Impact of Teaching an Interdisciplinary Course *Introduction of Applied Mathematics for the Life and Social Sciences* on High School Students' Skills and Attitudes Towards Mathematics in a JBMSHP Summer Program.

by

Linda Agoune

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved March 2020 by the
Graduate Supervisory Committee:

Carlos Castillo-Chavez, Co-Chair
Carlos Castillo-Garsow, Co-Chair
Anuj Mubayi

ARIZONA STATE UNIVERSITY

May 2020

ABSTRACT

Research shows that the subject of mathematics, although revered, remains a source of trepidation for many individuals, as they find it difficult to form a connection between the work they do on paper and their work's practical applications. This research study describes the impact of teaching a challenging introductive applied mathematics course on high school students' skills and attitudes towards mathematics in a college Summer Program. In the analysis of my research data, I identified several emerging changes in skills and attitudes towards mathematics, skills that high-school students needed or developed when taking the mathematical modeling course. Results indicated that the applied mathematics course had a positive impact on several students' attitudes, in general, such as, self-confidence, meanings of what mathematics is, and their perceptions of what solutions are. It also had a positive impact on several skills, such as translating real-life situations to mathematics via flow diagrams, translating the models' solutions back from mathematics to the real world, and interpreting graphs. Students showed positive results when the context of their problems was applied or graphical, and fewer improvement on problems that were not. Research also indicated some negatives outcomes, a decrease in confidence for certain students, and persistent negative ways of thinking about graphs. Based on these findings, I make recommendations for teaching similar mathematical modeling at the pre-university level, to encourage the development of young students through educational, research and similar mentorship activities, to increase their inspiration and interest in mathematics, and possibly consider a variety of sciences, technology, engineering and mathematics-related (STEM) fields and careers.

DEDICATION

To my loving grandfather, Dr. Vasil O. Vinnik

(1.1.1928 – 3.24.2020)

ACKNOWLEDGMENTS

I owe my deepest gratitude to my co-advisors Dr. Carlos Castillo-Chavez and Dr. Carlos Castillo-Garsow, as well as, to Dr. Anuj Mubayi for their tremendous guidance, support, and advising. I am incredibly grateful and honored to have them all in my committee.

I would also like to thank the dean of Graduate College, Dr. Alfredo Artiles, for his trust, encouragement, and advising.

Finally, my appreciation goes out to my family, my father Madjid, my mother Olga, my brother Karim, especially my husband Mahdi, in supporting me throughout the years of my doctorate studies, and to our children, Adam, Noah, Daniel, and Sammy, for their love, patience and understanding.

TABLE OF CONTENTS

Page

LIST OF TABLES vi

LIST OF FIGURES vii

CHAPTER

1 STATEMENT OF THE PROBLEM 1

2 LITERATURE REVIEW 5

Mathematical Modeling 5

Reseach on the Necessary Skills for Modeling 14

Dynamical Systems 19

Attitudes Towards Mathematics 25

Past Research about MTBI and High School Programs 28

Summary of Literature Review and Link to Reseach Questions 31

3 METHODS 33

Background and Subjects 33

Data Collection 35

Data Analysis 40

Hypothetical Trajectory 46

Summary of Methods 48

4 STORY OF THE CLASS AND TEST RESULTS 49

Story of the Class and Project Timeline 49

PCA Test Results 52

ATMI Test Results 53

CHAPTER Page

Interview Summary 54

5 SKILLS 56

Equation Writing and Explanation 56

Graph Analysis and Interpretation 84

Result Interpretation 111

Pure vs. Applied 127

6 ATTITUDES 143

Academic Freedom and Enjoyment....144

Productive Struggle 159

Changing the Meaning of Mathematics....175

Changing the Meaning of Doing Mathematics 189

7 CONCLUSION 202

REFERENCES 208

APPENDIX

A ORIGINAL TEST AND SURVEY PROTOCOLS 218

B INTERVIEW PROTOCOLS 232

C SAMPLE ASSENT FORMS 238

D HUMAN SUBJECT EXEMPTIONS 241

LIST OF TABLES

Table Page

1. Summary of Mathematical Modeling Categories 12
2. Summary of Students' Pretest and Posttest PCA Answers and Scores 52
3. Summary of Students ATMI Test Significant Changes 53
4. Final Project Interview Summary 54
5. Predator-Prey Interview Summary 54
6. Students' Emerging Ways of Reasoning Related to Literature 126
7. Pure vs. Applied PCA Improved Problems 141
8. Examples of Student 2's ATMI Results Related to Mathematics Enjoyment. 145
9. Examples of Student 3's ATMI Results Related to Mathematics Enjoyment. 153
10. Examples of Student 5's ATMI Results Related to Mathematics Enjoyment. 156
11. Significant Confidence ATM Changes 160
12. Examples of Student 2's ATMI Results Related to Confidence..................... 161
13. Example of Student 3's ATMI Results Related to Confidence 169
14. Examples of Student 3's ATMI Results Related to Math Usefulness. 171
15. Examples of Student 1's ATMI Results Related to Math Usefulness. 176
16. Examples of Student 3's ATMI Results Related to Math Usefulness. 184
17. Examples of Student 5's ATMI Significant Changes in Confidence Items ... 190
18. Examples of Student 5's ATMI Results Related to Math Usefulness............. 194
19. Examples of Student 5's ATMI Results Related to Math Usefulness............. 197

LIST OF FIGURES

Figure Page

1. The Real World and the Model World (Powell and Baker, 2009) 7
2. Components of a Modeling Process (GAIMME, 2016) 15
3. Stages of Mathematical Modeling (Edward and Hamson, 1996) 16
4. Stages of Mathematical Modeling (GAIMME, 2016; GAIMME, 2019) 16
5. Stages in the Mathematical Modeling Process (Haines et al., 2001) 18
6. Student 1's Group's vs. Student 1's Written Flow Diagram and Equations. 59
7. The Predator-Prey Problem.. 61
8. Student 2's Group's vs. Student 2's Written Flow Diagram and Equations. 66
9. Student 3's Group's vs. Student 3's Written Flow Diagram and Equations. 72
10. Student 5's Group's vs. Student 5's Written Flow Diagram and Equations. 79
11. Student 1's PCA Pretest Solution for Question 6 86
12. Student 1's PCA Posttest Solution for Question 6.. 86
13. Student 1's Graph for the Predator-Prey Interview Problem............................. 87
14. Student 2's PCA Posttest Solution for Question 8.. 91
15. Student 2's Graph for the Predator-Prey Interview Problem............................. 93
16. Student 3's PCA Pretest Solution for Question 15 98
17. Student 3's PCA Posttest Solution for Question 15.. 99
18. Student 3's Graph for the Predator-Prey Interview Problem........................... 101
19. Student 4's PCA Posttest Solution for Question 9.. 103
20. Student 4's PCA Posttest Solution for Question 10.. 104
21. Student 4's PCA Posttest Solution for Question 24.. 104

Figure Page

22. Student 5's PCA Pretest Solution for Question 19 105

23. Student 5's PCA Posttest Solution for Question 19.. 106

24. Student 5's Graph for the Predator-Prey Interview Problem........................... 108

25. Student 1's PCA Pretest Solution for Question 4 127

26. Student 1's PCA Posttest Solution for Question 4... 127

27. Student 5's PCA Pretest Solution for Question 4 131

28. Student 5's PCA Posttest Solution for Question 4... 132

29. Student 4's PCA Posttest Solution for Question 17.. 135

30. Student 2's PCA Pretest Solution for Question 1 136

31. Student 2's PCA Posttest Solution for Question 1... 136

32. Student 3's PCA Pretest Solution for Question 1 137

33. Student 3's PCA Posttest Solution for Question 1... 138

34. Graph for the Predator-Prey Problem as Provided by Instructor 1 235

35. Phase Plane Predator-Prey Problem Graph... 237

CHAPTER 1

STATEMENT OF THE PROBLEM

Learning mathematics has become a necessity for an individual's full development in today's complex society, and despite its usefulness and importance, mathematics is perceived by most students as difficult, boring, not practical, and too abstract (Ignacio, Nieto & Barona, 2006). Moreover, Joubert (2009) when analyzing students' attitudes towards mathematics suggests, that the subject of mathematics, although revered, remains a source of trepidation for many individuals, as they find it difficult to form a connection between the work they do on paper and their work's practical applications. Similarly, mathematicians view the world as divided into two parts: mathematics and everything else, or the 'real world', with the belief that the mathematics learned at schools and the mathematics used in their daily lives are completely independent and unrelated subjects. (Hernández et al., 2017).

A number of studies have also indicated that many children begin schooling with positive attitudes towards mathematics. These attitudes tend to change negatively as students grow up, and often become negative at the high school level (Ma & Kishor, 1997). As a result, many U.S. campuses struggle to attract students into mathematics beyond the required courses at the undergraduate level – about 1 percent of students major in mathematics at the undergraduate level, and over half the graduate students in mathematics are foreign (Tapia & Marsh, 2002). Because students' attitudes towards mathematics determine their mathematical success (Köğce et al., 2009), this negativity towards mathematics is important to address, since mathematics can be a foundational pathway to many sciences, technology, engineering and mathematics-related (STEM) fields and careers.

Mathematical modeling comes to help bridge the gulf between reasoning in the mathematics class and reasoning about a situation in the real world. (Pollak, 2011, Frejd, 2014, Hernández et al., 2017). Mathematical modeling can also be used to motivate curricular requirements and can highlight the importance and relevance of mathematics in answering important questions. It can also help students gain transferable skills, such as habits of mind that are pervasive across subject matter. (Hernández et al., 2017). In this perspective, mathematical modeling plays a great role in engaging students' curiosity, and the problems they are investigating can be highly motivating for students, when they can relate the mathematics that solves real-life problems to their own interests. Stylianides and Stylianides (2008) suggest that mathematical problems involving embedded real-life contexts have received increased attention by mathematics researchers, in part because of the considerable levels of student engagement often triggered by their feelings of motivation.

Mathematical modeling can be defined by a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena. (GAIMME 2019). Other researchers' definitions and perspectives will be discussed in the Literature Review section.

Haines et al. (2001) describe seven modeling skills for the students to develop. The authors presented them in stages: Transitioning from real world problem, formulating a model, solving mathematics, interpreting outcomes, evaluating the solution, and refining the model or reporting results. Dan and Xie (2011) group students' essential skills required for assessing mathematical modeling into: Simplifying assumptions, clarifying the goal, formulating the problem, assigning variables, parameters, and constraints, formulating mathematical statements, selecting a model, graphical representations, and relating back to the real world. Sodhi and Son (2007) also suggest that, the skills required for mathematical modeling future jobs are “soft” skills, pertaining to problem solving, teamwork, and communication.

To these skills, another key conception in developing mathematical models is the concept of functions. To understand mathematical modeling, a robust understanding of functions, both algebraically and graphically, is necessary, since mathematical models rely on the creation and the analysis of dynamical systems, i.e. functions of time, as they change from one state to another.

There have been several calls for school curricula to place greater emphasis on the concept of functions. More recently, researchers and mathematics educators have continued to recognize the need to improve the way we understand and teach functions (Hauger, 1995, 1997; Carlson, 1998; Bezuidenhout, 1998; Carlson et al., 2002; Oehrtman et al., 2008). However, it appears that students still lack a conceptual understanding of what a function is, or how it can be applied to real-life situations. Instead, it has been shown that students are still limited to procedural computations and skills, and view functions as mechanical calculation procedures rather than relations between the quantities involved (Monk & Nemirovsky, 1994; Kaput, 1992; Monk, 1992; Carlson, 1998; Davis, 2005; Oehrtman et al., 2008).

It is important to address and assess this problem at the pre-university level, since the understanding of the concept of a function is one of the most central understandings that a student is expected to gain in undergraduate mathematics, and is a key feature in developing mathematical skills necessary for many future mathematics courses such as, mathematical modeling.

Merging these two important ideas that focus on students' mathematical modeling performance, and beliefs towards mathematics, this study aims to examine and learn about the *changes* in students' skills and attitudes towards mathematics, when a mathematical modeling course, is taught in a truly interdisciplinary way, that includes application-driven group research projects and multiple group assignment activities. The details of this course and the community model it follows in terms of group research and mentoring will be discussed in the methodology and the history of the class chapters.

My research questions combine two major ideas of: (a) students' changes in mathematical skills, and (b) students' changes in attitudes towards mathematics, into the following two questions:

1. What skills do high-school students need, or will gain/develop, when taking a specific mathematical modeling course?
2. How do these students' beliefs and attitudes towards mathematics change, when taking an applied mathematics course, that is taught in an interdisciplinary way that includes an application-driven group research project and multiple group assignment activities?

CHAPTER 2

LITERATURE REVIEW

1. Mathematical Modeling

1.1. Introduction

Modeling is the process of creating a simplified representation of reality and working with this representation in order to understand or control some aspect of the world (Powell and Baker, 2009). The authors add that modeling itself is an omnipresent human activity. It is one of a few fundamental ways in which humans understand the world. Models take many different forms, such as mental, visual, physical, mathematical, and spreadsheet.

Mathematical modeling, as defined by GAIMME (2019) is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena. Mathematical models can take many forms, such as statistical models, differential equations, dynamical systems, or game theoretic models. These and other types of models may also overlap.

1.2. Different Perspectives about Modeling

Pollak (2011) suggests that, different researchers define mathematical modeling differently, in part because the mathematical modeling requires reasoning in the outside world differently than from reasoning in mathematics. He adds, an important point, that real situations have so many “angles” that you cannot take everything into account: you need to decide which aspects are most important and keep these in consideration when creating the model.

There are different categories of definitions of mathematical modeling and different types of mathematical modeling. In reading the literature, I found that researchers' definitions of modeling fell into some of the following categories,

Category 1. Mathematics and the Real World

- Mathematics and the real world are separate: where the real world is translated to mathematics

Pollak (2011), Frejd (2014) and Hernández et al. (2017) claim that mathematicians, or most people, view the world as divided into two parts: mathematics and everything else, or the 'real world', with the belief that the mathematics learned at schools and the mathematics used in their daily lives are completely independent and unrelated subjects.

While some authors believe that the mathematics and the real world are separate, mathematics can still be applied to the real world, by translating a real-world context to mathematics, and then working the mathematics separately. Pollak (2011) claims that once an idealized version of the real-world situation is translated into mathematical terms, a mathematical model of the idealized question is produced. When adding mathematical instincts, knowledge to the model, and get interesting insights, examples, approximations, theorems, and algorithms, you translate all this back into the real-world situation, verifying that the results are practical with reasonable answers and acceptable consequences. If this is not the case, then another look is required at the choices made at the beginning of this process, and another trial is required. Pollak calls this entire process mathematics modeling.

As example, to understand the difference between the idealized version and the real-world situation mentioned above, Powell and Baker (2009) explain that, “a model is an abstraction, or simplification, of the real world. It is a laboratory—an artificial environment—in which we can experiment and test ideas without the costs and risks of experimenting with real systems and organizations”. Figure 1 shows how modeling creates an artificial world.

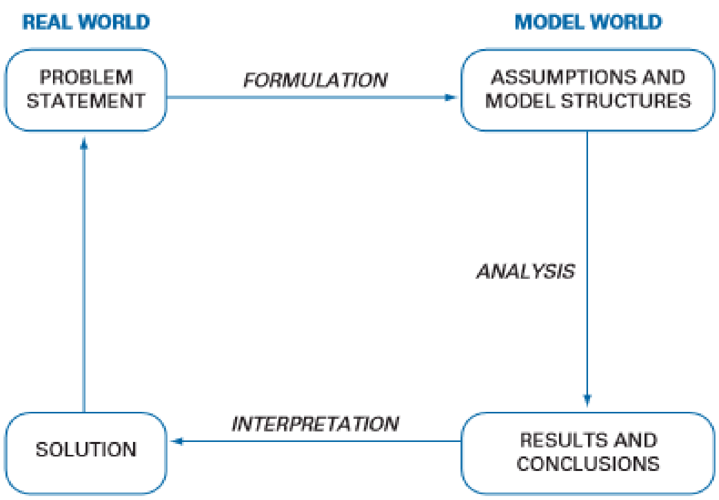


Figure 1. *The Real World and the Model World (Powell and Baker, 2009)*

Powell and Baker (2009) suggest that, we begin in the real world, with a problem statement. In order to move into the model world, we abstract the important features of the real world, leaving behind all the inessential details. Next, we construct our laboratory by combining our abstractions with assumptions, and building a model of the essential aspects of the real-world. (Powell and Baker, 2009). Other authors with similar perspectives or diagrams include Zbiek (1998), Schoenfeld (1991), one of Gravemeijer’s models (1994), Edwards and Hamson (1996), Haines et al. (2001), and Pollak (2011).

- <u>Mathematics and the real world are interleaved</u>

This is where every step of mathematics has meaning in the real-world context. For example, Thompson believes in modeling as an interleaved process, rather than one that keeps the mathematics and the real-world separate. Thompson's (2011) perspective about mathematical modeling relies on quantities and the relationship between the involved quantities. He suggests that mathematical modeling is the use of mathematical notation and methods to express and to reason about relationships among quantities so that the inputs and output of every mathematical operation has a corresponding measurement in the world. This way of reasoning, contrasts with what Thompson calls, customary algebra, where the relationship between the quantities is ignored, and the mathematics is manipulated purely symbolically until the final result is interpreted.

An interleaved perspective is also used to teach mathematics. Vos (2013) describes mathematics education in other countries, such as the Netherlands, where mathematics is interleaved with applications. This movement is based on the work of Hans Freudenthal (2006) and colleagues, and Gravemeijer's (1994) ideas of RME development. Freudental and colleagues developed a treatise known as Realistic Mathematics Education (RME). This reformed method of education relies on the students' use of the context to reinvent mathematical algorithms, so to these students, every step of the algorithm has meaning within the context.

Gravemeijer (1994) notes that, "In RME, modeling as an activity is further elaborated. The idea is that informal ways of modeling emerge when students are reorganizing their activity while solving contextual problems. Later, these ways of modeling may serve as a basis for developing more formal mathematical knowledge. At first a model is constituted as a context specific model of acting in a situation, then the model is generalized over situations. The model changes character, it becomes an entity of its own, and as such it can function as a model for more formal mathematical reasoning. The shift from model-of to model-for concurs with a shift in the students' thinking, from thinking about the modeled situation, to a focus on mathematical relations". In this sense, the role of the model gradually changes as it takes on a life of its own. As a consequence, the model can become a referential base for more formal mathematical reasoning. In short, his describes a "model of" where every step has a content, while in "model for," the algorithm is separated from the context and can be applied to new contexts.

Category 2. Algorithmic vs. Artistic process

Some authors view the modeling process to be a formalization of a mechanical or algorithmic process, others view it as a creative or artistic process:

To Ellis (2007), a mathematical model is a generalization of one's understanding of a situation's inner mechanics—of "how it works", as taken from one's perception of the process. For example, this mechanistic, or 1st principle process involves Ellis' approach of using an exponential function when an organism doubles at some number of hours.

On the other hand, Edwards and Hamson (1996) suggest that "mathematical modeling in by its very nature a practical and creative process". They state that the development of a model requires a creative imagination, as different modelers might approach the creation of the models differently.

In addition, Zbiek (1998) agrees with Niss' (1988) definition of a mathematical problem, they define it as – "a combination of one or more mathematical entities and the relationships among them that are chosen to represent aspects of a real-world situation". However, Zbiek (1998) views this process as requiring the application of mathematics to unstructured problems in real-life situations (Galbraith & Clatworthy, 1990), as an art (Kapur, 1982), not an algorithmic process, despite an underlying formal structure that includes: formulating the problem situation, choosing the appropriate variables, determining the relationships among these variables, devising a mathematical model embodying these variables and relationships, and testing that model and its implications (Galbraith, 1987, Zbiek, 1998). Zbiek (1998) also adds, that in some ways, modeling "seems more art than science", because often it is based on personal experience and insights.

Category 3. Types of models (Descriptive vs. Theoretical, and ad hoc vs. 1st principle)

Smith et al. (1997) describe the perspectives of one of their study's participant, an applied mathematician named Carlos, where he suggests some ways to classify models. In understanding the notion of a 'good' biological model, Carlos describes different purposes and types of modeling: the descriptive model, describes 'everything' in a biological system, where it tries to mimic the reality within a restricted biological framework, while the theoretical model is related to a specific question where the modeler tries to disregard all aspects that are insignificant to the question.

Carlos also describes ad hoc models are based on a phenomenological approach and are built from observations, curve fitting, and shape matching, such as logistic models, while first principle models try to directly tie a mathematical form to a biological understanding of a given phenomenon, with the consequence that all individuals are tracked and quantities are preserved, for example, when an organism doubles every number of hours, we use an exponential function. This type seems to describe the mechanistic process. Therefore, I will merge Carlos' 1st principle together with the mechanistic category.

Category 4. Types of roles for models as Epistemological objects

There is a number of possible roles for models as 'epistemological objects' in the practice of biology as Smith et al. (1997) described: (1) Model as a caricature of a biological system, in which we explore dynamic patterns of interaction. (2) Model as experiment in which evidence from several models is gathered. (3) Model as a first principle, where the relationship between biological dynamics and the mathematical structure creates opportunities to rule out possibilities. (4) Model as empiricist, where the model suggest what data needs to be collected in relation to a particular problem, and finally (5) model as dialectic, where the interaction between results from the model and the knowledge of the biologist forms the basis for the development of new biological knowledge.

Summary of results

Here is a table that summarizes the major categories for modeling:

Table 1

Summary of Mathematical Modeling Categories

Category Name	Category Description	Papers
Mathematics and the Real World interleaved	A model where every step of mathematics has meaning in the real-world context	Thompson, 2011; Gravemeijer (model of), 1994; GAIMME
Mathematics and the Real World separated	The real world is converted to mathematics, modelers do the mathematics, then math results are converted back to real world results.	Zbiek,1998; Powell and Baker (2009); Gravemeijer; (model for), 1994; Schoenfeld, 1991; Pollack, 2011; Hanes et a. (2001); Edwards and Hamson (1996)
Algorithmic/mechanistic, Carlos' 1st principle	The model ties a mechanical form to a biological understanding of a phenomena	Ellis, 2007; Thompson, 2011; Schoenfeld, 1991; Gravemeijer (model for), 1994; Smith et al. 1997 GAIMME
Artistic/Creative	A model that is creative is a model involving the imagination or original ideas/experiences, in the production of an artistic outcome.	Kapur, 1982; Edwards & Hamson, 1996; Zbiek, 1998; Niss 1988; Gravemeijer (model of), 1994;
Carlos' Descriptive	The model describes everything, mimicking the reality within a restricted framework	Smith et al. 1997; Sodhi & Son, 2007;
Carlos' Theoretical	The model is related to a specific question, modeler rejects all unimportant things unrelated to the question.	Thompson, 2011; GAIMME; Gravemeijer (model of), 1994; Ellis, 2007;
Carlos' Ad hoc	The model is built from data observations and shape matching.	Smith et al. 1997; Gravemeijer (model of), 1994; Edwards & Hamson, 1996; Sodhi & Son, 2007.
Epistemological object - Dialectic	A model where the interaction between results and the knowledge of the biologist forms the basis for the development of new biological knowledge	Smith et al. 1997 Gravemeijer (model of), 1994; Thompson (2011); GAIMME.

Epistemological object - Caricature	A model where we explore dynamic patterns of interaction.	GAIMME; Gravemeijer (model for), 1994;
Epistemological object - Experiment	A model with evidence from several models is gathered	Smith et al. (1997)
Epistemological object - Empiricist	The model suggests what data needs to be collected in relation to a particular problem	Smith et al. (1997)
Epistemological object - Smith's 1st Principle	A model where the relationship between biological dynamics and the mathematical structure creates opportunities to rule out possibilities.	Smith et al. (1997)

Conclusion

Mathematical modeling can be defined and categorized into different categories, for example, some authors such as Powell and Baker (2009), Edwards and Hamson (1996), Schoenfeld (1991), see mathematics modeling as a separate process where someone translates the real world to mathematics, does the mathematics, and then translates it back to the real world. Other authors see the mathematics and the real world as interleaved, such as Thompson (2011), and Gravemeijer's (1994) model-of.

Beguilingly, in his work, Gravemeijer (1994) described two ways: students start modeling in an interleaved manner and then eventually develop the model into a separate perspective.

GAIMME authors, on the other hand see the two as separate but with a many back and forth links and translation between the two worlds, which shows attempts as viewing thinks interleaved.

Carlos discusses four types of models, theoretical versus descriptive, adhoc vs. 1st principle: for example, Ellis seems to have theoretical, 1st principles models, while Edwards and Hamson's (1996) models fall into the ad-hoc category.

In addition, some authors describe the development of the mathematical process as a mechanistic process (Ellis, 2007; Thompson, 2011), while others as a creative or artistic process, such as Zbiek (1998), Edward and Hamson (1996), and Gravemeijer's (1994) "model-of".

2. Research on the Necessary Skills for Modeling

Several authors researched the skills that are required for mathematical modeling, for instance, the GAIMME team describe the chronological order of the required skills to reach a mathematical model. Haines et al.'s (2011) interests involved understanding specifically *students*' modeling skills and also followed a chronological order. Other authors, such as Thompson (2011) relied on students' quantitative reasoning to address the conceptual skills for modeling, i.e., how might the students identify and organize quantities, and finally Sodhi and Sons (2007)'s presented the skills required for mathematical modeling future professions, such as, what they call, the "soft" skills pertaining to problem solving and communication, attained through competitive team exercises, group discussions, presentations, and case analysis.

2.1. Chronological Order of Skills

In 2015, mathematics leaders and instructors from the Society for Industrial and Applied Mathematics (SIAM), the Consortium for Mathematics and Its Applications (COMAP), with input from NCTM, came together to define Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) report. The six components of a modeling Process they identified, can be viewed on Figure 2 below, and represent a breakdown of modeling into chronologically ordered steps.

Components of a Modeling Process

Identify the problem: We identify something in the real world we want to know, do, or understand. The result is a question in the real world.

Make assumptions and identify variables: We select “objects” that seem important in the real-world question and identify relations between them. We decide what we will keep and what we will ignore about the objects and their interrelations. The result is an idealized version of the original question.

Do the math: We translate the idealized version into mathematical terms and obtain a mathematical formulation of the idealized questions. This formulation is the model. We do the math to see what insights and results we get.

Analyze and assess the solution: We consider: Does it address the problem? Does it make sense when translated back into the real world? Are the results practical, the answers reasonable, and the consequences acceptable?

Iterate: We iterate the process as needed to refine and extend our model.

Implement the model: For real world, practical applications, we report our results to others and implement the solution.

Figure 2. *Components of a Modeling Process (GAIMME, 2016)*

Edwards and Hamson (1996) view mathematical modeling as the activity of translating a real problem into mathematics for subsequent analysis. A Mathematical problem will be created, and its solution will usually provide information useful in dealing with the original real problem. Researchers such as Edwards and Hamson (1996) and Haines et al. (2011) represent these, or similar to these steps, more of a cycle, see Figure 3 below, note that the “solve the mathematical problem” on this figure, as a separate item shows evidence of Edward and Hamson’s (1996) view of the mathematics as being separate from the real world.

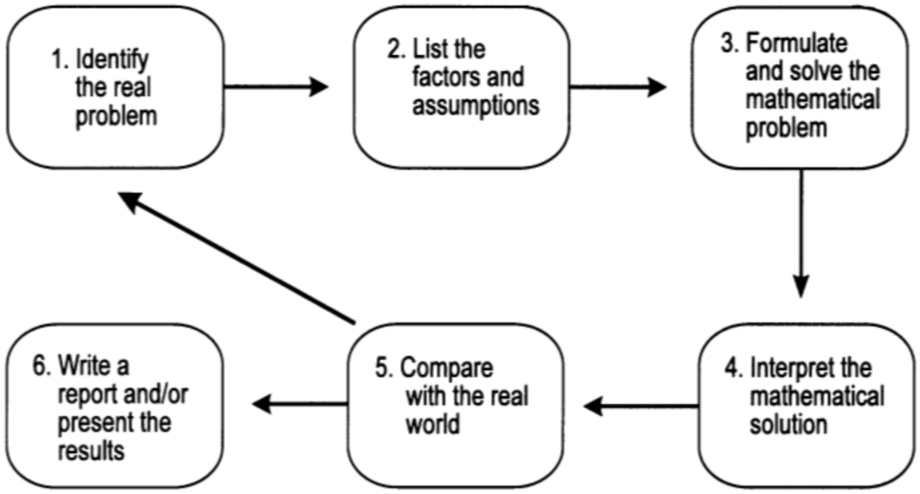


Figure 3. *Stages of Mathematical Modeling (Edward and Hamson, 1996)*

However, the GAIMME report, as the authors claim, intentionally use the specific relationship between these components in the way as shown on Figure 4, which reflects the fact that in practice "a modeler often bounces back and forth through the various stages" of mathematical modeling. (GAIMME 2016).

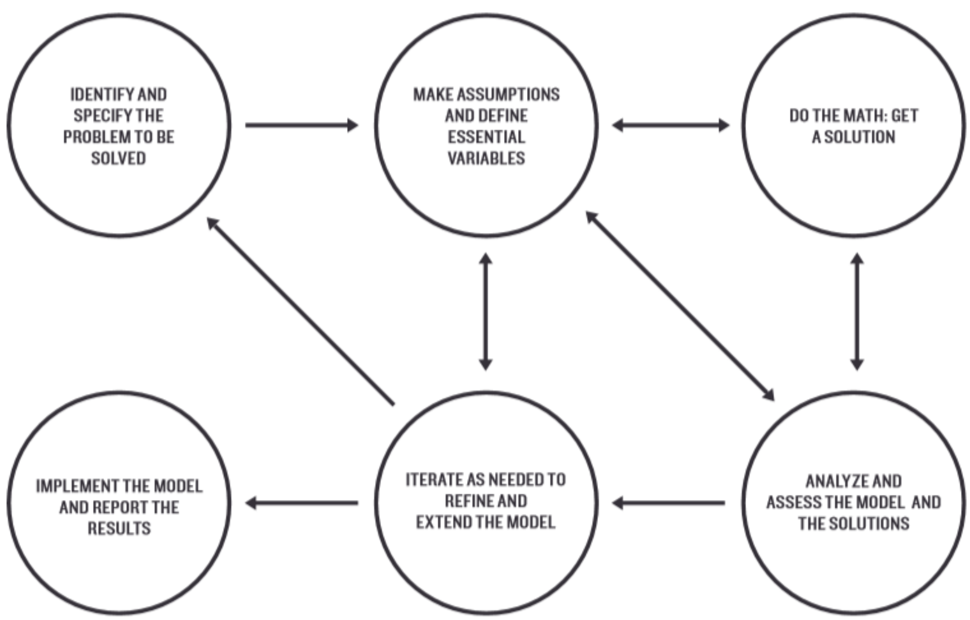


Figure 4. *Stages of Mathematical Modeling (GAIMME, 2016; GAIMME, 2019)*

Figure 3 and Figure 4 show some of the skills necessary in mathematical modeling. GAIMME reports that these skills are gradually gained through practice, experience, teamwork, and practical work with example models. Once a problem is identified, it is essential to list assumptions, choose a mathematical approach, produce function(s), get a solution, interpret the mathematical solution, and assess it for usefulness and accuracy by comparing it with the real world, and then re-adjust the model as needed until it provides an accurate and predictive enough understanding of the situation. This process concludes with either a written report and/or the presentation of the main findings and results. Figure 4 is an example of an attempt at seeing the mathematics and the real world interleaved, because people keep moving back and forth between doing the mathematics and interpreting it (as the arrows show).

Edwards and Hamson (1996) suggest that "mathematical modeling in by its very nature a practical and creative process involving a number of stages, all of which demand a range of skills", which are gradually gained through practice, experience, and practical work with example models. These authors add that some of the necessary skills for mathematical modeling involve collecting and interpreting data, setting up models, developing models, checking models, and fitting models to data. This shows evidence of compatibility with previously mentioned ad hoc models.

Haines et al. (2001) on understanding students' modeling skills, recognized seven important stages, see Figure 5, Authors found, that these skills are best developed through group projects and investigations. As shown on the following figure, Haines et al., these authors view the mathematics and the real world as separated entities.

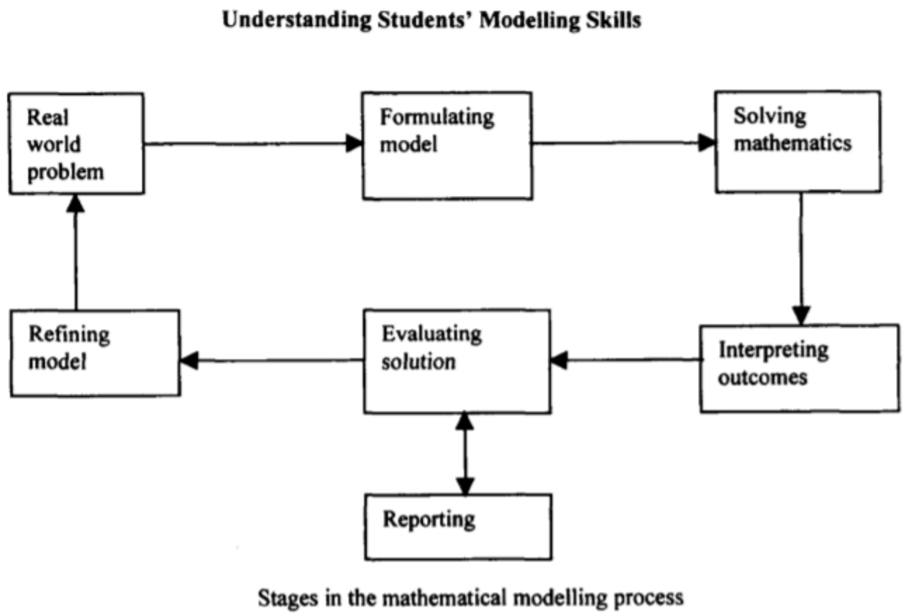


Figure 5. *Stages in the mathematical modeling process (Haines et al., 2001)*

2.2. Soft Skills

Sodhi and Son's (2007)'s paper analyzed advertisements for operations research jobs. They define operation research as a mathematical basis for making decisions. Their analysis shows that skills required for mathematical modeling future jobs include "soft" skills pertaining to problem solving and to communication, which are not covered by ordinary mathematical modeling curricula. These skills can be acquired by adopting innovative teaching methods such as competitive group exercises, discussions, presentations, and case analysis.

There is also strong evidence that particular kinds of small group discussions are important for improving the learning of general mathematical ideas (Schoenfeld, 1987; Ikeda and Stephens 1998). When students work in small groups, Bielaczyc et al. (1994) suggest that support from group members improves individual learning, because the students are fully engaged in cohesive group behavior.

2.3. Summary

In this paper, by modeling, I will follow GAIMME's definition, which is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena.

In order to solve modeling problems, it is necessary to make assumptions, choose a mathematical approach, get a solution, assess the solution for usefulness and accuracy, and then re-adjust the model as needed until it provides an accurate and predictive enough understanding of the situation. Communicating the model and its implications in a compelling way, as well as working in groups in order to develop shared project ideas, may be as essential to a model's success as the solution itself. Even young students can engage in mathematical modeling.

3. Dynamical Systems

3.1. Introduction

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a space. Newton's fluent and fluxions were early versions of formalizing this idea. Fluents and Fluxions were introduced by Isaac Newton in 1665, to describe his form of a time derivative, fluents were functions of time, and a fluxion was a derivative with respect to time. Fluxions and fluents made up Newton's early calculus (Edwards, 2012).

Dynamical systems, as explained by Dean & Leach (1995), are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time. The answer what a dynamical system is, requires that we are precise about the notions of "system state" and "evolution of the system state over time." The evolution of the system state refers to a sequence or continuous trajectory through the space of possible system states.

These models are used in financial and economic forecasting, environmental modeling, medical diagnosis, industrial equipment diagnosis, and a host of other applications, to predict, explain, and understand physical phenomena.

The characteristics of dynamical systems are the characteristics of mathematical models, e.g., linear, nonlinear, deterministic, stochastic, discrete, continuous (Castillo-Garsow and Castillo-Chavez, 2016). One of the most important characteristics of dynamical systems concerns the notion of observability.

3.2. Discrete vs. Continuous

Discrete and continuous systems behave differently - One basic type of dynamical system is a discrete dynamical system, where the state variables evolve in discrete time steps. Discrete dynamical systems are used to model population growth, from simple exponential growth of bacteria to more complicated models, such as logistic growth and harvesting populations. A continuous dynamical system is a dynamical system whose state evolves over state space continuously over according to a fixed rule.

Castillo-Garsow and Castillo-Chavez (2016) state that the distinction between discrete and continuous has a huge impact on the behavior of dynamical systems. Also, work with students has shown that the distinction between discrete and continuous can be difficult for students who are used to plotting points and connecting the dots. Even successful students will favor whole number counting, and regularly spaced numbers, that restricts their ability to draw conclusions about continuous systems.

Bassock and Olseth (1995) found that students were far more likely to use a discrete method for a continuous problem than they were to use a continuous method for a discrete problem. They suggested that students are more likely to use a discrete method because they are reasoning discretely, and a discrete method is what is generated by that discrete reasoning. They also add, that applying a continuous method to a discrete problem would fill in gaps that do not necessarily exist.

Castillo-Garsow (2010), when investigating his research questions through a teaching experiment, focused on Algebra II students' understanding of change, identified two major ways of thinking about change in general: chunky, and smooth within completed chunks: (1) Chunky thinking is thinking in terms of equal intervals, but not about the intervals. Students thinking this way, reason in discrete points, where the intermediate values within the chunk exist but have little attention. (2) Smooth thinking is imagining change in progress (Castillo-Garsow 2010). The change has a beginning point but no end point, and as soon as an endpoint is reached, the change is no longer in progress, which is different from getting smaller and smaller chunky images of change (Castillo-Garsow et al., 2013).

Castillo-Garsow (2010) suggests, that when working with linear functions, there is very little difference in the conclusions reached by students who begin with a completed chunk and divide it up and students who reach that chunk through a process of imagining smooth change in progress. When dealing with non-linear functions, such as the step function solution, smooth thinking becomes more important. When dealing with ideas of calculus, and specifically exponential growth, smooth thinking becomes critical, because smooth reasoning intuitively introduces the idea of passing through every point in-between the beginning and ending of a change without the much more sophisticated limit reasoning required by a discrete only approach. This intuition enables students to solve problems that would otherwise require much more elaborate mathematical toolkits.

3.3. Stochastic vs. Deterministic

Random behavior is non-deterministic, i.e., even if everything is known about a system at a given time in perfect detail, there would still not be possible to predict the state at a future time. Chaotic behavior on the other hand, is fully deterministic if the initial state is known in perfect detail, but any imprecision in the initial state, no matter how small, grows quickly (exponentially) with time.

Castillo-Garsow and Castillo-Chavez (2016) explain, that stochastic processes, such as agent-based models, have been shown to have tremendous applications in helping students develop a research modeling mindset. Because the high degree of complexity can be managed by computers, little programming training enables the students to explore complex systems, that they would not be well equipped to handle with algebraically.

Dynamical systems are deterministic, if there is a unique consequent to every state, or stochastic (random) if there is a probability distribution of possible consequents.

3.4. Perception of a Solution

Rasmussen (2001) explained, that the idea of a solution to a differential equation requires a different meaning of solution for students. Students need to think of a solution as a function, instead of a number. Also, Rasmussen discussed another type of solution, that he called an “equilibrium solution,” in which students study equilibria as a method of examining classes of behavior of the system. This distinction between types of solutions parallels the earlier distinction made between descriptive modeling (for which a function solution is more useful) and theoretical modeling (for which a qualitative behavioral solution is more useful).

Based on these views of "solution", Rasmussen (2001) highlights the importance of the student's understanding of function and quantity, as well as students' images of stability and numerical approximation.

3.5. Functions

Dynamical systems represent systems in which functions describe the time dependence of a point in a space.

Research on functions

The understanding of the concept of a function is one of the most major understandings that a student is expected to gain in undergraduate mathematics, and is a key feature in developing mathematical skills necessary for many future mathematics courses such as, mathematical modeling. It is trivial that in mathematical modeling, without a robust understanding of functions, formulating a mathematical problem with the use of mathematical notation and effectively reason about the relationships among the involved quantities might not lead to desired and accurate results. Therefore, function skills are crucial to have and gain for mathematical modeling.

Research has shown that many students do not reason about functions in a conceptual way and are still limited to procedural algorithmic skills. They view functions as mechanical calculation procedures rather than relations between the quantities involved (Monk & Nemirovsky, 1994; Kaput, 1992; Monk, 1992; Davis, 2005; Oehrtman et al., 2008).

Various studies in the literature have also identified the importance of analyzing the effectiveness of a course by giving pre- and post-evaluation tests. PCA (the Precalculus Concept Assessment) is one of the tools that has been partly used to identify characteristics of a course for evaluating the effectiveness of specific courses. Typically, the studies compare results from the PCA tool with results from traditional ones. The PCA serves as a model for others who take on the challenge of developing multiple-choice, conceptually focused instruments to assess students' knowledge, and performance in specific content areas. PCA also aims at provoking reflection on the role of cognitive research in informing instrument development that contributes to higher standards for the processes of instrument development.

Carlson, Oehrtman, and Engelke (2010) developed a Precalculus Concept Assessment (PCA) instrument, a 25-item multiple-choice test with conceptual questions (see Appendix A_1). The reasoning abilities and understandings crucial to precalculus and foundational for beginning calculus were identified and characterized in a series of research studies and were articulated in the PCA Taxonomy. These include a strong understanding of ideas of functions in applied math contexts, a process view of function, and the ability to examine how two varying quantities change together. This co-variational reasoning guided the PCA development and now provides the theoretical basis for interpreting and reporting PCA results. Among the uses of PCA questions may include assessing student learning in a classroom setting, which I will be primarily using as a framework to assess possible students' changes in performance. The uses of PCA may also involve (b) comparing the effectiveness of various curricular treatments, and (c) determining student readiness for calculus.

Summary and Analysis of Skills

The skills that seem to be most common in research papers are related to the chronological order required in the development of a mathematical modeling process, such as formulating the model, solving and interpreting the solutions. Other skills are organized in a conceptual way, that require ways of reasoning about quantities that are within real-life contexts. Next, other authors emphasize the importance of group work, good communication, and group analysis as important skills for engaging into a mathematical modeling process. Research shows that all these various skills are necessary to have, or gain for a successful mathematical modeling experience.

4. Attitudes Towards Mathematics

4.1. Introduction

Lockhard (2009) openly criticizes the way mathematics is typically taught in US schools and argues for an intuitive, artistic, and problem-oriented approach to teaching motivated entirely by pure mathematics. He gives examples that he calls the saddest part of reform, such as when attempts are made to "make math interesting" and "relevant to kids' lives." Lockard believes that mathematics is already interesting, attempts to present mathematics as relevant to daily life inevitably appear forced and contrived: "You see kids, if you know algebra then you can figure out how old Maria is if we know that she is two years older than twice her age seven years ago!" (As if anyone would ever have access to that ridiculous kind of information, and not her age.) Algebra is not about daily life, it's about numbers and symmetry".

However, not all mathematics students go on to become pure mathematicians, and many of them are motivated by applications. Negative feelings of emotions towards mathematics may prevent a great number of potential students from pursuing careers in technology, science, business, statistics, physics, biology, computer science, and several engineering, and mathematics related fields. Courses such as the one studied here answer Lockhart's criticisms not by removing applications, but by teaching better applications. Therefore, it is important to study the effects of applied mathematics courses on students' attitudes toward mathematics.

4.2. Research about Discomfort Towards Mathematics

There has been extensive research about students' discomfort towards mathematics. The research about mathematics discomfort seem to be divided into mathematics anxiety (Hembree, 1990; Ma, 1999, Dew & Galassi, 1983, Gurin, Jeanneret, Pearson, Salinas, Castillo-Garsow, 2017) and mathematics attitudes (Tapia, 1996; Tapia & Marsh, 2004; Hyde, Fennema, Ryan, Frost, & Hopp, 1990; Fennema, & Sherman, 1976, Ben, 1991). While some researchers view these two as being "relatives" (Aiken, 1960), other more recent researchers differentiate the (negative) attitudes towards mathematics from the anxiety, as the latter being "a more intense feeling" that the students exhibit while taking mathematics courses" (McLeod, 1992). In this paper, the focus will be solely on students' attitudes towards mathematics.

4.3. Attitudes Towards Mathematics

Kulm (1980) suggests that "It is probably not possible to offer a definition of attitude toward mathematics that would be suitable for all situations, and even if one were agreed on, it would probably be too general to be useful".

Zan and Di Martio (2007) add, that the positive attitude meaning varies depending on whether ‘positive’ refers to emotions, beliefs, or behavior. For example, when it refers to an emotion, ‘positive’ would mean pleasurable. When referring to beliefs, ‘positive’ is generally a meaning ‘shared by the experts’, and finally, when referring to behaviors, ‘positive’ is regarded as ‘successful’. All these meanings overlap.

Tapia (1996) suggests, that attitudes toward mathematics are very important in the achievement and participation of students in mathematics. Declining national test scores in mathematics and dislike of mathematics have increased recognition of the problem of student attitudes.

Majeed et al. (2013) also agrees, that students’ attitudes toward mathematics have been known to influence students’ participation, engagement, and achievement in mathematics. A variety of instruments have been developed to measure students’ attitudes toward mathematics for example Mathematics Attitude Scale (Aiken, 1976), Fennema-Sherman Mathematics Attitudes Scales (Fennema & Sherman, 1976), and Attitudes Toward Mathematics Inventory (ATMI) (Tapia & Marsh, 2004)

Tapia and Marsh (1996, 2002, 2004) designed an instrument to measure high school and college students attitudes towards mathematics - The Attitudes Toward Mathematics Inventory (ATMI), is a survey designed to be brief while also capturing multiple factors that contribute to a student’s attitude about mathematics. Items were constructed to assess confidence (Goolsby, 1988; Linn & Hyde, 1989; Randhawa, Beamer, & Lundberg, 1993), anxiety (Hauge, 1991; Terwilliger & Titus, 1995), value (Longitudinal Study of American Youth (1990), enjoyment (Ma, 1997; Thorndike-Christ, 1991), and motivation (Singh, Granville, & Dika, 2002; Thorndike-Christ, 1991).

These authors relied on these cited researches for each category, resulting in a 40 question, 4-factor survey.

Autonomy (taking control of your own project and interests)

During the same year, Yackel and Cobb (1996) wrote about the importance of the development of intellectual and social autonomy as a major objective in the current educational reform movement, in mathematics education (NCTM, 1989), which they mention is in agreement with Piaget (1973). Prior analysis shows that one of the benefits of establishing the social norms implicit in the inquiry approach to mathematics instruction is that they foster students' development of social autonomy (Yackel, et al., 1989, 1991; Kamii, 1985; Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990).

"Autonomy is defined with respect to students' participation in the practices of the classroom community. In particular, students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in these practices" (Kamii, 1985). These students can be contrasted with those who are intellectually heteronomous and who rely on the pronouncements of an authority to know how to act appropriately. When autonomous, students construct specifically mathematical beliefs and values that help them reach their solutions and form their own judgment.

5. Past Research about MTBI and High School Programs

The Mathematical, Computational and Modeling Sciences Center (MCMSC) at Arizona State University is home to two Presidential Award for Excellence in Science, Mathematics and Engineering Mentoring programs:

1) the Mathematical and Theoretical Biology Institute (MTBI), and

2) the Strengthening the Understanding of Mathematics and Science (SUMS) Institute, which houses the Joaquín Bustoz Math-Science Honors Program (JBMSHP).

The MTBI, established in 1996 by Dr. Carlos Castillo-Chavez, encourages the development of undergraduate students through educational, research and mentorship activities. This summer research program involves intensive 8-week research training, support from alumni, and continuous research opportunities for students.

The SUMS Institute was founded in 1995 by Dr. Joaquín Bustoz Jr. The JBMSHP is an intense academic program that provides motivated high-school students a great opportunity to begin university mathematics studies before even graduating high school. This summer 6-week research program is designed to provide a university experience for high-school students who are underrepresented to improve their future academic goals.

> MTBI and SUMS (MTBI/SUMS) foster the development of mathematical scientists. MTBI and SUMS carry out their efforts through educational, research and mentorship programs often tailored at the individual level from graduate school to the postdoctoral level. MTBI/SUMS programs are aimed at increasing the participation of students from diverse educational, cultural, racial and socioeconomic backgrounds in the scientific enterprise.
>
> (Castillo-Chavez et al., 2007).

> Students choose these topics because they are personally meaningful. Nearly every one of these off-topic applications is chosen because a group member has personal experience with the problem. Either they themselves, or a family member, or a close friend has run afoul of a gang, or has dropped out, or struggles with a mental illness. This personal experience both motivates the group and supplies them with valuable insider expertise … Student leader with consulting mentor approach is a critical factor to the success of MTBI/SUMS. Without students choosing their own projects, a vital portion of the research experience is missing, and students are not truly doing research.
>
> (Castillo-Chavez et al., 2007).

In her study, Evangelista (2015) also stressed the importance of applied mathematics research modeling as a motivator, as well as student leadership and autonomy.

Escontria's (2012) study researched the MTBI and other high school programs. His study was designed to answer what factors related to the JBMSHP did the participants perceive as influencing their continued interest in pursuing the mathematical sciences as a course of study. Among his major findings were that the JBMSHP program experiences, further exposed the students to higher mathematical courses and immersed them in a higher and postsecondary education environment, and the JBMSHP alumni resources, were integral to the current undergraduate experiences of most participants.

When researching similar programs for high school level of students, there were a few existing programs:

The Meyerhoff Program, established in 1988, whose mission was to increase success among black students in STEM, emphasizes on four areas: knowledge and skills, motivation and support, monitoring and advising, and academic and social involvement (Hrabowski and Maton, 1995). This program was associated with higher grade point averages than those who did not participate in the program (Hrabowski and Maton, 1995; Maton, Hrabowski, and Schmitt, 2000).

The Academic Investment in Math and Science (AIMS), created in 2001, which primarily focused on increasing the number and quality of STEM-based bachelor's degree recipients, specifically on women and students of color. Gilmer's (2007) study, concluded that, AIMS help build a support system and encourage students to remain in

STEM-based disciplines, strive for academic excellence, graduate, and further consider graduate studies.

Another difference between these programs and the JBMSHP, is that the latter is the only program in which high school participants earn university-level mathematics course credits.

6. Summary of Literature and Link to Research Questions

Mathematical models are used in the natural sciences (such as physics, biology, earth science, chemistry) and engineering disciplines (such as computer science, electrical engineering), as well as in the social sciences (such as economics, psychology, sociology, political science).

Generally, researchers agree that mathematical modelers use ideas from mathematics to tackle big, messy, real-life problems. Rather than finding a perfect answer, the solutions are “good enough” for the real-life requirements. A model may help to explain a system and study the effects of different components and make predictions about behavior.

These problems can be highly motivating for students, who can relate to mathematics that solve matters that are important to them and their interests.

Researchers define the mathematical modeling similarly, with some varying viewpoints and focus interests. Whether they view this process as a creative art or a mechanical algorithmic process or other, they agree on most necessary stages of the modeling process, yet interpret the relationships between these stages differently. Researchers also define the previously discussed types of models differently.

In this study, my emphasis was on investigating high-school students' changes in skills and attitudes towards mathematics, such as their ability to formulate equations when modeling real-life situations, or interpret their outcomes in context, and assess changes in their confidence or motivation attitudes, comparing these changes from before and after taking an applied mathematics course, a course that was taught in a way that included an application-driven group research project and multiple group assignment and analysis activities of several mathematical deterministic and discrete models in population biology and epidemiology.

CHAPTER 3

METHODS

This chapter describes the methods and the methodology that I used to gain insight into the major questions presented in my statement of the problem. For this study, I used mixed methods research designs that refer to a set of designs that purposively mix both quantitative and qualitative data. A constructivist approach provides the grounding for this study. Specifically, I used von Glasersfeld's (1995) idea of conceptual analysis as a method for generating models to describe how the students may have changed their attitudes or skills from the start of their learning journey until the end. The subsequent sections focus on the setting, my anticipated data collection and data interpretations:

1. Background and Subjects

1.1. The Background of the Summer Program

The Joaquin Bustoz Math-Science Honors Program (JBMSHP), as introduced in Chapter 2, is an intensive residential academic program at Arizona State University in Tempe, AZ, that provides high school students an opportunity to learn university level mathematical science studies before graduating high school. The JBMSHP is intended for highly motivated students who are interested in academic careers requiring mathematics, science, or engineering based coursework, and who are typically underrepresented in those fields of study.

This summer program offers several courses, such as Calculus with Analytic Geometry I and II, Pre-Calculus, College Algebra, and Introduction to Applied Mathematics in the Life and Social Sciences. The latter course, AML100, is the subject of this study. It is a three-credit course offered for a duration of a six-week session, from June 9, 2019 to July 19, 2019.

The recruitment of participants for the AML100 course was done by the program manager for Joaquin Bustoz Math-Science Program, Simon A. Levin Mathematical Computational and Modeling Science Center (Levin Center), ASU. Nine students were selected to participate in this Applied Mathematics course. Among them, two males and seven females.

The AML100 course offered all the students two-to-three hours of daily lectures with two alternating instructors, and five hours of laboratory and homework practice with two tutors, for a total of six intensive weeks. The course used a textbook, Elizabeth S. Allman and John A. Rhodes (2004) Mathematical Models in Biology: An Introduction, Cambridge University Press.

1.2. The Subjects of My Study

From this pool of nine students, five volunteers agreed to participate in my research. My written child assent with parental permission form (for minors) and student consent form (for adults) were sent to students' mailing address. These forms can be viewed in Appendix C_1. There were no financial, or any other, compensations for these students' volunteer participation.

2. Data Collection

2.1. Relevance of Data Collection Methods

Among the assessment method guidelines, adapted from University System of Georgia: Task Force on Assessing Major Area Outcomes, Assessing Degree Program Effectiveness (1992); and Western Carolina University, Assessment Resources Guide

(1999), are to collect information that will answer research questions, use multiple methods to assess each student learning outcome, include both indirect and direct assessment methods, and include both qualitative and quantitative methods. It is important to use multiple methods, because responses from students' test results may be informative, however, when combined with student reflective interview results, they will be more meaningful, valid, and reliable.

Goldin (1997) suggests some principles and relevance for interview design. He states, clinical interviews have found greater acceptance in mathematics education. Clinical interviews, as their name suggest, derive largely from the clinical method of Jean Piaget, who used a flexible style of questioning that gave him an opportunity to observe children's problem-solving behaviors as they worked on tasks, then ask questions tailored to their observed behavior. Task-based interviews have importance both as research instruments and as potential research-based tools for assessment and evaluation. "Since tests are almost always written from the point of view of the academic and are designed to detect standard forms of academic knowledge, they can fail to detect key elements in students' thinking. Clinical interviews, on the other hand, can be designed to elicit and document naturalistic forms of thinking" (Clement, 2000). Patton M.Q. (1987) suggests that powerful qualitative methods consist of three important types of data collection – (a) in-depth open-ended interviews, (b) direct observation, and (c) written documents.

2.2. Methods of Data Collection: Overview of the Methods

In this study, I examined and assessed the changes in these AML100 students' mathematical skills and beliefs towards mathematics, afforded from specific tests - the

Precalculus Concept Assessment test (PCA), and the Attitudes Towards Mathematics Inventory survey (ATMI), see Appendix A, and from videotaped, reflective, task-based interviews conducted with each student that volunteered to participate in my study. In addition, as a researcher, I was involved in attending all daily lectures, observing students' and instructors' interactions, and taking notes relative to my research questions.

Each of these methods will be described in more details in the following sections.

2.2.1. PCA Instrument

Carlson, Oehrtman, and Engelke (2010) developed a Precalculus Concept Assessment (PCA) instrument, a 25-item multiple-choice test with conceptual questions related to student understanding of function and interpretation of function notation (see Appendix A_1). Carlson et. al. identified and characterized the reasoning abilities and understandings crucial to students at the precalculus level, and foundational for beginning calculus and articulated these abilities in the PCA Taxonomy. Tested abilities include a strong understanding of ideas of functions in applied contexts, and the ability to examine how two varying quantities change together, known as co-variation. This co-variational reasoning guided the PCA development and now provides the theoretical basis for interpreting and reporting PCA results.

Carlson et al. (2010) add that "the correlation coefficient between the initial PCA score and final course grade was 0.47, which can be compared with Math SAT's correlation of 0.19 with calculus grades, and the MAA's placement tests correlation with calculus grades weighted average of 0.37". Despite the treatment of the PCA score as an emergent variable, the authors noted that 592 of the 600 pairwise correlation coefficients were positive.

Among the uses of PCA questions are, (a) assessing student learning in a classroom setting, (b) comparing the effectiveness of various curricular treatments, and (c) determining student readiness for calculus. I am primarily using it as a framework to assess the students' changes in performance and skills as a result of taking AML 100. The PCA was given to students on the first day of class, and during the last week of class.

2.2.2. ATMI Survey

The Attitudes Toward Mathematics Inventory (ATMI) is a 40-item questionnaire, 4 factor survey, initially designed by Tapia (1996) and later improved by Tapia and Marsh (2004). The survey measure high school and college students' attitudes toward mathematics, on a five-point Likert scale (see Appendix A_2). These questions are grouped into self-confidence; value of mathematics; enjoyment of mathematics; and motivation themes. Tapia and Marsh (2004) argue that the use of the ATMI is important for teachers and researchers, "because success or failure in math performance is greatly determined by personal beliefs. Regardless of the teaching method used, students are likely to exert effort according to the effects they anticipate, which is regulated by personal beliefs about their abilities, the importance they attach to mathematics, enjoyment of the subject matter, and the motivation to succeed".

Many researchers have used ATMI as their framework to study students' attitudes towards mathematics, from different parts of the world, (Asante, 2012; Yee, 2010; Afari, 2013; Majeed et al. 2013), or to analyze the framework's validity (Majeed et al. 2013; Afari, 2013). For example, Majeed et al. (2013) reported the validation of the ATMI instrument, as it was administered to 699 Year 7 and 8 students in 14 schools in South Australia. Confirmatory factor analysis (CFA) supported the original four factor correlated structure based on several fit indices. Precisely, the study used confirmatory factor analysis to validate the hypothesized factor structure of ATMI. The ATMI items were treated as observed or measured variables. First, the data were subjected to tests of multivariate normality. As Majeed et al. explain, the symmetry (skewness) and the flatness (kurtosis) of the distribution, ranged respectively between -1.169 and -0.153, and between -0.730 and 1.543. These results satisfy the guideline that the absolute values of skewness and kurtosis are less than 3 and 8 respectively (Kline, 1998). The authors add that this validation provided evidence that ATMI can be a promising instrument and a viable scale to measure students' attitudes toward mathematics across cultures.

Along with the PCA test, the ATMI questionnaire was given to students on the first day, and the last week of class.

2.2.3. Notetaking

Another method of data collection was notetaking. with the general aim of recording information and aid reflection. When observing a culture, setting, or social situation, field notes are created by the researcher to remember and record the behaviors, activities, events and other features of the setting being observed. Also, triangulation, as it combines two kinds of data, in this case, interview and test data, is of critical importance in qualitative research (Dubinsky et al.1999; Schoenfeld, 2010).

Notetaking for my research consisted of a mix of the Outline, and Cornell methods. These field notes were taken during my daily classroom observations. Using only paper and pen, my aim in taking notes was to manually record specific incidents happening during the AML100 class. These incidents focused on students freshly learned skills, noticeable changes in their engagements, student-teacher interactions, and different instructor methods, with some descriptive examples. These notes were also used with the aim to affect my future student interviews, by comparing other data sources with my notes, and testing my formed hypotheses during these final interviews.

2.2.4. Student Interviews

The student interview protocol (see Appendix B_1) consists of questions and tasks that were grouped into themes: questions about the final project, questions about changing attitudes towards mathematics, questions about noticeable individual changes in pre-and post- PCA tests, and a problem-solving task related to a predator-prey model – a topic covered during class. All student productions were accepted during the interviews without an indication of correctness, and follow-up questions were added. Due to time constraints, help was provided to students who were either having difficulties solving the problem or running out of time. The videotaped clinical task-based interviews were conducted individually with each student, during their last week of course attendance. These interviews did not exceed sixty minutes.

2.2.5. Summary

The methods of data collection for my study consisted of (1) the PCA test, (2) the ATMI questionnaire, (3) Notetaking, and (4) Students interviews. These four methods enabled the collection, transcription, and later analysis of the data.

3. Data Analysis

3.1. Overview of Data Analysis

My data analysis will rely on Strauss and Corbin’s (1998) Grounded Theory and Thompson’s (2008) conceptual analysis. Strauss and Corbin (1998) state that, Grounded Theory methodology and methods are now among the most influential and widely used models of carrying out qualitative research when generating theory is the researchers’ principle aim.

Grounded Theory is a systematic methodology in the social sciences involving the construction of theories through methodical gathering and analysis of data. A study using grounded theory is likely to begin with a question, or even just with the collection of qualitative data. As researchers review the data collected, repeated ideas, concepts or elements become apparent, and are highlighted as codes, which have been extracted from the data. As more data is collected, and re-reviewed, codes can be grouped into concepts, and then into categories. Grounded theory is different from the traditional research model, where the researcher chooses an existing theoretical framework, collects data in order to show how the theory is applicable to the phenomenon being studied.

The first step of grounded theory is a microanalysis of the data. Strauss and Corbin (1998) suggest, that microanalysis is an important step in theory development. It is through careful scrutiny of data line by line, that researchers are able to uncover new concepts and novel relationships and to systematically develop categories in terms of their properties and dimensions. In this study, the data will be microanalysed through the lens of conceptual analysis.

Thompson (2008) stated that "von Glasersfeld employed conceptual analysis in two ways. The first was to generate models of knowing that help us think about how others might know particular ideas. The second way is to devise ways of understanding an idea, that if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise". With respect to the first way, von Glasersfeld explained the conceptual analysis of an individual by asking "what mental operations must be carried out to see the presented situation in the particular way one is seeing it" (von Glasersfeld, 1995; Steffe, 1996; Thompson, 2008), in order to make the researcher's position in this question clear, I would be asking the question "what mental operations might my subject have carried out in order to see the presented situation in the particular way that he appears to me to be seeing it?" (Castillo-Garsow, 2010). It is these hypothesized mental operations that will comprise my models of the students' mathematics and how the interviews will be analyzed.

In my study, students' significant changes in PCA and ATMI pre and posttest results were analyzed for change, item by item, before student interviews, then individually addressed during these interviews, with the reflective aim to see how this course may have contributed to these attitude or skill changes. After collecting students interview data, I used the data to systematically construct models of the mathematics of each of my students in this study, which later resulted in skill and attitudes themes. This construction relied on the systematic collection of several sources of data and analytical procedures for developing grounded theory as described by Corbin & Strauss (1998). I used Conceptual Analysis, that resulted in proposed models of thinking that help distinguish the students' ways of thinking and also explain how these ways changed or persisted during the experiment.

Data from students' outcomes, in terms of attitudes and formed skills, were also compared with my own written notes. I used my notes as data for analyzing the interviews. These notes were also part of creating student models, to analyze the possible reasons for why a specific student answered a question the way they did.

The following sections describe in more details, the specific analysis for each of the methods used in my study:

3.2. Test Results Analysis

PCA and ATMI test analysis:

Pretest-posttest designs measure the degree of *change* occurring as a result of treatments or interventions. In my study, PCA pre- and post-data were collected and analyzed to examine these changes in each of the five students' performance relevant to a future calculus course. The ATMI pre and post surveys rank attitudes via a Likert scale. Therefore, in order to significantly rate each student's changes in attitudes towards mathematics as a result of the course, I relied on the difference interval value of two scales. After conducting and analyzing the interviews, I also looked at some one-scale changes.

Given the questions that I wished to have answered and analyzed, relied on students' changes in skills, I aimed at selecting one to three PCA tasks that showed a significant change from before and after taking the AML100 course, and addressed these during the final student interviews. I also highlighted one to three ATMI questions, and during the final students' interviews, for example, I asked about how this course changed their attitudes towards mathematics, their willingness to take more mathematics courses in the future, and self-confidence when it came to solving mathematical problems, comparing my hypothesis with their actual answers, and developing hypothetical reasons for these changes.

3.3. Note Analysis

Notetaking analysis fosters self-reflection, and self-reflection is crucial for understanding and meaning-making. Also, it reveals emergent themes, which gives me an opportunity to shift my attention in ways that can foster a more developed investigation of these emerging themes, that were not expected, and that could be addressed in my final interviews. Specifically, in my study, notetaking helped shape the student interviews for future analysis.

I used my notes to better understand data. For example, I used my notes to decide what test items were most similar to course content, so that I could focus on these specific questions when conducting their interviews. Also, to better understand the students when doing interview analysis, I relied on information in my notes that affect the models I form of the students, i.e., if I have notes on a student's increased participation in class, that affected how I interpreted their responses to attitude questions in the interview analysis.

3.4. Interview Analysis

One of my goals as a researcher, was to understand the students, and construct models of these students' mathematics, acknowledging that my mathematical reality is different from theirs; in other words, my aim was to establish possible living models of students' mathematics.

Interviewing provided me with only a mental memory of the interactions with students; this is where the importance of videotape analysis came into play. After watching, analyzing, and transcribing important parts of the videotapes, I became empowered to carry out further analyses of the students' mathematics, in this case, students' changes in their mathematical skills and attitudes towards mathematics.

Model building involves the researcher's creativity and self-flexibility. As such, attempting to think as the students do can facilitate the model construction process (Thompson, 2008), with the assumptions that the students say or do a specific thing for a purpose, Therefore, there are no random utterances.

These interview data were coded through open coding and axial coding. Strauss and Corbin (1990) describe some flexible guidelines for coding data when engaging in a Grounded Theory analysis: Open Coding is "The process of breaking down, examining, comparing, conceptualizing, and categorizing data" (p. 61), while Axial Coding forms "a set of procedures whereby data are put back together in new ways after open coding, by making connections between categories. This is done by utilizing a coding paradigm involving conditions, context, action/interactional strategies and consequences" (p. 96).

For the student interviews, my analysis started with discerning the observed skills that the students learned during this class, by gathering data from their learning own experience in the AML100 course, and observing their problem solving PCA and applied mathematics tasks. I also aimed at analyzing their changes in engagement and shifts in their attitudes towards mathematics, based on the ATMI questions that had the most significant changes.

Interviews were transcribed, coded, and the significant contents highlighted and categorized.

3.5. Summary

Data analysis for my study relied on Corbin and Strauss' Grounded Theory method, and von Glasersfeld's (and Thompson's) conceptual analysis, to construct models of students' mathematics, acknowledging that our mathematical reality is different from theirs.

Data analysis started with PCA and ATMI, comparing each student's PCA tasks one by one, and ATMI questions one by one, in order to carefully examine for student change before and after taking this course, to form relationships and to systematically develop categories. I then analyzed student interviews, transcribed, coded and examined the most significant changes I have noticed during these interviews, reviewed, reflected, and compared with the hypotheses I have formed and the categories I have developed.

4. Hypothetical Trajectory

4.1. PCA Outcome Predictions

I hypothesized that all five students would have general small positive PCA change after taking the AML100 course. Small because the AML100 course was not a Calculus course, and overall positive change because this course involved projects that relied on many graphical simulations and representations, especially during their evening laboratory sessions, where the students had to think about how the two quantities were changing and what certain points (such as equilibrium points) meant within their graphs, and relate it to the real-life context. Specifically, I hypothesized positive change (a wrong answer in pre-PCA that changed to a correct answer in post-

PCA) with PCA items related to graphical representations of the situations, such as problems 7, 15, 18, 19, 24, and #25. Also, because when solving equations, which at times involved finding a set of possible values, or domains of functions in R, the instructor spent time explaining concepts related to no zero-division rule, and the necessity of the expression under the square root to be positive, I hypothesized that some students may also have an increase for an item that required the students to find the domain of a function, item #21.

A guest instructor who participated in the study gave lectures about units, covering some aspects of variables, quantities and some function notation. This may have helped some students with PCA items that focus on what input quantities represent, and what units they are measured in, such as tasks #1, #10, and #20. I hypothesized that there would be no change for tasks related to inverse and composition functions.

4.2. ATMI Outcome Predictions

For the attitudes towards mathematics outcomes, I predicted that overall the students may have found mathematics as a subject that is much more useful than previously thought, not a boring nor a dull subject, with possible shifts in their believes of what a solution might mean, given the fact that this course gave these students an opportunity to apply mathematics to biology and relate it to real-life problems, therefore, now any AML100 graduate is able to develop a model related to their topics of interests. The ATMI items addressing these changes would be #1, #26, #31, #35.

I also hypothesized that the student would have more confidence in doing mathematical experiments, therefore, I predicted possible increase in questions #17, #18, #40, which would translate into having a lot of self-confidence when it comes to mathematics, being comfortable doing mathematics experiments without great difficulties, and believing that they are good at mathematics experiments.

4.3. Skills

Prior to the final interviews, the skills I hypothesized that the student would develop during this course are the stage skills that the GAIMME authors illustrated, and the "soft skills" such as communication, group work, and group analysis. In addition to these, there were some skills that they learned during class such as finding eigenvalues, nullclines and equilibrium points (see Appendix B_2), being able to translate the diagrams into equations and vice versa, being able to explain what the graphs and particular points represent in specific contexts, for example, in the given predator-prey model.

4.4. Summary

I hypothesized an increase in the skills acquired during this course, and a positive change and attitudes towards mathematics, specifically in terms of self-confidence towards their ability to tackle mathematical tasks, taking future mathematics courses, belief of how useful mathematics could be when applied to other subjects, and what solutions might mean to them. I also hypothesized that each student may have different outcomes from the other, simply because I noted the participating students being engaged, and facing challenges, differently in class.

5. Summary of Methods

This chapter described the methods and the methodology that I intend to use to gain insight into the major questions presented in my statement of the problem. I relied on mixed methods research design, following Goldin and Clement's suggestions for data collection, afforded from the PCA, ATMI tests, student interviews, and my personal notes. I also relied on Strauss and Corbin's ideas for data analysis, following a constructivist approach that provided the grounding for this study. Specifically, I based my analysis on von Glasersfeld's and Thompson's idea of conceptual analysis as a method for generating models to describe how these high-school students may have changed their attitudes towards mathematics, and skills during their summer learning journey, when taking an applied mathematics course, taught in an interdisciplinary way that included application-driven team research projects and group activities. My predictions about students' outcomes after taking this course, were certainly positive. I was predicting great shifts in skills these students would be learning these weeks, but no increases in other function and calculus related skills, that were less related to the course content. In addition, I was predicting positive changes in their attitudes towards mathematics, specifically related to their confidence, and ability to take on new challenging experiments and finding advanced mathematics as useful and inspiring.

CHAPTER 4

STORY OF THE CLASS AND TEST RESULTS

1. Story of the Class and Project Timeline

The AML 100 course was a 6-week program, where the first four weeks of the course consisted of lectures and homework in population biology and epidemiology, and where all students learned about modeling real-life situations, understanding and developing mathematical equations, simulating data, and finding solutions. During these weeks, they covered four major topics:

(1) Dynamic Modeling with Difference Equations, this is where students learned about the Malthusian Model, Nonlinear Models, Variations on the Logistic Model, and Discrete/Continuous Models. (Week 1)

(2) Linear Models of Structured Populations, this is where students covered Linear Models and Matrix Algebra, Projection Matrices, Eigenvectors and Eigenvalues. (Week 2)

(3) Nonlinear Models of Interactions, such as Simple Predator-Prey models, Equilibria of Multi-Population Models, Linearization and Stability, and Positive/Negative interactions. (Week 3)

(4) Infectious Disease Modeling, where students learned about Elementary Disease Models, The SIR models. (Week 4)

This course was taught by two instructors in the mornings from 9am-12pm and mentored by two tutors during evening lab lessons where they ran simulations using MATLAB and analyzed graphs. These lab lessons occurred from 1pm-3pm. Students often stayed until 5pm working on their homework.

During the evening sessions, students essentially did two types of graphs when they ran simulations:

(1) The first graph type showed the populations over time. For example, if students were looking at the spread of a disease, the graph would show the susceptible population, the infected population, and the recovered population, changing over time. This resulted in different outcomes of the simulations, like the disease is endemic, or it completely dies out.

(2) The second graph type was the limit cycles. An example would be graphing the susceptible versus the infected. Often times students analyzed these "cycles" in the graphs and analyzed its possible convergence to a point.

Once the students knew the purpose of the graphs and what kind of information they were telling them, then they were able to play with the parameter values and see the different outcomes: They "perturbed" the parameter values one at a time (they increased/decreased a parameter value by 1% for example), ran the simulations, and observed any significant differences in the long-term behavior of the model simulation. It was a simple version of sensitivity analysis. The most appropriate parameters to test were the ones in the reproduction number R0 because it was the threshold for whether or not the disease dies out.

For limit cycles, students would also change the initial conditions and see how the limit cycles changed too. The code that was provided to the students was very useful because they simply had to click on the graph, which would be the initial condition for that simulation, and compare many simulations at the same time, note and analyze any differences they observed.

For their final projects, students began to think about their topics, and initial ideas from week 2, and briefly present daily for about 10 minutes starting on week 3. Each student suggested two best topics, and instructors created groups. The students narrowed down the topics and merged them. When topics did not have enough research, instructors suggested modifications, for example changing from invasive frogs to invasive beavers. Some students wanted to do veterans and suicide, but the topic was too wide and problematic, so instructors guided the students on narrowing the topics down.

As the course was nearing the end, students were more focused on their final projects, presenting their findings to the whole class, and receiving feedback from instructors and mentors. Nine students were grouped into three groups according to their interests: Ecology, Technology and Health topics: Group 1 decided on “The Prevention of Chilean Forest Destruction Caused by Invasive Canadian Beavers”, Group 2 worked on “Social Media and Eating Disorders in Teenage Girls”, and Group 3 chose “The Roles of Substance Abuse Among the Homeless Population in California”. Five students initially agreed to be part of my research. Two male students were from Group 1, two female students were from Group 2, and one female student was from Group 3. All five also initially agreed to take the tests and be interviewed at the end of the course.

PCA and ATMI pretests were administrated on the first day of their AML 100 course and during last week of the course. Individual student interviews were conducted after the exams, on their last week, one after another. In these interviews, I also addressed some of the PCA and ATMI results where major changes occurred. Among these five students, four participated in these interviews, and one student (named Student 4 in this research) dropped out a day before their last day of attendance, due to a family emergency. I only collected pretest and posttest PCA and ATMI data for Student 4, and no interview data.

2. PCA Test Results

This table shows the five students' S1, S2, S3, S4, and S5, pretest and posttest PCA results. In bold/highlighted are the incorrect answers, and the last column shows the proportion of each student's correct answers over the total of questions, for either pre or posttest.

Table 2

Summary of Students' Pretest and Posttest PCA Answers and Scores

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	Score
S1	pre	C	B	D	B	B	C	C	B	E	C	B	A	D	D	B	D	C	B	B	A	C	D	D	D	B	23/25
	post	C	B	D	A	B	B	C	B	E	C	B	A	D	D	B	D	C	B	B	A	C	D	D	D	B	24/25
S2	pre	C	B	D	A	B	A	C	E	E	D	D	A	B	A	B	D	A	B	A	E	B	E	A	D	E	13/25
	post	E	B	D	A	B	A	C	E	E	D	D	A	B	A	C	D	A	B	A	E	B	A	E	D	E	11/25
S3	pre	D	B	D	A	B	A	C	E	E	C	D	A	D	D	C	D	A	B	B	A	D	D	D	D	B	19/25
	post	C	B	D	A	B	A	C	E	E	D	D	A	D	D	B	D	A	B	B	A	C	D	C	D	B	20/25
S4	pre	C	B	B	C	B	A	C	A	D	D	B	A	D	D	B	D	D	C	A	B	B	B	E	B	D	11/25
	post	C	B	B	C	B	A	B	E	E	C	B	A	D	B	B	D	C	D	A	E	B	B	E	D	B	14/25
S5	pre	C	D	D	C	D	A	C	E	E	D		B	A	B	B	D	A	B	A	D	C	B	C	D	A	10/25
	post	C	D	D	A	D	A	C	E	E	B	C	B	E	A	B	D	D	B	B	C	C	E	C	D	C	12/25

Before conducting student interviews, I analyzed students' results and mainly focused at PCA items that had either positive or negative changes.

All students who participated, except Student 2, had an overall slight increase in PCA tests: Student 1 had one point increase, Student 3 had one point increase, Student 4 had three points increase, and Student 5 had two points increase, see last column of the table above. Most of the PCA problems were not specifically related to this applied mathematics course, therefore, although there was an overall increase in scores, the changes were small.

3. ATMI Test Results

Before the interview, I looked at all the statements that had significant (two or more-interval) positive or negative changes, see next table, where a dark green box signifies a positive change of two-scale or more, and a dark red box signifies a negative one. I then addressed some of these changes in attitudes during the student interview sessions.

Table 3

Summary of Students ATMI Test Significant Changes

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
S1															+	+																-	-							-
S2																						+																+		+
S3						-																					+								-	-	-			
S4											-																					-					-			
S5											+					+																								

After conducting these interviews, I reflected on students' answers, and additionally attended at some 1-scale changes. These changes are shown on Table 3 as light green and light red, representing 1-scale positive and negative changes respectively. As illustrated on this table, the number of green boxes exceeds the number of the red boxes. I then analyzed these students' changes in attitudes towards mathematics, grouping them into four emerging themes, as illustrated on Chapter 6.

4. Interview Summary

Student interviews involved questions about their group projects, discussions about their changes in attitudes towards mathematics results, changes in PCA results, and their ability to solve a mathematical problem (The Predator-Prey Problem) that they had to explain as they solved, see Appendix B_1. Table 2 and Table 3 illustrated students' ATMI and PCA results.

The following two tables summarize the interviews, one table summarizes students' skills that emerged during interview about their final group project, and the other table summarizes students' skills that emerged during the Predator-Prey problem solving.

Table 4

Final Project Interview Summary

	Student 1	Student 2	Student 3	Student 4	Student 5
Ability to explain the research question	Yes	Yes	Yes	Absent	Yes
Ability to draw flow diagram	Yes	Yes	Yes	Absent	Yes
Ability to write equations	Yes	Yes	Yes	Absent	Yes
Ability to explain parameters correctly	Partial (missing time component))	Yes	Yes	Absent	Yes
Ability to explain results correctly	Partial (missing time component)	Yes	Yes	Absent	Yes

Table 5

Predator-Prey Interview Summary

	Student 1	Student 2	Student 3	Student 4	Student 5
Ability to identify predator existing in absence of prey	Yes	Yes	Yes	Absent	Yes
Ability to explain parameters	Yes	Yes	Yes (issues with some units)	Absent	Yes
Ability to find nullclines equations	Yes	No	No (but Explained the process)	Absent	No
Ability to draw nullclines	Yes	Yes	Yes	Absent	Yes
Ability to draw flow arrows on their own	Yes	No	No (copied)	Absent	No (copied)
Ability to interpret points of intersection correctly	Yes	Yes	Yes	Absent	Yes
Ability to identify behavior of orbits	Yes	Yes	No	Absent	Yes
Ability to identify stability of coexistence	Yes (tends)	Yes (most likely)	No	Absent	Yes

Analyzing these tables, it is clear that some abilities were developed successfully during this course, such as equation writing or interpretation of certain points in a graph, but others needed to be further addressed, for example, drawing flow arrows, interpreting the units of parameters, or identifying stability of coexistence.

CHAPTER 5

SKILLS

Based on pre and posttest results, notetaking and interview data I organized the students' emerging skill-related stories into several topics. I then narrowed these topics into five major themes:

(1) "Equation writing or interpretation", where students show evidence of their ability to theoretically write or produce mathematical equations, and their ability to transfer from real world to mathematics via mathematical equations or vice versa, and where students show evidence of their ability to explain components of their mathematical equations and what these produced equations represent in real-life context. Students' ability to write equations is tightly connected to their ability to explain these equations.

(2) "Graph Analysis and Interpretation", where students show evidence of their ability to analyze, explain, and translate graphs into specific or real-life contexts.

(3) "Result Interpretation", where students show evidence of their ability to explain results in real-life context, this is evidenced when they are comfortable going from mathematics to real-life and vice versa when explaining results, or give real-life examples to explain their mathematical solutions within a real-life context.

(4) "Pure vs. Applied", where students show positive results when the context of their problems is applied versus pure, requiring only pure mathematical notation.

1. Equation Writing and Explanation

All AML 100 students participated in research projects where they were expected to design their own projects, by first identifying and specifying a problem of their interest to be solved, then making assumptions, defining essential variables, and producing mathematical models in the form of recursive equations that represent their situations. These models were then refined, simplified or extended to fit the specific goals these students had for their problem's real-life solutions.

Formulating such mathematical equations to model real-life situations was a foreign concept to these high-school students. Equation formulating or equation writing was one of the important skills these students had to acquire or further develop in this course through learning and experience. It is also one of the most important stages mentioned in the literature review, see Figures 3 and 4.

The Equation Writing theme as defined here, could be either procedural or theoretical, where a context is not in focus or missing, or it could also be conceptual, where each component of the mathematical equation makes sense (in context) to the modeler, and where the modeler can bounce back and forth from the real world and model world, to perhaps eventually think of the two as one single interleaved world.

I hypothesized that this applied course would greatly help in developing students' skills to formulate mathematical equations - theoretically write or produce mathematical equations, and transfer from real world to mathematics via mathematical equations, given the fact that this was a central part of the applied mathematics course.

I also hypothesized that students test scores for questions related to equation writing might generally increase (PCA questions 1, 4, 10, 14, 17), and equation explanation as well (PCA questions 24 and 25). I expected the biggest changes specifically in problems regarding in-context situations (PCA questions 4 and 17). These questions will further be discussed.

Interview data also showed that equation writing, as illustrated in the following stories, was one of the important skills these students developed in the course of this AML 100 applied mathematics course. PCA testing showed that among all five students, all but one, showed some test increase in scores for problems related to writing equations.

1.1. The Story of Student 1

Based on my notetaking data, Student 1 was a very quick learner ─ quick to learn new materials and absorb new information. He was able to produce newly-learned equations that model real-life situations with great comfort and ease. He had great memory and did not rely on books to check previously learned lessons, as most students did. Student 1 was also able to describe and explain his mathematical equations interpreting the in-context meaning of each parameter, and easily defining its units. By the end of the course, Student 1 was comfortable producing flow diagrams that represented real-life situations covered in class or portrayed in his final project, and from these diagrams, he was able to easily formulate mathematical equations, i.e., transferring from the real world to mathematics via mathematical equations, and later transfer the results from the mathematics to the real world.

In this course, Student 1 developed his equation writing skill by learning to model real-life situations, such as the spread of the diseases, or specific population dynamics, initially producing flow diagrams and then formulating mathematical equations that represented these situations. Student 1 was also skilled at explaining and translating the mathematical models he created or was presented with. One in-class example, was the spread of HIV that the students worked on during class sessions. Student 1 was first to create the model and explain it.

For his group project, Student1 was able to quickly remember the flow diagram and the final equations he, and his group members, eventually agreed upon for the topic of The Prevention of Chilean Forest Destruction Caused by Invasive Canadian Beavers.

Student 1 explained that they initially considered and figured out their model "relatively quickly" presuming three species predator-prey model, Beavers (B), Trees (T), and wolves (W). They eventually agreed on a proposed simpler model that focused on two species instead, the number of Beavers (B) and the trees in acres (T), see Figure 6, that shows the group's produced model (on the left) as a reference, and Student 1's recreation of this model during the interview with no aid (on the right).

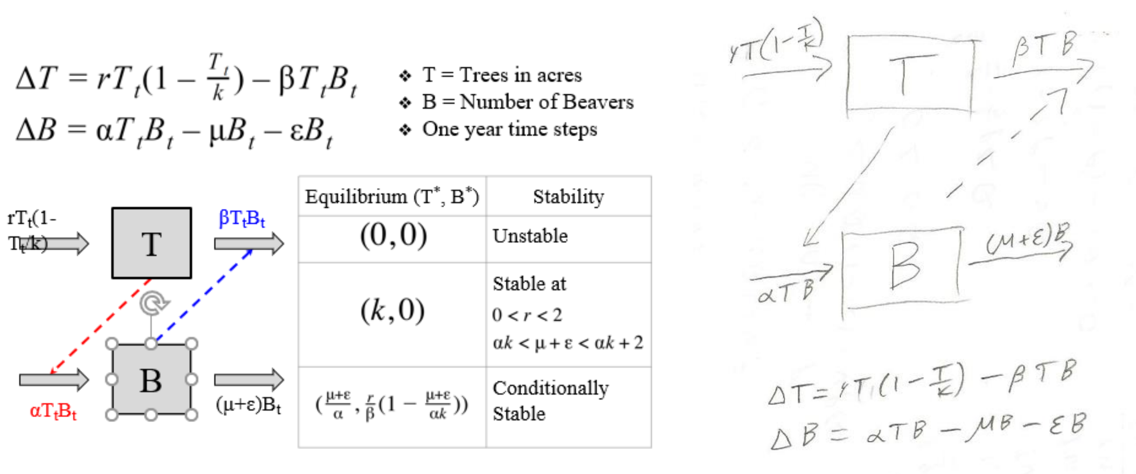


Figure 6. *Student 1's group final flow diagram and equations vs. Student 1's written flow diagram and equations.*

Student 1 was very quick at correctly producing his final project equations. He started with the flow diagram, from which he derived his mathematical equations, see the right side of Figure 6. Student 1 also explained his project equations in the following excerpt.

Excerpt 1

1 Student 1: For the population of the trees and then population of the beavers. So we had the birth of the trees which we said was a logistic growth function, it would be r times T times 1 minus T over k, then we said that the death of trees is caused by an interaction with the beavers and we said that was beta times T times B, the attraction with that. Then we said that the birth of the beavers was caused by the effect of the trees. That's a dotted line.

2 Researcher: Mm-hmm (affirmative)

3 Student 1: So, we described that as alpha T B.

4 Researcher: Okay

5 Student 1: Then the death of the beavers we said that was Mu, as the natural death rate plus epsilon, the harvesting rate times the beaver population. So, our equation was that the change in T was equal to … 1 minus T over k minus beta times B, and then the change in beavers is equal to alpha times T times B, minus Mu times B and then minus epsilon times B, and that was my equations, that we did.

Excerpt 1 illustrates that Student 1 explained the type of function they considered, the parameters of their function, and the dynamics involved between the populations. He focused on the changes in Beaver population, and the changes in Tree population.

During the interview, Student 1 was also presented with a predator-prey problem (see Figure 7 and Appendix B_1), where he had to explain the given equations within context:

<u>The Predator-Prey Problem</u>
A simple model for a predator-prey interaction where the predator has a source of food in addition to the prey is:

$\Delta P = r\,P\,(1 - P/K_1) - s\,P\,Q,$
$\Delta Q = u\,Q\,(1 - Q/K_2) + v\,P\,Q.$

a) Explain why these equations model the described situation.
b) Choosing units so that $K_1 = K_2 = 1$, find and plot the nullclines for this model. Draw arrows on your plot indicating the signs of ΔP and ΔQ.
c) What does your analysis of this model lead you to expect as typical behavior of orbits?

Figure 7. *The Predator-Prey Problem*

To explain why these equations model the described situation, Student 1 explained in the following excerpt,

Excerpt 2

1 Researcher: So, can you explain?

2 Student 1: Uhum. So, it says it's a predator prey interaction where the predator has a source of food that isn't the prey. So, this can be described as the ‘u’ times ‘Q’ times the one minus ‘Q’ over ‘k2’ term, where if there were no prey in the situation, that there would still be an increase in the predator population because they have an additional source of food other than the prey.

3 Researcher: Okay. Can you tell me what the ‘u’ represents in the equation?

4 Student 1: ‘u’ represents the growth rate for the predator without prey.

5 Researcher: Okay. What are the units of ‘u’?

6 Student 1: The units of ‘u’, would be one over time.

7 Researcher: Okay. So, can you tell me what ‘Delta Q’ represents?

8 Student 1: ‘Delta Q’ represents the change in the predator population.

9 Researcher: Okay. What is ‘k2’?

10 Student 1: ‘k2’ is the carrying capacity for the predator population.

This excerpt also illustrates Student 1’s ability to explain with ease what the equations represented, along with their parameters and units.

In this course, Student 1 developed his ability to produce, write and explain mathematical equations, and his ability to transfer back and forth from real world to mathematics via mathematical equations.

1.2. The Story of Student 2

Based on my notetaking data, Student 2 was eager to learn new materials. She started with a lower level of engagement and a smaller mathematical background than the rest of the class, which may have contributed to some of her little mathematical struggles. Student 2 claimed she has never taken a precalculus course. She was occasionally able to (roughly) produce newly-learned equations that model real-life situations, but often needed help in remembering how to proceed with the following steps, such as adjusting the exact parameters in her mathematical model.

Student 2 did not have a good memory and often relied on previous notes. Student 2 was able to describe situations with ease, for example, explaining the in-context meaning of each parameter, but had some little difficulty with formulating mathematical equations. She often was hesitant and unsure, but eventually, she was able to do the formulate her equations, taking her a longer time to achieve it.

For her interview, I hypothesized that Student 2 would partially remember her project-related mathematical equations but might require help in writing these mathematical equations, because although the AML 100 course involved a lot of work in terms of writing equations from real-life situations, thinking about what variables and expressions meant, checking units, and although Student 2 was very skilled at expressing the model's interactions in class, she was often unconfident about her mathematical translations, and required help in converting these interactions into mathematical equations.

For her group project, Student 2 was not able to remember her mathematical equations, see Line 2 of Excerpt 3. However, after suggesting she draws the flow diagram first, she was able to produce both the flow diagram and from it, formulate her mathematical equations, adding that there might be some "Greek letter" mix-up.

Excerpt 3

1 Researcher: Okay. Do you remember the equations that you have used?

2 Student 2: hmmm **I don't remember the equations off the top of my head**, but I do know that we have four equations for our non-homeless regular population that do not abuse drugs, our non-homeless substance abusers, our homeless substance abusers and our homeless non substance abusers, and in each of them they all have a variation between the variables for homeless substance abusers and homeless non substance abusers, they use beta one and beta

two to represent the positive and negative interaction between the two. And so, a positive interaction would be a homeless substance abuser recovering and getting support from a homeless non substance abuser and falling into that new category where they don't abuse drugs anymore, but unfortunately, they're still homeless, but they have a better chance of becoming non-homeless, non-substance abuser. And then, however, the negative interaction can be where a homeless non substance abuser meet say, a homeless substance abuser, and that influence and social interaction where they want to feel like they belong, causes them to fall into this homeless substance abuse, and it's a negative interaction.

3 Researcher: Do you remember how the diagram looked like?

4 Student 2: The diagram? We have...

5 Researcher: Do you want to try to draw it?

6 Student 2: Yeah, I can draw it. Like our model, right?

7 Researcher: Yeah.

8 Student 2: I can draw that [drawing]. So, we have our non-homeless regular non-substance abusers, our non-homeless substance abusers, homeless regular non-substance abusers, and homeless substance abusers. And we have the interaction between homeless non-substance abusers and homeless substance abusers, and as you can see, we have beta one, where they fall into this homeless

> substance abuse, and beta two, where they can recover. And the only way for a homeless substance abuser to get to Nr, which is the ideal state and where we want everyone to be, is through becoming a homeless regular to non-homeless regular. And the reason we don't have an arrow going back here is because the number of people who go from being a homeless substance abuser to back to being a non-homeless substance abuser is so small that we decided to consider it negligible. And so next we have our Nr to Hr, and that's our Lambda one, and our Lambda two is our non-substance abusers to non... Or non-homeless substance abusers and our non-homeless regular non-substance abusers. And what else would you like me to say about the model?

On Line 8 of this excerpt, Student 2 explained her variables and the interaction she was able to discuss after drawing the flow diagram, see Figure 8, where the top part shows her final project's flow diagram and equations, and the bottom part shows her remembered version of it.

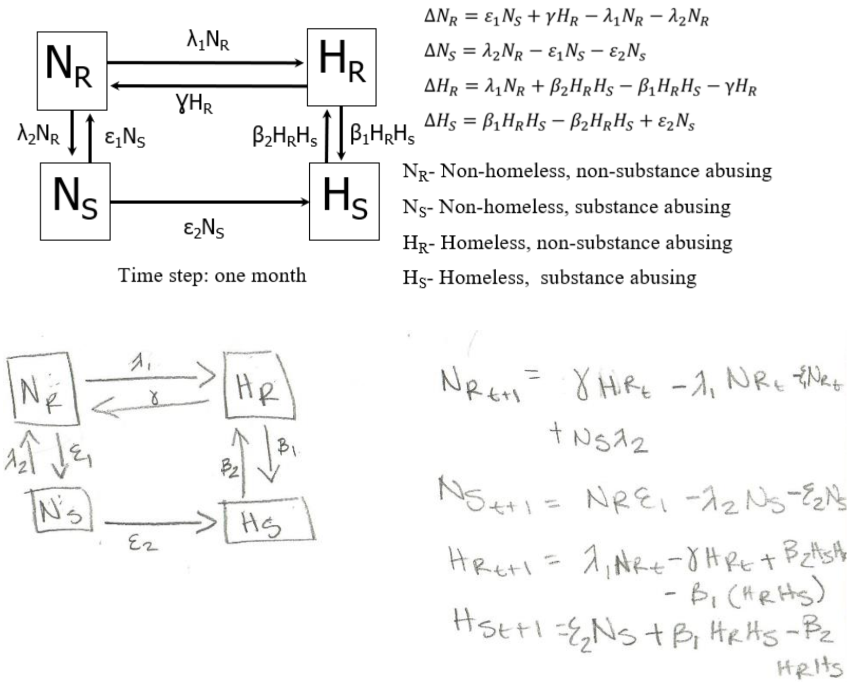


Figure 8. *Student 2's Group Final Flow Diagram and Equations (top) vs. Student 2's Written Flow Diagram and Equations (bottom).*

Comparing the two figures together, her hand-drawn flow diagram was slightly different, she switched λ_2 with ε_1 as she predicted, this is why her translations from the flow diagram to equations looked slightly different. Yet, her diagram-to-equation translation was accurate.

For her project's model, Student 2 did not initially remember her equations, yet her translation ability to turn interactions into equations was good. Once she remembered the positive and negative interactions between the four involved population Nr, Hr, Ns, and Hs used in her model and produced a flow diagram, Student 2 was then comfortable explaining her project's mathematical equations dynamics in real-life context.

The translation from the real world to mathematics was not as trivial and quick as it was with Student 1, but once the interactions were clearly translated into her flow diagram, she was able to produce her four equations, see Figure 8. In some ways, Student 2 demonstrated that same ability that student 1 did, translating a situation into a diagram into equations, but at a slower pace.

During the interview, Student 2 was also presented with a Predator-Prey Problem (see Figure 7 and its solution on Appendix B_2), where she had to explain the given equations within context:

Excerpt 4

1 Student 2: […] Okay, so the situation they give us… is where we have two populations, we have the predators and we have the prey.

2 Researcher: Mm-hmm (affirmative)

3 Student 2: And in this situation we have a predator who has another source of food.

4 Researcher: Mm-hmm (affirmative)

5 Student 2: So, when you look at the equations you have the original logistic model for, right, logistic equation for a normal one population.

6 Researcher: Okay.

7 Student 2: And then due to their being a predator, for a change of P, we have that minus SPQ which shows the negative interaction between the population of the prey and the population of the predator. And so, what we then have next or what we have next is the changing Q which is the predator equation.

8 Researcher: Mm-hmm (affirmative)

9 Student 2: And due to the fact they have a source of food, that means they have a caring capacity, a source of food and which they have, wait, wait, wait, which limits how many of them there can be.

10 Researcher: Mm-hmm (affirmative)

11 Student 2: But then we also have that positive interaction between the prey and the predator.

12 Researcher: Okay.

13 Student 2: Where the predator can eat the prey and it increases their carrying **capacity a little bit because they don't have to rely on berries and when they'll run out or something like that. Is that good or should I keep going more?**

14 Researcher: Let me see. Can the predator population still grow and sustain itself in the absence of prey?

15 Student 2: Yes.

In this Excerpt, Student 2 explained the interaction in the proposed predator and prey problem, named it a logistic model, and recognized that there was another source of food. She was initially concerned that the carrying capacity in the logistic model would limit the population of the predator (Line 9), but after she recognized a positive interaction between prey and predator, she came to the conclusions that the predator population can still grow and sustain itself in the absence of prey (Line 13).

Line 13 of Excerpt 5 illustrates that student 2 was thinking about the general effects of the model, not only the parameters. When stating that the carrying capacity was increasing, Student 2 was not thinking about the k2 parameter, but rather the effective maximum capacity increasing, i.e. that the population can grow beyond k2.

Student 2's interview data showed that she was more skilled at explaining the interactions between the involved populations than designing her model via mathematical equations. The equation writing also took a longer path and was not as easy as it was for Student 1, this could be due to the fact that either this translation was a new and complex concept to her which required longer thinking and analysis, or she had difficulties with her mathematical translations, given her lower mathematical background. Student 2 did not have a good memory and probably did not memorize her equations. She was quick at recognizing the interaction dynamics into a flow diagram and required more time and thinking to translate the dynamics of the flow diagram, to formulate her mathematical equations.

In this applied course, Student 2 developed her equation writing skills when learning to model real-life situations, such as the spread of the diseases, or specific population dynamics, initially by producing flow diagrams and then translating them into her mathematical equations to represent these situations.

1.3. The Story of Student 3

Based on my notetaking data, Student 3, as well as the other students in class, developed several skills, among them equation writing and equation explanation. When it came to mathematical equations, Student 3 was very skilled at and involved in procedural calculations (doing the theoretical mathematics part), and less involved in the design or analysis of the models they worked on (linking the mathematics with the real world). Student 3 was passive in his group. He also enjoyed biology too (separately from the mathematics) and taking the task of researching information about their topics of interest. Although not a leader, he had great communicative and group work skills. Student 3 and Student 1 were part of the same group, and shared the same final project.

By the end of the course, Student 3 was able to produce flow diagrams that represented real-life situations covered in class or portrayed in his final project, and from these diagrams, he was successful at formulating his mathematical equations, i.e., transferring from the real world to mathematics via mathematical equations (although not as quick as Student 1).

For his group project, Student 3, was able to explain the problem, the interaction and the behavior of predator and prey, and briefly explain the purpose of the two mathematical equations delta T and delta B in context. When asked if he remembered his equations, Student 3 initially said he did, but when it came to write them on paper, he did not remember, and asked for my help. I suggested he draw a (flow) diagram and try again,

Excerpt 5

1 Researcher: […] I would like to hear about your final project. Explain to me as much as you can please.

2 Student 3: So, the project is on the Canadian beaver and its invasion status in Chile, and how they consume trees, in this area called Tierra del Fuego. So, essentially, they cause economic and ecological damage. We decided to model their behavior of hunting the beavers through a **predator prey model with a harvesting component. And … so .. What we did is just a change in tree**

population with logistic growth for the trees, and then subtracted with the interaction between the beavers and the trees and the beta to signify the rate. Something like that... and for the beaver population it's an alpha since their benefit is different. And that's the alpha times the beaver population times the tree population and you subtract that by the natural death rate of the beavers, plus the harvesting rate of the hunters which are killing the beavers.

3 Researcher: Do you remember your equations?

4 Student 3: **Yeah**.

5 Researcher: Okay. Just so it's a little more… clearer, I'll give you a paper. You can write them down; you can explain them.

6 Student 3: [writing] **I don't think I remember**…

Line 2 from Excerpt 5 illustrates that Student 3 started with explaining his equations in context, he remembered his equations correctly. If equations were assembled based on Student 3's description, his equations would they look like: $rT(1\text{-}T/K)\text{-}\ \beta BT$ for trees and $\alpha BT - \mu B - \varepsilon B$ for beavers. However, he had some difficulties writing his mathematical equations on paper, stating he was "confused with the logistic rate". I suggested he draws his flow diagram and sees if that would help him formulate his equations, see Figure 9.

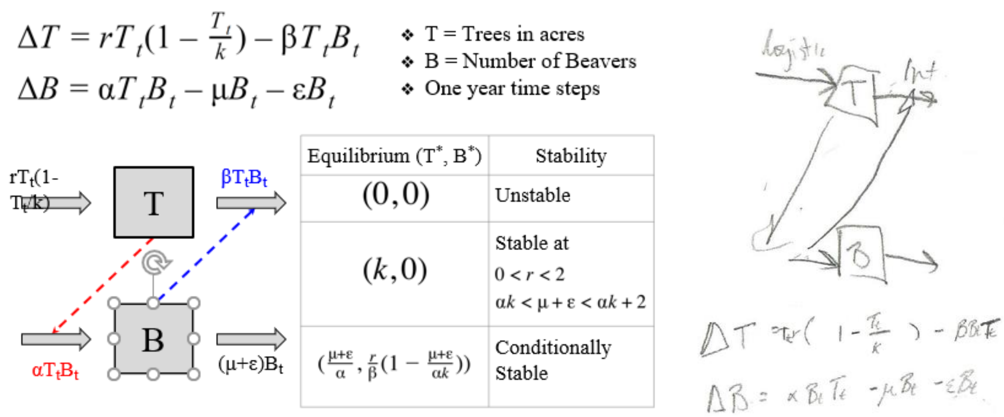


Figure 9. *Student 3's Group Final Flow Diagram and Equations vs. Student 3's Written Flow Diagram and Equations*

Figure 9 shows the group's produced model (on the left) as a reference, and Student 3's recreation of this model's equations during the interview (on the right). Student 3 was trying to correctly formulate his mathematical equations, he then moved to draw the flow diagram, see upper right portion of Figure 9. Student 3 then explained,

Excerpt 6

1 Student 3: I think I messed up on the logistic but, there's a T in front of it. Yeah, I think there might have been there. But, essentially, it's those two.

2 Researcher: Okay.

3 Student 3: and we're just varying the epsilon rate for the beavers since that signifies the hunting. We found the stability and the equilibria, which was like zero zero and it was unstable. Then K zero which was stable under conditions, like Alpha K is less than U plus epsilon is less than Alpha K plus two. And the third one was really long and complicated which was like coexistence.

4 Researcher: Do you remember how your diagram looked like?

5 Student 3: For the model itself?

6 Researcher: Yeah, the model.

7 Student 3: [writing and speaking] Well, it was an entering for the trees and an exit. The entering for the trees was the logistic, and the exit was the interaction. I think we drew arrows to signify, oh I'm not sure, to signify just that this is related to this [inaudible].

8 Researcher: Can I see your paper? [resulting image on Figure 9].

9 Student 3: Interacting with each other. I didn't draw that well. It's essentially just that but..

10 Researcher: DT equals to rT One minus T small t over k. B beta ... Yeah, the top one is correct

11 Student 3: Well then the bottom one.

12 Researcher: Bottom one, delta B is equal to alpha Bt Tt minus Mu, Bt minus epsilon Bt. Yeah, that is correct. Keep going.

13 Student 3: But yeah, our model was just the T on top and the B and the entry was the logistic, the exit was the interaction and this is the benefit with the same interaction, and exiting was both the Mu and epsilon which was the natural death rate and the hunting harvesting rate.

14 Researcher: So, epsilon is what?

15 Student 3: The harvest rate which we're varying.

16 Researcher: How about Mu?

17 Student 3: Mu is the natural death rate.

Student 3 finalized his equations, which he was not totally sure of: he was able to remember the equations, but was not totally clear on what some parts of the equations represented, or why the model was put together the way it was. He then discussed some of the results and drew a flow diagram. He also explained the interaction in terms of mu and epsilon, and the other parameters in his equations, in context. This excerpt illustrates Student 3's developed ability to explain what these equations represented, along with their parameters, developed ability to draw flow diagrams and formulate his project's mathematical equations.

During the interview, Student 3 was also presented with the predator-prey task (see Figure 7 and Appendix B_1), where he had to explain the given equations within context:

Excerpt 7

1 Student 3: So, it says that the predator has a source of food in addition to the prey. So, I just remember from the book, we did an example like this.

2 Researcher: Yeah.

3 Student 3: The source of food explains the logistic part, because the P is the predator, or the prey, right? Are you allowed to say?

[…]

4 Researcher: Yeah.

5 Student 3: The prey is the P?

6 Researcher: Okay. Yeah, it is.

7 Student 3: Okay. Well, since the prey's the P, they're going to have the carrying capacity since they can just eat the food, basically. And if the prey could be removed, and they wouldn't matter, then it would just be the logistic growth for the prey. But for the predator, they need the prey in order to exist. But with this question, the addition of food allows for them as well to have a logistic growth. And like the first part right here, is the food addition.

In this excerpt, Student 3 tried to translate his mathematical equations into the problem's context, and make sense of the situation, I then asked him to explain meanings for specific parts of the equations, see next excerpt.

Excerpt 8

1 Researcher: What does *s* represent and *v*?

2 Student 3: And *v*? And *p*?

3 Researcher: *v* and *p*. Sorry, *s* and *v*.

4 Student 3: *s* and *v*. I think it's what... Like honestly, I forgot what the book was. But like from our model, that's like the inter... Well, I think it's just the interaction rate between the two.

5 Researcher: What's the difference between minus *s* and plus *v*?

6 Student 3: The minus s is because the interaction is leading to the prey getting killed, and that's a benefit for the predator. So, it'd be a plus. And the v would be different since maybe because the

benefit in which the predator gets isn't the same as the number which the prey fall.

7 Researcher: What is s measured in?

8 Student 3: We could do like the dimensional analysis like they were telling us. I think, just since P and Q are both like animals, I think, like animal squared. I think so. Therefore, the s would be like one over the animal.

9 Researcher: Okay. And *v*?

10 Student 3: Oh, but it'd be like if there are two different animals, then it'd be like one over the other animal […]

11 Student 3: If it's the predator and prey, since the first one describing the prey, then your s would be one over predator, because you're trying to get your units in prey. And it'd be the opposite for the second where you'd have one over prey for the *v*.

This excerpt shows that Student 3 was able to recognize what his parameters meant in context, but showed some struggles identifying the prey equation and recognizing some units, as shown on Line 10 from this excerpt. Overall, Student 3 was able to explain the interactions within context, and correctly produce his mathematical equations only after drawing his flow diagram. From the mathematical equations, Student 3 was also able to recognize that the predator population could still grow and sustain itself in the absence of prey, given that there is a source of food.

Interview transcripts show Student 3's developed ability to produce, write and explain mathematical equations, and his ability to transfer from real world to mathematics via mathematical equations. This transfer was not trivial for Student 3, evidenced with his claims of uncertainty whether his mathematical equations exactly represented the situation in real-life, however, he was able to develop this newly learned skill to a successful functional degree in this course. Contrarily to Student 2, Student 3's mathematics was advanced, but both students found it challenging to translate from the real world to mathematics for different reasons, but eventually were successful at it. For Student 3, he thought of the situation in context, then tried to translate it to his mathematical equations, his translation difficulties are evidence that he viewed his mathematics and the real world as separate worlds, but had good results at attempting linking the two together.

1.4. The Story of Student 5

Based on my notetaking data, Student 5 greatly developed model explanation and interaction skills, she was able to transfer from real world to mathematics, and represent the given situation with a flow diagram, then formulate her mathematical equations, representing real-life situations. Student 5 often liked to be told what to do and follow teacher's instructions.

Student 5's group consisted of three female students whose topic was: "Social Media and Eating Disorders in Teenage Girls". She described her research question as "How social media impacted the development of eating disorders on teenage girls in the U.S." She explained that the goal of her model was to find solutions, specifically to "… what stage ... or place it would be best to intervene to prevent the development of an eating disorder?".

When asked to discuss her model, Student 5 began to explain it, having the model's flow diagram in mind, as her explanations involved boxes H and I. I then asked Student 5 whether she remembered her model, to which she quickly replied "yes". This followed with my suggestion to write her model on paper, she began by drawing the actual flow diagram she was having in mind, boxes H, I and E, explaining the meanings she had for these three boxes H, I and E, and explaining the interaction between them, see Excerpt 9.

Excerpt 9

1 Researcher: Do you remember what that model looked like? Okay.

2 Student 5: Yeah.

3 Researcher: Do you want to write down, here is a piece of paper here and a pencil?

4 Student 5: (Silence/drawing the flow diagram) Okay.

5 Researcher: Mm-hmm (affirmative).

6 Student 5: So, we had our H our incoming birth rate, that was the same as our death rates or our exit rates because there was a constant population (see right side of Figure 10). And, then we had from H was our healthy teenage girls and then they could transition to I, our insecure girls through an untargeted social media influence. So, it was kinda defined as girls who were just looking on social media, and saw posts and just compared themselves.

And, then, our insecure population, that went to E our teenage girls with eating disorders, diagnosed eating disorders, through a targeted influence. So, this was more the insecure girls seeking advice to developing an eating disorder or going to these sites that were pro-eating disorders and gave like tips on developing eating disorders, and losing weight and that sort of thing. And, then from each stage you ... We had our exit rates, and then from E, we had our recovery rate. And, I think that was 1/60 because we had 1/10 of girls who sought treatment, and 60% of those who sought treatment recovered. And, then we had our birth rate. That was a small decimal. I don't remember what it was exactly, but it was ... we found it from taking the amount of 12 year olds turning 13 and dividing it by the whole population of 13 to 18 year olds.

In this Excerpt 9, Student 5 explained with details the interactions between the three boxes, she then continued to discuss the results, which will be illustrated in section 3 of Chapter 5. Student 5 was able to remember her flow diagram with fair precision, and from it, she began to easily derive her mathematical equations, shown on Figure 10.

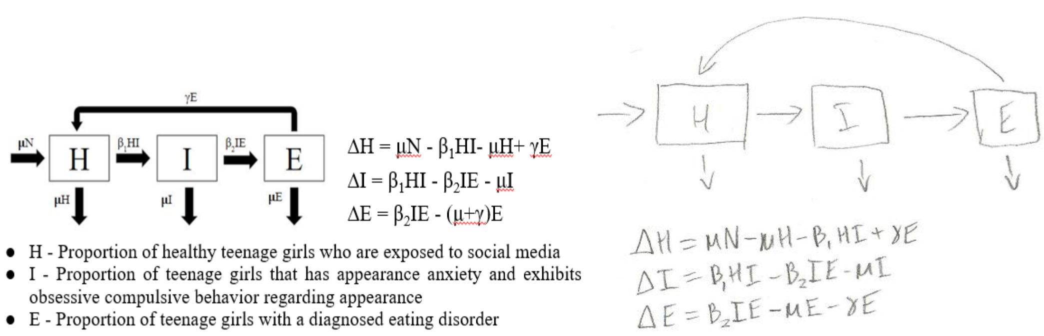


Figure 10. *Student 5's Group Final Flow Diagram and Equations vs. Student 5's Written Flow Diagram and Equations.*

Figure 10 shows how accurate Student 5's mathematical equations were, and how accurately, and within context, she was able to explain in her own words, the rates involved in her model: μ, which was the birth and exit rates of teens between the ages of 13-18 years old, β_1 which was the rate of direct and indirect untargeted interaction between healthy and insecure on social media, β_2 the rate of direct and indirect targeted interaction between insecure and eating disorder on social media, and γ, the rate of recovery from eating disorder.

Student 5 then followed by quickly writing her mathematical equations, as shown on Figure 10, explaining her equations in the following excerpt,

Excerpt 10

1 Student 5: Okay, so our change in H was equal to mu N, because mu was our birth rate and N was our total population. So, that was what was incoming to our H population, minus mu H, was the death rate from ... or the exit rate from the healthy stage. Beta1 H I was our interaction between the H and I stage, and those who transitioned to this stage I. And, then plus gamma E was the recovery rate. So, girls who came from eating disorder and went to healthy. And, then a change in I was beta1 H I because those were the ones transitioning to I. So, you add that to that, and then beta2 I E was subtracted because that's the interaction between I and E, and those who transitioned to E. Then, minus mu I was also the exit rate from stage I. And, then change in E was beta2 I E because those were the girls transitioning to E.

And, then, minus mu E, the death ... the exit rate. And, then minus gamma E was the recovery.

2 Researcher: Which one?

3 Student 5: This one from the recovery from E-

4 Researcher: to H?

5 Student 5: Yes.

6 Researcher: So, can you give me an example of H to I, what exactly happens?

7 Student 5: From H to I ...

8 Researcher: Mm-hmm (affirmative). That arrow ...

9 Student 5: Okay. So, a teenage girl would be on a phone or device looking at photo-based media. So, they would probably be looking probably like at Instagram and seeing all these pictures of girls and women, and comparing themselves and wanting to be like them or be thin. And, that's kinda what our culture is. It's kinda idolizing thin women, and so they would become insecure through that process, because they're looking at the social media and comparing themselves.

This excerpt demonstrates that Student 5 did not simply memorize her equations, she was able to give a real-life example of teenage girls, that would fit the situation where there would be an arrow going from box H to box I, that she discussed. This demonstrates her ability not only to formulate flow diagrams, mathematical equations, but also to explain with ease the interactions involved within her mathematical model, a skill she developed as a result of this applied mathematics course.

During the interview, Student 5 was also presented with a predator-prey task (see Figure 7 and Appendix B_2), where she had to explain why those equations model the described situation, see Excerpt 11.

Excerpt 11

1 Student 5: Well with the change in P equation, it equals the population of the prey.

2 Researcher: Mm-hmm (affirmative).

3 Student 5: Because you have the growth rate and the carrying capacity and then it's subtracted by the negative interaction that they get from in the prey and the predator interacting because the predator eat the prey, and with the changing Q, you have the positive or the growth rate of the predators and their carrying capacity and then the positive effect that they get from the interaction between the predator and the prey.

Student 5 correctly recognized the variables and their units.

Excerpt 12

1 Researcher: Can you read the question? The first one. A simple model

2 Student 5: A simple model for a predator and prey

3 Researcher: Mm-hmm (affirmative)

4 Student 5: For a predator, a predator-prey interaction where the predator has a food source in addition to the prey. **ohhh. Then yeah, they can, still grow.**

5 Researcher: Why's that?

6 Student 5: Because they still have the food source.

7 Researcher: Okay

8 Student 5: Other than the prey.

9 Researcher: Because they can sustain themselves

10 Student 5: Yeah

11 Researcher: In the absence of prey. Okay.

Excerpt 13 illustrates Student 5's ability to explain mathematical equations within context "they can still grow because they still have the food source […] other than prey", see Lines 4, 6, and 8.

1.5. Summary

All students that participated in this research had developed, during this applied mathematics course, a skill for writing mathematical equations that represented real-life situations. Data show a common important theme among all these students, they all felt the need to initially draw the flow diagram that summarized the interactions. They were then able to formulate equations, transferring from a context situation to mathematics using mathematical equations, and explain term by term meanings, units, and equation meanings as a whole. While some were able to do these interpretations instantly, viewing the mathematics and the real world as one, such as Student 1, others required more time to reflect and translate from real-life to mathematics and vice versa.

Students ability to write equations is in accordance with their positive changes in equation writing and explanation PCA scores, specifically for Students 1, 3, 4, and 5. Student 2 did not have any increases in this theme, her single decrease in score, although related to equation writing, was not in any applied context. Therefore, having a real-life context when formulating mathematical equations appears to be very helpful in students' abilities to create or validate their mathematical equations. PCA results are discussed in the "Pure vs. Applied" theme at section 4 from Chapter 5.

2. Graph Analysis and Interpretation

Among other common skills where changes emerged, were graphing skills, particularly graph interpretation. Students' graph interpretation ability was supported when they showed evidence of their ability to explain and translate graphs into specific or real-life contexts.

I hypothesized that this applied mathematics course would slightly help in developing students' graphing skills, which in turn would develop graphing precision and better interpretation.

I also hypothesized that graphing mathematical equations that represent real world situations, would give a better context to students, which would further develop their ability to interpret these graphs within their applied contexts.

For their PCA tests, I hypothesized that students test scores for questions related to graphs would slightly increase (PCA questions 6, 8, 9, 10, 15, 19, 24). I expected the biggest changes specifically in problems regarding in-context situations (PCA questions 10 and 15), and function behavior situations (PCA 19 and 24).

PCA data showed that questions related to graph analysis or interpretation represented an increase for certain students such as Student 3 (question 15), Student 4 (question 9, 10, 24), and Student 5 (question 19). Student 1 and Student 2 had no increases in their PCA scores in this category.

Interview data further showed that Graph Interpretation, as illustrated in the following stories, was cited among the most developed skill in this AML 100 applied mathematics course. Students mentioned that their graphical interpretation enhanced because of how this course was structured. This is most likely because evening lab lessons primarily focused on graph analysis and interpretation, specifically when running graph simulations. For example, students had to perturb parameter values one at a time and analyze the long-term behaviors on the model simulations, explaining parts of the graphs in real-life contexts and explaining how certain variables (which had a meaning in a specific context) would change the models' outcomes.

2.1. The Story of Student 1

Based on my notetaking data, Student 1 was already very good at graphing situations, given his strong background in Mathematics – Student 1 had already taken Calculus I and Calculus II prior to taking the AML 100 course. Student 1 was able to produce and explain graphs that model real-life situations with great comfort and ease. In addition, Student 1 was a great memorizer and a quick learner – quick to learn new materials and absorb new information presented in class.

In this course, Student 1 developed his model related Graph Interpretation skills by learning to further analyze new-to-him modeled graphs of real-life situations during evening lab lessons, such as the spread of the diseases, or specific population dynamics. As example, using a code, Student 1, along with his group, had to click on the graph, which would be the initial condition for that simulation, and then they could compare many simulations at the same time, note the differences they observed, and describe what these differences meant in terms of their contexts. Specific examples of these graphs were discussed in further details in section 4.1.

In his PCA pretest results, Student 1 had only two mistakes overall (question 4 and 6). Question 6 was related to graphs, see Figure 11.

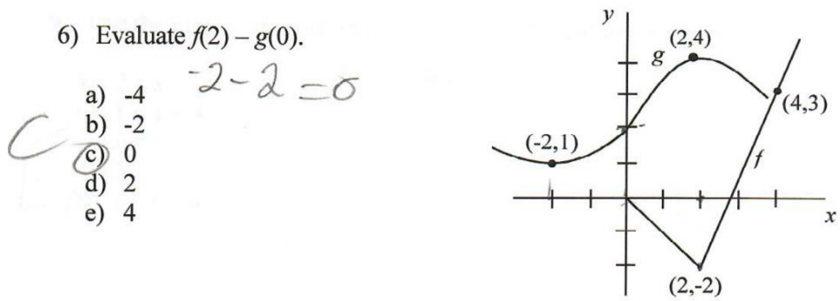


Figure 11. *Student 1's PCA Pretest Solution for Question 6*

Figure 11 shows a graphical composition problem that asked for finding the values of each function, then subtract from function's value from the other. In order to select the correct response, students need to demonstrate an action view of function when providing their rationale for their answer choices.

My hypotheses based on these observations and Student 1's pretest results were that he might have rushed resulting in a subtraction error. In his posttest, Student 1 changed his answer from 'c' to 'b', which was also an incorrect answer, see figure below.

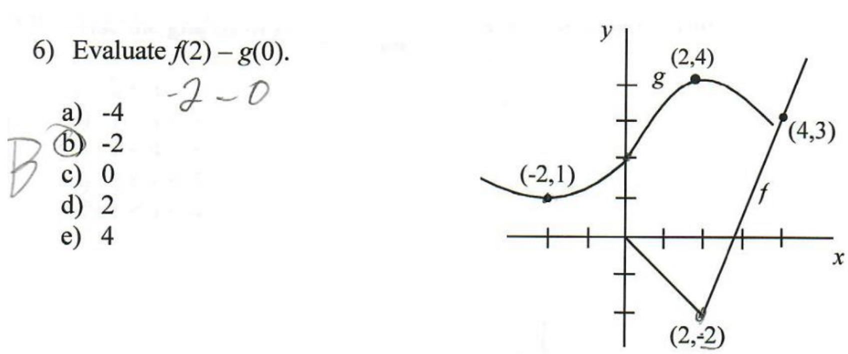


Figure 12. *Student 1's PCA Posttest Solution for Question 6*

Looking at Student 1's written work, it is clear that he focused on the *f* function instead of the *g* function when he had to find *g*(0). Surprised seeing these questions being wrong twice especially from Student 1, I decided to address this PCA question during our final interview. Student 1 answered the question correctly, showing his correct steps, demonstrating an action view of function, when providing his rationale for his answer 'a' and attributing his mistakes to rushing errors.

For his interview, I hypothesized that Student 1 would be able to analyze and interpret his predator-prey problem graphs with ease as he solved. (see Figure 7 and Appendix B_2), where he had to explain the produced-by-him graph within context:

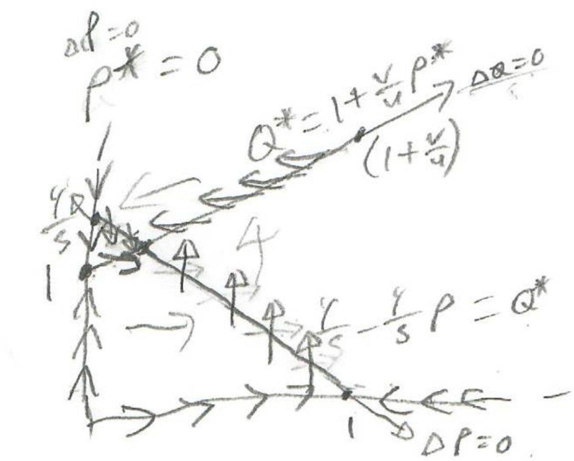


Figure 13. *Student 1's Graph for the Predator-Prey Interview Problem*

This figure shows Student 1's final P vs. Q graph, where he plotted two diagonal lines P*, and Q* that he calculated, shown on the graph. He also added the arrows on his own, after testing multiple possibilities via mathematics (for example, deriving his new P, and checking to what direction the arrows needed to go).

The following excerpt illustrates Student 1's graph explanation in terms of prey, and carrying capacity, as he solved his mathematical equations when finding his nullclines and drawing his graph,

Excerpt 13

1 Student 1: […] basically **if the prey is greater than one** [pointing at point (1, 0) on the graph], it goes back to the carrying capacity. These arrows go up, these arrows go down, back to one. And so, you see an arrow concept, anyway.

2 Researcher: Okay. So, what does your analysis of this model lead you to expect as the typical behavior of orbits?

3 Student 1: **That it spirals inward to the intersection of these two lines** [meaning the two diagonal lines].

4 Researcher: Okay. What is that intersection of two lines represent?

5 Student 1: **The equilibrium of the coexistence of the prey population and the predator population.**

6 Researcher: Okay. What about the other points? The ones that you circled or made them bigger dots?

7 Student 1: **Yeah. This [pointing at (0,1) on his graph] represents a predator population in which there is no prey.**

8 Researcher: Okay.

9 Student 1: **Which had reached its carrying capacity. And this point [pointing at (1, 0) on his graph] is the prey population. When there is no predator, as it reaches carrying capacity.**

There was P and Q mixed up in Student 1's graph. Both his diagonal lines were named Q*, instead of Q* and P*, which could be due to rushing errors, he also switched the axes from P vs. Q to Q vs. P, after drawing his graph.

The excerpt, however, shows that Student 1 had meanings in applied context for the predator and prey situation, while solving his equations, and explaining the meanings, such as shown on Lines 1, 5, and 7, for the points shown on his produced graph, see previous figure.

Although there were no PCA increases in Student 1's graph interpretation, interviews illustrated his ability to merge mathematics and real-life as one. This was evidenced when Student 1 when solving mathematical calculations and equations, drawing his graph, was able to interpret specific points in his graph in terms of predator and prey populations simultaneously, as he solved. Looking at his graph, he also recognized stability, he stated, that it tends to be stable around the point where his arrows were spiraling around, see Figure 13, a point where there would be co-existence between the two populations. Student 1 had a good ability to interpret graphs into a real-world situation.

2.2. The Story of Student 2

Based on my notetaking data, Student 2 was a student who was ready to learn new-to-her mathematics. She did not have a great memory and was not particularly good at analyzing or interpreting graphs. She often exhibited "shape thinking", a construct that Moore and Thompson (2015) characterized as individuals' ways of thinking about graphs, either entailing thinking of a graph as an object in and of itself, and as having properties that the student associates with learned facts, or entailing a vision of a graph in terms of what is made (a trace) and how it is made (covarying quantities).

In this course, all students worked at developing their models' related Graph Interpretation skills by learning to further analyze new-to-them modeled graphs of real-life situations during evening lab lessons, such as the spread of the diseases, or specific population dynamics. Shape thinking is an undesirable way of thinking that can hinder a student from correctly interpreting their graphs. Students exhibiting shape thinking often have difficulties correctly understanding and interpreting some of their graphs in context, as their attention is not always focused on the quantities as they vary, instead, their attention is on the "shape" of the graphs, graphing by shapes, without paying attention to the meaning of individual points from their graphs. This would result in a less successful and desirable Graph Analysis or Interpretation.

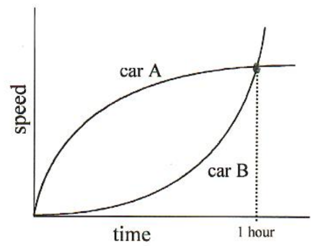
Despite this, I hypothesized that Student 2 might have a small increase in graph interpretation, since this applied mathematics course focused at graphing modeled situation and explaining the behavior of these graphs within context.

Student 2 had no increase in her PCA test results, regarding graph interpretation. In fact, she had one decrease in question 15, which surprisingly is an applied problem.

Student 2 also showed evidence of shape thinking regarding another question addressed during her interview (PCA question 8). When interpreting graphs such as the one in the following figure, students often confuse velocity for position (Monk, 1992), since the curves are laid out spatially and position refers to a spatial property. This confusion leads to false claims, such as: the two cars would collide at h or that Car B is catching up to Car A right before 1h. Students often think of the graph of a function as a picture of a physical situation rather than as a representation of how the time (input) and the speed (output) change together.

PCA Question 8 involves a situation where the *x*-axis is time and the *y*-axis is speed, not a distance. Student 2 answered both times incorrectly 'e'. To understand her reasoning when solving this problem, I addressed this problem 8 during her final student interview. Next figure shows her answer.

The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph below to answer item 8.



8) What is the relationship between the ***position*** of car A and car B at $t = 1$ hr.?

a) Car A and car B are colliding.
b) Car A is ahead of car B.
c) Car B is ahead of car A.
d) Car B is passing car A.
e) The cars are at the same position.

Figure 14. *Student 2's PCA Posttest Solution for Question 8*

During the interview, Student 2 explained that she thought the two cars would be at the same position, she added that they would not collide, simply because it was possible for one car to start a few hours ahead. Therefore, the location would be the same, but the colliding would not happen. After attempting to direct her attention toward the meaning of this graph when time was for example 1/5h, Student 2 recognized that the average speed of Car B would be higher than car A's, but went back to the same location answer, see Excerpt 14.

Excerpt 14

1 Researcher: Will the cars be at the same position, or did you change your mind, or still the same answer?

2 Student 2: This is so hard. Oh, my goodness [laughing].

3 Researcher: Just say what you think, how you think.

4 Student 2: Give me a second. [pause] **I still think the cars are going to be in the same position.**

5 Researcher: And the reason is because? Just give me a quick answer.

6 Student 2: Because, although they have different speed average, **car B still has this pickup and speed right here**, whereas car A is at a constant for a while.

7 Researcher: Constant what?

8 Student 2: A constant rate. Although it's increasing a little bit, it's still close enough to be constant, if that makes any sense. I don't know if that makes sense. I don't know.

9 Researcher: I see. So, at the end it's almost a straight line, is that what you're saying?

10 Student 2: Yeah.

This excerpt illustrates Student 2’s shape thinking. She was looking at the shape of the graph to explain the location of both cars. She also explained that Car B was catching up to Car A right before reaching 1 hour, where the cars would be at the same position at one hour. She was not able to recognize how the speed was changing as time increased.

For her interview, I hypothesized that Student 2 would be able to somehow analyze and interpret her predator-prey problem graphs as she solved. (see Figure 7 and Appendix B_2), where she had to explain the produced-by-her graph within context:

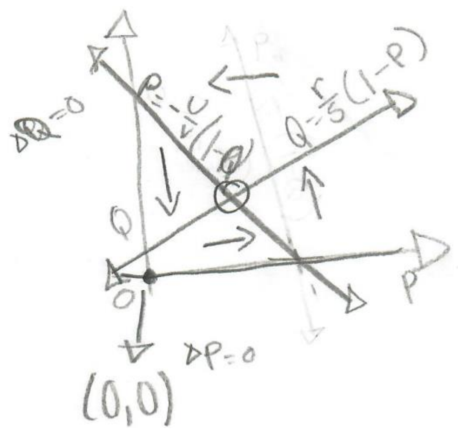


Figure 15. *Student 2's Graph for the Predator-Prey Interview Problem*

This figure shows Student 2's final P vs. Q graph showing nullclines, ΔP= 0 and ΔQ =0, where she plotted two diagonal lines P*, and Q*, whose equations were given to her. Looking at Student 2's graph, it appears that she was just shape thinking the flows, she knew that the problem needed to have cycles, so she drew a cycle around the intersection of the two lines, as she remembered it was done in class, but she drew them incorrectly, and in the wrong place. This also is a form of shape thinking, graphing by shapes instead of interpreting each point.

The book the students used in class had a slightly different notation for the arrows than how Student 1 did. In class, Instructor 1 presented some examples where the arrows for example were on the nullclines (such as Student 1's graph), and other examples from the book, where the arrows were next to the nullclines (similar to Student 3's graph, and the solution to this problem, see Appendix B_2), which is why some students had their arrows drawn differently, see Figure 13 and Figure 18. Following the book style, each region should have exactly two arrows: one vertical and one horizontal, and the combination of both arrows would show the general direction of flow, which was not the case for Student 2. She drew a single arrow in each region.

The following excerpt illustrates Student 2's instances of graph explanation in predator-prey-food context, as she looked at her produced graph. After providing her with the P and Q equations, her graph initially started with a vertical looking line, which she then corrected, explaining that she was trying to make sense of the situation before realizing that there was a food source. She soon realized that there was a food source, and quickly modified her P line and overall graph to look as it is illustrated on Figure 15. This is interesting because although Student 2 had instances of shape thinking, had the mathematical equations to draw, she was also able to draw lines according to the contextual situation in mind, "but there's source of food" before changing the direction of her line in the graph, not because the P equation mathematically looked like P= (-u/v) (1-Q), having a negative rate of change. She was drawing her lines qualitatively, based on the mechanisms of the given situation. Student 2 when asked to draw a graph (or do mathematics), she directly thought in terms of the real-life situation. To her mathematics and biology were thought of, as one. She was not doing the translation following the separate model from Figure 1 at Chapter 2 (translating the real world to mathematics to mathematics solution to real world interpretation). She viewed the two worlds as interleaved. In fact, she may have not even thought that there was a difference between doing mathematics and doing biology. This instance shows that she was doing her mathematics in terms of biology.

Excerpt 15

1 Student 2: Mm-hmm (affirmative)

Sorry, for some reason I felt like it needed to go vertically **but there's a food source**.

2 Researcher: Mm-hmm (affirmative)

3 Student 2: **So, it doesn't go vertically**.

4 Researcher: Okay.

5 Student 2: Because it changes, okay, okay. Sorry, I'm trying to watch if it makes sense...

Lines 1 and 3 from this excerpt illustrate how Student 2 attempted to make sense of her graph, interpreting her line direction. She then recognized the equilibrium points, added arrows without any calculations, evidence of shape thinking, then interpreted her graph, having in mind the context of the problem.

The following discussion followed, where Student 2, when asked, interpreted the equilibrium points of her graph within context:

Excerpt 16

1 Researcher: Okay. So, what does your analysis of this model lead you to expect as typical behavior of orbits?

2 Student 2: **They will all go around a singular equilibrium point. The one that is the most stable compared to the others.**

3 Researcher: Can you circle this stable equilibrium?

4 Student 2: [circles the intersection of the two new lines she drew]

5 Researcher: Okay. So, what does zero-zero represent?

6 Student 2: My zero... What?

7 Researcher: This one. [researcher pointing at point (0,0)]. Zero-zero, what does it-

8 Student 2: Oh, zero-zero represents-

9 Researcher: This point!

10 Student 2: ... **When there is no predator population or prey population. They will neither increase nor decrease or affect each other in any way.**

11 Researcher: How about one-zero?

12 Student 2: **That is when the prey lives, but the predator does not. So, they don't have an interaction necessarily, and the prey can increase as much as they would like.**

13 Researcher: Okay.

14 Student 2: And then zero-one?

15 Researcher: Uh-huh (affirmative)

16 Student 2: **Is when the predator population is there, but there's no prey population. So, because they have a food source they can still live after, even without the prey.**

17 Researcher: Okay. And what is this point representing? [pointing at the intersection of the two new lines drawn, her P=… and her Q=… lines]

18 Student 2: **That point represents the prey and predator population coexisting**.

19 Researcher: Okay.

20 Student 2: And the prey have a decrease here and there.

21 Researcher: Mm-hmm (affirmative)

22 Student 2: **But because they both have a food source; they both can coexist**.

23 Researcher: Okay. Perfect, thank you so much.

In this excerpt, Student 2 was able to partially interpret randomly selected points from her graph: (0,0), (1,0), (0,1) and the intersection of the two lines point, within real-life context. She explained the intersection of the two diagonal lines represented the coexistence between the prey and predator populations see Line 18, also that (0,1) represented no prey population but predator was existent, leading to prey living, since they would have other source of food. Also, her interpretation for point (1,0) for example, was incorrect, where the nullclines definitely intersect at (1,0), see Line 12 “they don't have an interaction necessarily”, I think she meant that the prey would live, while the predator would not, therefore there will not be any interaction between the two populations.

Although there were no PCA increases in Student 2’s graph interpretation, interviews illustrated a slight increase in her ability to interpret graphs in an applied context, and when asked to do mathematics, she often did the biology, not separating or translating in the manner that some authors described in the literature review, such as in Figure 1.

2.3. The Story of Student 3

Based on my notetaking data, I was not able to detect Student 3's graphing or graphing analysis and interpretation skills or changes in these skills. Student 3 was timid, and often let others speak in class, he rarely volunteered when there were problems related to graphical representations. However, because of the graphical interpretation exposure, I hypothesized that Student 3 might have a slight increase in his PCA test results related to graph explanation, and interviews related to the Predator-Prey problem graph interpretation.

In his PCA pretest results, Student 3 had an increase in question 15, and a decrease in question 10.

For PCA question 15, Student 3 initially chose answer 'c', see following figure.

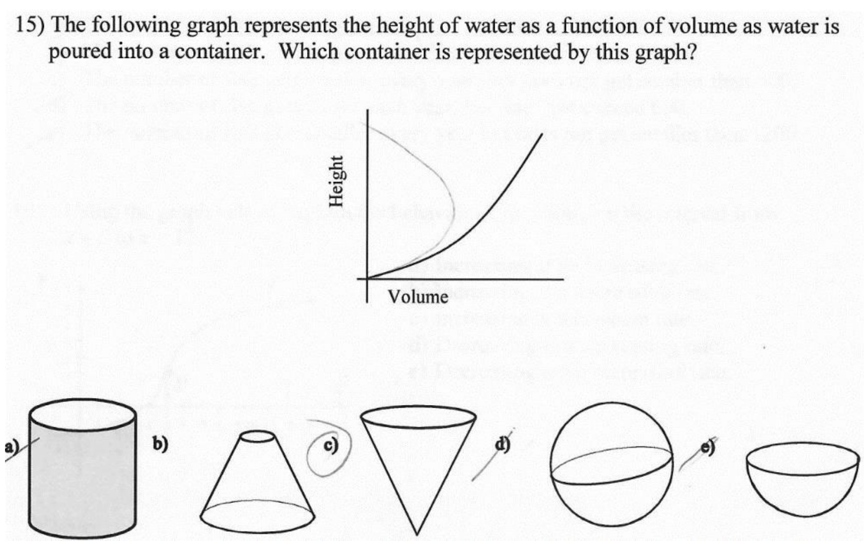


Figure 16. *Student 3's PCA Pretest Solution for Question 15*

This figure shows a bottle problem, where a volume vs. height graph needed to be correctly interpreted and linked to one of the five containers, as water is poured into that same container. To be able to correctly interpret this graph, Student 3 must consider *how* one variable was changing while imagining successive amounts of change in the other.

My hypotheses based on these observations and Student 3's pretest results were that he might select the correct answer in his posttest, since graph analysis and interpretation were the focus of the evening lessons.

In his posttest, Student 3 changed his answer from 'c' to 'b', which was the correct container, see next figure:

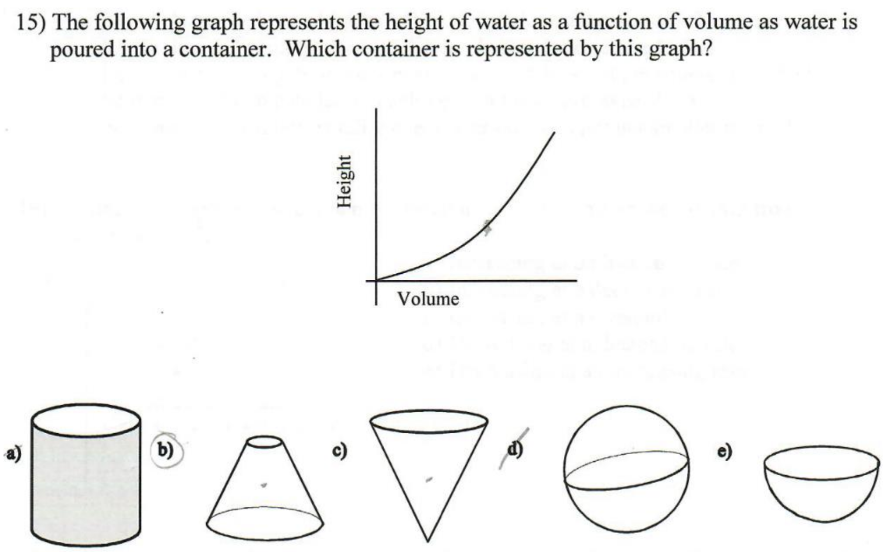


Figure 17. *Student 3's PCA Posttest Solution for Question 15*

Looking at Student 3's written work and the dots he drew on the graph, dividing the volume into two equal parts, and two more dots on two of the containers, it is clear that student 3 looked at what happened to the first half of the volume or height or both.

To understand more, I decided to address this PCA question during our final interview. Student 3 quickly eliminated containers 'a', 'd', and 'e'. He was hesitant and a confused about containers b and c, see the dots on containers 'b' and 'c', as he tried to relate which container would correspond to the graph.

Excerpt 17

Student 3: So**, in the first half, the volume's increasing slowly and then it goes quicker**. That's how I saw it.

This excerpt shows that Student 3 was thinking about segmenting the graph in two parts, then compared the two halves, while looking at what speed (slowly or fast) was the volume increasing in each half. Mentioning "slowly" or "faster" involves the idea of time quantity. However, time was not in any of the graph's axes. If time was involved in Student 3's thinking, it would have been experiential and discrete. From his explanations, he was imagining a volume vs. time graph, and probably attempting to relate it to how it would eventually affect the height. Thus, it can be said that Student 3 thought of volume varying independently from the container's height, because in his explanations, Student 3 did not involve height when explaining adding two (equal) amounts of volume. Instead, he looked at the steepness of the graph to explain how "slow" or "fast" the volume was increasing, rather than the graph's "values". This is also an example of shape thinking.

For his interview, I hypothesized that Student 3 would be able to analyze and interpret his predator-prey problem graph as he solved. (see Figure 7 and Appendix B_2), where he had to explain the produced-by-him graph within context:

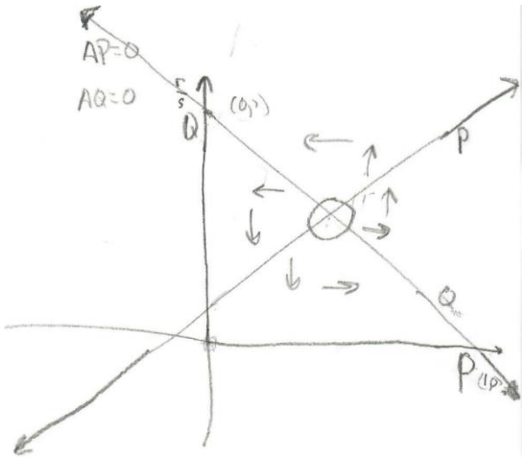


Figure 18. *Student 3's Graph for the Predator-Prey Interview Problem*

This figure shows Student 3's final P vs. Q graph, where he plotted the two diagonal lines P*, and Q* from the equations that I gave him, after he correctly explained how to find them. Student 3 also forgot how to draw the arrows "So, like the arrows, I don't really remember. I know you're supposed to set it up for like…. for this part up here…". Because of time constraints, I drew the arrows, following the solution manual for this problem. The following excerpt illustrates Student 3's explanation of some points on the graph within context,

Excerpt 18

1 Student 3: So, **you have your zero one up here, which is the carrying capacity**.

2 Researcher: Okay.

3 Student 3: It's just your carrying capacity of the second one, which I think would be unstable since that couldn't exist. Or, well no, it would exist here since you have the food.

4 Researcher: Okay.

5 Student 3: And then, it's the same thing for this, but instead it's the carrying capacity of-

6 Researcher: What point is that?

7 Student 3: One zero. **One zero is where the carrying capacity of the prey population**, and **this one in the center is the coexistence between the two populations**.

[…]

8 Researcher: Can I have your graph? So, what does this point represent over here? So, this is the ....

9 Student 3: That's coexistence, so.

10 Researcher: Can you say it in term of … the animals?

11 Student 3: **That's like your predator and your prey will both be able to sustain a population and it wouldn't lead to one of them dying and then the other one falling. It just ... The two existing in the same place**.

The excerpt shows that Student 3 had meanings to different points of his graph in applied context for the predator and prey situation. He recognized the coexistence from the graph and interpreted it as where the predator and prey would sustain a population, existing in the same place. However, Student 3 believed that there would be stability in points (0,1) which he explained as the carrying capacity, and point (1,0) on the graph, which he explained as the carrying capacity of the prey population. This is incorrect, since carrying capacity does not always guarantee stability. His interpretations seemed to result from his memory of similar problems where carrying capacity was stable.

Overall, Student 3 had one increase in his PCA test related to graphical interpretation. He also began to think of his graph within its predator-prey context, and interpret his graph according to his own understandings of the situation.

2.4. The Story of Student 4

Based on my notetaking data, Student 4 rarely volunteered to answer any questions, therefore, I was unable to identify her skills in class.

Student 4 took her pre and post PCA tests, and also scheduled her final interview in advance, but due to last minute family reasons, she was unable to attend. Therefore, my explanations of her changes in skills are purely hypothetical, since I was unable to conduct the interview and validate my hypothesis.

In her pretest, questions 9, 10, and 24 related to graph interpretation, were initially wrong, and in her posttest, she answered them correctly.

In her pretest, Student 4 chose answer ‘d’ for question 10, ‘d’ for questions 10, and ‘b’ for question 24. In her posttest, Student 4 picked ‘e’ for question 9, and ‘c’ for question 10, and ‘d’ for question 24, see next three figures.

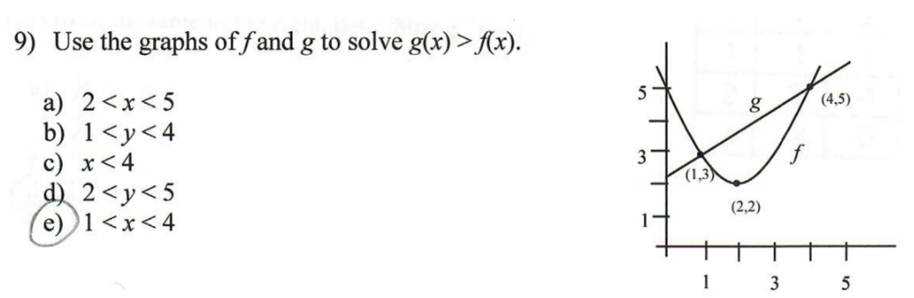
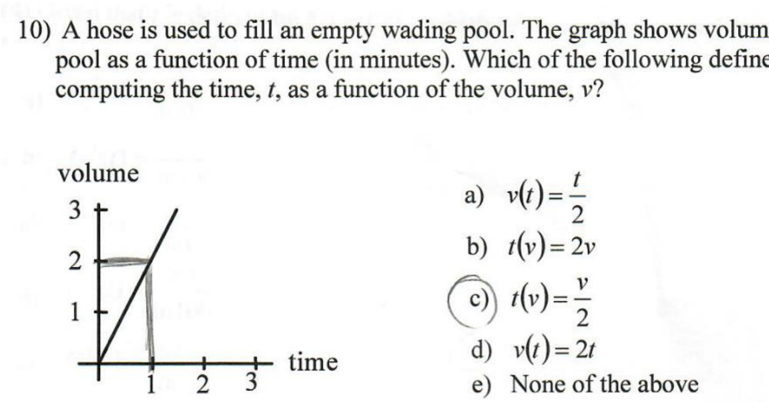


Figure 19. *Student 4’s PCA Posttest Solution for Question 9*

PCA question 9 addresses graph understanding without a real-life context, but the increase in score may have been due to the fact that the AML 100 extensively focused on graph analysis, and Instructor 1 often addressed constraint recognition during class.

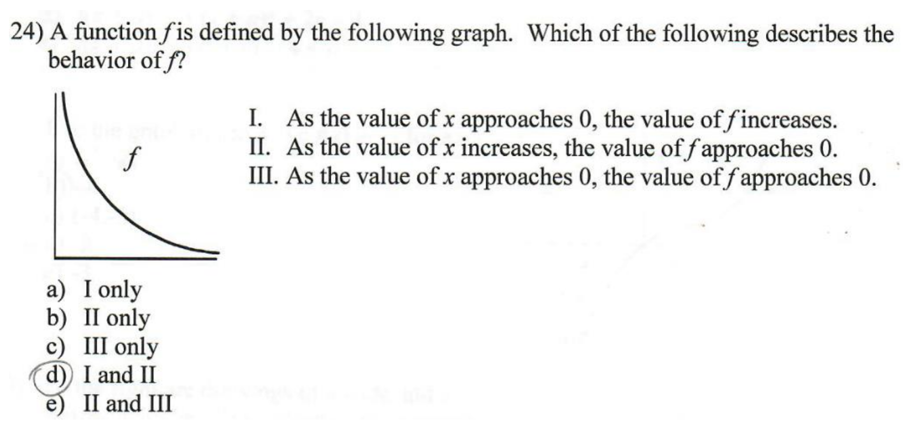
PCA question 10, on the other side, involves equation writing (function notation) given a graph. Problem 10 had an applied context, which explains the positive change – Student 4 was more attentive to the input and output variables required for the function to compute time as a function of the volume and not vice versa.


10) A hose is used to fill an empty wading pool. The graph shows volume (in gallons) in the pool as a function of time (in minutes). Which of the following defines a formula for computing the time, t, as a function of the volume, v?

a) $v(t) = \frac{t}{2}$
b) $t(v) = 2v$
c) $t(v) = \frac{v}{2}$
d) $v(t) = 2t$
e) None of the above

Figure 20. *Student 4's PCA Posttest Solution for Question 10*

PCA question 24 addresses graph analysis, which was often the focus of the evening lessons when analyzing the behavior of the graphical representation of the mathematical models. Focusing on how two quantities change at the same time, might have helped Student 4 in answering this question correctly.


24) A function f is defined by the following graph. Which of the following describes the behavior of f?

I. As the value of x approaches 0, the value of f increases.
II. As the value of x increases, the value of f approaches 0.
III. As the value of x approaches 0, the value of f approaches 0.

a) I only
b) II only
c) III only
d) I and II
e) II and III

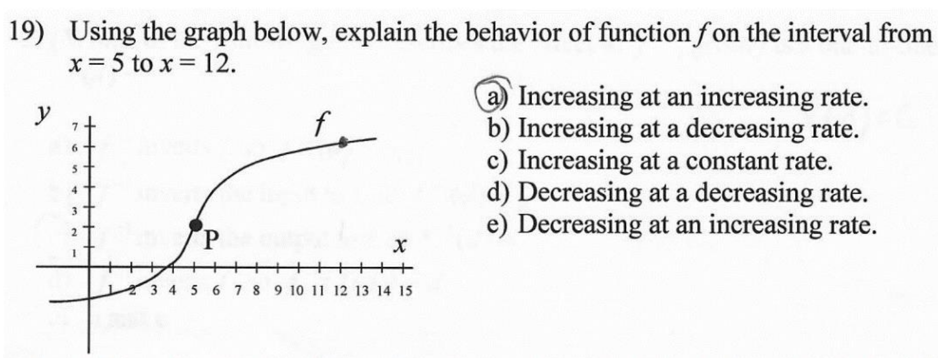
Figure 21. *Student 4's PCA Posttest Solution for Question 24*

Student 4 had the most PCA increases in the graph related problems.

2.5. The Story of Student 5

Based on my notetaking data, Student 5 greatly developed model explanation and interaction skills, she was able to transfer from real world to mathematics, and represent the given situation with a flow diagram, mathematical equations, graphs and most of the time interpret solutions and graphs within context accurately.

In her PCA pretest results, Student 5 chose answer ‘a’ for question 19, see next figure.


19) Using the graph below, explain the behavior of function *f* on the interval from $x = 5$ to $x = 12$.

a) Increasing at an increasing rate.
b) Increasing at a decreasing rate.
c) Increasing at a constant rate.
d) Decreasing at a decreasing rate.
e) Decreasing at an increasing rate.

Figure 22. *Student 5’s PCA Pretest Solution for Question 19*

This question aims at analyzing the behavior of the function from a specific interval, in terms of increasing/decreasing/constant rate. In order to have ways of thinking compatible with the correct response, students need to initially recognize whether the function is increasing or decreasing, and then analyze how this rate of increase itself is changing. What most likely Student 5 did, was thinking of the graph as a whole, shape thinking, and not separating out the rate as its own object that can have a different direction of change than the function output.

My hypotheses based on these observations and Student 5’s pretest results were that she would recognize that the function was increasing at a decreasing rate, since for equal amount of *x*, values of *y* were increasing less and less, and because function behavior analysis was often covered during evening lab activities.

In her posttest, Student 5 changed her answer from ‘a’ to ‘b’, which constitutes a correct answer, see next figure.

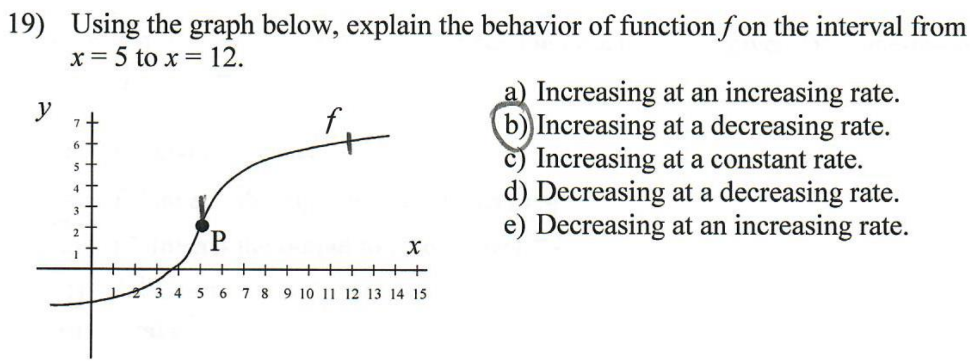


Figure 23. *Student 5’s PCA Posttest Solution for Question 19*

Looking at Student 5’s written work, it is clear that she was more attentive to the function’s behavior. Student 5 was able to realize that although the function was increasing, the way it was increasing in that interval was at a decreasing rate. Seeing this positive shift in Student 5’s ways of thinking about graphs, I decided to address this question 19 once again during the interview, see next transcript, where Student 5 explained the behavior of the graph, one Line 3:

Excerpt 19

1 Student 5: They're increasing at a decreasing rate, because the increase from 5 to 12 is getting smaller, but it is still-

2 Researcher: Increase of what is going from 5 to 12?

3 Student 5: **The increase of y is getting smaller from 5 to 12. From 5 until 6, it increases a lot more than say 11 to 12**. Because from 5 to 6, it goes up more than it does here.

4 Researcher: Okay.

5 Student 5: Like a whole number, and 11 to 12, it goes up maybe a fraction of the number.

6 Researcher: The increase in y, right?

7 Student 5: Yes. Sorry.

8 Researcher: Do you think the course helped you in answering this question somehow?

9 Student 5: **I think so, yes. Because we had to explain a lot of our answers**.

10 Researcher: Can you give me an example? Graphical answers?

11 Student 5: I don't remember any graphs. I knew that-

12 Researcher: Which answers you had to explain in the course?

13 Student 5: **From the book. We had our different problems and when we ran simulations on like MatLab and put in the equations, and that sort of thing, we had to explain what the graph was doing.**

14 Researcher: Okay.

15 Student 5: So, whether it was increasing or decreasing, if it was increasing a lot or a little, if it oscillated or stabilized at some point.

16 Researcher: Okay. So, was it in the mornings or in the afternoons?

17 Student 5: It was in the afternoon.

In this excerpt, Student 5 clearly believed that the course was helpful in answering these types of questions, because in the evening lab sessions, students were required to run simulations and interpret their graphs.

For her interview, I hypothesized that Student 5 would be able to interpret her predator-prey problem graph with ease as she solved. (see Figure 7 and Appendix B), where she had to interpret the produced graph within context:

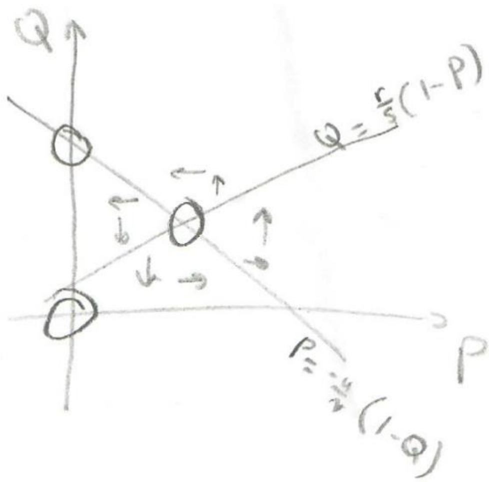


Figure 24. *Student 5's Graph for the Predator-Prey Interview Problem*

This figure shows Student 5's final P vs. Q graph, where she plotted two diagonal lines P*, and Q* whose equations were given to her, shown on the graph, and my arrow additions.

The following excerpt illustrates Student 5's graph explanation in terms of predator and prey context, after she finished drawing her graph,

Excerpt 20

Researcher: […] Okay. So, what does that point of the two lines represent? Or the P and Q line represent? In term of predator and prey.

Student 5: This one?

Researcher: Mm-hmm (affirmative)

Student 5: **Coexistence between the predator and prey**.

Due to time constraints, I did not have a chance to further research Student 5's graph interpretation within the predator and prey context. This transcript, however, shows that Student 5 was able to translate the point of intersection in context, she stated that it represented the coexistence between the predator and prey.

PCA results indicated an increase in Student 5's graph interpretation for the function behavior. Interview illustrated small instances where she was interpreting points of her graph in terms of specific applied contexts.

2.6. Summary

All students who participated in this research had some difficulties to either mathematically or graphically solve the Predator-Prey Problem, some incorrectly solved it, however, all of them recognized some of the points within the graph correctly and translated them within context, such as the coexistence between the two population from the graph. Data show, that students understood the intersection of nullclines are equilibria, but failed to understand the relationship between nullclines and the flows. Some were just shape thinking in the predator and prey problem, knowing that the problem needed to have cycles, so they drew them, sometimes, wrongly. Shape thinking here involved graphing by shapes, instead of interpreting each point. Shape thinking also occurred in PCA problems, such as with Student 2 and Student 5.

On a positive note, PCA test results also showed an increase in graph related problems. All but Student 2 had one to three score increases in problems related to graphing. Graphs with applied contexts, such as PCA questions 10, 15, 19, and 25 that had positive changes due to the applied characteristic of the course. Consequently, these students were overall, good with graph analysis and interpretations, but many had instances of shape thinking.

3. Result Interpretation

One of the skills that follows equation formation and explanation, and graphing, would be "Result Interpretation". This is when students show evidence of their ability to explain their proposed or formed model's results, interpret their mathematical solutions in a real-life applied context, or within the real world, or translate back from the mathematics to the real world. This is an important stage of the mathematical modeling mentioned in Chapter 2, see Figures 3 and 4.

For this theme, PCA testing was not a great tool for measuring or detecting the changes the students had before and after taking the AML 100 course, since the PCA test did not address such problems. To show evidence of result interpretation, student interviews were most helpful in illustrating these changes.

3.1. The Story of Student 1

In class, Student 1 was skilled at interpreting model results in context. One in-class example, was the analysis of the solutions to the Gonorrhea disease model of high school students. Student 1, when working in groups, was quick in finding and interpreting this model's results and meanings within its applied context.

For his interview about the group project, because the AML 100 course involved a lot of work in terms of analyzing the equations in context, finding then thinking about what models' solutions meant in real-life situations, I hypothesized that Student 1 would be able to easily remember, and explain his project-related mathematical model's solutions, i.e. translate the mathematical solutions that tells what actions would be necessary to prevent the destruction of Chilean forest by beavers.

Excerpt 21

1 Student 1: […] using estimations that we found through research and MATLAB, we were able to do some scenarios and see what would happen if we increased harvesting of these beavers, or hunting of the beavers, what would be an effective rate for killing off these beavers, and we found that **harvesting about one for every three beavers would be most … I mean an effective strategy for killing the beavers off in about twenty years, while preserving the forest**.

Student 1 explained that MATLAB and research were useful in reaching to their model's solution, they were searching for the best rate for killing off the beavers.

In this Excerpt, Student 1 interpreted the 0.3 harvesting rate within a context, however, the time component was missing. To Student 1, 0.3 meant that hunters would be expected to hunt 1 beaver for every 3 they encountered, and not 1/3 of beavers would be killed by hunting "every year". The units needed to be per time, rather than beavers/beaver. He added that this would result in the eradication of beavers from the Tierra del Fuego within twenty years, see Line 1. Student 1 explained, that with the beavers being hunted, this model's solution would allow the trees to prosper and grow to full capacity "preserving the forest".

For the Predator-Prey problem (see Figure 7 and Appendix B_2), that Student 1 had to solve during the final interview, and answer what this model's analysis lead him to expect as typical behavior of orbits, Student 1 solved the problem, found the equilibria points without any assistance with some minor calculation errors, drew the corresponding graph, and then reflected, analyzed and assessed his results.

When solving this problem, Student 1 was able to find his nullclines and the four equilibria points (0,0), (1,0), (0,1) and the point of intersection of the two diagonal lines he calculated, he also drew his arrows on his own, as shown on Figure 13. He then explained that that point of intersection of these lines meant the coexistence of both the predator and prey populations, see Excerpt 22.

Excerpt 22

1 Researcher: And then the other paper as well. Okay. So, what did you say about this point? What does it represent exactly? The intersection?

2 Student 1: **It represents the coexistence of both the predator and prey populations when they both exist.**

3 Researcher: What can you say about stability?

4 Student 1: That's … it is a ... I think I can't say anything about stability without like the specific numbers and then doing the whole matrix stuff. I know that eigenvalues and all that, but it appears as if it is spiraling around that point.

5 Researcher: Okay. Would you say they tend to be stable at the equilibrium point?

6 Student 1: Yeah, **they do tend to be stable**.

In this Excerpt, Lines 2 and 4, Student 1 talked about the populations, coexistence between the predator and prey, at a point at which he viewed the “spiraling around” it.

This Excerpt shows that Student 1 was able to translate the meaning of the line intersection within context. When asked to reflect about stability, Student 1 stated that without the mathematical calculations, he cannot be certain about stability, however, he added that the populations seemed to be stable to him, see Line 6, at the equilibrium point where both populations would co-exist. Instead of stating it is “stable” (which would be the correct answer) or “unstable”, Student 1 suggested they “tend to be stable”, which could be because he felt that without the mathematical calculations, the visual graph would only lead to an estimation for the stability. Another possibility would be the language used by Instructor 1, within the classroom culture, see Appendix B_2 for Instructor 1’s solution for the problem, it also suggests a “tendency for stability”.

For the Predator-Prey Problem, Student 1’s was able to interpret and explain his mathematical models’ results, applied to its specific real-life context. From all interviewed students, Student 1 was the most comfortable in finding the solutions to the problems with no assistance, analyzing, assessing, and explaining with ease what these solutions represented in a given specific context.

3.2. The Story of Student 2

Based on my notetaking data, Student 2 was a student who was ready to learn new-to-her mathematics. She did not have a great memory, but was able to derive equations from flow diagrams, and analyze equation results within context at a slower pace. Student 2 often relied on real-life situations, to explain pure mathematical situations. She often explained her ideas with contextual examples. She had great interest in finding and assessing solutions to mathematical models (interpreting results).

For her interview about their group project, I hypothesized that Student 2 might not be able to remember her group project's mathematical solutions exactly, instead she would be more able to explain these solutions within context, i.e. how would the population of homeless substance abusers decrease over time.

Student 2 explained that their hypothesis was correct, that adding a few shelters could potentially help people transition from homeless substance abusers to homeless non-substance abuser, eventually to non-homeless non-substance abusers. She summarized her findings in the following excerpt.

Excerpt 23

1 Student 2: The results? Oh, so after we found our parameters, we figured out that our hypothesis was correct that anybody who... If we were to put in place a few shelters or different methods that could potentially take people from being a homeless substance abuser to homeless non substance abuser and then all the way back to our non-homeless non substance abuser population**, we found in our simulations that with these shelters that is possible and that our homeless population would decrease and our non-homeless, non-substance abuser population would increase.** And we also did some other simulations using MATLAB and we figured out that in an unrealistic hypothetical situation, if both the betas were to increase, we would see a... the simulation was very

unstable and our homeless substance abuser population increased drastically, whereas our homeless non substance abuser population decreased.

In this Excerpt, Student 2 explained that having shelters, the homeless population would decrease, and the non-homeless non-substance abuser populations would increase, and that MATLAB was also helpful in simulating other scenarios, that were rather unrealistic and unstable. This is good evidence that Student 2 was interpreting her mathematical results into the real-life context her project was addressing, and her meaning of "solution" did not involve a number, it involved a situation, an equilibrium solution.

For the Predator-Prey problem (see Figure 7 and Appendix B_2), that Student 2 had to solve during the final interview, and answer what the presented model's analysis lead her to expect as typical behavior of orbits, Student 2 began to solve the problem, then asked to have access to the book because she was not sure whether she needed to find equilibrium points, she then stated she knew what nullclines meant but she "slightly forgot" how to find them. When presented with the P and Q equations, and later the equilibrium points, Student 2 drew her graph and gave meanings to her equilibrium points, and stability, see Excerpt 24.

Excerpt 24

1 Researcher: Okay. So, what does your analysis of this model lead you to expect as typical behavior of orbits?

2 Student 2: **They will all go around a singular equilibrium point. The one that is the most stable compared to the others.**

3 Researcher: Can you circle this stable equilibrium?

4 Student 2: [circles the intersection of the two new lines she drew]

5 Researcher: Okay. So, what does zero-zero represent? […]

6 Student 2: ... When there is no predator population or prey population. They will neither increase nor decrease or affect each other in any way.

7 Researcher: How about one-zero?

8 Student 2: That is when the prey lives, but the predator does not. So, they don't have an interaction necessarily, and the prey can increase as much as they would like.

9 Researcher: Okay.

10 Student 2: And then zero-one? is when the predator population is there, but there's no prey population. So, because they have a food source they can still live after, even without the prey.

13 Researcher: Okay. And what is this point representing? [pointing at the intersection of the two new lines drawn, her P=… and her Q=… lines]

14 Student 2: That point represents the prey and predator population coexisting.

15 Researcher: Okay.

16 Student 2: And the prey have a decrease here and there.

17 Researcher: Mm-hmm (affirmative)

18 Student 2: But because they both have a food source, they both can coexist.

In this excerpt, when asked to reflect about her analysis of this model leading her to expect as typical behavior of orbits, Student 2 interpreted these results as, they will all go around a singular equilibrium point, the one that is "the most stable" compared to the others, see Line 6, that point she later explained as the point where the prey and predator populations coexist, see Line 14. She did not check the stability for the other points and did not add arrows. Similarly to Student 1, her answer about stability was not firm such as stable or unstable. It is unclear to me, what Student 2 thought of, when stating "most stable". From my understanding of her thinking, she thought of it as the only stable looking point among the other non-stable points, or all could be stable but this point looked "more stable", which would be an inaccurate statement, since there is only one stable point in this problem. My interpretation of her claims is that, Student 2 drew arrows according to her memory of similar problems seen in class, once again, indicating shape thinking. Since she did not analyze the flows on the axes, and could not determine the stability for points (0,0), (0,1) or (1,0), she used "most stable" to describe what she felt looked mostly likely to be stable from her graphical representation of the arrows.

For the Predator-Prey Problem, with some thinking and reflecting, Student 2 was able to interpret her mathematical models' results, applied to a specific real-life context, just as she did with her project. Student 2 forgot the mathematical steps to solve the Predator-Prey problem, but once presented with the resulting equations and equilibrium points, she was able to translate the meanings of the mathematical equations' intersection back to the original context.

3.3. The Story of Student 3

Based on my notetaking data, Student 3 was very good at doing mathematics. Following class instructions, he often remembered "what" to do, rather than "why" we do. By the end of the course, Student 3 was able to produce flow diagrams, formulating mathematical equations with some assistance, doing the mathematics, and interpreting the results. In class, Student 3 was often doing the mathematical calculations, for example, finding complicated equilibrium points.

For his interview about the group project, I hypothesized that Student 3 would be able to interpret his project-related mathematical model's solutions, i.e. translate the mathematical solutions that tells what actions would be necessary to prevent the destruction of Chilean forest by beavers.

Student 3 clearly explained the summarized results in the following excerpt.

Excerpt 25

1 Student 3: We found that the epsilon rate of point three was an effective, I'm not sure what it is, point? effective rate. Which just means that **one in three beavers are killed per hunter. It means like a third of the population's killed every year basically.** And that leads to **in twenty years the beavers will pretty much fall, and the tree population will regrow.**

Student 3 correctly explained in this excerpt that with a 0.3 rate, or 1/3 of beavers' population needed to be killed yearly, so that within 20 years the beaver population would be eradicated, and the tree population would regrow, see Line 1. Unlike Student 1, Student 3's interpretation of the parameter was correct "it means a third of the population's killed every year", his result interpretation clearly involved the time, while with Student 1 it did not.

For the Predator-Prey problem (see Figure 7 and Appendix B_2) that Student 3 had to solve during his final interview, and answer what this model's analysis lead him to expect as typical behavior of orbits, due to time constraints, Student 3 was not able to solve, instead, he explained what he would do in order to find the nullclines and the equilibrium points, but he did not remember how to draw the arrows exactly. After I gave him some parts that were missing, such as the equations, equilibrium points, and added some arrows (according to the class book's version), I asked him to explain what the intersection point of the two lines meant, Student 3 explained that both the predator and prey would coexist, in the same place, see Line 5 from the following excerpt:

Excerpt 26

1 Student 3: **That's coexistence**, so.

2 Researcher: Can you say it in term of-

3 Student 3: That's like-

4 Researcher: ... the animals?

5 Student 3: **Your predator and your prey will both be able to sustain a population and it wouldn't lead to one of them dying and then the other one falling. It just ... The two existing in the same place.**

[…]

6 Researcher: And then ... Okay. So, can you circle the point of intersection of these two lines? Is there stability there?

7 Student 3: There doesn't seem to be stability.

8 Researcher: Okay.

9 Student 3: Since it's not ... It's going outwards.

10 Researcher: Okay.

11 Student 3: So, when we did the like phase planes you could see. When you start here, it's going to span outwards instead of staying here.

12 Researcher: Okay.

13 Student 3: Which means that I think just that it's unlikely for this to happen, right?

[…reflecting about stability…]

14 Student 3: Possibly just these two.

15 Researcher: Which ones?

16 Student 3: The ... No, just the zero zero, I think. Since it'd be leaning towards the left and the bottom. So, the one zero would not, and this would not, I don't think. Or well, this might since it's going up and out.

17 Researcher: Which one?

18 Student 3: Zero one.

19 Researcher: Zero one?

20 Student 3: Mm-hmm (affirmative).

21 Researcher: Do you think there's stability there?

22 Student 3: Yeah, in one zero.

23 Researcher: Zero one and one zero? How about zero zero?

24 Student 3: I don't think so.

25 Researcher: How about the point of intersection?

26 Student 3: No.

27 Researcher: No. Okay, because it's going outwards, you're saying?

28 Student 3: Mm-hmm (affirmative).

For the Predator-Prey Problem, Student 3 explained that the populations would co-exist, explaining that “the two existing in the same place”, however, he did not seem to understand stability. He said that it seemed to him that the arrows were going outwards, while they were not. Therefore, he saw no stability at the intersection of the two lines, but felt there was stability at point (1,0), and maybe (0,1). Stability in oscillating situations is sometimes difficult to determine. If one follows the arrows in the solution diagram (which would correspond to the book version of arrow drawing), one can see why someone might think that they are all pointing away from the co-existence equilibrium.

Student 3’s written work showed that when finding the solution, he did not understand the arrows or stability correctly, but interview data were evidence of his recognition of the meaning of the equations pertinent to population coexistence (recognition of positive values of the populations too) and ability to interpret some of the model’s results into a real-life context. Apart from stability, he was able to translate his mathematics back to a real-world context.

3.4. The Story of Student 5

Student 5’s group topic was: “Social Media and Eating Disorders in Teenage Girls”. She described her research question as “How social media impacted the development of eating disorders on teenage girls in the U.S.?” She explained that the goal of her model was to find solutions, specifically to “… what stage ... or place it would be best to intervene to prevent the development of an eating disorder?” As shown on previous section 1.4 from Chapter 5, they created a three-stage mathematical model to determine at what stage in the general progression of an eating disorder is best to limit social media use in teenage girls in the United States. As a result, it was best to limit social media use when teenage girls were healthy (before they developed anxiety about their body) so that there would be as few eating disorders as possible.

My notes on Student 5's project results also expand these results even further. Their group found that a low interaction rate between girls in H (proportion of healthy teenage girls who are exposed to social media) and I (proportion of teenage girls that has appearance anxiety and exhibits obsessive compulsive behavior regarding appearance, Insecure), caused the most desired results with a higher proportion of the population ending up in H with no eating disorders (low E) and little to no insecure girls (low I). However, if untargeted social influence was high, there would eventually be more girls with an eating disorder than insecure and healthy. Therefore, in order to decrease the prevalence of eating disorders among teenage girls, it would be best to intervene and limit social media use before they start to develop appearance anxiety. Lastly, they also concluded that intervening with social media use after they become "insecure" had little to no effect on the progression of the disease.

For the interview regarding their group project, I hypothesized that Student 5 might be able to explain these solutions within context, i.e. translate her results into the real-world context, and omit the mathematical part of her results.

When asked to discuss her final project results, Student 5 summarized her findings in the following excerpt.

Excerpt 27

1 Student 5: […] with our results, we found that it was best to intervene between H and I, because if there was less girls transitioning from healthy to insecure, then less girls can transition from insecure to E and develop an eating disorder.

In this excerpt, Student 5 did not give as many details as I expected about their results, however her response as shown on Excerpt 27, shows her comfort in using some of their mathematical variables and phases they used in her mathematical equations to explain her project results. Student 5's suggestion for an intervention between "H" and "I" as she mentioned in Line 1 of this excerpt, implies that she was able to interpret her results within context, but also explain the mechanism, and the reason for why this intervention, "because if there was less girls transitioning from healthy to insecure, then less girls can transition from insecure to E, and develop an eating disorder". Her results explanations were more applied to context than mathematical.

For the Predator-Prey problem (see Figure 7 and Appendix B_2), that Student 5 had to solve during the final interview, and answer what the presented model's analysis lead her to expect as typical behavior of orbits, Student 5 was able to explain the presented equations (part (a)), but she had difficulties solving the problem. With my intervention which consisted of giving her the equations to graph, Student 5 drew her graph and discussed her "three" equilibrium points (0,0), (0,1) and the point of intersection of the two lines, see Excerpt 28.

Excerpt 28

1 Researcher: It wouldn't? Okay. So, what does that point of the two lines represent? Or the P and Q line represent? In terms of predator and prey.

2 Student 5: This one?

3 Researcher: Mm-hmm (affirmative)

4 Student 5: **Coexistence between the predator and prey.**

5 Researcher: Okay. Do you know how to do the arrows? Do you remember?

6 Student 5: Not off the top of my head, no.

7 Researcher: Okay. Let me just help you out. Can I have this pencil?

8 Student 5: Yeah.

9 Researcher: Okay. So, it looks something like that. So my next question would be, do you see any stability in this graph? If yes, where? What do you think?

10 Student 5: Yes, it goes around the equilibrium point where the predator and prey coexist.

From only looking at the graph, Student 5 explained that the intersection of the two lines represented the coexistence between predator and prey. After I added some arrows to her graph, Student 5 interpreted that stability would be around the equilibrium point where predator and prey coexist.

For the Predator-Prey Problem, with some thinking and reflecting, Student 5 although had great difficulties doing the mathematics (forgot the mathematical steps to solve the problem), and not recognizing one equilibrium point, deriving her solutions, she was able to recognize coexistence and deduce the stability solution within context, and interpret her mathematical models' results, applied to a specific real-life context, just as she did with her project.

3.5. Summary

All interviewed students were able to show evidence of their developed ability to explain their projects' results, and the task's model of interaction's coexistence results in real-life context. While some had forgotten steps to solve the problem, and others showed some misunderstandings regarding arrows and stability, the translation of the two populations' coexistence was successful. All students showed a great progress in their ability to go from mathematics relating the intersection of the two mathematical equations, to a real-life context, or give real-life examples to explain this coexistence. The understanding of stability on the other hand, resulted in different answers, while two students thought would tend to be stable, one said it wouldn't and one mentioned it would, around that point of equilibrium where there would be co-existence. During this AML 100 course, not only these students began to develop a skill for result interpretation, their meanings for what a solution could represent, also changed, they began to discover equilibrium as possible solutions for their models. All students interpreted solution as being about finding an equilibrium, or the effects of parameters on an equilibrium (Table 6).

This table also summarizes students' ways of thinking about mathematical modeling (viewing mathematics and the real world as separate vs. interleaved), the emergence of shape thinking when dealing with graphs, and ways of reasoning about students' view of solutions.

Table 6

Students' Emerging Ways of Reasoning Related to Literature

	Student 1	Student 2	Student 3	Student 4	Student 5
Separate vs. Interleaved	Interleaved looking (because of instant translation)	Interleaved	Separate	N/A	Separate
Shape thinking	No	Yes	Yes	N/A	Yes
Solution: (Number, Function, Equilibrium)	Equilibrium	Equilibrium	Equilibrium	N/A	Equilibrium

Table 6 gives examples of students' ways of reasoning, for example, students 2, 3 and 5 showed some form of shape thinking when analyzing graphs, while Student 1 did not. Student 2 viewed the mathematics and biology as one (interleaved), while Student 3 translated step by step, from real life to mathematics, and then from mathematics back to real life, viewing mathematics and biology as separate.

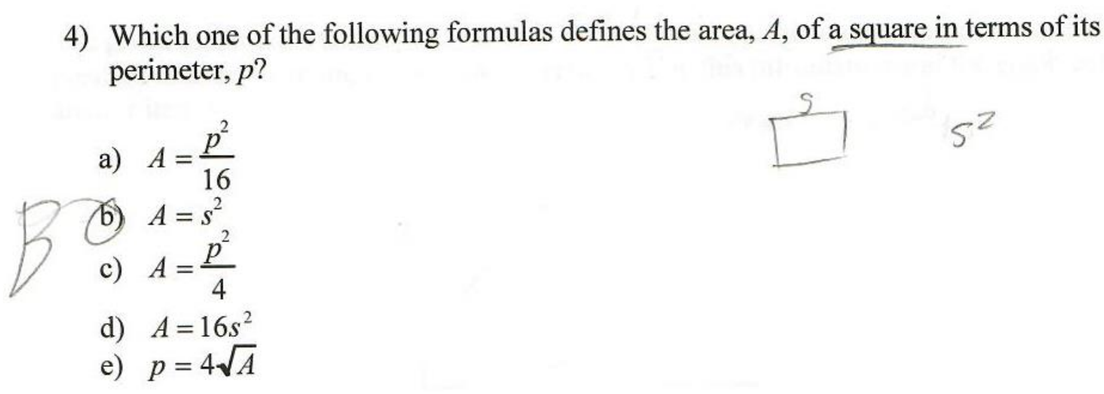
4. Pure vs. Applied

Pure versus Applied, is a where students showed positive results when the context of their problems was applied real-life contexts, but possibly fewer improvement on problems that required only pure mathematical notation.

4.1. Stories of Applied Mathematics Problems

Student 1 (PCA 4)

In his PCA pretest results, Student 1 only had two mistakes (question 4 and 6). Student 1 chose answer 'b' for question 4, see Figure 25.



4) Which one of the following formulas defines the area, A, of a square in terms of its perimeter, p?

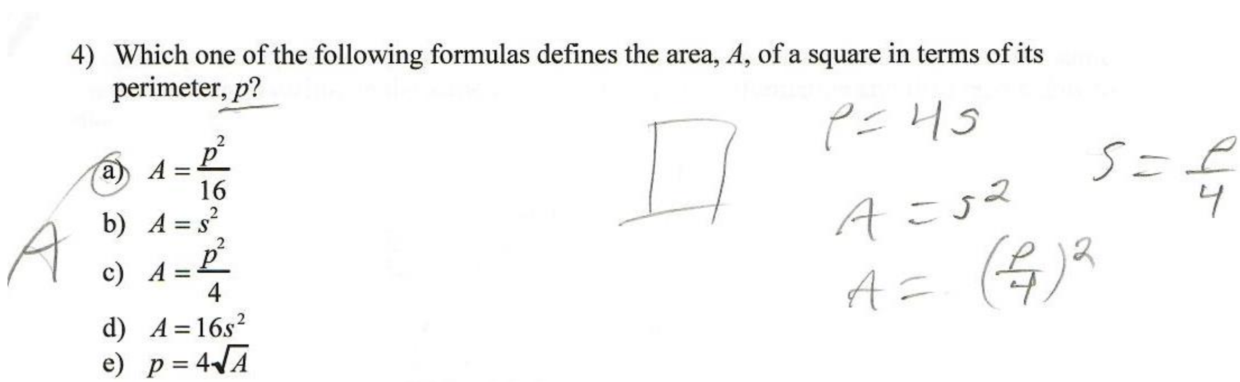
a) $A = \frac{p^2}{16}$
b) $A = s^2$
c) $A = \frac{p^2}{4}$
d) $A = 16s^2$
e) $p = 4\sqrt{A}$

Figure 25. *Student 1's PCA Pretest Solution for Question 4*

Figure 25 shows a question that asked to formulate an equation, that defines the area of a square in terms of its perimeter.

My hypotheses based on these observations and Student 1's pretest results were that, by the end of the course, he might have a correct score for question 4.

In his posttest, Student 1 changed his answer to 'a', see Figure 26.



4) Which one of the following formulas defines the area, A, of a square in terms of its perimeter, p?

a) $A = \frac{p^2}{16}$
b) $A = s^2$
c) $A = \frac{p^2}{4}$
d) $A = 16s^2$
e) $p = 4\sqrt{A}$

Figure 26. *Student 1's PCA Posttest Solution for Question 4*

To be able to answer this question correctly, Student 1 not only needed to view a function as a process that accepts input and produces output, but also apply appropriate algebraic manipulations to support reasoning about the creation of the model, where an understanding of function composition, inverse function processes (not notation), and understanding how to interpret a word problem in a different context (geometric context in this case) and represent this contextual information via mathematical equations, would be necessary.

Function composition and inverse functions, concepts that were not specifically addressed in class, were used in context, for the purpose of finding zero-growth isoclines, called nullclines, see solution to the Predator-Prey Problem in Appendix B_2, and where students experienced changing their variables of interest, and changing nullclines' input to outputs values, and where independent and dependent variables were flexible. Writing in-context mathematical equations was also one of the main focuses in this applied course, therefore, these may have helped Student 1 in correctly answering question 4, a question that involved a word problem in a geometric context to be translated into mathematics via mathematical equations, assumptions that I wanted to address during the end of the course interview session.

For his interview, I hypothesized that Student 1 would be able to easily remember, write and explain his project-related mathematical equations, and that because AML 100 course involved a lot of work in terms of writing equations from real-life situations, thinking about what variables and expressions mean, checking units, etc., he would be able to explain, in a more meaningful way, his reasoning about reaching his PCA correct answer for question 4, discussing the mathematical steps that involved finding other measurements via function composition, inverse functions, understanding of the geometric in-context situation, and forming the connections between these multiple geometrical measurements he had to initially find, in order to reach to his final answer 'a' $A = \frac{p^2}{16}$.

Regarding PCA results, I chose to address question 4 since there was evidence of a positive change in his answers, an increase in score related to this theme.

In the following excerpt, Student 1 explained his ways of thinking when solving question 4:

Excerpt 29

1 Researcher: So, what do you think the answer would be here?

2 Student 1: So, four, right here. Which of the following formulas to find area A of a square, in terms of its perimeter? And so I wrote the equation for the area. **First, I actually wrote the square, which all sides are the same**. So, I wrote S as a side, so the area which is P S squared. And then the perimeter would be four S**. And then it wants me to write the area, in terms of P. And so I want to substitute this S into P**. So, I would divide by four on both sides getting S is equal to P over four, and then substituting it in, I get area is equal to P over four squared. And then, distributing on the top and bottom, I get P squared over 16, and then that would be "a".

3 Researcher: Okay. Do you think it's possible that you had a different answer?

4 Student 1: It is possible.

5 Researcher: If yes, which one and why? What do you think?

6 Student 1: I would say that ... oh wow. Okay. So, I would not say 'e' because that's in terms of ... it's the perimeter. I ... that's just the

> area in terms of sides, it ... I think my mistake could have been at choosing answer “c”, and just believing the divide by four there, and not squaring it or something.

In this excerpt, it is evident that Student 1 became more attentive to the geometric context question’s necessity to find and connect multiple measurements. In his pretest, the answer he picked was not in terms of its perimeter (A=s^2). Although he squared his answer for the area. His writings and answer did not involve the perimeter *p*, while in the posttest, the perimeter *p*, was taken in consideration in his solution, see Line 2.

Student 1 was very good at answering questions that were asked of him. Therefore, I assumed that in both cases he understood the problem very well. In his pretest, he must have understood the question “differently”. He may have initially thought of mathematics, as simply writing familiar formulas, such as $A = s^2$, a way of thinking that, later, perhaps changed, after taking this course, where he started to think of mathematics a different way, form connections between variables and think of relating multiple measurements.

Question 4 was in a geometrical context, and geometry involves measuring physical objects in the real world. Taking this applied course where all modeling examples had a specific context, constantly working with real-life situations to model these same situations by formulating their mathematical equations, with context in mind, may have played a role in Student 1’s increase in score. For example, the AML 100 course often addressed non-linear models of two population interactions, such as simple predator-prey models, where students had to do similar substitutions, to explore the behavior of their models.

Student 5 (PCA 4)

In her PCA pretest results, Student 5 had several mistakes, such as question 4, 11, 17 and question 25, that I hypothesized she would probably score them correctly after taking this course, primarily because at the AML 100 course, students focused at representing real-life situations via mathematical equations, and then graphically analyzing the different behaviors of these equations using MATLAB during their evening activities.

PCA posttest results show that Student 5 did not have any negative changes in score, she had however 2 increases - question 4 and question 19. While question 4 is related to equation writing or explanation, question 19 refers to graphical interpretation, discussed in further details in section 5.3. Figure 27 shows Student 5's pretest answer for question 4, a question asking the students to write the area A in terms of its perimeter.

4) Which one of the following formulas defines the area, A, of a square in terms of its perimeter, p?

a) $A = \frac{p^2}{16}$

b) $A = s^2$

c) $A = \frac{p^2}{4}$

d) $A = 16s^2$

e) $p = 4\sqrt{A}$

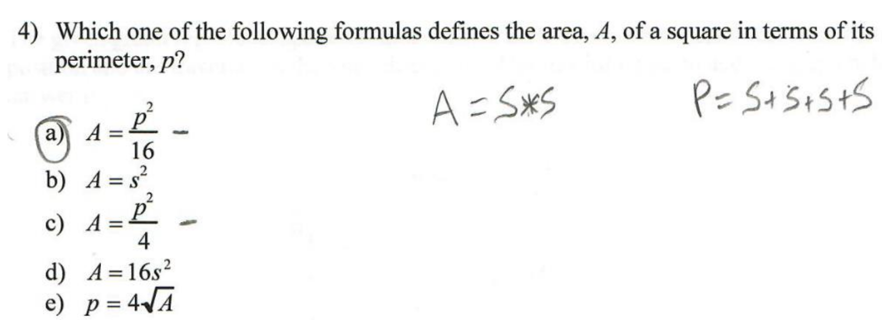
Figure 27. *Student 5's PCA Pretest Solution for Question 4*

Looking at Student 5's choice for this question, I can only speculate why she chose answer 'c', my hypothesis is that she may have thought that the perimeter has 4 sides, therefore the area in terms of the perimeter p (which would be *p* square in her way of thinking), is then divided by 4. Alternatively, she may have not correctly distributed the square, squaring the *p* only, and forgetting the 4 unsquared.

To be able to answer this question correctly, student 5 needed to apply appropriate algebraic manipulations to support reasoning about the creation of the model, where an understanding of function composition, inverse function, and understanding how to interpret a word problem in a different context (geometric context in this case) and represent this contextual information via mathematical equations, would be necessary.

Writing in-context mathematical equations was also one of the main concentrations in this applied course, therefore, these may have helped Student 5 in correctly answering question 4, a question that involved a word problem in a geometric context to be translated into mathematics via mathematical equations, assumptions that I wanted to address during the end of the course interview session.

In her posttest, Student 5 changed her answer to question 4 from ‘c’ to ‘a’, see Figure 28.


4) Which one of the following formulas defines the area, A, of a square in terms of its perimeter, p?

a) $A = \frac{p^2}{16}$ -

b) $A = s^2$

c) $A = \frac{p^2}{4}$ -

d) $A = 16s^2$

e) $p = 4\sqrt{A}$

$A = S*S$ $\quad$ $P = S+S+S+S$

Figure 28. *Student 5’s PCA Posttest Solution for Question 4*

In this Figure, Student 5 wrote on the side the two main equations she had to work with, in order to find her correct solution. She narrowed her possible answers to both ‘a’ and ‘c’, because these answers had an area A in terms of the perimeter p, and finally circled her answer ‘a’.

Constantly having exposure to real-life contexts, such as this geometric one, may have helped Student 5 to answer this question. To test my hypothesis, and to better understand her reasoning when solving this particular task, I decided to re-give this same question to Student 5 to solve during the interview, asking her to speak out loud and explain as she solved, and to better understand how her ways of thinking and solving.

Excerpt 30

1 Student 5: With the area, I know that for a square, it's the side times another side, and with perimeter is all the sides added together and they're all the same because it's a square. So, I was adding four S together. And then, I didn't know how to put those together to find an answer.

2 Researcher: Why did you pick A, and not B, and not C, or D?

3 Student 5: Well, I kinda narrowed the answers down to A and C, because it said to solve for the area in terms of the perimeter P.

4 Researcher: Okay. Okay. So, you saw the perimeter on the right side so that's why you picked either A or C for now?

5 Student 5: Yeah, because the others didn't have P.

6 Researcher: Okay. Okay. And then?

7 Student 5: Then...

8 Researcher: Then you picked A.

9 Student 5: Yeah, I'm not sure why though.

10 Researcher: Just guessed?

11 Student 5:   Yeah.

This Excerpt shows that Student 5 was attentive to the fact that the question had a perimeter as input and an area as output, this is why, she narrowed her answers to ‘a’ and ‘c’, see Line 3, however, later she was unable to explain why she chose a correct answer. It could be that she did not understand how to distribute an exponent, see Lines 8-11. In that case, it was the algebra rule that she was missing, which was not taught in class, therefore there is possibly no improvements here. Feeling her discomfort in explaining her ways of thinking for that question, I decided to accept her answer and switched to a different problem.

Student 4 (PCA 17)

Among all students who participated in this study, Student 4 had the most positive changes in her PCA tests. In her pretest, questions 9, 10, 17, 24 and 25 were wrong, and in her posttest, she answered them correctly. Student 4 also had two negative changes, questions 7 and 14, questions related to exponential and inverse functions, which are not relevant in this theme.

In her pretest, Student 4 chose answer ‘d’ for question 17. In her posttest, Student 4 circled ‘c’ for question 17, see Figure 29.

I hypothesized that Student 4 might have an increase in question 17, a question with an applied context, see next figure.

In her pretest, for question 17, Student 4 chose $A = 5\,\pi t^2$, she did not recognize the need to square 5 when squaring the expression for the radius.

17) A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, A, of the circle in terms of the time, t, (measured in seconds) since the ball hit the lake.

a) $A = 25\pi t$
b) $A = \pi t^2$
c) $A = 25\pi t^2$
d) $A = 5\pi t^2$
e) None of the above

Figure 29. *Student 4's PCA Posttest Solution for Question 17*

Question 17 requires the recognition of a need to compose two functions to create the new function, which expresses the area as a function of time since the ball hit the water. To reason in a way that produces a correct answer, students must recognize that the information in the problem allows them to establish a formula for the radius in terms of time that can then be used as an input to the standard relationship between area and radius of a circle. This involves algebraic constructions related to function creation and composition such as squaring to obtain $25t^2$. Students must be able to coordinate all of these manipulations and their products to recognize that the final answer would be $A = 25\pi t^2$. AML 100 covered several examples of similar situations mentioned earlier, therefore, it is not surprising to see, and fair to say, that this applied mathematics course played a role in Student 4's correct answer for this question, and developed her equation writing skills.

Student 3 (PCA 15)

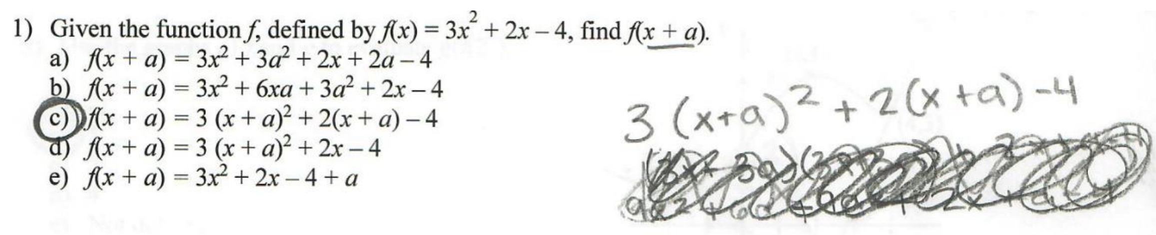
PCA problem 15 is also an illustration of an applied mathematical problem where Student 3 saw an increase in score when the context of the (container) problem was applied, see previously discussed section 2.3 (Chapter 5).

4.2. Stories of Pure Mathematics Problems

This section illustrates a pure theoretical problem, where two students had different outcomes. Student 2 did not have any increase in this pure mathematical problem, while Student 3 did.

Student 2 (PCA 1)

In her PCA pretest results, Student 2 had several mistakes related to formulation or explanation of equations, but there have been no positive nor negative changes in scores in this category, except for question 1, see next figure.


1) Given the function f, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.
a) $f(x + a) = 3x^2 + 3a^2 + 2x + 2a - 4$
b) $f(x + a) = 3x^2 + 6xa + 3a^2 + 2x - 4$
c) $f(x + a) = 3\,(x + a)^2 + 2(x + a) - 4$
d) $f(x + a) = 3\,(x + a)^2 + 2x - 4$
e) $f(x + a) = 3x^2 + 2x - 4 + a$

$3(x+a)^2 + 2(x+a) - 4$

Figure 30. *Student 2's PCA Pretest Solution for Question 1*

This figure shows a question that asked for writing an equation using function notation, where the functions input changes from x to $(x+a)$.

In her posttest, Student 2 also changed her answer to question 1 from ‘c’ to ‘e’, see following figure.

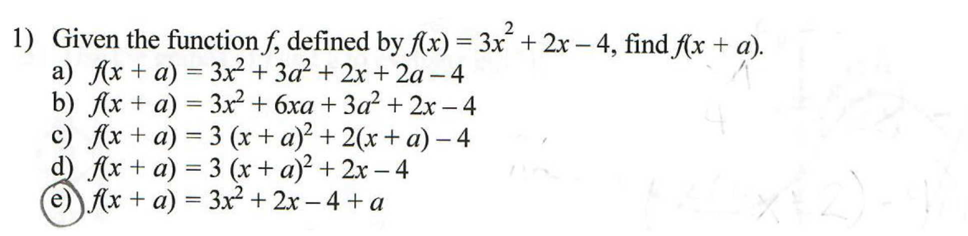

1) Given the function f, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.
a) $f(x + a) = 3x^2 + 3a^2 + 2x + 2a - 4$
b) $f(x + a) = 3x^2 + 6xa + 3a^2 + 2x - 4$
c) $f(x + a) = 3\,(x + a)^2 + 2(x + a) - 4$
d) $f(x + a) = 3\,(x + a)^2 + 2x - 4$
e) $f(x + a) = 3x^2 + 2x - 4 + a$

Figure 31. *Student 2's PCA Posttest Solution for Question 1*

To be able to answer this question correctly, Student 2 needed to have a fair knowledge of function notation. However, this applied mathematics course did not specifically focus on function notation topics. It focused more on variables within the equations, and their units. The lack of increase could have contributed to the lack of the function notation teachings in this course.

Some other examples confirmed her lack of function notation understanding, for example question 20 that also involved a function notation example, was also wrongly answered, in both pre and posttests, which confirms the idea that Student 2 did not have a robust understanding of the "function notation", a topic that was not particularly addressed in this course. Similarly to Student 1, Student 4 also answered question 20 wrongly, in her side notes, she wrote the function R(m) = S(m+12) = Sm+12S, which further confirms issues these students had with function notation.

Student 3 (PCA 1)

In his PCA pretest results, Student 3 had several mistakes. He had two positive changes in score, question 1, 15 and question 21.

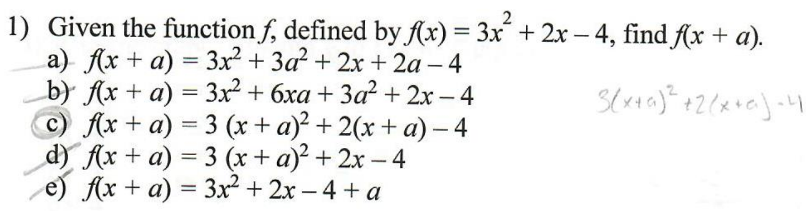
For question 1, Student 3 initially chose answer 'd', see next figure.

1) Given the function f, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.
 ~~a)~~ $f(x + a) = 3x^2 + 3a^2 + 2x + 2a - 4$
 b) $f(x + a) = 3x^2 + 6xa + 3a^2 + 2x - 4$
 ~~c)~~ $f(x + a) = 3(x + a)^2 + 2(x + a) - 4$
 d) $f(x + a) = 3(x + a)^2 + 2x - 4$
 ~~e)~~ $f(x + a) = 3x^2 + 2x - 4 + a$

Figure 32. *Student 3's PCA Pretest Solution for Question 1*

This figure illustrates a pure theoretical mathematical question that involved function notation with no real-life applications or contexts. As previously discussed, function notation, although indirectly used in class when changing input variables, was not specifically addressed (nor was the focus) in this applied mathematics course. All the changes in input or output variables in this course, when happened, were in specific contexts, and used subscript notation or variable substitution instead of function notation.

My hypotheses, however, based on these observations and Student 3's pretest results were that he would not have a positive change for this question. In his posttest, however, Student 3 changed his answer to 'c' (correct answer), see next figure.



1) Given the function f, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.

a) $f(x + a) = 3x^2 + 3a^2 + 2x + 2a - 4$

b) $f(x + a) = 3x^2 + 6xa + 3a^2 + 2x - 4$

c) $f(x + a) = 3\,(x + a)^2 + 2(x + a) - 4$

d) $f(x + a) = 3\,(x + a)^2 + 2x - 4$

e) $f(x + a) = 3x^2 + 2x - 4 + a$

$3(x+a)^2+2(x+a)-4$

Figure 33. *Student 3's PCA Posttest Solution for Question 1*

Only after viewing Student 3's increase in this theoretical problem, I began to hypothesize why Student 3 had a positive outcome. This might have been related to the course's focus on input and output variables, and their units of measurement, when the context was applied.

To be able to answer this question correctly, Student 3 had to abstract these learned concepts into theoretical problems such as problem 1. Specifically, Student 3 had to think about whether *a* would be an input or an output, and whether *a*'s units would be, the same as *x* units or the same as *y* or *f*(*x)* units, assuming there would be some real-life context involving specific units. To give a context to this idea, a similar and applied example, would be PCA question 20, where students needed to think of 12 as part of the function's input, which can only be measured in input's units, in other words, in months, adding 12 "months" to m months, which would be the new input, that would be measured in months. This unit involving way of reasoning would result in students' ability to realize that the input *x* in Problem 1, needed to become ($x+a$), replacing all input instances of *x* in the equation by ($x+a$), since a and x are of same units and added. While there have been instances where students worked on changing their input and output variables in class such as when finding nullclines, changing variables of interest, and being flexible with independent and dependent variables, these were always addressed in some type of context. This question was purely theoretical with no real-life applications or real-life contexts, yet had an increase, because Student 3 abstracted the contextual idea addressed in class, toward a pure mathematical problem.

Also, while some students may have grasped how changing the input modifies the equation (theoretically or in context), others did not (such as Student 2), perhaps because question 1 lacked a practical context, and function notation was not as developed in this applied course.

For his interview, I hypothesized that Student 3 would be able to give more insight into how he thought about solving question 1. I chose to address question 1 since there was evidence of a positive change in his answers, an increase in score although the problem was not applied.

In the following Excerpt, Student 3 explained how he was thinking when solving #1:

Excerpt 31

1 Researcher: […] Let's go to number one. Given the function *f* defined by *f* of *x* is equal to 3 *x* squared plus 2 x minus 4, find *f* of *x* plus *a*. What would you answer and why? And, if you can, explain the steps, as you write.

2 Student 3: So essentially, you're finding *f* of *x* plus a, so if it were a number, like if f is equal to one, then it would just be the one in the square root, or in the square, **but with the *x* plus a, it's essentially just you're adding something else in there, so you wouldn't be changing like anything out here with the a's, you'd just be adding another thing into every portion with *x*.**

3 Researcher: Okay.

4 Student 3: Something like that.

5 Researcher: What answer would you answer?

6 Student 3: C.

7 Researcher: C? Okay.

In this excerpt, it is evident that Student 3 acknowledged that the input changed from *x* to *x* plus a, he was thinking of "adding something else in there" see Line 2, adding 'another thing', a constant in the input, he explained, "you'd just be adding another thing into every portion with *x*", with the input, see Line 2 of Excerpt 31. He thought of adding it to every instance of input, which he called 'portion' where *x* was.

Question 1 was purely theoretical. Yet, it is possible that Student 3 had a prior experience with function notation that Student 2 did not, or that taking an applied course and constantly modeling situations that were experientially real to the students, by formulating their mathematical equations, where varying input and output variables during simulations (understanding function concepts like variable substitution), played a role in Student 3's increase in score.

4.3. Summary

Data show that this applied mathematics course had a positive impact on students' test performances on ten PCA questions, not only on problems with applied contexts, as I hypothesized, such as questions 4, 4, 10, 15, and 17, but surprisingly on those that did not have a specific real life context, such as 1, 9, 19, 21, 24, and 25, where the majority of these questions had a graphical context instead. Here is a table that summarizes the number of pure and applied PCA problems improved.

Table 7

Pure vs. Applied PCA Improved Problems

Number of applied and pure graphical problems improved	Number of pure problems improved	Number of applied problems improved	Number of pure non-graphical problems improved	Total number of all PCA problems improved
8	6	5	3	11

This table shows that for example 11 problems out of 25 PCA problems, had at an increase when comparing students answers from before and after taking this course. Among them, only three problems that had an increase involving pure problems (without a graphical representation), while applied or graphical context problems, had the highest increase, therefore, this course had a greater impact on applied or pure but graphical PCA problems, i.e., 8/11 or 72.73% of all improved PCA problems either had applied or graphical contexts.

Among the applied problems were, problems addressing the understanding of function composition (although not directly taught in class, yet used within applied contexts via variable substitution concepts) such as questions 4 and 17, or the understanding of input and output variables such as questions 10 and 15, and the understanding of how two quantities change together (container problem), which would also involve question 15.

Among the pure mathematical problems, the majority of the improved problems involved graphical representation or interpretation, which was one of the central focus of this course, such as questions 9, 19, 24, while the other two questions 21 and 25, involved domains of functions, with non-zero division rule (non-zero division was often addressed in class, both graphically and algebraically). Other than graphical or domain pure mathematical problems, other pure problems had no increases at all such as questions 13, 14, or 23, because this applied mathematics course had no direct or indirect effect on these pure (non-graphical) problems, as a result, students did not show any increases regarding these pure mathematical problems.

This applied mathematics modeling course involved situations that were experientially real to the students, involving specific contexts, therefore, these students developed the understanding of function concepts (such as variable substitution) but did not develop the understanding of function notation (such as function composition). They developed the understanding of certain concepts without learning the mathematical formal function notations that represent these concepts.

Overall, following this applied mathematics course, students showed positive results when the context of their problems was applied real-life contexts, and graphical non-applied, but fewer improvement on problems that required only pure mathematical notation.

CHAPTER 6

ATTITUDES

Based on pre and posttest results, notetaking and interview data I collected and carefully analyzed, I initially organized the students' stories into four emerging attitudes: enjoyment of mathematics, confidence in mathematics, usefulness of mathematics, and motivation/engagement towards mathematics, which I in turn categorized into four developing themes:

(1) "Academic freedom and Enjoyment", where students enjoy the freedom to choose their own topics of interest and do the mathematics significant to them.

(2) "Productive struggle", where their struggle may build confidence, but in some cases may also degrade it.

(3) "Changing the meaning of what *mathematics* is" and emerging problems from ATMI testing, where students' attitudes changed but their scores on the ATMI tests did not, simply because their actual meaning for the questions changed. ATMI did not work in this case because this applied mathematics course was about changing the meaning of *mathematics*, so the questions students were asked in the posttest were not interpreted in the same manner as those in their pretests.

(4) "Changing the meaning of *doing mathematics"* (which differs from not changing the meaning of "mathematics"), an important part of the course where evidence of students' interpretation of *doing mathematics* has changed.

1. Academic Freedom and Enjoyment

This section discusses the students' increase in enjoyment of mathematics in general. Many students related the academic freedom to be one source for their enjoyment attitude.

All AML 100 students participated in research projects where they were expected to design their own projects, choose varying topics of study from their own interests by applying the recently learned techniques to a broad range of interests they were given the freedom to choose from, and further investigate this topic. Contrarily to engaging in tasks presented by the teachers, I hypothesized that the students' freedom of topic choice would result in greater curiosity, interest, and achievement while the teachers acted as mentors, solely contributing their mathematical and modeling experience.

Interview data showed that academic freedom, as illustrated in these stories, played a positive role in shifting student attitudes toward mathematics—attitudes such as enjoyment of mathematics or even motivation—because when students explore topics close to their heart, their interests and engagement often increase. Students expressed their joy and cited benefits in choosing their topics of interest, and PCA results also deducted positive changes in enjoyment attitude.

1.1. The Story of Student 2

Student 2's interests evolved around health problems, specifically aimed at better understanding some factors that may have the potential to eradicate certain diseases and contribute to a healthier world. Given her interests, Student 2 was grouped with two students with similar initial health topics, such as PTSD, and veteran suicide, to understand and analyze how these topics could affect homelessness. This group eventually agreed on “The Roles of Substance Abuse Among the Homeless Population in California”.

Based on my notetaking data, Student 2 greatly enjoyed the applied mathematics course - her interaction in class was positive, she had a particular interest in finding solutions to world health problems, and her enjoyment was closely related to the usefulness of the mathematical modeling solutions into the real world. She enjoyed the applied side of the mathematics course. I also observed how Student 2 started with more of a shy personality but became more engaged as the lessons went on. By the end of the course, she was taking initiatives to perform tasks and answering questions with greater engaging roles.

My hypotheses based on these observations, were that Student 2 would have some small increases in her posttest items related to enjoying mathematics. The following table illustrates some of these one-scale changes.

Table 8

Examples of Student 2's ATMI Results Related to Mathematics Enjoyment

pre: pretest result
post: posttest result
*: my hypothetical posttest result for the student

	strongly disagree	disagree	neutral	agree	strongly agree
26. I like to do new experiments in mathematics				pre	post*
27. I would prefer to do an experiment in mathematics than to write an essay.			pre	post*	
30. I am happier in a mathematics class than any other class.			pre	post*	

As shown on this table, test results showed that there were some one-scale increases in Student 2’s enjoyment attitude statements. I interpret significant changes in attitudes, as two or more intervals of change.

Furthermore, her pre and post motivation answers had all been positive. Since she had already agreed or strongly agreed to items regarding motivation in her pretest, such as her willingness to take future advanced mathematics courses, I hypothesized that in her posttest, there would be no significant (2-scale or more) changes. Section 3.3 from Chapter 6 discusses Student 2's motivation is further details.

Overall, Student 2's pre- and posttest results were generally positive. There have been a few one-scale shifts, specifically regarding her enjoyment towards mathematics, but Student 2's score regarding enjoyment and motivation on the ATMI did not change significantly.

For the interview, I hypothesized that Student 2 would show more evidence of motivation and engagement, because she felt that what she was doing, was closely related to her interests, and the usefulness of the mathematical modeling's solutions to the real world.

In the following excerpt, Student 2 explained her group's motivation and interests towards their "problematic" topic of interest, homelessness, for which they were interested to find real-life solutions,

Excerpt 32

1 Student 2: … Homelessness is a very heavy issue, […] But when you see people on the streets, that's what they have to do every day. They have to beg for their money just so they can eat food or fulfill a certain basic need that they can't, because they don't have a job, or their life has been given up basically to substance abuse on the streets. And so, our motivation was we wanted to help these

people. We wanted to find a solution so that we could prevent it from happening to those who are below the poverty line. And so possibly we could just help everyone out of this homeless state and help them back to a stable, and having a home and all the above.

In Excerpt 1, Student 2 explained that homelessness, specifically due to substance abuse, is an important existing problem that needs to be solved. Her group wanted to find solutions to prevent this problem from happening, so that homeless people would leave the streets, have a job income, and a roof on top of them. Student 2 also described how, as a group, they decided on their final topic of interest, she added:

Excerpt 33

1 Student 2: So, we originally wanted to focus on homelessness, and we had all these different factors. We were going to focus on mental illnesses and substance abuse, and how both of those can contribute to the homeless population, and how these factors can lead to it. But we realized that homelessness itself is such a big, heavy, complicated issue that you can't focus on the factors themselves because those are a bunch of categories that are different cans of worms, and-

2 Researcher: Like what?

3 Student 2: Like there's PTSD veterans and they're on the street, and about, I'm pretty sure 1 in 12 homeless people are veterans. And so, that was one of our motivations was PTSD in veterans, and we were

like, we want to help them because they were fighting for our country and so they shouldn't be left on the streets.

4 Researcher: So how did you change it from veterans to homeless and just drop the veteran aspect?

5 Student 2: Well, okay, so that's where it was. Those were some of our factors. And so, we had the veterans, we had the substance abuse, we had the mental illness and we were trying to correlate all of them together and make sure it was one subject, but we couldn't fit it into that. And so, we tried to put them together like veterans, oh they have PTSD, so that goes with mental illness. And sometimes mental illnesses can cause substance abuse because the drugs given to them. Like schizophrenia, they have their pills and all of that. And then that can go around into homelessness. And then we realized that this was too much. And so, the one thing that was in the middle was substance abuse. So that was where we were like, well, these things can lead to substance abuse in the end, so we decided to stick with substance abuse.

In this excerpt, Student 2 explained that because homeless is a broad topic, the group felt the need to narrow it down from veterans' population, and their related PTSD factors that they initially were interested in, to simply substance abuse, for the whole homeless population. They later narrowed it down to specifically the state of California.

Because Student 2 had a few one-scale increases in her posttest results regarding enjoyment of mathematics, I decided in the interview, to address what she liked and disliked about this applied mathematics experience. I hypothesized that the interview results for her enjoyment would be slightly positive, with no negative feedback.

When asked whether Student 2 enjoyed the applied mathematics course, she excitedly answered “I really liked it”, see Line 2:

Excerpt 34

1 Researcher: What is your feedback, what was positive, negative about this course... that you'd like to share?

2 Student 2: **I really liked it.**

[…]

3 Researcher: What did you like about it, like two best things maybe that you liked?

4 Student 2: Okay. Two best things? **I liked the college experience.**

5 Researcher: Okay.

6 Student 2: And, being able to take a course, and, I don't know, understanding more how to prepare for college itself.

7 Researcher: Okay.

8 Student 2: Because now I know that, I don't know... I was expecting much different when I was like, 'Oh, it's probably going to be pretty easy, it is going to be okay'. And then I got here and three weeks in I was like, 'Okay, this is not as easy as I thought it'd be’.

9 Researcher: Okay

10 Student 2: Yeah. So, I'm more prepared than I could say I was a few months ago.

11 Researcher: Okay. Anything negative?

12 Student 2: Umm.. Not really. I mean... I kind of wish we had more time, but …

13 Researcher: Okay.

14 Student 2: I like how the course... Because I understand that the course had to be in the six-week limit kind of thing and I liked it over all.

15 Researcher: Okay.

16 Student 2: I don't really have any negative feedback.

Excerpt data show that Student 2 really enjoyed the applied mathematics course, mainly the college experience and liked feeling better prepared for college than she was “a few months ago”. She had no displeasure about her experience, and wished there was more time. My post-interview interpretation for wishing more time is that she either really enjoyed the applied mathematics course, and wanted more of it, since six weeks was a short period of time for her, or she possibly wished there was more time because she felt rushed, and she did not like feeling rushed during these six weeks.

During the interview, Student 2 also affirmed my observations about her changes in engagement, as illustrated on Lines 2 and 4,

Excerpt 35

1 Researcher: I also noticed, in the beginning, you were a little bit shy-

2 Student 2: **I was**.

3 Researcher: ... and then later, you were all the time volunteering, going overboard and everywhere. This is something that I noticed while I was there. Question 40 was, "I believe I'm good at mathematics

experiments." So, also “an increase”. Can you say more? How this course may have helped you?

4 Student 2: Yeah. Well, I feel like this course revolved around mathematical experiments. Before I was like, "Well, I don't really understand what that means." I guess I'm good at mathematical experiments.

This excerpt shows that Student 2’s engagement increased and her understanding of what mathematical experiments are, has changed. Although Student 2’s ATMI test results did not have significant changes regarding motivation, which were already positive in her pretests, therefore she would not have been able to have a significant increase to some items that were initially very positive (ATMI testing did not work well in these cases), Student 2;s interviews, on the other hand, showed more evidence confirming an increase in her enjoyment and motivation about mathematics.

Academic Freedom, where students like the freedom to choose their own topic and do mathematics that is important to them, may have laid ground to a positive impact on specifically Student 2’s enjoyment and motivation positive shifts in attitudes towards mathematics. Specifically, Student 2 enjoyed using mathematical modeling to find solutions to health-related problems that she, and members of her group, deeply cared about, topics that they had the freedom to choose from, based on their interests. Her willingness to take future advanced mathematics courses and increased observed classroom engagement were also evidence of her positive shifts in engagement and motivation towards mathematics.

1.2. The Story of Student 3

My notetaking data suggest that Student 3's interests evolved around Ecology issues that he wanted to find solutions to, specifically on invasive species that may cause economic and ecological damages. Student 3's initial topic suggestion was about invasive cane toads in Australia. Student 3 was then merged with two other students with similar "eco-system" topic interests. Together, they agreed on "The Prevention of Chilean Forest Destruction Caused by Invasive Canadian Beavers".

Based on my notetaking data, Student 3 enjoyed researching information about the Canadian Beavers, his groups' topic of interest, and was passively involved. Overall, he enjoyed the applied mathematics course, his interaction in class was generally positive, despite being reserved and taciturn, as a student. Student 3 was more of a follower rather than the leader of his group, with a generally pleasant and positive attitude (always smiling in class).

Based on these observations, I hypothesized that Student 3 would have a small increase in his ATMI posttest results regarding enjoyment attitudes towards mathematics. I hypothesized a positive shift in some items, such as item #27, "I would prefer to do an experiment in mathematics than to write an essay". As for motivation, I have also observed how Student 3 started with a shy personality and remained shy until the end of the course. But since he already strongly agreed to items regarding his motivation in his pretest, such as taking advanced math courses in the future, I hypothesized that in his posttest results, there would be no significant positive nor negative changes. Therefore, I decided to address these questions during his interview, to better understand if there were any significant changes in his attitudes.

ATMI test results have shown that Student 3 had a positive significant change in attitude towards the enjoyment of mathematics: In his pretest, Student 3 picked a neutral answer regarding preferring a math experiment over an essay. In his posttest, Student 3 strongly agreed to that statement, see item #27 from the next table. There were also a few one-scale positive changes, such as items #25 (a reversed statement), #26, and #29, and no evidence of any negative changes in enjoyment.

Table 9

Examples of Student 3's ATMI Results Related to Mathematics Enjoyment

pre: pretest result
post: posttest result
*: my hypothetical posttest result for the student

	strongly disagree	disagree	neutral	agree	strongly agree
27. I would prefer to do an experiment in mathematics than to write an essay.			pre		post*
25. Mathematics is dull and boring.	post*	pre			
26. I like to do new experiments in mathematics.		pre	post	*	
29. I really like mathematics.			pre	post	*

As for his motivation ATMI results, there have been no significant (two-scale) changes, Student 3’s motivation items in his pretest have been initially very positive. For example, he strongly agreed with #33 “I plan to take as much mathematics as I can during my education”, and #34, “The challenge of mathematics appeals to me”.

Overall, Student 3’s pre- and posttest results were generally positive. A significant positive enjoyment change has been recorded, specifically regarding mathematics experiments. I decided to address some of these questions to see whether this applied course, or the freedom of topic choice, further played a role in his changes in enjoyment attitudes towards mathematics.

For the interview, I hypothesized that Student 3 would show more evidence of enjoyment based on his posttest results, especially since his group's topic of study initiated from his ideas and his own interests, see the following excerpt for his answers regarding what he liked in this applied course.

Excerpt 36

1	Student 3:	Yeah, last year and the year before with the projects, they were kind of like the mentor or like the tutor picks them out and you basically have to settle for whatever they do. With this, I thought it was neat that you could just choose whatever… but also the math was more interesting.
2	Researcher:	You choose whatever what?
3	Student 3:	**Whatever topic you'd like.**
4	Researcher:	Oh, okay.
5	Student 3:	Yeah.
6	Researcher:	Yeah, that's a good thing.
7	Student 3:	**I liked that.**
8	Researcher:	So, you like the fact that you can choose your own experiment in math?
9	Student 3:	Yeah.

Student 3 stated that he overall liked mathematics "I like math", and that he enjoyed the fact that his group had the freedom to select a topic of interest for their mathematics experiment (specific to this course), as evidence on Lines 7-9.

Student 3 also added that he enjoyed the mathematical modeling course, and found it "fun" as soon as he grasped the concepts, see Excerpt 37.

Excerpt 37

1	Student 3:	So yeah, I really liked the course. It was interesting material and it definitely got me interested in math. At first, it was really kind of tough, since you had to think about it a different way. **But as soon as I grasped the concepts, or well, just the understanding, I think it was really fun.**
2	Researcher:	Okay.
3	Student 3:	And negatives, I don't think there were ... No, no negatives except maybe just wishing I could learn more about this.

Posttest and interview data show that Student 3 enjoyed the freedom of topic choice that this mathematical modeling course had to offer. He also enjoyed mathematics experiments and math in general. His posttest results align with the small, yet significant, increase in his enjoyment attitude towards mathematics.

1.3. The Story of Student 5

My notetaking data suggest that Student 5's interests evolved around "technology" influences on mental and physical health-related problems, such as self-esteem, anorexia, or suicide, contributing to a healthier world for women. Student 5 was grouped with two other female students who had similar “technology/health” interests, to eventually agreeing on “Social Media and Eating Disorders in Teenage Girls”.

Pretest data showed that Student 5’s enjoyment of mathematics was predominantly neutral, for items related to mathematics experiments, such as #26, and #27, and positive for item #31, viewing mathematics as an interesting subject. Other enjoyment items remained with no change.

Table 10

Examples of Student 5's ATMI Results Related to Mathematics Enjoyment

pre: pretest result
post: posttest result
*: my hypothetical posttest result for the student

	strongly disagree	disagree	neutral	agree	strongly agree
26. I like to do new experiments in mathematics			pre	post*	
27. I would prefer to do an experiment in mathematics than to write an essay.			pre	post*	
31. Mathematics is a very interesting subject.				pre*	post

Based on my notetaking data, Student 5, although very shy, initially enjoyed the applied mathematics course and even volunteered for my research, but as the course was getting closer to its end, Student 5 seemed to have some difficulties with some mathematical procedures (finding eigen values, equilibrium points, or even understanding the purpose of doing certain things in class etc...).

Based on my observations and pretest data, I hypothesized that Student 5 may have no-to-small increases in her posttest results for items related to enjoyment. I was attributing possible “no changes” to her struggles or lack of procedural and conceptual understandings that were evidenced at the end of the course, combined with possibly small increases, if any, to the first part of the course, where she may have enjoyed learning about the benefits of mathematical modeling.

By the end of the course, posttest data showed that Student 5 had a few one-scale positive changes for items #26, and #27 (from neutral to agree), and #31 (from agree to strongly agree), the other enjoyment items had no changes. Overall, Student 5's pre- and posttest results had no "significant" positive nor negative changes, regarding motivation nor enjoyment attitudes towards mathematics.

Prior to the interview, based on Student 5 posttest data (one-scale positive changes, specifically regarding mathematics experiments), I hypothesized that interview data for Student 5's enjoyment might be slightly positive, with no negative feedback. Similarly to some previous students, I decided to address what exactly Student 5 enjoyed about this applied mathematics experience in the following excerpt,

Excerpt 38

1 Student 5: I liked having the two instructors-

2 Researcher: The tutors?

3 Student 5: Yeah, the instructors-

4 Researcher: Two instructors, okay.

5 Student 5: It gave more opinions. They all were very smart and very helpful with the math, the models, and the projects. And they all gave us different feedback and ideas, **which sometimes was difficult because we had to pick what to do and which idea would be best for us to use**. But it was helpful so that we had more options and figured out what was best for us to use.

6 Researcher: Did you like working in groups?

7 Student 5: Yeah.

8 Researcher: Yeah?

9 Student 5: Yeah. I thought it was easier to bounce ideas and thoughts than it would be alone, because I have ... I don't know. I would probably try and figure it out on my own and I wouldn't have gotten very far by doing that [laughing].

10 Researcher: Did you like the fact that they allow you to choose your topics of interest rather than just give you, "Hey, study that."

11 Student 5: Yeah**. I thought it helps with researching so that you're more interested in the topic, and it's easier to learn about something and remember the information and present it when it's something you're more interested in, rather than just being given a topic.**

Interview data suggest that, Student 5 had a positive experience with the way the applied mathematics course was structured (two instructors, group work etc.).Although Student 5 suggested, that she found it helpful that the students were offered the freedom to pick a topic of interest [Academic Freedom], and had its benefits on the learning and remembering process, Student 5 also saw that choosing own project as making things harder, see Line 5, because she (and her group members) had to take some initiatives, by making decisions on what steps to do and what ideas to use. Student 5 preferred to be told what to do, therefore, she did not see a real benefit to choosing own project.

1.4. Summary

Test and interview data show that students enjoyed the applied mathematics course in general, including the academic freedom, citing various reasons and benefits:

While test results showed a positive shift in attitudes towards mathematics, specifically regarding the enjoyment of mathematics experiments, student interviews showed evidence of enjoying group work, others liked having multiple tutors/instructors, experiencing the college experience, doing mathematical experiments, and the ability to model real-life situations. The ability to choose their own project was another shared theme among the students, that resulted in their enjoyment and motivation towards the course. Students described that choosing their own topics of interest, was neat, made it easier to remember, easier to learn and present, it kept them motivated, and made mathematics more interesting.

2. Productive Struggle

Productive struggle is where students' struggles lead to progress and success, building confidence. However, in some uncommon cases, the challenges in the course may also decrease confidence. For all students who took part of this research, confidence was the attitude that had the most overall positive changes. Additionally, when comparing items from before and after taking this applied course, the ATMI test detected the most significant increases, in confidence-related statements. Among all students who participated in the study and accepted to be interviewed, the specific confidence items that had significant changes are listed in the following table:

Table 11

Significant Confidence ATM Changes

Positive confidence changes	11. Studying mathematics makes me feel nervous. 15. It makes me nervous to even think about having to do a mathematics experiment. 16. Mathematics does not scare me at all. 16. Mathematics does not scare me at all. 22. I learn mathematics easily. 38. I am comfortable answering questions in mathematics class. 40. I believe I am good at mathematics experiments
Negative confidence changes	37. I am comfortable expressing my own ideas on how to look for solutions to a difficult mathematics experiment.

This table shows the specific confidence items that had the most significant (at least two-interval) positive and negative changes among all students that were interviewed. Item 16 “Mathematics does not scare me at all” was selected twice, therefore repeated.

This table is evidence that the AML 100 course made the students more comfortable, less nervous or scared about mathematics, significantly increased their confidence in mathematics “experiments” and in answering questions, or simply added their confidence in learning mathematics in general.

Additional interview data shed more light into students’ struggles and how struggle helped them build their confidence, but in some cases decrease it too.

AML 100, a challenging course for the high-school level of students, covered topics that are mainly taught at the university level, therefore, student struggles were manifestly expected during the course.

Here are two stories related to student struggles, one that explains a situation where a student's struggle led to an increase in confidence (overcoming struggle), and another story of a student being overwhelmed by struggle.

2.1. The Story of Student 2

Student 2's story is about overcoming her struggles resulting in an increase in confidence. After analyzing her test results, Student 2 showed evidence of a significant increase in the following three confidence items 22, 38, and 40, see next table. She was initially neutral about learning mathematics easily, comfortable answering questions in mathematics classes, and believing she was good in mathematics experiments.

Table 12

Examples of Student 2's ATMI Results Related to Confidence (Productive Struggle)

pre: pretest result
post: posttest result
*: my hypothetical posttest results for the student

	strongly disagree	disagree	neutral	Agree	strongly agree
22. I learn mathematics easily	*		pre		post
38. I am comfortable answering questions in mathematics class			pre		post*
40. I believe I am good in mathematics experiments.			pre		post*

From my notetaking data, I witnessed an increase in Student 2's self-confidence. Student 2 started with a very shy personality, and by the end of the course, she was very comfortable asking and answering questions in class, often volunteering to participate or go to the board to present her thoughts and ideas. By the end of the course, she took on a leadership role in her group.

Initially, Student 2 was often hesitant when expressing ideas, and worried to be wrong, but by the end of the course, she was comfortable asking questions about topics she struggled with, and was no longer worried whether her answers were correct or not. Her confidence and engagement in class shifted dramatically.

Student 2 often claimed she was very proud to have finished what she called a difficult university course, being only sixteen years old, she was one of the youngest students among her classmates. She also mentioned not having taken any pre-calculus courses while most of her classmates did. Although bright, she had at times difficulties understanding concepts quickly, and required multiple explanations, that by the end of the course, she was no longer shy to ask for.

Despite her lower mathematics background and observed struggles during this course, her self-confidence significantly increased.

Two of my hypotheses regarding Student 2's ATMI confidence items where she scored positive changes in confidence, were predicted correctly, while one was not, i.e., I have (wrongly) predicted that she would no longer think she learns mathematics easily, but predicted (correctly), that she would be more comfortable answering questions in mathematics in class, and would believe she is much better at mathematics experiments after taking this course.

Prior to the interview, I hypothesized that despite observing her learning struggles, this applied mathematics course would have an overall positive impact on Student 2's confidence, and that the interview would give more data supporting this hypothesis:

Interview transcripts below show that Student 2 felt accomplished, and proud after taking this course, her confidence increased. She was excited she made it this far, without any negative experiences. Student 2 explained the positive change in learning mathematics as rooted in the conceptual understanding of things that she had to overcome, which rendered the learning process less difficult and less confusing, eventually realizing that she was much better at mathematics than she previously let herself think, which in turn increased her overall confidence towards mathematics.

Excerpt 39

1 Researcher: Okay. Describe to me how your confidence in mathematics has changed after taking this course?

2 Student 2: I know now I can brag to everyone and be like, I've taken a harder math class than you definitely. Just kidding. Just kidding. But still, I feel very accomplished after taking this course. I mean, I haven't finished it yet, but since I'm nearing the end and I'm like, wow, I've actually made it through six years-

3 Researcher: Tomorrow's the end.

4 Student 2: Or not six years, six weeks.

5 Researcher: Tomorrow's presentation day.

6 Student 2: Yeah. Tomorrow's presentation day. So, I made it this far and I haven't broken down crying.

7 Researcher: That's good.

8 Student 2: So, I have accomplished something.

In this Excerpt, Line 6, Student 2 showed evidence of her pride in making it this far, contrarily to some of her classmates who “broke down”, referring to two students who cried as a result of facing struggles, Student 2 felt accomplished, her overall experience was positive, and her confidence, despite these struggles, increased. The interview continued by discussing the specific confidence items that she had shown a significant change from before and after taking this course, see Excerpt 40:

Excerpt 40

9 Researcher: Okay, so do you remember the post and pretest that we did?

10 Student 2: Mm-hmm (affirmative).

11 Researcher: So, I would like to talk about four points actually where I saw a little bit of change. So, the first one was ‘I learn mathematics easily’.

12 Student 2: Mm-hmm (affirmative).

13 Researcher: You probably don't remember what you answered before, but you were neutral about it and now you're strongly agreeing about it. Can you tell me more?

14 Student 2: As to why that changed?

15 Researcher: Yeah. Or what you think, how did this course help you to change your opinion about that you learn mathematics easily?

16 Student 2: Well, I feel like this kind of goes into what I was saying before. I feel accomplished that I was able to learn this and almost finish it so far, because tomorrow's presentation day. I've made it this far. And so, I feel like when I was learning these things in class they seemed really hard, but as I started to get to understand them, I was like, oh wow. I’m getting this. And so, before this course I was a little like, I don’t know, maybe I’m a

little slow. Because sometimes I can be. But after I went through it I realized that I'm better at it than I let myself be.

17 Researcher: Okay. The second question is I am comfortable answering question in mathematics. So, same thing. Was ‘neutral’ and now it's ‘strongly agree’, which is positive. Can you say more, how is your level of comfort is when it comes to answering questions in mathematics changed or how this course helped you, if it did?

18 Student 2: I feel like, it kind of did. I don't know. I'm always afraid I'm going to be wrong. Whenever I do a math problem, I'm like, oh, I'm going to raise my hand. I'm going to say four, and the answer is actually 16 or something. Something simple like that. And so, I'm a very over thinker, and so when it comes to answering a simple question, I usually kind of sit there like, am I going to answer it? Should I answer it? Will I answer it? And then I end up not answering it because I'm too scared. And so I've realized that throughout this course **I started answering questions more because I had confidence to, I guess. Even when I was wrong, I was like, well, I mean, that's okay** because it's a, I guess, confusing topic. So being wrong ...

At the end of this Excerpt 40, Line 18, Student 2 explained that her level of comfort in answering questions in mathematics changed positively. During this course, she started answering questions more frequently, she developed more self-confidence even when she was hesitant or wrong, because she felt it was okay being wrong in these difficult and confusing topics.

In the following excerpt, Student 2 expressed pride in accomplishing this difficult course, a course that even undergraduates struggled with,

Excerpt 41

1 Student 2: It was okay to be wrong, if that make sense.

2 Researcher: I also noticed, in the beginning, you were a little bit shy-

3 Student 2: I was.

4 Researcher: ... and then later, you were all the time volunteering, going overboard and everywhere. This is something that I noticed while I was there. Question 40 was, "I believe I'm good at mathematics experiments." So also ‘an increase’. Can you say more? How this course maybe helped you?

5 Student 2: Yeah. Well, I feel like this course revolved around mathematical experiments. Before I was like, "Well, I don't really understand what that means." I guess I'm good at mathematical experiments. Is that two plus two is four? I mean, I get that. But I mean, after taking this course, I realized I was like, "Wow, I can do this. I can draw a model or figure out how things ... how people can interact and social influences will change things." I realized that, well, as we were doing our presentations and all of that, the professors kept telling us that this is undergrad research and there was this ...

I can't remember the name of it, but there's this group that's doing the same homelessness problem as us, and …

6 Researcher: MTBI?

7 Student 2: Huh?

8 Researcher: MTBI?

9 Student 2: **Yeah, MTBI. They were struggling with it, and they were telling us how we're doing great so far because I mean, we're getting somewhere at least. They made me feel more confident in it because, if this is something that I can do, then I guess I'm good at mathematical experiments in the end.**

The excerpt above shows that Student 2's increase in confidence attitude about being good at mathematics experiments was due to the fact that she was now considering herself confident about being able to create models, figure out interactions etc... See Line 5, where confidence was closely related to this course's mathematics modeling learned skills. She later compared her group achievements with another group's, of undergraduate students, that were part of the MTBI, who were working on a similar topic, and also struggling, see Line 9. Being able to do mathematics that undergraduates were doing, and struggling with, added to her confidence.

Student 2 ended this interview, by stating that she really enjoyed the college experience and felt more prepared to take on more mathematics college courses, which demonstrated an increase in her enjoyment and confidence attitudes towards mathematics.

Student 2's test results align with her interview data. In fact, all her significant test increases in attitudes towards mathematics from before and after taking this course, were related to her confidence attitude. Student 2 was able to overcome her learning struggles during the course of this applied mathematics experience, struggles that increased her confidence. Student 2 had no negative changes in attitudes towards mathematics in confidence or any other attitudes towards mathematics. Data confirmed my hypotheses about her increases in confidence and suggested that all her experiences were positive!

2.2. The Story of Student 3

In contrast to the previous student whose struggles led to an increase in confidence, this story describes a single student who was, in point of fact, overwhelmed by his struggles. This resulted in lessening his overall confidence. This student was also the only one, among all interviewed students, that showed a decrease in confidence. The other classmates showed significant (two-scale or more) increases in confidence.

From my notetaking data and observations, Student 3, as described in section 1.2. from Chapter 6, was timid, and rarely talked, presented or volunteered. His mentors often suggested he speaks louder and sounds more confident when presenting. Although he seemed to enjoy the course, I could not detect any increase, nor decrease in his confidence regarding this course or mathematics in general.

In his pretest, Student 3 was not too positive about his general confidence in mathematics. His confidence about learning mathematics has mainly been "neutral", no changes were recorded. However, with confidence attitude item 37, "I am comfortable expressing my own ideas on how to look for solutions to a difficult mathematics experiment”, he started with great comfort (strongly agreed), then later this comfort and confidence declined, see following table.

Table 13

Example of Student 3’s ATMI Result Related to Confidence

pre: pretest result
post: posttest result
*: my hypothetical posttest results for the student

	strongly disagree	disagree	neutral	agree	strongly agree
37. I am comfortable expressing my own ideas on how to look for solutions to a difficult mathematics experiment.	*		post		pre

As shown on this table, I hypothesized that Student 3 might have a decrease in confidence regarding mathematics experiments, see item 37, but no significant changes in other confidence items.

After seeing a decline in confidence regarding his discomfort expressing ideas related to experiments in his posttest results, I then hypothesized, prior to the interview, that this course may have not had such a positive impact on his confidence in general, but I did not expect having evidence of a decrease too.

When asked about Student 3’s confidence possible changes, he clarified in the following Excerpt,

Excerpt 42

1 Researcher: Describe to me how your confidence in mathematics has changed after taking this course. If, it did.

2 Student 3: Maybe is **lessened** it, since ..

3 Researcher: Lessened?

4 Student 3: **It was kind of confusing at first**. It's just something different, definitely.

5 Researcher: Overall do you think it increased your confidence?

6 Student 3: Yeah.

7 Researcher: Increased? Okay. Can you give me an example?

8 Student 3: Just understanding.

Student 3 suggested that his confidence in mathematics dropped, see Line 2, due to the fact that the (applied side of) mathematics he was exposed to was “just something different, definitely” and “It was kinda confusing at first”, see Line 4. In this Excerpt 42, I recognize a missed opportunity on my part - asking him to elaborate on what exactly was confusing to him. From what I can understand in this Excerpt, his confusion and the different mathematics he was exposed to, led to a decline in his confidence.

Seeing my surprised question “Lessened?”, see Line 3, Student 3 tried to explain, that his confidence lessened at first, but later, it slightly increased, when the course made more sense to him, see Line 8 for “understanding”. I then addressed the question he scored negatively, hoping to have some more clarifications. He discussed his lack of comfort change with expressing mathematics ideas, in Excerpt 43:

Excerpt 43

1 Researcher: Okay. Are you comfortable expressing your own ideas on how to look for solutions to a difficult mathematics experiment?

2 Student 3: Somewhat. Maybe.

3 Researcher: To a certain degree?

4 Student 3: Yes, **not too much**.

5 Researcher: Okay, did it change from the beginning of the course to the end of the course?

6   Student 3:   Not really.

Student 3 did not have any significant positive changes in confidence, evidenced in both his interview and test results. His struggles with understanding the (different type of) mathematics he was exposed to, and learning, being overwhelmed by these struggles led to a decrease in his confidence towards mathematics.

In addition, Student 3 had also negative changes in scores regarding "usefulness of mathematics" test items, such as items #6, #35, and #36, see following table:

Table 14

Examples of Student 3's ATMI Results Related to Math Usefulness
pre:	pretest result
post:	posttest result
*:	my hypothetical posttest results for the student

	strongly disagree	disagree	neutral	agree	strongly agree
6. Mathematics is one of the most imp. subjects for people to study.			post		pre*
35. I think studying advanced mathematics is useful.	post				pre*
36. I believe studying math helps me with problem solving in other areas.		post			pre*

As shown on this table, in his pretest, Student 3 strongly agreed with the following three items regarding the usefulness of mathematics:

#6. Mathematics is one of the most imp. subjects for people to study

#35. I think studying advanced mathematics is useful

#36. I believe studying math helps me with problem solving in other areas

Following the notes I took during lessons, despite seeing no confidence change in Student 3, I have observed how students in general started to see new ways mathematics can be used in other fields of interest, and that led me to hypothesize that Student 3 might show a small increase in "usefulness of math" posttest items, such as #, 6, 35, or 36. I was surprised to see the negative changes in his attitudes of mathematics usefulness. i.e., posttest data, see table, showed that Student 3 had three significant negative changes. As example, he no longer thought studying advanced mathematics was useful.

After seeing these results, I hypothesized that this decline in mathematics usefulness, may have been linked to his decline in confidence, as he no longer thinks of mathematics as useful to ‘him’ (personally), due to some struggles he encountered during this course which decreased his self-confidence, yet he would still think of it as useful in general, for others to study. Another reason for the decline could have been a human error, answering too fast when filling up the ATMI questionnaire. Therefore, the interview was an opportunity for me to address the specific items where the negative changes were apparent, see Excerpt 44:

Excerpt 44

1 Researcher: What do you think mathematics is useful for?

2 Student 3: For everything, I don't know. **It's useful in a lot of things**.

3 Researcher: Can you give me an example?

4 Student 3: just like all the sciences, you need math.

5 Researcher: What type of science are you thinking of?

6 Student 3: Physics and Chemistry. Even Biology too with the modeling.

7 Researcher: Okay. How has your understanding of, what mathematics is used for, changed, after taking this course?

8 Student 3: **It’s a lot more interesting seeing that you can model things** from (Instructor’s name) and the other teams’ works. **It’s pretty neat that you could do that.**

9 Researcher: Do what?

10 Student 3: **You can understand things differently through modeling**.

11 Researcher: What things? Can you say more?

12 Student 3: You could understand how with the homelessness is an issue I instead of it being just like ... I'm not sure how to say it. That you can just **use the math in a lot of ways and the modeling helps**. The different phases and stuff, that you can put it into. All the factors influence it subscribing through math.

These transcripts clearly show that Student 3 began to consider mathematics as a useful subject in general, useful in helping societies from undesirable issues such as homelessness, model situations, and analyze factors that could influence these models' outcomes. Student 3 also added,

Excerpt 45

1 Researcher: Okay. Can you say more about this sentence? "I think studying advanced mathematics is useful". Do you think it is or is not?

2 Student 3: I think it is. You could use it in a lot of different ways and it'll help society .. I don't know.. it's just good. I like math.

3 Researcher: Okay. Do you believe studying mathematics helps you with problem solving in other areas?

4 Student 3: Yes, because the math is used in other things like physics and stuff and you're just applying it in different ways.

In Excerpt 45, Student 3 suggested that studying advanced mathematics is useful, which is in contradiction to his test result answers. His interview explanations for its usefulness was aimed at others, not him, i.e., it is good for others to learn advanced mathematics because it will help society, it benefits the society, see Line 2.

When describing his views of what mathematics is useful for changed, he added,

In Excerpt 44, Line 8, Student 2 also explained that his understanding of what mathematics is used for has changed, as he finds mathematics more “interesting” (neat that you can model things), Student 3 gave examples of topics of interest other students worked on, and how factors could influence outcomes using mathematics. Clearly, his meanings of what mathematics *is*, also changed. This is discussed in further details in the next theme “Changing the meaning of what mathematics is”.

Interestingly, although Student 3 found mathematics as more useful in general (for the public), specifically after discovering the applied mathematics side of the course, and ways mathematical modeling can play a role in finding solutions to real-life problems, evidenced in his interviews, his posttest scores went down, i.e. he changed his opinions about its usefulness to him, personally, or his future.

To conclude, Student 3’s initial to mid-course “understanding” struggles he encountered in this course, contributed not only to a decrease in his confidence towards mathematics, but the latter also played a negative role in changing his beliefs and attitudes about mathematics, based on personal usefulness.

2.3. Summary

Most students’ significant positive changes in attitudes were related specifically to confidence. To these students, their struggles were productive enough, to lead to progress and success, building confidence. Yet, in one case, these cognitive (understanding and learning) struggles had a negative impact on self-confidence and personal usefulness for mathematics.

3. Changing the Meaning of "Mathematics" and Problems with the ATMI

The applied mathematics course AML 100 was a non-traditional mathematics course, that also aimed at changing students' meanings of what mathematics is. Here is a story where a student's attitude changed, but their score on the ATMI test did not.

In this perspective, the ATMI test was not a good working measure for this course, because the course's purpose was about changing the meaning of "mathematics", and the questions the students asked in their posttest were not interpreted the same as the questions in the pretest: their own meaning for the questions changed.

3.1. The Story of Student 1

This story is an example where a student's attitude towards mathematics changed, yet his ATMI scores did not:

Some of the ATMI items are related to the usefulness of mathematics, for these items, there have been no significant (+ 2-scale) changes. Below is a table that gives examples of students 1's pre and posttest selections:

Table 15

Examples of Student 1's ATMI Results Related to Math Usefulness
pre: pretest result
post: posttest result
*: my hypothetical posttest results for the student

	strongly disagree	disagree	neutral	agree	strongly agree
4. Mathematics helps develop the mind and teaches a person to think				pre*	post
5. Mathematics is important in everyday life.			pre post	*	
6. Mathematics is one of the most important subjects for people to study.				pre post	*
7. High school mathematics courses would be very helpful no matter what I decide to study.			post	pre	*

This table illustrates that Student 1's pre- and posttest selections show that both before and after the course, he equally agreed that mathematics is one of the most important subjects for people to study, and was equally neutral about mathematics being important in everyday life. Student 1 also had some one-scale changes for #4 and #7, changes that were less significant. He had a decrease specifically for high school mathematics (#7), that could be related to his changes in meaning of mathematics. High school mathematics seemed less useful, now that discovered how useful and applied college mathematics can be, i.e., he showed a loss of interest in high school mathematics after seeing the usefulness of college mathematics (other students did not), because he saw high school mathematics as too pure.

My notetaking data suggested that the students, in their class setting, started to see how mathematics may be merged with any subjects of interest, and that led me to hypothesize that Student 1 may have a significant increase in "usefulness of math" ATMI items, such as 4, 5, 6, or 7. However, this was not apparent in his test results. As table shows, there have been no significant changes in Student 1's believes about the usefulness of mathematics.

For the interview, I hypothesized that despite not having significant changes in "usefulness of math" test results, Student 1 might show evidence of newer considerations regarding mathematics usefulness, as a subject, because of his exposure to the mathematics modeling course, that was extensively applied, linking mathematics to other sciences, subjects of interest, involving real-life situations, resulting in useful real-life solutions. In this perspective, my hypothesis, as shown on Excerpt 46, was correct.

Here is an excerpt where Student 1 discussed the usefulness of mathematics, and how his view of how mathematics is used for has changed,

Excerpt 46

1 Student 1: I think mathematics is useful for understanding the kind of broader topics of kind of how ...

2 Researcher: Which topics?

3 Student 1: Broader.

4 Researcher: Broader, okay.

5 Student 1: Topics of kind of how things work or how things connect together. **When you're like connecting very specific ideas in like real-life applications**, a lot of that basis is founded on mathematics. And like architecture or science, a lot of those ideas are founded on what is mathematics. So there's a sort of knowledge that needs to be there.

6 Researcher: What type of sciences do you mean exactly?

7 Student 1: To a certain extent, a lot of them, to more extent, you have the more mathematical side like physics and chemistry. But through my experiences, you have sciences like biology, which still as you get into advanced biology, you get deeper and deeper and the mathematics is still there too, and I find that really interesting.

8 Researcher: Okay. How was your understanding of what mathematics is used for, changed, after taking this course?

9 Student 1: One of the biggest things was, how mathematics is used for modeling stuff. One of the biggest things that they were teaching us, was the difference between statistical models and mathematical models, rather than using a set of data and trying to just create a line of best fit and find the trend of the data, **a mathematical model can be used for kind of ... simulating kind of a real-life situation, and can be used in a way to understand things a lot better, such as populations, epidemics, and other things.**

10 Researcher: What type of questions can you answer? Or what would be a solution to a mathematical model? What does it represent?

11 Student 1: It would be like "How does a certain thing affect ... Like how does killing off beavers affect the tree populations?" Because all like statistical stuff, you just see over time, what has happened. Rather than, if you change something in real-life, what would happen? And how it affects it.

In the above excerpt Lines 5 and 8, Student 1 explained that mathematics is useful for modeling real-life phenomena: simulate situations, understand (population) dynamics, or epidemics, understand broader topics, and how things interconnect together. Student 1 also viewed mathematics as foundational to other sciences.

Student 1 began to think of mathematics, specifically after taking this applied mathematics course, as a much more valuable subject, in terms of its usefulness, but he also began to think of high school mathematics as less useful. This course changed his usefulness attitudes towards mathematics, it changes its meanings of what mathematics is. This is a significant change for him, since he did not know about mathematics modeling prior to taking this course.

Because Student 1's meanings have changed, the questions he asked in his posttest were not interpreted the same as the questions in his pretest, which explains the no change in his scores. In this perspective, the ATMI was not a good measuring tool for this course.

3.2. The Story of Student 2

Student 2's story carries some similarities to the previous story. There have been no significant (+ 2-scale) changes in her ATMI pre- and posttests for items related to mathematics usefulness, nor motivation.

Student 2's test results show that she equally agrees both before and after the course, with math usefulness, for example, item #6 “Mathematics is one of the most important subjects for people to study”.

From my notetaking data, I hypothesized, just like I did with the previous student, that Student 2 might have an increase in “usefulness of math” ATMI items, such as # 5, 6, or 7. However, this was not apparent in her test results. Student 2 posttest results did not show any significant changes as well.

For the interview, I hypothesized that despite not having significant changes in math usefulness ATMI items, Student 2 would show evidence of how mathematics could be viewed as more useful, after taking an applied mathematics course. In this regard, my hypothesis was also correct.

Excerpt 47

1 Researcher: […] How has your understanding of what mathematics is used for *changed* after taking this course?

2 Student 2: Let me think.

3 Researcher: … if it changed.

4 Student 2: I didn't know that applied mathematics and life sciences was a thing. I just kind of saw it as a part of the statistics group kind of thing**. But once I learned more about it, I understood more, and you can predict things and understand more deeply how you can prevent things that will happen in the future.** Like if you see a pattern and for example, before we all chose our subjects, I was going to do Alzheimer's because I don't know, that just had a personal connection to me. So when I was looking at all the data and I was like, oh wow, people can use these equations and statistics to see how it's going to increase in the new years, in the future years, I realized that if you have this information**, you know how to work harder to prevent it, or reduce it, or try**

> **and stop it, or prepare in a way that will make life easier in the future or something like that**.

The above Excerpt, Line 4, suggests that Student 2's understanding of what mathematics is used for has changed – she did not know about the applied mathematics to life sciences, that together, they could be used for prediction, deeper understanding of phenomena, or prevention/reduction of diseases such as Alzheimer, eventually making "life easier in the future".

Despite not recording any significant changes in her test result scores regarding mathematics usefulness, Student 2 views about mathematics changed. After being part of this course, she began to consider mathematics as very useful subject, especially for society goals (more than for her personal goals).

Regarding Student 2's motivation, Student 2 had no significant (+ 2-scale) changes in her pre- and posttests regarding motivation items.

Student 2's pre/posttest show that she equally agreed both before and after the course, with math motivation items, such as #33 "I plan to take as much mathematics as I can during my education", and equally strongly agreed on #34, "The challenge of mathematics appeals to me". Her motivation towards mathematics, has always been positive.

From my notetaking data, I have observed how Student 2 has started with more of a shy personality, but since became more and more motivated and engaged as the lessons went on. For example, by the end of the course, she was taking initiatives to perform tasks, and answering questions with great engagement.

But since she already agreed or strongly agreed to items regarding motivation in her pretest, such as taking advanced math courses in the future, I hypothesized that in her posttest, there would be no significant changes, i.e. there are no possible 2 interval increases to already “agree” or “strongly agree” items on the ATMI test. In this perspective, the ATMI testing was not a good working measure for this course and attitude.

For the interview, I hypothesized that despite not having significant changes in motivation related ATMI items, Student 2 would show evidence of an increase in motivation and engagement, because she felt that what she was learning and doing in class, was closely related to the usefulness of the solutions of the mathematical modeling, in the real world.

Student 2 affirmed my observations about her changes in engagement, as she was initially shy, see Excerpt 48.

Excerpt 48

1 Researcher: I also noticed, in the beginning, you were a little bit shy-

2 Student 2: **I was**.

3 Researcher: ... and then later, you were all the time **volunteering, going to the board** and everywhere. This is something that I noticed while I was there. Question 40 was, "I believe I'm good at mathematics experiments." So also ‘an increase’. Can you say more? How this course maybe helped you?

4 Student 2: Yeah. Well**, I feel like this course revolved around mathematical experiments**. Before I was like, "Well, I don't

really understand what that means." I guess **I'm good at mathematical experiments.**

In this Excerpt, Student 2 explained, that once her understanding of mathematical experiments increased, experiments that "this course revolved around", she felt she was better at experiments, and her engagement shifted positively, see Lines 2-4.

Student 2 also discussed her group's initial motivation in finding their project topic. Their motivation evolved around preventing and finding solutions to social problems, such as homelessness, so that all people, in this case, below the poverty line have a stable home.

Excerpt 49

1 Student 2: Okay. So, well, homelessness is a very heavy issue, and I don't know, that lifestyle, it's just not ideal. And well, when you come to America, it's about life, liberty and the pursuit of happiness. But when you see people on the streets, that's what they have to do every day. They have to beg for their money just so they can eat food or fulfill a certain basic need that they can't, because they don't have a job, or their life has been given up basically to substance abuse on the streets. And so**, our motivation was we wanted to help these people. We wanted to find a solution so that we could prevent it from happening to those who are below the poverty line. And so possibly we could just help everyone out of this homeless state and help them back to a stable and having a home and all the above.**

Clearly, Student 2's motivation to use mathematical modeling to find solutions to world problems was apparent, as evidenced in this Excerpt. Her willingness to take future advanced mathematics courses and increased observed classroom engagement were also evidence of her positive shifts in motivation towards mathematics, shifts that the ATMI testing, in this case, also failed to detect, because her idea of mathematical experiments changed and her meaning of mathematics also changed, her new motivation became rooted in her new idea of usefulness. Therefore, this test was not a good measure for the course and attitude.

3.3. The Story of Student 3

Student 3's story had an unexpected turn of events, his ATMI scores regarding mathematic usefulness decreased, yet his views about mathematics changed in two different ways: after taking this course, he began to think of mathematics as more useful, in general, but changed his opinion based on personal usefulness. This was briefly discussed in this chapter's section 2.2.

The following table summarizes Student 3's ATMI score decrease regarding math usefulness:

Table 16

Examples of Student 3's ATMI Results Related to Math Usefulness

pre: pretest result
post: posttest result
*: my hypothetical posttest results for the student

	strongly disagree	disagree	neutral	agree	strongly agree
6. Mathematics is one of the most imp. subjects for people to study.			post		pre*
35. I think studying advanced mathematics is useful.	Post				pre*
36. I believe studying math helps me with problem solving in other areas.		post			pre*

This table illustrates that in his pretest, Student 3 strongly agreed with some items regarding usefulness of mathematics. His posttest results had three negative changes. For example, he no longer thought studying advanced mathematics as useful at all, perhaps to him. For clarification purposes, this was addressed during his interview.

My notetaking data suggested that, despite seeing no confidence change in Student 3, students in general began to experience new ways mathematics can be used in other fields of interest, and that led me to hypothesize that Student 3 might have a small increase in “usefulness of math” items, such as # 6, 35, or 36, which was not the case, since ATMI did not have options to select positive changes to items that were previously selected as “strongly agree”. In this perspective, ATMI test did not work well, and my hypothetical answers to Student 3 items related to math usefulness, would have all remained the same “strongly agree”.

Anticipating positivity, I was surprised to see Students 3’s negative changes in his posttest results. Reflecting on these test results, prior to the interview, I hypothesized that this could have been linked to his decrease in confidence, or a due to a “rushing” error. Therefore, I decided to address these specific questions to understand more about Student 3’s views about mathematics usefulness:

Excerpt 50

1	Researcher:	What do you think mathematics is useful for?
2	Student 3:	**For everything**, I don't know. It's useful in a lot of things.
3	Researcher:	Can you give me an example?
4	Student 3:	just like all the sciences, you need math.
5	Researcher:	What type of science are you thinking of?

6 Student 3: Physics and Chemistry. Even Biology too with the modeling.

7 Researcher: Okay. How has your understanding of what mathematics is used for, changed, after taking this course?

8 Student 3: **It's a lot more interesting seeing that you can model things** from (Instructor's name) and the other teams' works. **It's pretty neat that you could do that**.

9 Researcher: Do what?

10 Student 3: **You can understand things differently through modeling.**

11 Researcher: What things? Can you say more?

12 Student 3: **You could understand how with the homelessness is an issue instead of it being just like ... I'm not sure how to say it. That you can just use the math in a lot of ways and the modeling helps. The different phases and stuff, that you can put it into. All the factors influence it subscribing through math.**

This excerpt illustrates that Student 3 considered mathematics as useful in general, see Lines 8 and 12. He explained that mathematics helps understand modeled real-life problems in societies, such as homelessness. Student 3 also found mathematics useful for “all sciences”, such as physics, chemistry, or biology with the use of mathematical modeling. He mentioned that after taking this course, his understanding of what mathematics is used for, changed. He began to find it "interesting" and "neat" that it is possible to model and better understand real-life situations.

In the following excerpt, I addressed items #35 and #36, where Student 3 had a significant decrease.

Excerpt 51 (repeated)

1 Researcher: Okay. Can you say more about this sentence? "I think studying advanced mathematics is useful". Do you think it is or is not?

2 Student 3: **I think it is. You could use it in a lot of different ways and it'll help society.** I don't know… it's just good. I like math.

3 Researcher: Okay. Do you believe studying mathematics helps you with problem solving in other areas?

4 Student 3: **Yes, because the math is used in other things like physics and stuff and you're just applying it in different ways.**

Student 3 gave different (positive) answers, citing mathematics usefulness for general purposes "help society", see Line 2. Student 3 did not mention about any math usefulness to his personal beliefs. When describing how his views of what mathematics is useful for changed, he added,

Excerpt 52 (repeated)

1 Researcher: Okay. How has your understanding of what mathematics is used for *changed*, after taking this course?

2 Student 3: It's a lot **more interesting** seeing that you can model things from (instructor's name) and the other teams' works. It's pretty **neat that you could do that**.

3 Researcher: Do what?

4 Student 3: You can understand things differently through modeling.

5 Researcher: What things? Can you say more?

6 Student 3: You could understand how with the homelessness is an issue instead of it being just like ... I'm not sure how to say it. That you can just use the math in a lot of ways and the modeling helps. **The different phases and stuff, that you can put it into. All the factors influence it subscribing through math**.

Student 3 claimed his meanings of mathematics have changed – he now finds it more "interesting", and "neat" what mathematics can do through modeling. Student 3 gave examples of social problems other students worked on, and talked about how different factors could influence outcomes, using mathematics. Although Student 3 mentioned only once "it'll help society", comparatively to Student 2 who mainly focused on finding solutions to real-life problems, Student 3 saw more value in the learning process (of mathematics), in the ability to model real-life situations and in deepening one's understanding and analysis of these modeled situations, all are values not to his own mathematics learning benefits, but to others.

Interview data showed that Student 3's value and meanings about mathematics have changed, specifically after discovering the mathematical modeling aspect that this course had to offer, and how mathematical modeling plays a role in enhancing the understandings of real-world problems.

Student 3 thought of mathematics as more useful in general. However, Student 3 changed his opinions based on personal usefulness (he did not see himself as becoming a scientist), he no longer viewed mathematics as useful to him, or his future personally, evidenced in his posttest ATMI results. As a result, Student 3's general mathematics usefulness increased, but personal usefulness decreased.

3.4. Summary

These three stories are evidence of students' changing meanings of mathematics. ATMI testing did not work well with this course because this course primarily aimed at changing meanings of mathematics, and the items these students answered in the posttest were not interpreted the same as in the pretest. This explains, that although these students' attitude changed positively, their ATMI scores failed to detect these changes.

4. Changing meanings of "doing mathematics"

Among another of this course's objectives was changing the meaning of "doing mathematics", which is different from changing meanings of "mathematics". In this section, I describe stories where a student did not change her meaning of "doing mathematics" as other students did. As a result, she received less benefit from this course.

4.1. Story 1 of Student 5

In this story, Student 5's confidence in mathematics increased:

In her pretest, Student 5 had some positive shifts, regarding her confidence in mathematics, as she no longer believed that mathematics made her feel nervous, and she began to agree that mathematics does not scare her at all, two significant increases in her confidence attitude, see following table.

Table 17

Examples of Student 5's ATMI Significant Changes in Confidence Items

pre: pretest result
post: posttest result
*: my hypothetical posttest results for the student

	strongly disagree	disagree	neutral	agree	strongly agree
11. Studying mathematics makes me feel nervous.	post	*	pre		
16. Mathematics does not scare me at all.		pre		post*	

My notetaking data also suggested an increase in Student 5 self-confidence: for example, Student 5 was very comfortable taking difficult (group) mathematical tasks, such as doing research for the group. She was confident about her mathematical skills in general and did not feel nervous in her mathematics course. Her confidence and engagement in class shifted positively. Despite encountering some struggles towards the middle-end with certain newly learned topics and procedures, such as finding equilibrium points, eigenvalues, she believed she was still good at mathematics.

For confidence items where Student 5 showed an increase, I hypothesized "disagree" for item #11, and "agree" for item #16, see previous table, which both would show an increase in confidence.

For the interview, I hypothesized that this course would have a positive impact on Student 5's confidence in mathematics, but the interview unfolded differently:

Excerpt 53

1 Researcher:… Describe to me how your confidence in mathematics has changed after taking this course.

2 Student 5: [pause] **I don't know that it has changed**. I've always-

3 Researcher: Is the confidence the same, improved, or decreased, after taking this course?

4 Student 5: **I think it's the same**.

5 Researcher: The same?

6 Student 5: **Yeah.**

In Excerpt 53, Student 5 whose confidence items significantly increased in her posttest results, suggested during her interview that her confidence had no changes, see Lines 4 and 6. Despite positive posttest ATMI scores regarding her confidence, Student 5 believed her confidence remained the same "I think it (confidence) is the same". Therefore, I decided to address the test items that she showed changes in confidence:

Excerpt 54

7 Researcher: Okay. So, now let's look at the answers that you gave before and after, and then let's compare them. So, "mathematics does not scare me at all", this is question number 16. So, in pretest, you probably don't forget that you said you disagreed and then you agreed. The question is negative. So, "mathematics does not scare me at all", you disagreed and then you agreed. So, I guess it scared you a little bit in the beginning and then does not scare you

anymore. Can you say more about it? How the course changed your way of being scared of mathematics, if it did.

8 Student 5: **I think seeing the math we were going to be doing in the very beginning and them showing us was kind of scary, because I didn't see how you could use it that way and apply it. And now, going through the course and learning how to actually apply it and use it that it's not as scary.**

9 Researcher: This was given before you started.

10 Student 5: Oh.

11 Researcher: So, you probably, just the topic of that time. So, was it scary in the beginning before you took the class? Were you scared or...

12 Student 5: I mean, sometimes. I mean, I know calculus and future math classes are going to be difficult and-

13 Researcher: So, you were just thinking that it might be difficult. So, that's why you were a little bit scared?

14 Student 5: Yeah.

15 Researcher: Is that what you mean exactly because I just wanna make sure\ that I'm understanding you correctly.

16 Student 5: [Nodding her head many times]

17 Researcher: Another question is, “studying mathematics makes me feel nervous”. Do you see any change before and after taking this class, when it comes to this question?

18 Student 5: I don't know.

19 Researcher: Does it make you nervous, mathematics? Studying mathematics?

20 Student 5: **Not...**

21 Researcher: A little? a lot? It's okay. Whatever you think, it's fine!

22 Student 5: I don't know. **It can**.

23 Researcher: It can? Okay.

24 Student 5: **It can just be difficult** [inaudible] a lot of things with it.

Excerpt 54 illustrated that Student 5 also anticipated the course to be difficult and scary. She explained that, studying mathematics can make her feel nervous, if the mathematics (course) is difficult, and she added that, it scared her at the beginning. She explained, “(it) was kind of scary, because I didn't see how you could use it that way and apply it. And now, going through the course and learning how to actually apply it and use it, that it's not as scary”, see Line 8.

In this story, my analysis of Student 5’s test results and interview led me to conclude that although she was not cognizant about her confidence changes, her confidence in mathematics, did in fact increase, because she saw how mathematics could be useful: Student 5 started to see a new purpose for mathematics, but her idea of how mathematics is used did not change. Her confidence increased because she thought formulas could not be used to solve these kinds of problems. Later, she realized that she could use formulas better.

From the perspective of this student, she discovered that there were new formulas that other people could memorize and use to analyze situations that mattered to them, and that she could do it and follow those formulas, when she previously thought they would be too complicated “difficult”.

From this perspective, it is not really discovering that mathematics is “useful” that increased her confidence. Instead, discovering that “useful” mathematics became accessible to her. She had no confidence in her ability to follow rules. She thought that because the problems were complicated, the rules would be too complicated for her to do and discovered that they were not.

4.2. Story 2 of Student 5

Here is a story, where Student 5’s meaning of mathematics usefulness did not change, or not significantly enough:

Test results showed that Student 5 had no significant changes in ATMI items related to mathematics usefulness. Here are some one-scale changes, see next table. For example, Student 5 believed, later strongly believed, that mathematics helps develop the mind and teaches a person to think.

Table 18

Examples of Student 5's ATMI Test Results Related to Math Usefulness

pre: pretest result
post: posttest result
*: my hypothetical posttest result for the student

	strongly disagree	disagree	neutral	agree	strongly agree
4. Mathematics helps develop the mind and teaches a person to think.				pre	post*
6. Mathematics is one of the most important subjects for people to study.			pre	post	*

From my notetaking data, I hypothesized that Student 5 might have an increase in “usefulness of math” ATMI items, which was not significant.

Prior to the interview, I also hypothesized that despite not having significant changes in usefulness of math ATMI questions, during the interview, Student 5 might show evidence of how mathematics is now being seen as more useful after taking an applied mathematics course to other fields.

Excerpt 55

1 Researcher: Okay. So, I'm going to go to the next set of questions. It's about your attitude towards mathematics in general. So, what do you think mathematics is useful for?

2 Student 5: Math is used in almost everything. Like to make different things, you have to just use math to create... I don't know, develop it.

3 Researcher: Can you give me an example what things mathematics is useful for?

4 Student 5: like… Buildings. Architects. They have to...

5 Researcher: Architecture.

6 Student 5: **Use formulas**, yeah.

7 Researcher: Okay. What else?

8 Student 5: (pause)

9 Researcher: So, you said “everything”, so you can give me at least three examples [laughing]

10 Student 5: [laughing] It's hard to like think of just things**. A lot of science you have to use formulas. You have to do math for it.**

11 Researcher: Okay. Like what type of sciences?

12 Student 5: **I know in chemistry you have different ratios, so you have to use math to figure out how much of each chemical you need. Physics, there's a lot of formulas and different calculations you have to do.**

13 Researcher: Okay. So, we got three examples here. How was your understanding of what mathematics is used for *changed*, after taking this course?

14 Student 5: I didn't know you could apply it in this biological way, I guess. So, taking a population size or a population, and putting it through, I guess, a model kind of like ours or any model.

15 Researcher: So, you didn't think that mathematics can be applied to other sciences? Like other sciences, not like physics and chemistry, but like studying the populations and stuff, right?

16 Student 5: Kind of. Yeah.

Although Student 5 suggested that her understanding of what mathematics is used for has changed, because she did not know mathematics could be applied "in this biological way", her discussions about how she views mathematics as useful for, revolved around its connection and usefulness to other subjects such as architecture, chemistry, and physics, where there is a need to use of formulas, ratios and calculations. Student 5 still thought that mathematics was mainly about formulas and calculations, even though this goes in contradiction with what students were doing in class. Therefore, in contrast to students 1, 2 and 3, Student 5's meanings of what mathematics is useful for, did not appear to have changed after taking this course, which is in accordance with her test results.

4.3. Story 3 of Student 5

In this story, Student 5's enjoyment attitude towards mathematics did not have significant positive changes. While she enjoyed certain aspects related to the course structure, she did not see a benefit in choosing her own project.

From my notetaking data, I observed how Student 5 enjoyed the AML 100 course, but as students were getting closer to the mid-end of the course, Student 5 started to have some difficulties with some mathematical procedures (finding eigen values, equilibrium points etc..). I hypothesized that this may contribute to a lack of positive changes in enjoyment of the mathematics course.

In her posttest results, Student 5 did not have significant changes. Her results about enjoyment of mathematics were predominantly neutral, such as for items #26, and #27, and later gained a 1-scale increase, see next table.

Table 19

Examples of Student 5's ATMI Test Results Related to Math Usefulness
pre: pretest result
post: posttest result
*: my hypothetical posttest result for the student

	strongly disagree	disagree	neutral	agree	strongly agree
26. I like to do new experiments in mathematics			pre	post*	
27. I would prefer to do an experiment in mathematics than to write an essay.			pre	post*	
31. Mathematics is a very interesting subject.				pre*	post

I hypothesized that she would have no to small increase (0 to 1-scale increase) in post ATMI test for items related to the enjoying of mathematics attitude.

This table shows that there were no significant (2-scale) changes from pre and posttest items regarding enjoyment. She had 1-scale changes for items related to math experiments, such as #26, and #27 (from neutral to agree), and #31 (from agree to strongly agree).

I decided to address what she enjoyed about this applied mathematics experience at the end of the interview. I hypothesized that interview results for [enjoyment] would be slightly positive, with no negative feedback.

Excerpt 56

1 Student 5: I liked having the two instructors-

2 Researcher: The tutors?

3 Student 5: Yeah, the instructors-

4 Researcher: Two instructors, okay.

5 Student 5: It gave more opinions. They all were very smart and very helpful with the math, the models, and the projects. And they all gave us different feedback and ideas, **which sometimes was difficult because we had to pick what to do and which idea would be best for us to use**. But, it was helpful so that we had more options and figured out what was best for us to use.

6 Researcher: Did you like working in groups?

7 Student 5: Yeah.

8 Researcher: Yeah?

9 Student 5: Yeah. I thought it was easier to bounce ideas and thoughts than it would be alone, because I have ... I don't know. I would probably

try and figure it out on my own and I wouldn't have gotten very far by doing that [laughing].

10 Researcher: Did you like the fact that they allow you to choose your topics of interest rather than just give you, "Hey, study that."

11 Student 5: Yeah. I thought it helps with researching so that you're more interested in the topic, and it's easier to learn about something and remember the information and present it when it's something you're more interested in, rather than just being given a topic.

This excerpt also shows that Student 5 did not see an advantage to choosing own project, see Line 5 of the transcript, Student 5 claimed, "… which sometimes was difficult because we had to pick what to do and which idea would be best for us to use.". Rather than exploring her own and her group's curiosity, taking initiatives, and together engaging in tasks, to develop solutions, that initiates from them, Student 5 wanted to be told what to do, evidence that she did not change her understandings of what "doing mathematics" meant to her. For this student, this applied mathematics course did not have an effect in developing (her) new meanings of "doing mathematics".

4.4. Story 4 of Student 5

In this story, Student 5 did not express any changes in understanding of what it means to do mathematics, and had no positive changes in motivation towards mathematics.

From my notetaking data, I observed how Student 5 has started with more of a timid personality, and became more engaged regarding her group's project presentation, but less engaged regarding the mathematical (procedural) part of the project. Since

Student 5 has already agreed to items regarding motivation in her pre-test, such as taking advanced math courses in the future, I hypothesized there would be no significant positive nor negative changes.

In her test results, Student 5's motivation items have already been positive and had no significant changes. For example, Student 5 strongly agreed before and after taking this course with #33 "I plan to take as much mathematics as I can during my education".

For the interview, I also hypothesized that because there were no significant changes in motivation ATMI items, Student 5 would not show evidence of these changes in motivation, nor in changes in understanding what it means to do mathematics.

Excerpt 57

1 Researcher: Do you think the course changed your way of thinking about studying mathematics? [pause] A little?

2 Student 5: I'm not sure.

This excerpt also hints to the fact that Student 5 was hesitant about thinking to study mathematics any further. Her interview answer regarding changes in ways of thinking about studying mathematics was "I'm not sure", see Line 2, but after the interview was performed, Student 5 expressed not preferring to take any future advanced mathematics courses, due to the fact that this course was difficult. Interestingly, the difficulty of this course played a role in increasing her confidence but not in changing her motivation. Student 5's motivations in tests and interviews did not show any significant changes, while post-interview discussions revealed, (although contradicting her ATMI answer), that she was not willing to take any advanced mathematics courses, because this course was too difficult for her. Overall, her understanding of what it means to do mathematics has not changed after taking this course, which in turn, explains her lack of motivation towards mathematics.

4.5. Summary

The stories in this section evolved around the same student ‘Student 5’. The applied mathematics course she had taken, had a different impact on her attitudes: her confidence in mathematics increased. Yet, her meaning of how mathematics can be useful, did not change.

Student 5 still thought of mathematics as being all about formulas, even though this course was aiming at changing these views. She also saw no benefit to choosing own project, because she wanted to be told what to do.

Student 5’s confidence came from learning that she had the ability to do the mathematics that mentors told her to do, while other students’ idea of usefulness came from discovering, that they could use mathematics to model situations that they wanted to do, and were interested in.

Contrarily to other students’ perspectives, who found a new way of using mathematics to study topics they were interested in and learn about, ways to make formulas and explore real-life problems, not just follow rules, their whole idea of doing mathematics changed in a way, that Student 5’s did not.

As a result, because Student 5 did not change her understanding of what “doing mathematics” meant, as a result, she received less benefit from his course.

CHAPTER 7

CONCLUSION

Although challenging, the AML 100 course was great at introducing new concepts for high-school students. Mathematical modeling came to help bridge the gulf between reasoning in the mathematics class and reasoning about situations in the real world. It gave life to the pure high-school mathematics they previously learned, resulting in positive shifts in their skills and attitudes towards mathematics.

Results indicated that the applied mathematics course had a positive impact on several students' attitudes, such as, self-confidence, meanings of what mathematics is, and their perceptions of what solutions are. It also had a positive impact on several skills, such as translating real-life situations to mathematics via flow diagrams, translating the models' solutions back from mathematics to the real world, and interpreting graphs. Students showed more positive results when the context of their problems was applied, or pure but graphical, and fewer improvement on problems that were not.

During this applied mathematics course, students worked primarily with discrete deterministic, theoretical models. When reasoning about these mathematical models that represented real life situations, some students thought of mathematics and the real world as separate, while others thought of mathematics and the real world, or biology, as interleaved. Students changed their perceptions of solutions too (Rasmussen, 2001), as they began to think of solutions, not as numbers or functions, but as equilibrium solutions, behavioral and qualitative solutions that parallel with theoretical models' description.

Their academic freedom involved choosing their own projects (Carlos Castillo-Chavez, Carlos Castillo-Garsow, 2007), often environmental or social, and the reversal of hierarchy resulted in their autonomy (Carlos Castillo-Chavez, Carlos Castillo-Garsow, 2007, Kamii 1985) which also added to the positive impacts on students attitudes towards mathematics, specifically on their enjoyment of mathematics and motivation towards it. Both Harel (2013) and Gravemeijer (2008) highlighted the importance of a problem first learning approach, where a problem is given before the tools needed to solve it, therefore, the students need to learn the tool or reinvent it, in order to solve the problem. Following this approach, the tool becomes useful and necessary to solve problems that students deeply care about, instead of a skill to memorize and practice. Following this perspective, the AML 100 course used a similar approach – the students studied a biological problem, that then required learning mathematics, in order to solve their problem, so the students became highly motivated to care about the mathematics.

Among the most emerging changes in attitudes that surfaced during this research study, were the mathematics enjoyment attitudes that resulted from the academic freedom, utility of mathematics, and confidence increases due to productive struggles, but in some cases confidence decreases. This course also had an impact on most students' meanings of mathematics.

Regarding the attitudes towards mathematics, I did not expect self-confidence to be the major change in their attitudes. I discovered how struggle increased students' confidence, but in some cases decreased it too, early instructor intervention and support would be highly recommended.

Student interviews also showed evidence for group work enjoyment, having multiple instructors and tutors, experiencing the college experience, learning about mathematical experiments, and modeling real-life situations. Data showed that the students' ability to choose their own project resulted in their enjoyment and motivation towards the course. Students described that choosing their own topics of interest kept them motivated and made mathematics more interesting.

Most students' meanings of what mathematics is, changed. They discovered the large-scale usefulness of mathematics, and the unlimited possibilities that real-life applications beyond their high school mathematics could offer, linking mathematics to other subjects of interest. Also, most students' meanings of doing mathematics also changed, mathematics was no longer all about formulas.

On the negative side, I was surprised that this course also frightened a student who was no longer looking at taking future advanced mathematics courses (felt overwhelmed with the challenges this course had to offer). Here, my recommendations involve attending to students' challenges within the first weeks, to be able to address them quickly, frequent one-to-one instructor-student meetings with each student, including timid ones, who would not volunteer or reach out for help when necessary (to avoid missing shy students, such as Student 5). This student had difficulties because her idea of math did not change. She was shy, did not ask for help and did not perform as well as the other students at the end.

I recommend for the instructors to spend time explaining the purpose of this course, give more examples of what mathematical modeling can do, so that this applied mathematics course makes more sense to students who initially feel lost or confused, such as Student 3 who showed a decrease in confidence (potentially) because his personal relationship with mathematics changed, and he did not see himself as becoming a scientist.

Among the most important emerging mathematical skills in this study, were the major modeling stage skills, described by several authors in the literature chapter, such as (1) equation writing and explanation, (2) graphical analysis and interpretation within real-life contexts, and (3) model result interpretation. Moreover, applied or graphical problems had higher positive outcomes than pure mathematics problems.

Regarding changes in students' skills, I did not expect the students to have the highest increase in test scores to be graph-related problems. Although most students did well when interpreting graphs within real world contexts recognizing equilibrium points, reducing the persistent shape thinking (Moore & Thompson, 2015) would be a useful teacher recommendation for improvement. This can be achieved by interpreting graphs point by point to reach a general conclusion, i.e., reasoning co-variationally (Carlson et al., 2001) which is by attending at how two quantities are changing in tandem. Increasing stability related examples would a great idea too, since many students had issues with stability recognition.

I also did not expect that all students would be able to formulate their mathematical equations for their models, however, data showed that this developed ability, necessitated the involvement of flow diagrams, which reminds me of Carlson and Bloom's (2005) problem solving paper where they emphasized the importance of drawing diagrams and sketching graphs during the orienting cycle (sense making), when describing the problem-solving behaviors of twelve mathematicians as they completed several mathematical tasks. Dray & Manogue (2008) also emphasized the importance of diagrams. In their paper, they stressed on the importance of diagrams, highlighting an important idea that mathematicians teach algebra while physicists do geometry. Mathematicians are mainly interested in rules (axioms and theorems) neglecting the importance of geometry and diagrams. Scientists, on the other side, think primarily in diagrams (geometric diagrams, or temporal diagrams like flow diagrams). As a result, students often need the diagrams that mathematicians tend to avoid. Here, my recommendation is to emphasize on diagrams in all modeling courses, and adjust Figure 2's components, by adding a new circle "Draw a diagram" to the GAIMME modeling process, see Figure 2, right before "Do the math" stage.

Students showed more positive results when the context of their problems was applied or graphical, and fewer improvement on problems that were not. There was very little improvement related to function notation (pure non-graphical problems), but students did improve in function concepts which some of them used on function notation problems, such as solving function composition problems by variable substitution). Similarly, many students did not know about inverse functions, but were able to reverse a process. Although this course aimed at applied contexts, it would be productive to add some pure problems that this course did not address, such as function notation (Oehrtman, Carlson, & Thompson, 2008). These authors suggested that students would benefit from explicit efforts to promote their understanding of function notation (p. 39).

I was pleasantly surprised at how much these students changed from the beginning and end of this challenging AML 100 course for the amount of skills that they developed within a short period of six-weeks. This modeling course can be for students who have skills, or to motivate the study of skills for students who do not have these skills.

I have described how my thinking and expectations changed during the course of this study. Courses such as these, necessitate small groups and intense environments, that allow research to identify emerging themes and issues. While it focused at understanding changes at the individual level, generating models of student thinking, and understanding each student's changes in skills and attitudes by forming common themes, there is a need for further future research to explore, for example, the impact of applied mathematics courses on a larger set of students, where the focus would be on a classroom level, rather than on individuals.

My other recommendation is to teach similar mathematical modeling at the pre-university level, taking in consideration ways to improve some of the negative outcomes that emerged in this study, to encourage the development of young students through educational, research and similar mentorship activities, to increase their inspiration and interest in mathematics, and possibly consider a variety of sciences, technology, engineering and mathematics-related fields and careers.

In order to accomplish this, teachers need to be better trained, so that they develop knowledge and experience necessary for them to teach programs like this one in their schools. Teachers and future teachers should be taking modeling classes themselves and learning about the benefits that these courses and perspectives offer their students.

REFERENCES

Afari, E. (2013). Examining the factorial validity of the attitudes towards mathematics inventory (ATMI) in the United Arab Emirates: Confirmatory factor analysis.

Aiken, L. R. (1960). *Mathemaphobia and mathemaphilia: An analysis of personal and social factors affecting performance in mathematics* (Doctoral dissertation, University of North Carolina at Chapel Hill).

Aiken Jr, L. R. (1976). Update on attitudes and other affective variables in learning mathematics. *Review of educational research*, *46*(2), 293-311.

Allman, E. S., Allman, E. S., & Rhodes, J. A. (2004). *Mathematical models in biology: an introduction*. Cambridge University Press.

Asante, K. O. (2012). Secondary students' attitudes towards mathematics. *IFE Psychologia: An International Journal*, *20*(1), 121-133.

Bassok, M., & Olseth, K. L. (1995). Object-based representations: Transfer between cases of continuous and discrete models of change. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *21*(6), 1522.

Ben, M. F. (1991). Attitudes and mathematics. *Comparative education*, *27*(2), 135-151.

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology*, *29*(3), 389-399.

Bielaczyc, K., Pirolli, P., & Brown, A. L. (1994, April). Collaborative explanations and metacognition: Identifying successful learning activities in the acquisition of cognitive skills. In *Proceedings of the sixteenth annual Conference of the Cognitive Science Society* (pp. 39-44). Lawrence Erlbaum Associates, Inc.

Brauer, F., Castillo-Chavez, C. (2001). Mathematical Models in Population Biology and Epidemiology. Texts in applied mathematics 40. Springer-Verlag, New York.

Camacho, E. T., Kribs-Zaleta, C. M., & Wirkus, S. (2013). The mathematical and theoretical biology institute–a model of mentorship through research. *Mathematical Biosciences and Engineering*, *10*(5/6), 1351-1363

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *CBMS Issues in Mathematics Education* (pp. 114-162). Providence, RI: American Mathematical Society.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational studies in Mathematics*, *58*(1), 45-75.

Carlson, M. P., Larsen, S., & Jacobs, S. (2001). An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation *Proceedings of the Twenty-third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 517-533). Snowbird, UT.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal of Research in Mathematics Education, 33*(5), 352-378.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113-145.

Castillo-Chávez, C. (2009). Undergraduate Mathematics: Promising Recruitment and Retention Strategies to Ensure Diversity in the STEM Pipeline. Retrieved from http://mcmsc.asu.edu/files/CastilloChavez_Congressional%20Briefing%2009-22-09.pdf

Castillo-Chavez, C., Castillo-Garsow, C. W., Chowell, G., Murillo, D., & Pshaenich, M. (2007). Promoting research and minority participation via undergraduate research in the mathematical sciences. MTBI/SUMS-Arizona State University. In *Proceedings of the Conference on Promoting Undergraduate Research in Mathematics"(ed. Joseph Gallian), American Mathematical Society, Washington, DC* (pp. 15-22).

Castillo-Garsow, C. (2010). Teaching the Verhulst Model: A Teaching Experiment in Covariational Reasoning and Exponential Growth. Unpublished Doctoral dissertation, Arizona State University.

Castillo-Garsow, C., & Castillo-Chavez, C. (2016). Why REUs Matter. In *Directions for Mathematics Research Experience for Undergraduates* (pp. 125-145).

Castillo-Garsow, C., Castillo-Chavez, C., & Woodley, S. (2013). A preliminary theoretical analysis of a research experience for undergraduate's community model. *PRIMUS*, *23*(9), 860-880.

Cifarelli, V. V., & Sevim, V. (2014). Examining the Role of Re-Presentation in Mathematical Problem Solving An Application of Ernst von Glasersfeld's Conceptual Analysis. *Constructivist Foundations*, *9*(3).

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education,* 547-589.

Creswell, J. W. (2009). Research Design: Qualitative, quantitative, and mixed methods approaches. Thousand Oaks, CA: SAGE Publications, Inc.

Dan, Q., & Xie, J. (2011). Mathematical modelling skills and creative thinking levels: An experimental study. In *Trends in teaching and learning of mathematical modelling* (pp. 457-466). Springer, Dordrecht.

Davis, J. D. (2005). Connecting procedural and conceptual knowledge of functions. *Mathematics Teacher*, *99*(1), 36-39.

Dean, T., & Leach, S. (1995). Introduction to Learning Dynamical Systems. Retrieved From http://cs.brown.edu/research/ai/dynamics/tutorial/Documents/DynamicalSystems.html

Dew, K. H., & Galassi, J. P. (1983). Mathematics anxiety: Some basic issues. *Journal of Counseling Psychology*, *30*(3), 443.

Dray, T., & Manogue, C. A. (2008, July). Bridging the gap between mathematics and physics. In *11th international congress on mathematics education, Topic Study Group* (Vol. 16).

Dubinsky, E. (1991). Reflective abstraction in mathematical thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers.

Dubinsky, E. (2000) Using a theory of learning in college mathematics courses. *Teaching and Learning Undergraduate Mathematics, 12,* 10-15. Retrieved from <http://ltsn.mathstore.ac.uk/newsletter/may2001/pdf/learning.pdf>

Dubinsky, E., Czarnocha, B., Prabhu, V., & Vidakovic, D. (1999). One theoretical perspective in undergraduate mathematics education research. In *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 65-73).

Dubinsky, E. and Harel, G. (1992). The nature of the process conception of function. In G. Harel and E. Dubinsky (Eds.) *The concept of function: aspects of epistemology and pedagogy*, 85-106. A publication of the Mathematical Association of America: MAA Notes.

Edwards, C. J. (2012). *The historical development of the calculus*. Springer Science & Business Media.

Edwards, D., & Hamson, M. (1996). *Mathematical modelling skills*. Macmillan International Higher Education.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. Cognition & Instruction, 25(4), 439-478.

Escontrias Jr, G. (2012). Enhancing the Math and Science Experiences of Latinas and Latinos: A Study of the Joaquín Bustoz Math-Science Honors Program. Doctoral dissertation, Arizona State University.

Evangelista, A. M. (2015). Characterization of the Mathematical Theoretical Biology Institute as a Vygotskian Holzman Zone of Proximal Development. Unpublished Doctoral dissertation, Arizona State University.

Fennema, E., & Sherman, J. A. (1976). Fennema-Sherman mathematics attitudes scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal for research in Mathematics Education*, *7*(5), 324-326.

Frejd, P. (2014). *Modes of mathematical modelling: An analysis of how modelling is used and interpreted in and out of school settings* (Doctoral dissertation, Linköping University Electronic Press).

Freudenthal, H. (2006). *Revisiting mathematics education: China lectures* (Vol. 9). Springer Science & Business Media.

GAIMME (2016). *Guidelines for assessment & instruction in mathematical modeling education*.

GAIMME (2019). *Guidelines for Assessment and Instruction in Mathematical Modeling Education, Second Edition,* Sol Garfunkel and Michelle Montgomery, editors, COMAP and SIAM, Philadelphia.

Galbraith, P. (1987). Modelling-teaching modelling. *The Australian Mathematics Teacher*, *43*(4), 6-9.

Galbraith, P. L., & Clatworthy, N. J. (1990). Beyond Standard Models--Meeting the Challenge of Modelling. *Educational Studies in Mathematics*, *21*(2), 137-63.

Gilmer, T. C. (2007). An Understanding of the Improved Grades, Retention and Graduation Rates of STEM Majors at the Academic Investment in Math and Science (AIMS) Program of Bowling Green State University (BGSU). *Journal of STEM Education,* 8(1-2).

Glasersfeld, E. von (1995). *Radical Constructivism: A way of knowing and learning:* London: The Falmer Press.

Goldin, G. A. (1997). Chapter 4: Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education. Monograph*, 40-177.

Goolsby, C. B. (1988). Factors affecting mathematics achievement in high risk college students. Research and Teaching in Developmental Education., 4(2), 18-27

Gravemeijer, K. P. (1994). *Developing realistic Mathematics Education (Ontwikkelen van realistisch reken/wiskundeonderwijs)*.

Gravemeijer, K. (2008). RME theory and mathematics teacher education. In *The Handbook of Mathematics Teacher Education: Volume 2* (pp. 283-302). Brill Sense.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111-129.

Gurin, A., Jeanneret, G., Pearson, M., Pulley, M., Salinas, A., & Castillo-Garsow, C. (2017). The Dynamics of Math Anxiety as it is Transferred through Peer and Teacher Interactions.

Haines, C., Crouch, R., & Davis, J. (2001). Understanding students' modelling skills. In *Modelling and mathematics education* (pp. 366-380). Woodhead Publishing.

Haladyna, T., Shaughnessy, J., & Shaughnessy, J. M. (1983). A causal analysis of attitude toward mathematics. *Journal for Research in mathematics Education*, 19-29.

Harel, G. (2013). Intellectual need. In *Vital directions for mathematics education research* (pp. 119-151). Springer, New York, NY.

Hauge, S. K. (1991). Mathematics Anxiety: A Study of Minority Students in an Open Admissions Setting.

Hauger, G. S. (1995). Rate of change knowledge in high school and college students. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco (April).

Hauger, G. S. (1997). Growth of knowledge of rate in four precalculus students. Paper presented at the Annual Meeting of the American Educational Research Association, Chicago (March 24-28).

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal for research in mathematics education*, 33-46.

Hernández, M. L., Levy, R., Felton-Koestler, M. D., & Zbiek, R. M. (2017). Mathematical modeling in the high school curriculum. *Mathematics Teacher*, *110*(5), 336-342.

Hrabowski III, F. A., & Maton, K. I. (1995). Enhancing the success of African-American students in the sciences: Freshman year outcomes. *School Science and Mathematics*, *95*(1), 19-27.

Hyde, J. S., Fennema, E., Ryan, M., Frost, L. A., & Hopp, C. (1990). Gender comparisons of mathematics attitudes and affect: A meta-analysis. *Psychology of women quarterly*, *14*(3), 299-324.

Ignacio, N.G., Nieto, L.J. B. & Barona,E.G., (2006). The affective domain in mathematics learning, international electronic Journal of Mathematics Education 1(1), 16-32.

Ikeda, T., & Stephens, M. (1998). Some Characteristics of Students' Approaches to Mathematical Modelling in Curriculum Based on Pure Mathematics. *Journal of science education in Japan*, *22*(3), 142-154.

Joubert, R. W. E., & Hargreaves, A. T. (2009). Reflective journals in service learning: a window into assessing learning and change in students' attitudes. *TD: The Journal for Transdisciplinary Research in Southern Africa*, *5*(2), 248-261.

Kamii, C. (1985). Leading primary education toward excellence: Beyond worksheets and drill. *Young children*.

Kaput, J. (1992). Patterns in students' formalization of quantitative patterns. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy. *MAA Notes, 25.* Washington D.C.: Mathematical Association of America.

Kapur, J. N. (1982). The art of teaching the art of mathematical modelling. *International Journal of Mathematical Education in Science and Technology*, *13*, 185-192.

Kline, R. B. (1998). Principle and practices of Structural Equation Modelling. New York, NY: The Guildford Press.

Köğce, D., Yıldız, C., Aydın, M., & Altındağ, R. (2009). Examining elementary school students' attitudes towards mathematics in terms of some variables. *Procedia-Social and Behavioral Sciences*, *1*(1), 291-295.

Kulm, G. (1980). Research on mathematics attitude. *Research in mathematics education*.

Linn, M. C., & Hyde, J. S. (1989). Gender, mathematics, and science. *Educational researcher*, *18*(8), 17-27.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form*. New York, NY: Bellevue Literary Press.

Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal of research in mathematics education, 30(5),* 520.

Ma, X., & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for research in mathematics education*.

Majeed, A. A., Darmawan, I. G. N., & Lynch, P. (2013). A confirmatory factor analysis of attitudes toward mathematics inventory (ATMI).

Maton, K. I., Hrabowski III, F. A., & Schmitt, C. L. (2000). African American college students excelling in the sciences: College and postcollege outcomes in the Meyerhoff Scholars Program. *Journal of Research in Science Teaching: The Official Journal of the National Association for Research in Science Teaching*, *37*(7), 629-654.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. *Handbook of research on mathematics teaching and learning*, *1*, 575-596.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of Epistemology and pedagogy, MAA Notes, Vol. 25*, 175-193. Washington, DC: Mathematical Association of America.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education. I. CBMS Issues in Mathematics Education* (pp. 139-168). Providence, RI: American Mathematical Society.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *proceedings of the 18th meeting of the MAA special interest group on research in undergraduate mathematics education* (pp. 782-789). Pittsburgh, PA: RUME.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston: National Council of Teachers of Mathematics. National Council of Teachers.

National Science Foundation (2011). President Obama Honors Outstanding Science, Math and Engineering Mentors. Press Release 11-246. December 14, 2011.

Nicholls, J. G., Cobb, P., Wood, T., Yackel, E., & Patashnick, M. (1990). Assessing students' theories of success in mathematics: Individual and classroom differences. *Journal for Research in Mathematics education*, 109-122.

Niss, M. (1988). Theme group 3: Problem solving, modeling, and applications. In *Proceedings of the sixth International Congress on Mathematical Education. Budapest, Hungary: János Bolyai Mathematical Society, 237_252*.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics,* 150-171. Washington, DC: Mathematical Association of America.

Patton, M. Q. (1987). *How to use qualitative methods in evaluation* (No. 4). Sage.

Piaget, J. (1973). Psychology and epistemology.

Pollak, H. O. (2011). What is Mathematical Modeling? *Journal of Mathematics Education at Teachers College*, *2*(1).

Powell, S. G., & Baker, K. R. (2009). *Management science: The art of modeling with spreadsheets*. Wiley.

Randhawa, B. S., Beamer, J. E., & Lundberg, I. (1993). Role of mathematics self-efficacy in the structural model of mathematics achievement. *Journal of educational psychology*, *85*(1), 41.

Rasmussen, C. L. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20,* 55-87.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition. *Cognitive science and mathematics education*, *189*, 215.

Schoenfeld, A. H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. *Informal reasoning and education*, 311.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. *Handbook of research on mathematics teaching and learning*, *334370*.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. Routledge.

Singh, K., Granville, M., & Dika, S. (2002). Mathematics and science achievement: Effects of motivation, interest, and academic engagement. *The journal of educational research*, *95*(6), 323-332.

Smith, E., Haarer, S., & Confrey, J. (1997). Seeking diversity in mathematics education: Mathematical modeling in the practice of biologists and mathematicians. *Science & Education*, *6*(5), 441-472.

Sodhi, M., & Son, B. G. (2007). Math Modelling: What Skills do Employers Want in Industry?. In *Mathematical Modelling* (pp. 212-220). Woodhead Publishing.

Steffe, L. P. (1996). Social-cultural approaches in early childhood mathematics education: A discussion. In *Mathematics for tomorrow's young children* (pp. 79-99). Springer, Dordrecht.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research techniques*. Thousand Oaks, CA: Sage publications.

Strauss, A. & Corbin, J. (1990). Basics of qualitative research: Grounded theory procedures and techniques. London: Sage.

Stylianides, A. J., & Stylianides, G. J. (2008). Studying the classroom implementation of tasks: High-level mathematical tasks embedded in 'real-life' contexts. *Teaching and Teacher Education*, *24*(4), 859-875.

Tapia, M. (1996). The Attitudes toward Mathematics Instrument.

Tapia, M., & Marsh, G. E. (2002). Confirmatory Factor Analysis of the Attitudes toward Mathematics Inventory.

Tapia, M., & Marsh, G. E. (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly*, *8*(2), 16-22.

Terwilliger, J. S., & Titus, J. C. (1995). Gender differences in attitudes and attitude changes among mathematically talented youth. *Gifted Child Quarterly*, *39*(1), 29-34.

Thompson, P.W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figuras, J.L. Cortina, S.

Alatorre, T. Rojno & A. Sepulveda (Eds.), *Proceedings of Annual Meeting of the International Group of the Psychology of Mathematics Education*, (1), 45-64. Morelia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In *New perspectives and directions for collaborative research in mathematics education.*

Thorndike-Christ, T. (1991). Attitudes toward Mathematics: Relationships to Mathematics Achievement, Gender, Mathematics Course-taking Plans, and Career Interests.

Von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical. In r. B. Davis, C. A. Mayer & N. Noddings (Eds.), Constructivist view on the teaching and learning of mathematics, 19-29). Reston, VA: National Council of Teachers of Mathematics.

Vos, P. (2013). The Dutch maths curriculum: 25 years of modelling. In *Modeling Students' Mathematical Modeling Competencies* (pp. 611-620). Springer, Dordrecht.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education*, 458-477.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for research in mathematics education*, *22*(5), 390-408.

Yee, L. S. (2010). Mathematics Attitudes and Achievement of Junior College Students in Singapore. *Mathematics Education Research Group of Australasia*.

Zan, R., & Di Martino, P. (2007). Attitude toward mathematics: Overcoming the positive/negative dichotomy. *The Montana Mathematics Enthusiast*, *3*, 157-168.

Zbiek, R. M. (1998). Prospective teachers' use of computing tools to develop and validate functions as mathematical models. *Journal for Research in Mathematics Education*, 184-201.

APPENDIX A

ORIGINAL TEST AND SURVEY PROTOCOLS

A$_1$. PCA Test

Name ____

Date ____

Please show all your work directly on this exam to the right or below each question.
THANK YOU IN ADVANCE FOR YOUR PARTICIPATION!

1) Given the function f, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.

a) $f(x + a) = 3x^2 + 3a^2 + 2x + 2a - 4$

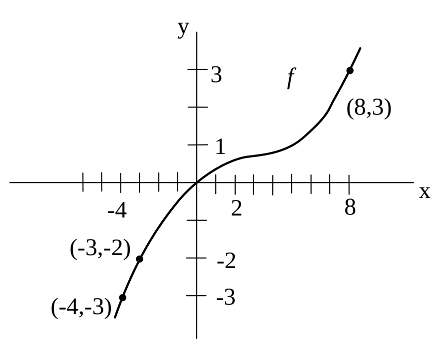
b) $f(x + a) = 3x^2 + 6xa + 3a^2 + 2x - 4$

c) $f(x + a) = 3\,(x + a)^2 + 2(x + a) - 4$

d) $f(x + a) = 3\,(x + a)^2 + 2x - 4$

e) $f(x + a) = 3x^2 + 2x - 4 + a$

2) Use the graph of f to solve $f(x) = -3$ for x.



a) (-3,-2)

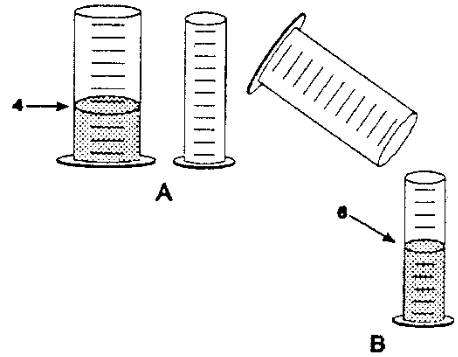
b) -4

c) (-4,-3)

d) -2

e) -3

3) To the right are drawings of a wide and a narrow cylinder. The cylinders have equally spaced marks on them. Water is poured into the wide cylinder up to the 4th mark (see A). This water rises to the 6th mark when poured into the narrow cylinder (see B).



Both cylinders are emptied, and water is poured into the narrow cylinder up to the 11th mark. How high would this water rise if it were poured into the empty wide cylinder?

a) To the $7\frac{1}{2}$ mark

b) To the 9th mark

c) To the 8th mark

d) To the $7\frac{1}{3}$ mark

e) To the 11th mark

4) Which one of the following formulas defines the area, A, of a square in terms of its perimeter, p?

a) $A = \dfrac{p^2}{16}$

b) $A = s^2$

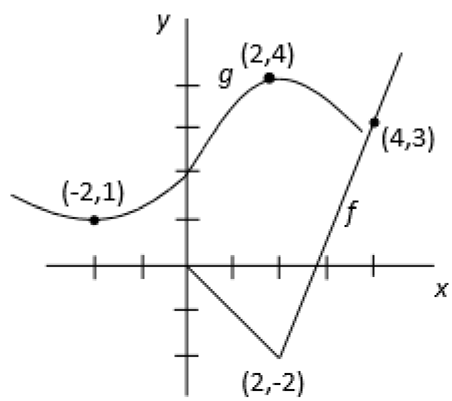
c) $A = \dfrac{p^2}{4}$

d) $A = 16s^2$

e) $p = 4\sqrt{A}$

Use the graphs of f and g to answer items 5 and 6.

5) Use the graphs of f and g to evaluate $g(f(2))$.



a) -2

b) 1

c) 3

d) 4

e) Not defined

6) Evaluate $f(2) – g(0)$.

a) -4

b) -2

c) 0

d) 2

e) 4

7) The model that describes the number of bacteria in a culture after t days has just been updated from $P(t) = 7(2)^t$ to $P(t) = 7(3)^t$. What implications can you draw from this information?

a) The final number of bacteria is 3 times as much of the initial value instead of 2 times as much.

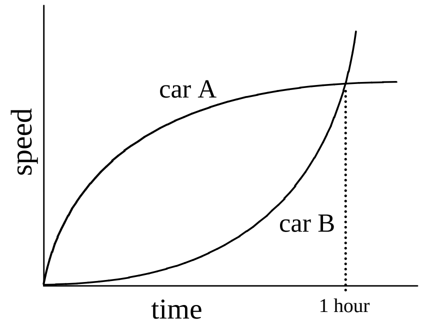
b) The initial number of bacteria is 3 instead of 2.

c) The number of bacteria triples every day instead of doubling every day.

d) The growth rate of the bacteria in the culture is 30% per day instead of 20% per day.

e) None of the above.

The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph below to answer item 8.



8) What is the relationship between the ***position*** of car A and car B at $t = 1$ hr.?

a) Car A and car B are colliding.

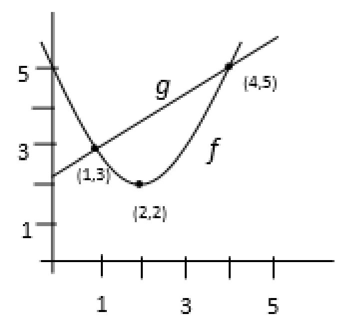
b) Car A is ahead of car B.

c) Car B is ahead of car A.

d) Car B is passing car A.

e) The cars are at the same position.

9) Use the graphs of f and g to solve $g(x) > f(x)$.



a) $2 < x < 5$

b) $1 < y < 4$

c) $x < 4$

d) $2 < y < 5$

e) $1 < x < 4$

10) A hose is used to fill an empty wading pool. The graph shows volume (in gallons) in the pool as a function of time (in minutes). Which of the following defines a formula for computing the time, t, as a function of the volume, v?

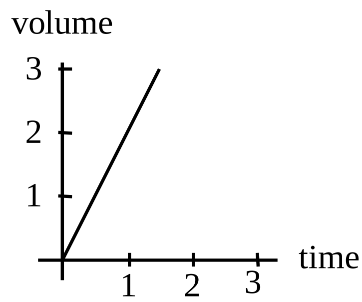
a) $v(t) = \frac{t}{2}$

b) $t(v) = 2v$

c) $t(v) = \frac{v}{2}$

d) $v(t) = 2t$

e) None of the above



11) The distance, s (in feet), traveled by a car moving in a straight line is given by the function, $s(t) = t^2 + t$, where t is measured in seconds. Find the average velocity for the time period from $t = 1$ to $t = 4$.

a) 5 ft/sec

b) 6 ft/sec

c) 9 ft/sec

d) 10 ft/sec

e) 11 ft/sec

12) Given the table to the right, determine $f(g(3))$.

a) 4

b) -1

c) 0

d) 1

e) 5

x	$f(x)$	$g(x)$
-2	0	5
-1	6	3
0	4	2
1	-1	1
2	3	-1
3	-2	0

13) Given the table to the right, determine $g^{-1}(-1)$.

a) $\frac{1}{2}$

b) $\frac{1}{3}$

c) 1

d) 2

e) 3

14) Given that f is defined by $f(t) = 100^t$, which of the following is a formula for f^{-1}?

a) $f^{-1}(t) = \dfrac{1}{100^t}$

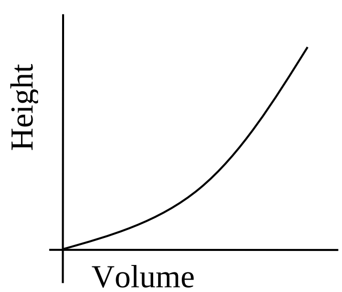
b) $f^{-1}(t) = \dfrac{t}{\ln 100}$

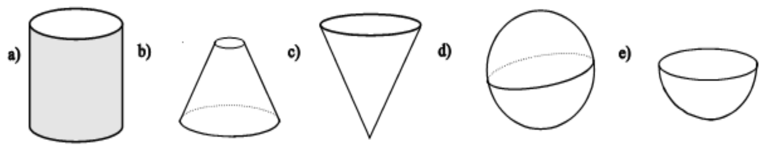
c) $f^{-1}(t) = \dfrac{t}{100^t}$

d) $f^{-1}(t) = \dfrac{\ln t}{\ln 100}$

e) $f^{-1}(t) = \dfrac{\ln t}{100}$

15) The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?





16) Given the function $h(x) = 3x - 1$ and $g(x) = x^2$, evaluate $g(h(2))$.

a) 10

b) 11

c) 20

d) 25

e) 36

17) A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, *A*, of the circle in terms of the time, *t*, (measured in seconds) since the ball hit the lake.

a) $A = 25\pi t$

b) $A = \pi t^2$

c) $A = 25\pi t^2$

d) $A = 5\pi t^2$

e) None of the above

18) The wildlife game commission poured 5 cans of fish (each can contained approximately 20 fish) into a farmer's lake. The function *N* defined by $N(t) = \dfrac{600t + 100}{.5t + 1}$ represents the approximate number of fish in the lake as a function of time (in years). Which one of the following best describes how the number of fish in the lake changes over time?

a) The number of fish gets larger each year, but does not exceed 500.

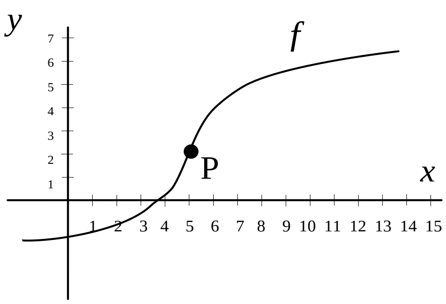
b) The number of fish gets larger each year, but does not exceed 1200.

c) The number of fish gets smaller every year, but does not get smaller than 500.

d) The number of fish gets larger each year, but does not exceed 600.

e) The number of fish gets smaller every year but does not get smaller then 1200.

19) Using the graph below, explain the behavior of function f on the interval from $x = 5$ to $x = 12$.



a) Increasing at an increasing rate.

b) Increasing at a decreasing rate.

c) Increasing at a constant rate.

d) Decreasing at a decreasing rate.

e) Decreasing at an increasing rate.

20) If $S(m)$ represents the salary (per month), in hundreds of dollars, of an employee after m months on the job, what would the function $R(m) = S(m + 12)$ represent?

a) The salary of an employee after $m + 12$ months on the job.

b) The salary of an employee after 12 months on the job.

c) \$12 more than the salary of someone who has worked for m months.

d) An employee who has worked for $m + 12$ months.

e) Not enough information.

21) What is the domain of the function, *f*, defined by $f(x)=\dfrac{\sqrt{x+2}}{x-1}$?

a) $(1,\infty)$

b) $x \neq 1$

c) $[-2,1)\cup(1,\infty)$

d) $[-2,\infty)$

e) All real numbers

22) A baseball card increases in value according to the function, $b(t) = \dfrac{5}{2}\, t + 100$, where b gives the value of the card in dollars and t is the time (in years) since the card was purchased. Which of the following describe what $\frac{5}{2}$ conveys about the situation?

I. The card's value increases by $5 every two years.

II. Every year the card's value is 2.5 times greater than the previous year.

III. The card's value increases by $\frac{5}{2}$ dollars every year.

a) I only

b) II only

c) III only

d) I and III only

e) I, II and III

23) Which of the following best describes the effect of f^{-1}, given *f* is a one-to-one function and $f(d)=c$?

a) f^{-1} inverts *f*, so $f^{-1}(d)=\frac{1}{f(d)}$

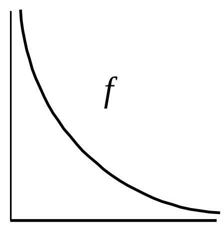
b) f^{-1} inverts the input to *f*, so $f^{-1}(d)=\frac{1}{d}$

c) f^{-1} inverts the output to *f*, so $f^{-1}(d)=\frac{1}{c}$

d) f^{-1} inverts f, so $f^{-1}(f(d))=d$

e) a and c

24) A function f is defined by the following graph. Which of the following describes the behavior of f?



I. As the value of x approaches 0, the value of f increases.

II. As the value of x increases, the value of f approaches 0.

III. As the value of x approaches 0, the value of f approaches 0.

a) I only

b) II only

c) III only

d) I and II

e) II and III

25) Which of the following best describes the behavior of the function f defined by,

$$f(x) = \frac{x^2}{x-2}.$$

I. As the value of x gets very large, the value of f approaches 2.

II. As the value of x gets very large, the value of f increases.

III. As the value of x approaches 2, the value of f approaches 0.

a) I only

b) II only

c) III only

d) I and III

e) II and III

A$_2$. ATMI Survey

Name:____

Date:____

Please indicate how strongly you agree or disagree with each of these statements. Please choose one answer per statement and write X in the appropriate box.

	strongly disagree	disagree	neutral	agree	strongly agree
1. Mathematics is a very worthwhile and necessary subject.					
2. I want to develop my mathematics skills.					
3. I get a great deal of satisfaction out of mathematics experiments.					
4. Mathematics helps develop the mind and teaches a person to think.					
5. Mathematics is important in everyday life.					
6. Mathematics is one of the most important subjects for people to study.					
7. High school mathematics courses would be very helpful no matter what I decide to study.					
8. I can think of many ways that I use mathematics outside of school.					
9. Mathematics is one of my most dreaded subjects.					
10. My mind goes blank and I am unable to think clearly when studying mathematics.					
11. Studying mathematics makes me feel nervous.					
12. Mathematics makes me feel uncomfortable.					
13. I am always under a terrible strain in a mathematics class.					
14. When I hear the word mathematics, I have a feeling of dislike.					
15. It makes me nervous to even think about having to do a mathematics experiment.					
16. Mathematics does not scare me at all.					
17. I have a lot of self-confidence when it comes to mathematics.					
18. I am able to do mathematics experiments without too much difficulty.					

19. I expect to do fairly well in any mathematics class I take.					
20. I am always confused in my mathematics class.					
21. I feel a sense of insecurity when attempting mathematics.					
22. I learn mathematics easily.					
23. I am confident that I could learn advanced mathematics.					
24. I have usually enjoyed studying mathematics in school.					
25. Mathematics is dull and boring.					
26. I like to do new experiments in mathematics.					
27. I would prefer to do an experiment in mathematics than to write an essay.					
28. I would like to avoid using mathematics in college.					
29. I really like mathematics.					
30. I am happier in a mathematics class than in any other class.					
31. Mathematics is a very interesting subject.					
32. I am willing to take more than the required amount of mathematics.					
33. I plan to take as much mathematics as I can during my education.					
34. The challenge of mathematics appeals to me.					
35. I think studying advanced mathematics is useful.					
36. I believe studying mathematics helps me with problem solving in other areas.					
37. I am comfortable expressing my own ideas on how to look for solutions to a difficult mathematics experiment.					
38. I am comfortable answering questions in mathematics class.					
39. A strong mathematics background could help me in my professional life.					
40. I believe I am good at mathematics experiments.					

APPENDIX B

INTERVIEW PROTOCOLS

B_1. Student Interview Protocol

In this interview, I would like to hear and learn about your experience from the AML100 course:

Questions about the final project

1. I would like to hear about your final project? Explain it to me please.
2. Discuss what motivated you to choose your topic of interest as your final project?
3. How did your group decide on this topic and how did your group divide up the work of the project?
4. How did working in groups (with your peers) influence your understanding of the topic?
5. Describe how was having different opinions and perspectives affected your learning experience during this project.

Questions about attitudes towards mathematics in general

6. What do you think mathematics is useful for?
7. How has your understanding of what mathematics is used for, changed after taking this course?
8. Describe to me how your confidence in mathematics has changed after taking this course.
9. *If there are significant changes in pre and posttests about student's ATMI answers, I will address those specific 3-4 questions that had the most changes.* *Examples of ATMI questions:*
    - *I am confident that I could learn advanced mathematics.*
    - *I feel a sense of insecurity when attempting mathematics.*
    - *I believe I am good at mathematics experiments.*

Questions about specific mathematical problems

Now let's look at a problem, similar to problems seen in class. I would like you to explain your thinking out loud as you solve,

10. <u>The Predator-Prey Problem</u>

    A simple model for a predator-prey interaction where the predator has a source of food in addition to the prey is:

    $\Delta P = r P (1 - P/K_1) - s P Q,$

    $\Delta Q = u Q (1 - Q/K_2) + v P Q.$

    a) Explain why these equations model the described situation.
    b) Choosing units of K_1 and K_2 so that $K_1 = K_2 = 1$, find and plot the nullclines for this model. Draw arrows on your plot indicating the signs of ΔP and ΔQ.
    c) What does your analysis of this model lead you to expect as typical behavior of orbits? *Hint: compute all equilibria for the model.*

Now, let's look at these three PCA questions you have already seen in your pre- and posttests, I would like you to think aloud and explain as you solve, *the following two exercises are just examples.*

11. (PCA question #1)

    Given the function f, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x + a)$.

    a. $f(x + a) = 3x^2 + 3a^2 + 2x + 2a - 4$
    b. $f(x + a) = 3x^2 + 6xa + 3a^2 + 2x - 4$
    c. $f(x + a) = 3\,(x + a)^2 + 2(x + a) - 4$
    d. $f(x + a) = 3\,(x + a)^2 + 2x - 4$
    e. $f(x + a) = 3x^2 + 2x - 4 + a$

12. (PCA question #4)

    Which one of the following formulas defines the area, A, of a square in terms of its perimeter, p?

    a. $A = \dfrac{p^2}{16}$
    b. $A = s^2$
    c. $A = \dfrac{p^2}{4}$
    d. $A = 16s^2$
    e. $p = 4\sqrt{A}$

B$_2$. Predator-Prey Problem Solution

<u>The Predator-Prey Problem</u>
Note: This problem was selected from the course book: Exercise 3.2.7.
A simple model for a predator-prey interaction where the predator has a source of food in addition to the prey is:

$\Delta P = r P (1 - P/K_1) - s P Q$,
$\Delta Q = u Q (1 - Q/K_2) + v P Q$.

a) Explain why these equations model the described situation.
b) Choosing units so that $K_1 = K_2 = 1$, find and plot the nullclines for this model. Draw arrows on your plot indicating the signs of ΔP and ΔQ.
c) What does your analysis of this model lead you to expect as typical behavior of orbits?

Solution & graph (provided to me by Instructor 1):

a) Without the interaction term between predator and prey, the predator population can still grow and sustain itself in the absence of prey.

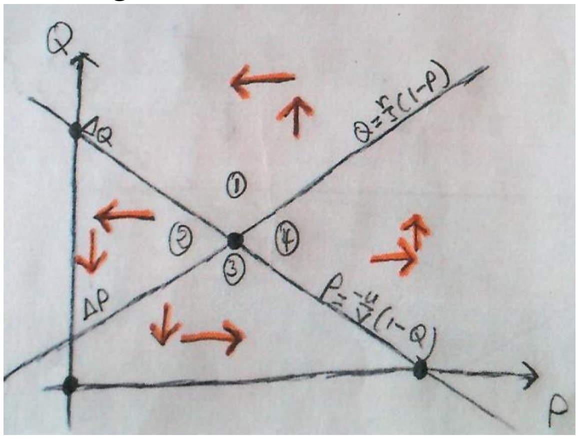


Figure 34. *Graph for the Predator-Prey Problem as Provided by Instructor 1*

b) When K_1=1 and K_2=1, we have the following four equilibria:
$(P^*, Q^*) = (0,0)$
$(P^*, Q^*) = (1,0)$
$(P^*, Q^*) = (0,1)$
$(P^*, Q^*) = (\frac{-us+ur}{vs+ur}, \frac{rv+ru}{sv+ru})$

c) They also tend to be stable at the equilibrium point where both populations co-exist.

Detailed solution:

We have,

$$\Delta P = rP\left(1 - \frac{P}{K_1}\right) - sPQ,$$
$$\Delta Q = uQ\left(1 - \frac{Q}{K_2}\right) + vPQ,$$

1. Model interpretation: Assuming *P* is prey and *Q* is predator, the first terms of model $rP\left(1 - \frac{P}{K_1}\right)$ and $uQ\left(1 - \frac{Q}{K_2}\right)$ means that the prey and predator populations grow to their maximum value (K_1 and K_2) through a logistic curve. Adding the terms $-sPQ$ and vPQ (that are interaction terms) means that the predator *Q* hunts the prey *P*, therefore, the prey population *P* decreases (through the term $-sPQ$) and the prey population *Q* increases (through the term $+vPQ$). This decrease in *P* and increase in *Q* become more intense when the population of *P* increases (because more food is being provided for the predator population), or the population of *Q* increases (because there is more predator to hunt the prey).

2. Nullclines: The *P*-nullcline is the set of points where $\Delta P = 0$:

$$\Delta P = rP(1 - P) - sPQ = 0,$$

Therefore, $P = 0$, or $P = 1 - \frac{s}{r}Q$.

Similarly, the *Q*-nullcline is the set of points where $\Delta Q = 0$:

$$\Delta Q = uQ(1 - Q) + vPQ = 0,$$

Therefore, Q = 0, or $Q = 1 + \frac{v}{u}P$.

3. Equilibria: Equilibria are all points for which $\Delta P = \Delta Q = 0$.
These intersections are within the following set:

$$(P^*, Q^*) = \{(0, 0),(K_1, 0),(0, K_2),(\frac{uK_1(r-sK_2)}{ru+svK_1K_2}, \frac{rK_2(u+vK_1)}{ru+sv\,K_1K_2})\}$$

Assuming $K_1=K_2=1$,

$$(P^*, Q^*) = \{(0, 0),(1, 0),(0, 1),(\frac{u(r-s)}{ru+sv}, \frac{r(u+v)}{ru+sv})\}$$

4. Analysis: Assuming $K_1 = K_2 = 1$,
The equilibrium point $(\frac{u(r-s)}{ru+sv}, \frac{r(u+v)}{ru+sv})$ which exists iff $r > s$ (both P^* and Q^* are nonzero or we say they co-exist) is a stable point, see phase plane illustration in next figure. Other equilibrium points do not show stability.

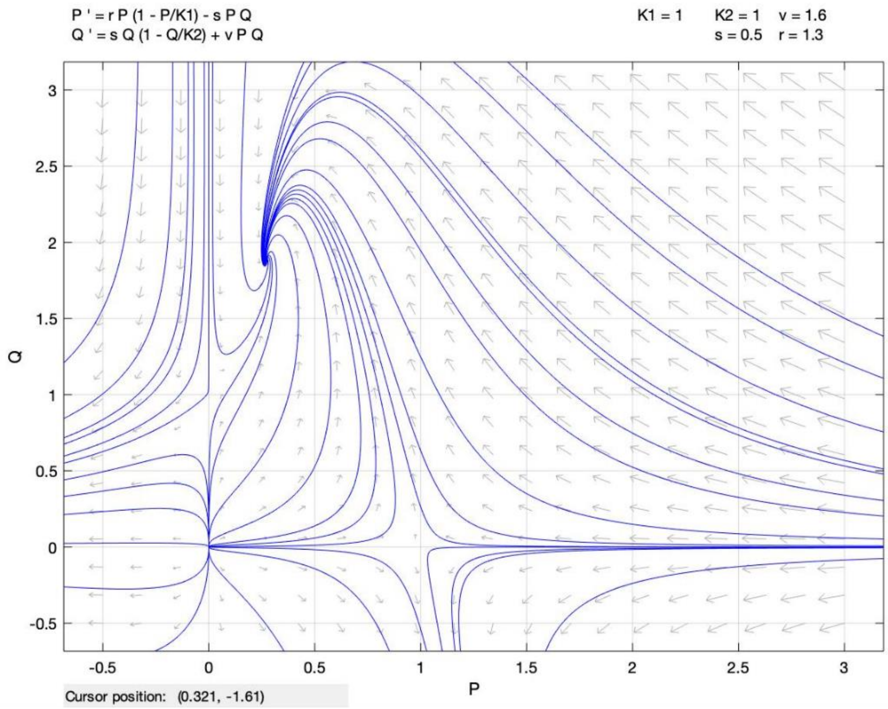


Figure 35. *Phase Plane Predator-Prey Problem Graph*

APPENDIX C

SAMPLE ASSENT FORMS

Student Recruitment and Consent Form

PARENTAL LETTER OF PERMISSION

Dear Parent and Student:

My name is Linda Agoune and I am currently a Ph.D. graduate student in Applied Mathematics for the Life and Social Science under the direction of Professor Carlos Castillo-Chavez at Arizona State University. I am conducting a research study to assess mathematics learning patterns among high school students through AML100 course contents and evaluate students' overall changing attitudes towards mathematics.

I am inviting your child's participation, which will involve an in-class pretest and a posttest (10 min), with a possible posttest interview (30-45 min) during class times at the Joaquin Bustoz Math-Science Honors Program (JBMSHP). These interviews will be audio and videotaped to preserve data accuracy for analysis purposes.

Your child's participation in this study is strictly voluntary and would be greatly appreciated. If you choose not to have your child participate or to withdraw your child from the study at any time, there will be no penalty or any effect on your child's participation in the JBMSHP. Likewise, if your child chooses not to participate or to withdraw from the study at any time, there will be no penalty or any effect on your child's participation in the JBMSHP. The results of the research study may be published, but your child's name will not be used. Your child's participation in this study is separate from the AML 100 course. Students can still participate in the course without participating in this study.

Although there may be no direct benefit to your child, the possible indirect benefit of your child's participation includes increasing personal knowledge and contributing to mathematics research. There are no foreseeable risks or discomforts to your child's participation beyond those of a typical mathematical class.

In order to protect confidentiality, common measures of practices will be implemented so that the participants' identities will be anonymous and will not be disclosed. If the student is videotaped, the participant's identities will be protected by blurring out the face. The students' responses will also be confidential, and all identifiable data will be kept for up to five years.

The results of this study will be used for a dissertation, possible reports, presentations, or publications but your child's name will never be known or used.

If you have any questions concerning the research study or your child's participation in this study, please contact me at lagoune@asu.edu.

Sincerely,
Linda Agoune

You may also contact the following professors involved in this research:

Dr. Carlos Castillo-Chavez (Project supervisor) ccchavez@asu.edu
Dr. Carlos Castillo-Garsow ccastillogarsow@ewu.edu
Dr. Anuj Mubayi amubayi@asu.edu

By signing below, you are giving consent for your child ___________________ (child's name) to participate in the above study and to their participation in a video recorded interview.

Parent signature _________________________
Printed Name ____________________________
Date _____________ (if student is under 18)

Student signature _________________________
Printed Name _____________________________
Date _____________

If you have any questions about you or your child's rights as a subject/participant in this research, or if you feel you or your child have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the Arizona State University Office of Research Integrity and Assurance, at (480) 965-6788.

APPENDIX D

HUMAN SUBJECT EXEMPTIONS

EXEMPTION GRANTED

Carlos Castillo-Chavez
CLAS-NS: Mathematical Computational Modeling Sciences Center, Simon A. Levin (MCMSC)
480/965-2115
ccchavez@asu.edu

Dear Carlos Castillo-Chavez:

On 4/15/2019 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Impact of Teaching an Interdisciplinary Course "Introduction of Applied Mathematics for the Life and Social Sciences" on High School Students' Performance and Beliefs Towards Mathematics in a JBMSHP Summer Program.
Investigator:	Carlos Castillo-Chavez
IRB ID:	STUDY00009979
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• citi Carlos Castillo-Chavez, Category: Other (to reflect anything not captured above); • Adult_Consent_04122019_LA.pdf, Category: Consent Form; • IBR-Attachment1-JBMSHP application 2019.pdf, Category: Recruitment Materials; • PCA, Category: Participant materials (specific directions for them); • Form-Social-Behavioral-Protocol-LA_04122019_LA.docx, Category: IRB Protocol; • Child_Assent_Minor_04152019_LA.pdf, Category: Consent Form; • citi Carlos Castillo-Garsow, Category: Other (to reflect anything not captured above); • Parental-Permission_04122019_LA.pdf, Category:

	Consent Form; • ATMI, Category: Participant materials (specific directions for them); • citi Linda Agoune, Category: Other (to reflect anything not captured above); • citi Anuj Mubayi, Category: Other (to reflect anything not captured above); • IRB-Attachment-JBMSHP Brochure 2019.pdf, Category: Recruitment Materials; • Interview Questions, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings on 4/15/2019.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Linda Agoune
Anuj Mubayi
Carlos Castillo-Garsow
Carlos Castillo-Chavez
Linda Agoune