Weak Measure-Valued Solutions to a Nonlinear Conservation Law Modeling a Highly Re-entrant Manufacturing System

by

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ABSTRACT

The main part of this work establishes existence, uniqueness and regularity properties of measure-valued solutions of a nonlinear hyperbolic conservation law with non-local velocities. Major challenges stem from in- and out-fluxes containing nonzero pure-point parts which cause discontinuities of the velocities. This part is preceded, and motivated, by an extended study which proves that an associated optimal control problem has no optimal L^1-solutions that are supported on short time intervals.

The hyperbolic conservation law considered here is a well-established model for a highly re-entrant semiconductor manufacturing system. Prior work established well-posedness for L^1-controls and states, and existence of optimal solutions for L^2-controls, states, and control objectives. The results on measure-valued solutions presented here reduce to the existing literature in the case of initial state and in-flux being absolutely continuous measures. The surprising well-posedness (in the face of measures containing nonzero pure-point part and discontinuous velocities) is directly related to characteristic features of the model that capture the highly re-entrant nature of the semiconductor manufacturing system.

More specifically, the optimal control problem is to minimize an L^1-functional that measures the mismatch between actual and desired accumulated out-flux. The focus is on the transition between equilibria with eventually zero backlog. In the case of a step up to a larger equilibrium, the in-flux not only needs to increase to match the higher desired out-flux, but also needs to increase the mass in the factory and to make up for the backlog caused by an inverse response of the system. The optimality results obtained confirm the heuristic inference that the optimal solution should be an impulsive in-flux, but this is no longer in the space of L^1-controls.

The need for impulsive controls motivates the change of the setting from L^1-controls and states to controls and states that are Borel measures. The key strategy is

to largely abandon the Eulerian point of view and first construct Lagrangian solutions. The final section proposes a notion of weak measure-valued solutions and proves existence and uniqueness of such. In the case of the in-flux containing nonzero pure-point part, the weak solution cannot depend continuously on the time with respect to any norm. However, using semi-norms that are related to the flat norm, a weaker form of continuity of solutions with respect to time is proven. It is conjectured that also a similar weak continuous dependence on initial data holds with respect to a variant of the flat norm.

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INTRODUCTION

1.1 Problem Setting

Hyperbolic conservation laws are commonly used to describe traffic flow, pedestrian motion, sedimentation models and many other applications. A continuum model was introduced in Armbruster *et al.* (2006) to describe highly re-entrant semiconductor manufacturing systems (see section (1.2)) which produce a large number of items in a large number of steps. Denote by $x \in [0, 1]$ the degree of completion in the semiconductor factory. That is, $x = 0$ represents the beginning of the production line and $x = 1$ the end. Let $\rho : [0, \infty) \times [0, 1] \to [0, \infty)$, $(t, x) \mapsto \rho(t, x)$ be the density variable which describes the work in progress (WIP) density of the product at stage x of the production at time t. A characteristic feature of the model is that the velocity at time t is non-local and depends on the the total load $W(t) = \int_0^1 \rho(t, x)dx$. This reflects the highly re-entrant nature of the product flow in semi-conductor manufacturing systems. The velocity is a positive, decreasing function $v = \alpha(W)$ of the total load. The time evolution of the product density ρ was described in Armbruster *et al.* (2006) by the scalar hyperbolic conservation law

$$0 = \partial_t \rho(t, x) + \partial_x(\alpha(W(t))\rho(t, x)) \quad \text{for} \quad (t, x) \in [0, T] \times [0, 1], \tag{1.1a}$$

$$W(t) = \int_0^1 \rho(t, x)\, dx \quad \text{for} \quad t \in [0, T], \tag{1.1b}$$

$$\rho_0(x) = \rho(0, x) \quad \text{for} \quad x \in [0, 1], \tag{1.1c}$$

$$u(t) = \rho(t, 0)\alpha(W(t)) \quad \text{for} \quad t \in [0, T], \quad \text{and} \tag{1.1d}$$

$$y(t) = \rho(t, 1)\alpha(W(t)) \quad \text{for} \quad t \in [0, T]. \tag{1.1e}$$

Here u is a natural boundary control in-flux at $x = 0$ and y is the actual out-flux $x = 1$. Motivated by business objectives of the semiconductor manufacturing company, for a given forcasted demand out-flux y_d and the actual out-flux $y(t) = \rho(t, 0)\alpha(W(t))$, denote by $Y_d(t) = \int_0^t y_d(s)\, ds$ and $Y(t) = \int_0^t y(s)\, ds$ the accumulated demand out-flux and the accumulated actual out-fluxes, respectively. Natural control objectives are to minimize the mismatch between the accumulated demand and accumulated actual out-flux. An alternative to this problem is to model a perishable demand and minimize in a suitable sense the size of the different error signal $\int_0^T |y_d(t) - y(t)|^p\, dt$, $p = 1$ or $p = 2$. In this thesis, we mainly consider the case when the backlog $\beta = Y_d - Y$ is eventually zero. The control problem associated to the nonlinear hyperbolic conservation law (1.1) is to find an optimal control u^* in a set of admissible controls such that the control objective functional

$$J(u) = \int_0^\infty |\beta(t)|^p\, dt, \text{ with typically } p = 1 \text{ or } p = 2 \tag{1.2}$$

is minimized by u^*.

1.2 Semi-conductor Manufacturing System

The descriptions of the semi-conductor manufacturing system in this section are mostly from the articles (Hsieh and Hsieh, 2018) and (Armbruster *et al.*, 2006).

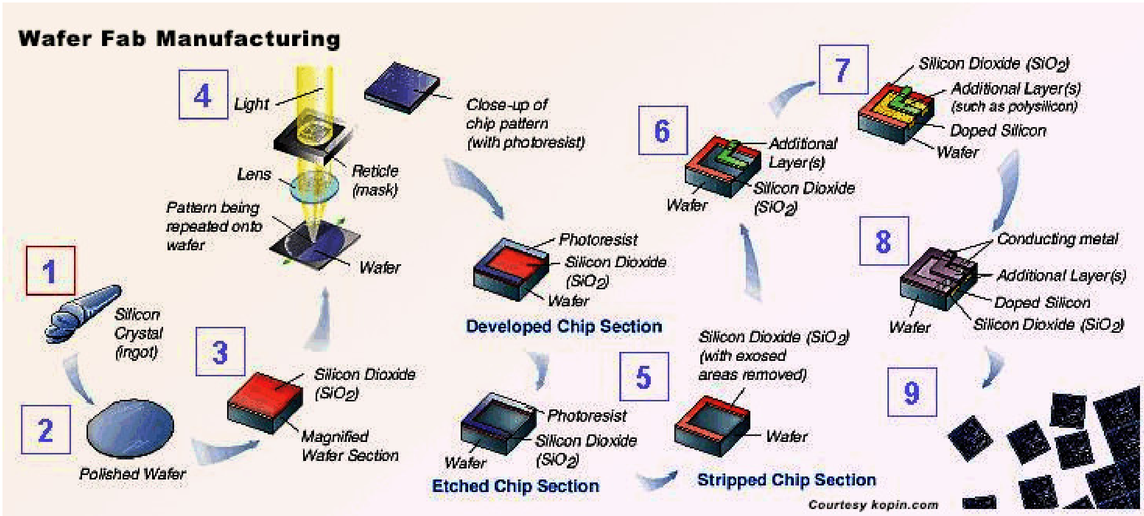
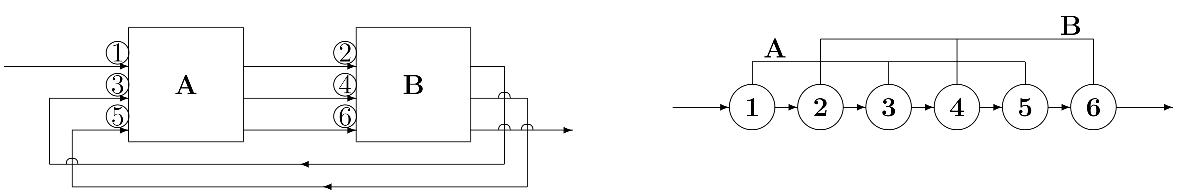


Figure 1.1: Wafer Fab Manufacturing

†www.semiwiki.com

The manufacturing process of semiconductor chips often involves hundreds of processing steps being executed layer by layer onto a bare wafer. The whole process is composed of a few repeating unit processes: thin film, photolithography, chemical mechanical planarization, diffusion, ion implantation, and etching. This nature of semiconductor manufacturing leads to long cycle times but also reduces production cost per chip.



The machine-based view

The process-based view

Figure 1.2: Machine-based and Process-based Views of Wafer Fab Manufacturing

In a modern semiconductor factory, we are interested in modeling and simulation on the order of 250 production steps executed on about 100 machines, with a re-entrant part of the production line that repeats about 15-20 times. In addition, the life cycle of a product is of the order of one year, whereas the through time lies between 40 and 60 days.

1.3 Literature Review and Distinctive Features of Our Problem

Various different choices of the space of admissible controls and objectives are of both practical and mathematical interest. Each space leads to distinct mathematical problems.

The original work Armbruster *et al.* (2006) validated the model using numerical simulations, comparing with discrete-event systems and with real factory data. In the context of L^2-data ($p = 2$ in equation (1.2)) $\rho_0,\ u,\ y_d$ and an L^2-control objective, the article La Marca *et al.* (2010) derived adjoint equations and computed the approximations of optimal controls numerically.

The article Coron *et al.* (2010) proved well-posedness for the Cauchy problem (1.1a)-(1.1d) (disregarding the out-flux) in the context of L^1-data (thus implying well-posedness for L^2-data), analyzed the regularity of solution curves $\rho\colon [0, T] \mapsto L^1([0, 1])$, and established existence of optimal controls for the original L^2-problem. Well-posedness of a elaborate multi-dimensional uncontrolled problem with unbounded spatial dimensions and was also demonstrated in Colombo *et al.* (2011), proving local existence of a weak entropy solutions and examining differentiability with respect to initial data.

It has been conjectured that for the more meaningful L^1-objective ($p = 1$ in equation (1.2)) and L^1-data ($u \in L^1((0,\infty))$ and $\rho_0 \in L^1([0, 1])$, optimal controls need not exist unless one requires the control to be bounded. Thus it is natural to recast the problem in the setting of controls and states being Borel measures.

In recent years the analysis of similar hyperbolic conservation laws in the setting of measures has seen substantial attention and progress. Here we briefly mention a few, and point the interested readers to many related references in these articles.

Motivated by earlier work on interactions of densities and point masses in the context of prey and predators Colombo and Lécureux-Mercier (2012), the article Crippa and Lécureux-Mercier (2013) established the well-posedness of similar nonlinear hyperbolic conservation laws (1.1) with non-local velocity in the setting of measure-valued data. The Wasserstein metric is a popular tool for models that use probability measures. Furthermore, in (Crippa and Lécureux-Mercier, 2013), the vector field v depends on time t, position x and the convolution product $\eta * \rho$ which represents the spatial average of the density ρ with convolution kernel η. This is more adapted to the case of panic in which pedestrains adapt their velocity to the average density rather than the nonlocal model in (Coron *et al.*, 2010) where the integral $\int_0^1 \rho(t,x)\,dx$ replaces the convolution product.

In order to allow for sources, and nonconstant total mass a generalized Wasserstein metric was introduced and studied in Piccoli and Rossi (2013, 2014). But it is restrictive since the article Piccoli and Rossi (2013, 2014) was only considering the measures on $\mathbb{R}$ that is absolutely continuous with respect the Lebesgue measure and is with compact support. Closely related are the Kantorovich-Rubinstein norm and the dual Lipschitz-norm or flat norm, see Gwiazda *et al.* (2018) for a careful study of continuity of semi-flows on the space of Borel measures endowed with the flat norm. The article Evers *et al.* (2016) introduces an innovative concept of sticky boundaries to deal with flux boundary conditions. Other very recent closely related articles Keimer and Pflug (2017); Keimer *et al.* (2018) consider system with the velocity being a *weighted* functional of the work in progress.

The problem addressed in this thesis has several distinctive features that significantly set it apart from the recent literature. Foremost, due to generally the influx being different from the outflux, the total mass is not constant. Consequently, most tools available for probability measures such as the Wasserstein metric do not apply here. Even more importantly, as a characteristic feature of the highly re-entrant semiconductor manufacturing system Armbruster *et al.* (2006), the velocity depends on the total load as in (1.1b), whereas in most popular traffic models it is governed by local interactions which are modeled by convolutions (naturally smoothen the velocity as a function of time). However, impulsive influxes and outfluxes cause the total load, and hence the velocity, to be discontinuous as functions of time. Consequently, weak measure-valued solutions of (1.1) are no longer meaningful in the usual distributional sense (formal integration by parts).

The thesis is organized as follows: We fix notation and discuss several commonly used distances of measures in Chapter 2. In chapter 3, we mainly study the case of transferring between equilibria with zero backlog at the terminal time and minimize the L^1-norm ($p = 1$ in equation (1.2)) of the backlog over a family of L^1-functions. In chapter 4, we reinterpret the hyperbolic conservation law (1.1a)-(1.1c) in the setting of Borel measures. Taking a Lagrangian point of view, we define a notion of Lagrangian solution to the system (1.1) and prove its existence and uniqueness. We also define a new notion of weak solution to the system (1.1) and establish its existence and uniqueness. In the case of the in-flux containing nonzero pure-point part, the weak solution can not depend continuously on the time with respect to any norm. We prove the continuity of solutions with respect to time by using a carefully crafted semi-norm that is a modification of the flat norm. We also conjecture that a similar weak continuous dependence on initial data holds by using a variant of the flat norm. In chapter 5, we make a conclusion on the thesis and outline plans for future research.

Chapter 2

GENERAL NOTATION AND COMMONLY USED DISTANCES OF MEASURES

In this section, we fix notation and discuss different distances defined on measure spaces.

2.1 General Notation

For the general notation this thesis uses, please refer the following table. In particaular, we take the production time $t \in [0, +\infty)$ and the degree of completion in the semiconductor wafer fab $x \in [0, 1]$. That is, $x = 0$ represents the beginning of the production line and $x = 1$ the end.

Table 2.1: General Notations

Symbols	Description
x	the degree of completion in the semiconductor wafer fab
t	time
ρ	density of work in process
W	total load
v	velocity field
u or μ	the boundary in-flux (control)
y	actual out-flux
y_d	demand out-flux
Y	accumulated actual out-flux
Y_d	accumulated demand out-flux
β	the backlog
(S, d)	metric space S with metric d
$\mathcal{M}^+(S)$	the set of positive finite regular Borel measures on S
$\mathcal{M}(S)$	the set of finite Borel measures on S
$\mathcal{P}(S)$	the set of probability measures
$BL(S)$	the set of bounded real-valued functions that are Lipschitz on S
λ	Lebesgue measure

2.2 Commonly Used Distances of Measures

We list and compare several different distances defined on measures in this subsection.

2.2.1 The Wasserstein Distance

The contents in this subsection generally follow the text book Villani (2008). We start with Monge's optimal transportation problem.

For every Borel measurable map $\gamma : \mathbb{R} \to \mathbb{R}$ and every finite Borel measure $\mu \in \mathcal{M}(\mathbb{R})$, the pushforward of μ by γ is defined as: for every Borel set $E \subset \mathbb{R}$,

$$\gamma \# \mu(E) := \mu(\gamma^{-1}(E)).$$

In this case, we also say that γ takes μ to $\gamma\#\mu$. And for every $\psi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \psi(y) \gamma \# d\mu(y) = \int_{\mathbb{R}} \psi(\gamma(x)) d\mu(x).$$

Note that $\gamma\#\mu \in \mathcal{M}(\mathbb{R})$ and the mass of μ is identical to the mass of $\gamma\#\mu$. Now, given two measures μ, $\nu \in \mathcal{M}(\mathbb{R})$ with the same mass, it is reasonable to ask if there exists a Borel measurable map $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\nu = \gamma\#\mu$. By the Jordan decomposition theorem, for any measure $\mu \in \mathcal{M}(\mathbb{R})$, there exist positive measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. Denote by $|\mu| = \mu^+(\mathbb{R}) + \mu^-(\mathbb{R})$, the total mass of μ. Now, we can add a cost to such γ if it exists as following:

$$I(\gamma) := |\mu|^{-1} \int_{\mathbb{R}} c(x, \gamma(x)) d\mu(x),$$

where $c : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is the cost function and informally $c(x, \gamma(x))$ tells how much it costs to transport one unit of mass from locaton x to location $\gamma(x)$. Also we assume that c is nonnegative and measurable.

The Monge's optimal transportation problem can be stated as:

$$\text{Minimize } I(\gamma) := |\mu|^{-1} \int_{\mathbb{R}} c(x, \gamma(x)) \mu(dx)$$

over the set of all Borel measurable maps γ such that $\gamma\#\mu = \nu$.

In fact, there exist examples of μ and ν for which such γ dose not exist. For example, on the real line, let $\mu = 2\delta_1$ and $\nu = \delta_0 + \delta_2$. There is no Borel measurable map γ which can take μ to ν since it can not separate masses. A simple condition that ensures the existence of a minimizing γ is that μ and ν are absolutely continuous with respect to Lebesgue measure.

This naturally suggests to generalize Monge's optimal transportation problem to Kantorovich's optimal transportation problem.

Given a probability measure π on the product space $\mathbb{R} \times \mathbb{R}$, one can interpret it as a way of transportation. Informally, for every $x, y \in \mathbb{R}$, $d\pi(x, y)$ measures the amount of mass transferred from location x to location y. The marginal measures μ and ν of a probability measure π are defined as

$$\int_{y \in \mathbb{R}} d\pi(x, y) = d\mu(x), \qquad \int_{x \in \mathbb{R}} d\pi(x, y) = d\nu(y),$$

That is to say, for every Borel measurable subsets $E, F \subset \mathbb{R}$, we have

$$\pi(E \times \mathbb{R}) = \mu(E), \qquad \pi(\mathbb{R} \times F) = \nu(F).$$

Equivalently, for all $\phi, \varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R} \times \mathbb{R}} (\phi(x) + \varphi(y)) d\pi(x, y) = \int_{\mathbb{R}} \phi(x)\mu(dx) + \int_{\mathbb{R}} \varphi(y) d\nu(y).$$

Such a π is called a admissible transference plan from μ to ν. The set of such transference plans is denoted as $\Pi(\mu, \nu)$.

The Kantorovich's optimal transportation problem can be stated as:

$$\text{Minimize } J(\pi) = \int_{\mathbb{R} \times \mathbb{R}} c(x, y) d\pi(x, y) \quad \text{for } \pi \in \Pi(\mu, \nu).$$

As stated in Piccoli and Rossi (2014), a minimizer of J in $\Pi(\mu, \nu)$ always exists. Note that Monge's problem is a special case of Kantorovich's problem. Indeed, the main restriction is that Monge's problem does not allow masses to be split. In other words, each location x is associated with a unique destination $\gamma(x)$ with $\gamma : \mathbb{R} \to \mathbb{R}$ being Borel measurable. In terms of transference plan, we can ask for π to be defined in a special from

$$d\pi_\gamma(x, y) := |\mu|^{-1} d\mu(x)\delta_0(y - \gamma(x)).$$

Then for every nonnegative and measurable function on $\mathbb{R} \times \mathbb{R}$, ζ, we have,

$$\int_{\mathbb{R}\times\mathbb{R}} \zeta(x, y) d\pi_\gamma(x, y) = |\mu|^{-1} \int_{\mathbb{R}} \zeta(x, \gamma(x)) d\mu(x).$$

In particular, we have

$$J(\pi_\gamma) = \int_{\mathbb{R}\times\mathbb{R}} c(x, y) d\pi_\gamma(x, y) = |\mu|^{-1} \int_{\mathbb{R}} c(x, \gamma(x)) d\mu(x) = I(\gamma).$$

Now let us define the Wasserstein distance between two measures with the same mass in $\mathcal{M}(\mathbb{R})$.

Denote the space of finite Borel measures on $\mathbb{R}$ with finite p -th moment by $\mathcal{M}^p(\mathbb{R})$, that is,

$$\mathcal{M}^p := \left\{ \mu \in \mathcal{M}(\mathbb{R}) \colon \int_{\mathbb{R}} |x|^p d\mu(x) < \infty \right\}.$$

This is a natural space on which J is finite. Consider a particular cost function $c(x, y) = |x - y|^p$, with $p \geq 1$ and two measures μ, ν of the same mass in $\mathcal{M}^p$, we define p-th order Wasserstein distance as follows:

$$W_p(\mu, \nu) := |\mu| \left(\min_{\pi\in\Pi(\mu,\nu)} J(\pi) \right)^{1/p} = |\mu| \left(\min_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x - y|^p d\pi(x, y) \right)^{1/p}.$$

for every $\phi \in C_c^0(\mathbb{R})$, the minimal Lipschitz constant for ϕ is defined as

$$\|\phi\|_{Lip} := \sup_{x\neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

The dual formulation for the first order Wasserstein distance W_1 can be stated as the following theorem:

Theorem 2.2.1. *(Kantorovich - Rubinstein Theorem, Villani (2008)) For all* μ*,* $\nu \in \mathcal{M}^1$ *with* $|\mu| = |\nu|$*,*

$$W_1(\mu, \nu) = \sup\left\{\int_{\mathbb{R}} \phi d(\mu - \nu)(x) \mid \phi \in C_c^0(\mathbb{R}), \|\phi\|_{Lip} \leq 1\right\}.$$

The Wasserstein distance on the subspace of measures in $\mathcal{M}^p$ with a given mass has revealed itself to be a powerful tool especially in dealing with the dynamics of measures, for example in the article Crippa and Lécureux-Mercier (2013). But the main restriction of this approach is that the Wasserstein distance is only defined between the measures with the same mass.

2.2.2 Total Variation Distance

Let $\mathscr{A}$ represent the Borel σ- algebra on $\mathbb{R}$. The total variation distance between μ and ν in $\mathcal{M}(\mathbb{R})$ is

$$\begin{aligned} \|\mu - \nu\|_{TV} &= 2 \sup_{A \in \mathscr{A}} |\mu(A) - \nu(A)| \\ &= \sup\left\{\int_{\mathbb{R}} \phi d(\mu - \nu)(x) \mid \phi \in C_c^0(\mathbb{R}), \|\phi\|_\infty \leq 1\right\}. \end{aligned} \tag{2.1}$$

The second equality in (2.1) is the dual formulation for the total variation distance. In some literature, for example in Piccoli and Rossi (2014), the total variation distance is also called as L^1 distance.

However the variation distance is too strong for the some applications. Consider the Dirac measures δ_1, δ_2, δ_5 in $\mathcal{B}(\mathbb{R})$. It is easy to see that

$$\|\delta_1 - \delta_2\|_{TV} = \|\delta_2 - \delta_5\|_{TV} = 2.$$

But this is not natural with respect to our intuition. In fact, the p-th order Wasserstein distance gives us a better model in this case. That is,

$$W_p(\delta_1 - \delta_2) = 1, \qquad W_p(\delta_2 - \delta_5) = 3.$$

2.2.3 Generalized Wasserstein Distance

Probability measures do not apply in the cases of nontrivial control in-flux and out-flux or source terms, since the total mass varies over time. As the intrinsic limitations of the Wasserstein distance and the total variation distance mentioned above, the generalized Wassertein distance was introduced in the article Piccoli and Rossi (2014) by combing the standard Wasserstein distance and total variation distance. Roughly speaking, given two measures μ, ν in $\mathcal{M}(\mathbb{R})$, an infinitesimal mass $\delta\nu$ of μ can either be removed at cost $a|\delta\mu|$, or moved from μ to ν at cost $bW_p(\delta\mu, \delta\nu)$. Formally, the generalized Wasserstein distance is defined as:

Given $a, b \in (0, \infty)$ and $p \geq 1$ and for every two measures μ, $\nu \in \mathcal{M}(\mathbb{R})$, the generalized Wasserstein distance between μ and ν is

$$W_p^{a,b}(\mu, \nu) = \inf_{\tilde{\mu}, \tilde{\nu} \in \mathcal{B}^p, |\tilde{\mu}| = |\tilde{\nu}|} \left(a(\|\mu - \tilde{\mu}\|_{TV} + \|\nu - \tilde{\nu}\|_{TV}) + bW_p(\tilde{\mu}, \tilde{\nu}) \right).$$

2.2.4 Dual Bounded Lipschitz Norm (Flat Norm)

The dual bounded Lipschitz norm $\|\cdot\|_{BL}^*$ on $\mathcal{M}(\mathbb{R})$ is defined as: for every $\mu \in \mathcal{M}(\mathbb{R})$,

$$\|\mu\|_{BL}^* = \sup \left\{ \int_{\mathbb{R}} f d\mu(x) \colon f \in BL(\mathbb{R}), |f|_{Lip} \leq 1, \|f\|_\infty \leq 1 \right\}.$$

Then the dual bounded Lipschitz distance between $\mu, \nu \in \mathcal{M}(\mathbb{R})$ is defined as:

$$\|\mu - \nu\|_{BL}^* = \sup \left\{ \int_{\mathbb{R}} f d(\mu - \nu)(x) \colon f \in BL(\mathbb{R}), |f|_{Lip} \leq 1, \|f\|_\infty \leq 1 \right\}.$$

It was proved in the article Piccoli and Rossi (2014) that the dual formulation of the generalized Wasserstein distance $W_1^{1,1}$ coincides with the dual bounded Lipschitz distance.

Theorem 2.2.2. *(Theorem 13, Piccoli and Rossi (2014)) for every* $\mu, \nu \in \mathcal{M}(\mathbb{R})$*,*

$$W_1^{1,1}(\mu, \nu) = \|\mu - \nu\|_{BL}^*.$$

Now let us consider the Dirac measure at $x \in \mathbb{R}$, δ_x. for every $a, b \in \mathbb{R}$, we have

$$\|\delta_a - \delta_b\|_{BL}^* = \min\{1, |a - b|\}.$$

Thus the dual bounded Lipschitz distance is in line with the intuition compared to the total variation distance in this case.

The dual bounded Lipschitz norm in also called flat norm, Fortet-Mourier norm, Dudley norm or Kantorovich-Rubinstein norm. For more versions of flat norm, we refer the interested reader to (Gwiazda *et al.*, 2018). In this thesis, we chose the version of flat norm $\|\cdot\|_\flat$ on $\mathcal{M}(S)$ that only involve nonnegative bounded Lipschitz continuous functions,

$$\|\mu\|_\flat = \sup\left\{\left|\int_S f\, d\mu\right|; f\colon S \mapsto [0,1], |f(x) - f(y)| \le d(x, y) \text{ for } x, y \in S\right\}. \quad (2.2)$$

We also collect some facts on $\mathcal{M}^+(S)$ endowed with the flat norm (2.2) from Gwiazda *et al.* (2018).

Theorem 2.2.3 (Theorem 4.22, (Gwiazda *et al.*, 2018))**.** *Let S be complete and separable and (μ_n) be a tight (inner regular) sequence in $\mathcal{M}^+(S)$ such that $\mu_n(S)$ is bounded. Then (μ_n) has a converging subsequence (with the limit measure being tight as well).*

Theorem 2.2.4 (Theorem 4.23, (Gwiazda *et al.*, 2018))**.** *If S is compact, then every bounded closed subset of $\mathcal{M}^+(S)$ is compact and $\mathcal{M}^+(S)$ is locally compact.*

Chapter 3

NON-OPTIMALITY OF A FAMILY OF L^1-CONTROLS

3.1 Introduction: L^1-setting

Compared with the work done by the articles (Armbruster *et al.*, 2006) and (Coron *et al.*, 2010) in the L^2-setting, both, more meaningful from the business point of view, and mathematically more challenging is the optimal control problem with an L^1-objective. From a business perspective, an important problem is the transfer between equilibria with zero backlog at the terminal time and minimizing the L^1-norm of the backlog, that is, the difference between the desired cumulative out-flux $Y_d(t) = \int_0^t y_d(\tau)\, d\tau$ and the actual cumulative out-flux $Y(t) = \int_0^t y(\tau)\, d\tau$. In this chapter, we demonstrate progress towards proving the conjectured non-existence of optimal L^1-controls for the optimal control problem with $p = 1$ in equation (1.2).

In this chapter, we consider a demand out-flux y_d that is piecewise constant and increases with a jump at time t^*, i.e.,

$$y_d(t) = \begin{cases} y_1 & \text{if} \quad t < t^* \\ y_2 & \text{if} \quad t \geq t^*, \end{cases} \tag{3.1}$$

with $y_1, y_2 \in [0, 1)$.

Additionally, we work with the fixed model $v = \alpha(W) = \frac{1}{1+W}$ for the velocity v as a function of the total load W, as in Coron *et al.* (2010). For arbitrary but fixed initial condition $\rho_0 \in L^1([0, 1])$ and in-flux $u \in L^1([0, T])$ (with T being large but fixed), the total load W is bounded above by $\int_{[0,1]} \rho_0(x)\, dx + \int_{[0,T]} u(t)\, dt$. This implies that v is bounded away from 0, i.e., there exists some $v_0 > 0$ such that the velocity $v \geq v_0$. This model assumes that all parts move through the factory with the

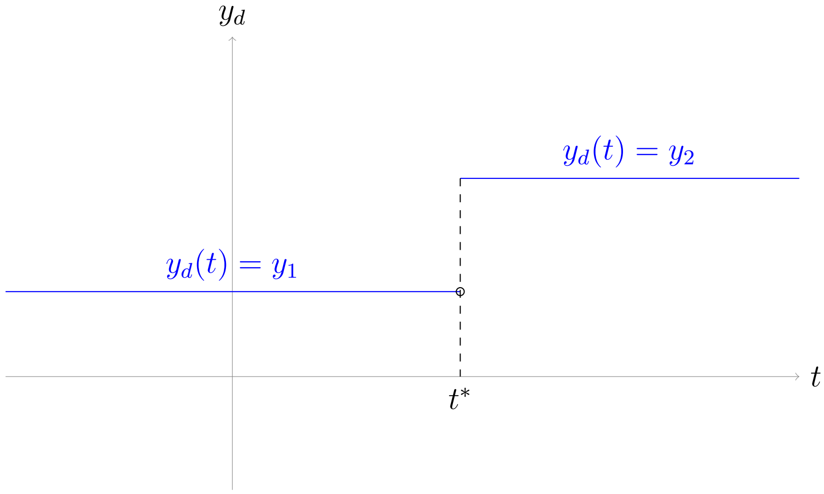


Figure 3.1: The Piecewise Demand Out-flux y_d

same velocity v and any increase of total load W at any stage slows down the entire production line. Denote the constant densities at the equilibrium states when $t < t^*$ and when $t \geq t^*$ by ρ_1 and ρ_2, respectively. (It will be clear from the context that these are not ρ_t at times $t = 1$ and $t = 2$.) Then for $k = 1, 2$, the constant densities at the equilibrium states are $\rho_k = \frac{y_k}{1-y_k}$.

This chapter is organized as follows: In section (3.2), we explicitly calculate the constant backlog under a natural control u when the system transfers from a smaller to a larger equilibrium. In section (3.3), we justify the existence of the additional mass M^* and prove the nonexistence of optimal control in a family of L^1-controls. In section (3.4), we numerically illustrate that a control taking zero in-flux from some time minimizes the cost functional J when the system transfers from a larger equilibrium to a smaller equilibrium.

3.2 Transfer from a Smaller to a Larger Equilibrium with Nonzero Backlog

In this section, we consider the case of the system transferring from a smaller to a larger equilibrium, i.e., $0 \leq y_1 < y_2 < 1$, with a nonzero backlog. To meet the requirement that the system arrives at another equilibrium at time $t = t^*$, the operator in factory needs to start to increase the total load W at some time $t_* < t^*$ such that

$$\int_{t_*}^{t^*} \alpha(W(t))\, dt = 1. \tag{3.2}$$

For our calculations, it is more convenient to instead consider the demand out-flux jump time t^* as a function of t_*, y_1, and y_2. Additionally, the densities at the equilibrium states when $t < t_*$ and when $t \geq t^*$ are

$$\rho(t,x) = \begin{cases} \rho_1 \text{ if } & t < t_*, \quad 0 \leq x \leq 1; \\ \rho_2 \text{ if } & t \geq t^*, \quad 0 \leq x \leq 1, \end{cases}$$

with $\rho_k = \frac{y_k}{1-y_k}$, $k = 1, 2$.

Intuitively, a reasonable control in-flux $u(t)$ to the special demand out-flux y_d is that $u(t) = \rho(t,0)\alpha(W(t))$ with

$$\rho(t,0) = \begin{cases} \rho_1 \text{ if } & t < t_*, \\ \rho_2 \text{ if } & t \geq t_*, \end{cases} \tag{3.3}$$

with $t_* \leq t^*$ and $\int_{t_*}^{t^*} \alpha(W(t))dt = 1$.

Remark. *By Theorem* (2.3) *in the article Coron* et al. *(2010), the hyperbolic conservation law* (1.1) *has a unique solution with the initial condition $\rho_0(x) = \rho_1$, $x \in [0,1]$ and the in-flux $u(t) = \rho(t,0)\alpha(W(t))$ with $\rho(t,0)$ defined as in equation* (3.3)*.*

Remark. *The control in-flux u coincides with the demand out-flux y_d for $t < t_*$ and $t > t^*$. That is, $u(t) = y_d(t)$ for $t < t_*$ and $t > t^*$.*

Correspondingly, the actual out-flux is $y(t) = \rho(t,1)\alpha(W(t))$ with

$$\rho(t,1) = \begin{cases} \rho_1 \text{ if } & t < t^*, \\ \rho_2 \text{ if } & t \geq t^*. \end{cases} \tag{3.4}$$

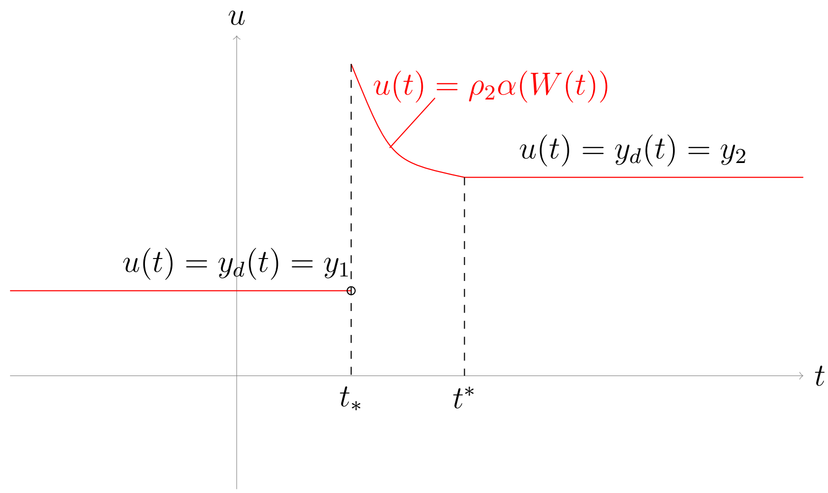


Figure 3.2: The In-flux u to Transfer from a Smaller to a Larger Equilibrium

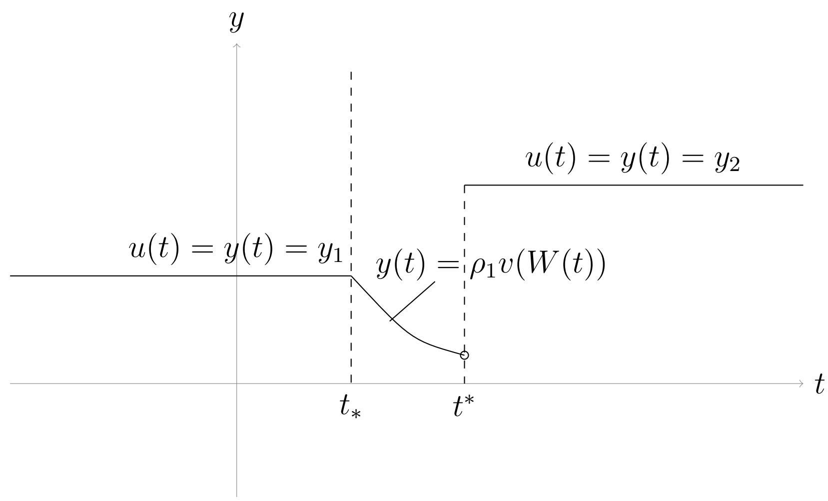


Figure 3.3: The Out-flux y When Transferring from a Smaller to a Larger Equilibrium

Remark. *The actual out-flux y coincides with the the demand out-flux y_d for $t < t_*$ and $t > t^*$. That is, $y(t) = y_d(t)$ for $t < t_*$ and $t > t^*$.*

With out loss of generality, we assume that $t_* > 0$. Taking the control in-flux u as in figure (3.2), We justify the existence of the demand out-flux jump time t^*.

Lemma 3.2.1. *For any $t_* > 0$, the demand out-flux jump time, t^*, is finite, i.e., $t^* < \infty$.*

Please find the proof of this lemma in appendix (B).

We derive the explicit expression of t^* through the relation defined in equation (3.2). For $t \in [0, t_*]$, the production line is at the equilibrium state with constant product density ρ_1. Thus the total load is $W(t) = \int_0^1 \rho_1 dx = \rho_1$ for any $t \in [0, t_*]$. Combined with equation (B.1), we have the flowing Cauchy problem:

$$W'(t) = \frac{\rho_2 - \rho_1}{1 + W(t)} \text{ for } t \in (t_*, t^*) \tag{3.5}$$

$$W(t_*) = \rho_1. \tag{3.6}$$

An explicit solution of the Cauchy problem (3.5) and (3.6) is

$$W(t) = -1 + \sqrt{(\rho_1 + 1)^2 + 2(\rho_2 - \rho_1)(t - t_*)}, \qquad t \in [t_*, t^*].$$

Thus the velocity field is

$$\alpha(W(t)) = \frac{1}{1 + W(t)} = \frac{1}{\sqrt{(\rho_1 + 1)^2 + 2(\rho_2 - \rho_1)(t - t_*)}}, \qquad t \in [t_*, t^*].$$

By equation (3.2) we have,

$$\begin{aligned}
\int_{t_*}^{t^*} \alpha(W(t))dt &= \int_{t_*}^{t^*} \frac{1}{\sqrt{(\rho_1 + 1)^2 + 2(\rho_2 - \rho_1)(t - t_*)}} dt \\
&= \frac{1}{\rho_2 - \rho_1} \left(\sqrt{(\rho_1 + 1)^2 + 2(\rho_2 - \rho_1)(t^* - t_*)} - (\rho_1 + 1) \right) \\
&= 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
t^* &= t_* + \frac{\rho_1 + \rho_2 + 2}{2} \\
&= t_* + \frac{(1 - y_1) + (1 - y_2)}{2(1 - y_1)(1 - y_2)}.
\end{aligned}$$

Note: Since $0 \le y_1 < y_2 < 1$, we have,

$$(1 - y_1) + (1 - y_2) \ge (1 - y_1)^2 + (1 - y_2)^2 \ge 2(1 - y_1)(1 - y_2)$$

which implies that $t^* \ge t_* + 1$.

Next, we find an expression for the backlog β. Recall that $Y_d(t)$ and $Y(t)$ represent the accumulated demand out-flux and the accumulated actual out-flux at time t respectively. The backlog of a production system at time t, $\beta(t)$, is defined as the accumulated demand out-flux minus the accumulated actual out-flux up to that time, i.e., $\beta(t) = Y_d(t) - Y(t)$.

The accumulated demand out-flux is

$$Y_d(t) = \int_0^t y_d(s)\, ds = \begin{cases} y_1 t \text{ if } \quad 0 \leq t < t^*, \\ y_1 t^* + y_2(t - t^*) \text{ if } \quad t \geq t^*. \end{cases}$$

The accumulated actual out-flux is

$$\begin{aligned} Y(t) &= \int_0^t y(s) ds = \int_0^1 \rho(s,1)\alpha(W(s))\, ds \\ &= \begin{cases} y_1 t \text{ if } \quad 0 \leq t \leq t_*, \\ y_1 t_* + \int_{t_*}^t \rho_1 \alpha(W(s)) ds \text{ if } \quad t_* < t \leq t^*, \\ y_1 t_* + \int_{t_*}^{t^*} \rho_1 \alpha(W(s)) ds + y_2(t - t^*) \text{ if } \quad t \geq t^*. \end{cases} \end{aligned}$$

Thus the backlog is

$$\beta(t) = \begin{cases} 0 \text{ if } \quad 0 \leq t \leq t_*, \\ y_1(t - t_*) - \int_{t_*}^t \rho_1 \alpha(W(s)) ds \text{ if } \quad t_* < t \leq t^*, \\ y_1(t^* - t_*) - \int_{t_*}^{t^*} \rho_1 \alpha(W(s)) ds \text{ if } \quad t \geq t^*. \end{cases}$$

Since $\int_{t_*}^{t^*} \alpha(W(t)) dt = 1$, the backlog is constant for $t \geq t^*$, that is,

$$\begin{aligned} \beta(t) &= y_1(t^* - t_*) - \rho_1 \\ &= y_1 \left(\frac{(1 - y_1) + (1 - y_2)}{2(1 - y_1)(1 - y_2)} \right) - \frac{y_1}{1 - y_1}, \\ &= \frac{y_1(y_2 - y_1)}{2(1 - y_1)(1 - y_2)} > 0. \end{aligned}$$

Conclusion: The control in-flux as in figure (3.2) causes an inverse response to the production system: as in-flux increases, out-flux decreases temporarily. This is because the velocity of the system decreases due to the increase of the total load and this results in nonzero backlog to the system. More specifically, for the piecewise constant demand out-flux y_d (3.1), the control in-flux

$$u(t) = \begin{cases} \rho_1 \alpha(W(t)) \text{ if } & 0 \leq t < t_*, \\ \rho_2 \alpha(W(t)) \text{ if } & t \geq t_*, \end{cases} \tag{3.7}$$

with $t_* = t^* - \frac{(1-y_1)+(1-y_2)}{2(1-y_1)(1-y_2)}$ produces a constant backlog for $t \geq t^*$ such that

$$\beta(t) = \frac{y_1(y_2 - y_1)}{2(1-y_1)(1-y_2)}.$$

3.3 Transfer from a Smaller to a Larger Equilibrium with Eventually Zero Backlog

In this section, we still consider the case when the system transferring from a smaller to a larger equilibrium, i.e., $0 \leq y_1 < y_2 < 1$, but with eventually zero backlog. To cancel the backlog produced by the control in-flux $u(t)$ (3.7), one needs to modify this control in-flux by increasing it, i.e., by adding additional mass $M > 0$ at $x = 0$ over some time interval $[0, \varepsilon)$. This results in an even larger inverse response due to the fact that velocity α further decreases as the total load W increases. In addition, the higher demand out-flux at a later time requires higher in-flux at some earlier time. Thus the additional mass M must not only make up for the further backlog caused by M itself, but must also make up for the missing out-flux due to the step up of the demand out-flux. In this thesis, we consider the case when $\varepsilon \in (0, 1]$ so that none of the in-flux over the time interval $[0, \varepsilon]$ exits from the system before time ε. It is not a prior clear that for every $\varepsilon \in (0, 1]$, such a mass M exists. Furthermore, the requirement that the system reaches another equilibrium at time t^* forces us to choose the control in-flux as $u(t) = \rho_2 \alpha(W(t))$ for $t > \varepsilon$, with $t_* < \varepsilon < t^*$. Without loss of generality, we assume that the action time $t_* = 0$, that is,

$$\int_0^{t^*} \alpha(W(t))\, dt = 1. \tag{3.8}$$

Note that since the velocity is bounded away from 0, that is, $\alpha(W(t)) \geq v_{\min} > 0$, there is a unique t^* that satisfies equation (3.8).

Note that in the situation with a modified control in-flux, the system reaches its new equilibrium state at time T^* defined by

$$\int_{\varepsilon}^{T^*} \alpha(W(t))\, dt = 1, \tag{3.9}$$

with zero backlog $\beta(T^*) = 0$.

The time T^* at which the backlog becomes zero also depends on both the shape of the control variation, and on the amount of the additional mass M. Given a direction

$$h \in L^1([0,1];[0,+\infty)) \text{ with } \int_0^1 h(t)\, dt = 1, \tag{3.10}$$

and for any $\varepsilon \in (0,1]$, we consider the curve of modified L^1-control in-fluxes

$$u_\varepsilon(t) = \begin{cases} \rho_2 \alpha(W(t)) + \frac{M^*(h,\varepsilon)}{\varepsilon} h(\frac{t}{\varepsilon}) & \text{if } 0 \leq t \leq \varepsilon; \\ \rho_2 \alpha(W(t)) & \text{if } t > \varepsilon, \end{cases} \tag{3.11}$$

with $M^*(h,\varepsilon) > 0$. Here the L^1-function h and the number $M^*(h,\varepsilon)$ represent the shape and the amount of the additional mass respectively.

3.3.1 Existence of $M^*(h,\varepsilon)$

One may want to ask: For every h satisfying equation (3.10) and every $\varepsilon \in (0,1]$, does there exist an additional mass $M^*(h,\varepsilon)$ such that the control in-flux u_ε (3.11) results in zero backlog in finite time? To show the existence of such $M^*(h,\varepsilon)$, we need to verify that the backlog β depends continuously on the additional mass. It looks like a very standard procedure, but much more tedious than expected due to the complexity of function h.

The accumulated demand out-flux $Y_d(t)$ is

$$Y_d(t) = \begin{cases} y_1 t & \text{if} \quad 0 \leq t < t^*, \\ y_1 t^* + y_2(t - t^*) & \text{if} \quad t \geq t^*, \end{cases}$$

where t^* is defined in (3.8).

The accumulated actual out-flux is

$$
\begin{aligned}
Y(t) &= \int_0^t y(s)ds \\
&= \begin{cases} \int_0^t \rho_1\alpha(W(s))\,ds & \text{if} \quad 0 \le t \le t^*, \\ \int_0^{t^*} \rho_1\alpha(W(s))\,ds + \int_{t^*}^t \rho(s,1)\alpha(W(s))\,ds & \text{if} \quad t^* < t \le T^*, \\ \int_0^{t^*} \rho_1\alpha(W(s))\,ds + \int_{t^*}^{T^*} \rho(s,1)\alpha(W(s))\,ds + \int_{T^*}^t \rho_2\alpha(W(s))\,ds & \text{if} \quad t \ge T^*. \end{cases}
\end{aligned}
$$

Thus for all times $t \ge T^*$ the backlog, $\beta(t)$, is

$$
\begin{aligned}
&\beta(t) = Y_d(t) - Y(t) \\
&\quad = y_1 t^* + y_2(t - t^*) - \int_0^{t^*} \rho_1\alpha(W(s))\,ds - \int_{t^*}^{T^*} \rho(s,1)\alpha(W(s))\,ds - \int_{T^*}^{t} \rho_2\alpha(W(s))\,ds.
\end{aligned}
$$

By equation (3.8), we have, $\int_0^{t^*} \rho_1\alpha(W(s))\,ds = \rho_1$. The integral $\int_{t^*}^{T^*} \rho(s,1)\alpha(W(s))\,ds$ represents the total out-flux over the time interval $[t^*, T^*]$ which is equal to the total in-flux from over the time interval $[0, \varepsilon]$. That is,

$$
\int_{t^*}^{T^*} \rho(s,1)\alpha(W(s))\,ds = \int_0^{\varepsilon} u(t)\,dt = \int_0^{\varepsilon} \rho_2\alpha(W(t))\,dt + M. \tag{3.12}
$$

Additionally, $\rho_2\alpha(W(t)) = y_2$ for $t \ge T^*$ implies that $\int_{T^*}^t \rho_2\alpha(W(s))\,ds = y_2(t - T^*)$. Therefore, for all $t \ge T$,

$$
\beta(t) = y_1 t^* + y_2(T^* - t^*) - \rho_1 - M - \int_0^{\varepsilon} \rho_2\alpha(W(s)). \tag{3.13}
$$

Lemma 3.3.1. *Given $\varepsilon > 0$, for any $M > 0$, $T^* - t^*$ is bounded above by $\left(\frac{1+\rho_2+M}{1+\rho_1}\right)\varepsilon$, i.e., $T^* - t^* \le \left(\frac{1+\rho_2+M}{1+\rho_1}\right)\varepsilon$.*

Please find the proof of this lemma in appendix (B).

Remark. *Rearranging the terms ,we have $T^* - t^* \le \frac{\varepsilon(1+\rho_2+M)}{1+\rho_1} = \frac{\frac{1}{1+\rho_1}}{\frac{1}{1+\rho_2+M}}(\varepsilon - 0)$.*

Remark. *Note that* $\frac{1}{1+\rho_1}$ *is the upper bound of the velocity on the time interval* $[0,\varepsilon]$ *and* $\frac{1}{1+\rho_2+M}$ *is the lower bound of the velocity on the time interval* $[t^*,T^*]$*. The ratio between* T^*-t^* *and* $\varepsilon-0$ *is bounded above by the ratio between the upper bound of the velocity at the time interval* $[0,\varepsilon]$ *and the lower bound of the velocity at the time interval* $[t^*,T^*]$*. That is,*

$$\frac{T^*-t^*}{\varepsilon-0} \leq \frac{\frac{1}{1+\rho_1}}{\frac{1}{1+\rho_2+M}}.$$

Lemma 3.3.2. *Given* $\varepsilon>0$*, for any* $M>0$*,* t^* *is bounded above by* $\varepsilon+\frac{1}{2}(\rho_2-\rho_1)+\left(1+\rho_1+M+\frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)$*, i.e.,* $t^*\leq\varepsilon+\frac{1}{2}(\rho_2-\rho_1)+\left(1+\rho_1+M+\frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)$.

Please find the proof of this lemma in appendix (B).

Lemma 3.3.3. *Given a function* h *as in* (3.10)*, for every* $\varepsilon\in(0,1]$*, if* M *is sufficiently large, then the backlog* $\beta(t)<0$ *for* $t\geq T^*$*.*

Proof. From lemma (3.3.1) and (3.3.2), we have, for $t\geq T$,

$$\begin{aligned}
\beta(t) \quad &\leq \quad \frac{\rho_1}{1+\rho_1}\left(\varepsilon+\frac{1}{2}(\rho_2-\rho_1)+\left(1+\rho_1+M+\frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)\right), \qquad (3.14)\\
&+ \quad \frac{\rho_2}{1+\rho_2}\frac{\varepsilon(1+\rho_2+M)}{1+\rho_1}-\rho_1-M,\\
&= \quad \frac{\rho_1}{1+\rho_1}\left(\varepsilon+\frac{1}{2}(\rho_2-\rho_1)+\left(1+\rho_1+\frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)\right)+\frac{\rho_2}{1+\rho_2}\frac{\varepsilon(1+\rho_2)}{1+\rho_1}-\rho_1,\\
&+ \quad \left(\frac{\rho_1}{1+\rho_1}+\frac{\varepsilon\rho_2}{(1+\rho_1)(1+\rho_2)}-1\right)M.
\end{aligned}$$

Note that for the coefficient of M,

$$\frac{\rho_1}{1+\rho_1}+\frac{\varepsilon\rho_2}{(1+\rho_1)(1+\rho_2)}-1=\frac{(\varepsilon-1)\rho_2-1}{(1+\rho_1)(1+\rho_2)}<0, \quad 0<\varepsilon\leq 1$$

so the right hand side of the inequality (3.14) is negative if M is large enough. Hence, for a fixed $0<\varepsilon\leq 1$, M large enough implies that $\beta(t)<0$ for $t\geq T$. □

The following lemma is from the Ordinary Differential Equation text book (Meiss, 2007). For sake of completeness, we recall the proof of the argument here.

Lemma 3.3.4 (Grönwall's Inequality). *Suppose the functions $g, k : [0, a] \mapsto \mathbb{R}$ are continuous, $a > 0$, $k(t) \geq 0$, and g obeys the inequality*

$$g(t) \leq G(t) \equiv c + \int_0^t k(s)g(s)\,ds \tag{3.15}$$

for all $t \in [0, a]$ and c being a constant. Then for all $t \in [0, a]$,

$$g(t) \leq ce^{\int_0^t k(s)\,ds}.$$

Proof. Since both g and k are continuous, then G is C^1 and $G(0) = c$. Furthermore, differentiating equation (3.15) gives

$$\dot{G}(t) = k(t)g(t) \leq k(t)G(t);$$

Consequently,

$$\dot{G} - kG \leq 0.$$

Multiplying both sides of the above inequality by the positive "integrating factor" $e^{-\int_0^t k(s)\,ds}$, we obtain

$$e^{-\int_0^t k(s)\,ds}(\dot{G} - kG) = \frac{d}{dt}\left(G(t)e^{-\int_0^t k(s)\,ds}\right) \leq 0.$$

Integrating the above inequality,

$$G(t)e^{-\int_0^t k(s)\,ds} \leq G(0),$$

which implies

$$g(t) \leq G(t) \leq ce^{\int_0^t k(s)\,ds}.$$

$\square$

Lemma 3.3.5. *Given a function h as in (3.10), for arbitrary but fixed $\varepsilon \in (0, 1]$, for every time $t \in [0, \varepsilon]$, the total load $W(t)$ depends continuously on the additional mass M.*

Proof. Over the time interval $[0, \varepsilon]$, the total load W satisfies the following Cauchy problem:

$$W'(t) = \frac{\rho_2 - \rho_1}{1 + W(t)} + \frac{M}{\varepsilon} h(\frac{t}{\varepsilon}) \tag{3.16}$$
$$W(0) = \rho_1.$$

Suppose that $W_k(t, M_k)$, $k = 1, 2$, is the solution to the Cauchy problem (3.16) with parameter $M = M_k$. We show that for every $\sigma > 0$, there exists some $\delta > 0$, such that if $|M_1 - M_2| < \delta$, then for every $t \in [0, \varepsilon]$, $|W_1(t, M_1) - W_2(t, M_2)| < \sigma$.

Note that for every $t \in [0, \varepsilon]$, $k = 1, 2$,

$$W_k(t, M_k) = \rho_1 + \int_0^t \frac{\rho_2 - \rho_1}{1 + W_k(\tau, M_k)} + \frac{M_k}{\varepsilon} h(\frac{\tau}{\varepsilon})\, d\tau. \tag{3.17}$$

Therefore, for every $t \in [0, \varepsilon]$,

$$|W_1(t, M_1) - W_2(t, M_2)| \tag{3.18}$$
$$= \left| \int_0^t \left(\frac{\rho_2 - \rho_1}{1 + W_1(\tau, M_1)} - \frac{\rho_2 - \rho_1}{1 + W_2(\tau, M_2)} \right) d\tau + \int_0^t \frac{1}{\varepsilon} h(\frac{\tau}{\varepsilon})\, d\tau (M_1 - M_2) \right| \tag{3.19}$$
$$\leq \int_0^t (\rho_2 - \rho_1) \left| W_1(\tau, M_1) - W_2(\tau, M_2) \right| d\tau + |M_1 - M_2| \int_0^{\frac{t}{\varepsilon}} h(s)\, ds \tag{3.20}$$
$$\leq |M_1 - M_2| + \int_0^t (\rho_2 - \rho_1) \left| W_1(\tau, M_1) - W_2(\tau, M_2) \right| d\tau. \tag{3.21}$$

By lemma (3.3.1), we have,

$$|W_1(t, M_1) - W_2(t, M_2)| \leq |M_1 - M_2|\, e^{(\rho_2 - \rho_1)t} \leq |M_1 - M_2|\, e^{(\rho_2 - \rho_1)\epsilon}.$$

Thus for every $\sigma > 0$, let $\delta = \sigma e^{-(\rho_2 - \rho_1)\varepsilon}$, then $|M_1 - M_2| < \delta$ implies that for every $t \in [0, \epsilon]$, $|W_1(t, M_1) - W_2(t, M_2)| < \delta$. □

Lemma 3.3.6. *Given a function h as in* (3.10)*, for arbitrary but fixed $\varepsilon \in (0, 1]$, and fixed $t \in [0, \varepsilon]$, the density $\rho(t, 0)$ over the time interval $[0, \varepsilon]$ is continuous with respect to the additional mass $M(h, \epsilon)$.*

Proof. The density at the location $x = 0$ at time $t \in [0, \varepsilon]$ is

$$\rho(t,0) = \frac{u(t)}{\alpha(W(t))}$$
$$= \rho_2 + \frac{\frac{M}{\varepsilon}h(\frac{t}{\varepsilon})}{\alpha(W(t))}.$$

By lemma (3.3.5) and the fact that α is bounded away from 0, we have, $\rho(t, 0)$ is continuous with respect to the additional mass M over the time interval $[0, \varepsilon]$. □

Lemma 3.3.7. *Suppose the function* $f\colon [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty); (t, x) \mapsto f(t, x)$ *is integrable with respect to the first variable* t *and continuous with respect to the second variable* x *uniformly in* t *and bounded away from zero, i,e., there exists some* $v_0 > 0$*, such that, for every* $t \in [0, \infty)$ *and* $x \in [0, \infty)$*,* $f(t, x) > v_0$*. Suppose that the function* $g\colon [0, \infty) \mapsto (a, \infty)$ *with* $a > 0$ *is bounded above by* $T > 0$*. If the following function is continuous:*

$$F\colon [0, \infty) \mapsto [0, \infty); x \mapsto \int_a^{g(x)} f(t, x)\, dt, \tag{3.22}$$

then, g *is also continuous with respect to* x*.*

Proof. For arbitrary but fixed $\varepsilon > 0$, let $\sigma = \frac{v_0}{T+1}\varepsilon$. Then the continuity of function F implies that there exists some $\delta_1 > 0$, such that, for every $x_1, x_2 \in [0, \infty)$, if $|x_1 - x_2| < \delta_1$, then

$$|F(x_1) - F(x_2)| = \left| \int_a^{g(x_1)} f(t, x_1)\, dt - \int_a^{g(x_2)} f(t, x_2)\, dt \right|$$
$$= \left| \int_a^{g(x_1)} f(t, x_1)\, dt - \int_a^{g(x_1)} f(t, x_2)\, dt + \int_a^{g(x_1)} f(t, x_2)\, dt - \int_a^{g(x_2)} f(t, x_2)\, dt \right|$$
$$= \left| \int_a^{g(x_1)} f(t, x_1) - f(t, x_2)\, dt + \int_{g(x_2)}^{g(x_1)} f(t, x_2)\, dt \right| < \sigma.$$

Furthermore, for all $t \in [a, T]$, there exist some $\delta_2 > 0$, such that, if $|x_1 - x_2| < \delta_2$, then $|f(t, x_1) - f(t, x_2)| < \sigma$. Let $\delta = \min\{\delta_1, \delta_2\}$, then by the triangle inequality and the fact that the function f is bounded below by $v_0 > 0$, we obtain if $|x_1 - x_2| < \delta$, then

$$\begin{aligned} v_0 \, |g(x_1) - g(x_2)| &\leq \left| \int_{g(x_2)}^{g(x_1)} f(t, x_2) \, dt \right| \leq \left| \int_{a}^{g(x_1)} f(t, x_1) - f(t, x_2) \, dt \right| + \sigma \\ &\leq (T+1)\sigma. \end{aligned}$$

Therefore,

$$|g(x_1) - g(x_2)| \leq \frac{T+1}{v_0} \sigma = \varepsilon$$

□

Lemma 3.3.8. *Given a function h as in (3.10), for arbitrary but fixed $\varepsilon \in (0, 1]$, the time t^* when the demand out-flux jumps depends continuously on the additional mass $M(h, \epsilon)$.*

Proof. From equation (3.8), we have

$$\int_0^{\varepsilon} \alpha(W(t)) \, dt + \int_{\varepsilon}^{t^*} \alpha(W(t)) \, dt = 1.$$

Let $F(M) = \int_0^{\varepsilon} \alpha(W(t)) \, dt$ and $G(M) = \int_{\varepsilon}^{t^*} \alpha(W(t)) \, dt$, then
$G(M) = 1 - F(M)$. Furthermore, from lemma (3.3.5), we have that the function F is continuous with respect to M, and thus the function G is also continuous.

Again by lemma (3.3.5), equation (B.3) and (B.9), we have, the total load W is continuous with respect to the additional mass M, thus the velocity $\alpha(W)$ is also continuous with respect to the additional mass M on the time interval $[\epsilon, t^*]$.

By lemma (3.3.7), the time t^* when the demand out-flux jumps depends continuously on the addition mass M. □

Lemma 3.3.9. *Given a function h as in* (3.10)*, for arbitrary but fixed $\varepsilon \in (0,1]$, the time T^* at which the system reaches its new equilibrium state is continuous with respect to the additional mass M.*

Please find the proof of this lemma in appendix (B).

Lemma 3.3.10. *Given a function h as in* (3.10)*, for arbitrary but fixed $\varepsilon \in (0,1]$, the backlog $\beta(t)$ for $t \geq T^*$ is continuous with respect to the additional mass M.*

Proof. By equation (3.13), lemma (3.3.5), lemma (3.3.8), and lemma (3.3.9), it is easy to see that the backlog $\beta(t)$ for $t \geq T^*$ depends continuously on the additional mass M.

□

Theorem 3.3.11. *Given a function h as in* (3.10)*, for arbitrary but fixed $\varepsilon \in (0,1]$, there exists a $M^*(h,\varepsilon) > 0$, such that if $M = M^*(h,\varepsilon)$, then the control in-flux u_ϵ as defined in* (3.11) *leads to zero backlog β in a finite time.*

Proof. If $M = 0$, then the modified control input u_ε (3.11) is the same as the natural control input (3.7). From section (3.2), we can see that the backlog is positive eventually. Hence by lemma (3.3.3), (3.3.10) and the intermediate value theorem, we have, given $0 < \varepsilon \leq 1$, there exists a $M^*(h,\varepsilon) > 0$, such that if $M = M^*(h,\varepsilon)$, then the backlog β is zero after certain time $t = T^*(\varepsilon)$. □

For $\varepsilon \in (0,1]$ sufficiently small and arbitrary but fixed, the objective functional $J(u_\varepsilon)$ for u_ε is defined in equation (3.11). For convenience, we use M^* to represent $M^*(h,\varepsilon)$.

We calculate the total load at time $t = \varepsilon$, $W(\varepsilon)$, and the distance traveled during the time interval $[0,\varepsilon]$, $\int_0^\varepsilon \alpha(W(t))\,dt$. For $\varepsilon \le t \le t^*$, by conservation of mass,

$$\begin{cases} W'(t) = \frac{\rho_2-\rho_1}{1+W(t)}, \\ W(\varepsilon) = \rho_1 + M^* + (\rho_2-\rho_1)\int_0^\varepsilon \alpha(W(t))dt. \end{cases} \tag{3.23}$$

Solving the Cauchy problem (3.23) yields

$$W(t) = \sqrt{2(\rho_2-\rho_1)(t-\varepsilon) + (1+W(\varepsilon))^2} - 1, \quad t \in [\varepsilon, t^*].$$

Thus the velocity is given by,

$$\alpha(W(t)) = \frac{1}{\sqrt{2(\rho_2-\rho_1)(t-\varepsilon) + (1+W(\varepsilon))^2}}, \quad t \in [\varepsilon, t^*].$$

By the definition of t^*, we have,

$$\begin{aligned} 1 &= \int_0^{t^*} \alpha(W(t))dt = \int_0^\varepsilon \alpha(W(t))dt + \int_\varepsilon^{t^*} \alpha(W(t))dt \\ &= \int_0^\varepsilon \alpha(W(t))dt + \int_\varepsilon^{t^*} \frac{1}{\sqrt{2(\rho_2-\rho_1)(t-\varepsilon)+(1+W(\varepsilon))^2}}dt \\ &= \int_0^\varepsilon \alpha(W(t))dt + \frac{\sqrt{2(\rho_2-\rho_1)(t^*-\varepsilon)+(1+W(\varepsilon))^2}}{\rho_2-\rho_1} - \frac{1+W(\varepsilon)}{\rho_2-\rho_1}. \end{aligned}$$

Now consider the initial condition in equation (3.23), from the last equality of the above relation we obtain,

$$\int_0^\varepsilon \alpha(W(t))dt + \frac{\sqrt{2(\rho_2-\rho_1)(t^*-\varepsilon)+(1+W(\varepsilon))^2}}{\rho_2-\rho_1} - \frac{1+\rho_1+M^*}{\rho_2-\rho_1} - \int_0^\varepsilon \alpha(W(t))dt = 1$$

which implies that

$$W(\varepsilon) = \sqrt{(\rho_2+M^*+1)^2 - 2(\rho_2-\rho_1)(t^*-\varepsilon)} - 1.$$

Again considering the initial condition in equation (3.23), we get,

$$\int_0^\varepsilon \alpha(W(t))dt = \frac{\sqrt{(\rho_2+M^*+1)^2 - 2(\rho_2-\rho_1)(t^*-\varepsilon)} - (\rho_1+M^*+1)}{\rho_2-\rho_1}. \tag{3.24}$$

Lemma 3.3.12. *For $\varepsilon \in (0,1]$ sufficiently small and arbitrary but fixed, we have*

$$\frac{\rho_1+\rho_2+2M^*+2}{2} - \frac{M^*}{1+\rho_1}\varepsilon + o(\varepsilon) \le t^* \le \frac{\rho_1+\rho_2+2M^*+2}{2} + o(\varepsilon), \tag{3.25}$$

and

$$\begin{aligned} M^* \ge& \frac{\rho_1(\rho_2-\rho_1)}{2} - \left(\frac{\rho_1^2(\rho_2-\rho_1)}{2(1+\rho_1)} + \rho_2\right)\varepsilon + o(\varepsilon) \\ M^* \le& \frac{\rho_1(\rho_2-\rho_1)}{2} + \left(\frac{\rho_1\rho_2(\rho_2-\rho_1)}{2(1+\rho_2)} + \rho_2\right)\varepsilon + o(\varepsilon). \end{aligned} \tag{3.26}$$

Please find the proof of this lemma in appendix (B).

Now we calculate the control objective functional $J(u_\varepsilon)$ when $\varepsilon > 0$ but sufficiently small. From the definition of J,

$$\begin{aligned} J(u_\varepsilon) &= \int_0^{T^*} (Y_d(t) - Y(t))\,dt = \int_0^{T^*} \left(\int_0^t y_d(s)ds - \int_0^t y(s)ds\right) dt \\ &= \int_0^{t^*} \left(y_1 t - \int_0^t \rho_1\alpha(W(s))ds\right) dt \\ &+ \int_{t^*}^{T^*} \left(y_1 t^* + y_2(t-t^*) - \rho_1 - \int_{t^*}^t \rho(s,1)\alpha(W(s))ds\right) dt. \end{aligned} \tag{3.27}$$

By lemma 3.3.1,

$$T^* - t^* \le \left(\frac{1+\rho_2+M^*}{1+\rho_1}\right)\varepsilon,$$

That is $T^* - t^*$ is at most of order $o(\varepsilon)$. Furthermore, for any $t \in [t^*, T^*]$, $y_1 t^* + y_2(t - t^*) - \rho_1 - \int_{t^*}^t \rho(s,1)\alpha(W(s))ds$ is bounded (since t^*, M^* and α is bounded above and below), which implies that

$$\int_{t^*}^{T^*} \left(y_1 t^* + y_2(t-t^*) - \rho_1 - \int_{t^*}^t \rho(s,1)\alpha(W(s))ds\right) dt$$

is of order $o(\varepsilon)$.

For the first integral in equation (3.27),

$$\begin{aligned}
&\int_0^{t^*}\left(y_1 t - \int_0^t \rho_1\alpha(W(s))ds\right)dt \qquad (3.28)\\
&= \int_0^{t^*} y_1 t dt - \int_0^{t^*}\int_0^t \rho_1\alpha(W(s))ds\,dt\\
&= \frac{1}{2}y_1(t^*)^2 - \int_0^{t^*} \rho_1\alpha(W(s))(t^*-s)ds\\
&= \frac{1}{2}y_1(t^*)^2 - \int_0^{\varepsilon} \rho_1\alpha(W(s))(t^*-s)ds - \int_{\varepsilon}^{t^*} \rho_1\alpha(W(s))(t^*-s)ds\\
&= \frac{1}{2}y_1(t^*)^2 - \int_{\varepsilon}^{t^*} \rho_1\alpha(W(s))(t^*-s)ds + o(\varepsilon),
\end{aligned}$$

The last equality in the above is because that the integral $\int_0^{\varepsilon} \rho_1\alpha(W(s))(t^*-s)ds$ is of order $o(\varepsilon)$.

Also recall that for $t \in (\varepsilon, t^*)$,

$$\alpha(W(t)) = \frac{1}{\sqrt{(M^*+\rho_2+1)^2 + 2(\rho_2-\rho_1)(t-\varepsilon)}},$$

therefore,

$$J(u_\varepsilon) = \frac{1}{2}y_1(t^*)^2 - \int_{\varepsilon}^{t^*} \frac{\rho_1(t^*-s)}{\sqrt{(M^*+\rho_2+1)^2 + 2(\rho_2-\rho_1)(s-\varepsilon)}}ds + o(\varepsilon). \qquad (3.29)$$

where M^* and t^* satisfy the inequalities (B.24) and (3.25).

3.3.2 Non-optimality of a Family of L^1-controls

For any $s \in \mathbb{R}$, denote by δ_s the Dirac delta centered at s. Formally consider the impulsive control

$$u_0(t) = \begin{cases} M^0\delta_0(t), & t = 0,\\ \rho_2\alpha(W(t)), & t > 0. \end{cases} \qquad (3.30)$$

where $M^0 > 0$ is such that the backlog β reaches 0 at time $t = T^* = t_0^*$ (from lemma (3.3.1), we obtain that $T^* = t_0^*$ by taking ε to be zero). Note that $u_0 \notin L^1([0,T])$.

We first find M^0 and t_0^*.

For $t \in (0, t_0^*)$, by the conservation of mass,

$$\begin{cases} W'(t) = \frac{\rho_2 - \rho_1}{1+W(t)}, \\ W(0) = M^0 + \rho_1. \end{cases} \tag{3.31}$$

Solving the Cauchy problem (3.31), for every $t \in (0, t_0^*)$,

$$W(t) = -1 + \sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)t}.$$

By the definition of t_0^*,

$$\begin{aligned} 1 &= \int_0^{t_0^*} \alpha(W(t))dt = \int_0^{t^*} \frac{1}{1+W(t)}dt \\ &= \int_0^{t_0^*} \frac{1}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)t}}dt \\ &= \frac{1}{\rho_2 - \rho_1}\left(\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)t_0^*} - (m^0 + \rho_1 + 1)\right), \end{aligned}$$

which implies

$$t_0^* = \frac{1}{2}(\rho_2 - \rho_1) + (M^0 + \rho_1 + 1). \tag{3.32}$$

Note that the backlog $\beta(t_0^*) = 0$ implies

$$y_1 t^* - \rho_1 = M^0. \tag{3.33}$$

Combine (3.32) and (3.33),

$$\begin{aligned} t_0^* &= \frac{1}{2}(1 + \rho_1)(\rho_2 - \rho_1 + 2), \\ M^0 &= \frac{\rho_1(\rho_2 - \rho_1)}{2}. \end{aligned}$$

Lemma 3.3.13. *For every function h as in (3.10), the modified control in-flux u_ϵ (3.11) converge to the impulsive control u_0 (3.30) in the sense of distribution as ε approaches to 0 from the right.*

Proof. For every function $\phi \in C_c^\infty((0,+\infty))$ that is smooth on $(0,+\infty)$ with compact support, we claim that

$$\lim_{\varepsilon\to 0^+}\int_0^\infty u_\varepsilon(t)\phi(t)\,dt = \int_0^\infty u_0(t)\phi(t)\,dt. \tag{3.34}$$

For the left hand side of equation (3.34),

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_0^\infty u_\varepsilon(t)\phi(t)\,dt\\
&=\lim_{\varepsilon\to 0^+}\left(\int_0^\varepsilon\left(\rho_2\alpha(W(t))+\frac{M}{\varepsilon}h\left(\frac{t}{\varepsilon}\right)\right)\phi(t)\,dt+\int_\varepsilon^\infty \rho_2\alpha(W(t))\phi(t)\,dt\right)\\
&=\int_0^\infty \rho_2\alpha(W(t))\phi(t)\,dt+\lim_{\varepsilon\to 0^+}\int_0^\varepsilon \frac{M}{\varepsilon}h\left(\frac{t}{\varepsilon}\right)\phi(t)\,dt\\
&=\int_0^\infty \rho_2\alpha(W(t))\phi(t)\,dt+\lim_{\varepsilon\to 0^+} M\lim_{\varepsilon\to 0^+}\int_0^\varepsilon \frac{1}{\varepsilon}h\left(\frac{t}{\varepsilon}\right)\phi(t)\,dt\\
&=\int_0^\infty \rho_2\alpha(W(t))\phi(t)\,dt+\lim_{\varepsilon\to 0^+} M\lim_{\varepsilon\to 0^+}\int_0^1 h\left(s\right)\phi(\varepsilon s)\,ds\\
&=\int_0^\infty \rho_2\alpha(W(t))\phi(t)\,dt+\lim_{\varepsilon\to 0^+} M\int_0^1 h\left(s\right)\phi(0)\,ds\\
&=\int_0^\infty \rho_2\alpha(W(t))\phi(t)\,dt+\lim_{\varepsilon\to 0^+} M\phi(0).
\end{aligned}$$

Here, $\lim_{\varepsilon\to 0^+} M$ is the amount of mass instantly added to the system at time $t = 0$ such that the backlog β reaches zero in finite time. Thus,

$$\lim_{\varepsilon\to 0^+} M = M^0.$$

Additionally, for the right hand side of equation (3.34),

$$\int_0^\infty u_0(t)\phi(t)\,dt = \int_0^\infty \rho_2\alpha(W(t))\phi(t)\,dt + M^0\phi(0).$$

Therefore, we have $u_\varepsilon \to u_0$ as $\varepsilon \to 0^+$ in the sense of distribution (see Definition (A.1.15)). □

Next, we study the control objective function $J(u_0)$. Since the mass M^0 leaves the system instantly at time $t = t_0^*$, we have

$$J(u_0) = \int_{[0,t_0^*)} Y_d(t) - Y(t)\,dt - M^0.$$

Note that for $t \in [0, t_0^*)$, the accumulated demand out-flux Y_d is

$$Y_d(t) = y_1 t,$$

and the accumulated actual out-flux Y is

$$\begin{aligned} Y(t) &= \int_0^t \rho_1 \alpha(W(s)) ds \\ &= \int_0^t \frac{\rho_1}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} ds. \end{aligned}$$

Thus, the control objective functional $J(u_0)$ is,

$$\begin{aligned} J(u_0) &= \int_{[0,t_0^*)} y_1 t dt - \int_{[0,t_0^*)} \int_0^t \frac{\rho_1}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} ds\, dt - M^0 \\ &= \int_{[0,t_0^*)} y_1 t dt - \int_0^{t_0^*} \int_s^{t_0^*} \frac{\rho_1}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} dt\, ds - M^0 \\ &= \int_{[0,t_0^*)} y_1 t dt - \int_0^{t_0^*} \frac{\rho_1 (t_0^* - s)}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} ds - M^0 \end{aligned}$$

with

$$\begin{aligned} t_0^* &= \frac{1}{2}(1 + \rho_1)(\rho_2 - \rho_1 + 2), \\ M^0 &= \frac{\rho_1(\rho_2 - \rho_1)}{2}. \end{aligned}$$

Next, we compare the control objective functional $J(u_\varepsilon)$ with $\varepsilon > 0$ being sufficiently small and the control objective $J(u_0)$. Let $M^1 = \frac{\rho_1^2(\rho_2 - \rho_1)}{2(1+\rho_1)} + \rho_2$ and $M^2 = \frac{\rho_1 \rho_2 (\rho_2 - \rho_1)}{2(1+\rho_2)} + \rho_2$. Then equation (B.24) implies

$$M^0 - M^1 \varepsilon + o(\varepsilon) \leq M^* \leq M^0 + M^2 \varepsilon + o(\varepsilon). \tag{3.35}$$

Furthermore, the range of t^* is

$$t_0^* - \left(M_1 + \frac{M^0}{1 + \rho_1} \right) \varepsilon + o(\varepsilon) \leq t^* \leq t_0^* + M_2 \varepsilon + o(\varepsilon). \tag{3.36}$$

Let $J_1 = \frac{1}{2} y_1 (t^*)^2$ and $J_2 = \int_\varepsilon^{t^*} \frac{\rho_1(t^*-s)}{\sqrt{(M^*+\rho_2+1)^2+2(\rho_2-\rho_1)(s-\varepsilon)}} ds$, then

$$J(u_\varepsilon) = J_1 - J_2 + o(\varepsilon).$$

By equation (3.36),

$$\begin{aligned} J_1 &= \frac{1}{2} y_1 (t^*)^2 \\ &\geq \frac{1}{2} y_1 \left(t_0^* - \left(M_1 + \frac{M^0}{1+\rho_1} \right) \varepsilon \right)^2 + o(\varepsilon) \\ &= \frac{1}{2} y_1 \left((t_0^*)^2 - 2 t_0^* \left(M_1 + \frac{M^0}{1+\rho_1} \right) \varepsilon \right) + o(\varepsilon) \\ &= \frac{1}{2} y_1 (t_0^*)^2 - y_1 t_0^* \left(M_1 + \frac{M^0}{1+\rho_1} \right) \varepsilon + o(\varepsilon). \end{aligned}$$

Note that there exist $\varepsilon_0 > 0$, such that, if $\varepsilon < \varepsilon_0$, then $y_1 t_0^* \left(M_1 + \frac{M^0}{1+\rho_1} \right) \varepsilon < \frac{M^0}{8}$. Thus, we have a lower bound for J_1, that is,

$$J_1 > \frac{1}{2} y_1 (t_0^*)^2 - \frac{M^0}{8}.$$

Therefore,

$$J(u_\varepsilon) > \frac{1}{2} y_1 (t_0^*)^2 - \frac{M^0}{8} - J_2 + o(\varepsilon).$$

Take $\varepsilon_1 > 0$, such that, if $\varepsilon < \varepsilon_1$, the term $o(\varepsilon)$ in the above equation is bounded above by $\frac{M^0}{2}$.

For J_2, we have,

$$\begin{aligned} J_2 &= \int_\varepsilon^{t^*} \frac{\rho_1(t^*-s)}{\sqrt{(M^*+\rho_2+1)^2+2(\rho_2-\rho_1)(s-\varepsilon)}} ds \\ &\leq \int_0^{t_0^*} \frac{\rho_1 (t_0^* + M_2\varepsilon + o(\varepsilon) - s)}{\sqrt{(M^0 - M_1\varepsilon + o(\varepsilon) + \rho_2 + 1)^2 + 2(\rho_2-\rho_1)(s-\varepsilon)}} ds \\ &\quad + \int_{t_0^*}^{t_0^*+M_2\varepsilon+o(\epsilon)} \frac{\rho_1 (t_0^* + M_2\varepsilon + o(\varepsilon) - s)}{\sqrt{(M^0 - M_1\varepsilon + o(\varepsilon) + \rho_2 + 1)^2 + 2(\rho_2-\rho_1)(s-\varepsilon)}} ds. \end{aligned}$$

Note that if $0 < \varepsilon < \frac{\rho_2-\rho_1}{M_1}$, then $-M_1\varepsilon + \rho_2 > \rho_1$.

Let

$$J_{2,1} = \int_0^{t_0^*} \frac{\rho_1 (t_0^* - s)}{\sqrt{(M^0 + \rho_1 + 1 + o(\varepsilon))^2 + 2(\rho_2 - \rho_1)(s - \varepsilon)}} ds,$$

$$J_{2,2} = \int_0^{t_0^*} \frac{\rho_1 M_2 \varepsilon + o(\varepsilon)}{\sqrt{(M^0 + \rho_1 + 1 + o(\varepsilon))^2 + 2(\rho_2 - \rho_1)(s - \varepsilon)}} ds,$$

and

$$J_{2,3} = \int_{t_0^*}^{t_0^* + M_2\varepsilon + o(\epsilon)} \frac{\rho_1 (t_0^* + M_2\varepsilon + o(\varepsilon) - s)}{\sqrt{(M^0 + \rho_1 + 1 + o(\varepsilon))^2 + 2(\rho_2 - \rho_1)(s - \varepsilon)}} ds,$$

then,

$$J_2 \leq J_{2,1} + J_{2,2} + J_{2,3}.$$

For $J_{2,1}$,

$$\lim_{\varepsilon \to 0} J_{2,1} = \int_0^{t_0^*} \frac{\rho_1 (t_0^* - s)}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} ds,$$

thus, there exists $\varepsilon_2 > 0$, such that, if $\varepsilon < \varepsilon_2$, then

$$J_{2,1} < \int_0^{t_0^*} \frac{\rho_1 (t_0^* - s)}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} ds + \frac{M^0}{8}.$$

For $J_{2,2}$ and $J_{2,3}$,

$$\lim_{\varepsilon \to 0} J_{2,2} = \lim_{\varepsilon \to 0} J_{2,3} = 0,$$

thus, there exists $\varepsilon_3 > 0$, such that if $\varepsilon < \varepsilon_3$, then $J_{2,2} < \frac{M^0}{4}$ and $J_{2,3} < \frac{M^0}{8}$. Hence, we have an upper bound for J_2, that is,

$$J_2 \leq \int_0^{t_0^*} \frac{\rho_1 (t_0^* - s)}{\sqrt{(M^0 + \rho_1 + 1)^2 + 2(\rho_2 - \rho_1)s}} ds + \frac{3M^0}{8}.$$

Let $\delta = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \frac{\rho_2 - \rho_1}{M}\}$, then if $\varepsilon < \delta$, then $J(u_\varepsilon) > J(u_0)$.

Theorem 3.3.14. *Given a function* h *as in* (3.10)*, there exists* $\varepsilon_h > 0$*, such that for every* $\varepsilon_1 \in (0, \varepsilon_h)$*, there is* $\varepsilon_2 < \varepsilon_1$*, such that the functional* J *satisfies* $J(u_0) < J(u_{\varepsilon_2}) < J(u_{\varepsilon_1})$.

Proof. From the discussion before, we have, given a function h as in (3.10), there exist $\varepsilon_h > 0$, such that the function J satisfies

$$J(u_\varepsilon) > J(u_0), \text{ for every } \varepsilon \in (0, \varepsilon_h). \tag{3.37}$$

By the squeeze theorem and inequalities (3.35) and (3.36), we have,

$$\lim_{\varepsilon \to 0} J(u_\varepsilon) = J(u_0). \tag{3.38}$$

Now suppose there exists some $\varepsilon_1 \in (0, \varepsilon_h)$, such that for very $\varepsilon_2 \in (0, \varepsilon_1)$, the function J satisfies $J(u_{\varepsilon_2}) \geq J(u_{\varepsilon_1})$. Then

$$\lim_{\varepsilon \to 0} J(u_\varepsilon) \geq J(u_{\varepsilon_1}),$$

which is a contradiction to equations (3.37) and (3.38). □

Conclusion: Theorem (3.3.14) formally analyzed the directional derivatives of the objective functional J at u_0 (a Borel measure on $[0, T]$) in the directions of absolutely continuous Borel measures that correspond to L^1-functions. Furthermore, theorem (3.3.14) strongly suggests that there is no optimal control in the class of L^1-functions defined as in equation (3.11) when ε is sufficiently small. Intuitively, there is no $J(u_\varepsilon)$ that is smaller than $J(u_0)$ for ε large. But this is not yet proved by theorem (3.3.14).

Heuristically the only reasonable candidate of L^1-controls u for which $J(u)$ is even close to $J(u_0)$ are of the above form u_ε with $\{t > 0 \colon u_\varepsilon(t) \neq \rho_2 \alpha(W(t))\}$ contained in an interval as short as possible. The impulsive control which can be understood as a Dirac measure (not in L^1) input would lead to a lower cost than any L^1-control of the form u_ε. This suggests us to consider controls and initial values in the setting of finite Borel measures, $\mathcal{M}^+$.

3.4 Transfer from a Larger to a Smaller Equilibrium

In this section, we work with the velocity $v = \alpha(W)$ that satisfies the conditions after equation (3.1) in section 3.1. Consider the case when the initial density of the factory is ρ_1, and the demand out-flux y_d is $y_d(t) = y_2 = \frac{\rho_2}{1+\rho_2}$, with $\rho_1 > \rho_2 \geq 0$. Then for $t \geq 0$, the accumulated demand out-flux Y_d is

$$Y_d(t) = y_2 t,$$

the accumulated actual out-flux Y is

$$Y(t) = \int_0^t \rho(s,1)\alpha(W(s))\,ds,$$

and the backlog β is

$$\beta(t) = Y_d(t) - Y(t).$$

We aim to minimize the control objective functional $J\colon = \int_0^\infty |\beta(t)|\,dt$ by choosing an optimal control u^*.

Note that the special demand out-flux forces us to choose the control in-flux as $u(t) = \rho_2\alpha(W(t))$ for some time $t > t_0 \geq 0$. Denote by t_1 the time at which all of the initial mass has exited the system and the characteristic curve by $\xi\colon [0,\infty] \mapsto [0,\infty]$ to track the position of the items that initially were located at $x = 0$ at $t = 0$. Thus the following relations hold:

$$\int_0^{t_1} \alpha(W(t))\,dt = 1, \quad \xi(t_1) = 1.$$

Denote by T^* the time at which the system reaches its new equilibrium and the backlog reaches zero, that is,

$$\int_{t_0}^{T^*} \alpha(W(t))\,dt = 1, \quad \beta(T^*) = 0.$$

3.4.1 A Conjecture of the Optimal Control

If the initial density of the factory ρ_1 is larger than the density in the new equilibrium ρ_2, then we can not use a "negative" Dirac delta. A natural candidate for the optimal solution is to apply zero in-flux for some time. In this section, we consider the case when $\rho_1 - \rho_2$ is not too large and thus the operator in the factory needs to take action at some time before all the initial mass exits from the factory, i.e., $t_0 < t_1$. We illustrate the following conjecture numerically.

Conjecture: The following control u^* leads to the minimal cost J^*:

$$u^*(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq t_0, \\ \rho_2 \alpha(W(t)) & \text{if} \quad t_0 < t \leq T^*. \end{cases} \tag{3.39}$$

We consider the case when $\rho_1 = 10$ and $\rho_2 = 8.7$. Correspondingly, $t_0 = 5.5983$, $t_1 = 6.1750$ and $T^* = 11.1494$.

Under the control of u^*, the total load W (figure (3.4)) is continuous on $[0, T^*]$, decreases with an increasing rate over the time interval $[0, t_1]$ and increases with a decreasing rate over the time interval $[t_1, T^*]$. Furthermore, the deceasing rate of the total load decreases at time t_0 since the operator started to put $u(t) = \rho_2 \alpha(W(t))$ at time t_0 into the system.

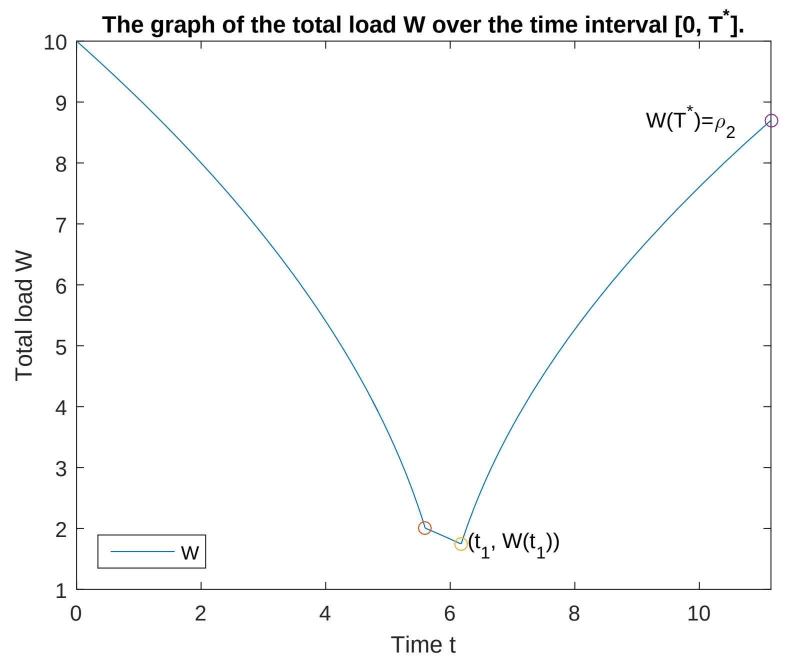


Figure 3.4: The Graph of the Total Load W Under the Control u^*

The characteristic curve ξ (figure (3.5)) is continuous and increasing over the time interval $[0, T^*]$, and reaches 1 at time t_1.

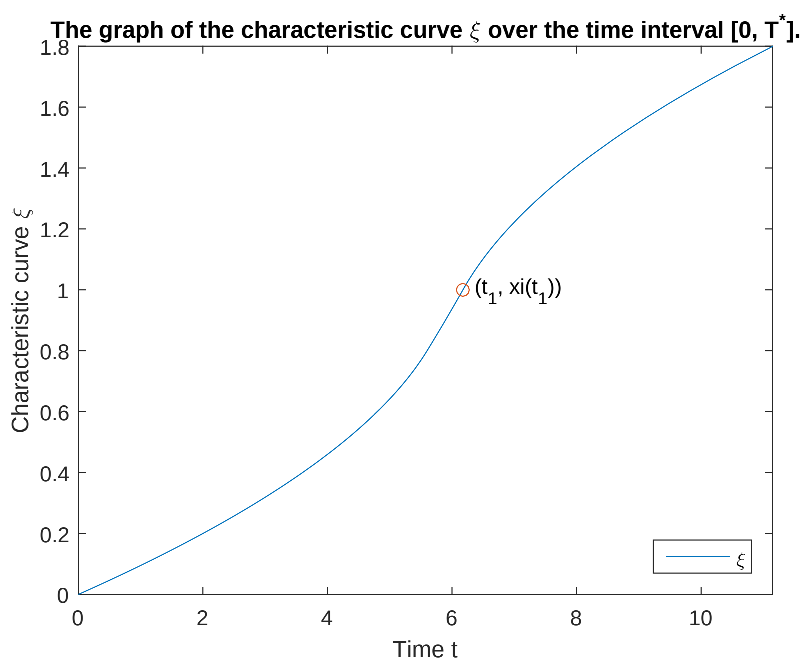


Figure 3.5: The Graph of the Characteristic Curve ξ Under the Control u^*

Over the time interval $[0, T^*]$, the system produces more than the demand until time T^*. That is, for every $t \in [0, T^*)$, $Y(t) > Y_d(t)$ and $Y(T^*) = Y_d(T^*)$ (figure (3.6)). Thus the backlog β (figure((3.7))) is negative for every $t \in [0, T^*)$ and $\beta(T^*) = 0$.

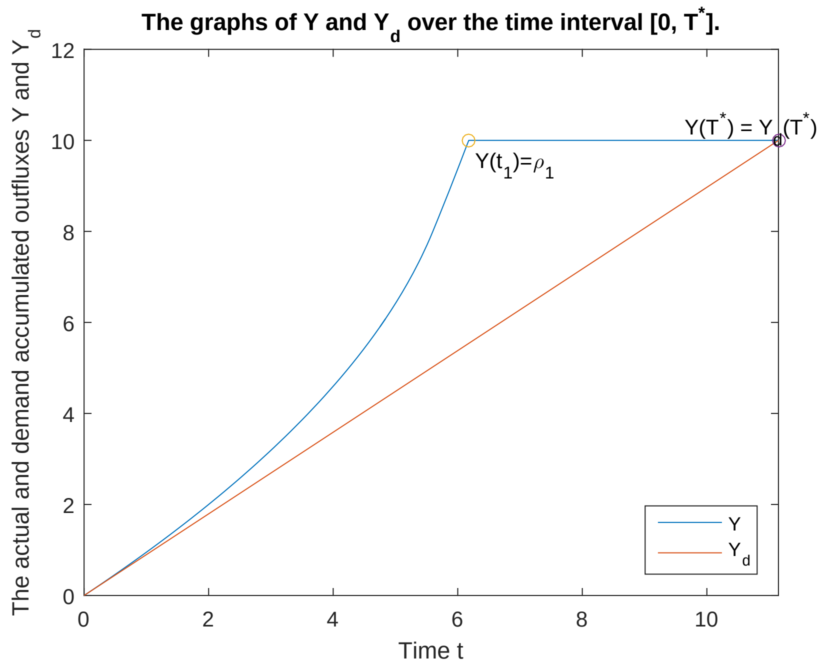


Figure 3.6: The Graph of the Accumulated Actual and Accumulated Demand O ut-flux Y and Y_d Under the Control u^*

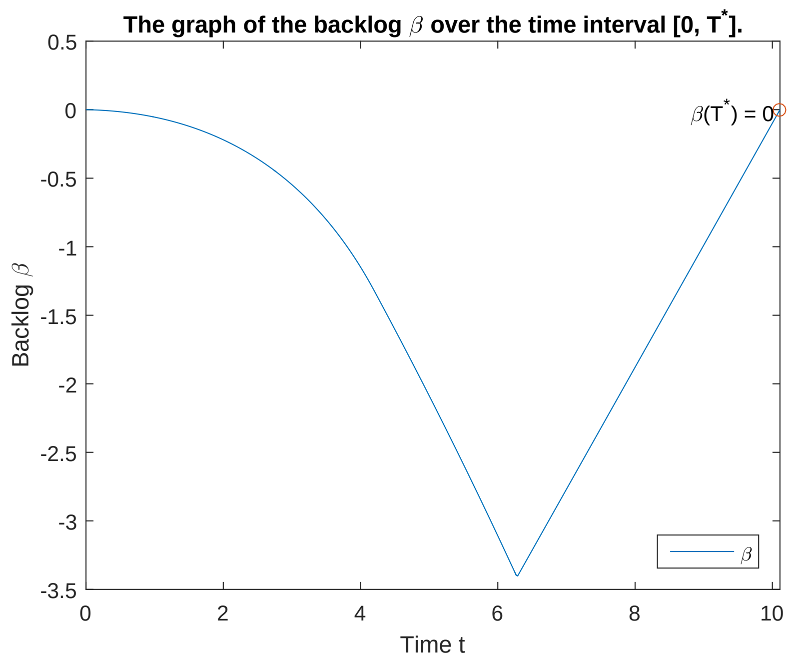


Figure 3.7: The Graph of the Backlog β Under the Control u^*

Furthermore, the value of the cost functional J at the control u^* is

$$J(u^*) = 17.2968.$$

3.4.2 A Control with Positive Impulsive Mass

Now to illustrate that the conjectured control, taking zero in-flux over the time interval $[0, t_0]$, is optimal in the sense of leading to minimal cost, we consider the control u with a positive impulsive mass M at some time t^0 with $t^0 \in [0, t_0]$. Specifically, the needle variation like control u is defined as

$$u(t) = M\delta_{t^0}(t) + \rho_2 \alpha(W(t))\chi_{(t_0, T^*]}, \tag{3.40}$$

where χ is the indicator function. Additionally, denote by t^1 the time when the impulsive mass M exits from the system. We consider the case when $\rho_1 = 10$, $\rho_2 = 8.7$, $M = 0.5$, $t^0 = 1.5$. Correspondingly, $t_0 = 6.1099$, $t_1 = 6.5161$, $t^1 = 7.0098$, and $T^* = 11.7069$.

Under the control of u, the total load W (figure (3.4.2)) is continuous over the time interval $[0, T^*]$ except at time t^0 and t^1 when the impulsive mass M entered into the system or exited from the system.

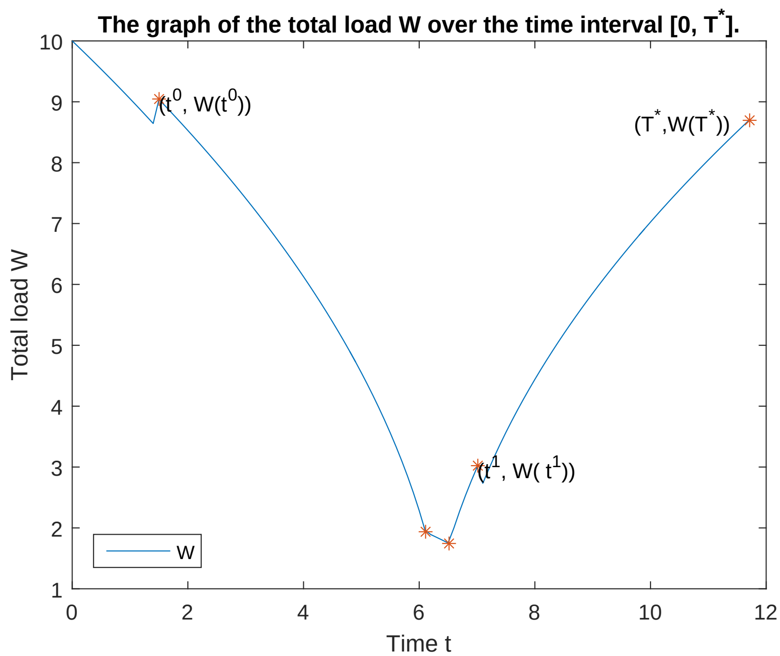


Figure 3.8: The Graph of the Total Load W Under the Control u

The characteristic curve ξ (figure (3.9)) is continuous and increasing on $[0, T^*]$ and reaches 1 at time t_1.

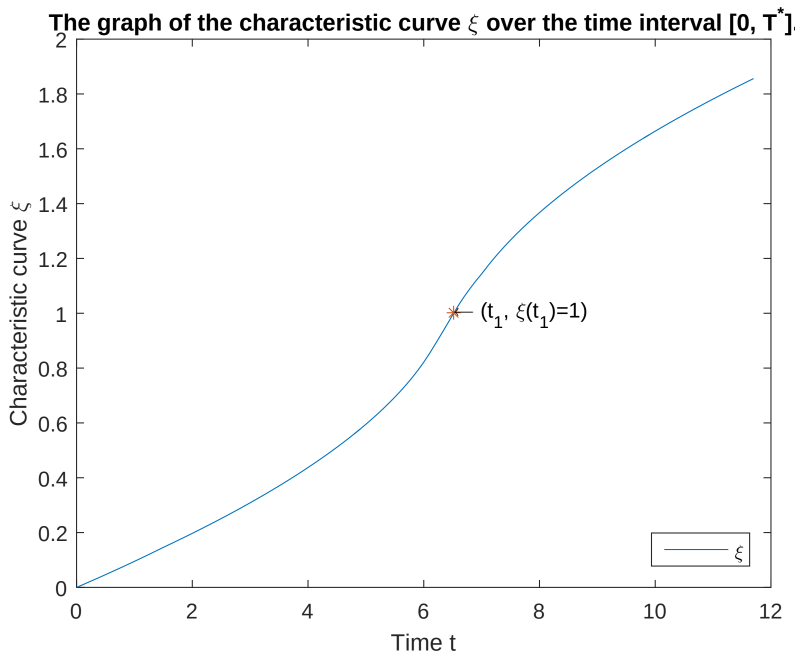


Figure 3.9: The Graph of the Characteristic Curve ξ Under the Control u

Over the time interval $[0, T^*]$, the accumulated actual out-flux Y is greater than the accumulated demand out-flux Y_d except at $t = T^*$ when $Y(T^*) = Y_d(T^*)$. But the accumulated actual out-flux Y has a jump at time t^1 when the impulsive mass exited from the system, thus, so does the backlog β.

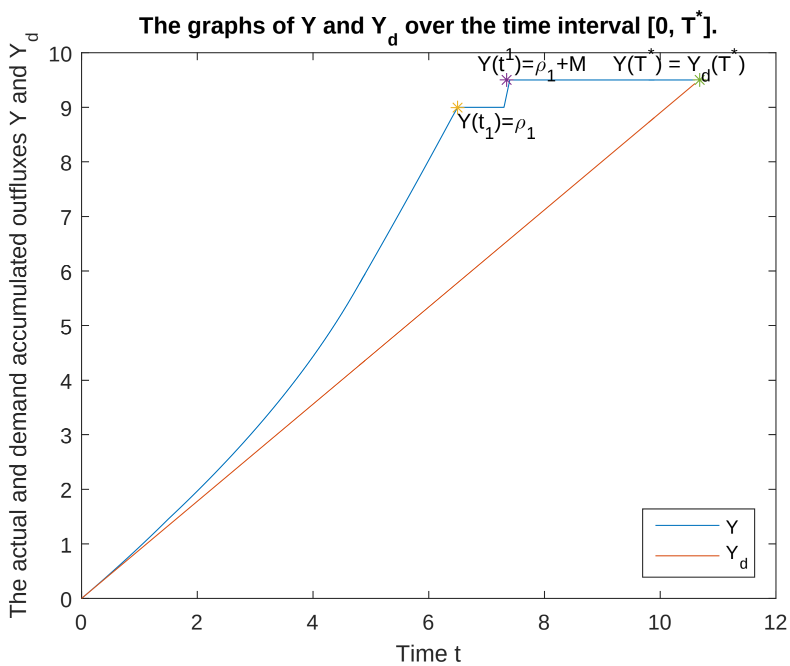


Figure 3.10: The Graph of the Accumulated Actual and Accumulated Demand out-fluxes Y, Y_d Under the Control u.

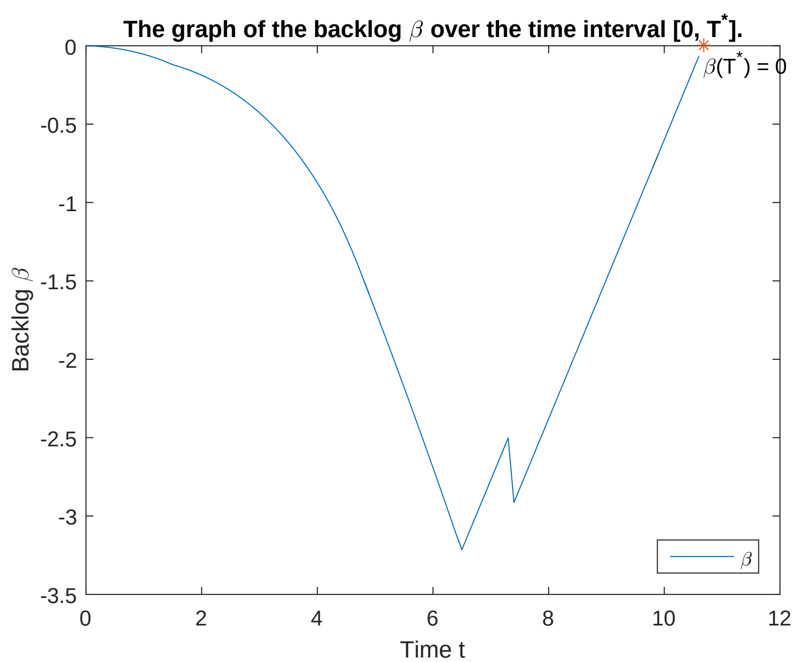


Figure 3.11: The Graph of the Backlog β Under the Control u.

The value of the cost functional J at the control u is

$$J(u) = 17.7575.$$

Furthermore, we sample more values for the densities, ρ_1, ρ_2, at the equilibria and compare the values of the cost function J with the above controls u^* and u.

Table 3.1: The Comparison of the Values $J(u^*)$ and $J(u)$

Densities \ Costs	$J(u^*)$	$J(u)$
$\rho_1 = 10,\ \rho_2 = 9.3$	11.5600	11.7195
$\rho_1 = 10,\ \rho_2 = 9.0$	16.3400	17.7156
$\rho_1 = 10,\ \rho_2 = 8.7$	17.2968	17.7575
$\rho_1 = 10,\ \rho_2 = 8.4$	14.3360	16.0046
$\rho_1 = 10,\ \rho_2 = 8.0$	12.9728	13.2942

Hence the numerical results suggest that the control u^* may minimize the cost functional J.

3.5 Discussion and Conclusion

For the case of transferring from a smaller to a larger equilibrium with nonzero backlog, we explicitly calculated the constant backlog β under the control u (figure (3.2)).

we analytically proved that no optimal control exists in a family of L^1-controls for the case of transferring from a smaller to a larger equilibrium with zero backlog. The minimizing sequences converges in distribution to an impulsive control (not in L^1). Thus it is natural to recast the problem in the setting of controls and states being Borel measures.

For the case of transferring from a larger to a smaller equilibrium with zero backlog, since we could not symmetrically take a negative impulsive control, a reasonable prediction would be that a control taking zero in-flux for some time minimizes the cost functional J. Numerically, we illustrated this conjecture by comparing the costs under the conjectured control u^* (3.39) and a control with positive impulsive mass u (3.40).

Chapter 4

WEAK MEASURE-VALUED SOLUTION

4.1 Introduction: Measure-setting

The conjectured nonexistence of optimal solutions in the L^1-setting and suggestive optimal impulsive controls might lead one to consider multiple, or even countably many of such point masses. A model for this, in Lagrangian form, might be a combination of hyperbolic conservation laws in L^1-setting as in Coron *et al.* (2010) and a sequence of ODEs coupled by total mass and velocity.

Throughout we will assume that the velocity $v = \alpha(\cdot)$ is positive (hence bounded away from zero on compact sets), decreasing, attains its maximum at $\alpha(0) = 1$, and is Lipschitz continuous with Lipschitz constant L: For all $W_1, W_2 > 0$, $\|\alpha(W_1) - \alpha(W_2)\| \leq L\|W_1 - W_2\|$. Whereas the first article Armbruster *et al.* (2006) used $\alpha(W) = \max\{0, 1 - \frac{W}{W_0}\}$, a much more common choice in subsequent work was $\alpha(W) = \frac{1}{1+W}$. The results presented here only use the above stated properties of α.

Staying close to the features of the original manufacturing system modeled by (1.1), provides both advantages that suggest carefully tailored approaches, but also leads to technical complications that prevent application of standard tools. In particular, we chose the underlying spaces for our measures to be the noncompact intervals $[0, 1)$ and $(0, T]$. This is essential for obtaining the desired contractions needed for the fixed point argument and for avoiding double counting. Another simple and strong argument for this choice is the common choice of the CONWIP dispatch policy (constant work in progress) for factories that are performing well: Use the most simple output feedback law imaginable, $u = y$, in-flux equals out-flux. With impulsive out-fluxes, if using $[0,1]$ instead, this would lead to awkward total loads which are constant except for possibly countably many jump discontinuities.

Suppose the initial condition ρ_0 consists of a function $\rho_{0L^1} \in L^1([0,1))$ and a sequence of point masses m_i located at $x_i \in [0,1)$, $i = 1,2,\cdots$ with $\sum_{i=1}^{\infty} m_i < \infty$. Similarly, the in-flux consists of a function $u_{L^1} \in L^1((0,T])$ and a sequence of point masses M_j entering the system at time $t_j \in (0,T]$, $j = 1,2,\cdots$ with $\sum_{j=1}^{\infty} M_j < \infty$. Furthermore, let $\xi_i \colon [0,T] \mapsto [0,+\infty)$ and $\eta_j \colon [0,T] \mapsto [0,+\infty)$ trace the location of the masses m_i and M_j, respectively. This suggests the following coupled model

$$0 = \partial_t \rho_{L^1}(t,x) + \partial_x(\alpha(W(t))\rho_{L^1}(t,x)) \text{ for } (t,x) \in [0,T] \times [0,1], \tag{4.1a}$$

$$\xi_i'(t) = \alpha(W(t)) \text{ for almost all } t \in [0,T] \text{ and } i = 1,2,\cdots, \tag{4.1b}$$

$$\eta_j'(t) = \alpha(W(t)) \text{ for almost all } t \in [t_j,T] \text{ and } j = 1,2,\cdots, \tag{4.1c}$$

$$W(t) = \int_0^1 \rho_{L^1}(t,x)\,dx + \sum_{\{i \colon \xi_i(t) \in [0,1)\}} m_i + \sum_{\{j \colon \eta_j(t) \in [0,1)\}} M_j \text{ for } t \in [0,T], \tag{4.1d}$$

$$\rho_{0L^1}(x) = \rho_{L^1}(0,x) \text{ for } x \in [0,1), \tag{4.1e}$$

$$\xi_i(0) = x_i \text{ for } i = 1,2,\cdots, \tag{4.1f}$$

$$\eta_j(t_j) = 0 \text{ for } j = 1,2,\cdots, \text{ and} \tag{4.1g}$$

$$u_{L^1}(t) = \rho_{L^1}(t,0)\alpha(W(t)) \text{ for } t \in [0,T]. \tag{4.1h}$$

Note that due to for every fixed $t \in [0,T]$ the velocity $\alpha(W(t))$ being constant with respect to the location $x \in [0,1]$, there really is only one single ordinary differential equation. All ξ_i and η_j are translates of each other.

This model, combining densities with point masses, convincingly reflects natural features of the original manufacturing system. Mathematically, it naturally suggests to combine the densities and point masses and informally write $\rho_0 = \rho_{0,L^1} + \sum_i m_i \delta_{x_i}$ and $u = u_{L^1} + \sum_j M_j \delta_{t_j}$ (with δ_s denoting the Dirac distribution centered at s). More satisfactorily, we combine the L^1-densities and point masses into a measure and consider a single hyperbolic conservation law like (1.1) for data and states that are Borel measures. From now on, we will assume that both the initial data ρ_0 and the control in-flux (we will write μ instead of u) are finite positive regular Borel measures in $\mathcal{M}^+([0,1))$ and $\mathcal{M}^+((0,T])$, respectively.

Furthermore, we will assume throughout that all initial data and in-fluxes have zero singular continuous part, i.e., they are sums of only an absolutely continuous measure (w.r.t Lebesgue measure) and a pure point measure (a countable sum of positive multiples of Dirac deltas). This assumption is motivated by the original industrial optimal control problem where singular continuous measures seem to not make much sense, and the desire to avoid unnecessary technical complications in the sequel. This is well in line with much of recent literature, e.g. Piccoli and Rossi (2013, 2014).

For any Borel measurable map $\gamma\colon S \subseteq \mathbb{R} \mapsto U \subseteq \mathbb{R}$ and any finite Borel measure $\nu \in \mathcal{M}^+(S)$, the push forward of ν by γ is defined as: for every Borel set $E \subseteq U$

$$\gamma\#\nu(E) \colon = \nu(\gamma^{-1}(E)).$$

Thus the push forward of the initial datum $\rho_0 \in \mathcal{M}^+([0,1))$ by a Borel measurable map

$X(t;0,\cdot)\colon [0,1) \mapsto [0,1)$ $(t \in [0,T])$ is defined for every Borel set $E \subseteq X(t;0,\cdot)([0,1))$ by

$$\left(X(t;0,\cdot)\#\rho_0\right)(E) = \rho_0\left(X(t;0,\cdot)^{-1}(E)\right) = \int_{[0,1)} \chi_E\left(X(t;0,x_0)\right)\, d\rho_0(x_0). \tag{4.2}$$

Similarly, the pushforward of the control in-flux $\mu \in \mathcal{M}^+((0,T])$ by a Borel measurable map

$X(t;\cdot,0)\colon [0,t] \mapsto [0,1)$ $(t \in [0,T])$ is for every Borel set $E \subseteq [0,1)$,

$$\left(X(t;\cdot,0)\#\mu\right)(E) = \mu\left(X(t;\cdot,0)^{-1}(E)\right) = \int_{[0,t)} \chi_E\left(X(t;s,0)\right)\, d\mu(s). \tag{4.3}$$

As a formal reference, we restate the problem (1.1) in the context of Borel measures. For every fixed $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$, a solution of the problem will be phrased in term of curves $\rho\colon [0,T] \mapsto \rho_0 \in \mathcal{M}^+([0,1))$. Their regularity will be addressed in subsection 4.3.3. We may interchangeably use both notation ρ_t and $\rho(t)$ depending on which is easier to read.

For every fixed $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$, consider the problem of finding a curve $\rho\colon [0,T] \mapsto \mathcal{M}^+([0,1))$ and a map $\tilde{\xi}\colon \{(t,r)\colon 0 \le r \le t \le T\} \times [0,1) \mapsto [0,\infty)$ that formally satisfy:

$$0 = \partial_t \rho(t) + \partial_x(\alpha(W(t))\rho(t)) \quad \text{for a.e.} \quad t \in [0,T], \tag{4.4a}$$

$$W(t) = \rho(t)([0,1)) \quad \text{for} \quad t \in [0,T], \tag{4.4b}$$

$$\rho_0 = \rho(0), \tag{4.4c}$$

$$\rho(t)(E) = \mu(\{r \in (0,T]\colon \tilde{\xi}(t;r,0) \in E\}) + \rho_0(\{x \in [0,1)\colon \tilde{\xi}(t;0,x) \in E\}),$$
$$\text{for} \quad t \in [0,T], \text{ and } E \subset [0,1) \text{ Borel set,} \tag{4.4d}$$

$$\frac{d}{dt}\tilde{\xi}(t;r,x) = \alpha(W(t)) \quad \text{for almost every} \quad 0 \le r \le t \le T, \quad \text{for } x \in [0,1) \text{ and} \tag{4.4e}$$

$$\tilde{\xi}(r;r,x) = x \quad \text{for} \quad r \in [0,T] \quad \text{for } x \in [0,1). \tag{4.4f}$$

Equation (4.4d) relates the in-flux $\mu \in \mathcal{M}^+((0,T])$ to the state $\rho_t \in \mathcal{M}^+([0,1))$ and best captures the sense of conservation of mass. However, this problem statement comes at the cost of presupposing part of the form of the Lagrangian solution defined in the next section.

Informally, to connect system (4.4) for Borel measures to system (1.1) for integrable functions, the boundary condition (4.4d) might be interpreted (in terms of the Lebesgue decomposition $\mu = \mu_{ac} + \mu_{pp}$ and $\rho_t = \rho_{t,ac} + \rho_{t,pp}$) as

$$\mu_{pp}(\{t\}) = \rho_{t,pp}(\{0\}) \tag{4.5}$$

$$\tilde{u}_{L^1}(t) = \tilde{\rho}_{t,ac}(0)\alpha(W(t)). \tag{4.6}$$

Here $\tilde{\rho}_{t,ac} = \frac{d\rho_{t,ac}}{d\lambda}$ and $\tilde{u_{acL^1}} = \frac{d\mu}{d\lambda}$ are the Radon-Nikodym derivatives of $\rho_{t,ac}$ and μ_{ac} with respect to Lebesgue measure λ. For Lebesgue measurable sets, and thus also Borel sets $E \subset (0,T]$ and $F \subset [0,1)$, these satisfy

$$\mu_{ac}(E) = \int_E \tilde{u}_{L^1} d\lambda \quad \text{and} \quad \rho_{t,ac}(F) = \int_F \tilde{\rho}_{t,ac} d\lambda. \tag{4.7}$$

Of course, as L^1 functions, the Radon-Nikodym derivatives only have values at Lebesgue points. It could well be that, e.g., $\tilde{\rho}_{t,ac}(0)$ is not defined for any t at all, i.e., if for no $t \in [0,T]$ is $x = 0$ a Lebesgue point of $\tilde{\rho}_{t,ac}$. Thus we consider this only an informal discussion, to motivate the precise statement of notions of solutions in the forthcoming sections. Note that from this point of view, the pure point part simply copies from the time to the space direction, whereas the velocity multiplies the L^1-functions associated to the absolutely continuous parts - which is commensurate with ρ_t being the push foreword of μ by the semi-flow as defined in the next section.

4.2 Lagrangian Solutions

In this section, we prove existence of unique solutions of a related scalar ordinary differential equation first. Then we define a Lagrangian solution to system (4.4) and establish its existence and uniqueness.

4.2.1 Existence of Unique Short Time Solutions

In this subsection, we prove local existence of unique solutions of a related scalar ordinary differential equation by using contraction mapping argument. For any fixed $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$ consider the Cauchy problem

$$\dot{\xi}(t) = \alpha(\mu((0,t]) + \rho_0([0, 1 - \xi(t)))) \;\text{ for }\; 0 \leq t \leq 1, \;\text{ together with }\; \xi(0) = 0. \tag{4.8}$$

While still involving the in-flux μ, the key difference is that this is a scalar ordinary differential equation with a single fixed measure $\rho_0 \in \mathcal{M}^+([0,1))$ as a parameter, rather than a hyperbolic conservation law and for which we are trying to find a solution, that is a curve $t \mapsto \rho_t \in \mathcal{M}^+([0,1))$. (The measure μ drops out in the contraction mapping argument for small times.) The price to pay for this is that instead of the vector field v that is constant in x, the fixed initial measure ρ_0 now enters this ordinary scalar differential equation as a parameter, which causes the velocity $v(t,x) = \alpha(\mu((0,t]) + \rho([0,1-x)))$ to generally be discontinuous in the space variable.

A key insight is that the usual contraction mapping argument can be modified to accommodate even an infinite number of discontinuities of the time-varying vector field $v(t,x)$. However, one will need to *restart* the argument at times when $v(t,x)$ has *large* discontinuities, caused by large point masses exiting the system.

The Hypotheses for the Contraction Mappings

Let $T > 0$, and $\alpha\colon [0,\infty) \mapsto (0,1]$ be a strictly decreasing Lipschitz continuous function with $\alpha(0) = 1$ and Lipschitz constant $L > 0$. Let $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$ be arbitrary but fixed measures with zero singular continuous part. Fix

$$v_{\min} = \alpha(1 + \rho_0([0,1]) + \mu([0,T])) > 0. \tag{4.9}$$

Denote the Lebesgue decomposition of the initial condition ρ_0 and the in-flux μ by

$$\rho_0 = \rho_{0,ac} + \rho_{0,pp} \quad \text{and} \quad \mu = \mu_{ac} + \mu_{pp}. \tag{4.10}$$

Thus there exist at most countably many $m_i,\ M_j > 0$, $x_i \in [0,1)$ and $t_j \in (0,T]$ such that

$$\rho_{0,pp} = \sum_i m_i \delta_{x_i}, \quad \text{and} \quad \mu_{pp} = \sum_j M_j \delta_{t_j}. \tag{4.11}$$

Since the measures ρ_0 and μ are bounded, there exist N_1, $N_2 \in \mathbb{N}$ such that

$$\sum_{i>N_1} m_i < \frac{v_{\min}}{4L} \quad \text{and} \quad \sum_{j>N_2} M_j < \frac{v_{\min}}{4L}. \tag{4.12}$$

Without loss of generality, after possible renumbering, we may assume that $x_{i+1} < x_i$ for all $i \le N_1$ and $t_j < t_{j+1}$ for all $j \le N_2$, the natural orderings in which the corresponding point masses will exit the system (if they do). Henceforth we will informally call the masses m_i, $i \le N_1$ and M_j, $j \le N_2$ *large masses.*

Choose $0 < t_{00} \le 1$ such that for every interval $I \subseteq [0, 1)$ of length less than t_{00}, and every interval $J \subseteq [0, T]$ of length less than $t_{00}/v_{\min}$

$$\rho_{0,ac}(I) < \frac{v_{\min}}{4L} \quad \text{and} \quad \mu_{ac}(J) < \frac{v_{\min}}{4L}. \tag{4.13}$$

Let Ω be the set of functions on $[0, t_{00}]$ that are Lipschitz continuous with Lipschitz constant bounded above by 1, and whose inverses are Lipschitz continuous with Lipschitz constant no larger than $v_{\min}^{-1}$, that is,

$$\Omega = \{\eta : [0, t_{00}] \to [0, 1] \colon \eta(0) = 0, \quad \text{and} \quad v_{\min} \le \frac{\eta(s) - \eta(t)}{s - t} \le 1 \text{ for all } 0 \le s < t \le t_{00}\}. \tag{4.14}$$

Since every $\eta \in \Omega$ is strictly increasing, each is absolutely continuous, and differentiable almost everywhere.

Lemma 4.2.1. *The set Ω defined as in (4.14) is closed under maxima and minima. That is, for every two functions η_1, η_2 in Ω, set $\hat{\eta}(t) = \max\{\eta_1(t), \eta_2(t)\}$ and $\check{\eta}(t) = \min\{\eta_1(t), \eta_2(t)\}$. Then both $\hat{\eta}$ and $\check{\eta}$ are in Ω.*

Proof. For two arbitrary but fixed two functions η_1, η_2 in Ω, set $\hat{\eta}(t) = \max\{\eta_1(t), \eta_2(t)\}$ and $\check{\eta}(t) = \min\{\eta_1(t), \eta_2(t)\}$. We will show first that $\hat{\eta} \in \Omega$.

It is obvious that $\hat{\eta}\colon [0,\tau_0] \mapsto [0,1]$ and $\hat{\eta}(0) = 0$. For arbitrary but fixed $s, t \in [0,\tau_0]$ such that $0 \le s < t \le \tau_0$, if $\hat{\eta}(s) = \eta_k(s)$ and $\hat{\eta}(t) = \eta_k(t)$ for $k = 1$ or $k = 2$, then clearly $\hat{\eta} \in \Omega$.

Without loss of generality, we assume that $\hat{\eta}(s) = \eta_1(s)$ and $\hat{\eta}(t) = \eta_2(t)$. Then

$$\frac{\hat{\eta}(s) - \hat{\eta}(t)}{s - t} = \frac{\eta_1(s) - \eta_2(t)}{s - t} \le \frac{\eta_1(s) - \eta_1(t)}{s - t} \le 1.$$

Furthermore,

$$\frac{\hat{\eta}(s) - \hat{\eta}(t)}{s - t} = \frac{\eta_1(s) - \eta_2(t)}{s - t} \ge \frac{\eta_2(s) - \eta_2(t)}{s - t} \ge v_{\min}.$$

Thus $\hat{\eta} \in \Omega$.

Similarly, to show $\check{\eta} \in \Omega$, we just need to consider the case when $\check{\eta}(s) = \eta_1(s)$ and $\check{\eta}(t) = \eta_2(t)$. Note that

$$\frac{\check{\eta}(s) - \check{\eta}(t)}{s - t} = \frac{\eta_1(s) - \eta_2(t)}{s - t} \le \frac{\eta_2(s) - \eta_2(t)}{s - t} \le 1$$

and

$$\frac{\check{\eta}(s) - \check{\eta}(t)}{s - t} = \frac{\eta_1(s) - \eta_2(t)}{s - t} \ge \frac{\eta_1(s) - \eta_1(t)}{s - t} \ge v_{\min}.$$

Hence $\check{\eta} \in \Omega$.

□

Lemma 4.2.2. *For every fixed $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$, the metric space $(\Omega, \|\cdot\|_\infty)$ with Ω defined as in equation (4.14) and the supremum norm defined by*

$$\|\eta\|_\infty := \sup_{t \in [0,\tau_0]} |\eta(t)| \tag{4.15}$$

is complete.

Proof. Let $(\eta_n)_{n=1}^\infty$ be a Cauchy sequence in $(\Omega, \|\cdot\|_\infty)$. We claim that $\eta_n \to \eta$ for some $\eta \in \Omega$. Since for every fixed $t \in [0,\tau_0]$, $(\eta_n(t))_{n=1}^\infty$ is a Cauchy sequence in $[0,1]$

and since $[0,1]$ is complete, for every $t \in [0, \tau_0]$ the pointwise limit $\lim_{n\to\infty} \eta_n(t)$ is in $[0,1]$. Denote the pointwise limit by $\eta : [0, \tau_0] \to [0,1]$, i.e. for all $t \in [0,T]$,

$$\eta(t) = \lim_{n\to\infty} \eta_n(t). \tag{4.16}$$

We first show that $(\eta_n)_{n=1}^{\infty}$ converges to η in norm, i.e., $\|\eta_n - \eta\|_\infty \to 0$ as $n \to \infty$. Let $\epsilon > 0$ and N be such that for all $n, m \geq N$, we have $\|\eta_n - \eta_m\|_\infty < \epsilon$. Then for all $n \geq N$, and for each $t \in [0, \tau_0]$,

$$|\eta(t) - \eta_n(t)| = \lim_{m\to\infty} |\eta_m(t) - \eta_n(t)| \leq \epsilon, \tag{4.17}$$

hence $\|\eta - \eta_n\|_\infty \leq \epsilon$.

To finish the proof, we need to show that $\eta \in \Omega$. Note that $\eta(0) = \lim\limits_{n\to\infty} \eta_n(0) = \lim\limits_{n\to\infty} 0 = 0$. For all fixed $0 \leq t < s \leq \tau_0$ and for every $\epsilon > 0$, there exists $N > 0$, such that $\|\eta_N - \eta\|_\infty < \frac{(s-t)\epsilon}{2}$. Thus

$$\frac{\eta(s) - \eta(t)}{s-t} = \frac{\eta(s) - \eta_N(s)}{s-t} + \frac{\eta_N(s) - \eta_N(t)}{s-t} + \frac{\eta_N(t) - \eta(t)}{s-t} < 1 + \epsilon. \tag{4.18}$$

Similarly, for every $\epsilon > 0$, we have $\frac{\eta(s)-\eta(t)}{s-t} > v_{\min} - \epsilon$. □

Lemma 4.2.3. *For fixed $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$ and $v_{\min}$, $0 < \tau_0 \leq 1$, and Ω as above, define the map $F : \Omega \to C([0,\tau_0])$ by*

$$F(\eta)(t) = \int_0^t \alpha\left(\rho_0([0, 1 - \eta(s))) + \mu((0,s])\right) ds. \tag{4.19}$$

Then for every $\eta \in \Omega$, $F(\eta) \in \Omega$.

Proof. Let $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$ be arbitrary but fixed, and $v_{\min}$, $0 < \tau_0 \leq 1$, Ω, and F as above as above. Clearly, for every $\eta \in \Omega$, $F(\eta)(0) = 0$. Note for every $\bar{s} \in [0,T]$,

$$v_{\min} \leq \alpha(\rho_0([0,1]) + \mu([0,T])) \leq \alpha\left(\rho_0([0, 1 - \eta(\bar{s}))) + \mu((0,\bar{s}])\right) \leq 1, \tag{4.20}$$

and thus, for all $s \neq t \in [0, \tau_0]$

$$v_{\min} \leq \frac{F(\eta)(s) - F(\eta)(t)}{s - t} \leq 1.$$

□

Now we show that for any fixed measures ρ_0 (or ρ^i) and μ (or μ^i) that contain zero singular continuous parts, there exists a time $0 < \tau_i \leq t_{00} < 1$ and a unique Lipschitz continuous function $\xi_i \colon [0, \tau_i] \mapsto [0, 1]$ such that for every $t \in [0, \tau_i]$,

$$\xi_i(t) = \int_0^t \alpha(\mu^i((0, s]) + \rho^i([0, 1 - \xi_i(t)))) \, ds. \tag{4.21}$$

The proof and this equation only involve a fixed measure ρ_0 (or ρ^i), no mention of a curve of measures $t \mapsto \rho_t$. In general, ξ_i is constructed as the restriction of a curve $\tilde{\xi}_i \in \Omega$ to a shorter time interval $[0, \tau_i] \subseteq [0, t_{00}]$, and thus it inherits the bi-Lipschitzness properties from $\tilde{\xi}_i \in \Omega$.

Theorem 4.2.4. *For every $\mu($ or $\mu^i) \in \mathcal{M}^+((0, T])$ and $\rho_0($ or $\rho^i) \in \mathcal{M}^+([0, 1))$ that contain zero singular continuous parts, there exists a unique characteristic curve $\xi_i : [0, \tau_i] \subseteq [0, t_{00}] \to [0, 1]$ that satisfies* (4.21).

The general strategy of the proof is a classic application of the contraction mapping theorem, similar to Coron *et al.* (2010). However, a naive argument breaks down over time intervals in which large point masses exit from the system. Thus we carefully demonstrate that the usual map is a contraction over intervals during which no large point masses exit from the system, and then restart the argument after the mass has left the system.

A small problem is that it is not a priori known when the large masses leave exit from the system. This can be overcome by a nice little trick: Replace the initial datum ρ_0 by a modified $\tilde{\rho}_0$ for which contractions can be established over a larger time interval, and whose characteristic curves agree with those for the original datum ρ_0 until the exactly computable time when the first large mass would have exited. Since there is only a finite number of large masses, one can guarantee that there is a positive lower bound for the lengths of the time intervals on which no large mass exits from the system. Such a lower bound can be easily calculated in terms of lower and upper bounds of the velocity α, and $\min\{x_{i-1} - x_i : i \leq N_1\}$ and $\min\{t_j - t_{j-1} : i \leq N_2\}$ where we conveniently added $x_0 = 1$ and $t_0 = 0$ to the sets of x_i and t_j defined below in (4.12). This lower bound is essential to guarantee a solution over the whole interval $[0, T]$ by using only finitely many restarts.

Proof. Let $\rho_0 \in \mathcal{M}^+([0, 1))$ and $\mu \in \mathcal{M}^+((0, T])$ (contain zero singular continuous part) be arbitrary but fixed, and $v_{\min}$, $0 < \tau_0 \leq 1$, Ω, and F as in (4.9), (4.13), (4.14), and (4.19).

A key innovation is to introduce a modification $\tilde{\rho}$ of the initial condition ρ_0 such that no large masses will leave the system in the time interval $[0, 1)$. Define the new initial condition which agrees mostly with ρ_0, except that all N_1 large masses have been moved to $x = 0$

$$\tilde{\rho}_{0,pp} = \left(\sum_{i=1}^{N_1} m_i \right) \delta_0 + \sum_{i > N_1} m_i \delta_{x_i} \quad \text{and} \quad \tilde{\rho}_0 = \rho_{0,ac} + \tilde{\rho}_{0,pp}. \tag{4.22}$$

Since the velocity $v = \alpha(W(t))$ only depends on the total load at time t, the characteristic curves ξ and $\tilde{\xi}$ corresponding to initial conditions ρ_0 and $\tilde{\rho}_0$ coincide over a small time interval until $[0, \tilde{\tau}_1]$ defined by $\tilde{\xi}(\tilde{\tau}_1) = 1 - x_1$ at which time the mass m_1 would leave the original system at $x = 1$.

Next, for the initial condition $\tilde{\rho}_0$ (and in-flux μ) define $v_{\min}$, τ_0, Ω, and F as in (4.9), (4.13), (4.14), and (4.19). We will demonstrate existence and uniqueness of a corresponding characteristic curve $\tilde{\xi}$ over the time interval $[0, \tau_0]$. For arbitrary but fixed $\eta_1, \eta_2 \in \Omega$, we will show that

$$\|F(\eta_1) - F(\eta_2)\|_\infty \le \frac{1}{2}\|\eta_1 - \eta_2\|_\infty. \tag{4.23}$$

Since the velocity α is a Lipschitz continuous function with Lipschitz constant L we have for every fixed $t \in [0, \tau_0]$

$$\begin{aligned}
&|F(\eta_1)(t) - F(\eta_2)(t)| = \\
&= \left|\int_0^t \alpha\left(\tilde{\rho}_0([0, 1-\eta_1(s))) + \mu((0,s])\right) ds - \int_0^t \alpha\left(\tilde{\rho}_0([0, 1-\eta_2(s))) + \mu((0,s])\right) ds\right| \\
&\le \int_0^t \left|\alpha\left(\tilde{\rho}_0([0, 1-\eta_1(s))) + \mu((0,s])\right) - \alpha\left(\tilde{\rho}_0([0, 1-\eta_2(s))) + \mu((0,s])\right)\right| ds \quad (4.24) \\
&\le L\int_0^t \left|\left(\tilde{\rho}_0([0, 1-\eta_1(s))) + \mu((0,s])\right) - \left(\tilde{\rho}_0([0, 1-\eta_2(s))) + \mu((0,s])\right)\right| ds \\
&= L\int_0^t \left|\tilde{\rho}_0([0, 1-\eta_1(s))) - \tilde{\rho}_0([0, 1-\eta_2(s)))\right| ds.
\end{aligned}$$

Choosing $\hat{\eta}(t) = \max\{\eta_1(t), \eta_2(t)\}$ and $\check{\eta}(t) = \min\{\eta_1(t), \eta_2(t)\}$, rewrite the last expression as a double integral

$$\begin{aligned}
&L\int_0^t \left|\tilde{\rho}_0([0, 1-\eta_1(s))) - \tilde{\rho}_0([0, 1-\eta_2(s)))\right| ds \\
&= L\int_0^t \left|\int_{[1-\hat{\eta}(s), 1-\check{\eta}(s))} 1\, d\tilde{\rho}_0(x_0)\right| ds \\
&= L\int_0^t \int_{[1-\hat{\eta}(s), 1-\check{\eta}(s))} 1\, d\tilde{\rho}_0(x_0) ds. \qquad (4.25)
\end{aligned}$$

Since the regions are bounded by bi-Lipschitz curves, we may change the order of integration as illustrated in the figure below (compare FIGURE 1 in Coron *et al.*

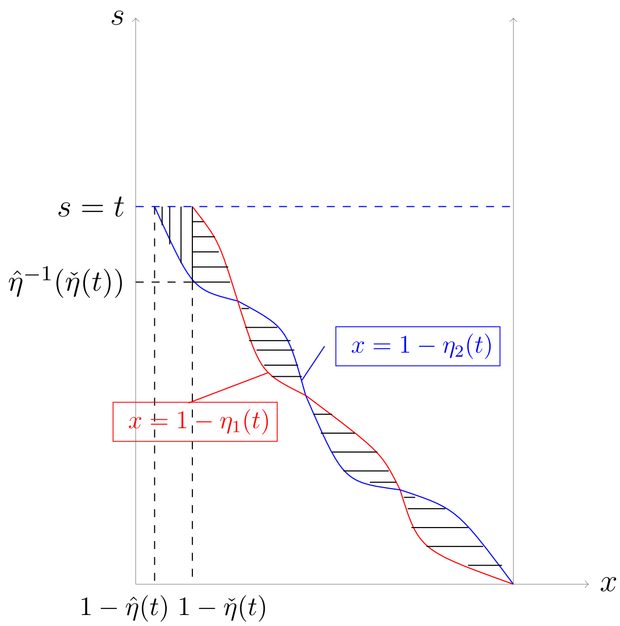


Figure 4.1: Change the Order of Integration

(2010))

$$
\begin{aligned}
L = & \int_0^t \int_{[1-\hat{\eta}(s),1-\check{\eta}(s))} 1 \, d\tilde{\rho}_0(x_0) ds \\
= \quad & L\left(\int_{[1-\hat{\eta}(t),1-\check{\eta}(t)]} \int_{[\hat{\eta}^{-1}(1-x),t]} 1 \, dt \, d\tilde{\rho}_0(x_0)\right. \\
+ \quad & \left.\int_{[1-\check{\eta}(t),1)} \int_{[\check{\eta}^{-1}(1-x),\hat{\eta}^{-1}(1-x))} 1 \, dt \, d\tilde{\rho}_0(x_0)\right) \\
\leq \quad & L\left(\int_{[1-\hat{\eta}(t),1-\check{\eta}(t)]} (t-\hat{\eta}^{-1}(1-x)) \, d\tilde{\rho}_0(x_0)\right. \\
+ \quad & \left.\int_{[1-\check{\eta}(t),1)} \left(\check{\eta}^{-1}(1-x)-\hat{\eta}^{-1}(1-x)\right) \, d\tilde{\rho}_0(x_0)\right) \\
\leq \quad & L\left(\int_{[1-\hat{\eta}(t),1-\check{\eta}(t)]} (\check{\eta}^{-1}(\check{\eta}(t)-\hat{\eta}^{-1}(\check{\eta}(t))) \, d\tilde{\rho}_0(x_0)\right. \\
+ \quad & \left.\int_{[1-\check{\eta}(t),1)} \left(\check{\eta}^{-1}(1-x)-\hat{\eta}^{-1}(1-x)\right) \, d\tilde{\rho}_0(x_0)\right) \\
\leq \quad & L\left(\tilde{\rho}_0([1-\hat{\eta}(t),1))\right) \sup_{0\leq y\leq\check{\eta}(t)} (\check{\eta}^{-1}(y)-\hat{\eta}^{-1}(y)).
\end{aligned}
\tag{4.26}
$$

To find an upper bound for the last term, use the Lipschitzness of the curves and their inverses, and simple geometric arguments relating the vertical offsets of the curves to their horizontal offsets. By the definition of $\hat{\eta}$, $\check{\eta}$, we have, for every $y \in [0, \check{\eta}(t)]$, (compare Equation (23) in Coron *et al.* (2010))

$$\begin{aligned}
0 &\le \check{\eta}^{-1}(y) - \hat{\eta}^{-1}(y) \\
&= \left(\check{\eta}^{-1}(y) - \frac{\hat{\eta}^{-1}(y) + \check{\eta}^{-1}(y)}{2}\right) + \left(\frac{\hat{\eta}^{-1}(y) + \check{\eta}^{-1}(y)}{2} - \hat{\eta}^{-1}(y)\right) \\
&\le \frac{1}{v_{\min}}\left(y - \check{\eta}\left(\frac{\hat{\eta}^{-1}(y) + \check{\eta}^{-1}(y)}{2}\right)\right) + \frac{1}{v_{\min}}\left(\hat{\eta}\left(\frac{\hat{\eta}^{-1}(y) + \check{\eta}^{-1}(y)}{2}\right) - y\right) \qquad (4.27) \\
&= \frac{1}{v_{\min}}\left(\hat{\eta}\left(\frac{\hat{\eta}^{-1}(y) + \check{\eta}^{-1}(y)}{2}\right) - \check{\eta}\left(\frac{\hat{\eta}^{-1}(y) + \check{\eta}^{-1}(y)}{2}\right)\right) \\
&\le \frac{1}{v_{\min}}\|\eta_1 - \eta_2\|_\infty.
\end{aligned}$$

Hence,

$$|F(\eta_1)(t) - F(\eta_2)(t)| \le \frac{L}{v_{\min}}\left(\tilde{\rho}_0([1 - \hat{\eta}(t), 1))\right)\|\eta_1 - \eta_2\|_\infty. \qquad (4.28)$$

By the choice of τ_0 (4.13), for the absolutely continuous part $\tilde{\rho}_{0,ac} = \rho_{0,ac}$, and for every $t \in [0, \tau_0)$,

$$\tilde{\rho}_{0,ac}([1 - \hat{\eta}(t), 1)) \le \tilde{\rho}_{0,ac}([1 - \hat{\eta}(\tau_0), 1)) \le \frac{v_{\min}}{4L}. \qquad (4.29)$$

Note that due to their relocation and $\tau_0 < 1$, none of the large point masses in $\tilde{\rho}_{0,pp}$ have exited in the interval $[0, \tau_0]$. Formally, since for every $t \in [0, \tau_0)$, $0 < 1 - \hat{\eta}(t)$, we conclude that $\tilde{\rho}_{0,pp}([1 - \hat{\eta}(t), 1)) < \frac{v_{\min}}{4L}$. Combining these,

$$\tilde{\rho}_0([1-\hat{\eta}(t), 1)) = \tilde{\rho}_{0,ac}([1-\hat{\eta}(t), 1)) + \tilde{\rho}_{0,pp}([1-\hat{\eta}(t), 1)) < \frac{v_{\min}}{2L}, \text{ for } t \in [0, \tau_0). \quad (4.30)$$

Hence $\frac{L}{v_{\min}}\tilde{\rho}_0([1 - \hat{\eta}(t), 1)) < \frac{1}{2}$ showing that F is a contraction on Ω. By the contraction mapping theorem, with the initial condition $\tilde{\rho}_0$, there exists a unique fixed point $\tilde{\xi}$ in Ω such that $\tilde{\xi} = F(\tilde{\xi})$ over the time interval $[0, \tau_0]$.

If $\tilde{\xi}(\tau_0) < 1 - x_1$ (the distance of the first large point mass of the initial condition from the exit point $x = 1$), then define $\xi = \tilde{\xi}$ on $[0, \tau_0]$ and let $\tau_1 = \tau_0$. On the other hand, if $\tilde{\xi}(\tau_0) \ge 1 - x_1$, there exists a unique time $\tau_1 \in (0, \tau_0]$ such that $\tilde{\xi}(\tau_1) = 1 - x_1$. In this case let ξ be the restriction of $\tilde{\xi}$ to the interval $[0, \tau_1]$. □

4.2.2 Existence of Unique Solutions for Large Times

We start this subsection by the following lemma.

Lemma 4.2.5. *Given a map*

$$X\colon \{(t,r)\colon 0 \le r \le t \le T\} \times [0,\infty) \mapsto [0,\infty); (t,r,x_0) \mapsto X(t;r,x_0)$$

that is monotone and bi-Lipschitz in terms of the first two variables and for fixed $(t,r) \in [0,T] \times [0,t]$ *and every* $x_0 \in [0,\infty)$*,* $X(t;r,x_0) = X(t;r,0) + x_0$ *and given arbitrary but fixed Borel measures* $\rho_0 \in \mathcal{M}^+([0,1))$ *and* $\mu \in \mathcal{M}^+((0,T])$ *with zero singular continuous part, that is,* $\rho_{0,sc} = 0$ *and* $\mu_{sc} = 0$*, the measure*

$$X(t;0,\cdot)\#\rho_0 + X(t;\cdot,0)\#\mu \tag{4.31}$$

also contains zero singular continuous part.

Proof. For arbitrary but fixed $t \in [0,T]$ and every Borel set $E \subset [0,1)$, with $\lambda(E) = 0$, Let $E_1 = \{X(t;0,x)\colon x \in \operatorname{supp} \rho_{0,pp}\}$ and $E_2 = \{X(t;\tau,0)\colon \tau \in \operatorname{supp} \mu_{pp}\}$. Set $\tilde{E} = E \setminus (E_1 \cup E_2)$. It is clear that $\lambda(\tilde{E}) = 0$. Furthermore,

$$\begin{aligned}(X(t;0,\cdot)\#\rho_0 + X(t;\cdot,0)\#\mu)(\tilde{E}) =& (X(t;0,\cdot)\#\rho_0)(\tilde{E}) + (X(t;\cdot,0)\#\mu)(\tilde{E}) \\ =& \rho_0\left(\left\{x \in [0,1)\colon X(t;0,x) \in \tilde{E}\right\}\right) \\ &+ \mu\left(\left\{\tau \in (0,t]\colon X(t;\tau,0) \in \tilde{E})\right\}\right).\end{aligned}$$

Let $F_1 = \left\{x \in [0,1)\colon X(t;0,x) \in \tilde{E}\right\} = \left\{x \in [0,1)\colon x + \xi(t) \in \tilde{E}\right\}$, then $F_1 = \tilde{E} - \xi(t)$. Since Lebesgue measure is translation invariant, $\lambda(F_1) = 0$. By the construction of $\tilde{E}$, $\rho_0(F_1) = \rho_{0,ac}(F_1) + \rho_{0,pp}(F_1) = 0 + 0 = 0$.

Let $F_2 = \left\{\tau \in (0,t]\colon X(t;\tau,0) \in \tilde{E})\right\}$, then F_2 is the preimage of the Lebesgue measure zero set $\tilde{E}$ under the map $X(t;\cdot,0) : (0,t] \mapsto [0,+\infty)$. Note that the map $X(t;\cdot,0) : (0,t] \mapsto [0,+\infty)$ is monotone and bi-Lipschitz, thus both itself and its inverse are absolutely continuous. Hence, $\lambda(F_2) = 0$. Again by the construction of $\tilde{E}$, $\mu(F_2) = \mu_{ac}(F_2) + \mu_{pp}(F_2) = 0 + 0 = 0$.

Therefore, $(X(t;0,\cdot)\#\rho_0 + X(t;\cdot,0)\#\mu)(\tilde{E}) = 0$ which implies that the measure $X(t;0,\cdot)\#\rho_0 + X(t;\cdot,0)\#\mu$ contains zero singular continuous part.

□

To prove the existence of unique solutions of ordinary differential equations for large time intervals, one customarily iterates the fixed-point argument, with suitably modified initial data. We will do this here, too. But before we can do this, after each iteration construct a new measure that serves as a parameter in the next iteration.

The key for this argument is that each iteration only involves a single fixed measure ρ^i. After a unique solution curve $\xi_i\colon [0,\tau_i] \mapsto [0,1]$ of the ordinary differential equation has been obtained, this is used to extend the curve $t \mapsto \rho_t$ from the interval $[0, T_{i-1}]$ to a larger interval $[0, T_i]$.

The curve ξ is constructed on each interval $[T_{i-1}, T_i]$ from the unique solutions of $\dot{\xi}_i = \alpha(\mu^i((0,s]) + \rho^i([0, 1 - \xi_i(t)))$. Afterwards, ρ_t is constructed from ξ for that same time interval. Thus one still needs to verify that indeed ξ_i also satisfies $\dot{\xi}_i = \alpha(\rho_t([0,1))$ on each new interval, or that the curve ξ satisfies the related equation $\dot{\xi}(t) = \alpha(\mu((\max\{0, \xi^{-1}(\xi(t) - 1)\}, t]) + \rho_0([0, 1 - \xi(t))))$.

The total number N of iterations needed to get a solution ρ for all of $[0, T]$ is a priori bounded above by $\mathrm{ceil}(T/t_{00}) + N_1 + N_2$ (only exiting masses stemming from μ matter). The maximal number N of iteration may be smaller than this bound, e.g., if at the end ρ_T still contains large point masses.

In the sequel we shall construct

- finite sequences $(\tau_i)_{i=0}^{N}$ and $(T_i)_{i=0}^{N+1}$ of nonnegative numbers,

- finite sequences of Lipschitz continuous functions $\xi_i \colon [0, \tau_i] \mapsto [0, 1)$, and $\xi \colon [0, T_i] \mapsto [0, \infty)$,
- a finite sequence of maps $X \colon \{(t, r, x) \colon 0 \le r \le t \le T_i, x \in [0, \infty)\} \mapsto [0, \infty)$,
- finite sequences of measures $\rho^i \in \mathcal{M}^+([0, 1))$ and $\mu^i \in \mathcal{M}^+((0, T - T_{i-1}])$, and
- a finite sequence of curves $\rho \colon [0, T_i] \mapsto \mathcal{M}^+([0, 1))$.

Strictly speaking, one should also index the curves ξ and ρ, and the maps X by i as they are defined on different domains. But it will be clear that they just denote the usual extensions of each other to larger domains. As is customary, we will omit such extra indexing. The members of these sequences and the curves will be shown to have the following properties for every $0 \le i \le N$ (or $1 \le i \le N$ for μ^i, ρ^i). Some of these properties actually will be used to construct these in the sequel.

(P1). If $i \ge 1$, then $0 < \tau_i < 1$.

(P2). $T_i = \sum_{j \le i} \tau_j$ and $T_{N+1} = T$.

(P3). For every $t \in [0, \tau_i]$,

$$\xi_i(t) = \int_0^t \alpha(\mu^i((0, s]) + \rho^i([0, 1 - \xi_i(s)))) \, ds. \tag{4.32}$$

(P4). For every $s \in [0, \tau_i]$, $\xi(T_{i-1} + s) = \xi(T_{i-1}) + \xi_i(s)$.

(P5). For all $0 \le r \le s \le t \le T_i$ and all $x \in [0, \infty)$, $X(t; s, X(s; r, x)) = X(t; r, x)$.

(P6). The measure μ^i is the push-forward (by a translation) of a restriction of the original in-flux, defined for any $F \in \mathcal{M}^+((0, T - T_i])$ by $\mu^i(F) = \mu(\{t \colon t - T_{i-1} \in F\})$

(P7). For every $0 \leq s \leq \tau_i$, $\rho_{T_{i-1}+s}$ is the sum of the push-forward (by a translation) of a restriction of the measure ρ^i, and by the push-forward by the map X of a restriction of the measure μ^i, defined for every Borel set $E \subseteq [0,1))$ by

$$\rho_{T_{i-1}+s}(E) = \mu^i(\{r \in (0,s] \colon X(T_{i-1}+s; T_{i-1}+r, 0) \in E\}) \quad (4.33)$$

$$+\rho^i(\{x \in [0,1) \colon X(T_{i-1}+s; T_{i-1}, x) \in E\}). \quad (4.34)$$

(P8). For every $0 \leq t \leq T_i$ and every Borel set $E \subseteq [0,1))$

$$\rho_t(E) = \mu(\{r \in (0,t] \colon X(t; r, 0) \in E\}) + \rho_0(\{x \in [0,1) \colon X(t; 0, x) \in E\}). \quad (4.35)$$

(P9). For every $t \in [0, T_i]$ and for every interval $I \subseteq [0,1)$, if the length of I is less than t_{00} then $\rho_{t,ac}(I) < \frac{v_{\min}}{4L}$.

(P10). The measure $\rho^i = \rho_{T_{i-1}}$ is used as the new initial condition ρ^i (if $1 \leq i \leq N$).

(P11). The measures μ^i, ρ^i, and ρ_t have zero singular continuous part (lemma 4.2.5).

(P12). For almost every $0 \leq t \leq T_i$, $\dot{\xi}(t) = \alpha(\rho_t([0,1)))$.

(P13). For almost every $0 \leq t \leq T_i$,

$$\dot{\xi}(t) = \alpha(\mu((\max\{0, \xi^{-1}(\xi(t)-1)\}, t]) + \rho_0([0, 1-\xi(t)))). \quad (4.36)$$

Note that if $\xi(t) \geq 1$ then $\rho_t([0, 1-\xi(t))) = \rho_t(\emptyset) = 0$).

For $i = 0$ set $\tau_0 = T_0 = 0$, take the trivial curves $\xi_0(0) = \xi(0) = 0$ and the identity $X(0; 0, x) = x$ for all $x \in [0,1]$. For $i = 1$ use the original measures as data $\rho^1 = \rho_0$ and $\mu^1 = \mu$.

Now suppose $0 < i \leq N$ is arbitrary but fixed and for all $0 \leq j < i$ all the above have been constructed, and have been shown to have the asserted properties **(P1)**–**(P13)**.

First define the new data $\rho^i = \rho_{T_{i-1}}$ and μ^i as in **(P11)** and **(P6)**. Both have zero singular continuous part and their combined total mass is less or equal to the combined mass of the original measures ρ_0 and μ. In particular, the estimate **(P9)** for the absolutely continuous part of $\rho_{T_{i-1}}$ still holds. Moreover, the combined number of *large point masses* of ρ^i and μ^i cannot exceed the combined number of *large point masses* of the original measures ρ_0 and μ. Thus using the same set Ω (with same $v_{\min}$ and same uniform initial choice for t_{00}, the fixed point theorem yields the existence of a $\tau_i > 0$ and a unique curve $\xi_i\colon [0,\tau_i] \mapsto [0,1)$ that satisfies **(P3)**.

Now use the formula in **(P4)** to extend the curve ξ from the interval $[0, T_{i-1}]$ to $[0, T_{i-1} + \tau_i] = [0, T_i]$.

Next extend the map X from $\{(r,t)\colon 0 \le r \le t \le T_{i-1}\} \times [0,\infty)$ to $\{(r,t)\colon 0 \le r \le t \le T_i\} \times [0,\infty)$ by first setting for all $0 \le r \le T_{i-1} \le t \le T_i$ and every $x \in [0,\infty)$, $X(t;r,x) = X(T_{i-1};r,x) + \xi_i(t - T_{i-1})$, and then, in a second step, for all $0 \le T_{i-1} \le r \le t \le T_i$ and every $x \in [0,\infty)$, $X(t;r,x) = x + \xi_i(t - T_{i-1}) - \xi_i(r - T_{i-1})$. Using the property **(P4)**, alternatively the above may be written in term is ξ. Indeed, for $0 \le r \le T_{i-1} \le t \le T_i$ and $x \in [0,\infty)$

$$
\begin{aligned}
X(t;r,x) &= X(T_{i-1};r,x) + \xi_i(t - T_{i-1}) \\
&= x + (\xi(T_{i-1}) - \xi(r) + (\xi(t) - \xi(T_{i-1})) = x + \xi(t) - \xi(r).
\end{aligned}
\tag{4.37}
$$

Similarly, for $0 \le T_{i-1} \le r \le t \le T_i$ and $x \in [0,\infty)$

$$
\begin{aligned}
X(t;r,x) &= x + \xi_i(t - T_{i-1}) - \xi_i(r - T_{i-1}) \\
&= x + (\xi(t) - \xi(T_{i-1}) - (\xi(r) - \xi(T_{i-1}) = x + \xi(t) - \xi(r).
\end{aligned}
\tag{4.38}
$$

The semi-group property of the map X on the larger domain follows immediately. (It is simply a consequence of the additivity of integral over disjoint intervals in equation (4.21).) Let $0 \leq r \leq s \leq t \leq T_i\}$ and $x \in [0,1]$ be arbitrary but fixed. Then calculate

$$X(t;s,X(s;r,x)) = (x+\xi(s)-\xi(r))+\xi(t)-\xi(s) = x+\xi(t)-\xi(r) = X(t;r,x). \quad (4.39)$$

Since the solution curves ξ_i (and their inverses) all satisfy the same Lipschitz bounds specified in the same set Ω (except for possible different final times), the curve ξ satisfies the same conditions. Thus the map X is Lipschitz and therefore absolutely continuous in each of its variables separately. Hence the push-forwards by scalar functions obtained from X (by holding two arguments fixed) of measures are well defined. Moreover, measures with zero singular continuous part are mapped to measures with zero singular continuous part (lemma 4.2.5), and absolutely continuous measures and pure point measures mapped to measures of the same kind.

Use the equation (4.33) in item **(P7)** to extend the curve $\rho\colon t \mapsto \rho_t$ from the interval $[0, T_{i-1}]$ to the interval $[0, T_i]$.

Note that, by hypothesis, (4.35) already holds for every $0 \leq t \leq T_{i-1}$ and for every Borel set $E \subseteq [0,1))$. In particular, $\rho_{T_{i-1}}(E) = \mu(\{r \in (0, T_{i-1}]\colon X(T_{i-1}; r, 0) \in E\}) + \rho(\{x \in [0,1)\colon X(T_{i-1}; 0, x) \in E\})$. Now let $t \in [T_{i-1}, T_i]$ and Borel set $E \subseteq [0,1))$ be arbitrary but fixed. The using the definitions of ρ_t for t in the new interval, the definitions of ρ^i and μ^i, and the induction hypothesis, calculate:

$$\begin{aligned}
\rho_t(E) &= \mu^i(\{s \in (0, t - T_{i-1}] \colon X(t; T_{i-1} + s, 0) \in E\}) \qquad (4.40)\\
&\quad + \rho^i(\{x \in [0,1) \colon X(t; T_{i-1}, x) \in E\})\\
&= \mu(\{r \in (T_{i-1}, t] \colon X(t; r, 0) \in E\}) + \rho_{T_{i-1}}(\{x \in [0,1) \colon X(t; T_{i-1}, x) \in E\})\\
&= \mu(\{r \in (T_{i-1}, t] \colon X(t; r, 0) \in E\})\\
&\quad + \mu(\{r \in (0, T_{i-1}] \colon X(t; T_{i-1}, X(T_{i-1}; r, 0) \in E\})\\
&\quad + \rho_0(\{x \in [0,1) \colon X(t; T_{i-1}, X(T_{i-1}, 0, x) \in E\})\\
&= \mu(\{r \in (0, t] \colon X(t; r, 0) \in E\}) + \rho_0(\{x \in [0,1) \colon X(t; 0, x) \in E\}). \qquad (4.41)
\end{aligned}$$

Note that there is no need to consider special cases, e.g., whether any of $\xi(T_{i-1}) \leq xi(t) \leq \xi(T_i)$ is less or larger or equal to 1. If $\xi(t) < 1$ then $\{r \in (0,t] \colon X(t; r, 0) \in [0,1)\} = (0,t]$ and $\{x \in [0,1) \colon X(t; 0, x) \in [0,1)\}$ is nonempty. If $\xi(t) \geq 1$ then $\{r \in (0,t] \colon X(t; r, 0) \in [0,1)\} = (t - \xi^{-1}(\xi(t) - 1), t]$ and $\{x \in [0,1) \colon X(t; 0, x) \in [0,1)\}$ is empty. Using slightly different notation , taking $E = [0,1)$, it is an immediate corollary that for all $t \in [0, T_i]$, $\rho_t([0,1)) = \mu(\max\{0, \xi^{-1}(\xi(t) - 1)\}, t) + \rho_0([0, 1 - \xi(t)))$.

Since the map $r \mapsto X(T_{i-1} + s; r, 0)$ reduces distances, intuitively $\Delta x = v\Delta t < \Delta t$, the push forward by this map of μ^i restricted to $(0, s]$ *concentrates* the absolutely continuous part of μ when becoming part of ρ_t. Now suppose $I \subseteq [0,1)$ is on interval of length at most t_{00}. Then $J = X(T_{i-1} + s; \cdot, 0)^{-1}(I)$ is an interval of length at most $t_{00}/v_{\min}$. Therefore $\mu(i(J) < \frac{v_{\min}}{4L}$, and thus $\rho_{T_{i-1}}(I) < \frac{v_{\min}}{4L}$.

The next to last item is to verify that for almost every $t \in [0, T_i]$ this curve satisfies $\dot{\xi}(t) = \rho_t([0,1))$. By hypothesis, this equation holds for for almost every $t \in [0, T_{i-1}]$. By construction, see item **(P4)**, for every $s \in [0, \tau_i]$, $\xi(T_{i-1} + s) = \xi(T_{i-1}) + \xi_i(s)$. Denoting differentiation by s again by a dot, using equation (4.32) in **(P3)**, it follows that at every $s \in [0, \tau_i]$ at which the integrand of (4.32) is continuous. Note that

$$\begin{aligned}\dot{\xi}(T_{i-1} + s) &= \dot{\xi}_i(s) \\ &= \alpha(\mu^i((0, s]) + \rho^i([0, 1 - \xi_i(s)))),\end{aligned}$$

and

$$\begin{aligned}\rho_{T_{i-1}+s}([0,1)) =& \mu^i\left(\{r \in (0, s] \colon X(T_{i-1}+s; T_{i-1}+r, 0) \in [0,1)\}\right) \\ &+ \rho^i\left(\{x \in [0,1) \colon X(T_{i-1}+s; T_{i-1}, x) \in [0,1)\}\right) \\ =& \mu^i\left(\{r \in (0, s] \colon \xi(T_{i-1}+s) - \xi(T_{i-1}+r) \in [0,1)\}\right) \\ &+ \rho^i\left(\{x \in [0,1) \colon \xi(T_{i-1}+s) - \xi(T_{i-1}) + x \in [0,1)\}\right) \\ =& \mu^i\left(\{r \in [0, s] \colon \xi(T_{i-1}) + \xi_i(s) - \xi(T_{i-1}) - \xi_i(t) \in [0,1)\}\right) \\ &+ \rho^i\left(\{x \in [0,1) \colon \xi_i(s) + x \in [0,1)\}\right) \\ =& \mu^i\left(\{r \in (0, s] \colon \xi_i(s) - \xi_i(r) \in [0,1)\}\right) + \rho^i([0, 1 - \xi_i(s)) \\ =& \mu^i((0, s]) + \rho^i([0, 1 - \xi_i(s)).\end{aligned}$$

Thus,

$$\dot{\xi}(T_{i-1} + s) = \alpha\left(\rho_{T_{i-1}+s}([0,1))\right). \tag{4.42}$$

This iterative procedure may be continued until $T_i = T$ naturally working with t_{00} replaced by $T - T_i$ if the latter is smaller. Since there still may be several *large point masses* exiting the system in these last intervals, there may be several such i such that $T - T_{i-1} < t_{00}$.

4.2.3 The Semi-flow and Lagrangian Solutions

It is convenient to formally fix notation for an ODE semi-flow for a vector field. In the sequel we are only interested in the special case of the time-varying vector field

$v = \alpha(W(t))$ that is constant in space (for every fixed time t), defined originally on $[0,T] \times [0,1)$, but naturally extended to $[0,T] \times [0,+\infty)$. As we will see, the vector field will be integrable and bounded. Thus we dispense stating the definition for more general regularity hypotheses.

Definition 4.2.1 (Semi-flow). *Suppose $v\colon [0,T] \times [0,\infty) \mapsto [0,1)$ is integrable with respect to the first variable, and constant with respect to the second variable. A map $X : \{(t,r)\colon 0 \le r \le t \ \le T\} \times [0,\infty) \to \mathbb{R}^+$ is called the semi-flow of the time-varying vector field v if it satisfies for all $r \in [0,T]$ and all $x_0 \in [0,\infty)$*

$$\dot{X}(t;r,x_0) = v(t,0) \ \textit{ for almost every } \ t \in [r,T], \quad \textit{and} \tag{4.43a}$$

$$X(r;r,x_0) = x_0 \tag{4.43b}$$

with $\dot{X}$ denoting the derivative of X with respect to the first variable t.

Note that in this special case the semi-flow of a vector field satisfying the stated hypotheses is clearly unique (since v is Lipschitz).

Definition 4.2.2 (Lagrangian Solution). *Suppose $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$ are fixed Borel measures. We say a function $\Phi : [0,T] \to \mathcal{M}^+([0,1))$ is a Lagrangian solution of the system* (4.4) *if for every $t \in [0,T]$ and every Borel set $E \subset [0,1)$*

$$\Phi_t(\rho_0,\mu)(E) = \int_{[0,1)} \chi_E(X(t;0,x_0))\, d\rho_0(x_0) + \int_{(0,t]} \chi_E(X(t;s,0))\, d\mu(s). \tag{4.44}$$

where the map $X : \{(t,r)\colon 0 \le r \le t \ \le T\} \times [0,1] \to \mathbb{R}^+$ is the semi-flow of the vector field $v(t,0) = \alpha(W(t))$ with $W(t) = \rho_t([0,1))$ and χ_E is the indicator function of set E.

Remark. *From definition* (4.2.2)*, the Lagrangian solution Φ of the system* (4.4) *can also be interpreted as the following:*

Given arbitrary but fixed Borel measures $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$, for every $t \in [0,T]$,

$$\Phi_t(\rho_0, \mu) = X(t;0,\cdot)\#\rho_0 + X(t,\cdot,0)\#\mu.$$

In addition, the procedure in subsection 4.2.2 yields a semi-flow $X\colon \{(r,t)\colon 0 \leq r \leq t \leq T\} \times [0,\infty) \mapsto [0,\infty)$ with the semi-group property, and a curve of positive measures $t \mapsto \rho_t \in \mathcal{M}^+([0,1))$ which satisfies for almost all $0 \leq r \leq t \leq T$ and all $x \in [0,\infty)$

$$\begin{aligned}
\frac{d}{dt}X(t,r,x) =& \frac{d}{dt}(x + \xi(t) - \xi(r)) = \dot{\xi}(t) = \alpha(\rho_t([0,1)) \\
=& \alpha(\mu\left(\{r \in (0,t]\colon X(t;r,0) \in [0,1)\}\right) \\
&+ \rho_0\left(\{x \in [0,1)\colon X(t;0,x) \in [0,1)\}\right)) \\
=& \alpha(\mu\left(\{r \in (0,t]\colon \ (\xi(t) - \xi(r)) \in [0,1)\}\right) \\
&+ \rho_0\left(\{x \in [0,1)\colon \ (x + \xi(t)) \in [0,1)\}\right)) \\
=& \alpha(\mu(\max\{0, \xi^{-1}(\xi(t) - 1)\}, t] + \rho_0([0, 1 - \xi(t)))).
\end{aligned} \tag{4.45}$$

In particular, by construction and Definition 4.2.2, the curve $t \mapsto \rho_t \in \mathcal{M}^+([0,1))$ is a Lagrangian solution of the system (4.4). Furthermore, the existence and uniqueness of the characteristic ξ implies that there is a unique semi-flow X that satisfies equation (4.45). Thus for fixed Borel measures $\rho_0 \in \mathcal{M}^+([0,1))$ and $\mu \in \mathcal{M}^+((0,T])$, $\rho_t = \Phi_t(\rho_0, \mu)$ is the unique Lagrangian solution of the system (4.4).

Now we will study the semi-group property of the Lagrangian solution of the system (4.4) as defined in Definition 4.2.2. For convenience, we temporarily change the notation of the Lagrangian solution of the system (4.4) as for $0 \leq r \leq t \leq T$, for arbitrary but fixed $\rho_r \in \mathcal{M}^+([0,1))$ and $\mu_r \in \mathcal{M}^+((r,T])$ (here we fix the in-flux by taking $\mu_0 = \mu$ and $\mu_r = \mu_0 \restriction_{(r,T]}$), and every Borel set $E \subset [0,1)$,

$$\begin{aligned} \rho_t(E) &= \Phi(t; r, \rho_r)(E) \\ &= \int_{[0,1)} \chi_E(X(t; r, x_0))\, d\rho_r(x_0) + \int_{(r,t]} \chi_E(X(t; \tau, 0))\, d\mu_r(\tau) \\ &= (X(t; r, \cdot)\#\rho_r)(E) + (X(t; \cdot, 0)\#\mu_r)(E). \end{aligned} \tag{4.46}$$

In particular, if $r = 0$, we have $\rho_t(E) = \Phi(t; 0, \rho_0)(E) = \Phi_t(\rho_0, \mu)(E)$.

Lemma 4.2.6. *For $0 \le r \le s \le t \le T$, the Lagrangian solution of the system* (4.4) *satisfies $\Phi(t; s, \Phi(s; r, \rho_r)) = \Phi(t; r, \rho_r)$.*

Proof. The proof uses by the notation of push-forward.

By equation (4.46),

$$\Phi(s, r, \rho_r) = X(s; r, \cdot)\#\rho_r + X(s; \cdot, 0)\#\mu_r.$$

Thus,

$$\begin{aligned} \Phi(t; s, \Phi(s; r, \rho_r)) =& X(t; s, \cdot)\#(X(s; r, \cdot)\#\rho_r + X(s; \cdot, 0)\#\mu_r) + X(t; \cdot, 0)\#\mu_s \\ =& X(t; s, \cdot)\#(X(s; r, \cdot)\#\rho_r) \\ &+ X(t; s, \cdot)\#(X(s; \cdot, 0)\#\mu_r) + X(t; \cdot, 0)\#\mu_s. \end{aligned}$$

Let $\Phi_{1,t} = X(t; s, \cdot)\#(X(s; r, \cdot)\#\rho_r)$, then for every Borel set $E \subset [0,1)$,

$$\begin{aligned} \Phi_{1,t}(E) &= X(t; s, \cdot)\#(X(s; r, \cdot)\#\rho_r)(E) \\ &= (X(s; r, \cdot)\#\rho_r)\left(\{x \in [0,1) \colon X(t; s, x) \in E\}\right) \\ &= \rho_r\left(\{x \in [0,1) \colon X(t; s, X(s; r, x)) \in E\}\right) \\ &= \rho_r\left(\{x \in [0,1) \colon X(t; r, x) \in E\}\right) \\ &= (X(t; r, \cdot)\#\rho_r)(E). \end{aligned}$$

The above second to last equality uses the semi-group property of the semi-flow X. Therefore, $\Phi_{1,t} = X(t; r, \cdot)\#\rho_r$.

Let $\Phi_{2,t} = X(t; s, \cdot)\#(X(s; \cdot, 0)\#\mu_r) + X(t; \cdot, 0)\#\mu_s$, then for every Borel set $E \subset [0, 1)$,

$$
\begin{aligned}
\Phi_{2,t}(E) =& X(t; s, \cdot)\#(X(s; \cdot, 0)\#\mu_r)(E) + X(t; \cdot, 0)\#\mu_s(E) \\
=& (X(s; \cdot, 0)\#\mu_r) (\{x \in [0, 1) \colon X(t; s, x) \in E\}) + X(t; \cdot, 0)\#\mu_s(E) \\
=& \mu_r (\{\tau \in (r, s] \colon X(t; s, X(s; \tau, 0)) \in E\}) + X(t; \cdot, 0)\#\mu_s(E) \\
=& \mu_r (\{\tau \in (r, s] \colon X(t; \tau, 0) \in E\}) \\
& + \mu_s (\{\tau \in (s, t] \colon X(t; \tau, 0) \in E\}) \\
=& \int_{(r,s]} \chi_E(X(t; \tau, 0)) d\mu(\tau) + \int_{(s,t]} \chi_E(X(t; \tau, 0)) d\mu(\tau) \\
=& \int_{(r,t]} \chi_E(X(t; \tau, 0)) d\mu(\tau) \\
=& (X(t; \cdot, 0)\#\mu_r)(E)
\end{aligned}
$$

The above fourth equality uses the semi-group property of the semi-flow X. Hence, $\Phi_{2,t} = X(t; \cdot, 0)\#\mu_r$.

Therefore, $\Phi(t; s, \Phi(s; r, \rho_r)) = X(t; r, \cdot)\#\rho_r + X(t; \cdot, 0)\#\mu_r = \Phi(t; r, \rho_r)$. □

4.3 Weak Measure-valued Solutions

4.3.1 Definition and Existence of Weak Measure-valued Solutions

This section defines a notion of weak solution to the hyperbolic conservation law (4.4), and proves that the Lagrangian solution is indeed a weak solution.

Let Ψ be the set of functions $\varphi \colon [0, T] \times [0, 1] \mapsto \mathbb{R}$ such that for every $t \in [0, T]$, $\varphi(t, \cdot)$ is differentiable and $\frac{\partial \varphi}{\partial x}$ is continuous (jointly in (t, x)) and for every $x \in [0, 1]$,

$\varphi(\cdot, x)$ is Lipschitz continuous. That is,

$$
\begin{aligned}
\Psi = \{\varphi\colon [0,T] \times [0,1] \mapsto \mathbb{R} \mid \ & \text{for every } t \in [0,T],\ \varphi(t,\cdot) \text{ is differentiable} \\
& \text{and } \partial_x \varphi \text{ is continuous (jointly in } (t,x)), \\
& \text{and for every } x \in [0,1], \varphi(\cdot, x) \text{ is Lipschitz continuous } \}.
\end{aligned}
$$

Definition 4.3.1 (Measure-Valued Weak Solution). *A measure-valued weak solution to equation* (4.4) *with the initial condition* $\rho_0 \in \mathcal{M}^+([0,1))$ *and the boundary condition* $\mu \in \mathcal{M}^+((0,T])$ *is a function* $\rho : [0,T] \to \mathcal{M}^+([0,1))$*, such that* $W\colon [0,T] \mapsto \rho_t([0,1))$ *is integrable and such that for every* $\tau \in [0,T]$ *and for every* $\varphi \in \Psi$ *that satisfies*

$$\varphi(t,1) = 0, \ \text{for all } t \in [0,\tau], \tag{4.47}$$

one has

$$
\begin{aligned}
&\int_{(0,\tau]} \int_{[0,1)} \left(\partial_t \varphi(t,x) + \alpha(W(t)) \partial_x \varphi(t,x)\right) \, d\rho_t(x)\, dt + \int_{(0,\tau]} \varphi(t,0)\, d\mu(t) \\
&- \int_{[0,1)} \varphi(\tau,x)\, d\rho_\tau(x) + \int_{[0,1)} \varphi(0,x)\, d\rho_0(x) = 0.
\end{aligned} \tag{4.48}
$$

Lemma 4.3.1. *Let* $\rho_0 \in \mathcal{M}^+([0,1))$ *and* $\mu \in \mathcal{M}^+((0,T])$ *be arbitrary but fixed. Let* $X : \{(t,r)\colon 0 \le r \le t \ \le T\} \times [0,1] \to \mathbb{R}^+$ *be the semi-flow of the vector field* $v = \alpha(W(t))$ *with* $W(t) = \Phi_t(\rho_0,\mu)([0,1))$*. Then for almost every* $t \in [0,T]$ *and every* $x_0 \in [0,1)$*, every test function* φ *in* Ψ *satisfies*

$$\frac{d\varphi(t, X(t;0,x_0))}{dt} = \partial_t \varphi(t, X(t;0,x_0)) + \alpha(W(t)) \partial_x \varphi(t, X(t;0,x_0)).$$

Proof. Fix a test function $\varphi \in \Psi$. We will show that for almost every $t \in [0,T]$, every $x_0 \in [0,1)$, arbitrary but fixed $\varepsilon > 0$, there exists $\delta > 0$, such that, if $|\Delta t| < \delta$, then

$$
\begin{aligned}
&\left| \frac{\varphi(t+\Delta t, X(t+\Delta t;0,x_0)) - \varphi(t, X(t;0,x_0))}{\Delta t} \right. \\
&\left. - \partial_t \varphi(t, X(t;0,x_0)) - \alpha(W(t)) \partial_x \varphi(t, X(t;0,x_0)) \right| < \varepsilon.
\end{aligned}
$$

Since for every $x \in [0,1]$, $\varphi(\cdot, x)$ is Lipschitz continuous and thus differentiable almost everywhere and the map $\xi\colon [0,T] \mapsto [0,\infty)$ is also differentiable almost everywhere, we can choose $(t, x_0) \in [0,T] \times [0,1]$ be arbitrary but fixed such that both $\partial_t \varphi(t, X(t;0,x_0))$ and $\partial_t X(t;0,x_0) = \dot{\xi}(t)$ exist. Let $\varepsilon > 0$ be arbitrary but fixed.

Since $\partial_t \varphi(t, X(t;0,x_0))$ exists, there exists $\delta_1 > 0$, such that, if $|\Delta t| < \delta_1$, then

$$\left| \frac{\varphi(t+\Delta t, X(t;0,x_0)) - \varphi(t; X(t;0,x_0))}{\Delta t} - \partial_t \varphi(t, X(t;0,x_0)) \right| < \frac{\varepsilon}{4}. \tag{4.49}$$

Since for every $t + \Delta t \in [0,T]$, the map $\varphi(t+\Delta t, \cdot)\colon [0,1] \mapsto \mathbb{R}$ is differentiable, there exists $\delta_2 > 0$, such that if $|\Delta x| < \delta_2$, then

$$\left| \frac{\varphi(t+\Delta t, X(t;0,x_0)+\Delta x) - \varphi(t+\Delta t, X(t;0,x_0))}{\Delta x} - \partial_x \varphi(t+\Delta t, X(t,0,x_0)) \right| < \frac{\varepsilon}{4}. \tag{4.50}$$

Note that if $|\Delta t| < \delta_2$, then $|X(t+\Delta t;0,x_0) - X(t;0,x_0)| \leq |\Delta t| < \delta_2$. Let $\Delta x = X(t+\Delta t;0,x_0) - X(t;0,x_0)$. From equation (4.50), we obtain

$$\left| \frac{\varphi(t+\Delta t, X(t+\Delta t;0,x_0)) - \varphi(t+\Delta t, X(t;0,x_0))}{X(t+\Delta t;0,x_0) - X(t;0,x_0)} - \partial_x \varphi(t+\Delta t, X(t,0,x_0)) \right| < \frac{\varepsilon}{4}. \tag{4.51}$$

In addition, since $\partial_x \varphi$ is continuous on $[0,T] \times [0,1]$ and hence bounded, from equation (4.51), we have, there exists some $U > 0$ such that

$$\left| \frac{\varphi(t+\Delta t, X(t+\Delta t;0,x_0)) - \varphi(t+\Delta t, X(t;0,x_0))}{X(t+\Delta t;0,x_0) - X(t;0,x_0)} \right| < U. \tag{4.52}$$

Since $\partial_t X(t;0,x_0) = \dot{\xi}(t)$ exists at t, there exists $\delta_3 > 0$, such that, if $|\Delta t| < \delta_3$, then

$$\left| \frac{X(t+\Delta t;0,x_0) - X(t;0,x_0)}{\Delta t} - \alpha(W(t)) \right| < \frac{\varepsilon}{4U}. \tag{4.53}$$

Since $\partial_x \varphi$ is continuous (jointly in (t,x)), there exists $\delta_4 > 0$, such that, if $|\Delta t| < \delta_4$, then

$$|\partial_x \varphi(t+\Delta t, X(t;0,x_0)) - \partial_x \varphi(t, X(t;0,x_0))| < \frac{\varepsilon}{4}. \tag{4.54}$$

Choose Δt such that $|\Delta t| < \min\{\delta_i \colon i = 1,2,3,4\}$. Then from equations (4.49), (4.51), (4.52), (4.53) and (4.54), and $\alpha(W(t)) \in [0,1]$, we obtain

$$
\begin{aligned}
&\left|\frac{\varphi(t+\Delta t, X(t+\Delta t;0,x_0)) - \varphi(t, X(t;0,x_0))}{\Delta t}\right.\\
&\left.-\partial_t\varphi(t, X(t;0,x_0)) - \alpha(W(t))\partial_x\varphi(t, X(t;0,x_0))\right|\\
\leq &\left|\frac{\varphi(t+\Delta t, X(t;0,x_0)) - \varphi(t; X(t;0,x_0))}{\Delta t} - \partial_t\varphi(t, X(t;0,x_0))\right|\\
&+\left|\left(\frac{\varphi(t+\Delta t, X(t+\Delta t;0,x_0)) - \varphi(t+\Delta t, X(t;0,x_0))}{X(t+\Delta t;0,x_0) - X(t;0,x_0)}\right)\alpha(W(t))\right.\\
&\left.-\ \partial_x\varphi(t+\Delta t, X(t,0,x_0))\alpha(W(t))\right|\\
&+\left|\frac{\varphi(t+\Delta t, X(t+\Delta t;0,x_0)) - \varphi(t+\Delta t, X(t;0,x_0))}{X(t+\Delta t;0,x_0) - X(t;0,x_0)}\right|\\
&\quad\left|\frac{X(t+\Delta t;0,x_0) - X(t;0,x_0)}{\Delta t} - \alpha(W(t))\right|\\
&+\left|\left(\partial_x\varphi(t+\Delta t, X(t;0,x_0)) - \partial_x\varphi(t, X(t;0,x_0))\right)\alpha(W(t))\right|\\
<&\varepsilon.
\end{aligned}
$$

By the definition of differentiability, we have for almost all $t \in [0,T]$,

$$
\frac{d\varphi(t, X(t;0,x_0))}{dt} = \partial_t\varphi(t, X(t;0,x_0)) + \alpha(W(t))\partial_x\varphi(t, X(t;0,x_0)).
$$

□

The following theorem guarantees the existence of weak measure-valued solutions.

Theorem 4.3.2. *Every Lagrangian solution of* (4.44) *is a measure-valued weak solution that satisfies* (4.48)*.*

Proof. We conveniently extend the functions φ, $\partial_t\varphi$ and $\partial_x\varphi$ to $[0,\infty)$ with value zero for $x > 1$.

Suppose Φ is a Lagrangian solution that satisfies (4.44). Evaluate the left hand side of equation (4.48) at $\rho = \Phi$,

$$\begin{aligned}
&\int_{(0,\tau]}\int_{[0,1)} (\partial_t\varphi(t,x) + \alpha(W(t))\partial_x\varphi(t,x))\, d\Phi_t(\rho_0,\mu)(x)\, dt \\
&= \int_{(0,\tau]}\int_{[0,1)} (\partial_t\varphi(t,X(t;0,x_0)) + \alpha(W(t))\partial_x\varphi(t,X(t;0,x_0)))\ d\rho_0(x_0)\, dt \\
&+ \int_{(0,\tau]}\int_{(0,t]} (\partial_t\varphi(t,X(t;s,0)) + \alpha(W(t))\partial_x\varphi(t,X(t;s,0)))\ d\mu(s)\, dt
\end{aligned} \tag{4.55}$$

Note

$$\begin{aligned}
&\int_{(0,\tau]}\int_{[0,1)} (\partial_t\varphi(t,X(t;0,x_0)) + \alpha(W(t))\partial_x\varphi(t,X(t;0,x_0)))\ d\rho_0(x_0)\, dt \\
&= \int_{(0,\tau]}\int_{[0,1)} \frac{d\varphi(t,X(t;0,x_0))}{dt}\, d\rho_0(x_0)\, dt \\
&= \int_{[0,1)}\int_{(0,\tau]} \frac{d\varphi(t,X(t;0,x_0))}{dt}\, dt\, d\rho_0(x_0) \\
&= \int_{[0,1)} (\varphi(\tau,X(\tau;0,x_0)) - \varphi(0,X(0;0,x_0)))\ d\rho_0(x_0) \\
&= \int_{[0,1)} \varphi(\tau,X(\tau;0,x_0))\, d\rho_0(x_0) - \int_{[0,1)} \varphi(0,X(0;0,x_0))\, d\rho_0(x_0) \\
&= \int_{[0,1)} \varphi(\tau,X(\tau;0,x_0))\, d\rho_0(x_0) - \int_{[0,1)} \varphi(0,x)\, d\rho_0(x).
\end{aligned} \tag{4.56}$$

And,

$$\begin{aligned}
&\int_{(0,\tau]}\int_{(0,t]} (\partial_t\varphi(t,X(t;s,0)) + \alpha(W(t))\partial_x\varphi(t,X(t;s,0)))\ d\mu(s)\, dt \\
&= \int_{(0,\tau]}\int_{(0,t]} \frac{d\varphi(t,X(t;s,0))}{dt}\, d\mu(s)\, dt \\
&= \int_{(0,\tau]}\int_{(s,\tau]} \frac{d\varphi(t,X(t;s,0))}{dt}\, dt\, d\mu(s) \\
&= \int_{(0,\tau]} (\varphi(\tau,X(\tau;s,0)) - \varphi(s,X(s;s,0)))\ d\mu(s) \\
&= \int_{(0,\tau]} \varphi(\tau,X(\tau;s,0))\, d\mu(s) - \int_{(0,\tau]} \varphi(t,0)\, d\mu(t).
\end{aligned} \tag{4.57}$$

In addition,

$$\int_{[0,1)} \varphi(\tau,x)\, d\Phi_\tau(\rho_0,\mu)(x)$$
$$= \int_{[0,1)} \varphi(\tau, X(\tau;0,x_0))\, d\rho_0(x_0) + \int_{[0,1)} \varphi(\tau, X(\tau;s,0))\, d\mu(s). \quad (4.58)$$

Thus, every Lagrangian solution (4.44) is a weak measure-valued solution. □

4.3.2 Uniqueness of the Weak Measure-valued Solution

Now we will show that every weak measure-valued solution is also a Lagrangian solution to the hyperbolic conservation law (4.4). From theorem (4.2.4) in section (4.2), we could obtain the uniqueness of the weak measure-valued solution.

Theorem 4.3.3. *Given the initial condition $\rho_0 \in \mathcal{M}^+([0,1))$ and the boundary condition $\mu \in \mathcal{M}^+((0,T])$, the weak measure-valued solution* (4.48) *to equation* (4.4) *is unique.*

Proof. We will show the uniqueness of the weak measure-valued solution $\hat{\rho}$ over the small time interval $[0,\tau_0]$ first, with τ_0 defined as in (4.13). By the definition of the weak measure-valued solution, for arbitrary but fixed $\tau \in (0,\tau_0]$ and every $\varphi \in \Psi$ such that $\varphi(t,1) = 0$, and for every $t \in [0,\tau]$,

$$\int_{(0,\tau]} \int_{[0,1)} \Big(\varphi_t(t,x) + \alpha(\hat{W}(t))\varphi_x(t,x)\Big)\, d\hat{\rho}_t(x)dt + \int_{(0,\tau]} \varphi(t,0)\, d\mu(t)$$
$$- \int_{[0,1)} \varphi(\tau,x)\, d\hat{\rho}_\tau(x) + \int_{[0,1)} \varphi(0,x)\, d\rho_0(x) = 0, \quad (4.59)$$

where $\hat{W}(t) = \hat{\rho}_t([0,1))$ is integrable.

Consider a C^1 function with compact support in $(0,1)$, φ_0, i.e., $\varphi_0 \in C_0^1(0,1)$ and let $\hat{\xi}(t) \colon = \int_0^t \alpha(\hat{W}(s))\, ds$, $t \in [0,\tau]$. Choose the test function

$$\varphi(t,x) = \begin{cases} \varphi_0(\hat{\xi}(\tau) - \hat{\xi}(t) + x), & \text{if } 0 \le x \le \hat{\xi}(t) - \hat{\xi}(\tau) + 1, 0 \le t \le \tau, \\ 0, & \text{if } 0 \le \hat{\xi}(t) - \hat{\xi}(\tau) + 1 \le x \le 1, 0 \le t \le \tau. \end{cases}$$

Note that the test function $\varphi \in \Psi$ and satisfies the following Cauchy problem

$$\begin{cases} \partial_t \varphi + \alpha(\hat{W}(t))\partial_x \varphi = 0, & \text{if } 0 \le t \le \tau, 0 \le x \le 1, \\ \varphi(\tau, x) = \varphi_0(x), & \text{if } 0 \le x \le 1 \\ \varphi(t, 1) = 0, & \text{if } 0 \le t \le \tau. \end{cases}$$

From equation (4.59), we obtain

$$\begin{aligned} &\int_{[0,1)} \varphi_0(x)\, d\hat{\rho}_\tau(x) \\ &= \int_{(0,\tau]} \varphi_0(\hat{\xi}(\tau) - \hat{\xi}(t))\, d\mu(t) + \int_{[0,1-\hat{\xi}(\tau))} \varphi_0(\hat{\xi}(\tau) + x)\, d\rho_0(x). \end{aligned}$$

Since $\varphi_0 \in C_0^1(0,1)$ and $\tau \in [0, \tau_0]$ were arbitrary, we have, for every Borel set $E \subset [0,1)$, and every $t \in [0, \tau_0]$,

$$\hat{\rho}_t(E) = \int_{(0,t]} \chi_E(\hat{\xi}(t) - \hat{\xi}(s))\, d\mu(s) + \int_{[0,1)} \chi_E(\hat{\xi}(t) + x)\, d\rho_0(x).$$

Therefore,

$$\begin{aligned} \hat{W}(t) &= \hat{\rho}_t([0,1)) \\ &= \int_{(0,t]} \chi_{[0,1)}(\hat{\xi}(t) - \hat{\xi}(s))\, d\mu(s) + \int_{[0,1)} \chi_{[0,1)}(\hat{\xi}(t) + x)\, d\rho_0(x) \\ &= \mu((0,t]) + \rho_0([0, 1 - \hat{\xi}(t)). \end{aligned}$$

Furthermore,

$$\begin{aligned} \hat{\xi}(t) &= \int_0^t \alpha(\hat{W}(s))\, ds \\ &= \int_0^t \alpha(\mu((0,s]) + \rho_0([0, 1 - \hat{\xi}(s)))\, ds \\ &= F(\hat{\xi})(t). \end{aligned}$$

It is easy to check that $\hat{\xi} \in \Omega$. Since ξ is the unique fixed point of function F in Ω, $\hat{\xi} = \xi$. Thus $\hat{\rho}_t = \rho_t$ over the time interval $[0, \tau_0]$ which implies the uniqueness of the weak measure-valued solution to equation (4.4) over the small time interval $[0, \tau_0]$.

Similar to the discussion in subsection 4.2.2, one can obtain the uniqueness of the weak measure-valued solution to equation (4.4) defined on all of $[0, T]$ in a finite number of iterations. □

4.3.3 Regularity of the Weak Measure-valued Solution

For the special case of finite signed Borel measures on the interval $[0, 1)$ briefly recall the definition of the flat norm. Denote by $\mathcal{F}$ the set of nonnegative Lipschitz continuous function with Lipschitz constant 1 that are bounded above by 1 by

$$\mathcal{F} = \{f : [0,1) \mapsto [0,1] : \text{ for all } x, y \in [0,1), |f(x) - f(y)| \leq |x - y|\}. \tag{4.60}$$

On the space $\mathcal{M}([0, 1))$ of signed measures, define for $\nu \in \mathcal{M}([0, 1))$ its flat norm

$$\|\nu\|_\flat = \sup_{f \in \mathcal{F}} \left| \int_{[0,1)} f \, d\nu \right|. \tag{4.61}$$

For applications and a careful discussions of properties of the flat norm in general settings see, e.g..

The following simple example shows that in general the solution $t \mapsto \rho_t$ need not be continuous under the flat norm $\| \cdot \|_\flat$.

Example. *Consider the case of trivial in-flux $\mu = 0$ and initial datum $\rho_0 = \delta_0$ consisting of a single unit mass at $x = 0$. Then for every $t < 2$, the solution $\rho_t = \delta_{t/2}$ consists of a single mass at $x = \frac{1}{2}t$, whereas for all $t \geq 2$, the solution $\rho_t = 0$ is the trivial measure. Using the function $f \equiv 1$ it is easily seen that for every $t < 2$ we have $\|\rho_t - \rho_2\|_\flat = \|\delta_{t/2}\|_\flat = 1$, and hence $t \mapsto \rho_t$ is not continuous at $t = 2$.*

In order to discount the importance of the exact time at which point masses enter the system or exit from it, we introduce a variation of the flat norm. First, denote by $g : \mathbb{R} \mapsto \mathbb{R}$, the piecewise linear *hat*-function defined by

$$h(x) = \begin{cases} \frac{1}{2} - |\frac{1}{2} - x| & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{4.62}$$

Define the map $\phi : \mathcal{M}([0,1)) \mapsto [0,\infty)$ by

$$\phi(\nu) = \sup_{f \in \mathcal{F}} \left| \int_{[0,1)} fh \, d\nu \right|. \tag{4.63}$$

Note: If f and h are both nonnegative Lipschitz continuous functions with Lipschitz constant 1 and both are bounded above by 1 then their product fh is Lipschitz with Lipschitz constant 2 and is also bounded by 1. It is immediate to see that:

Lemma 4.3.4. *The map ϕ defines a semi-norm on the space $\mathcal{M}([0,1))$.*

Example. *Continuing the example 4.3.3, it is easy to calculate that for all $t < 2$ one has $\phi(\rho_t - \rho_2) = \frac{1}{2} - \left|\frac{1}{2} - \frac{t}{2}\right|$ while for all $t \geq 2$ one has $\phi(\rho_t - \rho_2) = 0$, and for these special data the solution $t \mapsto \rho_t$ is a continuous curve in $\mathcal{M}([0,1))$ when endowed with the semi-norm ϕ.*

Theorem 4.3.5. *For every fixed $\mu \in \mathcal{M}^+((0,T])$ and $\rho_0 \in \mathcal{M}^+([0,1))$, the unique solution $\rho\colon [0,T] \mapsto \mathcal{M}^+([0,1)))$ of system* (4.4)*, and thus also of* (4.48)*, is continuous under the semi-norm ϕ.*

Proof. Let $T > 0$, $\mu \in \mathcal{M}^+((0,T])$, $\rho_0 \in \mathcal{M}^+([0,1))$ be arbitrary but fixed and let $\rho\colon [0,T] \mapsto \mathcal{M}^+([0,1)))$ be the unique solution of system (4.4), and thus also of (4.48), and let X be the associated semi-flow. Without loss of generality consider times $0 \leq t_2 < t_1 \leq T = 1$. (For times larger than 1, the continuity follows from the semi-flow property of $t \mapsto \rho_t$, via composition of continuous functions.)

By the choice of the time $T \leq 1$, there exists locations $0 \leq x_1 < x_2 < 1$ such that $X(t_1; 0, x_1) = X(t_2; 0, x_2) = 1$. We will show that for every $\varepsilon > 0$, if $t_1 - t_2$ is sufficiently small, then $\phi(\rho_{t_1} - \rho_{t_2}) < \varepsilon$. In particular for any arbitrary fixed $f \in \mathcal{F}$ we will find an upper bound for $\left|\int_{[0,1)} fh \, d(\rho_{t_1} - \rho_{t_2})\right|$. For those *parts* that are in the factory at both times t_2 and t_1, a simple Lipschitz estimate will do the job. However, for parts that entered, or exited from the factory between these times, we will use that for all $x \in [0, 1)$, $h(x) \leq x$ and $h(x) \leq 1 - x$. The first step uses that ρ_t is constructed from the push forwards of the data ρ_0 and μ.

$$\left|\int_{[0,1)} f \, d(\rho_{t_1} - \rho_{t_2})\right| = \left|\int_{[0,1)} f(x)h(x) \, d\rho_{t_1}(x) - \int_{[0,1)} f(x)h(x) \, d\rho_{t_2}\right| \tag{4.64a}$$

$$= \left|\int_{[0,x_1)} (fh)(X(t_1; 0, x_0)) \, d\rho_0(x_0) - \int_{[0,x_2)} (fh)(X(t_2; 0, x_0)) \, d\rho_0(x_0) \right.$$
$$\left. + \int_{(0,t_1]} (fh)(X(t_1; s, 0)) \, d\mu(s) - \int_{(0,t_2]} (fh)(X(t_2; s, 0)) \, d\mu(s)\right| \tag{4.64b}$$

$$\leq \int_{[0,x_1)} |(fh)(X(t_1; 0, x_0)) - (fh)(X(t_2; 0, x_0))| \, d\rho_0(x_0)$$
$$+ \int_{[x_1,x_2)} (fh)(X(t_2; 0, x_0)) \, d\rho_0(x_0)$$
$$+ \int_{(0,t_2]} |(fh)(X(t_1; s, 0)) - (fh)(X(t_2; s, 0))| \, d\mu(s)$$
$$+ \int_{(t_2,t_1]} (fh)(X(t_2; s, 0)) \, d\mu(s) \tag{4.64c}$$

$$\leq 2\int_{[0,x_1)} |X(t_1; 0, x_0) - X(t_2; 0, x_0)| \, d\rho_0(x_0) + \int_{[x_1,x_2)} h(X(t_1; 0, x_0)) \, d\rho_0(x_0)$$
$$+ 2\int_{(0,t_2]} |X(t_1; s, 0) - X(t_2; s, 0)| \, d\mu(s) + \int_{(t_2,t_1]} h(X(t_2; s, 0)) \, d\mu(s) \tag{4.64d}$$

In the last step, for the first and third integral in equation (4.64d) use the Lipschitz constant 2 for fh, whereas for the other two use that f is bounded above by 1. For the first and third integral in equation (4.64d) use that the semi-flow X is Lipschitz continuous (for fixed second and third variables) with Lipschitz constant 1, and hence the integrals are bounded above $(t_1 - t_2) \cdot \rho([0, x))$ and $(t_1 - t_2) \cdot \mu((0, T])$, respectively. For the second integral in equation (4.64d) note that for every $x \in [x_1, x_2)$, $X(t_2; 0, x) \geq 1-(t_1-t_2)$ and hence the integral is bounded above by $(t_1-t_2)\cdot\rho([0,x))$. For the fourth integral note that for every $s \in [t_2, t_1]$, $X(t_1; s, 0) < t_1 - t_2$n and hence the integral is bounded above by $(t_1 - t_2)\cdot\mu((0,T])$.

Thus given any $\varepsilon > 0$, choose $\delta = \varepsilon/(2\rho([0,1)) + 2\mu((0,T])$. Then for all $0 \leq t_2 \leq t_1 < 1$, if $t_1 - t_2 < \delta$, and for every $f \in \mathcal{F}$, $\left|\int_{[0,1)} fh\, d(\rho_{t_1} - \rho_{t_2})\right| < \varepsilon$ and hence $\phi(\rho_{t_1} - \rho_{t_2}) \leq \varepsilon$. □

Thus we have continuity of the solution $t \mapsto \rho_t$ using the semi-norm ϕ. The ultimately desirable joint continuity of the semi-flow with respect to time, the in-flux, and the initial conditions appears elusive. However, we present in theorem 4.3.8 that appears close to the continuity result of the semi-flow on initial conditions. It requires a slightly different semi-norm as illustrated in the following two examples.

Example. *Consider the case of* $\alpha(W) = \frac{1}{1+W}$, *trivial in-flux* $\mu = 0$ *and trivial initial datum* $\tilde{\rho}_0 = 0$, $0 < T \leq 1$, $\varepsilon = \frac{1}{2T}$ *and* $0 < \delta < 1$ *arbitrary but fixed. Set* $x_0 = \frac{\delta}{2}$ *and let* $\rho_0 \in \mathcal{M}^+([0,1))$ *be the measure consisting of the unit point mass at* x_0*. Then* $\phi(\rho_0 - \tilde{\rho}_0) = x_0 < \delta$, *yet the respective solutions at time* T *are* $\tilde{\rho}_T = 0$ *and* ρ_T *consisting of a unit point mass at* $(x_0 + \frac{1}{2}T)$ *and hence* $\phi(\rho_T - \tilde{\rho}_T) = x_0 + \frac{1}{2}T \not< \frac{1}{2}T = \varepsilon$.

As established in the example above, the semi-norm ϕ used (and needed) to establish continuity of the solution ρ_t with respect to time, will not provide continuity with respect to initial conditions using the semi-norm ϕ. However, using a similar semi-norm that only discounts variations close to the exit point $x = 1$ appears better suited. In analogy with (4.62) define $g\colon \mathbb{R} \mapsto \mathbb{R}$ by $g(x) = 1 - x$ and correspondingly to (4.63) define the variant of the flat norm ψ on $\mathcal{M}([0,1))$ by

$$\psi(\nu) = \sup_{f\in\mathcal{F}} \left| \int_{[0,1)} fg\, d\nu \right|. \tag{4.65}$$

Now, we recall the following lemma (Proposition 2.2 from Gwiazda *et al.* (2018)).

Lemma 4.3.6. *The indicator function of every closed (open) set in* $[0,1)$ *is the pointwise limit of a decreasing (respectively increasing) sequence of bounded Lipschitz continuous functions* (f_n)*, where each* f_n *has Lipschitz constant* n *and takes values between* 0 *and* 1*.*

Theorem 4.3.7. *The variant of the flat norm defined in equation* (4.65) *is a norm.*

Proof. Clearly, equation (4.65) defines a semi-norm. It remains to be verified that for every $\nu \in \mathcal{M}^+([0,1))$, $\psi(\nu)$ implies $\nu = 0$.

Let $\nu \in \mathcal{M}^+([0,1))$ be arbitrary but fixed such that $\psi(\nu) = 0$. Then for every Borel set $T \subseteq [0,1)$ and for every function f that is Lipschitz continuous with Lipschitz constant 1 and with values between 0 and 1, $\int_T f g \, d\nu = 0$, where $T \subseteq [0,1)$ is a Borel set. In addition, for arbitrary but fixed $\varepsilon > 0$, there exists $\delta \in (0,1)$, such that $\nu((1-\delta,1)) < \varepsilon$. Since ν is regular, for every Borel set $A \subseteq [0,1)$,

$$\nu(A) = \inf \left\{ \nu(G) \colon A \subseteq G \subseteq [0,1), G \text{ is an open set } \right\}.$$

Now fix Borel set $A \subseteq [0,1)$ and let $G \supseteq A$ be an open set in $[0,1)$. Then

$$\nu(G) = \nu(G \cap [0,1-\delta]) + \nu(G \cap (1-\delta,1)) < \nu(G \cap [0,1-\delta]) + \varepsilon.$$

By lemma 4.3.6, there exists an increasing sequence of bounded Lipschitz continuous functions (f_n) such that $f_n \to \chi_G$ pointwisely. Let $f_n^* = \frac{1}{g} f_n$. Since the function $\frac{1}{g}$ is continuously differentiable and bounded above by $\frac{1}{\delta}$ over the interval $[0,1-\delta]$, f_n^* is bounded above by $\frac{1}{\delta}$ and Lipschitz continuous with Lipschitz constant $n \cdot \frac{1}{\delta} + 1 \cdot \frac{1}{\delta^2}$ over the interval $[0,1-\delta]$. Note that $\frac{1}{\frac{1}{\delta}\left(n+\frac{1}{\delta}\right)} f_n^*$ is Lipschitz continuous with Lipschitz constant 1 and with values between 0 and 1 on $[0,1-\delta]$. Thus,

$$\int_{[0,1-\delta]} f_n^* g \, d\nu = 0.$$

By the monotone convergence theorem,

$$\int_{[0,1-\delta]} \chi_G \, d\nu = \lim_{n\to\infty} \int_{[0,1-\delta]} f_n \, d\nu = \lim_{n\to\infty} \int_{[0,1-\delta]} f_n^* g \, d\nu.$$

Thus,

$$\nu(G \cap [0, 1-\delta]) = \int_{[0,1-\delta]} \chi_G \, d\nu = 0.$$

Therefore,

$$\nu(G) < \varepsilon$$

which implies that $\nu(A) = 0$ and thus $\nu = 0$. □

The following example illustrates how this norm ψ avoids the problems of the semi-norm ϕ with regards to continuity with respect to initial conditions.

Example. *Consider the case of* $\alpha(W) = \frac{1}{1+W}$*, trivial in-flux* $\mu = 0$ *and initial data* ρ_{10} *and* ρ_{20} *consisting of point masses of sizes* $M \geq m \geq 1$ *located at* $0 \leq a \leq b < \frac{1}{2}$*, respectively. Then*

$$\psi(\rho_{20} - \rho_{10}) = M(1-a) - m(1-b) + m(1-b)(b-a) \tag{4.66}$$

which may be rewritten as

$$\psi(\rho_{20} - \rho_{10}) = (M-m)(1-a) + m(b-a)(2-b) \tag{4.67}$$

Suppose that $\delta > \psi(\rho_{20} - \rho_{10})$*. Then, in particular,* $M - m < 2\delta$ *and* $b - a < \delta$*. At any small time* $0 \leq t \leq 1$ *(before either mass exists the system), the measures* ρ_{1t} *and* ρ_{2t} *are point masses of sizes* M *and* m *at the locations* $(a + \frac{t}{1+M}) \leq (b + \frac{t}{1+m})$*, respectively. It is easily seen that*

$$\psi(\rho_{2t} - \rho_{1t}) = M(1-(a+\frac{t}{1+M})) - m(1-(b+\frac{t}{1+m}))(1-((b+\frac{t}{1+m}) - (a+\frac{t}{1+M}))). \tag{4.68}$$

Evaluating this at $m = 1$, $M = m + x$, $b = a + y$ gives a simple rational expression in x, y, t, δ whose numerator vanishes at $x = y = 0$ (and denominator bounded away from zero). In particular if $2|x|, |y| < \delta$ then

$$\psi(\rho_{2t} - \rho_{1t}) \leq \frac{2\delta(2\delta^2 + (4a + 2t - 2)\delta + \frac{1}{2}t^2 + (a + 2)t + 8a - 12)}{8 + 4\delta}. \tag{4.69}$$

Thus it is clear that for every $t > 0$ and every $\varepsilon > 0$ it is possible to choose $\delta > 0$ such that if the initial data $M, m = 1, a, b$ as above satisfy $\psi(\rho_{20} - \rho_{10}) < \delta$, then $\psi(\rho_{2t} - \rho_{1t}) < \varepsilon$.

This example shows that replacing the semi-norm ϕ by the semi-norm ψ on $\mathcal{M}([0, 1))$ provides some hope for continuity with respect to initial conditions. This semi-norm preserves the features of ϕ by discounting the importance of the specific exit times of large masses, but it avoids the trouble presented in the preceding example.

We have not been able to show that, in general, the semi-flow $(t, \rho_0) \mapsto \rho_t$ is continuous with respect to the initial datum ρ_0 and the semi-norm ψ. However, we have the following result which is weaker than continuity.

Theorem 4.3.8. *For every $\mu \in \mathcal{M}^+((0, T])$ and $\rho_0 \in \mathcal{M}^+([0, 1))$, the unique weak solution $\rho\colon [0, T] \mapsto \mathcal{M}^+([0, 1)))$ of system (4.4) satisfies: For every initial condition $\tilde{\rho}_0 \in \mathcal{M}^+([0, 1))$ and every $\varepsilon > 0$, there exist $\delta > 0$ and $\tau > 0$ such that if $\phi(\tilde{\rho}_0 - \rho_0) < \delta$, then for all $t < \tau$, $\phi(\tilde{\rho}_t - \rho_t) < \varepsilon$.*

Proof. Consider the control input in-flux μ and two initial conditions $\rho_0^1, \rho_0^2 \in \mathcal{M}^+([0, 1))$. For $k = 1, 2$, denote by ρ_t^k, W_k and X_k the weak measure-valued solution, the total load and the semi-flow to the initial condition ρ_0^k respectively. For convenience, for

$t \in [0,T]$, let $\xi_k(t) = X_k(t;0,0)$ be the characteristic as in section (4.2.3). Thus

$$\begin{cases} \xi_k'(t) &= \alpha(W_k(t)) \text{ for a. e. } t \in [0,T], \\ \xi_k(0) &= 0. \end{cases} \tag{4.70}$$

Since the velocity α_k is positive and bounded above by 1, we have for every $t \in [0,T]$,

$$|\xi_1(t) - \xi_2(t)| \leq \int_0^t |\alpha(W_1(s)) - \alpha(W_2(s))| \, ds \leq t.$$

Furthermore, for every $x_0 \in [0,1)$,

$$|X_1(t;0,x_0) - X_2(t;0,x_0)| = |\xi_1(t) - \xi_2(t)| \leq t,$$

and for every $s \in [0,t]$,

$$\begin{aligned} |X_1(t;s,0) - X_2(t;s,0)| &= |\xi_1(t) - \xi_1(s) - (\xi_2(t) - \xi_2(s))| \\ &= |\xi_1(t) - \xi_2(t) + \xi_2(s) - \xi_1(s)| \\ &\leq |\xi_1(t) - \xi_2(t)| + |\xi_1(s) - \xi_2(s)| \\ &\leq t + s \leq 2t. \end{aligned}$$

In addition, there exist $t_1, t_2 \in [0,1]$, such that $X_1\left(t_1; 0, \frac{1}{2}\right) = 1$ and $X_2\left(t_2; 0, \frac{1}{2}\right) = 1$. For an arbitrary but fixed $\varepsilon > 0$, consider the time interval $[0,\tau]$ where

$$\tau = \min\left\{1, t_1, t_2, \frac{\varepsilon}{15\left(\rho_0^1([0,1)) + \rho_0^2([0,1))\right)}, \frac{\varepsilon}{10\rho_0^2([0,1))}, \frac{\varepsilon}{20\mu([0,T])}\right\}.$$

For arbitrary but fixed $t \in [0,\tau]$ there exists locations $x_0^1, x_0^2 \in [0,1)$ such that $X_1(t;0,x_0^1) = 1$ and $X_2(t;0,x_0^2) = 1$. Without loss of generality, we assume that $x_0^1 < x_0^2$. Note that $x_0^1, x_0^2 \in (\frac{1}{2}, 1)$.

We will now show that if $\delta = \frac{\varepsilon}{5} > 0$, then for every $t \in [0,\tau]$, if $\phi(\rho_0^1 - \rho_0^2) < \delta$, then $\phi(\rho_t^1 - \rho_t^2) < \epsilon$. For arbitrary but fixed $f \in \mathcal{F}$, and for every $t \in [0, \tau]$, we have

$$\left|\int_{[0,1)} fh\, d(\rho_t^1 - \rho_t^2)\right| = \left|\int_{[0,1)} f(x)h(x)\, d\rho_t^1(x) - \int_{[0,1)} f(x)h(x)\, d\rho_t^2(x)\right| \tag{4.71a}$$

$$= \left|\int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^1(x_0) - \int_{[0,x_0^2)} (fh)(X_2(t;0,x_0))\, d\rho_0^2(x_0)\right.$$
$$\left. + \int_{[0,t)} (fh)(X_1(t;s,0))\, d\mu(s) - \int_{[0,t)} (fh)(X_2(t;s,0))\, d\mu(s)\right| \tag{4.71b}$$

$$= \left|\int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^1(x_0) - \int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^2(x_0)\right.$$
$$+ \int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^2(x_0) - \int_{[0,x_0^1)} (fh)(X_2(t;0,x_0))\, d\rho_0^2(x_0)$$
$$- \int_{[x_0^1,x_0^2)} (fh)(X_2(t;0,x_0))\, d\rho_0^2(x_0)$$
$$\left. + \int_{[0,t)} (fh)(X_1(t;s,0))\, d\mu(s) - \int_{[0,t)} (fh)(X_2(t;s,0))\, d\mu(s)\right| \tag{4.71c}$$

$$\leq \left|\int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^1(x_0) - \int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^2(x_0)\right| \tag{4.71d}$$

$$+ \int_{[0,x_0^1)} |(fh)(X_1(t;0,x_0)) - (fh)(X_2(t;0,x_0))|\; d\rho_0^2(x_0) \tag{4.71e}$$

$$+ \int_{[x_0^1,x_0^2)} (fh)(X_2(t;0,x_0))\, d\rho_0^2(x_0) \tag{4.71f}$$

$$+ \int_{[0,t)} |(fh)(X_1(t;s,0)) - (fh)(X_2(t;s,0))|\; d\mu(s). \tag{4.71g}$$

By the triangle inequality, we obtain,

$$\left|\int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^1(x_0) - \int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d\rho_0^2(x_0)\right|$$
$$\left|\int_{[0,x_0^1)} (fh)(X_1(t;0,x_0))\, d(\rho_0^1 - \rho_0^2)(x_0)\right| =$$
$$= \left|\int_{[0,x_0^1)} (fh)(x_0)\, d(\rho_0^1 - \rho_0^2)(x_0) + \int_{[0,x_0^1)} \left((fh)(X_1(t;0,x_0)) - (fh)(x_0)\right)\, d(\rho_0^1 - \rho_0^2)(x_0)\right|$$
$$\leq \left|\int_{[0,x_0^1)} (fh)(x_0)\, d(\rho_0^1 - \rho_0^2)(x_0)\right| + \left|\int_{[0,x_0^1)} \left((fh)(X_1(t;0,x_0)) - (fh)(x_0)\right)\, d(\rho_0^1 - \rho_0^2)(x_0)\right|.$$

Let $I_1 = \left|\int_{[0,x_0^1)}(fh)(x_0)\,d(\rho_0^1-\rho_0^2)(x_0)\right|$. Then

$$\begin{aligned}
I_1 &\le \left|\int_{[0,1)}(fh)(x_0)\,d(\rho_0^1-\rho_0^2)(x_0)\right| + \left|\int_{[x_0^1,1)}(fh)(x_0)\,d(\rho_0^1-\rho_0^2)(x_0)\right| \\
&\le \left|\int_{[0,1)}(fh)(x_0)\,d(\rho_0^1-\rho_0^2)(x_0)\right| + \left|\int_{[x_0^1,1)}(fh)(x_0)\,d\rho_0^1(x_0)\right| + \left|\int_{[x_0^1,1)}(fh)(x_0)\,d\rho_0^2(x_0)\right|.
\end{aligned}$$

Using that the function f is bounded above by 1, h is decreasing over the interval $(\frac{1}{2},1)$ and $x_0^1 \in (\frac{1}{2},1)$, we have,

$$\begin{aligned}
\left|\int_{[x_0^1,1)}(fh)(x_0)\,d\rho_0^1(x_0)\right| &\le \left|\int_{[x_0^1,1)}h(x_0)\,d\rho_0^1(x_0)\right| \\
&\le h(x_0^1)\rho_0^1([0,1)) \\
&= (1-x_0^1)\rho_0^1([0,1)) \\
&\le t\rho_0^1([0,1))
\end{aligned}$$

Similarly, we obtain

$$\left|\int_{[x_0^1,1)}(fh)(x_0)\,d\rho_0^1(x_0)\right| \le t\rho_0^2([0,1)).$$

By the expression of the semi-norm ϕ,

$$I_1 \le \phi(\rho_0^1-\rho_0^2) + t\left(\rho_0^1([0,1)) + \rho_0^2([0,1))\right).$$

Let $I_2 = \left|\int_{[0,x_0^1)}\left((fh)(X_1(t;0,x_0)) - (fh)(x_0)\right)\,d(\rho_0^1-\rho_0^2)(x_0)\right|$. Using that the function fh is Lipschitz continuous with Lipschitz constant 2, we have,

$$
\begin{aligned}
I_2 \leq & \left| \int_{[0,x_0^1)} ((fh)(X_1(t;0,x_0)) - (fh)(x_0))\ d(\rho_0^1)(x_0) \right| \\
& + \left| \int_{[0,x_0^1)} ((fh)(X_1(t;0,x_0)) - (fh)(x_0))\ d(\rho_0^2)(x_0) \right| \\
\leq & \int_{[0,x_0^1)} |(fh)(X_1(t;0,x_0)) - (fh)(x_0)|\ d(\rho_0^1)(x_0) \\
& + \int_{[0,x_0^1)} |(fh)(X_1(t;0,x_0)) - (fh)(x_0)|\ d(\rho_0^2)(x_0) \\
\leq & 2 \left(\int_{[0,x_0^1)} |X_1(t;0,x_0) - x_0|\ d\rho_0^1(x_0) + \int_{[0,x_0^1)} |X_1(t;0,x_0) - x_0|\ d\rho_0^2(x_0) \right) \\
\leq & 2t \left(\rho_0^1([0,1)) + \rho_0^2([0,1)) \right).
\end{aligned}
$$

Therefore, by the definition of the semi-norm ϕ, the expression (4.71d) is bounded above by $\phi(\rho_0^1 - \rho_0^2) + 3t\,(\rho_0^1([0,1)) + \rho_0^2([0,1)))$.

Using that the function fg is Lipschitz continuous with Lipschitz constant 2, we obtain the integral (4.71e) is bounded above by

$$
2 \int_{[0,1)} |X_1(t;0,x_0) - X_2(t;0,x_0)|\ d\rho_0^2(x_0) \leq 2t\rho_0^2([0,1)),
$$

and the integral in (4.71g) is bounded above by

$$
2 \int_{[0,t)} |X_1(t;s,0) - X_2(t;s,0)|\ d\mu(s) \leq 4t\mu([0,T)).
$$

For the integral (4.71f), use that the function f is bounded above by 1 we have

$$
\int_{[x_0^1,x_0^2)} (fh)(X_2(t;0,x_0))\, d\rho_0^2(x_0) \leq \int_{[x_0^1,x_0^2)} h(X_2(t;0,x_0))\, d\rho_0^2(x_0).
$$

Since the semi-flow X_2 is increasing with respect to the third variable, for $x_0 \in [x_0^1, x_0^2)$, $X_2(t;0,x_0) \in [X_2(t;0,x_0^1), 1) \subset (\frac{1}{2}, 1)$. Note also that the function h decreases over the interval $(\frac{1}{2}, 1)$. Thus

$$\int_{[x_0^1,x_0^2)} h(X_2(t;0,x_0))\,d\rho_0^2(x_0) \le \int_{[x_0^1,x_0^2)} h(X_2(t;0,x_0^1))\,d\rho_0^2(x_0)$$
$$= \int_{[x_0^1,x_0^2)} \left(1 - X_2(t;0,x_0^1)\right)\,d\rho_0^2(x_0) \le \int_{[x_0^1,x_0^2)} \left(1 - x_0^1\right)\,d\rho_0^2(x_0)$$
$$\le t\rho_0^2([0,1)).$$

The last inequality above is due to the fact that the velocity α_1 is bounded above by 1.

□

4.4 Discussion and Conclusion

We substantially relaxed the regularity hypothesis under which well-posedness is guaranteed for the model (1.1) from Armbruster *et al.* (2006) for highly re-entrant manufacturing systems, a model that has spawned much follow-up research. By closely adhering to the features of the original industrial problem, primarily by focusing on the Lagrangian point of view, we established well-posedness for Borel measure-valued data. Note that products of discontinuous velocities and atomic measures conflict with the usual distributional interpretation of weak solutions of the conservation law from a Eulerian point of view. In particular, if the initial condition ρ_0 and in-flux μ are Borel measures that are absolutely continuous with respect to Lebesgue measure, then our result reduces to the existence and uniqueness results proven in (Coron *et al.*, 2010) for L^1 data. Due to the discontinuity of the velocity v with respect to time t, we consider slightly different test functions for the definition of the weak solution.

Chapter 5

CONCLUSION AND FUTURE WORK

5.1 Conclusion

Transitions between equilibria of the hyperbolic conservation law model were analyzed in Chapter 3. In the case of transferring from a smaller to a larger equilibrium with nonzero backlog, due to the fact that the velocity of the system decreases with respect to the total load W, a control influx u defined in equation (3.7) results in an inverse response to the production system. In the case of transferring from a smaller to a larger equilibrium with zero backlog at the final time, the optimal control dose not exist in a family of L^1-controls, $\{\varepsilon \in [0,1) \colon \mu_\varepsilon$ defined in equation (3.11)$\}$. The suggestive impulsive control (not in L^1) leads us to generalize the set of controls and states to be Borel measures.

Staying close to the original manufacturing system modeled by (1.1), in Chapter (4), we reinterpret the hyperbolic conservation law (1.1a)-(1.1c) in the setting of Borel measures. The key is to temporarily abandon the Eulerian point of view, and instead focus on the Lagrangian point of view, which tracks the locations of *parts* (or *particles*), also known as method of characteristics. We establish the existence of unique solutions for the Cauchy problem from the Lagrangian point of view for the system with data that are Borel measures. The usual contraction argument breaks down due to the fact the large impulses leave the system at a-priori unknown time. We obtain a uniform lower bound for the lengths of time-intervals by an innovation that replace the initial datum ρ_0 by a modified $\tilde{\rho}_0$. The construction takes advantage of the characteristic curves being bi-Lipschitz, a key feature of the model. We also define a notion of weak measure-valued solution for the system (1.1) with data that are Borel measures, and demonstrates that the Lagrangian solutions are weak solutions. In particular, in the case of initial state and in-flux being absolutely continuous measures, the weak measure-valued solution for the system (1.1) reduces to the weak solution in L^1-setting as in Coron *et al.* (2010). Commensurate with the weaker regularity of the Lagrangian solution, the notion of weak solutions utilizes a class of the test functions that is slightly larger than usual. Finally, regularity properties of the solutions are established. In the case of the in-flux containing nonzero pure-point part, the weak solution cannot depend continuously on the time with respect to any norm. Given the noncompact domain, this is expressed using a weighted version of the flat metric, defined in terms of a semi-norm that accounts for impulses entering and leaving the system.

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APPENDIX A

REVIEW OF MEASURES AND MEASURABLE FUNCTIONS

A.1 Disambiguation of Terminology

In this section, we generally adopt the terminology in measure theory from the classical text book on Real Analysis by Royden and Fitzpatrick (1988). We list some related well-known facts without proof.

Definition A.1.1. *Let* (X, Σ) *be a measurable space. Let* $x \in X$ *be any point in* X*. Then the* ***Dirac measure at*** x*, denoted by* δ_x *is the measure defined by*

$$\delta_x : \Sigma \to \bar{\mathbb{R}}, \delta_x(E) := \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases}$$

where $\bar{\mathbb{R}}$ *denotes the extended set of real numbers.*

In fact, Dirac measure is a probability measure.

Definition A.1.2. *Let* (X, Σ, μ) *be a measure space. Then* μ *is a* ***discrete measure****, iff it is a countable linear combination of Dirac measures. That is, iff there exist a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *and a sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that*

$$\text{for every } E \in \Sigma : \mu(E) = \Sigma_{n\in\mathbb{N}}\lambda_n\delta_{x_n}(E).$$

Definition A.1.3. *Let* X *be a locally compact Hausdorff space.* ***The*** σ***-algebra of Borel sets*** $\mathcal{B}(X)$ *is the smallest* σ*-algebra that contains the open sets of* X*. A* ***Borel measure*** *is any measure* μ *defined on the* σ*-algebra of Borel sets.*

Definition A.1.4. *Let* μ *be a measure on a measurable space* (X, Σ)*. Then* μ *is a* ***finite measure*** *iff*

$$\mu(X) < \infty,$$

and is a σ***-finite measure*** *iff there is an increasing sequence* $(E_n)_{n\in\mathbb{N}}$ *of subsets of* X *in* Σ*, whose union is* X *such that*

$$\text{for any } n \text{ in } \mathbb{N} : \mu(E_n) < \infty.$$

Remark. *Any finite measure is also a* σ*-finite measure.*

Definition A.1.5. *Let* $(X, \mathcal{T})$ *be a Hausdorff topological space and let* Σ *be a* σ*-algebra on* X *that contains the topology* $\mathcal{T}$*. Then a measure* μ *on the measurable space* (X, Σ) *is called* ***inner regular*** *if, for every set* A *in* Σ*,*

$$\mu(A) = \sup\{\mu(F) \colon F \subset A, F \text{ compact and measurable}\},$$

and called ***outer regular*** *if for every set* A *in* Σ*,*

$$\mu(A) = \inf\{\mu(G) \colon A \subset G, G \text{ open and measurable}\}.$$

A measure is called ***regular*** *if it is both inner regular and outer regular. Mathematically, a regular measure on a topological space is a measure for which every measurable set can be approximated from above by open measurable sets and from below by compact measurable sets.*

A measure μ *defined on* Σ *is called* ***locally finite*** *if for every point* p *in* X*, there is an open neighborhood* N_p *of* p *such that the* μ*-measurable of* N_p *is finite.*

Definition A.1.6. *A Borel measure μ is called a **Radon measure** if it is both inner regular and locally finite.*

Definition A.1.7. *A measure λ is absolutely continuous with respect to another measure μ if for every set E with $\mu(E) = 0$, $\lambda(E) = 0$.*

Definition A.1.8. *A map $\gamma : X \to Y$ with X, Y being two topological spaces is called **Borel measurable** if for every open set $A \subset Y$, $\gamma^{-1}(A) \subset X$ is a Borel set.*

Remark. *For every Borel set B in Y, $\gamma^{-1}(B)$ is a Borel set in X; All monotone functions mapping from $\mathbb{R} \to \mathbb{R}$ are Borel measurable.*

Definition A.1.9. *Let $(X, \mathcal{M})$ be a measurable space and f a real-valued function on X. Then f is **Lebesgue measurable** if for each open set $\mathcal{O}$ in real numbers, $f^{-1}(\mathcal{O})$ is Lebesgue measurable.*

Remark. *Every Borel measurable function is Lebesgue measurable; If $f\colon \mathbb{R} \mapsto \mathbb{R}$ is Borel measurable and $g\colon \mathbb{R}^n \mapsto \mathbb{R}$ is Lebesgue (Borel) measurable, then the composition $f \circ g$ is Lebesgue (Borel) measurable since for every Borel set B of $\mathbb{R}$,*

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)),$$

is Lebesgue measurable; If $f\colon \mathbb{R} \mapsto \mathbb{R}$ is Lebesgue measurable and $g\colon \mathbb{R}^n \mapsto \mathbb{R}$ is Lebesgue or Borel measurable, then $f \circ g$ need not be Lebesgue or Borel measurable since $f^{-1}(B)$ need not be a Borel set even if B is a Borel set.

Definition A.1.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a non-negative real valued measurable function on X. Then f is **integrable** over X with respect to μ provided $\int_X f d\mu < \infty$.*

Remark. *Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be a continuous function and $g\colon [a, b] \mapsto \mathbb{R}$ a Lebesgue integrable function. If there exist constants c and d such that for every $x \in \mathbb{R}$, $|f(x)| < c + d\,|x|$, then $f \circ g$ is Lebesgue integrable over $[a, b]$.*

Definition A.1.11. *A function $f\colon S \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a **Lipschitz function** if there is a constant C such that for all $x, y \in S$,*

$$\|f(y) - f(x)\| \leq C\|y - x\|.$$

Definition A.1.12. *A function $f\colon [a, b] \mapsto \mathbb{R}$ is **absolutely continuous** on $[a, b]$ if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon,$$

whenever $\{[x_i, y_i] \colon i = 1, \cdots, n\}$ is a finite collection of mutually disjoint subintervals of $[a, b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

Remark. *Every absolutely continuous function on $[a, b]$ is uniformly continuous; Every Lipschitz continuous function on $[a, b]$ is absolutely continuous.*

Definition A.1.13. *The function $f : [a, b] \mapsto \mathbb{R}$ is of* ***bounded variation*** *on $[a, b]$ if and only if there is a constant $M > 0$ such that*

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M$$

for all partitions $\mathbb{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$.

Remark. *A function is of bounded variation if and only if it is the difference of two increasing functions; If a function is of bounded variation, then it is differntiable almost everywhere; Every absolutely continuous function is of bounded variation and hence is differentiable almost everywhere.*

Definition A.1.14. *The* ***Dirac delta distribution*** *δ is a linear functional from the space of all smooth functions on $\mathbb{R}$ with compact support. Specifically, for every smooth function f on $\mathbb{R}$ with compact support, and for arbitrary but fixed $a \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)\, dx = f(a).$$

Definition A.1.15. *A sequence of functions $\{f_n\}$ converges to function f on $\mathbb{R}$* ***in the sense of distribution*** *if for every function $\phi \in C_c^{\infty}(\mathbb{R})$ that is smooth on $\mathbb{R}$ with compact support,*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\phi(x)\, dx = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx.$$

A.2 Some Theorems from Measure Theory

We recall some theorems that are used in this thesis from measure theory in this section.

Theorem A.2.1 (Refined Lebesgue Decomposition Theorem)**.** *Every regular Borel measure μ on the real line can be decomposed in the following way,*

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$$

where μ_{ac} is the absolutely continuous part, μ_{sc} is the singular continuous part, and μ_{pp} is the pure point part (a discrete measure).

Theorem A.2.2 (Radon-Nikodym Theorem)**.** *Let (X, Σ) be a measurable space and μ, ν are two σ-finite measures. If ν is absolutely continuous with respect to μ, then there is a unique measurable function $f : X \to [0, \infty)$, such that for every measurable set $A \subset X$,*

$$\nu(A) = \int_A f d\mu.$$

The function f is called the Randon-Nikodym derivative and is denoted by $\frac{d\nu}{d\mu}$.

Theorem A.2.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. Then ν is absolutely continuous with respect to μ if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every set $E \in \mathcal{M}$,*

$$\text{if } \mu(E) < \delta, \text{ then } \nu(E) < \varepsilon.$$

Theorem A.2.4 (Regularity theorem for Lebesgue measure)**.** *The Lebesgue measure on the real line $\mathbb{R}$ is a regular measure. That is, for all Lebesgue measurable subsets A of $\mathbb{R}$, and $\epsilon > 0$, there exist closed subsets C and open subsets U of $\mathbb{R}$ such that $C \subset A \subset U$ and the Lebesgue measure of $U \setminus C$ is strictly less than ε.*

APPENDIX B

PROOFS OF SOME LEMMAS IN CHAPTER (3)

Proof of Lemma (3.2.1)

For any $t_* > 0$, the demand outflux jump time, t^*, is finite, i.e., $t^* < \infty$.

Combining equation (3.2) and a lower bound for the velocity field $\alpha(W(t))$ for very time $t \in (t_*, t^*)$, we calculate an upper bound for t^*.

Proof. Without loss of generality, we restrict the time to $t \in (t_*, t^*)$.

By the conservation of mass, for any time $t \in (t_*, t^*)$, the total load W satisfies

$$\begin{aligned} W'(t) &= u(t) - y(t) = (\rho(t,0) - \rho(t,1))\, \alpha(W(t)) \\ &= \frac{\rho(t,0) - \rho(t,1)}{1 + W(t)} = \frac{\rho_2 - \rho_1}{1 + W(t)} \\ &\leq \rho_2 - \rho_1. \end{aligned} \tag{B.1}$$

Thus for every $t \in (t_*, t^*)$,

$$W(t) \leq W(t_*) + (\rho_2 - \rho_1)t,$$

which implies

$$\alpha(W(t)) \geq \frac{1}{1 + W(t_*) + (\rho_2 - \rho_1)t} > 0.$$

Note that the velocity stays strictly positive and decays at most rationally during the time interval (t_*, t^*). From the relation defined in equation (3.2), we get, for any given $t_* > 0$, the demand outflux jump time $t^* < \infty$. Indeed,

$$\begin{aligned} 1 = \int_{t_*}^{t^*} \alpha(W(t))dt \;&\geq\; \int_{t_*}^{t^*} \frac{1}{1 + W(t_*) + (\rho_2 - \rho_1)t} dt, \\ &=\; \left. \frac{\log\left(1 + W(t_*) + (\rho_2 - \rho_1)t\right)}{\rho_2 - \rho_1} \right|_{t=t_*}^{t^*}, \\ &=\; \frac{\log\left(1 + W(t_*) + (\rho_2 - \rho_1)t^*\right) - \log\left(1 + W(t_*) + (\rho_2 - \rho_1)t_*\right)}{\rho_2 - \rho_1}. \end{aligned}$$

After simplification, an upper bounded of t^* is,

$$\begin{aligned} t^* \;&\leq\; \frac{e^{\rho_2 - \rho_1}\left(1 + W(t_*) + (\rho_2 - \rho_1)t_*\right) - (1 + W(t_*))}{\rho_2 - \rho_1}, \\ &=\; e^{\rho_2 - \rho_1} t_* + \left(\frac{e^{\rho_2 - \rho_1} - 1}{\rho_2 - \rho_1} \right) (1 + W(t_*)), \\ &<\; \infty. \end{aligned}$$

□

Proof of Lemma (3.3.1)

Given $\varepsilon > 0$, for any $M > 0$, $T^* - t^*$ is bounded above by $\left(\frac{1+\rho_2+M}{1+\rho_1}\right)\varepsilon$, i.e., $T^* - t^* \leq \left(\frac{1+\rho_2+M}{1+\rho_1}\right)\varepsilon$.

Proof. From equations (3.8) and (3.9),

$$\int_0^{\varepsilon} \alpha(W(s))\,ds = \int_{t^*}^{T^*} \alpha(W(s))\,ds. \tag{B.2}$$

The total load W increases over the time interval $[0, \varepsilon]$, i.e., for every $t \in [0, \varepsilon]$, $W(0) \leq W(t) \leq W(\varepsilon)$. By conservation of mass,

$$\begin{aligned} W(\varepsilon) &= W(0) + \int_0^{\varepsilon} u(t)\,dt - \int_0^{\varepsilon} y(t)\,dt \\ &= W(0) + \int_0^{\varepsilon} \rho_2\alpha(W(t))\,dt + M - \int_0^{\varepsilon} \rho(t,1)\alpha(W(t))\,dt \\ &= W(0) + \int_0^{\varepsilon} \rho_2\alpha(W(t))\,dt + M - \int_0^{\varepsilon} \rho_1\alpha(W(t))\,dt, \\ &= W(0) + M + \int_0^{\varepsilon} (\rho_2 - \rho_1)\alpha(W(t))\,dt. \end{aligned} \tag{B.3}$$

In addition, for every $t \in [0, \infty)$, the velocity $\alpha(W(t))$ is bounded above by 1 and $W(0) = \rho_1$. Thus

$$\begin{aligned} W(\varepsilon) &\leq W(0) + M + \varepsilon(\rho_2 - \rho_1) \\ &= \rho_1 + M + \varepsilon(\rho_2 - \rho_1). \end{aligned}$$

Thus for every $t \in [0, \varepsilon]$,

$$\rho_1 \leq W(t) \leq \rho_1 + M + \varepsilon(\rho_2 - \rho_1).$$

This implies

$$\frac{1}{1 + \rho_1 + M + \varepsilon(\rho_2 - \rho_1)} \leq \alpha(W(t)) \leq \frac{1}{1 + \rho_1}. \tag{B.4}$$

Hence

$$\int_0^{\varepsilon} \alpha(W(s))ds \leq \frac{\varepsilon}{1 + \rho_1}. \tag{B.5}$$

The total load W decreases over the time interval $[t^*, T^*]$, i.e., for every $t \in [t^*, T^*]$, $W(T^*) \leq W(t) \leq W(t^*)$. Thus.

$$\frac{1}{1 + W(t^*)} \leq \alpha(W(t)) \leq \frac{1}{1 + W(T^*)}. \tag{B.6}$$

Also by conservation of mass,

$$\begin{aligned}
W(t^*) &= W(T^*) + \int_{t^*}^{T^*} y(t)\,dt - \int_{t^*}^{T^*} u(t)\,dt \\
&= W(T^*) + \int_{t^*}^{T^*} \rho(t,1)\alpha(W(t))\,dt - \int_{t^*}^{T} \rho_2\alpha(W(t))\,dt \\
&= W(T^*) + \int_{0}^{\varepsilon} \rho_2\alpha(W(t))\,dt + M - \int_{0}^{\varepsilon} \rho_2\alpha(W(t))\,dt \\
&= W(T^*) + M.
\end{aligned}$$

Since at time $t = T^*$, the system is at an equilibrium state with constant density ρ_2, we have $W(T^*) = \rho_2$. Thus $W(t^*) = \rho_2 + M$. By equation $(B.6)$, for every $t \in [t^*, T^*]$,

$$\frac{1}{1+\rho_2+M} \le \alpha(W(t)) \le \frac{1}{1+\rho_2}.$$

Hence

$$\int_{t^*}^{T^*} \alpha(W(t))dt \ge \frac{T^* - t^*}{1+\rho_2+M}. \tag{B.7}$$

Combine equations $(B.2)$, $(B.5)$ and $(B.7)$,

$$\frac{T^* - t^*}{1+\rho_2+M} \le \frac{\varepsilon}{1+\rho_1},$$

which implies that

$$T^* - t^* \le \left(\frac{1+\rho_2+M}{1+\rho_1}\right)\varepsilon.$$

□

Proof of Lemma (3.3.2)

Given $\varepsilon > 0$, *for any* $M > 0$, t^* *is bounded above by* $\varepsilon + \frac{1}{2}(\rho_2 - \rho_1) + \left(1 + \rho_1 + M + \frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)$, *i.e.,* $t^* \le \varepsilon + \frac{1}{2}(\rho_2 - \rho_1) + \left(1 + \rho_1 + M + \frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)$.

Proof. For $\varepsilon \le t \le t^*$, from equation $(B.1)$ it follows that

$$W'(t) = (\rho_2 - \rho_1)\frac{1}{1+W(t)}. \tag{B.8}$$

Thus for every $t \in [\varepsilon, t^*]$, we have,

$$W(t) = \sqrt{2(\rho_2-\rho_1)(t-\varepsilon) + (1+W(\varepsilon))^2} - 1. \tag{B.9}$$

Also from $(B.3)$, $(B.4)$ and $w(0) = \rho_1$, we get the following inequality,

$$\rho_1 + M + (\rho_2-\rho_1)\frac{\varepsilon}{1+\rho_1+M+(\rho_2-\rho_1)\varepsilon} \le W(\varepsilon) \le \rho_1 + M + (\rho_2-\rho_1)\frac{\varepsilon}{1+\rho_1}.$$

Let

$$M_1 = \left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1 + \rho_1 + M + (\rho_2 - \rho_1)\varepsilon}\right)^2,$$

and

$$M_2 = \left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1 + \rho_1}\right)^2.$$

Then we get the range of the total load, $W(t)$, when $\varepsilon \le t \le t^*$,

$$\sqrt{2(\rho_2 - \rho_1)(t - \varepsilon) + M_1} - 1 \le W(t) \le \sqrt{2(\rho_2 - \rho_1)(t - \varepsilon) + M_2} - 1.$$

So the upper and lower bounds for the velocity $\alpha(W(t))$ for $\varepsilon \le t \le t^*$ are,

$$\alpha(W(t)) \ge \frac{1}{\sqrt{2(\rho_2 - \rho_1)(t - \varepsilon) + \left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1+\rho_1}\right)^2}}, \tag{B.10}$$

$$\alpha(W(t)) \le \frac{1}{\sqrt{2(\rho_2 - \rho_1)(t - \varepsilon) + \left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1+\rho_1+M+(\rho_2-\rho_1)\varepsilon}\right)^2}}.$$

From (3.8), $(B.4)$ and $(B.10)$, we have,

$$\int_0^{\varepsilon} \frac{1}{1 + \rho_1 + M + \varepsilon(\rho_2 - \rho_1)}\, dt + \int_{\varepsilon}^{t^*} \frac{1}{\sqrt{2(\rho_2 - \rho_1)(t - \varepsilon) + \left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1+\rho_1}\right)^2}}\, dt \le 1,$$

which implies

$$\frac{1}{1 + \rho_1 + M + \varepsilon(\rho_2 - \rho_1)}\varepsilon - \frac{1}{\rho_2 - \rho_1}\left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1 + \rho_1}\right) + \frac{1}{\rho_2 - \rho_1}\left(\sqrt{2(\rho_2 - \rho_1)(t^* - \varepsilon) + \left(1 + \rho_1 + M + (\rho_2 - \rho_1)\frac{\varepsilon}{1 + \rho_1}\right)^2}\right) \le 1$$

Therefore,

$$t^* \le \varepsilon + \frac{\left((1 + \rho_1 + M + \frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon) + (\rho_2 - \rho_1)(1 - \frac{\varepsilon}{1+\rho_1+M+(\rho_2-\rho_1)\varepsilon)})\right)^2}{2(\rho_2 - \rho_1)}, - \frac{\left(1 + \rho_1 + M + \frac{\rho_2-\rho_1}{1+\rho_1}\varepsilon\right)^2}{2(\rho_2 - \rho_1)}$$

$$\le \varepsilon + \frac{1}{2}(\rho_2 - \rho_1) + \left(1 + \rho_1 + M + \frac{\rho_2 - \rho_1}{1 + \rho_1}\varepsilon\right).$$

□

Proof of Lemma (3.3.9)

> *Given a function h as in* (3.10)*, for arbitrary but fixed $\varepsilon \in (0,1]$, the time T^* at which the system reaches its new equilibrium state is continuous with respect to the additional mass M.*

Proof. Similar to lemma (3.3.8), it is enough to show that for arbitrary but fixed $t \in (t^*, T^*]$, the total load $W(t)$ depends continuously on the additional mass M.

Let $t \in (t^*, T^*]$ be arbitrary but fixed. Then there exists $\tau(t) \in (0, \varepsilon]$, such that

$$\int_{\tau(t)}^{t} \alpha(W(s))\, ds = 1. \tag{B.11}$$

Differentiating both sides of equation (B.11) with respect to t,

$$\begin{aligned} \tau'(t) &= \frac{\alpha(W(t))}{\alpha(W(\tau(t)))} \\ \tau(t^*) &= 0. \end{aligned} \tag{B.12}$$

Thus

$$\tau(t) = \int_{t^*}^{t} \frac{\alpha(W(s))}{\alpha(W(\tau(s)))}\, ds. \tag{B.13}$$

Furthermore, for $t \in (t^*, T^*]$, the total load W satisfies

$$\begin{aligned} W'(t) &= -\frac{M}{\varepsilon} h\left(\frac{\tau(t)}{\varepsilon}\right) \tau'(t) \\ W(t^*) &= \rho_2 + M, \end{aligned} \tag{B.14}$$

which implies

$$\begin{aligned} W(t) &= \int_{t^*}^{t} -\frac{M}{\varepsilon} h\left(\frac{\tau(s)}{\varepsilon}\right) \tau'(s)\, ds + \rho_2 + M \\ &= \int_{0}^{\tau(t)} -\frac{M}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) dr + \rho_2 + M. \end{aligned}$$

Suppose that τ_k, W_k, $k = 1, 2$ are the solutions to the Cauchy problems (B.12) and (B.14) respectively with parameter $M = M_k$. Without loss of generality, we assume that $\tau_1(t) < \tau_2(t)$. Therefore,

$$|W_1(t) - W_2(t)| = \left| \int_0^{\tau_1(t)} -\frac{M_1}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) dr - \int_0^{\tau_2(t)} -\frac{M_2}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) dr + M_1 - M_2 \right| \tag{B.15}$$

$$\begin{aligned}
&\le |M_1 - M_2| + \left| \int_0^{\tau_2(t)} \frac{1}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) M_2 \, dr - \int_0^{\tau_1(t)} \frac{1}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) M_1 \right| \\
&\le |M_1 - M_2| + \int_0^{\tau_1(t)} \frac{1}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) dr \, |M_1 - M_2| + \left| \int_{\tau_1(t)}^{\tau_2(t)} \frac{1}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) M_2 \, dr \right| \\
&\le 2\,|M_1 - M_2| + \left| \frac{M_2}{\varepsilon} \int_{\tau_1(t)}^{\tau_2(t)} h\left(\frac{r}{\varepsilon}\right) dr \right| \\
&\le 2\,|M_1 - M_2| + \frac{M_2}{\varepsilon} |\tau_2(t) - \tau_1(t)| \,.
\end{aligned}$$

The third inequality is due to the fact that $\int_0^{\tau_1(t)} \frac{1}{\varepsilon} h\left(\frac{r}{\varepsilon}\right) dr < 1$ and the last inequality is due to the fact that h is bounded above by 1.

In addition, let t_k^*, $k = 1, 2$ be the time when all the initial mass exit from the system with parameter $M = M_k$. Without loss of generality, we assume that $t_1^* < t_2^*$.

$$|\tau_1(t) - \tau_2(t)| \tag{B.16}$$

$$= \left| \int_{t_1^*}^{t} \frac{\alpha(W_1(s))}{\alpha(W_1(\tau_1(s)))} ds - \int_{t_2^*}^{t} \frac{\alpha(W_2(s))}{\alpha(W_2(\tau_2(s)))} ds \right| \tag{B.17}$$

$$\begin{aligned}
&= \left| \int_{t_1^*}^{t_2^*} \frac{\alpha(W_1(s))}{\alpha(W_1(\tau_1(s)))} ds \right| + \left| \int_{t_2^*}^{t} \frac{\alpha(W_1(s))}{\alpha(W_1(\tau_1(s)))} - \frac{\alpha(W_2(s))}{\alpha(W_2(\tau_2(s)))} ds \right| \\
&\le \frac{1}{v_{\min}} |t_1^* - t_2^*| + \frac{1}{(v_{\min})^2} \left| \int_{t_2^*}^{t} \alpha(W_1(s))\alpha(W_2(\tau_2(s))) - \alpha(W_2(s))\alpha(W_1(\tau_1(s))) \, ds \right| .
\end{aligned}$$

Here $v_{\min}$ is the positive lower bound of the velocity α, that is, for every $t \in (t^*, T^*]$, $\alpha(W(t)) \ge v_{\min} > 0$.

Note that for every $s \in (t_2^*, T^*]$,

$$\begin{aligned}
&|\alpha(W_1(s))\alpha(W_2(\tau_2(s))) - \alpha(W_2(s))\alpha(W_1(\tau_1(s)))| \\
\le &|\alpha(W_1(s))\alpha(W_2(\tau_2(s))) - \alpha(W_1(s))\alpha(W_1(\tau_1(s)))| \\
&+ |\alpha(W_1(s))\alpha(W_1(\tau_1(s))) - \alpha(W_2(s))\alpha(W_1(\tau_1(s)))| \\
\le &|W_2(\tau_2(s)) - W_1(\tau_1(s))| + |W_1(s) - W_2(s)| \\
\le &|W_2(\tau_2(s)) - W_1(\tau_2(s))| + |W_1(\tau_2(s) - W_1(\tau_1(s)))| + |W_1(s) - W_2(s)| \,.
\end{aligned}$$

Furthermore, since for every $t \in [0, \varepsilon]$,

$$W'(t) = u(t) - y(t) = \rho_2 \alpha(W(t)) + \frac{M}{\varepsilon} h\left(\frac{t}{\varepsilon}\right) - \rho_1 \alpha(W(t)) \le \rho_2 + \frac{M}{\varepsilon},$$

we have,

$$|W_1(\tau_2(s) - W_1(\tau_1(s)))| \le \left(\rho_2 + \frac{M}{\varepsilon}\right)(\tau_2(s) - \tau_1(s)).$$

By equation (B.15), we obtain,

$$|W_1(s) - W_2(s)| \le 2\,|M_1 - M_2| + \frac{M_2}{\varepsilon}\,|\tau_2(s) - \tau_1(s)|\,.$$

Thus,

$$\begin{aligned} &|\tau_1(t) - \tau_2(t)| \\ \le& \frac{1}{v_{\min}}\,|t_1^* - t_2^*| + \frac{1}{(v_{\min})^2}\int_{t_2^*}^{t} |W_2(\tau_2(s)) - W_1(\tau_2(s))|\; ds \\ &+ \frac{2}{(v_{\min})^2}\int_{t_2^*}^{t} |M_1 - M_2|\; ds + \frac{1}{(v_{\min})^2}\int_{t_2^*}^{t}\left(\rho_2 + \frac{M}{\varepsilon} + \frac{M_2}{\varepsilon}\right)|\tau_2(s) - \tau_1(s)|\; ds. \end{aligned} \tag{B.18}$$

Recall that we have t^* depends continuously on the additional mass M (lemma (3.3.8)) and for every $s \in [0, \varepsilon]$, $W(s)$ depends continuously on the additional mass M (lemma (3.3.5)). Combine with Grönwall's Inequality (lemma (3.3.4)) and equation (B.18), we obtain that for arbitrary but fixed $t \in (t^*, T^*]$, $\tau(t)$ depends continuously on the additional mass M, do does the total load $W(t)$. □

Proof of Lemma (3.3.12)

For $\varepsilon \in (0, 1]$ sufficiently small and arbitrary but fixed, we have

$$\frac{\rho_1 + \rho_2 + 2M^* + 2}{2} - \frac{M^*}{1 + \rho_1}\varepsilon + O(\varepsilon) \le t^* \le \frac{\rho_1 + \rho_2 + 2M^* + 2}{2} + O(\varepsilon), \tag{B.19}$$

and

$$\begin{aligned} M^* &\ge \frac{\rho_1(\rho_2 - \rho_1)}{2} - \left(\frac{\rho_1^2(\rho_2 - \rho_1)}{2(1 + \rho_1)} + \rho_2\right)\varepsilon + O(\varepsilon), \\ M^* &\le \frac{\rho_1(\rho_2 - \rho_1)}{2} + \left(\frac{\rho_1\rho_2(\rho_2 - \rho_1)}{2(1 + \rho_2)} + \rho_2\right)\varepsilon + O(\varepsilon). \end{aligned} \tag{B.20}$$

Proof. We then find the range of t^*. Recall that when $t \in [0, \varepsilon]$,

$$\frac{1}{1 + \rho_1 + M^* + \varepsilon(\rho_2 - \rho_1)} \le \alpha(W(t)) \le \frac{1}{1 + \rho_1},$$

thus

$$\frac{\varepsilon}{1 + \rho_1 + M^* + \varepsilon(\rho_2 - \rho_1)} \le \int_0^{\varepsilon} \alpha(W(t))dt \le \frac{\varepsilon}{1 + \rho_1}. \tag{B.21}$$

Combining with equation (3.24),

$$\frac{\sqrt{(\rho_2+M^*+1)^2-2(\rho_2-\rho_1)(t^*-\varepsilon)}-(\rho_1+M^*+1)}{\rho_2-\rho_1} \geq \frac{\varepsilon}{1+\rho_1+M^*+\varepsilon(\rho_2-\rho_1)}$$
$$\frac{\sqrt{(\rho_2+M^*+1)^2-2(\rho_2-\rho_1)(t^*-\varepsilon)}-(\rho_1+M^*+1)}{\rho_2-\rho_1} \leq \frac{\varepsilon}{1+\rho_1}.$$

Thus, we can obtain a lower bound for t^*,

$$\begin{aligned}
t^* &\geq \frac{1}{2(\rho_2-\rho_1)}\left((\rho_2+M^*+1)^2-\left(\frac{\varepsilon(\rho_2-\rho_1)}{1+\rho_1}+(\rho_1+M^*+1)\right)^2\right)+\epsilon \\
&= \frac{1}{2(\rho_2-\rho_1)}\left(\left(\frac{\varepsilon(\rho_2-\rho_1)}{1+\rho_1}+\rho_1+\rho_2+2M^*+2\right)\left(\rho_2-\rho_1-\frac{\varepsilon(\rho_2-\rho_1)}{1+\rho_1}\right)\right)+\varepsilon \\
&= \frac{1}{2}\left(\left(\frac{\varepsilon(\rho_2-\rho_1)}{1+\rho_1}+\rho_1+\rho_2+2M^*+2\right)\left(1-\frac{\varepsilon}{1+\rho_1}\right)\right)+\varepsilon \\
&= \frac{1}{2}\left(\rho_1+\rho_2+2M^*+2+\varepsilon\left(\frac{\rho_2-\rho_1}{1+\rho_1}-\frac{\rho_1+\rho_2+2M^*+2}{1+\rho_1}\right)+O(\varepsilon)\right)+\varepsilon \\
&= \frac{1}{2}(\rho_1+\rho_2+2M^*+2)+\frac{\varepsilon}{2}\left(\frac{\rho_2-\rho_1}{1+\rho_1}-\frac{\rho_1+\rho_2+2M^*+2}{1+\rho_1}+2\right)+O(\varepsilon) \\
&= \frac{\rho_1+\rho_2+2M^*+2}{2}-\frac{\varepsilon M^*}{1+\rho_1}+O(\varepsilon).
\end{aligned}$$

For an upper bound of t^*,

$$\begin{aligned}
t^* \leq &\frac{1}{2(\rho_2-\rho_1)}\left((\rho_2+M^*+1)^2-\left(\frac{\varepsilon(\rho_2-\rho_1)}{1+\rho_1+M^*+\varepsilon(\rho_2-\rho_1)}+(\rho_1+M^*+1)\right)^2\right)+\varepsilon \\
=&\frac{1}{2}(\rho_1+\rho_2+2M^*+2) \\
&+\frac{1}{2}\varepsilon\left(\frac{\rho_2-\rho_1}{1+\rho_1+M^*+\varepsilon(\rho_2-\rho_1)}-\frac{\rho_1+\rho_2+2M^*+2}{1+\rho_1+M^*+\varepsilon(\rho_2-\rho_1)}+2\right)+O(\varepsilon) \\
=&\frac{\rho_1+\rho_2+2M^*+2}{2}+O(\varepsilon).
\end{aligned}$$

Therefore,

$$\frac{\rho_1+\rho_2+2M^*+2}{2}-\frac{M^*}{1+\rho_1}\varepsilon+O(\varepsilon) \leq t^* \leq \frac{\rho_1+\rho_2+2M^*+2}{2}+O(\varepsilon). \tag{B.22}$$

Next, we calculate the range of M^*. Recall that the backlog $\beta(T^*)=0$ indicates that

$$y_1t^*+y_2(T^*-t^*)-\rho_1-M^*=\int_0^{\varepsilon}\rho_2\alpha(W(s))ds, \tag{B.23}$$

and $T^* - t^* \geq 0$. Combine expressions $(B.21)$, (3.25) and $(B.23)$, we have,

$$y_1 \left(\frac{\rho_1 + \rho_2 + 2M^* + 2}{2} - \frac{M^*}{1+\rho_1}\varepsilon + O(\varepsilon) \right) - \rho_1 - M^* \leq \frac{\rho_2}{1+\rho_1}\varepsilon$$
$$\implies M^* \left(y_1 - \frac{y_1}{1+\rho_1}\varepsilon - 1 \right) + y_1 \left(\frac{\rho_1 + \rho_2 + 2}{2} + O(\varepsilon) \right) - \rho_1 \leq \frac{\rho_2}{1+\rho_1}\varepsilon.$$

Note that the coefficient of M^* is negative, thus,

$$\begin{aligned} M^* &\geq \frac{\frac{\rho_1+\rho_2+2}{2} y_1 - \rho_1 - \frac{\rho_2}{1+\rho_1}\varepsilon + y_1 O(\varepsilon)}{1 - y_1 + \frac{y_1 \varepsilon}{1+\rho_1}} \\ &= \frac{\frac{\rho_1+\rho_2+2}{2} y_1(1+\rho_1) - \rho_1(1+\rho_1) - \rho_2\varepsilon + O(\varepsilon)}{1 + \rho_1 - y_1(1+\rho_1) + y_1\varepsilon} \\ &= \frac{\frac{\rho_1(\rho_1+\rho_2+2)}{2} - \rho_1(1+\rho_1) - \rho_2\varepsilon + O(\varepsilon)}{1 + y_1\varepsilon} \\ &= \left(\frac{\rho_1(\rho_1+\rho_2+2)}{2} - \rho_1(1+\rho_1) \right) \left(\frac{1}{1+y_1\varepsilon} \right) - \rho_2\varepsilon \left(\frac{1}{1+y_1\varepsilon} \right) + O(\varepsilon). \end{aligned}$$

Note that by Taylor's theorem,

$$\frac{1}{1+y_1\varepsilon} = 1 - y_1\varepsilon + O(\varepsilon),$$

so,

$$\begin{aligned} M^* \geq &\left(\frac{\rho_1(\rho_1+\rho_2+2)}{2} - \rho_1(1+\rho_1) \right) (1 - y_1\varepsilon + O(\varepsilon)) - \rho_2\varepsilon(1 - y_1\varepsilon + O(\varepsilon)) + O(\varepsilon) \\ = &\left(\frac{\rho_1(\rho_1+\rho_2+2)}{2} - \rho_1(1+\rho_1) \right) \\ &+ \left(- \left(\frac{\rho_1(\rho_1+\rho_2+2)}{2} - \rho_1(1+\rho_1) \right) y_1 - \rho_2 \right) \varepsilon + O(\varepsilon) \\ = &\frac{\rho_1(\rho_2-\rho_1)}{2} - \left(\frac{\rho_1^2(\rho_2-\rho_1)}{2(1+\rho_1)} + \rho_2 \right) \varepsilon + O(\varepsilon). \end{aligned}$$

Again recall the restriction that the backlog β reaches zero at time $t = T^*$, i.e.,

$$\beta(T^*) = y_1 t^* + y_2(T^* - t^*) - \rho_1 - M^* - \int_0^\varepsilon \rho_2 \alpha(W(s)) ds = 0.$$

Consider the upper bound for t^* from the inequality (3.25), the upper bound for $T^* - t^*$, $T - t^* \leq \frac{1+\rho_2+M^*}{1+\rho_1}\varepsilon$, and for any t, $0 < \alpha(W(t)) < 1$. Therefore,

$$y_1 \left(\frac{\rho_1 + \rho_2 + 2M^* + 2}{2} + O(\varepsilon) \right) + y_2 \frac{1 + \rho_2 + M^*}{1+\rho_1}\varepsilon - \rho_1 - M^* \geq \beta(T^*) = 0,$$

which implies that

$$M^* \left(y_1 + \frac{y_2 \varepsilon}{1+\rho_1} - 1 \right) + y_1 \frac{\rho_1 + \rho_2 + 2}{2} + y_2 \frac{(1+\rho_2)\varepsilon}{1+\rho_1} - \rho_1 + O(\varepsilon) \geq 0.$$

Note that if $0 < \varepsilon < \frac{(1-y_1)(1+\rho_1)}{y_2}$, then the coefficient of M^* is negative. Thus we can obtain an upper bound of M^*,

$$\begin{aligned}
M^* \leq & \frac{-y_1 \frac{\rho_1+\rho_2+2}{2} - y_2 \frac{(1+\rho_2)\varepsilon}{1+\rho_1} + \rho_1 + O(\varepsilon)}{y_1 + \frac{y_2 \varepsilon}{1+\rho_1} - 1} \\
= & \frac{-\frac{\rho_1(\rho_2+\rho_1+2)}{2} - \rho_2 \varepsilon + \rho_1(1+\rho_1) + O(\varepsilon)}{\rho_1 + y_2 \varepsilon - (1+\rho_1)} \\
= & \frac{-\frac{\rho_1(\rho_2+\rho_1+2)}{2} + \rho_1(1+\rho_1) - \rho_2 \varepsilon + O(\varepsilon)}{y_2 \varepsilon - 1} \\
= & \left(-\frac{\rho_1(\rho_2+\rho_1+2)}{2} + \rho_1(1+\rho_1) \right) (-1) \left(\frac{1}{1 - y_2 \varepsilon} \right) + \rho_2 \varepsilon \left(\frac{1}{1 - y_2 \varepsilon} \right) + O(\varepsilon) \\
= & \left(\frac{\rho_1(\rho_2+\rho_1+2)}{2} - \rho_1(1+\rho_1) \right) (1 + y_2 \varepsilon + O(\varepsilon)) + \rho_2 \varepsilon \left(1 + y_2 \varepsilon + O(\varepsilon)\right) + O(\varepsilon) \\
= & \left(\frac{\rho_1(\rho_2+\rho_1+2)}{2} - \rho_1(1+\rho_1) \right) \\
& + \left(\left(\frac{\rho_1(\rho_2+\rho_1+2)}{2} - \rho_1(1+\rho_1) \right) y_2 + \rho_2 \right) \varepsilon + O(\varepsilon) \\
= & \frac{\rho_1(\rho_2 - \rho_1)}{2} + \left(\frac{\rho_1 \rho_2 (\rho_2 - \rho_1)}{2(1+\rho_2)} + \rho_2 \right) \varepsilon + O(\varepsilon).
\end{aligned}$$

Thus,

$$\begin{aligned}
M^* &\geq \frac{\rho_1(\rho_2 - \rho_1)}{2} - \left(\frac{\rho_1^2 (\rho_2 - \rho_1)}{2(1+\rho_1)} + \rho_2 \right) \varepsilon + O(\varepsilon) \\
M^* &\leq \frac{\rho_1(\rho_2 - \rho_1)}{2} + \left(\frac{\rho_1 \rho_2 (\rho_2 - \rho_1)}{2(1+\rho_2)} + \rho_2 \right) \varepsilon + O(\varepsilon).
\end{aligned} \tag{B.24}$$

□