Students' Interpretations of Expressions in the Graphical Register and Its Relation to Their Interpretation of Points on Graphs when Evaluating Statements about Functions from Calculus

by

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ABSTRACT

Functions represented in the graphical register, as graphs in the Cartesian plane, are found throughout secondary and undergraduate mathematics courses. In the study of Calculus, specifically, graphs of functions are particularly prominent as a means of illustrating key concepts. Researchers have identified that some of the ways that students may interpret graphs are unconventional, which may impact their understanding of related mathematical content. While research has primarily focused on how students interpret points on graphs and students' images related to graphs as a whole, details of how students interpret and reason with variables and expressions on graphs of functions have remained unclear.

This dissertation reports a study characterizing undergraduate students' interpretations of expressions in the graphical register with statements from Calculus, its association with their evaluations of these statements, its relation to the mathematical content of these statements, and its relation to their interpretations of points on graphs. To investigate students' interpretations of expressions on graphs, I conducted 150-minute task-based clinical interviews with 13 undergraduate students who had completed Calculus I with a range of mathematical backgrounds. In the interviews, students were asked to evaluate propositional statements about functions related to key definitions and theorems of Calculus and were provided various graphs of functions to make their evaluations. The central findings from this study include the characteristics of four distinct interpretations of expressions on graphs that students used in this study. These interpretations of expressions on graphs I refer to as (1) nominal, (2) ordinal, (3) cardinal, and (4) magnitude. The findings from this study suggest that different contexts may evoke different graphical interpretations of expressions from the same student. Further, some interpretations were shown to be associated with students correctly evaluating some statements while others were associated with students incorrectly evaluating some statements.

I report the characteristics of these interpretations of expressions in the graphical register and its relation to their evaluations of the statements, the mathematical content of the statements, and their interpretation of points. I also discuss the implications of these findings for teaching and directions for future research in this area.

To my parents, who first recognized and cultivated my love of mathematics and have provided unending support for my academic journey

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TABLE OF CONTENTS

CHAPTER Page

CHAPTER Page

CHAPTER Page

CHAPTER Page

CHAPTER Page

LIST OF TABLES

LIST OF FIGURES

Figure Page

Figure Page

CHAPTER 1
INTRODUCTION

This dissertation describes a study investigating undergraduate students' interpretations of graphs of functions while evaluating statements from Calculus. The goals of this study are: (1) to characterize students' interpretations of variables and expressions (after, both referred to as expressions) from statements in Calculus on graphs of real-valued functions and (2) to investigate the relationships between students' interpretations of expressions on graphs and (a) their evaluations of these Calculus statements, (b) the content of the expressions from these Calculus statements, and (c) students' interpretations of points on graphs. This study sought to achieve these goals using task-based clinical interviews with 13 undergraduate mathematics students.

While illustrations and diagrams of mathematical ideas date back to ancient Egypt and Greek geometers (Friendly, 2008), the use of graphs of functions as a means of representing relationships between two sets of values visually arose in the Western world within the mid-seventeenth century (Friendly, 2008). French mathematicians Pierre de Fermat and Rene Descartes are credited with formalizing a scheme for depicting relationships between two sets of varying values in a two-dimensional plane, uniting algebraic equations with geometric objects (Friendly, 2008). Representing algebraic relationships in geometric space as graphs (often referred to as Analytic Geometry) through what we now call a Cartesian plane, revolutionized the course of mathematics, leading to new advances in the fields of Calculus and Complex Analysis (Tall, 1991).

Today, the use of graphs remains central to mathematics curricula and instruction. Graphs and graphing activity (plotting points, graphing functions, etc.) comprise a significant portion of mathematics curricula at the 9-16 level, with the majority of graphs consisting of functions displayed in the Cartesian coordinate system (e.g., Finney, Thomas, Demana, & Waits, 1994; Stewart, 2012). Even as early as Grade 5, U.S. Common Core State Standards include the expectation that students graph points in the coordinate plane (5.G.A.1-A.2). USCCSS also include standards regarding high school students' ability to interpret graphs of functions throughout their study of mathematics (HSF.IF.B4).

While Cartesian graphs are found throughout students' mathematics education beginning as early as elementary school, they are particularly central to the study of Calculus at the undergraduate level. In line with reform efforts in Calculus (Hughes-Hallett, 1991), researchers have incorporated visual representations in recent curriculum and instruction. When surveyed, Calculus instructors reported a significant portion of their assessments included graphical interpretations of central ideas of Calculus (Burn & Mesa, 2015). Furthermore, Calculus courses and textbook titles often add the phrase "Analytic Geometry," to describe the graphs of the functional relationships that comprise their content (e.g., Arizona State University Course Catalog; Larson, Hostetler, & Edwards, 2002). In commonly used textbooks, the definitions and theorems of Calculus, including those related to limits, derivatives, and integrals of continuous functions, are often presented with accompanying graphs of relevant functions (e.g., Stewart, 2012). These graphs may be included in textbooks to support students' understanding of a given concept, such as illustrating the application of a theorem to a single function.

Researchers have also touted the benefits that students may receive from the inclusion of graphs in mathematics instruction (Arcavi, 2003; Hanna & Sidoli, 2007). Math educators have both proposed and tested interventions that rely on graphical representations of ideas from Calculus (e.g., Kidron & Tall, 2015; Roh, 2010). Several studies suggest that the use of graphical interventions may afford students the opportunity to construct rigorous meanings for mathematical concepts (e.g., Kidron & Tall, 2015; Roh & Lee, 2017). Furthermore, recent studies in the field of neuroscience also support the notion of incorporating visual reasoning into mathematics instruction. These studies reveal that when students solve mathematics problems, even those that do not include visual components, visual pathways in their brains are engaged (e.g., Boaler, Chen, Williams, & Cordero, 2016).

Given the call to include visualizations in mathematics instruction (Arcavi, 2003) and evidence indicating their potential power (e.g., Kidron & Tall, 2015), graphs have earned a prominent place in mathematics curriculum and instruction to support student learning. However, in order for such graphs to be effective in supporting student learning, students must interpret these graphs in the ways they are intended by instructors and curriculum designers, consistent with conventional use. In this study, I take the perspective that the use of Cartesian graphs, like any conventional system of communication, does not guarantee a shared understanding between creator and observer; that is, an observer may construe a meaning from the details of a Cartesian graph which is not equivalent to the meaning the creator of the graph intended to convey. For instance, instructors or experts may recognize that a graph of a linear equation represents the set of all ordered pairs that satisfy the equation. However, several studies have found that students may not recognize this connection between points on a graph and solutions to a corresponding algebraic equation (e.g., Knuth, 2000). Instead, students may associate the shape of a line on a graph with the algebraic structure of a linear equation, without a meaning for the points which comprise that line (e.g., Moore & Thompson, 2015). Because observers may interpret details of graphs in various ways, students may not interpret aspects of graphs found in class or in textbooks in ways that are intended by their instructors. Furthermore, the use of graphs by instructors or textbooks could potentially be detrimental to some students if these students have unconventional interpretations of the graphs. Unconventional interpretations of a graph may lead students to understand the related mathematical idea in a fundamentally different way that is not consistent with the mathematical community. Given both the potential benefits and risks of using graphs in mathematics instruction, I sought to further examine the ways in which students interpret graphs in the context of statements from Calculus, as a first step to support students' use of graphs.

Adopting the perspective that observers' interpretations of graphs may differ, researchers have begun the work of investigating students' interpretations of graphs presented to them (e.g., Moore & Thompson, 2015). In doing so, they seek to answer the question, how *do* students interpret graphs? Examining how students interpret graphs of functions holistically, Moore and Thompson (2015) and Moore (2016) detail two categories of student thinking they have observed. They observed that some students conceive of graphs as static shapes on the page with certain properties due to the shape (what they refer to as *static shape-thinking*), while others conceive of graphs as emergent shapes traced out by simultaneously tracking two quantities (what they refer to as *emergent shape-thinking*). Thus, while an instructor may view a graph in a textbook as a record of the covariation of two quantities (emergent shape-thinking), a student may not see a graph in the same way. Instead, a student may see an image that has certain mathematical properties because of its shape (static shape-thinking), which may impede the student's reasoning in some contexts. In fact, Frank (2017) found that static shape-thinking inhibits students' ability to engage in certain covariational reasoning tasks, while emergent shape-thinking supports them.

Researchers have also characterized ways that students may interpret outputs of functions and points on graphs of functions, as their interpretations of certain aspects of graphs may also differ from those of an instructor or expert. For instance, David, Roh, and Sellers (2019a) found a significant difference in the interpretations of graphs of two groups of students. In the context of the Intermediate Value Theorem (IVT), some students conventionally interpreted outputs to be on the y-axis and points on given graphs as a pair of values, which they refer to as *value-thinking*. On the other hand, other students interpreted points as designating a location in space and outputs as referring to the locations of these points, which they refer to as *location-thinking*. The students engaged in location-thinking labeled points with function notation for an output value, e.g., $f(a)$, corresponding to that given point. The students' unconventional interpretations of the graphs fundamentally altered their understanding of the IVT and other Calculus statements given to them in the study, even leading some to incorrectly evaluate statements (David et al., 2019a).

While previous studies (David et al., 2019; Moore & Thompson, 2015; Moore, 2016) have begun the work of understanding how students think about graphs, several questions remain regarding how undergraduate students think about graphs in Calculus contexts. Do students interpret certain aspects of graphs in ways that are similar to or differ from convention with other Calculus statements besides those related to the IVT? Do students' interpretations of aspects of graphs remain consistent across various contexts, or are their interpretations context-dependent? What other ways do students think about variables and expressions involving inputs or outputs on graphs of functions, that has yet to surface in previous studies? What ways of interpreting expressions on graphs of functions support students in making sense of statements from Calculus?

Based on the findings from existing literature previously described (e.g., David et al., 2019; Moore & Thompson, 2015), I conjecture that students may interpret graphs unconventionally in various contexts from Calculus in which graphs are used. Anecdotal evidence from my observations teaching Calculus students as well as evidence from pilot studies I conducted in Spring 2018 support this hypothesis. On numerous occasions, I found students interpreting output values of a function as points on the graph of a function, rather than as values on the y-axis. While David et al.'s (2019a) findings indicate that some students interpret outputs of the function as points on graphs in the context of the IVT, I hypothesize that this issue, or others, extends to other common Calculus statements beyond the IVT. If students' interpretations of a graph are unconventional, they may interpret important definitions, theorems, and results of Calculus in ways that are not intended. In essence, I conjecture that the ways in which students interpret graphs in the context of specific mathematical content impacts their understanding of such content. Research into this area may uncover which aspects of graphs students *are* interpreting and the ways in which they interpret them. If unconventional, characterizing these student interpretations of graphs and uncovering their effects may better inform instructional and curricular design decisions.

In an effort to build upon this growing body of research, I sought to further investigate and characterize students' interpretations of aspects of graphs. As I conducted this study, the importance of students' interpretations of *variables* and *expressions* on graphs, which were inputs and outputs of functions, and expressions containing them, emerged as a crucial component of describing differences among students' interpretations of graphs. Therefore, I focused my investigation in this study on students' interpretations expressions involving inputs and outputs of functions in graphs, to which Calculus statements often refer. The goal of this study, then, is to extend the current work that has been done in understanding students' interpretation of graphs by characterizing undergraduate students' interpretations of expressions on graphs in contexts that span secondary and tertiary mathematics curricula, namely from Calculus. This study aims to address the following questions:

1) *How do undergraduate students interpret expressions from statements in Calculus on graphs of functions in the Cartesian plane? Which aspects of*

graphs do these students attend to in the context of statements from Calculus when interpreting expressions, if any?

2) *How are undergraduate students' interpretations of expressions on graphs of functions related to their evaluations of statements from Calculus?*
3) *In what ways are the content of expressions in statements from Calculus related to undergraduate students' ways of interpreting these expressions on graphs? i.e. To what extent are students' interpretations of expressions on graphs consistent or inconsistent across different Calculus statements? If inconsistent, which statements evoke which interpretations for students?*
4) *How are undergraduate students' interpretations of expressions on graphs of functions related to their interpretation of points?*

In this study, the aspects of graphs I refer to include, but are not limited to: axes of the graph, points on graphs, portions of axes or traces of graphs, and graphs as a whole. The Calculus statements with which I will conduct this study are related to the following six Calculus topics: (1) the Intermediate Value Theorem, (2) the definition of an increasing function, (3) the definition of an injective function, (4) the definition of continuity at a point, (5) the difference quotient, (6) the Mean Value Theorem. I chose these six topics as they are all central ideas in traditional Calculus or advanced Calculus curriculum across several widely-used textbooks (e.g., Finney et al., 1994; Gaughan, 1998; Stewart, 2012).

To address these research questions, I conducted task-based clinical interviews with students who have various backgrounds in mathematics, with a minimum of one semester of Calculus at the undergraduate level. The tasks consisted of evaluating and interpreting propositional statements related to each of the six Calculus topics along with graphs of real-valued functions. The clinical interview setting provided me the opportunity to probe students' words and actions relative to aspects of the graphs they worked with, to better model student thinking. The tasks, along with the questions I asked, were designed to elicit students' interpretations of expressions on graphs in order to address the first research question. To answer the second research question, I have analyzed each student's interpretations of expressions alongside his or her evaluation of a statement with each provided graph. To address the third research question, I have compared each student's interpretations of expressions across each of the six statements chosen for this study. Finally, to answer the fourth research question, I have characterized each student's interpretation of points alongside his or her interpretation of expressions with each statement.

This dissertation is divided into seven chapters. Chapter 2 describes the theoretical perspective I have adopted for this study. Chapter 3 reviews literature related to students' interpretations of graphs and situates this study relative to previous studies. Chapter 4 details the methodology I have applied to carry out this study including data collection and analysis. Chapter 5: Results I contains my response to the first research question and includes the details of the characteristics of students' interpretations of expressions. Chapter 6: Results II contains my response to the second, third, and fourth research questions investigating the association between students' interpretations of expressions and their evaluations, the mathematical content of the statements, as well as the association with students' interpretations of points. Finally, Chapter 7 summarizes the findings of this study and outlines some implications of these findings for teaching as well as future research.

CHAPTER 2

THEORETICAL PERSPECTIVE

The overarching goal of this study is to characterize how students interpret expressions in the graphical register with Calculus statements, and what aspects of graphs students attend to when doing so. Because my goal is centered on understanding the cognitive activity of individual students, I have chosen to adopt a cognitive perspective to guide my theoretical considerations.

In mathematics education, theories that stem from a cognitive perspective may be general theories of learning or domain-specific theories that explain how students come to understand concepts in a particular mathematical context. In Cobb's (2007) explanation, theories from cognitive psychology, both general and domain-specific, account for differences in individual students' reasoning and are used to explain students' mathematical activity. The cognitive perspective places in the foreground an individual student's reasoning, rather than collective social dynamics. In this perspective, the individual is conceptualized as the locus of thinking (Cobb, ibid). Adopting a cognitive perspective supports the goal of understanding individual cognition and learning. As such, researchers in mathematics education who seek to characterize student thinking typically take a cognitive perspective, in line with their research goals (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

In this chapter, I make explicit the theoretical assumptions underlying this research study that stem from a cognitive perspective. I describe the general theoretical stance of radical constructivism I adopt as a lens to investigate student thinking. I contrast my choice of a cognitive perspective with sociocultural and emergent perspectives to highlight how a cognitive perspective serves my research purposes. I also detail the domain-specific theory to be used in this study, the theory of graphs as registers of representations, and David et al.'s (2019a) preliminary theoretical framework characterizing students' interpretations of outputs and points on graphs.

Radical Constructivism

For the purposes of this study, I adopt radical constructivism (Glasersfeld, 1995) as my guiding theoretical perspective. Radical constructivism is an epistemological theory that articulates how an individual comes to know something and what that knowledge consists of. One central principle of radical constructivism is the notion that all knowledge emerges out of a need (Glasersfeld, ibid). In other words, knowledge requires a purpose to stimulate its creation and arises out of experiences. Another key tenet of radical constructivism describes the nature of an individual's knowledge itself. Rather than viewing a person's knowledge as consisting of a direct representation of the world, constructivists view knowledge as consisting of a set of action schemes that become increasingly viable given an individual's experience (Glasersfeld, ibid).
As an individual gains experience, he or she refines his or her actions based on the favorability of the outcome. Constructivism also holds that individuals cannot know "Reality" in an objective sense. The reality that an individual experiences is that of his or her own construction. For example, if an individual perceives that he or she is petting a dog, this only because his or her mind has constructed so. The individual has no way of knowing whether this is "true" in an objective sense. Rather, his or her knowledge from

experiences to this point in time supports the idea the individual's visual and touch perception is consistent with other experiences the individual has perceived as petting a dog.

Instead of speaking in terms of the truth of knowledge, or being concerned with it, constructivists typically speak of knowledge in terms of its viability (Glasersfeld, ibid). Knowledge is considered more viable if the outcomes of the actions of an individual yield favorable results. (Glasersfeld, ibid). If unfavorable results occur, those which the individual does not expect given the current action scheme, the individual will adapt his or her knowledge to better match the situation, and thus become more viable. For instance, if the individual petting a dog expected to hear a bark from the dog, but instead hears a noise perceived to be a meow, the individual might adapt his or her image of the animal.

Knowledge, in this view, lies in the mind of the knower, rather than in the outside world around him or her (Glasersfeld, 1995). Consequently, in this view, signs and symbols, such as words, images, diagrams, and other notation, do not inherently contain a certain set of information (Glasersfeld, ibid). Any knowledge constructed from interpreting such symbols is relative to the experience of the observer who interprets these symbols. Because symbols or images do not inherently contain information, the way in which an observer might interpret them is subject to his or her action schemes, as well as where he or she places attention in his or her perceptual field (Glasersfeld, ibid).

In the context of teaching and learning mathematics, a constructivist perspective implies that the words, symbols, and images used to communicate mathematics do not contain or carry a fixed set of information readily transmitted between creator and observer. For instance, when a student reads a definition of a term or a symbol for the term, that an instructor writes on the board, he or she interprets the symbols on the board to assign meaning to the symbols based on the previous knowledge he or she has developed. The student's interpretation of the definition may differ from the interpretation of the instructor, even if they are both reading the same words or set of symbols. The possibility that different observers may ascribe different meaning to the same set of symbols implies that students may not understand the mathematics communicated to them in the way the instructor intends they do.

The constructivist perspective, as Thompson (2000) explains, operates as a backdrop upon which other theories in mathematics education may be built. In Thompson's (ibid) description, radical constructivism is neither sufficient for explaining phenomena nor for prescribing certain choices in research or in teaching. Rather, when a researcher subscribes to a perspective of radical constructivism, he or she is choosing a frame for (1) the types of research questions asked, (2) the types of answers offered to these questions from data, and (3) the role the researcher plays in the process of data collection and analysis. (Thompson, ibid).

Researchers who choose a constructivist perspective typically seek to investigate individual students' ways of reasoning, as opposed to social or cultural phenomena such as the social dynamics of a classroom or cultural trends in mathematics education (Cobb, 2007). In conducting a research study, researchers applying a constructivist lens seek to explain the actions of students while working in particular mathematical contexts. Thus, constructivist researchers typically pose research questions that center on understanding individuals' ways of thinking, ways in which knowledge develops for individuals, or obstacles that arise in individuals' learning.

While developing explanations for student behavior, constructivist researchers recognize that they can never truly know a student's own cognitive activity. In fact, from this perspective, even a student's perception of his or her own cognitive activity is seen a construct, rather than a direct reflection of reality. Steffe, Glasersfeld, Richards, and Cobb (1983) refer to such models that students construct to monitor their own cognition as *first-order models*. Constructivist researchers are careful to acknowledge that they are also a cognizing individual playing an active role in the development of data analysis. As such, they qualify their findings by admitting that their own explanations are a result of their own construction. Instead of claiming that they know what or how a student is thinking, these researchers frame their explanatory findings as *models* based in their observations of the student. Researchers refine and develop these models as they gather more data on the student's reasoning. The stronger the model, the better able it is to predict the outcomes of the student's behavior, consistent with the constructivist notion of viability. Steffe et al. (ibid) refer to such models built by researchers as *second-order models*. These models which are developed by researchers to explain student behavior include predictions of how a student may respond when presented with alternative situations (Steffe et al., ibid). Second-order models of student thinking made by a researcher may shed light on difficulties students face learning a particular concept and may even guide teaching or curricular design decisions (Thompson, 2000).

Subscribing to the perspective of radical constructivism has implications for the types of research questions to ask when students' interpretations of graphs, specifically. For instance, when a student interprets something which, from a researcher's perspective, is a point on a graph of a function in the Cartesian coordinate system, the way in which a student thinks about this point depends on the knowledge that the student has constructed. A constructivist perspective implies that what a student perceives when they look at a graph on paper is dependent upon what the student has formed from looking at that graph on paper. This perspective also implies that when an instructor chooses to show a student an image of a graph, he or she cannot assume that the student attributes the same set of action schemes to the image. My choice to investigate the characteristics of students' interpretations of expressions on graphs presupposes that students' interpretations may differ from those of other students or an instructor. From a radical constructivist perspective, which recognizes individual differences in interpretations, my attempt to characterize students' interpretations is a legitimate research goal.

The adoption of a radical constructivist perspective also has implications for both the type of analysis I plan to conduct and my role as a researcher. Specifically, adopting this perspective acknowledges the limitations on what I, as a researcher, can know about student thinking. The view of knowledge as constructed rather than as a direct representation of the external world implies that I, as a researcher, do not have direct access to a student's thinking. Given this limitation, my objective in this study is to build a model of students' thinking supported by evidence. This model may be tested and refined as I analyze a student's words, gestures, and graph labels (see 4.5). Essentially, I can never claim to know exactly what a student is thinking; I can only make claims of what I hypothesize a student is thinking, supported by what I take to be evidence in the students' actions. Based on these theoretical considerations, I view the relationships of knowledge in this study in the following way. When a student is presented with a graph of a function, he or she is building a model of that graph in his or her mind based on his or her perception. As the researcher attempting to analyze the student's thinking, I am building a model of the model that the student is building of the graph (Figure 1). In this example, the model of the graph that the student constructs is what Steffe et al. (1983) refer to as a first-order model (Thompson, 2000). My model of the student's interpretation of expressions on the graph, formed as the researcher, is consistent with their description of a second-order model (Steffe et al., 1983; Thompson 2000). In Figure 1, a student interprets a graph, and the model the student constructs, shown in the thought bubble above her head, is considered a first-order model. The student's gestures, words, and behavior all stem from this model the student has in mind. Also in Figure 1, a researcher studying the student's interpretation of the graph, through the student's words, gestures, and behavior, constructs a model, which is considered a second-order model. The researcher constructs this model bearing in mind two principles: (1) that the student has a constructed model in his or her mind that is the source of the student's behavior, and (2) that at best, the researcher can only construct a model of the student's interpretation that explains her behavior.

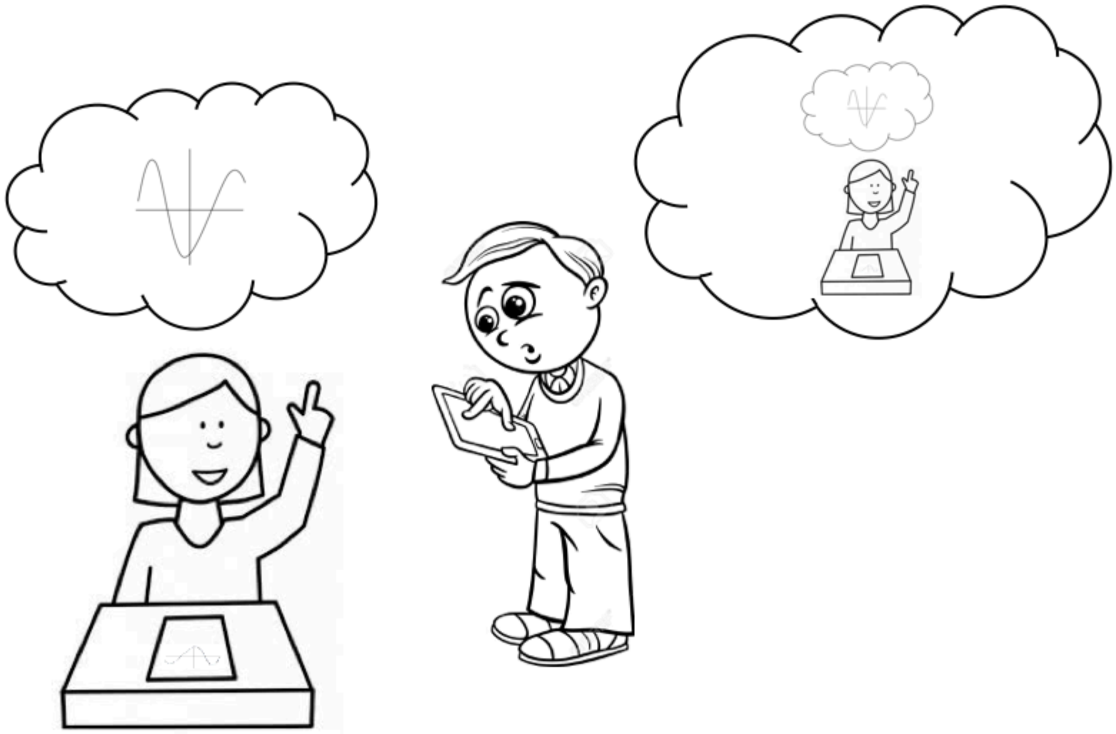


Figure 1. A student's first-order model and a researcher's second-order model. Inspired by Thompson's (2000) illustration of symbolic interactionist view of communication.

Sociocultural and Emergent Perspectives

While I adopt a constructivist perspective, I also acknowledge that this study involves students' interpretations of expressions on graphs, which are organized according to a set of conventions in the Cartesian Coordinate system. Although the development and use of conventions in mathematics is typically understood from a sociocultural or emergent perspective (Cobb & Yackel, 1996), my research focus remains on understanding students' cognitive activity. I will now describe the affordances of embracing a sociocultural or an emergent perspective, in contrast with a constructivist perspective, and explain why I have not chosen to adopt these in the design of this study.

Various perspectives for mathematics education research may be compared by examining two central criteria: how they characterize individuals and how they contribute to researchers' understanding of knowledge and learning (Cobb, 2007). Depending on the goals and objects of study of a researcher, other perspectives may afford certain advantages and be better suited for the researcher's goals.

One different theoretical stance commonly taken by mathematics education researchers is that of a sociocultural perspective. A sociocultural perspective differs from a cognitive perspective in that it views students' participation in cultural practices a direct means of intellectual development (Cobb, 2007). In this perspective, a student's cognition cannot be an attribute of the student alone, but must be framed in the context of the student's cultural surrounding. The sociocultural perspective views practices of the mathematical community as pre-established, and views the student as becoming a member of that community (Cobb & Yackel, 1996). In this perspective, social and cultural practices hold a primary place, while the individual student's cognition is viewed as part of and affected by the broader sociocultural context. The distinctions between a sociocultural and cognitive perspective may be described in terms of the foreground and background of a researcher's focus. A cognitive perspective focuses primarily on the individual, with the background understanding that this individual may be situated in certain social and cultural contexts (Cobb & Yackel, ibid). In contrast, a sociocultural perspective sees the social and cultural contexts an individual may find himself in, with a background understanding that the individual may be developing his or her knowledge in this setting (Cobb & Yackel, ibid).

One advantage of adopting a sociocultural perspective is that it allows researchers to investigate the influence of cultural practices such as those that occur outside the classroom, on the student's behavior. Those who subscribe to sociocultural theories in mathematics education research may be interested in studying the dynamics of institutions, such as math departments at universities across the U.S. They may even compare cultural differences in the teaching and learning of mathematics, such as the Third International Mathematics and Science Study TIMSS study (Stigler, Gonzales, Kawanaka, Knoll & Serrano, 1999). Studies embracing a sociocultural perspective may also examine such issues as equity in classrooms or across various demographics or the institutional practices that impact teachers' development (Cobb, 2007). The limitations of adopting a sociocultural perspective stem from the breadth of this perspective. Findings from these types of studies may be difficult to apply in guiding the learning of an individual student. Furthermore, they may not account for differences in individual students found within the same social or cultural context.

An alternative perspective, the emergent perspective is considered a middle-ground between a constructivist and a sociocultural perspective. The emergent perspective, also referred to by some as a social constructivist perspective (Cobb & Yackel, 1996), seeks to coordinate a cognitive perspective with a social perspective. Those that utilize this perspective seek to understand the dynamics of individuals in a more localized setting than sociocultural researchers. In this perspective, individuals are viewed as members of a local social group and influence and are influenced by the dynamics of the group. Specifically, researchers who adopt this perspective seek to investigate the relationship and interaction between the culture of a classroom and individuals' activity within the classroom. In the words of Cobb and Yackel (1996),

> the social constructivist, or emergent, approach to which we subscribe attempts to coordinate these two ways of analyzing classroom activity [analyzing collective processes and analyzing individual behavior] and treats them as complementary. In this joint perspective, classroom social norms are seen to evolve as students reorganize their beliefs, and conversely, the reorganization of these beliefs is seen to be enabled and constrained by evolving social norms (p. 178).

In their study, Yackel and Cobb (1996) describe the negotiation of what they refer to as sociomathematical norms, such as what constitutes a mathematically different solution to the same problem, within a single classroom. They describe how individual students contribute to the evolving idea of "mathematically different" as well as how they are influenced by this evolution. While the emergent perspective has advantages for understanding the dynamics of a classroom, it is still rooted in the activity of the students in it. In other words, findings from these types of studies may not generalize to other classrooms comprised of a different group of individuals.

In this study, I purposefully choose to investigate how students interpret graphs of functions which I have previously created. I made this choice because I recognize that mathematics students encounter graphs of functions in textbooks and in their instruction, which are arranged according to convention. Furthermore, the findings from this study may be used in a manner that supports "students' induction into practices that have emerged during the discipline's intellectual history" i.e., interpreting graphs of functions in the Cartesian coordinate system (Cobb, 2007, p. 23). While my main focus as a researcher is to characterize individual's interpretations from a constructivist perspective, I am doing so with an underlying acknowledgement of the conventional interpretations from the mathematics community in the background. Thus, my perspective remains constructivist, as investigating the cognitive activity of individual students lies in the foreground of my research purpose.

Graphing from a Constructivist Perspective

Due to the theoretical stance that I adopt in this study, described earlier in this chapter, I make use of terms related to graphing in a particular way. Adopting a perspective of radical constructivism necessitates that I articulate from whose perspective a graph is considered a graph. When I refer to a "graph" in this dissertation, I mean something which, from *my* perspective, I take to be a graph. Thus, when I describe a student interpreting a graph, I mean a student is interpreting that which I take to be a graph, without making any claims about the student's meaning for the graph.

The term visualization has a variety of meanings in the research literature. Arcavi (2003) defines visualization as,

> The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (p. 217).

When I refer to "visualization" in describing its role in curriculum or as part of an instructional intervention, I mean an image created by one who intends to illustrate a particular idea. By a "student's visualization," I mean a student's interpretation of an image, such as a graph, as evidenced by his or her words, gestures, or labels. This usage of visualization is consistent with Arcavi's (2003) definition, although his is much broader. For the purposes of this study, I also choose to separate a student's interpretation from the reasoning that may stem from or develop through their interpretation; however, I acknowledge that they are connected. To distinguish it from visualization, I will use the term "visual reasoning," to refer to a student's mathematical reasoning that makes use of some aspect of a graph the student is interpreting.

For example, if I claim a student is interpreting a graph of a polynomial function, I mean he or she is interpreting an image that I claim to be a graph of the polynomial function (consistent with the mathematics community). If a student seems to interpret a point on the graph, I would refer to the student's visualization of the graph in describing the student's interpretation. Suppose this student then reads a statement about a polynomial function from Calculus, and claims that the statement is false. If the student uses the graph of the polynomial function in his or her explanation of his or her evaluation, I would refer to his or her argument as his or her "visual reasoning" as it is reasoning which involves the student's visualization, i.e., his or her interpretation of a graph.

Theory for Characterizing Students' Graphical Interpretations

While a constructivist theoretical perspective lays a foundation for developing further theory about students' graphical interpretations, the goal of this study requires more fine-grained theory relative to details of students' interpretations of graphs in Calculus and beyond. In studying students' mathematical thinking, domain-specific theory is useful to characterize student thinking in a particular context or topic (Cobb, 2007).

Graphing as Semiotic: Theory of Registers

One theory related to graphing which I adopt is that the act of interpreting graphs, as an action of visualization, is semiotic in nature. By this, I mean that interpreting a graph involves cognitive activity of processing a system of symbols intended to represent certain mathematical properties. I use the term semiotic consistently with Duval's (1999) use of the term in relation to systems of graphing, such as the Cartesian coordinate system. Duval (ibid) discusses his theory of understanding semiotic representations, which he views as intimately connected with the notion of visualization. In his perspective, "visualization refers to a cognitive activity that is intrinsically semiotic, that is, neither mental nor physical." Duval (ibid) first explains several notions underlying his perspective. First, he acknowledges that in the study of mathematics, unlike other fields of study, such as biology, the objects of study are not readily accessible to the senses. Thus, mathematics relies on the use of semiotic representations in order for students to perceive these objects. However, Duval (ibid) notes, understanding mathematical ideas necessitates an understanding of the distinction between the representation and the mathematical object it represents. Duval (ibid) views visualization used in mathematics as relying on semiotic systems and in which the representation is intended to refer to the object being represented. While all semiotic systems require understanding of the specific "language" used in representation, understanding a graph, in which some figural components may represent different objects at different times, depends on context. In some cases, a graph may represent the path of a moving object, in which each point represents a location in two-dimensions, (x, y), such as in a parameterized curve. In other cases, a graph of the same shape may indicate the speed of a particle in terms of time elapsed, in which the graph itself does not represent any sort of physical path.

Duval (1999) explains that there are several "registers" of representations used in mathematics, such as formal language, graphs, tables, and diagrams. He also distinguishes between two types of cognitive activity relative to registers of representations, what he calls processing and conversion (Duval, ibid). *Processing* involves transforming a representation within the same register, such as simplifying an expression, or rotating or reflecting a geometric figure. *Conversions* involve changing a representation using one type of register into another, such as developing the graph of a function from the equation for the function rule. It is in the activity of *conversion* that Duval argues lies the central problem of learning mathematics. Furthermore, he argues that students must have the ability to distinguish between the representation and the mathematical object in order to successfully transition between registers of representation. Emphasizing the role of visualization, Duval (ibid) claims learning mathematics "always begins with the coordination of a register providing visualization" (p. 7). In investigating students' interpretation of expressions on graphs, I adopt the distinction in systems of symbolic representation between expressions of functions and graphs as different registers. To distinguish between students' interpretations with expressions on graphs from their reasoning with numerical values, I refer to the first as an interpretation in the graphical register and the second as an interpretation in the numerical register. While the graphical register broadly refers to several representation systems, in this study, I focus on students' interpretations of expressions on graphs in the Cartesian coordinate system (CCS).

Framework for Students' Interpretations of Points in the Graphical Register

In typical Calculus courses, students are asked to interpret and reason with graphs that are represented in the two-dimensional CCS (e.g., Stewart, 2012). The CCS follows certain conventions. In the CCS, two axes, with specified units, meet at a right angle, and points representing pairs of values are represented given distances from the intersection of these axes, referred to as the origin. For instance, a point represented generically as (x, y) will be graphed at a location x units to the right of the origin and y units above the origin. Thus, points in this system of representation are dual-natured, referencing a pair of values, as well as a location in the CCS.

Given the dual-nature of points in the CCS, a student may place his or her attention on one of these two aspects of a point, either the values represented by the point, or the location in the Cartesian plane. Where the student places his or her attention when he or she interprets a graph influences how he or she interprets the aspects of a graph.

For this study, I seek to employ and expand David, Roh, and Sellers's (2019a) preliminary framework as domain-specific theory to characterize students' interpretations of points on graphs in various contexts from Calculus. Using domain-specific theory, I plan to gain insight into students' interpretations of expressions as other aspects of graphs, which can then inform the design of instructional interventions (Cobb, 2007). This framework specifically characterizes students' interpretations in terms of where students place their attention when reasoning about points on graphs. David et al.'s (2019a) framework offers two ways in which students may interpret graphs, described in terms of how students interpret components of graphs of functions, such as inputs and outputs.

David et al.'s (2019a) initial framework, shown in Table 1, details two ways students may interpret aspects of graphs, referred to as *value-thinking* and *location-thinking*. This framework was developed to describe phenomena that were observed in undergraduate students' interpretations of graphs in the CCS in the context of the Intermediate Value Theorem. In this framework, if a student attends to the pairs of *values* that points represents, this way of thinking is referred to as *value-thinking*. On the other hand, if a student focuses on the *location* of points in the Cartesian plane, this way of thinking is referred to as *location-thinking*.

Table 1

Comparison of Characteristics of Value-Thinking and Location-Thinking

<table>
<tr><th colspan="2" rowspan="2"></th><th colspan="2">Value-Thinking</th><th colspan="2">Location-Thinking</th></tr>
<tr><th>Interpretations</th><th>Evidence</th><th>Interpretations</th><th>Evidence</th></tr>
<tr><td rowspan="3">Aspects of a Graph</td><td>Output of Function</td><td>The resulting value from inputting a value in the function</td><td>▪ Labels output values on output axis
▪ Speaks about output values</td><td>The resulting location in the Cartesian plane from inputting a value in the function</td><td rowspan="3">▪ Labels outputs on the graph
▪ Labels points as outputs
▪ Speaks about points as a result of an input into the function (e.g., “an input maps to a point on the graph”)</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs
▪ Speaks about points as the result of coordinating an input and output value</td><td>A specified spatial location in the Cartesian plane</td></tr>
<tr><td>Graph as a Whole</td><td>A collection of coordinated values of the input and output</td><td>A collection of spatial locations in the Cartesian plane associated with input values</td></tr>
</table>

(David, Roh, & Sellers, 2019a, p. 8)

These two ways of thinking, which characterize students' interpretation of graphs, are based on where students place their attention when interpreting points on graphs. The framework explains each way of thinking by detailing how a student engaged in that way of thinking thinks about three aspects of a given graph: outputs of the function, points on the graph, and the graph as a whole. Each of these aspects of graphs is described from the perspective of a researcher using conventional interpretations of the Cartesian coordinate system. The output of a function is conventionally represented as a magnitude of length in the direction of the y-axis. A point conventionally represents a pair of both input and output values, located a distance of the input value to the right of the origin, and a distance of the output value above the origin. Conventionally, a graph as a whole represents the set of all ordered pairs that satisfy the equation of the function. The framework also describes observable evidence indicative of thinking about aspects of the graph in a particular way. Using these descriptions of observable evidence in the framework, students' words, gestures, and markings on the graph can be used to characterize their way of thinking about graphs as either value-thinking or location-thinking.

Value-thinking.

In this framework, value-thinking refers to an attention to the values represented by a point in Cartesian space. Students engaged in value-thinking treat outputs as values associated with corresponding input values. This may be indicated by the student labeling output values on the output axis, or speaking about output *values*. Students engaged in value-thinking think of points as coordinated pairs of input and output values. These students may indicate this way of thinking by labeling points as ordered pairs, and speaking of simultaneous pairs of values when referring to points on a graph. Thus, students engaged in value-thinking treat graphs as a collection of points, each of which represents a pair of input and output values.

Location-thinking.

In contrast, location-thinking refers to an attention to the locations of the points in space. Students engaged in location-thinking treat points on the graph as outputs, confounding outputs of the function with points on the graph. This way of thinking about points may be indicated by referring to points solely as outputs or describing the output of a function as the location of the graph itself (e.g., "each input is mapped to a point on the graph"). Additionally, students engaged in location-thinking may label a point with an output value only, thus placing the output label at a point, rather than on the output axis. Thus, students engaged in location-thinking treat graphs as a collection of points that represent locations in the plane that correspond with input values.

To highlight the distinction between value-thinking and location-thinking, I provide two examples of sample student labeling on the same graph indicative of each of these ways of thinking, shown in Figure 2.

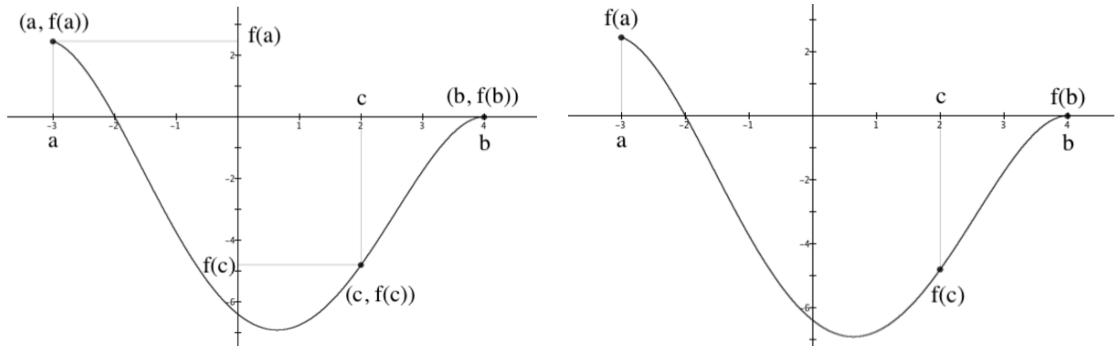


Figure 2. Example labels indicative of value-thinking, left, or location-thinking, right (David et al., 2019a, p. 2).

The labels on the graph in Figure 2, left indicate value-thinking. In this graph, output values are labeled on the output axis, and points are labeled as ordered pairs. In contrast, the labels on the graph in Figure 2, right indicate location-thinking. Output labels are not placed on the output axis but rather at the locations of points. Consequently, points are not labeled as ordered pairs but solely as outputs. While I plan to use a student's gestures and words in addition to the labels on a graph, these examples highlight distinctive characteristics of value-thinking and location-thinking. The classifications of value-thinking and location-thinking highlight previously undocumented phenomena in students' interpretations of points of graphs. Careful attention to students' interpretations, then, might yield further delineations of these ways of thinking, or other new classifications.

Building on David et al.'s (2019a) preliminary framework.

In this study, I apply David et al.'s (2019a) preliminary framework to initially classify each student's interpretation of points on the graphs that I present to students (see Chapter 4). Because I seek to investigate how students think about graphs with various

Calculus statements, I recognize that the same student may engage in value-thinking in one context and engage in location-thinking in a different context. Furthermore, a student may switch between these two ways of thinking while working in the same context. I will first attempt to classify students' interpretations according to this framework, bearing in mind that the framework likely may not account for all students' interpretations of graphs. Thus, I expanding this theoretical framework to more fully characterize students' interpretations of graphs.

Currently, David et al.'s (2019a) framework only addresses the way students who engage in either value-thinking or location-thinking think about outputs, points, and graphs as a whole. As is, it does not account for how students interpret other aspects of graphs, such as the inputs of a function, or the axes. It also does not account for ways students might interact with or operate on inputs, outputs, or portions of graphs in their reasoning. Additionally, the current framework does not account for student thinking that fails to attend to certain aspects of graphs, such as the input or output of a graph or points on a graph. Thus, this study was designed to build upon and expand on existing theory to account for these areas of students' interpretations of graphs.

CHAPTER 3
LITERATURE REVIEW

In the modern approach to the teaching and learning of Calculus, graphs of real-valued functions tend to figure prominently. Key results in Calculus, such as the Intermediate Value Theorem (IVT) or the definition of a derivative using the limit of the difference quotient are often illustrated with graphs as figures in textbooks and curricula (e.g., Stewart 2012). The inclusion of such graphs in Calculus texts is in line with initiatives to incorporate visual representations into mathematics curriculum (e.g., Arcavi, 2003), a key feature of reform efforts in Calculus (e.g., Hughes-Hallett, 1991). In fact, recent Calculus curriculum reform efforts have often sought to leverage the power of technology to illustrate the coherence of Calculus concepts using animated graphs of functions (e.g., Active Calculus- Boelkins, Austin, & Schlicker, 2018; DIRACC-Thompson, Byerley, & Hatfield, 2013). With these tools, teachers of Calculus may use graphs in their efforts to support students in moving beyond procedure-oriented mathematics. Graphs have been shown to afford students opportunities for reasoning about concepts in Calculus (e.g., Kidron & Tall, 2015; Roh, 2010) and may even promote an understanding of key ideas, definitions, and theorems in Calculus (Giaquinto, 1994; Roh & Lee, 2017). For instance, Swann (1997) offers a graphical illustration of the Mean Value Theorem as a way to expose Calculus students to proofs. Furthermore, Davis (1993) argues for the use and validity of visual displays of theorems using graphs in Calculus.

While graphs may provide a powerful support to students, they are only effective insofar as students interpret them in conventional ways. In fact, research has shown that students encounter a variety of issues when interpreting graphs of functions provided to them or those of their own construction. (e.g., Bell & Janvier, 1981; Frank, 2016). Students' interpretations of graphs, when unconventional, may have far-reaching consequences at the undergraduate level. As Dawkins and Epperson (2014) found, the students who withdrew from Calculus classes typically lacked an understanding of graphical representations. Because of the importance of graphs in undergraduate mathematics, researchers have begun to characterize students' interpretations of graphs in order to better understand how to support students in their study of mathematics (e.g., David et al., 2019a).

In this chapter, I will review literature related to visualization in the teaching and learning of Calculus as well as literature regarding students' interpretations of graphical representations in mathematics. I describe several studies that showcase the potential and documented benefits of incorporating graphs as visualizations of functions in the study of Calculus. I also review studies that describe the potential issues in students' interpretation of such visualizations at the secondary and postsecondary level in terms of (1) their attention to figural aspects of graphs, (2) their tendencies in reasoning with graphical examples, and (3) their interpretations of components of graphs. This review will serve to both situate my current study in the existing literature, as well as highlight how my study builds on previous findings.

Visualization in the Teaching and Learning of Calculus

The mathematics education community has called for the inclusion of visualization in the teaching of Calculus for several reasons. First, visualization has historically played a central role in the development of mathematical concepts (Arcavi, 2003). Second, visual information may be more readily perceived and recalled. Humans have been shown to tend to rely more heavily on visual information than information perceived from any other sense (Gal & Linchevski, 2010). Furthermore, memory of visual material has been shown to be superior to memory of verbal material (Paivio, 1990). Additionally, mathematics education research indicates that visual representations may support student learning (Davis, 1993; Dreyfus, 1991; Guzman, 2002; Vinner, 1989). Cognitive research also supports the inclusion of visualization in the teaching of mathematics. In the subject of Calculus, specifically, mathematics education researchers have recommended the use of visualization. In fact, Zimmermann (1991) claims, ‘‘the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject. This is especially true if the course is intended to stress conceptual understanding…” (p. 136). In line with such recommendations, researchers have developed instructional interventions that rely on graphs to support students in understanding key concepts from Calculus (e.g., Kidron & Tall, 2015; Roh, 2010; Tall, 2010; Thompson et al., 2013).

In this section, I will review studies that propose instructional interventions that utilize graphs to support students in learning Calculus concepts. I will also discuss studies that highlight the effectiveness of these interventions when used in practice either in classrooms or in teaching experiments.

Visualization as Instructional Interventions in Calculus

Instructional interventions that rely on visualizations of key concepts may be used in the teaching of Calculus and advanced Calculus to promote conceptual understanding or to help students ease the transition to working with formal definitions. For example, Roh's (2010) ε-strip activity was designed to support undergraduate students in understanding the formal definition of the limit of a sequence. In this activity, students use physical strips of clear paper of various widths of 2ε to place over the graphs of sequences to represent the various intervals (-ε, ε) around the potential limit of the sequence. As part of the activity, students are asked to test two possible definitions of the limit of a sequence. One definition claims a value is a limit of a sequence if infinitely many terms of the sequence lie within any ε-strip centered at this value, while the other definition claims only finitely many terms of the sequence lie outside any ε-strip centered at the target value. These ε-strips provided students the opportunity to visualize the interval created by single ε-values, and how many of the terms of a given sequence would lie inside or outside the given interval. As students work with ε-strips of various sizes, they could potentially visualize the impact of the value of ε on the number of terms of the sequence inside or outside the corresponding ε-strip. Through this activity, and testing ε-strips for various sequences, students may come to understand that a sequence has a limit when only finitely many terms of the sequence lie outside any ε-strip centered at the target value. Using their experience with ε-strips, students are then given the formal ε-*N* definition and guided through connecting this definition with the definition they found in the ε-strip activity.

In addition to Roh's (2010) activity, researchers have developed other interventions to support students in coming to understand formal definitions from advanced Calculus by using their intuitive images as a starting point. Some instructional interventions seek to combine students' intuitive, visual notion of a concept with its formal definition, for the purpose of supporting students in making the transition from the former to the latter. For instance, Tall (2010) describes his intervention for teaching the concept of continuous functions by explaining the natural, sense-based idea of continuity as drawing a curve without picking up the pen, which is a commonly held image of continuity among students. In order to connect this intuitive idea to the ε-δ definition of a continuous function at a value, x_0, two seemingly disparate ideas, Tall suggested that the idea of horizontally stretching a displayed graph of a function repeatedly around the point $(x_0, f(x_0))$ in question until the graph appears entirely horizontal. This process visually mirrors the formal definition involving δ and ε that shrink to zero. By repeatedly zooming in on a window of the graph, with a width of 2δ and a height of 2ε, students can observe the effects of shrinking ε and δ on the graph, and the values its points represent. Next, Tall (2010) suggests returning to the image of tracing a curve with a pencil, but this time thinking of the pencil point as having a certain thickness that covers over the function which does not have a finite thickness. Within the band created by the pencil trace around the function, it is possible for students to conceptualize the same rectangles with a width of 2δ and height of 2ε, even for increasingly finer pencil points. In this way, students may re-conceptualize tracing a function without lifting a pen with as creating an "error bound" for the function to be entirely contained within. Tall (2010) uses this visual process to connect the informal idea of not lifting a pencil with the meaning of the ε-δ definition of continuity. Tall's (2010) overall message with this suggested intervention is that the informal and intuitive images of continuity that students have are not necessarily harmful for students, and can even provide a foundation for the formal meaning, provided they are properly supported. Tall's (2010) work highlights the benefits of visualization in this context, as well as how visualization can be used to support students' thinking about the formalized version of a previously intuitive idea.

In addition to interventions that support students' conceptualization of formal definitions, researchers have also designed instructional tools to support students' understanding of other concepts from advanced Calculus, such as the limit of a sequence of functions. Kidron and Tall (2015) conducted a teaching experiment in which they used software with a group of calculus students to illustrate how a sequence of functions, such as those given by a function's Taylor series representation, converge to a target function. Using software, the teacher in the experiment presented students with a graphical animation of a series of polynomial functions that became increasingly better approximations of the sine function, pictured with the sine function on the same set of axes. In this way, students were given the opportunity to dynamically visualize graphs as objects that changed to eventually become indistinguishable from the target function. Kidron and Tall (2015) found that the visual imagery provided powerful support for students to move toward an understanding of the formal definition of a limit in terms of arbitrarily small error bounds.

The interventions described above, which all make use of the power of visualization, are designed to support students in developing their understanding of formal definitions of mathematical concepts such as limits and continuity. Still, other visual interventions have been developed to support students in understanding theorems and properties of Calculus concepts. For instance, Thompson, Byerly and Hatfield (2013) proposed an approach to teaching the Fundamental Theorem of Calculus conceptually using graphing software. Thompson and colleagues (ibid) describe the Fundamental Theorem of Calculus in terms of the relationship between accumulation functions, functions whose outputs are thought of as tracking the varying amount of some quantity, and their associated rate of change functions. One main goal of their instructional tasks is for students to consider accumulation functions written in open-form using integral notation as "first-class functions" including an input, output, and graph, rather than this notation simply triggering a procedure to re-write in closed form. In their approach, an instructor first presents students with rate of change functions, functions whose output tracks the value of the rate of change of some quantity for varying values of the input. From these rate of change functions, referred to as "exact" rate of change functions, students define approximate rate of change functions that have constant outputs for small intervals of the input. Using software, students visualize these approximate rate of change functions, which resemble the exact rate of change function, except consisting of small segments of constant functions that approximate the rate. From these approximate rate of change functions, students then construct approximate accumulation functions from constant rate of change functions, using their understanding of the relationship between rate of change values and the accumulation due to varying at a given rate value over an interval of the input. The approximate accumulation function developed from this process becomes more exact as students shrink the length of the interval over which they assume a constant rate value. Thompson et al.'s (2013) approach to teaching Calculus conceptually relies on students' interaction with the functions represented graphically. As Thompson et al. (2013) explain, the functions students work with, including open-form accumulation functions (e.g., $f(x) = \int_a^x t^2 dt$), become real functions as students see their graphs. In contrast, traditional Calculus approaches may simply encourage students to view functions of this form as requiring an application of the Fundamental Theorem of Calculus to re-write them in closed form (e.g., $y = \frac{x^3}{3} - \frac{a^3}{3}$).

Each of the interventions described above are designed to leverage the power of graphs as visualizations to support students in conceptualizing central ideas of Calculus, including limits of sequences of values, of functions, the formal definition of continuity, and the Fundamental Theorem of Calculus. Some of these proposed interventions may support conceptual learning for novice Calculus students, while others may assist students in the transition from intuitive images developed in introductory Calculus to the formal definitions Advanced Calculus. Not only have such instructional interventions that leverage the power of visualization been suggested, studies have also tested the effectiveness of such visual interventions in teaching experiments and in classroom use.

Benefits of Visualization in the Learning of Calculus

Several studies have demonstrated the benefit of interventions that rely on visualizations in students' understanding of Calculus topics, documented both by researchers (e.g., Cory & Garofalo, 2011; Kidron & Tall, 2015; Roh & Lee, 2017) and self-reported by students (e.g., Natsheh & Karsenty, 2014). Studies have also shown the advantages of students' reasoning from visualizations of their own construction (e.g., Pinto & Tall, 2002).

Interventions that utilize visualizations have been shown to support an understanding of important components of the formal definition of the limit of a sequence, such as the relationship between n, N, and ε in the definition. One study, conducted by Cory and Garofalo (2011), found that dynamic graphs supported preservice teachers' understanding of the formal definition of the limit of a sequence. In their tasks, teachers were able to select a band of width 2ε, a computerized version of Roh's ε-strip, and adjust the value of N. As they changed the values of ε and N, the teachers were able to observe the effects of such changes on whether the terms of the sequence lay inside or outside the ε–band dynamically. Cory and Garofalo (2011) found that the preservice teachers' images of the definition of a limit of a sequence grew to include a coherent understanding of "for every $n > N$, $|a_n - L| < \varepsilon$" through their engagement with these dynamical visualizations.

Research shows these instructional interventions that rely on visualization not only support students with understanding the given concept they are intended for, but also to support students' understanding of related concepts that they encounter later in their coursework. In an undergraduate introductory real analysis course, Roh and Lee (2017) found that the use of the ε-strip activity (Roh, 2010) supported students' development of an understanding of this ε–N definition of the limit of a sequence and the definition of a

Cauchy sequence. In their study, students worked with the ε-strip activity. Notably, the visualizations the students developed during the ε–strip activity supported students later in the course in their work with other formal definitions related to convergent sequences. Specifically, students recalled similar visualizations as they worked to develop their understanding of the relationships of the symbols involved in the ε–*N* definition of the limit of a sequence as well as the definition of a Cauchy sequence. Thus, the visualization found in the ε–strip activity, which uses graphs of sequences, supported students in their development of an understanding of formal definitions central to Advanced Calculus.

In addition to researchers who have espoused the benefits of visualization in instruction of Calculus, some researchers have also reported that students would prefer the inclusion of visual tasks in order to deepen their understanding of Calculus concepts. As a collective case study, Natsheh and Karsenty's (2014) conducted clinical interviews using a graphical display of a function and its derivative to investigate individuals' understandings of the concept of a derivative. High school seniors, college freshmen, and high school and university teachers were included in the study. The participants were shown two graphs and told one is the graph of a function, *f*, and the other is a graph of the function's derivative, *f'*. They were then asked to identify each graph as either *f* or *f'* and give as many reasons as they could supporting their claims. The majority of participants identified the graphs correctly. However, the researchers found some focused on different aspects of the graphs, such as inflection points, maximum and minimums, or intervals over which the functions were increasing or decreasing to make their decisions. When asked to reflect on the possible use of this task in Calculus classrooms, the participants self-reported the potential benefit of working with such visualizations in learning the concept of the derivative. They expressed that such tasks may deepen their understanding by applying concepts of the relationship of a derivative of a function to a function itself.

Besides the benefits offered by instructional interventions that use visualization, students who generate visual examples on their own to reason from may strengthen their own understanding of a concept. For instance, Pinto and Tall (2002) documented a case in which a high-performing undergraduate student who made sense of the formal definition of the limit of a sequence by first creating his own visual representation of the mathematical objects described in the statement. In their study, Pinto and Tall (2002) asked the student to provide his meaning for the limit of a sequence. The student drew an image of what he considered a generic function (a continuous function that was both increasing and decreasing in certain intervals of the domain of the function) and drew two horizontal lines that created bounds for the function to eventually stay within. The student reasoned from his image of a sequence and its corresponding limit value to re-construct his definition of the limit of the sequence. As the interviewers asked more about limits of sequences, such as the limit of a constant sequence, the student again used his image to refine his definition to include broader examples of sequences. This example from the literature highlights the power of visualizations for students of advanced Calculus. Rather than memorizing the verbal description of a definition, recalling a powerful image may support students in reconstructing the definition when necessary. In later mathematics courses, such as introduction to proof, researchers have found that students first need to develop their images of concepts through examples and graphs, before coming to an understanding of the formal definition (e.g., Moore, 1994).

Researchers have demonstrated the effectiveness of instructional interventions that rely on visualization to support students in learning concepts of Calculus, both the concept they are designed to teach (Cory & Garofalo, 2011; Kidron & Tall, 2015; Roh & Lee, 2017), as well as concepts they encounter later (Roh & Lee, 2017). Research has also shown that students would prefer learning Calculus concepts using visualization (Natsheh & Karsenty, 2014) and observed students successful draw on their own visual constructions (Pinto & Tall, 2002). From the above discussion, there is sufficient evidence to claim that reasoning using graphs may be a powerful tool for students in advancing their understanding of concepts in Calculus and Advanced Calculus. However, these interventions, and similar ones that utilize graphs as a basis for reasoning, all presuppose that students have conventional interpretations of the components that comprise a graph. In other words, how effective such interventions are in supporting students depends on the ways in which students interpret and reason from these graphs. In reality, many studies have shown that students often interpret graphs in unconventional ways (e.g., Duval, 1999).

Issues in Students' Understanding and Use of Graphs of Functions

A number of studies have documented the limitations and issues in students' interpretation and use of graphs in mathematics. Studies have shown that students may (1) attend to perceptual features of the image of a graph that are not relevant from the perspective of an expert while ignoring those features which an expert would consider relevant (e.g., Presmeg, 1986), (2) draw incorrect conclusions while reasoning from graphs (e.g., Harel & Sowder, 1998), and (3) interpret graphs as a whole or various aspects of graphs in ways that may be unconventional (e.g., David et al., 2019; Frank, 2017). I will describe the key findings relative to these issues that students may have with interpreting and using graphs, which could inhibit their learning with visual interventions.

Issues in Students' Attention to Figural Aspects in Graphs

Studies have shown that when interpreting or reasoning with graphs of functions, students may attend to figural aspects of the graph in ways that lead them to interpret the graphs in unconventional ways (e.g., Bell & Janvier, 1981; Duval, 1988; Goldenberg, 1988; Magidson, 1989). These include students attending (1) to a single figural aspect of these graphs, to the avoidance of other relevant features, or (2) to figural aspects which would be considered irrelevant by an expert while ignoring relevant ones. By figural aspects of a graph, I refer to aspects of the pictorial image of a graph that relate to the shape or form of the graph, such as the pixels used to create an image or diagram, the angle formed by the graph of the function and the axes, the scale of the axes, and the height of the local and global maxima and minima of a graph in the Cartesian plane. Such figural aspects are not intrinsic features of a function itself, but rather depend on the graphical depiction of that function. Thus, these features may change as the dimensions or scale of the graph change, while the function depicted remains the same.

Research has shown that certain figural aspects of graphs may actually become visual distractions for some students, undermining the potential support that graphs offer. In some instances, students tend to focus on a single figural aspect of a graph of a function, to the avoidance of other figural properties of the graph (Bell & Janvier, 1981; Goldenberg, 1988). Bell and Janvier (1981) found examples of students distracted by features of the graphs they presented in their study examining the graphical interpretations of 12 -year-old mathematics students in Britain. For instance, they found that when answering questions about the greatest difference between the outputs of two functions shown on the same axes, the students instead referred to the extrema of the functions on the graph. Additionally, when given a graph and asked where the greatest increase in the output quantity occurred, students also referenced the extrema of the functions on the graph. In both cases, the local or global maxima and minima captured students' perceptual attention when they were asked about extreme values of the function, either of a difference or an increase. In Bell and Janvier's (ibid) framing, the features of "peaks" and "valleys" on a graph, representing maximum or minimum values of the function, distracted students from other features of the graph that an expert might use to find the greatest difference between two functions' outputs or the greatest increase in a single function's output.

In his summary of findings of school-aged students' interpretation of graphs, Goldenberg (1988) reported that students often interpreted graphs by solely attending to either the scale of the axes, the angle formed by a line and the axes, or the position of graphs relative to each other. Some students also, when asked to model a provided graph of a parabola with an equation of their own construction, focused primarily on matching the "height" of the parabola in the Cartesian plane, rather than any other relevant feature such as the x-intercepts or y-intercepts. In attempting to solely match the height of the parabola, these students wrote equations for functions that were less similar than their previous attempts at the problem. Like Bell and Janvier's (1981) research, Goldenberg's (1988) findings of students' focusing on singular figural aspects also support the notion that students' attention to such features of graphs may be detrimental in some instances.

Other studies have shown that students may also be distracted by what an expert might consider entirely irrelevant features of graphs of functions (Duval, 1988; Magidson, 1989). Magidson (1989) found that while working with graphing software to graph lines, students attended to graph features that were products of the software that an expert would consider irrelevant, such as the pixels of the graphs. In Magidson's (ibid) study, pairs of students were asked to graph linear equations with the same y-intercept but different slopes ($y=2x+1$, $y=3x+1$, etc.) and discuss what they observed in comparing the graphs of the lines. Students' comments referred to features such as the x-intercept, noting that as the coefficient of x increased, the graphs appeared to move to the right. Other students commented on the pixelation of the graph as the slope increased. Due to the limitations in the graphing software, the picture produced by the computer appeared more jagged as the slope of the graph increased. These features of the graphs, which may seem irrelevant to an expert looking at a graph, aware of the limitations of displays using pixels, were relevant for the students as they reasoned about the graphs.

Similarly, Duval (1988), as cited by Duval (1999), also observed students attending to what the mathematics community would consider irrelevant features of a graph, such as the angle formed by the line and the horizontal axis, a feature which changes with the scale of the axes. In studying the role of visualization in the learning of mathematics, Duval (1988) found that some high school students, when asked to look at the graphs of two similar linear functions: $y=x+2$ and $y=2x$ were unable to distinguish between the two graphs. Instead of attending to the slopes or intercepts of the two lines, these students either focused on the angle formed by each line with each of the axes or the distance between certain points on the lines and the x-axis. However, these same students were able to construct the graph of a line and report coordinates of points when asked. Based on these and other findings, Duval (1999) concludes that learning to construct graphs alone seems insufficient for developing students' ability to interpret graphs conventionally. Similar to Magidson's (1989) findings, Duval (1988) also highlights ways in which the figural aspects of graphs of functions may influence students' reasoning.

The findings from the studies described above suggest that the figural properties of a graph, as an image, may distract students from what experts might consider to be relevant features. Students may attend to the pixels used to create the graph, the angle which the graph of a function forms, the scale of the axes, the height of the graph in the Cartesian plane, and the extrema of the graph of a function at times when attending to these features is not relevant from the perspective of an expert. Thus, while graphs have been shown to support student learning, researchers and instructors must acknowledge the nature of graphs as images with certain figural properties that draw students' attention in certain ways in the design and study of visualization for learning.

Students' Tendencies when Reasoning from Graphical Examples

Researchers in mathematics education have found that students may avoid using graphs to reason from when alternate information is provided (Dawkins & Epperson, 2014; Knuth, 2000), or when they do, they may focus on limiting examples or incorrect imagery of graphs from which to draw conclusions (Aspinwall, Shaw & Presmeg, 1997; Presmeg, 1986; Tall 1990). For instance, Knuth's (2000) study shows a strong tendency for high school students across levels to avoid using graphical representations when reasoning about linear relationships. Knuth (ibid) sought to investigate students' understanding of the "Cartesian connection," which refers to the relationship between a graph and an algebraic representation of the relation graphed. This relationship may be articulated as: A point lies on a graph, f, if and only if the coordinates of that point satisfy the algebraic representation of f. Knuth (ibid) surveyed 178 high school students across several levels of mathematics courses, from algebra through advanced placement Calculus. The survey contained tasks in which students were provided with both an equation of a line and a graph of that line. The tasks were designed in such a way that the most efficient solution could be found from the graph, although students could glean the same information from the equation. Students were asked to find the solutions to these tasks and to explain their reasoning, in order to discover which representation students would select to use in their solution. Knuth's (ibid) findings from the survey indicate a strong tendency for students to rely on an equation rather than a graph when drawing conclusions. Additionally, when the surveyed students did utilize graphical representations, it was mostly in familiar contexts, which suggests students' ability to connect graphs with their algebraic representation was largely superficial.

Dawkins and Epperson (2014) also observed students' reluctance to utilize graphs to solve tasks in the undergraduate Calculus course they studied. Investigating the relationship between Calculus learning and problem-solving, Dawkins and Epperson (ibid) studied two sections of a Calculus course over a semester and compared student performance on four "non-routine" pre-Calculus/Calculus tasks and their achievement in terms of grade. These tasks were chosen specifically because they allowed for solutions in various registers: algebraic, numeric, and graphical, while favoring graphical solutions. Their findings overwhelming indicate students' avoidance of reasoning in terms of graphs, as one task resulted in only 12 solutions utilizing graphs of the 346 total responses collected; in fact, the graphical solution to this task was considered by the researchers to be the most intuitive and elegant. One possible explanation for the avoidance of the graphical register that Dawkins and Epperson (ibid) offer was that the tasks found on the course homework and exams tended to reward algebraic rather than graphical reasoning.

One commonly acknowledged limitation of graphs as visual representations of concepts is that often one instance is intended to represent a broad range of possible cases. For instance, graphs of single functions are often used in Calculus textbooks to represent a set of functions, such as all continuous functions on a given interval (e.g., Stewart, 2012). For students who do choose to reason with graphs, the use of one image to represent many cases may become problematic if they overgeneralize from a single image. This overgeneralization may be especially problematic in Calculus if students have a single image in mind and counterexamples to statements may not readily occur to them. In the context of convergent sequences and series, Alcock and Simpson (2004) found that introductory real analysis students formed conclusions from their own visual imagery. In their study, the instructor asked students about the relationship between boundedness and convergence for sequences, as well as when a given sequence converged. The tasks did not present a visual component, but some students tended to reason from images in their minds or those which they drew. In doing so, these students sometimes neglected to consider other cases included in the idea of a bounded sequence. This is consistent with Harel's and Sowder's (1998) findings that students may overgeneralize from a single visual representation that they view as prototypical. Presmeg (1986) found that among high school seniors in math courses, those who tended to approach problems visually were often considered lower-achieving compared with students with non-visual tendencies. She attributes this difference in students' adherence to the single case represented by an image. For instance, Presemeg (1986) hypothesized that some students, when presented with a statement referring to a function, may have the graph of only one function in mind.

Beyond overgeneralizing from few examples, students' previously constructed images associated with types of functions may also lead them to incorrect conclusions (Aspinwall, Shaw & Presmeg, 1997; Tall, 1990). Aspinwall, Shaw, and Presmeg (1997) describe a case in which a Calculus student's "uncontrollable mental imagery" associated with the graph influenced his reasoning about graphing a derivative of a quadratic function. In their study, the student was presented with a graph of what appeared to be a quadratic function and was asked to graph the function's derivative. The student's interpretation of the graph of the parabola was that it extends in such a way it becomes asymptotic. His interpretation that the end behavior was asymptotic influenced his conception of what the graph of the derivative of the function should look like. He graphed a function which appeared to be a cubic function to describe the changes in the slopes of the original graph. However, he was unable to reconcile this graph with his algebraic solution that the derivative of a quadratic function is linear. In this instance, the student's interpretation of the end behavior of a graph led him to a conclusion that was not consistent with the results of his algebraic process, a discrepancy he was unable to resolve.

Relative to students' imagery of continuous functions, Tall (1990) noted that students' concept image[1] of a continuous function may hold inconsistencies with formalizations of continuity. First year mathematics university students in England, when surveyed, struggled to determine whether specified functions were continuous, particularly the function $f(x)=1/x$ for $x \neq 0$ given with its graph. Because the graph has a break in it at $x=0$ and thus is not "in one piece," students labeled this function as discontinuous, though mathematically according to the formal definition of continuity, this function is continuous on its domain. Tall explained this inconsistency as the students' concept image for continuous function was evoked, rather than the concept definition. When working with graphs of continuous functions, students may interpret

[1] Tall (1990) uses concept image in the sense of Tall & Vinner (1981) to refer to the entire cognitive structure that is associated with a particular concept, including all images, examples, ideas and connotations that may be evoked when a student works with a particular concept.

them as discontinuous due to their previously constructed images of what a continuous function ought to look like. Both Tall's (1990) and Aspinwall's et al. (1997) findings highlight the impact of students' previously constructed images associated with types of functions. These visual images may lead students to incorrect conclusions, as shown in the cases that Tall (1990) and Aspinwall et al. (1997) describe.

Issues in Students' Interpretations of Components of Graphs

Several researchers have begun the work of characterizing students' interpretations of axes, outputs of functions on graphs, points on graphs and graphs as a whole (e.g., Moore & Thompson, 2015; Frank, 2017). Students may interpret these components of graphs and reason from them in a variety of unconventional ways. If students' interpretations of graphs are unconventional, instructional interventions using graphs designed to support them may be ineffective, or even detrimental. In order to better support students in interpreting graphs, researchers further investigate students' interpretations of various components of a graph of a function.

Studies have shown that students often interpret graphs as a whole in unconventional ways. For instance, Bell and Janvier (1981) found that when a graph was related to a real-life situation, such as a graph of distance and time for a car driving on a racetrack, some students confused the shape of the graph with the shape of the racetrack. In Goldenberg's (1988) study, when students were asked to graph the relationship between two quantities, students did not model the relationship between these two quantities by tracking one on each axis of a Cartesian plane. Instead, some students created "event charts" in which the horizontal axis was time and the vertical axis was a place to list events rather than values of these quantities. I refer to such an interpretation as a place holder for an event as a nominal interpretation, in which values are used as labels rather than to represent a numeric value or magnitude.

In addition to thinking of graphs as records of events or related to a diagram of a physical situation, Moore and Thompson (2015) found that students may consider a graph as a static shape with certain figural properties. For example, a student may think that the graph of a quadratic must always be a parabolic shape. A student with such a conception of graphs as shapes may view the same functional relationship depicted in two different coordinate systems as different functions. Moore and Thompson (ibid) refer to this as *static shape-thinking.* Moore (2016) further explains that in the minds of students engaged in static shape-thinking, properties of the shape of the graph dominate their reasoning. In contrast, Moore and Thompson (2015) describe other students who view graphs as emerging from traces generated by coordinating corresponding values of two quantities, which they refer to as *emergent shape-thinking*. Moore (2016) explains that in the case of emergent shape-thinking, students' reasoning is characterized by *operative thought*, as students' thinking in this case is not subject to the figurative aspects of the graph, but rather the operative aspect of covarying quantities. Thus, a student who views the same function relationship depicted in two different coordinate systems as two representations of the same function, and sees the graphs as containing all points that satisfy the given relationship, may be said to engage in *emergent shape-thinking*. These two characterizations of students' interpretations of graphs emphasize students' thinking about graphs holistically.

Building on such previous studies (Goldenberg, 1988; Moore & Thompson, 2015), Frank (2017) found a total of six different meanings for graphs as a whole from the 10 undergraduates she recruited for her study, many of which may be considered unconventional. Most students even showed different meanings for graphs with different tasks or at different times. In Frank's (2017) study, all six of these meanings emerged from tasks in which students were asked to either construct or interpret graphs that represented these changing quantities in a real-life situation. I see these meanings as consisting of three categories: (1) as images, either as (a) depicting the movement of an object or (b) a shape with particular features, i.e., static shape-thinking (Moore & Thompson, 2015), (2) as a record of variation, either (a) a record of values of a *single* quantity kept as something changes, or (b) as a record of the changes in two quantities changing together, i.e., emergent shape-thinking (Moore & Thompson, 2015) or (3) as comprised of points, either (a) containing several key points and showing changes between these points, or (b) containing infinite isolated points that show a pair of measurements. Of these characterizations of observed students' interpretations of a graph as a whole, only some may be considered conventional, and even then only in certain contexts.

Lee, Hardison, and Paoletti (2018) propose one possible cause of students' unconventional meanings for graphs is their failure to distinguish between two ways in which coordinate systems are used in mathematics, which they refer to as *situational* and *quantitative coordination*. Lee et al. (ibid) describe the use of a coordinate system to spatially locate objects in the context of a problem as *situational coordination.* One example they provide of the use of a situational coordinate system is that students use coordinates to locate various positions of a cart around a Ferris wheel. In contrast, a quantitative use of coordinate system involves representing sets of values of quantities with products of measures. Thus, the construction of a sine function to record the variation in vertical distance above the horizontal diameter of the Ferris wheel with the variation in the angle measure swept out would entail the use of *quantitative coordination*. Lee et al. (ibid) claim that one source of students' issues in interpreting graphs may be that students use situational coordination for tasks which are intended to be represented with quantitative coordination. In failing to explicitly distinguish between these two uses of coordinate systems, instructors and curriculum designers may not be supporting students in constructing or interpreting graphs in the ways they are intended (Lee et al., ibid).

A limited amount of research has been conducted to investigate students' interpretation of other components of graphs, such as students' meanings for inputs, outputs, points, and segments of a graph of a function. For instance, previous research has shown that students may or may not perceive points on graphs as comprised of an identifiable ordered pair of values depending on the way in which they construct the graph (Goldenberg,1988). When constructing the graph of two lines by plotting points from tables of ordered pair values, students readily responded when asked to find the intersection point. However, once students had been taught the slope-intercept technique for graphing lines (plotting the intercept and another point found by using the slope), they struggled to identify points of intersection, often commenting that they could only identify the points that they had graphed. Instead, students pointed or gestured to the location of intersection, rather than report coordinates of the point. This finding from Goldenberg (1988) also supports the notion that a student's physical actions in constructing graphs may influence the way in which the student interprets the graph he or she has made.

Few studies have explicitly examined students' interpretations of axes on graphs of functions. However, because the convention of the Cartesian coordinate system involves the use of two number lines, findings from researchers who have examined primary students' interpretations of single number lines may provide insight into students' interpretations of axes on graphs. Students may interpret and use number lines in various ways, which may be unconventional depending on the context. For instance, Diezmann and Lowrie (2006) found that some of the fifth-grade students they interviewed used a number line as a measurement model, in which the placement of numbers on the line indicate lengths or magnitudes associated with those values. These students assigned values to positions on a number line using a conception of distance from reference points (either zero or other labeled points). In contrast, other students used the number line as simply a counting model, in which numbers appear in particular order based on their cardinality. These students placed values on a number line only attending to the ordering of the values and not the relative distance of these positions from others. Diezmann and Lowrie (ibid) recognized the limitation of using a number line (or an axis) as solely a counting model in their study as well as beyond. On the other hand, they recognized the power of interpreting values on a number line (or an axis) as indicating magnitudes in their study and for concepts beyond elementary mathematics.

Liang, Stevens, Tasova, and Moore's (2018) findings uphold the idea that conceiving of magnitudes on a graphical representation may support student thinking about ideas related to Calculus. In a dynamic covariation task, which involved motion around a circle, they found that conceiving of the magnitude of varying quantities' values (height above the horizontal diameter and arc length swept out) supported a student in constructing a graph to relate these two varying quantities. This student represented changes in the quantities as horizontal or vertical segments on the graph, and made comparisons of these magnitudes relative to each other to describe how the two quantities changed together. The authors emphasize that although the student was referencing numerical values at times, his reasoning was characterized by thinking about magnitudes (amount) of a quantity, which he connected to horizontal and vertical segments on graphs. This magnitude reasoning, as Liang et al. (ibid) refer to it, allowed him to accurately describe varying rates of change of the two quantities.

Further studies have shown that students may interpret outputs of functions and points on graphs unconventionally. David, Roh, and Sellers (2019a) found that some students conceived of a function's output as the physical location of the points along the graph. In their study, they asked undergraduate students who had completed at least one semester of Calculus to evaluate statements related to the Intermediate Value Theorem. One of these statements was not the IVT, but contained a reversal of the order of the variables from the IVT as follows: "Suppose that f is a continuous function on $[a, b]$.

Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$." David et al. (ibid) found that some students incorrectly evaluated this false statement as true. The students' incorrect evaluation was due to their interpretation of outputs of the function and points on the graph. Because these students interpreted $f(a)$ and $f(b)$ to refer not to values on the y-axis but to points on the graph, the students considered all points on the graph in $[a, b]$ to be between $f(a)$ and $f(b)$. These students even considered points to be between $f(a)$ and $f(b)$ if the output values associated with these points were not between the values of $f(a)$ and $f(b)$. The students who conceived of outputs as locations of points were said to be engaged in *location-thinking*, while those who conceived of outputs as values on the y-axis and points as coordinate pairs of values were said to be engaged in *value-thinking*, as described in Chapter 2. In David et al.'s (ibid) study, a total of four of the nine students interviewed engaged in location-thinking. Not only did four of the nine students conceive of outputs and points in such an unconventional manner, their interpretation of the graphs impacted their understanding and evaluation of the Calculus statements in the interview. David et al.'s (ibid) findings suggest a way to characterize students' interpretations of outputs of functions and points on graphs, and graphs as a whole. Their findings also indicate that students' interpretations of graphs may affect their understanding and evaluation of statements from Calculus.

Studies have also shown that in creating their own graphs, students may struggle to plot points to represent a pair of values that vary. For instance, Frank (2016) interviewed several undergraduate students to investigate students' conception of points.

She found that all the students interviewed could plot a point in the Cartesian plane and, when given an ordered pair on a graph with axes labeled with quantities, could interpret the pair of values at a point. However, all but one of the students could not construct a point on his own when the values of two quantities were represented on the *x* and *y*-axes. Frank (ibid) highlighted this discrepancy between students' interpretations of points and their ability to construct them, and explained that such a discrepancy occurs when students do not conceive of a point on a graph as simultaneously holding two values. Saldahna and Thompson (1998) and Thompson and Carlson (2017) call such objects *multiplicative objects*. A multiplicative object refers to an object which simultaneously possesses two or more aspects in one. Conceptualizing a point as a multiplicative object involves attending to the values of both coordinates of the ordered pair at the same time, which are represented by a single object, the point. A student who conceptualizes a point in this way bears in mind that each point holds both values, even as he or she reasons about an *x*-value or a *y*-value at any given time (Saldahna & Thompson, 1998; Thompson & Carlson, 2017). Researchers have theorized that a student's conception of a point as a multiplicative object may support his or her understanding of graphs as representations of co-varying quantities (Frank, 2016; Saldahna & Thompson, 1998; Thompson & Carlson, 2017). On the other hand, Frank's (2016) study indicates that students' inability to conceive of points as such contributes to the difficulties they experience in completing graphing tasks.

Researchers have also begun to theorize why students may struggle to conceive of points as multiplicative objects. Both Frank (2016), and earlier Goldenberg (1988), hypothesized that teaching students the action of plotting points in the Cartesian plane using the "over *x* and up *y*" technique does not support students in understanding a point as a simultaneous representation of two quantities, i.e., as a multiplicative object. The "over *x* and up *y*" technique for graphing a point (*x*, *y*) refers to the physical action of starting at the origin, moving right (over) to *x* on the *x*-axis, and then from this location, moving directly up to be parallel with *y* on the *y*-axis, and then plotting a point at the final location. In fact, they suggest the procedure of "over and up" may even support students in conceptualizing the graph of a function as a mapping of a value from the *x*-axis to a location in the Cartesian plane (Frank, 2016; Goldenberg, 1988).

Taken together, these studies (David et al., 2019a; Frank, 2016) indicate that students' interpretations of aspects of graphs, especially their interpretation of points, may be unconventional. Furthermore, students' interpretations of points on graphs may impact their subsequent mathematical activity in Calculus, not only completing graphing tasks, but also evaluating statements from Calculus. Because of the demonstrated issues and impact of students' interpretations of graphs, further research may be conducted to support students in their graphical reasoning. Prior to this, more research is needed to fully characterize how students conceive of other aspects of graphs of functions, such as the input represented on a graph.

Summary of Reviewed Literature and Discussion

Instructional and curricular resources in Calculus, such as the interventions described in 3.1, often make use of graphs to support students' understanding. Research has documented the potential benefits of incorporating visual interventions into classrooms (e.g., Roh & Lee, 2017). However, these interventions presuppose that students' interpretations of graphs is aligned with conventional interpretations. Therefore, instructors cannot assume these graphs clearly illustrate the intended concept to students. The research described in this chapter suggests students may often attend to the figural aspects of graphs, including the shape of the graph, as well as the location of the points in space, which may or may not be relevant in the given context, leading to unconventional interpretations of graphs (David et al., 2019a; Frank, 2016; Moore & Thompson, 2015). While figural aspects of a graph do relate to aspects of the relationship represented by the graph, students may be led astray by drawing conclusions based solely on these properties, devoid of a robust meaning for what they represent. Overcoming reliance on these perceptual cues is challenging. Frank (2016) noted that even students who were able to track how relevant quantities changed together in a situation still tended toward static shape-thinking when working with graphs of the relationship of those quantities.

Researchers have yet to fully characterize which aspects of graphs students interpret, and how they interpret these aspects, such as inputs, outputs, portions of graphs, graphs as a whole. A detailed investigation of these interpretations may reveal previously unknown issues in students' understanding of graphs and the information they are intended to depict. While previous studies have examined students' interpretation or construction of graphs in quantitative contexts, this study further investigates students' interpretations of graphs of functions with statements from Calculus.

This dissertation study aims to build on the research outlined in this chapter. While the benefits of and potential issues with graphs have been documented, there is a need for further investigation. Building on the findings of previous studies, I seek to characterize which aspects of graphs students attend to, how they interpret these aspects of graphs, especially the inputs of a function and other aspects that have yet to be described in the literature. Furthermore, examining students' interpretations of graphs in different Calculus contexts may reveal the dependence or independence of these interpretations on context. Characterizing students' interpretations of aspects of graphs will offer valuable insights into student thinking.

Findings that emerge from this study may directly support the development of instructional interventions designed to directly address the meanings I find. For instance, David et al.'s (2019a) findings laid the ground work for David, Roh, and Sellers's (2019b) development of an instructional intervention to support students in interpreting graphs in the context of the Intermediate Value Theorem. This intervention uses a comparison of two examples of student reasoning, one characterized by value-thinking and the other by location-thinking. The tasks in the intervention ask the student to compare these two ways of thinking about a graph and consider the differences between them. This intervention was a direct result of the findings from previous work highlighting location-thinking and value-thinking (David et al., 2019a). Furthermore, evidence from this study may further support the notion that students may benefit from direct instruction of interpreting graphs, rather than teaching concepts using graphs. The idea of teaching the registers of representations of concepts in mathematics as distinct from the concept itself, is novel, but may promote reform efforts in the area of Calculus and related courses that use graphs of functions.

CHAPTER 4
METHODOLOGY

In this section, I describe the methodological design that I employed to investigate the stated research questions for this study. In addition to explaining the methodological choices I have made, I also explain how the research questions, my chosen theoretical perspective, and previous findings have informed my decisions.

The goals of this study are: (1) to characterize students' interpretations of variables and expressions (after, both referred to as expressions) from statements in Calculus on graphs of real-valued functions and to investigate (2) its association with the expressions from these Calculus statements, (3) its relationship to their evaluations of these Calculus statements, and (4) its relationship to students' interpretations of points on graphs. In order to investigate these topics, I conducted tasked-based clinical interviews (Clement, 2000) with undergraduate students who have previously taken Calculus. I selected this population because students who have taken Calculus will likely be familiar with the concepts related to and the notations associated with functions that I used in the tasks in this study. Clinical interviews provided insights into individual students' thinking and evidence from which I modeled each student's interpretation of expressions on graphs in the contexts I have chosen. I will describe the details of the participant selection, which included a screening survey, the clinical interviews themselves, including the tasks and the questions I posed to students while they worked through them, as well as the rationale for my choices. Then, I will describe how I analyzed the data that I collected in order to address the research questions of this study.

Table 2, shows a schedule of research activity outlining when I conducted and analyzed the clinical interviews preceding this study and for the current study.

Table 2

Research Schedule

Timeline	Activity	Method
Spring 2017	Pilot Study I (2 students, 1 interview each)	Clinical Interviews, Content Logs
Spring 2018	Pilot Study II (3 students, 2 interviews each)	Clinical Interviews, Content Logs
Fall 2018	Recruit Participants from MAT300+ courses at ASU	Screening Survey & Theoretical Sampling
September-November 2018	Data Collection, 13 participants Initial Interview Analysis Refine Protocol	Clinical Interviews (~2 Interviews / Week), Content Log
November 2018-June 2019	Data Analysis and Writing Results	Transcription, Coding using framework, Open Coding & Axial Coding

Pilot Study I was conducted in Spring 2017 to investigate how students interpreted graphs relative to the definition of continuity at a point. For this first pilot study, I designed tasks related to this definition with various graphs. Using these tasks, I conducted clinical interviews with two undergraduate students asking them to interpret expressions from this definition on the graph and to evaluate the conclusion of this definition[2] for various functions presented as graphs. Pilot Study II was conducted in Spring 2018 with the goals of investigating students' interpretation of the graphs related to these statements, as well as determining which tasks and questions best elicit students' thinking about graphs. For Pilot Study II, I interviewed three undergraduate students

[2] The statement I used for Pilot Study I was: "For each real number $\varepsilon > 0$, there exists a real number $\delta>0$ such that, for all x in the domain of f with $|x-1|<\delta$, $f(x)-f(1)|< \varepsilon$."

using statements related to several Calculus concepts along with a variety of graphs. Both phases of pilot studies served to refine the interview tasks and protocol used in this study. For the current study, I recruited students to participate in the clinical interviews in Fall 2018, visiting upper-level math courses taught at Arizona State University. I used a screening survey (Lavrakas, 2008) to select eligible students for interviews who may represent a variety of interpretations of graphs. The goal of selecting students in this way, what Corbin and Strauss (2014) refer to as theoretical sampling, was to maximize the degree of variability in student interpretations of graphs in order to develop highly-detailed theory. Over the course of several months during Fall 2018, I conducted the clinical interviews and began preliminary data analysis both during and immediately following each interview. The data analysis following each interview also informed the subsequent interview participants I selected. This portion of research activity included a significant portion of data analysis, using David et al.'s (2019a) framework of value-thinking and location-thinking, as well as developing new categories to explain phenomena found in the data, to refine the current theory, which evolved throughout the data analysis process. Finally, I conducted later phases of data analysis and wrote the findings of this study throughout Spring and Summer 2019.

Grounded Theory and its Use in This Study

For this study, data collection and data analysis were conducted in the spirit of grounded theory (Corbin & Strauss, 2014). Grounded theory, developed by Glaser and Strauss (1967), describes a qualitative research methodology in which the theory to be used, both existing and developed in the process, is grounded in empirical data (Corbin &

Strauss, 2014). Unlike research that is designed to apply or verify existing theories, the purpose of grounded theory is to develop theory that is rooted, or “grounded” in data. Using the techniques of grounded theory, researchers can develop theories that explain particular phenomena and formulate hypotheses that can then be tested using quantitative means. As one of the main purposes of this study is to develop theory to explain how students interpret aspects of graphs when reasoning about statements from Calculus, I have chosen to use the methodology of grounded theory. To address the research questions of this study, in light of findings from previous research and the existing theoretical framework described in Chapter 2, I employed a method of theoretical sampling, specifically, from the tradition of grounded theory.

Theoretical Sampling

Theoretical sampling refers to a method of data collection from the tradition of grounded theory. Specifically, theoretical sampling is data driven (Corbin & Strauss, 2014). Unlike random sampling, which attempts to collect data from a group that is representative of a certain population, theoretical sampling seeks to “collect data from places, people, and events that will maximize opportunities to develop concepts in terms of their properties and dimensions, uncover variations, and identify relationships between concepts” (Corbin & Strauss, ibid, p. 134). In theoretical sampling, analysis takes place once data has begun to be collected, and each round of analysis informs subsequent data collection (Corbin & Strauss, ibid). The goal of conducting theoretical sampling as a sampling method is to develop and refine theory, which is why I employed it in this study.

In order for theoretical sampling to be an effective and reliable method of qualitative research, Emmel (2013), summarizing Glaser and Strauss (1967), describes three of its necessary components. The first is the influence of emerging theory on data collection. As theory develops, it guides the researcher in collecting the next sample. Thus, the collection of each sample is guided by theoretical considerations which emerge from analyzing previous samples (Figure 3). In fact, each round of data collection cannot continue until previous data has been initially analyzed. The second component of effective theoretical sampling is "an open and theoretically-sensitive researcher" (Emmel, 2013, p. 4). Because theory is emerging in the process of theoretical sampling, rather than pre-existing, the quality of samples collected is determined by the extent to which the researcher is open to new phenomena not previously accounted for. The observations made and codes developed by the researcher guide the process of theory development. The third component of theoretical sampling is that of constant comparison. In theoretical sampling, each event is compared to the others as they occur, in order to establish similarities and differences. These similarities and differences form the basis of categories which become theory. After these initial comparisons are made, events are compared against this theory in an effort to refine it. Throughout the stages of theoretical sampling, from data collection through data analysis, the continuously emerging theory is always the guiding consideration.

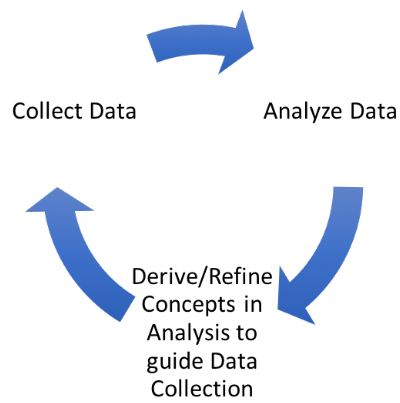


Figure 3. Cyclic nature of relationship between data collection and analysis in theoretical sampling. Adapted from Corbin and Strauss (2014).

The purpose of employing theoretical sampling in this study was to continue to build on evolving theory that characterizes students' interpretations of graphs related to concepts in Calculus. My intention with the use of theoretical sampling in this study was to allow me, as a researcher, to collect data that provided optimal opportunities for developing theory, while acknowledging the pre-existing theory guiding my design of the current study. As one of the goals of this study was to expand theory characterizing students' interpretations of expressions on graphs (see Chapter 2), I intended to collect data that represents a wide spectrum of possible student interpretations. In order to perform theoretical sampling and optimize the data collection process to uncover variability in students' interpretations, I screened potential participants and selected students with varied responses, which will be described later in this chapter.

Theoretical sampling not only guided the selection of students and the design of the tasks and questions, but also characterized the process of data collection during each interview, which was based on the findings of previous interviews, described in the next section. While I, as a researcher, sought to be sensitive to the nuances in student thinking brought to light by David et al.'s (2019a) framework, I also sought to remain open to other ways of thinking I had yet to read about or observe. In conducting this study, I acknowledged that findings from an interview may reveal a different variation on an interpretation of graphs not previously accounted for.

Recruitment, Screening, and Selection of Participants

In this section, I describe my methods for recruitment, screening, and selecting participants for this study, which were guided by theoretical sampling.

Recruitment of Participants

For this study, I recruited and surveyed 82 undergraduate students from Arizona State University (ASU) who were either enrolled in mathematics courses which have Calculus I as a pre-requisite course during Spring or Fall 2018. Out of these students, I selected 13 to interview for this study. Because this study involved investigating how students perceive graphs with Calculus statements, I purposely sought to select students with some degree of familiarity with the symbols and concepts referenced in the tasks. Specifically, I sought participants who had a familiarity with function notation (e.g., $f(a)$), prime notation for derivatives (e.g., $f'(c)$), and statements referencing functions that may include quantifier phrases ("for all…," or "there exists"). The statements that I have chosen to use as tasks in this study are either included in or related to traditional Calculus I content at ASU, so participants from a population of students who have successfully completed Calculus I will most likely have been exposed to similar content previously.

In order to recruit undergraduate mathematics students who had successfully completed Calculus I or later mathematics courses, I asked several ASU professors teaching MAT courses with Calculus I as a pre-requisite for permission to visit their classrooms to briefly describe the study in Fall 2018. During my classroom visit, I also asked that students complete a short screening survey, on which they indicated their interest in participating and their availability. In addition to my classroom visits to recruit students, I asked professors whose classes I was unable to attend, as well as the mathematics department, to send a recruitment email to their students describing the study and inviting them to participate. Interested students took the screening survey at designated times that I provided in exchange for entrance to a raffle for a gift card.

Screening Survey

In order to select participants who would be likely to give me a variety of responses, I used the screening method of a survey. Screening methods allow researchers to find and select participants who meet a list of eligibility requirements for participation in a research study (Lavrakas, 2008). Screening techniques are used when a select subsection of a population is to be studied. Often, screening survey items ask for demographical information to identify whether individuals are eligible or ineligible for participation in a given study. As recommended by Lavrakas (ibid), I designed the screening survey in this study to be concise and in such a way that respondents would not know the eligibility criterion for participation in the study, which could influence their responses. Of the 82 students surveyed who expressed interest in participating, 75 of them completed each of the survey tasks as directed. From these 75 students, interview participants were selected based on their responses to this screening survey, as well as the on-going data analysis, as described below.

The survey used to screen participants, appended to this document (see Appendix A), consisted of three items: (1) an availability and mathematical background item, (2) a function item, and (3) a graph labeling item. Students' responses to each of the items on the screening survey was used to inform my selection of the participants from this group. For the purposes of this study, I intended to interview students who had a meaning for function that includes the notion that each input is assigned to a single output. I also intended to collect a variety of responses to the graph labeling item and interview students whose thinking represented this variety, in line with the goals of theoretical sampling. I also planned to select students who had completed varying amounts of undergraduate mathematics coursework beyond Calculus, with a goal of increasing the variety of responses. I will explain each survey item, its purpose relative to my sampling goals, and the way it was used to select participants.

Screening survey function item.

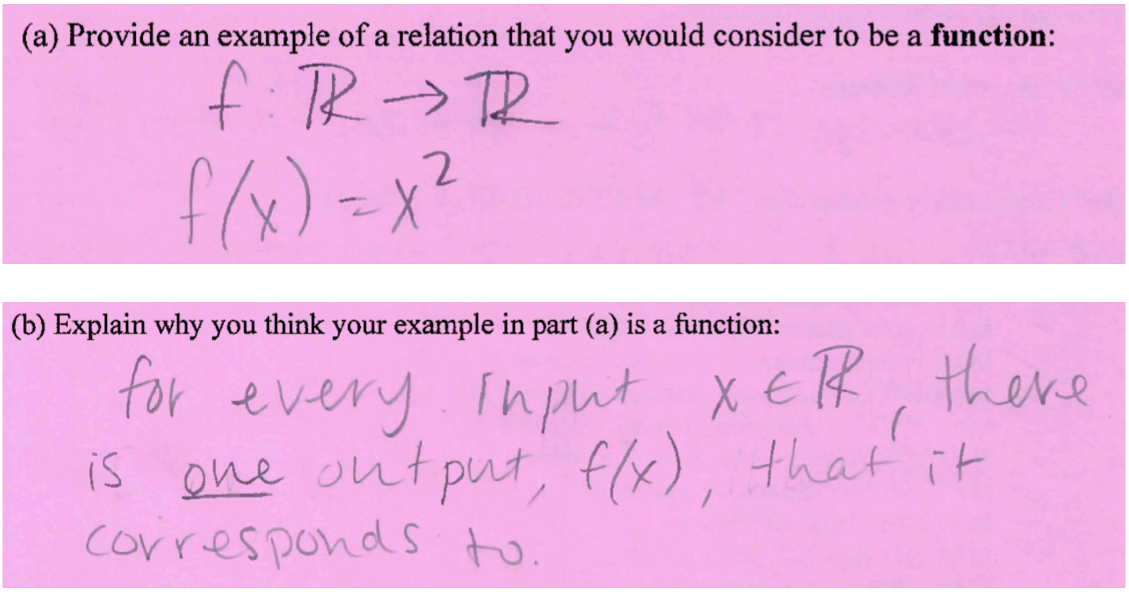
The function item asked students to provide an example of relations that are and that are not functions, and to explain their examples as shown below in Figure 4.

2. Please respond to all four parts of this item:
(a) Provide an example of a relation that you would consider to be a **function**.
(b) Explain why you think your example in part (a) is a function.
(c) Provide an example of a relation that you would consider to **NOT** be a **function**.

Figure 4. Screening Survey Function Item.

Students' responses to this item were analyzed with two goals in mind: to gauge a student's familiarity with the concept of function, as well as to indicate how well a student could describe his or her thinking. My goal was to interview students whose definition of function included a notion of an input-output relationship in which each input is assigned to exactly one output, and those who I determine explain their example clearly. Because all statements used in the clinical interview refer to real-valued functions and utilize function notation, I wanted to interview students who had a demonstrated familiarity with the concept and notation used for real-valued functions. Further, I did not want to select students who had unconventional meanings for function that might become confounded with their interpretation of the graphs and/or the statements in the interview. Thus, my goal with this item was to identify students who demonstrated a conventional meaning for functions and use of function notation, as well as students who could articulate their thinking and may be able to do so well in a clinical interview setting.

Indeed, the first nine students whom I selected to interview through the survey described the notion of a unique output for each input in their response to this item. Eight of these students directly described this in their response to parts (b) and (d) of this item, while one explained the vertical line test and gave correct examples of relations that were and were not functions. In some cases, I made judgement calls about students' intentions with their response. For example, in the response shown in Figure 5, I made the assumption that the student meant a unique output when she wrote "<u>one</u>."



(a) Provide an example of a relation that you would consider to be a **function**:

$f : \mathbb{R} \to \mathbb{R}$

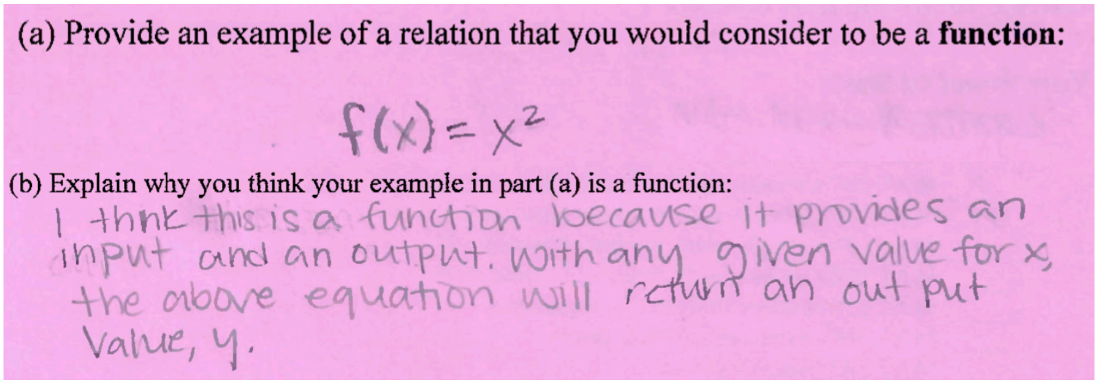
$f(x) = x^2$

(b) Explain why you think your example in part (a) is a function:

for every input $x \in \mathbb{R}$, there is <u>one</u> output, $f(x)$, that it corresponds to.

Figure 5. An example of a student response to the function item that was screened "in."

As it became more difficult to select and schedule students interested in participating and who met the other selection criteria, I broadened my screening with this item to include students who described an input –output relationship and used function notation in their examples, although they may not have clearly emphasized the uniqueness of the output. Thus, four students whom I selected near the end of the interview process described functions as an input and output relationship, and conventionally used function notation in their responses. For example, the following response in Figure 6 was considered acceptable for the purposes of the screening, as it showed a conventional use of function notation, as well as an explanation of an input-output relationship.



(a) Provide an example of a relation that you would consider to be a **function**:

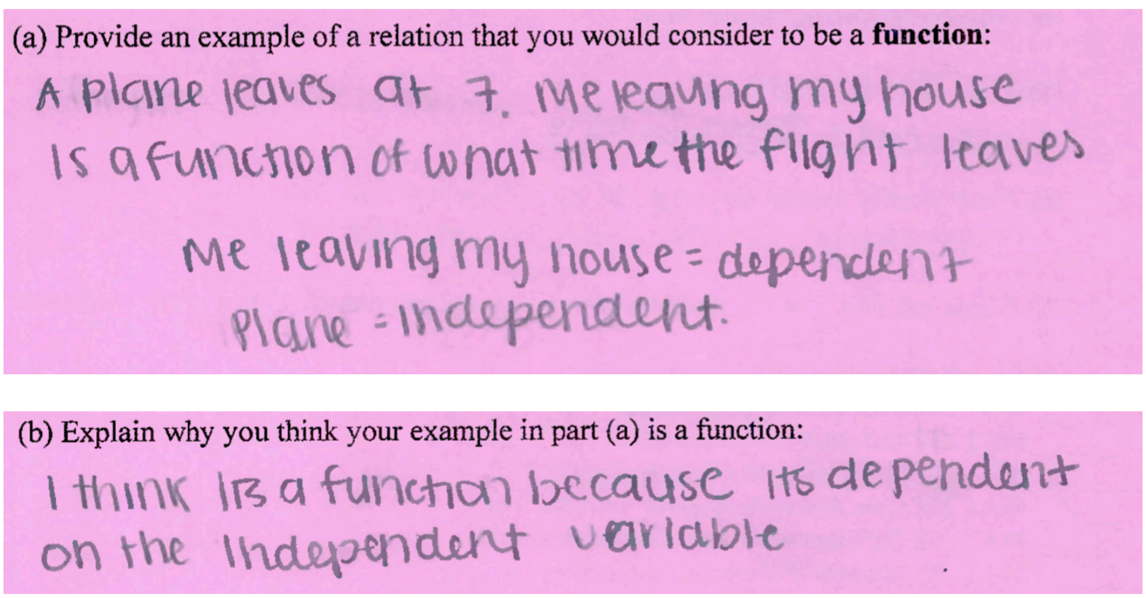
$f(x) = x^2$

(b) Explain why you think your example in part (a) is a function:

I think this is a function because it provides an input and an output. With any given value for x, the above equation will return an out put value, y.

Figure 6. An example of a student response that was considered screened "in" on the second pass.

For reference, Figure 7 below shows an example of a response to this item that was screened out, as it did not include a description of inputs mapping to unique outputs nor use function notation.



(a) Provide an example of a relation that you would consider to be a **function**:

A plane leaves at 7. Me leaving my house is a function of what time the flight leaves

Me leaving my house = dependent
Plane = independent.

(b) Explain why you think your example in part (a) is a function:

I think its a function because its dependent on the independent variable

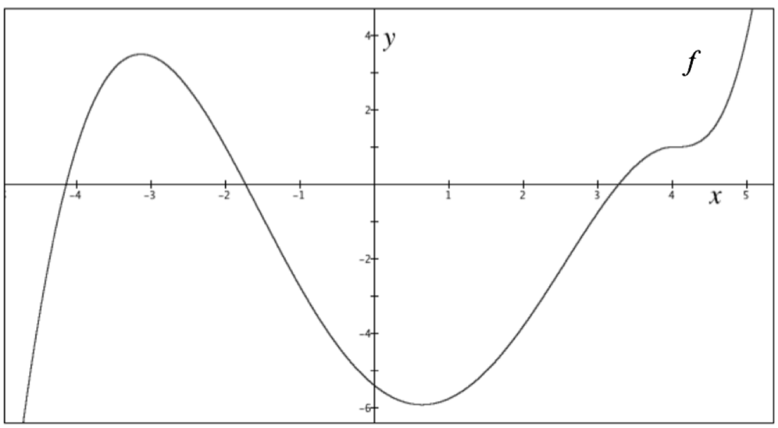
Figure 7. An example of a response to the function item that was screened "out."

Screening survey statement & graph labeling item.

The Statement & Graph Labeling Item asks students to read a statement that used function notation to describe output values and to place a list of values on a provided graph, as shown in Figure 8. This item also asks students to evaluate the statement as referring to the graph provided and to justify their evaluation.

3. For all parts of this question, **suppose *a*= –3 and *b* = 3.** Please respond to all three parts of this item:

   (a) Suppose *c* is a real number in the interval (–3, 3) and *N* is a real number between *f*(*a*) and *f*(*b*). Place the following labels on the graph of the function *f* below: *a*, *b*, *c*, *f*(*a*), *f*(*b*), *f*(*c*), and *N*. Use the table to make sure you labeled everything on the graph.



a	
b	
c	
f(*a*)	
f(*b*)	
f(*c*)	
N	

   (b) Is the following statement true or false for the function *f* shown in the graph above?

> For all real numbers *c* in (*a, b*), there exists a real number *N* between *f*(*a*) and *f*(*b*), such that *f*(*c*)=*N*.

Check one:

_____ True

_____ False

_____ Not enough information

   (c) Explain why you answered “True,” “False,” or “Not enough information” in part (b).

Figure 8. Screening Survey Statement & Graph Labeling Item.

The purpose of this item was to categorize students in terms of David et al.’s (2019a) framework, as well as to identify students who may be interpreting aspects of the graph in other ways. Once I began analyzing students’ responses to this item, I noticed a great deal of variety in the placement of labels on the graphs, the reasoning the students used in justifying their evaluation, and the combination of the two. Thus, I decided to separate these two criteria and categorized students' responses to this item according (1) the placement of their labels on the graph and (2) their justification their chosen evaluation separately. I created three categories to sort students' placement of graph labels: *Values*, *Locations*, and *Mix*. Students' graph labels were categorized as *Values* if each of the labels $f(a)$, $f(b)$, $f(c)$ and N were placed on the y-axis and if each of the labels a, b, and c were placed on the x-axis. Students' graph labels were categorized as *Locations* if each of the labels $f(a)$, $f(b)$, $f(c)$ and N were placed on points on the graph and if each of the labels a, b, and c were placed on the x-axis. Students' graph labels were categorized as *Mix* if the placement of the labels did not fall under the *Values* or *Locations* category. For example, one student whose label placement was categorized as *Mix* labeled N on the y-axis while labeling $f(a)$, $f(b)$, and $f(c)$ at points on the graph. Another student whose response I categorized as *Mix* only labeled a, b, c, and N, placing each of them at points along the graph.

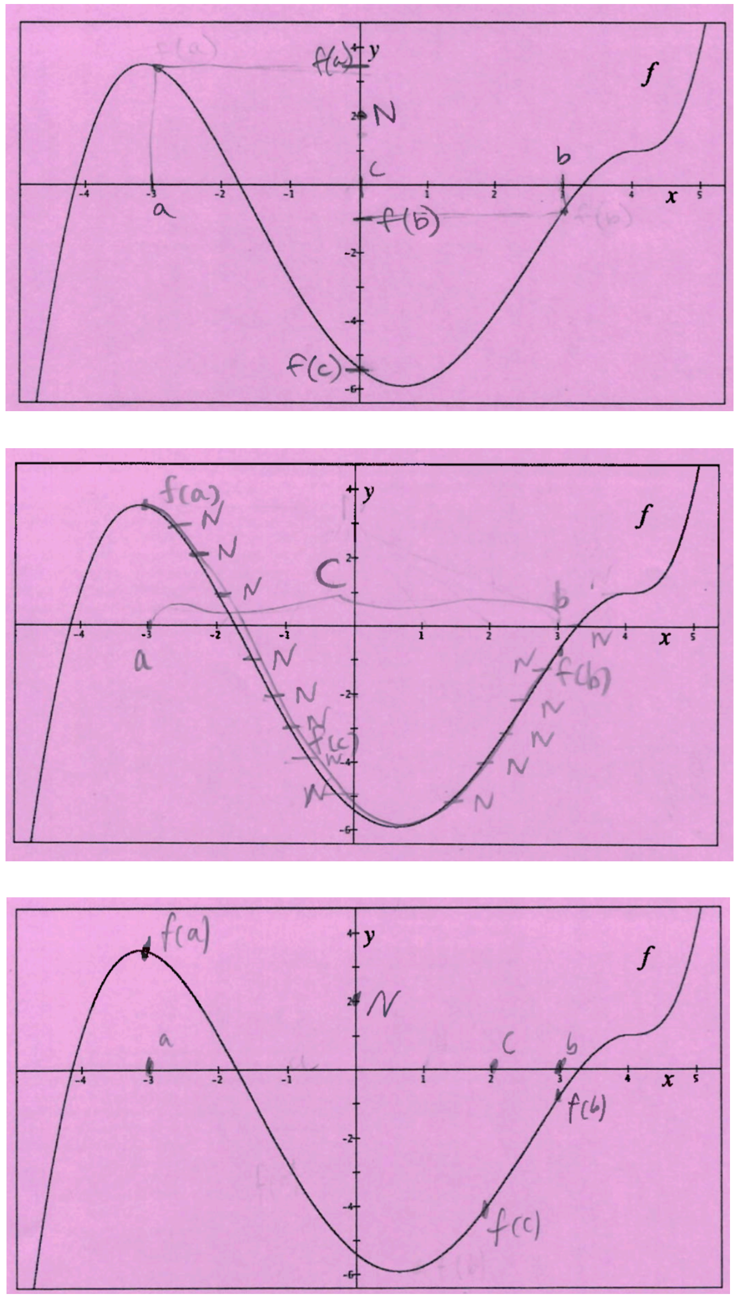


Figure 9. Student responses categorized as *Values, Locations,* and *Mix*, respectively.

I then used the categories described in the theoretical framework to model students' thinking in parts (b) and (c) of the item. Students' evaluations and justifications were categorized as indicative of: *Value-thinking*, *Location-thinking*, or *Other*. These categories were based on both the students' evaluation and how the student used $f(a)$, $f(b)$, $f(c)$ and N in their justification. Students who claimed the statement was false by finding a value of c for which the value of N and/or $f(c)$ was not between values they identified as $f(a)$ and $f(b)$ were classified as *Value-thinking* for this item. Students who claimed the statement was true by claiming all N's or $f(c)$'s were between the locations they identified as $f(a)$ and $f(b)$ were classified as *Location-thinking* for this item. Students who evaluated the statement as either true, false, or not enough information, and justified their evaluation of the statement in another way, were classified as *Other* for this item.

Based on this updated analysis, I sought to select at least five students whose responses were categorized as *Value-thinking* with a variety of label placement (*Values*, *Locations*, and *Mix*), at least five students whose responses were categorized as *Location-thinking* with a variety of label placement (*Values*, *Locations*, and *Mix*), and at least two students whose responses were categorized as both *Mix* and *Other* in order to maximize the likelihood of obtaining a variety of student responses during the interviews. Below, I summarize the students I selected and interviewed along with and the relevant categories of their survey responses.

Summary of Participant Selection

In summary, 13 students were selected to participate in this study based on their written description of their meaning for function, their graph labels, and their mathematical background. First, students were selected with responses to the function item categorized as: "each input is assigned to a single output" or "relationship between x

and $f(x)$." Preference was given to students with the first type of response, but I allowed students with the second type of response due to recruitment constraints and satisfying the other selection criteria.

Second, based on students' responses to the survey's statement and graph labeling item, I intended to select at least five *Value-thinking* students with a variety of label placement, at least five *Location-thinking* students with a variety of label placement, and at least two students whose responses were categorized as both *Mix* label placement and *Other* justification. However, during the first interview I conducted, I noticed that the student's response to Task 1 of the clinical interview differed from his response on the screening survey, even though this task was nearly identical to the statement and graphing item. Thus, I decided to use a student's response to Task 1 in the clinical interview to categorize his or her thinking when considering the variation in students' graphical interpretations, as this was more relevant to the student's thinking throughout the interview. As I selected participants, I sought to meet the quotas I had set described above (at least 5 students likely to engage in VT, at least 5 students likely to engage in LT, and at least two categorized as Other), and updated which category the student's thinking fit for this count following that student's interview.

To describe this selection process, I present an example of how a student, Micah, was categorized. On the survey, Micah placed graph labels at a mix of values and locations. He evaluated the statement as false and justified his evaluation by claiming $f(c)$ may not be defined, which I classified as *Other*. However, during the clinical interview, Micah evaluated the statement as false, citing an N value that was not between $f(a)$ and *f*(*b*) values. He also placed graph labels at *Locations*. Even though his survey response was initially categorized as *Mix* and *Other*, his response during Task 1 of the interview was categorized as *Locations* labeling and *Value-thinking*. Thus, Micah was included in the count of *Value-thinking* students whom I sought to interview. Table 3 shows the categorization of each participant's survey responses as well as the categorization of their thinking based on their clinical interview Task 1 response. If the student changed his or her interpretation during the interview with Task 1, I used the student's final response to categorize his or her thinking. In the final column of Table 3, VT and LT stand for *Value-thinking* and *Location-thinking* respectively, and are followed by L, V, or M which stand for *Locations, Values,* or *Mix* for their graph labeling placement. Only graph label placement which is distinct from value-thinking and location-thinking are denoted, so VT alone indicates *Values* label placement.

Finally, students were selected based on their mathematical background, with a goal to select students with a variety of backgrounds. By selecting students with varied mathematical backgrounds, who likely had worked with graphs in other courses to varying degrees, I intended to increase the variation in responses. I also intentionally selected two students who had taken Advanced Calculus because these students may have studied Calculus concepts in greater detail. These students were also selected with a goal of providing richer data through comparison to responses from students who have not taken this course. In summary, I selected four students enrolled in Calculus II (Calc II), one student who had recently completed Calculus II but who was not currently enrolled in a mathematics course, two students enrolled in Calculus III (Calc III), one student enrolled in Differential Equations (Diff EQ), three students enrolled in an Introduction to Proof course (ITP), and two students enrolled in a senior-level Introduction to Topology course, each of whom had completed a course entitled, Advanced Calculus (Adv. Calc), and was also enrolled in at least other senior-level mathematics course. Table 3 shows each student's self-reported highest math course(s) taken and the math course(s) in which he or she was currently enrolled.

Table 3

Summary of Students Selected in the Order Interviewed in this Study

Date Inter-viewed	Name	Mathematical Background		Survey				Interview
		Highest Course(s) Completed	*Current Course(s)*	*Function Meaning*	*Graph Labels*	*Eval.*	*Justification*	*First Task Category*
9/26/18	John	Calc I	Calc II	Each input unique output	Mix	F	VT	VT-L
10/3/18	Adam	Calc III	ITP	Each input unique output	Locations	T	LT	LT-V
10/10/18	Micah	Adv. Calc	Topology	Each input unique output	Mix	F	Other	VT-L
10/18/18	Jess	Adv. Calc	Topology	Each input unique output	Values	F	VT	VT
10/22/18	Carl	Calc II	ITP	Each input unique output	Mix	T	Other	VT-L
10/24/18	Tina	Calc II	ITP	Each input unique output	Locations	T	LT	LT
10/26/18	Jeremy	Calc II	Calc III	Each input unique output	Mix	T	LT	VT-L
10/31/18	Tim	Calculus in HS	Calc II	Relationship between x and $f(x)$	Locations	F[3]	LT	VT-L
10/31/18	Annie	Calc II	Calc III	Each input unique output	Locations	T	LT	LT
11/7/18	Abe	Calculus in HS	Calc II	Each input unique output	Mix	T	Other	VT-L
11/9/18	Martha	Calculus in HS	Calc II	Relationship between x and $f(x)$	Mix	T	Other	VT-M
11/12/18	Lola	Calc II	none	Relationship between x and $f(x)$	Mix	F	VT	VT-L
11/12/18	Kate	Calc I	Diff EQ	Relationship between x and $f(x)$	Locations	T	LT	LT

I emphasize that I did not select all 13 students to interview prior to conducting the first interview. Rather, I continuously returned to the survey responses and selected two to three students at a time, based on the categorization of the students in the previous interviews and the criteria described above. Table 3 contains a summary of each student selected in the order interviewed in terms of: their mathematical background, their function meaning, graph label placement, evaluation, and justification on the survey, and the updated categorization of their thinking after Task 1 in the clinical interview. Shading in the evaluation column indicates an incorrect evaluation.

Clinical Interviews

Through the screening process described above, I selected 13 participants throughout the interview process in Fall 2018. During this timeframe, I conducted clinical interviews (Clement, 2000; Hunting 1997) with the students I continued to select in order to address the research questions for this study (see Chapter 1). Each clinical interview lasted between 100-180 minutes with a short break if the student wanted one. Students who participated in the clinical interviews were compensated monetarily for their time. I will describe the method and purpose of clinical interviews, their relation to a constructivist theoretical stance, and their use in this study.

In a clinical interview, researchers use thought-eliciting tasks and interview questions to closely investigate an individual student's knowledge structure and reasoning. Typically, these tasks and questions are previously determined and described in an interview protocol. Additionally, clinical interviews provide the researcher the possibility of asking follow-up questions for clarification or further investigation (Clement, 2000; Hunting, 1997). Unlike surveys or testing, clinical interviews allow the researcher to more fully delve into a student's thinking by capturing evidence of a student's thinking in the moment that otherwise might not be witnessed. In other words, due to the nature of the interaction between the researcher and the student in the clinical interview setting, a student's verbal and written responses may provide stronger evidence of a certain way of reasoning than a survey response alone would. However, the researcher must be cautious in his or her interaction with the students; the goal of conducting clinical interviews is to explore a student's thought processes with regard to a certain domain, rather than to attempt to move the students' thinking forward in any way.

The use of a clinical interview presupposes that an individual student's thinking may not be clearly evidenced in a classroom setting or on a test (Clement, 2000). Furthermore, researchers who use clinical interviews acknowledge, either implicitly or explicitly, that an individual's way(s) of reasoning may be unique to that person and may differ from conventional practices taught in classrooms. Students' ways of reasoning are valuable objects of study, as students in classrooms may use these ways of thinking in their approach to problems and may influence the way they interpret the mathematics presented to them. This underlying principle, that an individual's reasoning may be unique, unconventional, and worthy of study, is consistent with the principles of constructivism described in Chapter 2. Through conducting and analyzing clinical interviews, a researcher may construct a model of a student's thinking. From a constructivist perspective, the researcher's analysis of a student's thinking in a clinical interview is at best an explanatory model of the student's behavior.

In this study, the intention of the clinical interviews I conducted was to develop a detailed model of various students' interpretation of aspects of graphs related to contexts from Calculus. Conducting clinical interviews allowed me as the researcher to investigate undergraduate students' ways of thinking relative to graphs in a detailed and focused manner. The tasks and questions I have developed and piloted for these interviews are intended to elicit a student's interpretation of aspects of graphs by asking him or her to use graphs to reason about statements from Calculus contexts. In the following section, I will detail the interview tasks used in this stdy and the ways in which they were intended to contribute to my goal of developing a model that characterizes students' interpretation of graphs with statements from Calculus.

Interview Tasks & Protocol

The tasks used in the clinical interview were designed to provide insight into students' interpretations of graphs with statements from Calculus. The design of these tasks was also informed by my initial theoretical framework, findings from previous studies (David et al., 2019a), as well as findings from my pilot studies (David, 2018).

I selected six topics from Calculus I material that are found in most traditional undergraduate Calculus curricula (e.g., Finney et al., 1994; Stewart, 2012). These topics are: (1) the Intermediate Value Theorem, (2) the definition of a strictly increasing function, (3) the definition of an injective function, (4) the definition of continuity at a point, (5) the difference quotient, and (6) the Mean Value Theorem. Within each of these topics, I designed a mathematical statement about real-valued functions associated with that topic. Each statement may be true or false, depending on the function to which it is referring. For instance, the first statement is not the IVT, but is related to the conclusion of the theorem. This statement is false for some functions, due to the reversal of the order of the variables compared to the IVT. The following table, Table 4, shows the statements presented to the student in order, including the context the statement is associated with, for the reader.

Table 4

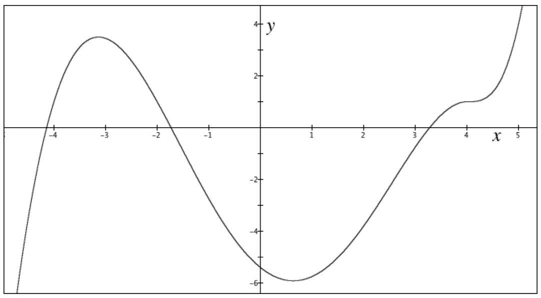
Statements Provided to Students in Interview

	Statement	**Related Topic**
1	For all real numbers *c* in (*a, b*), there exists a real number *N* between *f*(*a*) and *f*(*b*), such that *f*(*c*)=*N*.	IVT
2	For all real numbers *c, d* in (*a, b*), if *c* < *d,* then *f*(*c*) < *f*(*d*).	Strictly Increasing Function
3	For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*.	Injective Function
4	For all real numbers ε > 0, there exists a real number δ > 0 such that, for all *x* in the domain of *f* with –δ< *x*–1 <δ, – ε <*f*(*x*)–*f*(1) < ε.	Continuity at a Point
5	For all non-zero real numbers *h,* if 2+*h* is in (*a*, *b*), then $\frac{f(2+h)-f(2)}{h} > 0$.	Difference Quotient
6	There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.	MVT

I also selected eight graphs of real-valued functions in the Cartesian Coordinate System representing a variety of functions to accompany these statements. The graphs that accompany these six statements, shown in Figure 10, include functions for which a given statement is true, as well as those for which it is false. These eight graphs are: a polynomial with extrema beyond the endpoints of the displayed function (Graph 1), a constant function (Graph 2), a strictly decreasing function (Graph 3), a strictly increasing function (Graph 4), a function that is pointwise discontinuous on the interval shown and increasing (Graph 5), a function that has an asymptote on the interval shown (Graph 6) a function that is pointwise discontinuous on the interval shown and piecewise constant (Graph 7), and a piecewise continuous graph that is not differentiable at a point on the interval shown (Graph 8). Graphs 1–4 were chosen to represent four cases of continuous functions. Graphs 5–7 represent three cases of functions that are discontinuous on the interval shown. Graph 8 represents a continuous function that is not differentiable at *x*=1.

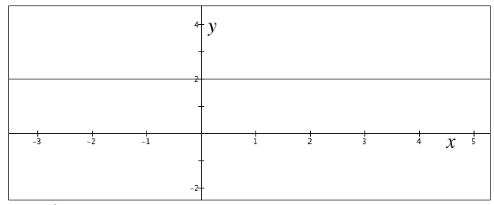
Because some statements refer to interval (a, b), values of a and b are provided on each graph for the student's reference.

Graph #1:



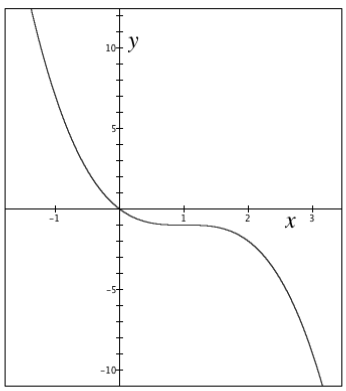
a = –3
b = 4

Graph #2:



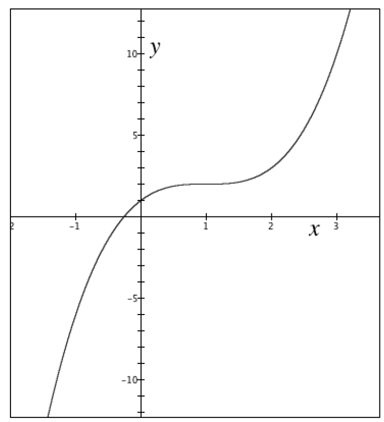
a = –3
b = 4

Graph #3:



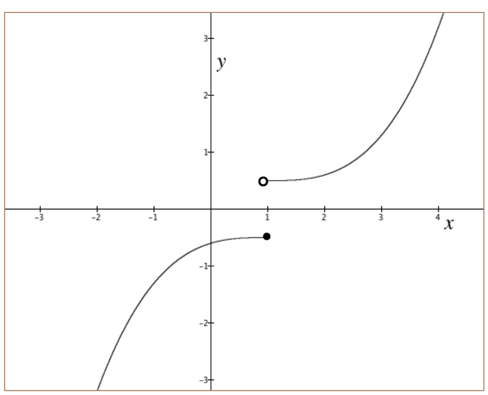
a = –1
b = 2

Graph #4:



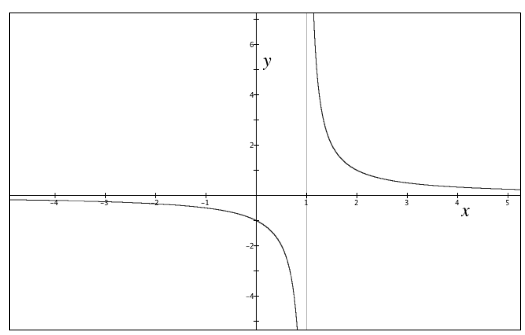
a = –1
b = 2

Graph #5:

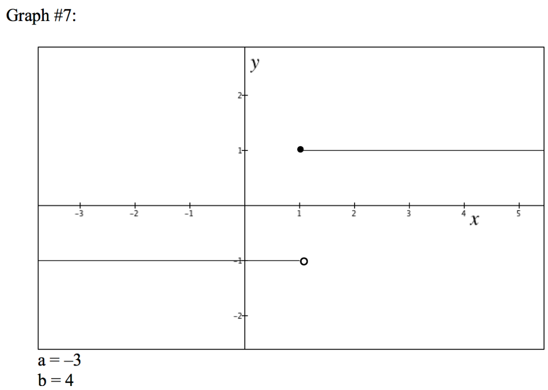


a = –3
b = 4

Graph #6:



a = –3
b = 4



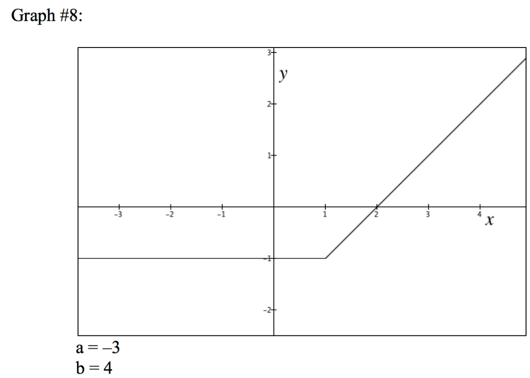


Figure 10. Graphs of functions provided to students in interview to accompany statements.

Statements 1–6 may be true or false, depending on the function to which the statement is referring. Table 5 shows the truth value for each statement/graph combination which will be presented to students in the interview when interpreted conventionally. Each statement and graph pairing was selected intentionally to elicit a student's interpretation of various aspects of graphs. Additionally, for each statement, at least one graph was selected that may be an instance of that statement being true and at least one graph was selected as an instance of the statement being false. Furthermore, the statements and graphs were chosen with hypothetical student responses in mind that were created from previous findings (David et al., 2019a), pilot studies (David, 2018), and the initial theoretical framework (David et al., 2019a) described in Chapter 2. I will describe some of the choices of tasks, including the choice of each statement and context, as well as describe these hypothetical student responses in more detail.

Table 5

Truth Evaluation for Each Statement & Graph Pair

	G1	G2	G3	G4	G5	G6	G7	G8
S1	F	T*	T					
S2	F	F	F	T				
S3	F	F	T	T				
S4			T		F	F	F	T
S5	F	F	F	T				
S6	T	T	T		F	F	F	F

Cells are shaded to indicate that a statement is ***false*** *for the function in the given graph.*
Blank cells indicate a pairing not given to the student in the interview.
**This truth value is based on an inclusive meaning of between.*

In the interview, I presented each student with each statement one-by-one. I first asked the student to read and explain the meaning of the statement. Then, I provided several graphs of real-valued functions to accompany this statement. As I presented each graph, I asked the student if the statement is true or false for the function shown. In total, there were 27 combinations of statements and graphs that I asked the student to consider. For reference, I include an outline of the interview protocol below in Figure 11.

Interview Protocol Outline

- Introduction:
  - Greeting
  - Consent form
  - Explain your thinking

- Statement 1: IVT Related
  - Read and Explain
  - Evaluate for G1
  - Evaluate for G2
  - Evaluate for G3

- Statement 2: Increasing Function Related
  - Read and Explain
  - Evaluate for G1
  - Evaluate for G2
  - Evaluate for G3
  - Evaluate for G4

- Statement 3: Injective Function Related
  - Read and Explain
  - Evaluate for G1
  - Evaluate for G2
  - Evaluate for G3
  - Evaluate for G4

-BREAK-

- Statement 4: Continuity at a Point Related
  - Read and Explain
  - Evaluate for G3
  - Evaluate for G5
  - Evaluate for G6
  - Evaluate for G7
  - Evaluate for G8

- Statement 5: Difference Quotient Related
  - Read and Explain
  - Evaluate for G1
  - Evaluate for G2
  - Evaluate for G3
  - Evaluate for G4

- Statement 6: MVT Related
  - Read and Explain
  - Evaluate for G1
  - Evaluate for G2
  - Evaluate for G3
  - Evaluate for G5
  - Evaluate for G6
  - Evaluate for G7
  - Evaluate for G8

- Follow-up as necessary

- Closing
  - Thank student for time
  - Remind them not to discuss interview with others

Figure 11. Outline of interview tasks & protocol.

As the student explained their evaluation, I asked the student to label the graph he or she was working with. The placement of these labels, as well as the student's words and gestures served as a basis for the model I began creating of the student's thinking as the interview progressed. I also asked the student follow-up questions as needed to refine this model, in response to the different possible responses from students. Some of these possible follow-up questions, which arose during the pilot study Phase II, are written into the interview protocol. For instance, if a student drew his or her own graph when first reading a statement, I asked the student to label the graph with the relevant parts of the statement, such as $f(a)$. Of course, each student I interviewed was a unique case and may have responded in a way I did not originally anticipate during the interview. Thus, I was open to asking other questions as necessary based on the model I was creating of the student's thinking in the moment.

Statement 1: Intermediate Value Theorem Context

Statement 1, which is related to the Intermediate Value Theorem[3], is "For all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$." The Intermediate Value Theorem (IVT) was chosen as a context for tasks in this study because of its centrality in Calculus material, as well as its utility in uncovering students' interpretations of outputs of functions and points on graphs in my earlier work (David et al., 2019a). In particular, Statement 1 is similar to the conclusion of the IVT, except that it contains a reversal of the variables, c and N. While the IVT conclusion, "for all real numbers N…there exists a real number c..." is true for continuous functions, Statement 1 is true only for intervals on which a continuous function is strictly increasing or decreasing.

Statement 1 was found to be particularly insightful in distinguishing between students who were engaged in value-thinking and those who engaged in location-thinking

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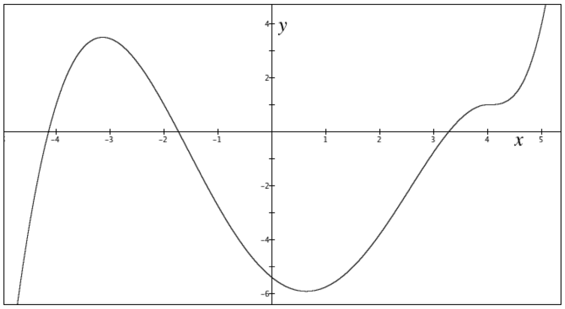
[3] The Intermediate Value Theorem may be stated as follows: Suppose f is a continuous function on the interval (a, b) and $f(a) \neq f(b)$. Then for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$. Adapted from Stewart (2012).

due to the differences in interpretations of the phrase "N between $f(a)$ and $f(b)$." Students engaged in value-thinking interpreted N between $f(a)$ and $f(b)$ as referring to a set of output values between two output values on the output axis. In contrast, students engaged in location-thinking interpreted N between $f(a)$ and $f(b)$ as referring to a set of points on a segment of the graph, between the points the student labeled $f(a)$ and $f(b)$, corresponding to a and b. Noticing these two different interpretations of this phrase with Statement 1 was what led me to further investigate the students meaning for outputs and points on graphs in previous studies (David et al., 2019a). Because Statement 1 contains the phrase "N between $f(a)$ and $f(b)$" and can be mathematically true or false depending on the characteristics of the function and interval it its referring to, students' responses to this statement will be an indicator of how the student is thinking about outputs of functions and points on graphs. A student's graph labels when evaluating this statement will allow me to initially characterize the student's way of interpreting graphs broadly in terms of *value-thinking*, if points are labeled as ordered pairs, or *location-thinking*, if points are labeled solely as outputs.

I presented the student with Statement 1 alone first, then with Graphs 1–3 individually, in that order (Figure 12). Statement 1 is false for the polynomial function in Graph 1, which is not injective. On the other hand, this statement is true for the constant function in Graph 2, if the term "between" is interpreted to be inclusive. Statement 1 is also true for Graph 3, a monotone decreasing function.

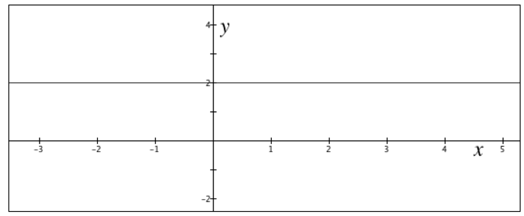
Statement 1: For all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.

Graph #1:



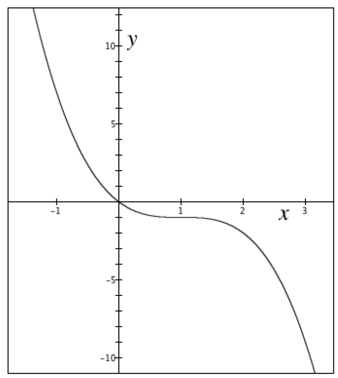
a = –3
b = 4

Graph #2:



a = –3
b = 4

Graph #3:



a = –1
b = 2

Figure 12. Graphs 1-3, used with Statement 1.

I will describe for the reader possible ways students may interpret Statement 1 with Graph 1. The first task of the interview involves students reading Statement 1 (IVT Context) and explaining whether the statement is true or false for the function shown in Graph 1. A similar task was used by David et al. (2019a) and students' responses to the task in their study indicated whether the student was engaged in value-thinking or location-thinking. Specifically, David et al. (2019a) found students who engaged in value-thinking evaluated the statement as false when looking at a graph similar to Graph 1. These students typically would indicate a limited interval of values on the y-axis that they interpreted as values of N between $f(a)$ and $f(b)$. Because some outputs of the function were outside of their selected interval, they claimed the statement was false. On the other hand, students engaged in location-thinking claimed the statement was true, and swept along the entire graph as they described N between $f(a)$ and $f(b)$. Because these students viewed $f(a)$ and $f(b)$ as points on the graph, they viewed all points along the graph as between these two endpoints. Because students engaged in value-thinking and students engaged in location-thinking evaluated Statement 1 differently when looking at Graph 1, students' response to this task may provide evidence needed to initially characterize their interpretation of graphs in terms of value-thinking or location-thinking.

Statement 2: Strictly Increasing Function Context

Statement 2, "For all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$," is the conclusion of the definition of a strictly increasing function[4]. This statement also contains a comparison of input values and output values. As one of the goals of this study was to refine and expand David et al.'s (2019a) framework described in Section 3 in terms of (1) students' interpretations of inputs and (2) whether students distinguish or confound inputs with outputs, I selected two statements that explicitly referred to both inputs and outputs. This statement provides an opportunity for students to work with both input and output values and may elicit the student's interpretation of both inputs and outputs at the same time. Students' interpretations and graph labels with this statement may offer more insight into how a student is thinking about inputs by explaining what it means for one input to be less than the other input. Similarly, a student that is confounding inputs and

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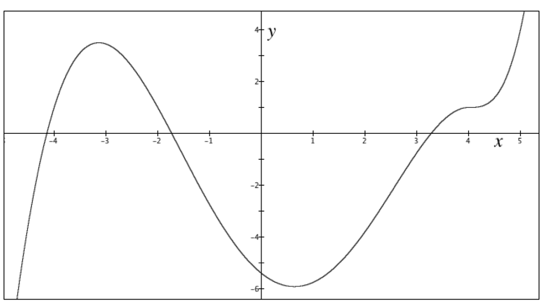
[4] The definition of a strictly increasing function may be stated as follows: A function f is said to be strictly increasing on the interval (a, b) if for all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$. Adapted from Gaughan (1998).

outputs may not consistently refer to inputs and outputs as distinct, or may exchange values of these parts of the function.

I presented to the student Statement 2 alone first, then with Graphs 1–4 individually, in that order (Figure 13). Statement 2 is false for the polynomial function in Graph 1, which is not increasing on the entire interval (a, b). It is also false for the constant function, Graph 2, and the monotone decreasing function, Graph 3. Of the four graphs shown with this statement, it is only true for Graph 4, a strictly increasing function.

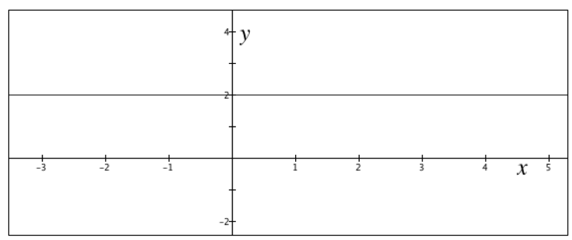
Statement 2: For all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$.

Graph #1:



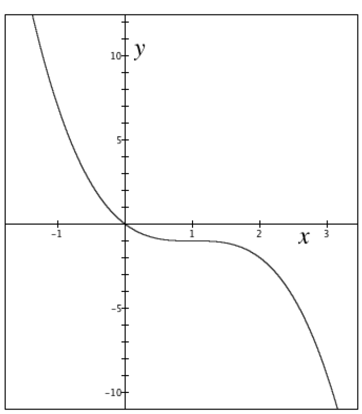
a = –3
b = 4

Graph #2:



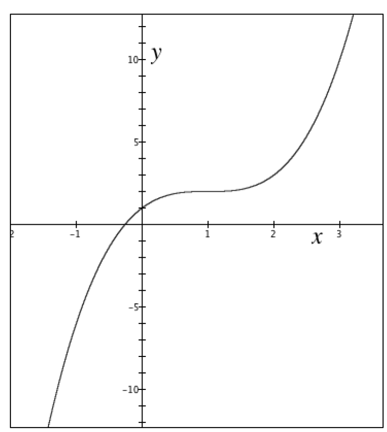
a = –3
b = 4

Graph #3:



a = –1
b = 2

Graph #4:



a = –1
b = 2

Figure 13. Graphs 1-4, used with Statements 2.

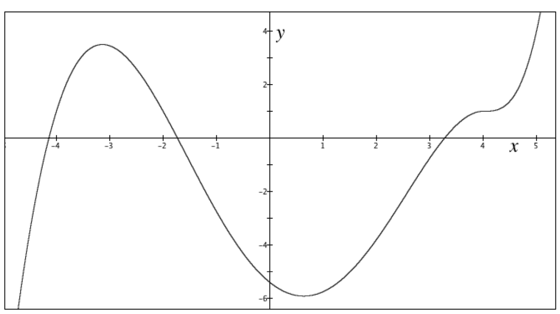
I will describe for the reader different ways students might possibly interpret Statement 2 with Graph 2. These hypothetical student responses are both theoretical, in that they are based on the initial theoretical framework, as well as on previous findings from students' work with the graph of a constant function (David et al., 2019a). When a student is presented with Statement 2 and Graph 2, a student engaged in value-thinking will likely say the statement is false. Such a student might justify this evaluation by selecting two distinct input values, and noting that the output values are equal to each other. On the other hand, a student engaged in location-thinking, may claim the statement to be true, and may compare outputs, which he or she treats as points, in terms of locations. For instance, a student may claim an output is "less than" another if it is a point to the left of the other point. This hypothetical student response is based on previous results found by David et al. (2019a) in which a location-thinking student, when working with the graph of a constant function, labeled the endpoints of the graph "$f(a)$" and "$f(b)$" and claimed that "$f(a)$ was not equal to $f(b)$" while pointing to these two endpoints. If a student views outputs as points, they may interpret "less than" as visually "to the left of," which is an interpretation that yields valid results for values on the horizontal axis. Thus, the goal of presenting the student with Statement 2 and Graph 2 is to elicit the student's interpretation of aspects of graphs, including inputs and outputs, as well as investigate further implications of value-thinking and location-thinking relative to the interpretation of "less than" in this context.

Statement 3: Injective Function Context

Statement 3, "For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*," is the conclusion of the definition of an injective, or one-to-one function[5]. Similar to Statement 2, Statement 3 also contains a comparison of input and output values, but with equality rather than inequality. As explained with Statement 2, Statement 3 was also selected as a task to elicit students' interpretations of inputs and outputs relative to a graph. I presented the student with Statement 3 alone first, then with Graphs 1–4 individually, in that order (Figure 14). Statement 3 is false for the polynomial function in Graph 1 and for the constant function in Graph 2. It is true for the functions in Graphs 3 and 4 which are both injective.

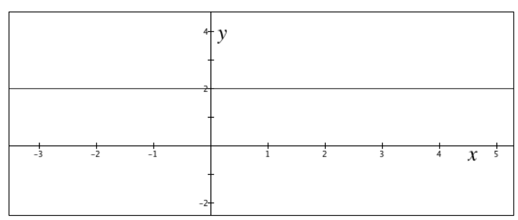
Statement 3: For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*.

Graph #1:



a = –3
b = 4

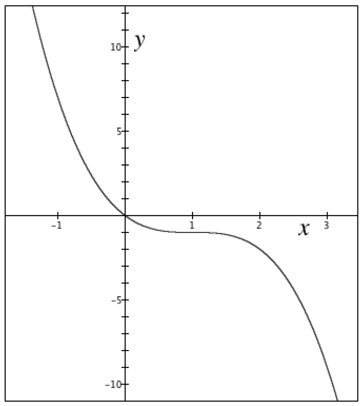
Graph #2:



a = –3
b = 4

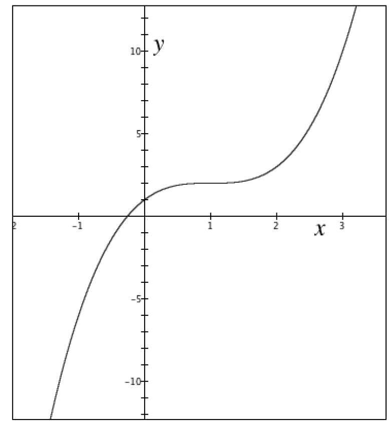
[5] The definition of an injective function may be stated as follows: A function *f* is said to be injective on its domain if, for all real numbers *a, b* in its domain, if *f*(*a*) = *f*(*b*), then *a* = *b*. Adapted from Gaughan (1998).

Graph #3:



a = –1
b = 2

Graph #4:



a = –1
b = 2

Figure 14. Graphs 1-4, used with Statements 3.

I will describe for the reader different ways students may interpret Statement 3 with Graph 1. Statement 3 is false for the function shown in Graph 1, as there are multiple pairs of distinct input values with corresponding output values that are equal. Students engaged in value-thinking are likely to evaluate the statement as true for this function and reference two points with equal output values. However, students engaged in location-thinking may claim the statement is true for this function due to treating outputs as points. For instance, David et al. (2019a) reported that a student engaged in location-thinking claimed two outputs were not equal, pointing to two points on the graph of a constant function. Statement 3 may reveal how students consider both inputs and outputs on graphs, and with Graph 1, on the graph of a function that is not injective.

Statement 4: Continuity Context

Statement 4, "For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $-\delta < x-1 < \delta$, $-\varepsilon < f(x)-f(1) < \varepsilon$," is the conclusion of the definition of continuity at $x = 1$ and thus is true for functions that are continuous at x

= 1[6]. The definition of continuity at a point was chosen as a Calculus context because of its ubiquity in Calculus and Advanced Calculus courses, as well as the difficulty students report with it (e.g., Jayakody & Zazkis, 2015). Relative to the research questions, this statement may also elicit how students think of inputs and outputs. For instance, a student in Pilot Study I confounded the input and output at $x=1$, labeling $f(1)$ at 1 on the output axis, although the output of the function at 1 was not 1.

Additionally, Statement 4 also may elicit a student's interpretation of terms in a statement that are not inputs and outputs of a function relative to a graph, such as $x-1$, $f(x)-f(1)$, δ, and ε. For instance, some students may represent $x-1$ as a distance between x and 1 on the input axis, while other students may represent $x-1$ as a single value on the input axis. In fact, a student I interviewed in Pilot Study I represented x, 1, and $x-1$ as separate values on x-axis on the graph of a function. The symbols δ and ε, which can vary in the definition of continuity at a point, are not values of inputs or outputs, but rather represent upper boundaries on the difference between two values. How a student represents these symbols, whether as values on the axes or not, and where, may offer insight into how students consider inputs and outputs of the function.

The wording of Statement 4 was informed by students' responses in pilot studies with similar tasks. Often the formal definition of continuity at a point is stated with absolute value inequalities. However, several students I interviewed in Pilot Study I, some of whom had even taken an Advanced Calculus course, were unable to relate the

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[6] The definition of continuity at a point may be stated as follows: "A function is said to be continuous at $x=c$ if, for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $|x-c| < \delta$, $|f(x)-f(c)| < \varepsilon$." Adapted from Gaughan (1998).

absolute value inequality "$|x-1|<\delta$" in any way to the graph of a function (David, 2019a). Previous research also suggests students may struggle to interpret absolute value inequalities in Calculus statements (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996). Due to these documented difficulties, I have written the absolute value inequalities in the statement as compound inequalities, such as $-\delta < x-1 < \delta$.

The sequencing of statement 4, before statements 5 and 6, was also informed by a student's response in Pilot Study II. In Pilot Study II, I was testing out these statements, and gave the student Statement 4 following Statement 6, the MVT context statement. Because of this sequencing, the student thought Statement 4 was related to the derivative of the function and explicitly stated it was due to the previous statement.

I presented Statement 4 to the student alone first, then with Graphs 3, 5–8 individually, in that order (Figure 15). The characteristics of Graphs 6–8 were inspired by tasks used by Tall and Vinner (1981) in their investigations of students' images of continuous functions. Statement 4 is true for continuous functions, Graphs 3 and 8, and false for discontinuous functions, Graphs 5–7.

Statement 4: For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $-\delta < x-1 < \delta$, $-\varepsilon < f(x)-f(1) < \varepsilon$.

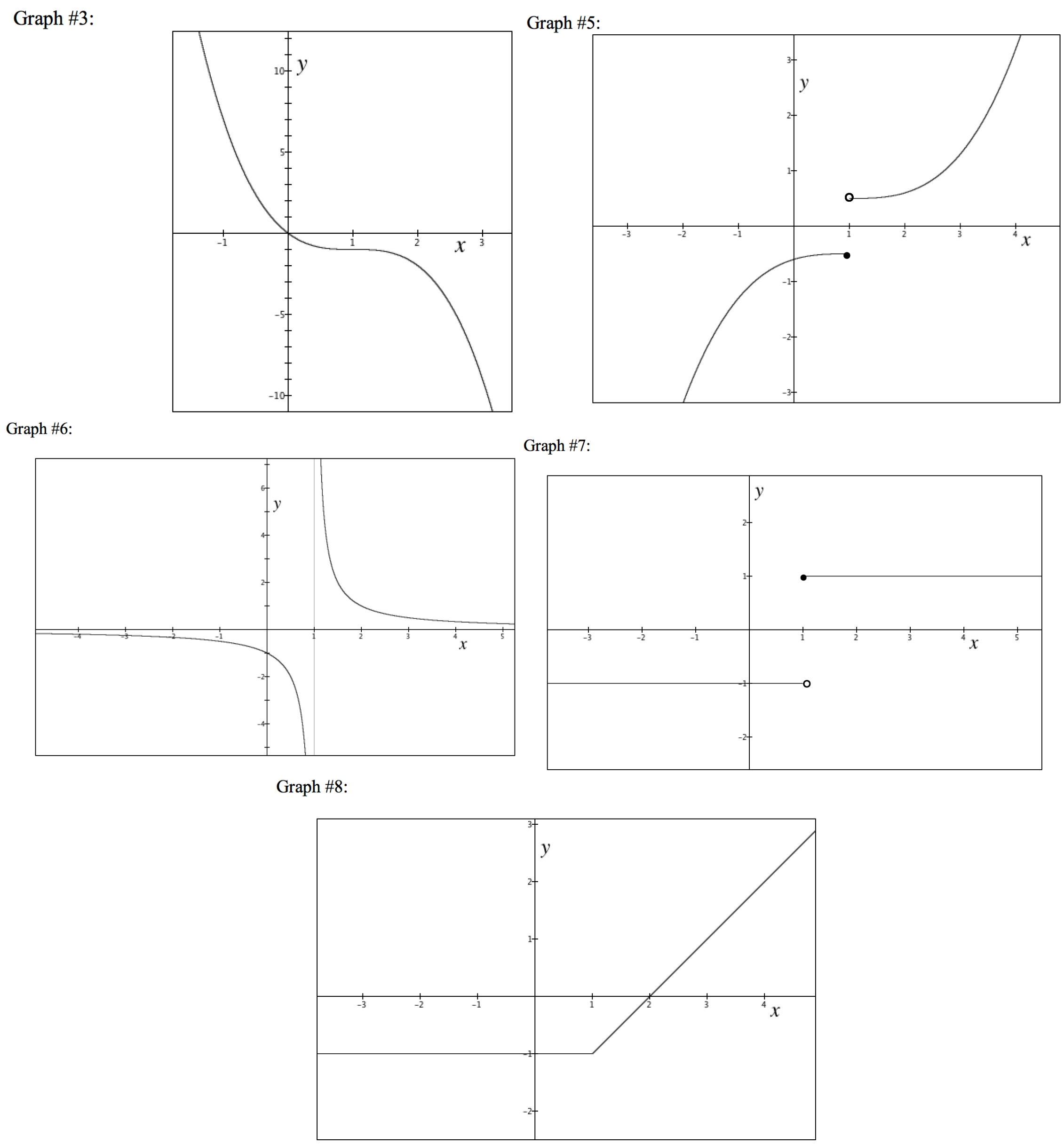


Figure 15. Graphs 3, 5-8 used with Statement 4.

I will describe for the reader different ways students may interpret Statement 4 with Graph 5. Statement 4 is false the function in Graph 5, which is discontinuous at x=1. However, this graph may elicit different interpretations from students, depending how the students thinks about the aspects of graphs, and the variables in the statement. For instance, one student in Pilot Study II, Rosa, interpreted $-\delta$, δ and $-\varepsilon$, ε as boundary values on the x and y-axes, centered at 0, respectively (Figure 16). To determine if $x–1$ was between $–\delta$ and δ for an x value she selected, Rosa calculated the value of the difference $x–1$ and compare it numerically to δ. On the graph, Rosa labeled $x–1$ at a point on the x-axis. She acknowledged that $x–1$ could represent a difference between 1 and x on the x-axis, as she indicated on the graph, but explained that it was easier for her to think of $x–1$ as a point along the x-axis, rather than a distance between two points on the x-axis. Her reasoning was analogous for $f(x)–f(1)$.

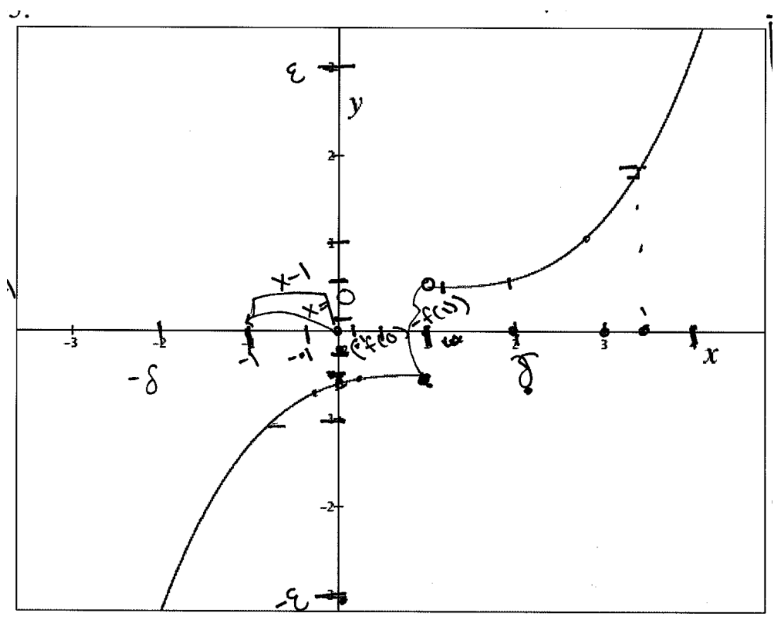


Figure 16. Rosa's work on Graph 5 with Statement 4.

Rosa argued that for any epsilon value she chose, she could find a δ value that would satisfy the statement. Because of Rosa's interpretation of δ and ε values as boundaries centered at zero, and her interpretation of differences as single points on the axes, she concluded the statement was true for this function. When I asked if ε could be 0.1, a value for which no δ exists that satisfies the statement, she said "yes" and chose to compare $f(0)$ to $f(1)$, which she estimated to be –0.9 and –0.8, respectively. She then proceeded to explain that she could find a δ value such that $f(x)–f(1)$ would be within ε and –ε. For Rosa, the differences in the statement were values that could be calculated from values on the graph and represented on axes, but themselves were represented on the graph of the function. Because the differences for Rosa did not represent horizontal or vertical distances between points on the graph, she did not find the discontinuity problematic in her search for δ. Because of my experience with Rosa, I believe Statement 4 with the various continuous and discontinuous graphs will provide opportunities for me to gain insights into students' interpretations of various aspects of graphs relative to the statement.

Statement 5: Difference Quotient Context

Statement 5, "For all non-zero real numbers *h,* if $2+h$ is in (a, b), then $\frac{f(2+h)-f(2)}{h} > 0$," involves the difference quotient and is true for functions which are strictly increasing functions. The difference quotient, in its generic form using x as the initial input value, represents a rate of change between two points on a function, and is the basis of the definition of the derivative.[7] Although the difference quotient is typically represented with a variable, x, I chose to use a set value of 2 as a way to reduce the number of variables in the statement. With this statement and various graphs, I can investigate how students think about 2, $f(2)$, h, $2+h$, and $f(2+h)$. Similar to Statement 4, Statement 5 involves a variable, h, which itself is not an input or output of the function,

[7] The difference quotient is typically used in the limit definition of the derivative function as $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (e.g., Finney et al., 1994; Stewart, 2012).

but is related to inputs and outputs of the function. I have also chosen to represent the denominator in its simplified form as h for two reasons. Typically, this is how the difference quotient is presented in textbooks (e.g., Finney et al., 1994; Stewart, 2012) and some students may not associate h with a difference in input values 2 and $2+h$. I have found evidence of this in Pilot Study II, in which several students indicated that h was a value on the input axis, rather than a distance between two input values. Based on my experience teaching Calculus for four semesters, I have also found that students often do not associate the difference quotient with a rate of change between two points on a graph. This statement may provide me with the opportunity to uncover students' interpretations of the symbols in the statement, including the difference quotient itself, on various graphs.

I presented Statement 5 to the student alone first, then with Graphs 1–4 individually, in that order (Figure 17). Statement 5 is false for the polynomial function in Graph 1, which is not increasing on the entire interval (a, b). It is also false for the constant function, Graph 2, and the monotone decreasing function, Graph 3. Of the four graphs shown with this statement, it is only true for Graph 4, a strictly increasing function.

Statement 5: For all non-zero real numbers h, if $2+h$ is in (a, b), then $\frac{f(2+h)-f(2)}{h} > 0$.

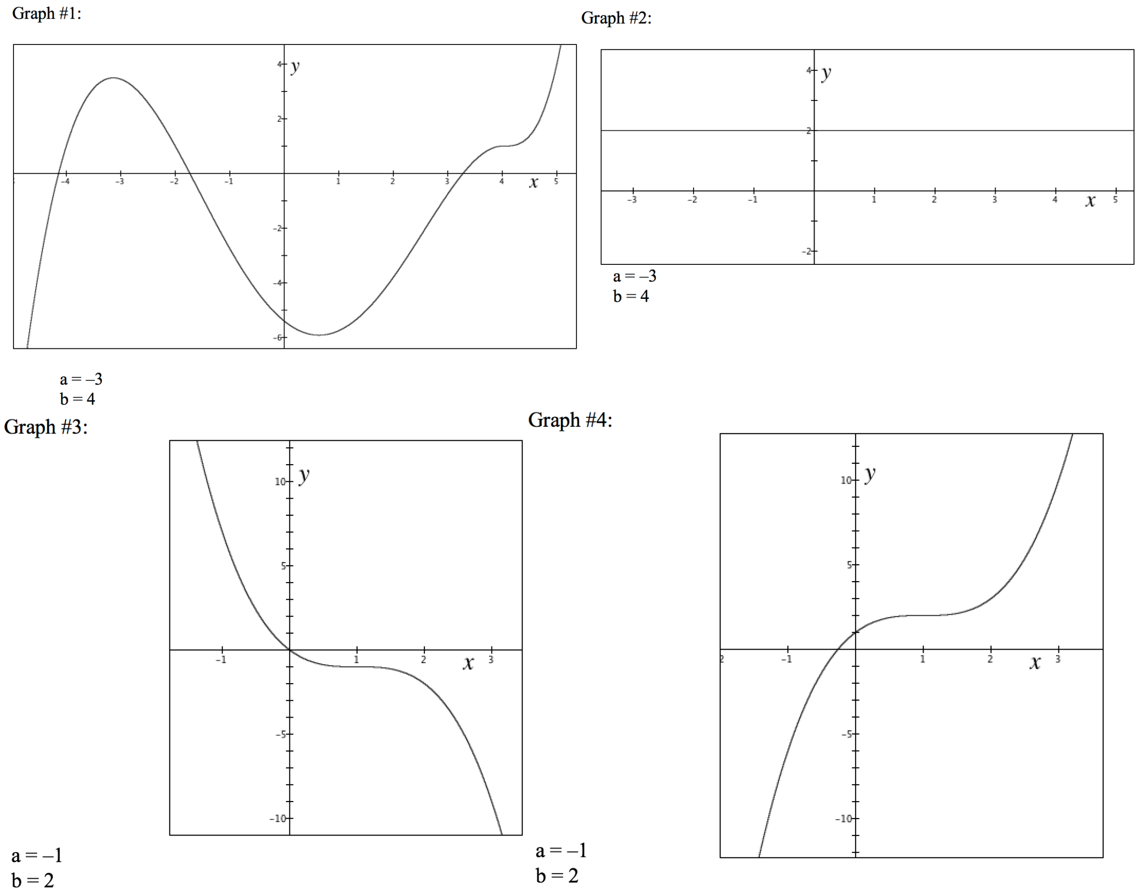


Figure 17. Graphs 1-4, used with Statement 5.

I will describe for the reader different ways students may interpret Statement 5 with Graph 1. How a student interprets *h* in the statement relative to the graphs of the functions provided may provide insight into how the student understands representations on graphs, as well as the difference quotient. For instance, with Graph 1, one of the students in Pilot Study II, Rosa, who I characterized as engaging in value-thinking, labeled *h* as a value on the input axis and stated that it "could be considered an independent value of the function." Later, she did acknowledge that *h* represented a distance between input values on the input axis, but encountered a conflict when she thought about a negative value for *h*. Rosa claimed that a distance could never be negative, even if the value of h was negative, and ended up calculating a positive value for the difference quotient, for two points with a negative rate of change. This caused conflict for her, because she recognized visually that the rate of change should be negative, because the line segment connecting the two points was sloped downward. Rosa's conflict, which she was unable to resolve, indicates that she did not conceive of h as representing a *signed* distance between two inputs on the x-axis. This finding may have implications for how students coordinate changes in two quantities on a graph and whether they allow for signed distances.

Statement 6: MVT Context

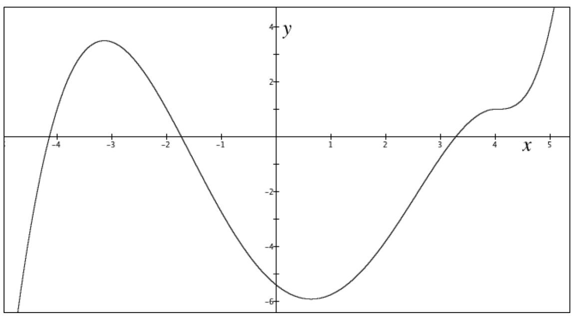
Statement 6, "There exists a real number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$," is the conclusion to the Mean Value Theorem[8] and is true for functions that are continuous on $[a, b]$ and differentiable on (a, b). I included this statement for several reasons. First, the statement involves derivative notation, and will allow me to investigate how students interpret "$f'(c)$" on the graph of a function f. Additionally, this statement also involves a quotient that represents a rate of change, but does so with reference to two different input values that are not related, unlike 2 and $2+h$ in the previous statement. Unlike the difference quotient in Statement 5, the quotient in Statement 6 also includes a difference of input values in the denominator. Students may interpret these expressions of rate of change differently based on how they are represented.

[8] The Mean Value Theorem may be stated as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a real number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Adapted from Stewart (2012).

I presented Statement 6 to the student alone first, then with Graphs 1–2, 5–8 individually, in that order (Figure 18). Statement 6 is true for continuous, differentiable functions, such as those in Graphs 1 and 2. The statement is false for functions which are discontinuous or not differentiable on the interval (*a*, *b*), which are graphs 5–8.

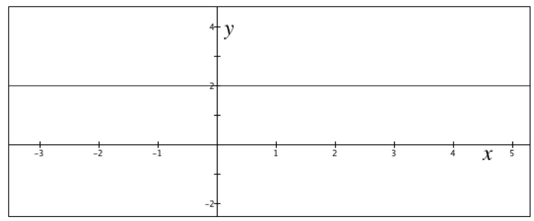
Statement 6: There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Graph #1:



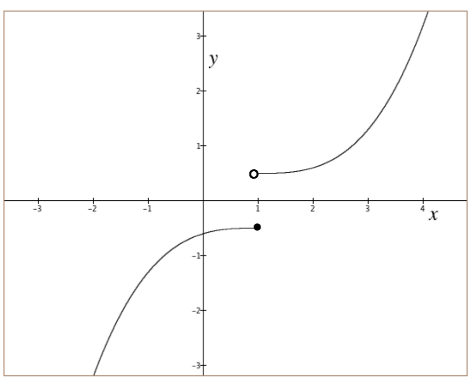
a = –3
b = 4

Graph #2:



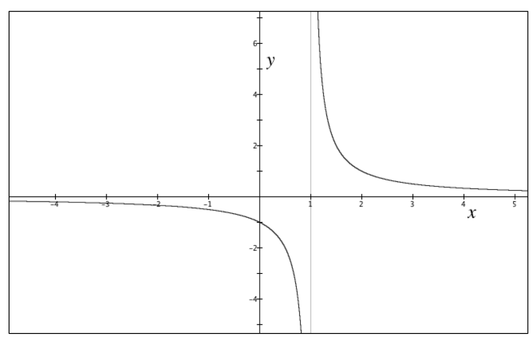
a = –3
b = 4

Graph #5:



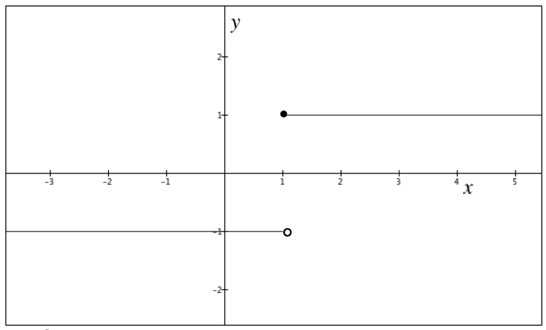
a = –3
b = 4

Graph #6:



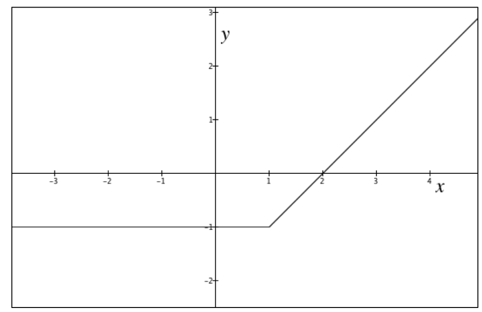
a = –3
b = 4

Graph #7:



a = –3
b = 4

Graph #8:



a = –3
b = 4

Figure 18. Graphs 1-2, 5-8 used with Statement 6.

I will describe for the reader different ways students may interpret Statement 6 with Graph 8. Statement 6 with Graph 8, which is continuous but not differentiable at $x = 1$, may elicit students' interpretations of $f'(c)$, differences in inputs and outputs, and the overall difference quotient. Statement 6 is false for the function in Graph 8, since $f'(1)$ is undefined. However, some students may not interpret the graph of the function in this way and understanding how they *are* interpreting $f'(c)$ is valuable insight into students' understanding of Calculus ideas. For instance, one student in the pilot study drew a graph of $f'(x)$ and claimed that because of the graph of $f'(x)$, which would be piecewise 0 then 1, there would not be a c such that would have the value of the quotient that she calculated, 3/7. However, when asked if she could represent values of $f'(x)$ in any way on the graph of $f(x)$, she said "no."

Implementation of Study

I will describe the logistical details I followed for the data collection and analysis to implement this study. I served as the interviewer in the clinical interviews with each student selected using the tasks described above. I used one video camera to capture a close-up view of the students' gestures and work during the interview. A second webcamera, connected to a monitor was set up to capture a wide view of the student while they work to record their words, gestures, and expressions throughout the interview. I also utilized an iPad, connected to the monitor, which had a PDF file of the graphs used in this study that students can write on. The statements were printed out on paper for the student to work with and write on during the interview, as well as have in front of them while labeling graphs on the iPad. Two video recordings were produced from this set-up. The first was the recording from the video camera showing close-up view of student work. The second was a screen capture of the monitor's screen using QuickTime, containing both the wide-view from the webcam and the screen capture of the iPad side-by-side. In this way, I was able to play back the student explaining their thinking and simultaneously view his or her writing and labeling on the graphs, without the student's hand obstructing the view. For each participant, these two video recordings and the students' work on the graphs on the iPad and on the papers containing the statements served as the data to be analyzed.

Data Analysis

The data analyzed in this study are all 13 students' responses to each of the tasks presented, including their words, gestures, labels on graphs and other markings on the statements. The data analysis was guided by the theoretical stances described in Chapter 2, and in part utilized the theoretical framework presented in that section. As described previously, in keeping with the method of theoretical sampling (Corbin & Strauss, 2014), data analysis was on-going throughout the process of data collection in order to analyze the data to create new categories, refine currently existing categories, and test the variability within categories. Ideally, data collection, in terms of sampling from the data set, will conclude at the point of theoretical saturation (Corbin & Strauss, 2014), upon which no new categories are emerging from the data set, and descriptions of categories has been sufficiently detailed. For practical considerations, the data collection ended after the 13 students were interviewed and categories of their interpretations emerged.

Each phase of data analysis contributed to answering the research questions stated in Chapter 1. As I describe the phases of data analysis, I will detail how these research questions were addressed with each phase.

Phase 1: Preliminary Data Analysis

The preliminary data analysis took place both during each interview as well as immediately following each interview. Both the analysis during the interviews and afterward was informed by findings from my pilot study, as well as the previously conducted interviews in this study, consistent with theoretical sampling.

As each interview was conducted, I constructed initial conjectures of the student's interpretations of aspects of graphs, including: inputs of the function, outputs of the function, points on the graph, segments of the graph, and other objects in the statement such as "h" in the difference quotient statement, or "$f'(c)$" in the MVT statement. To test these conjectures, I posed certain follow-up questions in each interview related to how the student was thinking about these aspects of the graph(s). Based on the student's response, I updated my model of the student's thinking as necessary. I continued to ask questions until I reached a model that was grounded in the student's words, gestures, or graph labels, or the student was unable to provide any new information in his or her response. Follow-up questions were different for different students or for the same student with different tasks.

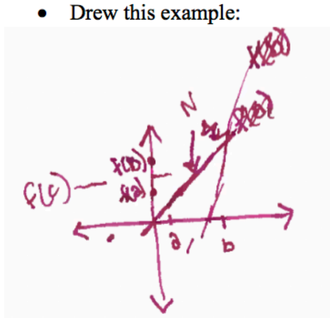
Within 48 hours of the interview, I documented my preliminary model of the student's thinking for each task through a content log. The content log consisted of notes of the student's thinking for each statement and graph combination after re-watching the interview, with special attention to the students' interpretation of aspects of graphs. I utilized the constructs of value-thinking and location-thinking, described in Chapter 2, as an initial coding scheme to characterize the student's interpretations of outputs and points on graphs for each statement and graph. In Figure 19, I provide an example of one such content log.

Martha Content Log

Overall: at first, primarily Value-thinking with some location labels (N). Inconsistent about x values on graph vs. on x-axis. Thinks f(x) is any y-value (regardless of range of function)

1:46 Statement 1

- There exists a number between f (a) and f(b) which is f(c) and that N number is that value, like a specific number
- Drew this example:



- Two numbers a,b on the x-axis
- F(a) f(b) more a y-axis type thing
- N labeled at the point
- Put f(c) in between f(a) and f(b) on the y-axis
- Thinks the statement is true for the example she drew

9:05 Statement 1 Graph 1: True (Quantifier issue)

- At first labels points a and b
- Draws vertical dotted lines from x-axis up to points
- Draws horizontal dotted lines from points to y-axis
- Labels f(a) and f(b) on y-axis
- F(c) would have to be in here like the "point value"
- Labels c on x-axis
- Labels N on graph
- Labels f(c) on y-axis
- Value-thinking interpretation of between f(a) and f(b)
- Found a point where N is not between f(a) and f(b) "below f(b)"
- Thinks its true because there is a number that exists between them (quantifier issue)
- Highlights range on graph that works between points

Figure 19. Sample of a content log as part of the preliminary data analysis.

In addition to coding students' interpretations of outputs and points, I also sought to identify how students interpret inputs to functions, as well as related expressions from the statements, such as differences or other expressions, on graphs. To investigate this, I analyzed how students represented these expressions on the graph and described them with their words and gestures. This begun the process of addressing Research Question 1, which asks about the characteristics of students' interpretations of expressions on graphs. To begin answering Research Question 2, I also recorded each student's final evaluation for each statement/graph pairing in a table similar to Table 5. Furthermore, I compared an individual student's interpretations of expressions on graphs across all six statements. In doing so, I began investigating the impact of context on the same student's interpretation of graphs to address Research Question 3. Also, comparing students' interpretations of expression with their interpretations of points began the process of addressing Research Question 4.

Through the content logs which documented how students are interpreting expressions on graphs, I began to look for commonalities and variations. Before conducting the next interview, I reviewed the content logs of all prior interviews. In conducting this review, I developed questions to further investigate how future students might think about a particular portion of a statement or graph. In this way, the questions asked in later interviews were informed by all prior interviews to gain increasing details on students' interpretations.

Phase 2: Search for Variation & Development and Refinement of Codes

Once all the interviews have been conducted and analyzed as described in Phase 1, I began to re-analyze each interview, drawing on the patterns observed in Phase 1. The observed patterns, developed through a process of open coding (Corbin & Strauss, 2014), in turn developed into a set of codes that I used to re-analyze each interview. For instance, a code of "difference of input values as distance" was used to categorize each instance in which a student shows evidence of interpreting an output of a function as a location on the graph. To be clear, I did not have a set list of pre-existing codes; rather the codes emerged and were refined in this phase, a process referred to as axial coding

(Corbin & Strauss, 2014). Axial coding took place through several types of comparisons, aligning with the research questions: (1) comparison of different students' interpretations of the same graphs, (2) comparison of different students' evaluations of the same statement/graph pairs, and (3) comparison of the same student's interpretation of graphs with different statements.

The nature of the development and use of these codes was cyclical. As more codes emerge from analyzing interviews, these codes were applied retroactively to previously analyzed interviews. As these codes emerged, I returned to the data to collect more samples relevant to the given code to test against other portions of data coded similarly. For instance, when I found several instances of a student referring to a difference from a statement as a single position on an axis, I returned to the portions of transcript flagged similarly to draw out more detailed comparisons. I asked myself questions such as, "do all students who identify differences this way describe the point in a similar manner?" or, "are there subtle differences in the way in which the student uses these points in their explanation or evaluations of the statements?" etc. In this way, I continued the process of theoretical sampling, returning to the data to answer newly-refined questions, develop from previous data analysis. Through this process, four codes for students' interpretations of expressions on graphs emerged.

As another component of this phase of data analysis, I compared each student's final evaluation of each statement and recorded these responses in a table, like the one below (see Table 6). I chose to use a student's final evaluation for analysis for several reasons. First, students' evaluations of statements changed throughout the interview as they considered the statements with various graphs and became increasingly familiar with the statements. Additionally, as I witnessed in previous interviews I have conducted, some students changed their mind about the truth-value of a statement several times, though they typically settled on an evaluation after working through all of the tasks associated with a statement. In order to account for such changes, I have chosen to use a student's final evaluation for each statement for comparison, as this represents the culmination of his or her reasoning about the statement. The findings from this comparison began to answer Research Question 2, examining the relationship between student's evaluations of given statements and their interpretations of the graphs.

Table 6

Sample List of Students' Evaluations of Each Statement & Graph Pairing

			Student 1	**Student 2**	**…**	**Student 13**
Statement & Graph Pairing	**S1**	**G1**	F	T	…	F
		G2	F	T	…	T
		⋮	⋮	⋮	…	⋮
	S2	**G1**	T	F	…	F
		⋮	⋮	⋮	…	⋮
	⋮	⋮	⋮	⋮	⋮	⋮
	S6	⋮	⋮	⋮	⋮	⋮
		G8	T	F	…	F

Shaded cells indicate mathematically incorrect evaluations

Using such a table to organize and illustrate student's evaluations, I looked for trends across items; for instance, many students incorrectly evaluating the same statement with a particular graph. Students who evaluated many tasks similarly were grouped together. Within these groupings, I searched for connections between these students' interpretations of expressions on graphs. Likewise, I compared students' interpretations of expressions on graphs who answered with opposite evaluations to uncover distinct ways in which students interpreted graphs, which may have explained the opposing evaluations.

Phase 3: Arrival at Theoretical Saturation & Analysis with Final Coding Scheme

Once I analyzed and re-analyzed each interview with the codes that emerged from Phase 2, and each student's interpretation of each graph and statement pair in terms of the existing codes, the data analysis will have reached the point of theoretical saturation (Corbin & Strauss, 2014). At this point in the data analysis, no new codes were needed to characterize students' thinking relative to the research questions. The final set of codes were used once more to re-analyze each interview to check for consistency in analysis and tweak the descriptions of each code as necessary to capture the details of student thinking. At this point, these codes for students' interpretations of expressions were named as: nominal, ordinal, cardinal, and magnitude.

The goal of this final phase of data analysis was (1) to use the set of codes to characterize students' interpretations of expressions on graphs, and (2) to describe the association between students' interpretations on graphs in terms of the codes and (a) their evaluations of the statements, (b) the content of the statements, and (c) their interpretation of points on graphs in terms of value-thinking and location-thinking. After the codes were finalized and each interview was re-coded, I began to write the results of this analysis in terms of the goals above, corresponding with the research questions.

CHAPTER 5

RESULTS I: FOUR WAYS STUDENTS INTERPRET EXPRESSIONS ON GRAPHS

The main purpose of this study is to investigate how undergraduate students interpret aspects of graphs of functions as they interpreted statements from Calculus with graphs, in accordance with Research Question 1. Students who participated in this study were asked to evaluate statements from Calculus for functions shown in various graphs. As I analyzed the interview data, I observed distinctions in the ways in which students engaged with the structure of the graphs in the Cartesian plane. Differences in students' interpretations of various components of the graphs relative to the statements began to emerge. I found that most of these distinctions were rooted in students' interpretations of the variables and expressions (e.g., $f(c) < f(d)$, $f(b) - f(a)$), henceforth referred to as "expressions," within the provided graphs. Students interpreted expressions in graphs in ways distinct from other students, even if they placed their labels for these expressions in the same places. Differences in students' interpretations of expressions in graphs were associated with differences in their evaluations of the statements provided to them during the interview.

Some students, at times, found individual values for variables or expressions from the graph and then evaluated the statement by operating on these values numerically. I refer to this type of reasoning as occurring in the *numerical register*, consistent with Duval's (1999) use of the term. Other students, at times, used features of the graphs of functions to evaluate the statements by assigning expressions to features of the graphs and operating on these features, which represented expressions, in the *graphical register*, a term I also intend in the manner of Duval (ibid). From the instances in which students operated with expressions in the graphical register with provided or self-drawn graphs, I observed four different ways in which students assigned and used these expressions, which I refer to as students' *interpretations* of expressions in the graphical register. These four ways of interpreting expressions in graphs I see as related to meanings for numerical values in mathematics. Thus, I refer to each of these four ways of interpreting expressions in graphs as follows: (1) *nominal*, referring to the use of an expression as a label; (2) *ordinal*, referring to the use of an expression as indicative of an ordering; (3) *cardinal*, referring to the use of an expression as a count of some quantity; and (4) *magnitude*, referring to the use of an expression to indicate an amount of a quantity regardless of unit.

In categorizing a student's interpretation of an expression in a certain episode, I describe the student's way of using the expression within the graphical register in his or her reasoning. Throughout the interviews, I observed each student engaging with the graphs in some or most of these four different ways of interpreting expressions with different statements. Some students even interpreted expressions in the graph in multiple ways for the same statement in their explanation of the statement with a given graph. Other students interpreted and reasoned with expressions in the graphical register in a certain way, while also using the graph to find numerical values of expressions and reasoning with them in the numerical register.

I consider the interpretations I am reporting in this study to be the students' *evoked* interpretations; they may not reflect other interpretations that students may use with other types of expressions or statements in the graphical register. The categories I ascribe to students' interpretations of expressions in this study are based on the data I observed, including students' words, gestures, and markings on the papers and graphs provided. Thus, that my categorization of a student's interpretation of an expression, based on the available data, may not fully capture the *meaning* for an expression the student may hold in mind. To be clear, the four interpretations reported in this chapter are used to describe aspects of a student's reasoning in a given moment. They are not intended to describe or assess a student's full meaning for expressions on graphs in all contexts.

I also acknowledge that the task at hand and the questions I asked as the interviewer may have created situations in which it was more favorable for a student to use a certain interpretation over another. Similarly, in some tasks, one interpretation rather than another, may have sufficed to justify an evaluation from the student's perspective.

In this chapter, I first detail these four ways that the students interpreted expressions in the graphical register. I provide students' work, including words, gestures, and written work on paper, selected from the interview data to highlight the nuances of each of these interpretations observed in this study. Using the data, I detail the characteristics of each of these four interpretations of expressions in graphs. I do not intend for each of the four interpretations to be viewed as more advanced or less advanced than the other interpretations. To this point, I clarify that the labeling of these interpretations with numerical values (1, 2, 3, 4) in this dissertation does not signify a ranking of these interpretations in any way. Still, these four interpretations of expressions in the graphical register, and which expressions students use them with, provide an insightful lens into students' visual reasoning and evaluations of the statements from Calculus used in this study. I conclude by providing an example of a student reasoning in the numerical register when completing one of the interview tasks, as a contrast to the four interpretations of expressions in the graphical register.

Interpretations of Expressions in the Graphical Register

Expressions as Nominal on Graphs

Throughout the interviews I conducted and analyzed, I observed some students in some instances place expressions on graphs to indicate a certain position within the graph. In these instances, students treated an expression as a label to indicate the position, typically on an axis or along the trace of the graph of the function. These students then reasoned about these expressions as referring to these positions they identified in the graph. In doing so, they tended to allow only a single label to be ascribed to one position; for two different positions on an axis or on the trace of the graph of the function, these students tended to use two different labels. I found that these students used such labels for comparison of the positions they had marked to see if these positions were the same or different, as a means of determining whether expressions were equal or unequal. In these cases, the students did not refer to the value of these expressions numerically, or as indicating a certain ordering, an amount of units, or a measurement, but purely as a label. When students reasoned about expressions as labels for positions on the graph in this way, I refer to their interpretation of expressions as *nominal*. I use the term nominal consistent with Merriam-Webster Dictionary's definition of nominal as "of, relating to, or constituting a name" ("nominal," Merriam-Webster, 2019). For students interpreting and reasoning with expressions nominally on graphs, a variable x or expression containing the variable x, was at once a symbolic expression, a label for a position, and a position itself on the graph. In other words, they tended to fail to distinguish these three things (symbols, labels, and positions) from each other in graphs.

I observed several instances of students interpreting expressions nominally on graphs, in both conventional and unconventional ways. Many of these instances occurred when students were evaluating Statement 3 for various graphs of functions. Statement 3, the conclusion of the definition of an injective function, reads as follows: "For all real numbers c, d in (a, b), if $f(c) = f(d)$, then $c=d$." I will describe several examples from students using a nominal interpretation of expressions in the graphs with this statement, both in cases I consider conventional and those I consider unconventional, to highlight the characteristics of this interpretation.

Nominal interpretation of expressions and its conventional use.

Micah was one such student who interpreted expressions nominally in the graphical register while completing one of the tasks. At the time of the interview, Micah had completed Advanced Calculus and was enrolled in a senior-level Topology course. When explaining his evaluation of Statement 3 with Graph 2, Micah used expressions as labels of positions, and then compared these positions as a way of comparing the expressions. Micah explained that in his counterexample labeled on the graph (Figure 20), "$f(c)$ does indeed equal $f(d)$, but c does not equal d. If they [c and d] were [equal], they would have to be the exact same point. So *d* would have to equal *c* (*writes in '=c' next to d on the x-axis*). They would both have to be here, but that's not true."

Statement 3: For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*.

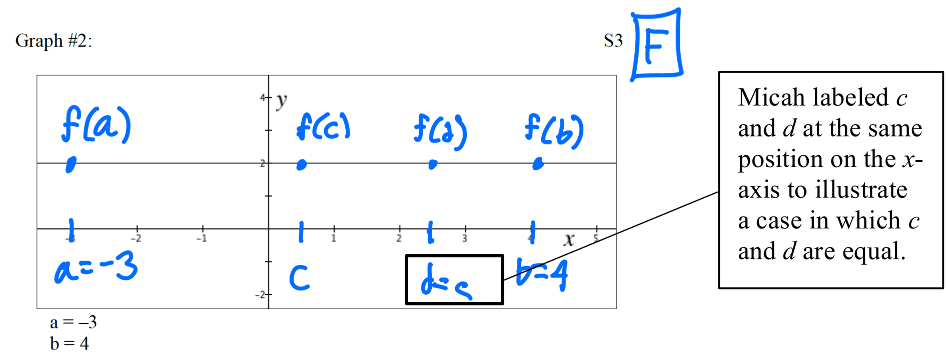


Figure 20. Micah's labels on Graph 2 when evaluating Statement 3.

Based on his explanation of his counterexample, which he labeled on the graph, Micah evaluated Statement 3 as false for the function shown in Graph 2. Micah claimed that for the *f*(*c*) and *f*(*d*) he chose, *c* and *d* were not equal to each other. The justification that he used to arrive at this claim was that *c* and *d* were in different positions along the *x*-axis. He confirmed that this was the reason for his conclusion by explaining that the only way for *c* and *d* to be considered equal would be for the two values to be located at "the exact same point." To show what he meant by being at "the exact same point," Micah used the label of "*d*=*c*" to indicate the case when these two expressions would be the same, i.e., when they would refer to the same position on the *x*-axis. In reasoning this way, Micah equated two expressions being placed on the same position on the *x*-axis with the expressions being equal to each other. Because Micah used the location of the expression labels to make his determination of equality, I categorized his interpretation of the values *c* and *d* as nominal. Because he interpreted the inputs *c* and *d* from Statement 3 as referring to positions *on the x-axis,* his nominal interpretation of the expressions supported him in concluding that the two chosen inputs, *c* and *d*, were unequal. The first choices of *c* and *d* Micah labeled on the graph that he claimed were unequal would, in fact, be considered unequal according to the conventions of the Cartesian coordinate system. Thus, I consider Micah's use of a nominal interpretation of expressions to be conventional.

Another student, Adam, enrolled in an Introduction to Proof course at the time of the interview, also interpreted expressions nominally when making a comparison between values. Adam used a nominal interpretation of *c* and *d* when he was evaluating Statement 3 for the function in Graph 1. Adam gave an example of two points on the graph for which he claimed *c* was not equal to *d*. Adam labeled the graph, placing *c* and *d* on the *x*-axis, and *f*(*c*) and *f*(*d*) on the trace of the graph, as shown in Figure 21.

Statement 3: For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*.

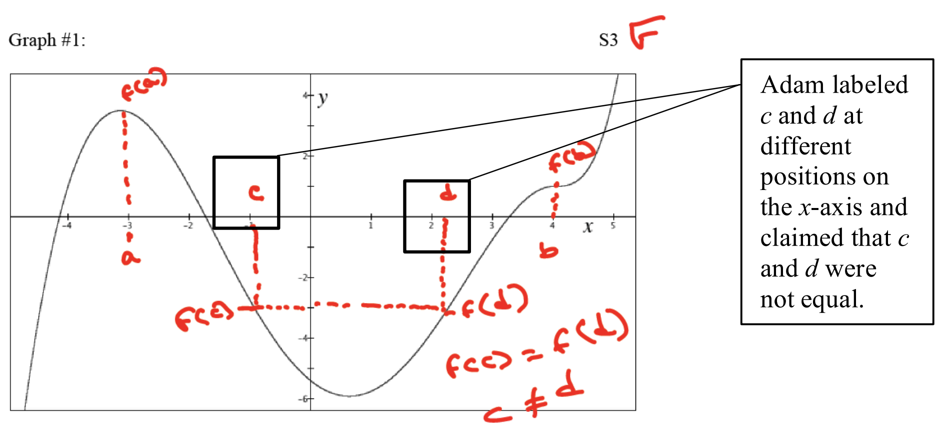


Figure 21. Adam's labels on Graph 1 when evaluating Statement 3.

When I asked him how he knew *c* was not equal to *d* in this instance, Adam explained that, "to judge whether they are equal to each other, I focus only on the *x*-axis and since they [*c* and *d*] are *not on the exact same spot*, they are not the exact same value." Like Micah, Adam equated not being "on the exact same spot" on the *x*-axis with not being "the exact same value." The explanation Adam provided for why *c* and *d* were not equal indicates that he was interpreting these expressions in the graph as referring to "spots" on the *x*-axis. His justification for the inequality of *c* and *d* rested on the positions of the two values which he labeled on the *x*-axis being different. For this reason, I categorized Adam's interpretation of expressions on the graph as nominal when determining the inequality of the two inputs of the function, *c* and *d*. Like Micah, Adam's interpretation of the inputs as referring to positions on the *x*-axis supported him in concluding that *c* was not equal to *d* in his example. Again, similar to Micah, Adam, used a nominal interpretation of expressions in a conventional sense regarding the inequality of these two chosen inputs, *c* and *d*, which he labeled on the *x*-axis.

Nominal interpretation of expressions and its unconventional use.

While some students interpreted expressions nominally on graphs and reasoned about the expressions in the statements in conventional ways, other students did not. In fact, some students who used a nominal interpretation of expressions on graphs were reluctant to use two different labels to describe the same place on the graph. As a result of this reluctance, these students often reasoned about the expressions in the statements in ways which would be considered unconventional in the Cartesian coordinate system.

One such instance was with a student, Annie, who was enrolled in Calculus III at the time of the interview. In this episode, Annie explained why she evaluated Statement 3 with Graph 3 as false (Figure 22).

Statement 3: For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*.

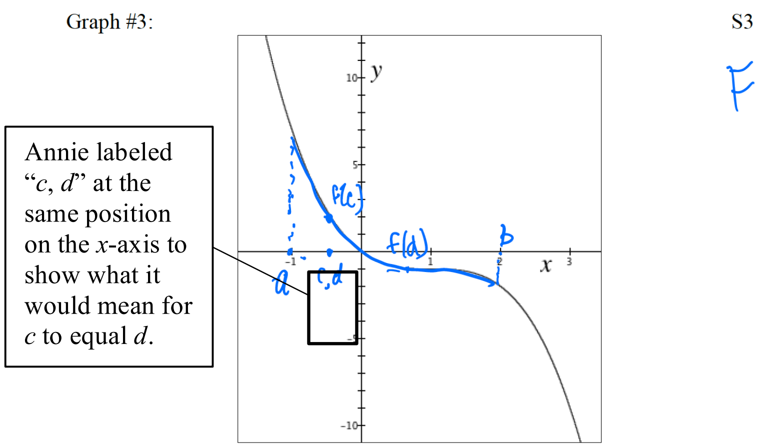


Figure 22. Annie's labels on Graph 3 when evaluating Statement 3.

She claimed that, "it doesn't look like your *f*(*c*) could even equal *f*(*d*) at any point on this graph, because no *y*-values are ever gonna be repeated." When I asked Annie if she could label *f*(*c*) and *f*(*d*) on the graph, she replied and we had the following exchange:

Annie Yeah, like you can kind of put anywhere your *f*(*c*) and your *f*(*d*). But even if
I put it, like, the only way is if it would be on top of each other, for them to
have the same *y* values because it [the function] is decreasing the whole
entire time.
Int So can *f*(*c*) and *f*(*d*) be on top of each other? Is that okay?
Annie I don't, no, I don't think so. I don't think I've ever ran into a case where
there was…It's like having two ordered pairs on top of each other, (Int:
Okay), which I guess, it could happen, yes, but I don't think we really, like
you don't notice it as much because you just see one ordered pair there.
Int Okay.
Annie So it's [Statement 3] like saying if you had like *c* and *d* here, (*places dot on
x-axis and writes 'c, d' below it*) (Int: mm-hmm), then *f*(*c*) and *f*(*d*) there
(*points to dot on graph labeled f(c)*).
Int mm-hmm. So, if you did have it that way, would the statement [3] be true?
Annie Yes in that case it [Statement 3] would be true … I don't think I've ever
really looked at that before like having them be the same [*f*(*c*) and *f*(*d*)]

In her nominal interpretation of $f(c)$ and $f(d)$, Annie viewed these as two distinct labels; for this reason, she did not wish to use them to label the same position along the graph. Annie initially placed labels for $f(c)$ and $f(d)$ along the graph of the function as shown in Figure 22. Regarding the possibility of $f(c)$ equaling $f(d)$, Annie claimed that would only occur if $f(c)$ and $f(d)$ were "on top of each other" (Line 2). I infer that Annie meant that $f(c)$ and $f(d)$ could only equal each other if the labels were placed at the exact same spot along the trace of the graph. Labeling the same spot as $f(c)$ and $f(d)$ on the graph conflicted with Annie's previous experience because she typically "just see[s] one" (Lines 8-9) label at each spot on the graph, and never "really looked at that before, like having [$f(c)$ and $f(d)$] be the same" (Line 16). In other words, in Annie's mind, each spot ought to be labeled with a single label, rather than "two ordered pairs on top of each other" (Line 7). In order to determine whether $f(c)$ equaled $f(d)$, Annie used the criterion of the two expressions being in the same spot on the graph, or, in her words, "on top of each other" (Lines 2, 7). That is, if the two expressions referred to the same location on the graph, she considered the expressions to be equal. Annie claimed that it was not possible for the two expressions to refer to the same place on the graph; for this reason, she concluded that $f(c)$ did not equal $f(d)$ for any c's and d's in this graph.

In her reasoning, Annie used $f(c)$ and $f(d)$ as expressions, labels, and positions, simultaneously. In the dialogue above, Annie equated the symbolic expressions of $f(c)$ and $f(d)$ being equal with $f(c)$ and $f(d)$ referring to the same spot on the graph as labels. Because having two distinct labels in the same spot conflicted with her experience, she concluded there was no way for $f(c)$ to equal $f(d)$ on Graph 3. Annie claimed the only way $f(c)$ and $f(d)$ could be equal would be if they were in the same position. However, she saw $f(c)$ and $f(d)$ as distinct labels, and did not wish to use distinct labels to refer to the same position. Because Annie thought of $f(c)$ and $f(d)$ as distinct labels, and thus as distinct positions, the expressions were unequal. Her reluctance to refer to the same spot on the graph with both $f(c)$ and $f(d)$ indicates that she was thinking about $f(c)$ and $f(d)$ as names (labels) for positions. Based on her reasoning in this episode, I categorized her interpretation of $f(c)$ and $f(d)$ as nominal when explaining whether $f(c)$ was equal to $f(d)$ in Graph 3.

While Annie used a nominal interpretation of expressions, she did not arrive at a conventionally correct conclusion regarding the evaluation of Statement 3 for either of the functions in Graphs 3 or 4. Unlike Micah and Adam, Annie used a nominal interpretation for expressions on the graph in an unconventional way. For Annie, the labels $f(c)$ and $f(d)$ were distinct and thus could not possibly refer to the exact same spot. In her mind, those two expressions referring to the same position on the graph would be the equivalent of "two ordered pairs on top of each other," which she dismissed. Annie concluded that on Graph 3, a monotone decreasing graph, $f(c)$ and $f(d)$ from Statement 3 could not equal each other because there were no y values that were repeated on the graph. Although she recognized that two expressions would have to be in the "same spot" in order to be equal, she did not consider two different labels ($f(c)$ and $f(d)$) as referring to the same position.

Annie was not the only student I interviewed who did not want to consider two different symbolic expressions as labels for the same position. Another student, Martha, who was enrolled in Calculus II at the time of the study, also did not want to label the same position with two different labels. Martha evaluated Statement 3 as false for the function in Graph 3 and labeled the graph as shown below in Figure 23. Martha labeled Graph 3 with *c* and *d* labeled at points on the graph and *f*(*c*) and *f*(*d*) labeled on the *y*-axis.

Statement 3: For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*.

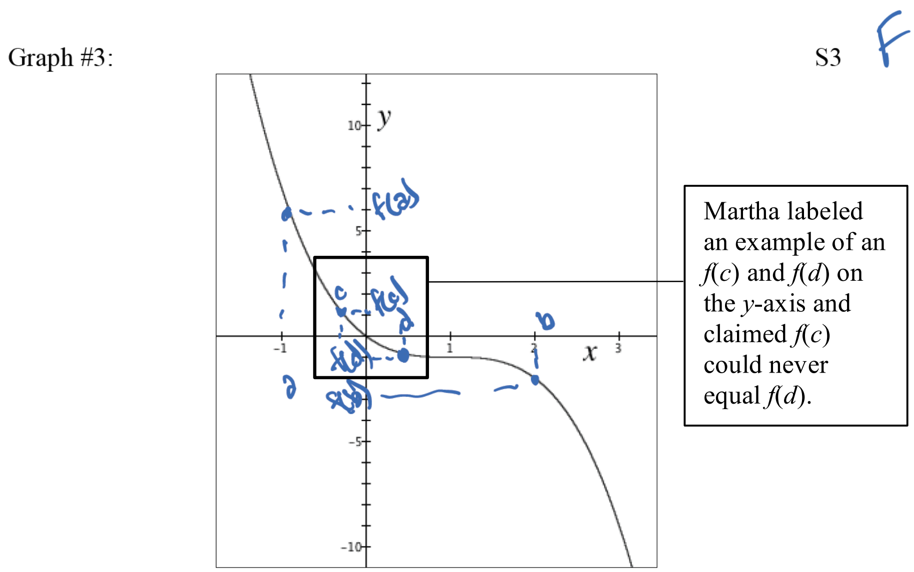


Figure 23. Martha's labels on Graph 3 when evaluating Statement 3.

Martha claimed that the *f*(*c*) and *f*(*d*) she labeled were not equal and could never be equal. She explained this claim by saying, "*c* and *d* aren't equal, and then *f*(*c*) and *f*(*d*) aren't equal, either. And *f*(*c*) and *f*(*d*) couldn't be equal on this graph because it's [*f*'s] always decreasing." Further, she claimed that *f*(*c*) and *f*(*d*) could never be equal on this graph because *f* is decreasing. In fact, Martha evaluated Statement 3 as false for all four functions shown in Graphs 1-4 (see Figure 15). I asked Martha if she could think of an example of a function where Statement 3 would be true. She thought for a few seconds and replied,

> I think it's [Statement 3] always going to be false because $f(c)$ and $f(d)$ are separate values. So I know I can't, they could be… maybe,?... No I don't think so. Yeah, I think f of yeah… c and d are gonna be, if they're [c and d] separate values, like they're [c and d] labeled separately I think they're [$f(c)$ and $f(d)$] gonna be separate. Yeah, yeah, I think it's [Statement 3] going to be [always] false.

Martha's explanation for why she thought Statement 3 was always false reveals how she was thinking about c and d. I infer from Martha's claim in this episode that, in her mind, if c and d were different, $f(c)$ and $f(d)$ would not be the same. Her claim implies that she was not considering the possibility for c and d to be equal. In her justification of why the statement is false for any function f, Martha associated c and d being "separate values" with being "labeled separately" on the graph. I take Martha's wording that c and d as "separate values" to mean unequal. Her language of "labeled separately" indicates that she was thinking of c and d being labeled in two different places on the graph, presumably on the x-axis, as she did in Figure 23. Martha considered c and d to be unequal ("separate values") because they would be labeled in different places on the graph. In this instance, Martha was thinking about c and d as labels for positions on the graph. For this reason, I categorize Martha's interpretation of expressions c and d as nominal, referring to labels on the graph. Considering c and d as distinct and, thus the labels as distinct, Martha, like Annie, did not consider cases where c and d both might refer to the same spot.

Summary of students' nominal interpretation of expressions on graphs.

The examples above from instances with four students (Micah, Adam, Annie, and Martha) highlight the characteristics of an interpretation of expressions as nominal on graphs. Students who interpreted expressions as nominals on graphs all placed expressions as labels somewhere on the graph and then referred to and reasoned with those positions in the graph as a proxy for the expression, to conclude whether expressions were equal or unequal.

Table 7 shows the pseudonyms of students whom I observed interpreting expressions as nominals on graphs while evaluating the statements in the interview, and the expressions in these statements which they interpreted nominally.

Table 7

Students Who Interpreted Expressions as Nominal in the Graphical Register

Statement	**Expressions**	**Students Using Nominal Interpretations of Expressions on Graphs**	**Count**
S1	N between $f(a)$ and $f(b)$		0
S2	$c<d$		0
	$f(c) < f(d)$		0
S3	$c=d$	Abe, Adam, Annie, John, Kate, Martha, Micah, Tim	8
	$f(c) = f(d)$	Abe, Annie, John, Kate, Martha	5
S4	$-\delta < x-1 < \delta$		0
	$-\varepsilon < f(x)-f(1) < \varepsilon$		0
S5	h		0
	$f(2+h)-f(2)$		0
S6	$f(b)-f(a)$		0
	$b-a$		0

Table 7 shows that students who used a nominal interpretation of expressions in some instance did so while interpreting "$c=d$" or "$f(c) = f(d)$" from Statement 3. In other words, students in this study who used a nominal interpretation of expressions, did so to compare two inputs or outputs for equality.

Students in the instances which I categorized as using a nominal interpretation of expressions all treated expressions as labels for positions, positions themselves, and symbols interchangeably. For these students, the sameness of position implied the equality of expression (what I take to be sameness of value) and vice versa. For some, the sameness of position implied not only the sameness of value (equality) but also the sameness of label. Figure 24 illustrates a student's nominal interpretation for *x*, in which *x* is viewed simultaneously as a value, a position on the graph, and a label for the position on a graph.

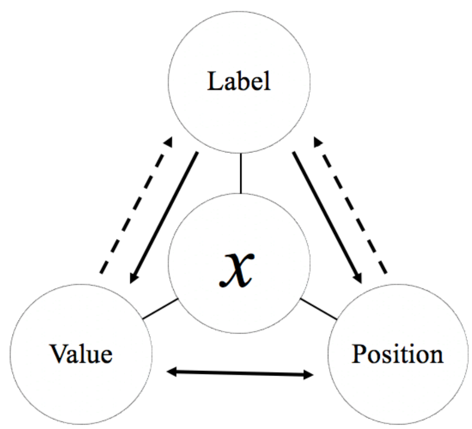


Figure 24. Variations on nominal interpretations of *x* in a graph.

The arrows in Figure 24 represent an implication of sameness that students used when interpreting expressions nominally. Specifically, the solid, double-sided arrow between "value" and "position" denotes that all students whom I categorized as using a nominal interpretation expressed that the sameness of position implied the equality of expressions and vice versa. All students using a nominal interpretation of expressions reasoned that the same label would imply the equality of value, or the same position. The dashed arrows denote unconventional uses of nominal interpretation of expressions on graphs, used by some students in this study. These dashed arrows represent that the equality of value implies the sameness of label, or symbol used, and that the same position implies the same label. Both of these implications were expressed by Annie and Martha, and are considered to be unconventional in the Cartesian coordinate system.

Expressions as Ordinal on Graphs

In addition to interpreting expressions nominally on graphs, I observed other instances in which students used other distinct interpretations of expressions when reasoning with them in the graphical register. Some students ordered expressions by using a spatial relation of positions in the graph. By spatial relation of positions, I mean vertical or horizontal relations, i.e., a position is above/below a position, or a position is to the left of/to the right of another position. Students using this interpretation would label expressions at positions in the graph and then spatially relate these positions to each other to describe the order of the expressions. Often, students used this interpretation of expressions on the graph to claim that one expression was less than, equal to, or greater than another expression. Unlike students using a nominal interpretation of expressions, students interpreting expressions in this way described the *ordering* of expressions using the positions labeled in relation to each other on the graphs. In this way, students used expressions as relative reference points for other expressions in the graph. I refer to this way of interpreting expressions on graphs as *ordinal*. I consider the use of this term consistent with Merriam-Webster Dictionary's definition of ordinal as "a number designating the place (such as first, second, or third) occupied by an item in an *ordered* sequence" ("ordinal," Merriam-Webster, 2019). While these assignments of expressions in the graph do not denote rank per se, they *do* denote an ordering and a means of determining ordering through the spatial relations the student used on the graph. Students using an ordinal interpretation of expressions on graphs relied on the spatial relationship they perceived between the positions of the expressions labeled on the graph, rather than numerical values, amounts or measurements in the graphs.

In this study, some students in some instances interpreted expressions ordinally on the graphs when comparing two or more expressions. This interpretation was most prevalent when students evaluated Statements 1, 2, and 4 for various functions. These statements contained expressions involving inequalities or ranges of values. Statement 1 reads, "For all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$," Statement 2 reads, "For all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$" and Statement 4 reads, "For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $-\delta < x-1 < \delta$, $-\varepsilon < f(x)-f(1) < \varepsilon$." With these statements, many of the students I interviewed used some form of a spatial comparison on the graph to order a pair or triple of expressions from these statements. Some students used both justifications from the graphical and numerical registers in their explanations of the expressions in the statements. In this section, I will provide examples from students who used an ordinal interpretation of expressions in graphs in different ways, using both spatial and temporal relations to compare positions on graphs and order expressions. While students used the nominal interpretation of expressions in conventional and unconventional ways, in this study, the students who used an ordinal interpretation of expressions did so in a conventional way.

Ordinal interpretation of expressions: spatial comparison.

Jess, a student in a senior-level Topology course at the time of the interview, interpreted expressions on the graph through a spatial relation of positions when evaluating Statement 2 for the function in Graph 1. Jess first labeled *c* and *d* on the *x*-axis, and then labeled the corresponding *f*(*c*) and *f*(*d*) on the *y*-axis (Figure 25).

Statement 2: For all real numbers *c, d* in (*a, b*), if $c < d$, then $f(c) < f(d)$.

.

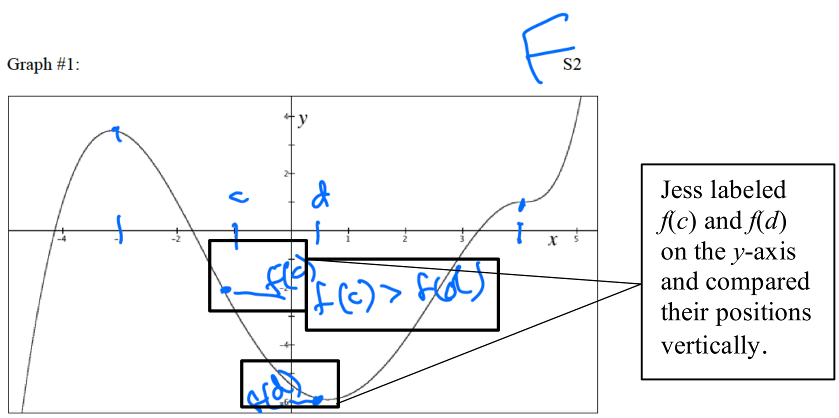


Figure 25. Jess' labels on Graph 1 when evaluating Statement 2.

Jess claimed that the example of *f*(*c*) and *f*(*d*) she chose disproved Statement 2 because *f*(*c*) was greater than *f*(*d*). Jess explained that if she chose *c* and *d* as she did in Figure 25, *f*(*c*) would be greater than *f*(*d*). As she explained this, she drew in two tick marks on the *x*-axis and labeled them *c* and *d*, marked points on the graph with a dot and then drew horizontal segments to the y-axis.

Jess At these points, you have an *f*(*c*) that is greater than *f*(*d*) (*labels f(c) and f(d) on the y-axis*).
Int … how do you know that?
Jess uhh vertically [f of] *c* is higher than [f of] *d*.

In this episode, Jess first identified the positions of $f(c)$ and $f(d)$ and labeled them on the y-axis. To make a comparison of $f(c)$ and $f(d)$, Jess compared the vertical orientation of these positions on the y-axis to each other. Specifically, Jess concluded that $f(c)$ was greater than $f(d)$ because the position labeled $f(c)$ was higher "vertically" on the graph than $f(d)$. In reasoning about the expressions in this way, Jess interpreted $f(d)$ as a reference point from which she compared the position of $f(c)$. While Jess did not use the origin as a reference point in comparing the values of $f(c)$ and $f(d)$, the spot she labeled as $f(d)$ became a relative reference point to spatially locate $f(c)$. By describing $f(c)$ as higher than $f(d)$, and thus, greater, Jess used a spatial ordering of these expressions. Jess made this visual comparison about the positions she had labeled as $f(c)$ and $f(d)$ to draw conclusions about the comparison of two expressions, rather than referring to numerical values of $f(c)$ and $f(d)$ to draw her conclusion. For this reason, I classify her interpretation of expressions in the graphical register in this instance as ordinal.

Ordinal interpretation of expressions: temporal comparison.

While some students used spatial language, like Jess, in their ordinal interpretation of expressions, other students used temporal wording (e.g., before/after, earlier/later) to describe the order of expressions on graphs. John, enrolled in Calculus II at the time of the interview, was one such students who used temporal wording to describe the ordering of expressions on Graph 1 with Statement 2. When evaluating Statement 2 for the function in Graph 1, John labeled the graph (Figure 26) and explained how he interpreted $c < d$ and $f(c) < f(d)$. John first labeled c on the x-axis and $f(c)$ on the graph. Then, he explained that it is possible for “*d* to come after *c*,” but for *f*(*d*) to be less than *f*(*c*). As he said this, he labeled *d* on the *x*-axis and *f*(*d*) on the graph (Figure 26). The following transcript contains John’s explanation of his interpretation of the expressions $c < d$ and $f(c) < f(d)$.

Statement 2: For all real numbers *c, d* in (*a, b*), if $c < d$, then $f(c) < f(d)$.

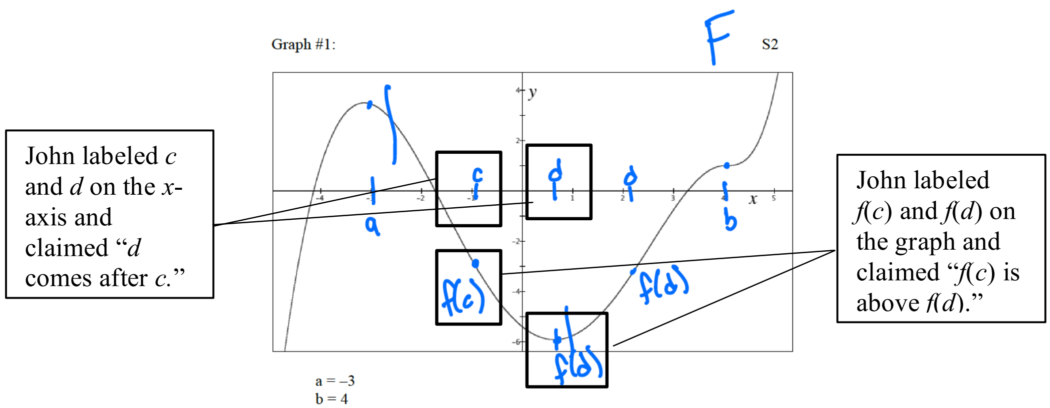


Figure 26. John’s labels on Graph 1 when evaluating Statement 2.

Int Okay so when you said “*d* comes after *c*” what are you thinking about with that?
John Um I was just thinking about …so if *f* is … always decreasing then *f*(*d*) here then *d* would have to come before *c* in order to make this statement true so if *f*(*c*) comes first and is above *f*(*d*), then the statement will be false.
Int Okay and when you say “comes first” what do you mean by that?
John I just mean that it [*c*] like, it’s [*c*] earlier in the interval over, if we’re just taking a look at like this section where it’s [*f*] only decreasing (*marks off decreasing portion of graph with vertical segments on graph*) since it comes first in the interval where it's [*f*] only decreasing…*c* lies before *d* which means that if it's [*f*] only decreasing …then *f*(*d*) will be lower than *f*(*c*).

In this portion of the interview, John explains how he is interpreting the expressions when comparing *c* to *d* and *f*(*c*) to *f*(*d*). John labeled *c* and *d* on the *x*-axis and claimed that “*d* comes after *c*” or that *c* is “earlier in the interval” than *d* (Line 7). Implicit in John’s explanation is a notion of ordering that is related to the spatial configuration of the graph. For John, this ordering was related to notions of time, as indicated by the words "earlier" and "after." I infer that John's notion of one value coming before or earlier than another value to mean as one visually scans the graph from left to right. In other words, John was using a horizontal comparison of the positions of c and d. In claiming "d comes after c," John used c as a relative reference point to describe d. Similarly, his claim that c is "earlier in the interval" than d used d as a relative reference point for c. Even when I asked him to further explain what he meant by the ordering phrases he was using ("comes after," "comes first"), John did not appeal to numerical values of c and d, but instead relied on the ordering of the expressions in relation to each other temporally or spatially. Thus, while John used temporal words to describe the ordering of c and d in order to claim that c was less than d, his explanation was indicative of an ordinal use of expressions in the graph.

In the excerpt above, John also used spatial language to describe the order of $f(c)$ and $f(d)$ in the graph. John labeled $f(c)$ and $f(d)$ at positions along the trace of the graph and compared the spatial orientation of these positions. John claimed that for the expressions he labeled on the graph, $f(d)$ would be less than $f(c)$ because $f(c)$ "is above $f(d)$" (Line 5), or, equivalently, "$f(d)$ will be lower than $f(c)$" (Line 11). In his explanation for why $f(d)$ would be less than $f(c)$ in his example labeled on the graph, John appealed to the spatial orientation of the positions he labeled as $f(c)$ and $f(d)$. John used spatial language of "above" and "lower" to describe the relationship between the positions. John explained that $f(d)$ was less than $f(c)$ using the vertical spatial comparisons that $f(c)$ was above $f(d)$, and that $f(d)$ was lower than $f(c)$. Again, John used the relative positions of $f(c)$ to $f(d)$ and vice versa to compare the two. Although John labeled positions along the graph with output labels, he used the spatial structure of the graphical register conventionally by comparing these two positions, labeled as expressions, vertically. In making a vertical comparison between $f(c)$ and $f(d)$, John described the order of these two expressions. Due to his spatially comparative language, I claim that John used an ordinal interpretation of expressions in this instance.

Summary of ordinal interpretation of expressions on graphs.

The examples above from instances with Jess and John highlight the characteristics of an interpretation of expressions as ordinal on graphs. Students who interpreted expressions ordinally on graphs all related values by first assigning expressions to each position, and then relating the positions of these expressions on the graph. In doing so, students use the position of one expression as a relative reference point for the positions of the other expressions. By comparing these two positions to each other, students describe an ordering among the positions, and by extension, the values of the expressions. The following table shows the names of students whom I observed interpreting expressions ordinally on graphs while evaluating the statements in the interview, and the expressions in these statements which they interpreted ordinally.

Table 8

Students Who Interpreted Expressions as Ordinal in the Graphical Register

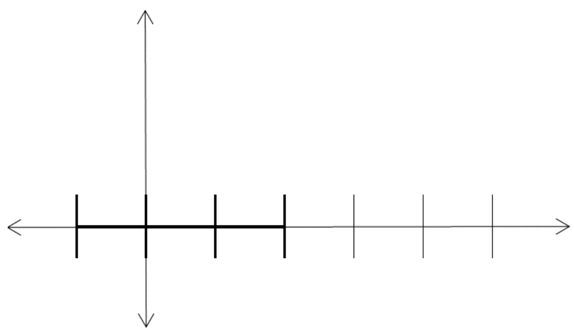
Statement	Expressions	Students Using Ordinal Interpretations of Expressions on Graphs	Count
S1	N between $f(a)$ and $f(b)$	Abe, Adam, Annie, Carl, Jeremy, Jess, John, Kate, Lola, Martha, Micah, Tim, Tina	13
S2	$c<d$	Abe, Carl, Jeremy, Jess, John, Lola, Tim	7
	$f(c) < f(d)$	Abe, Adam, Carl, Jeremy, Jess, John, Lola, Tim	8
S3	$c=d$	Carl, Adam	2
	$f(c) = f(d)$	Carl	1

S4	$-\delta < x-1 < \delta$	Adam, Martha, Tina	3
	$-\varepsilon < f(x)-f(1) < \varepsilon$	Abe, Annie, Kate, Martha	4
S5	h		0
	$f(2+h)-f(2)$		0
S6	$f(b)-f(a)$		0
	$b-a$		0

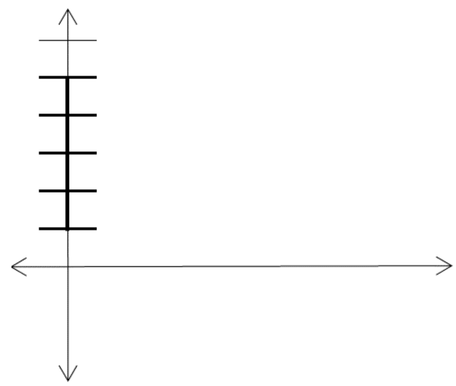
Table 8 shows that students predominantly used an ordinal interpretation of expressions on graphs with Statements 1, 2, and 4. In Statement 1, students described the spatial ordering of positions labeled $f(a)$, $f(b)$, and N to describe their interpretation of "between." To make determinations about this statement, students would describe whether or not a chosen N was spatially between $f(a)$ and $f(b)$ in the ordering they created. With Statement 4, students treated $x-1$ as a single position, and spatially compared the relative positions of $-\delta$, $x-1$, and δ to each other, in order to determine if the inequality would hold. Students who used an ordinal interpretation of expressions with $-\delta < x-1 < \delta$ often reasoned similarly with $-\varepsilon < f(x)-f(1) < \varepsilon$.

Expressions as Cardinal on Graphs

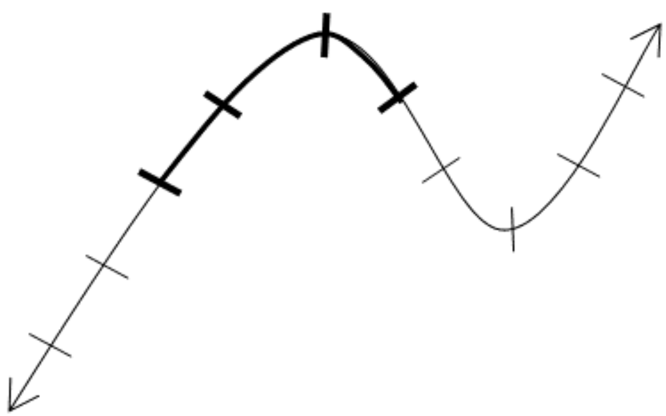
In addition to interpreting expressions nominally and ordinally in the graphical register, which focuses on the positions of expressions in relation to others, I also observed some students interpreting expressions as numerical values representing a measurement in the graph. Some of these students interpreted expressions as measuring a portion of an axis or trace of the graph between two reference points, which they measured by counting discrete units on that portion of the graph. By discrete units on the graph, I refer to the units that students perceived or ascribed to an axis or the trace of the graph to count. To some students, the units they perceived and then counted were the tick marks on the axes on the provided graphs. Other times, students ascribed units which were discrete points along the trace of the graph that they could count. Figure 27a-c shows ways students may have ascribed units and counted along a horizontal axis, a vertical axis, or the trace of the graph.



(a) Units ascribed along the horizontal axis



(b) Units ascribed along the vertical axis



(c) Units ascribed along the trace of the graph

Figure 27(a)-(c). Units ascribed and counted along a portion of an axis or graph–A key characteristic of a cardinal interpretation of expressions on graphs.

Students interpreting expressions in this way counted their chosen units to measure a portion of an axis or graph when describing their interpretation of an expression in the graph. I refer to this interpretation of expressions on graphs as a *cardinal* interpretation. Referring to this interpretation of expressions on graphs as cardinal is consistent with the Merriam-Webster definition of cardinal number as “a number (such as 1, 5, 15) that is used in simple counting and that indicates how many elements there are in an assemblage” (“cardinal,” Merriam-Webster, 2019). Students using a cardinal interpretation of expressions in graphs: (1) described a portion of an axis or graph with fixed reference points, (2) ascribed units to this portion, and (3) measured this portion by means of counting the ascribed units.

In ascribing and counting units in a graph, students reasoned additively to measure a portion of the graph. Because these students reasoned additively to interpret these expressions, which they often associated with numerical values, these students often only interpreted whole number values on the graph and did not connect non-whole number values to the graph. Students’ additive reasoning, a feature of their cardinal interpretation of expressions, presented some limitations in their interpretations of expressions on graphs in this study.

Students’ use of a cardinal interpretation of expressions in graphs was most apparent when they were interpreting expressions involving the difference of two inputs or outputs on the graph, such as those found in Statements 5 and 6. In this section, I provide examples from both students, Kate and Annie, who used a cardinal interpretation of expressions on some of the graphs they worked with.

Expressions as Cardinal: A Difference of Inputs Measured in Additive Units

Students who used a cardinal interpretation described an expression in a graph as indicating a count of units on a portion of the graph between two fixed reference points. Kate, who was enrolled in Differential Equations at the time of the interview, was one such student who interpreted an expression involving a difference this way when she was evaluating Statement 6. Statement 6 is the conclusion of the MVT and reads as follows: "There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$." When Kate was evaluating Statement 6 with the function shown in Graph 5, she interpreted "*b*–*a*" as a measurement of a portion of the *x*-axis, indicating a count of the number of intervals on the *x*-axis between *a* and *b*. Figure 28 below contains Kate's graph labels in which she labeled *b*–*a* as referring to the three squiggles on the *x*-axis. The transcript of our conversation about her interpretation of the expressions in Statement 6 on Graph 5 follows.

Statement 6: There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

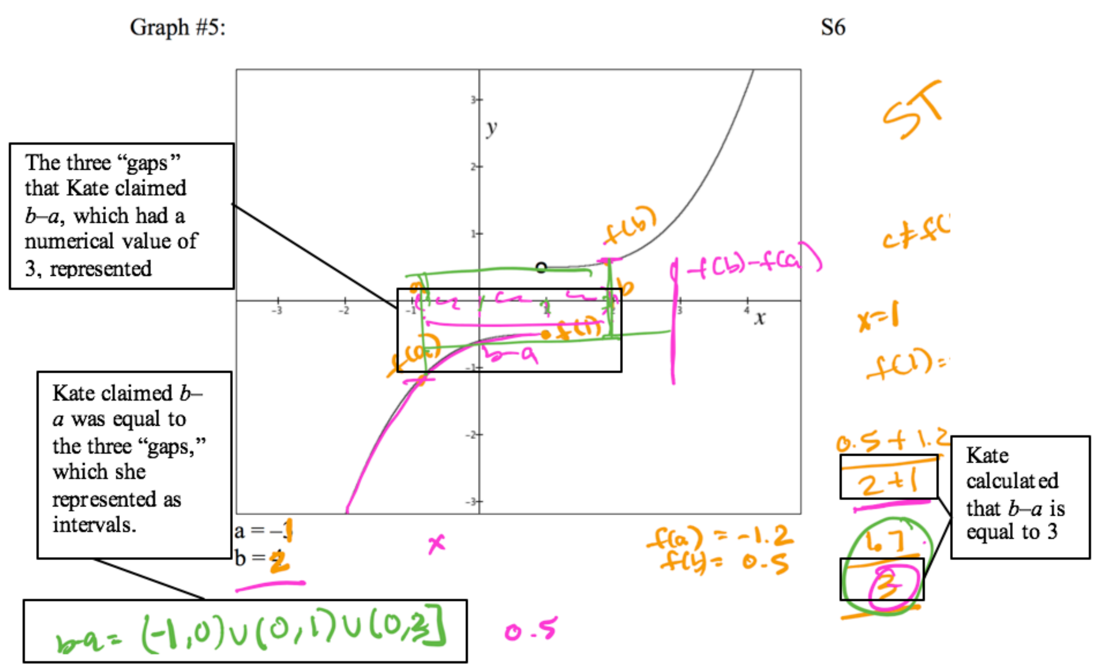


Figure 28. Kate's labels on Graph 5 when evaluating Statement 6.

Int Is there any way on the graph that you would want to label if you could…
what "*b*–*a*" was?

Kate So "*b–a*" I would say is the domain, like any numbers between those values [*b* and *a*] … I would say that *b–a* is for the *x* values…and then this part is *b–a* [*draws horizontal segment in pink parallel to x-axis from a to b and Labels it 'b–a'*].

When I asked Kate to label "*b–a*" on the graph, she described "*b–a*" as referring to the *x*-values, and drew and labeled a horizontal segment along the *x*-axis starting at *a* and extending to *b*. At this point in the interview, I hypothesized that Kate may have been thinking about "*b–a*" as the distance along the *x*-axis, as other students had described in prior interviews. To test this hypothesis, I asked Kate to explain how she was thinking about the numerical value she had calculated for the difference between *a* and *b*, which was 3. Instead, I found that Kate was thinking about *b–a* quite differently than I originally anticipated.

Int Okay so when you got the calculation …that *b–a* was 3, what does that '3' tell you?
Kate … so the 3 right here I would say is like the numbers… there's like three, one, two, three, so like there's three little (*draws in three pink squiggles on the x-axis, one between each of the tick marks from –1 to 2*), that part is like your domain for what you're like specifically looking at.
Int Okay… so these little like pink like squiggles here that you were marking off (*pointing to squiggles she drew on x-axis*), do you remember what you meant by doing that when you were doing that?
Kate Oh, yeah. I was just sort of, because, basically there's three tick marks between those [–1 and 2 on the *x*-axis], well there's three gaps because you have –1 to 0, and you have 0 to 1, and then 0 to 2, (*writes (–1,0) U (0,1) U (0,2)*[9]) so that's kind of how I thought about it right there that's I guess if I were to give those little squiggles the definition this [referencing (–1,0) U (0,1) U (0,2)] would be it.
Int Okay, and what did that have to do with *b –a*?
Kate Um because I feel like this [(–1,0) U (0,1) U (0,2)] is what *b-a* is equal to.

[9] I infer Kate's intention here as the intervals (–1,0), (0,1) and (1,2)

In this episode, Kate interpreted *b–a* as a count of unit intervals between positions she had marked as *a* and *b* along the *x*-axis. Kate first labeled "*b–a*" as a portion of the *x*-axis between two fixed reference points labeled *a* and *b*. Kate then described her units which she ascribed to this portion of the *x*-axis as "gaps" (Line 11). These "gaps" were Kate's units to measure the segment of the *x*-axis which she labeled as "*b–a*." Kate counted these units, "one, two, three" (Line 4) and denoted with 3 squiggle marks (Lines 4-5) the three gaps (Line 11) she was referring to between *a* and *b*. Kate used the "three tick marks" (Line 10) provided on the graph at integer values from –1 to 2 to ascribe units to the *x*-axis, splitting it into three equal sized "gaps" from "–1 to 0", "0 to 1," and "[1] to 2" (Lines 12-13).

For Kate, *b–a* (with a numerical value of 3) counted the number of "gaps" or spaces on the *x*-axis between *a* and *b*. I refer to these gaps that Kate counted as *intervals* on the *x*-axis, consistent with her use of interval notation (Lines 12-13). The expression *b–a*, for Kate, represented a segment on the *x*-axis in units of "gaps," which she measured by counting, starting at *a* and ending at *b*. In Kate's mind this *b–a* represented a count of units in the graph, rather than represent a position, ordering, or measurement in multiplicative units. Kate's actions of counting verbally, marking three distinct objects on the graph between her chosen reference points, and writing out three intervals on her paper, all contributed to my classification of Kate's interpretation of expressions on the graph in this episode as cardinal. While in this instance, Kate was able to interpret the meaning of a numerical value in a graph using a cardinal interpretation, this interpretation was limiting for students in other instances.

Expressions as cardinal: limitations of the use of units only additively.

Annie, enrolled in a Calculus III course, encountered the limitation of a cardinal interpretation of expressions in the graphical register when evaluating Statement 6. Like Kate, Annie interpret a difference in inputs as a count of units along the *x*-axis. Annie also attempted to interpret a difference in outputs as a count of units along the graph, which is when she expressed some confusion and was unable to interpret these expressions in a similar manner. When evaluating Statement 6 with Graph 1, Annie described her interpretation of the differences "$f(b) - f(a)$" and "$b–a$" in the statement. Figure 29 shows Annie's labels on Graph 1 when interpreting Statement 6.

Statement 6: There exists a real number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

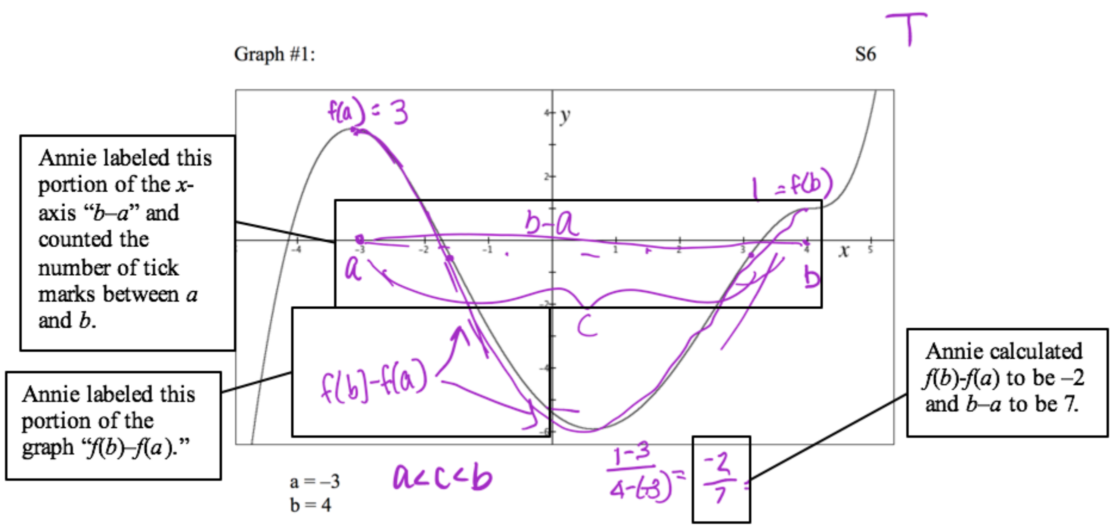


Figure 29: Annie's labels on Graph 1 with Statement 6, showing her labels of differences on the graph.

When interpreting Statement 6 with Graph 1, Annie spoke about the differences "$f(b) – f(a)$" and "$b–a$" in the statement as "changes in the values." When I asked her to represent those changes on the graph, Annie drew a line from b to a on the horizontal axis and claimed "this is $b–a$." She also traced over the graph from the point she had labeled $f(a)$ to the point she had labeled $f(b)$ and claimed "this whole thing is $f(b) – f(a)$." She described the values she calculated for the quotient in the statement, as –2/7. She explained this was related to slope and that "usually when you see slope, it would be like down 2, over 7" but claimed that was "not the idea here." Instead, Annie referred back to the "changes in y values over the changes in the x values" and motioned to the portions of the graph she had marked off. She then explained that the changes in the x values were "all of these values between our b and a [*motioning over x-axis from a to b*], which is 7" Then, we had the following exchange:

Int Okay so what does the 7 have to do with all those values that you're referencing between a and b?
Annie Well there's 7 of them, like, in between, like, if you were to count them out there's 7 (*tapping on the tick marks on the x-axis between a and b, –3 and 4*).
Int Okay
Annie But that wouldn't really make sense here… I'm confused.
Int so yeah, …then is there a way to think about –2 where you have it [labeled], between like $f(a)$ and $f(b)$?
Annie Well…since there's, well it all depends off of your x-value.
Int Okay.
Annie So, like, yeah we can get that 7 because there's like 7 real numbers in between that um but for this (*points to f(b)–f(a) label on graph*) it's not like there's –2 numbers in between… I don't know.

Annie used a cardinal interpretation of the value of the difference "$b–a$," which she calculated to be 7, as a count of units along a portion of the x-axis. Annie used a and *b* as reference points when referring to a portion of the *x*-axis which she labeled as "*b–a*." and claimed that *b–a* was "all of these values between our *b* and *a*." Annie then motioned along the units she ascribed to this portion of the *x*-axis and explained that she was counting them: "there's 7 of them… if you were to count them out there's 7" while motioning as though she was counting the tick marks on the *x*-axis (Lines 3-5). Her words and motioning indicate that she was thinking of the *x*-axis as consisting of units which she could count. The units Annie was verbally describing were, in her words, "real numbers" (Line 12). However, the units Annie counted were the discrete tick marks along the *x*-axis between *a* at –3 and *b* at 4. Annie's explanation implies that she would count up from 0 to count these tick marks which represented "real numbers." While she claimed to be counting a number of "real numbers in between" –3 and 4, Annie was really only counting the integer values located at the tick marks along the *x*-axis. Because Annie's interpretation of a difference was a measurement of a portion of an axis between two reference points in additive units, I categorized Annie's interpretation of the expression as cardinal.

Annie's explanation of "$f(b) - f(a)$" indicated an attempt to interpret this difference on the graph in a similar manner, but was met with conflict. In this case, Annie interpreted $f(b)$ and $f(a)$ at positions on the trace of the graph. While she placed these output labels on the graph, Annie found the numerical values associated with these outputs by projecting the points to the *y*-axis, and calculated the difference to be –2. Annie was confused when trying to interpret $f(b) - f(a)$ similarly to her interpretation of *b–a* described above. Annie claimed "it's not like there's –2 numbers in between" and trailed off in thought while pointing to the portion of the graph she had labeled "$f(b)$ – $f(a)$" (Lines 26-27). Annie was trying to make sense of "a change in values" which she claimed to be –2 as related to a number of units, between the positions she labeled $f(a)$ and $f(b)$ on the graph. Annie expressed conflict when she attempted to interpret the difference, $f(b) - f(a)$, on the graph with additive units of numbers between two positions on the graph. Annie had previously interpreted $b–a$ in terms of additive units, which were represented by tick marks, and did not express any conflict in explaining the meaning of the numerical value of "7" in that instance. However, Annie was unable to reconcile her interpretation of $f(b) - f(a)$ on the graph with a negative value. Because she was attempting to think about the units additively along the graph, which she anticipated counting up from 0, a numerical value of –2 was a source of conflict. At the end of the episode, Annie was unable to say any more about what the "–2" meant in the graph beyond "the change in the values [$f(b)$ and $f(a)$]." Thus, Annie's cardinal interpretation of the expression of a difference on the graph was limited to whole number values.

Annie's interpretation of a difference as cardinal on the graph was limiting in other instances in the interview as well. For example, when she was interpreting Statement 5 with Graph 1, she labeled a portion of the trace of the graph as "$f(2+h)–f(2)$" (Figure 30) and explained that she thought this difference "ha[d] to do with the y values along this part of the graph." As she said this, she pointed to the portion of the graph she had traced over between the positions she had labeled as $f(2+h)$ and $f(2)$. When I asked her to explain what exactly about that portion of the graph $f(2+h)–f(2)$ was referencing, she replied "all these little points I guess in between the $f(2+h)$ and $f(2)$ on the graph." As she said this, Annie motioned along the portion of the trace of the graph she had highlighted. She also gestured, moving her hand to the right and tapping on the table at spots equally spaced apart as she went. Annie had also calculated this difference to be –1.5. When I asked Annie what –1.5 would represent on the graph, she did not describe any portion of the graph but instead replied, "it would just be the change in *y* from *f*(2+*h*) to *f*(2)."

Statement 5: For all non-zero real numbers *h,* if 2+*h* is in (*a*, *b*), then $\frac{f(2+h)-f(2)}{h} > 0$.

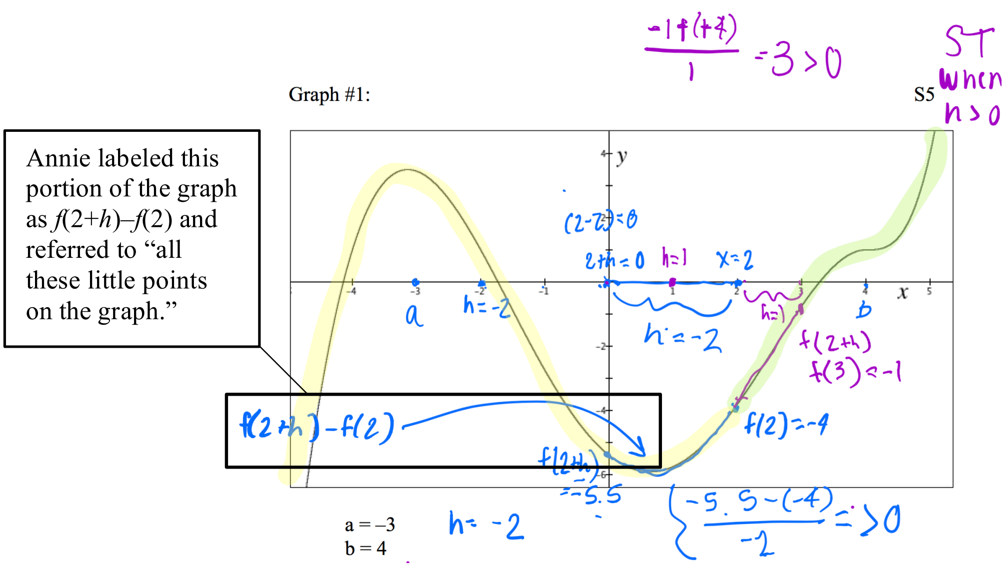


Figure 30. Annie's labels on Graph 1 with Statement 5, showing her label of a difference on the graph.

In this episode, Annie used a cardinal interpretation with the expression *f*(2+*h*) – *f*(2) on the graph. When describing how she was interpreting *f*(2+*h*) – *f*(2), she referenced "all these little points" along the trace of the graph, between the positions she had labeled *f*(2+*h*) and *f*(2). As she was describing "these little points," she motioned on the table, which I infer as identifying discrete points, or units, that she was visualizing when looking at the portion of the graph. Annie was interpreting $f(2+h) - f(2)$ as referring to a discrete number of points equally spaced along the portion of the graph from what she took to be $f(2)$ to $f(2+h)$. Because of her words and gestures, I classified Annie's interpretation of this expression as cardinal in the graph. However, her interpretation was limited as she was unable to connect "–1.5," the numerical value she calculated for this expression, to her interpretation of the expression in the graphical register.

Summary of cardinal interpretation of expressions on graphs.

While Annie and Kate were the only students to interpret expressions as cardinals on graphs in some instances, their interpretations were distinct from other episodes with other students. In the episodes described in this section, Annie and Kate both referenced a portion of an axis or graph, identified a unit along that portion, and then measured that portion by counting with that unit to interpret an expression in the graph. For Kate, the units she counted were the number of "gaps" or intervals between the tick marks along the *x*-axis between two reference points. For Annie, the units she counted were the number of whole number values or "points" between a starting point and ending point on an axis or on the trace of the graph. By interpreting expressions in the graphs as the result of a count of units in the graph, Annie and Kate reasoned about expressions and their numerical values additively on the graph. Neither Kate nor Annie expressed an attention to directionality of their counts and only interpreted whole number values of expressions in this manner. Table 9 summarizes the expressions for which Annie and Kate used a cardinal interpretation in the graphs.

Table 9

Students Who Interpreted Expressions as Cardinal in the Graphical Register

Statement	Expressions	Students Using Cardinal Interpretations of Expressions in Graphs	Count
S1	N between $f(a)$ and $f(b)$		0
S2	$c<d$		0
	$f(c) < f(d)$		0
S3	$c=d$		0
	$f(c) = f(d)$		0
S4	$-\delta< x-1 <\delta$		0
	$-\varepsilon <f(x)-f(1) < \varepsilon$		0
S5	h		0
	$f(2+h)-f(2)$	Annie	1
S6	$f(b)-f(a)$	Annie	1
	$b-a$	Annie, Kate	2

Expressions as Magnitudes on Graphs

The previous sections detail students' interpretation of expressions on the graphs as nominal, indicating locations on the graph, as ordinal, indicating an ordering, or as cardinal, indicating a measurement in additive units in the graph. Unlike these other interpretations, some students in some instances interpreted an expression as representing an amount of distance on a portion of a graph that could be measured from a fixed reference point on the graph. Unlike students using a cardinal interpretation of expressions, these students did not reason additively or count to measure this distance. Rather than identifying discrete units along a portion of the graph and counting these units, these students used distance or measurement language and labeled segments which they interpreted as having the length of the value to which they were referring. Students in these instances labeled expressions at the end of the portion of the graph with the specified length, or with a curly bracket to indicate a portion of the graph. They described the expression as the length of a whole segment, without reference to an additive unit, not breaking the segment up into units to count. Some students even expressed that the interpreted an expression as a directed distance, in which the sign of the numerical value of the expression indicated a direction between a starting and ending position on the graph. In these instances, I refer to these students as interpreting expressions as a *magnitudes* on the graph. I use magnitude consistent with Merriam-Webster's definition as "spatial quality: size" ("Magnitude," Merriam-Webster, 2019). In total, I observed six of the interviewed students, Micah, John, Jess, Carl, Tina, and Jeremy, interpret expressions on the graph as magnitudes with one or more expressions from the statements. In this section, I provide data from some of these students who interpreted expressions as magnitudes in different instances to highlight the characteristics of this interpretation.

Expressions as magnitudes: expressions as distances between two positions.

Students using what I refer to as a magnitude interpretation of expressions on graphs often described expressions as referring distances or lengths of segments of portions of axes. John, a student enrolled in Calculus II at the time of the interview, was one such student who interpreted differences in a graph as magnitudes. When evaluating Statement 4 with Graph 3, John described $x–1$ and $f(x)–f(1)$ as distances on the graph. On the graph, he drew in a horizontal segment on the x-axis from x to 1 and labeled it x-1 and a vertical segment labeled $f(x)–f(1)$ (Figure 31). John explained that the distances represented by these segments were $x–1$ and $f(x)–f(1)$, respectively.

Statement 4: For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $–\delta< x–1 <\delta$, $– \varepsilon <f(x)–f(1) < \varepsilon$.

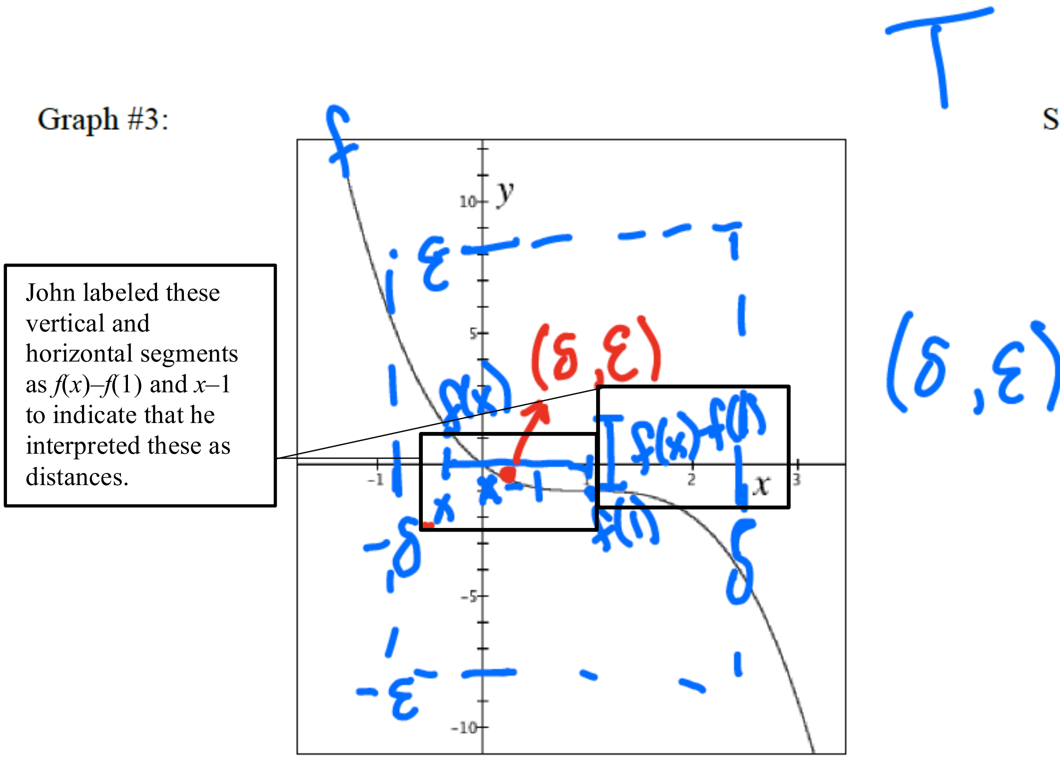


Figure 31. John's labels on Graph 3 when evaluating Statement 4

Int Um okay so you have x and 1 labeled, you have $f(x)$ and $f(1)$ labeled. Is there any way that you could indicate in the graph $x - 1$ and $f(x)–f(1)$?

John So let's say $x–1$ is just this distance and $f(x)–f(1)$ is that distance (*labels x–1 on segment of x axis and f(x) –f(1) and parallel to y axis*).

John explained that $x–1$ was a "distance" (Line 11) "in between" (Line 6) the where position he marked x and the position where he marked 1 on the x-axis. John used a horizontal segment to indicate the distance he was referring to when marking $x–1$ on the graph. Similarly, John explained that $f(x)–f(1)$ was a "distance" (Line 11) "in between" (Line 5) where he had marked $f(1)$ on the graph and a height corresponding to where he had marked off $f(x)$. He drew in a vertical segment to indicate this distance. John interpreted the differences, $x–1$ and $f(x)–f(1)$, as distances on the graph. For John, the magnitude of these segments corresponded with the differences from the statement. These distances had clear starting and ending points on the graph related to the variables involved in the difference. Unlike students like Kate who used a cardinal interpretation for a difference, and broke a segment into units which she measured additively, John referenced the distance as a singular amount. Further, John represented this distance with a segment that was not broken up into additive units. Based on his words and graph labels, I claim that John used a magnitude interpretation of differences on the graph in this instance.

Expressions as magnitudes: expressions as distances from a reference point, 0.

Not only did students use a magnitude interpretation of expressions on the graphs while interpreting a difference, but some also used this interpretation for single inputs and outputs on the graph. In doing so, they interpreted these variables as distances from a position labeled 0 on their respective axes. Micah was one such student who conceived of expressions as distances from 0 along the axes when interpreting Statement 2, which reads, "For all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$." When explaining his interpretation of Statement 2, Micah claimed that the statement "may hold for some values of *c* and *d* but definitely not for all." To illustrate his claim, Micah drew a graph of a linear function, labeled *c* and *d* on the *x*-axis and $f(c)$ and $f(d)$ on the graph, as shown in Figure 32. He explained that in his example, "*c* is less than *d* but $f(c)$ is not less than $f(d)$." When I asked him to further explain how he knew this, he replied "so *c* is less than *d* so I guess the distance here from 0 to *c* [*draws in horizontal curly bracket from the origin to c on x-axis*] is less than the distance from 0 to *d* [*draws in horizontal curly bracket from the origin to d on x-axis*] I guess so this distance is smaller than this distance right here." To explain that *f*(*c*) is not less than *f*(*d*), he drew two vertical curly brackets on his graph, one from the origin to *f*(*c*) on the *y*-axis, and one to the right of the point he labeled *f*(*d*) from the origin up to this point. He explained, "then now *f*(*c*) we're, we're doing vertically so this distance from 0 to *f*(*c*) is …smaller than the distance from 0 to *f*(*d*), because we go from 0 when we count."

Statement 2: For all real numbers *c, d* in (*a, b*), if *c* < *d,* then *f*(*c*) < *f*(*d*).

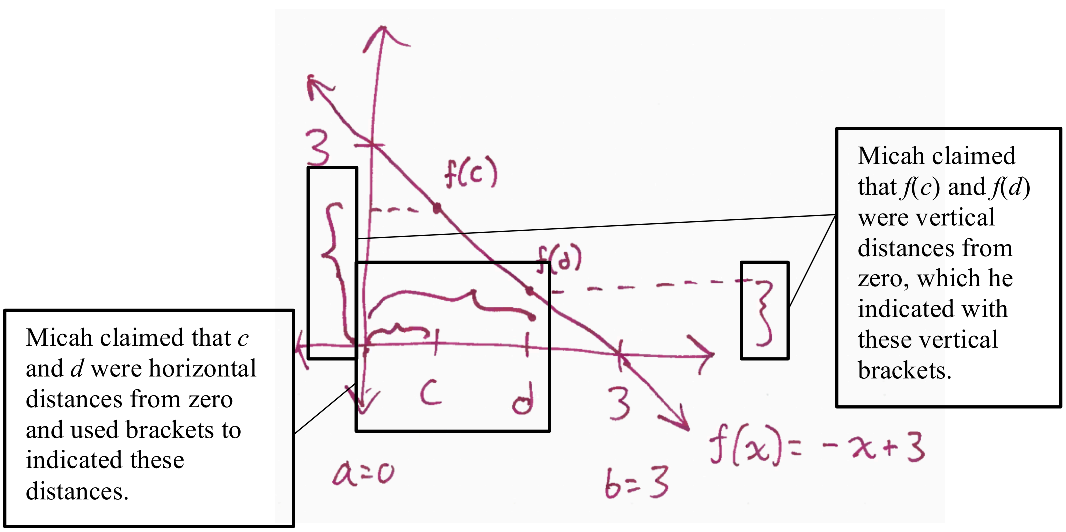


Figure 32. Micah's self-drawn graph to illustrate an example where he claimed Statement 2 was false, with distances marked as curly brackets.

To compare the pairs, *c* and *d* and *f*(*c*) and *f*(*d*), Micah compared the distances from 0 on their respective axes. In doing so, Micah revealed that he was reasoning about these expressions as magnitudes on the graph. Micah drew horizontal curly brackets from the origin, which he referenced as 0, to *c* and *d*, and explained that he was comparing the size of these distances to illustrate that *c* was less than *d*. Similarly, Micah drew in vertical curly brackets to compare the distances from 0 at the origin to $f(c)$ and 0 to $f(d)$. He also connected the reference point for measuring distance on the graph from 0 to a purely numeric reference point of 0 by commenting, “because we go from 0 when we count.” Although Micah mentioned the notion of counting when describing 0 as a reference point, Micah did not count by units when describing the distances in the graph. Instead, Micah compared the magnitudes of the distances, represented as segments on the graph. For Micah, comparing the value of pairs of inputs and outputs was connected to comparing distances from 0 in the direction of the axes on the graph. Thus, when working with the graph of a function in the Cartesian coordinate system, Micah conceived of inputs and outputs as representing a distance from a reference point of the origin, in the direction of either the *x* or *y*-axis, respectively. For this reason, I classified Micah’s interpretation of expressions in the graphical register as magnitudes when explaining his understanding of Statement 2.

Expressions as magnitudes: expressions as directed distances.

Some students who interpreted expressions as magnitudes and described them as distances on graphs also interpreted these distances as having direction, indicated by the signs of the expressions’ numerical values. Jess was one such student who described the meaning of negative values as directed distances on the graph in her examples she drew to explain Statement 5. Jess drew an example of a function for which she claimed Statement 5 was false (Figure 33). Jess labeled each graph and explained why she thought each function made the statement true or false. Below is the transcript from her explanation of why the second function she drew made the statement false.

Statement 5: For all non-zero real numbers h, if $2+h$ is in (a, b), then $\frac{f(2+h)-f(2)}{h} > 0$.

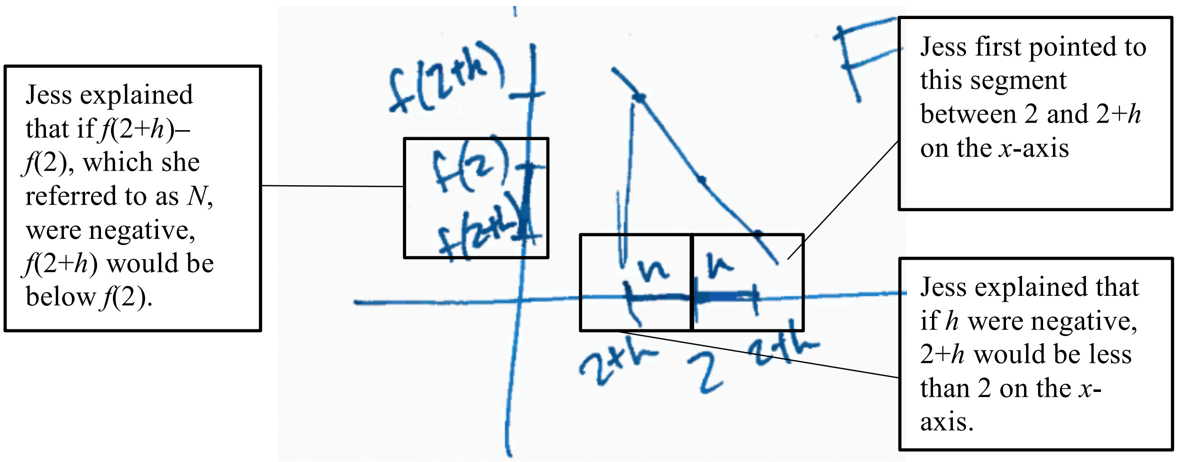


Figure 33. Jess' self-drawn graphs showing examples of negative values as distances.

Jess: 2, 2+*h*, *f*(2), *f*(2+*h*), and then this value here is negative, [*pointing to segment highlighted between f(2) and f(2+h) on y-axis*] this value here [*pointing to segment highlighted between 2 and 2+h*] is positive, so it's greater than zero.

Int: Okay… Is it okay if a distance is negative?

Jess: yeah it would just be *h*, *h* here [*labels h to the left of 2 above the x-axis*] 2, 2+*h* [*labels 2+h to the left of 2*] where *h* is a negative number.

Jess' words and labels on the graph indicate that she was thinking of expressions as directed distances on the graph in this instance. Jess first labeled *f*(2+*h*) on the *y*-axis, below *f*(2) and explained that the value of their difference, which she highlighted as the segment on the *y*-axis would be negative. When I asked Jess explicitly if "it was okay for a distance to be negative," she responded affirmatively and gave an example of *h* being negative, which resulted in 2+*h* being placed on the *x*-axis to the left of 2. While Jess did not explicitly state that her interpretation of *h* included direction, her examples in which she labeled *h* did include an attention to direction. Jess labeled *h* as a distance to the right of 2 when *h* was positive and labeled *h* as a distance to the left of 2 when *h* was negative. Jess interpreted differences on the graph as magnitudes, which, for her, included a notion of direction, along with distance.

Summary of expressions as magnitudes.

Students who used a magnitude interpretation of expressions on the graph indicated that an expression specified a distance on a graph from a reference point. Students using this interpretation would mark a starting point or reference point on the graph, from where they would measure said distance. These students spoke about lengths, distances, or measurements of segments on the graph when referring to expressions. Typically, they would draw segments to illustrate the distance they were referring to. The segments they drew were not divided up into equally spaced portions but rather reasoned about as a whole. In comparing the value of expressions, students using a magnitude interpretation compared the size or amount of these segments. Some students even expressed the notion of direction with distances, especially when describing negative values as distances on graphs. Not all students who used a magnitude interpretation of expressions explicitly stated that the expressions they were interpreting as distances could have numerical values that were negative. Yet, this was a distinguishing factor among students using a magnitude interpretation from those using a cardinal interpretation, who could not interpret negative values in the graphical register.

Table 10 summarizes the expressions for which students used a magnitude interpretation of expressions. Notably, the expressions involving a difference evoked most of the magnitude interpretations on graphs. Also, four students were observed using a magnitude interpretation with expressions from more than one statement: Micah, Jess, John, and Tina.

Table 10

Students Who Interpreted Expressions as Magnitude in the Graphical Register

Statement	Expressions	Students Using Magnitude Interpretations of Expressions on Graphs	Count
S1	N between $f(a)$ and $f(b)$		0
S2	$c<d$	Micah	1
	$f(c) < f(d)$	Micah	1
S3	$c=d$		0
	$f(c) = f(d)$		0
S4	$-\delta< x-1 <\delta$	Carl, Jess, John, Micah, Tina	5
	$-\varepsilon <f(x)-f(1) < \varepsilon$	Carl, Jess, John, Micah, Tina	5
S5	h	Jess, John, Micah, Tina	4
	$f(2+h)-f(2)$	Jess, John, Micah, Tina	4
S6	$f(b)-f(a)$	Jeremy, John, Micah	3
	$b-a$	Jeremy, John, Micah	3

Reasoning in the Numerical Register with Graphs

In contrast with many of the students described above, some students in some instances reasoned about values in the graphs in the numerical register rather than the graphical register. These students used the graph to find numerical values for variables or expressions in the statements and then reasoned about these numerical values when evaluating the statements. One example of a student reasoning in this way was Martha, a student enrolled in Calculus II at the time of the interview, when interpreting Statement 2, which reads as follows: “For all real numbers c, d in (a, b), if $c < d,$ then $f(c) < f(d)$.” When Martha read Statement 2, she drew the graph shown in Figure 34, and labeled it.

She then claimed that for the graph of the function that she drew, Statement 2 would be false. Martha explained her reasoning by comparing numerical values of *f*(*c*) and *f*(*d*).

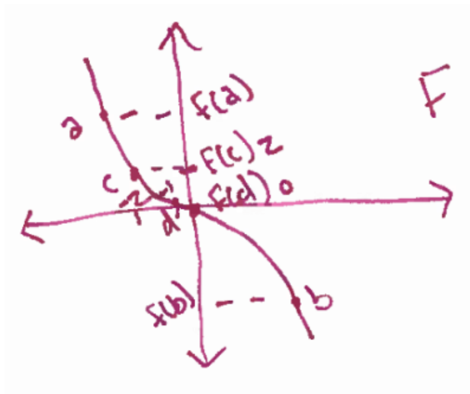


Figure 34. Martha's labels on a graph of *f* she drew to claim Statement 2 was false.

Martha	Yeah, all right in this case it wouldn't work…*c* might be less than *d* but then *f*(*c*) is greater than *f*(*d*).
Int	Okay so on that picture how can you tell that *c* is less than *d*?
Martha	Um because … d's at zero and it, c's at like negative two.

Martha explained that in the graph that she drew, *c* was less than *d* because *c* was negative two and *d* was zero. Unlike other students who used a spatial comparison of positions, or distances in the graph to make conclusions about expressions, Martha used the graph to find numerical values of *c* and *d* and then drew conclusions while reasoning in the numerical register. For the students who consistently reasoned about expressions by finding numerical values, the graphs provided were not a structured system to reason within, but simply a means of generating numerical values.

Summary of the Four Interpretations of Expressions in the Graphical Register

The four interpretations of expressions on graphs that I observed in this study, described in this chapter are: nominal, ordinal, cardinal, and magnitude. Each of these four interpretations describes how students interpret and use expressions in the graphical register. I view these four categories as answering Research Question 1 of this study.

A nominal interpretation of expressions is characterized by using a value as a label of a position in a graph, typically on an axis or along the trace of the graph. Students using a nominal interpretation of expressions typically label values on the graph, use locating language to describe the expression, such as “*x* is here.” These students then reason with *x* as a position in making determinations about it. In this study, I observed students using a nominal interpretation of expressions in order to compare two values for equality by comparing their positions on the graphs for sameness. Expressions in the same position were determined to be equal while expressions in different positions were determined to be unequal.

An ordinal interpretation of expressions is characterized by describing the order of values using the spatial arrangement of two or more expressions in a graph. Students using an ordinal interpretation typically label two or more expressions on a graph and compare the locations of these expressions horizontally and vertically. Expressions found to be to the left of other expressions and those found to be below are considered less than, while expressions to the right of or above are considered greater than. Like a nominal interpretation, an ordinal interpretation of expressions is concerned with the position of expressions in relation to each other. While a nominal interpretation of expressions may only be used to compare sameness of position, an ordinal interpretation of expressions uses positions of expressions as relative reference points to describe the positions of other expressions. This comparison of positions supports a student in ordering the expressions.

A cardinal interpretation of expressions is characterized by using expressions to indicate a measurement of additive units on a portion of the graph. The key characteristic of this interpretation is that an expression is the result of a counting process, as indicated by verbal or gestured counting. For one student, expressions counted a number of intervals on an axis. For another student, expressions counted a number of integer values on an axis or a number of points along the trace of the graph. In this study, students used a cardinal interpretation of expressions to describe a difference as a number of units between two reference points associated with these two expressions comprising the difference. Unlike students who used a nominal or ordinal interpretation of expressions and did not refer to numerical values, these students described the numerical value of a variable or expression as an additive amount of units on the graph. A cardinal interpretation of expressions may present a limitation, as this interpretation only supports the use of whole number values of expressions on graphs.

Finally, a magnitude interpretation of expressions is characterized by using expressions to indicate an amount that can be measured with the quantity of length or distance. Unlike students using a cardinal interpretation, students using a magnitude interpretation of expressions on graphs typically draw segments to indicate lengths or distances from a reference point on a graph, which they do not break up into countable units. They speak about distances or lengths from a reference point on the graph when describing expressions. Students may use this interpretation of expressions to describe an expression as an amount of distance on a graph from a fixed reference point, 0, or to describe a difference by measuring the size of the distance between the positions of the two expressions. Some students even refer to expressions on graphs as distances with direction; typically an expression with a negative value indicates a distance down or to the left, while an expression with a positive value indicates a distance up or to the right.

Table 11 shows a summary of each of the four interpretations of expressions on graphs, including a description of the graphical interpretations, the comparisons each interpretation was commonly used for in this study, and the observable evidence used to categorize a student's interpretation as such.

Table 11

Four Interpretations of Expressions in the Graphical Register

	Graphical Interpretation	*Comparison of two values*	*Observable Evidence*
Nominal: Expression as label	An expression indicates a position on the graph	Sameness: Two expressions are equal or unequal	▪ Labels expressions on the axes or graph ▪ Speaks about a location on an axis or the graph (e.g., "*x* is here")
Ordinal: Expression as ordered	An expression's position on the graph is relative to others' positions	Relative Position: One expression is greater than another expression	▪ Speaks using spatial orientation language such as left/right, before/after, above/below, or higher/lower ▪ Gestures horizontally or vertically on graph when comparing two or more values
Cardinal: Expression as indicative of count	An expression indicates a measurement in additive units on a portion of an axis or graph	Difference: One expression differs from another expression by a number of unit counts	▪ Speaks about a number of units (tick marks, points, values) on an axis or along the trace of the graph ▪ Counts a number of units on an axis or along the trace of the graph ▪ Reasons additively about these units on the graph ▪Uses whole number values exclusively on graphs
Magnitude: Expression as indicative of amount	An expression indicates a measurement of distance on a portion of an axis or graph	Difference: One expression differs from another expression by an amount of distance between the two	▪ Draws segments to indicate lengths of value of given expression ▪ Speaks about distances, lengths from a fixed reference point called 0 ▪May consider distances as directed ▪May consider non-whole number and negative values on graphs ▪May reason multiplicatively about these lengths

CHAPTER 6

RESULTS II: THE RELATIONSHIPS BETWEEN STUDENTS' INTERPRETATIONS OF EXPRESSIONS, STATEMENT EVALUATIONS, AND INTERPRETATIONS OF POINTS ON GRAPHS

In the previous chapter, I described students' interpretations of expressions on graphs, which varied from student to student and across the six statements. In this chapter, I will describe the association between their interpretations of expressions and their evaluations of the statements, the mathematical content of the statements, and their interpretations of points on graphs, to address Research Questions 2-4. I first describe students' evaluations of the six statements, and its association to their interpretations of expressions. Then, I discuss students' interpretations of expressions and points and its association with the content of each statement. Finally, I describe the cases of three students, who interpreted expressions and points in distinct ways with each of the statements, and describe their interpretations and associated evaluations.

Students' Evaluations of Statements

The second research question of this study asked what, if any, relationship exists between students' interpretations of expressions on graphs and their statement evaluations. In order to address this question, I first report students' evaluations for each of the Statement & Graph pairings and describe findings from these evaluations. I then report students' interpretations by expression in each statement. Using the evaluations and interpretations together, I discuss trends in students' interpretations and their associated evaluations of the statements.

Table 12 shows students' evaluations of Statements 1-6 with each of the graphs it was paired with in the tasks for this study. These evaluations are reported by student by mathematical experience, then by number of correct evaluations. Below each student's name is his or her mathematical level, based on the course the student was enrolled in at the time of the interview. I considered the interviewees to be in three groups in regards to mathematical experience. Group 1, with the highest level of mathematical experience, consists of Micah and Jess. These two students had both successfully completed Advanced Calculus and were enrolled in a senior-level Topology course. Advanced Calculus and Topology are courses which heavily rely on proof. Group 2, with an intermediate level of mathematical experience, consists of Carl, Adam, and Tina, who were in the middle of an Introductory Proof course. The interviews in this study were conducted in the mid to late part of the semester, so these students had at least half of the course prior to the interview and had been introduced to proof, logic, and propositional statements. Group 3, with a beginner level of mathematical experience, consists of Martha, John, Abe, Tim, Jeremy, Tina, Annie, Lola, and Kate. These students were interviewed while taking a course from the Calculus sequence or Differential equations, courses which are more calculation-based rather than proof-based.

In the interview, students evaluated statements for functions shown in graphs with one of five different responses, as shown in Table 12. T indicates an evaluation of "true," F an evaluation of "false," ST an evaluation of "sometimes true." In some cases, students claimed that the statement did not apply to the function shown in a certain graph, for various reasons. These responses are recorded as "DNA" in Table 12. On a few occasions, some students also claimed that they did not have enough information to evaluate the statement. These responses are recorded in Table 12 as “NEI.” Incorrect evaluations are shown in shaded cells. The total number of correct evaluations (out of 27) for each student is shown in the final row below each student’s evaluations. The number of correct evaluations by task is shown in the second column from the right. The number of correct evaluations out of the total evaluations by statement is shown in the final column. Because some statements were paired with more graphs than others, the total number of times students evaluated each statement differed.

Table 12 *Students' Evaluations of Statements with Graphs*

		Group 1		Group 2			Group 3								Correct	Total
Statement & Graph Pairing		**Jess**	**Micah**	**Carl**	**Adam**	**Tina**	**Martha**	**Abe**	**Jeremy**	**John**	**Tim**	**Annie**	**Lola**	**Kate**	**(/13)**	**Correct**
S1	**G1**	F	F	F	T	T	T	T	T	F	T	T	T	T	4	28/39
	G2	T	T	T	T	T	F	T	T	F	T	T	T	T	11	
	G3	T	T	T	T	T	T	T	T	T	T	T	T	T	13	
S2	**G1**	F	F	F	F	ST	F	F	F	F	F	F	F	F	12	50/52
	G2	F	F	F	F	F	F	F	F	F	F	F	F	F	13	
	G3	F	F	F	F	F	F	F	F	F	F	F	F	F	13	
	G4	T	T	T	T	T	T	T	T	T	T	T	T	ST	12	
S3	**G1**	F	F	F	F	F	F	F	F	F	F	F	T	F	12	42/52
	G2	F	F	F	F	NEI	F	F	F	F	F	F	F	F	12	
	G3	T	T	T	T	T	F	T	DNA	T	T	F	T	F	9	
	G4	T	T	T	T	T	F	T	DNA	T	T	F	T	F	9	
S4	**G3**	T	T	F	T	NEI	T	F	F	T	T	ST	F	ST	7	30/65
	G5	F	F	F	T	NEI	F	T	T	T	T	ST	F	F	6	
	G6	F	F	F	T	NEI	F	T	T	T	T	ST	T	T	4	
	G7	F	F	F	T	ST	F	F	T	ST	T	ST	F	T	6	
	G8	T	T	F	T	NEI	T	T	T	ST	T	ST	F	ST	7	
S5	**G1**	F	T	F	T	ST	T	T	F	ST	T	ST	T	ST	2	34/52
	G2	F	F	F	F	F	F	F	F	F	F	F	F	F	13	
	G3	F	F	F	F	F	F	T	F	T	T	F	F	F	10	
	G4	T	T	T	T	T	T	F	T	F	F	T	T	F	9	
S6	**G1**	T	T	T	T	T	T	T	T	T	T	T	F	ST	11	71/91
	G2	T	T	T	T	T	F	T	T	T	T	T	T	T	12	
	G3	T	T	T	T	T	T	T	T	T	T	F	NEI	ST	10	
	G5	F	F	NEI	T	T	F	T	T	T	T	F	NEI	ST	4	
	G6	F	F	F	F	F	F	F	F	F	F	F	NEI	F	12	
	G7	F	F	F	F	F	F	F	F	F	F	F	NEI	F	12	
	G8	F	F	F	F	F	T	F	F	F	F	ST	NEI	F	10	
Correct (/27)		27	26	24	21	17	20	19	19	19	19	16	15	14		

Trends in Students' Evaluations of Statements by Task

In this study, students correctly evaluated some of the statement and graph pairings more often than others. Table 12 shows the number of students (out of 13) who correctly evaluated each statement for each graph in the second column from the right. In total, there were four tasks which all 13 students evaluated correctly, regardless of differences in students' interpretations of expressions or points on graphs. These were: S1G3, S2G2, S2G3, and S5G2.

All students who were interviewed evaluated Statement 1 with Graph 3 correctly as true. Statement 1, the statement related to the IVT is true for Graph 3, a monotone increasing function. The fact that every student correctly evaluated Statement 1 for Graph 3 may have been due to the nature of the function in Graph 3; the function's graph has its extrema at the endpoints of the interval $[a, b]$. Some students engaged in value-thinking and interpreted "N between $f(a)$ and $f(b)$" in Statement 1 as referring to a subset of values on the y-axis between $f(a)$ and $f(b)$ on the y-axis. Others engaged in location-thinking and interpreted this phrase as referring to a subset of points along the graph between points they labeled $f(a)$ and $f(b)$. Regardless of which interpretation of "N between $f(a)$ and $f(b)$" students used on Graph 3, all students considered the entire portion of the graph between the x-values of a and b because the function was monotone increasing. For this reason, all students evaluated this statement as true for Graph 3.

All students also correctly evaluated Statement 2 for Graphs 2 and 3 as false. Statement 2 is the conclusion of the definition of an increasing function, and is false for constant functions such as the one in Graph 2, and decreasing functions, such as the one in Graph 3. Overall, Statement 2 was the most correctly evaluated statement of the six statements in the interview. Out of all the tasks asking students to evaluate Statement 2, there were only two instances in which students evaluated the statement for a function in a graph incorrectly, one with Graph 1 and one with Graph 4. The success of students on Statement 2 may have been related to how students used ordinal interpretations of expressions, or reasoned within the numerical register with this statement. The reasons for this trend will be discussed further in Section 2.2 of this chapter.

Finally, every student correctly evaluated Statement 5 with Graph 2 as false. Statement 5 is a statement involving a difference quotient, similar to the difference quotient commonly used in the definition of the derivative and is only true for increasing functions. Graph 2 is the graph of a constant function. Whether students used a magnitude, cardinal interpretation of expressions in Graph 2, or simply found numerical values from the graph, all students claimed that the numerator of the expression $(f(2+h)-f(2))/h$ would equal zero for all values of h, since Graph 2 depicted a constant function. Using this as a warrant, students evaluated Statement 5, which states that for all values of h, $(f(2+h)-f(2))/h > 0$, as false.

Some statement and graph pairings were correctly evaluated by the students in the interviews less frequently, as shown in the second column from the right of Table 12. The tasks which students correctly evaluated the least included: S5G1 (2 of 13 students correctly evaluated), S1G1(4 of 13 students correctly evaluated), S4G6 (4 of 13 students correctly evaluated), and S6G5 (4 of 13 students correctly evaluated). Students' evaluations of Statement 1 with Graph 1 were accounted for by differences in value-thinking and location-thinking, discussed in Section 3 of this chapter.

Overall, Statements 4 and 5 had the two lowest rates of correct evaluations: 30/65 (46.2%) and 34/52 (65.4%), respectively. These lower rates of correct evaluations compared to the other statements may have been caused by the relatively increased complexity of Statements 4 and 5. Statement 4, with the highest number of incorrect evaluations, was related to the definition of continuity, a definition likely studied only by students who had already taken Advanced Calculus. Statement 5 contained a difference quotient, which students reasoned about with various levels of success. For instance, some students failed to recognize either the difference quotient in Statement 5 as related to the slope of a line passing between two points. Failure to recognize this expression as related to the concept of slope, and issues with quantifiers and "if-then" structure may have influenced some students' evaluations of the statements, resulting in some of the unconventional "sometimes true" evaluations.

Comparison of Students by Mathematical Experience

Table 12, which is grouped by students' level of mathematical experience, shows an association between experience and the number of correct evaluations. Across the 13 students interviewed, students with more undergraduate mathematical experience tended to evaluate more statements correctly.

The two students in Group 1, Micah and Jess, performed the best on these tasks. Jess evaluated all statements correctly, and Micah evaluated all but one correctly for each of the graph pairs. The next two highest performing students, Carl and Adam, were two of the three students in Group 2, enrolled in an Introduction to Proof course at the time of the interview. Students in Group 3 fared worse on these tasks by comparison to the students with more experience. One reason for this difference in performance by experience level may be the instruction and practice students received in Introduction to Proof and proof-oriented courses related to statements with quantifiers and “if-then” logical structure.

In this study, the students with more mathematical experience with proof also tended to evaluate statements with conventionally appropriate responses of true and false more often than those with less mathematical experience. Table 12 also shows that of the seven students who provided unconventional responses to the evaluations, such as “does not apply,” “sometimes true,” and “not enough information,” five were from Group 3, with the least mathematical experience. In fact, only three of the eight students in Group 3 responded with only “true” or “false” evaluations when asked to evaluate the statements.

Relationship between Students’ Interpretations and Their Evaluations of Statements

In order to examine the relationship between students’ interpretations of expressions on graphs and their evaluations of the statements for each pairing, I first examined which registers students reasoned in when interpreting value with each statement. I report the register of students’ reasoning, either graphical or numerical, for each statement in Table 13. I categorized students who interpreted expressions as pertaining to some aspect of the graph as using the graphical register. Students who used a graph only to obtain numerical values and then reasoned with these numerical values were classified as using the numerical register. Because students’ interpretation and use of expressions in graphs tended to remain consistent for each statement across various graphs, I have chosen to report by statement rather than each statement / graph pairing. In cases when students reasoned about expressions and interpreted them within the graphical register and also within the numerical register to support their claims, I have only reported their interpretation of expressions as being in the graphical register. Instances in which students *only* reasoned within the numerical register are reported as such. Cases in which there was not enough evidence to classify a student's interpretation have been left blank.

Table 13

Students' Use of Registers When Interpreting Expressions on Graphs by Statement

	Jess 1	Micah 1	Carl 2	Adam 2	Tina 2	Martha 3	Abe 3	Jeremy 3	John 3	Tim 3	Annie 3	Lola 3	Kate 3
S1	G	G	G	G	G	G	G	G	G	G	G	G	G
S2	G	G	G	G		Nu	G	G	G	G	Nu	G	Nu
S3	Nu	G	G	G	Nu	G	G	Nu	G	G	G	Nu	G
S4	G	G	G	Nu	G	G	G	Nu	G	Nu	G	Nu	Nu
S5	G	G	Nu	Nu	G	Nu	Nu	Nu	G	Nu	G	Nu	Nu
S6		G	Nu	Nu		Nu	Nu	Nu	G		G	Nu	G

G=Graphical Register, Nu=Numerical Register

Next, I examined students' interpretations of expressions within the graphical register used with each statement. I have color-coded students' use of interpretations of expressions for statements as follows: nominal interpretations are yellow, ordinal interpretations are blue, cardinal interpretations are red, and magnitude interpretations are green. There was one instance in which a student, Adam, used one interpretation for inputs and another for outputs with the same statement, in which case both are reported. As in Table 13, cases in which there was not enough evidence to classify a student's interpretation have been left blank.

Table 14

Students' Interpretations of Expressions on Graphs by Statement

	Jess 1	Micah 1	Carl 2	Adam 2	Tina 2	Martha 3	Abe 3	Jeremy 3	John 3	Tim 3	Annie 3	Lola 3	Kate 3
S1	O	O	O	O	O	O	O	O	O	O	O	O	O
S2	O	M	O	O			O	O	O	O		O	
S3		N	O	N O		N	N		N	N	N		N
S4	M	M	M		M	O	O		M		O		O
S5	M	M			M				M		C		
S6		M						M	M		C		C

Nominal = Yellow, Ordinal = Blue, Cardinal = Red, Magnitude = Green

To describe the association between students' interpretations of expressions on graphs and their evaluations of the six statements, I first examined instances in which the students used each of the four interpretations of expressions for certain statements. Considering which students used a certain interpretation for which statement, I then examined those students' evaluations of those statements for observable trends. This section is organized by each of the four student interpretations of expressions on graphs. With each interpretation, I describe which students used that interpretation, which statements they used that interpretation with, and any observable trends in their associated statement evaluations.

Nominal Interpretation of Expressions and Students' Evaluations of Statement 3

Table 13 shows that students predominantly used a nominal interpretation of expressions (in yellow) while evaluating Statement 3. Eight of the 13 students used this interpretation for Statement 3: Micah, Adam, Martha, John, Abe, Tim, Annie, and Kate. Of these eight students, five students (Micah, Adam, John, Abe, and Tim) all correctly evaluated Statement 3 with Graphs 1-4. These five students interpreted expressions nominally in a conventional way, similar to Micah's and Adam's interpretations described in Chapter 5. For these five students, their nominal interpretation of expressions supported them in evaluating the statements correctly.

While most students using a nominal interpretation correctly evaluated Statement 3 for the functions shown in Graphs 1-4, three students, Annie, Martha, and Kate incorrectly evaluated Statement 3 for Graphs 3 and 4 as false. One of the reasons for their evaluation was related to their nominal interpretation of expressions on the graphs. As described in Chapter 5, both Martha and Annie determined that the same position could not be labeled with what they interpreted as two distinct value labels, $f(c)$ and $f(d)$. Kate reasoned similarly about $f(c)$ and $f(d)$ in Statement 3, that two different labels could not be used for the same position. Their nominal interpretation contributed to their incorrect evaluations of Statement 3 with these graphs as false.

The findings from this study show that a nominal interpretation of expressions on graphs with Statement 3 may have helped students in correctly evaluating Statement 3 for Graphs 1-4 when this interpretation was conventional. On the other hand, students incorrectly evaluated Statement 3 when using a nominal interpretation of expressions unconventionally. The key difference between a conventional and unconventional use of a nominal interpretation was whether students considered it possible to denote the same position with two different symbols, such as $f(c)$ and $f(d)$. Students who allowed two different value labels to apply to the same position I described as using a nominal interpretation of expressions conventionally; those who did not allow two different value labels to apply to the same position were considered to be using a nominal interpretation of expressions unconventionally. This result shows that an unconventional nominal interpretation of expressions accounts for three students' (Annie, Kate, and Martha) incorrect evaluations of Statement 3 with Graphs 3 and 4.

Ordinal Interpretation of Expressions and Students' Evaluations of Statement 2

Table 14 shows that students predominantly used an ordinal interpretation of expressions (in orange) while evaluating Statements 1 and 2. With Statement 1, all students were observed using an ordinal interpretation of expressions. Distinctions among these students' interpretations, which led to different evaluations, are best described in terms of value-thinking and location-thinking (see Section 3 of this Chapter). With Statement 2, nine of the 13 students used an ordinal interpretation: Jess, Carl, Adam, John, Abe, Tim, Annie, Lola, and Kate. Of these nine students, all but one correctly evaluated Statement 2 for Graphs 1-4. Kate was the only student of this group to incorrectly evaluate Statement 2 for Graph 4. Her incorrect evaluation was related to her understanding of the logical structure in this statement and not related to her interpretation of expressions on the graph. Thus, in this study, I conclude that an ordinal interpretation of expressions contributed to students correctly evaluating Statement 2.

Notably, there were only two instances of incorrect evaluations of Statement 2 with Graphs 1-4. Thus, while an ordinal interpretation of expressions supported students in correctly evaluating Statement 2, other interpretations, such as magnitude, used by Micah, and reasoning within the numerical register, as Martha and Jeremy did, also contributed to students in correctly evaluating this statement.

Cardinal Interpretation of Expressions and Students' Evaluations of Statements 5 & 6

Table 14 shows that the students who used a cardinal interpretation of expressions (in red), Annie and Kate, did so while evaluating Statement 5 and/or Statement 6. These students used a cardinal interpretation of expressions on graphs when describing a difference on graphs. Annie, who used a cardinal interpretation of expressions with Statements 5 and 6, evaluated these statements incorrectly for five of the graphs in the interview. Kate incorrectly evaluated Statement 6 for three of the graphs. While their cardinal interpretation did not entirely account for their correct or incorrect evaluations with these statements, it is notable that these students were not able to correctly evaluate all statements with this interpretation. Additionally, Annie and Kate were two of the three students who correctly evaluated the least number of statements in the interviews.

Magnitude Interpretation of Expressions and Students' Evaluations of Statements 4, 5, & 6

Table 14 shows that the students who used a magnitude interpretation of expressions (in green) primarily did so while evaluating Statements 4, 5, and 6. With Statement 4, five students used a magnitude interpretation of expressions on graphs: Jess, Micah, Carl, John, and Tina. Of these five students, Jess and Micah correctly evaluated Statement 4 with all five accompanying graphs. Carl, John, and Tina incorrectly evaluated Statement 4 for at least one of the accompanying graphs. With Statement 5, four students used a magnitude interpretation of expressions on graphs: Jess, Micah, John, and Tina. Of these four students, Jess evaluated Statement 5 correctly with Graphs 1-4. Micah and Tina evaluated Statement 5 correctly with Graphs 2-4. With Statement 6, two students used a magnitude interpretation of expressions on graphs: Micah and John. Micah correctly evaluated Statement 6 with all seven graphs, and John evaluated Statement 6 correctly for six of the seven graphs. While a magnitude interpretation of expressions did not determine students' evaluations as correct or incorrect, this interpretation was often used by students who correctly evaluated Statements 4-6. In fact, the three students who evaluated the highest number of statements correctly, Jess, Micah, and Carl, used a magnitude interpretation of expressions with at least one of the statements in the interview.

Summary of Students' Use of Four Interpretations and Their Associated Evaluations

The previous four sections detail, for each of the four interpretations of expressions on graphs, which students used each interpretation, which statement they used the interpretation with, and their evaluations for the statements with which they used that interpretation. To summarize, a nominal interpretation of expressions was associated with students evaluating Statement 3 correctly, when used conventionally. An ordinal interpretation of expressions was associated students evaluating Statement 2 correctly. A cardinal interpretation of expressions may have contributed to some students incorrectly evaluating Statements 5 or 6, as no student using this interpretation with these statements evaluated them correctly in all cases. This interpretation was used by two of the lowest performing students in the interviews. Finally, a magnitude interpretation of expressions may have contributed to some students evaluating some statements correctly. This interpretation was used by the three highest performing students in the interview, as well as some students who were not the highest performing in the interview.

Relationship between Students' Interpretations of Expressions, Points, and their Evaluations of Statements

The goals of this study also included examining the relationship between students' interpretations of expressions and their interpretation of points, as well as the association of both of these interpretations with their evaluations of each statement. In this section, I report students' interpretations of points with their interpretations of expressions evoked by each statement. Then, I highlight the cases of three students, Annie, Micah, and Lola to describe the associations between students' interpretations of expressions, interpretations of points, and their evaluations.

Students' Interpretations of Expressions and Points Evoked by Each Statement

In this study, I observed numerous instances of students' interpretations of points on graphs in ways that were consistent with either value-thinking or location-thinking as described in David et al.'s (2019a) framework. As explained in the description of my data analysis (Chapter 4), I categorized each student's interpretations of points on graphs for each statement-graph combination (e.g., S1G1) by using the David et al.'s (2019a) framework. I found that each student consistently engaged in value-thinking or location-thinking while working with each statement, even with different graphs. For different statements, some students engaged in either value-thinking or location-thinking, depending on the statement. Below I report students' interpretations of output values from statements and their interpretations of points on graphs in terms of value-thinking and location-thinking for each of the Statements 1-6 (Figure 35). Because the distinction in value-thinking and location-thinking lies in distinctions in students' interpretations of outputs, and students' interpretations of input and output values may have differed within a statement, I have chosen to only report students' interpretations of expressions related to *outputs* of the function in the tables below. For instance, Kate interpreted $b–a$ from Statement 6, which was related to inputs, as cardinals in graphs, but interpreted $f(b)–f(a)$ entirely in the numerical register, and did not interpret this expression relative to graphs she worked with. For this reason, Kate's interpretations of expressions in the table below for Statement 6 is recorded as in the "numerical" register. Students in this study who reasoned in the numerical register I categorized as engaging in value-thinking. This categorization was due to the fact that these students found output values from the output axis on the graph and thus were attending to the values of the points on the graph rather than reasoning about values as indicating locations on the graph.

Statement 1

		VT	LT
Graphical	N		
	O	Abe, Adam, Carl, Jeremy, Jess, John, Lola, Martha, Micah, Tim	Annie, Kate, Tina
	C		
	M		
Numerical			
Lack of Data			

Statement 2

		VT	LT
Graphical	N		
	O	Abe, Adam, Carl, Jeremy, Jess, John, Lola, Tim	
	C		
	M	Micah	
Numerical		Annie, Kate, Martha	
Lack of Data		Tina	

Statement 3

		VT	LT
Graphical	N	Abe, Annie, John, Kate, Martha, Micah, Tim	
	O	Adam, Carl	
	C		
	M		
Numerical		Jeremy, Jess, Lola, Tina	
Lack of Data			

Statement 4

		VT	LT
Graphical	N		
	O	Annie, Kate, Martha	
	C		
	M	Carl, Jess, John, Micah, Tina	
Numerical		Abe, Adam, Jeremy, Lola, Tim	
Lack of Data			

Statement 5

		VT	LT
Graphical	N		
	O		
	C		Annie
	M	Jess, John, Micah, Tina	
Numerical		Abe, Adam, Carl, Jeremy, Kate, Lola, Martha, Tim	
Lack of Data			

Statement 6

		VT	LT
Graphical	N		
	O		
	C		Annie
	M	Jeremy, John, Micah	
Numerical		Abe, Adam, Carl, Kate, Lola, Martha	
Lack of Data		Jess, Tim, Tina	

Figure 35. Students' interpretations of expressions involving output values and points on graphs by statement

The tables in Figure 35 highlight the differences in students' interpretations of expressions and points on graphs with each of the six statements used in this study. In particular, the tables show that the content of the statements evoked different interpretations of expressions and points on graphs from the same students. The tables also show that some statements evoked certain interpretations of expressions as well as certain interpretations of points from students than others. I will describe the interpretations of expressions and points evoked by each statement and comment on the association between the content of the statement and the associated interpretations.

Statement 1 and its evoked student interpretations of expressions and points.

Statement 1, "For all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$," evoked an ordinal interpretation of expressions in the graphical register for all 13 students interviewed. This interpretation of expressions on graphs is characterized by students interpreting expressions as ordered based on their positions on an axis or along the trace of the graph. I hypothesize that all students used an ordinal interpretation of expressions, due to the language "N between $f(a)$ and $f(b)$," which suggests ordering, in the statement. All students described the positions of $f(a)$, $f(b)$ and N, and the ordering of these positions somewhere in the graphs when concluding whether a chosen N value was between $f(a)$ and $f(b)$ while evaluating Statement 1.

Further, all students who were interviewed reasoned in the graphical register while evaluating this statement, explaining their interpretations of "N between $f(a)$ and $f(b)$" on the graph. The phrase "N between $f(a)$ and $f(b)$" may have encouraged students to reason within the spatial structure of the graphs provided. The word "between," itself, carries a spatial connotation as referring to the "time, *space*, or interval that separates" ("between," Merriam-Webster, 2019), which may have contributed to all students' reasoning in the graphical, rather than only in the numerical, register when evaluating this statement.

Statement 1 was also the statement which evoked location-thinking the most among the six statements. Three of the students interviewed, Annie, Kate, and Tina engaged in location-thinking with Statement 1, interpreting points on the graphs as output values of the function. Due to this interpretation of points, these students considered the entire portion of Graph 1 between *a* and *b* on the *x*-axis as possible *N*'s between *f*(*a*) and *f*(*b*). In using an ordinal interpretation of expressions, these students ordered the positions of expressions which they placed along the trace of the graph. As a result of their location-thinking, each of these students incorrectly evaluated Statement 1 as true for Graph 1. The other ten students interviewed engaged in value-thinking with this statement, considering points as pairs of values and output values as on the *y*-axis. These students interpreted *N* between *f*(*a*) and *f*(*b*) as referring to a portion of the *y*-axis. They ordered these expressions by ordering positions on the *y*-axis or parallel to the *y*-axis. Of the ten, four correctly evaluated Statement 1 as false for Graph 1. The remaining six incorrectly evaluated Statement 1 as true for Graph 1, due to issues beyond their interpretations of expressions and points on graphs.

Statement 2 and its evoked student interpretations of expressions and points.

Statement 2, "For all real numbers *c, d* in (*a, b*), if $c < d$, then $f(c) < f(d)$," evoked either ordinal and magnitude interpretations of expressions in the graphical register from some students, as well as reasoning in the numerical register from other students interviewed in this study. Eight of the 13 students interpreted expressions as ordinals in the graphs they worked with while evaluating Statement 2. These students used horizontal or vertical spatial comparisons of positions of expressions in the graphs to draw conclusions about the expression "$f(c) < f(d)$." This expression is a comparison of two outputs, which might explain why the majority of students who did reason in the graphical register used an ordinal interpretation to compare expressions. One student

Micah, who reasoned in the graphical register, but did not use an ordinal interpretation, explained his interpretation of '$f(c) < f(d)$'using a magnitude interpretation, comparing two distances in the graph as a means of comparing expressions (See Chapter 5).

Three other students, Annie, Kate, and Martha, did not reason about the positions of the expressions in the graphs to draw conclusions about $f(c) < f(d)$ but instead found numerical values from the graph and compared them in the numerical register while evaluating this statement. The use of an inequality symbol of less than "<" may have evoked the use of the numerical register for these three students. These students each reasoned in the graphical register with other statements, so they have found it more convenient to reason within the numerical register and justify their claims while making a comparison of values using a "less than" relation with this statement. Furthermore, I acknowledge that these students may have been able to reason in the graphical register with this statement, but did not verbalize this ability. I emphasize that these are evoked interpretations for each statement based upon the available evidence.

Finally, all students engaged in value-thinking with Statement 2. Even students who labeled points as outputs on the trace of the graph did not interpret these points as outputs in their reasoning. Instead, these students referenced the vertical axis or made vertical comparisons when interpreting "$f(c) < f(d)$," indicating that they interpreted output values on the output axis, not on the graph. As described earlier in this chapter, Statement 2 was the most correctly evaluated statement of the six. Students' interpretations of expressions, whether ordinal or magnitude, and their interpretations of points in terms of value-thinking in the graphical register or in the numerical register, was associated with students correctly evaluating Statement 2 in the vast majority of instances.

Statement 3 and its evoked student interpretations of expressions and points.

Statement 3, "For all real numbers *c, d* in (*a, b*), if *f*(*c*) = *f*(*d*), then *c*=*d*," evoked nominal and ordinal interpretations of expressions in the graphical register, as well as reasoning in the numerical register from the students interviewed in this study. Seven of the students reasoned in the graphical register and used a nominal interpretation of expressions, comparing the positions of *f*(*c*) and *f*(*d*) for sameness in order to conclude that these were equal. Two students used an ordinal interpretation of expressions with this statement and compared whether *f*(*c*) and *f*(*d*) lay on the same horizontal line, comparing the relative positions of these expressions.

Four students reasoned within the numerical register to claim *f*(*c*) and *f*(*d*) were equal or unequal. Statement 3 may have evoked reasoning in the numerical register due to the use of the equal sign "=." Again, the students who reasoned in the numerical register may have also interpreted expressions in the graphical register, but chose to justify their claims regarding the equality of *f*(*c*) and *f*(*d*) using numerical values rather than aspects of the graph.

All students engaged in value-thinking with Statement 3. In order to categorize students' interpretations of points with this statement, I looked at students' interpretations of expressions with Graph 2, the graph of a constant function. All students claimed that *f*(*c*) and *f*(*d*) were equal for this function, even if they placed these labels at different positions on the trace of the graph. This behavior was unlike the behavior of students who engaged in location-thinking in David et al.'s (2019a) study, such as Zack. Zack claimed two outputs placed at two different positions along the graph of a constant function to be unequal because they were in different positions (David et al., ibid).

Statement 4 and its evoked student interpretations of expressions and points.

Statement 4, "For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all x in the domain of f with $-\delta < x-1 < \delta$, $-\varepsilon < f(x)-f(1) < \varepsilon$," evoked ordinal and magnitude interpretations of expressions in the graphical register, as well as reasoning in the numerical register from the students interviewed in this study. Three students used an ordinal interpretation of expressions and described $-\varepsilon$, $f(x)-f(1)$, ε as ordered vertically in the graphs. These students tended to treat $f(x)-f(1)$ as a single value placed somewhere in the graph, rather than to reason about $f(x)$ and $f(1)$ separately. Similarly to Statement 2, the inequality sign, "<" may have evoked an ordinal interpretation of expressions in the graphs from students. Five students used a magnitude interpretation of expressions and considered $f(x)-f(1)$ to be a vertical distance in the graph between positions they labeled $f(x)$ and $f(1)$. These students then described ε as distances to which to compare the distance $f(x)-f(1)$. The difference in this statement may have evoked a magnitude interpretation of expressions on graphs for these students.

Five students reasoned within the numerical register with Statement 4. These students predominantly selected values for x, δ, and ε, found values from the graph for $f(x)$ and $f(1)$, and tested these values to see if the inequalities were satisfied in order to evaluate this statement. Four of the five students who reasoned in the numerical register with Statement 4 were from Group 3, who had only taken Calculus courses. These students may not have been as familiar with the content of Statement 4, which is the conclusion of the definition of continuity at a point, a topic typically introduced in more advanced courses. Their lack of familiarity with the content and how it may apply to the graph of a function, may have contributed to their reasoning in the numerical, rather than graphical register with this statement. The subtraction sign in the expressions may also have encouraged students to reason in the numerical register with this statement. Finally, all students engaged in value-thinking with Statement 4, interpreting points as pairs of values and output values on the output axis.

Statement 5 and its evoked student interpretations of expressions and points.

Statement 5, "For all non-zero real numbers *h,* if 2+*h* is in (*a*, *b*), then $\frac{f(2+h)-f(2)}{h} > 0$," evoked cardinal and magnitude interpretations of expressions in the graphical register, as well as reasoning in the numerical register from the students interviewed in this study. Four students, Jess, John, Micah, and Tina, all of whom used a magnitude interpretation of expressions with Statement 4, also did so with this statement. This finding supports the claim that a difference may have evoked a magnitude interpretation, as both statements contain a difference of outputs. Annie was the only student who used a cardinal interpretation of expressions with this statement, as described in Chapter 5. For her, the difference of output values referred to a count of "points" along a portion of the graph between positions she had labeled *f*(2) and *f*(2+*h*).

Eight students reasoned in the numerical register with Statement 5, the most of any of the six statements. Many of these eight students expressed hesitation with interpreting the quotient in this statement and resorted to a method of selecting values for *h* and reasoning about the quotient numerically. As the quotient in Statement 5 is typically expressed using *x*, rather than 2, the lack of familiarity of the quotient for some students may have evoked reasoning in the numerical register. In terms of students' interpretations of points, Annie was the only student to engage in location-thinking with this statement, which will be described further in section 3.2.1 of this chapter. All other students engaged in value-thinking with this statement. As with Statement 4, the use of a subtraction sign may have encouraged some students to consider outputs as values with this statement and thus points as pairs of values.

Statement 6 and its evoked student interpretations of expressions and points.

Statement 6, "There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$," evoked cardinal and magnitude interpretations of expressions in the graphical register, as well as reasoning in the numerical register from the students interviewed in this study. Three students, Jeremy, John, and Micah, used a magnitude interpretation of expressions with this statement, describing the differences of expressions as distances between these expressions on the graphs either horizontally or vertically. Annie was again the only student to use a cardinal interpretation of expressions with this statement, described further in section 3.2.1 of this chapter. Six students reasoned in the numerical register with this statement. Since the values of *a* and *b* were provided, these students used the graphs to find values for *f*(*b*) and *f*(*a*) and calculating the numerical value of the quotient to evaluate the statement. Three students reasoned about the quotient as a whole indicating a slope, but did not provide enough evidence for me to categorize how they were thinking about the individual values. In some cases, time constraints also factored into this lack of data, as this was the last statement in the interview.

Summary of evoked interpretations of expressions and points by statement.

As detailed above, different statements evoked different interpretations of expressions and points on graphs. For instance, statements involving differences of expressions were more likely to evoke a magnitude interpretation of expressions on graphs from students than those that did not. Statement 1, containing the phrase "N between $f(a)$ and $f(b)$," evoked location-thinking from the most students compared to the other statements. This finding suggests that students use graphs differently depending the on the mathematical content they are reasoning about and that students' interpretations of expressions and points are context-dependent. Furthermore, the same statement evoked different interpretations from different students in the interview. For instance, students used an ordinal and magnitude interpretation of expressions on graphs with Statement 2, as well as reasoning in the numerical register. This finding suggest that not only are interpretations of expressions and points on graphs context-dependent, they also vary between students.

Students' Interpretations of Expressions, Points, and Evaluations in Cases

To provide more detail on the relationship between students' interpretations of expressions, points, the content of the statements and their evaluations of statements, I review the cases of three students with distinct interpretations of expressions and points. I will describe the case of Annie, who engaged in location-thinking with three of the six statements, the most of any student. Next, I will describe the case of Micah, who consistently engaged in value-thinking, and used a magnitude interpretation of expressions more than any other student. Finally, I will describe the case of Lola, who reasoned in the numerical register for the majority of the statements in the interview.

The case of Annie: inconsistent location-thinking & cardinal interpretations of expressions.

Annie's interpretations of expressions and points on graphs changed depending on the statement. Annie's interpretations for each statement are shown in Table 15.

Table 15

Annie's Interpretations of Expressions and Points on Graphs with Each Statement

Annie		**VT**	**LT**
Graphical Register	*Nominal*	S3	
	Ordinal	S4	S1
	Cardinal		S5, S6
	Magnitude		
Numerical Register		S2	
Lack of Data			

Annie interpreted expressions as nominals, ordinals, and cardinals on graphs with various statements, indicating that her interpretation of expressions was flexible given the context of the statements. Notably, the only interpretation of expressions on graphs which Annie did not use was a magnitude interpretation. She also reasoned in the numerical register with one statement. Annie's interpretation of points was also context-dependent. Annie engaged in value-thinking with Statements 2-4, interpreting points as pairs of values and output values on the *y*-axis, and location-thinking with Statements 1, 5, and 6, interpreting points as outputs.

As described earlier, Annie's incorrect evaluation of Statement 1 with Graph 1 was attributed to her location-thinking in this context. While Statement 1 evoked location-thinking from two other students, Annie's location-thinking also persisted in the other contexts of Statements 5 and 6. Both Annie's location-thinking with these statements, as well as her cardinal interpretation of expressions, contributed to some of her incorrect evaluations of Statements 5 and 6. With both Statement 5 and 6, Annie used a cardinal interpretation of expressions and a location-thinking interpretation of points when reasoning about the difference of two outputs in each of these statements. These interpretations led her to place outputs at positions along the trace of the graph and to reason that a difference of two outputs counted in additive units, some aspect of a portion of a trace of the graph. As described in section 4.2 of Results I, Annie was confronted with the limitations of these interpretations on the graphs. When asked to further describe her interpretations she responded with confusion and by saying "I don't know" (see transcripts in 4.2 of Chapter 5). Subsequently, Annie reasoned in the numerical register to make sense of Statements 5 and 6, or appealed to prior knowledge related to slope. Annie's interpretations of expressions as cardinals, coupled with her location-thinking, limited her reasoning in the graphical register.

For instance, when Annie was evaluating Statement 6 with Graph 8, Annie was unable to explain what $f(b)–f(a)$ and $b–a$ represent in the graph. Instead, she reasoned in the numerical register and found that the quotient of these two differences was 3/7 as shown in Figure 36. Annie recalled that $f'(c)$ indicated a derivative, which was related to whether the graph was increasing or decreasing. Because Annie explained that $f'(c)=3/7$ which is positive, she said the statement would be true for any portion of the graph for which the graph was increasing. For this reason, she evaluated the statement as "Sometimes True" for $c > 1$, which was for the portion of the graph that was increasing.

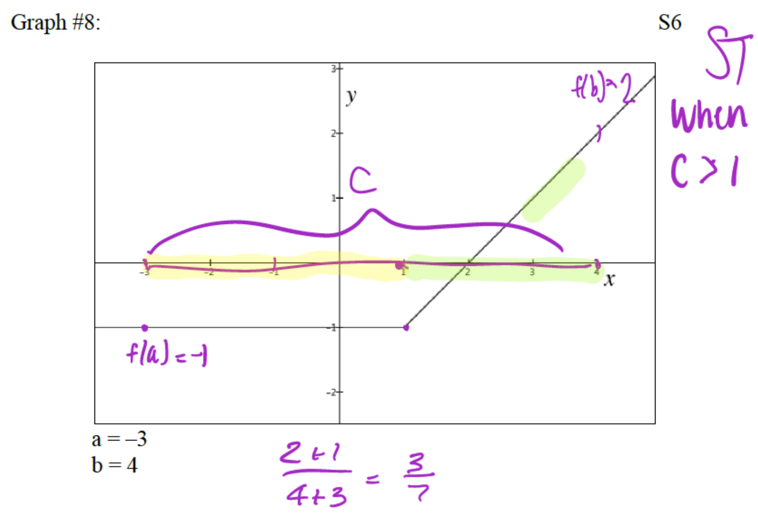


Figure 36. Annie's graph labels and work in the numerical register on Graph 8 when evaluating Statement 6.

In addition to the logical issues embedded in Annie's evaluation of "sometimes true," Annie was unable to reason about the difference quotient in the graph. As a result, Annie was left to reason about the statement in the numerical register, and recalling information about the derivative of a function. Had Annie interpreted the differences as magnitudes in the graphs, and the points on the graph as ordered pairs of values, as other students did with this statement, she may have been able to reason about the quotient in the graphical register. A magnitude interpretation of expressions and value-thinking may have supported Annie in interpreting the quotient as representing the slope of a secant line passing through $(a, f(a))$ and $(b, f(b))$. While interpreting the quotient in this way does not guarantee a correct evaluation, it may have supported Annie in persisting in reasoning within the graphical register.

Annie reasoned in the numerical register after attempting to reason in the graphical register with a cardinal interpretation of expressions and a location-thinking interpretation of points, with some success in correctly evaluating the statements. However, Annie had the third lowest number of correct evaluations of the students interviewed. Although Annie engaged in location-thinking with these statements, Annie engaged in value-thinking in other contexts, suggesting this way of interpreting points was not fixed for her. Annie's failure to interpret differences as magnitudes may have contributed to her inability to persist in reasoning in the graphical register.

The case of Micah: consistent value-thinking & magnitude interpretation of expressions.

Micah's interpretations of expressions also differed depending on the statement. However, Micah consistently engaged in value-thinking with all six statements throughout the entire interview. Micah's interpretations for each statement are shown in Table 16.

Table 16

Micah's Interpretations of Expressions and Points on Graphs with Each Statement

Micah		**VT**	**LT**
Graphical Register	*Nominal*	S3	
	Ordinal	S1	
	Cardinal		
	Magnitude	S2, S4, S5, S6	
Numerical Register			
Lack of Data			

Micah interpreted expressions as nominals, ordinals, and magnitudes on graphs with various statements, indicating that his interpretation of expressions was flexible given the context of the statements. Micah used a magnitude interpretation of expressions with four of the six statements, the most of any student in the interview. Notably, the only interpretation of expressions on graphs which Micah did not use was a cardinal interpretation. Micah also consistently reasoned in the graphical register with all six statements, rather than only in the numerical register. Micah's interpretation of points was also consistent with all six statements. Micah engaged in value-thinking with all six statements, interpreting points as pairs of values and output values on the *y*-axis, although he consistently labeled outputs at points along the graph.

Micah's value-thinking, coupled with his interpretation of expressions as magnitudes for many of the statements, supported him in consistently reasoning in the graphical register and correctly evaluating almost every statement. In fact, the only statement/graph pairing which Micah did not correctly evaluate was Statement 5 with Graph 1. While Micah described the numerator and denominator of the quotient in Statement 5 using a magnitude interpretation of expressions in the graphs, Micah did not describe the quotient of these two magnitudes as a slope between two points on the graph. Instead, Micah compared the signs of the differences found on the graph as distances to determine whether the quotient was positive or negative. He then reasoned from the sign of the quotient to evaluate Statement 5. Micah's graph labels and work reasoning in the numerical register about the quotient in Statement 5 is shown in Figure 37.

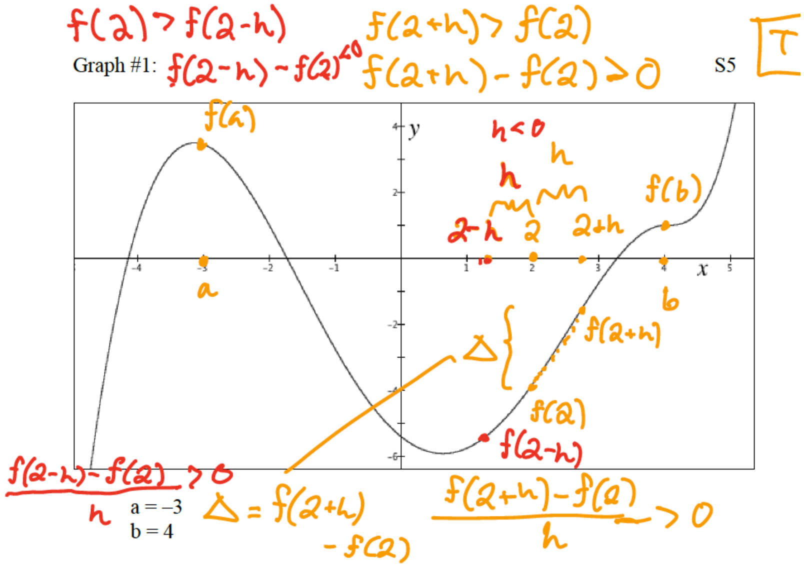


Figure 37. Micah's graph labels and reasoning in the numerical register on Graph 1 when evaluating Statement 5

In his evaluation of Statement 5 with Graph 1, Micah considered two situations: one in which *h* was positive and one in which *h* was negative. He labeled both of these possible *h* values as distances on the *x*-axis as shown in Figure 37. He then explained that in the case of *h* being positive, the difference of *f*(2+*h*) and *f*(2) would be the distance on the graph he labeled as delta or "Δ," and positive, making the quotient positive. In the case of *h* being negative, he explained that the difference of the outputs would be negative, also making the quotient positive. In either case, Micah concluded that the quotient would be positive, leading him to evaluate Statement 5 as true. Although Micah thought he was considering all values of *h* by considering two cases of *h*, the portion of the graph over which Micah reasoned was increasing, guaranteeing a positive difference quotient. Had Micah interpreted the quotient of the two magnitudes as a ratio of the relative sizes of the two magnitudes, which gives slope, he may have more readily recognized that there are some h values for which the quotient is negative, namely on portions of the graph which would yield decreasing secant lines. Because all other graphs shown with this statement were monotone increasing, decreasing, or constant, Micah was able to correctly evaluate Statement 5 with the other graphs using this same line of reasoning. While Micah's interpretation of expressions as magnitudes with many of the statements, and his consistent value-thinking supported him in evaluating almost all of the statements correctly for all the graphs, it alone was not enough to guarantee a correct evaluation with every statement/graph pairing.

The case of Lola: a tendency to avoid the graphical register.

Unlike Annie and Micah who reasoned or attempted to reason in the graphical register with most of the statements, Lola reasoned primarily in the numerical register with most statements in the interview. Table 17 shows Lola's interpretations of expressions and points with each statement.

Table 17

Lola's Interpretations of Expressions and Points on Graphs with Each Statement

Lola		**VT**	**LT**
Graphical Register	*Nominal*		
	Ordinal	S1, S2	
	Cardinal		
	Magnitude		
Numerical Register		S3, S4, S5, S6	
Lack of Data			

Lola interpreted expressions in the graphical register as ordinals with Statements 1 and 2. However, Lola reasoned about values of expressions in the numerical register for Statements 3-6. In all cases, Lola interpreted points on the graph in ways consistent with value-thinking. Her reasoning in the numerical register, especially with Statement 6, contributed to some of her incorrect evaluations.

Lola's tendency to reason in the numerical register and avoid reasoning in the graphical register, contributed to her incorrect evaluations. Lola correctly evaluated 15 of the statements with graphs in the interview, the second lowest of all students interviewed. Strikingly, Lola incorrectly evaluated Statement 6 with six of the seven graphs it was paired with. With five of those graphs, Lola evaluated the statement by saying she did not have enough information to evaluate the statement. For example, with Statement 6 and Graph 3, Lola said, "I constantly mess up with this wanting it to be *f* 'but this [pointing to the graph] is *f*(*x*)… I'm not sure where I would pinpoint *c*…I feel like I don't have enough information." Lola explained that she did not have a graph of *f* 'but only a graph of "*f*(*x*)" and so was unable to find a *c* value to satisfy the equation "*f*'(*c*)= –3," which she arrived at from calculating the value of the difference quotient, based on the provided values of *a* and *b* and the graph.

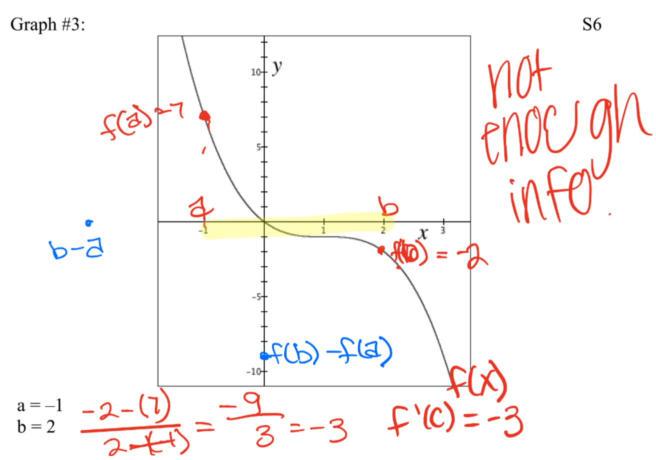


Figure 38. Lola's written work and labels on Graph 5 while evaluating Statement 6.

Lola subsequently evaluated the remaining graphs with Statement 6 by claiming she did not have enough information. While other students were able to reason about the derivative of *f* by reasoning about slopes of lines tangent to the graph of *f*, Lola, who was reasoning in the numerical register, did not think she had been provided sufficient information in order to reason about the derivative of *f* to evaluate Statement 6. Lola's reasoning in the numerical register limited her from reasoning about the derivative of *f*, and contributed to her concluding that she did not have enough information to evaluate this statement with the majority of the graphs provided.

Summary of students' interpretations of expressions, points, and evaluations.

From the cases of Annie, Micah, and Lola, described above, these students' interpretations of expressions and points on graphs were related to their evaluations. However, in many cases, they were not the only deciding factor in their evaluations. Annie's location-thinking and her interpretation of expressions as cardinals contributed to some of her incorrect evaluations. While Micah engaged in value-thinking and used a magnitude interpretation of expressions, he was not able to reason about the difference quotient in Statement 5 in the graphical register. Finally, Lola's avoidance of reasoning in the graphical register led her to conclude that she was not given enough information to evaluate Statement 6.

CHAPTER 7:

CONCLUSION & DISCUSSION

The findings of this study characterize four ways students interpret expressions on graphs of functions with statements from Calculus: nominal, ordinal, cardinal, and magnitude. The distinctions among these four ways of interpreting expressions in the graphical register highlight differences in ways that students may interact with graphs of functions in undergraduate mathematics courses. The findings of this study suggest that certain mathematical content may evoke some interpretations of expressions in the graphical register rather than others. Further, which interpretation a student uses has implications for the student's understanding of related content. In this chapter, I describe the significance of these findings, situate these findings in relation to existing literature, discuss implications for curriculum and instruction using graphs, and overview directions for future research.

Summary of Findings in Relation to Research Questions

I focused my investigation on students' interpretations of expressions in the graphical register in this study through the following research questions:

5) *How do undergraduate students interpret expressions from statements in Calculus on graphs of functions in the Cartesian plane? Which aspects of graphs do these students attend to in the context of statements from Calculus when interpreting expressions, if any?*
6) *How are undergraduate students' interpretations of expressions on graphs of functions related to their evaluations of statements from Calculus?*

7) *In what ways are the content of expressions in statements from Calculus related to undergraduate students' ways of interpreting these expressions on graphs? i.e. To what extent are students' interpretations of expressions on graphs consistent or inconsistent across different Calculus statements? If inconsistent, which statements evoke which interpretations for students?*
8) *How are undergraduate students' interpretations of expressions on graphs of functions related to their interpretation of points?*

I will summarize my findings relative to each research question, and then discuss the significance of these findings.

Response to Research Question 1

Research Question 1 asked: *How do undergraduate students interpret expressions from statements in Calculus on graphs of functions in the Cartesian plane? Which aspects of graphs do these students attend to in the context of statements from Calculus when interpreting expressions, if any?* To investigate this question, I analyzed students' interpretations of expressions while evaluating six statements related to ideas of Calculus with various graphs of functions. From this analysis of students' words, gestures, and markings on graphs, four distinct interpretations of expressions on the graphs emerged. Each of the four interpretations of expressions on graphs that emerged in this study may be understood to support students in various graphing activities. I summarize the characteristics of each of the four interpretations of expressions in the graphical register and the aspects of graphs to which the students using these interpretations attended.

A nominal interpretation of expressions on graphs refers to interpreting and using expressions on graphs as a label denoting a position. Students using this interpretation of expressions do not attend to the size of the numerical value of the expression (if they even associate a numerical value with the expression), but rather the placement of the label in a graphical arrangement. When relating expressions to each other using this interpretation, expressions are either the same or different, if the placement of these expressions is the same or different. In some contexts, a nominal interpretation may align with a conventional interpretation of expressions on graphs. In this study, a nominal interpretation was used by some students, like Adam and Micah, in making sense of a statement related to the definition of an injective function, in which input expressions and output expressions were equal to each other. However, not all students who used a nominal interpretation of expressions on graphs did so conventionally. Some students, like Annie and Martha used a nominal interpretation of expressions, but claimed two different expressions could not refer to the same position. In this study, the students who used a nominal interpretation of expressions on the graph attended to positions on an axis or positions along the trace of the graph when comparing expressions for sameness.

An ordinal interpretation of expressions in the graphical register refers to interpreting expressions as having an order when compared with other expressions, based on their positions on the graph. Students using this interpretation also may not attend to the numerical value of an expression but instead use the positions of expressions labeled in the graph in relation to each other to order expressions. Students in this study used horizontal and vertical comparisons, parallel to the axes, to compare the positions of expressions. Conventionally, to the left of (horizontally) or below (vertically) indicates a less than ordering. Many students in this study used this interpretation in evaluating a statement related to the definition of an increasing function. Students who used this interpretation made conclusions about the ordering of pairs of inputs and outputs of functions using spatial comparisons. In doing so, these students attended to the spatial relationship (vertically or horizontally) between pairs of positions on an axis or along the trace of the graph.

A cardinal interpretation of expressions in the graphical register refers to interpreting expressions as a measurement in additive units of a portion of the graph. Unlike a nominal or ordinal interpretation, students using a cardinal interpretation of expressions did ascribe meaning to the numerical value of an expression. Students using this interpretation of expressions counted out units along a portion of an axis or along the trace of the graph from a starting position to an ending position. They ascribed the numerical value of an expression to the result of this counting process. These students often attended to the tick marks on an axis, or gestured to indicate attention to discrete units along the trace of the graph. The students who used a cardinal interpretation of expressions in this study did so with expressions which were differences of inputs or outputs.

A magnitude interpretation of expressions in the graphical register refers to interpreting expressions as measurements indicating amounts of distance along a portion of the graph. Like a cardinal interpretation of expressions, a magnitude interpretation also included an attention to the numerical value of an expression. Unlike students using a cardinal interpretation, students using a magnitude interpretation described expressions as measuring a length of a segment of an axis. They explained that the numerical value of an expression gave the distance between two reference points along a portion of an axis. These students attended to what they considered to be reference points in the graph. For a single variable, one student described the origin of the graph as "0," which served as a reference point to measure a distance from. For a difference of variables, students described the reference points as the positions they had labeled as the variables.

Some students, when evaluating the statements in the interview with the provided graphs, did not interpret expressions from the statements in the graphical register. Instead, these students used the graph to obtain numerical values for these expressions, by finding a relevant position on an axis and determining the numerical value associated with that position. Then, these students reasoned about the expressions using the numerical values, rather than aspects of the graph. In this sense, they reasoned in the numerical register, rather than the graphical register, with these expressions.

Response to Research Questions 2 and 3

Research Question 2 asked: *How are undergraduate students' interpretations of expressions on graphs of functions related to their evaluations of statements from Calculus?* Research Question 3 asked: *In what ways are the content of expressions in statements from Calculus related to undergraduate students' ways of interpreting these expressions on graphs? i.e. To what extent are students' interpretations of expressions on graphs consistent or inconsistent across different Calculus statements? If inconsistent, which statements evoke which interpretations for students?*

In order to address Research Question 2, I first reported students' evaluations of each statement graph pair in Table 12 of Chapter 6. I then reported students' use of the graphical register in Table 13, and usage of the four interpretations of expressions within the graphical register by statement in Table 14. I then described the relationship between each interpretation, the statements for which students used this interpretation, and the evaluations provided by those students for that statement. I found that certain statement evaluations were associated with certain interpretations of expressions, as well as more general trends in association between students' interpretations and the number of correct evaluations. In doing so, I also responded to Research Question 3, by highlighting the statements which evoked certain interpretations from students in my descriptions of the associations between students' interpretations and evaluations. Some statements consistently evoked the same interpretation of expressions from students (Statement 1 and ordinal, for instance), while other statements evoked a variety of interpretations of expressions from different students.

Response to Research Questions 3 and 4

Finally, Research Question 4 asked: *How are undergraduate students' interpretations of expressions on graphs of functions related to their interpretations of points?* In order to address this research question, I analyzed students' interpretations of points using the value-thinking and location-thinking framework (David et al., 2019a) and considered their interpretations of outputs as well to make my determinations. In Figure 35 in Chapter 6, I report students' interpretations of expressions and interpretations of points on graphs for each statement. In creating these tables, I restricted my report to students' interpretations of expressions involving output values, to analyze the same data as I did for students' interpretations of points. Since the value-thinking and location-thinking framework does not address students' interpretations of inputs, I did not consider that portion of the data for the tables in this figure. Then, I described, for each statement, the interpretations of expressions that the statement evoked, as well as associated interpretations of points the statement evoked. Because I also considered the association between students' interpretations of expressions and points by statements, I also addressed Research Question 3, which asks about which statements evoke which interpretations for students. I found that in most instances, students engaged in value-thinking, interpreting points as ordered pairs of values and outputs as found on the *y*-axis. However, there were some instances in which students engaged in location-thinking, which was associated with an ordinal interpretation of expressions for three students, and a cardinal interpretation of expressions for one of these three. I conclude my description of the association between students' interpretations of expressions, their interpretation of points, and their evaluations by highlighting the cases of three students: Micah who engaged in value-thinking, Annie who engaged in location-thinking, and Lola who tended to reason only in the numerical register.

Relationship to Existing Literature and Theory

Next, I describe the relationship between the findings in this study to findings in existing literature, as well as situate my findings with existing theory. The four interpretations of expressions in the graphical register which emerged in this study extend existing literature on students' interpretation of graphs and their graphing activity. While previous studies have examined students' creation of graphs, views of graphs as a whole, or their interpretation of points, these four interpretations characterize how students operate within the graphical register in order to make sense of expressions in Calculus statements. The four interpretations of expressions in the graphical register may help explain other phenomena in students' graphing activity previously reported. Additionally, the findings of this study support findings of previous' researchers in other contexts.

The distinctions in the cardinal and magnitude interpretations of expressions in the graphical register aligns with the findings of Diezmann and Lowrie (2006), who found similar distinctions in fifth-grade students' use of a number line. These researchers found that some students tended to use a number line (a graphical register in one-dimension) as a counting model, labeling points on a number line as the result of a counting process, while others used it as a what they referred to as a measuring model, in which points were labeled to indicate a length from a reference point. In my framing, the first group of students would be said to use a cardinal interpretation of expressions, while the latter group would be said to use a magnitude interpretation of expressions. The connection between these findings offers an idea that students may have opportunities to develop multiple interpretations of expressions at the graphical register at the elementary level with number lines, before learning with graphs in the Cartesian plane.

Frank's (2016) hypothesis that teaching students to plot points on graphs by a motion of "over and up," (over some amount of units on the *x*-axis and up some amount of units on the *y*-axis) may be problematic in students' covariational reasoning may also apply to students' interpretation of expressions. While plotting a point using the "over and up" technique, units may be additively counted on an axis or on the graph, which typically has tick marks or grid lines at integer values. Thus, in addition to being a potential inhibitor of covariational reasoning (Frank, ibid), a potential cause of location-thinking (David et al., 2019a), plotting points using the "over and up" technique may encourage a cardinal interpretation of expressions on graphs. That is, this action may reinforce the idea that labels of variables or expressions are placed on the graph at the conclusion of a counting process. Goldenberg's (1988) claim that students' physical actions in constructing graphs may influence a students' interpretation may also apply to the connection between the action of counting and a students' cardinal interpretation.

The finding that some students in this study tended to avoid the graphical register with statements from Calculus, preferring to reason in the numerical register supports earlier findings of similar tendencies in Calculus (Dawkins & Epperson, 2014). The findings from this study extend previous findings by suggesting that students' avoidance of the graphical register may be context-dependent, and could even depend on students' familiarity with the mathematical content.

Finally, the four interpretations of expressions detailed in this study are another dimension by which to characterize students' interpretations of graphs. While David et al.'s (2019a) framework characterizes how students interpret what a graph is comprised of (either points which are outputs or points which are ordered pairs of values), these four interpretations characterize how a student operates within the graphical register. If the value-thinking and location-thinking framework is one dimension by which to characterize a students' graphical interpretations, the interpretations in this study are another. In this study, students who were observed engaging in value-thinking were also observed using a nominal, ordinal, or magnitude interpretation of expressions. Students who were observed engaging in location-thinking were observed using an ordinal or cardinal interpretation of expressions. However, hypothetically, a student may engage in either value-thinking or location-thinking and may use any of the four interpretations of expressions on graphs, or other interpretations yet to emerge.

The findings of this study also shed light on the observable evidence used to categorize students as engaged in value-thinking or location-thinking. Although I was able to categorize each student's interpretations of points according to David et al.'s (2019a) framework, there were some important nuances in some students' interpretations with some tasks which were not well-captured by the labels of expression-thinking and location-thinking. One of these phenomena was that many students who conceived of points as pairs of values consistently labeled outputs of the function at points along the graph rather than on the *y*-axis. According to David et al. (ibid), this behavior of placing output labels at points on the graph is typically associated with location-thinking, while placing output labels at expressions on the *y*-axis is a behavior associated with expression-thinking. Among the ten students engaged in value-thinking with Statement 1, seven students consistently labeled outputs at points while three consistently labeled outputs on the *y*-axis. One reason for this prevalence may be due to the screening survey and who I chose to interview to increase the variety in the sample of student responses. While I attempted to get an equal number of students who might engage in value-thinking and those who might engage in location-thinking, many students who I suspected as location-thinking students merely labeled points as outputs but did not show other signs of engaging in location-thinking.

Implications for Curriculum and Instruction

The four interpretations of expressions on graphs reported in this study, as well as its association with evaluations, content, and interpretation of points, have several implications for the teaching of mathematics using graphs of functions at the secondary and undergraduate level.

For the teaching and learning of Calculus in particular, which often includes graphs of functions, the findings of this study have implications. The results of this study indicate that students who are enrolled in or have completed Calculus courses may not interpret expressions on graphs in ways that support them in understanding a related mathematical idea. Many ideas in traditional Calculus curriculum are presented using graphs to illustrate them. Such examples include Riemann sums, the Fundamental Theorem of Calculus, the Intermediate Value Theorem, and the volume of solids of revolution. These graphical illustrations often suppose that students interpret expressions place on the graphs in a particular way, in order to interpret the entire illustration in the way intended by the designer. One implication of this study is that students' ways of interpreting expressions on graphs may differ from those tacitly assumed to be evoked by instructors or curriculum designers making use of such images. To describe this in more detail, I will focus on students' interpretations of expressions involving differences in the graphical register in this study and its implications for teaching ideas of Calculus.

Differences of Expressions on Graphs in the Study of Calculus

The findings from this study have implications for the teaching and learning of mathematical topics involving differences of expressions on graphs, specifically. Differences of expressions, and their representation on a graph, figure prominently in the study of Algebra and Calculus. The formula for the slope between two points on a function, which is the foundation for the limit definition of a derivative, involves a ratio of differences of inputs and outputs. In Calculus, the integral formulation of the area between two curves, as well as formulas for the volume of solids of revolution rely on differences. In advanced Calculus, differences within expressions of absolute expression inequalities are used to express notions of arbitrary closeness among expressions, the basis of the formal definition of continuity and many other concepts related to limits and convergence.

In this study, the only students who correctly evaluated each of the statements involving a difference were those who used a magnitude interpretation of expressions. Although a magnitude interpretation of expressions on graphs may have supported some students in interpreting a difference of expressions on a graph, this interpretation was not used by every student. Students who did not use a magnitude interpretation of the difference on the graph interpreted differences as cardinal expressions or not at all, instead focusing solely on the numerical value of these expressions. The students who interpreted differences of expressions as cardinals on graphs did not at any time interpret these as magnitudes on graphs, and vice versa. One possible explanation for the mutual exclusion of the use of these interpretations is that students who used a cardinal interpretation of expressions may have had no other interpretation of expressions to draw on. Another possible explanation is that the tasks provided in the interview did not evoke this interpretation for some students. In either case, students may benefit from opportunities to reason about magnitudes on graphs in their instruction. These reasoning opportunities ought to encourage students to conceive of measurements in the Cartesian plane beyond additive measurements, such as those used in a cardinal interpretation of expressions. The use of numerical values that are non-whole number values, including fractional and irrational values, may extend students' interpretations of measurement beyond the result of a counting process. Such opportunities may support the development of their understanding of the Cartesian coordinate system as being used to indicate distances in order for this interpretation to become one that is readily evoked in context.

Directions for Future Research

The findings of this study also lead to further areas of study in relation to students' interpretations of expressions. One possible area of research may be to examine the role of each of the four interpretations of expressions in the graphical register in other contexts. For instance, a nominal interpretation of expressions may or may not be intended by a graph's creator. One such context in which it is intended might be a histogram in which the *x*-axis is labeled with expressions or symbols to indicate groups. In this instance, a student wishing to interpret this image in a conventional way could only use a nominal interpretation of these expressions on the graph as the numerical values of the expressions have no significance beyond the labeling of groupings.

Not only might a context impact students' interpretations of expressions, the nuances of these interpretations and the operations in the graph that are associated with these interpretations may also change. An ordinal interpretation of expressions on graphs may support students in ordering expressions on graphs when in accordance with the conventions of the coordinate system they are working in. However, the types of spatial comparisons may change depending on the coordinate system. For instance, students comparing outputs of functions in a polar coordinate system may compare position in relation to the angle of rotation from the three o'clock position counter clockwise, rather than comparing positions vertically or horizontally.

Another area of mathematics which heavily relies on the graphical register to communicate trends is statistics. In statistical contexts, some interpretations of expressions and points on graphs may be more readily evoked than others. Investigating how students interpret variables and expressions relevant to a statistical context on a distribution curve may shed light on students' understanding and use of this representation.

Another direction for future research may be to develop interventions to support certain interpretations of expressions in graphs for contexts in Calculus. For instance, an intervention may be developed to support students to reason in the graphical register for specific contexts for which students may tend to avoid the graphical register. Other interventions may be designed to support students in interpreting expressions as magnitudes as a means of supporting their understanding slope in the graphical register as a ratio of magnitudes.

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APPENDIX A
SCREENING SURVEY

Name: ______________________________

1. If you are interested in participating in this study for monetary compensation, please provide the following information.

   (a) Your email address:

   (b) Your weekly availability (e.g., M 12-2, Th 9-1, F after 3):

   (c) Please **check** all math courses you have **previously taken** at the college level:

______ MAT 242 Elementary Linear Algebra
______ MAT 243 Discrete Mathematical Structures
______ MAT 265 Calculus for Engineers I
______ MAT 266 Calculus for Engineers II
______ MAT 267 Calculus for Engineers III
______ MAT 270 Calculus with Analytic Geometry I
______ MAT 271 Calculus with Analytic Geometry II
______ MAT 272 Calculus with Analytic Geometry III
______ MAT 275 Modern Differential Equations
______ MAT 300 Mathematical Structures
______ MAT 342 Linear Algebra
______ MAT 343 Applied Linear Algebra
______ MAT 370 Intermediate Calculus
______ MAT 371 Advanced Calculus 1
______ MAT 410 Introduction to General Topology
______ MAT 415 Introduction to Combinatorics
______ Other: __________________

   (d) Please **check** all math courses you are **currently taking** at the college level:

______ MAT 242 Elementary Linear Algebra
______ MAT 243 Discrete Mathematical Structures
______ MAT 266 Calculus for Engineers II
______ MAT 267 Calculus for Engineers III
______ MAT 271 Calculus with Analytic Geometry II
______ MAT 272 Calculus with Analytic Geometry III
______ MAT 275 Modern Differential Equations
______ MAT 300 Mathematical Structures
______ MAT 342 Linear Algebra
______ MAT 343 Applied Linear Algebra
______ MAT 370 Intermediate Calculus
______ MAT 371 Advanced Calculus 1
______ MAT 410 Introduction to General Topology
______ MAT 415 Introduction to Combinatorics
______ Other: __________________

2. Please respond to all four parts of this item:

(a) Provide an example of a relation that you would consider to be a **function**:

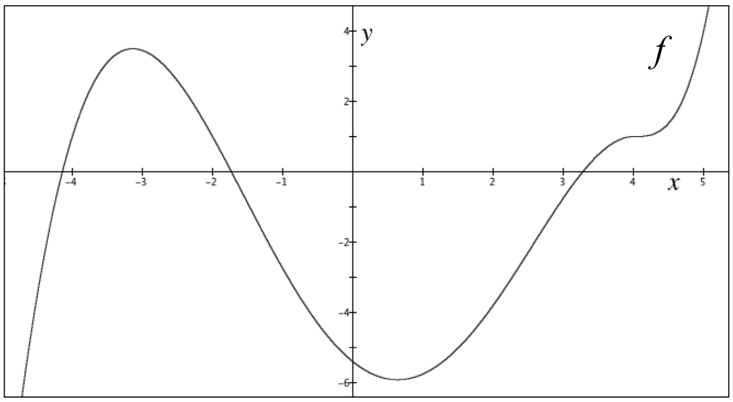
(b) Explain why you think your example in part (a) is a function:

(c) Provide an example of a relation that you would **NOT** consider to be a **function**:

(d) Explain why you think your example in part (b) is NOT a function:

3. For all parts of this question, **suppose *a*= –3 and *b* = 3.** Please respond to all three parts of this item:

   (a) Suppose c is a real number in the interval (–3, 3) and N is a real number between $f(a)$ and $f(b)$. Place the following labels on the graph of the function f below: a, b, c, $f(a)$, $f(b)$, $f(c)$, and N. Use the table to make sure you labeled everything on the graph.



a	
b	
c	
$f(a)$	
$f(b)$	
$f(c)$	
N	

   (b) Is the following statement true or false for the function f shown in the graph above?

> For all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.

Check one:
_____ True

_____ False

_____ Not enough information

   (c) Explain why you answered "True," "False," or "Not enough information" in part (b).

APPENDIX B
CLINICAL INTERVIEW PROTOCOL

Interview Protocol Dissertation Study

Have participant sign consent forms and give him or her a copy to take home.

Int: Thank you for taking the time to meet with me today. The purpose of this interview is to gain insight into student thinking. I ask that you try as best as you can to think out loud, explain everything as well as you can. I may ask you to clarify things you say or write down, or ask you how you arrived at particular conclusions. This does not mean that you are wrong, I just want to gain insight as to what is going on inside your head. For example, if I asked you what 2 times 3 was and you said 6, I may ask you how you got that answer and see if you could explain it as if to a child. Do you have any questions before we begin?

Stu: Responds

Notes:
-If at any point the student says this statement sounds like the previous statement, show them previous statements and ask them if they are similar or different
-If at any point the student says anything about continuity, probe.
-If student struggles with the quantifier "for all," pick values for the variables so the student can engage with the statements. Then, return to original statement after working with specific values.

Int: I will be presenting you with several statements about a **function** f, one by one. Please read each statement carefully and explain what the statement means. Here is the first statement.

Interviewer presents Statement 1 to student

Statement 1: For all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.

Note- the following questions will be asked after presenting each of the statements:
Stu: Reads statement and explains it.

(*If needed*) Int: Can you explain what you meant by ____
Stu: Responds

Int: When you're thinking about this statement, do you have a particular function in mind?
Purpose-to gauge whether students naturally use visual reasoning with graphs

If Stu: Yes
Int: Can you show me?
Stu: Draws mental image

Int: Can you explain how this image relates to the statement?
Stu: Responds

(*If needed*) Int: Can you label your image?

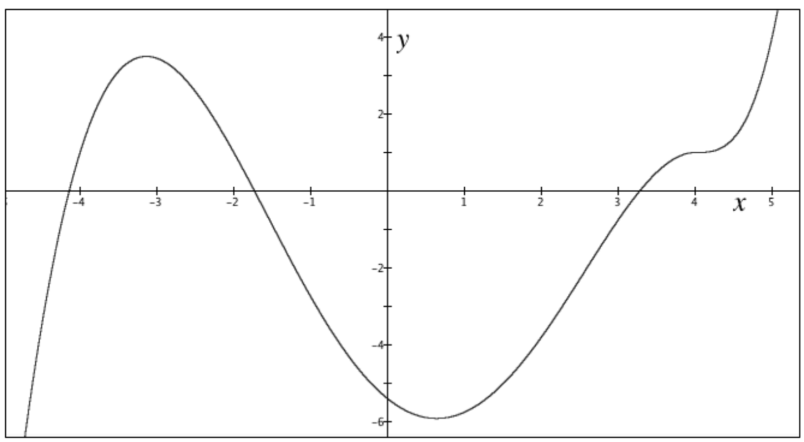
If student draws a graph, ask the questions corresponding to graphs

If Stu: No
Interviewer continues to next question

Int: I'm going to show you some graphs of a function *f* where *y* is a function of *x*. I would like you to tell me whether the statement is true or false for this graph.

Interviewer presents graph 1 unlabeled to Student:

Graph #1:



a = –3
b = 4

Note-the following questions will be asked for each graph:

Int: Suppose Graph #1 is the graph of the function in Statement 1. Is Statement 1 true for this function?
Stu: Responds

Int: Can you please label *c* from the statement on the graph?
Stu: *Labels graph*

Int: Can you please label *N* from the statement on the graph?
Stu: *Labels graph*

Int: Can you label the rest of them?
Stu: *Labels graph*

If student misses any of: a, b, c, f(a), f(b), f(c), N follow-up as needed:
Int: Can you show me where ___ is on the graph?
Stu: *Labels graph*

Int: Why is Statement 1 _____ (true or false) in this case?
Stu: Responds

Int: Can you explain why you placed these labels where you did? (May go individually)
Stu: Responds

Only ask with this graph and statement:

Interviewer marks point on the graph at x=4.5 on Graph 1.

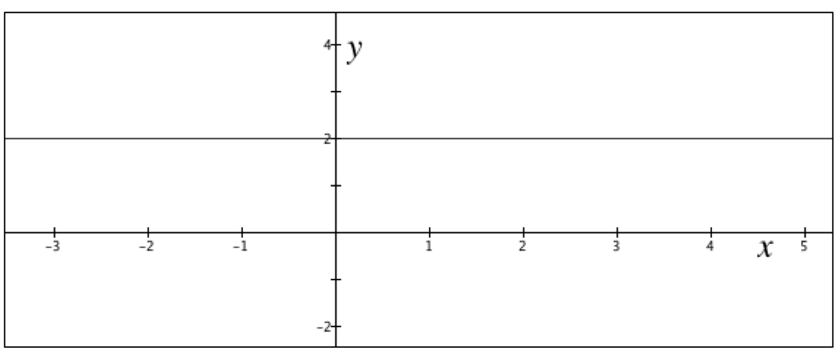
Int: Let's focus right here on the graph. Would you say here (*points to the point*) that *N* is between *f*(*a*) and *f*(*b*)?
Stu: Responds

**Students may not distinguish inputs and outputs*

Int: Is it possible to think of a function for which Statement 1 is true/false? (Only ask for Statement 1, opposite of whatever they responded)
Stu: Responds

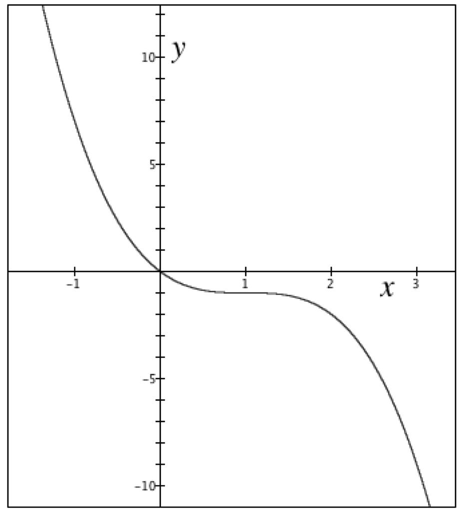
Interviewer presents Graphs 2-3 to the Student individually

Graph #2:



a = –3
b = 4

Graph #3:



a = –1
b = 2

Repeat graph questions for Graph 2-3 for Statement 1

Interviewer presents Statement 2 to Student

Statement 2: For all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$.

Interviewer repeats statement questions.

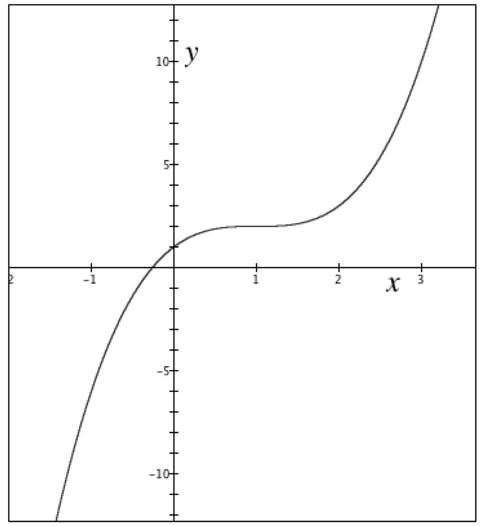
Interviewer presents graphs #1-4 with Statement 2 and asks graph questions.

Interviewer presents Statement 3 to Student

Statement 3: For all real numbers c, d in (a, b), if $f(c) = f(d)$, then $c=d$.

Interviewer repeats statement questions.
Interviewer presents graphs #1-4 with Statement 3 and asks graph questions.

Graph #4:



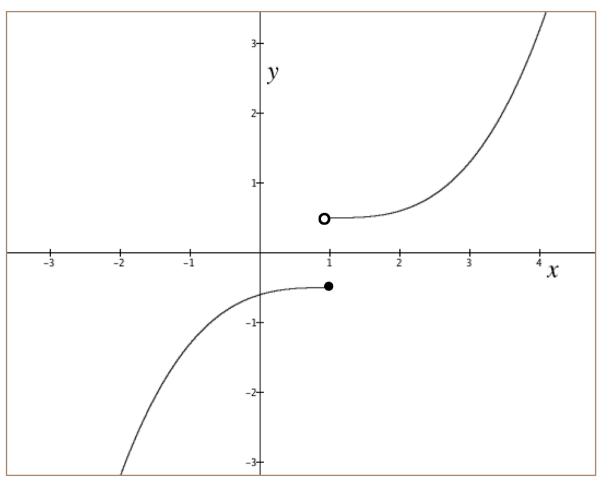
a = –1
b = 2

Interviewer presents Statement 4 to Student

Statement 4: For all real numbers ε > 0, there exists a real number δ > 0 such that, for all x in the domain of f with $-\delta < x-1 < \delta$, $-\varepsilon < f(x)-f(1) < \varepsilon$.

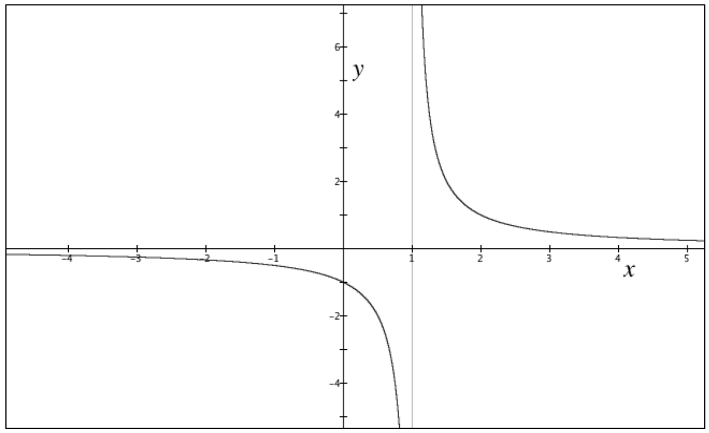
Interviewer repeats statement questions.
Interviewer presents graphs #3, 5-8 with Statement 4 and asks graph questions.

Graph #5:



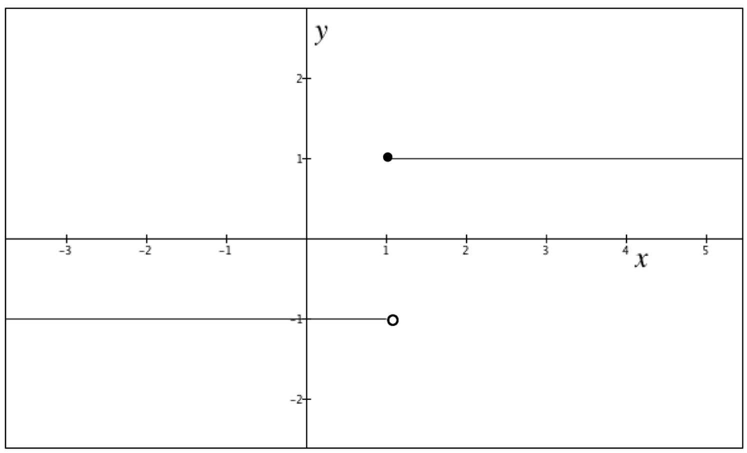
a = –3
b = 4

Graph #6:



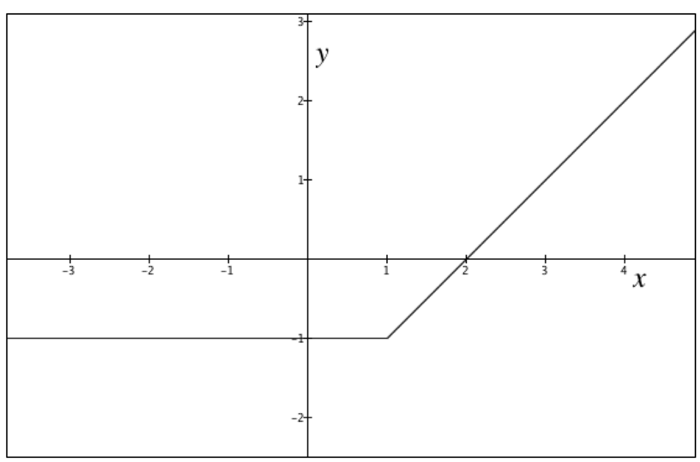
a = –3
b = 4

Graph #7:



a = –3
b = 4

Graph #8:



a = –3
b = 4

Interviewer presents Statement 5 to Student

Statement 5: For all non-zero real numbers *h,* if 2+*h* is in (*a*, *b*), then $\frac{f(2+h)-f(2)}{h} > 0$.

Interviewer repeats statement questions.
Interviewer presents graphs #1-4 with Statement 5 and asks graph questions.

Interviewer presents Statement 6 to Student

Statement 6: There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Interviewer repeats statement questions.
Interviewer presents graphs #1-3, 5-8 with Statement 6 and asks graph questions.

Interviewer asks follow up questions as necessary and may return to any statements/graphs if students ask to do so.

APPENDIX C
STUDENT CONSENT FORM

Consent/Assent Form

Title of research study: Students' Interpretations of Graphs of Real-Valued Functions and their Effects in Calculus Contexts

Investigator: Erika David

Why is this research being done?

The goal of this study is to better understand students' visual reasoning in Calculus contexts in an effort to improve support to students in understanding this topic in the future.

How long will the research last?

I expect that individuals will spend about 90-120 minutes per interview with a short break in the middle. Students may be asked to participate in between one and five such interviews, depending on how long it takes the student to work through the tasks.

How many people will be studied?

The number of participants may vary, but I anticipate between 10-15 students total will be participating in the both the pilot and main portion of this study.

What happens if I say yes, I want to be in this research?

If you are 18 or over and you agree to be apart of this research study you will take part in a video-recorded interview. In the interview, you will be asked to complete several mathematical tasks, as well as to verbalize your thinking. You may also be asked follow-up questions to further explain how you thought about a particular part of the task. You are free to decide whether you wish to participate in this study. You will receive extra credit points towards your quiz grade in your mathematics class. You will have the opportunity to earn extra credit through other opportunities throughout the semester.

What happens if I say yes, but I change my mind later?

You can leave the research at any time and it will not be held against you. You will have the opportunity to earn extra credit through other opportunities throughout the semester.

Is there any way being in this study could be bad for me?

Your participation will entail completing mathematical tasks while describing aloud how you are thinking. All efforts will be made to conceal your identity. It may take a minute or two to overcome initial discomfort related to being recorded and talking about your instruction.

Will being in this study help me in any way?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include improving your mathematical reasoning abilities in topics related to concepts from Calculus. You will also be compensated with extra credit points towards your quiz grade in your mathematics class.

What happens to the information collected for the research?

We cannot promise complete confidentiality, but every effort will be made to keep your data as anonymous as possible. The results of this study may be used in reports, presentations or publications but you will be given a pseudonym instead of using your personal name. A video recording of the interview may be shared at an academic conference presentation. ***Electronic copies of your recorded interview and hand-written work will be stored on a password-protected web-based cloud service and will only be shared with the researcher and the researcher's PhD advisor. Paper copies of your handwritten work along with your consent form will be kept in a locked filing cabinet in the researcher's office.***

Who can I talk to?

If you have questions, concerns, or complaints, talk to the research team by emailing Erika David at ejdavid@asu.edu or Kyeong Hah Roh at khroh@asu.edu

This research has been reviewed and approved by the Social Behavioral IRB. You may talk to them at (480) 965-6788 or by email at research.integrity@asu.edu if:

- Your questions, concerns, or complaints are not being answered by the research team.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You have questions about your rights as a research participant.
- You want to get information or provide input about this research.

I ______________agree to be videotaped during the interview, and to have these video clips used in public presentations. Data reported from this interview, including transcripts of the interview, or handwritten work, will use a pseudonym in place of my name. I understand my confidentiality cannot be guaranteed in this case because the public may be able to identify me.

Name (Printed):______________________________

____ __________________

Name (Signature) Date

APPENDIX D
HUMAN SUBJECTS APPROVAL LETTER

ASU Knowledge Enterprise Development

EXEMPTION GRANTED

Kyeong Roh
Mathematics and Statistical Sciences, School of
480/965-3792
khroh@asu.edu

Dear Kyeong Roh:

On 10/23/2017 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Investigating Undergraduate Students' Visual Reasoning in Contexts from Calculus
Investigator:	Kyeong Roh
IRB ID:	STUDY00007165
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Visual Reasoning Protocol, Category: IRB Protocol; • Interview Tasks and Questions, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Visual Reasoning Consent Form, Category: Consent Form; • Visual Reasoning Recruitment Statement, Category: Recruitment Materials;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings, (2) Tests, surveys, interviews, or observation on 10/23/2017.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Erika David
Erika David
Kyeong Roh