Undergraduate Students' Conceptions of Multiple Analytic Representations of Systems (of Equations)

by

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A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved October 2019 by the
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December 2019

ABSTRACT

The extent of students' struggles in linear algebra courses is at times surprising to mathematicians and instructors. To gain insight into the challenges, the central question I investigated for this project was: What is the nature of undergraduate students' conceptions of multiple analytic representations of systems (of equations)?

My methodological choices for this study included the use of one-on-one, task-based clinical interviews which were video and audio recorded. Participants were chosen on the basis of selection criteria applied to a pool of volunteers from junior-level applied linear algebra classes. I conducted both generative and convergent analyses in terms of Clement's (2000) continuum of research purposes. The generative analysis involved an exploration of the data (in transcript form). The convergent analysis involved the analysis of two student interviews through the lenses of Duval's (1997, 2006, 2017) Theory of Semiotic Representation Registers and a theory I propose, the Theory of Quantitative Systems.

All participants concluded that for the four representations in this study, the notation was varying while the solution was invariant. Their descriptions of what was represented by the various representations fell into distinct categories. Further, the students employed visual techniques, heuristics, metaphors, and mathematical computation to account for translations between the various representations.

Theoretically, I lay out some constructs that may help with awareness of the complexity in linear algebra. While there are many rich concepts in linear algebra, challenges may stem from less-than-robust communication. Further, mathematics at the

level of linear algebra requires a much broader perspective than that of the ordinary algebra of real numbers. Empirically, my results and findings provide important insights into students' conceptions. The study revealed that students consider and/or can have their interest piqued by such things as changes in register.

The lens I propose along with the empirical findings should stimulate conversations that result in linear algebra courses most beneficial to students. This is especially important since students who encounter undue difficulties may alter their intended plans of study, plans which would lead them into careers in STEM (Science, Technology, Engineering, & Mathematics) fields.

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CHAPTER 1

INTRODUCTION & RATIONALE

The research and resulting literature on the teaching and learning of undergraduate linear algebra frequently shares a common theme: linear algebra is hard. Student results and the extent of students' struggles in the course are at times surprising to mathematicians and instructors. Tucker (1993) discussed the issue while making an enthusiastic case for the importance of linear algebra, stating that linear algebra's "theory is so well structured and comprehensive, yet requires limited mathematical prerequisites" (p. 3). In addition, he stated "Linear algebra is … appealing because it is so powerful yet simple" (p. 4).

The limited number of prerequisites and the simplicity described by Tucker often does not translate into ease for students (Dorier & Sierpinska, 2001; Hillel, 2000; Stewart, 2018; Wawro, Sweeney, & Rabin, 2011). In introductory linear algebra courses, students encounter a seemingly invisible and inexplicable wall which often turns them away from their intended plans of study, plans which would lead them into careers in STEM (Science, Technology, Engineering, & Mathematics) fields. The purpose of this study was to explore the mystery, to some degree, of the challenges in linear algebra by focusing on the ubiquitous topic of systems of linear equations, often referred to simply as "systems" or "linear systems" without reference to the equations and/or their linearity.

I attribute the bewilderment about students' difficulties in linear algebra to practitioners' beliefs that if they themselves know and they share what they know with students, then students will know in the way that was intended, in a way similar to the practitioners' knowing. This view neglects many important influencing factors. The difficulty of linear algebra cannot be fully explained based on the content of the subject, especially from the perspective of an expert for whom it is cohesive and magnificent. Other factors include the intricacies of the teaching and learning processes and the complexities of human cognition. For these reasons RUME (Research in Undergraduate Mathematics Education) is vital in addressing the issue, ensuring that the needs of the technical and scientific communities are met. This project specifically considers viewing systems from the students' perspective rather than from the perspective of experts or through the consideration of content alone.

A consideration of undergraduate students' conceptions of systems is of particular interest since students are introduced to them fairly early in their academic careers, in grade 8 mathematics (National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010) if not earlier. They probably have seen linear systems many times before they reach their first linear algebra course. After years of exposure and multiple opportunities to work with systems, some students may have developed desirable and productive meanings that are not surprising given the curricular emphasis on systems. Alternatively, students may have idiosyncratic and/or unproductive conceptions. Most likely is a range of conceptions between those two possibilities. An investigation of students' culminated conceptions after years of exposure may provide new kinds of insights.

Deconstructing "Algebraic"

This study specifically targets *analytic representations* of systems, here understood as the algebraic symbolism used to denote them. In mathematics education research, a dichotomous view often emerges: mathematics is either procedural or conceptual (Hiebert & Lefevre, 1986). As a result, focus is often placed on what is considered to be conceptual mathematics and the symbolic is neglected. I believe this may result from not distinguishing *ordinary algebra*, the familiar algebra of real numbers, from other contexts like matrix algebra. While ordinary algebra may be a sufficient grain size in some contexts and at some levels of mathematics, I assert that the complexity of linear algebra requires the adoption of an alternative perspective.

With this project, I parse analytic notation into several distinctive categories, each with its own means of notation and rules of engagement. Lumping symbolic notation into a single category leaves implicit the number of moving parts involved; the multitude of parts and their numerous associations remain unspecified. Breaking down analytic notation into smaller components should aid in the deconstruction and illumination of the inexplicable wall described earlier. Facets that may go unrecognized by experts in their everyday practice are identified. Where others acknowledge the variety of mathematical representations as a source of students' struggles in mathematics, I impose a more finely-grained structure on analytic representations and organize them into systems of representations.

In considering the multifaceted nature of analytic notation in linear algebra, an important factor is students' potentially increased capacity to consider the abstract. By the time they reach a first course in linear algebra, undergraduates may be more developmentally capable of abstract thinking than at earlier points in their education. A study focused on analytic notation seems more appropriate at the undergraduate level than at previous grade levels. That is, given students' evolving ability in managing the increased complexity and abstraction that linear algebra demands, a study focused on analytic notation seems warranted.

A More Positive Perspective

Research in undergraduate mathematics education has often focused on students' misconceptions, ways that student understanding does not align with formal concept definitions generally accepted in the mathematics community (Tall & Vinner, 1981). Certainly, such studies highlight areas that need to be addressed; they serve the purpose of diagnosing problems and of sounding alarms. I see those studies as necessary in the history of RUME to illuminate that students' conceptions and institutionalized meanings differ. However, I believe studies that establish negative distinctions should yield to investigations that are more constructive in nature. This project has the potential to be an exemplar of a different kind of framing that could be beneficial to the RUME community at this point in the evolution of the field.

In the data analysis for this dissertation, I attempted to take students' conceptions at face-value without holding them up to a standard to evaluate whether they fit or not. Rather than simply addressing whether students had the right conception, their conceptions are characterized more fully. Further, the mature nature of the students' thinking is highlighted. That is, in the data analysis, I adopted the perspective that nonstandard conceptions need not be characterized as misconceptions (Smith, diSessa, & Roschelle, 1994); facets of students' conceptions, even when informal, can provide valuable insights that can be used to support learning.

The title of this chapter (Introduction & Rationale) is evidence of my attempt at a more positive approach. I have deliberately avoided a common practice of using the terminology "the problem" or "problem statement" in the title of my chapter. From my perspective, an opportunity for investigation need not be framed as a problem.

Theoretical Results

As described in Chapter 2, I have delved into the details of Duval's Theory of Semiotic Representation Registers (1999, 2006, 2017), specifying features that go beyond a common general characterization of it as a theory of "multiple representations". Working with the theory beyond a "multiple representations" description was challenging due to the abstract nature of the ideas involved and what I believe are complications arising from the translation of those ideas from French into English. A more detailed consideration of Duval's theory, along with the consideration of students' issues in solving systems of equations (Zandieh & Andrews-Larson, 2015), guided the creation of my own theory, the Theory of Quantitative Systems (TQS). Both theories are detailed in Chapter 2 Theoretical Perspective; in addition, an illustration of the theories appears as Appendix A. I see the deeper explication of Duval's theory as one result of my study, a result that might appeal to a broad range of mathematics scholars and mathematics education researchers. Further, I see my Theory of Quantitative Systems as another result of my study, a result more specifically of interest to linear algebra instructors and those that instruct students along the path to linear algebra.

My claim about the two aforementioned contributions is based on a definition of research as “the systematic investigation into and study of material and sources in order to establish facts and reach new conclusions” (“Research”, n.d.). To that definition of research, I add my personal perspective that research can be theoretical in nature, where I take *theoretical* to mean philosophical and potentially removed from any immediate, obvious application. Such findings can serve as catalysts in the scholarly community with an impact yet to be determined.

Empirical Considerations

To the two previously described results (new exposition of an existing theory and development of a new theory), I add empirical findings supported by clinical observations as described below.

The Research Questions

The central question I investigated for this project was: What is the nature of undergraduate students’ understandings of multiple analytic representations of systems (of equations)? The question involved a population for which little work is documented in literature (see Chapter 3 Literature Review). The specific consideration of linear systems of equations may be helpful in providing insight into 1) linear algebra more globally (topics other than systems), and 2) how students think about a topic after repeated exposure over several years. Further, while Duval (1999, 2006, 2017) often explicated his theory in geometric and/or visual contexts, I have adapted the theory to strictly analytic contexts. My use of parentheses in the question and in the title of this document suggests that I considered more than one kind of system. One kind of system is a linear system of equations in a conventional sense; another kind of system is one that I theorize. Additionally, the study takes into account various systems of notation.

In support of my primary question, I also address the question: What unified thing, if any, do students have in mind as the entity represented by various representations for systems? Further, I considered: How do students account for similarities and differences between the representations? Specifically, I was interested in how their accounting for the similarities and differences might be characterized in terms of Duval's theory and the Theory of Quantitative Systems. In other words, in what ways do students speak (from my researcher's perspective) in terms of translations and registers of representation in their accounting?

In summary, against the backdrop of a finer-grained consideration of Duval's Theory of Semiotic Representation Registers (TSRR, 1999, 2006, 2017) and my development of the Theory of Quantitative Systems (TQS), I analyzed video-recorded clinical interviews in which students were directly asked about their conceptions of four analytic representations of a system as shown in Questions 11 and 12 of Appendix B.

The Results

Based on aggregate data for all 10 participants, I report general findings that establish a baseline for this area of inquiry with a demographic for which little data exists. Catalogs of student conceptions were documented, and categories of student conceptions were identified. Further, two particular cases were analyzed; interviews with

Peter (Student 5) and Felix (Student 3) allowed for the application of Duval's and my theories.

Implications

The empirical analysis and documentation of undergraduate students' conceptions of systems may provide valuable and productive insights while contributing to the literature in the field (RUME). Taken alongside the theoretical results, the empirical results make contributions toward the consideration of ways to support students in their learning of linear algebra in general and in their learning of systems specifically. The findings offer some insight into the otherwise seemingly inexplicable wall (described earlier) which presents challenges to students and instructors alike.

About This Document

Chapters 2-4 are similar in structure, moving from general to specific. Each of those chapters begins with very general discussions of theory, literature, and methodology, respectively. Subsequent to each general discussion, I describe specifics more applicable to the current project. Chapter 5 Results & Findings includes observations supported by the clinical interview data. In Chapter 5, I first report overall findings for all participants; thereafter, I look at interactions with two particular students. In Chapter 6 Discussion, I make observations related to the study that are more general and less closely tied to empirical data. As a result, the chapter is suggestive of areas and topics that are candidates for future studies.

CHAPTER 2

THEORETICAL PERSPECTIVE

This chapter gives an account of theoretical perspectives from which I performed this investigation. I describe how, with constructivism as a backdrop, various meanings for notation are possible. Notational considerations in linear algebra in general and in systems of equations more specifically can account for some of the complications experienced by learners and teachers in linear algebra courses. My assertion is based on Duval's (1999, 2006, 2017) Theory of Semiotic Representation Registers, a theory about the cognitive complexity of mathematics in general. My analysis of students' written exam data and a subsequent thought experiment I performed coincided with my continued study of Duval's theory. As a result of that work, I have articulated a new theory, the Theory of Quantitative Systems. The aforementioned are lenses that orient my study, either directly or indirectly; I describe them in greater detail in what follows. I see the more detailed discussions as serving two purposes: 1) to acknowledge and more fully disclose theories that influence me as a researcher, and 2) to aid the reader in aligning their perspective in order to understand the design and results of the study.

Constructivism

Below I describe my perspective on constructivism which guides and influences this study. While theoretically constructivism can be taken as an objective research perspective, constructivism itself posits that we cannot detach from our experience of the world. That is, while hypothetically any researcher can attempt to conduct an unbiased investigation which uses constructivism as a tool for framing the study, constructivism asserts our conceptions are influenced by personal experience, negating claims of no bias. Therefore, I outline my ways of thinking about the theory to assist others in viewing the study through the theoretical lens that influences it.

Epistemology, Cognitive Science, & Constructivism

Epistemology is the study of knowledge. Two questions elicited by epistemology are: 1) What is the genesis of knowledge? and 2) What is the nature of knowledge? That is, 1) How does one come to know? and 2) How can that knowledge be characterized? Taken together, these considerations may be referred to as ways of knowing.

Cognitive science is the study of the mind and its operations. Thagard (2014) states "… cognitive science is just the sum of the fields mentioned: psychology, artificial intelligence, linguistics, neuroscience, anthropology, and philosophy" (p. 1). He describes his definition as weak since each field has its own theories and methods that lead in various directions. I take cognitive science to mean consideration of workings of the mind; I take cognitive theories to be potential explanations of those workings of the mind. We can use cognitive theories to describe how we come to know and the nature of that knowing. One way of doing this is to theorize about mental actions and constructions.

Constructivism is an epistemological theory where learners are not considered to be passive receivers of information; rather, learning involves active participation of the learner. When this perspective is paired with cognitive science, the result is a focus on individuals' mental activities. I think of the intersection of epistemology, cognitive science, and constructivism in the following way: constructivism allows for the consideration of how we come to know and the nature of that knowing described in terms of mental structures and processes that vary from person to person based on each individual's experiences. For me, a claim that the mental structures (constructions) are real is unnecessary; a sufficient perspective is that analyses in terms of mental structures may provide insights which that are productive in navigating the human experience.

Trivial & Radical Constructivism

According to von Glasersfeld, trivial constructivists adopt the tenet, "Knowledge is not passively received either through the senses or by way of communication, but it is actively built up by the cognizing subject" (1988, p. 83). Constructivism in the Piagetian sense is an epistemological theory based on Swiss psychologist Jean Piaget's framework for cognitive development of children. Piaget (1954) described cognizing individuals as adapting in one of two ways: assimilating new experiences to what is already known (*assimilation*) and accommodating what is already known to accept new experiences (*accommodation*). I think of assimilation as placing an item in a closet or pantry when space is readily available; I think of accommodation as rearranging things in a closet or pantry to make space for an additional item that does not fit otherwise. Piagetian constructivism was given a new interpretation referred to as *radical constructivism* in the 1970s by Ernst von Glasersfeld (1988).

In addition to the tenet that cognizing individuals actively build up knowledge, von Glasersfeld added a second tenet resulting in *radical constructivism*: "The function of cognition is adaptive and serves the subject's organization of the experiential world, not the discovery of an objective ontological reality" (Glasersfeld, 1988, p. 83). For me, this statement (and radical constructivism in general) does not address the existence of an objective reality. Rather, I take von Glasersfeld's tenet as a commentary on the nature of cognition; cognition is a variable and individual thing influenced by experience despite how one describes reality. Von Glasersfeld's tenet is a commentary on both Piaget's background in biology and von Glasersfeld's interest in communication. We are biological creatures and communication is a reflexive, subjective process.

While radical constructivism foregrounds an individual's cognitive activities, to assume that it gives no consideration to social considerations is inaccurate (Thompson, 2000, 2002). Communication, a facet of von Glasersfeld's studies, is necessarily social and interpersonal. Von Glasersfeld's promotion of *conceptual analysis*, a method he developed for studying cognition, further illuminates his perspective; he describes the goal of conceptual analysis as modeling "what mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (1995, p. 78). Thus, the assumption is that a subject's actions make sense to them and the goal is to account for whatever sense the subject is making. This idea frames both the methodology and data analysis in this study. Clinical interviews are a means of accounting for the sense of the participants, and the assumption that a subject's actions are sensible to them frames both the interview protocol and the analysis of data.

Constructivism as a Background Theory

I acknowledge that for me, constructivism operates as a background theory, a general theory that influences this project. Broadly, I tend to think of anything that allows for individual differences as "constructivist". In doing so, I concede that I am keenly focused on the learners' personal experiences without rigorous consideration of how a learner comes to know or builds up their knowledge. When I give greater consideration to epistemology, I reflect on von Glasersfeld's statement "knowledge is not a transferable commodity and communication is not a conveyance" (1983, p. 67). Additionally, he asserts that a teacher's role should not be "to dispense 'truth', but rather to help and guide the student in the conceptual organization of certain areas of experience" (1983, p. 67). Overall, I consider the perspective that teaching is a direct transmission of knowledge from teacher to student and any ideology related to direct transmission as *not* being constructivist. These two criteria (individual differences and no direct transmission) mean that my view of constructivism is quite broad; however, my perspective avoids some of the controversial, philosophical aspects of the epistemological theory. To me, this seems sufficient for a background theory.

Thompson (2002) explains that background theories frame the types of questions asked, how data is analyzed, and the role of the researcher. I largely take this to mean that background theories inform methodology; methodological considerations are discussed in greater detail in Chapter 4. Very generally, though, my research questions are focused on students' understandings and individual students' activity. I documented the nature of students' understandings rather than whether some kind of transmission of knowledge has occurred. I presume that a student's perceptions differ from my own and acknowledge that my own personal conceptions mean that I cannot objectively know what a student is thinking. I can, however, offer viable explanations that may prove to be pedagogically productive.

Theory or Methodology or Teaching Method

At times, “cognitive constructivism” is used interchangeably with “individual constructivism”. This is a misnomer because Vygotsky, the originator of social constructivism, was a cognitivist. I see the difference between Vygotsky’s social constructivism and Piaget’s individual constructivism as differing perspectives on the genesis of knowledge: does knowledge originate from our individual interactions with the environment or does knowledge originate from social interactions? Individual constructivism focuses on knowledge *construction* while social constructivism focuses on knowledge *co-construction*. Philosophically, I do not see these as mutually exclusive. That is, one need not make a personal commitment to either perspective. One can imagine a researcher who, making their best attempt to control personal bias in their work, is capable of analyzing a set of data from either one of the perspectives, being held to a single perspective only as a theoretical research lens. In this way, the issue is which factors are controlled for, and what can be learned with the reduced complexity that results from focusing on particular aspects of learning. Of course, whichever lens the researcher chooses will frame the kinds of research questions they ask, their methods of data collection and analysis, and the explanations they offer as viable given their adopted lens. In this way, constructivism shapes data collection and analysis based on the individual as a focal point or based on the individual as a participant in a more complex system of social and cultural variables. We can learn things and have insights when focusing on the individual that may not be apparent otherwise. A similar thing can be said of socio-cultural perspectives that allow for social and cultural factors to be considered.

Theoretically, whether one focuses exclusively on an individual or more broadly on other factors is an objective choice—a tool for framing a study. That is, constructivism in this way becomes a tool in the hands of the researcher rather than a personal, philosophical conviction.

Cobb (2007) discusses differing theoretical perspectives based on the kinds of research that can be addressed with each perspective and the potential usefulness of each perspective in supporting learning. He describes the kinds of research that can be conducted in terms of how each tradition 1) characterizes the individual, and 2) the limitations of each tradition. He describes cognitive theory as placing the individual at the forefront while socio-cultural theories foreground the activities of the group/culture. That is, the difference is what it taken as foreground or background; there is not a wholesale dismissal of other factors. My perspective does not fit neatly into Cobb's classification system since I consider both individual and social constructivism to be cognitive theories; however, my views are consistent with his concerning the differing focal points for the two types of constructivism.

Confusion seems to arise when one does not distinguish between theory, research methodology, and teaching methods. Maintaining a focus on constructivism as an epistemological theory is crucial; constructivism as a theory of knowledge does not necessitate its use as a methodological tool in a study. That is, constructivism can be an influencing epistemological theory without leading to an analysis of data and classification of student responses, say, in terms of assimilation or accommodation. Additionally, as a theory of knowledge, constructivism serves as a framing for research and not as a teaching prescription (Thompson, 2002). Taking constructivism as an epistemological theory will necessarily influence one's teaching approaches. For example, a teacher persuaded by constructivist theory may allow for more opportunities for active learning. However, those who mistakenly take constructivism as a theory of teaching have come to invalid conclusions. For example, some have taken constructivism to mean that students should be allowed to wander directionless until they somewhat spontaneously come to know through discovery. That is, they take the tenets of constructivism as prescriptions for teaching rather than taking them as an epistemological theory which frames research, which in turn may influence teaching. I think that many of the seemingly controversial aspects of constructivist theory can be mitigated by clearly distinguishing between epistemology, methodology, and teaching methods.

Constructivist Perspectives on Symbols & Language

From a constructivist perspective, symbols do not carry a fixed set of information that is directly transmitted between senders and receivers in the communication process (Glasersfeld, 1995). I take signs and symbols to be human constructs created for communication; their meanings are relative to and dependent on the users. However, according to mathematical conventions which are socially and culturally established, symbols represent certain objects and are connected through logical, deductive means. To a formalist mathematician, a symbol represents an entity that may interpreted and manipulated only according to some logical formulation. In that context, there is a limit on ambiguity. In broader contexts, what symbols are, what they represent, and what they mean, particularly to students, are more ambiguous considerations that need further study.

An Objective-Like Reality

When radical constructivism is properly framed as an epistemological theory, I find a discussion of whether an objective reality exists unnecessary. I do often, however, think in terms of an “objective-like” reality when I conduct mathematics education research. I take the formal concept definition (Tall & Vinner, 1981) to be an objective-like reality that represents what we intend for students to understand. Tall and Vinner describe the *formal concept definition* as “a concept definition which is accepted by the mathematical community at large”; they define a *concept definition* as “a form of words used to specify” a concept (p. 152). To be clear, I do not view the formal concept definition as an objective reality; I acknowledge that formal concept definitions are socially constructed within the mathematics community. However, I find taking the formal concept definition as a basis against which to compare various ways of thinking to be a valuable research approach; it establishes a baseline against which varied individual conceptions can be compared.

Concluding Remarks on Constructivism

My perspective on constructivism guides and influences this study. In this study I am less concerned with knowledge genesis than with the nature of students’ knowledge; I did not address how knowledge comes to be per se. Rather, I looked at the nature of students’ knowledge of systems of linear equations; this area is under-investigated in undergraduate mathematics education, a matter detailed in Chapter 3 Literature Review. Future studies may address the genesis of knowledge related to linear systems.

Duval's Theory of Semiotic Representation Registers

Duval (1999, 2006, 2017) theorized that semiotic systems are a major source of complexity in mathematical cognition. Broadly, a semiotic system is a system of signs and symbols used to communicate. While Duval's Theory of Semiotic Representation Registers (TSRR) is often described as a theory of "multiple representations", that level of generality leaves many interesting and productive aspects of the theory unexamined. I have conducted a more rigorous study of TSRR, identifying facets of the theory that I find compelling and useful for my research. I find the theory valuable in considering the complexity of learning linear algebra in general and of working with systems of equations more specifically. Further, in literature that reduces Duval's theory to a theory of "multiple representations", distinctions between the mathematics and the thinking about the mathematics are not always clear. The following discussion may help those who use the theory (TSRR) in their research to communicate more effectively by clearly distinguishing between the cognition and the mathematics.

Specifying Basic Terminology

Consideration of Duval's theory requires that his notions of "mathematical object", "representation", and "register" be specified. Duval characterized a *mathematical object* as an entity that is inaccessible in the physical world other than through a semiotic system (a system of notation). In simple terms, he described a *representation* as "something that stands for something else" (Duval, 2006, p. 103), though he offered contrasting perspectives as described below. He used *register of representation* to mean a system of signs and symbols used to communicate along with the ways those signs and symbols are used to represent and process mathematical thinking; *registers of representation* are semiotic systems with specific means of representation AND specific means of processing (Duval, 1999). Representations reside within registers, groupings of symbols coupled with a set of rules for working with the symbols. An example in my study is a systems register; ordinary rules of algebra (over the real numbers) are used to manipulate equations in systems of linear equations. Another example is the matrix register; rules of matrix algebra may be used to manipulate the matrix representation of the system.

Mathematical Activity from a Cognitive Point of View

Duval (1999, 2006, 2017) conjectured that the kind of thinking required in mathematics is distinct. His assertion that a mathematical entity is something that we can access through semiotic representations alone foregrounds the importance of semiotic representations; this differs from the hard sciences where the things that are studied are perceptible by other means. A complicating factor, which he referred to as the *cognitive paradox* (Duval, 1999), is that although we only have access to mathematical entities through their semiotic representations, we should not confuse the semiotic representation with the entity that it is intended to represent. That is, it is cognitively taxing to make a distinction between *the represented* and *the representation* when the only access to *the represented* is through its representation. Mathematical activity from a cognitive point of view is further complicated by the variety of semiotic representations available to express mathematical ideas.

Analysis of Mathematical Thinking Processes

Duval (2006) contended that due to the nature of mathematics, an analysis of mathematical cognition necessarily requires the consideration of the semiotic representation systems that are used. He maintained that two ways to analyze mathematical thinking processes are to consider: 1) transformations of semiotic representations, and 2) how one can recognize the same mathematical entity in multiple semiotic systems with varying content. That is, an analysis of mathematical thinking processes can involve the consideration of how semiotic representations are transformed; alternatively, an analysis of mathematical thinking processes can involve the consideration of ability to look at differing representations and recognize them as denoting the same thing when each method of representation has its own signs and rules.

Duval's (2006) first way of analyzing mathematical thinking processes, transformations of semiotic representation, breaks into what he viewed as two sources of incomprehension in mathematics. One source of incomprehension is the thinking required to perform *treatments*, which are transformations of representations within the same register; another source of incomprehension is the thinking required to perform *conversions*, which are transformations of representations between two different registers. These two types of transformations are described in greater detail in what follows. Duval's (2006) second proposed method for analyzing mathematical thinking processes, how to recognize the same mathematical entity in multiple semiotic systems, is a third source of incomprehension that involves the ability to distinguish what is mathematically relevant (or not) from any particular representation of a mathematical entity.

Varying Views on *Representation*

What is meant by *representation* varies greatly. Duval discussed one view of *representation* that resulted from Piaget's studies in 1923 and 1926. From a Piagetian perspective, Duval (2006) explained "Representations can be individuals' beliefs, conceptions, or misconceptions to which one gets access to the individuals' verbal or schematic productions" (p. 104). This perspective on representation has become a major methodological and theoretical framework for investigating how one gains new knowledge; for instance, one may use this characterization of *representation* to theorize and characterize thinking in terms of assimilation and accommodation as discussed earlier. Alternatively, Duval (2006) stated "representations can also be signs and their complex associations, which are produced according to rules and which allow the description of a system, a process, a set of phenomena" (p. 104). Duval placed semiotic representations, including language, within this second group; while this characterization of *representation* may involve the production of new knowledge, the perspective frames representations as a means of communication.

Partitions of *representation*. Frequently, the most basic division of *representation* in cognitive science is between whether a representation is internal (mental constructs) or external (in the world outside the mind). Duval (1999) viewed a more beneficial partition of representations as: 1) intentional semiotic ones which may be mental or external, and 2) "causal" ones that result from an organic system or physical device. Sentences, graphs, diagrams, and drawings are intentional semiotic representations, while dreams, memory, and photographs are causal representations. (C*ausal* may be an unfortunate and awkward translation from the original French manuscript. For me *organic* comes closer to capturing what I take Duval to mean, and I will use the word *organic* rather than *causal* hereafter. Other words that capture what I believe to be Duval's intended meaning for "causal" are *autonomic*, *perceptual*, *impulsive*, and *unconscious*.) Internal and external designations focus on the representation itself, while intentional or organic designations focus on the activity of the learner. For me, this means distinguishing between representations that are deliberately made and those that are autonomic and/or unconscious. Additionally, I see whether acts are deliberate or unconscious as dependent on the cognizing subject; that is, I see whether the subject's actions are intentional or organic as varying from person to person and as varying for a given person across time. Actions that may take deliberate intention for one person may be more automatic for another person; actions that may take deliberate intention when a person is first learning an idea may become automatic with more exposure and experience.

Duval's (2006) shift away from where a representation resides is based on his conjecture that semiotic representations, whether considered to be mental or external, work together with mental representations. He asserted that something more than getting access to knowledge is at work. Thus, he proposed that considering semiotic representations in the context of the mind's structure is essential in considering the complexity of mathematical cognition; that is, the analysis of mathematical cognition requires an analysis of the (cognitive) systems mobilized in working with the semiotic representations.

Discursive & Visual Registers

As mentioned above, Duval used *register of representation* to refer to a system of signs and symbols and the ways the signs and symbols are used for communicating and processing mathematical thinking. Further, Duval (1999, 2006) distinguished between registers for discursive representations and registers for visual representations. Discursive registers are those related to discourse and may be oral or written, like natural language and symbolic systems. Visual registers, which are non-discursive, are of two types. They may be iconic, like drawings or sketches, or they may be mathematical such as graphs or to-scale drawings. In this study I focus on discursive registers of symbolic representations.

Two Kinds of Transformations

Representations can be transformed so that the same concept is signified in a different way. Duval (2006) described two types of transformations which he asserted are two distinctive sources of complexity in mathematical cognition: treatments and conversions. A *treatment* is the transformation of a representation that yields a resulting representation within the same register (system of notation). An example of a treatment is a linear system of equations that is rewritten using ordinary rules of algebra. More specifically, multiplying one equation in a linear system by a non-zero constant is an example of a treatment; the result is a representation similar in form and notation to the original linear system. A *conversion* is a transformation of a representation that yields a representation of the signified object in a different register. An example of a conversion is rewriting a system of linear equations as a matrix equation. One register consists of equations, constants, and variables coupled with properties of real numbers. The other register consists of matrices coupled with the properties of matrix algebra. The objects in each register differ, as do the rules for manipulating those objects. While a linear system and a matrix equation can indicate the same mathematical entity, the two representations reside in different registers; transforming a representation from one of the registers to the other is an example of a conversion. I further discuss the distinctions in describing my theory, the Theory of Quantitative Systems, later in this chapter.

Duval (1999) asserted that the cognitive activity required for treatments differs from the cognitive activity required for conversions, compounding the complexity of mathematical thinking. While Duval used the term *transformation* to speak of both treatments and conversions, I will use the term *translation* in the more specific context of external symbolic inscriptions of the linear systems which are the focus of this study. While Duval's use of *transformation* applies to a variety of contexts, including graphical, I will use *translation* to refer to transformations where the context is strictly algebraic. I distinguish between translations within registers and between registers as treatments and conversions, respectively, as Duval did.

Congruence & Incongruence of Transformations

Duval described how transformations that are incongruent are more cognitively challenging than transformations that are congruent. He went further to claim that incongruent transformations may be an "impassable barrier" for some students (p. 123, 2006). He described a *congruent* transformation as one where "the representation of the starting register is transparent to the representation of the target register", where "conversion can be seen like an easy translation from unit to unit" (p. 9, 1999). I take *transparent* to mean visually similar and acknowledge that visual similarity is a subjective criterion. I take *units* to mean sub-pieces of notation contained within algebraic expressions and/or equations. Following Duval, I use *source* to refer to originating representation, and I use *target* to refer to the resulting representation.

From my perspective, an example of a congruent transformation is writing an augmented matrix for a system of linear equations. Consider the following.

$$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - 1z = 0 \end{cases} \quad \blacktriangleright \quad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 3 & -1 & 2 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

Writing an augmented matrix for a linear system is basically rewriting the same form without the variables. For me, this qualifies as a congruent translation since the source register is transparent to the target register and the translation happens unit by unit. That is, the target representation is visible from the source representation and units are not rearranged in the process of translation. I see the following as an example of an incongruent transformation.

$$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - 1z = 0 \end{cases} \quad \blacktriangleright \quad x\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

For me, writing a vector equation from a system of equations seems incongruent since one may focus on equations, and therefore (horizontal) rows, in the linear system;

however, the vector equation consists of (vertical) column vectors. I see the source and target representations as visually dissimilar. Further, a reassignment of units occurs when variables in the source representation take the role of scalars in the source representation; a commutative rearrangement also occurs. Since units of the equations differ and are rearranged, this is not a unit-by-unit translation. Thus, this translation is incongruent by both of Duval's criteria (transparency and unit-by-unit translation).

My Use of Duval's Theory of Semiotic Representation Registers (TSRR)

Considering Duval's TSRR, I analyzed the potential complexity involved in working with systems of equations; a description of my theory follows. Contemplating the theory allowed me to parse out details that may often go undetected in working with linear systems. Thinking in terms of Duval's theory along with analysis of students' written exam data led me to complete a thought experiment related to changes of register. The result was an interview protocol aimed at capturing students' conceptions of the multiple ways of representing a system; clinical interviews were conducted thereafter.

Comprehending Linear Systems: Foundations for a Theory

Reflecting on Duval's Theory of Semiotic Representation Registers, TSRR (1999, 2006, 2017), I analyzed working with systems of equations. My decomposition through the lens of Duval's theory illustrates potential cognitive complexity of comprehending systems as described below.

Hereafter, I use "linear systems" or "systems of linear equations" to refer to the common algebraic listing of equations for which simultaneous solutions are sought, if solutions exist. I use the term "quantitative system" or simply "system" to refer to the set of quantitative relationships represented by a linear system. I devised the idea of a *quantitative system* to be consistent with Duval's notion that a mathematical object is an abstract entity accessible only through semiotic systems. In that case, linear systems like the one shown in Figure 1 are but one way of denoting the quantitative systems under consideration. Other representations of the quantitative system are possible; I focus on four possibilities. Theoretically, I do not give the linear systems representation primacy over any other representation for a quantitative system.

$$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - z = 0 \end{cases}$$

Figure 1. A linear system, one way to represent a quantitative system.

Four symbolic representations of a quantitative system are: 1) a linear system, 2) a vector equation, 3) an augmented matrix, and 4) a matrix equation; examples are shown in Figure 2. Of the four ways to denote a quantitative system, the linear system has often been taken as the object of study; the other representations are taken as alternative means of expression in the study of the linear system. Payton (2017) took a slightly different but similar perspective by considering the linear system to be the primary representation. He called his perspective "systems-centric", but ultimately he concluded that the four denotations are "representations of one another" (Payton, 2017, p. 90). Either way, a study of quantitative systems is often initiated with the linear systems representation. Larson and Zandieh (2013) undertook a related study that looked at students' interpretations of the matrix equation $A\vec{x} = \vec{b}$. They found that one way that students interpret equations of the form $A\vec{x} = \vec{b}$ is to think of such equations in terms of a linear system; further details are given in Chapter 3 Literature Review. Rather than focus on any particular representation, I find it useful to think in terms of a quantitative system as the object of study and to consider each representation of the quantitative system as highlighting different facets of the set of quantitative relationships.

Linear System	Vector Equation	Augmented Matrix	Matrix Equation
$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - z = 0 \end{cases}$	$x\begin{bmatrix}2\\3\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\\-1\end{bmatrix} + z\begin{bmatrix}-1\\2\\-1\end{bmatrix} = \begin{bmatrix}5\\-1\\0\end{bmatrix}$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 5 \\ 3 & -1 & 2 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right]$	$\begin{bmatrix}2 & 1 & -1\\3 & -1 & 2\\1 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\-1\\0\end{bmatrix}$

Figure 2. Four ways to represent a quantitative system.

Duval (2006) conjectured that comprehension in mathematics requires the coordination of at least two registers of semiotic representation; he theorized that at any time, we have at least two registers engaged. (One of these is often natural language, which is not accounted for in this investigation.) If no particular representation of a quantitative system is given prevalence, six pairings of representations are possible from the four representations shown in Figure 2. However, research has shown that reversibility is not automatic (Krutetskii, 1976; Pavlopoulou, 1994), so translation between the pairings should be considered in each direction. The 12 resulting types of translations are shown in Table 1.

Table 1

Possible Translations of Representations of a Quantitative System

Linear System	►	Augmented Matrix
Linear System	►	Vector Equation
Linear System	►	Matrix Equation
Augmented Matrix	►	Vector Equation
Augmented Matrix	►	Matrix Equation
Vector Equation	►	Matrix Equation
Augmented Matrix	►	Linear System
Vector Equation	►	Linear System
Matrix Equation	►	Linear System
Vector Equation	►	Augmented Matrix
Matrix Equation	►	Augmented Matrix
Matrix Equation	►	Vector Equation

Analyses of Translations

The first two translations listed in Table 1, the **linear system ► augmented matrix** translation and the **linear system ► vector equation** translation, were used to exemplify a congruent transformation and an incongruent transformation, respectively, earlier in this chapter. Taken together, they illustrate the dissimilar nature of different translations. I will briefly restate the discussion of those translations and make some additional comments on them when appropriate. In addition, I discuss **matrix equation ► vector equation** translation and the **matrix equation ► linear system** translation in greater detail to further illustrate how translations differ. Further, the latter two translations merit further commentary since they at times are taken as definitions.

Linear system ► augmented matrix translation.

$$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - 1z = 0 \end{cases} \quad \blacktriangleright \quad \left[\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 3 & -1 & 2 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

An augmented matrix for a linear system as shown above is basically the same form without the variables. I see this as a congruent translation since the source register is transparent to the target register and the translation happens unit by unit (Duval, 1999). (See the discussion of Congruence and Incongruence of Transformations for full details.) I note that in his dissertation study, Payton (2017) documented that students gave verbal descriptions consistent with viewing the augmented matrix as "shorthand" (p. 106); this may not be surprising since he had used the word and related perspective in his teaching. Nonetheless, he provides empirical evidence that some students were operating with the idea of "shorthand" in mind.

Linear system ► vector equation translation.

$$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - 1z = 0 \end{cases} \quad \blacktriangleright \quad x\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

Earlier I used the translation above to exemplify what I see as an incongruent translation. (See the discussion of Congruence and Incongruence of Transformations for full details.) I see the source and target representations as visually dissimilar. Further, variables become scalars and a commutative rearrangement occurs. Thus, this translation is incongruent by both of Duval's criteria (transparency and unit-by-unit translation).

Matrix equation ► vector equation translation.

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} \quad \blacktriangleright \quad x\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

For this translation components of the variable matrix become coefficients (scalars) of column vectors and a rearrangement occurs; I see this as an incongruent translation. If this translation is performed directly in a single step, it relies on the definition of matrix-vector product as specified by Lay, Lay, and McDonald (2016a); theirs is a less traditional way of defining matrix multiplication. Lay et al. (2016a) refer to the matrix-vector product as demonstrated above as a "modern approach to matrix multiplication" (p. ix). They use the approach for a columnar focus which allows for earlier introduction of applications and is suggestive of vector space notions.

Matrix equation ► linear system translation.

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} \quad \blacktriangleright \quad \begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - 1z = 0 \end{cases}$$

To me, the source representation for this translation is not transparent to the target representation. Further, I do not see unit-by-unit translation since one instance of *x* in the source representation becomes three instances of *x* in the target representation. I chose to describe this translation since it mathematically requires the application of matrix multiplication in the traditional dot-product sense.

Taking six pairings of representations and accounting for reversibility, 12 translations are possible. Analyzing those translations in terms of congruence and/or incongruence is one way to illustrate the disparate nature of various translations. I have provided an analysis of four of the translations as a demonstration; similar analysis of the eight remaining translations would further elucidate the complexity involved in working with quantitative systems.

Concluding Remarks on the Complexity of Working with Systems

The preceding analysis illustrates the complexity involved in working with systems. Fine-grained analysis, allowable through consideration of Duval's theory (TSRR), distinguishes implicit, often taken-for-granted intricacies. Numerous translations are possible, and the nature and demands of each translation differ. Considering translations may be important in understanding the challenges that linear algebra students encounter. I incorporate the preceding analysis into a new theory, the Theory of Quantitative Systems (TQS). I summarize and rationalize the TQS at the end of this chapter.

Preliminary Analysis of Written Data

The following task appeared on the first exam of the semester for two sections of a junior-level undergraduate introductory linear algebra class. Grading the exams revealed that, surprisingly, students struggled to successfully complete the task.

> State the vector solution for the following equation. If you row reduce on your calculator, list the matrices before and after row reducing. If you use another method, indicate the method you use.
>
> $$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Figure 3. A "simple" exam question.

When initially considering students' responses to the question, I decided that coding the responses according to whether the student had used a matrix dot product (MDP), a matrix vector product (MVP), or no product (NP) might provide some insight. By matrix dot product (MDP) I mean the usual binary operation (of multiplying rows by columns) where the product of two matrices results in a matrix; that is, by matrix dot product I mean the operation associated with the closed mathematical system known as matrix algebra. By matrix vector product (MVP), I mean the subcase of the matrix dot product where one of the matrices is a column matrix which may be referred to as a vector and where computations are rearranged so that components of the vector are taken as scalars (coefficients) of columns of the matrix. The computational result is the same as taking the matrix dot product; however, the approaches differ structurally. (See the discussion of incongruent transformations for details.) The matrix vector product can be characterized as a linear combination of columns, highlighting a principal idea in linear algebra (linear combination) and allowing for the consideration of broader, more abstract vector space ideas.

Coding the written data revealed that 26.3% of students used notation that seemed consistent with using a matrix dot product (MDP), 31.6% of students used notation that seemed consistent with using a matrix-vector product (MVP), and 42.1% had work that did not seem to indicate any type of product use. I was aware that students had been required to demonstrate that they could multiply matrices on a quiz that came just before Exam 1. That fact along with my data analysis led me to hypothesize that students might approach the problem differently and/or with more success given more time to incorporate their knowledge of matrix multiplication. As a result, I asked the instructor to run the same question again on Exam 2. The coding of the results on Exam 2 revealed that 10.5% of students used notation that seemed consistent with using a matrix dot product (MDP), 26.3% of students used notation that seemed consistent with using a matrix vector product (MVP), and 63.2% had work that did not indicate that any type of product (NP) was used.

Comparing Exam 1 data with Exam 2 Data showed a migration away from the use of any type of product. This led me to take a closer look at the few instances of product use on the second exam. Upon closer inspection of the written responses that I coded as matrix-vector product (MVP), I became increasingly convinced that the students, for the most part, had not used the matrix-vector product. The notation students used which looked like notation associated with a matrix-vector product seemed to align with their attempts to express their answers in vector form as required by the instructions. Largely, their written work did not support attempts to use the matrix-vector product to solve the problem.

Upon closer inspection of the few responses that I had coded as matrix dot product (MDP) on Exam 2, I deduced that moves from the row-reduced matrix to a system to the final solution could indicate that a student was simply writing an equation to correspond to each row of the matrix. While it is true that each row of an augmented matrix corresponds to an equation in a system of equations, not knowing the underlying mathematical reasons could be potentially problematic. Passing back through the data with this in mind left me with the impression that not only had the students failed to use the matrix-vector product (MVP) in their solutions (as described above), but also, they likely had not used the matrix-dot product (MDP). Additionally, I realized that the same mental shortcut (rows correspond to equations) could be used as the initial step of solving the problem, and the written result would look the same as if they had applied a matrix dot product as an initial step. Thus, I became doubtful about responses that I had coded as MDP. The possibility existed that students were using a “rows correspond to equations” heuristic rather than the matrix dot product to solve the equation. I use *heuristic* to mean a rule of thumb or shortcut that allows one to solve a problem without doing all the computing or intermediate processing. In summary, all of my analysis of the written data revealed no clear evidence that students had used products of any kind. Further, the analysis raised the question: do students even see or conceive of a product when presented with a matrix equation like the one that appeared on the exam?

In a discussion, the instructor of the course concurred that the students were likely not using the matrix-vector product. The instructor also concurred that some students were likely using a heuristic like “rows correspond to equations” to arrive at their answers. I see this as an issue: when experts apply heuristics, they are often aware of the underlying mathematical processes and computations that allow for the shortcut. However, students who attempt to use shortcuts packaged by experts often apply them incorrectly in ways that seem anomalous to the expert; this is a result of the students’ lack of awareness of the underlying mathematical justifications for the shortcut. I believe expert knowledge needs to be unpacked and an investigation of the conceptions underlying students’ use of heuristics is needed to provide better support for students in their learning. Such efforts could inform moving students beyond thoughtless applications of "rules".

Analysis of the written exam data provided some insights; in addition, discussions with the highly-experienced instructor provided perspectives on student understanding that went beyond content knowledge. However, questions arose that could not be answered by the written data, so I began to consider conducting clinical interviews with the students. Further, since students' difficulty with the exam question could not be accounted for based on matrix multiplication, I formed a new hypothesis that changes of register might be contributing factor. Following up on my conjecture about registers, I analyzed the exam question through the lens of registers of representation; that analysis is described below.

A Thought Experiment with Registers

My analysis of students' written work for the exam question (Figure 3) led me to ask: What would an analysis of this problem look like through the lens of registers of representation? I had the expectation that the analysis could provide insights into productive ways to think about systems. The following thought experiment emerged in answer to my question.

Consider the following registers that can be used in working with systems of equations: systems of linear equations, vector equations, augmented matrices, and matrix equations. Giving differing degrees of attention to maintaining register, the following three scenarios are possible. To be clear, the following are not typical ways of solving;

they are hypothetical. The scenarios show contrasts that emerge when varying degrees of attention are given to maintaining or changing registers.

Scenario 1: Working in the Matrix Register with Row Reduction

I see the equation in the exam question (Figure 3) that I analyzed as a matrix equation. For minimal changes of register using a row reduction approach, one would need to know that the matrix equation can be represented by an augmented matrix that can be row reduced as follows.

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

If we use this result while staying in the matrix register, we get the following.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

How we proceed depends on our previous knowledge. Say we only know the definition of a matrix-vector product (MVP) as defined by Lay, Lay, and McDonald (2016a); that is, say we are unfamiliar with the matrix dot product (MDP). Then our next move must be into the vector register as shown below.

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Computing and simplifying as follows yields the solution in vector form.

$$\begin{bmatrix} 1x + 0y \\ 0x + 1y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This solution path consists of moving through the various registers in the following way.

MATRIX → AUGMENTED MATRIX → MATRIX → VECTOR

This approach is especially advantageous since the solution is in vector form as specified by the instructions.

Scenario 2: Working in the Matrix and Systems Registers with Row Reduction

At the point above where we assumed we were only equipped with the definition of matrix-vector product (MVP), now assume that we are equipped with matrix multiplication as a binary operation (MDP). (Lay et al. (2016a) delay this idea until chapter 2). Then we can write the matrix equation in the systems register as follows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 1x + 0y = 0 \\ 0x + 1y = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

We must now express the solution in vector form as instructed.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This solution path consists of moving through the various registers in the following way.

MATRIX → AUGMENTED MATRIX → MATRIX → SYSTEMS → VECTOR

Scenario 3: Solving without Row Reduction

If we are equipped with matrix multiplication as a binary operation (MDP) in the more traditional sense than Lay et al.'s "modern definition" (p. ix, 2016a) of the matrix-vector product, we can move out of the matrix register right away, rewriting the original matrix equation as a linear system which can be solved without row reduction (using substitution or elimination methods).

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 1x - 1y = 0 \\ 1x + 0y = 0 \end{cases}$$

$$\begin{cases} 1x - 1y = 0 \\ \qquad\quad x = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This solution path consists of moving through the various registers in the following way.

MATRIX → SYSTEMS → VECTOR

Discussion of the Thought Experiment.

The three cases above are not exhaustive, but they allow for consideration of the potential complexity in a seemingly simple task. They also provide a contrast to typical ways of solving. Distinguishing between registers of representation and the scenarios above may seem inconsequential and/or frivolous to the expert; however, empirical studies are necessary to establish the nature of students' conceptions while working with such problems. After developing the various scenarios, I enacted this thought experiment with the course instructor. Subsequently, I incorporated my analysis and findings from interactions with the instructor into an interview protocol (Appendix B) which can be used as a tool in the investigation of students conceptions; further details about the development of the interview protocol are given in Chapter 4 Methodology.

The Theory of Quantitative Systems (TQS)

My analysis of student exam data and subsequent thought experiment with registers (both described earlier) occurred alongside my continued study of Duval's theory. I made an effort to get more specific about how I was thinking about the Theory of Quantitative Systems and how it depended on and meshed with Duval's theory (1999, 2006, 2017). The result is the following description of my ways of thinking about the theories. My discussion includes a distinction between the mathematics and the cognition which I do not see as apparent in existing literature that appropriates Duval's theory.

What Is the TQS?

The Theory of Quantitative Systems is a perspective that all semiotic representations that may arise in working with a linear system have equal precedence; no particular representation is given primacy. That is, no particular representation is chosen as the object of study while all other representations are taken to be pointers to that particular representation. Rather, all representations are taken as pointers. (Figure 2 shows the four representations I focus on for this study.) This raises the question: what are the pointers signifying? I assert that the object of study (the signified) is the set of quantitative relationships that could be regarded as underlying the various written semiotic representations; I refer to the underlying set of quantitative relationships as a *quantitative system*.

Identifying a quantitative system as the object of study and making all representations equitable allows for a clearer analysis of the complexity involved in navigating the various semiotic representations when solving linear systems. Further, the construct *quantitative system* establishes an invariant that is missing from perspectives that take one of the representations as the object of study (*the represented*) and assert that another representation (*the representation*) signifies the first; those perspectives often allow variation in what may be chosen as the object of study. Further, perspectives that all the semiotic representations are representations of one another may leave one to wonder if we are signifying anything beyond another collection of notation. (This topic is addressed more fully in Chapter 3 Literature Review.) Identifying an invariant object, even an abstract one like a *quantitative system*, pinpoints a single object of study and may allow for new, productive ways to think about linear systems.

An Underlying Assumption

Underpinning the TQS is an assumption about what is meant by *mathematical object*. My development of the theory rested on adoption of Duval's (1999, 2006, 2017) notion of a mathematical object; he described a mathematical object as an entity that we only have access to in the physical world through a semiotic system. Duval's ideology necessarily means that a semiotic representation itself, as something concrete and perceptible, is not a mathematical object. Then what is the mathematical object involved in the study of linear systems? My answer, consistent with Duval's notion of mathematical object, is a set of quantitative relationships that underlie the notation, a quantitative system.

One Level of Cognitive Complexity

Considering all representations as equitable allows for the consideration of all translations between the representations rather than a focus on certain translations. Given the four representations targeted in this study and accounting for reversibility, 12 translation types are possible. (For details, see the previous section "Comprehending Linear Systems: Foundations for a Theory".) Duval's (1999) constructs *congruence* and *incongruence* (of transformations) allow for distinctions to be made between what the various translations entail. I consider my analysis of the numerous translations to be an analysis of the mathematics. While the analysis supports an argument for cognitive complexity in working with systems, much more can be said in terms of cognition. A discussion of additional cognitive complexity requires a keen focus on the definition of *register* and applying the definition at a level of granularity that I have not found in literature. I give the details below.

New Categories for *Registers of Representation*

When Duval's theory is described simply as a theory of "multiple representations", the discussion rarely includes specificity about what constitutes a *register of representation*. As a result, what are taken to be categories for registers are quite broad. An example is Pavlopoulou's (1994) dissertation study which is detailed in Chapter 3 Literature Review. For her, categories of registers were: graphical, symbolic, and tabular; the distinctions allowed her to obtain important and convincing results in linear algebra. Similar categories, while not explicitly discussed in terms of "registers", have been productive for analysis in other contexts; the function concept is an example. I assert, however, that a greater level of specificity is possible, and even necessary, to better understand challenges in the learning and teaching of linear algebra.

As I stated when discussing Duval's theory (TSRR), he used *register of representation* to mean a system of signs and symbols used to communicate along with the ways those signs and symbols are used to represent and process mathematical thinking; *registers* include both representations and a means of processing (Duval, 1999). This means each of the four representations for a quantitative system (linear system, vector equation, augmented matrix, and matrix equation) is suggestive of a *register*; each representation resides within a different mathematical system with its own rules of association and means of computation. For example, vector equations involve scalars and vectors along with the rules of vector addition and scalar multiplication, whereas matrix equations are composed of matrices that can be manipulated according to the rules of matrix algebra. Rather than adopting register categories like graphical, symbolic, and tabular, I see declaring each mathematical system to be a *register of representation* as consistent with Duval's notion. For example, matrix algebra is a register since it involves specific representations (matrices) and rules for associating them (operations and properties of matrix algebra). As a result, broad categories previously identified as "symbolic" or "algebraic" or "analytic" can be broken down into subcategories for closer consideration. This is what my study does, and I argue that linear algebra demands the additional specificity.

Another Level of Complexity

Now that I have established that, for me, each mathematical system represents a different register, I am positioned to speak about additional cognitive complexity. The 12 translations that I have discussed are translations across registers (conversions).

Treatments within a register, like manipulating a vector equation using properties of vector spaces, are not addressed by the 12 translations. If we appropriate Duval's (1999, 2006, 2017) cognitive hypothesis that the kind of cognition required for conversions differs from the kind of cognition required for treatments, we can imagine an intensely complex scenario. The complexity goes beyond acknowledging that various semiotic systems and combinations of representations make working with systems challenging. The hypothesis that cognitive mechanisms for working within a register differ from the cognitive mechanisms for working across registers takes the discussion of complexity to a new level. While variety and combinations of representations reveal complexity, much more can be said if Duval's cognitive hypothesis is considered.

An Open Question

The TQS provides a new way to think and speak about systems of equations. While the idea of a quantitative system is abstract, it establishes an invariant object of study which is not a semiotic representation. Thus far, the theory has allowed for new explication of complexity involved in working with linear systems; the TQS may also have pedagogical implications.

One could argue that, from a practical perspective, students only need knowledge of a few of the translations to solve linear systems. However, a good command of all translations is likely an asset rather than a liability. In addition, one who argues for the need for only a few translations while solving may not recognize the implicit expertise involved in knowing which translations are optimal. That is, the sequence of translations along a solution trajectory may seem quite natural to experts because of their experience and proficiency. Their expertise has focused them on a small subset of the translations in ways that they may not realize. Is the same true of the student experience? Is the learner able to focus on the required subset of translations that come up in usual solution trajectories, or does the learner have a murkier experience of multiple representations which are not related in any particular way? I argue that we do not know; a tool like the TQS may help us answer such questions.

Conclusion

The preceding details the theories that influence my study, including a theory I developed. To clarify the similarities and differences between Duval's theory and my own, a one-page illustration of the two appears as Appendix A. Taken together, the theories, analysis of student written work, and thought experiment described in the chapter provoked questions about students' conceptions of systems, including those that have already been documented in the literature in the field. The conceptions that have been documented are described in Chapter 3 Literature Review. Details of my investigation are described in Chapter 4 Methodology. Only through consideration of the numerous facets described in this chapter could I conceive of the research question: What is the nature of undergraduate students' understandings of multiple analytic representations of quantitative systems?

CHAPTER 3

LITERATURE REVIEW

RUME (Research in Undergraduate Mathematics Education) is a rather new field of academic study. Situated within RUME, studies in linear algebra are even more current. As a result, the volume of literature is somewhat limited. In what follows, I first discuss literature which is general in nature, then I discuss literature that is specifically relevant to systems of equations.

General Literature

This discussion of general literature begins with a broad overview and increases in specificity. First, I discuss overall work in RUME of which linear algebra was a part. Next, I discuss work by done by Hillel (2000) specifically focused on linear algebra. Hillel (2000) addressed overall issues in the learning and teaching of linear algebra, and he more specifically addressed issues related to notation. Finally, I discuss two educational studies based on Duval's Theory of Semiotic Representation Registers (1999, 2006, 2017) in the context of linear algebra. I discuss these works of literature to situate my study about systems within them.

The State of Linear Algebra Literature

Artigue's Summary of Post-Secondary Math Ed Research. In their 2007 report, Artigue, Batanero, and Phillip discussed developments in post-secondary mathematics research from 1992 through 2005. Their bases for describing developments in linear algebra research were works from a volume edited by Dorier (2000) about the teaching of linear algebra. Dorier's (2000) volume is divided into two parts. Part I gives a thorough history of how the theory of vector spaces developed; part II consists of eight chapters that address the teaching and learning of linear algebra. Rasmussen and Wawro (2017) described Dorier's volume as "more historical-epistemological" (p. 552) than work that has been done more recently. Rasmussen and Wawro (2017) classified the works in Dorier's (2000) volume as either (1) categorizing students' thinking, (2) discussing the use of geometric reasoning, or (3) discussing the difficulties students experience with the formalism of linear algebra.

Rasmussen's and Wawro's summary of RUME research. Moving forward to discuss post-calculus RUME (PC-RUME, Post-Calculus Research in Undergraduate Mathematics Education) from 2005 through 2016, Rasmussen and Wawro (2017) found 36 articles on the teaching and learning of linear algebra that they considered to be of sufficient research quality; 24 of those made student thinking their focus. In the time frame 2005-2016, several frameworks were developed, several studies used APOS theory to investigate students understanding of linear dependence and linear independence, and a small number of studies investigated students' understandings of eigenvectors and eigenvalues. Rasmussen and Wawro (2017) questioned the pedagogical implications of some of the studies they identified, echoing Radu's and Weber's (2011) argument that deficit accounts of students' conceptions do little to inform pedagogy. They joined Radu and Weber in calling for research approaches that document students' advancing conceptions and for pedagogical approaches that build from the students' understandings. An example of one such study was conducted by Wawro, Sweeney, and Rabin (2011);

they investigated how students productively incorporated the formal definition of subspace into their concept image (Tall & Vinner, 1981).

Artigue's retrospective analysis on RUME. Nine years after Artigue et al. (2007) reported on post-secondary mathematics research, she provided her updated retrospective analysis of how RUME developed (Artigue, 2016). She stated, "research first focused on discontinuities, but progressively became more sensitive to the essential role played by connections and flexibility in teaching and learning processes" (2016, p. 7). Specific to linear algebra, Artigue cited the work of Dorier and Sierpinska (2001) for clarifying the various connections necessary in linear algebra: "connections between different languages (geometrical, algebraic, abstract), between different registers of representations (graphical, algebraic, symbolic representations, tables), between Cartesian and parametric points of view, and between synthetic-geometric, analytic-arithmetic, and arithmetic-structural modes of reasoning" (Artigue, 2016, p. 7). Artigue (2016) proceeded to conclude that while connections and flexibility are not specific to university mathematics: (1) their intensity increases significantly at the university level, and (2) students are often for the first time given the autonomy to manage the flexibility and connections.

Empirical studies in linear algebra 2009-2018. I participated in a research project that conducted a review of linear algebra literature published 2009-2018 in the top twenty mathematics education journals as identified by Williams and Leatham (2017); we identified 54 papers that included some form of empirical results. Of the works we identified, three were relevant to systems of equations; they are described more fully in the systems of equations section of this chapter.

Hillel's Work in Linear Algebra

General sources of difficulty in linear algebra. After teaching linear algebra for many years, Hillel (2000) was able to identify three sources of difficulty specific to the learning of linear algebra. The sources of difficulty that he identified are: (1) the existence of several languages or modes of description, (2) the problem of representations, and (3) the applicability of the general theory. I discussed issues related to *representation* in chapter 2; his *modes of description* as described below are also especially relevant to the present study on notation. Briefly, in terms of this study, a representation is a written indicator of a particular mathematical entity, whereas a mode of description is an entire language for expressing the mathematics of linear algebra. A representation exists within a mode of description as described below.

Hillel's modes of description. Hillel (2000) described three modes of description used in linear algebra: the geometric, the algebraic, and the abstract. He described each mode of description as a language allowing for expression. He used *geometric mode* to refer to the language and concepts of 2- and 3-space, lines, and planes. He used *algebraic mode* to refer to language and concepts specific to $\mathbb{R}^n$, the general Euclidean spaces. He described the *abstract mode* as dealing with language and concepts of the general theory of vector spaces; the abstract mode of description is the language of vector spaces, subspaces, span, dimension, operators, and kernels. At times Hillel's modes of description are confused with Sierpinska's (2000) modes of reasoning. While the theories of the two researchers are related and each theory has three categories, they are different lenses. Hillel's (2000) modes of description are more relevant to a study on notation and representation than Sierpinska's (2000) modes of reasoning. Further, Hillel (2000) noted that each of his modes of description may be used within any one of Sierpinska's modes of thinking. That is, while engaging in, say, analytic-arithmetic thinking, one may use geometric, algebraic, and abstract modes of description to represent and consider ideas. I see Hillel's categories as helpful in illuminating the complexity in linear algebra; registers of representation are another way to partition the content of linear algebra for different, but complementary, insights.

Hillel's study of linear algebra lecturers. Experts often move fluidly between modes of description and various representations giving little notice to the nuances of notation and meaning. Hillel (2000) documented this in a study of five videotaped sessions of lecturers teaching on the topic of eigenvectors and eigenvalues. Hillel's (2000) review of the video showed that much content remained implicit during the lectures, and he conjectured that this was a potential obstacle to students' learning processes. He described his modes of description as a means for experts to decompose their expert knowledge so they can better address challenges students may face. That is, the framework may equip experts with a means of better connecting with students.

Students' Transformations Between Registers

Two linear algebra studies used Duval's TSRR (1999, 2006, 2017) as their theoretical basis. One was Pavlopoulou's (1994) dissertation study, a classical experiment based on Duval's early work written in French. Another is Sandoval's and Possani's

(2016) more recent qualitative study of students' conceptions in $\mathbb{R}^3$. Both are described in greater detail below.

Pavlopoulou's Classical Experiment. Pavlopoulou (1994) studied students' coordination of changes in register in linear algebra for her dissertation project. She used classical experimental design along with a teaching sequence she created to conduct her investigation. Following a pretest, experimental groups were instructed using an eight-hour teaching module focused on changes in register. Pavlopoulou (1994) identified three categories which for her were registers. Her analysis considered the coordination of the following:

1) The graphical register where a vector is represented by an arrow in $\mathbb{R}^2$ or $\mathbb{R}^3$.
2) The symbolic register where a vector is represented by the linear combination of any two or three vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.
3) The table register where a vector is represented by a column matrix with two or three rows.

After administering a post-test and analyzing her data, she found that while students in the control group showed little improvement between pretest and post-test, students in the experimental group reached near total success on post-test questions related to change of register. Further, while not as marked as the questions related to change of register, the experimental group's success with other types of questions was statistically significant when comparted with the control group. Pavlopoulou (1994) found that transformations between two registers are often more difficult in one direction than in the opposite direction. That is, the direction of a transformation between registers is significant; this finding was discussed in the previous chapter as an important facet in the Theory of Quantitative Systems. In that theory, I asserted that each direction of a translation should be given separate consideration.

My discussion of Pavlopoulou's study is based on Artigue's, Chartier's, and Dorier's (2000) review of her work, which is written in French. Artigue et al.'s (2000) thorough review and critique of Pavlopoulou's study discussed the convincing nature of her results and analysis. They called for investigations that would check the consistency of her results in contexts that are richer and more complex. I am aware of only one such follow-up study; that study by Sandoval and Possani (2016) is described below.

Sandoval's & Possani's Study in $\mathbb{R}^3$. Sandoval and Possani (2016) designed activities to help them evaluate students' flexibility in moving between registers, which they categorized as being either verbal, algebraic, or geometric. While Pavlopoulou (1994) asked students direct questions about specific conversions, Sandoval's and Possani's (2016) tasks allowed for some choice of register; this was an answer to Artigue et al.'s (2000) call for studies on registers in richer contexts. Their study focused on students' conceptions of vectors and planes in $\mathbb{R}^3$ and centered strongly on the geometric.

Sandoval and Possani (2016) administered their activities to 60 undergraduate students in Mexico who were enrolled in introductory one-semester linear algebra courses. Students were allowed up to one hour to work individually prior to group discussions. While they noted a couple of instances of a student exhibiting cognitive flexibility, they noted many instances of students' difficulties in coordinating registers. Additionally, they found that students rarely used a combination of algebraic and geometric registers in completing an activity, noting that once a student had chosen a register, they tended to stay within the register. While they credited Duval's theory with allowing them to identify the issues of working within and across registers, they acknowledged that the categories of analysis are ambiguous. That is, they conceded that "unifying registers" (Sandoval & Possani, 2016, p. 125) for Duval's theory have not been delineated and merit further investigation. Further, they noted the limitations of using written work to infer register use and called for interviews with students. My study addresses the gaps they highlighted in their efforts to answer Artigue et al.'s (2000) call to complement and build on Pavlopoulou's work.

Literature on Systems of Linear Equations

Research into undergraduate students' conceptions of linear systems is sparse. I am aware of work done by Harel (2017), Larson & Zandieh (2013), Payton (2017), and Zandieh & Andrews-Larson (2015). Two areas related to the present study were addressed, albeit incompletely, by Payton (2017) and Zandieh & Andrews-Larson (2015). Those areas, respectively, include students' conceptions of translating between the various representations of a system of equations (Payton, 2017) and students' determination of solutions to a system of equations (Zandieh & Andrews-Larson, 2015). Additionally, in their compilation *Challenges and Strategies in Teaching Linear Algebra*, editors Stewart, Andrews-Larson, Berman, & Zandieh (2018) included three chapters which addressed issues related linear systems; those chapters by Oktaç, Trigueros, and Pauer are discussed below.

Harel's Linear Systems Approach

Harel (2017) offered a broad discussion of obstacles in learning linear algebra as described in literature. One of the six sources of students' difficulties that Harel (2017) was described earlier in this document: one source of difficulty is instruction that "blends various contexts, modes of description, and notation, resulting in difficulties recognizing the same concept in different contexts" (Harel, 2015, p. 71; Dorier & Sierpinska, 2001; Hillel & Sierpinska, 1994). Heeding Hillel's (2000) conjecture that such blends might be detrimental to students' learning, Harel (2017) designed a teaching experiment on the theory of systems of equations in which he strictly emphasized the algebraic listing of linear equations; during the teaching experiment he did not introduce matrices as is customary. Referring to Hillel's findings, Harel states, "It was crucial, thus, for our planning to separate *the represented* (system of equations) from *the representing* (matrix representations of linear systems)" (2017, p. 81).

During his teaching experiment, Harel made sixteen observations of students' lacking conceptions and/or inabilities. One example related to this study was his observation that "the idea that a system of equations represents a set of quantitative constraints did not seem to have been self-evident for the participants, an indication of weak quantitative reason" (Harel, 2017, p. 91). I contend that highlighting the system of equations as the object of study in the lessons may contribute to this lacking conception. Removing the matrix representation as a consideration and making the systems of equations the object of study may leave unspecified (for some learners) that the algebraic listing of the system of equations may also be considered to be a representation; it may be a representation of what I have called a *quantitative system* in the Theory of Quantitative systems. In the direct quotes above, Harel refers to the system of equations as *the represented* but also describes it as a representation. From a traditional perspective, assigning the linear system dual roles is natural, especially when the content is framed as the *theory of the systems of equations*. However, I assert that the dual nature of the system of equations suggests that a new way to think and speak about systems may be productive, especially in applied courses. The Theory of Quantitative Systems addresses this issue.

Stated more directly, I see the Theory of Quantitative Systems as differing from Harel's approach in a significant way. In Harel's approach, one could take the algebraic listing of the linear system as the object of study without attending to potential underlying quantitative relationships (as his students seemed to do). That is, one might think that the study is about a set of notation; for me, this does not seem unreasonable in a theoretical course. However, in my theory (TQS), the system of equations in analytic form is simply another representation on par with the matrix representation; this highlights underlying quantitative relationships as the object of study rather making the algebraic representation the focal point. The distinction seems important in promoting a mathematics concerned with relationships and concepts rather than a mathematics of calculation and symbol manipulation.

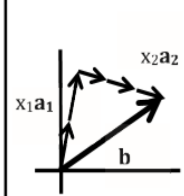
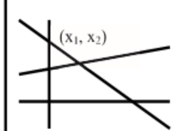
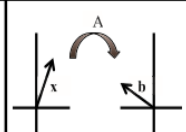
While my study approaches systems from a different perspective than Harel's (2017), I see his approach as a potentially powerful deviation from conventional methods. In conversations, Harel told me of his work on a textbook that mirrors the approach he used in his teaching experiment; that is, he is working on a text which addresses some fundamentals of linear algebra by focusing solely on systems of equations without reference to matrices. Harel's work and my own are important given the increasing demands for linear algebra courses which address the needs of a broadening demographic of students. That is, the applications for linear algebra have proliferated; as a result, linear algebra content has become appropriate for a much larger demographic than mathematics majors. Alternative perspectives like Harel's and my own may help address that changes.

Interpretations of $\mathbf{A\vec{x}} = \vec{\mathbf{b}}$

In working with linear algebra students, Larson and Zandieh (2013) identified three views of the equation $A\vec{x} = \vec{b}$ in students' conceptions: the linear combination interpretation, the systems of equations interpretation, and the transformation interpretation. A linear combination interpretation involves thinking and speaking of the components of the vector $\vec{x}$ as weights on column vectors. A system of equations interpretation involves thinking and speaking of the vector $\vec{x}$ as a set of values satisfying a system of equations. A transformation interpretation involves considering the vector $\vec{x}$ to be an entity that is transformed by the matrix *A*. The researchers took the three interpretations and aligned them with ways they might be denoted in both symbolic and graphical notation as shown in the Table 2 below. The researchers included conventional notation in the "Interpretation" column as a way to help experts connect their knowledge to the various interpretations; however, their study focused on students' thinking. The conventional notation in the "Interpretation" column was simply intended to be evocative for the expert and should not be taken as indicative that the study centered on notation.

Table 2

Views of $A\vec{x} = \vec{b}$

Interpretation of A**x**=**b**	Symbolic Representation	Geometric Representation
Linear combination (LC) interpretation $x_1\mathbf{a_1} + x_2\mathbf{a_2} = \mathbf{b}$	A: set of column vectors $(\mathbf{a_1}, \mathbf{a_2})$ **x**: weights (x_1, x_2) on column vectors of A **b**: resultant vector	
System of equations interpretation $a_{11}x_1 + a_{12}x_2 = b_1$ $a_{21}x_1 + a_{22}x_2 = b_2$	A: entries viewed as coefficients $(a_{11}, a_{12}, a_{21}, a_{22})$ **x**: solution (x_1, x_2) **b**: two real numbers (b_1, b_2)	
Transformation interpretation T:**x**↦**b**, T(**x**)=A**x**	A: matrix that transforms **x**: input vector **b**: output vector	

From my perspective, the symbolic descriptions in the second column of the framework can be used to analyze the symbols students use to determine whether their written notation is most aligned with a linear combination interpretation, a systems interpretation, or a transformation interpretation. Notation aligned with a linear combination interpretation would include indications that the matrix A consists of columns along with indications that $\vec{x}$ consists of individual components and that $\vec{b}$ is regarded as a vector. Notation aligned with a systems interpretation would include indications that the matrix A consists of many individual entries a_{ij} while $\vec{x}$ and $\vec{b}$ consist of individual components. Notation aligned with a transformation interpretation would include indications that the entire matrix A is taken a single unit, as is the vector $\vec{x}$ and the vector $\vec{b}$. To be clear, I am saying that the notation can be taken as an indicator of an interpretation; however, notation alone is insufficient in categorizing students' thinking. I detailed my experience with analyzing written work in Chapter 2; that analysis revealed the need for clinical interviews to determine the nature of students' thinking.

In addition to functioning as a tool to help experts deconstruct their knowledge, Larson and Zandieh (2013) describe their framework as a diagnostic tool to help make sense of linear algebra students' seemingly idiosyncratic responses that may be a blend of interpretations and contexts. In addition, while other frameworks may require a high level of inference, I see the framework as allowing for analysis that is more objective since it clearly delineates the symbolic notation correlated with each particular interpretation. Most relevant to my study is the researchers' focus on the equation $A\vec{x} = \vec{b}$, which I take to be one of several representations of a quantitative system. Further, the task for my study is an equation of the same form.

Payton's Study of Connections Between Symbolic Representations

Payton (2017) devoted part of his dissertation project to investigating symbolic representations of linear systems. He considered students' work related to four symbolic representations of systems of linear equations: linear system, vector equation, augmented matrix, and matrix-vector equation. Citing time considerations, he limited himself to investigating students' understandings of converting between a linear system and an augmented matrix, a vector equation and a linear system, and a matrix-vector equation and a vector equation. His choices were based on his assumptions that the linear systems representation is central and that the ideal chain of translations in solving a system of equations is: matrix-vector equation to vector equation to linear system to augmented matrix. (I assume Payton's ideal chain of translations resulted from the solution trajectory presented by Lay (2011), the only linear algebra textbook that he cited.) Whether he considered bi-directionality of the translations is unclear. That is, Payton did not seem to explicitly address reversibility of translations; in class presentations, he presented linear systems and augmented matrices side-by-side. Potentially implicit in a side-by-side presentation are the assumptions that students will grasp the bi-directional relationship between the symbolic representations and that students can navigate conversion in either direction. In essence, Payton conducted a brief investigation of three of the 12 translations noted in the Theory of Quantitative Systems; much more work remains to be done.

Payton (2017) identified his perspective as "systems-centric" (p. 110), meaning that he sees the linear system of equations as the primary representation of a system. He concluded that in their study described above, Larson and Zandieh (2013) considered the matrix equation $A\vec{x} = \vec{b}$ to be the primary representation of a system. He referred to a student who took the vector equation as the primary representation as "vector-centric" (p. 108). Payton rectified the differing perspectives by claiming that all four representations he considered are "representations of one another" (2017, p. 90).

Payton's (2017) study was based on action research that he conducted over three semesters of teaching linear algebra. Action research consists of taking planned actions followed by critical reflection to develop additional actions to be taken. He used the first semester, a summer course, to familiarize himself with inquiry-oriented teaching and with performing action research. The data collection (other than his own critical reflections) occurred over two long semesters. His classes had 40 students and 60 students during those semesters, and he conducted clinical interviews with seven volunteers and nine volunteers, respectively.

One of Payton's interesting findings relevant to my study was some students' descriptions of the augmented matrix as a sort of shorthand for a linear system. One student stated, "You're just excluding the variables" (Payton, 2017, p. 89). Some students also described the augmented matrix as shorthand for a vector equation, though one student (Martin) could not explain why we use an augmented matrix to represent a vector equation. Other findings include: 1) Several students were able to write a vector equation as a linear system, 2) a couple of students could translate between matrix and vector equations, and 3) one student struggled with the definition of matrix-vector product. Payton concluded that a good understanding of two individual representations is insufficient for understanding the relationship between the two (2017, p. 91); this highlights the need for investigations involving translations. I further assert that the translations need to be considered in each direction as it is not obvious that Payton considered this.

Related to views of $A\vec{x} = \vec{b}$ discussed above (Larson & Zandieh, 2013), which Payton might refer to as *matrix-centric*, one of Payton's students took "infinitely many solutions" for a matrix equation to mean that the equation is true for infinitely many $\vec{b}$ rather than for infinitely many $\vec{x}$. I note that the students' conception is concerning since it contradicts the uniqueness of a matrix-vector product and could be potentially indicative of a belief that mathematics is inconsistent. Ultimately, Payton (2017) asserted that his study suggests flexibility in using the various representations of systems significantly influences a students' ability to "develop and evoke logical implication connections" (p. 216).

Students' Solution Trajectories for Systems of Equations

Zandieh & Andrews-Larson (2015) documented students' solution trajectories when solving linear systems, noting that students experience difficulty in determining solutions to systems despite success in writing and row-reducing the corresponding augmented matrix. Their analysis of students' written final exam data revealed a decrease in success at the point where students were to determine solutions to a system from the row-reduced echelon form of the augmented matrix for the system. Questions remain about what we can learn about the issue they identified by conducting clinical interviews with students. I see the issue they isolated as potentially related to changes in registers of representation; my study may provide important insights into whether registers of representation contribute to the issue.

Book Chapters on Teaching Linear Systems

Important, but less pertinent to my proposed project, are the works by Oktaç, Trigueros, and Pauer included in *Challenges and Strategies in Teaching Linear Algebra* (Stewart, Andrews-Larson, Berman, & Zandieh, 2018). The purpose of that compilation was stated as: a multinational project focused on promoting conceptual understanding in linear algebra through the use of challenging problems that support students in their learning (p. ix). Chapters that include a discussion of system of linear equations are discussed below.

Studies in Mexico and Uruguay. The studies that Oktaç (2018) detailed in her chapter provide insight into non-US students' conceptions of linear systems. Since the studies are largely focused on students' geometric conceptions of systems, they are marginally related to my study of multiple analytic representations of systems. However, Oktaç's findings serve as a justification for additional studies like mine. Apparently, the study of systems in Mexico and Uruguay is motivated by geometric reasoning about two intersecting, parallel, or coincident lines as is often the case in the United States. This seems to be a reasonable way to introduce systems to younger students earlier in the curriculum.

Oktaç (2018) investigated students' conceptions of "system" and "solution" across different school levels. At the undergraduate level, she worked with two groups of students: seven students at a public university in Mexico, and, subsequently, 27 students at a public university in Mexico. Oktaç (2018) found that, geometrically, students considered the intersection of two lines to be a solution to the system represented by a graph of three lines that formed a triangle. That is, rather than looking for the intersection of three lines on the coordinate plane, the students maintained their conceptions for the case of two lines: that a solution is the intersection of two lines. This conception persisted even after instruction. Additionally, Oktaç (2018) found that students could rarely write a systems of equations representation for a system from its graphical representation. If one takes "graphical" and "algebraic" as categories of registers, the two findings highlight the challenge of working within a register (the graphical register) and in working between registers (the algebraic and graphical registers). Further, while there seems to be an assumed proficiency for moving from the algebraic to the graphical, moving from the graphical to the algebraic seemed to present a different kind of challenge. This highlights the need to directly address reversibility in transformations. For me, this indicates the challenge (cognitive complexity) of working with different contexts, whether those contexts are called interpretations, modes of description, or registers of representation. More study needs to be done to pinpoint productive ways to address the complexity.

Trigueros' model approach. Trigueros (2018) discussed work that has been done on undergraduate students' understandings of linear systems, asserting that it is well known that many students experience difficulty in interpreting the solution set of a system of linear equations. Note that this echoes the findings by Zandieh & Larson-Andrews (2015) discussed earlier. Trigueros used APOS (Action, Process, Object, Schema) theory (Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, Trigueros, & Weller, 2014) to develop a cognitive progression (genetic decomposition) that may support learners in developing an increasingly enriched linear systems schema. From her analysis of student work on a traffic-flow modeling problem, she concluded that many students benefited from their work with the problem: "most of them enriched their schema by constructing relations between variables, equations, functions, solutions procedures and solution sets, with differences among them" (Trigueros, 2018, p. 38). However, she also reported "We found that persistent difficulties with variables limited students' engagement in the solution of the modeling situation. They could use some methods but could neither explain them nor apply them to complex problems" (Trigueros, 2018, p. 38). In summary, Trigueros found that student thinking was enriched by working with the modeling problem, but her analysis also revealed that students often applied methods without understanding. Her results are related to my supposition that students may use heuristics in solving systems; my study is designed to provide insight into that hypothesis.

Trigueros' chapter builds on previous work she did with other researchers. In 2010, Possani, Trigueros, Preciado, and Lozano used the traffic flow problem to elucidate the use of models in teaching linear algebra. Their approach was further explicated by Trigueros and Possani (2013) using an economic model. For the economic model, Trigueros and Possani observed: "Although the expected model consists of a system of linear equations and they had experience with this topic, students found it difficult to select variables and interpret possible relations between them" (Trigueros & Possani, 2013, p. 1790). The researchers noted even after the students arrived at a linear model to describe the economic context, the students often did not appeal to their models to answer questions about the context. That is, students tended to use raw data to make connections rather the linear models they had created. Some students were able to deduce linearity from the raw data they were provided and connect it back to the linear model they constructed given continued work with the economic application. This is a promising result; however, this situation highlights a possible gap between the experts (researchers) and students which I will now discuss.

The researchers seem to use the word "model" interchangeably to refer to the application and/or context and the semiotic representation of the situation. Students, however, seemed to have disconnected conceptions of the situation, the data, and the linear models they constructed. That is, experts may enmesh an application and its representation while students have disconnected conceptions of the various representations. Studies premised on Duval's theory and the Theory of Quantitative Systems may help clarify communication between experts and learners.

Pauer's computational approach. Pauer (2018) described a computational approach to systems of linear equations at the undergraduate level. He downplayed geometric interpretations in two and three dimensions since he views geometric reasoning as demanding and since more than three dimensions are frequently necessary in describing phenomena. His approach was a departure from the recommendations put forth by the Linear Algebra Curriculum Study Group (LACSG; Carlson, Johnson, Lay, & Porter, 1993), which emphasized leveraging geometry in a first course in linear algebra. As such, it parted from usual approaches in the United States over the last three decades, thereby, providing an alternative perspective.

Conclusion

The preceding is a discussion of the rather recent and scant literature on undergraduate linear algebra from an educational perspective. Those works exist within a developing field of academic study, RUME (Research in Undergraduate Mathematics Education). Two studies specifically addressed linear algebra issues through the lens of Duval's Theory of Semiotic Representation Registers. In addition, a few studies specifically addressed systems of linear equations. The implications of Duval's theory and the consideration of linear systems at the undergraduate level are far from being fully explicated. Very specifically, I reiterate that my study addresses the gaps Sandoval and Possani (2016) highlighted in their efforts to answer Artigue et al.'s (2000) call to complement and build on Pavlopoulou's (1994) work. More generally, my study contributes to what is written about Duval's theory and to literature documenting undergraduate students' conceptions of linear systems. A discussion about how the latter was achieved is detailed in the next chapter, Chapter 4 Methodology.

CHAPTER 4

METHODOLOGY

This chapter is structurally similar to previous chapters, progressing from general to specific. First, I discuss general methodological perspectives that were influential in shaping my study. Next, I discuss the specifics of the study's implementation. In my discussion of the study's implementation, I describe the interview protocol that I developed and used as a tool for data collection, the method used to select study participants, the manner in which the interviews were conducted, how I prepared the data for analysis, and my methods for analyzing the data.

General Methodological Considerations

This section is a general discussion of the researchers' role, the use of clinical interviews for data collection, and two differing but complementary purposes for research studies.

The Role of the Researcher

Hunting (1997) made important distinctions between the role of researcher and the role of practitioner, emphasizing that those distinctions are based on expectations. First, the researcher is expected to add to the professional body of knowledge while the practitioner is expected to facilitate learning; this requires researchers to adopt a more global focus of contributing to a broad community while practitioners effect change more locally with students in classrooms. Next, Hunting explained that while a researcher isolates a particular question and familiarizes herself with the existing knowledge base, a practitioner deals with multiple dynamic social, emotional, and cognitive factors. "A student's response to a mathematical task or question, and the teacher's interpretation of that response, is embedded in the thick soup of the classroom environment and community" (p. 147). Consequently, teachers must often make instant responses while a researcher has more time to reflect on data. Further, Hunting supposed that a researcher, with their focus on a specific topic, is likely quite well-prepared in terms of mathematical sophistication, at least on the topic of interest in theoretical terms. In contrast, he asserted that the practitioner needs to be aware of links between and within concepts.

With these distinctions in mind, this project was designed to move the field forward in its understanding of the complexity of linear algebra notions and students' perceptions of them. The ever-relevant and increasingly ubiquitous topic *systems of equations* was chosen for closer consideration at the undergraduate level. "The thick soup" of classroom and community influences (Hunting, 1997) was mitigated through the use of video recorded clinical interviews as described below. I note that in a school setting, my initial hypothesis (described in Chapter 2) that students were using matrix multiplication to answer their exam question would likely have gone uninvestigated; as a result, the new conjecture about registers of representation would not have emerged.

Clinical Interviews

A Historical Account. Clinical interviews were developed to amend for the inadequacies of experimental (quantitative) studies that did little to address the process of learning. The method's development in the Soviet Union resulted from bans on mental testing in 1936 (Krutetskii, 1976, xii). Achievement tests were allowed in schools to measure progress, but other forms of testing were discontinued. The Soviets viewed tests as only giving an idea of existing status without providing information on students' potential or the process of learning. Further, they believed tests led to labeling of students and setting of norms for both content and expectations while doing little to inform about effective instructional practices. In response to the bans on testing and related experimental (quantitative) studies, Soviet educational psychologists developed new methods such as having students think aloud as they worked problems. Students would be interviewed repeatedly over several months so that they became accustomed to being interviewed, and educational psychologists worked with teachers and their students to administer individual achievement tests. Their work resulted in a large collection of research on the learning of school mathematics. Thus, the affordances of their qualitative methods addressed the concerns that resulted in bans on testing; the new methods allowed mental processes to be traced and for instructional practices to be considered.

Krutetskii was critical of Western researchers and educators at the time who relied on tests and who treated test results rather than studying learning processes; he had insights that have more recently been addressed in Western mathematics education research. For one, he thought a classroom emphasis on the result (the answer) instead of the process would give students a false conception of mathematics. Further, he contended that beyond biological processes, abilities are created and developed through activity.

Clement (2000) highlighted the strengths of the clinical interview as a research method as 1) the ability to collect and analyze data on a subject's authentic meanings and ideas, and 2) the ability to expose structures and techniques that might go undetected through other more formal and restrictive techniques. From Hunting's (1997) perspective the method recognizes the important role of language and clarification of meanings. Constructivist clinical interview methods like those described by Clement and Hunting originated with Piaget. Acknowledging the influence of the Soviets, Steffe and Thompson (2000) attribute the serious consideration of the methods (clinical interviews and think-aloud protocols) to reports of research in the USSR that "provided academic respectability for what was then a major departure in the practice of research in mathematics education" (p. 272). The methods began to be seriously considered in the United States in the 1970s.

Characteristics of the Ideal Interview Protocol. Hunting (1997) described ideal interview protocol questions as those that are open-ended, require reflection, and allow for discussion. He described the clinical interview as an opportunity to make inferences from students' communication and emphasized that the goal is not the successful completion of the task or tasks. Additionally, he suggested that it is good practice to video record interviews for two reasons: 1) so that reflection, discussion, and the consideration of various perspectives can occur post-interview, and 2) in case the objectivity of the clinical interview is questioned. Hunting proposed that following the subject's initial response, the interview may proceed in several ways and should be allowed to unfold. He mentioned that good practices include asking for explanations for responses, solutions, and/or gestures; giving neutral responses whether or not the subject is correct; and encouraging the student to keep talking. Ultimately, Hunting described the work of the interviewer as considering the student's mathematical knowledge while pondering the question: what might explain why the student acts and responds in this way?

Investigating Students' Notational Conceptions. If we could infer students' meanings from their inscriptions alone, clinical interviews would be unnecessary. However, acknowledging that the mind of the learner is not the logical entity that mathematics is (Tall & Vinner, 1981) requires that we investigate more deeply if we want to support students in their learning of mathematics. My analysis of students' written work alone, as described in Chapter 2, was insufficient to characterize their thinking. Clinical interviews seem essential to investigating the nature of students' understanding of the various analytic representations of systems. As Krutetskii stated: "Although the mathematical structure of the solution process, the sequence of operations of which the solution is made up, may obtain what is usually a complete enough record of solution, the mental processes that characterize the solution process—consideration, reflection, comparison of different possibilities, and so on—do not find objective expression in the record" (p. 92). Video-recorded clinical interviews with students reveal more about their thinking related to linear systems than the written records of their solutions.

Generative and Convergent Studies

Clement (2000) described two major purposes of educational research as being 1) generative, and 2) convergent. Generative studies are used to identify new categories and elements of models when theory on a topic is scarce; they require a high level of researcher inference. Convergent studies focus on classifying observations according to pre-determined categories. Rather than offering a dichotomous classification, Clement

(2000) described a spectrum of research purposes from generative to convergent as follows (pp. 575-576).

(1) Exploratory Studies: These studies can be described as case studies to explore new constructs based on what stands out to the researcher. These types of studies may not be coded and or appear in journal publications. Exploratory studies are foundational for future work.

(2) Grounded Model Construction Studies: These studies use descriptions generated in exploratory studies to form theoretical models connected to specific observations in order to refine interviews.

(3) Explicit Analysis Studies: These studies give detailed connections between theory and observations.

(4) Independent Coder Studies: These studies involve coding observational patterns and calculating interrater reliabilities.

Clement advocated for a balance between theoretical work and empirical work, asserting that research based on clinical interviews is a scientific undertaking that need not be at odds with quantitative measurement methods. He concluded that “generative, convergent, and quantitative measurement methods are seen as linked complementary techniques for generation, supporting, and testing models of students’ thinking, rather than as rival approaches. Thus, both generative and convergent clinical methods have roles to play as essential elements of a scientific approach to educational research” (p. 589).

My study includes both generative and convergent aspects. In general, many aspects of undergraduate students’ conceptions of systems of equations are uninvestigated; I have documented some aspects of students’ thinking. More specifically, my study is a more microscopic view of analytic changes of register for systems of equations than has been conducted. These considerations have allowed for the generation of new realizations which may serve groundwork for further investigation. Additionally, the discussion of students' notions according to Duval's theory and my own theory allowed for distinctions to be made that would have otherwise remained implicit. In terms of Clement's (2000) spectrum of research purposes, the different aspects of the study can more specifically be described as *exploratory*, *grounded model*, and *explicit analysis* as detailed below.

Implementation of Study

In this section I discuss the development of the interview protocol (Appendix B) I used to collect data for this project, the process of obtaining research participants (volunteers who met selection criteria), the manner in which clinical interviews were conducted, how the data was assembled, and the way that the data was analyzed. These activities occurred subsequent to the analysis of written exam data and the thought experiment described in Chapter 2 Theoretical Perspective which were important stepping stones for my articulation of the Theory of Quantitative Systems.

The Interview Protocol (Development & Explanation)

The interview protocol (Appendix B) is largely a product of two years of linear algebra grading (one semester of theoretical linear algebra and three semesters of applied linear algebra) and analysis of students' written work early in the Fall 2018 semester. Other influences included reviews of numerous textbooks and attending the workshop *National Pedagogical Initiatives on Linear Algebra* at the University of Oklahoma in Fall 2018. In Chapter 2 Theoretical Perspective, I described how analyzing student exam data with a focus on matrix multiplication shaped the present study; that analysis led to a new hypothesis that the challenge for students might lie in changes of register of representation (TSRR), a theory of Duval (1999, 2006, 2017) which I had been studying. I will briefly reiterate some of the details of the written data analysis and subsequent thought experiment to more fully describe how they contributed to the development of the interview protocol. Note that I use "task" and "chosen task" to refer to solving the matrix equation $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Students in an introductory applied linear algebra course appeared to be especially challenged by a seemingly simple problem on their first exam of the semester. They had trouble successfully answering an exam question that involved a matrix equation like the one shown above, the "task" in my study. Students experienced greater success with the question when they encountered it again on their second exam, and I noted that most students used a different approach on their second attempt. Having become especially familiar with the content related to the exam question (which Hunting (1997) asserted is a characteristic distinguishing a researcher from a practitioner), I challenged myself to consider all the ways I could use the task to investigate students' conceptions. That is, I sought to maximize the ways the task could be used to elicit student feedback in a clinical interview setting. I more specifically considered how the task could be used to explore students' conceptions by building on the thought experiment described in Chapter 2 Theoretical Perspective. The thought experiment involved solving a similar task in three different ways, each which required a different combination of translations within and between registers of representation. Recall that the thought experiment was enacted with the instructor, a mathematics educator, to validate its potential for productive data collection; I made adjustments based on that interaction.

Initiating an interview by directly asking a participant to comment on multiple analytic representations of systems did not seem promising without some kind of warm up. While Hunting (1997) suggested that open-endedness is a characteristic of a good interview question, asking such an open-ended question too early seemed to carry a high likelihood that the interview would terminate prematurely while yielding minimal results. Admittedly, asking the question directly at any time had the potential to be unproductive since students may not have considered all the representations at one particular moment during their coursework; however, building up to the question through intermediate tasks and conversations seemed like a favorable approach for eliciting quality responses. Therefore, I constructed the interview protocol so that students would have the opportunity to talk aloud and engage with relevant content in a context that is commonplace for them, the context of solving an equation. This is in line with Hunting's (1997) suggestions that useful tasks are novel, serve as a stimulus for conversation, engage student interest, and require the students to engage in mathematical thinking.

Since the interview protocol (Appendix B) was designed to allow for maximal data collection of students' engagement with the chosen task, the tool admits the consideration of a number of topics and considerations. Those topics and considerations include "solution", row reduction, matrix multiplication, multiple solution procedures, the use of heuristics, and the "special" cases when a system has infinite solutions or no solutions. (I refer to the cases as "special" since they are possibly "special" to the student who has the expectation of finding a unique solution in every instance.) For me, all these matters come up in considering the various solution trajectories for solving the equation. Thus, each topic is related, at least marginally, to the central consideration of this project. (What is the nature of undergraduate students' conceptions of multiple analytic representations of systems?) My primary focus, however, as I developed the interview protocol was that during the interview, a participant would work with each of the four representations of a quantitative system that I have targeted (linear system, vector equation, augmented matrix, and matrix equation). As a result, the protocol prescribes that interview participants engage with the task in each of the ways described in my thought experiment in Chapter 2: by using row reduction, by using the matrix dot product (MDP), and by using the matrix-vector product (MVP). (See A Thought Experiment with Registers in Chapter 2 for complete details.) The approach allows for participant engagement with the various forms of notation just prior to a discussion focused directly on multiple representations. I anticipated that the activities would serve to refresh ideas for some participants; for other participants, the activities may allow for exposure to perspectives they had not considered previously. Either way, the protocol was designed to provide participants with the opportunity to work with relevant content prior to engaging more directly with matters related to my research question.

In summary, the interview protocol as it appears in Appendix B was designed to accomplish two things: 1) to maximize what I might learn by using the chosen task, and 2) to induce participants into activity and conversations that might result in richer data for addressing my research question. That is, I designed the protocol to leverage the chosen task in as many ways as possible while allowing participants to acclimate to the interview process and to the relevant content prior to focusing on my primary research question.

Interview Sub-Protocol (A More Focused Description)

The interview protocol as it appears in Appendix B is comprehensive; the thematic organization of the protocol is indicated by headings and subheadings. The protocol (Appendix B) was my best anticipation of all the questions that might be addressed using the chosen task. However, the questions most relevant to the current project are questions 1, 4, 6, 7, 8, 11, 12, and 18. Those select questions are organized and presented in Appendix C, Interview Sub-Protocol. The Interview Sub-Protocol and my descriptions of each of its questions (given below) are intended to more clearly demonstrate how the data collected using the protocol provided answers to my research questions. I reiterate that the Comprehensive Interview Protocol (Appendix B) was designed to stimulate conversation and to get the essence of students' conceptions before asking them direct questions about multiple representations. I argue that asking the subset of questions in the Sub-Protocol (Appendix C) in isolation would likely result in premature interview termination and/or data of lesser quality.

Now I give brief thoughts on the questions which appear in the Interview Sub-Protocol (Appendix C) to help establish their relevance to the current project. For Question 1, I think of the representation in the task as a matrix equation (one way to denote a quantitative system) where finding a solution means finding a $2x1$ matrix that makes the equation true. Further, I clearly see multiplication in the equation, but I am not convinced that all students see multiplication when they consider the equation. The question allows for investigating whether participants have a name for the (matrix equation) object and how they tend to think about it. After some intermediate discussion, Question 4 allows participants to solve the equation however they wish; the question allows for the documentation of a participant's spontaneous solution method. Thereafter, the protocol prescribes coaching participants to solve the equation in ways other than their spontaneously chosen method; the plan includes coaching participants so that they work with row reduction, the matrix dot product (MDP), and the matrix-vector product (MVP).

Question 6 allows for the consideration of how students think about changes based on row-reduction; I see those changes as both similar and dissimilar to translation-related changes. (I say more about this in Chapter 6 Discussion.) The question also allows for exploring how students think about the augmented matrix representation of a system. Question 7 was designed to investigate whether students' translation activities have mathematic foundations or whether they use heuristics like "rows stand for equations" for solving, a phenomena described in Chapter 2. Question 8 was devised to provide deeper insight into the participant's translation activities by having the participant consider the "special cases" of no solutions and infinite solutions. As such, Question 8 addresses an important facet of Zandieh's and Andrews-Larson's (2015) findings.

Question 11 finally elicits participant responses directly related to my research question; by design the question occurs only after the participant has been given opportunity to work with each of the three representations targeted by the question (augmented matrix, matrix equation, and vector equation). I planned to present each participant with the three representations on an otherwise blank sheet of paper (see page 3 of Appendix D) and then ask if s/he thinks of the representations as being the same or as being different. For Question 12, an extension of Question 11; I planned to show the participant the linear systems representation on a separate and otherwise blank sheet of paper (see page 4 of Appendix D). I planned to ask the participant to comment on the 4th representation (linear system) relative to the three representations (augmented matrix, matrix equation, and vector equation) addressed in the previous question. Question 12 was devised to allow for the documentation of students' conceptions of what is often taken as the primary representation of a quantitative system (the linear systems representation), but only after they had been presented with the three other representations without reference to the linear systems representation. This was important to see if the student spoke in terms of a linear system prior to its presentation, a helpful detail in the identification of categories of "the thing" represented as outlined in Chapter 5 Results and Findings. Finally, Question 18 was included to allow participants to share any additional thoughts and to reflect on their overall experience of the interview. The expectation was that if participants are especially motivated to need to say more, such strongly-motivated comments might provide particularly unique and/or valuable insights.

In summary, the Comprehensive Interview Protocol (Appendix B) was designed to be a tool for collecting data on students' conceptions of systems in the natural (or at least commonplace) context of solving. Since the protocol addresses a broad range of topics and considerations, I isolated the questions most relevant to the present study into a separate document, Interview Sub-Protocol (Appendix C). The isolation and explicit descriptions of the questions in the Interview Sub-Protocol (Appendix C) are intended to clarify how the Comprehensive Interview Protocol (Appendix B) was a suitable tool for collecting data with the potential to address my primary research question. (What is the nature of students' conceptions of multiple analytic representations of systems?)

Selection of Participants

Three weeks prior to the end of the Fall 2018 semester, students in two sections of a junior level applied linear algebra course were notified by email of an opportunity to contribute to a research study through participation in a clinical interview. The classes were those for which written exam data had been analyzed earlier in the semester as described in Chapter 2. Timing the interviews at the end of the semester was essential for two reasons: 1) to ensure that students had been given a number of opportunities to work with content addressed in the interview protocol, and 2) to position students to reflect on the entirety of their coursework. Just before the last week of classes, I followed up by going to the two classes and extending an in-person invitation for students to participate in the clinical interviews. During that visit, I asked students to communicate their willingness to volunteer by responding to the email sent out prior to my visit or by noting my email address on the board. Nineteen students emailed me to volunteer to be interviewed *if selected* and to notify me of their scheduling availability during the last week of class and/or final exam week.

In order to select participants, I used the preliminary analysis of written exam data described in Chapter 2; recall that the exam question that was analyzed became the basis for the task in my study. Upon receiving emails from students, I entered volunteers' names into a spreadsheet along with 1) a brief description of their response on Exam 1; 2) a brief description of their response on Exam 2; 3) their course grade; and 4) their scheduling availability. Information for volunteers for whom I had previously set aside written work as interesting or distinctive in some way was highlighted in green on the spreadsheet, and I immediately sent those volunteers a proposed interview time for their confirmation. Examples of work I had set aside included exam responses with more than one solution method on the same exam (whether it was on Exam 1 or Exam 2). Also, since I noted few potential uses of matrix multiplication on the exam question, and none that I could confirm based on students written work, any response with a potential use of matrix multiplication was of interest. Further, some students changed their solution method between Exam 1 and Exam 2 while others maintained their solution method between Exam 1 and Exam 2; examples of each of these with complete and/or detailed responses had been set aside. In summary, students' responses meeting any of the aforementioned criteria had been set aside; when one of those students happened to volunteer, I immediately attempted to confirm a mutually agreeable interview time.

For volunteers whose work had not already piqued my interest in some way, I considered the notes in my spreadsheet to select additional participants. Spreadsheet data for each volunteer was highlighted in green or red to indicate whether I wanted to attempt to schedule an interview or not, respectively. If volunteers had not responded to the exam question or if their responses were incomplete, their data was highlighted in red; a volunteer's data was also highlighted in red if we were not mutually available to meet. Otherwise, a volunteer's data in the spreadsheet was highlighted in green, and I attempted to schedule volunteers so there was variety in solution methods and variety according to whether the volunteer had changed or maintained solution methods between exams. Further, I also considered performance in the course; students with grades in the A, B, and C ranges were chosen as participants.

As students confirmed my proposed meeting times, I recorded our appointment in the spreadsheet with the rest of the volunteer information. I continued to receive volunteer emails while also receiving confirmation emails. I managed to schedule 10 participants who seemed to be likely to contribute to a rich set of interview data. The evening or morning prior to each scheduled interview, I emailed the volunteer a friendly reminder of our appointment. Each of the 10 scheduled students appeared for the interview, thereby becoming participants. In summary, participants were volunteers who were vetted with selection criteria and subsequently scheduled according to availability.

Conducting the Interviews

From a group of 19 volunteers from two introductory linear algebra classes with a total of 52 students, I conducted one-hour, task-based clinical interviews with 10 selected students at the end of a semester. I video- and audio-recorded each interview and had participants make a written record of their solutions and/or thoughts when applicable.

The setting. I conducted the interviews in a conference room with the door slightly ajar. A video camera on a tripod was set up in the corner of the room to capture a record of each interview. A laptop on the conference table was used to capture the audio of each interview as a backup to the video recording. I also placed bottled water, an assortment of snacks, pencils, paper, and a graphing calculator on the table.

The process. Prior to beginning to work through the interview protocol, I read the following statement to each participant: “I am interested in learning about the ways you think about linear algebra. This is more important than getting correct answers or doing correct calculations. As we work through some activities, I would like you to think out loud. I will do any extensive calculations so you can focus on telling me about your thoughts.” By informing students that I would do any (intense) calculations for them, I specifically had in mind to provide them with the row-reduced matrix for the task when they needed it; more generally, I intended to assist them with any arithmetic or calculations that seemed to frustrate them or were too time-consuming.

The interviews can be described as semi-structured. I used the protocol (Appendix B) as a guide while following up on responses with clarifying questions when it seemed necessary or productive to do so. For instance, I often followed up on pronoun use like the word “it” by asking students to be more specific. I also followed up on interesting comments and adjusted the order of the interview in relation to students’ answers. I allowed conversations to be more authentic than strict adherence to the protocol would have; the approach is compatible with Hunting’s (1997) view that an interview should be allowed to unfold naturally with the researcher adapting the line of questioning based on the participant’s responses. The idea was to promote more feedback.

While conducting the interviews, I was mindful of Hunting’s (1997) suggestion that the researcher should respond impartially during an interview. At times that objectivity was challenged when a participant wanted to know if they were correct or not; other participants seemed content to proceed with the interview without such feedback.

One particular participant requested my evaluation of his overall understanding of linear algebra as we concluded the interview. While answering the question toward the end of the interview might have had minimal effect on the data, I still chose to provide a vague response to limit influence on any additional response the student might provide to the final question, question 18: “What do you think about what we have talked about today? Do you have any questions you would like to ask or any other comments that you would like to make?”

As anticipated, the interviews generally did not proceed in a linear fashion; only one interview proceeded linearly through the entire protocol with time to spare. Other interviews went over the original plan of one hour when students were available and amenable to continue past the hour. Still other interviews left some facets of the comprehensive interview protocol unaddressed. In all circumstances, I managed time so that the questions in the Interview Sub-Protocol (Appendix C) were addressed. Other questions in the Comprehensive Interview Protocol (Appendix B) played a secondary role to those related to eliciting quality responses to direct questions about the multiple representations of the task. That is, however an interview unfolded, I focused on ensuring that Questions 11 and 12 were addressed.

During the interviews, participants did any written work on printouts that I provided for them. To address Questions 11 and 12 during the interviews, I presented each participant with a sheet of paper with the three representations (augmented matrix, matrix equation, vector equation) in Question 11 on it and which was otherwise blank. For a visual of exactly what the participants saw, refer to page 3 of Appendix D, Instrument for Collecting Written Work. I then asked each participant if s/he thought of the representations as being the same or as being different and how so. In addition, I asked each participant if all three had the same solution and what thing the three represented. Next, I presented each participant with an additional sheet of paper with only the linear systems representation on it. To view exactly what the participants saw, refer to page 4 of Appendix D, Instrument for Collecting Written Work. I then asked the student to compare and contrast the linear systems representation with the first three representations. In summary, participants were presented with page 3 of Appendix D when I asked the Question 11, then page 4 of Appendix D was presented as I asked Question 12.

Preparation of the Raw Data

To prepare the data for analysis, I designated the participant in the first interview to be Student 1; I identified other participants similarly based on the order in which I interviewed them. For instance, the participant in the 8^{th} interview was identified as Student 8. I downloaded the video recordings from the video camera a few at a time during data collection. Each video recording was labeled Student 1, or Student 2, or Student 3, etc. During data collection, I also saved the audio files captured by laptop as Student 1, or Student 2, or Student 3, etc. After all interviews had been conducted, I scanned the written work for each participant to an electronic file, naming each file Student 1, or Student 2, or Student 3, etc. For each participant I created an electronic folder containing 1) the video-recorded interview, 2) the audio-recorded interview, and 3) a scan of the participant's written work. I then collected the 10 folders labeled Student 1, Student 2, …, Student 10 into a master folder so that the data was organized and ready for analysis.

Data Analysis

While it did not seem sensible to initiate interviews with direct questions about multiple representations of systems (see the section "The Interview Protocol: Development & Explanation" in this chapter for details), participant responses to those types of questions seemed like a natural place to initiate data analysis for this project. More specifically, participants' responses to Questions 11 and 12 of the interview protocol seemed to be an ideal place to begin analyzing the data in order to address my primary research question: What is the nature of undergraduate students' conceptions of multiple analytic representations of systems? Further, Questions 11 and 12 address a question that supports my primary research question: What unified thing, if any, do students have in mind as the represented entity?

Responses to Questions 11 and 12 also held potential for addressing the additional question: How do students account for similarities and differences between the representations in terms of translations and registers of representation? Though the questions provided students with the opportunity to address translations, I did not directly ask them about translations. I had in mind wanting to analyze student responses through the lenses of Duval's Theory of Semiotic Representation Registers and the Theory of Quantitative Systems; however, I left my questions open enough that I could not be sure of the outcome. This was intentional to answer Artigue et al.'s (2000) call for studies which would yield richer results than those from Pavlopoulou's (1994) experimental study. Shown below for convenience as Figure 4, Questions 11 and 12 are described in greater detail in the "Interview Protocol: Development & Explanation" section of this chapter, and both questions appear in both Appendix B and Appendix C.

11. If we think about augmented matrices and both types of matrix multiplication, we have the three following. Do you see them as the same or different? Explain. Will they all have the same solution? What is the thing they represent?

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & -1 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad x\begin{bmatrix} 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

12. Do those three have anything in common with the following system?

$$\begin{cases} x + 2y = 7 \\ 4x - y = 1 \end{cases}$$

Figure 4. Interview protocol questions 11 & 12, the focus of data analysis.

First Pass Through the Data. To begin data analysis, I made a spreadsheet listing Student 1, Student 2, … , Student 10 and assigned each student a pseudonym. I viewed each video, making notes and documenting direct quotes for each interview. In addition, I noted in the spreadsheet the time stamps for when Questions 11 and 12 were addressed. Following the review of each video, I wrote a summary description of the interview. Table 3 is an example of the early analysis that I did for one student. I give only a brief example from the spreadsheet and an abbreviated summary description since the early analysis includes my initial impressions which had not been given rigorous consideration. I created a spreadsheet entry similar to the one shown for Ken (Student 9) in Table 3 for each of the 10 students who participated in an interview.

Table 3

Early Analysis of Ken's Interview

Student	Pseudonym	Start Time	End Time	Quotes from Questions 11 & 12	Summary Description
Student 9	Ken	40:45	43:48	They are the same. I think they're just different in the way they are set up to be solved. Oh, yeah. Yeah, they are the same.	Early in the interview, Ken struggled to enact standard matrix multiplication, but was eventually able to reconstruct it for himself. He also struggled with the matrix-vector product, stating that he had never seen it. He later commented that he would not be likely to use the linear combination of columns even after seeing it.
		46:32	49:20	What's the thing? What's the thing that makes them all the same? Is there like a word problem attached to this? I haven't really thought about it. I think I mentioned that it's just about notation. It doesn't really make a difference to me. I just see a problem.	

Note that Student 9 was given the pseudonym "Ken". In the video of Ken's interview, Questions 11 and 12 were addressed during the time periods 40:45-43:48 and 46:32-49:20. The beginning time stamps indicate when I posed Questions 11 and 12 to Ken (Student 9), while the ending time stamps indicate when he made his last statement that I judged to be relevant to the question. In the earlier segment (40:45-43:48), Ken indicated that to him the representations are the same, differing only in the way that they were set up for solving. In the latter segment (46:32-49:20), Ken indicated that he had not put much thought into "the thing that makes them all the same". Further, he indicated that he was not too concerned about it, stating "It really doesn't make a difference to me".

Other seemingly important observations that I made while reviewing the video of my interview with Ken (Student 9) appear in the "Summary Description" column. Namely, Ken (Student 9) struggled to enact standard matrix multiplication and matrix-vector multiplication. He claimed that he had never seen the matrix-vector product and commented that he would not be likely to use the linear combination of columns even after seeing it. I use Ken’s data as shown in Table 3 in the discussion of my findings in Chapter 5. The purpose for including it here is to serve as an example of early exploratory analysis conducted for each of the 10 study participants.

Analysis for Generative Purposes. I introduce this section with a caveat. The exploratory nature of generative analysis makes separating the methods (the present chapter, Chapter 4) from the results and findings (Chapter 5) a delicate undertaking. Since consideration of the data may lead to one question with results that lead to the next question and so on, the results are closely tied to the methods. In what immediately follows, I attempt to describe my methods while avoiding statements about results and findings; I delay discussing the results and findings until Chapter 5. However, restating some of what I say in this chapter is necessary for framing the results and findings reported in the next chapter. I also note that, from a constructivists perspective, I view exploratory data analysis as a reflexive process between the data and the researcher where a researcher’s experience is an influence on the findings.

After completing the preliminary work exemplified in Table 3, I transcribed the segments of the interviews indicated by the time stamps for each of the 10 participating students. Thereafter, I highlighted the parts of the transcripts focused directly on the portion of Questions 11 and 12 about whether the representations were the same or different; I logged that data into a spreadsheet. Subsequently, I looked for themes and/or categories in the students' noted responses. The results are documented in Table 4 of Chapter 5 and a discussion of trends follows thereafter.

What is varying and what is invariant are important considerations in mathematical contexts (Lo, 2012). The aforementioned pass through the data and creation of Table 4 in Chapter 5 left me with the impression that the students somewhat universally considered the notation to be varying and the solution to be invariant. To test my impression, I went through the transcripts for each student and noted examples of student responses that indicated a conception of 1) varying notation, and 2) an invariant solution; I recorded my findings in a spreadsheet. That documentation appears as Table 5 of Chapter 5. Next, I used the data in the spreadsheet to synthesize and discuss the students' conceptions of "same solution". In addition, I looked at how the students characterized "different notation"; those findings are reported in Chapter 5.

Another important facet to Questions 11 and 12 is the consideration of "What is the thing that they represent?" This is related to Duval's (1999, 2006, 2017) suggested 3rd source of incomprehension in mathematics: the necessity of seeing multiple differing representations as indicating a unified whole. (Recall that Duval suggested that the first two sources of incomprehension in mathematics are treatments and conversions. See Chapter 2 and/or Appendix A for details.) I highlighted students' responses to this question on the transcripts I created and made a record of them in a spreadsheet (Table 6, Chapter 5). I subsequently looked for categories of "the thing" represented which were evident (to me, as a researcher) in the students' responses and documented them (Table 7, Chapter 5). I note that at this point in the data analysis, I moved down Clement's spectrum of research purposes from exploratory analysis toward grounded model analysis. In some circles (psychology, healthcare, and social science), the work I did to explore the data and identify categories might be described as *thematic analysis*, one of many tools used in grounded theory methodology (Braun and Clarke, 2006). Braun and Clarke (2006) described *thematic analysis* as the qualitative method of identifying "some level of patterned response or meaning within the data set" (p. 82).

Note that the question "What is the thing that they represent?" allowed for the exploration of Duval's 3rd proposed source of incomprehension in the context of systems; however, I used Duval's theory more as a motivation for collecting data to explore than as a lens for data analysis. The data generated allows for groundbreaking findings on whether students conceive of a unified whole and what that unified whole is. In other words, while Duval's theory was a guiding influence, the analysis was conducted to generate categories rather than to conduct a stringent analysis of student comprehension (or incomprehension). The results and findings generated by the aforementioned exploration and categorization of the data appear in Chapter 5.

Convergent Analysis. The interviews provided students with potential opportunities to translate in the context of solving; I did not, however, explicitly ask students to perform particular translations. Recall that Pavlopoulou's (2004) study was an experimental design, and her tasks asked students to perform particular translations. My approach created a richer context than Pavlopoulou's as called for by Artigue, et al. (2000). The open nature of my approach coupled with the semi-structured nature of the clinical interviews, however, meant obtaining data which could analyzed in terms of my new theory (the Theory of Quantitative Systems) and Duval's theory (the Theory of Semiotic Representation Registers) was not guaranteed. Thoughtful development of the interview protocol allowed for rich responses that complement and extend the results and findings in Pavlopoulou's (1994) experimental study; truly, a number of theoretical lenses are available for analyzing the rich data collected. However, as I explored the data, a couple of good candidates for analysis through the lenses of Duval's and my theories emerged.

When familiarizing myself with the data as described in the section "First Pass Through the Data" above, I noted that Peter (Student 5) had spent a major portion of his interview discussing the level of difficulty of various translations. In addition, I noted Felix's admission that he did not know how to work with the vector equation; I also noted that Felix (Student 3) had stated that getting a linear system from the vector equation "seems kind of weird". These observations persuaded me that the interviews with Peter (Student 5) and Felix (Student 3) were promising candidates for the application of my and Duval's theories. The insightful results and findings of the convergent analysis of the two interviews appear in Chapter 5.

I will more overtly lay out the constructs used for the analysis of Peter's and Felix's conceptions. (The discussion is largely a summary of the theoretical perspective I laid out in Chapter 2; complete details can be found there or in summary form in Appendix A.) First of all, Duval's (1999, 2006, 2017) theorized three sources of incomprehension in mathematics. Those three theorized sources of incomprehension are 1) treatments, 2) conversions, 3) and the necessity of seeing representations from differing registers as indicating the same mathematical object. Duval also conjectured about the difficulty of performing translations in terms of *transparency* and *unit-by-unit* translation. I used these constructs in my analysis.

My reshaping of Duval's theory in the context of systems (the Theory of Quantitative Systems) resulted in the idea of *translations*, which are transformations specifically between analytic representations. I also developed the idea that each mathematical system is a different register of representation. Further, I proposed that reversibility is an important consideration in translation activities, and especially so in linear algebra. I designated a *quantitative system* as the mathematical object represented by all the representations in this study so the representations could be given equal consideration. The result of *equitizing* the representations and considering reversibility led to my identification of twelve translations between the representations in this study.

My results and findings are written in terms of translations and in terms of mathematical systems as registers. I address the ideas of treatments, conversions, transparency, and unit-by-unit translation. In addition, I write in terms of reversibility and the twelve different translations I identified (Table 1). In other words, I used Duval's Theory of Representation Registers (1999, 2006, 2017) and the Theory of Quantitative systems as lenses for conducting convergent analysis of the data for Peter (Student 5) and Felix (Student 3). In doing so, I operated toward the lower end of Clement's (2000) spectrum of research purposes, conducting explicit analysis that connects theory with observations. The analysis differs in nature from the report of aggregate data for generative purposes and is convergent to the degree that a pioneering study can be.

Summary of Methodology

In terms of data collection, my methodological choices for this study included the use of one-on-one, task-based clinical interviews which were video and audio recorded and for which students' written work was collected using the instrument appearing as Appendix D. The interview protocol was drafted and went through several stages of revision. The 10 participants were chosen on the basis of selection criteria applied to a pool of volunteers from two junior-level applied linear algebra classes.

In terms of data analysis, my methodological choices included conducting both generative and convergent analyses in terms of Clement's (2000) continuum of research purposes. The generative analysis involved an exploration of the data (in transcript form) and the identification of categories of student responses. The convergent analysis involved consideration of students' responses through the lenses of the Theory of Quantitative Systems (TQS) and Duval's Theory of Semiotic Representation Registers (TSRR); constructivism operated as a background theory.

In summary, data was collected and analyzed to document students' conceptions of a common and recurring topic in mathematics (systems of linear equations) with a group of students (undergraduates) for whom little data exists. (See Chapter 3 Literature Review.) Further, I applied Duval's theory in a new way and in a new context and also employed my newly posed Theory of Quantitative Systems. The results and findings are reported in Chapter 5. Additional discussion less closely tied to data appears in Chapter 6.

CHAPTER 5

RESULTS & FINDINGS

In this chapter I report the results and findings from my analysis of the 10 one-hour, individual clinical interviews as described Chapter 4 Methodology. First, I present results and findings from generative analysis of the data. Next, I present results of the convergent analysis I conducted using Duval's Theory of Semiotic Representation Registers and the Theory of Quantitative Systems as lenses for analysis.

Results of Generative Analysis

In this section I discuss results and findings from the analysis I conducted for generative purposes. Exploration of the data generated realizations about 1) the students' conceptions of sameness of representations, 2) their identification and descriptions of the varying and invariants, and 3) their thoughts about "the thing" represented by the various representations.

Sameness of Representations

A peer and I have been discussing *sameness* in mathematics. We have been using the term in a non-technical, natural language sense to refer to mathematical objects, contexts, or situations that might be similar. Truly, developing rigorous, research-based descriptions of students' conceptions of sameness in various mathematical contexts is a promising direction for research. While the current project is filled with the essence of *sameness,* developing a rigorous framework is beyond its scope. For this project, I use *sameness* to mean not only the ways in which things are similar, but also the ways in which they differ. The contrasts apparent in the way things differ serve to give greater definition to the ways in which they are same (Lo, 2012). The work documented herein is a start toward a more rigorous characterization of *sameness* in the context of systems of linear equations through the eyes of undergraduate linear algebra students.

Following my first pass through the data as described in Chapter 4 Methodology and as exemplified in Table 3, I created Table 4 based on the transcripts I produced. The table records the students' direct responses to the first part of questions 11 and 12 of the interview protocol (Appendices C and D): whether the various representations are the same or different.

Like the various representations under consideration in this study, the student responses documented in Table 4 themselves have similarities and differences and demonstrate a range of conceptions. From analysis of Table 4, I find that seven participants (Students 3-9) concluded that the four representations were the same; they did so by concluding that the first three representations are the same and then concluding that the linear systems representation was yet another instance of the same thing. While Students 3-9 all essentially concluded that the four things are the same, their justifications and explanations reveal a variety of notions. I will now describe a few of their conceptions as evidenced by their verbal responses.

Ken (Student 9) referred to the first three representations as "alike". His justification was that the augmented matrix and/or the matrix equation could serve as "step 1". When presented with the fourth representation, he decisively concluded that they are "the same", qualifying that they differ in the way they are set up to be solved.

Table 4

Responses about Similarities and Differences Between the Various Representations

	Are these three things the same or different? (Question 11)	**Is the fourth thing the same or different? (Question 12)**
Student 1 Zeb	These three things are *not* the same. See, these three equations, they all have the same relationship. They all have *x*, *y*, and the same solution, and the same relationship, but they are *not* the same.	It's not the same equation as these. It has the same values, it has the same variables, and the same solution. But it's not the same thing. They all express the same variables, the same relationship, and the same solution but in different ways. The mathematical objects you use to express the equations are not the same.
Student 2 Mike	Umm…all the same to me.	I might not like that one.
Student 3 Felix	This is just another way of writing this and that (pointing to all 3 representations).	All of this is basically just linear systems of equations. It's just expressed different. It's kind of like writing, you know, instead of Cartesian, polar coordinates.
Student 4 Nick	Yeah, my first thought is okay, they're all the same. I think in algebra, you can always represent things different ways, but they mean the exact same thing.	It's the same. They all represent the same thing. They're all pointing to the same exact, you know, function or equation, I suppose. They all point to the same solution.
Student 5 Peter	I would say that they are the same because all the coefficients are the same, and these are just different notations for the same thing.	This should be the same. That's (pointing to the systems representation) these two (pointing to the matrix equation and the vector equation) if you distribute it. It's the easiest way for me to think about it.
Student 6 Sam	I would say if you understand the math behind it, then well, I guess… Regardless of whether you understand the math, they're the same as long as it's in the same context.	I would say it's the same thing. They're just different forms.
Student 7 Myra	They all represent the same relationship between *x* and *y*. They just represent it in different ways.	(Laughs) It's the same. It's the same thing.
Student 8 Anna	They're all the same to me. They're just written differently.	Well, it's the same. It looks like those will come up to the same solution. Just different notation, I guess.
Student 9 Ken	I can look at those and be like, "Oh, those are alike because this is technically my step 1" (points to augmented matrix). And this (points to matrix equation) could have been a step 1 as well.	They're the same. I think they're just different in the way they are set up to be solved. But, yeah, they're the same.
Student 10 Jake	I would say they're all *equivalent*. I probably wouldn't use the word *same*.	I would say they are all four *equivalent*. But I would say they are *not equal* because they don't all look exactly the same.

Note that I used Ken's interview data in Table 3 of Chapter 4 as an exemplar for how I began my data analysis. There I documented that Ken (Student 9) struggled to carry out matrix multiplication. He stated he had never seen the matrix-vector product, and commented that he would not likely use the linear combination of columns even after seeing it. As shown in Table 3, he indicated that he has not thought about "what makes them the same". Further, he did not seem to be committed to considering their sameness, stating "It really doesn't make a difference to me. I just see a problem." Given Ken's reference to steps and the general tenor of his interview, he seemed to conclude that the representations are "alike" or the "same" because they could exist along the same solution trajectory and/or because they could be alternative starting points for solving.

Myra (Student 7) gave a concise response about whether the first three representations were the same, stating "They all represent the same relationship between *x* and *y*. They just represent it in different ways." She was amused when I presented her with the 4^{th} representation, and rather emphatically stated through laughter, "It's the same. It's the same thing!" For context, consider that earlier in the interview, Myra (jovially) alleged that I was asking trick questions. I (jovially) denied the accusation but conceded that I might be asking her to think about things in an uncustomary way. I suspect that Myra's amusement and amazement resulted from what for her was the unquestionable transparency of the representations which she summed up toward the end of our interview by stating, "I'm going to tell you they're the same every time because I just see a system of equations".

Felix (Student 3) and Nick (Student 4) also provided interesting insights into their determination of sameness. Felix (Student 3) compared the four representations to using polar coordinates instead of Cartesian ones. Nick (Student 4) gave his universal perspective on algebra, stating "I think in algebra, you can always represent things different ways, but they mean the exact same thing." He went on to state that the four things are "all pointing to the same exact, you know, function or equation", describing the representations as pointers.

While seven (Students 3-9) of the 10 students concurred that the representations were the same and offered a variety of ways to describe why they thought so, the consideration of the other three students' responses (Jake, Mike, and Zeb) provide a different kind of insights. Jake (Student 10) said of the first three representations, "I would say they are all *equivalent*. I probably wouldn't use the word *same*." With the addition of the fourth representation, he stated, "I would say they are all four equivalent. But I would say they are *not equal* because they don't all look exactly the same." When I questioned Jake (Student 10) about whether he was intentionally avoiding the use of my word "same", he confirmed my impression. He stated, "*Same* is an ambiguous term. *Same* doesn't necessarily mean equal or equivalent. It just kind of means either one you want it to mean. That's the way I recognize it." Since Jake (Student 10) was exercising precision in the way he used the words "equal", "not equal", and "equivalent", I saw an opportunity to take a slight diversion from the interview protocol to investigate his thoughts on row equivalence. My findings on that topic were limited as indicated by the record of our dialogue shown in Figure 5.

1	JS:	We use equivalent, and we use row-equivalent.
2	Jake:	Yeah! *Row equivalent*. (smiling)
3	JS:	Something's going on there. If row-reduced matrices were just *equivalent*, we would call them *equivalent*. But we don't call them *equivalent*. We call them *row equivalent*.
4	Jake:	(Smiling) Yeah. They're called *row equivalent*.
5	JS:	So what's up with that?
6	Jake:	I remember when I was doing the homework, recognizing the distinction. But at the same time, I didn't really think about it too much.
7		(Silence)
8	Jake:	Because it didn't seem like a necessary step for me to acknowledge, like mentally. So, yeah.

Figure 5. Jake's discussion of row equivalence.

Jake's statement in line 8, "it didn't seem like a necessary step for me to acknowledge", fits with my overall impression of his interview. Jake (Student 10) often gave responses that seemed to indicate that he had thought deeply about an idea. For instance, at one point in the interview we discussed Platonic philosophy. He also communicated that at times he mitigated his effort based on what he needed to do for the homework and/or exams. That is, at times Jake (Student 10) seemed to have engaged in reflection and deep thought, but he suggested that at other times, he only exerted as much effort as he deemed necessary to meet the homework's demands. I would describe Jake (Student 10) as independent and reflective given his selective word choices and his deliberate management of his effort. In summary, he would not concede that the representations were the same, preferring to describe them as "equivalent" but "not equal".

Mike (Student 2) gave a rather quick and non-descriptive response to whether the first three representations were the same: "Umm…all the same to me". He had an interesting and unique response when I added the fourth (linear systems) representation:

“I might not like that one”. Apparently the brace I used to denote the system presented an obstacle for Mike (Student 2). He apparently had not seen that symbol in the context of systems of equations; his only immediate memory of seeing a brace was in the context of piecewise functions. My experience of students struggling with piecewise functions and given that the topic (piecewise functions) inhibited our progress through the interview protocol as planned, I argue that Mike (Student 2) had negative associations with the symbol that inhibited his ability to focus in productive ways.

Finally, Zeb (Student 1) was the only student of the 10 who said that the representations were definitely not the same; he was rather adamant in his conviction. Of the first three representations, Zeb (Student 1) stated “These three things are not the same. See, these three equations, they have the same relationship. They all have *x*, *y*, and the same solution, but they are *not* the same”. Of the fourth representation he stated, “It has the same values, the same variables, and the same solution. But it’s *not* the same thing.” Though Zeb (Student 1) catalogued the similarities between the representations, a distinguishing characteristic for him was the expression “in different ways”. Zeb’s conclusion that the representations are not the same seems to be based on the mathematical structure of the representations since he claimed, “the mathematical objects you use to express the equations are not the same”. I say more about this distinction in my discussion of “same solution” which follows.

In summary, while seven students described the various representations as the same, the notation I used presented an obstacle for Mike (Student 2) in reaching the conclusion that the representations are the same, and Jake (Student 10) preferred to use more precise terminology. Zeb (Student 1) was the only student who explicitly declared the representations are not the same.

A Universal Varying and a Universal Invariant

My work with the videos and the results documented in Table 4 left me with the impression that students somewhat universally thought of the notation as varying and the solution as invariant. Because what is varying and what is invariant are important considerations in mathematical contexts (Lo, 2012), I tested my impression by analyzing the transcripts and identifying responses that indicated a conception of 1) varying notation, and 2) an invariant solution. The record of those findings appears as Table 5.

A note about the quality of the students' responses documented in Table 5 seems appropriate. Students may have provided indications of varying notation at a number of points in the interview. However, the indications of an invariant solution seem to have occurred in one of two instances. Some were instances of a direct response to my question "Do you think they have the same solution?". Other indications of an invariant solution were made in response to my question "Do you see them as the same or different?". In other words, some indications of an invariant solution were an answer to a direct yes or no question while others were statements about a characteristic that they observed was shared by the representations. I make this distinction since the quality of the two types of responses differs. A student response of "Yeah" or "Yes" to the direct question provides less insight than a student's identification of a common characteristic of the representations without the prompting of the direct question. Granted, even the quality of the responses that identified an invariant solution as a common characteristic of the representations is dependent on when it was stated in the interview relative to the direct question. Nevertheless, each student gave some indication of thinking of the solution as the same and the notation as different as I had suspected. Below I point out some specific quotes suggestive of “same solution” and “different notation” conceptions and provide a discussion for each of the two notions.

Table 5

Evidence of “Different Notation” and “Same Solution” Conceptions

	Indication that notation is varying	**Indication that the solution is invariant**
Student 1 Zeb	The mathematical objects you use to express the equations are not the same.	They all have … the same solution.
Student 2 Mike	I mean, it's again variables. You know, the columns represent the variables (referring to the augmented matrix representation).	Uh…Yeah, I believe so. If they were larger, then maybe not.
Student 3 Felix	This is just basically another way of writing this and that (indicating each representation from Question 11).	Yeah. I would imagine so. (Long pause.) Unless there's something I haven't known about linear algebra.
Student 4 Nick	I suppose differences in the way that you represent it. I mean, I think in algebra, you can always represent things in different ways, but they mean the exact same thing. I guess they have the same solutions.	They have the same solution. They point to the same solution.
Student 5 Peter	These are just different notations of the same thing.	They should. Because all the coefficients are the same.
Student 6 Sam	They're just different forms.	They all have the same solution.
Student 7 Myra	Yeah, they are just written differently.	Yeah. All three.
Student 8 Anna	Just different notation I guess. But, yeah, it's just a different way of writing these two equations (referring to the linear equations of the systems representation).	Yes. They all have the same solution. It looks like those will come up to the same solution.
Student 9 Ken	They're the same. They are just different in the way they are set up to be solved. Yeah.	Yeah. (Nods head). Yeah.
Student 10 Jake	They don't all look exactly the same.	They all lead to the same answer.

Descriptions of "Same Solution". Based on Table 5, six students (Students 1, 6, 7, 8, 9, and 10) gave rather direct indication that the solution is the same without being too descriptive. For instance, Sam (Student 6) stated "they all have the same solution", while Ken (Student 9) and Myra (Student 7) simply responded "Yeah" when I ask them if the solutions are the same. Students 2, 3, and 5 offered a little more insight.

Mike (Student 2) indicated that he reached his conclusion of "same solution" because of the small 2x2 system. He stated "I believe so. If they were larger, then maybe not." Felix (Student 3) qualified his conclusion of "same solution" with a bit of hesitation, stating "unless there's something I haven't known about linear algebra". Nick (Student 4) stated cleanly "they all have the same solution", but he also added "they point to the same solution". The additional statement with the use of "point" leads me to wonder if Nick (Student 4) had an idea of representation similar to mine as described in the Theory of Quantitative Systems where I take all representations to be pointers or indicators of an abstract mathematical object. Peter (Student 5) concluded that "they should" have the same solution; he supported his conclusion with the statement "because all the coefficients are the same". Based on my experience of students' imprecise use of mathematical terminology, I believe he may have used "coefficients" to refer to both the values of the coefficients AND the values of the constants, basically concluding that all values involved were the same.

An additional review of the transcripts did not reveal much more for those students (Students 1, 6, 7, 8, 9, and 10) who gave direct and undescriptive answers with a couple of exceptions. When asked, "Do all these things have the same solution?", Sam

(Student 6) replied “Let me check”. He then scanned over the two pages of representations while using his pencil to point as he scanned. His conclusion was, “Yeah, pretty sure.” Similarly, only after scanning the two pages of representations with her eyes did Anna (Student 8) answer the question decisively, “Yes. They all have the same solution.” Sam’s and Anna’s scanning suggests that their conclusions of “same solution” may be supported by reasoning similar to that involved in Peter’s “same coefficient” criteria. They were possibly quickly checking whether all involved values were the same.

Discussion of “Same Solution”. One point of discussion is that the students’ universal affirmation of an invariant solution suggests that they were not distinguishing between values and objects. That is, the students did not make a distinction between values satisfying the system and a matrix satisfying the matrix equation. For the task in this study, I would conclude that for the linear system a proper solution might be expressed as “x=1 and y= 3” or (1, 3), though the latter may evoke graphical images rather than identification as an analytic ordered pair. In contrast, I conclude that for the matrix equation, a proper solution should be expressed as $\begin{bmatrix}1\\3\end{bmatrix}$.

Solutions to linear systems of equations are values of *x* and *y* that satisfy each equation in the system. Solutions to matrix equations are matrices that satisfy the matrix equation; the solution matrix consists of entries which are the values of *x* and *y* satisfying the associated linear system of equations. The representations involve different mathematical objects and, therefore, the solutions should be stated in terms of objects used to state the problem. Concisely, the solution should be stated in the same register in which the problem is posed. I believe this is what Zeb (Student 1) was indicating in his statement (Tables 4 and 5) “The mathematical objects you use to express the equations are not the same” and was the basis for his unique, emphatic conclusion that the four representations are not the same. I think Zeb may have had in mind that matrix equations are comprised of matrices, vector equations are comprised of vectors and scalars, and that linear systems of equations are comprised of values and variables. In essence, the representations reside in different registers, and therefore, cannot possibly be the same. While I am arguing that differences in register were the basis for Zeb’s unique conclusion that the representations are not the same, even he did not make the distinction between “x=1 and y=3” and $\begin{bmatrix}1\\3\end{bmatrix}$. This topic of distinguishing between values and matrices is highlighted in my work with Felix (Student 3) as documented in the section of this chapter entitled “Felix’s Emerging Conception of Translations”.

Although clear distinction between values and matrices is largely absent from the data, a couple of instances beyond Zeb’s suggest that students may have been considering the distinction. Sam (Student 6) and Peter (Student 5) seemed to make distinctions based on the registers (mathematical systems) of the representations. I refer to the attention given to the mathematical objects used in an analytic representation as *structure sense*, though the term has been appropriated in other similar but very specific ways in the mathematics education literature. For instance, Linchevski and Livneh (1999) used the phrase “structure sense” to describe the ability “to use equivalent structures of an expression flexibly and creatively” (191). Their study comparing algebraic contexts with numeric ones necessarily means that their sole focus was on ordinary algebra, which I have argued is an insufficient lens for considering the challenges that linear algebra presents. Musgrave, Hatfield, and Thompson (2015) performed an investigation of teachers' structure sense. The researchers concluded that giving teachers opportunities to develop structure sense and encouraging them to reflect on structure with awareness will position them to provide better support to students in their development of structure sense.

When I asked Sam (Student 6) if the four representations were of the same thing, he moved his head from side-to-side and rolled his eyes before stating, "It depends, yes and no. I mean, they all have a different definition, so in that sense they don't represent the same thing". Sam (Student 6) went on to say "They all have separate names: augmented matrix, matrix equation, linear combination of vectors, and system of equations. So they're all defined differently, but they can be used to get to the same solution." I believe Sam (Student 6) was making a distinction between various registers with his naming of each representation and with his use of "different definition" and "defined differently". Further, "they can be used to get to the same solution" is different from "they have the same solution". The nuance in his phrasing allows for the solution to the matrix equation to be a matrix from which the solution for the system could be extracted. I believe Sam (Student 6) had a structural conception similar to Zeb's though he was not as direct and succinct in communicating his perspective.

Peter (Student 5) also gave what I see as an indication of the distinction between values and matrices in the following exchange. Toward the end of our interview, I had Peter (Student 5) reflect on the work he had produced on the exam question (Figure 3) that was the catalyst for this study (see Chapter 2). Referring back to his work on the exam question (which came with the instruction that the solution should be stated in vector form), he stated: “We very often go for this vector solution, which just tells you what *x* and *y* are equal to in vector form because that’s the math and that’s the notation we’re working with this semester.” Peter had apparently taken notice of the distinction between the values of *x* and *y* and the vector (column matrix) composed of the values of *x* and *y*. While work in his prior schooling had involved the linear systems register made up of values and variables, he summed up the work of the present semester as a study in the mathematics of the vector and/or matrix registers.

Another point of discussion is necessary and sufficient conditions. Most student responses provide evidence (Tables 4 and 5) that the students (correctly) deemed “same solution” to be a necessary condition rather than (incorrectly) determining “same solution” to be a sufficient condition. In other words, the students seemed to (correctly) consider “same solution” as a shared characteristic amongst the representations rather than (incorrectly) regarding “same solution” as enough to conclude sameness of the representations. For instance, Zeb (Student 1) concluded that the representations were *not* the same while identifying a list of similarities, including “same solution”, between the representations. Certainly for Zeb (Student 1), “same solution” was not enough to declare sameness.

Granted, the students’ avoidance of the incorrect determination of “same solution” as sufficient could be the result of first determining that all the representations were the same and reaching the logical conclusion that they, therefore, must have the same solution. A different kind of outcome might have resulted if the interview questions had been ordered differently. For instance, without first determining that the representations were the same, some students might have assumed the reverse implication that having the same solution was enough for determining that the representations were the same. Assuming the reverse implication could be detrimental in the context of row equivalence, where row equivalent systems have the same solution but are not indicative of the same quantitative system. Unfortunately, my attempt to gain insight into this issue with Felix (Student 10) was unsuccessful as documented in Figure 5 and discussed earlier in this chapter.

Characterizations of "Different Notation". Based on Table 4, Jake (Student 10) gave the most visual description of the differing notation; he stated, "They don't all look exactly the same". Several students' responses (Students 2, 3, 4, 5, 7, 8) characterize "different notation" as alternative means of expression. Rather than appealing to visual qualities, they (Students 2, 3, 4, 5, 7, 8) seemed to appeal to the idea of communication. Phrases the students used were: "different notation(s)" (Students 5 and 8), "written differently" (Student 7), "differences in the way you represent it" (Student 4), and "another way of writing" (Student 3).

Two students' responses (Students 1 and 6) seem suggestive of mathematical structure within differing mathematical systems, which I designated as registers in the Theory of Quantitative Systems. Zeb (Student 1) stated "the mathematical objects you use to express the equations are not the same", while Sam (Student 6) stated "they're just different forms". (Note that in the section "Descriptions of 'Same Solution'", I discussed the potential similarity between Sam's and Zeb's conceptions in terms of mathematical structure.) In summary, students described “different notation” in visual terms, in terms of communication, and in terms of differing registers.

“The Thing” Represented

Another important facet to Questions 11 and 12 is the consideration of “What is the thing that they represent?”; the question asked the students to pinpoint a particular referent for all the representations. My goal was to gain insight into Duval’s hypothesized third source of incomprehension in mathematics, the necessity of recognizing representations from various registers as indicative of the same mathematical object. My record of the students’ responses appears as Table 6.

In working with the data, categories of “the thing” became evident in the students’ responses as shown in Table 6. When asked of the various representations “What is thing that they represent?”, the students’ responses clustered into three groups: 1) a system of equations-definitely, 2) a system of equations-less definitely, and 3) a quantitative system. One student response provided little evidence of a unified conception of the various representations, and there is no data for another student since our interview took an unexpected turn into other topics. Each of the categories is discussed and exemplified with data in the following sections; a summary appears as Table 7.

Systems of Equations—Definitely. Two students’ responses (Felix’s and Sam’s) unequivocally identify systems of equations as the thing represented by the various representations. Felix (Student 3) stated, “The same system of equations. All of this is basically just linear systems of equations”. Sam (Student 6) stated, “I would say it’s representing the system. It’s the same system in different forms.”

Table 6

Descriptions of "The Thing" Represented by the Four Representations

	What is the thing they represent?
Student 1 Zeb	They all have the same relationship.
Student 2 Mike	No data. Our discussion turned to piecewise functions.
Student 3 Felix	The same system of equations. All of this is basically just linear systems of equations.
Student 4 Nick	I'm thinking back to high school, like the Platonic idea. That's the first thing that comes to mind. You can represent them different ways that you want. But ideally, they're all pointing to the same exact, you know, function or equation, I suppose. They point to the same solution.
Student 5 Peter	I want to jump and say, what's the thing you're after? So the thing I would say is the numeric values of *x* and *y* such that these systems of equations, which are all equivalent, are satisfied.
Student 6 Sam	I would say it's representing the system. It's the same system in different forms.
Student 7 Myra	(Referring to the systems representation.) That's a representation of like, a mathematical truth or a mathematical statement that we learned first. It's just that that's the first way we learn to write it. These others are just different ways… Math is more than our representations of it. (She laughs.) I don't know how to say it… Even THIS isn't the thing! (She circles the graph of the system she created earlier in the interview.) It's a graphical representation of "the thing" (she does air quotes), whatever the thing is.
Student 8 Anna	I would say two equations. A different way to write two equations. They all mean the same. They have the same concept. Maybe they all, each are trying to represent this (pointing to the systems representation). But I would say that they're all trying to represent this (pointing to the systems representation) but like in a matrix or in a vector. (Long pause) But I don't want to make it seem like this (pointing to the systems representation) is the sole thing that all of them are trying to represent. Because you could just be given each one (pointing to each representation) and think of it like *this*, or think of it like *this*, or think of it like *that*.
Student 9 Ken	I never really thought about it. I mentioned I think it's just notation. It doesn't really make a difference. I just kind of see a problem.
Student 10 Jake	Mentally I think of this (the systems representation) as kind of like, I would use it as a baseline. But at the same time, I wouldn't say that this is *definitively* the baseline to use. I wouldn't say these (pointing to the first three representations) are all like reflections of this (pointing to the systems representation). I would say like, they're all reflections of each other.

Peter's (Student 5) response requires a bit more scrutiny to conclude that he viewed the representations as indicating a system. Initially, he spoke in terms of a common solution as the goal relating all the representations; he stated, "I want to jump and say, what's the thing you're after? So the thing I would say is numeric values of *x* and *y* such that these systems of equations (pointing to each of the four representations), which are all equivalent, are all satisfied". While Peter's response indicates that he had in mind the goal of solving and the idea that each representation would yield the same solution, he gave indication that for him, each of the representations indicated a system of equations. He did so by pointing to each of the four representations one after the other and stating, "these systems of equations which are all equivalent…". The representations may have been so obviously systems for Peter that my question prompted him to go after some other thing represented; as a result his direct answer was that a solution was represented. That he conceived of a system as a focal point, however, is evident in his complete response.

Felix (Student 3), Sam (Student 6), and Peter (Student 5) all appeared to have a singular focus on the linear system of equations (and/or the linear systems representation). In other words, the three seemed anchored to the linear systems representation in an unwavering way.

Systems of Equations—Less Definitely. Two students' responses (Anna's and Jake's) indicate that they anchored to the linear systems representation, but not exclusively. Anna (Student 8) stated "I would say they are all trying to represent this (pointing to the systems representation) but like in a matrix or in a vector. (Long pause)

But I don't want to make it seem like this is the sole thing that all of them are trying to represent." Similarly, Jake (Student 10) stated, "I think of this (pointing to the systems representation)…as the baseline. But at the same time, I wouldn't say that this (pointing to the systems representation) is definitively the baseline. … They're (the four representations) all reflections of each other." Anna (Student 8) and Jake (Student 10) drew the same conclusion that the systems representation might be primary in some way which is not absolute. However, broader consideration of their dialogue reveals differences in the nature of their responses. I discuss the details in what follows.

Table 7

Categories of "The Thing" Represented

"The Thing" Represented	Students Exemplifying the Conception
System of Equations—Definitely	Felix (Student 3), Peter (Student 5), Sam (Student 6)
System of Equations—Less Definitely	Anna (Student 8), Jake (Student 10)
Quantitative System	Zeb (Student 1), Nick (Student 4), Myra (Student 7)
No Unified Thing	Ken (Student 9)
No Data	Mike (Student 2)

During our interview, Anna (Student 8) seemed to be talking aloud as a means of processing her thinking; her complete response as documented in Table 6 supports this conclusion. She indicated uncertainty by using "maybe", often qualified her responses by using "but", and frequently took long pauses between her statements. In particular she stated (emphasis added), "*Maybe* they all, each are trying to represent this (pointing to the systems representation). *But* I would say that they're trying to represent this (pointing to the systems representation) *but* like in a matrix or vector. (Long pause) *But* I don't want to make it seem like this is the sole thing that all of these are trying to represent." In addition, Anna's interview was punctuated by periods of silence, squinting, and sighing; I take those to mean that she was engaged in deep reflection.

Anna (Student 8) concluded that all the representations could be indicative of the linear systems representation; she acknowledged, however, that any one of the representations could stand alone by continuing (Table 6) "Because you could just be given each one (pointing to each representation) and think of it like *this*, or think of it like *this*, or think of it like *that* (pointing to each representation)". Anna's statement allows that she may have been reflecting deeply about the differing meanings for the representations given the context (i.e., register) of each. This is consistent with her previous statement, which while undetailed, indicates that she may have had something deeper than symbols and notation in mind: "They all mean the same. They have the same concept." On the other hand, the earlier part of her statement, "you could be given each one", could indicate a problem-solving mentality rather than deep reflection. She may have been referring to the problem posing of school mathematics rather than thinking about the meaning of each representation in terms of its register.

While Anna's (Student 8) statements suggest that she may have been in the process of thinking things through, Jake (Student 10) gave a rather clean and decisive answer. Concise and matter-of-fact answers were characteristic of Jake's interview. In this case he stated (Table 6), “Mentally I think of this (the systems representation) as kind of like, I would use it as a baseline. But at the same time, I wouldn’t say it’s definitively the baseline”. He went on to say, “I wouldn’t say these (pointing to the sheet of paper with the three representations from Question 11) are all reflections of this (pointing to the sheet of paper with the linear systems representation from Question 12). I would say like, they’re all reflections of each other.” Interestingly, Jake (Student 10) did not use the term “represent”; instead he used the word “reflections”, concluding that the representations were “all reflections of one another” (Table 6). I see this as similar to his avoidance of the word “same” as documented in Table 4; Jake (Student 10) preferred to use “equal”, “equivalent”, and “not equal” to compare and contrast the representations. Jake (Student 10) seemed to answer in ways that suggest prior deep reflection and metacognition; he, however, seemed to be reporting rather than processing in the moment of the interview. I note that Jake’s concise way of answering questions meant that we made it through the entire comprehensive interview protocol (Appendix B) in our allotted time, a rare occurrence.

I note that both Anna’s and Jake’s responses, and Jake’s in particular, seem consistent with Payton’s (2018) view that all the representations are “representations of each other” and that any one of the representations could be used for problem posing and/or problem solving. Anna’s response suggests construction and/or reconstruction of her thoughts. In contrast, during his interview Jake (Student 10) seemed to be reporting on his prior cognitive activities and realizations rather than processing.

A Quantitative System. Three students' responses (Zeb's, Nick's, and Myra's) indicate that they could have had a quantitative system in mind; that is, they may have conceived of a set of quantitative relationships indicated by the various representations. To answer the question "What is the thing that they represent?", Zeb (Student 1) concisely stated "They all have the same relationship". Because his response is so to the point, it may not seem especially convincing and/or illustrative of a quantitative system on the surface. However, relationships were a recurring theme in my interview with Zeb (Student 1). He gave answers that from my perspective were mathematically sophisticated, and he often anticipated the direction of the interview. Recall that Zeb (Student 1) uniquely and decisively stated that the four representations of a system were *not* the same despite identifying a number of common characteristics between them (Table 4), and I maintain the difference for him was based on mathematical structure. (See Sameness of Representations earlier in this chapter for a more complete discussion.)

At the very least, Nick (Student 4) seemed to have an abstract mathematical object in mind, perhaps a quantitative system. I take an abstract mathematical object, potentially a quantitative system, to be the antecedent for his use of the pronoun "them" when he stated, "You can represent *them* different ways that you want". Whatever "them" refers to has something to do with Platonic philosophy because my question "What is the thing that they represent?" elicited the response "I'm thinking back to high school, like the Platonic idea. That's the first thing that comes to mind." Nick went on to say, "You can represent them different ways that you want. But ideally they're all pointing to the same exact, you know, function or equation". Nick was potentially referring to functions and equations as examples of abstract mathematical objects for which we have a variety of means of representation. If so, for Nick those functions and equations seem to have served as additional examples of the focus of this study: a quantitative system for which a variety of representational *pointers* exist.

Of the 10 participants, Myra (Student 7) gave the most explicit and complete description consistent with a conception of a quantitative system. Rather than concluding that the first three representations were alternatives to the systems representation, she supposed that a tendency to anchor to the systems representation can be attributed to the systems representation being the one used to introduce students to systems, stating " It's just that that's the first way we learn to write it." The *it* and/or "the thing" for her was a mathematical truth or statement because of the systems representation she stated, "That's a representation of like, a mathematical truth or a mathematical statement". Myra (Student 7) asserted "math is more than our representations of it", though she struggled to put her ideas into words, stating "I don't know how to say it".

Note that in the interviews I generally discouraged prolonged calculation. However, Myra's thought processes seemed tied to writing things out as she talked, so I allowed her to proceed with a preponderance of calculation. As a result Myra (Student 7) performed the computations on each representation to demonstrate why the representations were of the same thing. Further, she graphed the system of equations so she could talk about the system in terms of its graph in the two-dimensional coordinate plane. When she struggled to describe the esoteric thing indicated by the various representations, she went back to her work on the graph of the system and exclaimed, “Even THIS isn’t the thing! It’s a graphical representation of “the thing” (doing air quotes), whatever the thing is”. Perhaps Myra (Student 7) considered the graphical representation to be more concrete or real that the analytic ones that I have focused on in this study; she still, however, saw the graph as simply a representation of some other thing. I take that thing to be a quantitative system since, as noted in Table 1, Myra (Student 7) had also stated “They all represent the same relationship between *x* and *y*. They just represent it in different ways.” Further, relationship was a common and recurring theme during our interview.

No Unified Conception. When asked “What is the thing that they represent?”, Ken’s response (Table 6) was “I never really thought about it. I mentioned I think it’s just notation.” While Ken (Student 9) indicated that he had never really thought about it, he also seemed to express a lack of interest in thinking about it. He stated, “It doesn’t really make a difference. I just kind of see a problem.” The later part of his statement, “I just kind of see a problem” could indicate that Ken (Student 9) was focused on getting answers to whatever exercise he encountered. That perspective seems to be supported by his disclosure earlier in his interview that he had not really practiced matrix multiplication and vector space operations (multiplying a vector by a scalar, adding vectors, etc.) since he could always use the augmented matrix representation along with his calculator to solve problems. During the interview, I was able to coach Ken (Student 9) to reconstruct matrix multiplication from his limited practice with it, but doing so was a laborious undertaking for both of us. I note that Ken (Student 9) was agreeable to work with me even though he had devised other ways to productively deal with his coursework, so his possible lack of interest referenced earlier does not seem universal.

When I pressed Ken (Student 9) about what might be represented by the representations, he said, “I don’t know, is there a word problem around here somewhere?” His comment is insightful and could indicate that Ken (Student 9) had something like a quantitative system in mind. More consistent with his responses discussed in the previous paragraph is that Ken (Student 9) was acknowledging that some problems in school mathematics, which is often largely a mathematics of exercises, may be stated in terms of word problems.

Consideration of Ken’s data documented in Tables 4 and 5 provides an image that is different but complementary to this discussion of the data about “the thing” represented (Table 6). At one point in our interview, Ken (Student 9) stated, “Those are alike because this is technically my step 1 (pointing to the augmented matrix). And this (pointing to the matrix equation) could have been a step 1 as well” (Table 4). In addition, Ken made the statement (Table 5), “They’re the same. They are just different in the way they are set up to be solved.” For Ken (Student 9) each representation seemed to be a prompt for a particular algorithm; the representations seem compartmentalized, each calling for its own solution process. This hypothesized compartmentalization along with the discussed uncertainty of how Ken thought about word problems leads to the conclusion that Ken (Student 9) may have had no unified thing in mind as the object of the various representations in this study.

A Case of No Data. I never posed the question "What is the thing that they represent?" to Mike (Student 2). My use of a brace in the linear systems representation was an obstacle for him. He was accustomed to seeing systems as equations stacked one on top of the other without the use of any kind of grouping symbol. The only mathematical context where Mike could recall seeing a brace was piecewise functions, a topic which had been challenging for him. His negative associations with brace as used with piecewise functions seemed to inhibit our progress through the interview protocol.

Summary of Generative Analysis

In summary, a broad exploration of the data was performed consistent with Clement's (2000) description of research conducted for generative purposes. The analysis allowed for the documentation of students' conceptions of a common and recurring topic (systems of linear equations) with a group of students (undergraduates, generally, and students enrolled in a junior-level applied linear algebra course more specifically) for whom little data exists (See Chapter 3 Literature Review.)

In general, the 10 participants concluded, in varying ways, that the four representations in the study are the same. The three students that did not reach the conclusion that the representations were the same were influenced by 1) a keen focus on mathematical structure, 2) hesitancy about word use, and 3) a conflict related to the notation used. The participants universally concluded that for all the representations, the solution was invariant while the notation was varying; they described the varying and the invariant in insightful ways.

When asked to reflect on the represented thing indicated by the four representations, participants provided responses that fell into three categories: 1) a system of equations without question, 2) a system of equations but with some flexibility, and 3) a quantitative system. I also found that one student likely had no conception of a unified object indicated by the representations.

Peter's Ranking of Translation Difficulty (A Convergent Analysis)

My interview with Peter (Student 5) lends itself well to analysis through the lens of translations with consideration given to congruence and incongruence. A discussion of congruence and incongruence of translations was given in Chapter 2. As a reminder, the two characteristics of a congruent translation per Duval (1999, 2006) are: 1) transparency between the starting register and the target register, and 2) unit-by-unit conversion. I take *transparent* to mean visually similar, admitting that this is subjective; I take *units* to mean sub-pieces of notation within algebraic expressions and/or equations.

Results and Findings

The dialogue between Peter (Student 5) and myself when I presented him with the first three representations that appear in Question 11 is shown in Figure 6. Peter (Student 5) indicated (line 3) that the augmented matrix and vector equation representations are transparent to him, stating “these two I can immediately see”. He indicated that the matrix equation is the “hardest one to see”, suggesting that the representation was not transparent to him. Note that while transparency is subjective, Peter's conclusion is consistent with mine when I used the matrix representation as an example of incongruence of translations in Chapter 2; we both viewed the translation involving matrix multiplication to be incongruent. Further, Peter's statement (line 7) that "*x* and *y* are stacked over on one side" seems indicative of a violation of Duval's second criteria for congruence, unit-by-unit conversion. The "stacking" that Peter (Student 5) described results from the creation of a single math object, a column matrix, with entries that correspond to the separated uses of *x* and *y* in the other representations. Peter's statement is an acknowledgement that the coupling of the two separated objects into a single mathematical object is not a unit-by-unit translation for him.

1	Peter:	This (pointing to the matrix equation) is the hardest one to see as the same thing, right off the bat. Because what I don't see is, … again, it's this junk…
2	JS:	Is it the matrix multiplication that's still messing with your mind?
3	Peter:	Yeah, like I would, … I don't see it immediately as the same thing. This one (pointing to the matrix equation) would take my brain the longest to work out as being equivalent. These two (pointing to the augmented matrix and the vector equation) I can immediately see.
4	Peter:	Because it's easy to see this is *x* & *y*. (He writes an *x* above the first column of the augmented matrix; he writes a *y* above the second column of the augmented matrix. See Figure 8) These things get multiplied by *x*. These things get multiplied by *y*. Which is what this is (pointing to the vector equation).
5	Peter:	(Peter works from left to right to describe the three representations.) This is an augmented matrix. This is something; I don't know what it's called. And this would be the linear combination?
6	JS:	Yes.
7	Peter:	So this one is the one (pointing to the matrix equation). Again, it's because the *x* and *y* are stacked over on one side that I don't see this linearity. (Motioning horizontally across the paper with his pencil.) Like this one (pointing to the augmented matrix), you just take what's on top.
8	JS:	Okay, I get what you're saying.
9	Peter:	This one (pointing to the vector equation), it's very easy to split this like, lengthwise. I don't know why. But I can switch it very easily in my mind. This one (pointing to the augmented matrix), I'm like, okay, these get *x*'s next to them, these get *y*'s next to them.
10	JS:	And then here (pointing to the matrix equation)…

11	Peter:	Yeah, here, it should probably be obvious, and it is still easy. You're like, oh okay, *x* goes here. This entire thing gets multiplied by *x* (pointing to the left column of the coefficient matrix), and this entire thing gets *y* next to it (pointing to the second column of the coefficient matrix). That's the jump you hafta make, but it's least evident because of the orientation of *x* and *y*. That's it. It's purely visual. The visual quality of the notation is what takes my brain the longest to see.
12	JS:	So mathematically, would you say these things are the same?
13	Peter:	Yeah, I would. Based on what we just talked about, I would say that they're the same.
14	JS:	Would they have the same solution?
15	Peter:	They should. Because all the coefficients are the same, and these are just different notations for the same thing.

Figure 6. Peter's discussion of the three representations in Question 11.

In line 4 Peter (Student 5), described the translation **augmented matrix ► vector equation**. He introduced the variables *x* and *y* into the augmented matrix representation by writing an *x* above the first column and a *y* above the second column. Peter (Student 5) easily saw that multiplying the first column by *x* and the second column by *y* suggests the vector equation, stating "which is what this is" (line 4). Peter (Student 5) described "splitting" the vector equation "lengthwise" (line 9) as easy, stating "I don't know why. But I can switch it very easily in my mind." What he wrote on the page (shown in the upper right-hand corner of Figure 8) is intriguing, as he grouped *x* with the first horizontal loop he drew, and he grouped *y* with the second horizontal loop he drew. This "lengthwise splitting" is suggestive of the "rows correspond to equations" heuristic that I described in Chapter 2; I say more about this in the "Horizontal Orientation" part of the "Discussion" section that follows.

Peter (Student 5) again described (line 9) how he easily introduces the variables *x* and *y* into the augmented matrix: "I'm like, okay, these get *x*'s next to them, these get *y*'s next to them." Then he reiterated his challenge with the matrix equation; he acknowledged that it is “easy”, but there is a “jump you hafta make”. Said more technically, Peter (Student 5) acknowledged that the translation is incongruent. He identified the difficulty as being “the orientation of *x* and *y*”, describing the difficulty as “purely visual”. Peter (Student 5) acknowledged both of the criteria for incongruence in describing the difficulty with the matrix equation. For one, the translation was not transparent to Peter (Student 5) since he saw it as visually complicated. In addition, the arrangement (orientation) of the units of the matrix equation does not allow for unit-by-unit translation. To sum up he stated, “the visual quality of the notation is what takes my brain the longest to see” (line 11).

Peter’s focus on the visual characteristics of the notation led me to ask if the three “things” are “*mathematically*” the same (line 12) and if “they have the same solution” (line 14). Peter (Student 5) asserted that his previous descriptions (as documented in Figure 6) supported that they are the same thing (line 13). Further, he believed they all had the same solution based on two criteria: 1) since “all the coefficients are the same” (line 15), and 2) since he viewed the three representations as “different notations for the same thing” (line 15). For additional context, recall that in the generative data reported earlier in this chapter, Peter (Student 5) claimed that the thing being represented was “numeric values of *x* and *y* such that these systems of equations (he pointed to each representation) which are all equivalent, are all satisfied”. Peter’s statements suggest that the representations are so obviously transparent to him, that he took all the representations to be systems of equations. This conception involves the translations between each of the representations in Question 11 and the systems representation in

Question 12, three of the possibilities I discussed in the Theory of Quantitative Systems. Though Peter did not explicitly identify his source and target representations, given his anchoring to the linear systems representation, I infer that it was the target in all cases while the other three representations each served as sources.

Figure 7 shows my dialogue with Peter (Student 5) when I showed him the fourth representation, the linear systems representation, from Question 12 of the interview protocol. He described the ease with which he gets the two equations in the linear system from the vector equation, stating “That’s (the linear systems representation from Question 12) these two if you distribute it”, seemingly referring to the process of multiplying the scalars by the vectors in the vector equation. Specifically, he was describing the **vector equation ► linear system** translation. Next, he ranked the translations between each of the three representations (from Question 11) and the systems representation (from Question 12) in order of difficulty for him; his written work appears as Figure 8.

16	JS:	What if I add a fourth thing? (I give him a sheet of paper with the linear systems representation as shown in Appendix D page 4.) Are those all the same? Are they different?
17	Peter:	This should be the same, because this is just… That’s (pointing to the vector equation) these two (pointing to the two equations in the system) if you distribute it. It’s the easiest way for me to think about it.
18	JS:	So you make this connection easiest? (Pointing to the systems representation then the vector equation.)
19	Peter:	Yeah, that's very, very quick. And then next would be this one. So if I rank these, this is easiest. This is one, this is two, and this is three. (Peter labels the 3 representations from Question 11 in the order: 2, 3, 1)
20	Peter:	As far as these ones, these ones (pointing to the augmented matrix and vector equation) are already set up in the same way. It's across this way. (Motioning from side to side with his pencil.)
21	Peter:	This one (pointing to the augmented matrix), you just have to remember that the columns are associated with variables. That's not a hard jump; I just remember that.

22	Peter:	This one (pointing to the matrix equation), again, is least evident. What this one makes me want to do is write this way again (motions from side to side), so *x* plus two *x* equals seven. That's why it takes a little longer because of just defaulting to rows.
23	JS:	You're trying to override something that pops up that's not the right thing?
24	Peter:	Right! And then I hafta go "No, no, no. That's not true! Because remember these hafta be in two variables for this to make sense." So that's why this takes the longest. I would want to think of this as 4*y* minus *y* equals one, which is not correct. This is not good! (Marking through the inscription on his paper.)

Figure 7. Peter's discussion of the four representations in Question 12.

Peter hinted (line 21) at the incongruence between the representations that he ranked second in difficulty, stating "this one, you just have to remember that the columns are associated with variables". In other words, the variables are implicit in the augmented matrix representation and variables must be introduced. I described this translation as incongruent in Chapter 2 since the translation is not unit-by-unit. Truly, the variable units are absent from the augmented matrix representation. For Peter (Student 5), nevertheless, "that's not a hard jump" (line 21, Figure 7). Finally, Peter (Student 5) indicated that the hardest translation for him to make between the systems representation and three other representations involves the matrix equation. He labeled the arrow he had drawn between them with a 3 in his written work (Figure 8); he described the most challenging translation for him by stating "this one, again, is least evident" (line 22).

Of the augmented matrix and vector equation, Peter (Student 5) stated (line 20, Figure 7) "these ones (referring to the augmented matrix and the vector equation) are already set up in a way; it's across this way" (he motions from side-to-side with his pencil). He indicated he was inclined to do the something similar with the matrix equation (line 7), "What this one (pointing to the matrix equation) makes me want to do is write it this way again (motions from side to side), so x + 2x = 7" (line 22). He then wrote the linear equation above the matrix equation as shown in Figure 8. Likewise, he was inclined to (incorrectly) conclude that 4y – y = 1 (line 24), which he wrote below the matrix equation as shown in Figure 8. Peter (Student 5) stated that in practice he reminds himself, however, that his equations need to be in two variables (line 24), a fact he uses to re-orient himself. Subsequent to writing 4y – y = 1 under the matrix equation, he crossed out his inscription with a large x and exclaimed "This is not good!" (line 24).

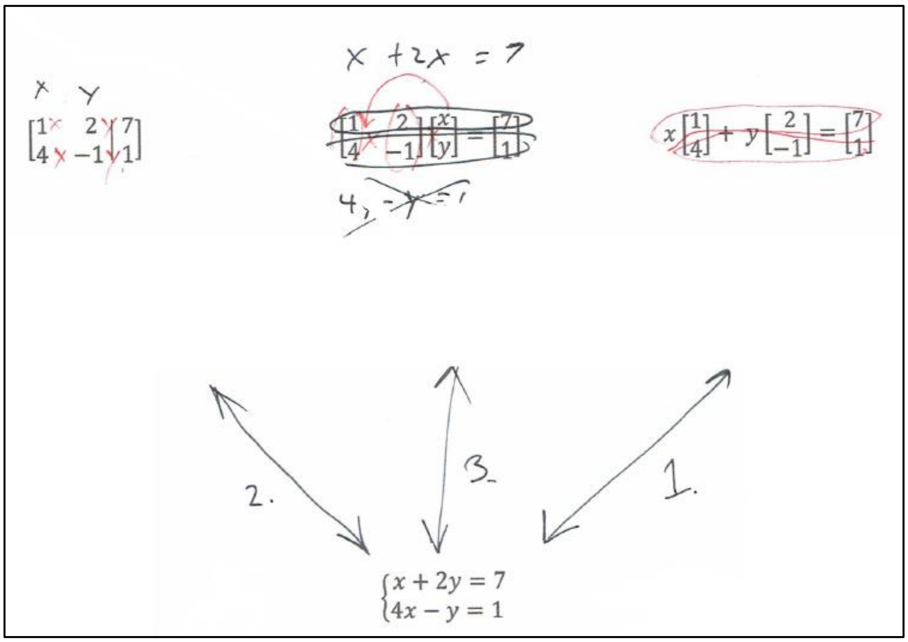


Figure 8. Peter's written work ranking translation difficulty.

Discussion

Peter's case allows for the consideration of 1) his horizontal orientation, and 2) how he manages an incorrect inclination, 3) whether his approach was "purely visual", and 4) which translations he performed and how he characterized each of them. I discuss the details of each in what follows.

Horizontal Orientation. Peter's description of the ease with which he moved between the augmented matrix and the linear system (ranked second most difficult) and the vector equation and the linear system (ranked the least difficult) seems to be based on a horizontal orientation. Peter (Student 5) explained the ease with which he worked with the two representations by stating (line 20, Figure 7), "They are already set up in a way. It's across this way (he motioned with his pencil from side to side)". Further indications of a horizontal orientation can be found in line 7, Figure 6, where he made horizontal motions with his pencil and described working with the augmented matrix by stating "you just take what's on top" (to form an equation). In addition (line 9, Figure 6), Peter (Student 5) spoke of working with the vector equation in terms of splitting it "lengthwise", which he illustrated as shown in the upper-right-hand corner of Figure 8. In addition, he motioned horizontally while referring to "defaulting to rows" (line 22, Figure 7).

Peter's communication, both verbal and non-verbal, is consistent with the "rows correspond to equations" heuristic that I hypothesized in Chapter 2 based on students' written exam data. In Chapter 2, I defined a heuristic as a shortcut or rule of thumb that allows one to solve a problem or achieve a result without all the computing or intermediate processing. The "rows correspond to equations" heuristic is the practice of taking anything horizontal and making an equation out of it. Perhaps the tendency is due to the horizontal way we write equations and/or because we read left to right. While the "rows correspond to equations" heuristic worked with the augmented matrix and the vector equation, it failed in the context of the matrix equation. That is, when Peter

(Student 5) tried to account for the translation from the matrix register to the systems register using the heuristic rather than mathematical processing via the dot product, the result was incorrect. Fortunately, Peter (Student 5) was able to bring his mathematics knowledge to bear in order to avoid applying the "rows correspond to equations" heuristic in the wrong context.

Overriding an Inclination. Peter's attempt to use the "rows correspond to equations" heuristic failed in the context of the matrix equation. Peter (Student 5) reported that, in practice his first tendency is to (incorrectly) make a linear equation using all the upper components of the matrix equation, and then to (incorrectly) make a linear equation using all the lower components. He demonstrated his tendency by writing x + 2x = 7 above the matrix equation and by writing 4y – y = 1 under the matrix equation as shown in Figure 8. I observe that his tendency is, of course, counter to typical dot-product multiplication which requires moving across a row while moving down a column. Peter's tendency results in single-variable equations which he, in practice, knows is wrong because he expects systems to be multivariable. As a result, he uses his expectation to do something different that will yield a result consistent with his multivariable expectation.

In constructivist terms, Peter's account of his thought processes might be framed by concluding that for Peter (Student 5), the matrix equation evoked the "rows correspond to equations" heuristic, a heuristic that frequently gives the correct result in working with systems. When Peter tried to apply the heuristic to the matrix equation, he encountered a cognitive obstacle. He expects multi-variate equations but applying the heuristic results in single-variable equations. The single-variable result does not fit with his systems of equations scheme, which includes the expectation of equations of more than one variable. He, therefore, adapts his approach so that the result would fit his expectation. Since Peter (Student 5) could not assimilate the result of the "rows correspond to equations" heuristic to his linear systems scheme, he rejected the heuristic and adapted his approach. In other words, he managed the unexpected result of the heuristic by rejecting the heuristic rather than adapting his linear systems scheme. He knows he needs some other combination of symbols to get a result that fit his scheme for systems of linear equations. Thus, he looks for an arrangement of symbols that does not produce a cognitive conflict for him. I note that the way Peter reported his practices leaves the impression that he goes through the sequence of thought processes frequently. One could hope that he would adapt his scheme related to the "rows correspond to equations" heuristic to reject its application in the context of matrix equations.

Peter (Student 5) and I discussed his thought processes in terms that were lighter than my constructivist framing of them; the approach seems to have been effective. We, as I see it, used a computer processing metaphor to discuss his thinking processes. In our dialogue, Peter (Student 5) acknowledged his primary instinct in practice is "just defaulting to rows" (line 22, Figure 7). His use of what I considered to be technological jargon, "defaulting", led me to pose a question in similar terms (line 23, Figure 7): "You're trying to *override* something that *pops up* that's not the right thing?" Peter (Student 5) seemed to connect with the metaphor, responding (line 24, Figure 7), "Right! And then I hafta go, 'No, no, no. That's not true!'." The metaphor could be extended to conclude that Peter used his mathematics to debug his process, thereby overriding his first inclination.

I take Peter's energetic "No, no, no!" and animated crossing out of his incorrect conclusion (Figure 8) as suggestive that overriding one's primary (invalid) impulse to reach a valid conclusion requires concentrated effort and energy. His reactions also, in my view, legitimize the use of the phrase "cognitive obstacle" with its potentially negative connotations. Further, the episode highlights that bringing mathematics to bear is important when one has the tendency to operate visually and/or according to a heuristic. Interestingly, the mathematics that Peter (Student 5) brought to bear on the situation is not particularly matrix multiplication (since it appeared to be an obstacle for him) but the expectation of a multi-variate result. Nevertheless, he was able to override his horizontal inclinations and reorient himself in order to get the correct result.

Purely visual? Peter (Student 5) identified his difficulty with the matrix equation as being "purely visual" (line 11, Figure 6), concluding "The visual quality of the notation is what takes my brain the longest to see" (line 11, Figure 6). Further, he used the word "see" seven times in the dialogue in Figure 6 (lines 1, 3, 4, 7, and 11), and largely communicated about the representations in horizontal terms in the dialogue in Figure 7 (lines 20, 22, 24). However, to take Peter's own analysis of his issue as being "purely visual" or to extend the diagnosis with the matrix equation context to the other representations is not supported by the data as I see it.

Peter's difficulty with matrix equation does not seem to be purely visual; his self-diagnosis addresses only the first criteria for congruence of translations: transparency between the source and target representations. His difficulty also potentially involves the second criteria of congruence of translations: unit-by-unit translation. I described the translation **matrix equation ► linear system** as an example of incongruence in Chapter 2 and discuss it further in the "Translations Used" section of that chapter. Since Peter struggled with matrix multiplication, his reliance on the visual qualities of the notation could be rooted in a deficit with the operation. I addressed this in the interview by asking Peter, "Is it the matrix multiplication that's still messing with your mind?" (line 2, Figure 6). Peter responded, "Yeah, I don't see it immediately as the same thing. This one (pointing to the matrix multiplication) would take my brain the longest to work out as being equivalent" (line 3, Figure 6). I conclude, therefore, that factors were involved that go beyond the visual qualities of the notation.

While I have argued that Peter's self-diagnosis was inadequate from my perspective, I will now discuss why it is invalid to extend Peter's self-diagnosis of "purely visual" to contexts other than the matrix equation. In other words, I will highlight several statements that Peter (Student 5) made that indicate that he combined whatever visual and heuristic approaches he used with mathematics. At times it is evident that mathematics supported and underpinned Peter's visual and heuristic approaches. Expecting multi-variate linear equations was one such instance. Another is what I see as a discussion of vector space operations. Peter stated (line 17, Figure 7) "That's (pointing to the vector equation) just these two (pointing to the two equations in the system) if you distribute it". "Distribute it" seems to indicate multiplying the vectors in the vector equation by the variable scalars; the phrase could further extend to include the vector addition required subsequent to the scalar multiplication in order to obtain the equations in the system. In his statement, I see an understanding of vector space operations.

Another indication of mathematics in Peter's dialogue is his mention of linear combinations. He made an attempt to give a name to each of the representations in Question 11. Specifically, he stated, "This is an augmented matrix. This is something; I don't know what it's called. And this would be a linear combination?" (line 5, Figure 6). The intonation of the last part of his statement suggested uncertainty, and technically, only the left-hand side of the vector equation would be referred to as a linear combination. Nevertheless, Peter (Student 5) acknowledged a concept fundamental to linear algebra, the concept of a linear combination.

A third indication of the mathematics that Peter (Student 5) employed is his justification for why the various representations would have the same solution. He stated, "Because all the coefficients are the same, and these are just different notations of the same thing" (line 15, Figure 6). I asserted earlier that Peter's use of "coefficients" may encompass both the coefficients and constants in the representations. I see Peter's "same coefficients" imply "same solutions" logic as especially insightful. Of course, the implication could be closely related to and/or an extension of his "different notations of the same thing" statement. I also see another possibility where the implication stands separate as a different kind of justification than the "different notations of the same thing" statement. Peter (Student 5) may have had in mind that since each representation had the same constraining values, he would expect the solutions to be the same. Either way, I see Peter's statement as evidence that his visual and heuristic approaches were supplemented with mathematics and logic.

I note that I went back and studied Peter's transcript for evidence of mathematics after working more with the data from Felix (Student 3). At first pass, I might have been inclined to concur with Peter's assertion of "purely visual" and to extend it beyond the context in which he used it, the matrix equation. Consideration of Felix's metaphor for translations led me to further scrutinize the term "purely visual" and brought into focus that mathematics might underpin Peter's conception. Perhaps the reader will appreciate the distinctions more after reading about Felix's emerging conception of translations later in this chapter.

Translations Used. What served as source and target representations for Peter (Student 5) was often implicit in his discussion of the first three representations (Figure 6). "This one (pointing to the matrix equation) would take my brain the longest to work out as equivalent. These two (pointing to the augmented equation and the vector equation) I can immediately see" (line 3, Figure 6). Apparently, Peter (Student 5) was trying to establish "equivalence" (his wording) of the three representations; however, I argue that he did not do so by translating between the three representations themselves with one obvious exception.

In Peter's dialogue documented in Figure 6, I find Peter's attempts to form equations out of horizontals as described in the "Horizontal Orientation" section above and as had been his practice throughout the interview. In a number of instances beyond those shown in Figure 8, Peter (Student 5) would draw horizontal loops prior to writing equations. In addition, Peter was one of the three students in this study who seemed anchored to the linear system of equations in an unwavering way. (See "'The Thing'

Represented" section of this chapter.) I argue, therefore, that Peter (Student 5) seemed to have in mind that if he could conclude that the three representations gave rise to the same equations, then he could conclude that they were "equivalent" to one another. That is, I see the linear systems representation as Peter's target representation in the majority of cases in the dialogue documented in Figure 6. This seems true even though I had not yet presented Peter (Student 5) with the linear systems representation. Peter's statement about trying to work out equivalence between the three representations, in my estimation, does not carry with it that he had in mind working with translations between the three representations themselves.

While I argue that Peter (Student 5) was trying to establish the "equivalence" of the three representations in Question 11 by using the linear system as a link, I find one exception where he identified a target representation other than the linear system. The data indicates that Peter (Student 5) clearly addressed the **augmented matrix ► vector equation** translation. He stated (line 4, Figure 6), "Because it's easy to see this is *x* & *y*. (He writes an *x* above the first column of the augmented matrix; he writes a *y* above the second column of the augmented matrix.) These things get multiplied by *x*. These things get multiplied by *y*. Which is what this is (pointing to the vector equation)." He essentially equated multiplying the columnar elements of the augmented matrix by variables to multiplying the column matrices (vectors) in the vector equation by variable scalars. I assert, however, Peter seemed to be adding variables to the augmented matrix as shown in Figure 8 to help him envision equations, especially since he subsequently said of the augmented matrix (line 7, Figure 6), "you just take what's on top" (to form an equation). Right after making his statement comparing the augmented matrix with the vector equation, Peter jumped to trying to name and/or label each representation working from left to right. From this I conclude that his acknowledgement of the translation **augmented matrix ► vector equation** was parenthetical in nature, supporting my assertion that translating each representation to a system was his central means of establishing their "equivalence".

Peter (Student 5) was more explicit (Figure 7) in identifying source representations and their intended targets after I introduced the fourth representation, the linear systems representation. He ranked the ease of translation between each of the three representations from Question 11 and the linear system as shown in Figure 8. He indicated that it was easiest for him to work between the vector equation and linear system. In one instance he described the translation **vector equation ► linear system** by stating (line 9, Figure 6), "This one (pointing to the vector equation), it's very easy to split like this, lengthwise"; he did so while pointing to the lengthwise loops he had made on the vector equation as shown in Figure 8. In another instance he described the translation by stating, "That's (pointing to the vector equation) these two (pointing to each of the two equations in the linear systems) if you distribute it" (line 17, Figure 7). In other words, Peter acknowledged that performing treatments on the vector equation would result in the two equations of the linear system.

Next Peter (Student 5) identified that working between the augmented matrix and linear system was the second easiest for him. He described the translation that he ranked second in difficulty, the **augmented matrix ► linear system** translation, by stating (line

7, Figure 6), “Like this one (pointing to the augmented matrix and motioning horizontally), you just take what’s on top” (to form an equation). At another point he said of the translation, “These ones (pointing to the augmented matrix and vector equation) are already set up in the same way. (Motioning horizontally) It’s across this way” (line 20, figure 7). Here, I take Peter to be calling upon the “rows correspond to equations” heuristic to envision equations resulting from each row of the augmented matrix.

Peter (Student 5) described the translation he ranked most challenging for him, working between the matrix equation and the linear system, by stating (line 11, Figure 6), “This entire thing gets multiplied by *x* (pointing to the left column of the coefficient matrix), and this entire thing gets *y* next to it (pointing to the second column of the coefficient matrix. That’s the jump you hafta make, but it’s the least evident because of the orientation of *x* and *y*”. Note that for me Peter’s description elicits the **matrix equation ► vector equation** translation since he spoke of multiplying the columnar elements of the coefficient matrix by variables. However, based on my analysis of his dialogue documented elsewhere in this chapter, I am convinced that for Peter (Student 5) this was an instance of the **matrix equation ► linear system** translation. In addition to my previous analysis and arguments, note that when Peter (Student 5) wrote an additional *x* and *y* into the matrix equation, he did so on the right-hand-side like they are written in the linear system and not on the left-hand-side like they are written in the vector equation.

While Peter used double-ended arrows (Figure 8) between the representations to rank his ease with translation, his verbal descriptions only indicate translating in one direction: from each representation as a source to the linear systems representation as the target. In his dialogue he addressed what happened with each of the three representations that would result in the linear system. For instance, he spoke about splitting the vector equation "lengthwise" (line 9, Figure 6) and about the augmented matrix by stating "this one … you just take what's on top" (line 7, Figure 6). He said of the matrix equation "these hafta be in two variables for this to make sense" (speaking of linear equations that can be extracted from the matrix equation; line 24, Figure 7). I do not find evidence that Peter (Student 5) addressed the translations where the linear system was the source with the other representations serving as the targets. He seemed to have in mind a *bi-directional conception* where translating in one direction subsumes translating in the opposite direction. As a result, the reverse translations were unaddressed. I assert that practice in translating in reverse and in translating with differing combinations of the representations is potentially instructive; I will discuss two examples in terms of Peter's conceptions. Specifically, I will address the **linear system ► matrix equation** translation and the **vector equation ► augmented matrix** translation.

If Peter (Student 5) were to concretely consider the translation **linear system ► matrix equation**, three groupings would need to occur: the grouping of the coefficients into a matrix, the grouping required to form the variable matrix, and the grouping required to form the constant matrix. Potentially, the acts of grouping the various components of the linear system would aid Peter's ability to make sense of how to work with the various symbols in the translation **matrix equation ► linear system**, the translation that he ranked as most challenging for him. Further, the formation of the various matrices might help with his tendency to think in terms of rows since both the variable and constant matrices are columnar. In short, working with the translation **linear system ► matrix equation**, the reverse of the translation he discussed, could help Peter (Student 5) mitigate the lack of transparency between the two representations.

In addition to mitigating the lack of (visual) transparency between the representations, working with the translation **linear system ► matrix equation** could help Peter (Student 5) address the second facet of incongruence: lack of unit-by-unit translation. Recall that Peter described (line 11, Figure 6) his difficulty with the matrix equation as "purely visual" because of "the orientation of *x* and *y*". His self-diagnosis neglects consideration of the incongruence which can be attributed to a lack of unit-by-unit translation rather than (visual) transparency. Working with the **linear system ► matrix equation** could help Peter's awareness of how to deal with the two occurrences of *x* and *y* in the linear systems representation relative to the single occurrence of each variable in the matrix equation; that is, working with the translation might help Peter manage the lack of unit-by-unit translation between the representations. In summary, working with a reverse translation might bring distinctions to the forefront which would allow Peter (Student 5) to address both the visual and unit-by-unit aspects of the incongruence between the representations.

Another potential instance of neglected reversal of a translation is the **vector equation ► augmented matrix** translation. I do not find an explicit description of the translation in the dialogue documented in Figure 6, and I only find a suggestive but ambiguous indication of the translation in the dialogue documented in Figure 7. While Peter speaks of both representations in line 2 of Figure 7, I have argued that he compared them to each other only in terms of the linear equations they yield. While he made a side comment about the **augmented matrix ► vector equation**, the reverse translation went unaddressed. Once again, Peter (Student 5) seemed to have a bi-directional conception where translating in one direction subsumes translating in the opposite direction. I argue that forming the augmented matrix from the vector equation might ameliorate Peter's tendency to think in terms of rows.

Imagine if Peter (Student 5) had explicitly considered the translation **vector equation ► augmented matrix**. Granted, his horizontal orientation (with the augmented matrix on the left and the vector equation on the right) and bi-directional conception likely inhibited his consideration of it. Possibly, in translating, he would have taken each column vector to form the augmented matrix rather than working row-by-row. Given the vertical nature of the column vectors, the exercise could amend his tendency to think in horizontal, row-like terms. In other words, the exercise could make Peter (Student 5) aware of looking with a vertical orientation rather than his usual horizontal orientation. A vertical orientation would be important for any work he might do with column spaces, a context where his horizontal orientation might serve as an obstacle.

Of the twelve translations that I identified in the Theory of Quantitative Systems, Peter (Student 5) explicitly addressed the ones in Table 8 for which I supply line numbers from his dialogue. Most were discussed in the previous exposition. Observe that of the twelve possible translations, Peter (Student 5) seems to have addressed four of them. I argue that breaking down Peter's bi-directional conception and having him work with a greater variety of the translations would allow him to recognize nuances that might otherwise remain implicit. Specifically, such activities could help Peter (Student 5) manage his horizontal orientation and his challenges with the matrix equation.

Table 8

Translations Evident in Peter's Dialogue

Source Representation	►	Target Representation	Line Numbers for Evidence
Linear System	►	Augmented Matrix	
Linear System	►	Vector Equation	
Linear System	►	Matrix Equation	
Augmented Matrix	►	Vector Equation	Line 4, Figure 6
Augmented Matrix	►	Matrix Equation	
Vector Equation	►	Matrix Equation	
Augmented Matrix	►	Linear System	Line 7, Figure 6 Line 20, Figure 7
Vector Equation	►	Linear System	Line 9, Figure 6 Line 17, Figure 7 Line 20, Figure 7
Matrix Equation	►	Linear System	Line 11, Figure 6* Line 24, Figure 7
Vector Equation	►	Augmented Matrix	
Matrix Equation	►	Augmented Matrix	
Matrix Equation	►	Vector Equation	

Peter's case has allowed for the consideration of 1) his horizontal orientation, 2) how he manages an incorrect inclination, 3) whether his approach was "purely visual", and 4) which translations he performed and how he characterized them. In contrast, Felix's story gives a view into 1) how a student navigated, with support, treatments in the vector space register; and 2) how a student came to an awareness and mathematical understanding of an often taken-for-granted conversion from the matrix register to the linear systems register.

Felix's Emerging Conception of Translations (A Convergent Analysis)

I address two particular findings from my interview with Felix: 1) his metaphor for translations, and 2) his developing conception of translations. I address Felix's development first in the context of a treatment, and then in the context of a conversion. That is, after establishing Felix's metaphorical basis for considering translations, I report on a portion of the interview that dealt with a translation within a single register, the vector space register. Next, I discuss a portion of the interview that dealt with a translation between two registers, the matrix register and the linear systems register. Note that the three excerpts from our dialogue are presented in chronological order, and they are numbered continuously for ease of reference. However, Figure 10 did not follow directly after Figure 9 in terms of our dialogue.

Felix's Metaphor

Felix described the various representations in terms of differing *units* (sub-pieces of algebraic expressions; see Chapter 2) that get put together in various ways; I have documented his descriptions in Figure 9. Felix's descriptions were difficult to following during the interview and are still a bit challenging given the annotations I have made. I, however, want to consider the overall nature of his comments more so than the details.

Felix largely described the various representations in terms of how they appear on paper and translations between the various representations in terms of dynamic actions. For instance, Felix (Student 3) described the augmented matrix representation as "before *x* and *y* is placed" (line 2, Figure 9). He described the matrix equation as "where it was before we multiply it" (referring to the linear system that results from multiplying). He described the vector equation as "showing" multiplication not element-by-element, but column-by-column (line 2, Figure 9). In addition, Felix (Student 3) described the representations in terms of "columns", "individual pieces", and "one piece" (lines 7 and 8, Figure 9). He was apparently describing the coefficient vectors in the vector equation as "columns", the individual coefficients in the augmented matrix as "pieces", and the matrix as a whole in the matrix equation as "one thing". In other words, Felix (Student 3) acknowledged various *units* in the representations. In addition to his visual descriptions, Felix's referenced various actions involved in translating between representations, and those actions were not particularly mathematical. In at least four instances, Felix (Student 3) referred to the action of *cutting* into different pieces (lines 2, 6, and 7, Figure 9). Other actions he described are *gluing*, *throwing*, *tilting*, and *multiplying downwards* (lines 6 and 7, Figure 9).

Overall, I would say that Felix (Student 3) was using a *cut-and-paste metaphor* for transitioning between representations. That is, he seemed to group symbols into different pieces, some larger and some smaller, while describing the action of transitioning in terms of rearrangement. This regrouping and rearranging of units might be referred to as imagistic (Clement, 1994) since Felix (Student 3) seemed to act (mentally) on the representations using dynamic imagery. That is, Felix (Student 3) used dynamic imagery as a means to account for what happens between the source representation and the target representation. The cut-and-paste metaphor allowed Felix (Student 3) to view the translations as transparent, where the target representation is obvious from the source representation, given his mental activities; in essence, translations between the representations seemed obvious to him given his metaphor.

1	JS:	(Speaking of the 3 representations in Question 11). So in what ways do you see those things as being the same? And what ways do you see them as different?
2	Felix:	Basically, this (indicating the augmented matrix) is before the *x* and *y* is placed. This (indicating the matrix equation) is where it was before we multiply it. (I believe the "it" Felix is referring to here is the system of equations. When I follow up on this later Felix states, "All this is basically just linear systems of equations".) And this (indicating the vector equation) is just showing where it's been multiplied into the numbers. But instead of each individual element, they just cut it up into columns, and then multiply it.
3	JS:	So do you think all of those will have the same solution?
4	Felix:	Yeah, I think so. Because this is just basically another way of writing this and that. (Indicating each of the three representations.)
5	JS:	So you're seeing these (the 3 representations in Question 11) as different ways to write the same thing?
6	Felix:	Yeah, we just kind of like cut it up into different pieces. (Next Felix describes the representations in Question 11, working from left to right.) Like this (pointing to the augmented matrix), we just glued them together. And then cut this out (referring to an *x* and *y* that he had written above each column of the augmented matrix), and then just left it off on top to put it back when we take it apart again (referring to writing the system from the augmented matrix).
7	Felix:	This one (pointing to the matrix equation), you just gotta know, so there's a vector over there. Multiply down, which is basically this (pointing to the vector equation). Yeah, kind of like, throw it up, and then tilt it and then multiply downwards (still referring to the matrix equation). And for this one (pointing to the vector equation), we do the same thing. But instead of like just throw it up and multiply it down (as with the matrix equation), we just cut it into columns instead of individual pieces.
8	Felix:	Or ONE piece, I'd say (pointing to the matrix equation) because it'd be a matrix.

Figure 9. Felix's descriptions of translations.

Felix (Student 3) gave some indications of mathematical foundations for his cutting and pasting by frequently referring to multiplying and using mathematically correct terminology like "vector" (line 7), "columns" (line 7), and "element" (line 2).

However, while I was conducting the interview, I was uncertain about what mathematics might or might not underlie his metaphor. Opportunities allowing me to address my uncertainty presented themselves. As a result, I was able to investigate the mathematical underpinnings for his cut-and-paste metaphor in two ways. First, I set out to address the question "What is the nature of his understanding of vector space operations?" Next, I considered "How might I influence his thinking about changes in register?" Each question is addressed in what follows.

Felix's Developing Conceptions

While Felix (Student 3) did not attend to mathematical rigor in his descriptions of the various translations, I got the impression that reflection and deep thinking were mathematical practices for him. For instance, early in our interview Felix (Student 3) had made a pedagogical recommendation. He mentioned that his experience of being introduced to row reduction left him with the impression that it was an entirely new topic. In trying to make sense of the morass of calculations involved in row reduction, he was able to tie the process back to the elimination method he had learned earlier in his schooling. In time, he came to see row reduction as an extension of the elimination method. He suggested that teaching row reduction by explicitly having students recall what they had learned in the past, and specifically the elimination method, might help them progress more productively than introducing it more like an isolated new topic.

Another instance that seemed particularly insightful was Felix's comparison between coordinate systems and the four representations in this study. He compared the four representations, which he viewed as representative of a linear system, to using polar coordinates instead of Cartesian ones; he seemed have in mind how varying systems of notation can be used to indicate the same thing. The two instances I have described along with my overall experience of our interview suggested to me that Felix (Student 3) could provide potentially valuable insights into students' thinking about vector space operations and changes of register. I followed up on the possibilities as described in the following two sections of this chapter.

Working with Treatments (Considering Vector Space Operations). Possibly, the four representations under consideration were so obviously transparent to Felix (Student 3) that it did not occur to him to describe the translations in terms of mathematics. In the moment of the interview, though, his descriptions did not make it apparent that he was doing more than visually manipulating symbols. An opportunity arose to get a more complete picture of Felix's conceptions by discussing the vector space operations of scalar multiplication and vector addition. (Our latter interactions support that transparency *and* visual symbol manipulation were both parts of Felix's conception in all likelihood.) Of the three representations in Question 11, Felix (Student 3) stated (line 9, Figure 10): "This way (pointing to the augmented matrix) you can solve with row-reduced echelon form. And this one (pointing to the matrix equation), if it's invertible, we can just do an inverse matrix. And for this one (pointing to vector equation)…" (long pause) "… I don't think I know of a way to solve it with this notation" (line 10, Figure 10). Our conversation proceeded as shown in Figure 10.

I asked Felix (Student 3) (line 11, Figure 10) to solve the system starting with the vector equation after he expressed doubt about his ability to do so. (I take his admission as evidence that, to some extent, he had previously been engaging in visual symbol manipulation based on his cut-and-paste metaphor.) In general, I had asked interview participants to suppress lengthy calculating and just talk with me, but this seemed like a promising opportunity for relaxing that instruction. With my framing of the problem by describing *x* as a scalar (line 11, Figure 10) and my question "how do you multiply it (the *scalar x*) by the vector?" (line 12, Figure 10), Felix (Student 3) responded "just multiply element by element" (line 13, Figure 10). The written work he produced appears in line 2 of Figure 11. Note that I felt it might be necessary to described *x* as a scalar in case Felix (Student 3) had a surface view of *x* as a variable without attributing to it the additional role of scalar within a vector space; further, the possibility existed that Felix (Student 3) had only seen scalars that were real numbers.

9	Felix:	This way (pointing to the augmented matrix), you can solve with row-reduced echelon form. And this one (pointing to the matrix equation), if it's invertible, we can just do an inverse matrix, and for this one (pointing to the vector equation)…
10	Felix:	…I don't think I know a way to solve it with this notation.
11	JS:	Yeah! Can you do that for me? How would you do that? You know how to multiply by a scalar.
12	JS:	Okay, that doesn't look like a scalar (pointing to the *x* in the vector equation). But if *x* is a scalar, how do you multiply it by the vector?
13	Felix:	Just multiply by each element. (Mumbling and writing line 2 of Figure 11.)
14	JS:	And then how do you add two vectors? You need to add these two vectors, right? (pointing to the right-hand side of line 2 of his written work)
15	Felix:	Yeah. So just add the corresponding elements. (Mumbling and writing line 3 of Figure 11.) You could even go as far as like…
16	JS:	Oh, what are you doing?
17	Felix:	Like maybe even just like, go one step further. (Mumbling and writing 4 of Figure 11.)
18	JS:	Okay, I gotta see what you are doing here.
20	JS:	Okay, you could do that. Right?

21	Felix:	Yeah.
22	JS:	Can you see how to get a system from this point? (pointing to line 3 of Figure 11.)
23	Felix:	Yeah. Because like with this one both terms of… hmm… (pauses)
24	Felix:	At the point we're adding them together ... we add them element by element. So it seems like the two equations are already sectioned off there, because they only add them by the same elements. We’re not adding it this way, or that way. They’re separate, but they are tethered to each other. They have a section, like a partition.

Figure 10. A dialogue about vector space operations.

Next, I framed his written result in line 2 of Figure 11 as the sum of two vectors and asked, “how do you add two vectors?” (line 14, Figure 10). Felix (Student 3) stated, “so just add the corresponding elements” (line 15, Figure 10) and wrote line 3 of Figure 11. My expectation was that Felix would next write a system of equations; however, he did not. Rather, he stated, “You could even go as far as like…go one step further” (lines 15 &17, Figure 11). He then mumbled while writing line 4 of Figure 2. In terms of my and Duval’s theories, Felix performed a treatment in the vector space register rather than performing, as I expected, a conversion between the vector space register and the linear systems register. I note that his work in line 4 is evidence of a propensity to continue working in the same register rather than changing registers. An open question is whether students in general exhibit such an inclination to perform treatments rather than conversions. A similar question could be asked about experts.

After his unexpected action, I asked Felix (Student 3) (line 22, Figure 10) if he could see how to get a system directly from line 3 of his written work. Note that Felix’s answer (line 24, Figure 10) provides additional indications of an imagistic, cut-and-paste mentality. He described adding vectors element-by-element as a means of “sectioning off" (line 24, Figure 10) two different equations while at the same time indicating that they are "tethered to each other" (line 24, Figure 10). I deduce Felix (Student 3) viewed a single vector as a unifying object for its differing entries. That is, the vector indicates a coupling of the elements that comprise it; however, vector addition which occurs element-by-element indicates a "section" or "partition" (line 24, Figure 10), in this case indicating two different linear equations. Felix's vivid images of sectioning, tethering, and partitioning capture the essence of the translation from line 3 of Figure 11 to a linear system of equations in an impressive way. Whether Felix (Student 3) could describe the translation in mathematical terms according to mathematical logic was still an open question, however. In fact, Felix's responses could have been influenced by the way I asked the question, "Can you *see* how to get a system from this point?" (line 22, Figure 10). My investigation into Felix's formation of a mathematically logical conception is discussed in the next section, "Mathematically Solidifying a Conversion".

In summary, with my framing of the problem (line 11, Figure 10) by suggesting that *x* is a scalar and that the result of the scalar multiplication (line 14, Figure 10) is the sum of two vectors, along with my encouragement that "you know how to multiply by a scalar" (line 11, Figure 10) and my question "how do you add two vectors?" (line 14, Figure 10), Felix (Student 3, Figure 10) accurately described and enacted the vector space operations of scalar multiplication and vector addition. In other words, Felix (Student 3) was able to perform a series of treatments in the vector space register as shown in Figure 11. This suggests that, while Felix's descriptions tended to be metaphorical and imagistic, his conceptions were not without mathematical foundations.

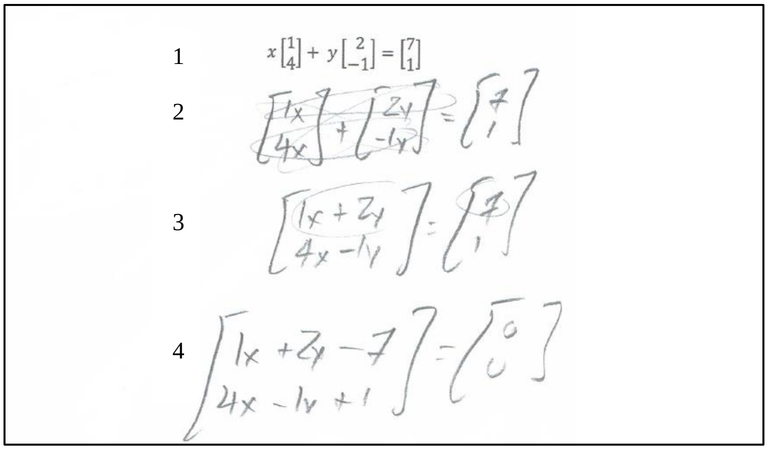


Figure 11. Felix's written work (with line numbers).

Supporting a Conversion Mathematically. My expectation was that Felix (Student 3) might write a system of equations from line 3 of his written work (Figure 11). Since he did not write a system on his own, I took the opportunity to investigate if Felix (Student 3) could develop a mathematically logical foundation for the required change from the matrix register to the systems register. I note that Felix (Student 3) was interested in, and therefore able to, engage in the conversation; this is in contrast with Ken (Student 9) described earlier in this chapter. While Ken (Student 9) seemed to have little interest in investigating connections, Felix's interest in making connections had become apparent during our interview as evidenced by the explanations he often constructed for himself. Recall, for instance, his pedagogical recommendation and his comparison of the representations in this study to Cartesian versus polar coordinates. My attempt to influence his thinking is documented in our dialogue shown in Figure 12.

When I suggested (line 25, Figure 12) to Felix (Student 3) that some people go directly from what he wrote in line 3 to writing a system of equations, Felix (Student 3) acknowledged that the translation "seems kind of weird" (line 30, Figure 12) since "we're just kind of getting rid of the vector". While I described the translation as "getting rid of the brackets" (line 31, Figure 12), Felix (Student 3) spoke in terms of "getting rid of the vector" (lines 30 and 32, Figure 12). Felix (Student 3) had seemingly become uncomfortable with the "disappearing brackets" approach. Subsequently, he rather energetically asked a question in three ways (line 32, Figure 12): "Why did you get rid of the vector? What happened to the vector? Why is the vector not there anymore?"

25	JS:	So some people go from this (pointing to line 3 of Figure 11) straight to writing a system, which we have written here somewhere.
26	Felix:	Yeah, like this. (pointing back to the systems representation on some of his earlier work.)
27	JS:	Why can I write a system from this? (pointing to line 3 of Figure 11) When I multiply these out, I should be getting matrices, right? Or these are actually vectors? Do you see those as vectors? (I tend to think in terms of matrices, but Felix had been calling them vectors.)
28	Felix:	Um-hmm.
29	JS:	How can I go from having a vector to having a system? How can I explain that mathematically?
30	Felix:	Yeah, trying to explain it from just looking at this (pointing to line 3 of Figure 11) seems kind of weird because we're just kind of getting rid of the vector.
31	JS:	You just get rid of the brackets, right?
32	Felix:	Yeah, you just get rid of the vector. But then the question comes up, why did you get rid of the vector? What happened to the vector? Why is the vector not there anymore?
33	JS:	(I laugh.) Yeah, that's my question!
34	Felix:	Yeah!
35	JS:	What if you think about it as a matrix rather than a vector? Does that help any?
36	JS:	What makes two matrices equal? How do you know two matrices are equal?
37	Felix:	If the corresponding elements equal to each other.
38	Felix:	So this would equal that!
39	Felix:	Yeah, but that IS the definition of a matrix, and if they know how to identify if they are the same.
40	JS:	What does *matrix equality* mean?
41	Felix:	Yeah, *matrix equality*.

Figure 12. A dialogue about a change of register.

Next, I asked Felix (Student 3) if thinking in terms of matrices would help resolve the issue (line 35, Figure 12). I followed up by asking, “what makes two matrices equal?” (line 36, Figure 12); Felix (Student 3) responded “if the corresponding elements equal to each other” (line 37, Figure 12). At that point, Felix (Student 3) seemed to have an epiphany. He circled the first entry of the matrix/vector on the left and then circled the first entry of the matrix/vector on the right as shown in line 3 of Figure 11. While circling the top two components, he stated “So this would equal that!” (line 38, Figure 12). Next, Felix (Student 3) drew an arrow from line 3 of his written work and extended his written work by writing the first equation of the system as shown in Figure 13. Felix had apparently realized that the equations for the linear systems representation result from equating corresponding elements of the matrices (vectors). In his next statement, Felix (Student 3) seemed to be describing the essence of matrix equality (line 39, Figure 12): “but that IS the definition of a matrix, and if they know how to identify if they are the same”. I gave Felix (Student 3) a name for his description by asking (line 40, Figure 12): “What does *matrix equality* mean?”. He latched onto the terminology, stating “Yeah, *matrix equality*.”

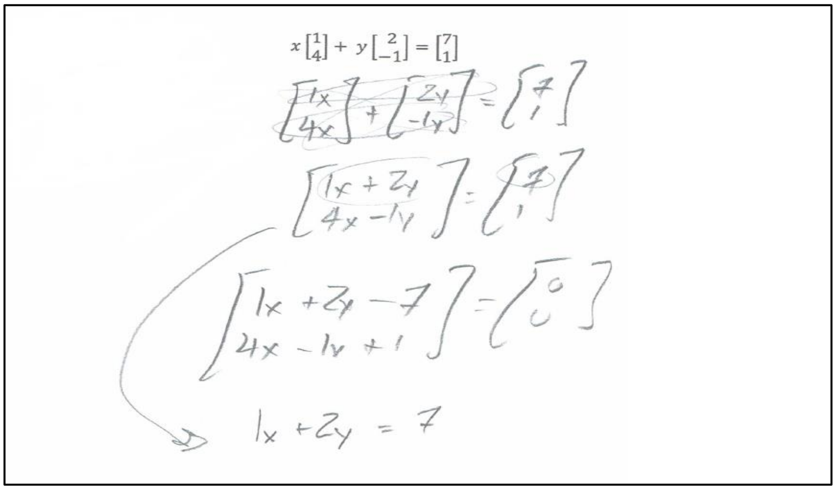


Figure 13. Felix’s addendum to his written work.

Earlier in this chapter I described how Felix’s cut-and-paste metaphor for translations led me to consider the question: “How might I influence his thinking about changes in register?” I did three key things to support Felix’s mathematical realization. First, I drew his attention to the change in register: “How can I go from having a vector to having a system? How can I explain that mathematically?” (line 29, Figure 12). Next, I changed my language to match his language “Do you see those as vectors?” (line 27, Figure 12). While I was thinking in terms of matrices, he was thinking in terms of vectors. Third, I attempted to help him shift from his vector perspective to my matrix perspective (line 35, Figure 12). I did this since I supposed that he might have more experience with the tenets of matrix algebra than with the tenets of vector spaces. Specifically, I thought the idea of “matrix equality” might be a more familiar idea than that of “vector equality”.

As a result of my influence, Felix (Student 3) seemed to have constructed mathematical support for the conversion from the matrix register to the linear systems register. In other words, Felix (Student 3) became able to mathematically justify why a system of equations can be extracted from a representation involving only matrices (or vectors depending on one's perspective/language). I assert that for Felix (Student 3), an unreflected, taken-for-granted step in a solution process had become a translation rooted in mathematical logic.

Summary of Results and Findings

A summary of results from the broad exploration of the data collected was outlined in the section "Summary of Generative Analysis". Briefly, the 10 participating students concluded that for the four representations in this study, the notation was varying while the solution was invariant. Further, categories of "the thing" represented were 1) a system of equations without question, 2) a system of equations but with some flexibility, and 3) a quantitative system. In addition, one student likely had no conception of a unified object indicated by the representations.

In addition to the analysis conducted for generative purposes, I conducted convergent analysis through the lenses of Duval's theory and the Theory of Quantitative Systems. The steps that I took to ensure richer results than Pavlopoulou's (1994) as called for Artigue (2000) along with the semi-structured nature of the interviews meant that data that lent itself to analysis through those theoretical lenses was not guaranteed. Peter's (Student 5) and Felix's (Student 3) interviews emerged as prominent examples of data suitable for analysis using Duval's and my theories.

From Peter (Student 5) we learned that he perceived of some translations as more difficult than others. Further, he used a combination of visual techniques, heuristics, and mathematical computation to establish connections between representations. Of the twelve possible translations that I outlined in Chapter 2 (Table 1), I found evidence that Peter (Student 5) addressed four of them. He seemed to have a bi-directional conception of translations, giving no apparent notice to reverse translations. Peter (Student 5) was without question tethered to the linear systems representation, referring to linear algebra as “the math of systems of equations”.

Felix (Student 3), in contrast to Peter (Student 5), accounted for translations primarily by using dynamic imagery which I have designated the *cut-and-paste metaphor*. With support, however, Felix (Student 3) was able to voice his knowledge of vector space operations and to carry them out. Further, a change of register that he had seemingly taken for granted prior to our interview became a matter of importance for him. With support, Felix (Student 3) was able to construct mathematical justification for the conversion and resolve the conflict he encountered.

The results and finding in this chapter address my primary research question for this project: What is the nature of undergraduate students’ conceptions of multiple analytic representations of systems? For instance, students thought in terms of varying notation and an invariant solution and gave various descriptions of the two. In support of my primary research question, the results and findings address the question: What unified thing, if any, do students have in mind as the represented entity? Some students thought of the linear system as the absolute target of all the representations, while others thought of the systems representation as more equitable with the other representations. Others seemed to have a quantitative system in mind, while one participant did not provide strong evidence of having a unified object in mind as the target of the various representations. Finally, the interviews unfolded in a way that allowed for the consideration of an additional question that supports my primary research question: How do students account for similarities and differences between the representations in terms of translations and registers of representation? Students employed visual techniques, heuristics, metaphors, and mathematical computation to account for translations between the various representations.

Given the results and findings discussed in this chapter, I address ideas that are relevant but less closely tied to documented observations in Chapter 6 Discussion. Chapter 6 addresses additional issues, questions, and realizations that emerged during the study along with pedagogical implications and potential topics for future research.

CHAPTER 6

DISCUSSION

This chapter provides a discussion of what I see as topics relevant, but perhaps auxiliary, to my study. I present the discussion in sections based on my findings. I claim the delineation of the Theory of Quantitative Systems as my first significant result and discuss its utility at greater length. Next, I provide additional discussion related to my generative findings. I conclude with further discussion related to my convergent findings.

The Theory of Quantitative Systems

The Theory of Quantitative Systems accomplishes at least two things. First, the designation of a *quantitative system* provides a mechanism for clearly distinguishing between *the represented* and *the representation* in the context of linear systems. Second, the *equitizing* of the various representations made possible through the designation of a quantitative system alongside the consideration of reversibility allowed for the identification of 12 possible translations. This brings an awareness to perhaps taken-for-granted moving parts involved in linear algebra. I discuss the two contributions at greater length in what follows.

Decoupling *The Represented* and *The Representation*

Duval's 3rd theorized source of incomprehension in mathematics (treatments and conversions are the 1st and 2nd sources) is the necessity in mathematics to see a variety of representations, perhaps from differing registers, as representative of the same entity. Closely tied to Duval's 3rd source of incomprehension is his idea of the cognitive paradox. He described the cognitive paradox by asserting that mathematics comprehension requires that we not confuse *the represented* with *the representation* even though the only (perceptible) access we have to the represented mathematical object is through the symbols we use to denote it. I propose the descriptor *enmeshed* for situations where *the represented* is not distinct from *the representation*; in essence, *enmeshed* is descriptive of the confounding of *the representation* and *the represented*. In that case, one may have an *enmeshed conception* where there is no distinction in thought between *the represented* and *the representation*. One may also engage in *enmeshed communication* where the communicator may on some level be aware of the distinction between *the represented* and *the representation*, but their communication does not make the distinction explicit. In other words, one may cognitively separate *the represented* and *the representation* while communicating in such a way that there is no clear boundary between the two.

In Chapter 3 Literature Review, I noted how Harel (2017) referred to the linear system as both *the represented* and *the representation*. I would say that doing so is a case of enmeshed communication which could be problematic for students; indeed, as described in Chapter 3, it seems to have been problematic for his research participants. In Chapter 3 I also noted that Trigueros, et al. (2018) seemed to use the word "model" interchangeably to refer to the application being studied (traffic flow) and the symbolic representation of the situation; I see this as another instance of enmeshed communication. Enmeshed communication is also likely what Hillel (2000) observed in his study of videotaped sessions of lecturers teaching on the topic of eigenvectors and eigenvalues.

He noted that the experts moved fluidly between various representations giving little notice to the nuances of notation and meaning.

The construct *quantitative system* is a mechanism which provides an individual (on an intrapersonal level) with a way of thinking which clearly distinguishes between *the representation* and *the represented*. As a mechanism, it allows for the identification of an object which is not (nor can be taken to be) notation. On an interpersonal level, the idea of a quantitative system provides a language for communicating which avoids the confounding of *the represented* and *the representation*. The construct should allow experts to consider whether their communication is enmeshed in ways that may influence students to fall victim to the cognitive paradox, taking the notation itself to be the object of study, *the represented*.

I would say that I have identified a potential *expert blind spot* (Nathan, Koedinger, and Alibali, 2001) where experts' content knowledge prevents them from viewing the content in terms of students' development and learning processes. Consider the situation where an instructor, who clearly knows the difference between *the represented* and *the representation*, communicates in such a way that the distinction is not clear; students may conclude that they are studying the *notation*, the linear systems representation itself. In the case of a quantitative system, what is often considered to be the primary representation, the linear systems representation, cannot be taken as the thing under consideration, and working with the linear system potentially takes on a quality other than moving symbols around according to properties. Given the idea of a quantitative system, experts can reflect on the cognitive paradox and consider ways their communication might promote the confounding of *the representation* with *the represented* amongst students. In other words, experts can consider a potential expert blind spot.

While in my estimation referring to the notation of the system (the linear system representation) as the object of study (*the represented*) is acceptable in a theoretical course, doing so becomes potentially more problematic in an applied course, which the bulk of undergraduate students experience. As I understand theory related to systems of equations, we have systems of equations in the real number system which we can translate to matrix algebra; working in the matrix algebra informs us about the system of equations. This a common practice in mathematics: reframing a problem in a different context that may be easier, better understood, or have different affordances in order to gain insight into the original problem. One example is linearizing. In the case of linear systems and matrices, the solution to the matrix equation can inform us about the solution to the system. I would not say the system of linear equations and the matrix equation have the same solution. If a solution exists, the matrix that solves the matrix equation informs us about the solution to the system. More specifically, the values of the entries in the solution matrix are precisely the values that solve the system. We are able to solve the system by casting it in a different light.

When we recast a linear system to consider it in terms of matrices, we need not always be specifically focused on solving. We can use characteristics of the matrices to inform us about the nature of the solutions to the system. The rank of a matrix can be used to determine the existence and uniqueness of solutions for the system. That is, by knowing the rank of the matrix we can know if a solution exists, and if so, we can determine whether the system has a unique solution or infinite solutions. In this way, the matrix algebra becomes a tool for informing us about the system. Matrix algebra as a mathematical system is a tool for analyzing the system of equations, so in a theory course thinking of the linear system only in terms of notation may be sufficient. We are simply moving between mathematical systems to leverage the affordances of the new mathematical system (the matrix algebra) in the exploration and understanding of the linear system (in the algebra of real numbers).

In the case of a theoretical course, considering the system of equations to be *the represented* and the matrix equation to be *the representation* (as Harel 2017 does) means the idea of a quantitative system or a real-world application is placed extremely far into the background. The quantitative system or real-world application may always be present in the mind of the expert and may get mentioned every now and then. However, an occasional mention is likely inadequate to make the quantitative system or applied context real and material for students. I note here, as I did in Chapter 3 Literature Review, that Trigueros et al. (2018) found that a well-planned, immersive experience was necessary for students to connect the linear systems model they created (the notation) with the real-world model (the applied context) they studied. A similar type immersive experience may be necessary for students to connect the various representations in meaningful ways.

What happens, though, when we move into an applied course? I note that the bulk of undergraduates take a single course in linear algebra, and the course is most frequently applied in nature. In the case of the top-selling linear algebra text in the United States (Lay, Lay, and McDonald, 2016a), writing a system as a linear combination of columns (in essence, what I have called a vector equation) is given a place of prominence in a way that is uncharacteristic in other texts. More traditional texts (e.g., Hoffman and Kunze, 1971) often make the observation about the linear combination of columns in passing, if at all. This puts an additional representation alongside the linear systems representation, the matrix representation, and the augmented matrix, in my estimation complicating the consideration of what the object of study is. In that case, the idea of a quantitative system seems more crucial than with more traditional theoretical approaches. While I assume the authors (Lay, et al., 2016a) have laid out a plan of study in terms of what they view as logical and/or productive in terms of content, I believe the effect of the alternative approach on student outcomes and on the quality of students' understanding is largely unresearched. In an applied course, we likely want students to have a more concrete conception of what is represented than symbols that we move around according to mathematical properties as a theoretical course might; however, the pedagogical implications of the alternative approach may need further investigation especially in terms of student thinking. I say more about this in my discussion of "same solution".

Distinguishing (Otherwise Implicit) Moving Parts

The construct *quantitative system* also allows for the *equitizing* of the various representations, where all representations are given equal precedence. Most vividly, the linear systems representation is not taken as primary as is customary. When the consideration of reversibility is joined with the equitizing, 12 translations become evident. While translating in one direction may be enough to establish that the representations are of the same entity, the process and nature of translating differs depending on the direction of the translation. The identification of the many combinations brings an awareness to perhaps taken-for-granted moving parts involved in linear algebra.

The way equitable representations are often presented side-by-side in instruction and in textbooks neglects that the process of translating differs depending on the direction of translation. This may be because representations are inappropriately given the mathematical characteristic of “equality”; if we think of representations as occurring within different languages, we can see that “equality” is not an appropriate descriptor. Languages are not equal; things get lost in translation when moving between languages. Different languages, as means of expression, may foreground or background different features or characteristics of what it is we are trying to communicate. (I say more about the pitfall of thinking in terms of “equivalent representations” in my discussion of the convergent findings of this study.) Considering all the various translations allows experts to reflect on their own conceptions. They can also become more careful analysts of students’ conceptions. Thereby, they can position themselves to better support students in their learning.

In summary, the Theory of Quantitative Systems makes enmeshed conceptions and enmeshed communication distinguishable and allows for reflection on the complexity in working with linear systems by pointing out distinctions that otherwise might not be taken into consideration. Further, the approach of applied linear algebra texts may make the distinctions afforded by the Theory of Quantitative Systems more important than approach of more traditional textbooks.

Discussing the Generative Findings

My generative analysis resulted in findings related to “same solution”, “different notation”, and “the thing” represented by the four representations in this study. I discuss additional considerations of each in this section.

The Invariant: Same Solution

Some students were more focused on “solution” than others—several could have a conversation about something other than solving. My structuring of the interview protocol to help them acclimate to the interview setting and content prior to posing my primary questions most likely had some effect. In the section “Discussion of ‘Same Solution’” of Chapter 5, my second point of discussion was that the participants in this study seemed to have the correct idea that “same solution” was a common characteristic of the four representations (a necessary condition for sameness) while avoiding the incorrect conclusion that “same solution” was enough to determine sameness (a sufficient condition). I discussed how it seems rather natural to conclude that the representations have the same solution subsequent to describing them as representing the same thing. The observation brings an interesting lens to the consideration of a question that I posed earlier: What happens when we move into an applied course and expand our scope beyond the theoretical idea of using matrices as a tool for the analysis of linear systems?

While most traditional linear algebra texts comment, often in passing, that a system of equations can be written as a linear combination of columns (what I have called the vector equation representation), the linear combinations of columns can take a more prominent role. Consider the statement of the following theorem. The theorem states (for me) that the different representations are connected because they have the same solution.

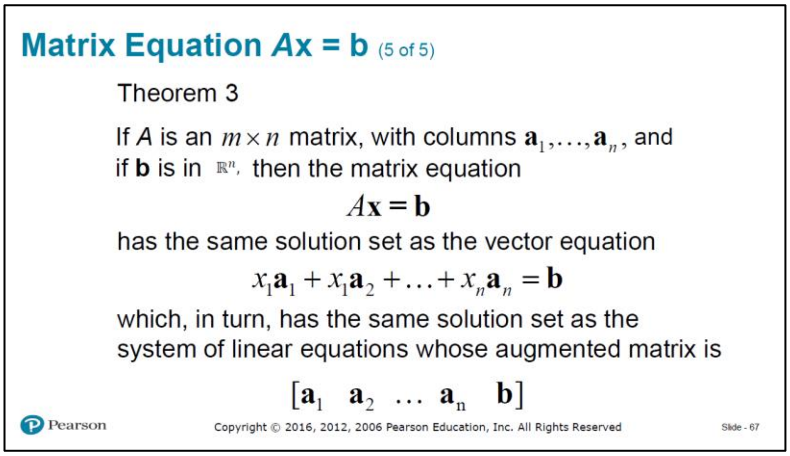


Figure 14. A presentation of the representations in this study (Lay, Lay, & McDonald, 2016b).

Consider the theorem through the lens of the Theory of Quantitative Systems. In that case, all the representations are representative of the same quantitative system; as I see it, the idea that all the representations are of the same entity is not highlighted in the theorem. I have argued that it seems natural to conclude that “representations of the same entity have the same solution”, a claim supported by my empirical findings in this study. (All participants seemed to concur.) Thus, the theorem seems to miss an important opportunity to capture *the degree of sameness* between the representations (they represent the same entity) by simply appealing to a shared characteristic among them: they have the same solution. That they are representative of the same entity is in the background.

Appealing to the shared characteristic, “same solution”, is interesting in another way; it could lead one to confound the idea of how the representations in the theorem are alike and how row-equivalent systems are alike. As representations of the same quantitative system, they all have the same solution (up to differences in register). However, another major idea in linear algebra is that row-equivalent systems have the same solution. The row-equivalent systems may not, however, represent the same quantitative system. Row operations have the effect of holding the solution constant (the solution is invariant) while potentially changing the quantitative system (the quantitative system is varying). The theorem’s appeal to the “same solution” characteristic could present obstacles to students’ conceptual understanding of row equivalence.

Introducing the linear combination of columns alongside the matrix and the linear systems representations, I believe, complicates things more than it may appear on the surface. While the idea of a quantitative system can be used in addressing the complications, more work needs to be done toward supporting students of single-semester applied linear algebra courses in ways that do not create potential obstacles; further, studies to test the effectiveness of any such approaches are needed. This is especially true given the burgeoning demand for workers conversant in linear algebra.

The Varying: Differing Notation

In Chapter 5 Results, I reported the participants’ universal conclusion that the solution was the same for all four representations in this study. I also noted their universal inattention to the difference between values and matrices that had those values as entries when discussing the solution(s). My theory provides a way to describe the situation more precisely: the participants were inattentive to considering the solution in terms of the register used to denote the quantitative system. If the quantitative system was represented in the matrix register, the solution should be matrix; however, if the quantitative system was represented in the real number system, the answer should be real number values.

While the majority of students described “differing notation” as looking different or as a different way to communicate, I discussed how two participants gave consideration to the structure of the representations. While attending to the characteristic “same solution”, however, they seemed to lose focus on the idea of “different notation”. That is, they did not acknowledge that the solution to a system of equations is a set of values while the solution to a matrix equation is a matrix. Perhaps getting the answer (which many see as the point of mathematics) short-circuited the consideration of anything else.

One curious observation that I make is Jake’s statement “I would say they are not equal because they don’t all look exactly the same”. Of course Jake (Student 10) was guarded in the way he used “equal”, “equivalent”, and “not equal”, and we do not know precisely what he had in mind for each of those terms. However, at face value, his statement raises the question: Then do equal things look exactly the same? Without a better idea of his meaning for “not equal”, we cannot know if the question is the logical negation of what he had in mind. However, the question is a good one to ponder in a study of representations such as this one.

I make another observation based on recognizing mathematical structure. Each register suggests what the appropriate units (sub-pieces of algebraic notation) are for parsing the representations residing within the register. Participants in this study did not distinguish between “x=1 and y=3” and $\begin{bmatrix}1\\3\end{bmatrix}$. Further, Felix’s (Student 3) cut-and-paste metaphor did not make it clear that he was thinking in terms of mathematical units; he seemed to just be rearranging various pieces of notation. Consider this: for $\begin{bmatrix}1\\3\end{bmatrix}$ the units of the notation could be taken to be a left bracket, the number one, the number three, and a right bracket. If we take all those units of notation together and think of them as a single unit, we could simply refer to the unit as a *vector* like Felix did at times. (Or the unit could be thought of as a matrix like I did.) How students parse algebraic notation is an area of interest for me that I call *structure sense*.

Are students aware of the combinations of real numbers and variables which are representative of real numbers in ordinary algebra? Are they aware of the matrices comprising a matrix equation? Are they aware of the scalars and vectors that comprise the vector equation? Or do they parse the notation into different bits? Certainly an instructor who makes explicit the appropriate parsing of the notation while also using precise terminology will support students in their parsing of the notation into appropriate mathematical units. When students can group several individual symbols to form a particular mathematical object, perhaps the packaging makes the situation less visually and cognitively taxing.

More study of student understanding of mathematical structure according to mathematical system (register) needs to be done. The area may be insufficiently researched as a result of grouping everything symbolic under the umbrella *symbolic algebra*, an area often neglected for research focused on what is described as conceptual understanding. Truly, linear algebra may bring to the forefront, in ways that other areas of mathematics have not, the need for attending to the various registers of representation in undergraduate mathematics education research. Just as there are many graphical coordinate systems, the category "symbolic algebra" is inadequate for analyzing teaching and learning in linear algebra.

"The Thing" and Constructivism

My findings related to students' conceptions of "the thing" represented in this study include the categories 1) a system of equations—definitely, 2) a system of equations—less definitely, and 3) a quantitative system. I also found that one student did not appear to have an integrated represented object in mind. I discuss the findings from two perspectives that I see as falling within Piagetian constructivist theory (Piaget, 1954, 1976; von Glasersfeld, 1983, 1988, 1995). First, the categories of "the thing" represented, along with the ideas of enmeshed conceptions and enmeshed communication, allow for a discussion of whether students have cognitively constructed "the thing". Next, I discuss the quality and range of the participants' conceptions in terms of Piaget's stage theory (Piaget, 1954, 1976).

Have Students Constructed "The Thing"? Considering whether students have cognitively constructed a *represented* as the referent of the representations in this study is easy in one case: one student did not seem to have a particular object in mind as the target of all the representations. For Ken (Student 9), each representation seemed to serve as a prompt for a particular solution technique. I would say that he had not cognitively constructed "the thing" represented. I also see discussing the idea of whether students had constructed “the thing” as rather straightforward in the cases of those students who appeared to have a quantitative system or similar abstract object in mind. While Zeb (Student1), Nick (Student 4), and Myra (Student 7) described a thing that was very abstract in nature, I would say decisively that they had cognitively constructed a “thing” represented by all the representations.

Considering the categories where the students identified the system of equations as “the thing”, whether they appeared to think so in flexible or inflexible ways, is a bit more complicated. They could have had a constructed “thing” in mind for which the linear systems representation was one among a number of available means of expression for the object. On the other hand, the linear systems representation as a set of notation may have been the only object the participants had constructed cognitively. Making the determination about which was the case is difficult and speculative. In short, when students identified the “system of equations” as “the thing” represented, their dialogue does not make it obvious if they were referring to the notation or to some other entity.

Delineating *enmeshed conception* and *enmeshed communication* allowed me to give closer consideration to whether students appeared to have a cognitively constructed “thing”. More specifically, the constructs allowed me to consider what “thing” it was that participants had constructed. I would say the students who identified the system of equations as “the thing” represented, whether their conception was flexible or inflexible in nature, had a cognitively constructed “thing”. However, determining what thing they had constructed is harder when considering the cognitive paradox. Questioning students more extensively and/or in different ways than I did is likely necessary for making such determinations. The idea of enmeshment, whether it applies to conceptions or communication, illuminates that it is not always clear what represented "thing" one may have in mind.

What appears to me, having a quantitative system in mind, as a fusion between *the represented* and *the representation* could be because there is no cognitive construct beyond the notation; enmeshed communication could be the result of an enmeshed conception. On the other hand, there may be some cognitive construct of a "thing" other than a set of notation; one might have something constructed besides the notation while communicating in enmeshed ways. The distinction between *the represented* and *the representation* may have been reflected upon at some point, but knowing the distinction becomes so natural that attending to it becomes unnecessary and/or tedious. Visual techniques, heuristics, and algorithms may be used where concepts were at one time rigorously thought out but no longer require reflection. Participants in my study suggested at times, when I made my questioning more incisive, that at one time they had thought about details that they no longer considered (except that I asked) at the time of the interview.

Consider the case where an instructor has cognitively constructed "the thing" to be something other than the notation, but its consideration no longer merits attention. What happens when the instructor's constructed object (a quantitative system or applied context) is implicit in communication and the students' only constructed object is the notation? Students may inherit, in a sense, the instructors' inattention to the distinction, leaving them incapable of constructing anything other than the notation. In such cases, the resulting student conceptions are likely less vivid and rich than we would prefer. Harel described this by claiming “the idea that a system of equations represents a set of quantitative constraints did not seem to have been self-evident for the participants, an indication of weak quantitative reason” (Harel, 2017, p. 91).

In terms of constructivism, my findings related to “the thing” represented indicate that students may not have had a specific cognitive structure for “the thing”. Some may have cognitively constructed “the thing” as set of notation. Lastly, others’ cognitive structure of “the thing” represented may have been an abstract mathematical entity like a quantitative system.

Stage Theory. The quality and range of participants’ conceptions as evidenced by their dialogue varied; the participants exhibited a broad range of conceptions. Some participants (Ken, Student 9) thought of mathematics in a very goal-oriented “find the solution” way, perhaps an artifact of schooling. While others acknowledged that more is going on than solving, at times they did not investigate too deeply since they did not see doing so as supporting their fundamental purpose of performing in class and passing the course (Jake, Student 10). Some described the various representations as interchangeable, while others had developed their own philosophy of mathematics. While a range and variety of conceptions are evident in the participants’ responses, I contend that students at this level (university juniors) often answered with a level of sophistication uncharacteristic of younger students. That is, many of the participants’ descriptions reflect cognitive abilities superseding that of younger students. My observation is influenced by stage theory.

By stage theory I am referring to theories such as Piaget's (1954, 1976) which avow that cognitive development progressively develops in stages. People and research disagree with the age bands that Piaget outlined based on his empirical observations of children and adolescents. Others criticize appealing to stage theories since they often result in putting a cap on expectations of students' abilities. I am only espousing that cognitive abilities are developmental in the same sense that our physical bodies are. From my perspective, development of cognitive abilities likely mirrors the development and maturation of our physical (biological) bodies, though not in lockstep. While cognitive abilities are not strictly age dependent, one would expect that older students are a natural place to look for upper stages of cognitive maturation.

The results I have documented suggest that several students may have neared what could be considered to be some kind of end point of cognitive development: the ability to reason abstractly and philosophically. Note that I am not appealing to stage theory in a way that puts a limit on expectations for students; rather, I am embracing the idea of progressive cognitive development and using it to highlight the rich nature of the participants' conceptions. Examples of notable conceptions were reported in "'The Thing' Represented" section of Chapter 5 Results and Findings. In particular, I see responses from participants who thought of the representations in flexible ways or whose dialogue suggested a quantitative system as notable. Other conceptions I see as noteworthy are reported in the "Discussion of 'Same Solution'" section of Chapter 5, where I reported how three students seemed to attend to differences in register. Some brief indicators of what I see as rich conceptions include Zeb's (Student 1) and Myra's

(Student 7) constant appeal to relationships between *x* and *y* during their interviews, Jake's (Student 10) description of the representations as "reflections" of one another, and Nick's (Student 4) description of the representations as "pointers". Further, I find noteworthy Myra's (Student 7) claim that "math is more than our representations of it". I argue that this study of upper-level undergraduate students gives a view into not only a different student demographic, but also (as a result of the demographics), a view into conceptions which are more cognitively complex than many presently documented in mathematics education literature.

Discussing the Convergent Findings

In the convergent analysis of the data, my finding that Peter (Student 5) engaged with four of the 12 translations that I identified to the neglect of considering reverse translations allows for two points of discussion: 1) gauging flexibility, and 2) my use of the word "equitizing" to describe giving the representations equal precedence. In addition, I discuss how Felix's interview unfolded like a teaching experiment.

Gauging Flexibility

The visual aspects alone of Table 1 (12 possible translations) may make one aware of previously unconsidered intricacies involved in working with systems of equations. Numerous translations are possible, and the nature and demands of each translation differ. The table, with its variety and combinations of representations, makes implicit complexity explicit. Further, Table 8 illustrates how I used Table 1 to analyze Peter's dialogue, noting that he engaged with four of the 12 translations I identified. I suggest that experts can use the table to consider their own flexibility in moving between registers; further, it provides them with an instrument for assessing students' flexibility. Increased awareness on part of experts means students' flexibility can be evaluated and developed, especially in the direction of good structure sense — the appropriate parsing of algebra expressions into mathematical units.

I suggest that exercising all translations could be pedagogically valuable, bringing the students' attention to structure. I gave two hypothetical examples of how translating in reverse might help Peter (Student 5) in Chapter 5. In addition, improved flexibility with all the translations could serve to solidify those that are more common in practice. Further, translation exercises could help students recognize structural nuances that might otherwise remain implicit.

What can be realized when we reflect on the various translations and how students work with them? Considering the nuances allowed me to see something I might have otherwise missed; I describe my observation in the discussion of my choice of the word *equitizing* below.

Discussing *Equitizing*

Noting Peter's neglect of reverse translations (Table 8) allows me to discuss a possible aspect of his conception other than his flexibility. In my discussion "Distinguishing Moving Parts" earlier in this chapter, I used the word *equitizing* to describe giving all representations equal precedence. I did so to avoid using the words "equal" or "equivalent" as a result of studying Peter's clinical interview data.

Peter (Student 5) answered the question "What is the thing they (the four representations) represent?" by stating "numeric values of x and y such that these systems of equations, which are equivalent, are satisfied” (Table 6). I gave an extensive analysis and discussion of Peter’s conceptions, including this statement, in the convergent analysis reported in Chapter 5. Here I want to focus on another part of his statement as a point of discussion: his use of the word “equivalent” as he pointed to each of the four representations. I suggest that each of the double-ended arrows that Peter (Student 5) used between representations (Figure 8) may have functioned like an equals sign to him; this could explain his neglect of reverse translations.When I think of registers of representation as if they are languages, equality does not seem like an appropriate descriptor. Mathematics can be equal; language is a different kind of phenomenon where some representations capture aspects of an entity in ways that another might not. Thus, representations are not “equal”. Truly, from the perspective of the Theory of Quantitative Systems, the mathematical object in this study is the quantitative system, and there is only one quantitative system. Thus, equality is appropriate only in the sense of identity, like 7=7. Since we see notation for a mathematical object, not a mathematical object itself, the idea of *equity* of representations (as language) seems more fitting than the idea of equality.

Additionally, I do not think of mathematical equality in terms of direction; equality just is, in a sense. However, the process of translating from representation A to representation B differs from the process of translating from representation B to representation A. (See Analyses of Translations in Chapter 2).Translations are bi-directional in that the process and nature of translating differs depending on direction. With Peter (Student 5) a (quantitative) mathematical characteristic, equality, has possibly been applied to a representation. An enmeshed conception (where *the representation* and *the represented* are not distinct) may have led to an inappropriate application of a mathematical property, equality, to representations. In other words, mathematics has possibly been confounded with language.

Representations of systems are often presented side-by-side in instruction and in textbooks without acknowledgment that translating differs in nature depending on the direction of translation. Does such a presentation of corresponding representations lead students to think of representation in terms of mathematical equality? Students may apply the mathematical property of equality to the representations rather than characterizing the representations in a way more appropriate for notation and/or language. I see applying a mathematical property like equality to representations as an indication of an enmeshed conception; the mathematical object and the notation used to communicate it are indistinct.

While we do not know specifically what Peter meant by his use of the word “equivalent”, I have pointed out an important distinction in case he was thinking in terms of quantitative equality. Perhaps Peter could benefit from having an explicit represented other than the notation so that he can (mentally) work with a mathematical object which is not a representation.

I reflected on Peter’s use of the word “equivalent” and continued to ponder his neglect of reverse translations and use of double-ended arrows. As a result, I went back and gave the name “equitizing” to my decision to give all representations the same precedence in the Theory of Quantitative Systems. For me, *equitable* is a better descriptor for translations than *equivalent* and possibly avoids confounding the idea of mathematical equality with the idea of alternative representation of the same entity.

A Mini Teaching Experiment

Students' mathematical conceptions may not be mathematically rigorous for a variety of reasons. However, while students may use visual techniques, heuristics, metaphors, and rehearsed algorithms, experts can investigate and support such approaches with mathematics when they recognize that students are using them. This is what I did with Felix (Student 3), though I was motivated in two distinct ways.

First, I used the clinical interview setting to explore Felix's mathematical understanding. I did this to address what I saw as a less-than-impressive display of formal mathematics in his cut-and-paste metaphor. To be clear, I see Felix's metaphor as enlightening and a potentially productive way of thinking. However, in the moment of the interview his descriptions were confusing mixtures inconsistent with other profound statements he had made. Framing the problem with mathematical terminology and ideas that Felix connected to, I uncovered indications of mathematical understanding absent in his metaphor.

Second, I would describe my next stage of interactions with Felix (Student 3) as an impromptu, mini teaching experiment. By *impromptu*, I mean I acted on an opportunity that presented itself during the clinical interview and took a diversion away from my interview protocol, a practice encouraged by Hunting (1997). I use *teaching experiment* in the sense that Steffe & Thompson (2000) prescribed; I tend to think of these experiments as being individual and/or small group (not whole class experiments) and cognitive in nature. Steffe and Thompson (2000) elaborated the idea of a teaching experiment as a living methodology; the experiment may occur over several sessions with adaptions taking place between sessions. Such a teaching experiment involves a great deal of thinking and planning to conduct a conceptual analysis of a math concept prior to the development of an experimental teaching sequence. The process includes considering how to influence the student's thinking in productive ways, and the researcher must be adaptive both within and between sessions when the student responds in ways that are not expected. Clearly, my work with Felix (Student 3) occurred within a single session. While I did not have a formal conceptual analysis and experimental instructional sequence documented, I had ones in mind from several years of repetitively teaching multiple sections of College Algebra.

As I see it, the greatest value of the teaching experiment methodology is that inherent in it are pedagogical suggestions. While documenting students thought processes through individual clinical interviews is a productive undertaking in itself, pedagogical recommendations can only be hypothesized subsequent to the investigation. In contrast, the teaching experiment puts on display a pedagogical approach that data from clinical interviews will support as productive or unproductive and/or needs improvement.

Since I argue that teaching experiments have direct pedagogical implications, I briefly recap the discussion of my interaction with Felix (Student 3) which I detailed in Chapter 5 Results and Findings. I described how Felix's cut-and-paste metaphor for translations led me to wonder about the state of his mathematical understanding. As a result, I considered two questions: 1) What is the nature of his understanding of vector space operations? and 2) How might I influence his thinking about changes in register?

In addressing the second question, I did three key things to support Felix's mathematical realization. First, I drew his attention to the change in register (line 29, Figure 12). Apparently, he had not previously acknowledged the distinction, but I sensed that he might be oriented toward giving it serious consideration. Next, I changed my language to match his language (lines 27 and 29, Figure 12). Third, I attempted to help him shift from his vector perspective to my matrix perspective (line 35, Figure 12), believing that my language might enlighten him in ways that his language did not. The key, I believe, was in navigating between the student's language and conception (acknowledging where he was) and what I expected might be a more productive perspective based on my experience with teaching students about matrices. As a result of my actions, Felix (Student 3) seemed to have constructed mathematical support for the conversion from the vector register to the linear systems register.

Discussion Summary

In this chapter, I have provided additional discussion of ideas I see as relevant to my project, ideas that merit further consideration given the results of the study. I introduced my study in Chapter 1 by describing how a mathematician's appraisal of linear algebra as powerful and yet simple seems to be at odds with how students experience the course. Many students seem to experience a seemingly inexplicable wall when they first encounter linear algebra. I argue for the value of seeing small things that are often taken for granted as a possible explanation.

Hillel (2000) observed that the lecturers he videotaped moved in and out of registers of representation (my terminology, not his) seemingly without awareness, and thereby, were unable to cue students about the actions they were taking. I think of this in terms of an instructor who speaks Spanglish (a mix of Spanish and English) to a room full of students who exclusively know English or exclusively know Spanish. The communication connects with some students in one way and other students in another way. The instructor, all the while, is unaware s/he is mixing languages. Students cannot discern with any certainty what the expert is doing, and the expert has no clue why the students are not following. While the expert may have the language and related conceptions to see the content of linear algebra as simple, the student cannot even connect to the expert's language (much less conceptions) in basic ways.

How often do we brush aside seemingly simple distinctions to eventually arrive at a linear algebra that is coherent to us (experts) and a mystery to students? While there are many rich concepts in linear algebra to research and explore in terms of students' thinking, perhaps several little mysteries add up to incomprehension, and the resulting frustration causes students to give up. In other words, the cumulative effect of perhaps years of less-than-robust communication add up. Linear algebra, with its richness, may well be the point where making progress becomes untenable for students. Dissecting enmeshed communication and considering students' potential enmeshed conceptions may be a productive path for addressing the disconnect I mentioned in my introduction to this document.

Theoretically, I have laid out some constructs that may help with awareness of complexity in linear algebra. (See The Theory of Quantitative Systems in Chapter 2 and/or Appendix A.) An instructor's awareness of the registers of representation should allow them to, in general, promote good structure sense amongst students and to, more specifically, support students in their work with systems. The lens I have provided becomes more critical when the linear combination of columns (vector equation) takes a more prominent role alongside the linear systems and matrix representations. Further, considering mathematics at the level of linear algebra requires the decomposition of the category "symbolic algebra".

Empirically, my results and findings provide important insights into students' conceptions as described in the Chapter 5. In particular, the study has revealed that students consider and/or can have their interest piqued by such things as changes in register. Ultimately, I envision conversations between all stakeholders where crosstalk is minimized, resulting in linear algebra courses most beneficial to students. This is especially important since students who encounter undue difficulties may alter their intended plans of study, plans which would lead them into careers in STEM (Science, Technology, Engineering, & Mathematics) fields.

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APPENDIX A

THE THEORY OF QUANTITATIVE SYSTEMS

The Theory of Quantitative Systems

Duval's Three Sources of Incomprehension in Mathematics
1) Treatments, 2) Conversions, and 3) The need to recognize many representations as indicating the same mathematical entity.
The Cognitive Paradox
Duval's suggestion that although we only have (perceptual) access to mathematical objects through the symbol systems that we use to represent them, mathematics comprehension requires that we not confound the representations with the represented mathematical object.

↓

Duval's Constructs	**My Constructs**
A mathematical *object* is an abstract entity that we only have access to through a symbol system.	The set of quantitative relationships represented by a number of linear equations, the *quantitative system*, is the mathematical object.
Transformations are changes to a representation both within a register (symbol system) and between registers (symbol systems) including graphical ones.	Translations are changes to a representation both within and between analytic registers.
Treatments are *transformations* within a register.	Treatments are *translations* within a register.
Conversions are *transformations* between registers.	Conversions are *translations* between registers.
Congruence/Incongruence of a transformation takes into account two criteria: transparency and unit-by-unit translation.	I take transparency to mean visual similarity and units to be sub-pieces of algebraic notation.
Registers of representation are differing symbol systems in which representations can be expressed.	Each mathematical system is designated to be a register of representation. A vector space is an example of one register of representation; representations are formed from scalars and vectors.
Transforming in reverse is a relevant consideration.	*Translating* in reverse is a relevant consideration.
Distinguishing between *the representation* and *the represented* is fundamental to mathematics comprehension.	The quantitative system is clearly defined to be the represented mathematical object. All representations, including the linear systems representation, are given equal precedence.

↓

The Theory of Quantitative Systems designates a set of quantitative relationships as the object of study in the context of systems of equations. The four representations involved are given equal precedence, and acknowledgement is given to the differing nature of translating between two representations depending on the direction of translation. Equitizing the representations and taking reverse translations into account allows the identification of 12 possible translations between the representations.

APPENDIX B

COMPREHENSIVE INTERVIEW PROTOCOL

Comprehensive Interview Protocol

A Fresh Problem.

1. How do you think about the following? What would you call it?

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

2. If asked to solve the equation, what do you see as your goal? How do you think about *x* and *y*? What are you trying to find?
3. Is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ a solution to the equation? How do you know?
4. Solve the equation. Telling me why you made each step in your solution.
5. Have students solve the equation by each of the following methods.
    - a. Row Reduction
    - b. Matrix Dot Product (MDP)
    - c. Matrix Vector Product (MVP)

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

For row reduction.

6. Why do we row reduce? What does the row-reduced form have in common with the original equation? When do we use an augmented matrix?
7. If they use “rows stand for equations”, question further. Can they explicitly write a detailed system and justify it? How does it relate to the original statement of the problem? (Can you apply it first rather than last?)
8. If rows correspond to equations, what equations are related to each of the following row-reduced matrices? What can you tell me about solutions?

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right]$$

For Matrix Dot Product (MDP) or Matrix Vector Product (MVP).

9. What if you multiply things out as a first step rather than using row reduction?
10. Note which form of matrix multiplication they use and ask them to do the other way. Provide them with the definitions if needed.

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & -1 & 1 \end{array}\right]$$

11. If we think about augmented matrices and both types of matrix multiplication, we have the three following. Do you see them as the same or different? Explain. Will they all have the same solution? What is the thing they represent?

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad x \begin{bmatrix} 1 \\ 4 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

12. Do those three have anything in common with the following system?

$$\begin{cases} x + 2y = 7 \\ 4x - y = 1 \end{cases}$$

A Look Back at Old Exams.

13. Did you feel obligated to use row-reduction since the instructions suggested it?
14. What do x and y mean to you? Does your answer check in the original equation? Please demonstrate.
15. Can you write the matrix equation that corresponds to the row-reduced matrix?
16. Can you tell us about what might have happened between Exam 1 7a and Exam 2 2a?
17. What similarities do you see between Exam 1 7a and Exam 1 8b?

Wrap Up.

18. What do you think about what we have talked about today? Do you have any questions you would like to ask or any other comments that you would like to make?

APPENDIX C

INTERVIEW SUB-PROTOCOL

Interview Sub-Protocol

1. How do you think about the following? What would you call it?

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

4. Solve the equation. Telling me why you made each step in your solution.

6. Why do we row reduce? What does the row-reduced form have in common with the original equation? When do we use an augmented matrix?

7. If they use “rows stand for equations”, question further. Can they explicitly write a detailed system and justify it? How does it relate to the original statement of the problem? (Can you apply it first rather than last?)

8. If rows correspond to equations, what equations are related to each of the following row-reduced matrices? What can you tell me about solutions?

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right]$$

11. If we think about augmented matrices and both types of matrix multiplication, we have the three following. Do you see them as the same or different? Explain. Will they all have the same solution? What is the thing they represent?

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & -1 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad x\begin{bmatrix} 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

12. Do those three have anything in common with the following system?

$$\begin{cases} x + 2y = 7 \\ 4x - y = 1 \end{cases}$$

18. What do you think about what we have talked about today? Do you have any questions you would like to ask or any other comments that you would like to make?

APPENDIX D

INSTRUMENT FOR COLLECTING WRITTEN WORK

Instrument for Collecting Written Work

1

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

2

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right]$$

3

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & -1 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad x\begin{bmatrix} 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

4

$$\begin{cases} x + 2y = 7 \\ 4x - y = 1 \end{cases}$$

APPENDIX E

INTERNAL REVIEW BOARD APPROVAL LETTER

APPROVAL: MODIFICATION

Michelle Zandieh
CISA: Polytechnic Science and Mathematics
480/727-5014
zandieh@asu.edu

Dear Michelle Zandieh:

On 11/16/2018 the ASU IRB reviewed the following protocol:

Type of Review:	Modification
Title:	Teaching & Learning in Inquiry-Oriented Linear Algebra
Investigator:	Michelle Zandieh
IRB ID:	STUDY00003047
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Keene 4075 Approval Letter (1).pdf, Category: Off-site authorizations (school permission, other IRB approvals, Tribal permission etc); • VT IRB-14-638 Approval Letter.pdf, Category: Off-site authorizations (school permission, other IRB approvals, Tribal permission etc); • FSU-IRB-ApprovalLetter-150611.pdf, Category: Off-site authorizations (school permission, other IRB approvals, Tribal permission etc); • FSU-TIMES-IRB-ApprovalUpdate-160316.pdf, Category: Off-site authorizations (school permission, other IRB approvals, Tribal permission etc); • Zandieh_PROTOCOL_Teaching&Learning)inIOLA_160414.pdf, Category: IRB Protocol; • LinearAlgebraAssessment.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);

The IRB approved the modification.

When consent is appropriate, you must use final, watermarked versions available under the “Documents” tab in ERA-IRB.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc:

APPENDIX F

PERMISSION FOR SLIDE USE

PERMISSIONS
200 Old Tappan
Road Old Tappan,
NJ 07675
globalpermissions@pearson.com

PE Ref # 210130

18 September 2019
JANET SIPES
c/o Arizona State University
[redacted]

Dear Janet

You have our permission to include content from our text, ***POWERPOINT SLIDES (DOWNLOAD ONLY) FOR LINEAR ALGEBRA AND ITS APPLICATIONS, 5th Ed. by LAY, DAVID C.; LAY, STEVEN R.; MCDONALD, JUDI J.,*** in your dissertation or masters thesis at Arizona State University.

Content to be included is:
Power point slide 7 of Section 1.4 based on Theorem 3 on page 36

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Sincerely,
Allison Bulpitt, Permissions Analyst