Representing Certain Continued Fraction AF Algebras as C^*-algebras of Categories of Paths and non-AF Groupoids

by

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ABSTRACT

C*-algebras of categories of paths were introduced by Spielberg in 2014 and generalize C*-algebras of higher rank graphs. An approximately finite dimensional (AF) C*-algebra is one which is isomorphic to an inductive limit of finite dimensional C*-algebras. In 2012, D.G. Evans and A. Sims proposed an analogue of a cycle for higher rank graphs and show that the lack of such an object is necessary for the associated C*-algebra to be AF. Here, I give a class of examples of categories of paths whose associated C*-algebras are Morita equivalent to a large number of periodic continued fraction AF algebras, first described by Effros and Shen in 1980. I then provide two examples which show that the analogue of cycles proposed by Evans and Sims is neither a necessary nor a sufficient condition for the C*-algebra of a category of paths to be AF.

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Chapter 1

INTRODUCTION

Directed graphs and their associated C^*-algebras have been objects of study for several decades (see, e.g., (12) for an introductory treatment). As it turns out, many C^*-algebras can be realized as graph algebras including, at least up to Morita equivalence, all AF algebras (2). One of the advantages of graph algebras is the ability to see certain structure in the associated C^*-algebra simply by observing structure in the graph, and one such example is that the presence or lack of a cycle in the graph completely determines whether or not the C^*-algebra is AF (8, Theorem 2.4).

In (7), Kumjian and Pask introduced the concepts of higher rank graphs and their C^*-algebras, where by "higher rank", we might think of the paths in the graph as being multi-dimensional. This generalized the concept of graph algebras, broadening the class of algebras which can be represented, but the more complicated structure meant many questions were much less tractable than in the 1-graph case. One such question is when the C^*-algebra of a higher rank graph is AF, which was addressed by Evans and Sims in (6) in which they give a necessary condition, the lack of a "generalized cycle" in the graph, but it was unclear if this condition was sufficient.

More recently, Spielberg introduced the idea of categories of paths and their C^*-algebras in (16) generalizing (among other things) higher rank graph C^*-algebras (see also (17)). It's natural to then ask when such a C^*-algebra is AF and it was this question, together with the work in (6) which was the original motivation for the this thesis.

This question is, for now, beyond our reach, although we show in the penultimate section that the notion of a generalized cycle, in the sense of Evans and Sims, is not the appropriate characterization in the setting of categories of paths. The bulk of this thesis is dedicated to a class of examples of categories of paths which give rise to a large number of continued fraction AF algebras (in the sense of (3)).

Chapter 2

PRELIMINARIES

In this section, we will establish some preliminary definitions and standard notation. We shall use $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T}$ for the integers, reals, complex numbers, and unit circle in the complex plane respectively. By $\mathbb{N}$ we mean the non-negative integers, and will use $\mathbb{Z}^+$ for the positive integers.

The C^*-algebra of a category of paths will be defined as that of an appropriate groupoid, and we will define both categories of paths and groupoids in terms of small categories.

Definition 2.0.1. A set Λ is called a **small category** if the following hold:

1. There is a subset $\Lambda^0 \subseteq \Lambda$, called the set of **units**.

2. There are **source** and **range** maps s, $r : \Lambda \to \Lambda^0$ such that $s|_{\Lambda^0} = \mathrm{id}_{\Lambda^0} = r|_{\Lambda^0}$.

3. Let $\Lambda^2 = \{(\alpha, \beta) \in \Lambda \times \Lambda : s(\alpha) = r(\beta)\}$. There is a map $\Lambda^2 \to \Lambda$, denoted $(\alpha, \beta) \mapsto \alpha\beta$, such that $r(\alpha\beta) = r(\alpha)$, $s(\alpha\beta) = s(\beta)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ whenever $s(\alpha) = r(\beta)$ and $s(\beta) = r(\gamma)$.

4. For all $\alpha \in \Lambda$, we have $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$.

We can now standardize some of the notation which will be used with groupoids and categories of paths.

Notation 2.0.2. Let Λ be a small category and $v, u \in \Lambda^0$. Then $v\Lambda = \{\mu \in \Lambda : r(\mu) = v\}$, $\Lambda u = \{\mu \in \Lambda : s(\mu) = u\}$, and $v\Lambda u = v\Lambda \cap \Lambda u = \{\mu \in \Lambda : r(\mu) = v \text{ and } s(\mu) = u\}$. For $\mu \in \Lambda$, let $\mu\Lambda = \{\mu\nu \in \Lambda : \nu \in s(\mu)\Lambda\}$ and $\Lambda\mu = \{\nu\mu \in \Lambda : \nu \in \Lambda r(\mu)\}$.

Definition 2.0.3. A **category of paths** is a small category Λ which contains no non-trivial inverses, but is left- and right-cancellative. That is, for any $\mu \in \Lambda \setminus \Lambda^0$, and any $\rho, \sigma, \tau \in \Lambda$

1. $\mu\nu, \nu\mu \notin \Lambda^0$ for any $\nu \in \Lambda$.

2. $\rho\sigma = \rho\tau$ implies $\sigma = \tau$.

3. $\rho\tau = \sigma\tau$ implies $\rho = \sigma$.

Definition 2.0.4. A category of paths Λ is **finitely aligned** if for every pair of elements $\sigma, \rho \in \Lambda$, there is a finite subset $F \subseteq \Lambda$ such that $\sigma\Lambda \cap \rho\Lambda = \cup_{\tau \in F} \tau\Lambda$.

Definition 2.0.5. A **groupoid** is a small category G in which every element has an inverse; i.e., for every $\mu \in G$, there is a $\nu \in G$ such that $\nu\mu = s(\mu)$ and $\mu\nu = r(\mu)$. We will denote such a ν as μ^{-1} (which is unique).

Definition 2.0.6. Let G be a groupoid, and $x \in G^0$. We define the **orbit** of x to be the set $r(Gx) = \{y \in G^0 : y = r(\mu) \text{ some } \mu \in Gx\}$. G is **transitive** if for any two $y, z \in G^0$, there is a $\mu \in G$ such that $s(\mu) = y$ and $r(\mu) = z$; equivalently, $r(Gw) = G^0$ for all $w \in G^0$. The **isotropy at x** is the set $xGx = \{\mu \in G : s(\mu) = r(\mu) = x\}$. The **isotropy of** G is $\mathrm{Iso}(G) := \cup_{x \in G^0} xGx \supseteq G^0$. The point x has **trivial** isotropy if $xGx = \{x\}$, and G is **principal** if all isotropy is trivial.

Chapter 3

A FIRST EXAMPLE

For a detailed treatment of categories of paths and their C^*-algebras, see (16). In this paper, we will be concerned with a number of examples.

3.1 Defining the Category of Paths Λ

Our first example will be the category of paths defined by the 1-graph in figure 3.1 together with the identification

$$\alpha_i \beta_{i+1} = \beta_i \alpha_{i+1}$$

for $i \geq 1$. We wish to make the above description more precise (and hopefully more clear), and we do so below. Let

$$\begin{aligned}
\Sigma_i &= \{\alpha_i, \beta_i, \gamma_i\} \text{ for } i \geq 1 \\
\Sigma &= \cup_{i \geq 1} \Sigma_i \\
\Sigma^* &= \{ \ \sigma_m \sigma_{m+1} \ldots \sigma_n \ : \ \sigma_i \in \Sigma_i, \ 1 \leq m \leq n \ \} \\
\Lambda^0 &= \{ \ v_i \ : \ i \geq 1 \ \}
\end{aligned}$$

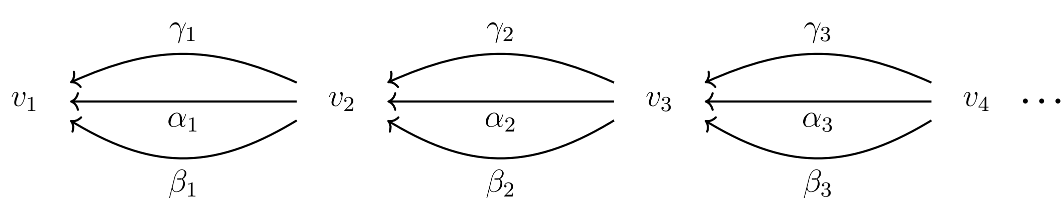

Figure 3.1: The 1-Graph of the First Example

Define $r', s' : \Sigma^* \to \Lambda^0$ by $r'(\sigma_m \dots \sigma_n) = v_m$, and $s'(\sigma_m \dots \sigma_n) = v_{n+1}$. For $\mu, \nu \in \Sigma^*$, define $\mu \approx \nu$ if one of the following holds:

$$(1)\ \mu = \sigma\alpha_i\beta_{i+1}\tau \text{ and } \nu = \sigma\beta_i\alpha_{i+1}\tau$$

$$(2)\ \mu = \sigma\beta_i\alpha_{i+1}\tau \text{ and } \nu = \sigma\alpha_i\beta_{i+1}\tau$$

$$(3)\ \mu = \nu$$

where $\sigma, \tau \in \Sigma^*$ with $s(\sigma) = v_i$ and $r(\tau) = v_{i+2}$.

Now define $\mu \sim \nu$ if there are $\Theta_1, \dots, \Theta_p \in \Sigma^*$ with $\Theta_1 = \mu$, $\Theta_p = \nu$, and $\Theta_i \approx \Theta_{i+1}$ for $1 \leq i < p$. It is clear that if $\mu \approx \nu$ then $r'(\mu) = r'(\nu)$ and $s'(\mu) = s'(\nu)$, and so if $\mu \sim \nu$, then $r'(\mu) = r'(\nu)$ and $s'(\mu) = s'(\nu)$. Further, for $\mu \in \Sigma^*$, we can define the **length** of μ to be $|\mu| = |\mu_m \dots \mu_n| = n - m + 1$ where $\mu_m \in \Sigma_m$ and $\mu_n \in \Sigma_n$. By convention, we will have $|v_i| = 0$ for $v_i \in \Lambda^0$. It is clear from the definitions of $\approx$ and $\sim$ that $|\mu| = |\nu|$ if $\mu \sim \nu$.

Proposition 3.1.1. *As defined above, $\sim$ is an equivalence relation.*

Proof. Suppose $\mu \sim \nu$ and $\nu \sim \sigma$. Then there are $\Theta_1, \dots, \Theta_p, \Theta'_1, \dots, \Theta'_r \in \Sigma^*$ such that $\mu = \Theta_1$, $\nu = \Theta_p = \Theta'_1$, and $\sigma = \Theta'_r$, with $\Theta_i \approx \Theta_{i+1}$ $1 \leq i < p$ and $\Theta'_i \approx \Theta'_{i+1}$ $1 \leq i < r$. For $1 \leq i < r$, let $\Theta_{p+i-1} = \Theta'_i$. Then $\mu = \Theta_1$, $\sigma = \Theta_{p+r-1}$, and $\Theta_i \approx \Theta_{i+1}$ for $1 \leq i < p + r - 1$ so that $\mu \sim \nu$, and $\sim$ is transitive

Reflexivity is clear so suppose that $\mu \sim \nu$. Then there are $\Theta_1, \dots, \Theta_p \in \Sigma^*$ with $\Theta_1 = \mu$, $\Theta_p = \nu$, and $\Theta_i \approx \Theta_{i+1}$ for $1 \leq i < p$. Now if $\Theta_i \approx \Theta_{i+1}$ we have three possibilities:

$$\text{(i) } \Theta_i = \Theta_{i+1}$$

$$\text{(ii) } \Theta_i = \mu_1\alpha_j\beta_{j+1}\mu_2$$
$$\text{and } \Theta_{i+1} = \mu_1\beta_j\alpha_{j+1}\mu_2$$

$$\text{(iii) } \Theta_i = \mu_1\beta_j\alpha_{j+1}\mu_2$$
$$\text{and } \Theta_{i+1} = \mu_1\alpha_j\beta_{j+1}\mu_2$$

In the case of (i), we have $\Theta_{i+1} \approx \Theta_i$ by (3) in the definition of $\approx$. In the cases of (ii) and (iii), we have $\Theta_{i+1} \approx \Theta_i$ by (2) and (1) respectively in the definition of $\approx$. Then $\Theta_{i+1} \approx \Theta_i$ for $1 \leq i < p$, so let $\Theta'_j = \Theta_{p-j+1}$ for $1 \leq j < p$. Then $\nu = \Theta'_1$, $\mu = \Theta'_p$, and $\Theta'_i \approx \Theta'_{i+1}$ for $1 \leq i < p$ and $\nu \sim \mu$ so that $\sim$ is an equivalence relation. □

Next, we look to characterize the equivalence classes in Σ^* under $\sim$. If $\mu \approx \nu$, it's clear from the definition that $\mu_i = \gamma_i$ iff $\nu_i = \gamma_i$ and hence if $\mu \sim \nu$, then $\mu_i = \gamma_i$ iff $\nu_i = \gamma_i$. Now suppose $\mu \approx \nu$ with $\mu \neq \nu$ and without loss of generality, $\mu = \eta_1 \alpha_i \beta_{i+1} \eta_2$ and $\nu = \eta_1 \beta_i \alpha_{i+1} \eta_2$. Further suppose that $\mu_j = \gamma_j = \nu_j$, $\mu_k = \gamma_k = \nu_k$ and $\mu_n \neq \gamma_n \neq \nu_n$ for $j < n < k$. Then $\mu = \rho_1 \gamma_j \Theta \gamma_k \rho_2$, and $\nu = \sigma_1 \gamma_j \Theta' \gamma_k \sigma_2$ where Θ and Θ' are words in α and β. We have three possibilities:

$$\begin{aligned}
(1) \quad \mu &= \rho'_1 \alpha_i \beta_{i+1} \rho''_1 \gamma_j \Theta \gamma_k \rho_2 \\
\nu &= \rho'_1 \beta_i \alpha_{i+1} \rho''_1 \gamma_j \Theta \gamma_k \rho_2 \\
(2) \quad \mu &= \rho_1 \gamma_j \theta_1 \alpha_i \beta_{i+1} \theta_2 \gamma_k \rho_2 \\
\nu &= \rho_1 \gamma_j \theta_1 \beta_i \alpha_{i+1} \theta_2 \gamma_k \rho_2 \\
(3) \quad \mu &= \rho_1 \gamma_j \Theta \gamma_k \rho'_2 \alpha_i \beta_{i+1} \rho''_2 \\
\nu &= \rho_1 \gamma_j \Theta \gamma_k \rho'_2 \beta_i \alpha_{i+1} \rho''_2
\end{aligned}$$

In any case, we have

$$\begin{aligned}
|\{ i : \mu_i = \alpha_i,\ j < i < k \}| &= |\{ i : \nu_i = \alpha_i,\ j < i < k \}| \\
\text{and } |\{ i : \mu_i = \beta_i,\ j < i < k \}| &= |\{ i : \nu_i = \beta_i,\ j < i < k \}|
\end{aligned}$$

That is, in (imprecise) words, if $\mu \approx \nu$, then between two edges which are γ's, and for which there are no other γ's in between, then μ and ν must have the same number of edges which are α's and the same number which are β's. Similar arguments show the following:

If there is a j such that $\mu_j = \gamma_j = \nu_j$ and $\mu_i \neq \gamma_i \neq \nu_i$ for $i > j$, then

$$\begin{aligned} |\{\ i\ :\ \mu_i = \alpha_i,\ j < i\ \}| &= |\{\ i\ :\ \nu_i = \alpha_i,\ j < i\ \}| \\ \text{and } |\{\ i\ :\ \mu_i = \beta_i,\ j < i\ \}| &= |\{\ i\ :\ \nu_i = \beta_i,\ j < i\ \}| \end{aligned}$$

If there is a j such that $\mu_j = \gamma_j = \nu_j$ and $\mu_i \neq \gamma_i \neq \nu_i$ for $i < j$, then

$$\begin{aligned} |\{\ i\ :\ \mu_i = \alpha_i,\ i < j\ \}| &= |\{\ i\ :\ \nu_i = \alpha_i,\ i < j\ \}| \\ \text{and } |\{\ i\ :\ \mu_i = \beta_i,\ i < j\ \}| &= |\{\ i\ :\ \nu_i = \beta_i,\ i < j\ \}| \end{aligned}$$

and if $\mu_i \neq \gamma_i \neq \nu_i$ for any i, then

$$\begin{aligned} |\{\ i\ :\ \mu_i = \alpha_i\ \}| &= |\{\ i\ :\ \nu_i = \alpha_i\ \}| \\ \text{and } |\{\ i\ :\ \mu_i = \beta_i\ \}| &= |\{\ i\ :\ \nu_i = \beta_i\ \}| \end{aligned}$$

Roughly speaking, if $\mu \approx \nu$, then μ and ν must have the same number of α's and β's as edges before the first γ and after the last γ, and if no edges in μ or ν are γ's, then μ and ν must agree on their numbers of α's and β's. Since the above holds when $\mu \approx \nu$, it is clear that it holds when $\mu \sim \nu$.

On the other hand, consider the equivalence class of

$$\mu = \rho\gamma\alpha^{m_1}\beta^{n_1}\alpha^{m_2}\beta^{n_2}\ldots\alpha^{m_k}\beta^{n_k}\gamma\sigma$$

where we suppress subscripts for ease of notation, and where $\alpha^m\beta^n = \alpha\ldots\alpha\beta\ldots\beta$ with $m \geq 0$ α's and $n \geq 0$ β's. Now let $\nu \in \Sigma^*$ with

$$\nu = \rho\gamma\alpha^{i_1}\beta^{j_1}\alpha^{i_2}\beta^{j_2}\ldots\alpha^{i_{k'}}\beta^{j_{k'}}\gamma\sigma \text{ where}$$
$$\textstyle\sum_{p=1}^{k'} i_p = \sum_{p=1}^{k} m_p$$
$$\textstyle\sum_{p=1}^{k'} j_p = \sum_{p=1}^{k} n_p$$

(that is, $\alpha^{i_1}\beta^{j_1}\ldots\alpha^{i_{k'}}\beta^{j_{k'}}$ is a permutation of $\alpha^{m_1}\beta^{n_1}\ldots\alpha^{m_k}\beta^{n_k}$). Then it is clear from the definition of $\approx$ and $\sim$, and the presentations of μ and ν above, that ν is in the equivalence class of μ.

Similarly, if $\mu = \alpha^{m_1}\beta^{n_1}\ldots\alpha^{m_k}\beta^{n_k}\gamma\mu'$, then

$$\nu = \alpha^{i_1}\beta^{j_1}\ldots\alpha^{i_{k'}}\beta^{j_{k'}}\gamma\mu' \text{ where}$$
$$\textstyle\sum_{p=1}^{k'} i_p = \sum_{p=1}^{k} m_p$$
$$\textstyle\sum_{p=1}^{k'} j_p = \sum_{p=1}^{k} n_p$$

is in the class of μ. If $\mu = \mu'\gamma\alpha^{m_1}\beta^{n_1}\alpha^{m_2}\beta^{n_2}\ldots\alpha^{m_k}\beta^{n_k}$, then

$$\nu = \mu'\gamma\alpha^{i_1}\beta^{j_1}\alpha^{i_2}\beta^{j_2}\ldots\alpha^{i_{k'}}\beta^{j_{k'}} \text{ where}$$
$$\textstyle\sum_{p=1}^{k'} i_p = \sum_{p=1}^{k} m_p$$
$$\textstyle\sum_{p=1}^{k'} j_p = \sum_{p=1}^{k} n_p$$

is in the class of μ. Finally, if $\mu = \alpha^{m_1}\beta^{n_1}\ldots\alpha^{m_k}\beta^{n_k}$, then

$$\nu = \alpha^{i_1}\beta^{j_1}\ldots\alpha^{i_{k'}}\beta^{j_{k'}} \text{ where}$$
$$\textstyle\sum_{p=1}^{k'} i_p = \sum_{p=1}^{k} m_p$$
$$\textstyle\sum_{p=1}^{k'} j_p = \sum_{p=1}^{k} n_p$$

is in the class of μ. These observations, together with the previous arguments show:

Proposition 3.1.2. *For $\mu, \nu \in \Sigma^*$, $\mu \sim \nu$ if and only if the following conditions hold:*

1. $r'(\mu) = r'(\nu)$ *and* $s'(\mu) = s'(\nu)$
2. $\mu_i = \gamma_i$ *iff* $\nu_i = \gamma_i$
3. *If* $\mu_j = \gamma_j = \nu_j$, $\mu_k = \gamma_k = \nu_k$, *and* $\mu_i \neq \gamma_i \neq \nu_i$ *for* $j < i < k$, *then*

$$|\{\ i\ :\ \mu_i = \alpha_i,\ j < i < k\ \}| \;=\; |\{\ i\ :\ \nu_i = \alpha_i,\ j < i < k\ \}|$$

$$\textit{and}\ |\{\ i\ :\ \mu_i = \beta_i,\ j < i < k\ \}| \;=\; |\{\ i\ :\ \nu_i = \beta_i,\ j < i < k\ \}|$$

4. If $\mu_j = \gamma_j = \nu_j$*, and* $\mu_i \neq \gamma_i \neq \nu_i$ *for* $j < i$*, then*

$$\begin{aligned} |\{ i : \mu_i = \alpha_i,\ j < i \}| &= |\{ i : \nu_i = \alpha_i,\ j < i \}| \\ \textit{and } |\{ i : \mu_i = \beta_i,\ j < i \}| &= |\{ i : \nu_i = \beta_i,\ j < i \}| \end{aligned}$$

5. If $\mu_j = \gamma_j = \nu_j$*, and* $\mu_i \neq \gamma_i \neq \nu_i$ *for* $i < j$*, then*

$$\begin{aligned} |\{ i : \mu_i = \alpha_i,\ i < j \}| &= |\{ i : \nu_i = \alpha_i,\ i < j \}| \\ \textit{and } |\{ i : \mu_i = \beta_i,\ i < j \}| &= |\{ i : \nu_i = \beta_i,\ i < j \}| \end{aligned}$$

6. If $\mu_i \neq \gamma_i \neq \nu_i$ *for any* i*, then*

$$\begin{aligned} |\{ i : \mu_i = \alpha_i \}| &= |\{ i : \nu_i = \alpha_i \}| \\ \textit{and } |\{ i : \mu_i = \beta_i \}| &= |\{ i : \nu_i = \beta_i \}| \end{aligned}$$

We can now define our category of paths.

Definition 3.1.3. Let $\Lambda = \Lambda^0 \sqcup \Sigma^*/_\sim$ (recall that $\Lambda^0 = \{v_i : i \geq 1 \}$). The set of units in Λ will be Λ^0. Define $r, s : \Lambda \to \Lambda^0$ as

$$\begin{aligned} r([\mu]) &= r'(\mu) \text{ for } \mu \in \Sigma^* \\ s([\mu]) &= s'(\mu) \text{ for } \mu \in \Sigma^* \\ r(v_i) &= v_i \text{ for } v_i \in \Lambda^0 \\ s(v_i) &= v_i \text{ for } v_i \in \Lambda^0 \end{aligned}$$

noting that we have shown the first line in the definition to be well-defined. Set $\Lambda^2 = \{ (\mu, \nu) \in \Lambda \times \Lambda \ : s(\mu) = r(\nu) \}$ (dropping the equivalence class notation for ease). For $(\mu, \nu) \in \Lambda^2$, define composition as follows: If $\mu = [\mu']$, $\nu = [\nu']$ with $\mu', \nu' \in \Sigma^*$, then $\mu\nu = [\mu'][\nu'] = [\mu'\nu']$ where $\mu'\nu'$ is concatenation in Σ^*. If $\mu = r(\nu)$, then $\mu\nu = \nu$, and if $\nu = s(\mu)$, then $\mu\nu = \mu$.

We check that composition is independent of the choice of μ' and ν' (noting that there is nothing to check if μ or ν is a unit). Suppose we have $\mu', \mu'', \nu'\nu'' \in \Sigma^*$ with $\mu' \sim \mu''$ and $\nu' \sim \nu''$. Then there are $\Theta_1, \ldots, \Theta_p, \Theta'_1, \ldots, \Theta'_r \in \Sigma^*$ with $\mu' = \Theta_1, \mu'' = \Theta_p, \nu' = \Theta'_1, \nu'' = \Theta'_r$ and $\Theta_i \approx \Theta_{i+1}$ for $1 \leq i \leq p$ and $\Theta'_i \approx \Theta'_{i+1}$ for $1 \leq i \leq r$. Let

$$\Phi_i \ = \ \begin{cases} \Theta_i\Theta'_1 & 1 \leq i \leq p \\ \Theta_p\Theta'_{i-p} & p < i \leq p+r \end{cases}$$

Then $\mu'\nu' = \Phi_1$, $\mu''\nu'' = \Phi_{p+r}$, and $\Phi_i \approx \Phi_{i+1}$ for $1 \leq i \leq p+r$, so that $\mu'\nu' \sim \mu''\nu''$.

Associativity of composition is clear since concatenation in Σ^* is associative, and the definition of composition forces $r(\mu)\mu = \mu = \mu s(\mu)$ for all $\mu \in \Lambda$, so that Λ is a small category.

Theorem 3.1.4. *Λ is a category of paths.*

Proof. We must show that Λ contains no inverses and is left- and right-cancellative. For $[\mu], [\nu] \in \Lambda \setminus \Lambda^0$, since $|\mu\nu|$ is independent of the choice of representatives, and clearly $|\mu\nu| = |\mu| + |\nu| \geq 2$, we conclude that $[\mu\nu]$ is not a unit.

Now suppose $\mu, \nu, \sigma \in \Sigma^*$ with $\mu\sigma \sim \nu\sigma$. Proposition 3.1.2 applied to $\mu\sigma$ and $\nu\sigma$ shows that conditions (2), (3), and (5) must hold for μ and ν. We must check that (1), (4), and (6) hold as well. To check (4), suppose there is a j such that $\mu_j = \gamma_j = \nu_j$ and $\mu_i \neq \gamma_i \neq \nu_i$ for and $i > j$. First suppose $\sigma_k = \gamma_k$ for some k with $\sigma_i \neq \gamma_i$ for and $i < k$. Then Proposition 3.1.2(3) applied to $\mu\sigma$ and $\mu\nu$ tells us that

$$|\{\ i\ :\ (\mu\sigma)_i = \alpha_i,\ j < i < k\ \}| \ = \ |\{\ i\ :\ (\nu\sigma)_i = \alpha_i,\ j < i < k\ \}|$$

$$\text{and } |\{\ i\ :\ (\mu\sigma)_i = \beta_i,\ j < i < k\ \}| \ = \ |\{\ i\ :\ (\nu\sigma)_i = \beta_i,\ j < i < k\ \}|$$

But clearly,

$$\begin{aligned} |\{\ i\ :\ (\mu\sigma)_i = \alpha_i,\ j < i < k\ \}| \ &= \ |\{\ i\ :\ \mu_i = \alpha_i,\ j < i\ \}| \\ &+ \ |\{\ i\ :\ \sigma_i = \alpha_i,\ i < k\ \}| \end{aligned}$$

and

$$\begin{aligned} |\{\ i\ :\ (\nu\sigma)_i = \alpha_i,\ j < i < k\ \}| \ &= \ |\{\ i\ :\ \nu_i = \alpha_i,\ j < i\ \}| \\ &+ \ |\{\ i\ :\ \sigma_i = \alpha_i,\ i < k\ \}| \end{aligned}$$

so that $|\{\ i\ :\ \mu_i = \alpha_i,\ j < i\ \}| = |\{\ i\ :\ \nu_i = \alpha_i,\ j < i\ \}|$. A nearly identical argument shows that $|\{\ i\ :\ \mu_i = \beta_i,\ j < i\ \}| = |\{\ i\ :\ \nu_i = \beta_i,\ j < i\ \}|$ so that (4) holds in this case.

Now if $\sigma_k \neq \gamma_k$ for any k, then a similar argument (applying (4) instead of (3) to $\mu\sigma$ and $\nu\sigma$) again shows that $|\{\ i\ :\ \mu_i = \alpha_i,\ j < i\ \}| = |\{\ i\ :\ \nu_i = \alpha_i,\ j < i\ \}|$ and $|\{\ i\ :\ \mu_i = \beta_i,\ j < i\ \}| = |\{\ i\ :\ \nu_i = \beta_i,\ j < i\ \}|$ so that (4) holds for μ and ν.

Showing that Proposition 3.1.2(6) holds for μ and ν is similar; assume $\mu_i \neq \gamma_i \neq \nu_i$ for any i. If $\sigma_k = \gamma_k$ with $\sigma_i \neq \gamma_i$ for $i < k$, we apply (5) to $\mu\sigma$ and $\nu\sigma$ and argue as above. If $\sigma_i \neq \gamma_i$ for any i, we apply (6) to $\mu\sigma$ and $\nu\sigma$ and again argue as above, so that (6) holds for μ and ν.

It's clear that $r'(\mu) = r'(\mu\sigma) = r'(\nu\sigma) = r'(\nu)$ using Proposition 3.1.2(1) for the middle equality and the definition of r' for the other two. Further, $s'(\mu) = r'(\sigma) = s'(\nu)$ so that (1) holds for μ and ν and hence $\mu \sim \nu$ and therefore Λ is right-cancellative. A symmetric argument shows that left-cancellation holds and we conclude that Λ is a category of paths. $\square$

Before moving on to $C^*(\Lambda)$, we give some preliminary definitions and prove a corollary to Proposition 3.1.2, which will be useful in the sequel.

Definition 3.1.5. Given a category of paths Γ, for $\lambda, \sigma \in \Gamma$, we say that λ is an **initial segment** of σ if there exists a $\sigma' \in s(\lambda)\Gamma$ such that $\lambda\sigma' = \sigma$, i.e. $\sigma \in \lambda\Gamma$. In such a case, we also say that σ **extends** λ. We can define a partial order $\leq$ on Γ by declaring $\lambda \leq \sigma$ if λ is an initial segment of σ. A subset $C \subseteq \Gamma$ is **directed** if it is directed under this partial order; that is, for all $\lambda, \sigma \in C$, $\lambda\Gamma \cap \sigma\Gamma \cap C \neq \emptyset$. C is said to be **hereditary** if for each $\lambda \in C$ and each initial segment λ' of λ, we have $\lambda' \in C$. Finally, $\mu, \nu \in \Gamma$ are said to have a **common extension** if there exist $\mu' \in s(\mu)\Gamma$ and $\nu' \in s(\nu)\Gamma$ such that $\mu\mu' = \nu\nu'$, i.e. if $\mu\Gamma \cap \nu\Gamma \neq \emptyset$.

Corollary 3.1.6. *Suppose $\mu, \nu \in \Lambda$ have a common extension and neither extends the other. Let $p = \min\{|\mu|, |\nu|\}$ and $q = \max\{\ j\ :\ \mu_j = \gamma_j = \nu_j\ \}$, where $q = 0$ if this set is empty (note that $q \leq p$). Then $\mu_j \neq \gamma_j$ and $\nu_j \neq \gamma_j$ for any $j > q$.*

Before proving the corollary, we will try to clarify its contents. Given μ and ν as in the corollary, if, say, μ is longer than ν, then it's conceivable that μ has a γ as an edge somewhere after the p^{th} edge (say $\mu_{p+1} = \gamma_{p+1}$). The corollary states that this can't happen (in fact, it makes a slightly stronger statement; neither path can have a γ as an edge after the q^{th} edge).

Proof. If $|\mu| = |\nu|$, the result is immediate from Proposition 3.1.2, so without loss of generality, assume $|\mu| > |\nu|$. Since μ and ν have a common extension, there are $\sigma, \tau \in \Lambda$ such that $\mu\sigma = \nu\tau$. By Proposition 3.1.2, $(\mu\sigma)_j = \gamma_j$ iff $(\nu\tau)_j = \gamma_j$ so that $\mu_j \neq \gamma_j \neq \nu_j$ for any $q < j \leq p = |\nu|$. Now suppose $\mu_k = \gamma_k$ for some $k > p$ with $\mu_j \neq \gamma_j$ for any $p < j < k$. By Proposition 3.1.2, $(\nu\tau)_k = \gamma_k$ and $(\nu\tau)_j \neq \gamma_j$ for any $q < j < k$.

Now if

$$\begin{aligned} |\{\ i\ :\ \nu_i = \alpha_i,\ q < i\ \}| &\leq |\{\ i\ :\ \mu_i = \alpha_i,\ q < i < k\ \}| \\ \text{and} \quad |\{\ i\ :\ \nu_i = \beta_i,\ q < i\ \}| &\leq |\{\ i\ :\ \mu_i = \beta_i,\ q < i < k\ \}|, \end{aligned}$$

then μ extends ν. Therefore we may suppose without loss of generality that

$$|\{\ i\ :\ \nu_i = \alpha_i,\ q < i\ \}| \ > \ |\{\ i\ :\ \mu_i = \alpha_i,\ q < i < k\ \}|$$

Then

$$\begin{aligned} |\{\ i\ :\ (\nu\tau)_i = \alpha_i,\ q < i < k\ \}| &\geq |\{\ i\ :\ \nu_i = \alpha_i,\ q < i\ \}| \\ &> |\{\ i\ :\ \mu_i = \alpha_i,\ q < i < k\ \}| \\ &= |\{\ i\ :\ (\mu\tau)_i = \alpha_i,\ q < i < k\ \}| \end{aligned}$$

and Proposition 3.1.2 implies $\nu\tau \neq \mu\sigma$. □

Recall from Definition 2.0.4 that Λ is finitely aligned if for every pair of elements $\sigma, \rho \in \Lambda$, there is a finite subset $F \subseteq \Lambda$ such that $\sigma\Lambda \cap \rho\Lambda = \cup_{\tau \in F} \tau\Lambda$.

Proposition 3.1.7. *Λ is finitely aligned.*

Proof. Fix $\sigma, \rho \in \Lambda$. If σ and ρ have no common extension then there is nothing to show, so assume $\sigma\Lambda \cap \rho\Lambda \neq \emptyset$. Suppose first that one extends the other, say, σ extends ρ. Then $\sigma\Lambda \subseteq \rho\Lambda$ so that $\rho\Lambda \cap \sigma\Lambda = \sigma\Lambda$.

Now suppose neither σ nor ρ extend the other and let $i \geq 1$ be such that $r(\sigma) = v_i = r(\rho)$. Let p and q be as in the statement of Corollary 3.1.6 so that, for some $\eta, \tau \in v_{i+q}\Lambda$, we have

$$\begin{aligned} \sigma &= \sigma_i \ldots \sigma_{i+q-1}\eta \\ \text{and } \rho &= \rho_i \ldots \rho_{i+q-1}\tau = \sigma_i \ldots \sigma_{i+q-1}\tau, \end{aligned}$$

where the second equation follows from an application of Proposition 3.1.2. By Corollary 3.1.6, η and τ have only α's or β's as edges. Set

$$\begin{aligned} r &= \max\{|\{j : \eta_j = \alpha_j\}|, |\{j : \tau_j = \alpha_j\}|\} \\ \text{and } s &= \max\{|\{j : \eta_j = \beta_j\}|, |\{j : \tau_j = \beta_j\}|\} \end{aligned}$$

If λ extends σ and ρ, then an application of Proposition 3.1.2 shows that $\lambda = \sigma_i \dots \sigma_{i+q-1}\alpha_{i+q} \dots \alpha_{i+q+r-1}\beta_{i+q+r} \dots \beta_{i+q+r+s-1}\lambda'$ for some $\lambda' \in v_{i+q+r+s}\Lambda$. Then $\sigma\Lambda \cap \rho\Lambda = \sigma_i \dots \sigma_{i+q-1}\alpha_{i+q} \dots \alpha_{i+q+r-1}\beta_{i+q+r} \dots \beta_{i+q+r+s-1}\Lambda$. □

3.2 Defining the C^*-algebra $C^*(\Lambda)$

To build a C^*-algebra from Λ, we will build a (Hausdorff, ètale) groupoid G from the data in Λ and then define $C^*(\Lambda)$ to be $C^*(G)$, constructed in the usual way. We will be following the construction and applying the results in (16, Section 7), which applies to finitely aligned categories of paths.

To build a groupoid G, we begin with the following definitions:

Definition 3.2.1. For $v \in \Lambda^0$, let X_v^* be the collection of all directed hereditary subsets of $v\Lambda$ and X_v^{**} the maximal elements of X_v^*. Set $X^* = \cup_{v\in\Lambda^0} X_v^*$ and $X^{**} = \cup_{v\in\Lambda^0} X_v^{**}$ and define a topology on X^* by taking as a basis the collection of sets

$$\mathcal{B}^* = \{\ Z(\lambda) \setminus \cup_{i=1}^n Z(\sigma_i)\ :\ \lambda, \sigma_i \in \Lambda \text{ and } \sigma_i \text{ extends } \lambda\ \}$$

where $Z(\lambda) = \{\ x \in X^*\ :\ \lambda \in x\ \}$. Let X be the closure of X^{**} in X^* and write X_v for $X \cap X_v^*$.

Remark 3.2.2. Each X_v is locally compact and Hausdorff. See the discussion preceding (16, Lemma 4.1) together with (16, Definition 7.5) and (16, Theorem 7.6) which identify the X_v.

For the infinite directed hereditary sets (those without a maximal element), we find it convenient to identify them with "infinite paths" in Λ; more precisely, with infinite words

$$x = x_i x_{i+1} \dots \text{ where } x_j \in v_j \Lambda v_{j+1}$$

Definition 3.2.3. Given an infinite word $x = x_i x_{i+1} \dots$, we define the **range** of x to be $\tilde{r}(x) := r(x_i) = v_i$. We will typically drop the tilde and simply write $r(x)$. Given two infinite words x and y, we say x is **equivalent** to y if the following conditions hold:

1. $r(x) = r(y)$

2. $x_i = \gamma_i$ iff $y_i = \gamma_i$

3. If $x_j = \gamma_j = y_j$, $x_k = \gamma_k = y_k$, and $x_i \neq \gamma_i \neq y_i$ for $j < i < k$, then

$$\begin{aligned} |\{\ i\ :\ x_i = \alpha_i,\ j < i < k\ \}| &= |\{\ i\ :\ y_i = \alpha_i,\ j < i < k\ \}| \\ \text{and } |\{\ i\ :\ x_i = \beta_i,\ j < i < k\ \}| &= |\{\ i\ :\ y_i = \beta_i,\ j < i < k\ \}| \end{aligned}$$

4. If $x_j = \gamma_j = y_j$, and $x_i \neq \gamma_i \neq y_i$ for $j < i$, then

$$\begin{aligned} |\{\ i\ :\ x_i = \alpha_i,\ j < i\ \}| &= |\{\ i\ :\ y_i = \alpha_i,\ j < i\ \}| \\ \text{and } |\{\ i\ :\ x_i = \beta_i,\ j < i\ \}| &= |\{\ i\ :\ y_i = \beta_i,\ j < i\ \}| \end{aligned}$$

5. If $x_j = \gamma_j = y_j$, and $x_i \neq \gamma_i \neq y_i$ for $i < j$, then

$$\begin{aligned} |\{\ i\ :\ x_i = \alpha_i,\ i < j\ \}| &= |\{\ i\ :\ y_i = \alpha_i,\ i < j\ \}| \\ \text{and } |\{\ i\ :\ x_i = \beta_i,\ i < j\ \}| &= |\{\ i\ :\ y_i = \beta_i,\ i < j\ \}| \end{aligned}$$

6. If $x_i \neq \gamma_i \neq y_i$ for any i, then

$$|\{ i : x_i = \alpha_i \}| = |\{ i : y_i = \alpha_i \}|$$
$$\text{and } |\{ i : x_i = \beta_i \}| = |\{ i : y_i = \beta_i \}|$$

We are identifying infinite words in a way analogous to the way we identify elements of Σ^* as in Proposition 3.1.2 with some slight changes. There is no reasonable sense of a source, and the sets in (4) and (6) may be infinite; in fact, at least one pair of equal sized sets must be. This clearly defines an equivalence relation and we will denote the class of x as $[x]$ when necessary, but will frequently write x when (we hope) there is no risk of confusion.

We make the identification between infinite words and infinite elements of X^* as follows: Given a class of an infinite word $[x]$ as above with $x = x_i x_{i+1} \dots$, we identify this class with the element $x' \in X^*$ where

$$x' = \{ y_i y_{i+1} \dots y_n : y \in [x], n \geq i \}$$

That is, x' is the collection of all initial segments of representatives of $[x]$. It is clear that x' is hereditary and infinite, both by construction. This set must also be directed since any two $\mu, \nu \in x'$ must be initial segments of representatives of the same equivalence class. By the way that class is defined, we can always find a long enough initial segment (of any representative) that extends both μ and ν. Therefore, $x' \in X^*$.

To realize an infinite directed hereditary set as an infinite word, fix such a set x and let $S = \{ i : \exists \sigma \in x, \sigma_i = \gamma_i \} \subseteq \mathbb{N}$. Given any two paths μ and ν in x with $p = |\mu| \leq |\nu|$, for these two paths to have a common extension, an application of Proposition 3.1.2 shows that $\{ i : \mu_i = \gamma_i \; i \leq p\} = \{ i : \nu_i = \gamma_i \; i \leq p\}$. Moreover, for any $i < j \in S$ with $k \notin S$ for $i < k < j$, the values $|\{ \ k \ : \ i < k < j, \sigma_k = \alpha_k \ \}|$ and $|\{ \ k \ : \ i < k < j, \sigma_k = \beta_k \ \}|$ are independent of the choice of σ, so long as $s(\sigma) = v_\ell$ with $\ell > j$. If S is infinite, then these observations define an infinite word z and we can realize x as z' as above.

If S is finite, let $k = \max S$. Then any $\sigma \in x$ with $s(\sigma) = v_\ell$ and $\ell > k + 1$ will have $\sigma_i \in \{ \ \alpha_i, \beta_i \ \}$ for $k < i < \ell$. Since x is infinite, we have three possibilities:

1. There is some $m \in \mathbb{N}$ such that for any $\sigma \in x$, $|\{i : \sigma_i = \alpha_i, i \geq k\}| \leq m$ and for all $n \in \mathbb{N}$ there is a $\sigma \in x$ with $|\{i : \sigma_i = \beta_i, i \geq k\}| = n$. That is, x contains only paths with at most m α's and arbitrarily many β's after γ_k.

2. There is some $m \in \mathbb{N}$ such that for any $\sigma \in x$, $|\{i : \sigma_i = \beta_i, i \geq k\}| \leq m$ and for all $n \in \mathbb{N}$ there is a $\sigma \in x$ with $|\{i : \sigma_i = \alpha_i, i \geq k\}| = n$. That is, x contains only paths with at most m β's and arbitrarily many α's after γ_k.

3. For all $c, d \in \mathbb{N}$, there exists $\sigma \in x$ with $s(\sigma) = v_\ell$, $\ell > k + 1$, and such that $|\{i : \sigma_i = \alpha_i, i \geq k\}| = c$ and $|\{i : \sigma_i = \beta_i, i \geq k\}| = d$. That is, x contains paths with arbitrarily many α's and β's after γ_k.

Each of these three define an infinite word, unique up to our identification. Take any $\sigma \in x$ with $s(\sigma) = v_{k+1}$. In the first case above, take the infinite word $z = \sigma\alpha_{k+1} \dots \alpha_{k+m}\beta_{k+m+1}\beta_{k+m+1} \dots$ and we can realize x as z' as done above. The second case is similar, and in the third case, we realize x as z' where

$z = \sigma\alpha_{k+1}\beta_{k+2}\alpha_{k+3}\beta_{k+4} \dots$

We can make a similar identification between finite directed hereditary sets and finite words, and under this identification, the cylinder set $Z(\lambda)$ represents the set of all classes of words which contain λ as an initial segment. The classes of infinite words where S (as above) is infinite, or as in case (3) when S is finite, represent the maximal elements. To see this, fix an infinite word x and S as above with either S infinite or as in (3). Suppose y is a directed hereditary set with $x \subsetneq y$ and fix $\sigma \in y \setminus x$. Then either there exists an $i \notin S$ such that $\sigma_i = \gamma_i$, or without loss of generality (the other cases follow similarly) there are $i < j \in S$ such that $\sigma_i = \gamma_i$, $\sigma_j = \gamma_j$, $\sigma_k \neq \gamma_k$ for any $i < k < j$ and for any $\tau \in x$ with $|\tau| \geq |\sigma|$,

$$|\{\ k\ :\ \sigma_k = \alpha_k,\ i < k < j\ \}| \neq |\{\ k\ :\ \tau_k = \alpha_k,\ i < k < j\ \}|.$$

In either case, pick $\tau \in x$ with $|\tau| \geq |\sigma|$. Since $x \subset y$, $\tau \in y$ and hence, there in some $\lambda \in y$ extending σ and τ, say $\sigma\sigma' = \lambda = \tau\tau'$. In the first case, Proposition 3.1.2 implies $\gamma_i = \sigma_i = \lambda_i = \tau_i \neq \gamma_i$, a contradiction. In the second case, the same proposition implies

$$\begin{aligned} |\{\ k\ :\ \sigma_k = \alpha_k,\ i < k < j\ \}| &= |\{\ k\ :\ \lambda_k = \alpha_k,\ i < k < j\ \}| \\ &= |\{\ k\ :\ \tau_k = \alpha_k,\ i < k < j\ \}| \end{aligned}$$

which is again a contradiction.

Now fix an infinite word x with S (as above) finite and suppose (1) holds. Let $k = \max S$ and $m = \min\{n \in \mathbb{N} : |\{i : \sigma_i = \alpha_i, i \geq k\}| \leq n \text{ for all } \sigma \in x\}$. Letting $r(x) = v_r$, define $y = x \cup \{x_r \dots x_{k-1}\gamma_k\alpha_{k+1} \dots \alpha_{k+m+2}\beta_{k+m+3} \dots \beta_{k+m+j+2} : j \geq 0\}$ so that $x \subsetneq y$. It is clear that y is hereditary; fix $\sigma \in y \setminus x$ and σ' an initial segment of σ. If $|\{i : \sigma'_i = \alpha_i, i \geq k\}| \leq m$, then $\sigma' \in x$ and if $|\{i : \sigma'_i = \alpha_i, i \geq k\}| = m+1$, then $\sigma' \in y \setminus x$. To see that y is directed, fix $\sigma, \tau \in y$. If $\sigma, \tau \in x$ then they have a common extension in $y \supset x$ since x is directed. If $\sigma, \tau \in y \setminus x$ and, without loss of generality, $|\sigma| \geq |\tau|$, then σ extends τ so they have a common extension in y. Suppose $\sigma \in x$ and $\tau \in y \setminus x$. If $|\sigma| \leq k - r$, then σ is an initial segment of $x_r \dots x_{k-1}\gamma_k$ so τ extends

σ. Then suppose $|\sigma| > k - r$ and that τ does not extend σ. Let

$$d = |\{ \ i \ : \ \sigma_i = \beta_i, \ k < i \ \}| - |\{ \ i \ : \ \tau_i = \beta_i, \ k < i \ \}|.$$

Then $d > 0$ since otherwise, τ would extend σ. Letting $s(\tau) = v_s$, the element $\tau\beta_s \ldots \beta_{s+d-1}$ extends σ (and τ) and is in y. Therefore y is directed and x is not a maximal element of X^*. The case where S is finite and (2) holds is similar, and finally, if x is a finite word with, say, $s(x) = v_p$, then the word $y = x\gamma_p$ represents a directed hereditary set which properly contains x.

Proposition 3.2.4. *The set X, which is the closure in X^* of the maximal elements described above, is the set of all infinite directed hereditary sets.*

Proof. Given an infinite word x and any $Z(\lambda) \setminus \cup_{i=1}^n Z(\sigma_i)$ containing x, $Z(\lambda) \setminus \cup_{i=1}^n Z(\sigma_i)$ must also contain one of the maximal directed hereditary sets as described above. To see this, note that $\lambda \in x$ but $\sigma_i \notin x$ for any i. Then there must be some $x_j \ldots x_m \in x$ which extends λ but none of the σ_i and which is longer than all of the σ_i. Then the infinite word $y = x_j \ldots x_m\gamma_{m+1}\alpha_{m+2}\beta_{m+3}\alpha_{m+4}\beta_{m+5} \ldots$ represents a maximal element (of the form (3) on page 18) which is contained in $Z(\lambda) \setminus \cup_{i=1}^n Z(\sigma_i)$. This follows since we have $\lambda \in y$ by construction, and if $\sigma_k \in y$, then σ_k and $x_j \ldots x_m\gamma_{m+1}$ have a common extension. Then there exist $\mu, \nu \in \Lambda$ such that $\sigma_k\mu = x_j \ldots x_m\gamma_{m+1}\nu$. Since $|x_j \ldots x_m| > |\sigma_k|$, $\mu_{m+1} = \gamma_{m+1}$ and Proposition 3.1.2 implies $\sigma_k\mu_n \ldots \mu_m = x_j \ldots x_m \in x$ where $v_n = s(\sigma_k)$, but this implies $\sigma_k \in x$.

On the other hand, if $x = x_i x_{i+1} \ldots x_m$ is a finite word representing a finite directed hereditary set, then $Z(x) \setminus (Z(x\gamma_{m+1}) \cup Z(x\alpha_{m+1}) \cup Z(x\beta_{m+1}))$ is an open set containing x but no infinite directed hereditary set, and hence, no maximal ones. Thus, the set X defined previously is precisely all infinite directed hereditary sets (or all infinite words under our identification). □

We make one final note which will be needed to define our groupoid. Given $\mu \in \Lambda$, we think of μ as inducing a map $\tau^\mu : X_{s(\mu)} \to X_{r(\mu)}$ where $x \in X_{s(\mu)}$ is sent to $\mu x \in X_{r(\mu)}$. Similarly, there is a map $\sigma^\mu : \mu X_{s(\mu)} \to X_{s(\mu)}$ defined by the equation $\sigma^\mu(\mu y) = y$.

We are now ready to finish defining our groupoid.

Definition 3.2.5. Let

$$G' = \cup_{v \in \Lambda^0} \Lambda_v \times \Lambda_v \times X_v$$

Define a relation $\sim$ on G' by $(\mu, \nu, x) \sim (\mu', \nu', x')$ if there exists a $z \in X$ and $\delta, \delta' \in \Lambda_{r(z)}$ such that

1. $x = \delta z$
2. $x' = \delta' z$
3. $\mu\delta = \mu'\delta'$
4. $\nu\delta = \nu'\delta'$

This is an equivalence relation by (16, Lemma 4.15).

Example 3.2.6. $[\alpha_1, \alpha_1, \beta_2 z]_\sim = [\beta_1, \beta_1, \alpha_2 z]_\sim$

Definition 3.2.7. The **groupoid** of Λ is the set $G = G'/_\sim$ with composable pairs

$$G^2 = \{ \ ([\mu, \nu, x], [\sigma, \tau, y]) \ : \ \nu x = \sigma y \ \}$$

with multiplication given as follows: For $([\mu, \nu, x], [\sigma, \tau, y]) \in G^2$, since $\nu x = \sigma y$, by (16, Lemma 4.12) there exist z, ξ, η such that $x = \xi z$, $y = \eta z$ and $\nu\xi = \sigma\eta$. Then

$$[\mu, \nu, x][\sigma, \tau, y] = [\mu\xi, \tau\eta, z]$$

Inversion is given by $[\mu, \nu, x]^{-1} = [\nu, \mu, x]$ with source and range maps defined by $s([\mu, \nu, x]) = [\nu, \nu, x]$ and $r([\mu, \nu, x]) = [\mu, \mu, x]$. The units are elements of the form $[\mu, \mu, x] = [r(\mu), r(\mu), \mu x]$, and we will frequently identify the unit $[r(x), r(x), x]$ with the element $x \in X$.

Example 3.2.8. (of composition)

$$[\gamma_1, \beta_1, \alpha_2 x][\alpha_1, \beta_1, \beta_2 x] = [\gamma_1\alpha_2, \beta_1\alpha_2, x][\alpha_1\beta_2, \beta_1\beta_2, x] = [\gamma_1\alpha_2, \beta_1\beta_2, x]$$

Definition 3.2.9. For $v \in \Lambda^0$, let $\mathcal{E}_v$ denote the collection of all sets of the form $Z(\mu) \setminus \cup_{i=1}^n Z(\nu_i)$ where $r(\mu) = v$, and $\nu_i \in \mu\Lambda$. Let $\mathcal{A}_v$ be the collection of all finite disjoint unions of sets in $\mathcal{E}_v$. Let $\mathcal{B} = \{[\mu, \nu, E] \ : \ s(\mu) = s(\nu), E \in \mathcal{A}_{s(\mu)}\}$ where $[\mu, \nu, E] = \{[\mu, \nu, x] \ : \ x \in E\}$.

Remark 3.2.10. $\mathcal{B}$ is a base for a topology on G making G a Hausdorff, ample, étale groupoid by (16, Proposition 4.10)

We proceed to build a C^*-algebra from G in the usual way: We make $C_c(G)$ into a *-algebra by defining multiplication as convolution:

$$f * g(a) = \textstyle\sum_{bc=a} f(b)g(c)$$

and involution is given by:

$$f^*(a) = \overline{f(a^{-1})}$$

We then take $C^*(G)$ to be the completion of $C_c(G)$ in the norm $||f|| = \sup_\pi ||\pi(f)||$ where the supremum is taken over all representations π of $C_c(G)$.

Definition 3.2.11. The **C^*-algebra of Λ** is defined as $C^*(\Lambda) \equiv C^*(G)$.

To analyze $C^*(G)$, we make a simplification which will allow us to determine $C^*(G)$ up to Morita equivalence. First, we let $X_1 = \{ \ [v_1, v_1, x] \ : \ x \in Z(v_1) \ \} \subseteq G^0$

and we will consider $C^*(G|_{X_1})$ where $G|_{X_1} = X_1 G X_1$, i.e., the subgroupoid of G whose elements have range and source in X_1. The reason for this is the following from (10, Example 2.7):

Theorem 3.2.12. *Given a Hausdorff, second-countable, locally compact topological groupoid G and $T \subseteq G^0$. If T is closed and transversal, i.e., for all $x \in G^0$, there is a $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) \in T$, and if $r|_{G_T}$ and $s|_{G_T}$ are open maps (where $G_T = \{\gamma \in G : s(\gamma) \in T\}$), then $C^*(G)$ is Morita equivalent to $C^*(G_{|T})$.*

We can see that X_1 is closed since $G^0 \setminus Z(v_1) = \cup_{i=2}^{\infty} Z(v_i)$ and to check that it is transversal, fix $x \in G^0$ with $r(x) = v_i$. Then $[\alpha_1 \ldots \alpha_{i-1}, v_i, x]$ has source x and range $\alpha_1 \ldots \alpha_{i-1} x \in X_1$. Since G is étale, r and s are open, and since $X_1 = Z(v_1)$ is clopen, G_{X_1} is open and therefore the restrictions of r and s to G_{X_1} are open maps. To keep notation clean, we will drop the subscript and simply write G for the groupoid with this now restricted unit space.

We will realize $C^*(G)$ as an inductive limit of sub-algebras. To this end, we let

$$G_1 := \langle\ [\sigma, \tau, x]\ :\ |\sigma| = |\tau| \leq 1\ \rangle.$$

Proposition 3.2.13. *The elements of G_1 are, up to equivalence, of the form:*

$$[\alpha_1 \ldots \alpha_k, \beta_1 \ldots \beta_k, w] \quad [\gamma_1 \alpha_2 \ldots \alpha_\ell, \beta_1 \ldots \beta_\ell, x]$$

$$[\alpha_1 \ldots \alpha_m, \gamma_1 \beta_2 \ldots \beta_m, y] \quad [\gamma_1 \alpha_2 \ldots \alpha_n, \gamma_1 \beta_2 \ldots \beta_n, z]$$

together with inverses (and the units from X_1).

Proof. Let G_1' denote the set of elements of G of the above form and their inverses. We will show that $G_1' = G_1$.

Direct computations show that

$$[\alpha_1, \beta_1, x_1][\alpha_1, \beta_1, x_2] \ldots [\alpha_1, \beta_1, x_n] = [\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, x]$$

for appropriate $x, x_1, \ldots, x_n$ (i.e., such that the multiplication is defined). Further, we have:

$$\begin{aligned}
[\gamma_1, \alpha_1, x][\alpha_1 \ldots \alpha_m, \beta_1 \ldots, \beta_m, y] &= [\gamma_1\alpha_2 \ldots \alpha_m, \beta_1 \ldots, \beta_m, y] \\
[\alpha_1 \ldots \alpha_m, \beta_1 \ldots, \beta_m, x][\beta_1, \gamma_1, y] &= [\alpha_1 \ldots \alpha_m, \gamma_1\beta_2 \ldots, \beta_m, x] \\
[\gamma_1, \alpha_1, x][\alpha_1 \ldots \alpha_m, \beta_1 \ldots, \beta_m, y][\beta_1, \gamma_1, z] &= [\gamma_1\alpha_2 \ldots \alpha_m, \gamma_1\beta_2 \ldots, \beta_m, y]
\end{aligned}$$

again, when the multiplication is defined. Similar computations give inverses, and hence $G_1' \subseteq G_1$.

On the other hand, it's clear that generators of G_1 are elements of G_1'. Suppose inductively that any product of n generators of G_1 is an element of G_1', $[\mu, \nu, y] \in G_1$ is such a product, and $[g, h, x]$ is a generator (and not a unit). To compute $[g, h, x][\mu, \nu, y]$, first suppose μ extends h. Then there is $\mu' \in \Lambda$ with $h\mu' = \mu$, so we may as well assume $\mu_1 = h$. By (16, Lemma 4.12), there are σ, τ, z such that $x = \sigma z$, $y = \tau z$, and $h\sigma = \mu\tau = h\mu_2 \ldots \mu_{|\mu|}\tau$. Cancellation then gives $\sigma = \mu_2 \ldots \mu_{|\mu|}\tau$ so that

$$\begin{aligned}
[g, h, x][\mu, \nu, y] &= [g, h, \sigma z][\mu, \nu, \tau z] \\
&= [g, h, \mu_2 \ldots \mu_{|\mu|}\tau z][\mu, \nu, \tau z] \\
&= [g\mu_2 \ldots \mu_{|\mu|}, h\mu_2 \ldots \mu_{|\mu|}, \tau z][\mu, \nu, \tau z] \\
&= [g\mu_2 \ldots \mu_{|\mu|}, \mu, y][\mu, \nu, y] \\
&= [g\mu_2 \ldots \mu_{|\mu|}, \nu, y]
\end{aligned}$$

which has one of the forms claimed (up to equivalence) since $\mu_i \neq \gamma_i$ and $\nu_i \neq \gamma_i$ for $i > 1$. To belabor the point about equivalence, note that if, say, $g = \alpha_1$ and $\nu_2 \ldots \nu_{|\nu|} = \alpha_2 \ldots \alpha_{|\nu|}$, then

$$
\begin{aligned}
[g\mu_2 \dots \mu_{|\mu|}, \nu, y] &= [\alpha_1\mu_2 \dots \mu_{|\mu|}, \nu, y] \\
&= [\mu_1 \dots \mu_{|\mu|-1}\alpha_{|\mu|}, \nu_1\alpha_2 \dots \alpha_{|\mu|}, y] \\
&= [\mu_1 \dots \mu_{|\mu|-1}, \nu_1\alpha_2 \dots \alpha_{|\mu|-1}, \alpha_{|\mu|}y]
\end{aligned}
$$

Now suppose μ does not extend h. For the multiplication to be defined, μ and h must have a common extension, which implies $h \neq \gamma_1$ and $\mu_1 \neq \gamma_1$; if both edges were γ_1 then μ would extend h, and if one were but not the other, then Proposition 3.1.2 would imply they have no common extension. Then $h = \alpha_1$ and $\mu = \beta_1 \dots \beta_{|\mu|}$ or $h = \beta_1$ and $\mu = \alpha_1 \dots \alpha_{|\mu|}$.

In the first case, we have σ, τ, z such that $x = \sigma z$, $y = \tau z$, and $\alpha_1\sigma = h\sigma = \mu\tau = \beta_1 \dots \beta_{|\mu|}\tau$. Fix k such that $(h\sigma)_k = \gamma_k = (\mu\tau)_k$ and $(h\sigma)_j \neq \gamma_j \neq (\mu\tau)_j$ for $j < k$, taking $k = |h\sigma| + 1 = |\mu\tau| + 1$ if no such k exists. Proposition 3.1.2 implies

$$
\begin{aligned}
|\{\ j\ :\ (h\sigma)_j = \alpha_j,\ j < k\ \}| &= |\{\ j\ :\ (\mu\tau)_j = \alpha_j,\ j < k\ \}| \geq 1 \\
\text{and } |\{\ j\ :\ (h\sigma)_j = \beta_j,\ j < k\ \}| &= |\{\ j\ :\ (\mu\tau)_j = \beta_j,\ j < k\ \}| \geq |\mu|.
\end{aligned}
$$

Then, again by Proposition 3.1.2, there exists σ', τ' such that

$$
\begin{aligned}
\alpha_1\beta_2 \dots \beta_{|\mu|+1}\sigma' &= \alpha_1\sigma \\
&= \beta_1 \dots \beta_{|\mu|}\tau \\
&= \beta_1 \dots \beta_{|\mu|}\alpha_{|\mu|+1}\tau' \\
&= \alpha_1\beta_2 \dots \beta_{|\mu|+1}\tau'
\end{aligned}
$$

and by cancellation, $\sigma' = \tau'$, $\sigma = \beta_2 \dots \beta_{|\mu|+1}\sigma'$, and $\tau = \alpha_{|\mu|+1}\tau'$. Then we have

$$
\begin{aligned}
[g,h,x][\mu,\nu,y] &= [g,h,\sigma z][\mu,\nu,\tau z] \\
&= [g\sigma,h\sigma,z][\mu\tau,\nu\tau,z] \\
&= [g\sigma,\mu\tau,z][\mu\tau,\nu\tau,z] \\
&= [g\sigma,\nu\tau,z] \\
&= [g\beta_2\ldots\beta_{|\mu|+1}\sigma',\nu\alpha_{|\mu|+1}\tau',z] \\
&= [g\beta_2\ldots\beta_{|\mu|+1}\tau',\nu\alpha_{|\mu|+1}\tau',z] \\
&= [g\beta_2\ldots\beta_{|\mu|+1},\nu\alpha_{|\mu|+1},\tau' z]
\end{aligned}
$$

which has the form claimed (noting that $\nu_i = \alpha_i$ for $i > 1$ by assumption, and $g \neq \alpha_1$ since $[g,h,x]$ is not a unit).

The case where $h = \beta_1$ is similar, as is the case for multiplication by a generator on the right. By induction, $G_1 \subseteq G_1'$ and hence $G_1 = G_1'$. □

3.3 Analysis of $C^*(G_1)$

To analyze $C^*(G_1)$, we partition the unit space X_1 into the following two sets:

$$
\begin{aligned}
U_1 &= \{x \in X_1 : x_i = \gamma_i, \text{ some } i > 1\} \\
F_1 &= X_1 \setminus U_1 = \{x \in X_1 : x_i \neq \gamma_i, \text{ any } i > 1\}
\end{aligned}
$$

We first observe that U_1 is open: given $x \in U_1$, we can write $x = x_1 \ldots x_{j-1}\gamma_j x_{j+1} \ldots$ where $j > 1$. Then $x \in Z(x_1 \ldots x_{j-1}\gamma_j) \subseteq U_1$. Moreover, U_1 is invariant for G_1, in the sense that any element in G_1 whose source is in U_1 also has range in U_1. This is easily seen given our observations in Proposition 3.2.13 about the elements of G_1. Then $F_1 = U_1^c$ is closed and invariant, so by (15, Proposition 4.3.2), $C^*(G_{1|U_1})$ is an ideal in $C^*(G_1)$ and the quotient by that ideal is $C^*(G_{1|F_1})$ yielding the short-exact sequence:

$$0 \longrightarrow C^*(G_{1|U_1}) \longrightarrow C^*(G_1) \longrightarrow C^*(G_{1|F_1}) \longrightarrow 0$$

We'll see below that the ideal is AF and the quotient type I so that $C^*(G_1)$ is nuclear and hence G_1 is amenable by (15, Theorem 4.1.5). We first analyze the ideal.

For each $\ell \geq 2$, let $\Omega_\ell = \{\ x_1 \dots x_{\ell-1}\ :\ x_i \neq \gamma_i,\ 1 < i < \ell\ \}$. Then define

$$\begin{aligned} E_\ell &:= \{x \in U_1 : x_\ell = \gamma_\ell, \text{ and } x_i \neq \gamma_i \text{ for any } 1 < i < \ell\} \\ &= \sqcup_{\sigma \in \Omega_\ell} Z(\sigma\gamma_\ell) \end{aligned}$$

which is evidently compact-open since the union is finite. Furthermore the E_ℓ are each invariant for G_1 and are pairwise disjoint, so we have $U_1 = \bigsqcup_{\ell \geq 2} E_\ell$. Now for $f \in C_c(G_{1|U_1})$ and $\ell \geq 2$ we have

$$\begin{aligned} f * \chi_{E_\ell}(a) &= \textstyle\sum_{bc=a} f(b)\chi_{E_\ell}(c) \\ &= \textstyle\sum_{bs(b)=a} f(b)\chi_{E_\ell}(s(b)) \\ &= f(a)\chi_{E_\ell}(s(a)) \\ &= \begin{cases} f(a) & s(a) \in E_\ell \\ 0 & else. \end{cases} \end{aligned}$$

Then we may write $f = \sum_{\ell \geq 2} f * \chi_{E_\ell}$ where the sum is finite since the support of f is compact. From this we can deduce that

$$C^*(G_{1|U_1}) \cong \textstyle\bigoplus_{\ell \geq 2} C^*(G_{1|E_\ell}).$$

Proposition 3.3.1. *For each* $\ell \geq 2$

$$C^*(G_{1|E_\ell}) \cong M_{2\ell-1} \otimes C(X_{\ell+1}),$$

where $X_{\ell+1} = Z(v_{\ell+1})$.

Proof. For $\ell \geq 2$, write

$$E_\ell = \{x_1 \dots x_{\ell-1} : x_i \neq \gamma_i, i > 1\} \times \{\gamma_\ell\} \times X_{\ell+1}.$$

For $\mu_0, \mu, \nu \in \{x_1 \dots x_{\ell-1} : x_i \neq \gamma_i, i > 1\}$ and $F \subseteq X_{\ell+1}$, compact-open, the maps

$$\chi_{[\mu,\nu,\{\gamma_\ell\}\times X_{\ell+1}]} \mapsto e_{\mu\nu} \otimes 1_{C(X_{\ell+1})}$$
$$\chi_{[\mu_0,\mu_0,F]} \mapsto e_{\mu_0\mu_0} \otimes \chi_F$$

define a *-isomorphism from a dense *-sub-algebra of $C^*(G_{1|E_\ell})$ to a dense *-sub-algebra of $M_{2\ell-1} \otimes C(X_{\ell+1})$. Note that there are $2\ell - 1$ choices for $\mu \in \{x_1 \dots x_{\ell-1}\}$. There are ℓ choices when $x_1 \neq \gamma_1$ since there are evidently ℓ words of the form $\mu = \alpha_1 \dots \alpha_j \beta_{j+1} \dots \beta_{\ell-1}$ (including one with no α's). Similarly, there are $\ell - 1$ choices for μ when $x_1 = \gamma_1$. □

Now we can conclude that

$$C^*(G_{1|U_1}) \cong \bigoplus_{\ell \geq 2}(M_{2\ell-1} \otimes C(X_{\ell+1}))$$

Next we turn our attention to the quotient $C^*(G_{1|F_1})$ and our first step is to decompose F_1. We first introduce some notation.

Definition 3.3.2. For each $i \geq 1$ and $j, k \geq 0$ with $j + k = \infty$, let $\eta^i(j,k)$ denote the (class of) infinite word(s) whose range is v_i and which have j edges which are α's, k edges which are β's, and no appearances of γ's.

As above, we will frequently identify $\eta^i(j,k)$ with the unit $[v_i, v_i, \eta^i(j,k)] \in G^0$, which we do now. Let

$$
\begin{aligned}
F_1^\infty &= \{\eta^1(\infty,\infty), \gamma_1\eta^2(\infty,\infty)\} \\
F_1^0 &= F_1 \setminus F_1^\infty = \{\eta^1(j,k), \gamma_1\eta^2(m,n) : j,k \text{ not both } \infty,\ m,n \text{ not both } \infty\}
\end{aligned}
$$

Note that F_1^∞ is closed (being finite) and invariant for G_1 so that F_1^0 is open (relatively in F_1) and invariant. Then similarly as above, we get a short exact sequence:

$$0 \longrightarrow C^*(G_{1|F_1^0}) \longrightarrow C^*(G_{1|F_1}) \longrightarrow C^*(G_{1|F_1^\infty}) \longrightarrow 0$$

We begin by examining the quotient $C^*(G_{1|F_1^\infty})$. Each of the two units has isotropy group isomorphic to $\mathbb{Z}$:

$$[v_1, v_1, \eta^1(\infty,\infty)]G_{1|F_1^\infty}[v_1, v_1, \eta^1(\infty,\infty)] = \{[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n : n \in \mathbb{Z}\}$$

and

$$[\gamma_1, \gamma_1, \eta^2(\infty,\infty)]G_{1|F_1^\infty}[\gamma_1, \gamma_1, \eta^2(\infty,\infty)] =$$
$$\{[\gamma_1, \alpha_1, \eta^2(\infty,\infty)][\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n[\alpha_1, \gamma_1, \eta^2(\infty,\infty)] : n \in \mathbb{Z}\}$$

We can see the first equation (and claimed isomorphism) above as follows: inductively, since

$$
\begin{aligned}
&[\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \eta^{n+1}(\infty,\infty)][\alpha_1, \beta_1, \eta^2(\infty,\infty)] \\
= \ &[\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \alpha_{n+1}\eta^{n+2}(\infty,\infty)][\alpha_1, \beta_1, \beta_2 \ldots \beta_{n+1}\eta^{n+2}(\infty,\infty)] \\
= \ &[\alpha_1 \ldots \alpha_{n+1}, \beta_1 \ldots \beta_n\alpha_{n+1}, \eta^{n+2}(\infty,\infty)][\alpha_1\beta_2 \ldots \beta_{n+1}, \beta_1 \ldots \beta_{n+1}, \eta^{n+2}(\infty,\infty)] \\
= \ &[\alpha_1 \ldots \alpha_{n+1}, \beta_1 \ldots \beta_{n+1}, \eta^{n+2}(\infty,\infty)],
\end{aligned}
$$

(and similarly if multiplying in the opposite order) it follows that

$$[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n = [\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \eta^{n+1}(\infty,\infty)]$$

for all $n \geq 0$. Similarly, for $n \leq -1$,

$$[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n = [\beta_1, \alpha_1, \eta^2(\infty,\infty)]^{-n} = [\beta_1 \ldots \beta_{-n}, \alpha_1 \ldots \alpha_{-n}, \eta^{-n+1}(\infty,\infty)]$$

For $0 \leq m \leq n$, we compute

$$\begin{aligned}
& [\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \eta^{n+1}(\infty,\infty)][\beta_1 \ldots \beta_m, \alpha_1 \ldots \alpha_m, \eta^{m+1}(\infty,\infty)] \\
= \; & [\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \eta^{n+1}(\infty,\infty)][\beta_1 \ldots \beta_m, \alpha_1 \ldots \alpha_m, \beta_{m+1} \ldots \beta_n \eta^{n+1}(\infty,\infty)] \\
= \; & [\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \eta^{n+1}(\infty,\infty)][\beta_1 \ldots \beta_n, \alpha_1 \ldots \alpha_m \beta_{m+1} \ldots \beta_n, \eta^{n+1}(\infty,\infty)] \\
= \; & [\alpha_1 \ldots \alpha_n, \alpha_1 \ldots \alpha_m \beta_{m+1} \ldots \beta_n, \eta^{n+1}(\infty,\infty)] \\
= \; & [\alpha_1 \ldots \alpha_{n-m}, \beta_1 \ldots \beta_{n-m}, \alpha_{n-m+1} \ldots \alpha_n \eta^{n+1}(\infty,\infty)] \\
= \; & [\alpha_1 \ldots \alpha_{n-m}, \beta_1 \ldots \beta_{n-m}, \eta^{n-m+1}(\infty,\infty)]
\end{aligned}$$

so that

$$[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{-m} = [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{n-m}$$

Similar computations show that

$$\begin{aligned}
[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{-m} [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n &= [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{n-m} \\
[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{m} [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{-n} &= [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{m-n} \\
\text{and } [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{-n} [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^m &= [\alpha_1, \beta_1, \eta^2(\infty,\infty)]^{m-n}
\end{aligned}$$

Now if $r[\mu, \nu, \eta^{|\mu|+1}(\infty,\infty)] = s[\mu, \nu, \eta^{|\mu|+1}(\infty,\infty)] = \eta^1(\infty,\infty)$, then $\mu_i, \nu_i \in \{\alpha_i, \beta_i\}$ for all $i \leq |\mu| = |\nu|$ so for some $j, k \geq 0$, up to equivalence,

$$\begin{aligned}
[\mu, \nu, \eta^{|\mu|+1}(\infty,\infty)] &= [\alpha_1 \ldots \alpha_j \beta_{j+1} \ldots \beta_{|\mu|}, \alpha_1 \ldots \alpha_k \beta_{k+1} \ldots \beta_{|\mu|}, \eta^{|\mu|+1}(\infty,\infty)] \\
&= \begin{cases} [\alpha_1 \ldots \alpha_{j-k}, \beta_1 \ldots \beta_{j-k}, \eta^{j-k+1}(\infty,\infty)] & j \geq k \\ [\beta_1 \ldots \beta_{j-k}, \alpha_1 \ldots \alpha_{j-k}, \eta^{k-j+1}(\infty,\infty)] & k > j. \end{cases}
\end{aligned}$$

Then

$$\begin{aligned} \mathbb{Z} &\cong \{[\alpha_1, \beta_1, \eta^2(\infty,\infty)]^n : n \in \mathbb{Z}\} \\ &= [v_1, v_1, \eta^1(\infty,\infty)] G_{1|F_1^\infty} [v_1, v_1, \eta^1(\infty,\infty)] \end{aligned}$$

The other claimed equivalence and isomorphism follow similarly noting that, using computations similar to those above

$$\begin{aligned} &[\gamma_1, \alpha_1, \eta^2(\infty,\infty)][\alpha_1, \beta_1, \eta^2(\infty,\infty)][\alpha_1, \gamma_1, \eta^2(\infty,\infty)] \\ (\star) \quad = \quad &\begin{cases} [\gamma_1\alpha_2 \ldots \alpha_{n+1}, \gamma_1\beta_2 \ldots \beta_{n+1}, \eta^{n+2}(\infty,\infty)] & n \geq 0 \\ [\gamma_1\beta_2 \ldots \beta_{n+1}, \gamma_1\alpha_2 \ldots \alpha_{n+1}, \eta^{n+2}(\infty,\infty)] & n < 0, \end{cases} \end{aligned}$$

and that if $r[\mu, \nu, \eta^{|\mu|+1}(\infty,\infty)] = s[\mu, \nu, \eta^{|\mu|+1}(\infty,\infty)] = \gamma_1\eta^2(\infty,\infty)$, then $\mu_1 = \nu_1 = \gamma_1$ and $\mu_i, \nu_i \in \{\alpha_i, \beta_i\}$ for all $1 < i \leq |\mu| = |\nu|$. Then reasoning similar to the previous case shows this element must one of the forms in $(\star)$ above.

In light of this, we might expect to find two copies of $C(\mathbb{T})$ in the quotient, and indeed this is the case:

Proposition 3.3.3. $C^*(G_{1|F_1^\infty}) \cong M_2 \otimes C(\mathbb{T})$

Proof. We first note that $G_1|_{F_1^\infty}$ is transitive since $[\gamma_1, \alpha_1, \eta^2(\infty,\infty)]$ has source $\eta^1(\infty,\infty)$ and range $\gamma_1\eta^2(\infty,\infty)$. Letting $u = \eta^1(\infty,\infty)$, so that $uG_1|_{F_1}u \cong \mathbb{Z}$, the claim follows from (10, Theorem 3.1). □

We can realize this isomorphism explicitly with the maps

$$\chi_{[\alpha_1,\beta_1,\eta^2(\infty,\infty)]} \mapsto e_{11} \otimes z$$
$$\chi_{[\gamma_1,\alpha_1,\eta^2(\infty,\infty)]} \mapsto e_{21} \otimes 1.$$

Next we turn to the ideal.

Proposition 3.3.4. $C^*(G_{1|F_1^0}) \cong \mathcal{K} \oplus \mathcal{K}$

Proof. Let $F_1^{0,1} = \{\eta^1(\infty, k), \gamma_1\eta^2(\infty, k) : k \in \mathbb{N}\}$ and $F_1^{0,2} = \{\eta^1(k, \infty), \gamma_1\eta^2(k, \infty) : k \in \mathbb{N}\}$ so that $F_1^0 = F_1^{0,1} \sqcup F_1^{0,2}$. Since $Z(\beta_1 \ldots \beta_k) \setminus Z(\beta_1 \ldots \beta_{k+1}) \cap F_1^{0,1} = \{\eta^1(\infty, k)\}$ and $Z(\gamma_1\beta_2 \ldots \beta_k) \setminus Z(\gamma_1\beta_2 \ldots \beta_{k+1}) \cap F_1^{0,1} = \{\gamma_1\eta^2(\infty, k)\}$, $F_1^{0,1}$ is discrete (likewise with $F_1^{0,2}$) and hence closed. Each is invariant for G_1, so that $C^*(G_1|_{F_1^0}) \cong C^*(G_1|_{F_1^{0,1}}) \oplus C^*(G_1|_{F_1^{0,2}})$.

Now let $u = \eta_1(\infty, 0)$ so that $uG_1|_{F_1^0}u = \{u\}$ (see Claim 3.5.12 for a full characterization of isotropy in G). The element $[\alpha_1 \ldots \alpha_k, \beta_1 \ldots \beta_k, \eta^{k+1}(\infty, j)]$ has source $\eta^1(\infty, j + k)$ and range $\eta^1(\infty, j)$ and $[\gamma_1\alpha_2 \ldots \alpha_k, \beta_1 \ldots \beta_k, \eta^{k+1}(\infty, j)]$ has source $\eta^1(\infty, j+k)$ and range $\gamma_1\eta^2(\infty, j)$ so that $G_1|_{F_1^0}$ is transitive. Then (10, Theorem 3.1) implies $C^*(G_1|_{F_1^0}) \cong \mathbb{C} \otimes \mathcal{K} \cong K$. An analogous argument shows $C^*(G_1|_{F_1^{0,2}}) \cong \mathcal{K}$ and the claim follows. □

Again, we can realize this isomorphism (or really $C^*(G_{1|F_1^0}) \cong M_2 \otimes (\mathcal{K} \oplus \mathcal{K})$) explicitly with the maps

$$\begin{aligned}
\chi_{[\beta_1 \ldots, \beta_j, \alpha_1 \ldots \alpha_j, \eta^{j+1}(0,\infty)]} &\mapsto e_{11} \otimes (0 \oplus e_{0j}) \\
\chi_{[\beta_1, \gamma_1, \eta^2(0,\infty)]} &\mapsto e_{12} \otimes (0 \oplus e_{00}) \\
\chi_{[\alpha_1 \ldots \alpha_j, \beta_1 \ldots, \beta_j, \eta^{j+1}(\infty,0)]} &\mapsto e_{11} \otimes (e_{0j} \oplus 0) \\
\chi_{[\alpha_1, \gamma_1, \eta^2(\infty,0)]} &\mapsto e_{12} \otimes (e_{00} \oplus 0).
\end{aligned}$$

Thus our previous short-exact sequence becomes:

$$0 \longrightarrow M_2 \otimes (\mathcal{K} \oplus \mathcal{K}) \xrightarrow{i} C^*(G_{1|F_1}) \xrightarrow{\pi} M_2 \otimes C(\mathbb{T}) \longrightarrow 0$$

This induces the six-term exact sequence

$$\begin{array}{ccccc} \mathbb{Z}^2 & \xrightarrow{i_{0*}} & K_0(C^*(G_{1|F_1})) & \xrightarrow{\pi_{0*}} & \mathbb{Z} \\ \delta_1 \uparrow & & & & \downarrow \\ \mathbb{Z} & \xleftarrow[\pi_{1*}]{} & K_1(C^*(G_{1|F_1})) & \xleftarrow[i_{1*}]{} & 0 \end{array}$$

The element $u = e_{11} \otimes z + e_{22} \otimes 1$ is a unitary whose class generates $K_1(M_2 \otimes C(\mathbb{T}))$. By (14, Proposition 9.2.2), if we can find a partial isometry v in $C^*(G_{1|F_1})$ such that $\pi(v) = u$ then $\delta_1[u]_1 = [1 - v^*v]_0 - [1 - vv^*]_0$. The element u corresponds to

$$u = \chi_{[\alpha_1, \beta_1, \eta^2(\infty,\infty)]} + \chi_{[v_1, v_1, \gamma_1 \eta^2(\infty,\infty)]}$$

and the desired partial isometry is

$$v = \chi_{[\alpha_1, \beta_1, \{\eta^2(j,k) : j+k=\infty\}]} + \chi_{[v_1, v_1, Z(\gamma_1) \cap F_1]}$$

It's easily checked that v is a partial isometry, and determining its image in the quotient amounts to checking the intersection of $[\alpha_1, \beta_1, \{\eta^2(j,k) : j + k = \infty\}]$ and $[v_1, v_1, Z(\gamma_1) \cap F_1]$ with $G_{1|F_1^\infty}$ and this indeed gives us the sets defining u.

We compute

$$1 - v^*v = \chi_{[v_1, v_1, \eta^1(\infty,0)]}$$
$$1 - vv^* = \chi_{[v_1, v_1, \eta^1(0,\infty)]}$$

representing rank-one projections in the left and right summands of $\mathcal{K} \oplus \mathcal{K}$ respectively. Then $\delta_1[u]_1 = (1, -1)$ and hence the image of the index map is $\mathbb{Z}(1, -1)$. This also shows that the image of π_{1*} is 0, so

$$K_1(C^*(G_{1|F_1})) = \ker(\pi_{1*}) = \mathrm{im}(i_{1*}) = 0.$$

We take $[\chi_{[v_1, v_1, \eta^1(\infty,0)]}]_0 = (1, 0)$ to be our second generator of $K_0(C^*(G_{1|F_1^0}))$ and since $\mathbb{Z}(1, -1)$ is the kernel of i_0^* we get a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(C^*(G_{1|F_1})) \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the normal sub-group in the sequence is generated by $(1,0) = [\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0$, and the quotient is generated by the rank-one projection $e_{11} \otimes 1 = [v_1, v_1, \eta^1(\infty,\infty)]$. Since the quotient is free abelian, the sequence splits, and the middle term is generated by the left generator and a lift of the right generator. An appropriate lift in this case is $[\chi_{[v_1,v_1,Z(\beta_1\beta_2)\cap F_1]}]_0$ since $Z(\beta_1\beta_2)\cap F_1 \cap F_1^\infty = \eta^1(\infty,\infty)$. The reason for this particular choice will become clear later.

We now turn our attention back to the short-exact sequence

$$0 \longrightarrow \bigoplus_{\ell \geq 2}(M_{2\ell-1} \otimes C(X_{\ell+1})) \longrightarrow C^*(G_1) \longrightarrow C^*(G_{1|F_1}) \longrightarrow 0$$

and investigate the corresponding K-theory. We saw that the quotient has trivial K_1 group, and as for the ideal, $X_{\ell+1}$ is a Cantor set, so that $C(X_{\ell+1})$ is AF and therefore the ideal is AF. Thus K_1 of the ideal is trivial, and therefore $K_1(C^*(G_1))$ is trivial. We know the K_0 group of the quotient, and as for the ideal, again since $X_{\ell+1}$ is a Cantor set, $K_0(C(X_{\ell+1})) \cong C(X_{\ell+1}, \mathbb{Z})$. We then have a short-exact sequence of the K_0 groups:

$$0 \longrightarrow \bigoplus_{\ell \geq 2} C(X_{\ell+1}, \mathbb{Z}) \longrightarrow K_0(C^*(G_1)) \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

Again we have a free abelian group in the quotient, so $K_0(C^*(G_1))$ is isomorphic to the direct sum of the left and right groups, generated by the generators of the left, and lifts of the generators on the right. Our lift of $[\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0$ is $[\chi_{[v_1,v_1,Z(\alpha_1)\setminus Z(\alpha_1\beta_2)]}]_0$ (since $Z(\alpha_1) \setminus Z(\alpha_1\beta_2) \cap F_1 = \eta^1(\infty, 0)$) and our lift of $[\chi_{[v_1,v_1,Z(\beta_1\beta_2)\cap F_1]}]_0$ is $[\chi_{[v_1,v_1,Z(\beta_1\beta_2)]}]_0$.

Our final step in analyzing $C^*(G_1)$ is to compute the positive cone of $K_0(C^*(G_1))$. First we establish some notation. Let

$$a \in K_0(C^*(G_1)) \cong (\bigoplus_{\ell \geq 2} C(X_{\ell+1}, \mathbb{Z})) \oplus \mathbb{Z}^2$$

We write

$$a = \sum_{\ell \geq 2} \sum_{i=1}^{m_\ell} c_{\ell,i}[\chi_{\{z_{\ell,i}\} \times F_{\ell,i}}]_0 + m[\chi_{Z(\alpha_1) \setminus Z(\alpha_1 \beta_2)}]_0 + n[\chi_{Z(\beta_1 \beta_2)}]_0$$

where

$$\{z_{\ell,i} : 1 \leq i \leq m_\ell\} = \{\ x_1 \ldots x_{\ell-1}\gamma_\ell \ :\ x_i \neq \gamma_i \text{ for } 1 < i < \ell\ \}$$

and $F_{\ell,i} \subseteq X_{\ell+1}$ is compact-open. We adopt the convention that for each ℓ, the collection $\{F_{\ell,i} \ :\ 1 \leq i \leq m_\ell\ \}$ form a partition of $X_{\ell+1}$, taking $c_{\ell,i} = 0$ if necessary, and refining the collection if $F_{\ell,i} \cap F_{\ell,j} \neq \emptyset$ for any $i \neq j$.

We also must make a few observations which will be used shortly when characterizing positive elements, as well as other times in the sequel. First note that, for any $j \geq 0$ and $k > j$,

$$\begin{aligned}
& Z(\alpha_1 \ldots \alpha_j) \setminus Z(\alpha_1 \ldots \alpha_j \beta_{j+1}) \\
= \quad & Z(\alpha_1 \ldots \alpha_j \gamma_{j+1}) \sqcup (Z(\alpha_1 \ldots \alpha_{j+1}) \setminus Z(\alpha_1 \ldots \alpha_{j+1}\beta_{j+2})) \\
= \quad & Z(\alpha_1 \ldots \alpha_j \gamma_{j+1}) \sqcup Z(\alpha_1 \ldots \alpha_{j+1}\gamma_{j+2}) \\
\sqcup \quad & (Z(\alpha_1 \ldots \alpha_{j+2}) \setminus Z(\alpha_1 \ldots \alpha_{j+2}\beta_{j+3})) \\
= \quad & \cdots \\
= \quad & (\sqcup_{\ell=j+1}^{k} Z(\alpha_1 \ldots \alpha_{\ell-1}\gamma_\ell)) \\
\sqcup \quad & (Z(\alpha_1 \ldots \alpha_k) \setminus Z(\alpha_1 \ldots \alpha_k \beta_{k+1}))
\end{aligned}$$

Also note that

$$\begin{aligned}
Z(\alpha_1 \ldots \alpha_j) &= Z(\alpha_1 \ldots \alpha_j \gamma_{j+1}) \sqcup (Z(\alpha_1 \ldots \alpha_{j+1}) \cup Z(\alpha_1 \ldots \alpha_j \beta_{j+1})) \\
&= Z(\alpha_1 \ldots \alpha_j \gamma_{j+1}) \sqcup Z(\alpha_1 \ldots \alpha_j \beta_{j+1}) \\
&\sqcup (Z(\alpha_1 \ldots \alpha_{j+1}) \setminus Z(\alpha_1 \ldots \alpha_j \beta_{j+1})) \\
&= Z(\alpha_1 \ldots \alpha_j \gamma_{j+1}) \sqcup Z(\alpha_1 \ldots \alpha_j \beta_{j+1}) \\
&\sqcup (Z(\alpha_1 \ldots \alpha_{j+1}) \setminus Z(\alpha_1 \ldots \alpha_{j+1} \beta_{j+2}))
\end{aligned}$$

Now, if $\mu = \mu_1 \ldots \mu_n$ and $\nu = \nu_1 \ldots \nu_n$ with $\mu_i, \nu_i \in \{\alpha_i, \beta_i\}$ for $i > 1$, then for $x, y \in z(v_{n+1})$, $[\mu, \nu, x], [\nu, \mu, y] \in G_1$ by Proposition 3.2.13 and $\chi_{Z(\mu)} \sim \chi_{Z(\nu)}$ in $C^*(G_1)$ via the partial isometry $\chi_{[\mu,\nu,Z(v_{n+1})]}$. Then together with the observations above, we have

$$(\dagger\dagger)\ [\chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)}]_0 = \textstyle\sum_{\ell=2}^{k} [\chi_{Z(\alpha_1 \ldots \alpha_{\ell-1}\gamma_\ell)}]_0 + [\chi_{Z(\alpha_1 \ldots \alpha_k)\setminus Z(\alpha_1 \ldots \alpha_k \beta_{k+1})}]_0$$

and

$$
\begin{aligned}
[\chi_{Z(\beta_1\beta_2)}]_0 &= [\chi_{Z(\alpha_1\alpha_2)}]_0 \\
&= [\chi_{Z(\alpha_1\alpha_2\gamma_3)}]_0 + [\chi_{Z(\alpha_1\alpha_2\alpha_3)\setminus Z(\alpha_1\alpha_2\alpha_3\beta_4)}]_0 + [\chi_{Z(\alpha_1\alpha_2\beta_3)}]_0 \\
&\quad \text{(by partitioning } Z(\alpha_1\alpha_2) \text{ as above)} \\
&= [\chi_{Z(\alpha_1\alpha_2\gamma_3)}]_0 + [\chi_{Z(\alpha_1\alpha_2\alpha_3)\setminus Z(\alpha_1\alpha_2\alpha_3\beta_4)}]_0 \\
&+ [\chi_{Z(\alpha_1\alpha_2\alpha_3)}]_0 \text{ (using equivalence as noted in the paragraph above)} \\
&= [\chi_{Z(\alpha_1\alpha_2\gamma_3)}]_0 + [\chi_{Z(\alpha_1\alpha_2\alpha_3)\setminus Z(\alpha_1\alpha_2\alpha_3\beta_4)}]_0 \\
&+ [\chi_{Z(\alpha_1\alpha_2\alpha_3\gamma_4)}]_0 + [\chi_{Z(\alpha_1\alpha_2\alpha_3\alpha_4)\setminus Z(\alpha_1\alpha_2\alpha_3\alpha_4\beta_5)}]_0 + [\chi_{Z(\alpha_1\alpha_2\alpha_3\beta_4)}]_0 \\
&= \cdots \\
&= \textstyle\sum_{\ell=3}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 + \sum_{j=3}^{k}[\chi_{Z(\alpha_1\ldots\alpha_j)\setminus Z(\alpha_1\ldots\alpha_j\beta_{j+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
&= \textstyle\sum_{\ell=3}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ \textstyle\sum_{j=3}^{k}(\sum_{\ell=j+1}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0) \\
&\quad \text{(partitioning } Z(\alpha_1\ldots\alpha_j)\setminus Z(\alpha_1\ldots\alpha_j\beta_{j+1}) \text{ as above)} \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0
\end{aligned}
$$

$$\begin{aligned}
&= \textstyle\sum_{\ell=3}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ \textstyle\sum_{\ell=4}^{k}(\ell-3)[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ (k-2)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
&= \textstyle\sum_{\ell=3}^{k}(\ell-2)[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ (k-2)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0
\end{aligned}$$

Then for $m, n \in \mathbb{Z}$, and for $\ell \geq 2$, $z_\ell = \alpha_1 \ldots \alpha_{\ell-1}\gamma_\ell$ and $X_{\ell+1} = \sqcup_i F_{\ell,i}$ a finite union of compact-open sets, we have

$$m[\chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)}]_0 + n[\chi_{Z(\beta_1\beta_2)}]_0$$

$$\begin{aligned}
= \quad & m\textstyle\sum_{\ell=2}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
+ \quad & m[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0
\end{aligned}$$

(by using (††))

$$\begin{aligned}
+ \quad & n\textstyle\sum_{\ell=3}^{k}(\ell-2)[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
+ \quad & n(k-2)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 + n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0
\end{aligned}$$

(from the previous computations)

$$
\begin{aligned}
&= \textstyle\sum_{\ell=2}^{k}(m+(\ell-2)n)[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ (m+(k-2)n)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
&+ n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
&= \textstyle\sum_{\ell=2}^{k}\sum_i(m+(\ell-2)n)[\chi_{\{z_\ell\}\times F_{\ell,i}}]_0 \\
&+ (m+(k-2)n)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 + n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0
\end{aligned}
$$

We are now ready to give a necessary and sufficient condition for positivity in K_0.

Proposition 3.3.5. *Fix $a \in K_0(C^*(G_1))$ and write*

$$
a = \sum_{\ell\geq 2}\sum_{i=1}^{m_\ell} c_{\ell,i}[\chi_{\{z_{\ell,i}\}\times F_{\ell,i}}]_0 + m[\chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)}]_0 + n[\chi_{Z(\beta_1\beta_2)}]_0
$$

with the conventions outlined above. Then $a \geq 0$ if and only if for all ℓ and i we have $c_{\ell,i} + m + (\ell-2)n \geq 0$.

Proof. First suppose that for all ℓ, i, we have $c_{\ell,i}+m+(\ell-2)n \geq 0$. Since the double sum in a must be finite, we know $c_{\ell,i} = 0$ for large enough ℓ. Then also we must have $n \geq 0$ since otherwise, we would have $m + (\ell-2)n < 0$ for large enough ℓ. Choose k such that $c_{\ell,i} = 0$ for all $\ell \geq k$ and using our computations above, write

$$
\begin{aligned}
a &= \textstyle\sum_{\ell\geq 2}\sum_{i=1}^{m_\ell} c_{\ell,i}[\chi_{\{z_{\ell,i}\}\times F_{\ell,i}}]_0 + m[\chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)}]_0 + n[\chi_{Z(\beta_1\beta_2)}]_0 \\
&= \textstyle\sum_{\ell=2}^{k}\sum_{i=1}^{m_\ell}(c_{\ell,i}+m+(\ell-2)n)[\chi_{\{z_{\ell,i}\}\times F_{\ell,i}}]_0 \\
&+ (m+(k-2)n)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 + n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0
\end{aligned}
$$

If $n > 0$, we may also choose k such that $m + (k-2)n \geq 0$ and the above shows that $a \geq 0$. If $n = 0$ then our initial assumption implies $m \geq 0$ (since it must hold when $c_{\ell,i} = 0$) and again, the above shows $a \geq 0$.

Now suppose we have $a \in K_0(C^*(G_1))_+$ and write a as above with the outlined conventions. Fix $x \in U_1$ and write

$$x = x_1 \ldots x_{\ell(x)-1} \gamma_{\ell(x)} x'$$

where $x_i \neq \gamma_i$ for $1 < i < \ell(x)$ and $x' \in X_{\ell(x)+1}$. Define a $*-$homomorphism

$$\pi_x : C^*(G_1|_{U_1}) \longrightarrow M_{2\ell(x)-1}$$

by taking

$$f \in C^*(G_1|_{U_1}) \cong \bigoplus\nolimits_{\ell \geq 2} (C(X_{\ell+1}, M_{2\ell-1}))$$

where $f = (f_2, f_3, \ldots, f_j, \ldots)$ with $f_j \in C(X_{j+1}, M_{2j-1})$ to $\pi_x(f) = f_{\ell(x)}(x')$. Since $C^*(G_1|_{U_1})$ is an ideal, π_x extends to a representation

$$\tilde{\pi}_x : C^*(G_1) \longrightarrow M_{2\ell(x)-1}$$

Let

$$\begin{aligned} \Omega &= \{ \ \mu_1, \ldots, \mu_{2\ell(x)-1} \ \} \\ &= \{ \ \alpha_1 \ldots \alpha_i \beta_{i+1} \ldots \beta_{\ell(x)-1} \gamma_{\ell(x)} \ : \ 0 \leq i < \ell(x) \ \} \\ &\sqcup \ \{ \ \gamma_1 \alpha_2 \ldots \alpha_i \beta_{i+1} \ldots \beta_{\ell(x)-1} \gamma_{\ell(x)} \ : \ 1 \leq i < \ell(x) \ \} \end{aligned}$$

and for each $1 \leq k \leq 2\ell(x) - 1$, define

$$p_k = \chi_{\{\mu_k\} \times X_{\ell(x)+1}}$$

so that $\tilde{\pi}_x(p_k) = e_{kk} \in M_{2\ell(x)-1}$. Let $i(x) \in \{1, \ldots, m_{\ell(x)}\}$ be such that $x' \in F_{\ell(x),i(x)}$ and let $k(\alpha) \in \{1, \ldots, 2\ell(x) - 1\}$ be such that $\mu_{k(\alpha)} = \alpha_1 \ldots \alpha_{\ell(x)-1} \gamma_{\ell(x)}$. Then since $a \geq 0$, we have

$$
\begin{aligned}
0 &\leq \tilde{\pi}_x(a) \\
&= \tilde{\pi}_x(\textstyle\sum_{\ell\geq 2}\sum_{i=1}^{m_\ell} c_{\ell,i}[\chi_{\{z_{\ell,i}\}\times F_{\ell,i}}]_0 + m[\chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)}]_0 + n[\chi_{Z(\beta_1\beta_2)}]_0) \\
&= \textstyle\sum_{\ell\geq 2}\sum_{i=1}^{m_\ell} c_{\ell,i}[\pi_x(\chi_{\{z_{\ell,i}\}\times F_{\ell,i}})]_0 \\
&+ m[\pi_x(\textstyle\sum_{k=1}^{2\ell(x)-1} p_k * \chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)})]_0 \\
&+ n[\pi_x(\textstyle\sum_{k=1}^{2\ell(x)-1} p_k * \chi_{Z(\beta_1\beta_2)})]_0 \\
&= c_{\ell(x),i(x)}[\pi_x(\chi_{\{z_{\ell(x),i(x)}\}\times F_{\ell(x),i(x)}})]_0 \\
&+ m[\pi_x(p_{k(\alpha)})]_0 \\
&+ n[\pi_x(\textstyle\sum_{\{k:\mu_k\in Z(\beta_1\beta_2)\}} p_k)]_0 \\
&= c_{\ell(x),i(x)} + m + (\ell(x)-2)n
\end{aligned}
$$

noting that there are $\ell(x)-2$ choices for $\mu_k \in Z(\beta_1\beta_2)$; namely,

$$\mu_k \in \{\ \alpha_1 \ldots \alpha_i \beta_{i+1} \ldots \beta_{\ell(x)-1}\gamma_{\ell(x)}\ :\ 0 \leq i < \ell(x)\ \}$$

where $0 \leq i < \ell(x)-2$. Finally, if $a \in K_0(C^*(G_1))$ is positive, then for any choice of i and ℓ there exists $x \in X$ such that $i = i(x)$ and $\ell = \ell(x)$, so the inequalities above hold. □

For now, this completes our analysis of $C^*(G_1)$ and we turn our attention to $C^*(G_i)$ for $i \geq 1$.

3.4 Definition and Analysis of $C^*(G_i)$

Our definition and analysis of $C^*(G_i)$ will follow a very similar path as that of G_1. First, define

$$G_i = \langle\ [\sigma, \tau, x]\ :\ |\sigma| = |\tau| \leq i\ \rangle$$

We will adopt the convention that for $[\sigma, \tau, x] \in G_i$, if

$$|\{\ j\ :\ \sigma_j = \alpha_j \text{ and } \sigma_k \neq \gamma_k, k > j\ \}| = m > 0$$
$$|\{\ j\ :\ \tau_j = \alpha_j \text{ and } \tau_k \neq \gamma_k, k > j\ \}| = n > 0$$

then we write $[\sigma, \tau, x] = [\sigma', \tau', \alpha_j \ldots \alpha_{j+p-1}x]$ where $p = \min\{m, n\}$. In words, if σ and τ both have α's as edges after the last γ, we take the representative of $[\sigma, \tau, x]$ where we "factor" the common number of these edges into the tail. We do the same for shared edges which are β's. Now, define

$$G_i' = \{\ [\mu\alpha_{|\mu|+1} \ldots \alpha_m, \nu\beta_{|\nu|+1} \ldots \beta_m, x], [\sigma\beta_{|\sigma|+1} \ldots \beta_n, \tau\alpha_{|\tau|+1} \ldots \alpha_n, y]\ :\ |\mu|, |\nu|, |\sigma|, |\tau| \leq i, \mu_{|\mu|} = \gamma_{|\mu|}, \nu_{|\nu|} = \gamma_{|\nu|}, \sigma_{|\sigma|} = \gamma_{|\sigma|}, \tau_{|\tau|} = \gamma_{|\tau|}\ \}$$

Proposition 3.4.1. *Given our convention for elements in G_i, we claim that $G_i = G_i'$.*

Proof. Direct computations show that

$$[\alpha_1, \beta_1, x_1][\alpha_1, \beta_1, x_2] \ldots [\alpha_1, \beta_1, x_n] = [\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, x]$$

for appropriate $x, x_1, \ldots, x_n$ (i.e., such that the multiplication is defined). Moreover,

$$[\sigma, \alpha_1 \ldots \alpha_j, x][\alpha_1 \ldots \alpha_n, \beta_1 \ldots, \beta_n, y][\beta_1 \ldots \beta_k, \tau, z] = [\sigma\alpha_{j+1} \ldots \alpha_n, \tau\beta_{k+1} \ldots, \beta_n, y]$$

for $j, k \leq i, n$ (again, when the multiplication is defined). Similar computations give inverses, and hence $G_i' \subseteq G_i$.

Conversely, it's clear that generators of G_i belong to G_i'. We will show that G_i' is closed under multiplication by generators of G_i. Suppose $[\sigma, \tau, x] \in G_i'$ and let $[\mu, \nu, y]$ be a generator of G_i. Consider $[\mu, \nu, y][\sigma, \tau, x]$ (assuming throughout that the product is defined).

If σ extends ν then we may as well assume $\sigma = \nu\sigma_{|\nu|+1} \ldots \sigma_{|\sigma|}$ and we have

$$\begin{aligned}[\mu,\nu,y][\sigma,\tau,x] &= [\mu\sigma_{|\mu|+1}\ldots\sigma_{|\sigma|},\nu\sigma_{|\mu|+1}\ldots\sigma_{|\sigma|},x][\sigma,\tau,x] \\ &= [\mu\sigma_{|\mu|+1}\ldots\sigma_{|\sigma|},\tau,x].\end{aligned}$$

Similarly, if ν extends σ, then writing $\nu = \sigma\nu_{|\sigma|+1}\ldots\nu_{|\nu|}$ we have

$$[\mu,\nu,y][\sigma,\tau,x] = [\mu,\tau\nu_{|\tau|+1}\ldots\nu_{|\nu|},y].$$

After factoring common α's or β's after the last γ into the tail, both of these have the desired form. Namely, in the first two coordinates, neither has γ as an edge after the i^{th} edge since neither μ,ν,σ, nor τ do.

Now suppose neither of σ or ν extends the other. Let $p = \min\{|\sigma|,|\nu|\}$ and $q = \max\{\ j\ :\ \sigma_j = \gamma_j = \nu_j, j \leq p\ \}$. For the product to be defined, σ and ν must have a common extension, and this fact together with an application of Proposition 3.1.2 implies that

$$\sigma = \sigma_1\ldots\sigma_q\sigma'$$
$$\nu = \nu_1\ldots\nu_q\nu'$$

with $\sigma_1\ldots\sigma_q \sim \nu_1\ldots\nu_q$ in Σ^* (i.e. Proposition 3.1.2 applies) and $|\sigma'|,|\nu'| > 0$ (or else one of μ or ν would extend the other). Corollary 3.1.6 implies that $\sigma'_j \neq \gamma_j$ and $\nu'_j \neq \gamma_j$ for any j; that is, σ' and ν' are words in α's and β's only. Since neither of σ nor ν extend the other, we must have

$$\begin{aligned}\sigma_\alpha &:= |\{j : \sigma'_j = \alpha_j\}| < \nu_\alpha := |\{j : \nu'_j = \alpha_j\}| \\ \text{and } \sigma_\beta &:= |\{j : \sigma'_j = \beta_j\}| > \nu_\beta := |\{j : \nu'_j = \beta_j\}|\end{aligned}$$

or $\sigma_\alpha > \nu_\alpha$ and $\sigma_\beta < \nu_\beta$. Without loss of generality, assume the first pair of inequalities holds. Then

$$
\begin{aligned}
[\mu,\nu,y][\sigma,\tau,x] &= [\mu,\nu_1\dots\nu_q\nu',y][\sigma_1\dots\sigma_q\sigma',\tau,x] \\
&= [\mu\beta_{|\mu|+1}\dots\beta_{|\mu|+\nu_\beta-\sigma_\beta},\nu\beta_{|\mu|+1}\dots\beta_{|\mu|+\nu_\beta-\sigma_\beta},z] \\
&\quad\cdot[\sigma\alpha_{|\sigma|+1}\dots\alpha_{|\sigma|+\sigma_\alpha-\nu_\alpha},\tau\alpha_{|\sigma|+1}\dots\alpha_{|\sigma|+\sigma_\alpha-\nu_\alpha},z] \\
&= [\mu\beta_{|\mu|+1}\dots\beta_{|\mu|+\nu_\beta-\sigma_\beta},\tau\alpha_{|\sigma|+1}\dots\alpha_{|\sigma|+\sigma_\alpha-\nu_\alpha},z].
\end{aligned}
$$

After factoring any common α's or β's into the tail, this has the form of elements in G_i' since $\mu_j \neq \gamma_j$ and $\tau_j \neq \gamma_j$ for $j > i$. Then everything in G_i has the form of some element of G_i', so that $G_i \subseteq G_i'$ and therefore $G_i = G_i'$. □

Now we analyze $C^*(G_i)$.

Theorem 3.4.2. *There are positive integers $m(i)$, and $n(\ell)$ for $\ell > i$, a closed invariant set $F_i \subseteq X_1$, and exact sequences*

$$(1)\ 0 \longrightarrow \bigoplus_{\ell>i}(M_{n(\ell)} \otimes C(X_{\ell+1})) \longrightarrow C^*(G_i) \longrightarrow C^*(G_{i|F_i}) \longrightarrow 0,$$

$$(2)\ 0 \longrightarrow M_{m(i)} \otimes (\mathcal{K} \oplus \mathcal{K}) \to C^*(G_{i|F_i}) \to M_{m(i)} \otimes C(\mathbb{T}) \longrightarrow 0$$

Proof. Similarly to the case when $i = 1$, we let

$$
\begin{aligned}
U_i &= \{x \in X_1 : x_j = \gamma_j, \text{ some } j > i\} \\
F_i &= X_1 \setminus U_i = \{x \in X_1 : x_j \neq \gamma_j, \text{ any } j > i\}.
\end{aligned}
$$

With Proposition 3.4.1 in mind, it is evident that U_i is invariant for G_i. If $x \in U_i$, with $x_j = \gamma_j$ for some $j > i$, then $x \in Z(x_1 \dots x_{j-1}\gamma_j) \subseteq U_i$, so that U_i is open. Then F_i is closed and invariant, and similarly to the $i = 1$ case, we have

$$0 \longrightarrow C^*(G_{i|U_i}) \longrightarrow C^*(G_i) \longrightarrow C^*(G_{i|F_i}) \longrightarrow 0.$$

As before, we begin by looking at the ideal, $C^*(G_{i|U_i})$.

For $\ell > i$, let $\Omega_\ell = \{\ x_1 \dots x_{\ell-1}\ :\ x_j \neq \gamma_j,\ i < j < \ell\ \}$. Then define

$$\begin{aligned} E_\ell &:= \{x \in U_i : x_\ell = \gamma_\ell, \text{ and } x_j \neq \gamma_j \text{ for any } i < j < \ell\} \\ &= \sqcup_{\sigma \in \Omega_\ell} Z(\sigma\gamma_\ell). \end{aligned}$$

Ω_ℓ is finite so E_ℓ is compact-open, and we observe that by Proposition 3.4.1 each E_ℓ is invariant for G_i. Also, $U_i = \bigsqcup_{\ell > i} E_\ell$ and the same argument as the case when $i = 1$ shows

$$C^*(G_{i|U_i}) \cong \bigoplus_{\ell > i} C^*(G_{i|E_\ell}).$$

A nearly identical proof as in the previous case shows that

$$C^*(G_{i|E_\ell}) \cong M_{n(\ell)} \otimes C(X_{\ell+1})$$

where $n(\ell)$ is some integer depending on ℓ. (Recall that $X_{\ell+1} = Z(v_{\ell+1})$). Then we conclude that

$$C^*(G_{i|U_i}) \cong \bigoplus_{\ell > i} (M_{n(\ell)} \otimes C(X_{\ell+1})).$$

We now turn to $C^*(G_{i|F_i})$ and similarly to before, we decompose F_i as follows: let

$$\begin{aligned} F_i^\infty &= \{\sigma\eta^{|\sigma|+1}(\infty, \infty)\ :\ |\sigma| \leq i, \sigma_{|\sigma|} = \gamma_{|\sigma|}\ \} \\ F_i^0 &= F_i \setminus F_i^\infty = \{\sigma\eta^{|\sigma|+1}(j,k)\ :\ |\sigma| \leq i, \sigma_{|\sigma|} = \gamma_{|\sigma|},\ j,k \text{ not both } \infty\ \}. \end{aligned}$$

Then F_i^∞ is closed, being finite, and invariant for G_i so that F_i^0 is open in $G_{i|F_i}$ and invariant. This gives us the short exact sequence

$$0 \longrightarrow C^*(G_{i|F_i^0}) \longrightarrow C^*(G_{i|F_i}) \longrightarrow C^*(G_{i|F_i^\infty}) \longrightarrow 0.$$

Similarly to the case for G_1, the maps

$$\chi_{[\alpha_1,\beta_1,\eta^2(\infty,\infty)]} \mapsto e_{11} \otimes z$$

$$\chi_{[\sigma,\alpha_1\ldots\alpha_{|\sigma|},\eta^{|\sigma|+1}(\infty,\infty)]} \mapsto e_{\sigma 1} \otimes 1$$

where $|\sigma| \leq i$ and $\sigma_{|\sigma|} = \gamma_{|\sigma|}$ define a $*-$isomorphism from $C^*(G_{i|F_i^\infty})$ to $M_{m(i)} \otimes C(\mathbb{T})$, where $m(i)$ is the number of choices for σ. Further, the maps

$$\begin{aligned}
\chi_{[\beta_1\ldots\beta_j,\alpha_1\ldots\alpha_j,\eta^{j+1}(0,\infty)]} &\mapsto e_{11} \otimes (0 \oplus e_{0j}) \\
\chi_{[\beta_1\ldots\beta_{|\sigma|},\sigma,\eta^{|\sigma|+1}(0,\infty)]} &\mapsto e_{1\sigma} \otimes (0 \oplus e_{00}) \\
\chi_{[\alpha_1\ldots\alpha_j,\beta_1\ldots,\beta_j,\eta^{j+1}(\infty,0)]} &\mapsto e_{11} \otimes (e_{0j} \oplus 0) \\
\chi_{[\alpha_1\ldots\alpha_{|\tau|},\tau,\eta^{|\tau|+1}(\infty,0)]} &\mapsto e_{1\tau} \otimes (e_{00} \oplus 0)
\end{aligned}$$

where $|\sigma|, |\tau| \leq i$, $\sigma_{|\sigma|} = \gamma_{|\sigma|}$, and $\tau_{|\tau|} = \gamma_{|\tau|}$ define a $*-$isomorphism from $C^*(G_{i|F_i^0})$ to $M_{m(i)} \otimes (\mathcal{K} \oplus \mathcal{K})$. This yields

$$0 \longrightarrow M_{m(i)} \otimes (\mathcal{K} \oplus \mathcal{K}) \xrightarrow{\iota} C^*(G_{i|F_i}) \xrightarrow{\pi} M_{m(i)} \otimes C(\mathbb{T}) \longrightarrow 0$$

□

Theorem 3.4.3. *$K_1(C^*(G_i|_{F_i})) = 0$ and $K_0(C^*(G_i|_{F_i})) \cong \mathbb{Z}^2$, with generators $[\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0$ and $[\chi_{[v_1,v_1,Z(\beta_1\ldots\beta_{i+1})\cap F_i]}]_0$.*

Proof. The exact sequence (2) in Theorem 3.4.2 induces the six-term exact sequence

$$\begin{array}{ccccc}
\mathbb{Z}^2 & \xrightarrow{\iota_{0*}} & K_0(C^*(G_{i|F_i})) & \xrightarrow{\pi_{0*}} & \mathbb{Z} \\
\delta_1 \uparrow & & & & \downarrow \\
\mathbb{Z} & \xleftarrow[\pi_{1*}]{} & K_1(C^*(G_{i|F_i})) & \xleftarrow[\iota_{1*}]{} & 0.
\end{array}$$

The element

$$\begin{aligned}
u &= e_{11} \otimes z + \textstyle\sum_{j=2}^{m(i)} (e_{jj} \otimes 1) \\
&= \chi_{[\alpha_1,\beta_1,\eta^2(\infty,\infty)]} + \textstyle\sum_\sigma \chi_{[v_1,v_1,\sigma\eta^{|\sigma|+1}(\infty,\infty)]}
\end{aligned}$$

where $|\sigma| \le i$, $\sigma_{|\sigma|} = \gamma_{|\sigma|}$ is a unitary generating $K_1(M_{m(i)} \otimes C(\mathbb{T})) \cong \mathbb{Z}$. Now

$$v = \chi_{[\alpha_1,\beta_1,\{\eta^2(j,k):j+k=\infty\}]} + \sum_\sigma \chi_{[v_1,v_1,Z(\sigma)\cap F_i]}$$

where $|\sigma| \le i$, $\sigma_{|\sigma|} = \gamma_{|\sigma|}$ is a partial isometry such that $\pi(v) = u$. We compute

$$1 - v^*v = \chi_{[v_1,v_1,\eta^1(\infty,0)]}$$
$$1 - vv^* = \chi_{[v_1,v_1,\eta^1(0,\infty)]}$$

which, like the G_1 case, are rank-one projections in the left and right summands of $\mathcal{K} \oplus \mathcal{K}$. Then $\delta_1[u]_1 = (1, -1)$ so $\delta_1(\mathbb{Z}) = \mathbb{Z}(1, -1)$. As before, $K_1(C^*(G_{i|F_i}) = 0$, we take $[\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0 = (1, 0)$ to be our second generator of $K_0(C^*(G_{i|F_i^0}) \cong \mathbb{Z}$, and we again have a short-exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(C^*(G_{i|F_i}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Just as before, we conclude $K_0(C^*(G_{i|F_i}) \cong \mathbb{Z}^2$, generated by $(1, 0) = [\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0$, the generator of the left term, and a lift of $[e_{11} \otimes 1]_0 = [\chi_{[v_1,v_1,\eta^1(\infty,\infty)]}]_0$, the generator of the right term

$$\mathbb{Z} \cong K_0(C^*(G_{i|F_i^\infty}) \cong K_0(M_{m(i)} \otimes C(\mathbb{T})).$$

The lift we choose is $[\chi_{[v_1,v_1,Z(\beta_1 \ldots \beta_{i+1})\cap F_i]}]_0$. □

Theorem 3.4.4. *$K_1(C^*(G_i)) = 0$ and $K_0(C^*(G_i)) \cong (\bigoplus_{\ell>i} C(X_{\ell+1}, \mathbb{Z})) \oplus \mathbb{Z}^2$.*

Proof. Recall the exact sequence (1) from Theorem 3.4.2:

$$0 \longrightarrow \bigoplus_{\ell>i}(M_{n(\ell)} \otimes C(X_{\ell+1})) \longrightarrow C^*(G_i) \longrightarrow C^*(G_{i|F_i}) \longrightarrow 0$$

As we saw in the earlier case, the ideal is AF, so has trivial K_1-group, and its K_0-group is $\bigoplus_{\ell>i} C(X_{\ell+1}, \mathbb{Z})$. We saw in Theorem 3.4.3 that $K_1(C^*(G_i|_{F_i})) = 0$ so that $K_1(C^*(G_i)) = 0$ and we have a short-exact sequence of K_0-groups

$$0 \longrightarrow \bigoplus_{\ell>i} C(X_{\ell+1}, \mathbb{Z}) \longrightarrow K_0(C^*(G_i)) \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

and again, $K_0(C^*(G_i))$ is the direct sum of the left and right K_0-groups. We pick $[\chi_{[v_1,v_1,Z(\alpha_1\ldots\alpha_i)\setminus Z(\alpha_1\ldots\alpha_i\beta_{i+1})]}]_0$ and $[\chi_{[v_1,v_1,Z(\beta_1\ldots\beta_{i+1})]}]_0$ as lifts of the generators $[\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0$ and $[\chi_{[v_1,v_1,Z(\beta_1\ldots\beta_{i+1})\cap F_1]}]_0$ of $K_0(C^*(G_i|_{F_i})) \cong \mathbb{Z}^2$ to be generators of

$$K_0(C^*(G_i)) \cong (\bigoplus_{\ell>i} C(X_{\ell+1}, \mathbb{Z})) \oplus \mathbb{Z}^2$$

together with the generators of the left term. □

Positive elements in K_0 will have a characterization similar to the previous case, and we first make some observations generalizing those from before.

First, if $\mu = \mu_1 \ldots \mu_n$ and $\nu = \nu_1 \ldots \nu_n$ with $\mu_j, \nu_j \in \{\alpha_j, \beta_j\}$ for $j > i$, then the partial isometry $\chi_{[\mu,\nu,Z(v_{n+1})]}$ gives an equivalence between $\chi_{Z(\mu)}$ and $\chi_{Z(\nu)}$ in $C^*(G_i)$. Then using the observations preceding Proposition 3.3.5,

$$
\begin{aligned}
&[\chi_{Z(\beta_1\ldots\beta_{i+1})}]_0 = [\chi_{Z(\alpha_1\ldots\alpha_{i+1})}]_0 \\
= \ & [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\gamma_{i+2})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+2})\setminus Z(\alpha_1\ldots\alpha_{i+2}\beta_{i+3})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\beta_{i+2})}]_0 \\
= \ & [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\gamma_{i+2})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+2})\setminus Z(\alpha_1\ldots\alpha_{i+2}\beta_{i+3})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\alpha_{i+2})}]_0 \\
= \ & [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\gamma_{i+2})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+2})\setminus Z(\alpha_1\ldots\alpha_{i+2}\beta_{i+3})}]_0 \\
+ \ & [\chi_{Z(\alpha_1\ldots\alpha_{i+2}\gamma_{i+3})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+3})\setminus Z(\alpha_1\ldots\alpha_{i+3}\beta_{i+4})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_{i+2}\beta_{i+3})}]_0 \\
= \ & \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{\ell=i+2}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 + \sum_{j=i+2}^{k}[\chi_{Z(\alpha_1\ldots\alpha_j)\setminus Z(\alpha_1\ldots\alpha_j\beta_{j+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
&= \textstyle\sum_{\ell=i+2}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ \textstyle\sum_{j=i+2}^{k}(\sum_{\ell=j+1}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0) \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
&= \textstyle\sum_{\ell=i+2}^{k}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ \textstyle\sum_{\ell=i+3}^{k}(\ell-(i+2))[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ (k-(i+1))[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
&= \textstyle\sum_{\ell=i+2}^{k}(\ell-(i+1))[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell)}]_0 \\
&+ (k-(i+1))[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0.
\end{aligned}
$$

Then for $m, n \in \mathbb{Z}$, and for $\ell > i$, $z_\ell = \alpha_1 \ldots \alpha_{\ell-1}\gamma_\ell$ and $X_{\ell+1} = \sqcup_j F_{\ell,j}$ a finite union of compact-open sets, we have

$$m[\chi_{Z(\alpha_1\dots\alpha_i)\setminus Z(\alpha_1\dots\alpha_i\beta_{i+1})}]_0 + n[\chi_{Z(\beta_1\dots\beta_{i+1})}]_0 = m\sum_{\ell=i+1}^{k}[\chi_{Z(\alpha_1\dots\alpha_{\ell-1}\gamma_\ell)}]_0$$

$$+ \quad m[\chi_{Z(\alpha_1\dots\alpha_k)\setminus Z(\alpha_1\dots\alpha_k\beta_{k+1})}]_0$$

$$+ \quad n\sum_{\ell=i+2}^{k}(\ell-(i+1))[\chi_{Z(\alpha_1\dots\alpha_{\ell-1}\gamma_\ell)}]_0$$

$$+ \quad n(k-(i+1))[\chi_{Z(\alpha_1\dots\alpha_k)\setminus Z(\alpha_1\dots\alpha_k\beta_{k+1})}]_0$$

$$+ \quad n[\chi_{Z(\alpha_1\dots\alpha_k)}]_0$$

$$= \quad \sum_{\ell=i+1}^{k}(m+(\ell-(i+1))n)[\chi_{Z(\alpha_1\dots\alpha_{\ell-1}\gamma_\ell)}]_0$$

$$+ \quad (m+(k-(i+1))n)[\chi_{Z(\alpha_1\dots\alpha_k)\setminus Z(\alpha_1\dots\alpha_k\beta_{k+1})}]_0$$

$$+ \quad n[\chi_{Z(\alpha_1\dots\alpha_k)}]_0$$

$$= \quad \sum_{\ell=i+1}^{k}\sum_j(m+(\ell-(i+1))n)[\chi_{\{z_\ell\}\times F_{\ell,j}}]_0$$

$$+ \quad (m+(k-(i+1))n)[\chi_{Z(\alpha_1\dots\alpha_k)\setminus Z(\alpha_1\dots\alpha_k\beta_{k+1})}]_0$$

$$+ \quad n[\chi_{Z(\alpha_1\dots\alpha_k)}]_0.$$

We can now prove

Proposition 3.4.5. *Fix $a \in K_0(C^*(G_i))$ and write*

$$a = \sum_{\ell>i}\sum_{j=1}^{m_\ell} c_{\ell,j}[\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0 + m[\chi_{Z(\alpha_1\dots\alpha_i)\setminus Z(\alpha_1\dots\alpha_i\beta_{i+1})}]_0 + n[\chi_{Z(\beta_1\dots\beta_{i+1})}]_0$$

where each $F_{\ell,j} \subseteq X_{\ell+1}$ is compact-open and for each ℓ, $\{\ F_{\ell,j}\ :\ 1 \le j \le m_\ell\ \}$ is a partition of $X_{\ell+1}$. As on page 35,

$$\{z_{\ell,j} : 1 \le j \le m_\ell\} = \{\ x_1 \dots x_{\ell-1}\gamma_\ell\ :\ x_j \neq \gamma_j \text{ for } i < j < \ell\ \}.$$

Then $a \geq 0$ if and only if for all ℓ and j we have $c_{\ell,j} + m + (\ell - (i+1))n \geq 0$.

The proof is nearly identical to the case when $i = 1$, making some obvious changes to indices.

Proof. Suppose for all ℓ, j, we have $c_{\ell,j} + m + (\ell - (i+1))n \geq 0$. Similar to before, we must have $c_{\ell,j} = 0$ for large enough ℓ, and therefore $n \geq 0$. Choose k such that $c_{\ell,j} = 0$ for all $\ell \geq k$ and using our computations above, write

$$\begin{aligned} a &= \textstyle\sum_{\ell > i} \sum_{j=1}^{m_\ell} c_{\ell,j} [\chi_{\{z_{\ell,j}\} \times F_{\ell,j}}]_0 + m[\chi_{Z(\alpha_1 \ldots \alpha_i) \setminus Z(\alpha_1 \ldots \alpha_i \beta_{i+1})}]_0 + n[\chi_{Z(\beta_1 \ldots \beta_{i+1})}]_0 \\ &= \textstyle\sum_{\ell=i+1}^{k} \sum_{j=1}^{m_\ell} (c_{\ell,i} + m + (\ell - (i+1))n)[\chi_{\{z_{\ell,j}\} \times F_{\ell,j}}]_0 \\ &+ \ (m + (k - (i+1))n)[\chi_{Z(\alpha_1 \ldots \alpha_k) \setminus Z(\alpha_1 \ldots \alpha_k \beta_{k+1})}]_0 + n[\chi_{Z(\alpha_1 \ldots \alpha_k)}]_0. \end{aligned}$$

If $n > 0$, also choose k such that $m + (k - (i+1))n \geq 0$ and the above shows that $a \geq 0$. If $n = 0$ then our initial assumption implies $m \geq 0$ and the above shows $a \geq 0$.

Now suppose we have $a \in K_0(C^*(G_i))_+$ and write a as above with the outlined conventions. Fix $x \in U_i$ and write

$$x = x_1 \ldots x_{\ell(x)-1} \gamma_{\ell(x)} x'$$

where $x_i \neq \gamma_i$ for $i < j < \ell(x)$ and $x' \in X_{\ell(x)+1}$. Define a $*-$homomorphism

$$\pi_x : C^*(G_i|_{U_i}) \longrightarrow M_{n(\ell(x))}$$

as follows: For

$$f \in C^*(G_i|_{U_i}) \cong \textstyle\bigoplus_{\ell > i} (C(X_{\ell+1}, M_{n(\ell)}))$$

with $f = (f_{i+1}, f_{i+2}, \ldots, f_k, \ldots)$ where $f_k \in C(X_{k+1}, M_{n(k)})$, we define $\pi_x(f) = f_{\ell(x)}(x')$. We saw that $C^*(G_i|_{U_i})$ is an ideal in $C^*(G_i)$, so π_x extends to a representation

$$\tilde{\pi}_x : C^*(G_i) \longrightarrow M_{n(\ell(x))}$$

Let

$$\begin{aligned}
\Omega &= \{\ \mu_1, \ldots, \mu_{n(\ell(x))}\ \} \\
&= \{\ \alpha_1 \ldots \alpha_j \beta_{j+1} \ldots \beta_{\ell(x)-1} \gamma_{\ell(x)}\ :\ 0 \le j < \ell(x)\ \} \\
&\sqcup\ \{\ \sigma \alpha_{|\sigma|+1} \ldots \alpha_j \beta_{j+1} \ldots \beta_{\ell(x)-1} \gamma_{\ell(x)}\ :\ 0 < |\sigma| \le i, \sigma_{|\sigma|} = \gamma_{|\sigma|}, |\sigma| \le j < \ell(x)\ \}
\end{aligned}$$

and for each $1 \le k \le n(\ell(x))$, define

$$p_k = \chi_{\{\mu_k\} \times X_{\ell(x)+1}}$$

so that $\tilde{\pi}_x(p_k) = e_{kk} \in M_{n(\ell(x))}$. Let $j(x)$ be such that $x' \in F_{\ell(x),j(x)}$ and $p_{k(\alpha)} = \chi_{\{\mu_{k(\alpha)}\} \times X_{\ell(x)+1}}$ where $\mu_{k(\alpha)} = \alpha_1 \ldots \alpha_{\ell(x)-1} \gamma_{\ell(x)}$. Since $a \ge 0$,

$$\begin{aligned}
0 &\le \tilde{\pi}_x(a) \\
&= \tilde{\pi}_x(\textstyle\sum_{\ell > i} \sum_{j=1}^{m_\ell} c_{\ell,j} [\chi_{\{z_{\ell,j}\} \times F_{\ell,j}}]_0 + m[\chi_{Z(\alpha_1 \ldots \alpha_i) \setminus Z(\alpha_1 \ldots \alpha_i \beta_{i+1})}]_0 + n[\chi_{Z(\beta_1 \ldots \beta_{i+1})}]_0) \\
&= \textstyle\sum_{\ell > i} \sum_{j=1}^{m_\ell} c_{\ell,j} [\pi_x(\chi_{\{z_{\ell,j}\} \times F_{\ell,j}})]_0 \\
&+ m[\pi_x(\textstyle\sum_{k=1}^{n(\ell(x))} p_k * \chi_{Z(\alpha_1 \ldots \alpha_i) \setminus Z(\alpha_1 \ldots \alpha_i \beta_{i+1})})]_0 \\
&+ n[\pi_x(\textstyle\sum_{k=1}^{n(\ell(x))} p_k * \chi_{Z(\beta_1 \ldots \beta_{i+1})})]_0 \\
&= c_{\ell(x),j(x)} [\pi_x(\chi_{\{z_{\ell(x),j(x)}\} \times F_{\ell(x),j(x)}})]_0 \\
&+ m[\pi_x(p_{k(\alpha)})]_0 \\
&+ n[\pi_x(\textstyle\sum_{\{k : \mu_k \in Z(\beta_1 \ldots \beta_{i+1})\}} p_k)]_0 \\
&= c_{\ell(x),j(x)} + m + (\ell(x) - (i+1))n.
\end{aligned}$$

We note that there are $\ell(x)-(i+1)$ choices for $\mu_k \in Z(\beta_1 \ldots \beta_{i+1})$; those $\mu_k \in \{\ \alpha_1 \ldots \alpha_j \beta_{j+1} \ldots \beta_{\ell(x)-1}\gamma_{\ell(x)}\ :\ 0 \le j < \ell(x)\ \}$ where $0 \le j < \ell(x)-(i+1)$ (since μ_k must have at least $i+1$ edges which are β). If $a \in K_0(C^*(G_i))$ is positive, then for any j and ℓ, there is $x \in X$ with $j = j(x)$ and $\ell = \ell(x)$ so the inequalities above hold. □

We are now equipped to analyze $C^*(G)$, the limit of $\{C^*(G_i), \iota\}$.

3.5 Analysis of $C^*(G)$

Our first step in identifying $C^*(G)$ is to consider the connecting maps

$$K_0(C^*(G_i)) \mapsto K_0(C^*(G_{i+1}))$$

and to do so, we will need a lemma, the argument for which was provided by Spielberg. For ease of notation, we will suppress subscripts and write $\alpha^j\beta^k = \alpha_1 \ldots \alpha_j \beta_{j+1} \ldots \beta_{j+k}$ and $\alpha^j\beta^k\gamma = \alpha_1 \ldots \alpha_j \beta_{j+1} \ldots \beta_{j+k}\gamma_{j+k+1}$.

Lemma 3.5.1. *Fix $n \ge 0$ and let*

$$W_n = \{\ x \in Z(v_0)\ :\ x_i \ne \gamma_i \text{ for } i \le n\ \}.$$

Put

$$\begin{aligned}
P_n^{(1)} &= \{Z(\alpha^{n+1}\beta^j) \setminus Z(\alpha^{n+1}\beta^{j+1}) : j \le n\} \\
P_n^{(2)} &= \{Z(\alpha^j\beta^{n+1}) \setminus Z(\alpha^{j+1}\beta^{n+1}) : j \le n\} \\
P_n^{(3)} &= \{Z(\alpha^j\beta^k\gamma) : j+k \ge n,\ k \le j \le n\} \\
P_n^{(4)} &= \{Z(\alpha^k\beta^j\gamma) : j+k \ge n,\ k < j \le n\} \\
P_n^{(5)} &= \{Z(\alpha^{n+1}\beta^{n+1})\} \\
P_n &= \cup_{r=1}^5 P_n^{(r)}.
\end{aligned}$$

Then P_n is a partition of W_n that refines $Z(\nu)$ for every $\nu \in v_0\Lambda$ with $|\nu| = n$ and $\nu_i \neq \gamma_i$ for $i \leq n$.

Proof. First we show that $W_n = \cup P_n$. It's clear that $\cup P_n \subseteq W_n$. To show the other inclusion, fix $x \in W_n$ and let $p = \min\{j : x_j = \gamma_j\}$ so $n+1 \leq p \leq \infty$. We have several cases to consider

1. Suppose $p \leq 2n+1$. Then $x \in Z(\alpha^j\beta^k\gamma)$ where $n \leq j+k \leq 2n$.

   (a) If $k \leq j \leq n$, then $Z(\alpha^j\beta^k\gamma) \in P_n^{(3)}$.

   (b) If $j > n$, then $k < n$. Then $Z(\alpha^j\beta^k\gamma) \subseteq Z(\alpha^{n+1}\beta^k) \setminus Z(\alpha^{n+1}\beta^{k+1})$, which is a set in $P_n^{(1)}$.

   (c) If $j < k \leq n$, then $Z(\alpha^j\beta^k\gamma) \in P_n^{(4)}$.

   (d) If $k > n$, then $j < n$. Then $Z(\alpha^j\beta^k\gamma) \subseteq Z(\alpha^j\beta^{n+1}) \setminus Z(\alpha^{j+1}\beta^{n+1})$, which is a set in $P_n^{(2)}$.

2. Suppose $p > 2n+1$. If $x \notin Z(\alpha^{n+1}\beta^{n+1})$, a set in $P_n^{(5)}$, then there are two possibilities.

   (a) If $\ell = |\{j < p : x_j = \beta_j\}| \leq n$, then $x \in Z(\alpha^{n+1}\beta^\ell) \setminus Z(\alpha^{n+1}\beta^{\ell+1})$, a set in $P_n^{(1)}$.

   (b) If $\ell = |\{j < p : x_j = \alpha_j\}| \leq n$, then $x \in Z(\alpha^\ell\beta^{n+1}) \setminus Z(\alpha^{\ell+1}\beta^{n+1})$, a set in $P_n^{(2)}$.

Then in all cases, we have $x \in \cup P_n$ so $W_n = \cup P_n$.

Next let $\nu \in v_0\Lambda$ with $|\nu| = n$ and $\nu_j \neq \gamma_j$ for $j \leq n$. Let $S \in P_n$ be such that $S \cap Z(\nu) \neq \emptyset$. We will show that $S \subseteq Z(\nu)$. Let $\nu = \alpha^k\beta^{n-k}$ so that $Z(\nu) = Z(\alpha^k) \cap Z(\beta^{n-k})$.

First suppose $S \in P_n^{(1)}$ so $S = Z(\alpha^{n+1}\beta^j) \setminus Z(\alpha^{n+1}\beta^{j+1})$ for some $j \leq n$. Then $S = Z(\alpha^{n+1}) \cap Z(\beta^j) \setminus Z(\beta^{j+1})$. Since $Z(\nu) \cap S \neq \emptyset$, we know $Z(\beta^{n-k}) \cap (Z(\beta^j) \setminus Z(\beta^{j+1})) \neq \emptyset$, hence $n-k \leq j$. Therefore $Z(\beta^j) \setminus Z(\beta^{j+1}) \subseteq Z(\beta^{n-k})$, and since $k \leq n$, $Z(\alpha^{n+1}) \subseteq Z(\alpha^k)$. Therefore $S = Z(\alpha^{n+1}) \cap Z(\beta^j) \setminus Z(\beta^{j+1}) \subseteq Z(\alpha^k) \cap Z(\beta^{n-k}) = Z(\nu)$. If $S \in P_n^{(2)}$, the argument is analogous.

Now suppose $S \in P_n^{(3)}$ so $S = Z(\alpha^j\beta^\ell\gamma)$ with $j + \ell \geq n$ and $\ell \leq j \leq n$. Then $S = (Z(\alpha^j) \setminus Z(\alpha^{j+1})) \cap Z(\beta^\ell) \setminus Z(\beta^{\ell+1}))$. Since $Z(\nu) \cap S \neq \emptyset$, $k \leq j$ and $n - k \leq \ell$, hence $S \subseteq Z(\nu)$. The argument where $S \in P_n^{(4)}$ is analogous.

Finally, since $k, n-k < n+1$, we have $Z(\alpha^{n+1}\beta^{n+1}) \subseteq Z(\nu)$. Therefore P_n refines $Z(\nu)$.

Lastly we show the elements of P_n are pairwise disjoint. For $S \in P_n$ we consider how many occurrences of α's and β's, before the first occurrence of a γ, there must be in elements of S.

Elements of $P_n^{(1)}$ are distinguished from each other by the number of β's, and elements of $P_n^{(2)}$ are distinguished from each other by the number of α's. Elements of $P_n^{(1)}$ are distinguished from elements of $P_n^{(2)}$ by the fact that the former have at most n β's while the latter have at least $n + 1$. For an element in $P_n^{(3)}$ or $P_n^{(4)}$ the number of α's and β's are exact, and different for different elements. Both are less than $n + 1$ which makes these sets disjoint from $P_n^{(1)}$ and $P_n^{(2)}$. Finally, every set in $\cup_{r=1}^4 P_n^{(r)}$ has at most one occurrence of at least one of α or β, which makes it disjoint from $Z(\alpha^{n+1}\beta^{n+1})$ $\square$

Recall for $\mu \in \Lambda$ the map $\tau^\mu : X_{s(\mu)} \to X_{r(\mu)}$ that takes $x \in X_{s(\mu)}$ to $\mu x \in X_{r(\mu)}$. Lemma 3.5.1 implies the following:

Proposition 3.5.2. *Fix $n \geq 0$ and let*

$$Q_n = \cup\{\tau^\mu(P_m) : |\mu| \leq n \ \mu_{|\mu|} = \gamma_{|\mu|}, \ |\mu| + m = n\}$$

where P_m is as in Lemma 3.5.1. Then Q_n is a partition of $Z(v_0)$ that refines $Z(\nu)$ for all $\nu \in v_0\Lambda$ with $|\nu| \leq n$.

Proof. Fix $n \geq 0$ and $\mu \neq \nu \in v_0\Lambda$ with $|\mu| \leq |\nu| \leq n$ and suppose $x \in \mu P_m$, and $y \in \nu P_{m'}$. If $|\mu| = |\nu|$, then $\mu \notin y$ and $\nu \notin x$ (and hence $x \notin \nu P_{m'}$ and $y \notin \mu P_m$) since otherwise they have a common extension, say $\mu\mu' = \lambda = \nu\nu'$. But then Proposition 3.1.2 implies $\mu = \lambda_1 \dots \lambda_{|\mu|} = \nu$. If $|\mu| < |\nu|$, then $x_{|\nu|} \neq \gamma_{|\nu|}$ while $y_{|\nu|} = \gamma_{|\nu|}$ so that again, $x \notin \nu P_{m'}$ and $y \notin \mu P_m$. In either case, μP_m and $\nu P_{m'}$ are pairwise disjoint. Any two sets in μP_m are disjoint since any two sets in P_m must be.

To see that Q_n refines $Z(\nu)$, first suppose $|\nu| = n$ and write $\nu = \eta\theta$ where $\eta_{|\eta|} = \gamma_{|\eta|}$ and $\theta_i \neq \gamma_i$ for any i. Define θ' by $\theta'_i = \theta_{i-|\eta|}$. Then by Lemma 3.5.1,

$$Z(\theta') = \cup\{w \in P_{|\theta|} : w \subseteq Z(\theta')\}$$

and therefore

$$Z(\nu) = \cup\{\eta w : w \in P_{|\theta|}, w \subseteq Z(\theta')\}$$

where $|\eta| + |\theta| = n$ so each $\eta w \in Q_n$. Now if $|\nu| < n$ and $m = n - |\nu|$, then $Z(\nu) = \cup_{|\rho|=m} Z(\nu\rho)$. For each such ρ, $Z(\nu\rho)$ is a disjoint union of sets in Q_n, by the above, and hence

$$Z(\nu) = \cup\{w \in Q_n : w \subseteq Z(\nu\rho), |\rho| = m\}$$

which we can take to be disjoint by throwing out repeated w's. That Q_n partitions $Z(v_0)$ follows readily. □

Recall that

$$K_0(C^*(G_i)) \cong (\textstyle\bigoplus_{\ell > i} C(X_{\ell+1}, \mathbb{Z})) \oplus \mathbb{Z}^2$$

where a typical generator of the leftmost direct sum is $[\chi_{\{z_{\ell,j}\} \times F_{\ell,j}}]_0$.

Lemma 3.5.3. *For each $a = [\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0 \in K_0(C^*(G_i))$, there is $h > i$ such that under the induced map $K_0(C^*(G_i)) \to K_0(C^*(G_h))$, the image of a lies in the summand $\mathbb{Z}^2$ for $K_0(C^*(G_h))$.*

Proof. For $\ell \geq i+2$, it is clear that

$$[\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0 \in K_0(C^*(G_i)) \mapsto [\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0 \in K_0(C^*(G_{i+1}))$$

If $\ell = i+1$ and $F_{\ell,j} = X_{\ell+1}$, then in $K_0(C^*(G_i))$,

$$\begin{aligned} [\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0 &= [\chi_{Z(\alpha_1\ldots\alpha_i\gamma_{i+1})}]_0 \\ &= [\chi_{Z(\alpha_1\ldots\alpha_i\gamma_{i+1})\setminus Z(\alpha_1\ldots\alpha_i\gamma_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\alpha_1\ldots\alpha_i\gamma_{i+1}\beta_{i+2})}]_0 \end{aligned}$$

Now in $K_0(C^*(G_{i+1}))$

$$\begin{aligned} [\chi_{Z(\alpha_1\ldots\alpha_i\gamma_{i+1})\setminus Z(\alpha_1\ldots\alpha_i\gamma_{i+1}\beta_{i+2})}]_0 &= [\chi_{Z(\alpha_1\ldots\alpha_{i+1})\setminus Z(\alpha_1\ldots\alpha_{i+1}\beta_{i+2})}]_0 \\ \text{and } [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\gamma_{i+1}\beta_{i+2})}]_0 &= [\chi_{Z(\beta_1\ldots\beta_{i+2})}]_0 \end{aligned}$$

so we have

$$[\chi_{\{z_{\ell,j}\}\times X_{\ell+1}}]_0 \in K_0(C^*(G_i)) \mapsto [\chi_{Z(\alpha_1\ldots\alpha_{i+1})\setminus Z(\alpha_1\ldots\alpha_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\beta_1\ldots\beta_{i+2})}]_0 \in K_0(C^*(G_{i+1}))$$

where the elements on the right are generators of the summand $\mathbb{Z}^2$ in $K_0(C^*(G_{i+1}))$.

Now suppose $\ell = i+1$ and take $F_{\ell,j} \subsetneq X_{\ell+1}$, compact-open. Since $F_{\ell,j}$ is compact, it suffices to consider the case when $F_{\ell,j} = Z(\nu) \setminus \cup_{k=1}^m Z(\nu_k)$ where the ν_k extend ν. Fix $n \geq |\nu|, |\nu_k|$ for all k. Then Proposition 3.5.2 implies that for each k,

$$\begin{aligned} Z(\nu_k) &= \sqcup\{w \in Q_n : w \subseteq Z(\nu_k)\} \\ \text{and } Z(\nu) &= \sqcup\{w \in Q_n : w \subseteq Z(\nu)\} \\ \text{so } Z(\nu)\setminus \cup_k Z(\nu_k) &= \sqcup\{w \in Q_n : w \subseteq Z(\nu), w \nsubseteq Z(\nu_k) \text{ any } k\}. \end{aligned}$$

Each $w \in Q_n$ has the form

$$Z(\mu\alpha^m\beta^k) \setminus Z(\mu\alpha^m\beta^{k+1}),$$
$$Z(\mu\alpha^k\beta^m) \setminus Z(\mu\alpha^{k+1}\beta^m),$$
$$Z(\mu\alpha^\ell\beta^k\gamma), \quad Z(\mu\alpha^\ell\beta^k\gamma),$$
$$\text{or } Z(\mu\alpha^m\beta^m)$$

so that $\{z_{\ell,j}\} \times w$ has the form

$$Z(z_{\ell,j}\mu\alpha^m\beta^k) \setminus Z(z_{\ell,j}\mu\alpha^m\beta^{k+1}),$$
$$Z(z_{\ell,j}\mu\alpha^k\beta^m) \setminus Z(z_{\ell,j}\mu\alpha^{k+1}\beta^m),$$
$$Z(z_{\ell,j}\mu\alpha^\ell\beta^k\gamma), \quad Z(z_{\ell,j}\mu\alpha^\ell\beta^k\gamma),$$
$$\text{or } Z(z_{\ell,j}\mu\alpha^m\beta^m).$$

Then for each $\{z_{\ell,j}\} \times w$, there is a k such that $\chi_{\{z_{\ell,j}\}\times w} \sim \chi_E$ where

$$E \in \{Z(\alpha_1 \dots \alpha_k) \setminus Z(\alpha_1 \dots \alpha_k\beta_{k+1}), Z(\beta_1 \dots \beta_{k+1})\}.$$

Therefore, for some large enough h, $[\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0$ is a sum of generators of the $\mathbb{Z}^2$ summand in $K_0(C^*(G_h))$. $\square$

Lemma 3.5.4. *The induced map $K_0(C^*(G_i)) \to K_0(C^*(G_{i+1}))$ carries the summand $\mathbb{Z}^2$ of $K_0(C^*(G_i))$ into the summand $\mathbb{Z}^2$ of $K_0(C^*(G_{i+1}))$, and this restriction is implemented by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.*

Proof. We have

$$[\chi_{Z(\alpha_1\dots\alpha_i)\setminus Z(\alpha_1\dots\alpha_i\beta_{i+1})}]_0 = [\chi_{Z(\alpha_1\dots\alpha_{i+1})\setminus Z(\alpha_1\dots\alpha_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\alpha_1\dots\alpha_i\gamma_{i+1})}]_0$$

$$= \quad [\chi_{Z(\alpha_1\dots\alpha_{i+1})\setminus Z(\alpha_1\dots\alpha_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\alpha_1\dots\alpha_i\gamma_{i+1})\setminus Z(\alpha_1\dots\alpha_i\gamma_{i+1}\beta_{i+2})}]_0$$

$$+ \quad [\chi_{Z(\alpha_1\dots\alpha_i\gamma_{i+1}\beta_{i+2})}]_0.$$

Since

$$[\chi_{Z(\alpha_1\ldots\alpha_i\gamma_{i+1})\setminus Z(\alpha_1\ldots\alpha_i\gamma_{i+1}\beta_{i+2})}]_0 \;=\; [\chi_{Z(\alpha_1\ldots\alpha_{i+1})\setminus Z(\alpha_1\ldots\alpha_{i+1}\beta_{i+2})}]_0$$

$$\text{and } [\chi_{Z(\alpha_1\ldots\alpha_{i+1}\gamma_{i+1}\beta_{i+2})}]_0 \;=\; [\chi_{Z(\beta_1\ldots\beta_{i+2})}]_0$$

in $K_0(C^*(G_{i+1}))$, we see that

$$[\chi_{Z(\alpha_1\ldots\alpha_i)\setminus Z(\alpha_1\ldots\alpha_i\beta_{i+1})}]_0 \mapsto 2[\chi_{Z(\alpha_1\ldots\alpha_{i+1})\setminus Z(\alpha_1\ldots\alpha_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\beta_1\ldots\beta_{i+2})}]_0.$$

Finally,

$$[\chi_{Z(\beta_1\ldots\beta_{i+1})}]_0 = [\chi_{Z(\beta_1\ldots\beta_{i+1})\setminus Z(\beta_1\ldots\beta_{i+2})}]_0 + [\chi_{Z(\beta_1\ldots\beta_{i+2})}]_0$$

so that

$$[\chi_{Z(\beta_1\ldots\beta_{i+1})}]_0 \mapsto [\chi_{Z(\alpha_1\ldots\alpha_{i+1})\setminus Z(\alpha_1\ldots\alpha_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\beta_1\ldots\beta_{i+2})}]_0,$$

again, using equivalences in $K_0(C^*(G_{i+1}))$. □

Theorem 3.5.5. *$K_0(C^*(G)) \cong \mathbb{Z}^2$.*

Proof. If we let $\{\iota_{i*}\}$ be the connecting maps in K_0 induced by the inclusions $C^*(G_i) \hookrightarrow C^*(G_{i+1})$ and $C^*(G_i) \hookrightarrow C^*(G)$, then for

$$a \in K_0(C^*(G)) = \cup_{i\geq 1}\iota_{i*}(K_0(C^*(G_i)))$$

with

$$a = \sum_{\ell>i}\sum_{j=1}^{m_\ell} c_{\ell,j}[\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0 + m[\chi_{Z(\alpha_1\ldots\alpha_i)\setminus Z(\alpha_1\ldots\alpha_i\beta_{i+1})}]_0 + n[\chi_{Z(\beta_1\ldots\beta_{i+1})}]_0$$

for some i, note that

$$\sum_{\ell>i}\sum_{j=1}^{m_\ell} c_{\ell,j}[\chi_{\{z_{\ell,j}\}\times F_{\ell,j}}]_0$$

is a finite sum. Given our observations about the maps

$$K_0(C^*(G_i)) \mapsto K_0(C^*(G_{i+1}))$$

namely that generators of $\mathbb{Z}^2$ in $K_0(C^*(G_i))$ map to generators of $\mathbb{Z}^2$ in $K_0(C^*(G_{i+1}))$, and generators of $\bigoplus_{\ell>i} C(X_{\ell+1}, \mathbb{Z})$ in $K_0(C^*(G_i))$ must map to generators of $\mathbb{Z}^2$ in $K_0(C^*(G_{i+k}))$ for sufficiently large k, we conclude that for some $p \geq i$,

$$a = m_p[\chi_{Z(\alpha_1 \ldots \alpha_p) \setminus Z(\alpha_1 \ldots \alpha_p \beta_{p+1})}]_0 + n_p[\chi_{Z(\beta_1 \ldots \beta_{p+1})}]_0$$

in $K_0(C^*(G_p))$. Thus,

$$K_0(C^*(G)) \cong \mathbb{Z}^2$$

the limit of the inductive sequence

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \cdots$$

with the connecting maps given by the matrix

$$B = \begin{pmatrix} 2 & 1 \\ \\ 1 & 1 \end{pmatrix}.$$

by Lemma 3.5.4. $\square$

Using the above, we will compute $K_0(C^*(G))_+$.

Theorem 3.5.6. *Let* $\tau = \frac{1+\sqrt{5}}{2}$. *Then*

$$K_0(C^*(G))_+ = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 : y + \tau x \geq 0\}.$$

Proof. The i^{th} group in the inductive sequence defining $K_0(C^*(G))$ is generated by

$$a_i \quad := \quad [\chi_{Z(\alpha_1 \ldots \alpha_i) \setminus Z(\alpha_1 \ldots \alpha_i \beta_{i+1})}]_0$$

$$\text{and } b_i \quad := \quad [\chi_{Z(\beta_1 \ldots \beta_{i+1})}]_0$$

If $a = ma_i + nb_i$, then Proposition 3.4.5 implies $a \geq 0$ if and only if $m, n \geq 0$, so that the positive cone in each term of the inductive sequence is the standard $\mathbb{N}^2$. Letting

$$A = B^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

we will compute

$$\cup_{n \geq 1} A^n(\mathbb{N}^2) \cong K_0(C^*(G))_+$$

using the fact that A is an isomorphism of $\mathbb{Z}^2$.

To compute $A^n(\mathbb{N}^2)$, we first diagonalize $A = XDX^{-1}$ where

$$D = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{2}{1+\sqrt{5}} & -\frac{2}{1-\sqrt{5}} \\ 1 & 1 \end{pmatrix}$$

$$X^{-1} = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{2}{1-\sqrt{5}} \\ -1 & -\frac{2}{1+\sqrt{5}} \end{pmatrix}.$$

Let $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$ so that

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Straightforward computations show

$$X \begin{pmatrix} 1 & 0 \\ 0 & (\frac{\lambda_2}{\lambda_1})^n \end{pmatrix} X^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{2}{1+\sqrt{5}} - \frac{2}{1-\sqrt{5}}(\frac{\lambda_2}{\lambda_1})^n \\ -1-(\frac{\lambda_2}{\lambda_1})^n \end{pmatrix}$$

and

$$X \begin{pmatrix} 1 & 0 \\ 0 & (\frac{\lambda_2}{\lambda_1})^n \end{pmatrix} X^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{5}} \begin{pmatrix} 1+(\frac{\lambda_2}{\lambda_1})^n \\ \frac{2}{1-\sqrt{5}} - \frac{2}{1+\sqrt{5}}(\frac{\lambda_2}{\lambda_1})^n \end{pmatrix}.$$

Since $0 < \frac{\lambda_2}{\lambda_1} < 1$,

$$X \begin{pmatrix} 1 & 0 \\ 0 & (\frac{\lambda_2}{\lambda_1})^n \end{pmatrix} X^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{2}{5+\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

and

$$X \begin{pmatrix} 1 & 0 \\ 0 & (\frac{\lambda_2}{\lambda_1})^n \end{pmatrix} X^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{5-\sqrt{5}} \end{pmatrix}.$$

These vectors lie on the same line, namely the line through the origin with slope $\frac{2}{1-\sqrt{5}} = -\frac{1+\sqrt{5}}{2}$. Therefore, $K_0(C^*(G))_+$ is the upper half-plane defined by this line, which we determine since, say, $(1,1) \in K_0(C^*(G_i))_+$ for every $i \geq 1$. □

To identify $C^*(G)$ we will appeal to classification results, and to do so we will need to show it has a unique tracial state. Our first step will be showing that there is a unique invariant Borel probability measure on G^0. To this end, we first create a sequence of partitions of G^0 (which we will frequently identify with v_1X). We begin with an observation:

$$
\begin{aligned}
Z(v_1) &= Z(\alpha_1) \cup Z(\beta_1) \cup Z(\gamma_1) \\
&= (Z(\alpha_1) \setminus Z(\alpha_1\beta_2)) \sqcup (Z(\beta_1) \setminus Z(\beta_1\beta_2)) \sqcup Z(\beta_1\beta_2) \sqcup Z(\gamma_1) \\
&= (Z(\alpha_1) \setminus Z(\alpha_1\beta_2)) \sqcup (Z(\beta_1) \setminus Z(\beta_1\beta_2)) \sqcup (Z(\gamma_1) \setminus Z(\gamma_1\beta_2)) \\
&\sqcup \; Z(\beta_1\beta_2) \sqcup Z(\gamma_1\beta_2).
\end{aligned}
$$

In general, we can make the following decompositions:

$$Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}) = Z(\mu\gamma_{|\mu|+1}) \sqcup (Z(\mu\alpha_{|\mu|+1}) \setminus Z(\mu\alpha_{|\mu|+1}\beta_{|\mu|+2}))$$

$$\text{and } Z(\nu) = (Z(\nu) \setminus Z(\nu\beta_{|\nu|+1})) \sqcup Z(\nu\beta_{|\nu|+1}).$$

Now define

$$
\begin{aligned}
A_1 &= \{Z(\alpha_1) \setminus Z(\alpha_1\beta_2),\ Z(\beta_1) \setminus Z(\beta_1\beta_2),\ Z(\gamma_1) \setminus Z(\gamma_1\beta_2)\ \} \\
B_1 &= \{\ Z(\beta_1\beta_2),\ Z(\gamma_1\beta_2)\ \},
\end{aligned}
$$

and for $i \geq 1$, we will recursively define collections A_i and B_i where each $E \in A_i$ has the form $E = Z(\mu) \setminus Z(\mu\beta_{|\mu|+1})$ and each $E \in B_i$ has the form $E = Z(\nu)$ where $\nu_{|\nu|} = \beta_{|\nu|}$ and such that the elements in A_i and B_i are pairwise disjoint. It's clear that the collections A_1 and B_1 have these properties, so for $i \geq 1$, define A_{i+1} and B_{i+1} as follows.

Observe that for $Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}) \in A_i$, we can write

$$Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}) = Z(\mu\gamma_{|\mu|+1}) \sqcup (Z(\mu\alpha_{|\mu|+1}) \setminus Z(\mu\alpha_{|\mu|+1}\beta_{|\mu|+2})),$$

$$Z(\mu\gamma_{|\mu|+1}) = (Z(\mu\gamma_{|\mu|+1}) \setminus Z(\mu\gamma_{|\mu|+1}\beta_{|\mu|+2})) \sqcup Z(\mu\gamma_{|\mu|+1}\beta_{|\mu|+2}),$$

and for $Z(\nu) \in B_i$,

$$Z(\nu) = (Z(\nu) \setminus Z(\nu\beta_{|\nu|+1})) \sqcup Z(\nu\beta_{|\nu|+1}).$$

Now let

$$B_i' = \{\ Z(\mu\gamma_{|\mu|+1})\ :\ Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}) \in A_i\ \}$$

Since the elements of A_i and B_i are pairwise disjoint, it's clear that this is true of B_i and B_i' as well. Then define

$$\begin{aligned} A_{i+1} &= \{Z(\nu) \setminus Z(\nu\beta_{|\nu|+1})\ :\ Z(\nu) \in B_i \sqcup B_i'\ \} \\ &\sqcup \ \{\ Z(\sigma\alpha_{|\sigma|+1}) \setminus Z(\sigma\alpha_{|\sigma|+1}\beta_{|\sigma|+2})\ :\ Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}) \in A_i\ \} \\ B_{i+1} &= \{\ Z(\mu\beta_{|\mu|+1})\ :\ Z(\mu) \in B_i \sqcup B_i'\ \}. \end{aligned}$$

Since the elements of A_i and B_i are pairwise disjoint, the decompositions above used to define A_{i+1} and B_{i+1} show that the elements of A_{i+1} and B_{i+1} are also pairwise disjoint.

We make several observations about A_i and B_i for $i \geq 1$:

First, for $Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}) \in A_i$ and $Z(\nu) \in B_i$, we have $|\mu| = i$ and $|\nu| = i+1$. This is clearly true for A_1 and B_1. Suppose it holds for some $i \geq 1$. For $Z(\mu\gamma_{|\mu|+1}) \in B_i'$, $|\mu| = i$ since $Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}) \in A_i$. Then for $Z(\nu\beta_{|\nu|+1}) \in B_{i+1}$, $|\nu\beta_{|\nu|+1}| = |\nu|+1$ where $Z(\nu) \in B_i \sqcup B_i'$, so $|\nu| = i+1$ and hence $|\nu\beta_{|\nu|+1}| = i+2$.

Further, for $Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) \in A_{i+1}$ where $Z(\nu) \in B_i \sqcup B_i'$, then $|\nu| = i+1$. On the other hand, if $Z(\sigma\alpha_{|\sigma|+1}) \setminus Z(\sigma\alpha_{|\sigma|+1}\beta_{|\sigma|+2}) \in A_{i+1}$ where $Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}) \in A_i$, then $|\sigma| = i$ so that $|\sigma\alpha_{|\sigma|+1}| = i+1$.

Also note that for each $i \geq 1$, $A_i \sqcup B_i$ is a partition of $Z(v_1)$. This is clearly true for $A_1 \sqcup B_1$, and is true for $i > 1$ since each $A_{i+1} \sqcup B_{i+1}$ refines $A_i \sqcup B_i$. To see this, let $Z(\sigma) \in B_i$. Then $Z(\sigma\beta_{|\sigma|+1}) \in B_{i+1}$ and $Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}) \in A_{i+1}$ so $Z(\sigma) = Z(\sigma\beta_{|\sigma|+1}) \sqcup Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1})$. Now let $Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) \in A_i$. Then $Z(\nu\gamma_{|\nu|+1}) \in B_i'$ so $Z(\nu\gamma_{|\nu|+1}) \setminus Z(\nu\gamma_{|\nu|+1}\beta_{|\nu|+2}) \in A_{i+1}$ and $Z(\nu\gamma_{|\nu|+1}\beta_{|\nu|+2}) \in B_{i+1}$. Also $Z(\nu\alpha_{|\nu|+1}) \setminus Z(\nu\alpha_{|\nu|+1}\beta_{|\nu|+2}) \in A_{i+1}$. Then

$$Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) =$$
$$(Z(\nu\alpha_{|\nu|+1}) \setminus Z(\nu\alpha_{|\nu|+1}\beta_{|\nu|+2})) \sqcup Z((\nu\gamma_{|\nu|+1}) \setminus Z(\nu\gamma_{|\nu|+1}\beta_{|\nu|+2})) \sqcup Z(\nu\gamma_{|\nu|+1}\beta_{|\nu|+2}).$$

This shows that $\cup(A_{i+1} \sqcup B_{i+1}) = \cup(A_i \sqcup B_i)$.

We can also see that for $i \geq 1$,

$$|A_i| = f_{2i+2}$$

$$\text{and } |B_i| = f_{2i+1}$$

where f_j is the j^{th} Fibonacci number (using here and throughout the convention that $f_0 = 0$). Observe that this claim is true for $|A_1| = 3 = f_4$ and $|B_1| = 2 = f_3$. Now suppose $|A_i| = f_{2i+2}$ and $|B_i| = f_{2i+1}$ for some $i \geq 1$. Then

$$\begin{aligned}
|A_{i+1}| &= |B_i| + |B_i'| + |A_i| \\
&= |B_i| + |A_i| + |A_i| \\
&= f_{2i+1} + f_{2i+2} + f_{2i+2} \\
&= f_{2i+3} + f_{2i+2} \\
&= f_{2i+4} \\
&= f_{2(i+1)+2}
\end{aligned}$$

and

$$\begin{aligned}|B_{i+1}| &= |B_i| + |B_i'| \\ &= |B_i| + |A_i| \\ &= f_{2i+1} + f_{2i+2} \\ &= f_{2i+3} \\ &= f_{2(i+1)+1}.\end{aligned}$$

Now, suppose μ is an invariant Borel probability measure on G^0. Note that for $Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}), Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) \in A_i$,

$$\begin{aligned} s([\sigma, \nu, Z(\beta_{|\nu|+1})^c]) &= Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) \\ \text{and } r([\sigma, \nu, Z(\beta_{|\nu|+1})^c]) &= Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}) \end{aligned}$$

so that

$$\mu(Z(\nu) \setminus Z(\nu\beta_{|\nu|+1})) = \mu(Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}))$$

since μ is invariant. Similarly, for $Z(\sigma), Z(\nu) \in B_i$,

$$\begin{aligned} s([\sigma, \nu, Z(v_{|\nu|+1})]) &= Z(\nu) \\ \text{and } r([\sigma, \nu, Z(v_{|\nu|+1})]) &= Z(\sigma) \end{aligned}$$

so that $\mu(Z(\nu)) = \mu(Z(\sigma))$.

Now for each $i \geq 1$, pick $Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}) \in A_i$ and $Z(\nu) \in B_i$ and let $a_i = \mu(Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}))$ and $b_i = \mu(Z(\nu))$. These are well-defined by the previous observations. Observe

$$\begin{aligned} Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1}) &= Z(\sigma\gamma_{|\sigma|+1}) \sqcup Z(\sigma\alpha_{|\sigma|+1}) \setminus Z(\sigma\alpha_{|\sigma|+1}\beta_{|\sigma|+2}) \\ &= Z(\sigma\gamma_{|\sigma|+1}) \setminus Z(\sigma\gamma_{|\sigma|+1}\beta_{|\sigma|+2}) \\ &\sqcup Z(\sigma\alpha_{|\sigma|+1}) \setminus Z(\sigma\alpha_{|\sigma|+1}\beta_{|\sigma|+2}) \\ &\sqcup Z(\sigma\gamma_{|\sigma|+1}\beta_{|\sigma|+2}) \end{aligned}$$

so that $a_i = 2a_{i+1} + b_i$ for every $i \geq 1$. Similarly,

$$Z(\nu) = Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) \sqcup Z(\nu\beta_{|\nu|+1})$$

so that, for each $i \geq 1$, $b_i = a_{i+1} + b_{i+1}$. Recalling the maps

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A = B^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

we have

$$B \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } A \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix}$$

We will make extensive use of the following lemma.

Lemma 3.5.7. *Given*

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and $n \geq 1$,

$$A^n = \begin{pmatrix} f_{2n-1} & -f_{2n} \\ -f_{2n} & f_{2n+1} \end{pmatrix}$$

where f_m *is the* m^{th} *Fibonacci number (with* $f_0 = 0$*).*

Proof. The proof is by induction. Clearly,

$$A^1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} f_1 & -f_2 \\ -f_2 & f_3 \end{pmatrix},$$

so suppose for some $n \geq 1$ that

$$A^n = \begin{pmatrix} f_{2n-1} & -f_{2n} \\ -f_{2n} & f_{2n+1} \end{pmatrix}.$$

Then

$$\begin{aligned} A^{n+1} &= AA^n \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} f_{2n-1} & -f_{2n} \\ -f_{2n} & f_{2n+1} \end{pmatrix} \\ &= \begin{pmatrix} f_{2n-1} + f_{2n} & -f_{2n} - f_{2n+1} \\ -f_{2n-1} - 2f_{2n} & f_{2n} + 2f_{2n+1} \end{pmatrix} \\ &= \begin{pmatrix} f_{2n+1} & -f_{2n+2} \\ -f_{2n+1} - f_{2n} & f_{2n+2} + 2f_{2n+1} \end{pmatrix} \\ &= \begin{pmatrix} f_{2n+1} & -f_{2n+2} \\ -f_{2n+2} & f_{2n+3} \end{pmatrix} \\ &= \begin{pmatrix} f_{2(n+1)-1} & -f_{2(n+1)} \\ -f_{2(n+1)} & f_{2(n+1)+1} \end{pmatrix}. \end{aligned}$$

$\square$

We will now compute a_1 and b_1, and subsequently, a_i and b_i for $i \geq 1$. Fix $n \geq 1$. Then by Lemma 3.5.7

$$\begin{pmatrix} a_{n+1} \\ \\ b_{n+1} \end{pmatrix} = A^n \begin{pmatrix} a_1 \\ \\ b_1 \end{pmatrix} = \begin{pmatrix} f_{2n-1} & -f_{2n} \\ \\ -f_{2n} & f_{2n+1} \end{pmatrix} \begin{pmatrix} a_1 \\ \\ b_1 \end{pmatrix} = \begin{pmatrix} f_{2n-1}a_1 - f_{2n}b_1 \\ \\ -f_{2n}a_1 + f_{2n+1}b_1 \end{pmatrix}$$

Since $A_1 \sqcup B_1$ partitions $Z(v_1)$ and μ is a probability measure, we have $3a_1 + 2b_1 = 1$ so $b_1 = \frac{1-3a_1}{2}$. Then

$$\begin{pmatrix} a_{n+1} \\ \\ b_{n+1} \end{pmatrix} = A^n \begin{pmatrix} a_1 \\ \\ b_1 \end{pmatrix}$$

$$= \begin{pmatrix} f_{2n-1}a_1 - f_{2n}\dfrac{1-3a_1}{2} \\ -f_{2n}a_1 + f_{2n+1}\dfrac{1-3a_1}{2} \end{pmatrix} = \begin{pmatrix} \dfrac{2a_1 f_{2n-1} + 3a_1 f_{2n} - f_{2n}}{2} \\ \dfrac{-2a_1 f_{2n} - 3a_1 f_{2n+1} + f_{2n+1}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{a_1(2f_{2n-1} + 3f_{2n}) - f_{2n}}{2} \\ \dfrac{-a_1(2f_{2n} + 3f_{2n+1}) + f_{2n+1}}{2} \end{pmatrix} = \begin{pmatrix} \dfrac{a_1(f_{2n+1} + f_{2n-1} + 2f_{2n}) - f_{2n}}{2} \\ \dfrac{-a_1(f_{2n+2} + f_{2n} + 2f_{2n+1}) + f_{2n+1}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{a_1(2f_{2n+1} + f_{2n}) - f_{2n}}{2} \\ \dfrac{-a_1(2f_{2n+2} + f_{2n+1}) + f_{2n+1}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{a_1(f_{2n+1} + f_{2n+2}) - f_{2n}}{2} \\ \dfrac{-a_1(f_{2n+2} + f_{2n+3}) + f_{2n+1}}{2} \end{pmatrix} = \begin{pmatrix} \dfrac{a_1 f_{2n+3} - f_{2n}}{2} \\ \dfrac{f_{2n+1} - a_1 f_{2n+4}}{2} \end{pmatrix}.$$

Then

$$\begin{aligned} a_{n+1} &= \tfrac{a_1 f_{2n+3} - f_{2n}}{2} \\ \text{and } b_{n+1} &= \tfrac{f_{2n+1} - a_1 f_{2n+4}}{2}. \end{aligned}$$

Since μ is a probability measure, we have

$$0 \leq \frac{a_1 f_{2n+3} - f_{2n}}{2} \leq 1$$

and

$$0 \leq \frac{f_{2n+1} - a_1 f_{2n+4}}{2} \leq 1,$$

so that

$$\frac{f_{2n}}{f_{2n+3}} \leq a_1 \leq \frac{f_{2n} + 2}{f_{2n+3}}$$

and

$$\frac{f_{2n+1}}{f_{2n+4}} \geq a_1 \geq \frac{f_{2n+1} - 2}{f_{2n+4}},$$

and therefore

$$\frac{f_{2n}}{f_{2n+3}} \leq a_1 \leq \frac{f_{2n+1}}{f_{2n+4}}. \quad (\star)$$

We will now apply Binet's formula, which states $f_j = \frac{1}{\sqrt{5}}(\tau^j - (1-\tau)^j)$ where $\tau = (1+\sqrt{5})/2$. Then $(\star)$ becomes

$$\frac{\tau^{2n} - (1-\tau)^{2n}}{\tau^{2n+3} - (1-\tau)^{2n+3}} \leq a_1 \leq \frac{\tau^{2n+1} - (1-\tau)^{2n+1}}{\tau^{2n+4} - (1-\tau)^{2n+4}} \quad (\dagger)$$

We will use these inequalities to show $a_1 = \frac{1}{\tau^3}$ but first we need the following observation. Note that

$$1 < \frac{1+\sqrt{4}}{2} < \tau < \frac{1+\sqrt{9}}{2} = 2$$

so

$$-1 > -\tau > -2$$

and

$$0 > 1 - \tau > -1$$

Since $\tau > 1$, $0 < \frac{1}{\tau} < 1$, so $-\frac{1}{\tau} > -1$. Therefore,

$$0 > \frac{1-\tau}{\tau} > -\frac{1}{\tau} > -1$$

and hence

$$\left(\frac{1-\tau}{\tau}\right)^n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then returning to (†), beginning with the left term, we see

$$\begin{aligned}\frac{\tau^{2n} - (1-\tau)^{2n}}{\tau^{2n+3} - (1-\tau)^{2n+3}} &= \frac{1 - \left(\frac{1-\tau}{\tau}\right)^{2n}}{\tau^3 - (1-\tau)^3\left(\frac{1-\tau}{\tau}\right)^{2n}} \\ &\longrightarrow \frac{1}{\tau^3},\end{aligned}$$

and on the right

$$\begin{aligned}\frac{\tau^{2n+1} - (1-\tau)^{2n+1}}{\tau^{2n+4} - (1-\tau)^{2n+4}} &= \frac{1 - \left(\frac{1-\tau}{\tau}\right)^{2n+1}}{\tau^3 - (1-\tau)^3\left(\frac{1-\tau}{\tau}\right)^{2n+1}} \\ &\longrightarrow \frac{1}{\tau^3}\end{aligned}$$

and from this we conclude $a_1 = \frac{1}{\tau^3}$.

Before computing b_1 and then a_i and b_i for all $i \geq 1$, we give a lemma which will be used several times.

Lemma 3.5.8. *For $n \geq 0$ and $\tau = (1+\sqrt{5})/2$,*

$$\frac{1}{\tau^n} = (-1)^n(f_{n+1} - f_n\tau)$$

where f_j is the j^{th} Fibonacci number.

Proof. The proof is by induction on n. When $n = 0$,

$$\begin{aligned}\frac{1}{\tau^0} &= 1 \\ &= (-1)^0(1 - 0\tau) \\ &= (-1)^0(f_1 - f_0\tau)\end{aligned}$$

Now suppose

$$\frac{1}{\tau^n} = (-1)^n (f_{n+1} - f_n \tau)$$

for some $n \geq 0$. Then, using $1/\tau = \tau - 1$ and $\tau^2 = \tau + 1$,

$$\begin{aligned}
\tfrac{1}{\tau^{n+1}} &= \tfrac{1}{\tau^n}\tfrac{1}{\tau} \\
&= (-1)^n (f_{n+1} - f_n \tau)\tfrac{1}{\tau} \\
&= (-1)^n (f_{n+1} - f_n \tau)(\tau - 1) \\
&= (-1)^n (f_{n+1}\tau - f_{n+1} - f_n \tau^2 + f_n \tau) \\
&= (-1)^n (f_{n+1}\tau - f_{n+1} - f_n(\tau + 1) + f_n \tau) \\
&= (-1)^n (f_{n+1}\tau - f_{n+1} - f_n \tau - f_n + f_n \tau) \\
&= (-1)^n (f_{n+1}\tau - f_{n+1} - f_n) \\
&= (-1)^n (f_{n+1}\tau - f_{n+2}) \\
&= (-1)^{n+1} (f_{n+2} - f_{n+1}\tau). \quad \square
\end{aligned}$$

We can now compute

$$
\begin{aligned}
b_1 &= \tfrac{1-3a_1}{2} \\
&= \tfrac{1-3\left(\frac{1}{\tau^3}\right)}{2} \\
&= \tfrac{1-3((-1)^3(f_4-f_3\tau))}{2} \\
&= \tfrac{1-3(-(3-2\tau))}{2} \\
&= \tfrac{1-3(2\tau-3))}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1-6\tau+9}{2} \\
&= \tfrac{10-6\tau}{2} \\
&= 5-3\tau \\
&= (-1)^4(f_5-f_4\tau) \\
&= \tfrac{1}{\tau^4}
\end{aligned}
$$

and we are now ready to prove:

Proposition 3.5.9. *Let μ be an invariant Borel probability measure on G^0 and $\tau = (1+\sqrt{5})/2$. For $n \geq 0$ and any $\sigma = \sigma_1 \ldots \sigma_n \in v_1\Lambda$ let*

$$
\begin{aligned}
a_n &= \mu(Z(\sigma) \setminus Z(\sigma\beta_{|\sigma|+1})) \\
b_n &= \mu(Z(\sigma\beta_{|\sigma|+1}))
\end{aligned}
$$

Then $a_n = 1/\tau^{2n+1}$ and $b_n = 1/\tau^{2n+2}$.

Proof. The proof will be in two parts; we first prove the claim for $n \geq 1$ by inducting on n, and then compute directly for the case when $n = 0$. We saw above that

$$a_1 = \frac{1}{\tau^3} \text{ and } b_1 = \frac{1}{\tau^4}$$

so suppose that

$$a_n = \frac{1}{\tau^{2n+1}} \text{ and } b_n = \frac{1}{\tau^{2n+2}}$$

for some $n \geq 1$. Since

$$\begin{aligned}
\begin{pmatrix} a_{n+1} \\ \\ b_{n+1} \end{pmatrix} &= A \begin{pmatrix} a_n \\ \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a_n \\ \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} a_n - b_n \\ \\ 2b_n - a_n \end{pmatrix}
\end{aligned}$$

then (using $(\tau - 1) = \frac{1}{\tau}$)

$$
\begin{aligned}
a_{n+1} &= a_n - b_n \\
&= \tfrac{1}{\tau^{2n+1}} - \tfrac{1}{\tau^{2n+2}} \\
&= \tfrac{1}{\tau^{2n+1}}\left(1 - \tfrac{1}{\tau}\right) \\
&= \tfrac{1}{\tau^{2n+1}}\left(\tfrac{\tau-1}{\tau}\right) \\
&= \tfrac{1}{\tau^{2n+1}}\left(\tfrac{1}{\tau}\cdot\tfrac{1}{\tau}\right) \\
&= \tfrac{1}{\tau^{2n+3}} \\
&= \tfrac{1}{\tau^{2(n+1)+1}}
\end{aligned}
$$

Since $\tau^2 = \tau + 1$, we have

$$\frac{1}{\tau^2} = (\tau - 1)^2 = \tau^2 - 2\tau + 1 = \tau + 1 - 2\tau + 1 = 2 - \tau$$

so that

$$
\begin{aligned}
b_{n+1} &= 2b_n - a_n \\
&= \tfrac{2}{\tau^{2n+2}} - \tfrac{1}{\tau^{2n+1}} \\
&= \tfrac{1}{\tau^{2n+2}}\left(2 - \tau\right) \\
&= \tfrac{1}{\tau^{2n+2}}\cdot\tfrac{1}{\tau^2} \\
&= \tfrac{1}{\tau^{2(n+1)+2}}
\end{aligned}
$$

For the case when $n = 0$, note that

$$Z(\beta_1) = Z(\beta_1) \setminus Z(\beta_1\beta_2) \sqcup Z(\beta_1\beta_2)$$

so

$$
\begin{aligned}
b_0 &= a_1 + b_1 \\
&= \frac{1}{\tau^3} + \frac{1}{\tau^4} \\
&= \frac{1}{\tau^4}(\tau + 1) \\
&= \frac{1}{\tau^4}\tau^2 \\
&= \frac{1}{\tau^2}
\end{aligned}
$$

Since μ is a probability measure, $\mu(Z(v_1)) = 1$ and therefore

$$
\begin{aligned}
a_0 &= \mu(Z(\beta_1)^c) \\
&= \mu(Z(v_1)) - \mu(Z(\beta_1)) \\
&= 1 - \frac{1}{\tau^2} \\
&= \frac{\tau^2 - 1}{\tau^2} \\
&= \frac{1}{\tau^2}(\tau^2 - 1) \\
&= \frac{1}{\tau^2}(\tau + 1 - 1) \\
&= \frac{1}{\tau^2}\tau \\
&= \frac{1}{\tau}
\end{aligned}
$$

□

The proposition above shows that there is at most one invariant Borel probability measure on G^0. Defining a measure μ_0 on cylinder sets as above gives a finitely additive measure on the algebra $\mathcal{A}$ generated by cylinder sets. Since the sets in $\mathcal{A}$ are compact, any countable disjoint union in $\mathcal{A}$ is equal to a finite disjoint union, so

μ_0 is a premeasure. Then μ_0 extends to a measure μ on the Borel subsets of G^0, so we have a unique Borel probability measure on G^0. The following argument provided by Spielberg is based on the argument in (11, Proposition 1.1) (see also (9, Corollary 1.2)) and shows that traces on $C^*(G)$ are in one-to-one correspondence with invariant Borel probability measures on G^0 (so that $C^*(G)$ has a unique tracial state). We will see shortly (Claim 3.5.12 and the remarks following it) that G satisfies the conditions of the following proposition.

Proposition 3.5.10. *Let G be a Hausdorff étale groupoid. Suppose*

1. *the orbits with nontrivial isotropy are infinite (i.e. if $xGx \neq \{x\}$ then $r(Gx)$ is an infinite set);*
2. *$Iso(G) \setminus G^0$ is a discrete subset of G.*

Then every trace on $C_r^*(G)$ factors through the conditional expectation E.

Proof. Let τ be a trace on $C_r^*(G)$. Define a measure μ on G^0 by $\tau|_{C_0(G^0)} = \int_{G^0} \cdot d\mu$. We first show that μ is invariant. Let U be an open bisection with compact closure and choose $\phi_n \in C_c(G^0)$ with $0 \leq \phi_n \prec \chi_{r(U)}$ and $\phi_n \nearrow \chi_{r(U)}$. Put $\psi_n = \chi_{U^{-1}} * \phi_n * \chi_U = \phi_n(U \cdot U^{-1})$. Then $0 \leq \psi_n \prec \chi_{s(U)}$ and $\psi_n \nearrow \chi_{s(U)}$. Let $f_n = \phi_n^{1/2} * \chi_U$ and $g_n = \psi_n^{1/2} * \chi_{U^{-1}}$. Then $f_n * g_n = \phi_n$, $g_n * f_n = \psi_n$, and

$$\mu(r(U)) = \lim_n \int \phi_n d\mu = \lim_n \tau(f_n * g_n) = \lim_n \tau(g_n * f_n) = \lim_n \int \psi_n d\mu = \mu(s(U)).$$

Since μ is invariant, for $\alpha \in G$, $\mu(\{s(\alpha)\}) = \mu(\{r(\alpha)\})$ so that if $\Omega \subseteq G^0$ is an orbit, then $\mu(\{x\}) = \mu(\{y\})$ for all $x, y \in \Omega$. If the isotropy on Ω is nontrivial, then Ω is infinite by (1) so that $\mu(\Omega) = 0$ since μ is a probability measure (since τ is a trace).

Now suppose $h \in C_c(G)$ with $\text{supp}(h) \cap G^0 = \emptyset$. Using a standard partition of unity argument, we may as well assume $\text{supp}(h) \subseteq U$ for some open bisection U with compact closure and $U \cap G^0 = \emptyset$, and also that $h \geq 0$. By (2), U contains only finitely many elements of $\mathrm{Iso}(G)$, say $\alpha_1, \ldots, \alpha_m$. Write $h = \rho * \chi_U$ with $\rho \in C_c(r(U))$. Choose a sequence (ρ_n) in $C_c(G^0)$ with $0 \leq \rho_n \leq \rho$, $\rho_n = 0$ near $\alpha_1, \ldots, \alpha_m$, and such that $\rho_n \to \rho$ pointwise on $r(U) \setminus \{r(\alpha_1), \ldots, r(\alpha_m)\}$. Since orbits with nontrivial isotropy have measure zero, it follows that $\rho_n \to \rho$ in $\mu-$measure. Let $h_n = \rho_n * \chi_U$. We show that $\tau(h_n) = 0$. Cover $\mathrm{supp}(h_n)$ with finitely many open bisections $W_{n,i} \subseteq U$ such that $W_{n,i} \cap \mathrm{Iso}(G) = \emptyset$. Using a partition of unity we may write h_n as a sum $\sum_i h_{n,i}$ with $\mathrm{supp}(h_{n,i}) \subseteq W_{n,i}$. Since $W_{n,i} \cap \mathrm{Iso}(G) = \emptyset$, for $\alpha \in W_{n,i}$, $r(\alpha) \neq s(\alpha)$, so that α has a neighborhood V such that $r(V) \cap s(V) = \emptyset$. Then we may as well assume $r(W_{n,i}) \cap s(W_{n,i}) = \emptyset$. Defining $\rho_{n,i}$ by $h_{n,i} = \rho_{n,i} * \chi_{W_{ni,}}$ with $\mathrm{supp}(\rho_{n,i}) \subseteq r(W_{n,i})$ we have

$$\begin{aligned}
\tau(h_{n,i}) &= \tau(\rho_{n,i}^{1/2} * \rho_{n,i}^{1/2} * \chi_{W_{n,i}}) \\
&= \tau(\rho_{n,i}^{1/2} * \chi_{W_{n,i}} * \rho_{n,i}^{1/2}) \\
&= \tau(\rho_{n,i}^{1/2} \rho_{n,i}({W_{n,i}}^{-1} \cdot W_{n,i})^{1/2} * \chi_{W_{n,i}}) \\
&= \tau(0) \\
&= 0,
\end{aligned}$$

since $\rho_{n,i}|_{s(W_{n,i})} = 0$. Therefore $\tau(h_n) = 0$. Recall the Cauchy-Schwartz inequality for states: if ω is a state on a C^*-algebra, then $|\omega(ab)| \leq \omega(aa^*)^{1/2}\omega(b^*b)^{1/2}$. Now we have

$$
\begin{aligned}
h - h_n &= (\rho - \rho_n) * \chi_U \\
&= (\rho - \rho_n)^{1/2} * ((\rho - \rho_n)^{1/2} * \chi_U) \\
&=: F_n * G_n; \\
|\tau(h - h_n)| &= |\tau(F_n * G_n)| \\
&\leq \tau(F_n * F_n^*)^{1/2} \tau(G_n^* * G_n)^{1/2} \\
&= \tau(\rho - \rho_n)^{1/2} \tau(\chi_{U^{-1}} * (\rho - \rho_n) * \chi_U)^{1/2}.
\end{aligned}
$$

But

$$
\begin{aligned}
\tau(\chi_{U^{-1}} * (\rho - \rho_n) * \chi_U)^{1/2} &= \tau((\rho - \rho_n)(U \cdot U^{-1})) \\
&= \int_{G^0} (\rho - \rho_n)(UxU^{-1}) d\mu(x) \\
&= \int_{G^0} (\rho - \rho_n) d\mu, \text{ by invariance} \\
&= \tau(\rho - \rho_n).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\tau(h - h_n)| &\leq \tau(\rho - \rho_n) \\
&= \int (\rho - \rho_n) d\mu \\
&\to 0
\end{aligned}
$$

and therefore $\tau(f) = 0$ so that τ must factor through E. □

We are nearly ready to show that $C^*(G)$ is isomorphic to the continued fraction AF algebra (in the sense of (3)) for the continued fraction expansion of

$$
\tau = (1 + \sqrt{5})/2 = [1, 1, 1, \dots]
$$

Let $\mathcal{A}_\tau$ denote this algebra. For a concise analysis of $\mathcal{A}_\tau$, see (1, Section VI.3) which we reference here.

Theorem 3.5.11. *$C^*(G) \cong \mathcal{A}_\tau$*

Proof. We will show that $Ell(C^*(G)) \cong Ell(\mathcal{A}_\tau)$ (where Ell denotes the Elliot invariant) and then appeal to recent classification results. To this end, as in (1), we note that

$$K_0(\mathcal{A}_\tau) = \mathbb{Z}^2$$

with positive cone

$$K_0(\mathcal{A}_\tau)_+ = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \ : \ y + \tau x \geq 0 \},$$

$[1_{\mathcal{A}_\tau}]_0 = \binom{1}{0}$, and with a unique state σ_* (coming from a unique trace σ) given by

$$\sigma_* \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = y + \tau x.$$

To this point, we have realized $K_0(C^*(G))$ as an isomorphic copy in the first term of the inductive sequence

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \cdots$$

with connecting maps $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ between the terms and where the first term is generated by

$$\begin{aligned} \tbinom{1}{0} &= [\chi_{Z(\alpha_1)\setminus Z(\alpha_1\beta_2)}]_0 \\ \text{and } \tbinom{0}{1} &= [\chi_{Z(\beta_1\beta_2)}]_0 \end{aligned}$$

Note that since

$$Z(v_1) = Z(\alpha_1) \setminus Z(\alpha_1\beta_2) \sqcup Z(\beta_1) \setminus Z(\beta_1\beta_2) \sqcup Z(\gamma_1) \setminus Z(\gamma_1\beta_2)$$

$$\sqcup Z(\beta_1\beta_2) \sqcup Z(\gamma_1\beta_2$$

The position of the unit in the first term is $\binom{3}{2}$. Since the connecting maps are constant and isomorphisms, we can "prepend" two more terms to the beginning of the inductive sequence, the new first term being $\mathbb{Z}^2$ generated by

$$\begin{aligned} \tbinom{1}{0} &= 2[\chi_{Z(\beta_1)^c}]_0 + [\chi_{Z(\beta_1)}]_0 \\ \text{and } \tbinom{0}{1} &= [\chi_{Z(v_1)}]_0 \end{aligned}$$

and the second being $\mathbb{Z}^2$ generated by

$$\begin{aligned} \tbinom{1}{0} &= [\chi_{Z(\beta_1)^c}]_0 \\ \text{and } \tbinom{0}{1} &= [\chi_{Z(\beta_1)}]_0 \end{aligned}$$

with the same connecting maps $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Since the connecting maps are all the same and isomorphisms, we can realize $K_0(C^*(G))$ as an isomorphic copy now in the first term of our new inductive sequence. The same analysis as before gives us a positive cone of

$$\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \ : \ y + \tau x \geq 0 \ \}.$$

We also have

$$\begin{aligned} [1]_0 &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-2} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= [\chi_{Z(v_1)}]_0 \end{aligned}$$

Now if tr is the unique trace on $C^*(G)$ and tr_* the induced state on $K_0(C^*(G))$, then

$$\begin{aligned} tr_*\binom{1}{0} &= tr(2\chi_{Z(\beta_1)^c} + \chi_{Z(\beta_1)}) \\ &= \tfrac{2}{\tau} + \tfrac{1}{\tau^2} \\ &= 2(\tau-1) + (\tau-1)^2 \\ &= 2\tau - 2 + \tau^2 - 2\tau + 1 \\ &= -1 + (\tau+1) \\ &= \tau \end{aligned}$$

and $tr_*\binom{0}{1} = tr(\chi_{Z(v_1)}) = 1$ so that $tr_* = \sigma_*$ and thus, $Ell(C^*(G)) \cong Ell(\mathcal{A}_\tau)$.

It remains to check that $C^*(G)$ is classifiable. We will do this by verifying that $C^*(G)$ is separable, simple, unital, nuclear, $Z-$stable, and UCT, and then it will follow from the final theorem of (19) that $C^*(G)$ is classified by $Ell(C^*(G))$. Our first step will be to show that $C^*(G)$ has finite nuclear dimension.

Recall for $i > 0$ the short-exact sequences

$$0 \to \bigoplus_{\ell > i} (M_{n(\ell)} \otimes C(X_{\ell+1})) \to C^*(G_i) \to C^*(G_i|_{F_i}) \to 0$$

and

$$0 \to M_{m(i)} \otimes (\mathcal{K} \oplus \mathcal{K}) \to C^*(G_i|_{F_i}) \to M_{m(i)} \otimes C(\mathbb{T}) \to 0$$

By (20, Remark 2.2(iii)),

$$\dim_{nuc} M_{m(i)} \otimes (\mathcal{K} \oplus \mathcal{K}) = 0$$

and by (20, Proposition 2.4 and Corollary 2.8(i)),

$$\dim_{nuc} M_{m(i)} \otimes C(\mathbb{T}) = \dim \mathbb{T} = 1$$

(where dim denotes covering dimension). Then by (20, Proposition 2.9),

$$\dim_{nuc} C^*(G_i|_{F_i}) \leq 2$$

Now since each $X_{\ell+1}$ is a Cantor set, $\oplus_{\ell>i}(M_{n(\ell)} \otimes C(X_{\ell+1}))$ is AF, so by (20, Remark 2.2(iii)),

$$\dim_{nuc} \bigoplus_{\ell>i} (M_{n(\ell)} \otimes C(X_{\ell+1})) = 0$$

Another application of (20, Proposition 2.9) then shows that

$$\dim_{nuc} C^*(G_i) \leq 3$$

Since this holds for all $i > 0$, (20, Proposition 2.3(iii)) shows that $\dim_{nuc} C^*(G) \leq 3$. By (20, Remark 2.2(i)), $C^*(G)$ is nuclear, so by (15, Theorem 4.1.5 and Theorem 4.1.7) G is amenable and $C^*(G)$ satisfies the UCT.

To see that $C^*(G)$ is simple, we will show that G is topologically free (that is $\{ x \in G^0 : xGx = \{x\} \}$ is dense in G^0) and minimal (for every $x \in G^0$, the orbit of x is dense in G^0). Then applications of (15, Lemma 4.2.3 and Theorem 4.3.6) will show that $C^*(G)$ is simple.

To check topological freeness, we first make the following claim:

Claim 3.5.12. *For any $\mu \in v_0\Lambda$, the element $\mu\eta^{|\mu|+1}(\infty, \infty)$ has non-trivial isotropy, and $xGx = \{x\}$ for any unit x such that $x \neq \mu\eta^{|\mu|+1}(\infty, \infty)$ for any $\mu \in \Lambda$.*

To see this claim holds, first fix some $\mu \in v_0\Lambda$ and $m > |\mu|$ and consider

$$a = [\mu\alpha_{|\mu|+1} \dots \alpha_m, \mu\beta_{|\mu|+1} \dots \beta_m, \eta^{m+1}(\infty, \infty)]$$

Then $s(a) = r(a) = \mu\eta^{|\mu|+1}(\infty, \infty)$ so that any such element is in the isotropy group of $\mu\eta^{|\mu|+1}(\infty, \infty)$.

Now consider a unit x not of the form $\mu\eta^{|\mu|+1}(\infty, \infty)$. Then, thinking of x as an infinite word as in Definition 3.2.3, we have three possibilities; $x_j = \gamma_i$ for infinitely many j, $|\{j : x_j = \alpha_j\}|$ is finite, and $|\{j : x_j = \beta_j\}|$ is finite.

Suppose first that $x_j = \gamma_j$ for infinitely many j, and let $[\sigma, \tau, y] \in G$ with $\sigma y = \tau y = x$. Choose $p > |\sigma| = |\tau|$ such that $x_p = \gamma_p$ and write

$$\begin{aligned}\sigma y = \tau y &= \sigma y_1 \gamma_p y' \\ &= \tau y_1 \gamma_p y'\end{aligned}$$

Definition 3.2.3 applied to $\sigma y_1 \gamma_p y' = \tau y_1 \gamma_p y'$ shows that Proposition 3.1.2 applies to σy_1 and τy_1 so that $\sigma y_1 = \tau y_1$ and hence $\sigma = \tau$ and $[\sigma, \tau, y]$ is a unit.

Now suppose $|\{j : x_j = \alpha_j\}|$ is finite and again let $[\sigma, \tau, y] \in G$ with $\sigma y = \tau y = x$. If there exists a p with $y_p = \gamma_p$ then the previous argument applies, so suppose $y_j \in \{\alpha_j, \beta_j\}$ for all $j > |\sigma| = |\tau|$. Let $p = \max\{j : \sigma_j = \gamma_j = \tau_j\}$, taking $p = 0$ if the defining set is empty. Then

$$\begin{aligned}|\{j : (\sigma y)_j = \alpha_j, j > p\}| &= |\{j : \sigma_j = \alpha_j, j > p\}| \\ &+ |\{j : y_j = \alpha_j\}| \\ \text{and } |\{j : (\tau y)_j = \alpha_j, j > p\}| &= |\{j : \tau_j = \alpha_j, j > p\}| \\ &+ |\{j : y_j = \alpha_j\}|\end{aligned}$$

Since $|\{j : y_j = \alpha_j\}| < \infty$, $|\{j : \sigma_j = \alpha_j, j > p\}| = |\{j : \tau_j = \alpha_j, j > p\}|$. Since $|\sigma| = |\tau|$ (and $r(\sigma) = r(\tau)$), we must also have $|\{j : \sigma_j = \beta_j, j > p\}| = |\{j : \tau_j = \beta_j, j > p\}|$. Definition 3.2.3 applied to $\sigma y = \tau y$ and then an applications of Proposition 3.1.2 shows that $\sigma_1 \dots \sigma_p = \tau_1 \dots \tau_p$ so that $\sigma = \tau$ by another application of Proposition 3.1.2.

The argument when $|\{j : x_j = \beta_j\}|$ is finite is analogous, and we conclude $xGx = \{x\}$ so that Claim 3.5.12 holds.

Claim 3.5.12 above and the first paragraph of its proof show that G satisfies condition (1) of Proposition 3.5.10. To see that condition (2) is satisfied, suppose we have

$$[\sigma, \tau, x] \in [\sigma, \tau, Z(v_{|\tau|+1})] \cap \text{Iso}(G) \setminus G^0$$

so that $\sigma x = \tau x$. If $x_j = \gamma_j$ for some j, then Definition 3.2.3 and Proposition 3.1.2 imply $\sigma x_{|\sigma|+1} \dots x_j = \tau x_{|\tau|+1} \dots x_j$ so that $[\sigma, \tau, x]$ is a unit. This together with Claim 3.5.12 imply

$$[\sigma, \tau, Z(v_{|\tau|+1})] \cap \text{ Iso}(G) \setminus G^0 = \{[\sigma, \tau, \eta^{|\tau|+1}(\infty, \infty)\}.$$

We now show that G is topologically free.

Claim 3.5.13. *Fix $x = \mu\eta^{|\mu|+1}(\infty, \infty) \in G^0$. Then x is a limit point of $\{ x \in G^0 : xGx = \{x\} \}$.*

To prove this claim, fix any $Z(\lambda) \setminus \cup_{j=1}^n Z(\sigma_j)$ containing x. Since $\lambda \in x$ but $\sigma_j \notin x$ for any j, there must be some $x_1 \dots x_m \in x$ which extends λ but none of the σ_j and which is longer than all of the σ_j. Then $y = x_1 \dots x_m \gamma_{m+1} \gamma_{m+2} \dots$ is a unit which is contained in $Z(\lambda) \setminus \cup_{j=1}^n Z(\sigma_j)$. This follows since we have $\lambda \in y$ by construction, and if $\sigma_k \in y$, then σ_k and $x_1 \dots x_m \gamma_{m+1}$ have a common extension. Then there exist $\mu, \nu \in \Lambda$ such that $\sigma_k \mu = x_1 \dots x_m \gamma_{m+1} \nu$. Since $|x_1 \dots x_m| > |\sigma_k|$, $\mu_{m+1} = \gamma_{m+1}$ and Proposition 3.1.2 implies $\sigma_k \mu_n \dots \mu_m = x_1 \dots x_m \in x$ where $n = s(\sigma_k)$, but this implies $\sigma_k \in x$. Claim 3.5.12 shows that y has trivial isotropy, and the conclusion of Claim 3.5.13 follows.

We next show that G is minimal. Fix $x \in G^0$ and consider its orbit $r(Gx)$. Note that for any $\mu \in v_1\Lambda$ and $x' \in X_{|\mu|+1}$ such that $x = \mu x'$, and any $\nu \in v_1\Lambda$ with $|\mu| = |\nu|$, we have $\nu x' \in r(Gx)$ since $[\mu, \nu, x']$ has range $\nu x'$ and source x. Now fix any $Z(\lambda) \setminus \cup_{j=1}^n Z(\sigma_j) \neq \emptyset$ and $y \in Z(\lambda) \setminus \cup_{j=1}^n Z(\sigma_j)$. Since $\lambda \in y$ and $\sigma_j \notin y$ for any j, choose $y_1 \dots y_m \in y$ which extends λ and is longer than all the σ_j. Let $x' \in X_{m+1}$ be such that $x = x_1 \dots x_{m+1} x'$ and let $z = y_1 \dots y_m \gamma_{m+1} x'$. The same argument as in the proof of Claim 3.5.13 shows that $z \in Z(\lambda) \setminus \cup_{j=1}^n Z(\sigma_j)$ and the remarks at the beginning of this paragraph show that $z \in r(Gx)$ so that $r(Gx)$ is dense in G^0.

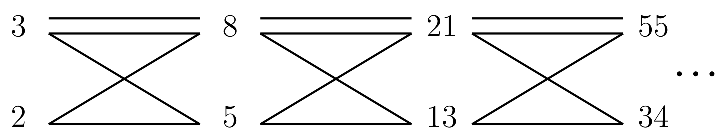

Figure 3.2: The Bratteli Diagram of A

Thus we conclude that $C^*(G)$ is simple. It is unital since G^0 is compact and separable since Λ is countable. Finally, $C^*(G)$ is $Z-$stable by (19, Theorem A) since, as we saw, it has finite nuclear dimension. Then $C^*(G)$ is classified by its Elliot invariant by the final theorem in (19) so that $C^*(G) \cong \mathcal{A}_\tau$. □

This mostly concludes our analysis of $C^*(G)$ but we make two final observations.

Remark 3.5.14. While $C^*(G)$ is AF, G is not an AF groupoid. This follows since G has nontrivial isotropy groups, and AF groupoids must be principal (see, e.g., (13, III.1)).

Remark 3.5.15. There exists an AF algebra A which is a proper subalgebra of and isomorphic to $C^*(G)$, and is the limit of a sequence of finite-dimensional subalgebras of $C^*(G)$. We show this below.

Proof. Recall the sets A_i and B_i defined beginning on page 63. We saw that each A_i contained f_{2i+2} sets of the form $Z(x_1 \dots x_i) \setminus Z(x_1 \dots x_i\beta_{i+1})$ and each B_i contained f_{2i+1} sets of the form $Z(x_1 \dots x_i\beta_{i+1})$. Now, for $Z(\mu) \setminus Z(\mu\beta_{|\mu|+1}), Z(\nu) \setminus Z(\nu\beta_{|\nu|+1}) \in A_i$ and $Z(\sigma\beta_{|\sigma|+1}), Z(\theta\beta_{|\theta|+1}) \in B_i$, the maps

$$\begin{aligned} \chi_{[\mu,\nu,Z(\beta_{|\mu|+1})^c]} &\mapsto e_{\mu\nu} \oplus 0 \\ \text{and } \chi_{[\sigma,\theta,Z(\beta_{|\sigma|+1})]} &\mapsto 0 \oplus e_{\sigma\theta} \end{aligned}$$

define an isomorphism from a subset of $C_c(G)$ onto $M_{f_{2i+1}} \oplus M_{f_{2i+2}}$. This defines an inductive sequence with associated Bratteli diagram in figure 3.2 where the multiplicities follow from our observations about A_{i+1} and B_{i+1} refining A_i and B_i (noting that the diagonal elements are of the forms χ_D where $D \in A_i \sqcup B_i$). The generators of the K_0 group of each term are identical to those of the corresponding term in the inductive sequence we used to define $K_0(C^*(G))$ and essentially the same analysis shows that the AF algebra defined above is isomorphic to $C^*(G)$.

To see that this is a proper subalgebra, we first show the following:

Claim 3.5.16. *For any $i > 0$, and any $\mu, \nu, \sigma, \theta \in \Lambda$ with $|\mu| = |\nu| = |\sigma| = |\theta| = i$ such that $Z(\mu) \setminus Z(\mu\beta_{i+1}), Z(\nu) \setminus Z(\nu\beta_{i+1}) \in A_i$ and $Z(\sigma\beta_{i+1}), Z(\theta\beta_{i+1}) \in B_i$, we have*

$$[\alpha_1, \beta_1, \eta^2(\infty, \infty)] \notin [\mu, \nu, Z(\beta_{i+1})^c], [\sigma, \theta, Z(\beta_{i+1})]$$

To see this, first, since $\eta^{i+1}(\infty, \infty) \notin Z(\beta_{i+1})^c$, $[\alpha_1, \beta_1, \eta^2(\infty, \infty)] \notin [\mu, \nu, Z(\beta_{i+1})^c]$ for any i and μ, ν such that $Z(\mu) \setminus Z(\mu\beta_{i+1}), Z(\nu) \setminus Z(\nu\beta_{i+1}) \in A_i$.

To show that $[\alpha_1, \beta_1, \eta^2(\infty, \infty)] \notin [\sigma, \theta, Z(\beta_{i+1})]$ for any σ, θ such that $Z(\sigma\beta_{i+1}), Z(\theta\beta_{i+1}) \in B_i$, we will induct on i. When $i = 1$, $B_i = \{\ Z(\gamma_1\beta_2), Z(\beta_1\beta_2)\ \}$. Since $[\alpha_1, \beta_1, \eta^2(\infty, \infty)]$ is not a unit, $[\alpha_1, \beta_1, \eta^2(\infty, \infty)] \notin [v_1, v_1, Z(\sigma\beta_2)]$ for either $Z(\sigma\beta_2) \in B_1$. It's also clear that $[\alpha_1, \beta_1, \eta^2(\infty, \infty)]$ is in neither $[\beta_1, \gamma_1, Z(\beta_2)]$ nor $[\gamma_1, \beta_1, Z(\beta_2)]$ since the source and range of $[\alpha_1, \beta_1, \eta^2(\infty, \infty)]$ contain no γ's.

Now suppose for some $i \geq 1$ and any $Z(\sigma), Z(\theta) \in B_i$ that $[\alpha_1, \beta_1, \eta^2(\infty, \infty)] \notin [\sigma, \theta, Z(v_{i+2})]$. Following our construction beginning on page 63, we have

$$B_{i+1} = \{\ Z(\sigma\beta_{i+2}), Z(\mu\gamma_{i+1}\beta_{i+2})\ :\ Z(\sigma) \in B_i, Z(\mu) \setminus Z(\mu\beta_{i+1}) \in A_i\ \}$$

Again, since $[\alpha_1, \beta_1, \eta^2(\infty, \infty)]$ is not a unit, it is not in $[\sigma, \sigma, Z(\beta_{i+2})]$ for any $Z(\sigma\beta_{i+2}) \in B_{i+1}$. Consider the remaining possible bisections; they are of the form

$$\begin{array}{lll} (1) & [\sigma\beta_{i+2}, \theta\beta_{i+2}, Z(v_{i+3})] & Z(\sigma), Z(\theta) \in B_i \\ (2) & [\sigma\beta_{i+2}, \mu\gamma_{i+1}\beta_{i+2}, Z(v_{i+3})] & Z(\sigma) \in B_i, Z(\mu) \setminus Z(\mu\beta_{i+1}) \in A_i \\ \text{or} & [\mu\gamma_{i+1}\beta_{i+2}, \sigma\beta_{i+2}, Z(v_{i+3})] & \\ (3) & [\mu\gamma_{i+1}\beta_{i+2}, \nu\gamma_{i+1}\beta_{i+2}, Z(v_{i+3})] & Z(\mu) \setminus Z(\mu\beta_{i+1}), Z(\nu) \setminus Z(\nu\beta_{i+1}) \in A_i \end{array}$$

Since the source and range of $[\alpha_1, \beta_1, \eta^2(\infty, \infty)]$ contain no γ's, it is not in any of the sets of the form in (2) or (3). Since

$$[\sigma\beta_{i+2}, \theta\beta_{i+2}, Z(v_{i+3})] = [\sigma, \theta, Z(\beta_{i+2})] \subseteq [\sigma, \theta, Z(v_{i+2})]$$

and $[\alpha_1, \beta_1, \eta^2(\infty, \infty)] \notin [\sigma, \theta, Z(v_{i+2})]$ by assumption, $[\alpha_1, \beta_1, \eta^2(\infty, \infty)]$ is not in any of the sets of the form in (1). This proves the claim.

We can now show that A is a proper subalgebra. Let $u = [v_1, v_1, \eta^1(\infty, \infty)]$ and π_u be the regular representation of $C_c(G)$ on $\ell^2(G_u)$. Let $h = [\alpha_1, \beta_1, \eta^2(\infty, \infty)] \in G_u$, $f = \chi_{[\alpha_1, \beta_1, Z(v_2)]} \in C_c(G)$, and g be any characteristic function in the sequence defining A (so that $g(h) = 0$, by Claim 3.5.16 above). Then

$$\begin{aligned} \langle \pi(f-g)\delta_u, \delta_h \rangle &= \langle (f-g) * \delta_u, \delta_h \rangle \\ &= \textstyle\sum_{x \in G_u} (f-g)(xu^{-1}) \langle \delta_x, \delta_h \rangle \\ &= (f-g)(hu) \\ &= (f-g)(h) \\ &= 1 \end{aligned}$$

Then

$$
\begin{aligned}
||f-g||_{C^*(G)} &\geq ||\pi_u(f-g)||_{op} \cdot ||\delta_u|| \cdot ||\delta_h|| \\
&\geq 1
\end{aligned}
$$

for any g in the inductive sequence and therefore $f \notin A$. □

Chapter 4

GENERALIZING THE PREVIOUS EXAMPLE

Our next goal is to generalize the previous example. Consider the category of paths Λ given by the 1-graph in figure 4.1 where for some $m \in \mathbb{Z}^+$, $k_i = k_{m+i}$ for every $i \geq 1$ and where $\alpha_i\beta_{i+1} = \beta_i\alpha_{i+1}$.

Much of the work in defining and analyzing $C^*(\Lambda)$ will closely resemble that of the first example. Many analogues of the previous results hold with fairly obvious changes. To mitigate repetition, we will try to point out these without spending much or any time on the details, but also slow down when the differences are significant.

The precise definition of Λ is done nearly the same as in chapter 3, with the modification that

$$\Sigma_i = \{\alpha_i, \beta_i, \gamma_i^j \ 1 \leq j \leq k_i\} \text{ for } i \geq 1$$

We define equivalence of paths in the same way, and following the arguments preceding it, adding superscripts where appropriate, we have the analogue of Proposition 3.1.2:

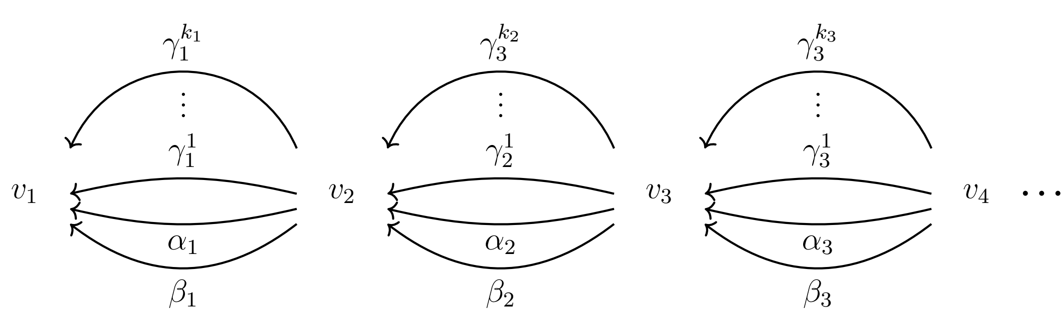

Figure 4.1: The 1-Graph Defining the General Example

Proposition 4.0.1. *For $\mu, \nu \in \Sigma^*$, μ and ν define the same element of Λ if and only if the following conditions hold:*

1. $r(\mu) = r(\nu)$ *and* $s(\mu) = s(\nu)$

2. $\mu_i = \gamma_i^j$ *iff* $\nu_i = \gamma_i^j$

3. *If* $\mu_j = \gamma_j^k = \nu_j$, $\mu_{j'} = \gamma_{j'}^{k'} = \nu_{j'}$, *and* $\mu_i \neq \gamma_i^\ell \neq \nu_i$ *for* $j < i < j'$ *and any* ℓ, *then*

$$|\{\ i\ :\ \mu_i = \alpha_i,\ j < i < j'\ \}| = |\{\ i\ :\ \nu_i = \alpha_i,\ j < i < j'\ \}|$$

$$\text{and } |\{\ i\ :\ \mu_i = \beta_i,\ j < i < j'\ \}| = |\{\ i\ :\ \nu_i = \beta_i,\ j < i < j'\ \}|$$

4. *If* $\mu_j = \gamma_j^k = \nu_j$, *and* $\mu_i \neq \gamma_i^\ell \neq \nu_i$ *for* $j < i$ *and any* ℓ, *then*

$$|\{\ i\ :\ \mu_i = \alpha_i,\ j < i\ \}| = |\{\ i\ :\ \nu_i = \alpha_i,\ j < i\ \}|$$

$$\text{and } |\{\ i\ :\ \mu_i = \beta_i,\ j < i\ \}| = |\{\ i\ :\ \nu_i = \beta_i,\ j < i\ \}|$$

5. *If* $\mu_j = \gamma_j^k = \nu_j$, *and* $\mu_i \neq \gamma_i^\ell \neq \nu_i$ *for* $i < j$ *and any* ℓ, *then*

$$|\{\ i\ :\ \mu_i = \alpha_i,\ i < j\ \}| = |\{\ i\ :\ \nu_i = \alpha_i,\ i < j\ \}|$$

$$\text{and } |\{\ i\ :\ \mu_i = \beta_i,\ i < j\ \}| = |\{\ i\ :\ \nu_i = \beta_i,\ i < j\ \}|$$

6. *If* $\mu_i \neq \gamma_i^\ell \neq \nu_i$ *for any* i, ℓ, *then*

$$|\{\ i\ :\ \mu_i = \alpha_i\ \}| = |\{\ i\ :\ \nu_i = \alpha_i\ \}|$$

$$\text{and } |\{\ i\ :\ \mu_i = \beta_i\ \}| = |\{\ i\ :\ \nu_i = \beta_i\ \}|$$

Using this and arguing as in the proof of Corollary 3.1.6, we have its analogue:

Corollary 4.0.2. *Suppose $\mu, \nu \in \Lambda$ have a common extension and neither extends the other. Let $p = \min\{|\mu|, |\nu|\}$ and $q = \max\{\ j\ :\ \mu_j^k = \gamma_j^k = \nu_j^k, j \leq p,\ some\ k\ \}$, where $q = 0$ if this set is empty. Then $\mu_j^k \neq \gamma_j^k$ and $\nu_j^k \neq \gamma_j^k$ for any $j > q$ and any k.*

We define a groupoid G in the same way as before, including identifying directed hereditary sets with (possibly infinite) words in Λ. We identify infinite words with the analogue of Definition 3.2.3:

Definition 4.0.3. Given an infinite word $x = x_i x_{i+1} \ldots$, we define the **range** of x to be $\tilde{r}(x) := r(x_i) = v_i$, typically written $r(x)$. Given two infinite words x and y, we say x is **equivalent** to y if the following conditions hold:

1. $r(x) = r(y)$

2. $x_i = \gamma_i^j$ iff $y_i = \gamma_i^j$

3. If $x_j = \gamma_j^k = y_j$, $x_{j'} = \gamma_{j'}^{k'} = y_{j'}$, and $x_i \neq \gamma_i^\ell \neq y_i$ for $j < i < j'$ and any ℓ, then

$$|\{\ i\ :\ x_i = \alpha_i,\ j < i < j'\ \}| = |\{\ i\ :\ y_i = \alpha_i,\ j < i < j'\ \}|$$

$$\text{and } |\{\ i\ :\ x_i = \beta_i,\ j < i < j'\ \}| = |\{\ i\ :\ y_i = \beta_i,\ j < i < j'\ \}|$$

4. If $x_j = \gamma_j^k = y_j$, and $x_i \neq \gamma_i^\ell \neq y_i$ for $j < i$ and any ℓ, then

$$|\{\ i\ :\ x_i = \alpha_i,\ j < i\ \}| = |\{\ i\ :\ y_i = \alpha_i,\ j < i\ \}|$$

$$\text{and } |\{\ i\ :\ x_i = \beta_i,\ j < i\ \}| = |\{\ i\ :\ y_i = \beta_i,\ j < i\ \}|$$

5. If $x_j = \gamma_j^k = y_j$, and $x_i \neq \gamma_i^\ell \neq y_i$ for $i < j$ and any ℓ, then

$$|\{\ i\ :\ x_i = \alpha_i,\ i < j\ \}| \ = \ |\{\ i\ :\ y_i = \alpha_i,\ i < j\ \}|$$

$$\text{and } |\{\ i\ :\ x_i = \beta_i,\ i < j\ \}| \ = \ |\{\ i\ :\ y_i = \beta_i,\ i < j\ \}|$$

6. If $x_i \neq \gamma_i^\ell \neq y_i$ for any i, ℓ, then

$$|\{\ i\ :\ x_i = \alpha_i\ \}| \ = \ |\{\ i\ :\ y_i = \alpha_i\ \}|$$

$$\text{and } |\{\ i\ :\ x_i = \beta_i\ \}| \ = \ |\{\ i\ :\ y_i = \beta_i\ \}|$$

We will make the same simplification as before, namely, consider $G|_{X_1}$, for the same reason as before, and will drop the subscript and simply writing G henceforth. We will again realize $C^*(G)$ as an inductive limit of sub-algebras $C^*(G_i)$. Unlike before, we will only treat the general i^{th} case. First we define

$$G_i = \langle\ [\sigma, \tau, x]\ :\ |\sigma| = |\tau| \leq i\ \rangle$$

and adopt the convention that for $[\sigma, \tau, x] \in G_i$, if

$$|\{\ j\ :\ \sigma_j = \alpha_j \text{ and } \sigma_k \neq \gamma_k^\ell, k > j \text{ and any } \ell\}| = m > 0$$
$$|\{\ j\ :\ \tau_j = \alpha_j \text{ and } \tau_k \neq \gamma_k^\ell, k > j \text{ and any } \ell\}| = n > 0$$

then we write $[\sigma, \tau, x] = [\sigma', \tau', \alpha_j \ldots \alpha_{j+p-1}x]$ where $p = \min\{m, n\}$ and likewise for β's (again, we are "factoring" the common number of α's and β's after the last γ's in σ and τ into the third coordinate).

Proposition 4.0.4. *Given the convention outlined above,*

$$G_i = \{\ [\mu\alpha_{|\mu|+1} \ldots \alpha_m, \nu\beta_{|\nu|+1} \ldots \beta_m, x], [\sigma\beta_{|\sigma|+1} \ldots \beta_n, \tau\alpha_{|\tau|+1} \ldots \alpha_n, y]\ :$$
$$|\mu|, |\nu|, |\sigma|, |\tau| \leq i, \mu_{|\mu|} = \gamma^j_{|\mu|}, \nu_{|\nu|} = \gamma^k_{|\nu|}, \sigma_{|\sigma|} = \gamma^\ell_{|\sigma|}, \tau_{|\tau|} = \gamma^m_{|\tau|} \textit{ some } j, k, \ell, m\}.$$

Proof. The proof is nearly identical to the proof of Proposition 3.4.1, making some obvious changes in notation. □

We now begin our analysis of $C^*(G_i)$.

Theorem 4.0.5. *There are positive integers $m(i)$, $n(\ell)$ and k_ℓ for $\ell > i$, a closed invariant set $F_i \subseteq X_1$, and exact sequences*

$$(1)\ 0 \longrightarrow \bigoplus_{\ell>i}(\bigoplus_{j=1}^{k_\ell}(M_{n(\ell)} \otimes C(X_{\ell+1}))) \longrightarrow C^*(G_i) \longrightarrow C^*(G_{i|F_i}) \longrightarrow 0,$$

$$(2)\ 0 \longrightarrow M_{m(i)} \otimes (\mathcal{K} \oplus \mathcal{K}) \to C^*(G_{i|F_i}) \to M_{m(i)} \otimes C(\mathbb{T}) \longrightarrow 0$$

Proof. Let

$$\begin{aligned} U_i &= \{x \in X_1 : x_j = \gamma_j^k, \text{ some } j > i, 1 \le k \le k_j\} \\ F_i &= X_1 \setminus U_i = \{x \in X_1 : x_j \ne \gamma_j^k, \text{ any } j > i\}. \end{aligned}$$

Given Proposition 4.0.4, it is evident that U_i is invariant for G_i and it's open by an argument similar to before. Then F_i is closed and invariant so we have

$$0 \longrightarrow C^*(G_{i|U_i}) \longrightarrow C^*(G_i) \longrightarrow C^*(G_{i|F_i}) \longrightarrow 0.$$

As before, we begin by looking at the ideal, $C^*(G_{i|U_i})$.

For $\ell > i$, let $\Omega_\ell = \{\ x_1 \dots x_{\ell-1}\ :\ x_j \ne \gamma_j^k,\ i < j < \ell\ \}$. Then define

$$\begin{aligned} E_\ell &:= \{x \in U_i : x_\ell = \gamma_\ell^m, \text{ some } 1 \le m \le k_\ell \text{ and } x_j \ne \gamma_j^k, \text{ any } i < j < \ell, \text{ any } k\} \\ &= \sqcup_{\sigma\in\Omega_\ell}(\sqcup_{j=1}^{k_\ell} Z(\sigma\gamma_\ell^j)). \end{aligned}$$

Each E_ℓ is compact-open, and by Proposition 4.0.4, is invariant for G_i. Also, $U_i = \bigsqcup_{\ell>i} E_\ell$ and again we have

$$C^*(G_{i|U_i}) \cong \bigoplus_{\ell > i} C^*(G_{i|E_\ell}).$$

For $\mu_0, \mu, \nu \in \Omega_\ell$, $F \subseteq X_{\ell+1}$ compact-open, and $1 \le j \le k_\ell$, the maps

$$\chi_{[\mu,\nu,\{\gamma_\ell^j\}\times X_{\ell+1}]} \mapsto e^j_{\mu\nu} \otimes 1_{C(X_{\ell+1})}$$
$$\chi_{[\mu_0,\mu_0,\{\gamma_\ell^j\}\times F]} \mapsto e^j_{\mu_0\mu_0} \otimes \chi_F$$

give an isomorphism

$$C^*(G_{i|E_\ell}) \cong \bigoplus_{j=1}^{k_\ell} (M_{n(\ell)} \otimes C(X_{\ell+1}))$$

where $n(\ell) \in \mathbb{N}$ depends on ℓ, so that

$$C^*(G_{i|U_i}) \cong \bigoplus_{\ell > i} (\bigoplus_{j=1}^{k_\ell} (M_{n(\ell)} \otimes C(X_{\ell+1}))).$$

For $C^*(G_{i|F_i})$ decompose F_i into

$$\begin{aligned} F_i^\infty &= \{\sigma\eta^{|\sigma|+1}(\infty,\infty) \ : \ 0 \le |\sigma| \le i\} \\ \text{and } F_i^0 &= F_i \setminus F_i^\infty \\ &= \{\sigma\eta^{|\sigma|+1}(j,k) \ : \ 0 \le |\sigma| \le i, \ j,k \text{ not both } \infty \ \}. \end{aligned}$$

Again, we get a short exact sequence

$$0 \longrightarrow C^*(G_{i|F_i^0}) \longrightarrow C^*(G_{i|F_i}) \longrightarrow C^*(G_{i|F_i^\infty}) \longrightarrow 0.$$

The maps

$$\chi_{[\alpha_1,\beta_1,\eta^2(\infty,\infty)]} \mapsto e_{11} \otimes z$$
$$\chi_{[\sigma,\alpha_1\ldots\alpha_{|\sigma|},\eta^{|\sigma|+1}(\infty,\infty)]} \mapsto e_{\sigma 1} \otimes 1$$

where $0 \le |\sigma| \le i$ define a $*-$isomorphism from $C^*(G_{i|F_i^\infty})$ to $M_{m(i)} \otimes C(\mathbb{T})$, where $m(i)$ is the number of choices for σ. The maps

$$\chi_{[\beta_1\ldots\beta_j,\alpha_1\ldots\alpha_j,\eta^{j+1}(0,\infty)]} \mapsto e_{11}\otimes(0\oplus e_{0j})$$

$$\chi_{[\beta_1\ldots\beta_{|\sigma|},\sigma,\eta^{|\sigma|+1}(0,\infty)]} \mapsto e_{1\sigma}\otimes(0\oplus e_{00})$$

$$\chi_{[\alpha_1\ldots\alpha_j,\beta_1\ldots,\beta_j,\eta^{j+1}(\infty,0)]} \mapsto e_{11}\otimes(e_{0j}\oplus 0)$$

$$\chi_{[\alpha_1\ldots\alpha_{|\tau|},\tau,\eta^{|\tau|+1}(\infty,0)]} \mapsto e_{1\tau}\otimes(e_{00}\oplus 0)$$

where $0\le|\sigma|,|\tau|\le i$ define a $*-$isomorphism from $C^*(G_{i|F_i^0})$ to $M_{m(i)}\otimes(\mathcal{K}\oplus\mathcal{K})$. This yields

$$0\longrightarrow M_{m(i)}\otimes(\mathcal{K}\oplus\mathcal{K})\xrightarrow{\iota} C^*(G_{i|F_i})\xrightarrow{\pi} M_{m(i)}\otimes C(\mathbb{T})\longrightarrow 0$$

□

Nearly identical reasoning as in the first example, beginning on page 46, show that $K_1(C^*(G_i|_{F_i}))=0$ and $K_0(C^*(G_i|_{F_i}))\cong\mathbb{Z}^2$, generated by $[\chi_{[v_1,v_1,\eta^1(\infty,0)]}]_0$ and $[\chi_{[v_1,v_1,Z(\beta_1\ldots\beta_i)\cap F_i]}]_0$. Then, again, we have

$$K_0(C^*(G_i))\cong(\bigoplus_{\ell>i}(\bigoplus_{j=1}^{k_\ell}C(X_{\ell+1},\mathbb{Z}))\oplus\mathbb{Z}^2$$

where the right summand is generated by $[\chi_{[v_1,v_1,Z(\alpha_1\ldots\alpha_{i-1})\setminus Z(\alpha_1\ldots\alpha_{i-1}\beta_i)]}]_0$ and $[\chi_{[v_1,v_1,Z(\beta_1\ldots\beta_i)]}]_0$. We should point out that the indices on the generators were chosen to be one lower than those in the previous example (see, for example, the proof of Theorem 3.4.4). This was done to facilitate the computation of $K_0(C^*(G))_+$ (in particular, so that the connecting maps B_i defined on page 103 are given by $\begin{pmatrix} k_i+1 & 1 \\ k_i & 1\end{pmatrix}$ and not $\begin{pmatrix} k_{i+1}+1 & 1 \\ k_{i+1} & 1\end{pmatrix}$).

Positive elements in $K_0(C^*(G_i))$ have a similar characterization to those in our first example, but we feel some of the details are significant enough to treat carefully. Like before, for $a\in K_0(C^*(G_i))$ write

$$a = \sum_{\ell > i} \sum_{j=1}^{k_\ell} \sum_{k=1}^{m_{j,\ell}} c_{\ell,j,k} [\chi_{\{z_{\ell,j,k}\} \times F_{\ell,j,k}}]_0 + m[\chi_{Z(\alpha_1 \dots \alpha_{i-1}) \setminus Z(\alpha_1 \dots \alpha_{i-1}\beta_i)}]_0 + n[\chi_{Z(\beta_1 \dots \beta_i)}]_0$$

where each $F_{\ell,j,k} \subseteq X_{\ell+1}$ is compact-open and for each ℓ and $1 \leq j \leq k_\ell$, $\{ F_{\ell,j,k} : 1 \leq k \leq m_{j,\ell} \}$ is a partition of $X_{\ell+1}$. Now, for each ℓ, we take $\{F_{\ell,k}\}$ to be a common refinement of $\{ F_{\ell,j,k} : 1 \leq k \leq m_{j,\ell} \}_{j=1}^{k_\ell}$ and we may write

$$a = \sum_{\ell > i} \sum_{j=1}^{k_\ell} \sum_{k=1}^{m_\ell} c_{\ell,j,k} [\chi_{\{z_{\ell,j,k}\} \times F_{\ell,k}}]_0 + m[\chi_{Z(\alpha_1 \dots \alpha_{i-1}) \setminus Z(\alpha_1 \dots \alpha_{i-1}\beta_i)}]_0 + n[\chi_{Z(\beta_1 \dots \beta_i)}]_0$$

Observe now that

$$\begin{aligned}
& Z(\alpha_1 \dots \alpha_j) \setminus Z(\alpha_1 \dots \alpha_{j-1}\beta_j) \\
= \quad & (\sqcup_{r=1}^{k_j} Z(\alpha_1 \dots \alpha_{j-1}\gamma_j^r)) \sqcup Z(\alpha_1 \dots \alpha_j) \setminus Z(\alpha_1 \dots \alpha_j \beta_{j+1}) \\
= \quad & (\sqcup_{r=1}^{k_j} Z(\alpha_1 \dots \alpha_{j-1}\gamma_j^r)) \\
\sqcup \quad & (\sqcup_{r=1}^{k_{j+1}} Z(\alpha_1 \dots \alpha_j \gamma_{j+1}^r)) \\
\sqcup \quad & Z(\alpha_1 \dots \alpha_{j+1}) \setminus Z(\alpha_1 \dots \alpha_{j+1}\beta_{j+2}) \\
= \quad & \dots \\
= \quad & \sqcup_{\ell=j}^{k} (\sqcup_{r=1}^{k_\ell} Z(\alpha_1 \dots \alpha_{\ell-1}\gamma_\ell^r)) \\
\sqcup \quad & Z(\alpha_1 \dots \alpha_k) \setminus Z(\alpha_1 \dots \alpha_k \beta_{k+1})
\end{aligned}$$

and

$$
\begin{aligned}
Z(\alpha_1 \dots \alpha_j) &= \sqcup_{s=1}^{k_{j+1}} (Z(\alpha_1 \dots \alpha_j \gamma_{j+1}^s)) \sqcup Z(\alpha_1 \dots \alpha_j \beta_{j+1}) \\
&\sqcup \quad Z(\alpha_1 \dots \alpha_{j+1}) \setminus Z(\alpha_1 \dots \alpha_{j+1}\beta_{j+2}).
\end{aligned}
$$

Then (similar to the work preceding Propositions 3.3.5 and 3.4.5) we have

$$
[\chi_{Z(\beta_1 \dots \beta_i)}]_0 = [\chi_{Z(\alpha_1 \dots \alpha_i)}]_0
$$

$$
\begin{aligned}
&= \sum_{r=1}^{k_{i+1}} [\chi_{Z(\alpha_1 \dots \alpha_i \gamma_{i+1}^r)}]_0 + [\chi_{Z(\alpha_1 \dots \alpha_{i+1}) \setminus Z(\alpha_1 \dots \alpha_{i+1}\beta_{i+2})}]_0 + [\chi_{Z(\alpha_1 \dots \alpha_i \beta_{i+1})}]_0 \\
&= \cdots \\
&= \sum_{\ell=i+1}^{k} (\sum_{r=1}^{k_\ell} [\chi_{Z(\alpha_1 \dots \alpha_{\ell-1}\gamma_\ell^r)}]_0) \\
&+ \sum_{j=i+1}^{k} [\chi_{Z(\alpha_1 \dots \alpha_j) \setminus Z(\alpha_1 \dots \alpha_j \beta_{j+1})}]_0 \\
&+ [\chi_{Z(\alpha_1 \dots \alpha_k)}]_0 \\
&= \sum_{\ell=i+1}^{k} (\sum_{r=1}^{k_\ell} [\chi_{Z(\alpha_1 \dots \alpha_{\ell-1}\gamma_\ell^r)}]_0) \\
&+ \sum_{j=i+1}^{k} (\sum_{\ell=j+1}^{k} (\sum_{r=1}^{k_\ell} [\chi_{Z(\alpha_1 \dots \alpha_{\ell-1}\gamma_\ell^r)}]_0) + [\chi_{Z(\alpha_1 \dots \alpha_k) \setminus Z(\alpha_1 \dots \alpha_k \beta_{k+1})}]_0) \\
&+ [\chi_{Z(\alpha_1 \dots \alpha_k)}]_0 \\
&= \sum_{\ell=i+1}^{k} (\sum_{r=1}^{k_\ell} [\chi_{Z(\alpha_1 \dots \alpha_{\ell-1}\gamma_\ell^r)}]_0) \\
&+ \sum_{\ell=i+2}^{k} (\ell - (i+1)) (\sum_{r=1}^{k_\ell} [\chi_{Z(\alpha_1 \dots \alpha_{\ell-1}\gamma_\ell^r)}]_0) \\
&+ (k-i) [\chi_{Z(\alpha_1 \dots \alpha_k) \setminus Z(\alpha_1 \dots \alpha_k \beta_{k+1})}]_0 \\
&+ [\chi_{Z(\alpha_1 \dots \alpha_k)}]_0
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{\ell=i+1}^{k}(\ell-i)(\sum_{r=1}^{k_\ell}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell^r)}]_0) \\
&+ (k-i)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
&+ [\chi_{Z(\alpha_1\ldots\alpha_k)}]_0.
\end{aligned}
$$

Then if we fix $m, n \in \mathbb{Z}$, $z_\ell^r = \alpha_1 \ldots \alpha_{\ell-1}\gamma_\ell^r$, $X_{\ell+1} = \sqcup_j F_{\ell,j}$ a finite union of compact-open sets, we have

$$
\begin{aligned}
& m[\chi_{Z(\alpha_1\ldots\alpha_{i-1})\setminus Z(\alpha_1\ldots\alpha_{i-1}\beta_i)}]_0 + n[\chi_{Z(\beta_1\ldots\beta_i)}]_0 \\
= \;& m\textstyle\sum_{\ell=i}^{k}\sum_{r=1}^{k_\ell}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell^r)}]_0 \\
+ \;& m[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
+ \;& n\textstyle\sum_{\ell=i+1}^{k}(\ell-i)(\sum_{r=1}^{k_\ell}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell^r)}]_0) \\
+ \;& n(k-i)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
+ \;& n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
= \;& \textstyle\sum_{\ell=i}^{k}(m+(\ell-i)n)\sum_{r=1}^{k_\ell}[\chi_{Z(\alpha_1\ldots\alpha_{\ell-1}\gamma_\ell^r)}]_0 \\
+ \;& (m+(k-i)n)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
+ \;& n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0 \\
= \;& \textstyle\sum_{\ell=i}^{k}(m+(\ell-i)n)\sum_{r=1}^{k_\ell}(\sum_j[\chi_{\{z_\ell^r\}\times F_{\ell,j}}]_0 \\
+ \;& (m+(k-i)n)[\chi_{Z(\alpha_1\ldots\alpha_k)\setminus Z(\alpha_1\ldots\alpha_k\beta_{k+1})}]_0 \\
+ \;& n[\chi_{Z(\alpha_1\ldots\alpha_k)}]_0.
\end{aligned}
$$

With these observations, we can prove the following.

Proposition 4.0.6. *Fix $a \in K_0(C^*(G_i))$ and write*

$$a = \sum_{\ell>i} \sum_{j=1}^{k_\ell} \sum_{k=1}^{m_\ell} c_{\ell,j,k}[\chi_{\{z_{\ell,j,k}\}\times F_{\ell,k}}]_0 + m[\chi_{Z(\alpha_1\ldots\alpha_{i-1})\setminus Z(\alpha_1\ldots\alpha_{i-1}\beta_i)}]_0 + n[\chi_{Z(\beta_1\ldots\beta_i)}]_0$$

where each $F_{\ell,k} \subseteq X_{\ell+1}$ is compact-open and for each ℓ, $\{ F_{\ell,k} : 1 \leq j \leq m_\ell \}$ is a partition of $X_{\ell+1}$. Then $a \geq 0$ if and only if for all ℓ, j, and k we have $c_{\ell,j,k} + m + (\ell - i)n \geq 0$.

Proof. The reverse direction is nearly identical to that in the proof of Proposition 3.4.5 using the computations above. The forward direction is similar as well: Suppose we have $a \in K_0(C^*(G_i))_+$ and write a as above with the outlined conventions. Fix $x \in U_i$ and write

$$x = x_1 \ldots x_{\ell(x)-1}\gamma_{\ell(x)}^{r(x)}x'$$

where $x_j \neq \gamma_j^s$ for $i < j < \ell(x)$ and any s, $1 \leq r(x) \leq k_{\ell(x)}$, and $x' \in X_{\ell(x)+1}$. Define a $*-$homomorphism

$$\pi_x : C^*(G_i|_{U_i}) \longrightarrow M_{n(\ell(x))}$$

as follows: For

$$f \in C^*(G_i|_{U_i}) \cong \bigoplus_{\ell>i}(\oplus_{j=1}^{k_\ell}(C(X_{\ell+1}, M_{n(\ell)})))$$

with $f = (f_{i+1}, f_{i+2}, \ldots, f_k, \ldots)$ and $f_n = (\tilde{f}_{n,1}, \ldots, \tilde{f}_{n,k_n})$, we define $\pi_x(f) = \tilde{f}_{\ell(x),r(x)}(x')$. The remainder of the proof is nearly identical, with some obvious additions of superscripts (e.g., when defining the set Ω) and noting that the index on the two rightmost summands of a have been shifted down by one when compared to those in Proposition 3.4.5 □

To identify $C^*(G)$, we first note than an analogue of Lemma 3.5.1 holds. In its statement, we will write $\gamma(j)$ for γ^j to prevent confusion with the shorthand superscript for the α's and β's.

Lemma 4.0.7. *Fix $n \geq 0$ and let*

$$W_n = \{\ x \in Z(v_0)\ :\ x_i \neq \gamma_i(j) \text{ for } i \leq n \text{ and any } j\ \}.$$

Put

$$\begin{aligned}
P_n^{(1)} &= \{Z(\alpha^{n+1}\beta^j) \setminus Z(\alpha^{n+1}\beta^{j+1}) : j \leq n\} \\
P_n^{(2)} &= \{Z(\alpha^j\beta^{n+1}) \setminus Z(\alpha^{j+1}\beta^{n+1}) : j \leq n\} \\
P_n^{(3)} &= \{Z(\alpha^j\beta^k\gamma(m)) : j+k \geq n,\ k \leq j \leq n, 1 \leq m \leq k_{j+k+1}\} \\
P_n^{(4)} &= \{Z(\alpha^k\beta^j\gamma(m)) : j+k \geq n,\ k < j \leq n, 1 \leq m \leq k_{j+k+1}\} \\
P_n^{(5)} &= \{Z(\alpha^{n+1}\beta^{n+1})\} \\
P_n &= \cup_{r=1}^5 P_n^{(r)}.
\end{aligned}$$

Then P_n is a partition of W_n that refines $Z(\nu)$ for every $\nu \in v_0\Lambda$ with $|\nu| = n$ and $\nu_i \neq \gamma_i^j$ for $i \leq n$ and any j.

Proof. The proof is nearly the same as the proof of Lemma 3.5.1. The only significant change is in showing that sets in $P_n^{(3)}$ and $P_n^{(4)}$ are disjoint. We now have $Z(\alpha^j\beta^k\gamma(m)), Z(\alpha^j\beta^k\gamma(m')) \in P_n^{(3)}$ i.e., the same number of α's and β's, but note that this only occurs when $m \neq m'$, so the sets are disjoint. The same is true for sets in $P_n^{(4)}$. □

This gives us the analogue of Proposition 3.5.2, which has a similar proof.

Proposition 4.0.8. *Fix $n \geq 0$ and let*

$$Q_n = \{\tau^{\mu}(P_m) : |\mu| \leq n \ \mu_{|\mu|} = \gamma^j_{|\mu|}, \ \textit{some } 1 \leq j \leq k_{|\mu|}, |\mu| + m = n\}$$

where P_m is as in Lemma 4.0.7. Then Q_n is a partition of $Z(v_0)$ that refines $Z(\nu)$ for all $\nu \in v_0\Lambda$ with $|\nu| \leq n$.

To compute $(K_0(C^*(G)), K_0(C^*(G))_+)$, note that for $n \geq 0$

$$\begin{aligned} & Z(\alpha_1 \dots \alpha_n) \setminus Z(\alpha_1 \dots \alpha_n \beta_{n+1}) \\ = \ & (\sqcup_{i=1}^{k_{n+1}} Z(\alpha_1 \dots \alpha_n \gamma^i_{n+1})) \\ \sqcup \ & Z(\alpha_1 \dots \alpha_{n+1}) \setminus Z(\alpha_1 \dots \alpha_{n+1}\beta_{n+2}) \\ = \ & (\sqcup_{i=1}^{k_{n+1}} Z(\alpha_1 \dots \alpha_n \gamma^i_{n+1}) \setminus Z(\alpha_1 \dots \alpha_n \gamma^i_{n+1}\beta_{n+2})) \\ \sqcup \ & (\sqcup_{i=1}^{k_{n+1}} Z(\alpha_1 \dots \alpha_n \gamma^i_{n+1}\beta_{n+2})) \\ \sqcup \ & Z(\alpha_1 \dots \alpha_{n+1}) \setminus Z(\alpha_1 \dots \alpha_{n+1}\beta_{n+2}) \end{aligned}$$

and

$$Z(\beta_1 \dots \beta_{n+1}) = Z(\beta_1 \dots \beta_{n+1}) \setminus Z(\beta_1 \dots \beta_{n+2}) \sqcup Z(\beta_1 \dots \beta_{n+2})$$

so that in $K_0(C^*(G_{i+1}))$,

$$\begin{aligned} [\chi_{Z(\alpha_1 \dots \alpha_{i-1}) \setminus Z(\alpha_1 \dots \alpha_{i-1}\beta_i)}]_0 &= (k_i + 1)[\chi_{Z(\alpha_1 \dots \alpha_i) \setminus Z(\alpha_1 \dots \alpha_i \beta_{i+1})}]_0 \\ &+ k_i[\chi_{Z(\beta_1 \dots \beta_{i+2})}]_0 \\ \text{and } [\chi_{Z(\beta_1 \dots \beta_i)}]_0 &= [\chi_{Z(\alpha_1 \dots \alpha_i) \setminus Z(\alpha_1 \dots \alpha_i \beta_{i+1})}]_0 \\ &+ [\chi_{Z(\beta_1 \dots \beta_{i+1})}]_0 \end{aligned}$$

These observations, together with Proposition 4.0.8 and reasoning similar to that following Proposition 3.5.2 in the previous example allow us to conclude that $K_0(C^*(G))$ is the inductive limit of the sequence

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \cdots$$

where each term has positive cone $\mathbb{N}^2$ (by Proposition 4.0.6) and the i^{th} connecting map is given by the matrix

$$B_i = \begin{pmatrix} k_i + 1 & 1 \\ k_i & 1 \end{pmatrix}$$

For each i, let

$$A_i = B_i^{-1} = \begin{pmatrix} 1 & -1 \\ -k_i & k_i + 1 \end{pmatrix}$$

and $A = A_1 \ldots A_m$. Let $\theta = \overline{[k_1, 1, k_2, 1, \ldots, k_m, 1]}$ (that is, the periodic continued fraction

$$k_1 + \cfrac{1}{1 + \cfrac{1}{k_2 + \ddots}}$$

and

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

be the matrix representation of the fractional linear transformation which fixes θ (see, e.g., (18) for a treatment of continued fractions); i.e.

$$\frac{P\theta + Q}{R\theta + S} = \theta$$

Lemma 4.0.9.

$$A = \begin{pmatrix} S & -R \\ -Q & P \end{pmatrix}$$

Proof. Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$\begin{aligned}
\begin{pmatrix} P & Q \\ R & S \end{pmatrix} &= T^{k_1}STST^{k_2}STS\cdots T^{k_m}STS \\
&= \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\cdots \begin{pmatrix} 1 & k_m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} k_m & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} k_1+1 & k_1 \\ 1 & 1 \end{pmatrix}\cdots\begin{pmatrix} k_m+1 & k_m \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

Now, if $m = 1$, then

$$\begin{pmatrix} P & Q \\ & \\ R & S \end{pmatrix} = \begin{pmatrix} k_1 + 1 & k_1 \\ & \\ 1 & 1 \end{pmatrix}$$

and

$$A = A_1 = \begin{pmatrix} 1 & -1 \\ & \\ -k_1 & k_1 + 1 \end{pmatrix} = \begin{pmatrix} S & -R \\ & \\ -Q & P \end{pmatrix}$$

Next, suppose

$$\begin{pmatrix} P & Q \\ & \\ R & S \end{pmatrix} = \begin{pmatrix} k_1 + 1 & k_1 \\ & \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} k_{m-1} + 1 & k_{m-1} \\ & \\ 1 & 1 \end{pmatrix}$$

and

$$A = A_1 \ldots A_{m-1} = \begin{pmatrix} 1 & -1 \\ & \\ -k_1 & k_1 + 1 \end{pmatrix} = \begin{pmatrix} S & -R \\ & \\ -Q & P \end{pmatrix}$$

Then

$$\begin{pmatrix} P & Q \\ & \\ R & S \end{pmatrix} \begin{pmatrix} k_m + 1 & k_m \\ & \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} P(k_m + 1) + Q & Pk_m + Q \\ & \\ R(k_m + 1) + S & Rk_m + S \end{pmatrix}$$

and

$$AA_m = \begin{pmatrix} S & -R \\ & \\ -Q & P \end{pmatrix} \begin{pmatrix} 1 & -1 \\ & \\ -k_m & k_m + 1 \end{pmatrix} = \begin{pmatrix} Rk_m + S & -(R(k_m + 1) + S) \\ & \\ -(Pk_m + Q) & P(k_m + 1) + Q \end{pmatrix}$$

so that the claim holds for all $m \geq 1$. $\square$

We will need the following lemma.

Lemma 4.0.10. *Let* $P, Q, R, S \geq 1$, *and*

$$\Gamma(z) = \frac{Pz + Q}{Rz + S}$$

and let $\theta > \theta' \in \mathbb{R}$ *satisfy* $\Gamma(\theta) = \theta$, *and* $\Gamma(\theta') = \theta'$. *Suppose*

$$A = \begin{pmatrix} S & -R \\ -Q & P \end{pmatrix}$$

has two real eigenvalues $\lambda_1 > \lambda_2$ *and* $\mu = \frac{y}{x}$ *where* $\binom{x}{y}$ *is an eigenvector of* λ_1. *If* $PS - QR = 1$, *then* $\theta = -\mu$.

Proof. Straightforward computations show that the eigenspace of λ_1 is spanned by

$$\begin{pmatrix} \frac{R}{S-\lambda_1} \\ 1 \end{pmatrix}$$

so that $\mu = (S - \lambda_1)/R$. Since

$$\begin{aligned} \det(A - \lambda) &= \lambda^2 - tr(A)\lambda + \det(A) \\ &= \lambda^2 - (S+P)\lambda + 1 \end{aligned}$$

we have

$$\lambda_1 = \frac{S + P + \sqrt{(S+P)^2 - 4}}{2}$$

hence

$$\begin{aligned} -\mu &= \tfrac{\lambda_1 - S}{R} \\ &= \tfrac{S+P+\sqrt{(S+P)^2-4}-2S}{2R} \\ &= \tfrac{P-S+\sqrt{(S+P)^2-4}}{2R}. \end{aligned}$$

On the other hand, θ is a root of $Rz^2 + (S-P)z - Q$ so that

$$\theta = \frac{P - S + \sqrt{(S-P)^2 + 4QR}}{2R}$$

Since $PS - QR = 1$, $4 = 4SP - 4QR$ and therefore

$$\begin{aligned} (S-P)^2 + 4QR &= S^2 - 2SP + P^2 + 4QR \\ &= S^2 + 2SP + P^2 - 4 \\ &= (S+P)^2 - 4 \end{aligned}$$

so that $\theta = -\mu$. □

We now show that for A as defined above and μ as in the previous lemma, $\mu \leq 0$. Note that $P > 1$ and $Q, R, S > 0$ which is easily seen (beginning on page 105) when computing $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$. Now

$$\mu = \frac{S - \lambda_1}{R} = \frac{S - P - \sqrt{(S+P)^2 - 4}}{2R},$$

so showing $\mu \leq 0$ amounts to showing $S - P \leq \sqrt{(S+P)^2 - 4}$. This is straightforward; since $S, P > 0$, $4 \leq 4SP$ so that

$$\begin{aligned} (S-P)^2 &= S^2 - 2SP + P^2 \\ &\leq S^2 + 2SP + P^2 - 4 \\ &= (S+P)^2 - 4 \end{aligned}$$

Computation of the positive cone is similar to the case when $k_1 = k_2 = \cdots = k_m = 1$. In particular, we first diagonalize $A = XDX^{-1}$ where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{R}{S-\lambda_1} & \frac{R}{S-\lambda_2} \\ & \\ 1 & 1 \end{pmatrix}$$

$$X^{-1} = \frac{1}{\sqrt{\det X}} \begin{pmatrix} 1 & \frac{R}{\lambda_2 - S} \\ & \\ -1 & \frac{R}{S-\lambda_1} \end{pmatrix}.$$

Straightforward computations show

$$X \begin{pmatrix} 1 & 0 \\ & \\ 0 & (\frac{\lambda_2}{\lambda_1})^n \end{pmatrix} X^{-1} \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} \longrightarrow \frac{1}{\det X} \begin{pmatrix} \frac{R}{S-\lambda_1} \\ \\ 1 \end{pmatrix}$$

and

$$X \begin{pmatrix} 1 & 0 \\ & \\ 0 & (\frac{\lambda_2}{\lambda_1})^n \end{pmatrix} X^{-1} \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \longrightarrow \frac{1}{\det X} \begin{pmatrix} \frac{R^2}{(\lambda_2 - S)(S-\lambda_1)} \\ \\ \frac{R}{\lambda_2 - S} \end{pmatrix}.$$

as $n \longrightarrow \infty$. These vectors lie on the same line; the one through the origin with slope $\frac{S-\lambda_1}{R} = \mu$. Since the maps B_i are positive, A maps $\mathbb{N}^2$ into the upper half plane defined by μ. Then we conclude that

$$\begin{pmatrix} x \\ y \end{pmatrix} \in K_0(G^*(G))_+$$

iff $y \geq \mu x$; i.e.

$$K_0(C^*(G))_+ = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \ : \ y - \mu x \geq 0 \}.$$

Now

$$K_0(\mathcal{A}_\theta)_+ = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \ : \ y + \theta x \geq 0 \}$$

where $\mathcal{A}_\theta$ is the continued fraction AF algebra associated to θ, so that

$$K_0(C^*(G))_+ = K_0(\mathcal{A}_\theta)_+.$$

Similarly to our first example, we wish to show that there is a unique trace on $C^*(G)$ and our first step will be to show there is at most one invariant Borel probability measure on G^0. Let μ be such a measure. As we did in the first example, for each $i \geq 0$, let

$$\begin{aligned} a_i &= \mu(Z(\alpha_1 \ldots \alpha_i) \setminus Z(\alpha_1 \ldots \alpha_i \beta_{i+1})) \\ \text{and } b_i &= \mu(Z(\beta_1 \ldots \beta_{i+1})) \end{aligned}$$

For a given i, we have seen that

$$\begin{aligned} Z(\alpha_1 \ldots \alpha_i) \setminus Z(\alpha_1 \ldots \alpha_i \beta_{i+1}) &= (\sqcup_{j=1}^{k_{i+1}} Z(\alpha_1 \ldots \alpha_i \gamma_{i+1}^j) \setminus Z(\alpha_1 \ldots \alpha_i \gamma_{i+1}^j \beta_{i+2})) \\ &\sqcup \ (\sqcup_{j=1}^{k_{i+1}} Z(\alpha_1 \ldots \alpha_i \gamma_{i+1}^j \beta_{i+2})) \\ &\sqcup \ Z(\alpha_1 \ldots \alpha_{i+1}) \setminus Z(\alpha_1 \ldots \alpha_{i+1} \beta_{i+2}) \end{aligned}$$

and

$$Z(\beta_1 \ldots \beta_{i+1}) = (Z(\beta_1 \ldots \beta_{i+1}) \setminus Z(\beta_1 \ldots \beta_{i+2})) \sqcup Z(\beta_1 \ldots \beta_{i+2})$$

Hence, $a_i = (k_{i+1} + 1)a_{i+1} + k_{i+1}b_{i+1}$ and $b_i = a_{i+1} + b_{i+1}$. Letting

$$\tilde{A}_i = \begin{pmatrix} k_{i+1} + 1 & k_{i+1} \\ 1 & 1 \end{pmatrix}$$

then, we have

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \tilde{A}_i \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix}$$

More generally, given our period of m, fix $i \geq 1$ and let

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \tilde{A}_i \ldots \tilde{A}_{i+m-1}$$

so that

$$\begin{pmatrix} a_i \\ \\ b_i \end{pmatrix} = \begin{pmatrix} P & Q \\ \\ R & S \end{pmatrix}^k \begin{pmatrix} a_{km+i+1} \\ \\ b_{km+i+1} \end{pmatrix}$$

for all $k \geq 1$ and

$$\begin{pmatrix} a_{km+i+1} \\ \\ b_{km+i+1} \end{pmatrix} = \begin{pmatrix} P & Q \\ \\ R & S \end{pmatrix}^{-k} \begin{pmatrix} a_i \\ \\ b_i \end{pmatrix}$$

Note that $\det(\tilde{A}_j) = 1$ for all j so that $\det \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right) = 1$ and hence

$$\begin{pmatrix} P & Q \\ \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} S & -Q \\ \\ -R & P \end{pmatrix}$$

Also, since $k_j \geq 1$ for all j, it is clear that $P \geq 2$ and $Q, R, S \geq 1$. Given these observations, we have $\lambda^2 - (P+S)\lambda + 1$ as the characteristic polynomial of $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$, which has real roots

$$\lambda_1 = \frac{P + S + \sqrt{(P+S)^2 - 4}}{2} > \lambda_2 = \frac{P + S - \sqrt{(P+S)^2 - 4}}{2} > 0$$

Diagonalizing, we find

$$\begin{pmatrix} P & Q \\ \\ R & S \end{pmatrix} = XDX^{-1}$$

where

$$X = \begin{pmatrix} \frac{Q}{\lambda_1 - P} & \frac{Q}{\lambda_2 - P} \\ \\ 1 & 1 \end{pmatrix}, \ X^{-1} = \frac{1}{\det X} \begin{pmatrix} 1 & \frac{Q}{P - \lambda_2} \\ \\ -1 & \frac{Q}{\lambda_1 - P} \end{pmatrix},$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ and } \frac{1}{\det X} = \frac{R}{\sqrt{(P+S)^2 - 4}}$$

so that

$$\begin{pmatrix} S & -Q \\ -R & P \end{pmatrix} = X \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix} X^{-1}.$$

Our goal moving forward is to compute the right side of

$$\begin{pmatrix} a_{km+i+1} \\ b_{km+i+1} \end{pmatrix} = \begin{pmatrix} S & -Q \\ -R & P \end{pmatrix}^k \begin{pmatrix} a_i \\ b_i \end{pmatrix}$$

and use the fact that μ is a probability measure to bound the resulting above 0. We will use this to show that a_i is bounded above and below by terms which both converge to the same limit as $k \to \infty$. The computations are laborious but straightforward, and follow now:

$$\begin{pmatrix} a_{km+i+1} \\ b_{km+i+1} \end{pmatrix} = \begin{pmatrix} S & -Q \\ -R & P \end{pmatrix}^k \begin{pmatrix} a_i \\ b_i \end{pmatrix} = XD^{-k}X^{-1} \begin{pmatrix} a_i \\ b_i \end{pmatrix}$$

$$= \frac{1}{\det X} XD^{-k} \begin{pmatrix} 1 & \dfrac{Q}{P-\lambda_2} \\ -1 & \dfrac{Q}{\lambda_1 - P} \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \frac{1}{\det X} X \begin{pmatrix} (\frac{1}{\lambda_1})^k & 0 \\ 0 & (\frac{1}{\lambda_2})^k \end{pmatrix} \begin{pmatrix} a_i + \dfrac{Qb_i}{P-\lambda_2} \\ \dfrac{Qb_i}{\lambda_1 - P} - a_i \end{pmatrix}$$

$$= \frac{1}{\det X} X \begin{pmatrix} (\dfrac{1}{\lambda_1})^k \left(a_i + \dfrac{Qb_i}{P-\lambda_2} \right) \\ (\dfrac{1}{\lambda_2})^k \left(\dfrac{Qb_i}{\lambda_1 - P} - a_i \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} \frac{Q}{\lambda_1 - P} & \frac{Q}{\lambda_2 - P} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(a_i + \frac{Qb_i}{P - \lambda_2} \right) \\ (\frac{1}{\lambda_2})^k \left(\frac{Qb_i}{\lambda_1 - P} - a_i \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(a_i + \frac{Qb_i}{P - \lambda_2} \right) \left(\frac{Q}{\lambda_1 - P} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i}{\lambda_1 - P} - a_i \right) \left(\frac{Q}{\lambda_2 - P} \right) \\ (\frac{1}{\lambda_1})^k \left(a_i + \frac{Qb_i}{P - \lambda_2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i}{\lambda_1 - P} - a_i \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(\frac{a_i Q(P - \lambda_2) + Q^2 b_i}{(\lambda_1 + \lambda_2)P - \lambda_1 \lambda_2 - P^2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Q^2 b_i - a_i Q(\lambda_1 - P)}{\lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)P + P^2} \right) \\ (\frac{1}{\lambda_1})^k \left(\frac{a_i(P - \lambda_2) + Qb_i}{P - \lambda_2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i - a_i(\lambda_1 - P)}{\lambda_1 - P} \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(\frac{a_i Q(P - \lambda_2) + Q^2 b_i}{(P + S)P - 1 - P^2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Q^2 b_i - a_i Q(\lambda_1 - P)}{1 - (P + S)P + P^2} \right) \\ (\frac{1}{\lambda_1})^k \left(\frac{a_i(P - \lambda_2) + Qb_i}{P - \lambda_2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i - a_i(\lambda_1 - P)}{\lambda_1 - P} \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(\frac{a_i Q(P - \lambda_2) + Q^2 b_i}{PS - 1} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Q^2 b_i - a_i Q(\lambda_1 - P)}{1 - PS} \right) \\ (\frac{1}{\lambda_1})^k \left(\frac{a_i(P - \lambda_2) + Qb_i}{P - \lambda_2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i - a_i(\lambda_1 - P)}{\lambda_1 - P} \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(\frac{a_i Q(P - \lambda_2) + Q^2 b_i}{QR} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Q^2 b_i - a_i Q(\lambda_1 - P)}{-QR} \right) \\ (\frac{1}{\lambda_1})^k \left(\frac{a_i(P - \lambda_2) + Qb_i}{P - \lambda_2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i - a_i(\lambda_1 - P)}{\lambda_1 - P} \right) \end{pmatrix}$$

$$= \frac{1}{\det X} \begin{pmatrix} (\frac{1}{\lambda_1})^k \left(\frac{a_i(P - \lambda_2) + Qb_i}{R} \right) + (\frac{1}{\lambda_2})^k \left(\frac{a_i(\lambda_1 - P) - Qb_i}{R} \right) \\ (\frac{1}{\lambda_1})^k \left(\frac{a_i(P - \lambda_2) + Qb_i}{P - \lambda_2} \right) + (\frac{1}{\lambda_2})^k \left(\frac{Qb_i - a_i(\lambda_1 - P)}{\lambda_1 - P} \right) \end{pmatrix}$$

Now, similarly to the first example, $Z(v_1)$ is partitioned by sets of the form $Z(x_1 \dots x_j) \setminus Z(x_1 \dots x_j \beta_{j+1})$ and $Z(x_1 \dots x_j \beta_{j+1})$ for each $j \geq 0$, so there exist

$f_i, g_i \in \mathbb{Z}^+$ such that $f_i a_i + g_i b_i = 1$ and hence $b_i = (1 - f_i a_i)/g_i$. Then equating top entries in the previous string of equalities, we have

$$a_{km+i+1} = \frac{1}{\det X}\left((\frac{1}{\lambda_1})^k \left(\frac{a_i(P-\lambda_2) + Qb_i}{R} \right) + (\frac{1}{\lambda_2})^k \left(\frac{a_i(\lambda_1 - P) - Qb_i}{R} \right) \right)$$

$$= \quad \frac{1}{\det X}\left((\frac{1}{\lambda_1})^k \left(\frac{a_i(P-\lambda_2) + Q(\frac{1-f_i a_i}{g_i})}{R} \right) + (\frac{1}{\lambda_2})^k \left(\frac{a_i(\lambda_1 - P) - Q(\frac{1-f_i a_i}{g_i})}{R} \right) \right)$$

$$= \quad \frac{1}{\det X}\left((\frac{1}{\lambda_1})^k \left(\frac{a_i g_i (P-\lambda_2) + Q - Qf_i a_i}{Rg_i} \right) \right)$$

$$+ \quad \frac{1}{\det X}\left((\frac{1}{\lambda_2})^k \left(\frac{a_i g_i (\lambda_1 - P) - Q + Qf_i a_i}{Rg_i} \right) \right)$$

$$= \quad \frac{1}{Rg_i \det X}\left(\frac{\lambda_2^k(a_i g_i (P-\lambda_2) + Q - Qf_i a_i) + \lambda_1^k(a_i g_i(\lambda_1 - P) - Q + Qf_i a_i)}{(\lambda_1\lambda_2)^k} \right)$$

$$= \quad \tfrac{1}{Rg_i \det X}(\lambda_2^k a_i(g_i(P-\lambda_2) - Qf_i) + \lambda_2^k Q + \lambda_1^k a_i(g_i(\lambda_1 - P) + Qf_i) - \lambda_1^k Q)$$

Since $0 \le a_{mk+i+1}$, we have

$$\frac{(\lambda_1^k - \lambda_2^k)Q}{Rg_i \det X} \le \frac{a_i}{Rg_i \det X}(\lambda_2^k(g_i(P-\lambda_2) - Qf_i) + \lambda_1^k(g_i(\lambda_1 - P) + Qf_i))$$

Since $\det X = (\sqrt{(P+S)^2 - 4})/R, R, g_i > 0$, we have

$$(\lambda_1^k - \lambda_2^k)Q \le a_i(\lambda_2^k(g_i(P-\lambda_2) - Qf_i) + \lambda_1^k(g_i(\lambda_1 - P) + Qf_i))$$

Note that

$$\lambda_1^k Qf_i - \lambda_2^k Qf_i = (\lambda_1^k - \lambda_2^k)Qf_i > 0$$

since $\lambda_1 > \lambda_2$ (and $Q, f_i > 0$). Moreover

$$P - \lambda_2 = \frac{P - S + \sqrt{(P+S)^2 - 4}}{2} > 0$$

since $P \geq 2$ and $S \geq 1$, and therefore $\sqrt{(P+S)^2 - 4} > S$, and similarly

$$\lambda_1 - P = \frac{S - P + \sqrt{(P+S)^2 - 4}}{2} > 0.$$

Then

$$\lambda_2^k(g_i(P - \lambda_2) - Qf_i) + \lambda_1^k(g_i(\lambda_1 - P) + Qf_i) > 0$$

and therefore

$$\begin{aligned} a_i &\geq \frac{(\lambda_1^k - \lambda_2^k)Q}{\lambda_2^k(g_i(P-\lambda_2) - Qf_i) + \lambda_1^k(g_i(\lambda_1 - P) + Qf_i)} \\ &= \frac{\lambda_1^k}{\lambda_1^k} \frac{(1 - (\lambda_2/\lambda_1)^k)Q}{(\lambda_2/\lambda_1)^k(g_i(P-\lambda_2) - Qf_i) + (g_i(\lambda_1 - P) + Qf_i)} \end{aligned}$$

Since $0 < \lambda_2 < \lambda_1$, the right side converges to

$$\frac{Q}{g_i(\lambda_1 - P) + Qf_i}$$

as $k \to \infty$.

Similarly, we have

$$
\begin{aligned}
b_{km+i+1} &= \frac{1}{\det X}\left((\frac{1}{\lambda_1})^k\left(\frac{a_i(P-\lambda_2)+Qb_i}{P-\lambda_2}\right)+(\frac{1}{\lambda_2})^k\left(\frac{Qb_i-a_i(\lambda_1-P)}{\lambda_1-P}\right)\right) \\
&= \frac{1}{\det X}\left((\frac{1}{\lambda_1})^k\left(\frac{a_i(P-\lambda_2)+Q(\frac{1-f_ia_i}{g_i})}{P-\lambda_2}\right)\right) \\
&+ \frac{1}{\det X}\left((\frac{1}{\lambda_2})^k\left(\frac{a_i(\lambda_1-P)-Q(\frac{1-f_ia_i}{g_i})}{P-\lambda_1}\right)\right) \\
&= \frac{1}{\det X}\left((\frac{1}{\lambda_1})^k\left(\frac{a_ig_i(P-\lambda_2)+Q-Qf_ia_i}{g_i(P-\lambda_2)}\right)\right) \\
&+ \frac{1}{\det X}\left((\frac{1}{\lambda_2})^k\left(\frac{a_ig_i(\lambda_1-P)-Q+Qf_ia_i}{g_i(P-\lambda_1)}\right)\right) \\
&= \frac{1}{g_i\det X}\left(\frac{\lambda_2^k(P-\lambda_1)(a_ig_i(P-\lambda_2)+Q-Qf_ia_i)}{(P-\lambda_1)(P-\lambda_2)}\right) \\
&+ \frac{1}{g_i\det X}\left(\frac{\lambda_1^k(P-\lambda_2)(a_ig_i(\lambda_1-P)-Q+Qf_ia_i)}{(P-\lambda_1)(P-\lambda_2)}\right)
\end{aligned}
$$

Since $0 \leq b_{mk+i+1}$, we have

$$
\begin{aligned}
0 &\leq \frac{1}{g_i\det X}\left[\frac{\lambda_2^k(P-\lambda_1)(a_ig_i(P-\lambda_2)+Q-Qf_ia_i)}{(P-\lambda_1)(P-\lambda_2)}\right] \\
&+ \frac{1}{g_i\det X}\left[\frac{\lambda_1^k(P-\lambda_2)(a_ig_i(\lambda_1-P)-Q+Qf_ia_i)}{(P-\lambda_1)(P-\lambda_2)}\right]
\end{aligned}
$$

Now, $g_i, \det X > 0$ and we saw that $(P-\lambda_2), (\lambda_1-P) > 0$ so that $(P-\lambda_1)(P-\lambda_2) < 0$

so

$$\begin{aligned} 0 &\geq a_i\lambda_2^k(P-\lambda_1)(g_i(P-\lambda_2)-Qf_i)+\lambda_2^k(P-\lambda_1)Q \\ &+ a_i\lambda_1^k(P-\lambda_2)(g_i(\lambda_1-P)+Qf_i)-\lambda_1^k(P-\lambda_2)Q \end{aligned}$$

Rearranging,

$$a_i(\lambda_2^k(P-\lambda_1)(g_i(P-\lambda_2)-Qf_i)+\lambda_1^k(P-\lambda_2)(g_i(\lambda_1-P)+Qf_i))$$

$$\leq Q(\lambda_1^k(P-\lambda_2)-\lambda_2^k(P-\lambda_1))$$

Since $P-\lambda_1<0, P-\lambda_2>0$, and $\lambda_2<\lambda_1$,

$$\lambda_2^k(P-\lambda_1)g_i(P-\lambda_2)+\lambda_1^k(P-\lambda_2)g_i(\lambda_1-P)$$

$$=\lambda_2^k(P-\lambda_1)g_i(P-\lambda_2)-\lambda_1^k(P-\lambda_2)g_i(P-\lambda_1)$$

$$=(\lambda_2^k-\lambda_1^k)(P-\lambda_1)(P-\lambda_2)g_i$$

$$>0.$$

Moreover,

$$-\lambda_2^k(P-\lambda_1)Qf_i+\lambda_1^k(P-\lambda_2)Qf_i>0$$

so that

$$\lambda_2^k(P-\lambda_1)(g_i(P-\lambda_2)-Qf_i)+\lambda_1^k(P-\lambda_2)(g_i(\lambda_1-P)+Qf_i)>0$$

and therefore

$$\begin{aligned} a_i &\leq \frac{\lambda_1^k(P-\lambda_2)Q-\lambda_2^k(P-\lambda_1)Q}{\lambda_2^k(P-\lambda_1)(g_i(P-\lambda_2)-Qf_i)+\lambda_1^k(P-\lambda_2)(g_i(\lambda_1-P)+Qf_i)} \\ &= \frac{(P-\lambda_2)Q-(\lambda_2/\lambda_1)^k(P-\lambda_1)Q}{(\lambda_2/\lambda_1)^k(P-\lambda_1)(g_i(P-\lambda_2)-Qf_i)+(P-\lambda_2)(g_i(\lambda_1-P)+Qf_i)} \end{aligned}$$

Since $0 < \lambda_2 < \lambda_1$, the right side converges to

$$\frac{(P-\lambda_2)Q}{(P-\lambda_2)(g_i(\lambda_1 - P) + Qf_i)} = \frac{Q}{g_i(\lambda_1 - P) + Qf_i}$$

so that

$$a_i = \frac{Q}{g_i(\lambda_1 - P) + Qf_i}$$

Then for each i, there is only one possibility for a_i and hence, the same holds for b_i and this forces μ to be unique (if it exists).

Next, we will produce a state on

$$(K_0(C^*(G)), K_0(C^*(G))_+, [1]_0)$$

Essentially the same argument as in the first example shows that $C^*(G)$ has finite nuclear dimension; namely, there is a uniform bound on the terms in the inductive sequence defining $C^*(G)$. Then $C^*(G)$ is nuclear and therefore exact which means this state must come from a trace (14, Theorem 5.2.2). Proposition 3.5.10 still applies in this more general case, so that traces on $C^*(G)$ are in one-to-one correspondence with invariant Borel probability measures on G^0, so that this trace must be unique.

Let $\delta = 1/(\theta+1)$ and define

$$T : K_0(C^*(G)) \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \delta(\theta x + y)$$

This map is clearly a homomorphism, and if $\theta x + y \geq 0$, then $\delta(\theta x + y) \geq 0$ so that it is also positive. We claim that $[1]_0 = \binom{1}{1}$, which we see by noting

$$Z(v_1) = Z(\beta_1) \sqcup Z(v_1) \setminus Z(\beta_1)$$

Now $T\binom{1}{1} = \delta(\theta + 1) = 1$ and therefore T is a state. Using classification results as we did before, we conclude $C^*(G)$ is isomorphic to the AF algebra with ordered K_0

group and order unit

$$(\mathbb{Z}^2, \{(x,y) \in \mathbb{Z}^2 : y + \theta x \geq 0\}, (1,1))$$

which is stably isomorphic to the continued fraction AF algebra A_θ which has the same ordered K_0 group and order unit $(0,1)$.

Chapter 5

TWO FINAL EXAMPLES

The previous examples were originally investigated in an attempt to shed light on the question of when the C^*-algebra of a category of paths is AF. In (6), the authors address this question in the context of $k-$graphs. They give a necessary condition (which we shall describe below) for the algebra of a higher-rank graph to be AF but leave open the question of whether or not the condition is sufficient. The examples that follow show that the condition is not sufficient in the setting of $k-$graphs, and not even necessary in the setting of categories of paths. Before stating this condition we need some definitions.

Definition 5.0.1. Let Λ be a higher rank graph. For $\mu, \nu \in \Lambda$, a common extension λ of μ and ν is **minimal** if $\lambda' \leq \lambda$ and $\lambda' \neq \lambda$ implies λ' is not a common extension of μ and ν. We denote the set of minimal common extension of μ and ν by $\mathrm{MCE}(\mu, \nu)$. We say Λ is **finitely aligned** if $\mathrm{MCE}(\mu, \nu)$ is finite for all $\mu, \nu \in \Lambda$. Let $v \in \Lambda^0$. A subset $F \subseteq v\Lambda$ is **exhaustive** if for every $\mu \in v\Lambda$, there is a $\nu \in F$ such that $\mathrm{MCE}(\mu, \nu) \neq \emptyset$. A subset $H \subseteq \Lambda^0$ is **hereditary** if $s(H\Lambda) \subseteq H$ and **saturated** if for all $v \in \Lambda^0$, if there is a finite exhaustive set F at v with $s(F) \subseteq H$, then $v \in H$.

The next definitions were first given in (6) in the context of $k-$graphs, and we use them in the more general setting of categories of paths.

Definition 5.0.2. Let Λ be a category of paths and $\mu, \nu \in \Lambda$ with $\mu \neq \nu$, $s(\mu) = s(\nu)$, and $r(\mu) = r(\nu)$. The pair (μ, ν) is called a **generalized cycle** if, given any $\tau \in s(\mu)\Lambda$ the elements $\mu\tau$ and ν have a common extension. A generalized cycle (μ, ν) is said to **have an entrance** if there exists $\tau \in s(\nu)\Lambda$ such that μ and $\nu\tau$ have

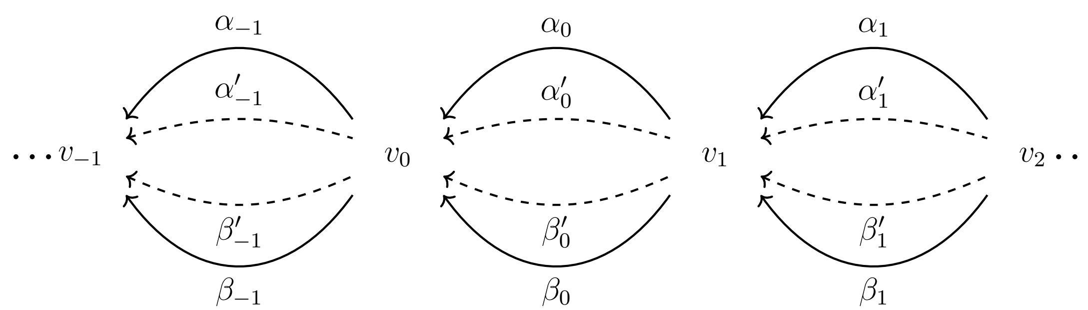

Figure 5.1: Skeleton of Λ'

no common extension (that is, a generalized cycle (μ, ν) has an entrance if (ν, μ) is not a generalized cycle).

In (6), Evans and Sims proved the following.

Theorem 5.0.3. *(6, Theorem 3.4) Let Λ be a finitely-aligned $k-$graph. If $C^*(\Lambda)$ is AF, then Λ contains no generalized cycles.*

In fact, they use this theorem to prove a slightly stronger corollary.

Corollary 5.0.4. *(6, Corollary 3.11) Let Λ be a finitely aligned $k-$graph. Suppose there exists a saturated hereditary subset $H \subseteq \Lambda^0$ such that $\Lambda \setminus \Lambda H$ contains a generalized cycle. Then $C^*(\Lambda)$ is not AF.*

Their proof of Theorem 5.0.3 is given in two parts, one for cycles with an entrance and one without. The argument in the proof of (16, Theorem 10.18) shows that in a category of paths, the presence of a generalized cycle with an entrance gives rise to an infinite projection in the associated C^*-algebra (so that it is not AF). Below we shall give an example of a category of paths which has a generalized cycle (without an entrance), but for which the C^*-algebra is AF (in fact, finite dimensional). We first show that the converses of Theorem 5.0.3 and Corollary 5.0.4 do not hold.

Example 5.0.5. Let Λ' be the 2-graph which has skeleton depicted in figure 5.1 and factorization rules $\sigma_i \tau'_{i+1} = \tau'_i \sigma_{i+1}$ where $\sigma, \tau \in \{\alpha, \beta\}$. We will use the convention that $d(\alpha_i) = d(\beta_i) = (1,0)$ and $d(\alpha'_i) = d(\beta'_i) = (0,1)$. Now fix $\mu, \nu \in \Lambda'$ with $r(\mu) = r(\nu)$, $s(\mu) = s(\nu)$, and $\mu \neq \nu$. Let $d(\mu) = (m,n)$ and $d(\nu) = (k,\ell)$ and fix $\rho, \sigma, \tau, \eta \in \Lambda'$ with $\mu = \rho\sigma$, $\nu = \tau\eta$, $d(\rho) = (m,0)$, $d(\sigma) = (0,n)$, $d(\tau) = (k,0)$, and $d(\eta) = (0,\ell)$.

If λ extends μ, then the factorization rules imply that any time we factor $\lambda = \gamma\xi$ with $d(\gamma) = (p,0)$ and $d(\xi) = (0,q)$, then $\gamma = \rho\gamma'$ for some γ'. In words, any extension of μ where we factor all solid edges to the front must begin with the unique sequence of solid α's and β's that is ρ, and the same is true for any extension of ν. Similarly, if λ extends μ and $\lambda = \gamma\xi$ with $d(\gamma) = (0,p)$ and $d(\xi) = (q,0)$ then $\gamma = \sigma'\gamma'$ with $d(\sigma') = (0,n)$ and where $\sigma'_i = \alpha'_i$ iff $\sigma_{m+i} = \alpha'_{m+i}$. That is, if we factor the first n dashed edges to the left in any λ extending μ, then the number and order of α''s and β''s making up the first n edges must be the same as the number and order in σ, and again the analogous statement is true for ν.

Now if $m = k$ (and hence $n = \ell$), then either $\rho \neq \tau$ or $\sigma \neq \eta$. If $\rho \neq \tau$ and λ extends μ and ν, then the observations above show λ has two factorizations $\rho\lambda'$ and $\tau\lambda''$ with $d(\rho) = d(\tau)$, $d(\lambda') = d(\lambda'')$, but $\rho \neq \tau$. Hence μ and ν have no common extension. If $\sigma \neq \eta$, then writing $\mu = \sigma'\rho'$, $\nu = \eta'\tau'$ with $d(\sigma') = d(\eta') = (0,n)$, then $\sigma' \neq \eta'$ and arguing as above shows μ and ν have no common extension.

If $m > k$ and, say, $\rho_{k+1} = \alpha_{k+1}$, then μ and $\nu\beta_{|\nu|+1}$ have no common extension. If $\lambda = \lambda'\lambda''$ were such an extension with $d(\lambda') = (p,0)$ and $d(\lambda'') = (0,q)$, then extending μ would imply $\lambda'_{k+1} = \alpha_{k+1}$ while extending ν would imply $\lambda'_{k+1} = \beta_{k+1}$. Analogously, if $\rho_{k+1} = \beta_{k+1}$, then μ and $\nu\alpha_{|\nu|+1}$ have no common extension. Now write $\mu = \sigma'\rho'$ and $\nu = \eta'\tau'$ with $d(\sigma') = (0,n)$, $d(\eta') = (0,\ell)$ (with $\ell > n$). If $\eta'_{n+1} = \alpha'_{n+1}$, then ν and $\mu\beta'_{|\mu|+1}$ have no common extension, and if $\eta'_{n+1} = \beta'_{n+1}$, then ν and $\mu\alpha'_{|\mu|+1}$ have no common extension. If $m < k$, we make an analogous argument and this shows that (μ,ν) is not a generalized cycle.

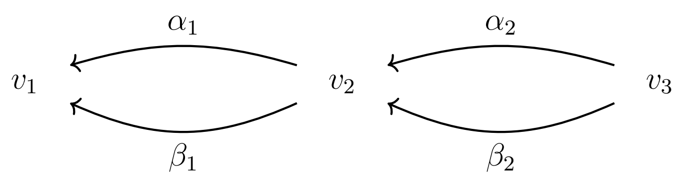

Figure 5.2: The 1-Graph Defining Λ''

Thus, Λ' is a 2-graph without generalized cycles. Now (5, Proposition 3.16) (see also (4, Corollary 5.1)) implies that two row-finite 2-graphs with no sinks or sources and the same skeleton have C^*-algebras with the same K_0 and K_1 groups. Using this result in the remarks following (4, Figure 4.3), Evans shows that the C^*-algebra of a 2-graph with the same skeleton as Λ' has K_1 group isomorphic to $\mathbb{Z}[\frac{1}{2}]$ and therefore is not AF. This shows that the converse of Theorem 5.0.3 does not hold. To show the converse of Corollary 5.0.4 does not hold, it remains to show that $(\Lambda')^0$ has no nontrivial saturated hereditary subsets.

Suppose $\emptyset \neq H \subseteq (\Lambda')^0$ is saturated and hereditary and fix $v_i \in H$. For any $j > i$, the path $\alpha_i \ldots \alpha_{j-1} \in H\Lambda'$ and $s(\alpha_i \ldots \alpha_{j-1}) = v_j$ so that $v_j \in H$ since H is hereditary. Now for $j < i$, let $F_j = \{\alpha_j, \alpha_j', \beta_j, \beta_j'\}$, which is clearly a finite exhaustive set at v_j. Since $s(F_{i-1}) = \{v_i\} \subseteq H$ and H is saturated, $v_{i-1} \in H$, and now by induction, $v_j \in H$ for every $j < i$ and hence $H = (\Lambda')^0$.

Our final example demonstrates that for categories of paths, even a generalized cycle need not preclude an AF algebra.

Example 5.0.6. Let Λ'' be the category of paths given by the 1-graph in figure 5.2 where $\alpha_1\beta_2 = \beta_1\alpha_2$ and $\alpha_1\alpha_2 = \beta_1\beta_2$. Explicitly,
$\Lambda'' = \{v_1, v_2, v_3, \alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_1\alpha_2, \alpha_1\beta_2\}$ where $(\Lambda'')^0 = \{v_1, v_2, v_3\}$, $(\Lambda'')^2$
$= \{(\alpha_1, \alpha_2), (\alpha_1, \beta_2), (\beta_1, \alpha_2), (\beta_1, \beta_2)\}$ with the obvious range and source maps, and ranges for multiplication. It's clear that this is a small category, without inverses since the commuting relations preserve length, and there is no (non-trivial) cancellation to

check. Note that (α_1, β_1) is a generalized cycle (without an entrance). Since the category is finite, the associated groupoid will be finite, and hence $C^*(\Lambda'')$ is finite dimensional.

Chapter 6

FURTHER RESEARCH

We close this paper with an outlook on further research, and there are some obvious avenues forward. First is broadening the class of examples in this paper to include continued fraction AF algebras $\mathcal{A}_\theta$ for arbitrary θ. We see two primary hurdles to overcome before this happens. One is to represent A_θ where $\theta = [k_1, 1, k_2, 1, k_3, 1, \ldots]$, i.e. θ's expansion has the same pattern of alternating 1's, but is not necessarily periodic. We are confident that this will arise from generalizing the example in chapter 4 in the obvious way, having k_i γ's in $v_i \Lambda v_{i+1}$. The reason this is so promising is the following: Given a continued fraction $\phi = [a_1, a_2, a_3, \ldots]$, the ordered $K-$theory of A_ϕ is determined entirely by the connecting maps $\left(\begin{smallmatrix} a_i & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (see, for example, (1, VI.3) for details) and then to realize the connecting maps in our example factor as

$$\begin{pmatrix} k_i + 1 & 1 \\ k_i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_i & 1 \\ 1 & 0 \end{pmatrix}$$

so that we are getting precisely the ordered $K-$theory of $A_{1+\theta}$. The main sticking point is in showing that the algebra we build has a unique trace; it was the fact that our connecting maps were periodic that allowed us to bound measures of cylinder sets and then show their measure must be unique. We believe a further approximation argument will solve this, but as yet we have not discovered it.

A method for eliminating the alternating 1's in $[1, k_1, 1, k_2, \ldots]$ is less clear. The factorization above makes it evident in one sense where the 1's are coming from, but it is not yet clear to this author what changes need to be made in the category of

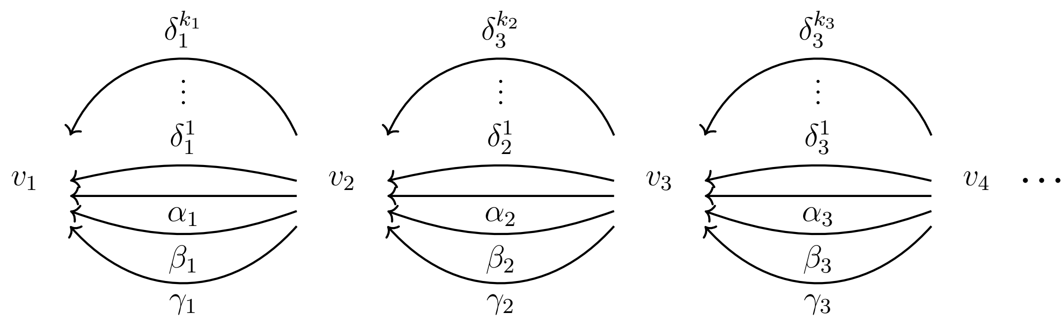

Figure 6.1: The 1-Graph of the Rank 3 Analogue

paths to eliminate them. Some ad-hoc attempts have been made, but the solution will take some more thought.

We would also like to generalize the examples in this paper to include the categories of paths defined by figure 6.1 where the $\alpha_i, \beta_i, \gamma_i$ form a 3-graph (and potentially analogues for $k-$graphs with $k > 3$). Some thought has been given to this and while it doesn't seem to be an obvious generalization of the work here, some similar techniques can be applied, and the problem appears to be tractable.

More broadly, we would like to apply techniques such as the ones demonstrated in this paper to problems involving $k-$graph algebras. For example, showing that $C^*(G)$ had finite nuclear dimension relied heavily on realizing it as an inductive limit of algebras of subcategories. There is a limited amount of freedom to do so in the context of $k-$graphs, since a subcategory need not be a sub $k-$graph, but we can still take advantage of the degree functor when appropriate. The γ's also seemed to play a critical role in generating subgroupoids whose C^*-algebras were manageable, and the key property seemed to be that they do not commute with any edges. If we could identify a structure in a $k-$graph with a similar property, it may be possible to develop a criterion for when a $k-$graph C^*-algebra has finite nuclear dimension. Unfortunately, it's not clear to this author that such a structure should even exist, and a problem like this will take some thought.

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