbors in P_f as P_f is an isometric path in C_f. Thus $|\,\mathrm{W}(v) \cap P_f| \leq 3$. From Lemma 1.2.6, $|\,\mathrm{W}(v) \cap P_h| \leq 5$. So $R \leq 9$.

Case 2: $v \in N(v'_k)$ and $v_k \neq v'_k$.

If $|\,\mathrm{W}(v) \cap P_h| \leq 4$ then $R \leq 10$ as $|\,\mathrm{W}(v) \cap P_f| \leq 5$. So assume that $|\,\mathrm{W}(v) \cap P_h| = 5$. Let $P \in \{w_h P_h v'_k, v'_k P_h w'_h\}$ such that $v_k \in P$. Assume that $\mathrm{W}(v) \cap P_h \subseteq P$. Let $u \in \mathrm{W}(v) \cap P_h$ such that $\mathrm{dist}_{P_h}(u, v'_k)$ is maximum, and let Q be a path witnessing $u \in \mathrm{W}(v)$. Since $|\,\mathrm{W}(v) \cap P_h| = 5$, $\mathrm{dist}_{P_h}(u, v'_k) \geq 4$. On the other hand, $v'_k v Q u$ is a path in C_h of length at most three, a contradiction. Thus there exists a vertex $w \in \mathrm{W}(v) \cap P_h$ such that $w \in P_h - P$. From Lemma 2.2.1, $V(v_k P_h v'_k) <_L w$. As $w \notin v_k P_h v'_k$, w is not a neighbor of v. So any path U witnessing $w \in \mathrm{W}(v)$ has to be of length two, say $U = vw'w$ where $w' \in v_k P_h v'_k$ or $w' \in z_k P_j z'_k$. As $V(v_k P_h v'_k) <_L w$, $w' \notin v_k P_h v'_k$. Thus $w' \in z_k P_j z'_k$. It means that $N(v'_k) \cap P_f = \phi$ as G is planar graph. So $|\,\mathrm{W}(v) \cap P_f| \leq 3$ (v_k has at most three neighbors in P_f). Thus $R \leq 9$. □

Claim 2.4.7. *If $v \neq r_k$ and $v_k = v'_k$ then $R \leq 10$.*

Proof. Recall that $|\,\mathrm{W}(v) \cap P_k| \leq 3$. If $v \notin N(v_k)$ then $|\,\mathrm{W}(v) \cap P_f| = 0$. With $|\,\mathrm{W}(v) \cap P_h| \leq 5$ we have $R \leq 8$. So assume $v \in N(v_k)$. The vertex v_k has at most three neighbors in P_f, so $|\,\mathrm{W}(v) \cap P_f| \leq 3$. If also $|\,\mathrm{W}(v) \cap P_h| \leq 4$ then $R \leq 10$ and we are done. So assume $|\,\mathrm{W}(v) \cap P_h| = 5$. If they exist, let t_1 and t'_1 be the neighbors of v_k in P_h. Let D'_4 be the cycle $w_k P_k w'_k v_k w_k$ and O'_4 be the interior of D'_4, see Figure 2.4 on page 38. Observe that $N(v) \cap P_h \subseteq \{v_k\}$. If $vv'z$ is a path witnessing $z \in \mathrm{W}(v) \cap P_h$ where $v' \in O'_4 \cup \{v^-, v^+, v_k\}$ then $z \in \{v_k, t_1, t'_1\}$. Since $|\,\mathrm{W}(v) \cap P_h| = 5$, there exist two vertices $z \in \mathrm{W}(v) \cap P_h \setminus \{v_k, t_1, t'_1\}$ such that the witnessing path is of the form $vv'z$ where $v' \in \{z_k, z'_k\}$. Call those two vertices t_2

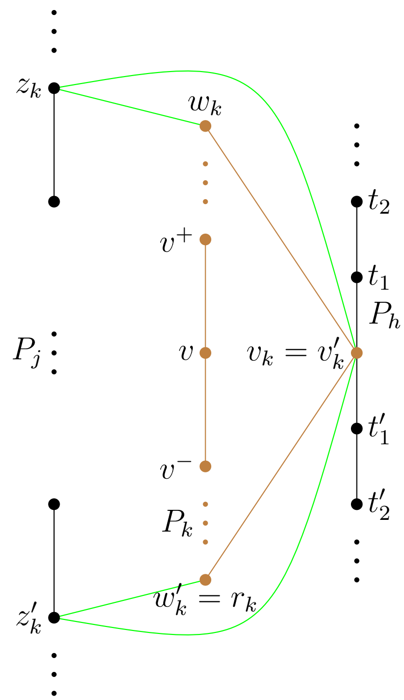

Figure 2.4. $v_k = v'_k$, D'_4 is the Brown Cycle in Claim 2.4.7

and t'_2. Clearly, $\mathrm{dist}_{P_h}(v_k, t_2), \mathrm{dist}_{P_h}(v_k, t'_2) \geq 2$. Assume without loss of generality that vz_kt_2 is the path witnessing $t_2 \in \mathrm{W}(v)$. Since $v_kP_ht_2$ is isometric and $v_kz_kt_2$ is a path in C_h, $\|v_kP_ht_2\| = 2$, fix notation so that $v_kt_1t_2 \subseteq P_h$. Thus $vz'_kt'_2$ must be the path witnessing $t'_2 \in \mathrm{W}(v)$ and $v_kP_ht'_2 = v_kt'_1t'_2$. Since G is planar, v_k cannot have neighbors in P_f. Thus $|\,\mathrm{W}(v) \cap P_f| = 0$ and $R \leq 8$. $\square$

If it exists, let $x_1 \in N(r_k) \cap w_kP_kr_k$, and if it exists, let $x_2 \in N(r_k) \cap r_kP_kw'_k$. If $x_i, i \in [2]$ exists then it is unique as P_k is isometric.

Claim 2.4.8. If $v \neq r_k$, $v_k \neq v'_k$ and $v \in x_2P_kw'_k$ *then* $R \leq 10$.

Proof. Observe that $v \notin N(v_k)$. If also $v \notin N(v'_k)$ then $|\,\mathrm{W}(v) \cap P_f| = 0$. As $|\,\mathrm{W}(v) \cap P_h| \leq 5$ and $|\,\mathrm{W}(v) \cap P_k| \leq 3$ we get $R \leq 8$. So assume that $v \in N(v'_k)$. The vertex v'_k has at most three neighbors in P_f, so $|\,\mathrm{W}(v) \cap P_f| \leq 3$. If also $|\,\mathrm{W}(v) \cap P_h| \leq 4$ then $R \leq 10$. So assume that $|\,\mathrm{W}(v) \cap P_h| = 5$. Let $P \in \{w_hP_hv'_k, v'_kP_hw'_h\}$ such that

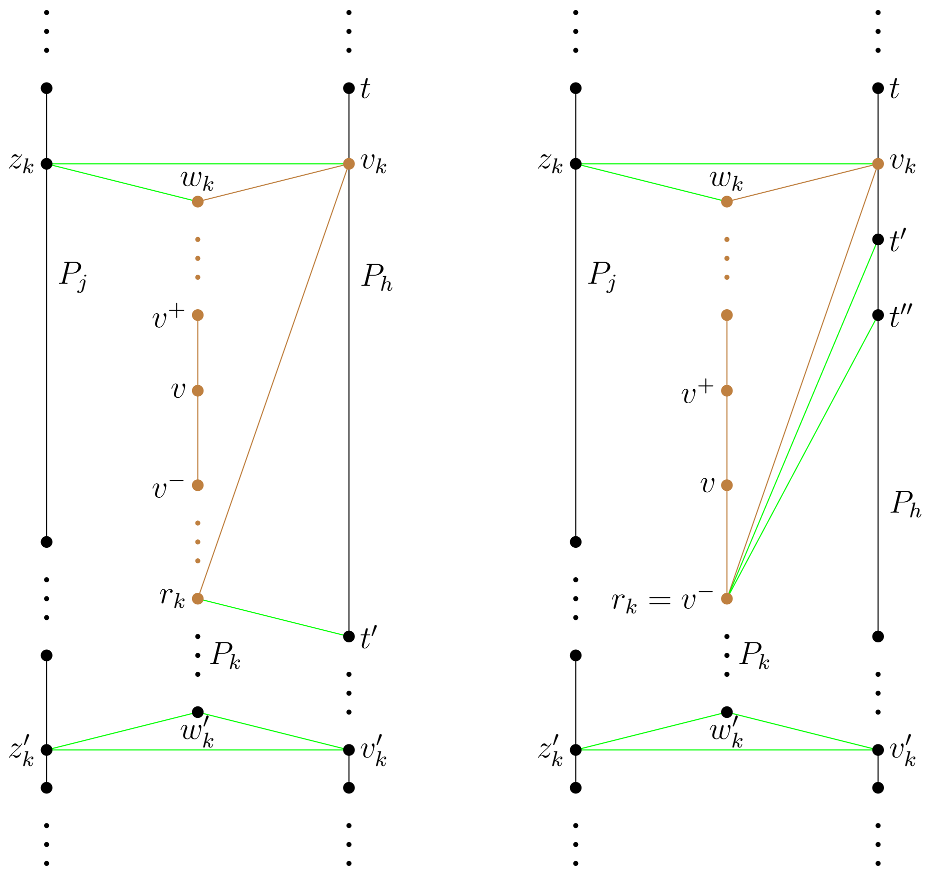

Figure 2.5. D'_4 is the Brown Cycle in the First Drawing, the Second Drawing Illustrates the Possible Neighbors of v^- if $v^- = r_k$

$v_k \in P$. As in Case 2 of Claim 2.4.6, there exists a vertex $w \in \mathrm{W}(v) \cap P_h$ such that $w \in P_h - P$, $v'_k <_L w$ and any path witnessing $w \in \mathrm{W}(v)$ is of the form $vw'w$ where $w' \in z_kP_jz'_k$. The graph G is a planar, so $N(v'_k) \cap P_f = \phi$. This gives $|\,\mathrm{W}(v) \cap P_f| = 0$, so $R \leq 8$. □

Claim 2.4.9. *If $v \neq r_k, v_k \neq v'_k$ and $v \in w_kP_kx_1$ then $R + |\,\mathrm{W}(v) \cap P_j| \leq 14$.*

Proof. Recall that $|\,\mathrm{W}(v) \cap P_j| \leq 4$ (Claim 2.4.3). So if we show that $R \leq 10$ then we are done. As G is planar and $r_kv_k \in G$, $v \notin N(v'_k)$. If also $v \notin N(v_k)$ then $|\,\mathrm{W}(v) \cap P_f| = 0$. As $|\,\mathrm{W}(v) \cap P_h| \leq 5$ and $|\,\mathrm{W}(v) \cap P_k| \leq 3$ we get $R \leq 8$. Assume that $v \in N(v_k)$. The vertex v_k has at most three neighbors in P_f, so $|\,\mathrm{W}(v) \cap P_f| \leq 3$. If

$|\,\mathrm{W}(v)\cap P_h| \leq 4$ or $|\,\mathrm{W}(v)\cap P_f| \leq 2$ then $R \leq 10$. So assume that $|\,\mathrm{W}(v)\cap P_h| = 5$ and $|\,\mathrm{W}(v)\cap P_f| = 3$. Clearly, $R \leq 5+2*3 = 11$. It suffices to prove that $|\,\mathrm{W}(v)\cap P_j| \leq 3$.

In this paragraph, we prove that v and z'_k are neighbors. Let D'_4 be the cycle $w_kP_kr_kv_kw_k$ and O'_4 be the interior of D'_4, see Figure 2.5 on page 39. Then $v \in D'_4$ and $N(v)\cap D'_4 = \{v^-, v^+, v_k\}$; note that in this case v^- exists and if $v \neq w_k$ then v^+ exists too. Let $z \in \mathrm{W}(v)\cap P_h$, and let Q be a path witnessing that $z \in \mathrm{W}(v)$. Then $\{v_k, v^-, v^+, z_k, z'_k\}\cap(Q-v) \neq \phi$. If it exists, let $t \in N(v_k)$ such that $t \in P_h - v_kP_hv'_k$. Let $t' \in N(v_k)\cap v_kP_hv'_k$, and if it exists, let $t'' \in N(t')\cap(P_h - v_k)$. If $v_k \in Q$ then $z \in \{v_k, t, t'\}$. If $v^- \in Q$ and $v^- \neq r_k$ then $z = v_k$. If $v^- \in Q$ and $v^- = r_k$ then $z \in \{v_k, t', t''\}$. If $v^+ \in Q$ then $z = v_k$, see Figure 2.5 on page 39. As $|\,\mathrm{W}(v)\cap P_h| = 5$ and $|\{v_k, t, t', t''\}| = 4$, there exists a vertex $z \in \mathrm{W}(v)\cap P_h \smallsetminus \{v_k, t, t', t''\}$ and a path Q witnessing $z \in \mathrm{W}(v)$ such that $\{z_k, z'_k\}\cap Q \neq \phi$. Since v_k has three neighbors in P_f and G is planar, $Q = vz'_kz$. So v and z'_k are neighbors.

If $z_k = z'_k$ then directly $|\,\mathrm{W}(v)\cap P_j| \leq 3$ (z_k and the two unique neighbors of z_k in P_j, if they exist). So assume that $z_k \neq z'_k$.

We show in this paragraph that $z_k <_L z'_k$. The strong parent of P_j is P_h, so $\{v_j, v'_j\} \subseteq V(P_h)$. From the construction of $\mathscr{P}$, $v_kP_hv'_k \subseteq v_jP_hv'_j$. Let $y \in \{v_j, v'_j\}$ such that $\mathrm{dist}_{P_h}(y, v_k) \leq \mathrm{dist}_{P_h}(y, v'_k)$ and let $y' \in \{v_j, v'_j\} \smallsetminus \{y\}$. So $\mathrm{dist}_{P_h}(y', v'_k) \leq \mathrm{dist}_{P_h}(y', v_k)$. See the example in Figure 2.6 on page 41. As v_k has three neighbors in P_f and G is planar, $y = v_k$. Since $y = v_k <_L v'_k$, Lemma 2.2.1 tells us that $v'_k \leq_L y'$. So $y <_L y'$, $y = v_j = v_k$ and $y' = v'_j$. Since $z_kv_k \in E(G)$, $z_k \in N(v_j)\cap P_j$ and $\mathrm{dist}_{P_j}(w_j, z_k)$ is maximum over all vertices in $N(v_j)\cap P_j$. So $z_k = r_j$, i.e., z_k is the minimum vertex in P_j with respect to L. Hence $z_k <_L z'_k$.

Assume that there exists $u \in \mathrm{W}(v)\cap P_j$ such that $u \in z'_kP_jw'_j - z'_k$. Then u is not a neighbor of v as $u \notin z_kP_jz'_k$. Let U be a path witnessing $u \in \mathrm{W}(v)$, then $U = vu'u$

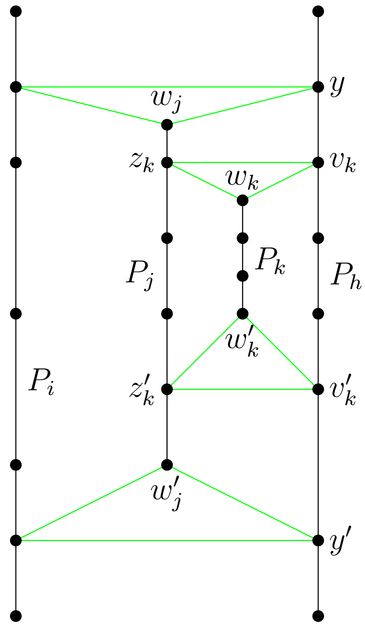

Figure 2.6. y and y'

where $u' \in z_k P_j z'_k \cup v_k P_h v'_k$. As $h < j$, $u' \in z_k P_j z'_k$. The path P_j is isometric so $u' = z'_k$. From Lemma 2.2.1, $z'_k <_L u$ which is in contradiction to the definition of weak reachability. Thus $\mathrm{W}(v) \cap P_j \subseteq w_j P_j z'_k$.

Let $l \in \mathrm{W}(v) \cap w_j P_j z'_k$ such that $\mathrm{dist}_{P_j}(z'_k, l)$ is maximum. Since $z'_k \in N(v)$ and $l \in \mathrm{W}(v)$, there exists a path U of length at most three between z'_k and l in $O_{j,1} \cup P_j$. If $v \notin U$ then U is of length less than three; if $v \in U$ then from Lemma 2.1.5 we have $\|l P_j z'_k\| \leq \|U\| - 1 \leq 2$. Thus $\mathrm{dist}_{P_j}(l, z'_k) \leq 2$ which means that $|\,\mathrm{W}(v) \cap P_j| \leq 3$. $\square$

Claim 2.4.10. $R + |\,\mathrm{W}(v) \cap P_j| \leq 14$.

Proof. Follows from Claims 2.4.3, 2.4.6, 2.4.7, 2.4.8 and 2.4.9. $\square$

From Lemma 1.2.6, Claims 2.4.5 and 2.4.10, $\mathrm{w}(v) \leq 4 + 14 + 5 = 23$. $\square$

Chapter 3

ON THE CHROMATIC NUMBER OF THE EXACT DISTANCE-3 GRAPHS OF PLANAR GRAPHS

In this chapter, we improve the best known upper bound of the chromatic numbers of the exact distance-3 graphs $G^{[\natural 3]}$ of planar graphs G, which is 105, to 95. We also improve the best known lower bound, which is 7, to 9.

Theorem 3.0.1. *Let G be a planar graph. Then $\chi(G^{[\natural 3]}) \leq 95$.*

Recall that $\chi(G^{[\natural 3]}) \leq \mathrm{dcol}_5(G)$ (Theorem 1.4.5), so it suffices to prove that $\mathrm{dcol}_5(G) \leq 95$ for every planar graph G.

We may assume that G is a maximal planar graph as $\mathrm{dcol}_r(G)$ does not increase by removing edges.

Recall the isometric-path decomposition $\mathscr{P} = (P_1, \ldots, P_s)$ constructed in Section 1.2. We define a liner ordering L of $V(G)$ as follows. For each $x \in P_i, y \in P_j$ with $i < j$, put $x <_L y$. For every $i \in [s]$, fix any end point r_i of P_i and make it the L-smallest vertex in $V(P_i)$, then order $V(P_i)$ toward this end, i.e., if $x, y \in P_i$ such that $\mathrm{dist}_{P_i}(r_i, x) < \mathrm{dist}_{P_i}(r_i, y)$ then put $x <_L y$.

Let $v \in G$. We denote the number $\mathrm{dcol}_5[G, L, v]$ and the set $\mathrm{Dcol}_5[G, L, v]$ by $\mathrm{dc}(v)$ and $\mathrm{Dc}(v)$ respectively.

Lemma 3.0.2. *Let $u \in P_i$ and $v \in P_k, i < k$. Assume that $Q_u := uu'u'' \ldots v$ is a path witnessing that $u \in \mathrm{Dc}(v)$. Then $u''Q_u v - u'' \subseteq C_k$.*

Proof. The definition of $\mathrm{Dc}(v)$ tells us that $v <_L V(\mathring{u}''Q_u\mathring{v})$ (if $V(\mathring{u}''Q_u\mathring{v}) \neq \phi$). As $V(D_k) <_L v$, $D_k \cap \mathring{u}''Q_u\mathring{v} = \phi$, and as $v \in C_k$, $u''Q_u v - u'' \subseteq C_k$. $\square$

Theorem 3.1.1. $\mathrm{dcol}_5[G, L] \leq 95$.

First we prove an easy case. Then we name and count the number of paths $P \in \mathscr{P}$ such that $\mathrm{Dc}(v) \cap P \neq \phi$ where v is a fixed vertex in G. We will see that the number of such paths is at most ten. After that, throughout a series of claims, we bound $|\mathrm{Dc}(v) \cap P|$ for each one of those paths. The proof will be a direct result from the claims and the fact that $\mathrm{dc}(v) = \sum_{P \in \mathscr{P}} |\mathrm{Dc}(v) \cap P|$.

Claim 3.1.2. Let $v \in G$. If $|\{P \in \mathscr{P} : \mathrm{Dc}(v) \cap P \neq \phi\}| \leq 9$ then $\mathrm{dc}(v) \leq 94$.

Proof. Let $P_\alpha \in \mathscr{P}$ such that $\mathrm{Dc}(v) \cap P_\alpha \neq \phi$. From the construction of $\mathscr{P}$, P_α is isometric in $G' := G[P_{\geq \alpha}]$, and from the definition of $\mathrm{Dc}(v)$, $\mathrm{Dc}(v) \cap P_\alpha \subseteq N_5^{G'}[v] \cap P_\alpha$. From Lemma 1.2.6, $|\mathrm{Dc}(v) \cap P_\alpha| \leq 11$. Let P_k be the path containing v. From the restriction of L on $V(P_k)$, $\mathrm{Dc}(v) \cap P_k \subseteq r_k P_k v$; let $u \in \mathrm{Dc}(v) \cap P_k$ such that $\mathrm{dist}_{P_k}(u, v)$ is maximum. Then $\mathrm{dist}_{P_k}(u, v) \leq 5$, and so $|\mathrm{Dc}(v) \cap P_k| \leq 6$. Thus $\mathrm{dc}(v) \leq 8 * 11 + 6 = 94$. $\square$

Claim 3.1.3. For every $v \in G$, $|\{P \in \mathscr{P} : \mathrm{Dc}(v) \cap P \neq \phi\}| \leq 10$.

Proof. Let $k \in [s]$ such that $v \in P_k$. Assume that $k \geq 3$. Let $u \in \mathrm{Dc}(v) \setminus \{v\}$, as $\mathrm{Dc}(v) \setminus \{v\} <_L v$, $u \in V(P_1 \cup \ldots \cup P_k)$. Let $Q_u := u_0 \ldots u_q$ be a path witnessing that $u \in \mathrm{Dc}(v)$ where $u_0 = u$ and $u_q = v$. Let $i = \max\{j \in [q-1] \cup \{0\} : u_j <_L v\}$, then i exists because $u_0 <_L v$, and from the definition of $\mathrm{Dc}(v)$, $i \leq 2$.

Case 1: $i = 0$.

As $k \geq 3$, P_k has a strong parent, call it P_h, and a weak parent, call it P_l (so $h < l < k$). Assume that $u \notin P_k$. From Lemma 2.1.2, uQ_uv intersects with $P_l \cup P_h$. Since u is the only vertex in uQ_uv satisfying that $u <_L v$, $u \in P_l \cup P_h$. Thus if $i = 0$ then $u \in P_k \cup P_l \cup P_h$.

Case 2: $i = 1$.

By replacing u with u_1 in Case 1, we get $u_1 \in P_k \cup P_l \cup P_h$. From the definition of $\mathrm{Dc}(v)$, $u(= u_0) <_L u_1$. If $u_1 \in P_k$ then again $u \in P_k \cup P_l \cup P_h$ (Lemma 2.1.2). Assume that $u_1 \in P_h$ and $h \geq 3$. Denote the strong parent of P_h by P_g, and the weak parent by P_f. Then either $u \in P_h$, or else from Lemma 2.1.2, $u \in P_f \cup P_g$. Assume that $u_1 \in P_l$ and $l \geq 3$. Recall that P_l and P_h are the parents of P_k and $h < l$. From Lemma 2.1.3, P_h is a parent of P_l. Denote the other parent of P_l by P_m. Then either $u \in P_l$ or , $u \in P_m \cup P_h$ (Lemma 2.1.2). Thus if $i = 1$ then $u \in P_j$ for some $j \in A := \{k, l, m, h, f, g\}$.

We need the following observation for the next case:

($\star$) If $m < h$ then P_m is a parent of P_h (Lemma 2.1.3). As P_g and P_f are the strong and weak parents of P_h, $m \in \{f, g\}$.

Case 3: $i = 2$.

By replacing u with u_2 in Case 1, we get $u_2 \in P_k \cup P_l \cup P_h$. From the definition of $\mathrm{Dc}(v)$, $u <_L \{u_1, u_2\}$. Assume that $u_2 <_L u_1$. By replacing u_1 with u_2 in Case 2, we get $u \in P_j$ for some $j \in A$. Assume now that $u_1 <_L u_2$, so $u <_L u_1 <_L u_2$. By replacing u_1 with u_2 and replacing u with u_1 in Case 2, we get $u_1 \in P_j$ for some $j \in A$. If $u_1 \in P_k \cup P_l \cup P_h$ then again $u \in P_j$ for some $j \in A$. Assume that $u_1 \in P_g$ and $g \geq 3$. Denote the strong parent of P_g by P_b, and the weak parent by P_c. From Lemma 2.1.2, $u \in P_g \cup P_c \cup P_b$. Assume that $u_1 \in P_f$ and $f \geq 3$. Recall that P_f and P_g are the parents of P_h and $g < f$, so P_g is a parent of P_f. Denote the other parent of P_f by P_d. From Lemma 2.1.2, $u \in P_f \cup P_d \cup P_g$.

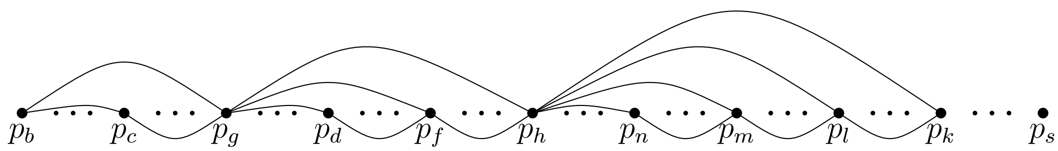

Figure 3.1. $p_j, j \in B$

Assume that $u_1 \in P_m$ and $m \notin \{f, g\}$, so $h < m$ (observation ($\star$)). Recall that P_m and P_h are the parents of P_l, so P_h is a parent of P_m. Denote the other parent of P_m by P_n. From Lemma 2.1.2, $u \in P_m \cup P_n \cup P_h$. Thus if $i = 2$ then $u \in P_j$ for some $j \in B := \{k, l, m, n, h, f, d, g, c, b\}$.

We end this proof with the following observation:

($\star\star$) if $j \leq 2$ for some $j \in \{k, l, m, h, f, g\}$ then the number of paths $P \in \mathscr{P}$ satisfying that $\mathrm{Dc}(v) \cap P \neq \phi$ is at most nine since P_j has at most one parent instead of two. $\square$

Remark 3.1.4. Recall that P_g and P_d are the parents of P_f. If $d < g$ then P_d is a parent of P_g; since P_b and P_c are the parents of P_g, $d \in \{b, c\}$. Recall that P_h and P_n are the parents of P_m. If $n < h$ then P_n is a parent of P_h; since P_g and P_f are the parents of P_h, $n \in \{g, f\}$. Thus if $m < h, d < g$ or $n < h$ (recall observation ($\star$)), then the number of paths $P \in \mathscr{P}$ satisfying that $\mathrm{Dc}(v) \cap P \neq \phi$ is at most nine, and we are done (Claim 3.1.2).

In the remainder of this proof, we assume that $h < m, g < d$ and $h < n$. We will estimate $|\mathrm{Dc}(v) \cap P_j|$ for every $j \in B$. Before we proceed, we provide a simple drawing to give a quick reference showing how the paths in $\mathscr{P}$ are related.

In Figure 3.1 on page 45, we contracted each path $P_j, j \in B$ to a single vertex p_j. The vertex p_j is to the left of $p_{j'}$ if and only if $j < j'$.

Claim 3.1.5. $|\mathrm{Dc}(v) \cap P_k| \leq 6$.

Proof. Follows from the restriction of L on $V(P_k)$ and from the fact that P_k is isometric in $G[P_{\geq k}]$. $\square$

Claim 3.1.6. $|\operatorname{Dc}(v) \cap P_l|, |\operatorname{Dc}(v) \cap P_f| \leq 10$.

Proof. We first show that $v \in O_{l,1} \cap O_{f,1}$ (recall the definition of $O_{i,1}$ from Section 1.2). The paths P_h and P_m are the parents of P_l, and as $h < m$, P_h is the strong parent of P_l. Thus $D_{l,1} = v_l P_h v'_l w'_l P_l w_l v_l$; as P_h and P_l are the parents of P_k, P_k is in the interior of $D_{l,1}$, i.e., P_k is in $O_{l,1}$, see Figure 3.2 on page 47. Similarly, the paths P_g and P_d are the parents of P_f, and as $g < d$, P_g is the strong parent of P_f. Thus $D_{f,1} = v_f P_g v'_f w'_f P_f w_f v_f$. The paths P_g and P_f are the parents of P_h, so P_h is in $O_{f,1}$, as P_h and P_k are adjacent and G is planar, P_k is also in $O_{f,1}$. As $v \in P_k$, $v \in O_{l,1} \cap O_{f,1}$ too.

Let $j \in \{l, f\}$, and let $x, y \in \operatorname{Dc}(v) \cap P_j$ such that $\operatorname{dist}_{P_j}(x, y)$ is maximum. To prove the claim, it suffices to show that $\|xP_j y\| \leq 9$. Let Q_x and Q_y be paths witnessing that $x, y \in \operatorname{Dc}(v)$ respectively. Then $||Q_x||, ||Q_y|| \leq 5$ and $Q_x, Q_y \subseteq C_j$. Starting from the v-end, let x' be the first vertex in Q_x contained in P_j, and starting from the v-end, let y' be the first vertex in Q_y contained in P_j.

Now the walk $W := x' Q_x v Q_y y'$ is in $P_j \cup O_{j,1}$ since $v \in O_{j,1}$, and $W \neq x' P_j y'$ as $v \in W - x' P_j y'$. From Lemma 2.1.5, $\|x' P_j y'\| \leq \|W\| - 1$. Therefore

$$\begin{aligned}
\|xP_j y\| &\leq \|xP_j x'\| + \|x' P_j y'\| + \|y' P_j y\| \\
&\leq \|xP_j x'\| + \|W\| + \|y' P_j y\| - 1 \\
&\leq_{(1)} \|xQ_x x'\| + \|W\| + \|yQ_y y'\| - 1 \\
&= \|xQ_x v Q_y y\| - 1 \\
&\leq 9,
\end{aligned}$$

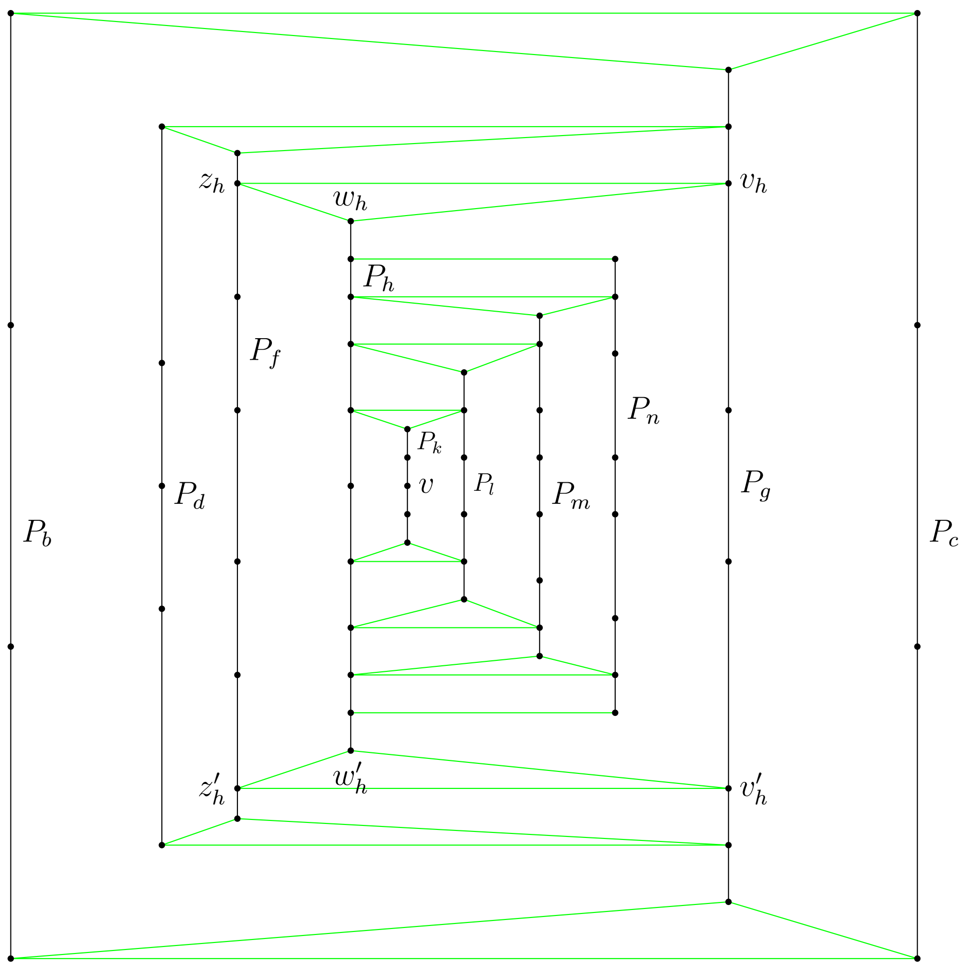

Figure 3.2. $v \in O_{h,1}$, $O_{h,1}$ is the Interior of $D_{h,1} = v_h P_g v'_h w'_h P_h w_h v_h$

where (1) follows from the fact that P_j is isometric in C_j and both paths Q_x and Q_y are in C_j. $\square$

Claim 3.1.7. $|\operatorname{Dc}(v) \cap P_m| \leq 9$.

Proof. Let $x, y \in \operatorname{Dc}(v) \cap P_m$. If we show that $\|xP_m y\| \leq 8$, we are done. Let Q_x be a path witnessing that $x \in \operatorname{Dc}(v)$, and Q_y a path witnessing that $y \in \operatorname{Dc}(v)$. Then $||Q_x||, ||Q_y|| \leq 5$ and $Q_x, Q_y \subseteq C_m$. As $m < k$, any path from $v \in P_k$ to a vertex in P_m must intersect with a parent of P_k, i.e., either P_h or P_l (Lemma 2.1.2). As $h < m$, $Q_x \cap P_h, Q_y \cap P_h = \phi$. So $Q_x \cap P_l, Q_y \cap P_l \neq \phi$. Starting from the v-end, let

x' be the first vertex in Q_x contained in P_l. Starting from the v-end, let y' be the first vertex in Q_y contained in P_l. Then the walk $W := x'Q_xvQ_yy'$ is in $P_l \cup O_{l,1}$ as $Q_x \cap P_h, Q_y \cap P_h = \phi$ and $v \in O_{l,1}$ (see Claim 3.1.6), and $W \neq x'P_ly'$. From Lemma 2.1.5, $\|x'P_ly'\| \leq \|W\| - 1$.

Starting from the x'-end, let x'' be the first vertex in $x'Q_xx$ contained in P_m. Starting from the y'-end, let y'' be the first vertex in $y'Q_yy$ contained in P_m. We show that the walk $W' := x''Q_xx'P_ly'Q_yy''$ is in $P_m \cup O_{m,1}$. As P_h is the strong parent of P_m, $D_{m,1} = v_mP_hv'_mw'_mP_mw_mv_m$. As P_h and P_m are the parents of P_l, P_l is in $O_{m,1}$, and as $Q_x \cap P_h, Q_y \cap P_h = \phi$, W' is in $P_m \cup O_{m,1}$.

Clearly, $W' \neq x''P_my''$. From Lemma 2.1.5,

$$\begin{aligned}
\|x''P_my''\| &\leq \|W'\| - 1 \\
&= \|x''Q_xx'\| + \|x'P_ly'\| + \|y'Q_yy''\| - 1 \\
&\leq \|x''Q_xx'\| + \|W\| + \|y'Q_yy''\| - 2 \\
&= \|x''Q_xvQ_yy''\| - 2.
\end{aligned} \tag{3.1.1}$$

Since P_m is isometric in C_m and $Q_x, Q_y \subseteq C_m$, $\|xP_mx''\| \leq \|xQ_xx''\|$ and $\|y''P_my\| \leq \|y''Q_yy\|$. Thus

$$\begin{aligned}
\|xP_my\| &\leq \|xP_mx''\| + \|x''P_my''\| + \|y''P_my\| \\
&\leq \|xQ_xx''\| + \|x''Q_xvQ_yy''\| + \|y''Q_yy\| - 2 \\
&= \|xQ_xvQ_yy\| - 2 \\
&\leq 8.
\end{aligned}$$

□

Claim 3.1.8. $|\operatorname{Dc}(v) \cap P_n| \leq 9$.

Proof. The proof is similar to the proof of Claim 3.1.7. Let $x, y \in \operatorname{Dc}(v) \cap P_n$. Let Q_x be a path witnessing that $x \in \operatorname{Dc}(v)$, and Q_y a path witnessing that $y \in \operatorname{Dc}(v)$. Then

$||Q_x||, ||Q_y|| \leq 5$ and $Q_x, Q_y \subseteq C_n$. Any path from a vertex in P_k to a vertex in P_n intersects with either P_h or P_l. As $h < n$, $Q_x \cap P_h, Q_y \cap P_h = \phi$, so $Q_x \cap P_l, Q_y \cap P_l \neq \phi$. Starting from the v-end, let x' be the first vertex in Q_x contained in P_l. Starting from the v-end, let y' be the first vertex in Q_y contained in P_l. Let $W := x'Q_xvQ_yy'$. Recall that $D_{l,1} = v_lP_hv_l'w_l'P_lw_lv_l$ and $v \in O_{l,1}$ (see Claim 3.1.6), and as $Q_x \cap P_h, Q_y \cap P_h = \phi$, W is a walk in $P_l \cup O_{l,1}$. Clearly, $W \neq x'P_ly'$, so from Lemma 2.1.5, $\|x'P_ly'\| \leq \|W\| - 1$.

Now both $x'Q_xx$ and $y'Q_yy$ are paths between a vertex in P_l and a vertex in P_n. Since $n < l$ (P_n is a parent of P_m, and P_m is a parent of P_l), both $x'Q_xx$ and $y'Q_yy$ each intersect with a parent of P_l, i.e., either P_h or P_m. We have just seen that $Q_x \cap P_h, Q_y \cap P_h = \phi$, so $x'Q_xx \cap P_m, y'Q_yy \cap P_m \neq \phi$. Starting from the x'-end, let x'' be the first vertex in $x'Q_xx$ that is in P_m. Starting from the y'-end, let y'' be the first vertex in $y'Q_yy$ that is in P_m. We show that the walk $W' := x''Q_xx'P_ly'Q_yy''$ is in $P_m \cup O_{m,1}$. As P_h is the strong parent of P_m, then $D_{m,1} = v_mP_hv_m'w_m'P_mw_mv_m$. The parents of P_l are P_h and P_m, so P_l is in $O_{m,1}$. As $Q_x \cap P_h, Q_y \cap P_h = \phi$, W' is in $P_m \cup O_{m,1}$. From Lemma 2.1.5, $\|x''P_my''\| \leq \|W'\| - 1$. As in Inequality (3.1.1), $\|x''P_my''\| \leq \|x''Q_xvQ_yy''\| - 2$. Both $Q_x, Q_y \subseteq C_n$, and as $n < m$, $C_m \subset C_n$ (Lemma 2.1.4). So $xQ_xx''P_my''Q_yy$ is a walk in C_n. As xP_ny is isometric in C_n,

$$\begin{aligned}
\|xP_ny\| &\leq \|xQ_xx''\| + \|x''P_my''\| + \|y''Q_yy\| \\
&\leq \|xQ_xx''\| + \|x''Q_xvQ_yy''\| + \|y''Q_yy\| - 2 \\
&= \|xQ_xvQ_yy\| - 2 \\
&\leq 8. \qquad \square
\end{aligned}$$

Claim 3.1.9. Let $u \in \mathrm{Dc}(v) \cap P_d$ and assume that $Q_u := u_0u_1u_2 \ldots u_q$ is a path witnessing that $u \in \mathrm{Dc}(v)$ where $u_0 = u$ and $u_q = v$. Then $u_1 \in P_f$ and $u_2 \in v_kP_hv_k'$.

Proof. Let $i = \max\{j \in [q-1] \cup \{0\} : u_j <_L v\}$. We recall some facts from the proof of Claim 3.1.3: If $i = 0$, then $u \in P_j$ for some $j \in \{k, l, h\}$, if $i = 1$ then $u \in P_j$ for some $j \in A$, if $i = 2$ and $u_2 < u_1$ then $u \in P_j$ for some $j \in A$, and if $i = 2$ and $u_1 < u_2$ then $u \in P_j$ for some $j \in B$. Thus $i = 2$ and $u_1 < u_2$ as $u \in P_d$. Also from the proof of Claim 3.1.3, $u_1 \in P_f$. From Lemma 2.1.2, the path $u_1 Q_u v$ intersects with either $v_k P_h v'_k$ or P_l. So either $u_2 \in v_k P_h v'_k$ or $u_2 \in P_l$. As P_f and P_l are not adjacent, $u_2 \in v_k P_h v'_k$. $\square$

Claim 3.1.10. $|\operatorname{Dc}(v) \cap P_d| \leq 10$.

Proof. Let $x, y \in \operatorname{Dc}(v) \cap P_d$. Let Q_x be a path witnessing that $x \in \operatorname{Dc}(v)$, and let Q_y be a path witnessing that $y \in \operatorname{Dc}(v)$. Then $\|Q_x\|, \|Q_y\| \leq 5$ and $Q_x, Q_y \subseteq C_d$. Observe also that Q_x and Q_y each have to be of length at least three. From Claim 3.1.9, $Q_x = xx'x'' \ldots v, Q_y = yy'y'' \ldots v$ where $x', y' \in P_f$ and $x'', y'' \in v_k P_h v'_k$. We show that the walk $W := x' Q_x v Q_y y'$ is in $P_f \cup O_{f,1}$. As P_g is the strong parent of P_f, $D_{f,1} = v_f P_g v'_f w'_f P_f w_f v_f$. As P_g and P_f are the parents of P_h, $C_h \subseteq O_{f,1}$. From Lemma 3.0.2, $x'' Q_x v - x'', y'' Q_y v - y'' \subseteq C_k$, and from Lemma 2.1.4, $C_k \subset C_h$. So $x'' Q_x v Q_y y'' \subseteq C_h$, and as $x', y' \in P_f$, $W \subseteq P_f \cup O_{f,1}$. From Lemma 2.1.5, $\|x' P_f y'\| \leq \|W\| - 1$.

Now the walk $xx' P_f y' y$ is in C_d since $x, y \in P_d$ and $x' P_f y' \subseteq C_f \subset C_d$ (Lemma 2.1.4). As $x P_d y$ is an isometric path in C_d,

$$
\begin{aligned}
\|x P_d y\| &\leq \|xx' P_f y' y\| \\
&= \|xx'\| + \|x' P_f y'\| + \|y' y\| \\
&\leq \|xx'\| + \|W\| + \|y' y\| - 1 \\
&\leq \|x Q_x v Q_y y\| - 1 \\
&\leq 9.
\end{aligned}
$$

$\square$

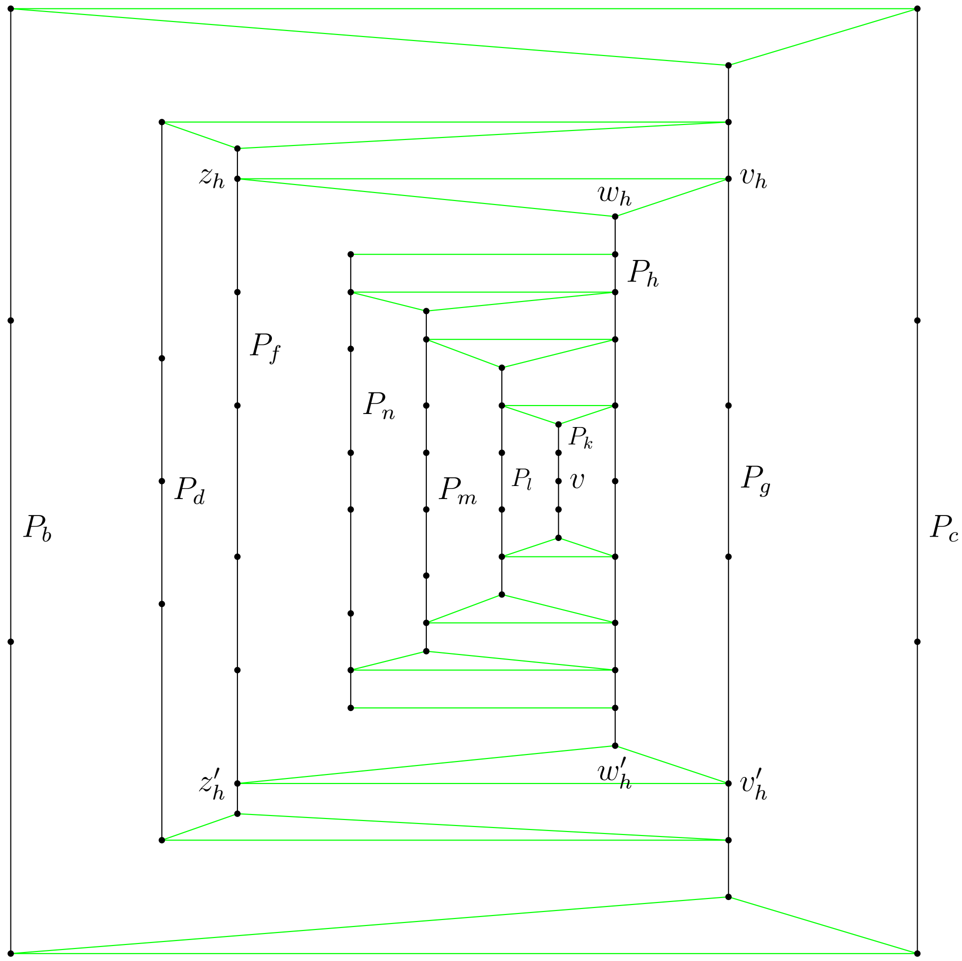

Figure 3.3. $v \in O_{h,2}$, $O_{h,2}$ is the Interior of $D_{h,2} = z_h P_f z'_h w'_h P_h w_h z_h$

Recall that P_b is the strong parent of P_g, and P_c is the weak parent. Then $D_{g,1} = v_g P_b v'_g w'_g P_g w_g v_g$ and $D_{g,2} = z_g P_c z'_g w'_g P_g w_g z_g$. As P_g is a parent of P_h, and P_h is a parent of P_k, $C_k \subset C_h \subset C_g$ (Lemma 2.1.4). So C_k (and then v) is either in $O_{g,1}$ or $O_{g,2}$. In both figures, Figure 3.2 on page 47 and Figure 3.3 on page 51, $v \subseteq O_{g,1}$.

Claim 3.1.11. Assume that $v \in O_{g,1}$. Let $u \in \mathrm{Dc}(v) \cap P_b$ and assume that $Q_u := u_0 u_1 \ldots u_q$ is a path witnessing that $u \in \mathrm{Dc}(v)$ where $u_0 = u$ and $u_q = v$. Then $u_1 \in \{v_h, v'_h\}$ and $u_2 \in v_k P_h v'_k$.

Proof. Let $i = \max\{j \in [q-1] \cup \{0\} : u_j <_L v\}$. From the proof of Claim 3.1.3: If

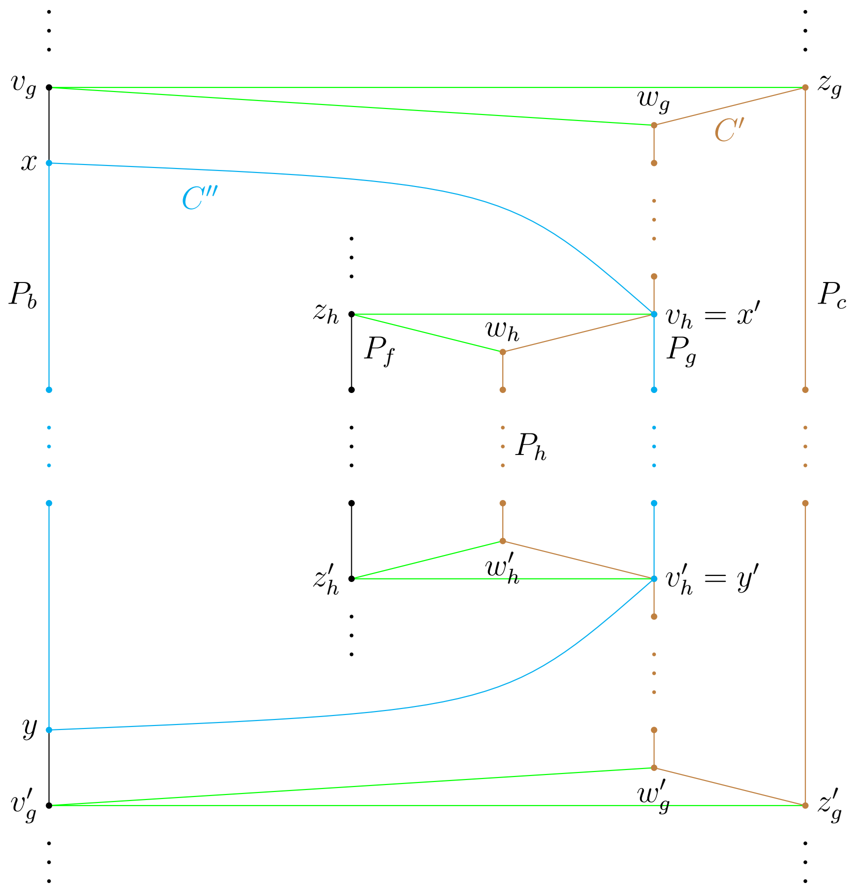

Figure 3.4. The Brown Cycle is C' in Claim 3.1.11 and the Cyan Cycle is C'' in Claim 3.1.13

$i = 0$, then $u \in P_j$ for some $j \in \{k, l, h\}$, if $i = 1$ then $u \in P_j$ for some $j \in A$, if $i = 2$ and $u_2 < u_1$ then $u \in P_j$ for some $j \in A$, and if $i = 2$ and $u_1 < u_2$ then $u \in P_j$ for some $j \in B$. Thus $i = 2$, $u_1 < u_2$ and $u_1 \in P_g$. From Lemma 2.1.2, $u_1 Q_u v$ intersects with either $v_k P_h v'_k$ or P_l. So either $u_2 \in v_k P_h v'_k$ or $u_2 \in P_l$. Since P_g and P_l are not a adjacent, $u_2 \in v_k P_h v'_k$.

It is left to show that $u_1 \in \{v_h, v'_h\}$. As $u_2 \in P_h$, u_2 is in the interior of D_h, while $P_g - v_h P_g v'_h$ is in the exterior of D_h. As $u_1 \in N(u_2) \cap P_g$, $u_1 \in v_h P_g v'_h$. Let the cycle $C' := D_{h,1} \cup D_{g,2} - \mathring{v}_h P_g \mathring{v}'_h$, see Figure 3.4 on page 52; we assumed in the figure that $\mathrm{dist}_{P_g}(v_h, w_g) \leq \mathrm{dist}_{P_g}(v'_h, w_g)$. The inner vertices of $v_h P_g v'_h$ are in the interior of C', while u is in the exterior ($u \in P_b$). As u and u_1 are neighbors, $u_1 \notin \mathring{v}_h P_g \mathring{v}'_h$, and therefore $u_1 \in \{v_h, v'_h\}$. $\square$

Claim 3.1.12. Assume that $v \in O_{g,2}$. Let $u \in \mathrm{Dc}(v) \cap P_c$ and assume that $Q_u := u_0 u_1 \dots u_q$ is a path witnessing that $u \in \mathrm{Dc}(v)$ where $u_0 = u$ and $u_q = v$. Then $u_1 \in \{v_h, v'_h\}$ and $u_2 \in v_k P_h v'_k$.

Proof. Replace the cycle $D_{h,1} \cup D_{g,2} - \mathring{v}_h P_g \mathring{v}'_h$ in the proof of Claim 3.1.11 with the cycle $D_{h,1} \cup D_{g,1} - \mathring{v}_h P_g \mathring{v}'_h$. $\square$

Claim 3.1.13. Assume that $v \in O_{g,1}$. If $|\,\mathrm{Dc}(v) \cap P_b| \geq 4$ then $|\,\mathrm{Dc}(v) \cap P_g| \leq 10$.

Proof. Let $x, y \in \mathrm{Dc}(v) \cap P_b$ such that $\mathrm{dist}_{P_b}(x, y)$ is maximum. Let Q_x be a path witnessing that $x \in \mathrm{Dc}(v)$, and Q_y a path witnessing that $y \in \mathrm{Dc}(v)$. Then $\|Q_x\|, \|Q_y\| \leq 5$ and $Q_x, Q_y \subseteq C_b$. Observe that both Q_x and Q_y each have to be of length at least three. From Claim 3.1.11, $Q_x = xx'x'' \dots v$ and $Q_y = yy'y'' \dots v$ where $x', y' \in \{v_h, v'_h\}$ and $x'', y'' \in v_k P_h v'_k$. Assume that $x' = y'$. The path P_b is isometric in C_b and $xx'y$ is a path in C_b, so $\|xP_b y\| \leq \|xx'y\| \leq 2$, which is a contradiction to the assumption $|\,\mathrm{Dc}(v) \cap P_b| \geq 4$. Thus $x' \neq y'$. We may assume that $x' = v_h$, $y' = v'_h$ and $v_h <_L v'_h$. We may also assume that $\mathrm{dist}_{P_g}(v_h, w_g) < \mathrm{dist}_{P_g}(v'_h, w_g)$, see Figure 3.4 on page 52. As $v_h <_L v'_h$, $V(v_h P_g v'_h) <_L V(v'_h P_g w'_g - v'_h)$. To finish the proof, it suffices to show that $|\,\mathrm{Dc}(v) \cap w_g P_g v'_h| \leq 10$ and $|\,\mathrm{Dc}(v) \cap (v'_h P_g w'_g - v'_h)| = 0$.

Let $z \in \mathrm{Dc}(v) \cap w_g P_g v'_h$. Let Q_z be a path witnessing that $z \in \mathrm{Dc}(v)$. Then $\|Q_z\| \leq 5$ and $Q_z \subseteq C_g$. From Lemma 3.0.2, $vQ_y y'' - y'' \subseteq C_k$, and as $g < h < k$, $C_k \subset C_h \subset C_g$. Therefore the walk $zQ_z vQ_y v'_h$ is in C_g and of length at most nine. As P_g is isometric in C_g, $\|zP_g v'_h\| \leq \|zQ_z vQ_y v'_h\| \leq 9$. Thus $|\,\mathrm{Dc}(v) \cap w_g P_g v'_h| \leq 10$.

Assume that $\mathrm{Dc}(v) \cap (v'_h P_g w'_g - v'_h) \neq \phi$. Let $w \in \mathrm{Dc}(v) \cap (v'_h P_g w'_g - v'_h)$ and assume that Q_w is a path witnessing $w \in \mathrm{Dc}(v)$. The vertex v is in the interior of the cycle $C'' := xP_byv'_hP_gv_hx$, while w is in the exterior, see Figure 3.4 on page 52. As G is a planar graph, Q_w intersects with C''. Note that $b < g$, so Q_w intersects with $v_hP_gv'_h$. But $V(v_hP_gv'_h) <_L V(v'_hP_gw'_g - v'_h)$, a contradiction to the definition of $\mathrm{Dc}(v)$. Thus $|\,\mathrm{Dc}(v) \cap (v'_hP_gw'_g - v'_h)| = 0$. $\square$

Claim 3.1.14. Assume that $v \in O_{g,2}$. If $|\,\mathrm{Dc}(v) \cap P_c| \geq 4$ then $|\,\mathrm{Dc}(v) \cap P_g| \leq 10$.

Proof. Replace each b by c, and use Claim 3.1.12 instead of Claim 3.1.11 in the proof of Claim 3.1.13. $\square$

Claim 3.1.15. $|\,\mathrm{Dc}(v) \cap (P_g \cup P_c \cup P_b)| \leq 32$.

Proof. Assume that $v \in O_{g,1}$. From Claim 3.1.13, $|\,\mathrm{Dc}(v) \cap (P_g \cup P_b)| \leq 21$. As $|\,\mathrm{Dc}(v) \cap P_c| \leq 11$, $|\,\mathrm{Dc}(v) \cap (P_g \cup P_c \cup P_b)| \leq 32$. Assume that $v \in O_{g,2}$. From Claim 3.1.14, $|\,\mathrm{Dc}(v) \cap (P_g \cup P_c)| \leq 21$. As $|\,\mathrm{Dc}(v) \cap P_b| \leq 11$, the statement holds. $\square$

Recall that $v \in C_k \subset C_h$, so $v \in O_{h,1}$ or $v \in O_{h,2}$, see the examples in Figure 3.2 on page 47 and Figure 3.3 on page 51.

Claim 3.1.16. If $v \in O_{h,1}$ then $|\,\mathrm{Dc}(v) \cap P_h| \leq 10$.

Proof. Let $x, y \in \mathrm{Dc}(v) \cap P_h$. Let Q_x be a path witnessing that $x \in \mathrm{Dc}(v)$, and let Q_y be a path witnessing that $y \in \mathrm{Dc}(v)$. Then $\|Q_x\|, \|Q_y\| \leq 5$ and $Q_x, Q_y \subseteq C_h$. Starting from the v-end, let x' be the first vertex in Q_x that is also in P_h. Starting from the v-end, let y' be the first vertex in Q_y that is also in P_h. Recall that $D_{h,1} = v_hP_gv'_hw'_hP_hw_hv_h$, so $V(D_{h,1}) \subseteq P_g \cup P_h$. As $g < h$, $Q_x \cap P_g, Q_y \cap P_g = \phi$. Thus the walk $W := x'Q_xvQ_yy'$ is in $P_h \cup O_{h,1}$ as $v \in O_{h,1}$. From Lemma 2.1.5, $\|x'P_hy'\| \leq \|W\| - 1$.

The path xP_hy is isometric in C_h and $W' := xQ_xx'P_hy'Q_yy$ is a walk in C_h ($Q_x, Q_y \subseteq C_h$). Thus

$$\begin{aligned}\|xP_hy\| &\le \|W'\| \\ &= \|xQ_xx'\| + \|x'P_hy'\| + \|y'Q_yy\| \\ &\le \|xQ_xx'\| + \|W\| + \|y'Q_yy\| - 1 \\ &\le \|xQ_xvQ_yy\| - 1 \\ &\le 9. \end{aligned}$$

□

Claim 3.1.17. If $v \in O_{h,1}$ then $|\operatorname{Dc}(v) \cap P_f| \le 9$.

Proof. Let $x, y \in \operatorname{Dc}(v) \cap P_f$, let Q_x be a path witnessing that $x \in \operatorname{Dc}(v)$, and let Q_y be a path witnessing that $y \in \operatorname{Dc}(v)$. Then $\|Q_x\|, \|Q_y\| \le 5$ and $Q_x, Q_y \subseteq C_f$. Note that v is in the interior of $D_{h,1}$ (assumption) while P_f is in the exterior of $D_{h,1}$, so Q_x and Q_y each intersect with $D_{h,1}$. Recall that $V(D_{h,1}) \subseteq P_g \cup P_h$; as $g < f$, $Q_x \cap P_g, Q_y \cap P_g = \phi$, so $Q_x \cap P_h, Q_y \cap P_h \neq \phi$.

Starting from the v-end, let x' be the first vertex in Q_x that is also in P_h. Starting from the v-end, let y' be the first vertex in Q_y that is also in P_h. Then the walk $W := x'Q_xvQ_yy'$ is in $P_h \cup O_{h,1}$ as $v \in O_{h,1}$ and $Q_x \cap P_g, Q_y \cap P_g = \phi$. From Lemma 2.1.5, $\|x'P_hy'\| \le \|W\| - 1$. Starting from the x'-end, let x'' be the first vertex in $x'Q_xx$ that is also in P_f. Staring from the y'-end, let y'' be the first vertex in $y'Q_yy$ that is also in P_f. We show that the walk $W' := x''Q_xx'P_hy'Q_yy''$ is in $P_f \cup O_{f,1}$. Since P_g is the strong parent of P_f, $D_{f,1} = v_fP_gv'_fw'_fP_fw_fv_f$. As P_g and P_f are the parents of P_h, P_h is in the interior of $D_{f,1}$. To see that also $x''Q_xx', y'Q_yy'' \subseteq P_f \cup O_{f,1}$, observe that $V(D_{f,1}) \subseteq P_g \cup P_f$ and recall that $Q_x \cap P_g, Q_y \cap P_g = \phi$. Thus $W' \subseteq P_f \cup O_{f,1}$.

From Lemma 2.1.5,

$$\begin{aligned}
\|x''P_fy''\| &\leq \|W'\| - 1 \\
&= \|x''Q_xx'\| + \|x'P_hy'\| + \|y'Q_yy''\| - 1 \\
&\leq \|x''Q_xx'\| + \|W\| + \|y'Q_yy''\| - 2 \\
&= \|x''Q_xvQ_yy''\| - 2.
\end{aligned}$$

The path xP_fy is isometric in C_f and the walk $W'' := xQ_xx''P_fy''Q_yy$ is clearly in C_f. Therefore

$$\begin{aligned}
\|xP_fy\| &\leq \|W''\| \\
&= \|xQ_xx''\| + \|x''P_fy''\| + \|y''Q_yy\| \\
&\leq \|xQ_xx''\| + \|x''Q_xvQ_yy'\| + \|y''Q_yy\| - 2 \\
&= \|xQ_xvQ_yy\| - 2 \\
&\leq 8. \qquad \square
\end{aligned}$$

Claim 3.1.18. If $v \in O_{h,1}$ then $|\operatorname{Dc}(v) \cap P_d| \leq 9$.

Proof. Let $x, y \in \operatorname{Dc}(v) \cap P_d$, let Q_x be a path witnessing that $x \in \operatorname{Dc}(v)$, and Q_y a path witnessing that $y \in \operatorname{Dc}(v)$. Then $\|Q_x\|, \|Q_y\| \leq 5$ and $Q_x, Q_y \subseteq C_d$. From Claim 3.1.9, $Q_x = xx'x'' \ldots v, Q_y = yy'y'' \ldots v$ where $x', y' \in P_f$ and $x'', y'' \in v_kP_hv_k'$. We show that the walk $W := x''Q_xvQ_yy''$ is in $P_h \cup O_{h,1}$. From Lemma 3.0.2, $W - \{x'', y''\} \subseteq C_k$. As $C_k \subseteq C_h$, either $C_k \subseteq O_{h,1}$ or $C_k \subseteq O_{h,2}$. From assumption, $v \in O_{h,1}$, so $C_k \subseteq O_{h,1}$. Thus $W \subseteq P_h \cup O_{h,1}$. From Lemma 2.1.5, $\|x''P_hy''\| \leq \|W\| - 1$.

Recall that $D_{f,1} = v_fP_gv_f'w_f'P_fw_fv_f$. As P_g and P_f are the parents of P_h, P_h is in the interior of $D_{f,1}$. Thus the walk $W' := x'x''P_hy''y' \subseteq P_f \cup O_{f,1}$. From Lemma 2.1.5,

$$\begin{aligned}\|x'P_fy'\| &\leq \|W'\| - 1 \\ &= \|x'x''\| + \|x''P_hy''\| + \|y''y'\| - 1 \\ &\leq \|x'x''\| + \|W\| + \|y''y'\| - 2 \\ &= \|x'Q_xvQ_yy'\| - 2.\end{aligned}$$

Now $W'' := xx'P_fy'y$ is a walk in C_d as $x, y \in P_d$ and $x'P_fy' \subseteq C_f \subset C_d$ (Lemma 2.1.4). Since xP_dy is isometric in C_d,

$$\begin{aligned}\|xP_dy\| &\leq \|W''\| \\ &= \|xx'\| + \|x'P_fy'\| + \|y'y\| \\ &\leq \|xx'\| + \|x'Q_xvQ_yy'\| + \|y'y\| - 2 \\ &= \|xQ_xvQ_yy\| - 2 \\ &\leq 8.\end{aligned}$$

□

Claim 3.1.19. Assume that $v \in O_{h,2}$. Let $u \in \mathrm{Dc}(v) \cap P_d$ and assume that $Q_u := u_0u_1u_2\dots u_q$ is a path witnessing that $u \in \mathrm{Dc}(v)$ where $u_0 = u$ and $u_q = v$. Then $u_2 \in \{v_k, v'_k\}$.

Proof. From Claim 3.1.9, $u_1 \in P_f$ and $u_2 \in v_kP_hv'_k$. We just need to show that $u_2 \notin \mathring{v}_kP_h\mathring{v}'_k$. If $\mathring{v}_kP_h\mathring{v}'_k = \phi$, we are done, so assume that $\mathring{v}_kP_h\mathring{v}'_k \neq \phi$. Clearly, $\mathring{v}_kP_h\mathring{v}'_k$ is in the interior of the cycle $C' := D_{k,1} \cup D_{h,1} - \mathring{v}_kP_h\mathring{v}'_k$, see Figure 3.5 on page 58. Both the interior of $D_{k,1}$ and the interior of $D_{h,1}$ are in C_h, and as $f < h$, Lemma 2.1.1 tells us that $P_f \cap C_h = \phi$. As $P_f \cap D_{k,1}, P_f \cap D_{h,1} = \phi$, P_f (and then u_1) is in the exterior of C'. As u_2 is a neighbor of $u_1 \in P_f$, $u_2 \notin \mathring{v}_kP_h\mathring{v}'_k$. □

Claim 3.1.20. Assume that $v \in O_{h,2}$. If $|\,\mathrm{Dc}(v) \cap P_d| \geq 6$ then $|\,\mathrm{Dc}(v) \cap P_h| \leq 9$.

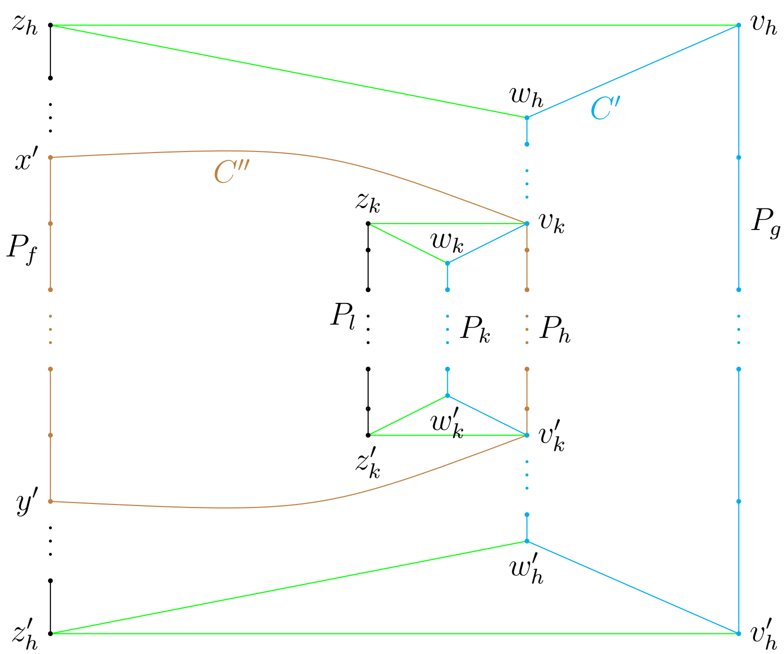

Figure 3.5. The Cyan Cycle is C' in Claim 3.1.19 and the Brown Cycle is C'' in Claim 3.1.20

Proof. Let $x, y \in \mathrm{Dc}(v) \cap P_d$ such that $\mathrm{dist}_{P_d}(x, y)$ is maximum. Let Q_x be a path witnessing that $x \in \mathrm{Dc}(v)$, and Q_y a path witnessing that $y \in \mathrm{Dc}(v)$. Then $\|Q_x\|, \|Q_y\| \leq 5$ and $Q_x, Q_y \subseteq C_d$. Observe that Q_x and Q_y each have to be of length at least three. From Claim 3.1.19, $Q_x = xx'x'' \ldots v, Q_y = yy'y'' \ldots v$ where $x', y' \in P_f$, $x'', y'' \in \{v_k, v'_k\}$. Assume that $x'' = y''$. Then $xx'x''y'y$ is a walk of length four in C_d. As xP_dy is an isometric path in C_d, $\|xP_dy\| \leq 4$, and so $|\mathrm{Dc}(v) \cap P_d| \leq 5$, which contradicts the assumption. Thus $x'' \neq y''$. We may assume without loss of generality that $x'' = v_k$ and $y'' = v'_k$. We also may assume that $v_k <_L v'_k$ and $\mathrm{dist}_{P_h}(v_k, w_h) \leq \mathrm{dist}_{P_h}(v'_k, w_h)$. From the definition of L, $V(v_kP_hv'_k) <_L V(v'_kP_hw'_h - v'_k)$. We finish the proof by showing that $|\mathrm{Dc}(v) \cap w_hP_hv'_k| \leq 9$ and $|\mathrm{Dc}(v) \cap (v'_kP_hw'_h - v'_k)| = 0$.

Let $z \in \mathrm{Dc}(v) \cap w_hP_hv'_k$, and let Q_z be a path witnessing that $z \in \mathrm{Dc}(v)$. Then $\|Q_z\| \leq 5$ and $Q_z \subseteq C_h$. The path $v'_kQ_yv - v'_k$ is in C_k (Lemma 3.0.2), as $C_k \subset C_h$ and

$\operatorname{Dc} P_i := \lvert\operatorname{Dc}(v) \cap P_i\rvert$	$v \in O_{h,1}$	$v \in O_{h,2}$
$\operatorname{Dc} P_k$	6	6
$\operatorname{Dc} P_l$	10	10
$\operatorname{Dc} P_m$	9	9
$\operatorname{Dc} P_n$	9	9
$\operatorname{Dc} P_h$	10	11
$\operatorname{Dc} P_f$	9	10
$\operatorname{Dc} P_d$	9	10

	$v \in O_{h,1}$	$v \in O_{h,2}$
$\operatorname{Dc} P_d + \operatorname{Dc} P_h$	19	19
$\operatorname{Dc} P_g + \operatorname{Dc} P_c + \operatorname{Dc} P_b$	32	32

Table 1. Upper Bounds for $\lvert\operatorname{Dc}(v) \cap P_i\rvert$

$v'_k \in P_h \subseteq C_h$, $v'_k Q_y v \subseteq C_h$. From the definition of $\operatorname{Dc}(v)$, $\|v'_k Q_y v\| \leq 3$, so $zQ_z v Q_y v'_k$ is a walk of length at most eight between z and v'_k in C_h, and $zP_h v'_k$ is an isometric path in C_h. Thus $\|zP_h v'_k\| \leq \|zQ_z v Q_y v'_k\| \leq 8$, so $\lvert\operatorname{Dc}(v) \cap w_h P_h v'_k\rvert \leq 9$.

Assume that $\operatorname{Dc}(v) \cap (v'_k P_h w'_h - v'_k) \neq \phi$. Let $w \in \operatorname{Dc}(v) \cap (v'_k P_h w'_h - v'_k)$ and assume that Q_w is a path witnessing that $w \in \operatorname{Dc}(v)$. From Lemma 2.1.2, Q_w intersects with either $v_k P_h v'_k$ or $z_k P_l z'_k$. As $V(v_k P_h v'_k) <_L V(v'_k P_h w'_h - v'_k)$, $Q_w \cap v_k P_h v'_k = \phi$, so $Q_w \cap z_k P_l z'_k \neq \phi$. Let the cycle $C'' := v_k P_h v'_k y' P_f x' v_k$, see Figure 3.5 on page 58. Clearly, w is in the exterior of C'', we show that $z_k P_l z'_k$ is in the interior of C''. The path $z_k P_l z'_k$ is adjacent to P_k, and P_k is in $O_{h,2}$ (assumption); as G is planar, $z_k P_l z'_k$ is in $O_{h,2}$ too. The cycle $D_{h,2}$ and the two edges $v_k x'$,$v'_k y'$ make up three cycles, one of them is C''. Observe that C'' is the only cycle among those three that contains $\{v_k, v'_k\}$. As $z_k P_l z'_k$ is adjacent to v_k and v'_k, $z_k P_l z'_k$ is in the interior of C''. Thus Q_w intersects with C''. As $V(C'') \subseteq P_f \cup v_k P_h v'_k$ and $Q_w \cap v_k P_h v'_k = \phi$, $Q_w \cap P_f \neq \phi$, but this is not possible since $V(P_f) <_L V(P_h)$. Thus $\operatorname{Dc}(v) \cap (v'_k P_h w'_h - v'_k) = \phi$. $\square$

We sum up Claims 3.1.5–3.1.20 in Table 1 on page 59. As $\operatorname{dc}(v) = \sum_{j \in B} \lvert\operatorname{Dc}(v) \cap P_j\rvert \leq 95$, Theorem 3.1.1 holds.

3.2 Lower Bound

An example of a planar graph G with $\chi(G^{[\natural 3]}) = 6$ is given in Exercise 11.4 [31]. Later, Van den Heuvel et al. [14] gave an example of an outerplanar graph G' with $\chi(G'^{[\natural 3]}) = 5$ and a planar graph G with $\chi(G^{[\natural 3]}) = 7$. Here we give an outerplanar graph G' with $\chi(G'^{[\natural 3]}) \geq 6$, and a planar graph G with $\chi(G'^{[\natural 3]}) \geq 9$.

Theorem 3.2.1.

1. *There exists an outerplanar G such that $\chi(G^{[\natural 3]}) \geq 6$.*
2. *There exists a planar graph G such that $\chi(G^{[\natural 3]}) \geq 9$.*

Proof. (a) Draw a path P of length six, say $P = w_1 \dots w_7$. Add a new vertex z, then for every $i \in \{1, \dots, 7\}$, add an edge between z and w_i. For every $i \in \{1, \dots, 6\}$, add a new vertex z_i such that z_i is a neighbor for w_i and w_{i+1}. For every w_i, add a new neighbor y_i where $i \in \{1, \dots, 7\}$, see Figure 3.7 on page 62. Let F be the graph shown in Figure 3.6 on page 61. We call the vertex w the *center* of F. We finish constructing G by making each w_i a center of a copy F_i of F where $i \in \{1, \dots, 7\}$. Let $x_1^i, x_2^i, x_3^i, x_4^i$ and x_5^i be the vertices in F_i corresponding respectively to the vertices x_1, x_2, x_3, x_4 and x_5 in F.

In the graph $G^{[\natural 3]}$ we have the odd cycle $C_i := x_1^i x_3^i x_5^i x_4^i x_2^i x_1^i$ for each $i \in \{1, \dots, 7\}$. So the vertices of C_i received at least three different colors. Moreover for each $i \in \{2, 3, \dots, 6\}$, the vertices z_{i-1}, z_i, y_i and z are neighbors for each vertex of C_i in $G^{[\natural 3]}$. So we cannot use a color appearing in $V(C_i)$ to color any vertex in $\{z_{i-1}, z_i, y_i, z\}$. Similarly, we cannot use a color appearing in $V(C_1)$ (or in $V(C_7)$) to color a vertex in $\{y_1, z_1, z\}$ (or in $\{z_6, y_7, z\}$). Assume without loss of generality that the color given to y_4 is the color 1. Then we have two cases.

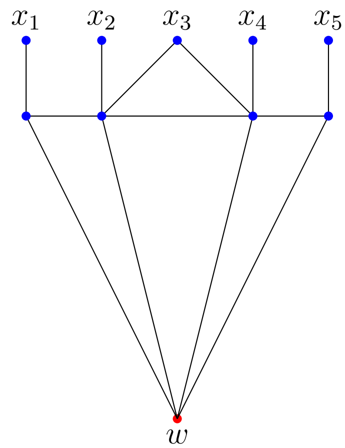

Figure 3.6. The Graph F

Case 1: z_4 is colored with color 1.

Since both y_5 and z_5 are at distance 3 from y_4, neither y_5 nor z_5 is colored with color 1. Assume without loss of generality y_5 is colored with color 2. If z_5 is colored with a color different than color 2 then the set $\{z_4, y_5, z_5\}$ received 3 different colors and we are done since there are six colors appeared in $V(C_5) \cup \{z_4, y_5, z_5\}$. So assume z_5 is colored with 2. Since y_6 and z_6 are at distance 3 from both z_4 and y_5, none of them can be colored with 1 or 2. Say y_6 is colored with color 3. If z_6 is colored with a color different than 3 then the set $\{z_5, y_6, z_6\}$ received 3 colors and we are done. So assume z_6 is colored with 3. Since y_7 is at distance 3 from both z_5 and y_6, it cannot be colored with 2 or 3. Thus y_7 has to be colored with either 1 or a new color, say 4. If z is colored with a color different than 2 and 3 then the set $\{z, z_5, y_6\}$ received 3 different colors and we are done. If z is colored with 2 then the set $\{z, z_6, y_7\}$ received 3 different colors. Similarly, if z is colored with 3 then the set $\{z, z_4, y_5\}$ received 3 different colors.

Case 2: z_4 is not colored with 1, say color 2.

If z_3 is colored with a color different than 1 and 2 then $\{z_3, y_4, z_4\}$ received 3 colors.

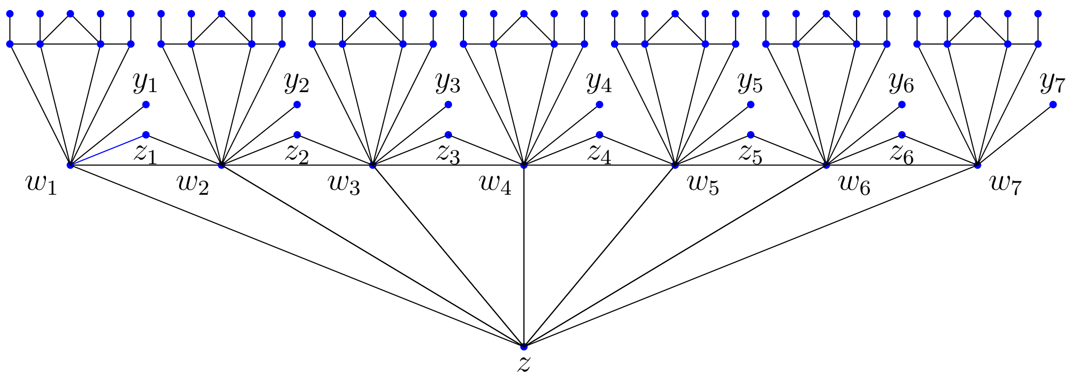

Figure 3.7. The Outerplanar Graph G

If z_3 is colored with 1, we have a case similar to Case 1 since G has a vertical line of symmetry and z_3 is facing z_4. So assume z_3 is colored with 2. Since both z_2 and y_3 are at distance 3 from both y_4 and z_4, they cannot be colored with 1 or 2. If z_2 and y_3 received different colors then $\{z_2, y_3, z_3\}$ received 3 colors. So assume z_2 and y_3 received the same color, say 3. Since y_2 and z_1 are at distance 3 from both y_3 and z_3, they cannot be colored with 2 or 3. If y_2 and z_1 received different colors then $\{z_2,y_2,z_1\}$ received three colors and we are done. Assume both z_1 and y_2 are colored with 1, or both are colored with a new color, say 4. If z is colored with a color different than 2 and 3 then the set $\{z, z_3, y_3\}$ received 3 colors. If z is colored with 2 then the set $\{z, z_2, y_2\}$ received 3 colors. If z is colored with 3 then the set $\{z, z_4, y_4\}$ received 3 colors.

(b) As in part (a) we start with a path P of length six, say $P = w'_1 ... w'_7$. Add a new vertex z', then add an edge between z' and w'_i for every $i \in \{1, \ldots, 7\}$. Add a new vertex $z'_i, 1 \leq i \leq 6$ such that z'_i is a neighbor for w'_i and w'_{i+1}. Then for every w'_i add a new neighbor y'_i where $i \in \{1, \ldots, 7\}$, see Figure 3.8 on page 63. Let G' be the outerplanar graph constructed in part (a). For each $i \in \{1, \ldots, 7\}$, add a copy G'_i of G', then add an edge between w'_i and every vertex in G'_i. This is possible to do without losing planarity because G'_i is an outerplanar. Now subdivide each edge

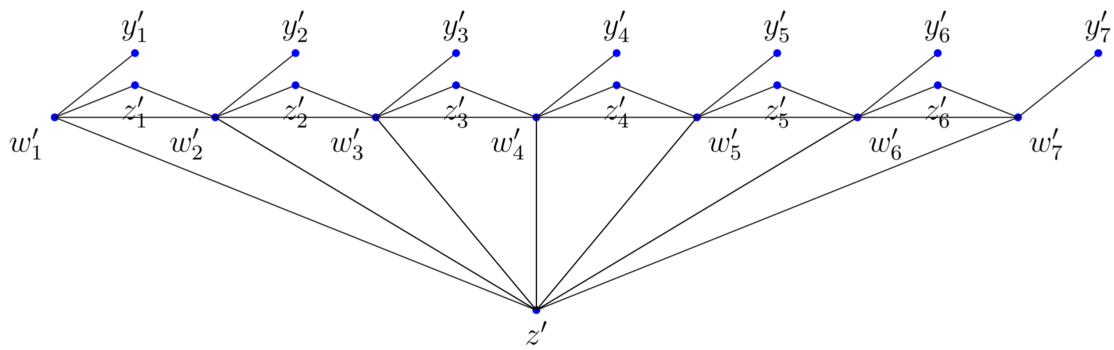

Figure 3.8. Part of the Construction of G in (b)

between $V(G'_i)$ and w'_i, so the distance between w'_i and any vertex in G'_i is two. The resulting graph is the desired planar graph G.

From part (a), there are six different colors appearing in the coloring of the subgraph of $G^{[\natural 3]}$ induced by $V(G'_i)$. Using the same argument used in part (a) we find that for any proper coloring of $G^{[\natural 3]}$, one of the sets $\{\{z'_{i-1}, z'_i, y'_i, z'\}_{i\in\{2,3,\ldots,6\}}, \{y'_1, z'_1, z'\}, \{z'_6, y'_7, z'\}\}$ received 3 different colors. Assume without loss of generality that the set $\{z'_1, z'_2, y'_2, z'\}$ received 3 colors. In the graph G, each vertex in $\{z'_1, z'_2, y'_2, z'\}$ is at distance 3 from each vertex in G'_2. Thus at least 9 colors are needed to properly color $G'^{[\natural 3]}$. $\square$

Chapter 4

GENERALIZED COLORING NUMBERS OF PLANAR GRAPHS

In this chapter, we recall a linear ordering L defined by Van den Heuvel et al. [14] to prove that $\mathrm{scol}_r(G) \leq 5r+1$ for planar graphs (Theorem 1.1.5). Then we show that for any planar graph G, $\mathrm{wcol}_2[G, L, v] \leq 26$. We end this chapter by giving an example for planar graph G such that $\mathrm{wcol}_2[G, L, v] = 26$ for some $v \in G$.

4.1 Breadth-first Trees

A *breadth-first search* is an algorithm for searching trees, it starts at the tree root or at an arbitrary vertex, and explores all the neighbor vertices at the present depth prior to moving to the vertices at the next depth level. The resulted tree is called a *breadth-first tree.*

Let T be a breadth-first spanning tree. Let $x, x' \in T$. We write $x <_T x'$ if x was found by T before x'; and we write $x \leq_T x'$ if $x = x'$ or $x <_T x'$.

Let T be a tree rooted at a vertex v. We define the *level* of a vertex $x \in T$ to be $l_x := \mathrm{dist}_T(v, x)$, and we denote the set of vertices $x \in T$ such that $l_x = d$ by D_d.

Lemma 4.1.1. *Let G be a graph, and let $T \subseteq G$ be a breadth-first spanning tree with root r. Then:*

1. *the path rTx is isometric for all vertices x;*
2. *if $xx' \in E(T)$, $yy' \in E(G)$ and $x \leq_T x' \leq_T y'$ then $x \leq_T y$;*
3. *if $P_x = x_0 x_1 \cdots x_t$ is a path in T where $x = x_0$ and $r = x_t$ and $Q = y_0 \cdots y_t$ is a path in G where $x_0 \leq_T y_0$ then $x_i \leq_T y_i$ for all $i \in [t]$.*

Proof. We argue by induction on $|T|$ to prove (1). The statement is trivial if $|T| = 1$. Assume that the statement holds for every breadth-first spanning tree T with $|T| \leq k$ where k is a positive integer.

Assume now that $|T| = k + 1$. Let u be the last vertex added to T. Clearly, u is a leaf and $T' := T - u$ is a breadth-first spanning tree with root r in $G' := G - u$. By induction hypothesis, the path $rT'x(= rTx)$ is isometric in G' for all vertices $x \in G'$. Assume for contradiction that rTx is not isometric in G for some $x \in G'$. Let P be an isometric path in G between r and x. Then $u \in P$ and $\|P\| < \|rTx\|$. This contradicts the fact that x is found by T before u. Assume that rTu is not isometric in G. Let $Q := v_0 \dots v_i$ be an isometric path in G where $v_0 = r$ and $v_i = u$, so $\|Q\| < \|rTu\|$. Let u' be the unique neighbor of u in T, then $\|v_0Qv_{i-1}\| < \|rTu'\|$. This is in contradiction to the fact that $rT'u'(= rTu')$ is isometric in G' as $v_0Qv_{i-1} \subseteq G'$.

For (2), observe that if $y <_T x$ then T would have explored every leftover neighbor of y before adding x' to T. This is in contradiction to the fact that $x' \leq_T y'$ and $xx' \in E(T)$. Thus $x \leq_T y$.

We prove (3) by induction on $|P_x|$. The statement is trivial if $|P_x| = 1$. Assume that the statement holds for every path P_x with $|P_x| \leq k$. Assume now that $|P_x| = k+1$. Since $rP_xx \subseteq T$, $r \leq_T x_{t-1} \leq_T \cdots \leq_T x_1 \leq_T x$. So $x_1 \leq_T x \leq_T y_0$. From (2), $x_1 \leq_T y_1$. By induction hypothesis, statement (3) holds for the paths $x_1P_xx_t$ and y_1Qy_t. Hence $x_i \leq_T y_i$ for all $i \in [t]$. $\square$

4.2 Bounds on the Strong r-coloring Numbers of Planar Graphs

Let G be a planar graph, and fix a plane drawing $\widetilde{G}$ of G. For simplicity, we write G for $\widetilde{G}$. Let $F = F(G)$ denote the set of faces of G, $G[f]$ denote the boundary of the face f, $V(f)$ denote the set of vertices of $G[f]$ and $G^* = (F, E^*)$ denote the dual of G. Set $F^* := F(G^*)$. It is well known that every connected plane graph has a connected plane dual.

Lemma 4.2.1. *(Proposition 4.6.1, Diestel [7]) For any connected planar graph G, an edge set $D \subseteq E(G)$ is the edge set of a cycle in G if and only if $D^* := \{e^* : e \in D\}$ is a minimal cut in G^*.*

Here we present the linear ordering L that witnesses the bound in the following theorem.

Theorem 4.2.2. *(Van den Heuvel et al. [14]) Let G be a planar graph and r be a positive integer. Then* $\operatorname{scol}_r(G) \leq 5r + 1$.

Without loss of generality, assume G is maximal with $|G| \geq 4$. Then G is not a cycle, and so no two faces have the same boundary.

4.2.1 Construction of L

Fix a vertex v, and a breadth-first spanning tree $T \subseteq G$ rooted at v. Let H be the spanning subgraph of G^* with $E(H) = \{e^* : e \in E(G) \smallsetminus E(T)\}$.

Lemma 4.2.3. *H is a spanning tree of G^*.*

Proof. It suffices to show that $\|H\| = |H| - 1$ and H is acyclic. As $|H| = |F(G)|$, Euler's Formula yields

$$\|H\| = \|G\| - \|T\| = (|G| + |F(G)| - 2) - (|G| - 1) = |F(G)| - 1 = |H| - 1.$$

Since T is a spanning tree in G, every cut in G must have an edge in T. So $E(G) \smallsetminus E(T)$ does not contain a cut. By Lemma 4.2.1, H is acyclic. $\square$

Order the vertices of H (faces of G) as $f_0, \ldots, f_h$ using a depth-first search of H, starting with the outer face f_0 of G. For all $i \in [h] \cup \{0\}$, let $V(f_i) = \{a_i, b_i, c_i\}$. For all $i \in [h]$, there is a unique index $j(i) < i$ with $f_i f_{j(i)} \in E(H)$, $a_i b_i \in E(G - T)$ and $a_i b_i = G[f_i] \cap G[f_{j(i)}]$. Then $V(f_i) \setminus V(f_{j(i)}) = \{c_i\}$.

Treating paths vTx as sequences of vertices starting with v, define a sequence σ (with repeated vertices) by:

$$\sigma = vTa_0^\wedge Tb_0^\wedge vTc_0^\wedge \ldots^\wedge vTa_h^\wedge vTb_h^\wedge vTc_h.$$

Finally, define an ordering L by $x <_L y$ if the first occurrence of x in σ comes before the first occurrence of y in σ. Set $X_{f_i} = vTa_i \cup vTb_i \cup vTc_i$. We sometimes write X_i instead of X_{f_i}.

4.2.2 Properties of L

Consider any $u \in V(G)$. Set $f(u) = f_i$, where $i = \min\{j : u \in X_j\}$. If $i \geq 1$ then $u \in vTc_i$ since $u \notin X_{j(i)}$ and $a_i, b_i \in V(f_{j(i)}) \subseteq X_{j(i)}$. Next we define a cycle C_u. Let $C_u = G[f_0]$ if $f(u) = f_0$; else $f(u) = f_i$ for some $i > 0$; put $e_u = a_i b_i$ and let C_u be the unique cycle in $T + e_u$. Then $C_u = a_i T x_u T b_i a_i$ where $x_u = \leq_T$ - $\max(vTa_i \cap vTb_i)$. Let O_u be the face of C_u not containing $f(u)$ if $f(u)$ is the outer face; else let O_u be the face of C_u containing $f(u)$. Finally, set $P_u = vTu$,$a(u) = a_i, b(u) = b_i$ and $c(u) = c_i$. See Figure 4.1 on page 68; the black edges are in T.

Lemma 4.2.4. *(Van den Heuvel et al. [14]) Let $u \in V(G)$ and $f(u) = f_i$. Then $V(X_i) <_L V(O_u \setminus X_i)$.*

Proof. If $f(u) = f_0$ then $X_i = X_0 = vTa_0 \cup vTb_0 \cup vTc_0$. as $vTa_0^\wedge vTb_0^\wedge vTc_0$ is the initial segment of the sequence σ, the conclusion holds. Suppose $f(u) = f_i$ for some

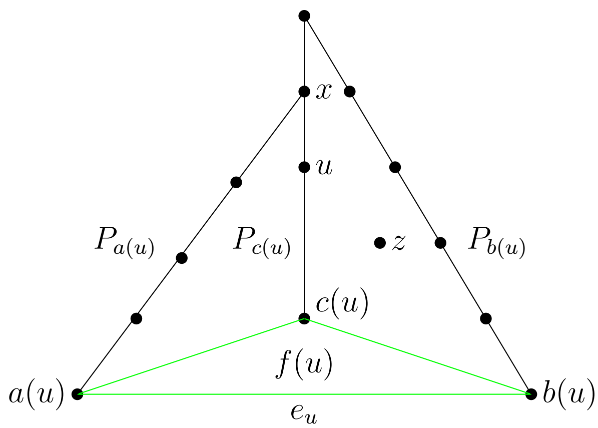

Figure 4.1. $f(u)$,e_u,$P_{a(u)}$, $P_{b(u)}$ and $P_{c(u)}$

$i > 0$. Let $z \in V(O_u \setminus X_i)$; say $f(z) = f_k$. Then $z \in vTa_k \cup vTb_k \cup vTc_k$, say $z \in vTc_k$. As $z \in X_k \setminus X_i$, $k \neq i$. It suffices to show that $i < k$. Observe that $v \in X_i \cap vTz$ and $z \in vTz \setminus X_i$; as T is acyclic, $X_i \cap zTc_k = \emptyset$. Recall that $C_u \subseteq X_i \cup a_ib_i$, so $zTc_k \subseteq O_u \setminus X_i$. Since G is planar, $a_kb_kc_ka_k \subseteq O_u \cup C_u$, and so $f_k \subseteq O_u$. Since $e_u = a_ib_i$ is the only edge of C_u in $G - T$, $f_i \in f_0Hf_k$. Thus $i < k$. $\square$

Lemma 4.2.5. *Let $z \in V(G)$ and $z', z'' \in \mathrm{Scol}_1[L, z] \setminus P_z$ with $z' <_L z''$ and $z' \notin P_{z''}$. If $f(z) \neq f_0$ then (a) $z \in O_{z''} \setminus P_{z''}$* and (b) *$z' \in X_{f(z'')}$.*

Proof. Assume that $f(z'') \neq f_0$, then $z'' \in O_{z''}$ and $z'' \in P_{c(z'')} - P_{a(z'')} \cup P_{b(z'')}$. As $P_{a(z'')}$ and $P_{b(z'')}$ appear before $P_{c(z'')}$ in the sequence σ, $V(P_{a(z'')} \cup P_{b(z'')}) <_L z''$. The vertices z and z'' are adjacent and G is planar, so $z \in O_{z''} \cup C_{z''}$. Recall that $C_{z''} = a(z'')Tx_{z''}Tb(z'')a(z'')$. As $V(P_{a(z'')} \cup P_{b(z'')}) <_L z'' <_L z$, $z \notin C_{z''} \cup P_{z''}$. Thus $z \in O_{z''} \setminus P_{z''}$. The vertices z and z' are adjacent in the planar graph G, so $z' \in O_{z''} \cup C_{z''}$. From the assumption $z' <_L z''$ and Lemma 4.2.4, $z' \notin O_{z''} \setminus X_{f(z'')}$. As $C_{z''} \subseteq X_{f(z'')}$, $z' \in X_{f(z'')}$. Assume now that $f(z'') = f_0$, then $C_{z''} = G[f_0]$. From the assumption $f(z) \neq f_0$, $z \notin C_{z''}$, and so $z \in O_{z''}$. Since $z'' <_L z$, $z \in O_{z''} \setminus vTz''$. As $z' <_L z''$ and $f(z'') = f_0$, $f(z') = f_0$. Thus $X_{f(z'')} = X_{f(z')}$, and so $z' \in X_{f(z'')}$. $\square$

Proof of Theorem 4.2.2

Theorem 4.2.6. $\mathrm{scol}_r[G, L] \leq 5r + 1$

Proof. Fix a vertex $u \in V(G)$ with $f(u) = f_i$. To avoid double subscripts, let $a = a_i, b = b_i, c = c_i$.

By Lemma 4.2.4, any path from u to an L-smaller vertex must contain a vertex in X_i. Assume that f_i is the outer face and fix notations so that $u \in P_c$. Then any vertex w that is strongly r-reachable from u must be in $P_a \cup P_b \cup P_u$. If f_i is an inner face, then any vertex w that is strongly r-reachable from u must be in $C_u \cup P_u$ as $u \in O_u$. So in both cases $w \in P_a \cup P_b \cup P_u$. By Lemma 4.1.1, the paths P_a, P_b and P_u are all isometric. As $\mathrm{Scol}_r[L, u] \subseteq N_r[u]$, Lemma 1.2.6 tells us that $|\mathrm{Scol}_r[L, u] \cap P_a|, |\mathrm{Scol}_r[L, u] \cap P_b| \leq 2r + 1$ and $|\mathrm{Scol}_r[L, u] \cap P_u| \leq r + 1$. Thus $\mathrm{scol}_r[L, u] \leq 5r + 3$. To finish the proof, we improve this bound to $5r + 1$ by showing:

$$\text{if } v_1, v_2 \in \mathrm{Scol}_r[L, u] \cap P_a \text{ (also } P_b\text{) with } \mathrm{dist}(v_1, v_2) \geq 2r \text{ and } v_1 <_L v_2, \text{ then } v_1 \in P_u. \tag{4.2.1}$$

Consider v_1, v_2 satisfying the hypothesis. As $|l_u - l_{v_1}| \leq r$ and $|l_u - l_{v_2}| \leq r$, this yields $l_{v_1} = l_u - r$ and $l_{v_2} = l_u + r$. Let Q_i be the path that witnesses $v_i \in \mathrm{Scol}_r[L, u]$ for $i \in [2]$. Since $l_{v_i} = l_u + (-1)^i r$, $\|Q_i\| = r$. Let $R = u_0 \dots u_r \subseteq P_u$ with $u_0 = u$. By Lemma 4.1.1(3) applied to $v_2 T v_1$ and $Q_2 u R$, $v_1 \leq_T u_r$. By Lemma 4.1.1(3) applied to R and Q_1, we have $v_1 \geq_T u_r$. Thus $v_1 = u_r$, and so $v_1 \in P_u$, proving (4.2.1). □

Corollary 4.2.7. *Let $u \in V(G)$ with $f(u) = f_i$. Then u has at most five L-smaller neighbors and they are all in X_i.*

Proof. Take $r = 1$ in the previous theorem. □

4.3 Bounds on $\mathrm{wcol}_2[G, L]$

In this section, we prove that $\mathrm{wcol}_2[G, L] \le 26$. We also give an example for a planar graph G such that $\mathrm{wcol}_2[G, L, u] = 26$ for some $u \in G$.

4.3.1 Upper Bound

Theorem 4.3.1. $\mathrm{wcol}_2[G, L] \le 26$.

Proof. Consider any vertex $u \in G$; set $l = l_u$. Assume first that $f(u)$ is the outer face. As $u \in X_0$, all the vertices that are L-smaller than u are in $X_0 = P_{a(u)} \cup P_{b(u)} \cup P_{c(u)}$. As $\mathrm{Wcol}_2[L, u] \subseteq N_2[u]$, Lemma 1.2.6 tells us that $|\mathrm{Wcol}_2[L, u] \cap P_{a(u)}|, |\mathrm{Wcol}_2[L, u] \cap P_{b(u)}|, |\mathrm{Wcol}_2[L, u] \cap P_{c(u)}| \le 5$. Thus $\mathrm{wcol}_2[L, u] \le 5 + 5 + 5 = 15$.

Assume that $f(u)$ is an inner face. By Lemma 4.2.4,

$$\mathrm{Scol}_1[L, u] \subseteq (C_u \cup P_u) \cap (D_{l-1} \cup D_l \cup D_{l+1})$$

We will use the estimate

$$\begin{aligned}
\mathrm{wcol}_2[L, u] &= \mathrm{scol}_2[L, u] + |\mathrm{Wcol}_2[L, u] \setminus \mathrm{Scol}_2[L, u]| \\
&\le \mathrm{scol}_2[L, u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L, u']) \setminus \mathrm{Scol}_2[L, u]| \\
&\le \mathrm{scol}_2[L, u] + \sum_{u' \in \mathrm{Scol}_1[L,u]} |\mathrm{Scol}_1[L, u'] \setminus \mathrm{Scol}_2[L, u]| \\
&\le \mathrm{scol}_2[L, u] + \sum_{u' \in \mathrm{Scol}_1[L,u]} |\mathrm{Scol}_1[L, u'] \setminus \mathrm{Scol}_1[L, u]|.
\end{aligned}$$

By Lemma 1.2.6 and (4.2.1),

$$|\mathrm{Scol}_1[L, u] \cap V(P_{a(u)} - P_u)|, |\mathrm{Scol}_1[L, u] \cap V(P_{b(u)} - P_u)| \le 2 \text{ and } |\mathrm{Scol}_1[L, u] \cap P_u| \le 2.$$

We consider three cases. In each case, we first suppose $|\mathrm{Scol}_1[L, u] \setminus \{u\}| = 5$, and then make easy modifications for the subcase $|\mathrm{Scol}_1[L, u] \setminus \{u\}| < 5$. Choose notation

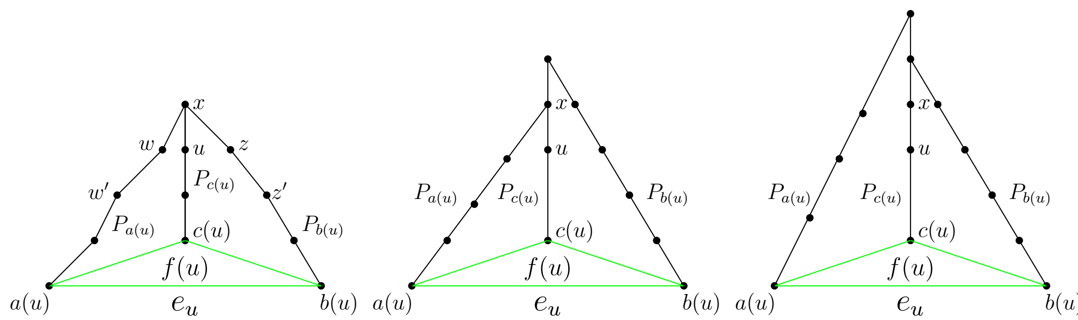

Figure 4.2. The Three Major Cases of Theorem 4.3.1

so that $ux \in P_u$ and $w, w' \in \mathrm{Scol}_1[L,u] \cap V(P_{a(u)} - P_u)$ with $w <_L w'$ and $z, z' \in \mathrm{Scol}_1[L,u] \cap V(P_{b(u)} - P_u)$ with $z <_L z'$. By (4.2.1), $ww', zz' \in E(T)$, if $x \in P_{a(u)}$ then $xw \in E(T)$, and if $x \in P_{b(u)}$ then $xz \in E(T)$. Let $w^\circ \in P_{a(u)}$ with $l_{w^\circ} = l_w - 1$, $z^\circ \in P_{b(u)}$ with $l_{z^\circ} = l_z - 1$ and let $y \in P_u$ with $l_y = l_x - 1$. If they exist, let $w'' \in P_{a(u)}$ with $l_{w''} = l_w + 2$ and $z'' \in P_{b(u)}$ with $l_{z''} = l_z + 2$.

Case 1: $x \in P_{\mathrm{a}(u)} \cap P_{b(u)}$, see the first drawing of Figure 4.2 on page 71.
Then $wxz \subseteq T$. So $l_w = l_u = l_z$ and $x = w^\circ = z^\circ$. If $u' \in \mathrm{Scol}_2[L,u]$ then $l_u - 2 \le l_{u'} \le l_u + 2$. Thus $\mathrm{Scol}_2[L,u] \subseteq \{u, x, y, w, w', w'', z, z', z''\}$, and so $\mathrm{scol}_2[L,u] \le 9$. Since $x, w \in \mathrm{Scol}_1[L,u]$, $|\,\mathrm{Scol}_1[L,w] \setminus \mathrm{Scol}_1[L,u]| \le 4$, and as $y \in \mathrm{Scol}_1[L,x]$, $\mathrm{scol}_2[L,u] + |\,\mathrm{Scol}_1[L,x] \setminus \mathrm{Scol}_1[L,u]| \le 9 + 4 = 13$.

Throughout the rest of the proof when we write $v_i < v_j$ we mean that v_i is L-smaller than v_j.

Assume without loss of generality that $w < z$; so $w < z < z'$. We have three possibilities regarding the L-order of w'. Assume first that $w < w' < z < z'$. The L-smaller neighbor of w' in $P_{w'}$ is w. Since $\{w, w'\} \subseteq \mathrm{Scol}_1[L,u]$, $|\,\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| \le 4$.

Now we estimate the size of $\mathrm{Scol}_1[L,z] \setminus \mathrm{Scol}_1[L,u]$. Note that $\mathrm{Scol}_1[L,z] \subseteq (P_{a(z)} \cup P_{b(z)} \cup P_{c(z)}) \cap (D_{l-1} \cup D_l \cup D_{l+1})$. Since $w' < z$, Lemma 4.2.5 tells us that

$P_{w'} \subseteq X_{f(z)}$. Since $w' \notin P_z \subseteq P_{c(z)}$ and $w' < z$, either $P_{w'} \subseteq P_{a(z)}$ or $P_{w'} \subseteq P_{b(z)}$, say $P_{w'} \subseteq P_{a(z)}$. Since $l_w = l_z$, all the possible L-smaller neighbors of z in $P_{w'}$ (x, w and w') are in $\mathrm{Scol}_1[L,u]$. The L-smaller neighbor of z in P_z is x. Thus $|\mathrm{Scol}_1[L,z] \setminus \mathrm{Scol}_1[L,u]| \leq 2$. By Lemma 4.2.5, $P_{w'} \subseteq X_{f(z')}$, say $P_{w'} \subseteq P_{a(z')}$. The L-smaller neighbor of z' in $P_{c(z')}$ is z. The vertices at levels $l_{z'} - 1$ and $l_{z'}$ in $P_{a(z')}$ are w and w'. As $\{w, w', z, z'\} \subseteq \mathrm{Scol}_1[L,u]$, $|\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 3$. Thus $\mathrm{wcol}_2[L,u] \leq 13 + 2 * 4 + 2 + 3 = 26$.

Now assume that $w < z < w' < z'$, the second possibility. By Lemma 4.2.5, $P_z \subseteq X_{f(w')}$, say $P_z \subseteq P_{a(w')}$. If they exist, let $t, r \in P_{a(w')}$ with $l_t = l_z + 1, l_r = l_z + 2$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z, t and r. Since $w, w' \in \mathrm{Scol}_1[L,u]$, $|\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| \leq 4$. Since $w < z$, $P_w \subseteq X_{f(z)}$, say $P_w \subseteq P_{a(z)}$. If it exists, let $s \in P_{a(z)}$ with $l_s = l_w + 1$. The possible L-smaller neighbors of z in $P_{a(z)}$ are x, w and s. Thus $|\mathrm{Scol}_1[L,z] \setminus \mathrm{Scol}_1[L,u]| \leq 3$.

By Lemma 4.2.5, $P_{w'} \subseteq X_{f(z')}$, say $P_{w'} \subseteq P_{a(z')}$. If it exists, let $s' \in P_{a(z')}$ with $l_{s'} = l_{w'} + 1$. Since $l_{w'} = l_{z'}$, the possible L-smaller neighbors of z' in $P_{a(z')}$ are w, w' and s'. The L-smaller neighbor of z' in $P_{z'}$ is z; since $\{w, w', z, z'\} \subseteq \mathrm{Scol}_1[L,u]$, $|\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 3$. In the rest of this paragraph, we show that $|\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| = 4$ and $|\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| = 3$ cannot occur at the same time. If $|\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| = 4$ then $w'r \in E(G)$; by Lemma 4.1.1(3) applied to $ztr \subseteq T$ and $ww'r \subseteq G$, $z \leq_T w$. If $|\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| = 3$ then $z's' \in E(G)$. By Lemma 4.1.1(3) applied to $ww's' \subseteq T$ and $zz's' \subseteq G$, $w \leq_T z$. So either $w'r \in E(G)$ or $z's' \in E(G)$ but not both. Thus $|\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| + |\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 6$. Therefore $\mathrm{wcol}_2[L,u] \leq 13 + 4 + 3 + 6 = 26$.

Lastly, assume that $w < z < z' < w'$. Since $z' < w'$, $P_{z'} \subseteq X_{f(w')}$; assume that $P_{z'} \subseteq P_{a(w')}$. If it exists, let $s \in P_{a(w')}$ with $l_s = l_{z'} + 1$. Since $l_{w'} = l_{z'}$, the possible

L-smaller neighbors of w' in $P_{a(w')}$ are z, z' and s. So $|\operatorname{Scol}_1[L,w'] \setminus \operatorname{Scol}_1[L,u]| \le 3$. Also as $w < z$, $P_w \subseteq X_{f(z)}$; assume that $P_w \subseteq P_{a(z)}$. If it exists, let $t \in P_{a(z)}$ with $l_t = l_w + 1$. Since $l_w = l_z$, the possible L-smaller neighbors of z in $P_{a(z)}$ are x, w and t. So $|\operatorname{Scol}_1[L,z] \setminus \operatorname{Scol}_1[L,u]| \le 3$. We show that $|\operatorname{Scol}_1[L,w'] \setminus \operatorname{Scol}_1[L,u]| = 3$ and $|\operatorname{Scol}_1[L,z] \setminus \operatorname{Scol}_1[L,u]| = 3$ cannot occur at the same time. If $|\operatorname{Scol}_1[L,w'] \setminus \operatorname{Scol}_1[L,u]| = 3$ then $w's \in E(G)$. By Lemma 4.1.1(3) applied to $zz's \subseteq T$ and $ww's \subseteq G$, $z \le_T w$. If $|\operatorname{Scol}_1[L,z] \setminus \operatorname{Scol}_1[L,u]| = 3$ then $zt \in E(G)$. By Lemma 4.1.1(2) applied to $wt \in E(T)$ and $zt \in E(G)$, $w \le_T z$. So either $w's \in E(G)$ or $zt \in E(G)$ but not both. Thus $|\operatorname{Scol}_1[L,w'] \setminus \operatorname{Scol}_1[L,u]| + |\operatorname{Scol}_1[L,z] \setminus \operatorname{Scol}_1[L,u]| \le 5$. With $|\operatorname{Scol}_1[L,z'] \setminus \operatorname{Scol}_1[L,u]| \le 4$ we get $\operatorname{wcol}_2[L,u] \le 13 + 5 + 2 * 4 = 26$.

Case 2: Without loss of generality $x \in P_{a(u)} - P_{b(u)}$.
So $x = w^\circ$. See the second drawing of Figure 4.2 on page 71. First we bound $\operatorname{scol}_2[L,u]$. By (4.2.1), $|\operatorname{Scol}_2[L,u] \cap V(P_{b(u)} - P_u)| \le 4$ and $|\operatorname{Scol}_2[L,u] \cap P_{c(u)}| \le 3$. Every $u' \in \operatorname{Scol}_2[L,u] \cap V(P_{a(u)} - P_u)$ satisfies that $l_u \le l_{u'} \le l_u + 2$. So $|\operatorname{Scol}_2[L,u] \cap V(P_{a(u)} - P_u)| \le 3$. Thus $\operatorname{scol}_2[L,u] \le 4 + 3 + 3 = 10$.

Assume that $w < z$, so $w < z < z'$. There are three possibilities regarding the L-order of w'; they are $w < w' < z < z'$, $w < z < w' < z'$ and $w < z < z' < w'$. In all those possibilities, $|\operatorname{Scol}_1[L,w] \setminus \operatorname{Scol}_1[L,u]| \le 4$ as $\{x, w\} \subseteq \operatorname{Scol}_1[L,u]$. Assume first that $w < w' < z < z'$. The L-smaller neighbor of x in P_x is y. Since $w, w' \in \operatorname{Scol}_1[L,u]$, $|\operatorname{Scol}_1[L,w'] \setminus \operatorname{Scol}_1[L,u]| \le 4$. Since $w' < z$, Lemma 4.2.5 tells us that $P_{w'} \subseteq X_{f(z)}$, say $P_{w'} \subseteq P_{a(z)}$. By Lemma 4.2.5, $P_{w'} \subseteq X_{f(z')}$, say $P_{w'} \subseteq P_{a(z')}$. If it exists, let $s \in P_{a(z')}$ with $l_s = l_{w'} + 1$.

Assume first that $l_z = l_u - 1$. From Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$, $x \le_T z$. The possible L-smaller neighbors of z in $P_{a(z)}$ are y, x and w. The L-smaller neighbor of z in P_z is z°. If $y, z^\circ \in \operatorname{Scol}_2[L,u]$

then $y, z^\circ \notin \mathrm{Scol}_1[L,z] \smallsetminus \mathrm{Scol}_2[L,u]$ and $y \notin \mathrm{Scol}_1[L,x] \smallsetminus \mathrm{Scol}_2[L,u]$. Thus $|\mathrm{Scol}_1[L,x] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 4$ and $|\mathrm{Scol}_1[L,z] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 2$. Assume that $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$. Then $|\mathrm{Scol}_1[L,x] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 4$ and $|\mathrm{Scol}_1[L,z] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 3$. We show that $\mathrm{scol}_2[L,u] \leq 9$. If it exists, let $t \in P_{b(u)}$ with $l_t = l_u + 2$. Assume that $t \in \mathrm{Scol}_2[L,u]$. Let $P := uu't$ be the witnessing path. From Lemma 4.1.1(3) applied to $zz'z''t \subseteq T$ and $xuu't \subseteq G$ we get $z \leq_T x$, a contradiction. So $t \notin \mathrm{Scol}_2[L,u]$. Thus $\mathrm{Scol}_2[L,u] \cap P_{b(u)} \subseteq \{z, z', z''\}$, proving that $\mathrm{scol}_2[L,u] \leq 9$. Assume that $z^\circ \in \mathrm{Scol}_2[L,u]$ and $y \notin \mathrm{Scol}_2[L,u]$. Clearly, $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 7$. Assume that $y, z^\circ \notin \mathrm{Scol}_2[L,u]$. Then $\mathrm{scol}_2[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 8$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 16$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are x, w and w'. Thus $|\mathrm{Scol}_1[L,z'] \smallsetminus \mathrm{Scol}_1[L,u]| \leq 2$. So $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \smallsetminus \mathrm{Scol}_2[L,u]| = 16 + 2 * 4 + 2 = 26$.

Now assume that $l_z = l_u$. By Lemma 4.1.1(3) applied to $z^\circ zz' \in\subseteq T$ and $xuz' \subseteq G$, $z^\circ \leq_T x$. The possible L-smaller neighbors of z in $P_{a(z)}$ are x, w and w', and the L-smaller neighbor of z in P_z is z°. If $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|\mathrm{Scol}_1[L,x] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 4$ and $|\mathrm{Scol}_1[L,z] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 2$. Assume that $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$. Then $|\mathrm{Scol}_1[L,x] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 4$ and $|\mathrm{Scol}_1[L,z] \smallsetminus \mathrm{Scol}_2[L,u]| \leq 3$. We show that $\mathrm{scol}_2[L,u] \leq 9$. Let $z^{\circ\circ} \in P_{b(u)}$ with $l_{z^{\circ\circ}} = l_u - 2$. Assume that $z^{\circ\circ} \in \mathrm{Scol}_2[L,u]$ and $z^{\circ\circ} \neq y$. Let $P := uu'z^{\circ\circ}$ be the witnessing path. From Lemma 4.1.1(3) applied to $yxu \subseteq T$ and $z^{\circ\circ}u'u \subseteq G$ we get $y \leq_T z^{\circ\circ}$, and so $x \leq_T z^\circ$, a contradiction. So $z^{\circ\circ} \notin \mathrm{Scol}_2[L,u]$. Thus $\mathrm{Scol}_2[L,u] \cap P_{b(u)} \subseteq \{z, z', z''\}$, proving that $\mathrm{scol}_2[L,u] \leq 9$. Assume that $z^\circ \in \mathrm{Scol}_2[L,u]$ and $y \notin \mathrm{Scol}_2[L,u]$. Clearly, $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 7$. Assume that $y, z^\circ \notin \mathrm{Scol}_2[L,u]$. Then $\mathrm{scol}_2[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq$

8. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 16$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, w' and s. Assume that $z's \in E(G)$; by Lemma 4.1.1(3) applied to $xww's \subseteq T$ and $z^\circ zz's \subseteq G$ we have $x \leq_T z^\circ$, a contradiction. Therefore $|\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 2$. Thus $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 16 + 2*4+2 = 26$.

Now assume that $w < z < w' < z'$. The L-smaller neighbor of x in P_x is y. Since $w, w' \in \mathrm{Scol}_1[L,u]$, $|\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| \leq 4$. By Lemma 4.2.5 we get $P_z \subseteq X_{f(w')}$, say $P_z \subseteq P_{a(w')}$. If they exist, let $s_1, s_2 \in P_{a(w')}$ with $l_{s_1} = l_z + 1, l_{s_2} = l_z + 2$. By Lemma 4.2.5 we have $P_w \subseteq X_{f(z)}$, say $P_w \subseteq P_{a(z)}$. If it exists, let $s \in P_{a(z)}$ with $l_s = l_w + 1$. By Lemma 4.2.5 we have $P_{w'} \subseteq X_{f(z')}$, say $P_{w'} \subseteq P_{a(z')}$. If it exist, let $s' \in P_{a(z')}$ with $l_{s'} = l_{w'} + 1$.

Assume that $l_z = l_u - 1$. From Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$ we get $x \leq_T z$. The possible L-smaller neighbors of z in $P_{a(z)}$ are y, x and w. If $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|\mathrm{Scol}_1[L,x] \setminus \mathrm{Scol}_2[L,u]| \leq 4$ and $|\mathrm{Scol}_1[L,z] \setminus \mathrm{Scol}_2[L,u]| \leq 2$. Assume that $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$. Then $|\mathrm{Scol}_1[L,x] \setminus \mathrm{Scol}_2[L,u]| \leq 4$ and $|\mathrm{Scol}_1[L,z] \setminus \mathrm{Scol}_2[L,u]| \leq 3$. Let $t \in P_{b(u)}$ with $l_t = l_u + 2$. We showed before that $t \notin \mathrm{Scol}_2[L,u]$ and that $\mathrm{scol}_2[L,u] \leq 9$. Assume that $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$ then clearly, $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 7$. If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 8$. Thus in all cases we have $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 16$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are x, w and w'. So $|\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 2$. Thus $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 16+2*4+2 = 26$.

Assume that $l_z = l_u$. From Lemma 4.1.1(2) applied to $zz' \in E(T)$ and $uz' \in E(G)$ we get $z \leq_T u$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z, s_1 and s_2.

Assume that $w's_2 \in E(G)$. By Lemma 4.1.1(3) applied to $zs_1s_2 \subseteq T$ and $ww's_2 \subseteq G$ we get $z \leq_T w$. The possible L-smaller neighbors of z in $P_{a(z)}$ are x, w and s. Assume that $zs \in E(G)$; by Lemma 4.1.1(2) applied to $ws \in E(T)$ and $zs \in E(G)$ we get $w \leq_T z$. So either $zs \in E(G)$ or $w's_2 \in E(G)$ but not both. Assume without loss of generality that $zs \in E(G)$ and $w's_2 \notin E(G)$. Then $|\operatorname{Scol}_1[L, w'] \setminus \operatorname{Scol}_1[L, u] \leq 3$. If $y, z^\circ \in \operatorname{Scol}_2[L, u]$ then $|\operatorname{Scol}_1[L, x] \setminus \operatorname{Scol}_2[L, u]| \leq 4$ and $|\operatorname{Scol}_1[L, z] \setminus \operatorname{Scol}_2[L, u]| \leq 3$. Assume that $y \in \operatorname{Scol}_2[L, u]$ and $z^\circ \notin \operatorname{Scol}_2[L, u]$. Then $|\operatorname{Scol}_1[L, x] \setminus \operatorname{Scol}_2[L, u]| \leq 4$ and $|\operatorname{Scol}_1[L, z] \setminus \operatorname{Scol}_2[L, u]| \leq 4$. We showed before that in this case $\operatorname{scol}_2[L, u] \leq 9$. Assume that $z^\circ \in \operatorname{Scol}_2[L, u]$ and $y \notin \operatorname{Scol}_2[L, u]$. Clearly, $\operatorname{scol}_2[L, u] \leq 9$ and $|(\operatorname{Scol}_1[L, x] \cup \operatorname{Scol}_1[L, z]) \setminus \operatorname{Scol}_2[L, u]| \leq 8$. Assume that $y, z^\circ \notin \operatorname{Scol}_2[L, u]$. Then $\operatorname{scol}_2[L, u] \leq 8$ and $|(\operatorname{Scol}_1[L, x] \cup \operatorname{Scol}_1[L, z]) \setminus \operatorname{Scol}_2[L, u]| \leq 9$. So in all cases $\operatorname{scol}_2[L, u] + |(\operatorname{Scol}_1[L, x] \cup \operatorname{Scol}_1[L, z]) \setminus \operatorname{Scol}_2[L, u]| \leq 17$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, w' and s'. Assume that $z's' \in E(G)$, by Lemma 4.1.1(3) applied to $xww's' \subseteq T$ and $z^\circ zz's' \subseteq G$ we get $x \leq_T z^\circ$, and so $u \leq_T z$, a contradiction. Thus $z's' \notin E(G)$, and so $|\operatorname{Scol}_1[L, z'] \setminus \operatorname{Scol}_1[L, u]| \leq 2$. Therefore $\operatorname{wcol}_2[L, u] \leq 17 + 4 + 3 + 2 = 26$.

Now assume that $w < z < z' < w'$. Lemma 4.2.5 tells us that $P_{z'} \subseteq X_{f(w')}$, say $P_{z'} \subseteq P_{a(w')}$. If they exist, let $s_1, s_2 \in P_{a(w')}$ with $l_{s_1} = l_{z'} + 1, l_{s_2} = l_{z'} + 2$. As $w < z$, $P_w \subseteq X_{f(z)}$, say $P_w \subseteq P_{a(z)}$. If it exists, let $t \in P_{a(z)}$ with $l_t = l_w + 1$. Since $w < z'$, Lemma 4.2.5 tells us that $P_w \subseteq X_{f(z')}$, say $P_w \subseteq P_{a(z')}$. If it exists, let $t_1, t_2 \in P_{a(z')}$ with $l_{t_1} = l_w + 1, l_{t_2} = l_w + 2$.

Assume first that $l_z = l_u - 1$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$ we get $x \leq_T z$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z', s_1 and s_2. Assume that $w's_2 \in E(G)$; by Lemma 4.1.1(3) applied to $zz's_1s_2 \subseteq E(T)$ and $xww's_2 \subseteq E(G)$ we get $z \leq_T x$, a contradiction. So $w's_2 \notin E(G)$. Thus

$|\operatorname{Scol}_1[L,w'] \smallsetminus \operatorname{Scol}_1[L,u]| \leq 3$. The L-smaller neighbor of x in P_x is y. The possible L-smaller neighbors of z in $P_{a(z)}$ are y, x and w, and the L-smaller neighbor of z in P_z is z°. Note that $x, w, z \in \operatorname{Scol}_1[L,u]$. If $y, z^\circ \in \operatorname{Scol}_2[L,u]$ then $|\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 4$ and $|\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 2$. If $y \in \operatorname{Scol}_2[L,u]$ and $z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 9$, $|\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 4$ and $|\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 3$. If $y \notin \operatorname{Scol}_2[L,u]$ and $z^\circ \in \operatorname{Scol}_2[L,u]$, then $\operatorname{scol}_2[L,u] \leq 9$, $|(\operatorname{Scol}_1[L,x] \cup \operatorname{Scol}_1[L,z]) \smallsetminus \operatorname{Scol}_2[L,u]| \leq 7$. If $y, z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 8$ and $|(\operatorname{Scol}_1[L,x] \cup \operatorname{Scol}_1[L,z]) \smallsetminus \operatorname{Scol}_2[L,u]| \leq 8$. Thus in all cases $\operatorname{scol}_2[L,u] + |(\operatorname{Scol}_1[L,x] \cup \operatorname{Scol}_1[L,z]) \smallsetminus \operatorname{Scol}_2[L,u]| \leq 16$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are x, w and t_1. Since $x, w, z' \in \operatorname{Scol}_1[L,u]$, $|\operatorname{Scol}_1[L,z'] \smallsetminus \operatorname{Scol}_1[L,u]| \leq 3$. Thus $\operatorname{wcol}_2[L,u] \leq \operatorname{scol}_2[L,u] + |(\cup_{u' \in \operatorname{Scol}_1[L,u]} \operatorname{Scol}_1[L,u']) \smallsetminus \operatorname{Scol}_2[L,u]| \leq 16 + 4 + 2 * 3 = 26$.

Now assume that $l_z = l_u$. By Lemma 4.1.1(3) applied to $z^\circ z z' \subseteq T$ and $xuz' \subseteq G$ we get $z^\circ \leq_T x$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z, z' and s_1. Since $w, w', z, z' \in \operatorname{Scol}_1[L,u]$, $|\operatorname{Scol}_1[L,w'] \smallsetminus \operatorname{Scol}_1[L,u]| \leq 3$. The possible L-smaller neighbors of z in $P_{a(z)}$ are x, w and t. Assume that $zt \in E(G)$; by Lemma 4.1.1(3) applied to $xwt \subseteq T$ and $z^\circ zt \subseteq G$ we get $x \leq_T z^\circ$, a contradiction. Thus $zt \notin E(G)$. If $y, z^\circ \in \operatorname{Scol}_2[L,u]$ then $|\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 4$ and $|\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 2$. If $y \in \operatorname{Scol}_2[L,u]$ and $z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 9$, $|\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 4$ and $|\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 3$. If $z^\circ \in \operatorname{Scol}_2[L,u]$ and $y \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 9$, $|\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 5$ and $|\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 2$. If $y, z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 8$, $|\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 5$ and $|\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 3$. So in all cases $\operatorname{scol}_2[L,u] + |\operatorname{Scol}_1[L,x] \smallsetminus \operatorname{Scol}_2[L,u]| + |\operatorname{Scol}_1[L,z] \smallsetminus \operatorname{Scol}_2[L,u]| \leq 16$. So The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, t_1 and t_2. Assume that $z't_2 \in E(G)$; by Lemma 4.1.1(3) applied to $xwt_1t_2 \subseteq T$ and $z^\circ z z' t_2 \subseteq G$ we get that $x \leq_T z^\circ$, a contradiction. Thus $z't_2 \notin E(G)$, and so $|\operatorname{Scol}_1[L,z'] \smallsetminus$

$\mathrm{Scol}_1[L,u]| \leq 3$. Therefore $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + \sum_{u' \in \mathrm{Scol}_1[L,u]} |\,\mathrm{Scol}_1[L,u'] \setminus \mathrm{Scol}_2[L,u]| \leq 16 + 4 + 2 * 3 = 26$.

Assume that $z < w$; so $z < w < w'$. We have three possibilities regarding the L-order of z'. They are $z < z' < w < w'$, $z < w < z' < w'$ and $z < w < w' < z'$. We discuss each possibility.

Assume that $z < z' < w < w'$. Since $z, z' \in \mathrm{Scol}_1[L,u]$, $|\,\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 4$. Since $z' < w$, Lemma 4.2.5 tells us that $P_{z'} \subseteq X_{f(w)}$, say $P_{z'} \subseteq P_{a(w)}$. If it exists, let $t \in P_{a(w)}$ with $l_t = l_{z'} + 1$. Since $z' < w'$, $P_{z'} \subseteq X_{f(w')}$, say $P_{z'} \subseteq P_{a(w')}$. If they exist, let $s_1, s_2 \in P_{a(w')}$ with $l_{s_1} = l_{z'} + 1, l_{s_2} = l_{z'} + 2$.

Assume first that $l_z = l_u - 1$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$ we get $x \leq_T z$. The possible L-smaller neighbors of w in $P_{a(w)}$ are z, z' and t. Assume that $wt \in E(G)$. By Lemma 4.1.1(3) applied to $zz't \subseteq T$ and $xwt \subseteq G$ we get $z \leq_T x$, a contradiction. Thus $wt \notin E(G)$, so $|\,\mathrm{Scol}_1[L,w] \setminus \mathrm{Scol}_1[L,u]| \leq 2$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z', s_1 and s_2. Assume that $w's_2 \in E(G)$. By Lemma 4.1.1(3) applied to $zz's_1s_2 \subseteq T$ and $xww's_2 \subseteq G$ we get $z \leq_T x$, a contradiction. Thus $|\,\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| \leq 3$.

Assume that $z < x$; then $P_z \subseteq X_{f(x)}$, say $P_z \subseteq P_{a(x)}$. If it exists, let $s \in P_{a(x)}$ with $l_s = l_z + 1$. The possible L-smaller neighbors of x in $P_{a(x)}$ are z°, z and s. Assume that $xs \in E(G)$; by Lemma 4.1.1(2) applied to $zs \in E(T)$ and $xs \in E(G)$ we get $z \leq_T x$, a contradiction. So $xs \notin E(G)$. Thus the possible L-smaller neighbors of x in $X_{f(x)}$ are z°, z, y and two more vertices in $P_{b(x)} - P_x$. The possible L-smaller vertices of z in $X_{f(z)}$ are z°, two vertices in $P_{a(z)} - P_z$ and two vertices in $P_{b(z)} - P_z$. If $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 6$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 7$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then

$\mathrm{scol}_2[L,u] \le 8$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \le 8$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \le 16$.

Now assume that $x < z$; then $P_x \subseteq X_{f(z)}$, say $P_x \subseteq P_{a(z)}$. If it exists, let $x' \in P_{a(z)}$ with $l_{x'} = l_x + 1$. The possible L-smaller neighbors of z in $P_{a(z)}$ are y, x and x'. So the possible L-smaller neighbors of z in $X_{f(z)}$ are y, x, x', z° and two vertices in $P_{b(z)} - P_z$. The possible L-smaller neighbors of x in $X_{f(x)}$ are y, two vertices in $P_{a(x)} - P_x$ and two vertices in $P_{b(x)} - P_x$. If $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \le 7$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \le 9$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \le 8$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \le 8$ and $|(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \le 9$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \le 17$. Thus in both cases (i.e., $z < x$ and $x < z$) we have $\mathrm{wcol}_2[L,u] \le \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \le 17+2+3+4 = 26$.

Assume that $l_z = l_u$. By Lemma 4.1.1(3) applied to $z^\circ z z' \subseteq T$ and $xuz' \subseteq G$ we get $z^\circ \le_T x$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z, z' and s_1. Since $w, w', z, z' \in \mathrm{Scol}_1[L,u]$, $|\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| \le 3$. The possible L-smaller neighbors of w in $P_{a(w)}$ are z°, z and z'. So the possible L-smaller neighbors of w in $X_{f(w)}$ are z°, z, z', x and two vertices in $P_{b(w)} - P_w$.

Assume that $z < x$. By Lemma 4.2.5 we get $P_z \subseteq X_{f(x)}$, say $P_z \subseteq P_{a(x)}$. Let $z^{\circ\circ} \in P_{a(x)}$ with $l_{z^{\circ\circ}} = l_x - 1$. The possible L-smaller neighbors of x in $P_{a(x)}$ are $z^{\circ\circ}, z^\circ$ and z. Assume that $z^{\circ\circ} \neq y$ and $xz^{\circ\circ} \in E(G)$. By Lemma 4.1.1(2) we get $y \le_T z^{\circ\circ}$, and so $x \le_T z^\circ$, a contradiction. So either $z^{\circ\circ} = y$ or $xz^{\circ\circ} \notin E(G)$. Thus the possible L-smaller neighbors of x in $X_{f(x)}$ are y, z°, z and two vertices in $P_{b(x)} - P_x$. The possible L-smaller neighbors of z in $X_{f(z)}$ are z°, two vertices in $P_{a(z)} - P_z$ and two vertices in $P_{b(z)} - P_z$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,w] \cup$

$\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 8$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 9$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_1[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 10$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 18$.

Assume that $x < z$. By Lemma 4.2.5 we get $P_x \subseteq X_{f(z)}$, say $P_x \subseteq P_{a(z)}$. If they exist, let $v_1, v_2 \in P_{a(z)}$ with $l_{v_1} = l_x + 1, l_{v_2} = l_x + 2$. Assume that $zv_2 \in E(G)$; by Lemma 4.1.1(3) applied to $xv_1v_2 \subseteq T$ and $z^\circ z v_2 \subseteq G$ we get $x \leq_T z^\circ$, a contradiction. So $zv_2 \notin E(G)$. Thus the possible L-smaller neighbors of z in $X_{f(z)}$ are z°, x, v_1 and two vertices in $P_{b(z)} - P_z$. The possible L-smaller vertices of x in $X_{f(x)}$ are y, two vertices in $P_{a(x)} - P_x$ and two vertices in $P_{b(x)} - P_x$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 9$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 10$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$) . If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 11$. Thus in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 19$. Thus in both cases (i.e., $x < z$ or $z < x$) we have $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 19 + 3 + 4 = 26$.

Assume that $z < w < z' < w'$. Since $z < w$, $P_z \subseteq X_{f(w)}$, say $P_z \subseteq P_{a(w)}$. If they exist, let $v_1, v_2 \in P_{a(w)}$ with $l_{v_1} = l_z + 1, l_{v_2} = l_z + 2$. Since $z' < w'$, Lemma 4.2.5 tells us that $P_{z'} \subseteq X_{f(w')}$, say $P_{z'} \subseteq P_{a(w')}$. If they exist, let $s_1, s_2 \in P_{a(w')}$ with $l_{s_1} = l_{z'} + 1, l_{s_2} = l_{z'} + 2$. Since $w < z'$, Lemma 4.2.5 tells us that $P_w \subseteq X_{f(z')}$, say $P_w \subseteq P_{a(z')}$. Let $t_1, t_2 \in P_{a(z')}$ with $l_{t_1} = l_w + 1, l_{t_2} = l_w + 2$.

Assume first that $l_z = l_u - 1$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$ we find that $x \leq_T z$. The possible L-smaller neighbors of w in $P_{a(w)}$ are z, v_1 and

v_2. Assume that $wv_2 \in E(G)$. By Lemma 4.1.1(3) applied to $zv_1v_2 \subseteq T$ and $xwv_2 \subseteq G$ we get $z \leq_T x$, a contradiction. So $wv_2 \notin E(G)$. Thus $|\operatorname{Scol}_1[L,w] \setminus \operatorname{Scol}_1[L,u]| \leq 3$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z', s_1 and s_2. Assume that $w's_2 \in E(G)$. By Lemma 4.1.1(3) applied to $zz's_1s_2 \subseteq T$ and $xww's_2 \subseteq G$ we get $z \leq_T x$, a contradiction. So $w's_2 \notin E(G)$. Thus $|\operatorname{Scol}_1[L,w'] \setminus \operatorname{Scol}_1[L,u]| \leq 3$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are x, w and t_1. So $|\operatorname{Scol}_1[L,z'] \setminus \operatorname{Scol}_1[L,u]| \leq 3$.

Assume that $z < x$. Then $P_z \subseteq X_{f(x)}$, say $P_z \subseteq P_{a(x)}$. If it exists, let $t \in P_{a(x)}$ with $l_t = l_z + 1$. The possible L-smaller neighbors of x in $X_{f(x)}$ are z°, z, t, y and two vertices in $P_{b(x)} - P_x$. The possible L-smaller neighbors of z in $X_{f(z)}$ are z°, two vertices in $P_{a(z)} - P_z$ and two vertices in $P_{b(z)} - P_z$. Therefore if $y, z^\circ \in \operatorname{Scol}_2[L,u]$ then $|(\operatorname{Scol}_1[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 7$. If $y \in \operatorname{Scol}_2[L,u]$ and $z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 9$ and $|(\operatorname{Scol}[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 8$ (same result is true if $y \notin \operatorname{Scol}_2[L,u]$ and $z^\circ \in \operatorname{Scol}_2[L,u]$). If $y, z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_1[L,u] \leq 8$ and $|(\operatorname{Scol}_1[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 9$. So in all cases $\operatorname{scol}_2[L,u] + |(\operatorname{Scol}_1[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 17$.

Assume that $x < z$. Then $P_x \subseteq X_{f(z)}$, say $P_x \subseteq P_{a(z)}$. If it exists, let $s \in P_{a(z)}$ with $l_s = l_x + 1$. The possible L-smaller neighbors of z in $X_{f(z)}$ are y, x, s, z° and two vertices in $P_{b(z)} - P_z$. The possible L-smaller neighbors of x in $X_{f(x)}$ are y, two vertices in $P_{a(x)} - P_x$ and two vertices in $P_{b(x)} - P_x$. Therefore if $y, z^\circ \in \operatorname{Scol}_2[L,u]$ then $|(\operatorname{Scol}_1[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 7$. If $y \in \operatorname{Scol}_2[L,u]$ and $z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_2[L,u] \leq 9$ and $|(\operatorname{Scol}[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 8$ (same result is true if $y \notin \operatorname{Scol}_2[L,u]$ and $z^\circ \in \operatorname{Scol}_2[L,u]$). If $y, z^\circ \notin \operatorname{Scol}_2[L,u]$ then $\operatorname{scol}_1[L,u] \leq 8$ and $|(\operatorname{Scol}_1[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 9$. So in all cases $\operatorname{scol}_2[L,u] + |(\operatorname{Scol}_1[L,z] \cup \operatorname{Scol}_1[L,x]) \setminus \operatorname{Scol}_2[L,u]| \leq 17$. Thus in both cases (i.e., $x < z$ or $z < x$) we have $\operatorname{wcol}_2[L,u] \leq \operatorname{scol}_2[L,u] + |(\cup_{u' \in \operatorname{Scol}_1[L,u]} \operatorname{Scol}_1[L,u']) \setminus \operatorname{Scol}_2[L,u]| \leq 17 + 3 * 3 = 26$.

Assume that $l_z = l_u$. By Lemma 4.1.1(3) applied to $z^\circ z z' \subseteq T$ and $xuz' \subseteq G$ we get $z^\circ \leq_T x$. The possible L-smaller neighbors of w' in $P_{a(w')}$ are z, z' and s_1. As $z, z', w, w' \in \mathrm{Scol}_1[L, u]$, $|\mathrm{Scol}_1[L, w'] \setminus \mathrm{Scol}_1[L, u]| \leq 3$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, t_1 and t_2. Assume that $z't_2 \in E(G)$. By Lemma 4.1.1(3) applied to $xwt_1t_2 \subseteq T$ and $z^\circ z z' t_2 \subseteq G$ we get $x \leq_T z^\circ$, a contradiction. So $z't_2 \notin E(G)$. Thus $|\mathrm{Scol}_1[L, z'] \setminus \mathrm{Scol}_1[L, u]| \leq 3$. The possible L-smaller neighbors of w in $P_{a(w)}$ are z°, z and v_1. Thus the possible L-smaller neighbors of w in $X_{f(w)}$ are z°, z, v_1, x and two vertices in $P_{b(w)} - P_w$.

Assume that $z < x$. By Lemma 4.2.5 we get $P_z \subseteq X_{f(x)}$, say $P_z \subseteq P_{a(x)}$. Let $z^{\circ\circ} \in P_{a(x)}$ with $l_{z^{\circ\circ}} = l_z - 2$. Assume that $z^{\circ\circ} \neq y$ and $xz^{\circ\circ} \in E(G)$. By Lemma 4.1.1(2) applied to $yx \in E(T)$ and $xz^{\circ\circ} \in E(G)$ we get $y \leq_T z^{\circ\circ}$, and so $x \leq_T z^\circ$, a contradiction. So either $z^{\circ\circ} = y$ or $xz^{\circ\circ} \notin E(G)$. Thus the possible L-smaller neighbors of x in $X_{f(x)}$ are z°, z, y and two vertices in $P_{b(x)} - P_x$. The possible L-smaller neighbors of z in $X_{f(z)}$ are z°, two vertices in $P_{a(z)} - P_z$ and two vertices in $P_{b(z)} - P_z$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L, u]$ then $|(\mathrm{Scol}_1[L, w] \cup \mathrm{Scol}[L, z] \cup \mathrm{Scol}_1[L, x]) \setminus \mathrm{Scol}_2[L, u]| \leq 9$. If $y \in \mathrm{Scol}_2[L, u]$ and $z^\circ \notin \mathrm{Scol}_2[L, u]$ then $\mathrm{scol}_2[L, u] \leq 9$ and $|(\mathrm{Scol}_1[L, w] \cup \mathrm{Scol}[L, z] \cup \mathrm{Scol}_1[L, x]) \setminus \mathrm{Scol}_2[L, u]| \leq 10$ (same result is true if $y \notin \mathrm{Scol}_2[L, u]$ and $z^\circ \in \mathrm{Scol}_2[L, u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L, u]$ then $\mathrm{scol}_1[L, u] \leq 8$ and $|(\mathrm{Scol}_1[L, w] \cup \mathrm{Scol}_1[L, z] \cup \mathrm{Scol}_1[L, x]) \setminus \mathrm{Scol}_2[L, u]| \leq 11$. So in all cases $\mathrm{scol}_2[L, u] + |(\mathrm{Scol}_1[L, w] \cup \mathrm{Scol}_1[L, z] \cup \mathrm{Scol}_1[L, x]) \setminus \mathrm{Scol}_2[L, u]| \leq 19$.

Assume that $x < z$. By Lemma 4.2.5 we get $P_x \subseteq X_{f(z)}$, say $P_x \subseteq P_{a(z)}$. If they exist, let $x_1, x_2 \in P_{a(z)}$ with $l_{x_1} = l_x + 1, l_{x_2} = l_x + 2$. Assume that $zx_2 \in E(G)$. By Lemma 4.1.1(3) applied to $xx_1x_2 \subseteq T$ and $z^\circ z x_2 \subseteq G$ we get $x \leq_T z^\circ$, a contradiction. So $zx_2 \notin E(G)$. Thus the possible L-smaller neighbors of z in $X_{f(z)}$ are x, x_1, z° and two vertices in $P_{b(z)} - P_z$. The possible L-smaller neighbors of x in $X_{f(x)}$ are y, two vertices in $P_{a(x)} - P_x$ and two vertices in $P_{b(x)} - P_x$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 10$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 11$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_1[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 12$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 20$. Thus in both cases (i.e., $x < z$ or $z < x$) we have $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 20 + 3 * 2 = 26$.

Assume now that $z < w < w' < z'$. Lemma 4.2.5 tells us that $P_z \subseteq X_{f(w)}$, say $P_z \subseteq P_{a(w)}$. If they exist, let $s_1, s_2 \in P_{a(w)}$ with $l_{s_1} = l_z + 1, l_{s_2} = l_z + 2$. Lemma 4.2.5 tells us that $P_{w'} \subseteq X_{f(z')}$, say $P_{w'} \subseteq P_{a(z')}$. If it exists, let $s \in P_{a(z')}$ with $l_s = l_{w'} + 1$.

Assume first that $l_z = l_u - 1$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$ we get $x \leq_T z$. The possible L-smaller neighbors of w in $P_{a(w)}$ are z, s_1 and s_2. Assume that $ws_2 \in E(G)$. By Lemma 4.1.1(3) applied to $zs_1s_2 \subseteq T$ and $xws_2 \subseteq G$ we get $z \leq_T x$, a contradiction. So $ws_2 \notin E(G)$. Thus $|\mathrm{Scol}_1[L,w] \smallsetminus \mathrm{Scol}_1[L,u]| \leq 3$. Since $w, w' \in \mathrm{Scol}_1[L,u]$, $|\mathrm{Scol}_1[L,w'] \smallsetminus \mathrm{Scol}_1[L,u]| \leq 4$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are x, w and w'. Thus $|\mathrm{Scol}_1[L,z'] \smallsetminus \mathrm{Scol}_1[L,u]| \leq 2$.

Assume that $z < x$. So $P_z \subseteq X_{f(x)}$, say $P_z \subseteq P_{a(x)}$. If it exists, let $v' \in P_{a(x)}$ with $l_{v'} = l_z + 1$. So the possible L-smaller neighbors of x in $P_{a(x)}$ are z°, z and v'. Assume that $xv' \in E(G)$. By Lemma 4.1.1(2) applied to $zv' \in E(T)$ and $xv' \in E(G)$ we get $z \leq_T x$, a contradiction. So $xv' \notin E(G)$. Thus the possible L-smaller neighbors of x in $X_{f(x)}$ are z°, z, y and two vertices in $P_{b(x)} - P_x$. The possible L-smaller neighbors of z in $X_{f(z)}$ are z°, two vertices in $P_{a(z)} - P_z$ and two vertices in $P_{b(z)} - P_z$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 6$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \smallsetminus \mathrm{Scol}_2[L,u]| \leq 7$

(same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_1[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 8$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 16$.

Assume that $x < z$. So $P_x \subseteq X_{f(z)}$, say $P_x \subseteq P_{a(z)}$. If it exists, let $t \in P_{a(z)}$ with $l_t = l_x + 1$. The possible L-smaller neighbors of z in $P_{a(z)}$ are y, x and t. So the possible L-smaller neighbors of z in $X_{f(z)}$ are y, x, t, z° and two vertices in $P_{b(z)} - P_z$. The possible L-smaller neighbors of x in $X_{f(x)}$ are y, two vertices in $P_{a(x)} - P_x$ and two vertices in $P_{b(x)} - P_x$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 7$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 8$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_1[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 9$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 17$. Thus in both cases (i.e., $x < z$ or $z < x$) we have $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 17+3+4+2 = 26$.

Assume that $l_z = l_u$. By Lemma 4.1.1(3) applied to $z^\circ z z' \subseteq T$ and $xuz' \subseteq G$ we get $z^\circ \leq_T x$. The possible L-smaller neighbors of w in $P_{a(w)}$ are z°, z and s_1. Assume that $z^\circ w \in E(G)$. By Lemma 4.1.1(2) applied to $xw \in E(T)$ and $z^\circ w \in E(G)$ we get $x \leq_T z^\circ$, a contradiction. Thus $|\,\mathrm{Scol}_1[L,w] \setminus \mathrm{Scol}_1[L,u]| \leq 3$. The possible L-smaller neighbors of w' in $X_{f(w')}$ are w, two vertices in $P_{a(w')} - P_{w'}$ and two vertices in $P_{b(w')} - P_{w'}$. Thus $|\,\mathrm{Scol}_1[L,w'] \setminus \mathrm{Scol}_1[L,u]| \leq 4$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, w' and s. Assume that $z's \in E(G)$. By Lemma 4.1.1(3) applied to $xww's \subseteq T$ and $z^\circ zz's \subseteq G$ we get $x \leq_T z^\circ$, a contradiction. So $z's \notin E(G)$. Thus $|\,\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 2$.

Assume that $z < x$. By Lemma 4.2.5 we get $P_z \subseteq X_{f(x)}$, say $P_z \subseteq P_{a(x)}$. Let $z^{\circ\circ} \in P_{a(x)}$ with $l_{z^{\circ\circ}} = l_z - 2$. The possible L-smaller neighbors of x in $P_{a(x)}$ are

$z^{\circ\circ}, z^\circ$ and z. Assume that $z^{\circ\circ} \neq y$ and $z^{\circ\circ}x \in E(G)$. By Lemma 4.1.1(3) applied to $yx \in E(T)$ and $z^{\circ\circ}x \in E(G)$ we get $y \leq_T z^{\circ\circ}$. So $x \leq_T z^\circ$, a contradiction. Thus either $z^{\circ\circ} = y$ or $z^{\circ\circ}x \notin E(G)$. So the possible L-smaller neighbors of x in $X_{f(x)}$ are z°, z, y and two vertices in $P_{b(x)} - P_x$. The possible L-smaller neighbors of z in $X_{f(x)}$ are z°, two vertices in $P_{a(z)} - P_z$ and two vertices in $P_{b(z)} - P_z$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 6$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 7$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_1[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 8$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 16$.

Assume that $x < z$. By Lemma 4.2.5 we get $P_x \subseteq X_{f(z)}$, say $P_x \subseteq P_{a(z)}$. If they exist, let $t_1, t_2 \in P_{a(z)}$ with $l_{t_1} = l_x + 1, l_{t_2} = l_x + 2$. Assume that $zt_2 \in E(G)$. By Lemma 4.1.1(3) applied to $xt_1t_2 \subseteq T$ and $z^\circ z t_2 \subseteq G$ we get $x \leq_T z^\circ$, a contradiction. So the possible L-smaller neighbors of z in $X_{f(z)}$ are x, t_1, z° and two vertices in $P_{b(z)} - P_z$. The possible L-smaller neighbors of x in $X_{f(x)}$ are y, two vertices in $P_{a(x)} - P_x$ and two vertices in $P_{b(x)} - P_x$. Therefore if $y, z^\circ \in \mathrm{Scol}_2[L,u]$ then $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 7$. If $y \in \mathrm{Scol}_2[L,u]$ and $z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_2[L,u] \leq 9$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 8$ (same result is true if $y \notin \mathrm{Scol}_2[L,u]$ and $z^\circ \in \mathrm{Scol}_2[L,u]$). If $y, z^\circ \notin \mathrm{Scol}_2[L,u]$ then $\mathrm{scol}_1[L,u] \leq 8$ and $|(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 9$. So in all cases $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,z] \cup \mathrm{Scol}_1[L,x]) \setminus \mathrm{Scol}_2[L,u]| \leq 17$. Thus in both cases (i.e., $x < z$ or $z < x$) we have $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 17+3+4+2 = 26$.

Case 3: $x \notin P_{a(u)} \cup P_{b(u)}$. See the third drawing of Figure 4.2 on page 71.
By Theorem 4.2.6, $\mathrm{scol}_2[L,u] \leq 11$. Assume without loss of generality that $w < z$. So $w < z < z'$. There are three possibilities regarding the L-order of w'. They are

$w < w' < z < z'$, $w < z < w' < z'$ and $w < z < z' < w'$. If they exist, let $s \in P_{a(u)}$ with $l_s = l_w + 3$ and $t \in P_{b(u)}$ with $l_t = l_z + 3$.

Assume that $w < w' < z < z'$. Since $w, w' \in \mathrm{Scol}_1[L, u]$, $|\mathrm{Scol}_1[L, w'] \setminus \mathrm{Scol}_2[L, u]| \leq 4$. Since $w' < z'$, Lemma 4.2.5 tells us that $P_{w'} \subseteq X_{f(z')}$, say $P_{w'} \subseteq P_{a(z')}$. If they exist, let $s', s'' \in P_{a(z')}$ with $l_{s'} = l_{w'} + 1, l_{s''} = l_{w'} + 2$. Assume first that $l_u = l_{w'} = l_{z'}$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $wu \in E(G)$ we get $x \leq_T w$. Similarly, we get $x \leq_T z$. Assume that $s \in \mathrm{Scol}_2[L, u]$. Let $P := uu's$ be the witnessing path. By Lemma 4.1.1(3) applied to $ww'w''s \subseteq T$ and $xuu's \subseteq G$ we get $w \leq_T x$, a contradiction. So $s \notin \mathrm{Scol}_2[L, u]$, analogously we get $t \notin \mathrm{Scol}_2[L, u]$. Let $A := \{y, x, u, w^\circ, w, w', w'', z^\circ, z, z', z''\}$. Clearly, $\mathrm{Scol}_2[L, u] \subseteq A$. Also $\mathrm{Scol}_1[L, x] \subseteq \{y, x, w^\circ, w, w', z^\circ, z, z'\} \subseteq A$. Therefore $\mathrm{scol}_2[L, u] + |\mathrm{Scol}_1[L, x] \setminus \mathrm{Scol}_2[L, u]| \leq 11$. The unique neighbor of w in P_w is w°, the unique neighbor of z in P_z is z° and $w^\circ, z^\circ \in A$. Thus $\mathrm{scol}_2[L, u] + |(\mathrm{Scol}_1[L, x] \cup \mathrm{Scol}_1[L, w] \cup \mathrm{Scol}_1[L, z]) \setminus \mathrm{Scol}_2[L, u]| \leq 11 + 2 * 4 = 19$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, w' and s'. So $|\mathrm{Scol}_1[L, z'] \setminus \mathrm{Scol}_2[L, u]| \leq 3$. Thus $\mathrm{wcol}_2[L, u] \leq \mathrm{scol}_2[L, u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L, u']) \setminus \mathrm{Scol}_2[L, u]| \leq 19 + 4 + 3 = 26$.

Assume that $l_u = l_w = l_z$. Let $w^{\circ\circ} \in P_{a(u)}$ with $l_{w^{\circ\circ}} = l_w - 2$ and $z^{\circ\circ} \in P_{b(u)}$ with $l_{z^{\circ\circ}} = l_z - 2$. By Lemma 4.1.1(2) applied to $ww' \in E(T)$ and $uw' \in E(G)$ we get $w \leq_T u$, analogously we get $z \leq_T u$. Assume that $w^{\circ\circ} \neq y$ and $w^{\circ\circ} \in \mathrm{Scol}_2[L, u]$. Let $P := w^{\circ\circ}u'u$ be the witnessing path. By Lemma 4.1.1(3) applied to $yxu \subseteq T$ and $w^{\circ\circ}u'u \subseteq G$ we get $y \leq_T w^{\circ\circ}$. So $x \leq_T w^\circ$ and then $u \leq_T w$, a contradiction. Thus $w^{\circ\circ} \notin \mathrm{Scol}_2[L, u]$, analogously we get $z^{\circ\circ} \notin \mathrm{Scol}_2[L, u]$ if $z^{\circ\circ} \neq y$. Then $\mathrm{Scol}_2[L, u] \subseteq A$. Assume that $w^{\circ\circ} \neq y$ and $w^{\circ\circ}x \in E(G)$. By Lemma 4.1.1(2) applied to $yx \in E(T)$ and $w^{\circ\circ}x \in E(G)$ we get $y \leq_T w^{\circ\circ}$ which leads to a contradiction as we have just seen. Thus $w^{\circ\circ}x \notin E(G)$, similarly, we get $z^{\circ\circ}x \notin E(G)$ if $z^{\circ\circ} \neq y$.

So $\mathrm{Scol}_1[L,x] \subseteq \{y, x, w^\circ, w, z^\circ, z\} \subseteq A$. The unique neighbor of w in P_w is w°, the unique neighbor of z in P_z is z° and $w^\circ, z^\circ \in A$. Thus $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 11 + 2*4 = 19$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w, w' and s'. So $|\,\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_2[L,u]| \leq 3$. Thus $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 19 + 4 + 3 = 26$.

Assume that $l_{w'} \neq l_{z'}$. In this case either $l_{w'} = l_u = l_{z'} - 1$ or $l_{w'} - 1 = l_u = l_{z'}$. By the two previous paragraphs we get $\mathrm{Scol}_2[L,u] \subseteq A$ and $\mathrm{Scol}_1[L,x] \subseteq A$. The unique neighbor of w in P_w is w° and the unique neighbor of z in P_z is z°. Thus $\mathrm{scol}_2[L,u] + |(\mathrm{Scol}_1[L,x] \cup \mathrm{Scol}_1[L,w] \cup \mathrm{Scol}_1[L,z]) \setminus \mathrm{Scol}_2[L,u]| \leq 11 + 2*4 = 19$. Assume that $l_{w'} = l_u = l_{z'} - 1$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $wu \in E(G)$ we get $x \leq_T w$. By Lemma 4.1.1(3) applied to $z^\circ z z' \subseteq E(T)$ and $xuz' \subseteq E(G)$ we get $z^\circ \leq_T x$. So $z^\circ \leq_T w$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are w', s' and s''. Assume that $z's'' \in E(T)$. By Lemma 4.1.1(3) applied to $ww's's'' \subseteq T$ and $z^\circ z z' s'' \subseteq G$ we get $w \leq_T z^\circ$, a contradiction. So the possible L-smaller neighbors of z' in $P_{a(z')}$ are w' and s', and then $|\,\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_1[L,u]| \leq 3$. Assume that $l_{w'} - 1 = l_u = l_{z'}$. By Lemma 4.1.1(2) applied to $xu \in E(T)$ and $zu \in E(G)$ we get $x \leq_T z$. By Lemma 4.1.1(3) applied to $w^\circ w w' \in E(T)$ and $xuw' \in E(G)$ we get $w^\circ \leq_T x$. So $w^\circ \leq_T z$. The possible L-smaller neighbors of z' in $P_{a(z')}$ are $w^\circ, w, w' \in A$. Therefore $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 11 + 3*4 + 3 = 26$.

Assume that $w < z < w' < z'$. Note that the facts that $\mathrm{Scol}_2[L,u], \mathrm{Scol}_1[L,x] \subseteq A$ were independent from the L-order of w, w', z, z'. So even in this case we have $\mathrm{Scol}_2[L,u], \mathrm{Scol}_1[L,x] \subseteq A$. In the previous case we used only the inequality $w' < z'$ to show that $|\,\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_2[L,u]| \leq 3$. Since the inequality $w' < z'$ is still valid in this case, we get $|\,\mathrm{Scol}_1[L,z'] \setminus \mathrm{Scol}_2[L,u]| \leq 3$. Therefore $\mathrm{wcol}_2[L,u] \leq \mathrm{scol}_2[L,u] + |(\cup_{u' \in \mathrm{Scol}_1[L,u]} \mathrm{Scol}_1[L,u']) \setminus \mathrm{Scol}_2[L,u]| \leq 11 + 3*4 + 3 = 26$.

Assume that $w < z < z' < w'$. Again $\mathrm{Scol}_2[L,u], \mathrm{Scol}_1[L,x] \subseteq A$. By interchanging the roles of w' and z' in the previous cases we get the same result.

The proof above was for the case $|\mathrm{Scol}_1[L,u] \setminus \{u\}| = 5$. Assume now that $|\mathrm{Scol}_1[L,u] \setminus \{u\}| \leq 4$. Let $x \in P_u$ with $l_x = l_u - 1$, and if it exists, let $y \in P_u$ with $l_y = l_u - 2$. Assume that $x \in P_{a(u)} \cap P_{b(u)}$. If they exist, let $w, w', w'' \in P_{a(u)}$ at levels $l_u, l_u + 1, l_u + 2$ respectively, and let $z, z', z'' \in P_{b(u)}$ at levels $l_u, l_u + 1, l_u + 2$ respectively. Let $B := \{u, x, y, w, w', w'', z, z', z''\}$. Then $\mathrm{Scol}_2[L,u] \subseteq B$. Note that $|\mathrm{Scol}_1[L,x] \setminus \{x\}| \leq 5$. Let $u' \in \mathrm{Scol}_1[L,u] \setminus \{x, u\}$. The unique neighbor of u' in $P_{u'}$ is in B. Therefore $\mathrm{wcol}_2[L,u] \leq 9 + 5 + 3 * 4 = 26$.

Assume without loss of generality that $x \in P_{a(u)} - P_{b(u)}$. If they exist, let $w, w', w'' \in P_{a(u)}$ at levels $l_u, l_u + 1, l_u + 2$ respectively. If they exist, let $z_1, z_2, z_3, z_4, z_5 \in P_{b(u)}$ at levels $l_u - 2, l_u - 1, l_u, l_u + 1, l_u + 2$ respectively. Let $C_1 := \{y, x, u, w, w', w'', z_1, z_2, z_3, z_4\}$ and $C_2 := \{y, x, u, w, w', w'', z_2, z_3, z_4, z_5\}$. Note that either $\mathrm{Scol}_2[L,u] \subseteq C_1$ or $\mathrm{Scol}_2[L,u] \subseteq C_2$. Assume that $\mathrm{Scol}_2[L,u] \subseteq C_1$. For each $u' \in \mathrm{Scol}_1[L,u]$, the unique neighbor of u' in $P_{u'}$ is in C_1. Thus $\mathrm{wcol}_2[L,u] \leq 10 + 4 * 4 = 26$. Assume that $\mathrm{Scol}_2[L,u] \subseteq C_2$. Assume that $z_5 \notin \mathrm{Scol}_2[L,u]$, then $\mathrm{scol}_2[L,u] \leq 9$. For each $u' \in \mathrm{Scol}_1[L,u] \setminus \{z_2\}$, the unique neighbor of u' in $P_{u'}$ is in C_2. Thus $\mathrm{wcol}_2[L,u] \leq 9 + 5 + 3 * 4 = 26$. Assume that $z_5 \in \mathrm{Scol}_2[L,u]$. Let $P := uzz_5$ be the witnessing path. By Lemma 4.1.1(3) applied to $z_2z_3z_4z_5 \subseteq T$ and $xuzz_5 \subseteq G$ we get $z_2 \leq_T x$. Assume that $z_2u \in E(G)$; by Lemma 4.1.1(2) applied to $xu \in E(T)$ and $z_2u \in E(G)$ we get $x \leq_T z_2$, a contradiction. So $z_2 \notin \mathrm{Scol}_1[L,u]$. Thus for every $u' \in \mathrm{Scol}_1[L,u]$, the unique neighbor of u' in $P_{u'}$ is in C_2. Therefore $\mathrm{wcol}_2[L,u] \leq 10 + 4 * 4 = 26$.

Assume that $x \notin P_{a(u)} \cup P_{b(u)}$. If they exist, let $w_1, \ldots, w_5 \in P_{a(u)}$ with $l_{w_i} = l_u + (i - 3)$ and let $z_1, \ldots, z_5 \in P_{b(u)}$ with $l_{z_i} = l_u + (i - 3)$. Assume that $w_5, z_5 \notin$

$\mathrm{Scol}_2[L,u]$. Let $D_1 := \{y, x, u, w_1, \ldots, w_4, z_1, \ldots, z_4\}$, then $\mathrm{Scol}_2[L,u] \subseteq D_1$ and $\mathrm{Scol}_1[L,x] \subseteq D_1$. For each $u' \in \mathrm{Scol}_1[L,u]$, the unique neighbor of u' in $P_{u'}$ is in D_1. Since $|D_1| \leq 11$, $\mathrm{wcol}_2[L,u] \leq 11 + 3 * 4 = 23$. Assume without loss of generality that $w_5 \notin \mathrm{Scol}_2[L,u]$ and $z_5 \in \mathrm{Scol}_2[L,u]$; then as we showed in the previous paragraph, $z_2 \leq_T x$ and $z_2 \notin \mathrm{Scol}_1[L,u]$. Let $D_2 := \{y, x, u, w_1, \ldots, w_4, z_2, \ldots, z_5\}$. By (4.2.1) if both $z_1, z_5 \in \mathrm{Scol}_2[L,u]$ then $z_1 \in P_u$ (i.e., $z_1 = y$). Thus $\mathrm{Scol}_2[L,u] \subseteq D_2$. Assume that $z_1 \neq y$ and $z_1x \in E(G)$. By Lemma 4.1.1(2) applied to $yx \in E(T)$ and $z_1x \in E(G)$ we get $y \leq_T z_1$, and so $x \leq_T z_2$, a contradiction. Thus $z_1 \notin \mathrm{Scol}_1[L,x]$ and so $\mathrm{Scol}_1[L,x] \subseteq D_2$. For every $u' \in \mathrm{Scol}_1[L,u]$, the unique neighbor of u' in $P_{u'}$ is in D_2. Therefore $\mathrm{wcol}_2[L,u] \leq 11 + 3 * 4 = 23$. Assume that $w_5, z_5 \in \mathrm{Scol}_2[L,u]$, then $w_2 \leq_T x$, $z_2 \leq_T x$ and $w_2, z_2 \notin \mathrm{Scol}_1[L,u]$. Let $D_3 := \{y, x, u, w_2, \ldots, w_5, z_2, \ldots, z_5\}$. By (4.2.1) if $w_1, w_5 \in \mathrm{Scol}_2[L,u]$ then $w_1 = y$, similarly, if $z_1, z_5 \in \mathrm{Scol}_2[L,u]$ then $z_1 = y$. Thus $\mathrm{Scol}_2[L,u] \subseteq D_3$. We have seen that assuming $z_1 \neq y$ and $z_1 \in \mathrm{Scol}_1[L,x]$ led to a contradiction. Analogously we also have $w_1 \notin \mathrm{Scol}_1[L,x]$ if $w_1 \neq y$. Thus $\mathrm{Scol}_1[L,x] \subseteq D_3$. Since $w_2, z_2 \notin \mathrm{Scol}_1[L,u]$, for every $u' \in \mathrm{Scol}_1[L,u]$, the unique neighbor of u' in $P_{u'}$ is in D_3. Therefore $\mathrm{wcol}_2[L,u] \leq 11 + 3 * 4 = 23$. □

4.3.2 Example

In the above technic of defining the ordering L, choosing the drawing of G and the vertex v are arbitrary. When a drawing of G and a vertex v are fixed, we perform a breadth-first search of G starting from v. The graph G could have several breadth-first trees rooted at v. Any one of those trees is a candidate for T. When T is fixed, the spanning tree H of G^* is determined in a unique way. Then we perform a depth-first search F of H starting from the outer face f_0. This could be done in several ways

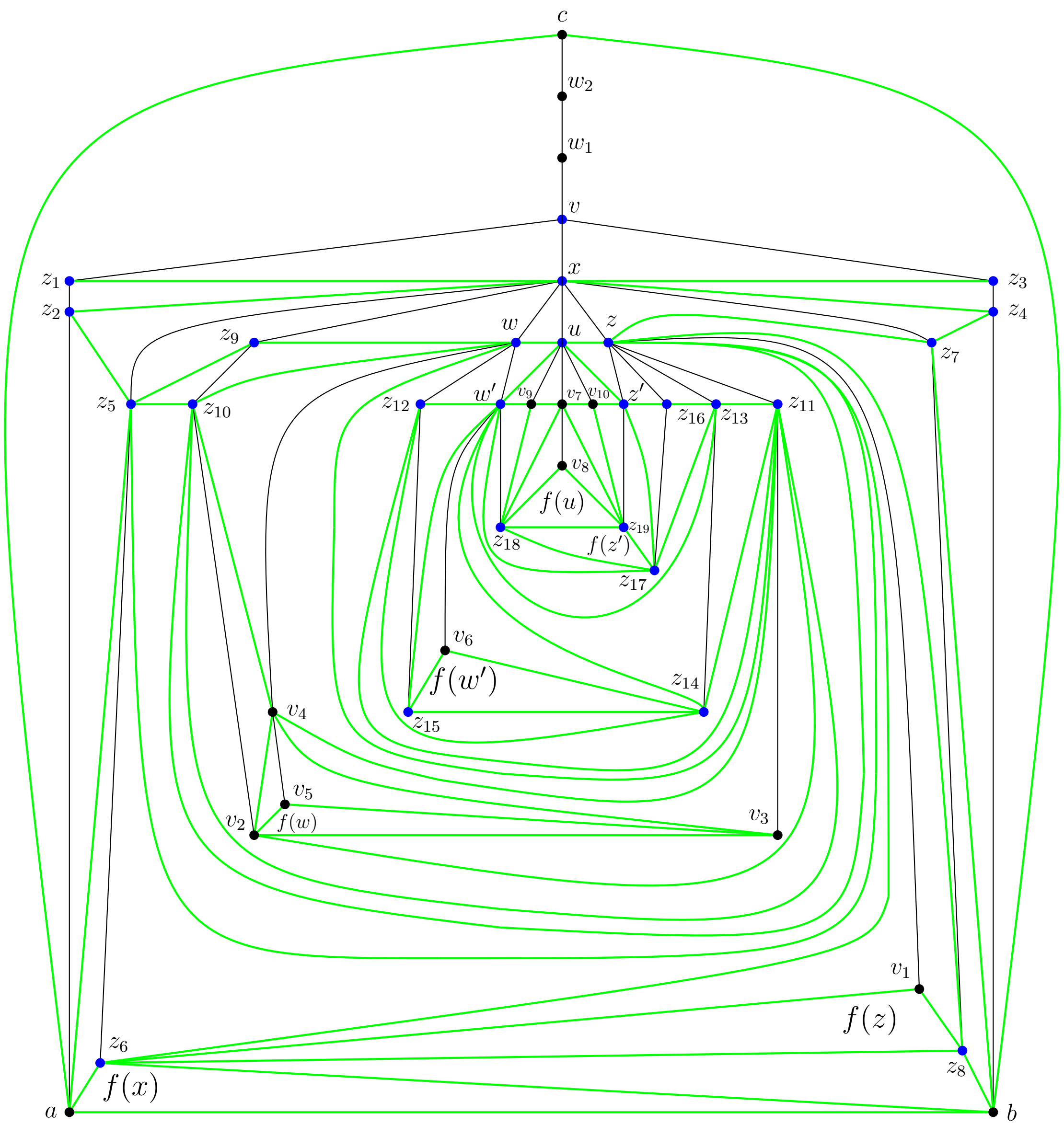

Figure 4.3. $G'[A' \cup \{c, w_1, w_2\}]$

(i.e., F is not unique). We give an example of a maximal planar graph G' with a fixed drawing, a vertex v, a breadth-first tree T' rooted at v and a depth-first search F' of H' rooted at the outer face f_0 such that $\mathrm{wcol}_2[G', L] = 26$ where L is a linear order constructed using the above technic.

We use Figure 4.3 on page 90 to illustrate the drawing $\widetilde{G'}$ of the graph G'. For simplicity, we write G' for $\widetilde{G'}$. The boundary of the outer face f_0 is the cycle $C := abca$. Let $P_1 := vz_1z_2a$, $P_2 := vz_3z_4b$ and $P_3 := vw_1w_2c$. The paths union $P_1 \cup P_2 \cup P_3$ divides the internal region of $R^2 \smallsetminus C$ into three identical regions: A (the region bounded by aP_1vP_2ba), B (the region bounded by bP_2vP_3cb) and C (the region bounded by aP_1vP_3ca). Let A' be the set of vertices of G' in region A or it's boundary (i.e., aP_1vP_2ba). Define the sets B' and C' in a similar way. We give in Figure 4.3 on page 90 the drawing of G' that is induced by $A' \cup \{c, w_1, w_2\}$, we denote the drawing induced by A' by $G'[A']$. The drawings $G'[B']$ and $G'[C']$ are each identical to $G'[A']$. Choose a breadth-first tree T' of G' such that T' satisfies the following. T' is rooted at v, it satisfies that $z_1 \leq_{T'} z_2 \leq_{T'} x \leq_{T'} w_1$, the subtree $T'[A']$ and the ordering $\leq_{T'}$ induced by A' coincide with Figure 4.4 on page 92. In Figure 4.3 on page 90 the edges $E(T')$ are colored black (the thin edges) and $E(G') \smallsetminus E(T')$ are colored green (the thick edges).

Let H' be the spanning subgraph of the dual graph $(G')^*$ of G' with $E(H') = \{e^* : e \in E(G') \smallsetminus E(T')\}$. Lemma 4.2.3 tells us that H' is a spanning tree of $(G')^*$. We will construct a depth-first search tree of H' starting from f_0. This tree is presented by an ordering F' of $V(H')$ (faces of G'). We write $f <_{F'} f'$ if f appears before f' in the ordering F'.

The root v is the first vertex in the ordering L. The vertices $z_1, z_2, z_3, z_4 \in X_0 = P_a \cup P_b \cup P_c$. It is not important which one of those four vertices comes first in L, so assume without loss of generality that $z_1 <_L z_2 <_L z_3 <_L z_4$. Let $f_1 \in V(H')$ be the face that is bounded by abz_6a. Note that $ab \in G'[f_0] \cap G'[f_1]$ and $ab \notin E(T')$, so $f_0f_1 \in E(H')$. Add f_1 to the ordering F' (i.e., f_0, f_1 is the first segment of F').

The vertices $x, z_5, z_6 \in X_1 \smallsetminus X_0 = V(P_a \cup P_b \cup P_{z_6}) \smallsetminus X_0$ and x is the closer vertex to v along P_{z_6} then z_5, z_6 in this order. Thus $f_1 = f(x) = f(z_5) = f(z_6)$ and

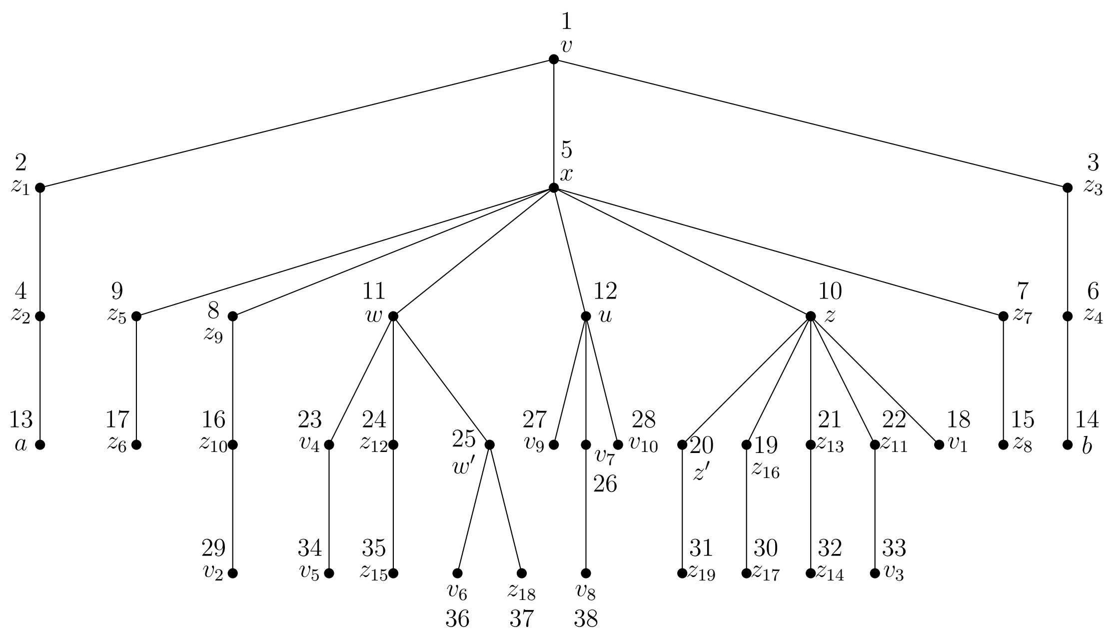

Figure 4.4. The Subtree $T'[A']$ and the Ordering $\leq_{T'}$

$z_4 <_L x <_L z_5 <_L z_6$. Let f_2 be the face of the cycle bz_6z_8b, so $X_2 = P_b \cup P_{z_6} \cup P_{z_8}$. Since $bz_6 \in (G'[f_1] \cap G'[f_2]) - T'$, $f_1f_2 \in E(H')$. Add f_2 to F'. The vertices $z_7, z_8 \in P_{z_8} - (X_0 \cup X_1)$. Thus $f_2 = f(z_7) = f(z_8)$ and $z_6 <_L z_7 <_L z_8$.

Let f_3 be the face of the cycle $z_6z_8v_1z_6$. The edge $z_6z_8 \in (G'[f_2] \cap G'[f_3]) - T'$, so $f_2f_3 \in E(H')$. Add f_3 to F'. As $z \in P_{v_1} - (X_0 \cup X_1 \cup X_2)$, $f_3 = f(z)$ and $z_8 <_L z$. Let f_4 be the face of the cycle zz_6v_1z. Since $z_6v_1 \in (G'[f_3] \cap G'[f_4]) - T'$, $f_3f_4 \in E(H')$. Add f_4 to F'. Let f_5 be the face of the cycle zz_5z_6z. The edge $zz_6 \in (G'[f_4] \cap G'[f_5]) - T'$, so $f_4f_5 \in E(H')$. Add f_5 to F'. Let f_6 be the face of the cycle $zz_5z_{10}z$. Since $zz_5 \in (G'[f_5] \cap G'[f_6]) - T'$, $f_5f_6 \in E(H')$. Add f_6 to F'. The vertices $z_9, z_{10} \in P_{z_{10}} - (X_0 \cup \ldots \cup X_5)$. So $f_6 = f(z_9) = f(z_{10})$ and $z <_L z_9 <_L z_{10}$.

Now we proceed in the same fashion with less details. Let f_7 be the face of the

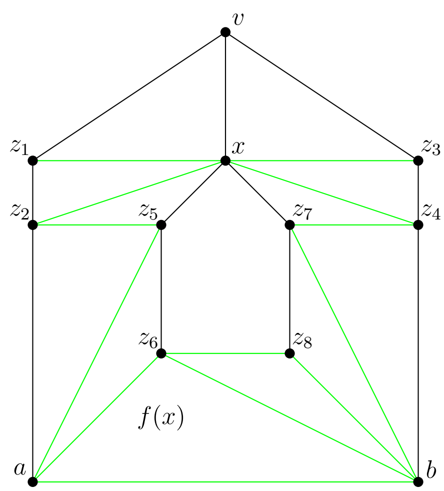

Figure 4.5. $f(x)$ and $\mathrm{Scol}_1[L, x]$

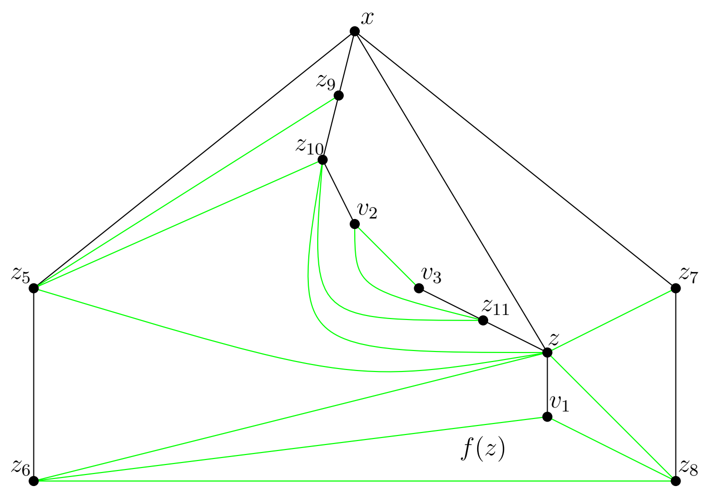

Figure 4.6. $f(z)$ and $\mathrm{Scol}_1[L, z]$

cycle $zz_{10}z_{11}z$. Add f_7 to F', this is possible because $f_6f_7 \in E(H')$. Then $f_7 = f(z_{11})$ and $z_{10} <_L z_{11}$. Let f_8 be the face of the cycle $v_2z_{10}z_{11}v_2$, f_9 the face of the cycle $v_2v_3z_{11}v_2$, f_{10} the face of the cycle $v_2v_3v_5v_2$. Add the segment f_8, f_9, f_{10} to F', this is possible because $f_7f_8f_9f_{10} \subseteq H'$. Then $f_{10} = f(w)$ and $z_{11} <_L w$. Let f_{11} be the face of the cycle $v_3v_4v_5v_3$, f_{12} the face of the cycle $v_3v_4z_{11}v_3$, f_{13} the face of the cycle $wv_4z_{11}w$, f_{14} the face of the cycle $wz_{11}z_{12}w$. Add the segment $f_{11}, f_{12}, f_{13}, f_{14}$ to F' (note that $f_{10}f_{11}f_{12}f_{13}f_{14} \subseteq H'$). Then $f_{14} = f(z_{12})$ and $w <_L z_{12}$.

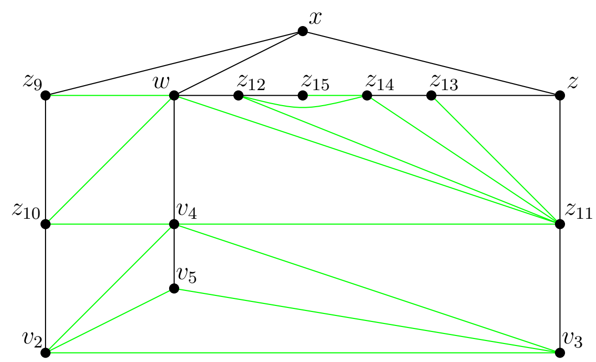

Figure 4.7. $f(w)$ and $\mathrm{Scol}_1[L, w]$

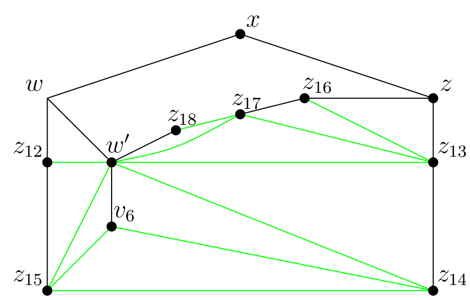

Figure 4.8. $f(w')$ and $\mathrm{Scol}_1[L, w']$

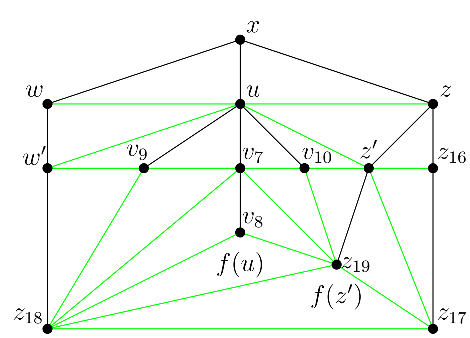

Figure 4.9. $f(z')$, $f(u)$, $\mathrm{Scol}_1[L, z']$ and $\mathrm{Scol}_2[L, u]$

Let f_{15} be the face of the cycle $z_{11}z_{12}z_{14}z_{11}$, f_{16} the face of $z_{12}z_{14}z_{15}z_{12}$, f_{17} the face of $v_6z_{14}z_{15}v_6$. Add the segment f_{15}, f_{16}, f_{17} to F'. Then $f_{15} = f(z_{13}) = f(z_{14})$, $f_{16} = f(z_{15})$, $f_{17} = f(w')$ and $z_{12} <_L z_{13} <_L z_{14} <_L z_{15} <_L w'$. Let f_{18} be the face of $w'v_6z_{14}w'$, f_{19} the face of $w'z_{13}z_{14}w'$, f_{20} the face of $w'z_{13}z_{17}w'$, f_{21} the face of $w'z_{17}z_{18}w'$. Add the segment $f_{18}, f_{19}, f_{20}, f_{21}$ to F'. Then $f_{20} = f(z_{16}) = f(z_{17})$, $f_{21} = f(z_{18})$ and $w' <_L z_{16} <_L z_{17} <_L z_{18}$.

Let f_{22} be the face of $z_{17}z_{18}z_{19}z_{17}$, f_{23} the face of $v_8z_{18}z_{19}v_8$. Add the segment f_{22}, f_{23} to F'. Then $f_{22} = f(z') = f(z_{19})$, $f_{23} = f(u)$ and $z_{18} <_L z' <_L z_{19} <_L u$. Now $F' = f_0, \ldots f_{23}$. Extend F' to a depth-first search tree of H'.

Now we are ready to show that $\mathrm{wcol}_2[L, u] = 26$. Theorem 4.3.1 tells us that $\mathrm{wcol}_2[L, u] \leq 26$. For each $s \in \{x, w, w', z, z'\}$, $s <_L u$ and $su \in E(G')$. So $\{u, x, w, w', z, z'\} \subseteq \mathrm{Scol}_1[L, u]$. For each $s \in \{z_{18}, z_{19}\}$, $s <_L u$ and there is a path of length two $stu \subseteq G'$ with $u <_L t$. So $\{z_{18}, z_{19}\} \subseteq \mathrm{Scol}_2[L, u]$, see Figure 4.9 on page 94. For each $s \in \{v, z_1, z_2, z_3, z_4\}$, $s <_L x$ and $sx \in E(G')$. So $\{v, z_1, z_2, z_3, z_4\} \subseteq \mathrm{Scol}_1[L, x] \subseteq \mathrm{Wcol}_2[L, u]$, see Figure 4.5 on page 93. For each $s \in \{z_5, z_6, z_7, z_8\}$, $s <_L z$ and $sz \in E(G')$. So $\{z_5, z_6, z_7, z_8\} \subseteq \mathrm{Scol}_1[L, z] \subseteq \mathrm{Wcol}_2[L, u]$, see Figure 4.6 on page 93. For each $s \in \{z_9, z_{10}, z_{11}\}$, $s <_L w$ and $sw \in E(G')$. So $\{z_9, z_{10}, z_{11}\} \subseteq \mathrm{Scol}_1[L, w] \subseteq \mathrm{Wcol}_2[L, u]$, see Figure 4.7 on page 94. For each $s \in \{z_{12}, z_{13}, z_{14}, z_{15}\}$, $s <_L w'$ and $sw' \in E(G')$. So $\{z_{12}, z_{13}, z_{14}, z_{15}\} \subseteq \mathrm{Scol}_1[L, w'] \subseteq \mathrm{Wcol}_2[L, u]$, see Figure 4.8 on page 94. For each $s \in \{z_{16}, z_{17}\}$, $s <_L z'$ and $sz' \in E(G')$. So $\{z_{16}, z_{17}\} \subseteq \mathrm{Scol}_1[L, z'] \subseteq \mathrm{Wcol}_2[L, u]$, see Figure 4.9 on page 94.

Thus $\{u, x, w, w', z, z', v, z_1, \ldots z_{19}\} \subseteq \mathrm{Wcol}_2[L, u]$, those vertices are colored blue in Figure 4.3 on page 90. This shows that $\mathrm{wcol}_2[L, u] = 26$.

Chapter 5

A NEW CHARACTERIZATION OF NOWHERE DENSE CLASSES

We give a new characterization of monotone nowhere dense classes by proving that the converse of Lemma 1.5.13 is also true. We will use a technique similar to the one used in Theorem 1.5.9.

Lemma. *(Lemma 1.5.13) Let $\mathcal{C}$ be a nowhere dense class of graphs. Then for every $r \in \mathbb{N}$ and $\epsilon > 0$ there exists an integer n_0 such that for every graph $G \in \mathcal{C}$ with $|V(G)| \geq n_0$ we have*

$$\mathrm{dom}_r(G) \leq |V(G)|^{2\epsilon}\alpha_{2r}(G).$$

We first construct a somewhere dense class of graphs $\mathcal{C}'$: For every positive integers $n \geq 4$ and $s \geq 1$, let $G_{\text{n,2s-1}}$ be the graph obtained from K_n by subdividing each edge $2s-1$ times. Let Y be the set of vertices with degree $n-1$ in $G_{n,2s-1}$; and let X be the set of the middle vertices of the paths corresponding to the edges of K_n in $G_{n,2s-1}$. Let $Q_{n,2s-1}$ be the graph obtained from $G_{\text{n,2s-1}}$ by adding a new vertex v adjacent to all the vertices in X. Dvorak [8] showed that $\mathrm{dom}_s(Q_{n,2s-1}) \geq \frac{n}{2}$ and $\alpha_{2s}(Q_{n,2s-1}) \leq 2$.

Let G and H be graphs. We write $H \preceq_r^t G$ if G contains a topological minor of H where each edge is subdivided at most r times. From Definition 1.5.1 and since every graph H is a subgraph of $K_{|H|}$, we have the following equivalent definition of somewhere dense class. A class of graphs $\mathcal{C}$ is somewhere dense if there exists $r \geq 1$ such that for all graphs H there exists a graph $G \in \mathcal{C}$ such that $H \preceq_r^t G$.

Now we are ready to prove the result.

Theorem 5.0.1. *Let $\mathcal{C}$ be a monotone class of graphs. The class $\mathcal{C}$ is nowhere dense if and only if (*) for every $k \in \mathbb{N}$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ with $|G| \geq n_0$ we have*

$$\mathrm{dom}_k(G) \leq |G|^{2\epsilon} \alpha_{2k}(G).$$

Proof. The forward implication is Lemma 1.5.13. We prove the backward implication. Assume for a contradiction that (*) holds, but $\mathcal{C}$ is somewhere dense. Then there exists $s \in \mathbb{N}$ such that for all graphs H there exists a graph $G \in \mathcal{C}$ such that $H \preceq_s^t G$.

Let $A = \{r(r+2) : r \in [s]\} \cup \{s\}$. For every $k \in A$ and $\epsilon = \frac{1}{20}$, (*) returns n_k. Let $n_0 = \max_{k \in A}\{n_k, 2s, 64\}$ and let $n' = \binom{n_0}{2} \cdot (2s-1) + n_0 + 1$; from Ramsey Theorem, there exists an $n'' \geq 2$ such that every n''-set X has a monochromatic n'-subset with respect to any $(s+1)$-coloring of $[X]^2$. Let $G \in \mathcal{C}$ such that $K_{n''} \preceq_s^t G$. Let G' be the depth-s topological minor of $K_{n''}$ contained in G. Since $\mathcal{C}$ is monotone, G' is also in $\mathcal{C}$.

Let P_{uv} denote the path corresponding to the edge $uv \in K_{n''}$ in G'. Define the coloring $c : E(K_{n''}) \to [s] \cup \{0\}$ such that $c(uv) = \|P_{uv}\| - 1$. Ramsey Theorem tells us that $K_{n''}$ contains a clique of size n' (i.e., $K_{n'}$) such that $c(e) = s'$ for every $e \in K_{n'}$ where $s' \in [s] \cup \{0\}$.

Case 1: $s' = 0$.

Then $K_{n'} \subseteq G' \in \mathcal{C}$; as $\mathcal{C}$ is monotone, $K_{n'} \in \mathcal{C}$ too. Since $|Q_{n_0,2s-1}| = \binom{n_0}{2} \cdot (2s-1) + n_0 + 1 = n'$, $Q_{n_0,2s-1} \subseteq K_{n'}$. Thus $Q_{n_0,2s-1} \in \mathcal{C}$. As $|Q_{n_0,2s-1}| \geq n_0$, $Q_{n_0,2s-1}$ satisfies (*). So

$$\begin{aligned}
\mathrm{dom}_s(Q_{n_0,2s-1}) &\leq |Q_{n_0,2s-1}|^{2\epsilon} \alpha_{2s}(Q_{n_0,2s-1}) \\
&\leq (n_0^3)^{2\epsilon} \alpha_{2s}(Q_{n_0,2s-1}) \\
&\leq \sqrt{n_0}\, \alpha_{2s}(Q_{n_0,2s-1}).
\end{aligned}$$

This contradicts the fact that $\alpha_{2s}(Q_{n_0,2s-1}) \leq 2$ and $\mathrm{dom}_s(Q_{n_0,2s-1}) \geq \frac{n_0}{2}$

Case 2: s' is a positive integer.

Since $|Q_{n_0,2s'-1}| = \binom{n_0}{2} \cdot (2s'-1) + n_0 + 1 \leq n'$, $Q_{n_0,2s'-1} \subseteq K_{n'}$. Let H' be the graph obtained from $K_{n'}$ by subdividing each edge s' times, and let Q' be the graph obtained from $Q_{n_0,2s'-1}$ by subdividing each edge s' times. Then $Q' \subseteq H' \subseteq G'$; since $\mathcal{C}$ is monotone, Q' is a graph in $\mathcal{C}$.

From assumption we have

$$\mathrm{dom}_{s'(s'+2)}(Q') \leq |Q'|^{2\epsilon} \alpha_{2s'(s'+2)}(Q').$$

Note that

$$\begin{aligned}
|Q'| &= |Q_{n_0,2s'-1}| + |E(Q_{n_0,2s'-1})| \cdot s' \\
&\leq \binom{n_0}{2} \cdot (2s'-1) + n_0 + 1 + \binom{n_0}{2} \cdot (2s'+1)s' \\
&\leq 2s'n_0^2 + 3s'^2 n_0^2 \\
&\leq n_0^3 + n_0^4 \\
&< 2n_0^4 \\
&< n_0^5
\end{aligned}$$

So $|Q'|^{2\epsilon} < \sqrt{n_0} \leq \frac{n_0}{8}$.

In the rest of the proof, we will show that $\alpha_{2s'(s'+2)}(Q') \leq 2$. and $\mathrm{dom}_{s'}(Q_{n_0,2s'-1}) \leq 2\,\mathrm{dom}_{s'(s'+2)}(Q')$ which gives a contradiction with the fact that $\mathrm{dom}_{s'}(Q_{n_0,2s'-1}) \geq \frac{n_0}{2}$.

Denote the set of vertices of degree $n_0 - 1$ in Q' by I; and let $I_{s'}$ be the set of vertices in Q' within distance at most s' from some vertex in I. The distance between any two vertices in $I_{s'}$ in the graph Q' is at most $2s'(s'+2)$. In addition, the distance between any two vertices in $V(Q')\backslash I_{s'}$ is at most $2s'(s'+1)$. Thus $\alpha_{2s'(s'+2)}(Q') \leq 2$.

Let $B \subseteq Q'$ such that $|B| = \mathrm{dom}_{s'(s'+2)}(Q')$ and every vertex $v \in Q'$ is within distance at most $s'(s'+2)$ from some vertex $u \in B$ in Q'. Let C be the set of vertices obtained from B by replacing each subdividing vertex w with the closest two branch vertices in Q'. Then $C \subseteq Q_{n_0,2s'-1}$ and $|C| \leq 2|B|$. In addition, every vertex

$v \in Q_{n_0,2s'-1}$ is within distance at most s' from some vertex $u \in C$. Hence,

$$\mathrm{dom}_{s'}(Q_{n_0,2s'-1}) \leq 2\,\mathrm{dom}_{s'(s'+2)}(Q'). \qquad \square$$

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