On the Existence of Loose Cycle Tilings and Rainbow Cycles

by

Roy Oursler

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Graduate Supervisory Committee:

Andrzej Czygrinow, Chair
H. A. Kierstead
S. Fishel
J. Jones
C. Colbourn

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ABSTRACT

Extremal graph theory results often provide minimum degree conditions which guarantee a copy of one graph exists within another. A perfect F-tiling of a graph G is a collection $\mathcal{F}$ of subgraphs of G such that every element of $\mathcal{F}$ is isomorphic to F and such that every vertex in G is in exactly one element of $\mathcal{F}$. Let C_t^3 denote the loose cycle on $t = 2s$ vertices, the 3-uniform hypergraph obtained by replacing the edges $e = \{u, v\}$ of a graph cycle C on s vertices with edge triples $\{u, x_e, v\}$, where x_e is uniquely assigned to e. This dissertation proves for even $t \geq 6$, that any sufficiently large 3-uniform hypergraph H on $n \in t\mathbb{Z}$ vertices with minimum 1-degree $\delta^1(H) \geq \binom{n-1}{2} - \binom{n-\lceil\frac{t}{4}\rceil\frac{n}{t}}{2} + c(t,n) + 1$, where $c(t, n) \in \{0, 1, 3\}$, contains a perfect C_t^3-tiling. The result is tight, generalizing previous results on C_4^3 by Han and Zhao. For an edge colored graph G, let the minimum color degree $\delta^c(G)$ be the minimum number of distinctly colored edges incident to a vertex. Call G rainbow if every edge has a unique color. For $\ell \geq 5$, this dissertation proves that any sufficiently large edge colored graph G on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$ contains a rainbow cycle on ℓ vertices. The result is tight for odd ℓ and extends previous results for $\ell = 3$. In addition, for even $\ell \geq 4$, this dissertation proves that any sufficiently large edge colored graph G on n vertices with $\delta^c(G) \geq \frac{n+c(\ell)}{3}$, where $c(\ell) \in \{5, 7\}$, contains a rainbow cycle on ℓ vertices. The result is tight when $6 \nmid \ell$. As a related result, this dissertation proves for all $\ell \geq 4$, that any sufficiently large oriented graph D on n vertices with $\delta^+(D) \geq \frac{n+1}{3}$ contains a directed cycle on ℓ vertices. This partially generalizes a result by Kelly, Kühn, and Osthus that uses minimum semidegree rather than minimum out degree.

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INTRODUCTION

Since the advent of computers, the NP-complete class of problems has been of interest as there is no known fast solution, and yet no proof that a solution must be slow. These problems are often encountered and provide major limitations on what is computable in practice. As a result, it is common to search for solutions that work effectively on a restriction of NP-complete problems.

The NP-complete problem of interest to us is the subgraph isomorphism problem: given hypergraphs F and G, is F isomorphic to a subgraph of G? Many problems in extremal graph theory attack a variant of the subgraph isomorphism problem: given hypergraphs F and G, does G have enough edges to guarantee F is isomorphic to a subgraph of G? Answering questions of this form provides a condition where the subgraph isomorphism problem may be easily answered, and additionally, the proofs may reveal fast algorithms that apply to a significant restriction of the subgraph isomorphism problem. In this dissertation, we continue work on this problem by attempting to answer questions of the form: if F is a collection of vertex disjoint loose 3-cycles on t vertices, what vertex degree conditions on a 3-graph G guarantee that F is isomorphic to a subgraph of G? In particular, we focus on two different minimum degree conditions.

The first minimum degree condition we consider is a minimum 1-degree condition. In Theorem 1.2.12, we prove a tight minimum 1-degree bound $\delta^1(n)$ for which all sufficiently large 3-graphs H on $n \in t\mathbb{Z}$ vertices with $\delta^1(H) \geq \delta^1(n)$ contain a perfect tiling with the loose cycles on $t \geq 6$ vertices. The introductory material for this result is contained in Section 1.2 and the proof in Chapter 2.

The second minimum degree condition we consider interprets finding a loose 3-cycle as finding a rainbow cycle, a cycle in which every edge has a unique color, in an edge colored graph. From Theorems 1.3.4 and 1.3.5, we obtain a minimum color degree bound $\delta^c(n)$ for which a sufficiently large edge colored graph G on n vertices with a minimum color degree at least $\delta^c(n)$ must contain a rainbow cycle. The bound we obtain is tight for all cycles with an odd number of vertices and cycles with an even number of vertices when the number of vertices is not divisible by three. Using the result on rainbow even length cycles, we obtain in Theorem 1.3.6 a minimum out degree condition on sufficiently large oriented graphs for the existence of directed cycles on at least 4 vertices. The directed graph result is tight for any directed cycle whose length is not divisible by 3 as well. The introductory material for these results is contained in Section 1.3. The proof for Theorem 1.3.4, which provides a color degree bound that is tight for odd length cycles, is contained in Chapter 3. The remaining results are proved in Chapter 4.

1.1 Definitions and Notation

This section gives an overview of the standard notation used throughout this dissertation. It is intended as a concise reference for when the reader encounters unfamiliar notation.

1.1.1 Standard Notation Paradigm

The notation we use is standard. We define $[n] := \{1, \ldots, n\}$ and for a set V we define $\binom{V}{k}$ as all subset of V of size k. For convenience, indices which run from 1 to ℓ are always considered modulo ℓ, e.g., if we have a sequence $v_1, \ldots, v_\ell$, then $v_1 = v_{\ell+1}$ and $v_0 = v_\ell = v_{-\ell}$. In addition, when a set $V = \{v\}$ has size one, we may refer to V as its element v instead. Throughout the dissertation, we write $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants α, β, and γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose $\beta \leq f(\gamma)$ and $\alpha \leq g(\beta)$, all calculations needed in our proof are valid. Longer hierarchies are defined in the obvious way.

1.1.2 Definitions of Various Graph Types

A hypergraph is a pair $H = (V, E)$ of vertices V and edges E where for all edges $e \in E$, $e \subseteq V$. A k-graph is a hypergraph where $E \subseteq \binom{V}{k}$. We refer to 2-graphs as graphs in this dissertation. Most results in this dissertation focus on graphs and 3-graphs. A multigraph is a graph where E is a multiset which allows duplicate edges. If a multigraph has no duplicate edges, then the graph is called simple. In particular, all graphs are simple multigraphs. A directed graph (digraph) is a graph with the additional restriction that there is an ordering associated with each edge $e \in E$. An oriented graph is a digraph such that there is no directed edge uv for which the directed edge vu also exists.

1.1.3 Graph Notation Transcending Graph Type

For all hypergraph/digraphs/multigraphs $H = (V, E)$ we define the following. Let $V(H) = V$, $|H| = |V|$, $E(H) = E$, and $||H|| = |E|$. If $\mathcal{M}$ is a collection of subgraphs of H, we use $V(\mathcal{M})$ and $E(\mathcal{M})$ to denote $\bigcup_{M\in\mathcal{M}} V(M)$ and $\bigcup_{M\in\mathcal{M}} E(M)$ respectively. The graph H is called k-partite if there exists a partition of $V(H)$ into k sets $V_1 \cdots V_k$ such that for all edges $e \in E(H)$, $|e \cap V_i| \leq 1$ for $i \in [k]$. Alternatively when $k = 2$, H is called bipartite. The notation $H[V]$ denotes the subgraph induced by edges of H contained in V, and if H is a graph, $H[V_1, V_2]$ denotes the bipartite subgraph induced in H with bipartition (V_1, V_2). Let $E(V) = E(H[V])$. An edge $e \in E(H)$ is incident to a vertex v if $v \in e$. If H' is a subgraph H, also denoted $H' \subseteq H$, we say that the graph H' spans H if $V(H') = V(H)$.

1.1.4 Hypergraph Notation

In addition, we define the following if H is a k-graph. An edge $e = \{x_1, \ldots, x_k\} \in H$ has form $(X_1, \ldots, X_k)$ if $x_i \in X_i$ for all $i \in [k]$. Define $E_H(X_1, \ldots, X_k) = \{e \mid e \in$

$E(H)$ has form $(X_1, \ldots, X_k)\}$. For a set $S \subseteq V(H)$, define the neighborhood of S to be $N_H(S) = \{e \setminus S \mid e \in E_H(S, V(H), \ldots, V(H))\}$ and the neighborhood of S in U as $N_H(S, U) = \{e \setminus S \mid e \in E_H(S, U, \ldots, U)\}$. Define the degree of S in H as $d_H(S) = |N_H(S)|$. When the graph H is obvious the H subscripts may be dropped. Define the minimum t-degree, $\delta^t(H) = \min_{S \subseteq \binom{V(H)}{t}} d(S)$. Similarly we define the maximum t-degree as, $\Delta^t(H) = \max_{S \in \binom{V(H)}{t}} d(S)$. One result of these definitions is that $||H|| = \delta^0(H) = \delta^0(H)$. Throughout this dissertation, we may drop the superscript t when $t = 1$. For a k-graph F, define $ex_t(F, n)$ to be the smallest integer such that if H is a k-graph satisfying $\delta^t(H) > ex_t(F, n)$, then F is a subgraph of H. Finally, if H is a bipartite 2-graph with bipartition (A, B) and minimum vertex cover W, the type of W is (a, b) if $|W \cap A| = a$ and $|W \cap B| = b$.

1.1.5 Edge Colored Graph Notation

Let G be a graph. We call a function c from $E(G)$ to another set an *edge-coloring of* G. For a graph G with edge coloring c, let the color degree of a vertex $d^c(v) = |c(E_G(v))|$ denote the number of distinct edge colors among edges incident to v. Define the minimum and maximum color degree of G, $\delta^c(G)$ and $\Delta^c(G)$ respectively, as the minimum/maximum over all vertices in G. An edge colored graph is called rainbow if all edges have a unique color. An edge-coloring c is *proper* if $d_G(v) = d^c_G(v)$ for every $v \in V(G)$.

1.1.6 Directed Graph Notation

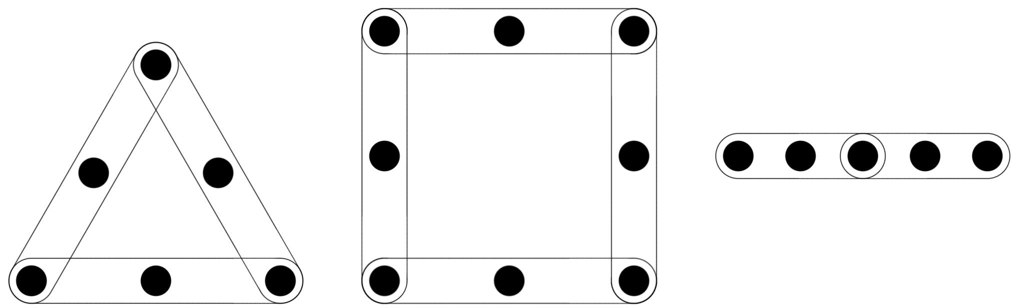
Let D be a digraph. The *simple underlying graph G* is the graph formed by removing the orientation from the edges of D, i.e., $V(D) = V(G)$ and $E(G) = \{\{u, v\} : (u, v) \in E(D)\}$. For a vertex $v \in D$, let the out neighborhood of v be $N^+_D(v) = \{u \in V(D) \ : vu \in E(D)\}$, the out degree of v be $d^+(v) = |N_D(v)|$, the in neighborhood of v be $N^-_D(v) = \{u \in V(D) \ : uv \in E(D)\}$, and the in degree of v be $d^-(v) = |N_D(v)|$. In addition, let

$N_D^+(v,U) = \{u \in U \mid vu \in E(D)\}$ and $N_D^-(v,U) = \{u \in U \mid uv \in E(D)\}$. When the directed graph D is obvious, the subscripts may be dropped. Define the semidegree of v to be $d^0(v) = \min(d^-(v), d^+(v))$. Similar to above, define the minimum out degree $\delta^+(D)$, the minimum in degree $\delta^-(D)$, and the minimum semidegree $\delta^0(D)$ as the minimum value of $d^+(v)$, $d^-(v)$, and $d^0(v)$ over all vertices v respectively. Also, define the maximum out degree $\Delta^+(D)$, the maximum in degree $\Delta^-(D)$, and the maximum semidegree $\Delta^0(D)$ as the maximum value of $d^+(v)$, $d^-(v)$, and $d^0(v)$ over all vertices v respectively.

1.1.7 Commonly Used Graphs

Along with the above graph properties we use the following notation for certain hypergraphs and digraphs that show up throughout this dissertation. A (di)graph G is a (directed) path on i vertices if there exists a sequence of distinct vertices $v_1v_2\cdots v_t$, and $E(G) = \{v_iv_{i+1}|i \in [t-1]\}$. A (di)graph G is (directed) cycles on t vertices if there exists a sequence of distinct vertices $v_1v_2\cdots v_t$ such that $E(G) = \{v_iv_{i+1}|i \in [t]\}$. We use (directed) P_t and (directed) C_t to denote the (directed) path and (directed) cycle on t vertices respectively. We call a 3-graph H loose if it can be constructed from a multigraph G by replacing each edge $e = \{u, v\} \in G$ with an edge triple $\{u, w_e, v\}$, where $w_e \notin V(G)$ is uniquely assigned to e. Let P_t^3 and C_t^3 denote the loose 3-graphs on t vertices obtained from a path and a cycle respectively. We refer to the graphs P_t^3 as loose paths and the graphs C_t^3 as loose cycles. In particular, we define C_4^3 which is obtained from the multigraph C_2, the multigraph on two vertices with exactly two edges. Because of the construction of P_t^3 and C_t^3, t must be odd for P_t^3 and t must be even for C_t^3. A graph is called complete if all possible edges exist and we use K_n^k to denote the complete k-graph on n vertices, K_n to denote K_n^2, $K_{a,b}$ the complete bipartite graph with partitions of size a and b, and finally $K_{a,b,c}^3$ the complete 3-partite 3-graph with partitions of size a, b, and c.

The *n-vertex blow-up of a directed ℓ-cycle* is the directed graph on n vertices for which



(a) C_6^3 (b) C_8^3 (c) P_5^3

Figure 1.1: Examples of Loose 3-Graphs

there exists a partition $V_1, \ldots, V_\ell$ such that, for $i \in \ell$, $|V_i| \in \{\lfloor \frac{n}{\ell} \rfloor, \lceil \frac{n}{\ell} \rceil\}$, $|E(V_i)| = 0$, and $E(V_i, V_{i+1}) = \{(u,v) : u \in V_i \text{ and } v \in V_{i+1}\}$. An ℓ-*walk* in a directed graph G is a sequence $v_1, \ldots, v_\ell$ of not necessarily unique vertices such that $v_i v_{i+1} \in E(G)$ for every $i \in [\ell - 1]$, and it is a *closed* ℓ-*walk* if $v_1 = v_\ell$. We use analogous terminology for paths and cycles in simple graphs. We call a 3-cycle a *directed triangle*, and a three vertex digraph with vertex set $\{u, v, w\}$ and edge set $\{uv, uw, vw\}$ is called a *transitive triangle*.

1.2 Previous Results on Hypergraph Tilings

For a k-graph F, an F-*tiling* of a k-graph $H = (V, E)$ is a partition of a set $W \subseteq V$ into $q := \frac{|W|}{|F|}$ sets $W_1, \ldots, W_q$, each of size $|F|$, so that for every $i \in [q]$, $H[W_i]$ contains F. We say that a vertex $v \in V(H)$ is covered by an F-tiling if v is contained in one of the sets W_i. We say that H has a *perfect* F-*tiling* (or is F-*tileable*) if H has an F-tiling for $W = V$, in particular a perfect F-tiling corresponds to spanning subgraph composed of vertex disjoint copies of F.

Questions on graph tilings are central questions in extremal graph theory. Some of the simplest results deal with finding K_2-tilings, also known as matchings. Two fundamental results, a theorem by Hall [11] characterizing K_2-tilings on bipartite graphs and a theorem by Tutte [29] characterizing K_2-tilings on all graphs, are especially notable in this case.

Theorem 1.2.1 (Hall, 1935). *A bipartite graph with bipartition (A, B) has a K_2-tiling covering all vertices in A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

Theorem 1.2.2 (Tutte, 1947). *A graph G has a 1-factor if and only if $q(G - S) \leq |S|$ for all $S \subseteq V(G)$, where the function $q(G)$ counts the number of connected components on an odd number of vertices.*

An exact minimum degree condition is known for K_2-tilings as well.

Theorem 1.2.3. *If G is a graph with $|G| \in 2\mathbb{Z}$ and $\delta(G) \geq \frac{n}{2}$, then G has a perfect K_2-tiling.*

The proof of Theorem 1.2.3 is often derived from a theorem of Dirac [7] which provides a minimum degree condition for G to contain a spanning cycle, also known as a Hamiltonian cycle.

Theorem 1.2.4 (Dirac, 1952). *If G is a graph on $n \geq 3$ vertices and $\delta(G) \geq \frac{n}{2}$, then G contains a Hamiltonian cycle.*

Theorem 1.2.3 then follows from noting that if $|G|$ is even, a Hamiltonian cycle contains a perfect K_2-tiling. Generalizations to other graphs are often considered, but these problems are fundamentally harder. There are efficient polynomial time algorithms for finding maximum K_2-tilings [8], but finding maximum tilings of larger graphs is NP-hard [20]. This fact has major implications as proving a useful characterization for tilings becomes more difficult. In addition, when a characterization exists, it is difficult to use since identifying a graph that satisfies the characterization is an NP-hard problem as well. Because of this, most research on larger graph tilings focuses on finding sufficient conditions, similar to the minimum degree condition in Theorem 1.2.3. One such example on larger graphs is the Corrádi-Hajnal theorem [4] which gives an exact bound for cycles on 3 vertices.

Theorem 1.2.5 (Corrádi & Hajnal, 1963)**.** *If G is a graph on $n \in 3\mathbb{Z}$ vertices such that $\delta(G) \geq \frac{2n}{3}$, then G has a perfect C_3-tiling.*

Generalizations to tilings in hypergraphs are also being researched. One result by Rödl, Ruciński, and Szemerédi [26] gives a tight generalization of Theorem 1.2.3 to hypergraphs.

Theorem 1.2.6 (Rödl, Ruciński & Szemerédi, 2009)**.** *If H is sufficiently large k-graph on $n \in k\mathbb{Z}$ vertices with $\delta^{k-1}(H) \geq \frac{n}{2} - k + C$, where $C \in \{3, \frac{5}{2}, \frac{3}{2}, 2\}$ and depends on the divisibility of n and k, then H contains a perfect K_k^k-tiling.*

Generalizations to other degree conditions also exist. One result by Treglow and Zhao [27] determines an exact condition $\delta(n, 4r, \ell)$ which is asymptotically close to $(\frac{1}{2}+o(1))\binom{n}{k-\ell}$ for which the following applies:

Theorem 1.2.7 (Treglow & Zhao, 2012)**.** *Let $r, \ell \in \mathbb{N}$ such that $2r \leq \ell \leq 4r - 1$. If H is a sufficiently large $4r$-graph on $n \in 4r\mathbb{Z}$ vertices with $\delta^{\ell}(H) > \delta(n, 4r, \ell)$, then H contains a perfect K_k^k-tiling.*

The problem for determining an exact minimum ℓ-degree with $\ell < k - 1$ for which all k-graphs H satisfying the minimum ℓ-degree contain a perfect K_k^k-tiling is still an open problem in many cases, although some other approximate results exist. The fact that the general ℓ-degree problem is still open, but that the $k - 1$-degree has been solved is a common situation for tiling problems in k-graphs as smaller degree bounds appear to be harder problems.

Generalizations of problems similar to Theorem 1.2.5 have also been considered. In the case of loose cycles on four vertices, Kühn and Osthus [21] prove the following asymptotic result.

Theorem 1.2.8 (Kühn & Osthus, 2006)**.** *Let H be a 3-graph on $n \in 4\mathbb{Z}$ vertices. If $\delta^2(H) \geq (\frac{1}{4} + o(1))n$, then H has a perfect C_4^3-tiling*

This was improved by Czygrinow, DeBiasio, and Nagle [6] who got rid of the $o(1)$ term and showed the following tight result:

Theorem 1.2.9 (Czygrinow, DeBiasio, & Nagle, 2014)**.** *There is an integer n_0 such that if H is a 3-graph on n vertices with $n \in 4\mathbb{Z}$, $n \geq n_0$, and*

$$\delta^2(H) \geq \begin{cases} \dfrac{n}{4} & \text{if } \frac{n}{4} \text{ is odd} \\ \dfrac{n}{4} + 1 & \text{if } \frac{n}{4} \text{ is even} \end{cases},$$

then H has a perfect C_4^3-tiling.

A more general tight result is also known and was proved in [5] (and independently by Mycroft in [25] with an $o(n)$ error term in the degree condition.)

Theorem 1.2.10 (Czygrinow, 2016)**.** *For every even integer $t \geq 6$, there is an integer n_0 such that if H is a 3-graph on n vertices with $n \in t\mathbb{Z}$, $n \geq n_0$, and $\delta^2(H) \geq \frac{\lceil \frac{t}{4} \rceil}{t} n$, then H has a perfect C_t^3 tiling.*

Analogous statements which involve $\delta^1(H)$ rather than $\delta^2(H)$ can be more difficult to prove similar to the K_k^k case. Han and Zhao [13] (and independently [6]) proved a best possible analog of Theorem 1.2.9 with δ^1 in lieu of δ^2.

Theorem 1.2.11 (Han & Zhao, 2015)**.** *There is an integer n_0 such that if H is a 3-graph on n vertices with $n \in 4\mathbb{Z}$, $n \geq n_0$, and $\delta^1(H) \geq \binom{n-1}{2} - \binom{\frac{3n}{4}}{2} + \frac{3n}{8} + \frac{1}{2}$, then H has a perfect C_4^3-tiling.*

Recently Han, Zang, and Zhao also proved an asymptotic minimum 1-degree bound for a perfect $K_{a,b,c}$-tiling [12]. A loose cycle is a 3-partite 3-graph, so this result implies a bound on C_t^3 which is also the asymptotic bound for C_t^3. In this dissertation we prove an analog of Theorem 1.2.10, we give an exact minimum degree condition for the existence of a C_t^3-tiling. For $t \in 2\mathbb{Z}$ and $n \in t\mathbb{Z}$, define the functions

$$c(t,n) = \begin{cases} 0 & \text{if } 4 \nmid t \\ 1 & \text{if } 4 \mid t \text{ and } 4 \nmid \frac{3}{4}n + 1 \\ 3 & \text{if } 4 \mid t \text{ and } 4 \mid \frac{3}{4}n + 1 \end{cases}$$

and

$$\delta(n) = \binom{n-1}{2} - \binom{n - \lceil \frac{t}{4} \rceil \frac{n}{t}}{2} + c(t,n) + 1.$$

The main result of Chapter 2 is the following theorem.

Theorem 1.2.12 (Czygrinow & Oursler). *For every even integer $t \geq 6$, there is an integer n_0 such that if H is a 3-graph on n vertices with $n \in t\mathbb{Z}$, $n \geq n_0$, and $\delta^1(H) \geq \delta(n)$, then H has a perfect C_t^3-tiling.*

Proposition 1.2.13. *Theorem 1.2.12 is best possible for sufficiently large n.*

Proof. Consider the following construction.

Construction 1.2.14. *Let $H = (V, E)$ be a 3-graph where $|V| = n$. Let V_1 and V_2 be a partition of H such that $|V_1| = \lceil \frac{t}{4} \rceil \frac{n}{t} - 1$ and $|V_2| = n - |V_1|$. Let H contain all edges $e \in \binom{V}{3}$ such that $e \cap V_1 \neq \emptyset$. Additionally let $H[V_2]$ contain edges as follows:*

- *If $4 \nmid t$, let $H[V_2]$ be the empty 3-graph.*
- *If $4 \mid t$ and $4 \nmid |V_2|$, let $H[V_2]$ contain $v_1, v_2 \in V_2$ and all edges of the form (v_1, v_2, V_2).*
- *If $4 \mid t$ and $4 \mid |V_2|$, let $H[V_2]$ be a perfect tiling of K_4^3.*

The minimum degree in the construction is achieved by a vertex v in V_2. Since v can be in at most $\binom{n-1}{2}$ edges and at most $\binom{n - \lceil \frac{t}{4} \rceil \frac{n}{t}}{2}$ edges are contained in V_2, we get that

$$\delta^1(H) = \binom{n-1}{2} - \binom{n - \lceil \frac{t}{4} \rceil \frac{n}{t}}{2} + \delta^1(H[V_2]) = \delta(n) - 1.$$

A minimum vertex cover of C_t^3 has size $\lceil \frac{t}{4} \rceil$. When $4 \nmid t$, $H[V_2]$ is empty so every C_t^3 in H contains at least $\lceil \frac{t}{4} \rceil$ vertices in V_1. Additionally when $4 \mid t$, the deletion of any matching from C_t^3 does not change the size of a minimum vertex cover. Since $H[V_2]$ contains no P_5^3, every C_t^3 in H still contains at least $\lceil \frac{t}{4} \rceil$ vertices in V_1. Thus a perfect C_t^3-tiling would use at least $\lceil \frac{t}{4} \rceil \frac{n}{t} > |V_1|$ vertices in V_1, a contradiction. $\square$

The proof of Theorem 1.2.12 uses the so-called absorbing method which usually consists of three components: finding a large tiling in a 3-graph which is non-extremal, proving an absorbing lemma, and finding a perfect tiling in the extremal case. The extremal case occurs when H is close to the graph in Construction 1.2.14. It is the first component of the proof, finding a large tiling, which requires the most substantial argument. The proof of these results is contained in Chapter 2.

1.3 Previous Results on Rainbow Cycles and Digraphs

Let H and F be k-graphs. We say that H is F-free if H does not contain a copy of F. Another central problem in extremal graph theory poses the question: if F is a fixed k-graph, under what conditions is H not F-free? One common example of this is calculating the value of $ex_t(F, n)$, which is the maximum t-degree such that H can be F-free. In particular if $\delta^t(H) > ex_t(F, n)$, then H contains a copy of F.

Since F is a fixed graph, this problem does not have the same algorithmic issues as tiling problems. A brute force polynomial time algorithm can be used to find F by iterating over $\binom{V(H)}{|F|}$ and search for a copy of F in each subset of $V(H)$. Determining these results is still necessary as they appear often in other proofs. An example of this occurs in the proof of Theorem 1.2.12 on loose cycle tilings as knowing the value of $ex_1(P_5^3, n)$ is required in the proof. This occurs since the $H[V_2]$ in Construction 1.2.14 must be a P_5^3-free 3-graph when 4 divides t. A seminal result in this line of research is a theorem by Turán [28] determining the value of $ex_0(K_r, n)$.

Theorem 1.3.1 (Turán, 1941). *If G is a K_{r+1}-free graph on n vertices, then $||G|| \leq (1 - \frac{1}{r})\frac{n^2}{2}$.*

In regards to this dissertation, the following result by Erdős [9] implies that C_t^3-free 3-graphs H on n vertices have $o(n^3)$ edges as C_t^3 is a 3-partite 3-graph.

Theorem 1.3.2 (Erdős, 1964). *Let $K^\ell_{a,\cdots,a}$ denote the complete k-partite k-graph with parts of size a. For a k-graph H on n vertices with n sufficiently large, if $||H|| \geq n^{k-\frac{1}{a^{k-1}}}$, then H contains $K^\ell_{a,\cdots,a}$.*

We consider a degree condition differing from $ex_t(F, n)$. Let H be a 3-graph with a partition of the vertices of H into sets V and C such that all edges in H are of the form (V, V, C). We can reinterpret the process of finding a copy of C_t^3 as follows: let G be a graph with vertex set V and edges vv' if there exists an edge of the form (v, v', C) in H. Associate with each edge vv' in G a list of colors with value $N_H(\{v, v'\}) \subseteq C$. Then H contains C_t^3 if and only if there is an edge coloring c of G, where every edge vv' is colored with $c(vv') \in N_H(\{v, v'\})$, such that G contains a rainbow C_ℓ, a cycle where every edge is colored uniquely, for $t = 2\ell$. We are focusing on the question: given an edge coloring of a graph G, what is the minimum color degree such that G contains a rainbow C_ℓ?

Work on rainbow subgraph problems has a very long and rich history through its connection to transversals of Latin squares. A transversal of a given $n \times n$ Latin square is equivalent to a rainbow perfect matching in a particular proper edge-coloring of the complete bipartite graph with parts of size n that uses n colors, and a Latin square has an orthogonal mate if and only if it can be decomposed into disjoint transversals. There has been substantial recent breakthrough work on closely related questions (see [1], [10], and [19]). There has also been work related to the rainbow Turán number of various graphs H (first considered in [16]), which, for $n \in \mathbb{N}$, is defined to be the maximum number r for which there exists an n-vertex graph G with r edges and a proper edge-coloring of G such that G does not contain a rainbow copy of H.

Our focus is different as we consider edge-colorings that may be far from proper. One of our motivations is the following result which was proved independently, by Li [24] and Li, Ning, Xu, & Zhang [23].

Theorem 1.3.3 (Li, 2013 and Li, Ning, Xu & Zhang, 2014)**.** *If G is a graph on n vertices, c is an edge-coloring of G, and $\delta^c(G) \geq \frac{n+1}{2}$, then G contains a rainbow 3-cycle.*

In fact, in [23], it was proved that G contains a rainbow triangle when only the weaker condition $\sum_{v\in V(G)} d^c_G(v) \geq \frac{n(n+1)}{2}$ holds, and also that G contains a rainbow triangle when $\delta^c(G) \geq \frac{n}{2}$ unless either G is a complete bipartite graph with parts of size $\frac{n}{2}$, G is K_4, or G is K_4 minus an edge.

In Chapter 3 we extend Theorem 1.3.3 with large n to the following theorem.

Theorem 1.3.4 (Czygrinow, Molla, Nagle, & Oursler)**.** *For every $\ell \geq 5$ and $n \geq 200\ell$, if G is an edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$, then G contains a rainbow cycle of length ℓ.*

By considering the complete bipartite graph and an edge-coloring in which every edge is given a unique color, Theorems 1.3.3 and 1.3.4 prove a tight bound in the minimum degree condition for all cycles with odd length. The following related theorem on even length cycles is our main result in Chapter 4.

Theorem 1.3.5 (Czygrinow, Molla, Nagle, & Oursler)**.** *For every even $\ell \geq 4$, there exists $\alpha > 0$ and n_0 such that for every $n \geq n_0$ the following holds. If G is a graph on n vertices* *and c is an edge-coloring of G such that*

$$\delta^c(G) \geq \begin{cases} (\frac{1}{3} - \alpha)n & \textit{if } \ell = 0 \pmod 3 \\ \dfrac{n+5}{3} & \textit{if } \ell = 1 \pmod 3 \\ \dfrac{n+7}{3} & \textit{if } \ell = 2 \pmod 3, \end{cases} \tag{1.1}$$

then G contains a rainbow ℓ-cycle.

Theorem 1.3.5 is sharp in the minimum color degree condition when ℓ is not divisible by 3. (See Subsection 1.3.1 for further discussion.) Previously, Čada, Kaneko,. Ryjáček, and Yoshimoto [30] proved that if G is triangle-free and $\delta^c(G) \geq \frac{n}{3} + 1$, then G contains a rainbow 4-cycle.

As we describe in detail in Subsection 1.3.1, problems of this type have a close connection to similar results on directed graphs. In fact, with a proof that shares many of its arguments with our proof of Theorem 1.3.5, we also have the following result in Chapter 4.

Theorem 1.3.6 (Czygrinow, Molla, Nagle, & Oursler)**.** *For every $\ell \geq 4$, there exists n_0 such that for every $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and $\delta^+(G) \geq \frac{n+1}{3}$, then G contains a directed ℓ-cycle.*

By considering the blow-up of a directed triangle, Theorem 1.3.6 is sharp for every $\ell \geq 4$ that is not divisible by 3. For sufficiently large n, Theorem 1.3.6 is a partial generalization of the following theorem of Kelly, Kühn & Osthus.

Theorem 1.3.7 (Kelly, Kühn & Osthus, 2010 [18])**.** *For every $\ell \geq 4$ and every $n \geq 10^{10}\ell$ the following holds, if G is an oriented graph on n vertices and $\delta^0(G) \geq \frac{n+1}{3}$, then G contains an ℓ-cycle. Moreover, for every vertex $u \in V(G)$, there exists an ℓ-cycle that contains u.*

Note that the statement of the famous triangle case of the Caccetta-Häggkvist conjecture [3] is the same as the statement of Theorem 1.3.6 with $\ell = 3$ and no lower bound on n. The following theorem of Hladký, Král' & Norin gives the current best lower bound on the minimum out-degree that implies the existence of a directed triangle in an oriented graph.

Theorem 1.3.8 (Hladký, Král' & Norin, 2017 [14])**.** *If G is an oriented graph on n vertices and $\delta^+(G) \geq 0.3465n$, then G contains a directed triangle.*

Combining Theorem 1.3.8 with Theorem 1.3.6 implies that, for every $\ell \geq 3$, if G is an oriented graph on n vertices, n is sufficiently large, and $\delta^+(G) \geq 0.3465n$, then G contains an ℓ-cycle.

The following conjecture of Kelly, Kühn, & Osthus is also of interest, because, by arguments described in Section 1.3.1, an asymptotic proof of the conjecture with minimum semidegree replaced by minimum out-degree would immediately imply an asymptotically best possible result for rainbow cycles in edge-colored graphs. The conjecture has been proved asymptotically when ℓ is sufficiently large compared to k (for $k \leq 6$, by Kelly, Kühn, & Osthus [18] and, for $k \geq 7$, by Kühn, Osthus, & Piguet [22]).

Conjecture 1.3.9 (Kelly, Kühn, & Osthus, 2010 [18])**.** *Let $\ell \geq 4$ be a positive integer and let k be the smallest integer that is greater than 2 and does not divide ℓ. Then there exists an integer n_0 such that for every $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and $\delta^0(G) \geq \lceil \frac{n}{k} \rceil + 1$, then G contains an ℓ-cycle.*

1.3.1 Relationship Between Digraphs and Rainbow Subgraphs

It turns out there is a major connection between directed graphs and rainbow subgraphs. To begin with, consider the following coloring, which is a slight modification of a coloring used by Li [24] for rainbow cycles. Let G' be a directed graph, let G be the simple graph underlying G', and let c be the edge-coloring of G defined as follows. For every edge

$uv \in E(G)$ with uv a directed edge in G', define $c(uv) = v$ if vu is not a directed edge in G' and define $c(uv) = uv$ when $vu \in E(G)$ is also a directed edge. We call the pair (G, c) the *simple edge-colored graph determined by* G'.

Additionally for a graph F, let F' be a directed graph with F the simple underlying graph of F' such that for every vertex $v \in V(F)$, $|N^-_{F'}(v) \setminus N^+_{F'}(v)| \leq 1$. We call F' a 1-*in direction of* F. Note that a directed cycle is an example of a 1-in direction of a graph cycle.

Proposition 1.3.10. *Let F be a graph and let (G, c) be the simple edge-colored graph determined by a directed graph G' on n vertices. Then G' contains a* 1*-in direction of F if and only if G has a rainbow (or properly colored) copy of F.*

Proof. Let uv and $u'v'$ be distinct edges in G. If $w = c(\{u, v\}) = c(\{u', v'\})$, then it must be that $w \in \{u, v\}$ and $w \in \{u', v'\}$, so without loss of generality we may assume that $w = v = v'$. Then uv and $u'v' = u'v$ are both directed edges in G'. Therefore every properly colored subgraph of G is a rainbow subgraph of G. The conclusion follows since a subdigraph F' of G' has $|N^-_{F'}(v) \setminus N^+_{F'}(v)| \leq 1$ if and only if the graph underlying F' is properly colored in G. □

If (G, c) is the simple edge colored graph determined by a directed graph G', then for every $v \in V(G)$,

$$d^c_G(v) = \begin{cases} d^+_{G'}(v) + 1 & \text{if } |N^-_{G'}(v) \setminus N^+_{G'}(v)| > 0 \\ d^+_{G'}(v) & \text{otherwise.} \end{cases}$$

Therefore, when $3 \leq \ell \leq n$, k is the largest positive integer that does not divide ℓ, and G' is the n-vertex blow-up of a directed k-cycle, for the simple edge colored graph (G, c) determined by G' we have that $\delta^c(G) \geq \lfloor \frac{n}{k} \rfloor + 1$ for $k \geq 3$ and $\delta^c(G) \geq \lfloor \frac{n}{2} \rfloor$ for $k = 2$. The construction of G' implies that all directed cycles in G' must have length that is a multiple of k. Since k does not divide ℓ, Proposition 1.3.10 implies that (G, c) does not have a rainbow

ℓ-cycle. This yields a sharpness example for Theorem 1.3.4 along with Theorem 1.3.5 when $\ell \equiv 1 \pmod 3$ and $n \pmod 3 \in \{0,1\}$. With slight modification for other cases we get the following.

Proposition 1.3.11. *Theorem 1.3.5 is the best possible for sufficiently large n when 3 does not divide ℓ.*

As the actual construction contains a number of small modifications on (G, c), the proof of Proposition 1.3.11 is delayed to Section 4.2.

If F is a graph and F' is a 1-in direction of F, with Proposition 1.3.10, results on F'-free digraphs can be used to deduce lower bound results for rainbow F graphs. In addition, we get that certain rainbow F-free edge colored graphs can be used to deduce lower bounds for F'-free digraphs. In fact, there is a stronger result which is that the minimum out degree condition for digraphs to contain a 1-in direction of F is asymptotically equivalent to the minimum degree bound for an edge colored graph to contain a rainbow F. For this we use the following definition. Let G an n-vertex graph and c an edge-coloring of G, we say a directed graph G' is *associated with* with the pair (G, c) if

- $V(G') = V(G)$;
- $uv \in E(G')$ implies that $\{u, v\} \in E(G)$;
- for every $v \in V(G)$, we have that $d^+_{G'}(v) = d^c_G(v)$; and
- the edge set $E_G(v, N^+_{G'}(v))$ is rainbow.

We can always construct a directed graph associated with the pair (G, c) by making the out-neighborhood of every vertex $v \in V(G)$ some subset $U \subseteq V(G)$ of order $d^c_G(v)$ such that $E(v, U)$ is rainbow. Note that there can be many different directed graphs G' that are associated with a particular pair (G, c), and that G' may contain 2-cycles and therefore may not be an oriented graph. The following proposition provides a connection between results on 1-in directions on directed graphs and results on rainbow subgraphs in edge-colored graphs.

Proposition 1.3.12. *For every graph F and $\alpha > 0$, there exists n_0 such that for every $n \geq n_0$ the following holds. Let G be a graph on n vertices, let c be an edge-coloring of G and let G' be a directed graph associated with (G, c). If F' is a 1-in direction of F and G' contains at least $\alpha n^{|F|}$ copies of F', then G contains a rainbow F.*

The proof of Proposition 1.3.12 is delayed to Section 4.2. Let $f(n)$ denote the minimum out degree condition such that a directed graph G' satisfying $d^+(G') \geq f(|G'|)$ contains a 1-in direction of a graph F. Similarly, let $g(n)$ denote the minimum color degree condition such that an edge colored graph G with $\delta^c(G) \geq g(|G|)$ contains a rainbow F. From Proposition 1.3.10, we know that $f(n) \leq g(n)$, as a directed graph with no 1-in direction of F can be used to generate an edge colored graph with no rainbow F. On the other hand, standard arguments give that if a directed graph G' on n vertices has minimum degree $f(n) + \epsilon n$ for $\epsilon > 0$, then G' contains at least $\alpha n^{|F'|}$ 1-in directions F' for some $\alpha > 0$ when n is sufficiently large. Proposition 1.3.12 then proves that for sufficiently large graphs $f(n) \leq g(n) \leq f(n) + \epsilon(n)$, giving that $g(n) = f(n) + o(n)$.

Chapter 2

LOOSE CYCLE TILINGS

2.1 Proof of Theorem 1.2.12

We prove Theorem 1.2.12 with the absorbing method which consists of three components: finding a large tiling if the graph is not extremal, finding an absorbing set, and finding a perfect tiling if the graph is extremal. In this chapter, we give the following definition for β-extremal.

Definition 2.1.1. *A 3-graph H on n vertices is β-extremal if $V(H)$ can be partitioned into sets A and B so that $|B| = n - \lceil \frac{t}{4} \rceil \frac{n}{t}$ and $||H[B]|| \leq \beta |V|^3$.*

For convenience, define the following 3 functions which are used throughout the rest of this chapter in the computation of minimum degree conditions.

$$c(t,n) = \begin{cases} 0 & \text{if } 4 \nmid t \\ 1 & \text{if } 4 \mid t \text{ and } 4 \nmid \frac{3}{4}n + 1 \\ 3 & \text{if } 4 \mid t \text{ and } 4 \mid \frac{3}{4}n + 1 \end{cases}$$

$$\delta(n) = \binom{n-1}{2} - \binom{n - \lceil \frac{t}{4} \rceil \frac{n}{t}}{2} + c(t,n) + 1,$$

$$\delta_\epsilon(n) = \frac{(2t - \lceil \frac{t}{4} \rceil) \lceil \frac{t}{4} \rceil}{t^2} \binom{n}{2} - \epsilon n^2.$$

and note that one can show

$$\delta_\epsilon(n) < \delta(n) - \frac{\epsilon}{2} n^2$$

We prove the following three lemmas to accomplish the components of the absorbing method.

Lemma 2.1.2. *(Large Tiling) For all $\beta > 0$, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there exists n_0 such that if H is a 3-graph, $|H| \geq n_0$, $\delta(H) \geq \delta_\epsilon(|H|)$, and $\mathcal{M}$ is a maximum C_t^3-tiling, then $|V(H) \setminus V(\mathcal{M})| \leq n_0$ or H is β-extremal.*

Lemma 2.1.3. *(Extremal) There exists a $\beta_0 > 0$ such that if $\beta < \beta_0$ and H is a β-extremal 3-graph satisfying $\delta(H) \geq \delta(|H|)$, then H has a perfect C_t^3-tiling.*

Lemma 2.1.4. *(Absorbing) For every integer $t \geq 6$ and $\nu > 0$, there is $\xi > 0$ and n_0 such that the following holds. If H is a 3-graph on $n \geq n_0$ vertices which satisfies $\delta(H) \geq \delta(n)$, then there is a set $A \subset V(H)$ with $|A| \leq \nu n$, such that $H[A]$ is C_t^3-tileable and for every set $B \subseteq V(H) \setminus A$ with $|B| \in t\mathbb{Z}$ and $|B| < \xi n$, $H[A \cup B]$ is C_t^3-tileable.*

The proof of Lemmas 2.1.2, 2.1.3, 2.1.4 will be in Sections 2.2, 2.3, and 2.4 respectively. Of the three proofs, the proof of Lemma 2.1.2 requires the most substantial argument. We now prove Theorem 1.2.12, the main theorem of this chapter.

Theorem 1.2.12 (Czygrinow & Oursler)**.** *For every even integer $t \geq 6$, there is an integer n_0 such that if H is a 3-graph on n vertices with $n \in t\mathbb{Z}$, $n \geq n_0$, and $\delta^1(H) \geq \delta(n)$, then H has a perfect C_t^3-tiling.*

Proof. Let H be such that $\delta(H) \geq \delta(|H|)$ and $|H| \geq n_0$. Fix $\beta > 0$ small enough so that it satisfies Lemma 2.1.3 and fix ϵ small enough so that Lemma 2.1.2 is satisfied with β and ϵ having values $\frac{\beta}{2}$ and ϵ respectively. By Lemma 2.1.4, there exists $\eta > 0$ and a set $S \subset V(H)$ such that $|S| \leq \epsilon|H|$ and such that S can absorb any set T with $|T| \leq \eta|H|$. Let $H' = H[V(H) \setminus S]$. Then

$$\delta(H') \geq \delta(|H|) - \frac{\epsilon}{2}|H|^2 \geq \delta(H') - \frac{\epsilon}{2}|H'|^2 \geq \frac{(2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil}{t^2}\binom{|H'|}{2} - \epsilon|H'|^2$$

since $n \geq n_0$. Then by Lemma 2.1.2, either H' is $\frac{\beta}{2}$-extremal or H' has a C_t^3-tiling $\mathcal{M}$ using all but n_1 vertices for some constant n_1. If H' is $\frac{\beta}{2}$ extremal, it can be partitioned into two sets (A', B') such that $|B'| = |H'| - \lceil \frac{t}{4} \rceil \frac{|H'|}{t}$ and $||H'[B']|| \leq \frac{\beta}{2}|H'|^3$. Since $|S| \leq \epsilon|H| < \frac{\beta}{2}|H|$, we can partition the vertices of H into two sets (A, B) such that $|B| = |H| - \lceil \frac{t}{4} \rceil \frac{|H|}{t}$ and $||H[B]|| \leq \beta|H|^3$. Thus H is β-extremal and by Lemma 2.1.3, H contains a perfect C_t^3-tiling. Otherwise let U be the set of at most n_1 vertices not contained in the tiling $\mathcal{M}$ on H'. Since $|H| \geq n_0$, $|U| \leq n_1 \leq \eta|H|$. Thus by the choice of S, $S \cup U$ is C_t^3-tileable with a tiling $\mathcal{M}'$. Therefore H contains a perfect C_t^3-tiling consisting of $\mathcal{M} \cup \mathcal{M}'$, completing the proof. □

2.2 Large Tiling

The goal of this section is to show that if a 3-graph H satisfies $\delta(H) \geq \delta_\epsilon(|H|)$ for $\epsilon > 0$, then there exists a "very large" C_t^3-tiling or H is in an extremal configuration. In particular, we prove Lemma 2.1.2. We accomplish this by determining the characteristics of maximum C_t^3-tilings. Let $\mathcal{M}$ be a C_t^3-tiling in a 3-graph H and let $\epsilon > 0$. Define the following structures associated with H, $\mathcal{M}$, and ϵ:

- Define $U_{\mathcal{M}} = V(H) \setminus V(\mathcal{M})$ to be the vertices not covered by $\mathcal{M}$.
- Let $S \subset V(H)$. Define G_S to be the graph with $V(G_S) = V(H)$ where $v_1v_2 \in E(G_S)$ iff $|N_H(v_1, v_2) \cap S| > \epsilon|S|$.
- Define $F_{\mathcal{M}}$ to be the graph with $V(F_{\mathcal{M}}) = \mathcal{M}$ where the edge C_1C_2 is in $E(F_{\mathcal{M}})$ if $||G_{U_{\mathcal{M}}}[V(C_1), V(C_2)]|| \geq (2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil$ and $||G_{U_{\mathcal{M}}}[V(C_1), V(C_2)]||$ admits a minimum vertex cover Y whose type is not $(\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil)$.
- Define $G_{\mathcal{M}}$ to be the graph with $V(G_{\mathcal{M}}) = \mathcal{M}$ where the edge C_1C_2 is in $E(G_{\mathcal{M}})$ if $||G_{U_{\mathcal{M}}}[V(C_1), V(C_2)]|| = (2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil$ and $||G_{U_{\mathcal{M}}}[V(C_1), V(C_2)]||$ only admits minimum vertex covers Y with type $(\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil)$.

When the C_t^3-tiling $\mathcal{M}$ is obvious, the $\mathcal{M}$ subscripts may be dropped.

For a maximum C_t^3-tiling $\mathcal{M}$ on a 3-graph H such that $\delta(H) \geq \delta_\epsilon(|H|)$, to prove Lemma 2.1.2 we successively refine where edges can exist. In particular, we show that if $|U_\mathcal{M}|$ is larger than a constant n_0, then

1. $||F_\mathcal{M}||$ is small(Lemma 2.2.6)
2. $|E_H(V, U_\mathcal{M}, U_\mathcal{M})|$ is bounded (Lemma 2.2.7)
3. $G_\mathcal{M}$ is almost complete (Lemma 2.2.8).

Using the structure in $G_\mathcal{M}$, we then deduce that H is in an extremal configuration. The most demanding part of this proof is item 1.

To prove item 1, we consider a maximum C_t^3-tiling $\mathcal{M}$ and attempt to extend it by analyzing G_S. When $|S|$ is large enough, each edge $xy \in C \subseteq S$, where C is a graph cycle on s edges with $t = 2s$, can be associated with a unique vertex $z_{xy} \in S - V(C)$ with $xyz_{xy} \in H$. These edges form a loose cycle C_t^3 in H. Using this and a similar, but more complicated, construction required when s is odd, we show that $||H[U_\mathcal{M}]||$ is small in Lemma 2.2.3 and that when $||F_\mathcal{M}||$ is large we can construct a C_t^3-tiling $\mathcal{M}'$ with $|\mathcal{M}'| > |\mathcal{M}|$ in Lemma 2.2.6. To find the required subgraphs we need the following facts:

Fact 2.2.1. *For all $\alpha > 0$ and positive integers s, there exists n_0 such that if Q is a graph with $|Q| \geq n_0$ and $||Q|| \geq \alpha|Q|^2$, then Q contains $K_{s,s}$.*

Fact 2.2.2. *Let $\alpha > 0$, $\epsilon > 0$, and c be a positive integer. Then there exist n_0 and $\alpha' > 0$ such that if H is a 3-graph with $|H| = n \geq n_0$, $S \subseteq V(H)$, $|S| = \epsilon n$, and $||G_S|| \geq \alpha n^2$, then there exists a set of edges $E \subseteq G_S$ and vertices $V \subseteq S$ with $|E| \geq \alpha' n^2$ and $|V| \geq c$ such that for every $e \in E$ and $v \in V$, $e \cup \{v\} \in E(H)$.*

Proof. There are exactly $\binom{|S|}{c}$ subsets of S of size c. Since every edge in G_S intersects at least $\epsilon|S|$ vertices of S, there are at least $\alpha n^2 \binom{\epsilon|S|}{c}$ edges $e \in G_S$ and subsets $T \subset S$ of size

c such that $e \cup \{t\} \in E(H)$ for all $t \in T$. Thus on average a set T of size c appears in at least

$$\frac{\alpha n^2 \binom{\epsilon|S|}{c}}{\binom{|S|}{c}} \geq \alpha' n^2$$

many edges for some α' with value approximately $\alpha\epsilon^c$. Letting V be one of the subsets T of at least average size provides the desired set E. $\square$

We use Fact 2.2.1 and Fact 2.2.2 in the following lemma.

Lemma 2.2.3. *For all $\alpha > 0$ and $\epsilon > 0$, there exists n_0 such that if H is a 3-graph, $S \subseteq V(H)$, $U \subseteq V(H)$, and $H[S \cup U]$ is C_t^3-free, then one of the following is true: $|S| < n_0$, $|U| < n_0$, or $||G_U[S]|| < \alpha|S|^2$.*

Proof. Assume to the contrary, that $|U| \geq n_0$, $|S| \geq n_0$, and that $||G_U[S]|| \geq \alpha|S|^2$. Consider the case when C_t^3 contains an even number of edges. Then by Fact 2.2.1, $G_U[S]$ contains a complete bipartite graph $K_{||C_t^3||/2,||C_t^3||/2}$ and thus a cycle on $||C_t^3||$ edges. Since $|U| \geq n_0$, this cycle can be extended to create a copy of C_t^3 in $H[S \cup U]$ contradicting that $H[S \cup U]$ is C_t^3-free.

Otherwise C_t^3 contains an odd number of edges which makes things more difficult since we cannot guarantee the existence of an odd cycle in $G_U[S]$. Instead we will find a path $P = v_1 v_2 \cdots v_s$ in $G_U[S]$ which has the same number of edges as C_t^3 with the restriction that $|N_H(\{v_1, v_2\}, U) \cap N_H(\{v_{s-1}, v_s\}, U)| \geq s + 1$. If u is a vertex in $N_H(\{v_1, v_2\}, U \setminus \{v_1, \cdots v_s\}) \cap N_H(\{v_{s-1}, v_s\}, U \setminus \{v_1, \cdots v_s\})$, the path $P - v_1 - v_s$ can the be transformed into a loose path P' in $H[S \cup U]$ by adding a unique vertex from $U \setminus \{u, v_1, \cdots, v_s\}$ to each edge in $P - v_1 - v_s$. Adding the edges $\{u, v_1, v_2\}$ and $\{u, v_s, v_{s-1}\}$ to P' forms a copy of C_t^3. From applying Fact 2.2.2 and then Fact 2.2.1, there exists a complete bipartite graph $K_{a,a}$, which contains two disjoint edges e_1 and e_2 satisfying $|N_H(e_1, U) \cap N_H(e_2, U)| > s$. Thus we can construct such a path P, contradicting that $H[S \cup U]$ is C_t^3-free. $\square$

The above lemma immediately implies the following corollary by greedily extending a tiling on unmatched vertices.

Corollary 2.2.4. *For any $\alpha > 0$ there exist $\beta > 0$ and n_0 such that if $|H| > n_0$ and $||H|| > \alpha |H|^3$, then the size of a maximum C_t^3-tiling $\mathcal{M}$ covers at least $\beta |H|$ vertices.*

From here we will consider subgraphs in $F_{\mathcal{M}}$ in an attempt to find a larger C_t^3-tiling. By Corollary 2.2.4, we may assume that $|F_{\mathcal{M}}|$ is sufficiently large. Define an ordered bipartite graph F with ordered bipartition (A, B) to be a bipartite graph such that A and B are totally ordered sets. We say that two ordered bipartite graphs F and F', with ordered bipartitions (A, B) and (A', B') respectively, are equivalent, denoted $F = F'$, if the graphs are isomorphic under the isomorphism ϕ that maps A to A', B to B', preserves the order between A and A', and preserves the order between B and B'. An edge $e \in E(F)$ is equivalent to an edge $e' \in E(F')$ if e is mapped to e' under ϕ.

Lemma 2.2.5. *Fix a total ordering on the vertices of C for every $C \in \mathcal{M}$. For all $\alpha > 0$ and for all positive integers a, b, and c, there exists n_0 such that if H is a 3-graph with $|H| > n_0$, $\mathcal{M}$ is a C_t^3-tiling of H, and $F_{\mathcal{M}}$ contains αn^2 edges, then $F_{\mathcal{M}}$ contains a complete bipartite subgraph $K_{a,b}$ with bipartition (A, B) satisfying the following:*

(i) There exists an ordered bipartite graph F such that for all $C \in A$ and $C' \in B$,

$$F = G_{U_{\mathcal{M}}}[V(C), V(C')].$$

(ii) For all $e \in E(F)$, let E_e be the set of edges between cycles in A and cycles in B which are equivalent to e, and let $V_e = \bigcap_{e' \in E_e} N(e', U_{\mathcal{M}})$. Then $|V_e| \geq c$.

Proof. Orient the edges of $F_{\mathcal{M}}$ and color the oriented edges CC' with the equivalence class of ordered bipartite graphs containing $G_{U_{\mathcal{M}}}[V(C), V(C')]$. Since there are finitely many equivalence classes of ordered bipartite graphs on $2t$ vertices, there must be a set E of at least $8\alpha' n^2$ oriented edges with equivalent color for some $\alpha' > 0$. Let $F'_{\mathcal{M}}$ be the directed graph with edge set E. Let $F''_{\mathcal{M}}$ be the largest bipartite subgraph of $F'_{\mathcal{M}}$ with bipartition (A, B), and with edges directed only from A to B. Note that $||F''_{\mathcal{M}}|| \geq 2\alpha' n^2$. Let $F = G_{U_{\mathcal{M}}}[V(C), V(C')]$ for some directed edge $CC' \in F''_{\mathcal{M}}$ and label the edges of F so that $E(F) = \{e_1, \cdots e_m\}$. Construct a sequence of graphs $F_0, \ldots, F_m$ as follows. Let $F_0 = F''_{\mathcal{M}}$ and for every edge $e_i \in E(F)$, let E_{e_i} be the set of edges e that are equivalent to e_i and are between cycles that form an edge in F_{i-1}. Let F_i be a subgraph of F_{i-1} with the maximum number of edges such that $|\bigcap_{e \in E_{e_i}} N(e, U_{\mathcal{M}})| \geq c$. From Fact 2.2.2, we get that each F_i must contain at least $\alpha_i n^2$ edges for some $\alpha_i > 0$. Thus F_m contains at least $\alpha_m n^2$ edges for some $\alpha_m > 0$. By Fact 2.2.1, we can find a complete bipartite $K_{a,b} \subseteq F_m$ which satisfies both conditions (i) and (ii). $\square$

We will exploit the $K_{a,b}$ in the previous lemma to find a larger C_t^3-tilings in H.

Lemma 2.2.6. *For all $\alpha > 0$, there exist integers n_0 and n_1 such that if H is a 3-graph and $\mathcal{M}$ is a maximum C_t^3-tiling of H covering at least n_0 vertices, then $|U_{\mathcal{M}}| < n_1$ or $F_{\mathcal{M}}$ does not contain $\alpha|\mathcal{M}|^2$ edges.*

Proof. Assume to the contrary, that $\mathcal{M}$ is maximum, $|U_{\mathcal{M}}| \geq n_1$, and $F_{\mathcal{M}}$ contains at least $\alpha|\mathcal{M}|^2$ edges. From Lemma 2.2.5 with $a, b = 3\lceil \frac{t}{4} \rceil$, and $c = 6\lceil \frac{t}{4} \rceil (t - 2\lceil \frac{t}{4} \rceil)$, there exists $K \subseteq F_{\mathcal{M}}$ satisfying conditions (i) and (ii). Let F be the ordered bipartite graph with bipartition (A, B) found in condition (i). Let W be a minimum vertex cover of F, $k = |W \cap A|$, and $\ell = |W \cap B|$. Without loss of generality assume $k > \ell$.

Let $K' \subseteq K$ be a graph which duplicates the vertices corresponding to A $3\lceil \frac{t}{4} \rceil$ times and the vertices corresponding to B $2\lceil \frac{t}{4} \rceil$ times. Let A_i and B_i refer to the ith copy of the vertices respectively. Let $\mathcal{A}_j = \{A_i : (j-1)\lceil \frac{t}{4} \rceil < i \leq j\lceil \frac{t}{4} \rceil\}$ for $j \in [3]$ and $\mathcal{B}_j = \{B_i : (j-1)\lceil \frac{t}{4} \rceil < i \leq j\lceil \frac{t}{4} \rceil\}$ for $j \in [2]$. Let M_1 be a maximum matching in F, let M_2 be a maximum matching so that $V(M_1) \cap V(M_2) \cap B = \emptyset$, and let M_3 be a maximum matching

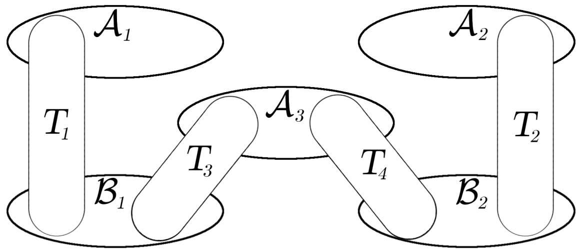


Figure 2.1: Intersection of $V(\mathcal{A}_i)$, $V(\mathcal{B}_i)$, and the Tilings T_j

so that $V(M_1) \cap V(M_3) \cap B = \emptyset$ and $V(M_2) \cap V(M_3) \cap A = \emptyset$. Let $p_i = |M_i|$ and note that $k + \ell = p_1 \geq k \geq p_2 \geq p_3$. Since $F = G_{U_\mathcal{M}}[V(C_1), V(C_2)]$ for all $C_1C_2 \in E(K')$, every edge $e \in E(F)$ induces a $K_{\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil}$ on edges equivalent to e in $G_{U_\mathcal{M}}[V(\mathcal{A}_i), V(\mathcal{B}_j)]$ for any pair $(\mathcal{A}_i, \mathcal{B}_j)$. Since M_1 is a matching, M_1 induces $K_{\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil}$-tilings T_i with p_1 elements in $G_{U_\mathcal{M}}[V(\mathcal{A}_i), V(\mathcal{B}_i)]$ for $i \in [2]$. Similarly for $i \in \{2, 3\}$, M_i induces tilings T_{i+1} with p_i elements in $G_{U_\mathcal{M}}[V(\mathcal{A}_3), V(\mathcal{B}_{i-1})]$. By construction of the matchings, $T = \bigcup_{i=1}^4 T_i$ is a $K_{\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil}$-tiling. Since $c = 6\lceil \frac{t}{4} \rceil (t - 2\lceil \frac{t}{4} \rceil)$, condition (ii) on K' implies that T can be extended using vertices of $U_\mathcal{M}$ to a $K_{\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil, t-2\lceil \frac{t}{4} \rceil}$-tiling $\mathcal{M}'$ of H with size $\min(|T|, 6\lceil \frac{t}{4} \rceil)$. Let $\mathcal{M}_i$ be the extension of T_i under the tiling $\mathcal{M}'$. As $C_t^3 \subseteq K_{\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil, t-2\lceil \frac{t}{4} \rceil}$, we can create a C_t^3-tiling by replacing the elements of $V(K') \subseteq \mathcal{M}$ which intersect with C_t^3-tilings induced by some set of the $\mathcal{M}_i$ in $\mathcal{M}$. The maximality of $\mathcal{M}$ implies that $\ell + k = |T_1| \leq 2\lceil \frac{t}{4} \rceil$, $\ell + k + p_2 = |T_1| + |T_3| \leq 3\lceil \frac{t}{4} \rceil$, and $2\ell + 2k + p_2 + p_3 = |T| \leq 5\lceil \frac{t}{4} \rceil$. Let $B' = B \setminus V(M_1)$, then we can bound the number of edges from A to B with

$$|E(A, B)| \leq t|W \cap B| + |W \cap A|^2 + |E(A, B')| \leq t\ell + k^2 + |E(A, B')|,$$

where the inequality follows by counting the maximum size of the sets $E(W \cap B, A)$, $E(W \cap A, V(M_1) \setminus W)$, and $E(A, B')$. We can also bound $E(A, B')$. Since M_2 is of maximum size and $V(M_3) \cap V(M_2) \cap A = \emptyset$ by construction, $V(M_3) \cap B' \subseteq V(M_2) \cap B'$. Therefore we obtain the bound

$$|E(A, B')| \leq p_2|B'| + p_2p_3 \leq p_2(t - (k + \ell)) + p_2p_3.$$

But then $|E(A,B)|$ is maximized when $k+\ell = 2\lceil \frac{t}{4} \rceil$, $p_2 = \lceil \frac{t}{4} \rceil$, and $p_3 = 0$, so

$$\begin{aligned}(2t - \lceil \tfrac{t}{4} \rceil)\lceil \tfrac{t}{4} \rceil \;\leq\; |E(A,B)| &\leq t\ell + k^2 + p_2(t-(k+\ell)) + p_2 p_3 \\ &\leq t(2\lceil \tfrac{t}{4} \rceil - k) + k^2 + \lceil \tfrac{t}{4} \rceil (t - 2\lceil \tfrac{t}{4} \rceil) \\ &= k^2 - tk + \lceil \tfrac{t}{4} \rceil (3t - 2\lceil \tfrac{t}{4} \rceil).\end{aligned}$$

Rearranging yields

$$0 \leq k^2 - tk + \lceil \frac{t}{4} \rceil (t - \lceil \frac{t}{4} \rceil) = (k - \lceil \frac{t}{4} \rceil)(k - (t - \lceil \frac{t}{4} \rceil)),$$

which is false for $\lceil \frac{t}{4} \rceil < k < (t - \lceil \frac{t}{4} \rceil)$ and only true when $t = 6$. But the calculation is exact, so $k = 2\lceil \frac{t}{4} \rceil$, $\ell = 0$, $p_2 = \lceil \frac{t}{4} \rceil$, and $p_3 = 0$. Then $F[V(M_1)]$ is a $K_{2\lceil \frac{t}{4} \rceil, 2\lceil \frac{t}{4} \rceil}$ and $F[V(M_2)]$ is a $K_{\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil}$. Create matchings M_1' and M_2' by adding the edges in M_2 to M_1 and removing the conflicting edges from M_1 into the matching M_2'. Under these matchings, $F[A, B \setminus V(M_1')]$ contains a $K_{2\lceil \frac{t}{4} \rceil, \lceil \frac{t}{4} \rceil}$. Thus we can find a matching M_3' on $F[A \setminus V(M_2'), B \setminus V(M_1')]$ with $|M_3'| = \lceil \frac{t}{4} \rceil$. But the argument above implies that the matchings M_1', M_2', and M_3' cannot exists, completing the proof. □

Thus we have completed the first step outlined at the beginning of this section, that if $|\mathcal{M}|$ is maximum and $|U_{\mathcal{M}}|$ is not bounded by a constant, then $||F_{\mathcal{M}}||$ must be small. Therefore when $|U_{\mathcal{M}}|$ is unbounded almost all pairs $C_1, C_2 \in \mathcal{M}$ have at most $(2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil$ edges, with equality when the minimum vertex cover has the same number of vertices in C_1 as in C_2 (which uniquely determines $G_{U_{\mathcal{M}}}[C_1, C_2]$). Now we move to limit the number of edges of the form $(V(\mathcal{M}), U_{\mathcal{M}}, U_{\mathcal{M}})$. Call a cycle $C \in \mathcal{M}$ α-big if there exist $(\lceil \frac{t}{4} \rceil + \alpha)\binom{|U_{\mathcal{M}}|}{2}$ edges of the form $(V(C), U_{\mathcal{M}}, U_{\mathcal{M}})$.

Lemma 2.2.7. *For all $\epsilon, \alpha > 0$ there exists n_0 such that if H is a 3-graph with $\mathcal{M}$ a maximum C_t^3-tiling, then $|U_{\mathcal{M}}| < n_0$ or there are fewer than $\epsilon|\mathcal{M}|$ α-big elements of $\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be a maximum C_t^3-tiling of H and assume that $|U_{\mathcal{M}}| \geq n_0$. Consider any α-big element $C \in \mathcal{M}$. Then there exists a set of vertices $A_C \subseteq V(C)$ such that $|A_C| = \lceil \frac{t}{4} \rceil$

and $|\bigcap_{v\in A_C} N(v,U_{\mathcal{M}})| \geq \binom{t}{\lceil \frac{t}{4}\rceil}^{-1}\binom{|U_{\mathcal{M}}|}{2}$. To show this count pairs (e,A) with $e \in \binom{U_{\mathcal{M}}}{2}$, $A \subseteq \binom{V(C)}{\lceil \frac{t}{4}\rceil}$, and $e \in \bigcap_{v\in A} N(v)$. Let s be the minimum possible value of this count over all possible 3-graphs. Note that if the count s is achieved there cannot exist edges $e_1, e_2 \in \binom{U_{\mathcal{M}}}{2}$ such that e_1 is in the neighborhood of more than $\lceil \frac{t}{4}\rceil$ vertices of C and e_2 is in the neighborhood of fewer than $\lceil \frac{t}{4}\rceil$ vertices since transferring a neighbor from e_1 to e_2 results in a smaller count. From the number of edges of the form $(V(C), U_{\mathcal{M}}, U_{\mathcal{M}})$, there exists an edge in $\binom{U_{\mathcal{M}}}{2}$ with at least $\lceil \frac{t}{4}\rceil + 1$ neighbors in C. Combined with the previous fact, this implies that $s > \binom{|U_{\mathcal{M}}|}{2}$. Since there are at most $\binom{t}{\lceil \frac{t}{4}\rceil}$ subsets of size $\lceil \frac{t}{4}\rceil$, there exists a set of $\lceil \frac{t}{4}\rceil$ vertices in C with intersection of size at least $\binom{t}{\lceil \frac{t}{4}\rceil}^{-1}\binom{|U_{\mathcal{M}}|}{2}$.

Since $|A_C| = \lceil \frac{t}{4}\rceil$, there also exists a vertex $v_C \in V(C)\setminus A_C$ such that $|N(v_C)\cap U_{\mathcal{M}}| \geq \alpha\binom{|U_{\mathcal{M}}|}{2}$. Let $B = \{v_C : C \in \mathcal{M}\}$. If the number of α-big vertices is at least $\epsilon|\mathcal{M}|$, then there are at least $\epsilon\alpha|\mathcal{M}|\binom{|U_{\mathcal{M}}|}{2}$ edges on $U_{\mathcal{M}} \cup B$. Since $|U_{\mathcal{M}}| \geq n_0$ by Lemma 2.2.3 and Corollary 2.2.4, there exists a copy C' of C_t^3 on $U_{\mathcal{M}} \cup B$. But for each $C \in \mathcal{M}$ intersecting with C', we can use each A_C and its intersection property to find disjoint copies of C_t^3 as well, contradicting the maximality of C. Thus there are fewer than $\epsilon|\mathcal{M}|$ α-big elements of $\mathcal{M}$. □

From this point on we start using assumptions about the minimum degree on a 3-graph H. As a reminder

$$\delta_\epsilon(n) = \frac{(2t - \lceil \frac{t}{4}\rceil)\lceil \frac{t}{4}\rceil}{t^2}\binom{n}{2} - \epsilon n^2.$$

Lemma 2.2.8. *For all $\epsilon > 0$, there exists n_0 such that if H is a 3-graph satisfying $\delta(H) \geq \delta_\epsilon(|H|)$ and $\mathcal{M}$ is a maximum C_t^3-tiling, then $|U_{\mathcal{M}}| \leq n_0$ or $||G_{\mathcal{M}}|| \geq (1-3\epsilon)\binom{|\mathcal{M}|}{2}$.*

Proof. Assume $|U_{\mathcal{M}}| \geq n_0$, let $n = |H|$, and let $W = V(\mathcal{M})$. Then it suffices to show that the claim holds if $||G_{U_{\mathcal{M}}}[W]|| \geq \delta_\epsilon(|W|) - \epsilon|\mathcal{M}|^2$. To see this, let $R_{\mathcal{M}}$ be the graph with vertices $\mathcal{M}$ and edges C_1C_2 such that $||G_{U_{\mathcal{M}}}[V(C_1), V(C_2)]|| < (2t - \lceil \frac{t}{4}\rceil)\lceil \frac{t}{4}\rceil$. From Lemma 2.2.6 with $\alpha = \frac{\epsilon}{4t}$, $||F_{\mathcal{M}}|| \leq \frac{\epsilon}{4t}|\mathcal{M}|^2$, so bounding the size of $||G_{U_{\mathcal{M}}}[W]||$ based on

$|\mathcal{M}|$, $||G_{\mathcal{M}}||$, $||F_{\mathcal{M}}||$, and $||R_{\mathcal{M}}||$ we get that

$$\begin{aligned}
||G_{U_{\mathcal{M}}}[W]|| &\leq ((2t - \lceil \tfrac{t}{4} \rceil)\lceil \tfrac{t}{4} \rceil)\tbinom{|\mathcal{M}|}{2} + t^2||F_{\mathcal{M}}|| + \tfrac{t^2}{2}|\mathcal{M}| - ||R_{\mathcal{M}}|| \\
&\leq \frac{(2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil}{t^2}\tbinom{|W|}{2} + \tfrac{1}{2}\epsilon t|\mathcal{M}|^2 - ||R_{\mathcal{M}}|| \\
&\leq \delta_\epsilon(|W|) + \tfrac{3}{2}\epsilon t|\mathcal{M}|^2 - ||R_{\mathcal{M}}||.
\end{aligned}$$

But $||G_{U_{\mathcal{M}}}[W]|| \geq \delta_\epsilon(|W|) - \epsilon|\mathcal{M}|^2$ implies that $||R_{\mathcal{M}}|| \leq \frac{5}{2}\epsilon t|\mathcal{M}|^2$. Since $F_{\mathcal{M}}$, $G_{\mathcal{M}}$, and $R_{\mathcal{M}}$ partition $\binom{\mathcal{M}}{2}$, we get that $||G_{\mathcal{M}}|| \geq (1 - 3\epsilon)\binom{|\mathcal{M}|}{2}$.

Consider the degree sum on $U_{\mathcal{M}}$ from which we obtain the following inequality

$$\delta_\epsilon(n)|U_{\mathcal{M}}| \leq |E(U_{\mathcal{M}}, W, W)| + 2|E(U_{\mathcal{M}}, U_{\mathcal{M}}, W)| + 3|E(U_{\mathcal{M}}, U_{\mathcal{M}}, U_{\mathcal{M}})|.$$

Because $|U_{\mathcal{M}}| > n_0$, by Lemma 2.2.3 we know that $|E(U_{\mathcal{M}}, U_{\mathcal{M}}, U_{\mathcal{M}})| \leq \epsilon\frac{|U_{\mathcal{M}}|^3}{3}$, and by Lemma 2.2.7 we know that

$$|E(U_{\mathcal{M}}, U_{\mathcal{M}}, W)| \leq (\frac{\lceil \frac{t}{4} \rceil}{t} + \epsilon)\frac{|U_{\mathcal{M}}|^2}{2}|W|.$$

Finally, we know that $|E(U_{\mathcal{M}}, W, W)| \leq |U_{\mathcal{M}}|(||G_{U_{\mathcal{M}}}[W]|| + \frac{\epsilon}{t^2}|W|^2)$. Using this information and rearranging the previous inequality to calculate a bound on $||G_{U_{\mathcal{M}}}[W]||$ gives:

$$\frac{1}{|U_{\mathcal{M}}|}(\delta_\epsilon(n)|U_{\mathcal{M}}| - \frac{\lceil \frac{t}{4} \rceil}{t}|U_{\mathcal{M}}|^2|W| - \frac{\epsilon}{t^2}|W|^2 - \epsilon|U_{\mathcal{M}}|^2|W| - \epsilon|U_{\mathcal{M}}|^3) \leq ||G_{U_{\mathcal{M}}}[W]||.$$

From the definition of δ_ϵ we get that

$$\begin{aligned}
\delta_\epsilon(n) &= \frac{(2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil}{t^2}\tbinom{n}{2} - \epsilon n^2 \\
&= \frac{(2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil}{t^2}(\tbinom{|W|}{2} + \tbinom{|U_{\mathcal{M}}|}{2} + |U_{\mathcal{M}}||W|) - \epsilon n^2 \\
&= \delta_\epsilon(|W|) + \delta_\epsilon(|U_{\mathcal{M}}|) + \frac{(2t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil}{t^2}|U_{\mathcal{M}}||W| - 2\epsilon|U_{\mathcal{M}}||W|.
\end{aligned}$$

Combined with the assumption $||G_{U_{\mathcal{M}}}[W]|| \leq \delta_\epsilon(|W|) - \epsilon|\mathcal{M}|^2 = \delta_\epsilon(|W|) - \frac{\epsilon}{t^2}|W|^2$, and canceling out like terms yields

$$\delta_\epsilon(|U_{\mathcal{M}}|) + \frac{(t - \lceil \frac{t}{4} \rceil)\lceil \frac{t}{4} \rceil}{t^2}|U_{\mathcal{M}}||W| - 3\epsilon|U_{\mathcal{M}}||W| - \epsilon|U_{\mathcal{M}}|^2 \leq 0.$$

The above inequality if obviously false, giving a contradiction and proving the claim. $\square$

From Lemma 2.2.8, we can see that if $|U_{\mathcal{M}}|$ is larger than a constant, $G_{\mathcal{M}}$ is almost a complete graph. To exploit this structure for a C_t^3-tiling $\mathcal{M}$, call a set $S \subseteq V(\mathcal{M})$ swappable if for any set $T \subseteq S$ with $|T| \leq t$ and $|T \cap V(C)| \leq 1$ for all $C \in \mathcal{M}$, then $H - T$ contains a C_t^3 tiling with at least $|\mathcal{M}|$ elements.

Lemma 2.2.9. *For all $(32\binom{t}{\lceil \frac{t}{4} \rceil})^{-1} > \epsilon > 0$, there exists n_0 such that if H is a 3-graph satisfying $\delta(H) \geq \delta_\epsilon(|H|)$ and $\mathcal{M}$ is a maximum C_t^3-tiling, then $|U_{\mathcal{M}}| \leq n_0$ or there exists a swappable set S such that $|S| \geq (t - \lceil \frac{t}{4} \rceil)(1 - 16\binom{t}{\lceil \frac{t}{4} \rceil}\epsilon)|\mathcal{M}|$.*

Proof. Let $\mathcal{M}$ be a maximum C_t^3-tiling, $|U_{\mathcal{M}}| \geq n_0$, and let $s = ||C_t^3|| = \frac{t}{2}$. As we are searching for a swappable set, we start by finding cycles on s vertices in $G_{U_{\mathcal{M}}}$ that allow us to generate alternative C_t^3-tilings. For an edge $C^*C_1 \in G_{\mathcal{M}}$, let W_{C^*,C_1} be the minimum vertex cover of $G_{U_{\mathcal{M}}}[V(C^*), V(C_1)]$. When s is even, since every vertex $w \in W_{C^*,C_1}$ satisfies $N(w) = V(C_1)$ or $N(w) = V(C^*)$ in $G_{U_{\mathcal{M}}}[V(C^*), V(C_1)]$ depending on whether $w \in V(C^*)$ or $w \in V(C_1)$ respectively, $G_{U_{\mathcal{M}}}[V(C^*), V(C_1)]$ contains two vertex disjoint cycles on s edges consisting of the s vertices in W_{C^*,C_1}, and any other subset of s vertices not in W_{C^*,C_1}.

When s is odd, consider the following structure in $G_{\mathcal{M}}$. Let C^*, C_1, and C_{s-1} be vertices in $G_{\mathcal{M}}$ such that $C_{s-1}C^*C_1$ is a path in $G_{\mathcal{M}}$ and $W^* = W_{C^*,C_1} \cap V(C^*) = W_{C^*,C_{s-1}} \cap V(C^*)$. In addition, let $C_1C_2 \cdots C_s$ be a cycle in $G_{\mathcal{M}}$ that does not contain the vertex C^*. For the vertices in the cycle, let $W_i^+ = W_{C_i,C_{i+1}} \cap V(C_i)$ and $W_i^- = W_{C_{i-1},C_i} \cap V(C_i)$. Let Y_i be a set such that $W_i^+ \cup W_i^- \subseteq Y_i \subseteq V(C_i)$ and such that $|Y_i| = 2\lceil \frac{t}{4} \rceil$ for $i \in [s]$. Then there is a matching of Y_i onto Y_{i+1} in $G_{U_{\mathcal{M}}}[V(C_i), V(C_{i+1})]$. To see this, since $|Y_i \setminus W_i^+| = \lceil \frac{t}{4} \rceil$, we can match $Y_i \setminus W_i^+$ with $W_{i+1}^- \subseteq Y_{i+1}$. In addition, since W_i^+ is in the minimum cover, we can match W_i^+ to the set $Y_{i+1} \setminus W_{i+1}^-$, providing the desired matching. These matchings provide $2\lceil \frac{t}{4} \rceil$ vertex disjoint paths on $s-1$ vertices between C_1 and C_{s-1}. Let $Y^- \subseteq Y_{s-1}$ be the $\lceil \frac{t}{4} \rceil$ end points of the paths that start with W_1^- and $Y^+ = Y_{s-1} \setminus Y^-$. Then there is a matching of Y^- into Y_s, and by the definition of W_1^-, we can close the paths starting in W_1^- to form $\lceil \frac{t}{4} \rceil$ vertex disjoint cycles on s vertices. Additionally, since $W^* = W_{C^*,C_{s-1}} \cap V(C^*)$, we can match Y^+ onto W^*. Since $W^* = W_{C^*,C_1} \cap V(C^*)$ as well, we can close the $\lceil \frac{t}{4} \rceil$ paths starting in $Y_1 \setminus W_1^-$ to find $\lceil \frac{t}{4} \rceil$ more vertex disjoint cycles. Thus there exist $2\lceil \frac{t}{4} \rceil$ vertex disjoint cycles in $G_{U_\mathcal{M}}$ on the $s+1 = 2\lceil \frac{t}{4} \rceil$ loose cycles C^*, C_1, …, C_s in $V(G_\mathcal{M})$ which intersect $V(C^*)$ only in W^*.

Now we will use the vertex disjoint cycles we found above to find a large swappable set. Fix $\alpha = (4\binom{t}{\lceil \frac{t}{4} \rceil})^{-1}$ and let X be the set of vertices in $G_\mathcal{M}$ with degree at least $(1-\alpha)|G_\mathcal{M}|$. Then using Lemma 2.2.8 we get that

$$2(1-3\epsilon)\binom{|G_\mathcal{M}|}{2} \leq 2||G_\mathcal{M}|| \leq |G_\mathcal{M}||X| + (1-\alpha)|G_\mathcal{M}||V(G_\mathcal{M})| \setminus X|.$$

Solving this relation for $|X|$ yields that

$$|X| \geq (1 - 4\frac{\epsilon}{\alpha})|G_\mathcal{M}| \geq (1 - 16\binom{t}{\lceil \frac{t}{4} \rceil}\epsilon)|G_\mathcal{M}|.$$

Then we claim that there exists a swappable set of vertices composed of at least $t - \lceil \frac{t}{4} \rceil$ vertices from every element in X. Let C^* be an element in X. As there are at most $\binom{t}{\lceil \frac{t}{4} \rceil}$ subsets of size t and $\epsilon \leq (32\binom{t}{\lceil \frac{t}{4} \rceil})^{-1}$, there exists a set W_{C^*} such that at least $\binom{t}{\lceil \frac{t}{4} \rceil}^{-1}(1-\alpha-4\frac{\epsilon}{\alpha})|G_\mathcal{M}| \geq 2\alpha|G_\mathcal{M}|$ neighbors $C_1 \in X$ of C^* satisfy that $W_{C^*} = W_{C^*,C_1} \cap V(C^*)$. Then the set $S = \bigcup_{C^* \in X} V(C^*) \setminus W_{C^*}$ is a swappable set. To see this, let $T \subseteq S$ with $|T| \leq t$ and $|T \cap V(C^*)| \leq 1$ for all $C^* \in X$. For each vertex $v \in T$, let C_v^* be the unique loose cycle with $v \in V(C_v^*)$, then we can associate each loose cycle C_v^* with a loose cycle $C_{v1} \in X$ such that $C_v^* C_{v1}$ is an edge in $G_\mathcal{M}$, $W_{C_v^*} = W_{C_v^*,C_{v1}} \cap V(C_v^*)$, and so that $C_{v1} \neq C_{v'1}$ and $C_v^* \neq C_{v'1}$ for all distinct vertices $v, v' \in T$. When s is even, there exist two vertex disjoint cycles on s vertices in $G_{U_{\mathcal{M}}}[V(C_v^*), V(C_{v1})]$ that do not contain v for each $v \in T$. Since $|U_{\mathcal{M}}|$ is sufficiently large, these cycles can be used to construct a new C_t^3-tiling $\mathcal{M}'$ with $|\mathcal{M}'| = |\mathcal{M}|$ and $T \cap V(\mathcal{M}') = \emptyset$, implying that S is swappable. Otherwise assume that s is odd. Since for $x \in X$ we have $d_{G_{\mathcal{M}}}(x) \geq (1-\alpha)|G_{\mathcal{M}}|$ and C_v^* has at least $2\alpha|G_{\mathcal{M}}|$ other neighbors with $W_{C_v^*}$ the cover in C_v^*, we can additionally associate C_v^* with the loose cycles C_{v2}, ..., C_{vs} such that $W_{C_v^*} = W_{C_v^*, C_{s-1}}$, such that $C_{v1} \cdots C_{vs}$ is a cycle in $G_{\mathcal{M}}$, and such that $C_{vi} \neq C_{v'j}$ and $C_v^* \neq C_{v'j}$ for all $i, j \in [s]$ and distinct $v, v' \in T$. In this case, there exist $2\lceil \frac{t}{4} \rceil$ vertex disjoint cycles on s vertices in $G_{U_{\mathcal{M}}}$ which do not intersect T since $T \cap W_{C_v^*} = \emptyset$. Since $|U_{\mathcal{M}}|$ is sufficiently large, these cycles can be used to construct a new C_t^3-tiling $\mathcal{M}'$ with $|\mathcal{M}'| = |\mathcal{M}|$ and $T \cap V(\mathcal{M}') = \emptyset$, implying that S is swappable. In both cases, we get that S is a swappable set with $|S| \geq (t - \lceil \frac{t}{4} \rceil)|X| \geq (t - \lceil \frac{t}{4} \rceil)(1 - 16\binom{t}{\lceil \frac{t}{4} \rceil}\epsilon)|G_{\mathcal{M}}|$. $\square$

We are now ready to prove Lemma 2.1.2, the main result of this section. For convenience we restate the definition of β-extremal and Lemma 2.1.2.

Definition 2.1.1. *A 3-graph H on n vertices is β-extremal if $V(H)$ can be partitioned into sets A and B so that $|B| = n - \lceil \frac{t}{4} \rceil \frac{n}{t}$ and $||H[B]|| \leq \beta|V|^3$.*

Lemma 2.1.2. *(Large Tiling) For all $\beta > 0$, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there exists n_0 such that if H is a 3-graph, $|H| \geq n_0$, $\delta(H) \geq \delta_\epsilon(|H|)$, and $\mathcal{M}$ is a maximum C_t^3-tiling, then $|V(H) \setminus V(\mathcal{M})| \leq n_0$ or H is β-extremal.*

Proof. Assume that $|U_{\mathcal{M}}| \geq n_0$. From Lemma 2.2.9 there exists a swappable set S with size at least $(1 - 16\binom{t}{\lceil \frac{t}{4} \rceil}\epsilon)(t - \lceil \frac{t}{4} \rceil)|\mathcal{M}|$. There cannot exist a copy of C_t^3 on $S \cup U_{\mathcal{M}}$ which intersects at most 1 vertex of any $M \in \mathcal{M}$, as that copy of C_t^3 can be used to create a larger C_t^3-tiling. Thus $||H[S \cup U_{\mathcal{M}}]|| \leq \frac{\beta}{2}|H|^3$. By adding at most $16\binom{t}{\lceil \frac{t}{4} \rceil}\epsilon(t - \lceil \frac{t}{4} \rceil)\frac{|H|}{t}$ vertices into $S \cup U_{\mathcal{M}}$ we can find a set $B \subseteq S \cup U_{\mathcal{M}}$ such that $|B| = (t - \lceil \frac{t}{4} \rceil)\frac{|H|}{t}$ and $||H[B]|| \leq \beta|H|^3$, implying that H is β-extremal.

$\square$

2.3 Extremal Case

In this section we show that if H is β-extremal and $\delta(H) \geq \delta(|H|)$, then H is C_t^3-tileable. The method of proof uses a stability strategy. In Lemma 2.3.5, we show that if a β-extremal partition of H behaves nicely, then H is C_t^3-tileable. Then in the main lemma of this section, Lemma 2.1.3, we find a small C_t^3-tiling $\mathcal{M}$ such that $H \setminus V(\mathcal{M})$ has a β-extremal partition that behaves nicely, implying H is C_t^3-tileable. To construct $\mathcal{M}$ we use the following two lemmas.

Lemma 2.3.1. *If H is a 3-graph satisfying $|H| = n \geq 8$ and $\delta(H) \geq \binom{n-1}{2} - \binom{n-k}{2} + 1$ for $0 \leq k \leq \frac{n}{20}$, then a maximum matching of edges has size at least k.*

Proof. Let A be the maximum sized set of vertices such that the maximum matching in $H' = H[V(H) \setminus A]$ is exactly $|A|$ less then the maximum matching in H. As $A = \emptyset$ satisfies the criteria, such a set exists. Then there is no vertex $v \in V(H')$ that is in all maximum matchings as v could be added to A. Let C be a minimum vertex cover of H' and let $U = V(H') \setminus C$. Let $\mathcal{M}$ be a maximum matching in H'. If $|\mathcal{M}| + |A| \geq k$, then H contains a matching of size k.

$$|\mathcal{M}| \leq k - |A| - 1.$$

Otherwise there exists a vertex $v \in C$. Fix v with $|E_{H'}(v, U, U)|$ maximum and note we can assume that $v \notin V(\mathcal{M})$ by the definition of A. Since $\mathcal{M}$ is a maximum matching, $V(\mathcal{M})$ is a vertex cover implying that

$$|C| \leq 3|\mathcal{M}|.$$

Since $v \notin V(\mathcal{M})$, all edges containing v are of the form $(v, V(\mathcal{M}), V(H'))$. But then,

$$|V(\mathcal{M}) \cap U||U| \geq |E_{H'}(v, U, U)| \geq \frac{|E_{H'}(C, U, U)|}{|C|}.$$

At the same time every edge has a vertex in C, so

$$\delta(H')|U| \leq 2|E_{H'}(C, U, U)| + |E_{H'}(C, C, U)|.$$

Since every edge has a vertex in $\mathcal{M}$,

$$|E_{H'}(C,C,U)| \leq |V(\mathcal{M}) \cap C||C||U| + |V(\mathcal{M}) \cap U|\binom{|C \setminus V(\mathcal{M})|}{2}.$$

Combining the inequalities above yields

$$\delta(H') \leq ((2 + \frac{\binom{|C \setminus V(\mathcal{M})|}{2}}{|C||U|})|V(\mathcal{M}) \cap U| + |V(\mathcal{M}) \cap C|)|C| \leq (15 + \frac{4|\mathcal{M}|}{|U|})|\mathcal{M}|^2$$

since the expression is maximized when $|V(\mathcal{M}) \cap C| = |\mathcal{M}|$, $|V(\mathcal{M}) \cap U| = 2|\mathcal{M}|$, and $|C| = 3|\mathcal{M}|$. Since $n \geq 6k$, this can be further simplified to

$$\delta(H') \leq 16|\mathcal{M}|^2 \leq 16(k - |A| - 1)^2$$

At the same time, since deleting a vertex from a 3-graph on n vertices drops the minimum degree by at most $n-2$,

$$\delta(H') > \binom{n-1}{2} - \sum_{i=0}^{|A|-1}(n-2-i) - \binom{n-k}{2} = \binom{n-|A|-1}{2} - \binom{n-k}{2}.$$

which is obviously false for sufficiently large n when $|A| + 1 < k$. One can show that $n \geq 20k$ suffices in this case. Therefore $|A| = k-1$, but $|\mathcal{M}| \geq 1$ since $\delta(H') > 0$. Therefore H' has a matching $\mathcal{M}$ with $|\mathcal{M}| > 1$ implying H has a matching of size at least k. $\square$

For the purposes of the upcoming results, define a k-star to be a 3-graph where there exists a set S with $|S| = k$ such that for every pair of edges $e_1, e_2 \in H$, $S \subseteq e_1 \cap e_2$.

Fact 2.3.2. *For all $n \geq 3$,*

$$ex_1(P_5^3, n) = \begin{cases} 3 & \textit{if } 4 \mid n \\ 1 & \textit{otherwise} \end{cases},$$

where P_5^3 is the loose path on 5 vertices.

Proof. Consider a connected P_5^3-free 3-graph H. Then H is either a 3-graph on at most 4 vertices or H is a 2-star. Since any P_5^3-free 3-graph is a tiling of connected P_5^3-free 3-graphs and the only connected P_5^3-free 3-graphs H with $\delta(H) > 1$ satisfy $|H| = 4$, the result follows. □

Lemma 2.3.3. *There exists $\epsilon_0 > 0$ such that for all ϵ with $0 < \epsilon \leq \epsilon_0$, there exists an n_0 such that if H is a 3-graph satisfying $|H| = n > n_0$, $k < \epsilon n$, and $\delta(H) \geq \binom{n-1}{2} - \binom{n-k}{2} + ex_1(P_5^3, n+1-k) + 1$, then there exists a P_5^3-tiling of H of size at least k.*

Proof. Since $\delta(H) \geq ex_1(P_5^3, n) + 1$, we may assume that $k \geq 2$. But then

$$\begin{aligned}
\delta(H) &\geq \tbinom{n-1}{2} - \tbinom{n-k}{2} + 1 \\
&= \tfrac{(n-1)(n-2)}{2} - \tfrac{(n-k)(n-k-1)}{2} + 1 \\
&= (k-1)n - \tbinom{k+1}{2} + 2.
\end{aligned}$$

Assume to the contrary and let $\mathcal{M}$ be maximum P_5^3-tiling with $|\mathcal{M}| \leq k-1 < \epsilon n$. Let $U := V(H) - V(\mathcal{M})$. Define $P \in \mathcal{M}$ to be *acceptable* if

$$|E(P, U, U)| \geq \epsilon \binom{|U|}{2}.$$

For every acceptable P, there is $v_P \in V(P)$ such that $|E(v_P, U, U)| \geq \epsilon\binom{|U|}{2}/5$.

Let $\mathcal{M}_0$ denote the set of unacceptable $P \in \mathcal{M}$, with $l := |\mathcal{M}| - |\mathcal{M}_0|$ and $W := \bigcup_{P \in \mathcal{M} \setminus \mathcal{M}_0} V(P) \setminus \{v_P\}$. We show that $\mathcal{M}_0$ must be empty by considering the number of edges with vertices in U. By the definition of an acceptable path we get that

$$|E(U, U, V(\mathcal{M}_0))| \leq \epsilon \binom{|U|}{2}(k-l-1) < \epsilon \frac{|U|^2}{2}(k-l-1).$$

The size of $V(\mathcal{M}_0)$ and W also implies that

$$|E(U, V(\mathcal{M}_0), V(\mathcal{M}_0))| \leq \binom{5(k-l-1)}{2}|U| < \frac{25}{2}(k-l-1)^2|U|$$

and

$$|E(U, W, V(\mathcal{M}_0))| \leq 5|U||W|(k-l-1) < 25|U|l(k-l-1).$$

Finally, the inequality

$$|E(U, U \cup W, U \cup W)| \leq n$$

is true because $H[U \cup W]$ must be P_5^3-free. If $H[U \cup W]$ contains a copy P of P_5^3, we can use P along with disjoint copies of P_5^3 obtained using the vertices v_{P_i} for those $P_i \in \mathcal{M}$ with $P \cap P_i \neq \emptyset$ to obtain a larger family than $\mathcal{M}$. Since all connected P_5^3-free 3-graphs are subgraphs of K_4^3 or are 2-stars, the upper bound on the number of edges follows.

Let $Q := \sum_{u \in U} |E(u, V \setminus \bigcup\{v_P \mid P \in \mathcal{M} - \mathcal{M}_0\}, V \setminus \bigcup\{v_P \mid P \in \mathcal{M} - \mathcal{M}_0\})|$. The above bounds and taking into account how many times an edge can be counted in Q imply

$$Q < \epsilon|U|^2(k-l-1) + \frac{25}{2}(k-l-1)^2|U| + 25|U|l(k-l-1) + 3n.$$

On the other hand, we get that

$$\begin{aligned} Q &\geq |U|\delta(H) - |U|ln \\ &\geq |U|\left((k-1)n - \tbinom{k+1}{2} + 2 - ln\right) \\ &= |U|((k-l-1)n - \tbinom{k+1}{2} + 2) \end{aligned}$$

implying

$$(k-l-1)n < \epsilon|U|(k-l-1) + \frac{25}{2}(k-l-1)^2 + 25l(k-l-1) + \binom{k+1}{2} - 2 + 4$$

because $\frac{3n}{|U|} < 4$ when ϵ is small enough. Since $k < \epsilon n$, this inequality is false when $k - l - 1 \geq 1$ for sufficiently small ϵ. Thus $l = k-1$ and $\mathcal{M}_0$ is empty. Also $H[U \cup W]$

does not contain a copy of P, so we get that $\delta(H[U \cup W]) \leq ex_1(P_5^3, |U \cup W|)$. Thus in this last case the minimum degree on U implies

$$\begin{aligned} \delta(H) &\leq \binom{n-1}{2} - \binom{n-1-|\mathcal{M}|}{2} + ex_1(P_5^3, |U \cup W|) \\ &\leq \binom{n-1}{2} - \binom{n-k}{2} + ex_1(P_5^3, n+1-k) \end{aligned}$$

since $k \geq 2$ and $ex_1(P_5^3, |U \cup W|) < 4$, contradicting the minimum degree of H. Thus $\mathcal{M}$ contains at least k elements. $\square$

We now show that if H has a β-extremal partition which behaves nicely, then there exists a perfect C_t^3-tiling. We will use the following theorem by Kühn and Osthus to accomplish this task.

Theorem 2.3.4. *For all positive constants $d, \nu_0, \eta \leq 1$, there is a positive $\epsilon = \epsilon(d, \nu_0, \eta)$ and an integer $N_0 = N_0(d, \nu_0, n)$ such that the following holds for all $n \geq N$ and all $\nu \geq \nu_0$. Let $G = (A, B)$ be a (d, ϵ)-superregular bipartite graph whose vertex classes both have size n and let F be a subgraph of G with $||F|| = \nu||G||$. Choose a perfect matching M uniformly at random in G. Then with probability $1 - e^{-\epsilon n}$ we have*

$$(1-\eta)\nu n \leq |M \cap E(F)| \leq (1+\eta)\nu n.$$

For a 3-graph H and set $S \subseteq V(H)$, we call a vertex v (γ, S)-good if $|N(v) \cap \binom{S}{2}| \geq (1-\gamma)\binom{|S|}{2}$, and we call a pair of vertices $\{v_1, v_2\}$, (γ, S)-good if $|N(\{v_1, v_2\}) \cap S| \geq (1-\gamma)|S|$.

Lemma 2.3.5. *There exist $\gamma > 0$ and n_0 such that if there is a partition (A, B) of a 3-graph H with $|H| = n > n_0$, $n \in 2t\mathbb{Z}$, $|A| = \lceil \frac{t}{4} \rceil \frac{n}{t}$, $|B| = n - \lceil \frac{t}{4} \rceil \frac{n}{t}$, where every vertex in A is (γ, B)-good, and for all $b_1 \in B$ all but at most $\gamma|B|$ vertices $b_2 \in B$ satisfy that $\{b_1, b_2\}$ is (γ, A)-good, then H is C_t^3-tileable.*

Proof. Let $d = 1$, $\nu_0 = \frac{15}{16}$ and $\eta = \frac{1}{16}$. Let ϵ be such that Theorem 2.3.4 holds and set $\gamma = \min(\frac{\epsilon^2}{3}, 1-(1-\eta)\nu)$. Let G be a graph on B where $E(G) = \{bb' \mid \{b, b'\}$ is (γ, A)-good$\}$. If $4|t$, partition B into 3 sets B_i, for $1 \leq i \leq 3$, with $|B_i| = \frac{n}{4}$. Otherwise partition B into 4 sets B_i, for $1 \leq i \leq 4$, such that $|B_1| = |B_2| = |B_3| = \frac{n}{4} - \frac{n}{2t}$ and $|B_4| = \frac{n}{t}$. This partition is possible since $2t|n$. Since $\delta(G) \geq (1-\gamma)|B| \geq (1-\epsilon^2/3)|B|$, $G[B_i, B_{i+1}]$ are all $(1, \epsilon)$-superregular bipartite graphs. Let $F_a^i = \{bb' \mid bb' \in G[B_i, B_{i+1}]$ and $\{a, b, b'\} \in H\}$. From Theorem 2.3.4 applied to all the F_a^i, there exists a perfect matching M_i on G from B_i onto B_{i+1} for $i \in [2]$ when 4 divides t and $i \in [3]$ otherwise. This matching is such that every vertex in A is in a 3-edge with at least $(1-\eta)\nu|M_i| = (1-\eta)\nu|B_{i+1}|$ edges of each M_i. Considered over all edges, every vertex A is a 3-edge with at least $(1-\eta)\nu\sum_i |M_i| \geq (1-\gamma)\sum_i |M_i|$ edges in all the matchings M_i. The set of M_i induce a tiling of B with paths on 2 edges when $4 \mid t$ and with paths on 2 and 3 edges otherwise. Partition the tiling into families of paths $\mathcal{P}_i = \{P_{i0}, \cdots P_{i(\lceil \frac{t}{4} \rceil - 1)}\}$ with $0 \leq i < \frac{n}{t}$ where P_{i0} is the only path on 3 edges when $4 \nmid t$. Let $P_{ij} = e_{ij0}e_{ij1}$ or $P_{ij} = e_{ij0}e_{ij2}e_{ij1}$ be the representation of P_{ij} in terms of its edges when the path has two or three edges respectively (note that e_{ij2} is the middle edge).

Let $\mathcal{S}$ be a set of sets of edges of the form $\{e_{ij1}, e_{i(j+1)0}\}$ and $\{e_{ij2}\}$. Construct a bipartite graph L with partition $(A, \mathcal{S})$. Let there be an edge from $a \in A$ to $S \in \mathcal{S}$ if for every edge $e \in S$, $a \cup e \in E(H)$. By the construction, every vertex in A has degree at least $(1-2\gamma)|B| \geq \frac{1}{4}|B| = \frac{1}{2}|S|$. From the definition of an edge in G, we also get that the minimum degree of an element of S is at least $|A| - 2\gamma|A| \geq \frac{1}{2}|A|$ as well. Thus there is a perfect matching in L using the vertices A. The matchings on L corresponds to a perfect C_t^3-tiling of H because each family $\mathcal{P}_i$ and its associated matchings induce a copy of C_t^3. Thus H is C_t^3-tileable. $\square$

We will now find a small C_t^3-tiling which when removed from a β-extremal 3-graph H

forms a subgraph H' on which we can apply Lemma 2.3.5, proving Lemma 2.1.3 which is, restated below for convenience.

Lemma 2.1.3. *(Extremal) There exists a $\beta_0 > 0$ such that if $\beta < \beta_0$ and H is a β-extremal 3-graph satisfying $\delta(H) \geq \delta(|H|)$, then H has a perfect C_t^3-tiling.*

Proof. Let H be a β-extremal 3-graph, let $n = |H|$, and let σ be sufficiently small. Then there exists a set B such that $|B| = \frac{t-\lceil \frac{t}{4} \rceil}{t} n$ and $||H[B]|| \leq \beta n^3$. Let $A = V \setminus B$. We have

$$\begin{aligned} |E(A,B,B)| &\geq \tfrac{1}{2}(\delta(n)|B| - |E(A,A,B)| - 3|E(B,B,B)|) \\ &\geq \tfrac{|B|}{2}(\delta(n) - \tbinom{|A|}{2}) - \tfrac{3}{2}\beta n^3 \\ &\geq (1-\sigma^4)|A|\tbinom{|B|}{2} \end{aligned}$$

Let A' be the set of (σ^2, B)-good vertices in A and $\bar{A}$ the rest, then

$$(1-\sigma^4)|A|\binom{|B|}{2} \leq |A'|\binom{|B|}{2} + (1-\sigma^2)|\bar{A}|\binom{|B|}{2} = (|A| - \sigma^2|\bar{A}|)\binom{|B|}{2}.$$

Simplifying the above inequality yields that there are few elements in $\bar{A}$ since

$$|\bar{A}| \leq \sigma^2|A|.$$

Similarly, let G be the set of (σ, A)-good pairs in B and $\bar{G}$ be the remaining pairs of vertices in B. Then we get a similar chain of inequalities with

$$(1-\sigma^4)|A|\binom{|B|}{2} \leq |G'||A| + (1-\sigma^2)|\bar{G}||A|.$$

Solving this inequality yields

$$|\bar{G}| \leq \sigma^2\binom{|B|}{2}.$$

Let $\bar{B}$ be the set of vertices with degree less than $(1-\sigma)|B|$ in G. From the number of edges in G, $|\bar{B}| \leq \sigma(|B|-1)$. Let $B' = B \setminus \bar{B}$, the set of vertices with degree at least $(1-\sigma)|B|$ in G.

We will now construct new sets A'' and B'' from $\bar{A}$ and $\bar{B}$ which will allow us to find a perfect tiling. The minimum degree in H implies that for every vertex $v \in V(H)$, $|E(v, A, B)| \geq \frac{3}{4}|A||B|$ or $|E(v, B, B)| \geq \frac{1}{16}\binom{|B|}{2}$ since

$$\delta(H) - \binom{|A|}{2} \geq \binom{n-1}{2} - \binom{|B|}{2} - \binom{|A|}{2} = |A||B| - (n-1) \geq \frac{|B|^2}{4}.$$

Call a vertex acceptable to A if the first inequality is true, and acceptable to B if the second is true. Construct a partition of $\bar{A} \cup \bar{B}$ into the sets A'' and B'' where A'' is the set of vertices which are acceptable to B, and B'' the rest. Then $A^* = A' \cup A''$ and $B^* = B' \cup B''$ is a partition of $V(H)$. We will find C_t^3-tilings $\mathcal{M}_1$ and $\mathcal{M}_2$ with $V(\mathcal{M}_1) \cap V(\mathcal{M}_2) = \emptyset$ such that $H \setminus (V(\mathcal{M}_1) \cup V(\mathcal{M}_2))$ satisfies the conditions of Lemma 2.3.5. Then $H \setminus (V(\mathcal{M}_1) \cup V(\mathcal{M}_2))$ has a perfect C_t^3-tiling $\mathcal{M}_3$ implying H has a perfect C_t^3-tiling $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$.

Under the partition (A^*, B^*), note that for any set S of size less than $\frac{1}{32}n$ and any pair of vertices $b_1, b_2 \in B^* \cap S$, there exists a copy of P_5^3 composed of edges of the form $(b_1, B^* \setminus S, A^* \setminus S)$ and $(b_2, B^* \setminus S, A^* \setminus S)$ which intersect in $A^* \setminus S$. This follows since for any $b \in B^*$,

$$|E(b, B, A)| \geq \delta(n) - \binom{|A|}{2} - \frac{1}{16}\binom{|B|}{2} \geq |A||B| - \frac{|B|^2}{32} - (n-1) \geq \frac{7}{8}|A||B|.$$

Consequently there exist at least $\frac{3}{4}|A|$ elements $a \in A$ for which $|N(a, b) \cap B| \geq \frac{1}{4}|B|$. Since $|S| + |\bar{A}| < \frac{1}{4}|A|$ and $|S| + |\bar{B}| < \frac{1}{8}|B|$, there exists an element $a \in A'$ and elements $b_1', b_2' \in B'$ for which such a path can be formed.

Consider the case where $k_1 = |B^*| - (n - \lceil \frac{t}{4} \rceil \frac{n}{t}) > 0$. Then we know that

$$\begin{aligned}
\delta(H[B^*]) &\geq \tbinom{n-1}{2} - \tbinom{|B^*|-k_1}{2} + c(t,n) + 1 - (\tbinom{n-1}{2} - \tbinom{|B^*|-1}{2}) \\
&= \tbinom{|B^*|-1}{2} - \tbinom{|B^*|-k_1}{2} + c(t,n) + 1.
\end{aligned}$$

When $4|t$, since $c(t,n) = ex_1(P_5^3, |B^*| - k_1 + 1)$, by Lemma 2.3.3 we can find k_1 disjoint copies of P_5^3. Since $k_1 t < 2\sigma n < \frac{n}{32}$, the just noted property on pairs of vertices in B^* implies that we can greedily construct a C_t^3-tiling $\mathcal{M}_1$ with $|\mathcal{M}_1| = k_1$ using the copies of P_5^3 and edges of the form (A^*, B^*, B^*) intersecting in A^*. Similarly when $4 \nmid t$, we get a matching of size k_1 by Lemma 2.3.1 which can be used to construct a C_t^3-tiling $\mathcal{M}_1$ with $|\mathcal{M}_1| = k_1$. Let $A_1^* = A^* \setminus V(\mathcal{M}_1)$ and $B_1^* = B^* \setminus V(\mathcal{M}_1)$. Then the size of A_1^* is

$$|A_1^*| = \lceil \frac{t}{4} \rceil \frac{n}{t} - k_1 - k_1(\lceil \frac{t}{4} \rceil - 1) = \lceil \frac{t}{4} \rceil \frac{n - k_1 t}{t},$$

the size of B_1^* is

$$|B_1^*| = n - \lceil \frac{t}{4} \rceil \frac{n}{t} + k_1 - k_1(t - \lceil \frac{t}{4} \rceil + 1) = n - k_1 t - \lceil \frac{t}{4} \rceil \frac{n - k_1 t}{t}.$$

Otherwise $k_1 = |A^*| - \lceil \frac{t}{4} \rceil \frac{n}{t} \geq 0$. Note that for any $b \in B^*$

$$|E(b, A, A)| \geq \delta(n) - |A||B| - \frac{1}{16}\binom{|B|}{2} \geq \binom{|A|}{2} - \frac{|B|^2}{32} \geq \frac{1}{4}\binom{|A|}{2}$$

implying that there exist k_1 disjoint edges $e_j = \{b_j, a_{1j}, a_{2j}\}$ of the form (B^*, A', A') for $j \in [k_1]$. In the case of C_6^3, since the a_{ij} are (σ^2, B)-good, we can greedily choose paths of length two in $N(a_{ij}) \cap N(a_{2j}) \cap \binom{B}{2}$ to create a C_6^3-tiling $\mathcal{M}_1$ with $|\mathcal{M}_1| = k_1$. Otherwise we can find two disjoint edges $\{a_{ij}, b_{1ij}, b_{2ij}\}$ for $i \in [2]$ to get a loose path P_j on 3 edges for $j \in [k_1]$. By the same method as the above case, we can greedily extend each P_j to a C_t^3-tiling $\mathcal{M}_1$ with $|\mathcal{M}_1| = k_1$ using edges of the form (A^*, B^*, B^*) with consecutive added edges intersecting in A^*. This is possible since fewer than $k_1 t < 2\sigma n < \frac{n}{32}$ vertices are used in this process. Once again let $A_1^* = A^* \setminus V(\mathcal{M}_1)$ and $B_1^* = B^* \setminus V(\mathcal{M}_1)$. Once again, the size of A_1^* is

$$|A_1^*| = \lceil \frac{t}{4} \rceil \frac{n}{t} + k_1 - k_1(\lceil \frac{t}{4} \rceil + 1) = \lceil \frac{t}{4} \rceil \frac{n - k_1 t}{t},$$

the size of B_1^* is

$$|B_1^*| = n - \lceil \frac{t}{4} \rceil \frac{n}{t} - k_1 - k_1(t - \lceil \frac{t}{4} \rceil - 1) = n - k_1 t - \lceil \frac{t}{4} \rceil \frac{n - k_1 t}{t}.$$

Let k_2 be the smallest integer such that it is larger than the number of remaining non-good vertices from $\bar{A} \cup \bar{B}$ such that $n - (k_1 + k_2)t$ is divisible by $2t$. Then $k_2 \leq \sigma n < \frac{n}{64}$. For all vertices v remaining in $\bar{A} \cup \bar{B}$ and up to one additional vertex, we can find disjoint edges $e_v = (v, x_v, y_v)$ such that $|e_v \cap A_1^*| = 1$ and $|e_v \cap B_1^*| = 2$. We can then greedily extend these edges with edges of the form (A_1^*, B_1^*, B_1^*) with consecutive edges intersecting in A^* to form a C_t^3-tiling $\mathcal{M}_2$ with $|\mathcal{M}_2| = k_2$, since fewer than $(k_1 + k_2)t < 2\sigma n < \frac{n}{32}$ vertices are used during this process.

Let $A_2^* = A_1^* \setminus V(\mathcal{M}_2)$ and $B_2^* = B_1^* \setminus V(\mathcal{M}_2)$. Then since $2t$ divides $n - (k_1 + k_2)t$

$$|A_2^*| = |A_1^*| - k_2 \lceil \frac{t}{4} \rceil = \lceil \frac{t}{4} \rceil \frac{n - (k_1 + k_2)t}{t},$$

and

$$|B_2^*| = |B_1^*| - k_2(t - \lceil \frac{t}{4} \rceil) = n - (k_1 + k_2)t - \lceil \frac{t}{4} \rceil \frac{n - (k_1 + k_2)t}{t}$$

This final partition was constructed by removing $(k_1 + k_2)t < 2\sigma n$ vertices from A^* and B^*, so $H[A_2^* \cup B_2^*]$ contains a perfect C_t^3-tiling $\mathcal{M}_3$ by Lemma 2.3.5. Thus H has a perfect C_t^3-tiling $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. $\square$

2.4 Absorption

To prove the absorbing lemma, we use the following facts:

Fact 2.4.1. *There exist $\alpha > 0$ and $n_0 > 0$ such that if H is a 3-graph with $|H| = n \geq n_0$ and $\delta(H) \geq \frac{7}{16}\binom{n}{2}$, then for any two distinct vertices u and v there exist at least αn^3 loose paths $uxyzv$.*

Proof. Consider the graphs $G_v = (V(H), N(v))$ and $G_u = (V(H), N(u))$. Let $0 < \gamma \leq 0.02$ and define $A_v = \{x : d_{G_v}(x) \geq \gamma n\}$ and $A_u = \{x : d_{G_u}(x) \geq \gamma n\}$. Then we get the inequality:

$$\frac{7}{16}\binom{n}{2} \leq \frac{|A_v|^2}{2} + \gamma n^2$$

since there are fewer than γn^2 edges containing a vertex in $V(H) \setminus A_v$. Solving for $|A_v|$ yields:

$$|A_v|^2 \geq \frac{7}{8}\binom{n}{2} - 2\gamma n^2 > (\frac{7}{16} - 3\gamma)n^2 \geq 0.35n^2$$

which immediately implies that $|A_v| \geq 0.59n$. But then $|A_v \cap A_u| \geq 0.09n$, so there are more than $0.08\gamma^2 n^3$ vertex triples (x, y, z) such that we get the loose path u, x, y, z, v. $\square$

Call a vertex coloring f of a 3-graph H proper if for all edges $e \in H$, there does not exist a pair of distinct vertices $v_1, v_2 \in e$ with $f(v_1) = f(v_2)$.

Fact 2.4.2. *If $||C_t^3||$ is even, there is a proper vertex coloring f of C_t^3 with the colors 1, 2, and 3 such that $|f^{-1}(1)| = \frac{t}{4} + 1$, $|f^{-1}(2)| = \frac{t}{2} - 1$, and $|f^{-1}(3)| = \frac{t}{4}$.*

Proof. Consider a loose cycle

$$C = v_1 w_1 x_1 w_2 v_2 \cdots v_{\frac{t}{4}} w_{\frac{t}{2}-1} x_{\frac{t}{4}} w_{\frac{t}{2}} v_1.$$

Let f be the proper coloring of C with $f(v_i) = 1$ and $f(x_i) = 3$ for $i = 1, \ldots, \frac{t}{4} - 1$ and with $f(w_j) = 2$ for $j = 1, \ldots, \frac{t}{2} - 3$. Finally let $f(w_{\frac{t}{2}-2}) = f(w_{\frac{t}{2}-1}) = 1$, $f(v_{\frac{t}{4}}) = 2$, $f(w_{\frac{t}{2}}) = 2$ and $f(x_{\frac{t}{4}}) = 3$. Then f is a proper coloring of C satisfying the above conditions. $\square$

Fact 2.4.3. *If $||C_t^3||$ is odd and $||C_t^3|| \geq 7$, there is a proper vertex coloring f of C_t^3 with the colors 1, 2, and 3 such that $|f^{-1}(1)| = \frac{t+2}{4} + 1$, $|f^{-1}(2)| = \frac{t}{2} - 2$, and $|f^{-1}(3)| = \frac{t+2}{4}$.*

Proof. Consider a loose path

$$P = v_1 w_1 x_1 w_2 v_2 \cdots v_{\frac{t-10}{4}} w_{\frac{t-10}{2}-1} x_{\frac{t-10}{4}} w_{\frac{t-10}{2}} v_{\frac{t-10}{4}+1}$$

and color it with function f_2 where $f_2(v_i) = 3$, $f_2(x_i) = 1$, and $f_2(w_i) = 2$. Then $|f_1^{-1}(1)| = \frac{t-10}{4}$, $|f_1^{-1}(2)| = \frac{t-10}{2}$, and $|f_1^{-1}(3)| = \frac{t-10}{4} + 1$. Now consider the loose path $P' = v_1 u_1 \cdots u_9 v_{\frac{t-10}{4}+1}$ on 11 vertices. Color the vertices of P with function f_1 where the color of the vertices listed in order is $3, 1, 2, 1, 3, 2, 1, 3, 2, 1, 3$. Note that $|f_2^{-1}(1)| = 4$,

$|f_2^{-1}(2)| = 3$, and $|f_2^{-1}(3)| = 4$. Now f_1 and f_2 form a proper coloring f of the cycle $C = P + P'$ with $|f^{-1}(1)| = \frac{t+2}{4} + 1$, $|f^{-2}(2)| = \frac{t}{2} - 2$, and $|f^{-1}(3)| = \frac{t+2}{4}$. □

We now build to proving the main theorem of this section. We start by showing that for every pair of vertices u, and v, there are a significant number of sets T of constant size such that $H[u+T]$ and $H[v+T]$ are C_t^3-tileable.

Lemma 2.4.4. *There exist $\delta > 0$ and $n_0 > 0$ such that if H is a 3-graph with $|H| = n \geq n_0$ and $\delta(H) \geq \delta(|H|)$, then for all pairs u, v there exist δn^{3t-1} sets T with $|T| = 3t - 1$ such that $H[u+T]$ and $H[v+T]$ are C_t^3-tileable.*

Proof. Let $0 < \gamma \leq 0.03$ and consider G_γ where $xy \in G_\gamma$ if $|N(x) \cap N(y)| \geq \gamma n^2$. Let $A = \{z \mid zu \in G_\gamma\}$ and $B = \{z \mid zv \in G_\gamma\}$. If $|A \cap B| \geq \gamma n$, then we can construct the required set T as follows. Pick a vertex $z \in A \cap B$ and vertices a, b, c, and d such that avb, cud, azb and czd are all edges in H. From the definition of G_γ and size of $|A \cap B|$, we get that there must be at least αn^5 such choices of a, b, c, d, and z for some $\alpha > 0$. For some $\alpha' > 0$, we can then find at least $\alpha' n^{2(t-3)}$ pairs of loose paths $P_1 = ax_1 \cdots x_{t-3}b$ and $P_2 = cy_1 \cdots y_{t-3}d$ on $V(H) \setminus \{u, v, z\}$ such that $V(P_1) \cap V(P_2) = \emptyset$ since such pairs of loose paths can be constructed by greedily extending loose paths starting at a and c until there are exactly two edge left to choose. By fact Fact 2.4.1 we can extend the paths so that that there are $\alpha' n^{2(t-3)}$ pairs of paths. These disjoint paths create cycles C and C' such that $z \in C$, $z \in C'$, $V(C) \cap V(C') = \{z\}$, $u, v \notin V(C) \cup V(C')$, but $C - z + u$ and $C' - z + v$ are loose cycles. Since H contains βn^t copies of C_t^3 for some $\beta > 0$, for some $\delta > 0$ we have at least δn^{3t-1} sets $T = V(C) \cup V(C') \cup V(C'')$ such that $H[T+u]$ and $H[T+v]$ are C_t^3-tileable, where C'' is any copy of C_t^3 vertex disjoint from C and C'.

Now all that is left is when $|A \cap B| \leq \gamma n$. To start this case, assume that $|N(u) \cap N(v)| \geq \gamma n^2$ and let $xy \in N(u) \cap N(v)$. By greedily extending a path and Fact 2.4.1, for some $\alpha > 0$ we can find αn^{t-1} loose paths P on $\frac{t}{2} - 1$ edges starting at x and ending at y. Once again since H contains βn^t copies of C_t^3 for some $\beta > 0$, for some $\delta > 0$ we have at least δn^{3t-1} sets $T = V(P) \cup V(C) \cup (C')$ for which $H[T+u]$ and $H[T+v]$ are C_t^3-tileable, where C and C' are disjoint copies of C_t^3 on $H[V(H) \setminus (V(P) \cup \{u, v\})]$.

Thus we may assume $|N(u) \cap N(v)| \leq \gamma n^2$ as well. Note that if $4 \nmid t$ and $t \leq 10$, $\frac{(t - \lceil \frac{t}{4} \rceil)^2}{t^2} \leq \frac{1}{2} - 3\gamma$. In that case

$$\delta(H) \geq \frac{(n-2)^2}{2} - \frac{((t - \lceil \frac{t}{4} \rceil)\frac{n}{t})^2}{2} \geq (\frac{1}{2} + \gamma)\binom{n}{2}.$$

Thus if $4 \nmid t$, then $t \geq 14$. Also note that for any vertex z, we have that $z \in A \cup B$ since the minimum degree forces $N(z) \cap N(u)$ or $N(z) \cap N(v)$ to be large. Without loss of generality we may assume that $|A| < (\frac{1}{2} + \gamma)n$, so then for $z \in A$, $|N(z) \cap \binom{A}{2}| < (\frac{1}{4} + 2\gamma)\binom{n}{2}$ implying $|N(z) \cap (A \times B \cup \binom{B}{2})| \geq (\frac{3}{16} - 2\gamma)\binom{n}{2}$. Thus $|E_H(A, A, B)| + |E_H(A, B, B)| \geq 2\gamma n^3$. Since there exists $\eta > 0$ such that there are ηn^{t+1} copies $K \subseteq H[A, A \cup B, B]$ of $K_{\frac{t}{4}+1, \frac{t}{2}-1, \frac{t}{4}+1}$ if $4|t$ or $K_{\lceil \frac{t}{4} \rceil+1, \frac{t}{2}-2, \lceil \frac{t}{4} \rceil+1}$ if $4 \nmid t$, we can now construct the desired sets T by the following method. Pick a copy K and then a vertex u' such that $u' \in V(K) \cap A$ and u' is in the partition class of size $\lceil \frac{t}{4} \rceil + 1$. Similarly pick $v' \in V(K) \cap B$ such that v' is in the partition class of size $\lceil \frac{t}{4} \rceil + 1$. Since $u' \in A$, there are γn^2 pairs of vertices xy such that uxy and $u'xy$ are edges. We can then find $\alpha' n^{t-1}$ paths P starting at x and ending at y on $\frac{t}{2} - 1$ edges like in the last case. Then for $T_u = \{x, y\} \cup V(P)$, $T_u + u'$ and $T_u + u$ are C_t^3-tileable. We can repeat this process to find a disjoint set T_v with the same properties. Let $T = V(K) \cup T_u \cup T_v$. Then $H[T+u]$ contains $T_u + u$, $T_v + v'$, and $H[V(K) - v']$. Thus $H[T+u]$ is C_t^3-tileable since $H[V(K) - v']$ is C_t^3 tileable by Facts 2.4.2 and 2.4.3. Similarly, $H[T+v]$ contains $T_u + u'$, $T_v + v$, and $H[V(K) - u']$ and is C_t^3-tileable. Thus there exist δn^{3t-1} sets T such that $T+u$ and $T+v$ are C_t^3-tileable for some $\delta > 0$. $\square$

Lemma 2.4.5. *For all sets S with $|S| = t$, there exists $\delta > 0$ and $\delta n^{t(3t-1)}$ sets T with $|T| = t(3t-1)$ such that $H[T]$ is C_t^3-tileable and $H[T \cup S]$ is C_t^3-tileable.*

Proof. Let $S = \{v_1, \ldots, v_t\}$. Pick a set of vertices $W = \{w_1, \ldots w_t\}$ such that $H[W]$

contains a copy of C_t^3. By the previous lemma there exist at least $\delta_1 n^{3t-1}$ sets T_i such that $T_i \subseteq V(H) \setminus S$, $|T_i| = 3t-1$, $T_i \cup w_i$ is C_t^3-tileable, and $T_i \cup v_i$ is C_t^3-tileable. Let $T = T_1 \cup \ldots \cup T_t \cup W$, where $|T_i \cap T_j| = 0$ for all $i, j \leq t$ with $i \neq j$. By the choice of T_i, T is C_t^3-tileable. Since W contains a cycle, we get that $T \cup S$ is C_t^3-tileable as well. Thus for a fixed W, there are at least $\delta_2 n^{3t-1}$ such sets T. Since there are least βn^t sets W which contain a copy of C_t^3 for some $\beta > 0$ we get that the number of possible sets T is at least $\delta n^{t(3t-1)}$, proving the lemma. □

And with the completion of the last proof we are now ready to prove the main lemma of this section.

Lemma 2.1.4. *(Absorbing) For every integer $t \geq 6$ and $\nu > 0$, there is $\xi > 0$ and n_0 such that the following holds. If H is a 3-graph on $n \geq n_0$ vertices which satisfies $\delta(H) \geq \delta(n)$, then there is a set $A \subset V(H)$ with $|A| \leq \nu n$, such that $H[A]$ is C_t^3-tileable and for every set $B \subseteq V(H) \setminus A$ with $|B| \in t\mathbb{Z}$ and $|B| < \xi n$, $H[A \cup B]$ is C_t^3-tileable.*

Proof. To begin with, we may assume that ν is sufficiently small. For $W \in \binom{V(H)}{t}$, let $\mathcal{A}(W)$ be the family of sets A of size $k = t(3t-1)$ such that A and $A \cup W$ is C_t^3-tileable. By Lemma 2.4.5, there exists $\alpha > 0$ such that $|\mathcal{A}(W)| \geq \alpha\binom{n}{k}$. Let $\beta = \min(\frac{\nu}{2k}, \frac{\alpha}{32k^2})$ and let $\mathcal{F}$ be obtained by adding every set $F \in \binom{V(H)}{k}$ independently, at random with probability $p = \beta n \binom{n}{k}^{-1}$. Then $E(|\mathcal{F}|) = p\binom{n}{k} = \beta n$ and for all $W \in \binom{V(H)}{t}$, $E(|\mathcal{A}(W) \cap \mathcal{F}|) \geq \alpha\beta n$. Let $\mathcal{E}$ be the set of pairs $\{F, F'\}$ such that $F, F' \in \mathcal{F}$ and $F \cap F' \neq \emptyset$. Then

$$E(\mathcal{E}) = kp^2\binom{n}{k-1}\binom{n}{k} = \frac{k^2\beta^2 n^2}{n-k+1} < 2k^2\beta^2 n.$$

Therefore, by the Chernoff and Markov inequalities, there exists a family $\mathcal{F}$ such that the following conditions hold: $|\mathcal{F}| \leq 2\beta n$, $|\mathcal{E}| \leq 4k^2\beta^2 n$, and for every $W \in \binom{V(H)}{t}$, $|A(W) \cap \mathcal{F}| \geq \frac{\alpha\beta n}{2}$. Let $\mathcal{G}$ be obtained from $\mathcal{F}$ by deleting all sets F which are in intersecting pairs and all sets F that do not absorb any W. Then $|\mathcal{G}| \leq 2\beta n$, for every $W \in \binom{V(H)}{t}$,

$|A(W) \cap \mathcal{F}| \geq \frac{\alpha\beta n}{2} - 8k^2\beta^2 n \geq \frac{\alpha\beta n}{4}$, and for $A = \bigcup_{F \in \mathcal{G}} F$, $|A| \leq 2k\beta n \leq \nu n$ by the choice of β. Also note that $H[A]$ is C_t^3-tileable with copies of C_t^3 since for every set $F \in \mathcal{G}$, there exists $W \in \binom{V(H)}{t}$ such that $F \in \mathcal{A}(W)$. Finally if $B \subseteq V(H) \setminus A$ is a set with $|B| \leq k\frac{\alpha\beta n}{4}$ and $|B| \in t\mathbb{Z}$, then B can be partitioned into disjoint k-sets B_j and absorbed by using the fact that $|\mathcal{A}(B_i) \cap G| \geq \frac{\alpha\beta n}{4}$. $\square$

Chapter 3

THE EXISTENCE OF RAINBOW CYCLES WITH ODD LENGTH

3.1 Proof of Theorem 1.3.4

In this chapter we prove a minimum color degree condition under which an edge colored graph G must contain a rainbow C_ℓ, for $\ell \geq 5$, which is tight when ℓ is odd. Call an edge-colored graph G *edge-minimal* if, for every edge $e \in E(G)$, $\delta^c(G) > \delta^c(G-e)$. Let $N(v,c)$ denote the neighbors u of v such that $c(uv) = c$ and let $\nu(G)$ be the maximum of $|N(v,c)|$ over all vertices v and colors c. For every $v \in V(G)$, let $N^*(v)$ be the set of vertices $u \in N(v)$ such that uv is the only edge incident to v that is given the color $c(uv)$, i.e., $N^*(v) := \{u \in N(v) : |N(v, c(uv))| = 1\}$. Let $v \in V(G)$ and $X \subseteq N(v)$. When $x \in X$, $xy \in E(G)$, and $y \neq v$ we say that xy is (X,v)-*bad for* y if

(B1) the path vxy is rainbow, and

(B2) $N(y, c(xy)) \subseteq X$.

Lemma 3.1.1. *Let G be an edge-minimal edge-colored graph on n vertices, let $v \in V(G)$, and let $X \subseteq N(v)$. If $Y \subseteq V(G) \setminus \{v\}$ is a nonempty set such that for every $y \in Y$ at most j different colors are used on the edges that are (X,v)-bad for y, then*

$$n \geq |X| + \delta^c(G) - \frac{|X \cap N^*(v)|}{|Y|}(\nu(G) - 1) - j.$$

Proof. Form a directed graph D on the vertex set $X \cup Y$ by setting $N_D^+(x) = N(x, c(vx)) \cap Y$ for every $x \in X$. Note that, because $v \notin Y$, for every $x \in X$, we have $d_D^+(x) \leq |N(x, c(vx))| - 1 \leq \nu(G) - 1$. If $x \in X \setminus N^*(v)$, then there exists $x' \in V(G) \setminus \{x\}$ such that $c(vx') = c(vx)$, so there cannot exist $y \in N^+(v)$ as otherwise the monochromatic path or triangle formed by the edges vx', vx, and xy would violate the edge-minimality of G. Therefore,

$$\sum_{y\in Y} d_D^-(y) = |E(D)| = \sum_{x\in X} d_D^+(x) = \sum_{x\in N^*(x)\cap X} d_D^+(x) \le |N^*(x)\cap X|(\nu(G)-1).$$

Fixing $y \in Y$ so that $d_D^-(y)$ is minimum then gives us that $|Y|d_D^-(y) \le |N^*(v)\cap X|(\nu(G)-1)$, so

$$d_D^-(y) \le \frac{|N^*(v)\cap X|}{|Y|}(\nu(G)-1). \tag{3.1}$$

Let $x \in N_G(y, X)$ and suppose that $N(y, c(xy)) \cap \overline{X} = \emptyset$. Then either xy is (X, v)-bad or, since xy satisfies (B2), $c(vx) = c(xy)$. If $c(vx) = c(xy)$, then $x \in N_D^-(y)$. Thus the number of colors used on edges in $E(y, \overline{X})$ is at least $d^c(y) - (d_D^-(y) + j)$. This means that $n - |X| = |\overline{X}| \ge d^c(y) - d_D^-(y) - j$, and, with (3.1), we have

$$n \ge |X| + \delta^c(G) - d_D^-(y) - j \ge |X| + \delta^c(G) - \frac{|N^*(v)\cap X|}{|Y|}(\nu(G)-1) - j. \qquad \square$$

Note that the condition (B2) is not needed for this proof, but with this condition we can quickly show that $\delta^c(G) > \frac{n}{2}$ implies a rainbow triangle. To see this, assume that G is an edge-minimal graph without rainbow triangles and let v be a vertex such that $d(v) = \Delta(G)$. Then $|N(v)| \ge \delta^c(G) + \nu(G) - 1$. The condition $\delta^c(G) \le \frac{n}{2}$ then follows from Lemma 3.1.1 with $j = 0$ and $N(v)$ and $N^*(v)$ playing the roles of X and Y, respectively, because, for every $y \in N^*(v)$, the fact that G has no rainbow triangles implies that there are no edges that are $(N(v), v)$-bad for y.

To apply Lemma 3.1.1 to longer cycles, we need to find a large set Y with a limited number of colors on the (X, v)-bad edges. By considering certain rainbow paths of length $\ell - 2$, the next lemma provides a condition under which a vertex y has few (X, v)-bad edges. We then use this result to find a large set Y.

Lemma 3.1.2. *Let G be an edge-minimal edge-colored graph on n vertices that does not contain a rainbow cycle of length ℓ. Let $v \in V(G)$, let $X \subseteq N(v)$, and let C be the set of* *colors which appear at least twice on the edge set $E(v, X)$. If $y \in V(G)$ is such that there exists a rainbow v, y-path of length $\ell - 2$ that avoids the colors in C, then the number of colors used on the edges yx such that $x \in X$ and yxv is rainbow is at most 3ℓ. In particular, there are at most 3ℓ colors used on the edges that are (X, v)-bad for y.*

Proof. Let F be the set of edges yx with $x \in X$ such that yxv is rainbow, and let P be a rainbow v, y-path of length $\ell - 2$ that avoids the colors in C. Let $F_1 \subseteq F$ be the set of edges xy in F such that $x \in V(P)$, so $|F_1| \leq |V(P) \setminus \{v\}| = \ell - 2$. Let $F_2 \subseteq F$ be the set of edges xy in F such that the color $c(vx)$ appears on the path P. Because P avoids the colors in C, for every $e \in E(P)$, we have that $|N(v, c(e)) \cap X| = 1$, so $|F_2| \leq |E(P)| = \ell - 2$. Because there does not exist a rainbow cycle of length ℓ in G, for each edge $e \in F \setminus (F_1 \cup F_2)$, we have $c(e) \in E(P)$. Therefore, at most $|E(P)| + |F_1| + |F_2| \leq 3\ell$ colors are used on the edges in F. $\square$

Lemma 3.1.3. *Let G be an edge-minimal edge-colored graph on n vertices such that $\delta^c(G) \geq \frac{n}{2}$ that does not contain a rainbow cycle of length ℓ. Let v be a vertex and c a color such that $|N(v, c)| = \nu(G)$. If there exists a non-empty set B such that for every $b \in B$ there exists a rainbow v, b-path of length $\ell - 2$ that avoids the color c, then*

$$\frac{n}{2} \geq \delta^c(G) + \left(1 - \frac{n+1}{2|B|}\right)(\nu(G) - 1) - 3\ell.$$

Proof. Let v be a vertex and c a color such that $|N(v, c)| = \nu(G)$. Because $d^c(v) \geq \delta^c(G) \geq \frac{n}{2}$, we can select $X' \subseteq N(v)$ so that $|X'| = \lceil \frac{n}{2} \rceil$, the color c appears on the set $E(v, X')$, and $E(v, X')$ is rainbow. Let $X := X' \cup N(v, c)$. Note that

$$|X| = |X'| + (\nu(G) - 1) \geq \frac{n}{2} + (\nu(G) - 1).$$

This with Lemmas 3.1.1 and 3.1.2 plus the fact that $|N^*(v) \cap X| \leq |X'| \leq \frac{n+1}{2}$ gives us that

$$n \geq |X| + \delta^c(G) - \frac{|X'|}{|B|}(\nu(G) - 1) - 3\ell \geq \frac{n}{2} + \delta^c(G) + \left(1 - \frac{n+1}{2|B|}\right)(\nu(G) - 1) - 3\ell,$$

which proves the lemma. $\square$

Using the inequality from Lemma 3.1.3, we can now restrict the minimum color degree to be near the desired value of $\frac{n+1}{2}$.

Lemma 3.1.4. *For $\ell \geq 3$, if G is an edge-minimal edge-colored graph on n vertices such that $\delta^c(G) > \frac{n}{2} + 3\ell$, then G contains a rainbow C_ℓ.*

Proof. Let v be a vertex and c a color such that $|N(v,c)| = \nu(G)$. For $0 \leq j \leq \ell - 2$, let B_j be the set of vertices b such that there exists a rainbow v,b-path of length j from v to b avoiding the color c. For $1 \leq j \leq \ell - 2$, there exists $b \in B_{j-1}$ and a rainbow v,b-path P of length $j-1$, so

$$|B_j| \geq d^c(b) - |E(P)| - |V(P)| - 1 > \frac{n}{2} + 3\ell - 2j - 1 \geq \frac{n+1}{2}.$$

This with Lemma 3.1.3 implies that

$$\frac{n}{2} \geq \delta^c(G) + \left(1 - \frac{n+1}{2|B_{\ell-2}|}\right)(\nu(G) - 1) - 3\ell > \frac{n}{2},$$

a contradiction. $\square$

In order to further use Lemma 3.1.3, we provide a condition under which the set B can be much larger than $\frac{n}{2}$.

Lemma 3.1.5. *For $\ell \geq 3$, let G be an edge-minimal edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n}{2}$ that does not contain a rainbow C_ℓ. Suppose T is a triangle in G, $v \in V(T)$, and C is a set of colors that is disjoint from the colors used on T. If $3 \leq k \leq \ell$ and B_k is the set of vertices for which there exists a rainbow v,b path of length k that avoids the colors in C, then $|B_k| \geq \frac{3n}{4} - \frac{3|C|}{2} - 6\ell$.*

Proof. By the edge-minimality of G, T is not monochromatic. Therefore we can label the vertices of T as $\{v, x_1, x_2\}$ so that $c(vx_1) \neq c(x_1x_2)$. For $1 \leq j \leq k-1$, let $\mathcal{P}_j$ be the set of rainbow paths of length j that start with the edge vx_1 and avoid the colors in $C \cup \{c(vx_2), c(x_1x_2)\}$ and the vertex x_2. Let A_j be the vertices a such that there exists a v, a-path in $\mathcal{P}_j$. Then, for every $2 \leq j \leq k-1$, there exists $a \in A_{j-1}$ and a v, a-path $P \in \mathcal{P}_{j-1}$, so

$$|A_j| \geq d^c(a) - (|C| + 2 + |E(P)|) - (|V(P)| + 1) \geq \delta^c(G) - |C| - 2k. \tag{3.2}$$

Let $A := A_{k-1}$ and note that, because of the rainbow path vx_2x_1, we have $A \subseteq B_k$. Fix $a \in A$ so that the color degree of a in $G[A]$ is $\delta^c(G[A])$, and let $P \in \mathcal{P}_{k-1}$ be a v, a-path of length $k-1$. Let A' be the set of vertices $a' \in N(a) \setminus A$ such that $a' \notin V(P)$, $c(aa') \notin C$, and $c(aa')$ does not appear on P. Note that $A' \subseteq B_k$, and by the selection of a and Lemma 3.1.4, we have that

$$|A'| \geq d^c(a) - (\frac{|A|}{2} + 3\ell) - |C| - |V(P)| - |E(P)| \geq \delta^c(G) - (\frac{|A|}{2} + 3\ell) - |C| - 2k. \tag{3.3}$$

Recalling that $k \leq \ell$ and combining (3.2) and (3.3) gives us that

$$|B_k| \geq |A| + |A'| \geq \frac{|A|}{2} + \delta^c(G) - 5\ell - |C| \geq \frac{3}{2}\delta^c(G) - \frac{3}{2}|C| - 6\ell \geq \frac{3n}{4} - \frac{3}{2}|C| - 6\ell. \quad \square$$

We are now ready to prove the main theorem of this chapter.

Theorem 1.3.4 (Czygrinow, Molla, Nagle, & Oursler)**.** *For every $\ell \geq 5$ and $n \geq 200\ell$, if G is an edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$, then G contains a rainbow cycle of length ℓ.*

Proof. Assume that G is an edge-minimal counterexample. Let v be a vertex and c be a color such that $|N(v, c)| = \nu(G)$. Let $X' \subseteq N(v)$ be such that $|X'| = \delta^c(G) - 1$, $E(v, X')$ is rainbow, and the color c does not appear on the edges $E(v, X')$.

First suppose that for every edge $e \in E(G[X'])$, we have that $c(e) = c$. Let $Y := V(G) \backslash (X' \cup \{v\})$, and note that, for every $x \in X'$, the only colors that could appear on the edges in $E(x,\overline{Y})$ are c and $c(vx)$, so $|Y| \geq \delta^c(G) - 2$. This with the fact that $|X'| \geq \delta^c(G) - 1$, $\delta^c(G) \geq \frac{n+1}{2}$ and $V(G) = X' \cup Y \cup \{v\}$ implies that

$$\delta^c(G) - 1 \geq |Y| \geq \delta^c(G) - 2 \qquad \text{and} \qquad \frac{n+1}{2} \geq |X'| \geq \delta^c(G) - 1. \tag{3.4}$$

Let $Y' \subseteq Y$ be the vertices $y \in Y$ for which there are at least four vertices $x \in N(y, X')$ such that $c(xy) = c(xv)$. For every $x \in X'$, by (3.4), we have that

$$|N(x, c(vx)) \cap Y| \leq |Y| - (d^c(x) - 2) \leq 1,$$

so $|Y'| \leq \frac{1}{4}|X'|$. Let $Y'' \subseteq Y$ be the set of vertices that send less than $3\ell+3$ different colors into X'. Then, using (3.4), the minimum color degree of $G[Y'']$ is at least

$$\delta^c(G) - (3\ell + 2) - |(Y \cup \{v\}) \setminus Y''| \geq |Y''| - (3\ell + 2).$$

Thus, by Lemma 3.1.4, we have $|Y''| \leq 12\ell + 4$. Let $Y''' = Y \setminus (Y' \cup Y'')$, so by (3.4)

$$|Y'''| \geq |Y| - \frac{1}{4}|X'| - 12\ell + 3 \geq \frac{3}{4}|Y| - 12\ell + 4 > \frac{3}{8}n - 12\ell.$$

If ℓ is even, let $u_{\ell-1}$ be an arbitrarily selected vertex in X' and let $P_0 := vu_{\ell-1}$. If ℓ is odd, let $u_{\ell-1}$ be a neighbor of v such that $c(vu_{\ell-1}) = c$. Recall that $u_{\ell-1}$ is in Y. By (3.4), $|(Y \cup \{v\}) \setminus \{u_{\ell-1}\}| < \delta^c(G)$, so there exists a neighbor $u_{\ell-2}$ of $u_{\ell-1}$ in X' such that $c(u_{\ell-1}u_{\ell-2}) \neq c$. Let $P_0 := vu_{\ell-1}u_{\ell-2}$. Construct a sequence of paths $P_0 \cdots P_{\ell-|P_0|}$. Let u_i denote the final vertex in P_i. If i is even then $u_i \in X'$, so let P_{i+1} be the path P_i plus the edge u_iy for some $y \in Y'''$ such that $y \notin P_i$ and $c(u_iy)$ is not in P_i. This is possible for $i \leq \ell - 4$ since $n \geq 40\ell$ implies that there are at least

$$|Y'''| - \frac{3i}{2} - 3 \geq \frac{3}{8}n - 15\ell > 0$$

ways P_i can be extended. Otherwise i is odd and $u_i \in Y'''$. Because $u_i \notin Y''$, we can select $u_{i+1} \in N(u_i) \cap X'$ so that the vertex u_{i+1} and the colors $c(u_iu_{i+1})$ and $c(u_{i+1}v)$ do not appear on the path P_i. Because $u_i \notin Y'$, we can also ensure that $c(u_i u_{i+1}) \neq c(u_{i+1} v)$. But then $P_{\ell-||P||}$ is part of a rainbow C_ℓ, a contradiction.

Therefore we can assume that there exists $e \in E(G[X'])$ such that $c(e) \neq c$ for the remainder of the proof. Then there exists a triangle that includes v and avoids the color c. If we then let B be the set of vertices b such that there exists a v, b-path of length $\ell - 2$ that avoid the color c, by Lemma 3.1.5,

$$|B| \geq \frac{3n}{4} - \frac{3}{2} - 6\ell \geq \frac{71(n+1)}{100}, \tag{3.5}$$

since $n \geq 200\ell$. By Lemma 3.1.3 and solving for $\nu(G)$, we have that

$$\nu(G) \leq \frac{3\ell}{1 - \frac{n+1}{2|B|}} + 1 \leq 11\ell. \tag{3.6}$$

Claim 3.1.6. *For every $w \in V(G)$, $d(w) < \frac{n+1}{2} + 2\nu(G) + 3\ell$.*

Proof. Assume there exists $w \in V(G)$ such that $d(w) \geq \frac{n+1}{2} + 2\nu(G) + 3\ell$. Let $s \in N^*(w)$, and note that s exist since $\delta^c(G) \geq \frac{n+1}{2}$. Let c be the number of colors incident to w which are duplicated, then $c \leq d(w) - \delta^c(G)$. Consider an edge st from s to $N(w)$ which avoids colors incident to w that are duplicated. Such an edge exists as otherwise,

$$n \geq d(w) + \delta^c(s) - c \geq d(w) + \delta^c(G) - (d(w) - \delta^c(G)) \geq 2\delta^c(G) > n.$$

Then the triangle wst is such that $c(ws) \neq c(wt)$. Since the colors that appear on the triangle wst occur at most $\nu(G) + 2$ times on the edges incident to w, we can select $U \subseteq N(w)$ of size $\lceil \frac{n}{2} \rceil + \nu(G) + 3\ell$ such that on the edge set $E(w, U)$ at least $\delta^c(G) \geq \frac{n+1}{2}$ different colors appear and the colors of the edges in the triangle wst each appear at most once. Let C be the set of colors that appear more than once on the edge set $E(w, U)$. By (3.6),

$$|C| \leq |U| - \delta^c(G) \leq \nu(G) + 3\ell \leq 14\ell. \tag{3.7}$$

By Lemma 3.1.5, (3.6), and (3.7), if B is the set of vertices b such that there exists a w, b-path of length $\ell - 2$ that avoids the colors in C, then since $n \geq 50\ell$

$$|B| + 2|C| \geq \frac{3n}{4} + \frac{|C|}{2} - 6\ell \geq \frac{3n}{4} + 2\ell \geq \left\lceil \frac{n}{2} \right\rceil + 14\ell \geq \left\lceil \frac{n}{2} \right\rceil + \nu(G) + 3\ell \geq |U|.$$

Since $|U \cap N^*(w)| \leq |U| - 2|C|$, Lemma 3.1.1 (with w, U, and B playing the roles of v, X, and Y, respectively, and $j = 3\ell$) implies that

$$n \geq |U| + \delta^c(G) - \frac{|U \cap N^*(w)|}{|B|}(\nu(G) - 1) - 3\ell \geq \frac{n}{2} + \nu(G) + 3\ell + \frac{n}{2} - (\nu(G) - 1) - 3\ell > n,$$

a contradiction. □

Let $X := X' \cup N(v, c)$, so $|X| \geq \delta^c(G)$. Let $\mathcal{P}$ be the set of rainbow paths vxu for some $x \in X$ and $u \in V(G)$. We have that

$$|\mathcal{P}| \geq |X|(\delta^c(G) - 1). \tag{3.8}$$

Note that, by Lemma 3.1.2, every $b \in B$ uses at most 3ℓ different colors on edges bx where $x \in X$ and the path bxv is rainbow. By Claim 3.1.6 and (3.6), this means that b appears on at most

$$d(b) - (\delta^c(G) - 3\ell) \leq 2\nu(G) + 6\ell \leq 28\ell$$

of the paths in $\mathcal{P}$. Therefore, with (3.5), we have that (using $|X| \geq \frac{n}{2} \geq 100\ell$)

$$|\mathcal{P}| \leq (n - |B|)|X| + |B| \cdot 28\ell = n|X| - (|X| - 28\ell)|B| \leq n|X| - \frac{5|X|}{7} \cdot \frac{71n}{100} = |X| \cdot \frac{69n}{140},$$

but this contradicts (3.8). □

Chapter 4

THE EXISTENCE OF RAINBOW CYCLES WITH EVEN LENGTH

4.1 Proof of Theorems 1.3.5 and 1.3.6

We prove Theorems 1.3.5, and 1.3.6 with the stability method, i.e., the proof contains two cases: the non-extremal case, where the graph is far from an extremal example, and the extremal case, where the graph is close to an extremal example. We make the following definitions to make this precise.

Definition 4.1.1. *A directed graph G on n vertices is λ-extremal if there exists a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that $e_G(V_i, V_{i+1}) \geq \frac{n^2}{9} - \lambda n^2$ for $i \in [3]$.*

Definition 4.1.2. *A graph G with an edge-coloring c is λ-extremal if there exists a digraph associated with (G, c) that is λ-extremal.*

In both cases, a partition $\{V_1, V_2, V_3\}$ that witnesses that a graph or digraph is λ-extremal is a λ-extremal partition. The following fact follows from the definition of λ-extremal.

Fact 4.1.3. *There exists $\lambda > 0$ such that for every ℓ that is a multiple of 3, there exists n_0 such that for every $n \geq n_0$ the following holds. If G is a λ-extremal directed graph (respectively, edge-colored graph) on n vertices, then G contains a directed C_ℓ (respectively, rainbow C_ℓ).*

When 3 does not divide ℓ, it is more difficult to show that a directed C_ℓ exists in a λ-extremal graph. To this end, we get the following proposition which follows from a standard application of the degree form of the digraph regularity lemma of Alon & Shapira [2], and its modification for oriented graphs by Kelly, Kühn, & Osthus (See Lemma 3.2 in [17]).

This lemma is similar to Lemma 22 in [18] and reduces the problem from finding ℓ-cycles to finding closed ℓ-walks.

Proposition 4.1.4. *For every $\ell \geq 3$, $\xi, \beta, \lambda > 0$ and n_0', there exists $\alpha > 0$ and n_0 such that for every $n \geq n_0$ the following holds. Suppose that every oriented graph G' on $n' \geq n_0'$ vertices with $\delta^+(G') \geq (\xi - \beta)n'$ either has a closed ℓ-walk or is $(\lambda - \beta)$-extremal. Then both of the following statements are true:*

- *Every oriented graph G on n vertices such that $\delta^+(G) \geq \xi n$ either has αn^ℓ directed cycles of length ℓ or is λ-extremal.*
- *If ℓ is even, then every directed graph G on n vertices such that $\delta^+(G) \geq \xi n$ either has αn^ℓ directed cycles of length ℓ or is λ-extremal.*

Proof sketch. If G is an oriented graph, then apply the degree form of the digraph regularity lemma for oriented graphs (Lemma 3.2 in [17]) to obtain a cluster oriented graph G' on n' vertices for some $n' \geq n_0'$ such that $d^+(G') \geq (\xi - \beta)n'$.

If ℓ is even and G is a directed graph, then apply the degree form of the digraph regularity lemma to obtain a cluster digraph G' on n' vertices for some $n' \geq n_0'$ such that $d^+(G') \geq (\xi - \beta)n'$. If G' contains a directed C_2, then, because ℓ is even, G' contains a closed ℓ-walk. Otherwise, G' is an oriented graph.

In either case, if G' has a closed ℓ-walk, then, by a standard argument, G has at least αn^ℓ directed C_ℓ. Otherwise, because G' is an oriented graph on $n' \geq n_0'$ vertices with $\delta^+(G') \geq (\xi - \beta)n'$, it must be that G' is $(\lambda - \beta)$-extremal. By a standard argument, this implies that G is λ-extremal. $\square$

We combine Lemma 4.1.4 with the following lemma, Lemma 4.1.5, to prove the non-extremal case.

Lemma 4.1.5 (Non-extremal lemma)**.** *Suppose $\lambda > 0$. For every $\ell \in \mathbb{N} \setminus \{1, 2, 3, 5\}$ there exists $\alpha > 0$ and n_0 such that every oriented graph G on $n \geq n_0$ vertices that does not contain a closed ℓ-walk and such that $\delta^+(G) \geq (\frac{1}{3} - \alpha)n$ is λ-extremal. Furthermore, when $\ell = 5$ and $\lambda > 0$, there exists n_0 such that every oriented graph G on $n \geq n_0$ vertices that does not contain a closed ℓ-walk and such that $\delta^+(G) \geq \frac{n+1}{3}$ is λ-extremal.*

In Lemma 4.1.5, it is necessary to treat the case when $\ell = 5$ in a special way, because the set of extremal examples in this case is more complicated. To see this, consider the n-vertex blow-up of a directed triangle with parts V_1, V_2 and V_3 and edges going from V_i to V_{i+1} for $i \in [3]$. Now split V_2 into two parts, V_2^1 and V_2^2, and put all possible edges from V_2^1 to V_2^2. This modified oriented graph still has no directed C_5 and has minimum out-degree $\lfloor \frac{n}{3} \rfloor$. Furthermore, for every $v \in V_2^1$ we can remove $|V_2^2|$ edges directed from v to V_3 and not decrease the minimum out-degree condition. If $\lambda > 0$ is small and $|V_2^2|$ large, for example $|V_2^2| = \lfloor \frac{n}{6} \rfloor$, then digraphs constructed in this way are not λ-extremal.

Combining Proposition 4.1.4 with Lemma 4.1.5 implies the following.

Lemma 4.1.6. *Suppose $\lambda > 0$. For every $\ell \in \mathbb{N} \setminus \{1, 2, 3, 5\}$ there exists $\alpha > 0$ and n_0 such that if G is an oriented graph on $n \geq n_0$ vertices such that $\delta^+(G) \geq (\frac{1}{3} - \alpha)n$, or ℓ is even and G is a directed graph on $n \geq n_0$ vertices such that $\delta^+(G') \geq (\frac{1}{3} - \alpha)n$, then either G' is λ-extremal or G' contains at least αn^ℓ directed C_ℓ.*

Proof. Let 2λ, 2α, and n_0' be the values of λ, α and n_0 in Lemma 4.1.5 respectively. Let α, α, and λ be the values of ξ, β, and λ in Proposition 4.1.4 respectively. Then for every oriented graph G' on $n' \geq n_0'$ either has a closed ℓ-walk or is 2α-extremal. But then Proposition 4.1.4 applies immediately implying the lemma. □

To prove the extremal case, we prove the following lemma.

Lemma 4.1.7 (Extremal lemma)**.** *For every $\ell \geq 4$ that is not divisible by 3, there exists n_0 and $\lambda > 0$ such that for every $n \geq n_0$ the following holds. Suppose that G is a graph on*

n vertices and c is an edge-coloring of G such that (G, c) is λ-extremal. If $\ell \neq 5$ and (1.1) *holds, i.e.,*

$$\delta^c(G) \geq \begin{cases} \frac{n+5}{3} & \text{if } \ell = 1 \pmod 3 \\ \frac{n+7}{3} & \text{if } \ell = 2 \pmod 3, \end{cases}$$

then G contains a rainbow C_ℓ. Furthermore, if $\ell \equiv 1 \pmod 3$ and $\delta^c(G) \geq \frac{n+4}{3}$, then G contains a properly colored C_ℓ. Finally, if $\ell \equiv 2 \pmod 3$, $\delta^c(G) \geq \frac{n+4}{3}$, and there exists an oriented graph G' such that (G, c) is the simple edge-colored graph determined by G', then G contains a properly colored C_ℓ.

We prove Lemma 4.1.5 in Section 4.3 and we prove Lemma 4.1.7 in Section 4.4. We now show how the above lemmas and facts along with Propositions 1.3.12 and Fact 1.3.10 from Subsection 1.3.1 imply Theorems 1.3.5 and 1.3.6.

Theorem 1.3.5 (Czygrinow, Molla, Nagle, & Oursler)**.** *For every even $\ell \geq 4$, there exists $\alpha > 0$ and n_0 such that for every $n \geq n_0$ the following holds. If G is a graph on n vertices and c is an edge-coloring of G such that*

$$\delta^c(G) \geq \begin{cases} (\frac{1}{3} - \alpha)n & \text{if } \ell = 0 \pmod 3 \\ \dfrac{n+5}{3} & \text{if } \ell = 1 \pmod 3 \\ \dfrac{n+7}{3} & \text{if } \ell = 2 \pmod 3, \end{cases} \tag{1.1}$$

then G contains a rainbow ℓ-cycle.

Proof. Let $\ell \geq 4$ be even. Fix λ and α so that Lemma 4.1.7, Lemma 4.1.6, and Proposition 1.3.12 apply. Let n_0 be the larger of the values produced by this choice. Let G be a graph on $n > n_0$ vertices and c an edge-coloring of G such that (G, c) satisfies (1.1). Let G' be a directed graph associated with (G, c). We have that $\delta^+(G') = \delta^c(G) \geq (\frac{1}{3} - \alpha)n$, so by Lemma 4.1.6, either G' is λ-extremal or G' contains αn^ℓ directed cycles C_ℓ. If G' contains αn^ℓ cycles, then Proposition 1.3.12 implies that G has a rainbow C_ℓ. If G' is λ-extremal, then (G, c) is also λ-extremal. If ℓ is divisible by 3, then Fact 4.1.3 implies that G has a rainbow C_ℓ. Otherwise, Lemma 4.1.7 implies that G has a rainbow C_ℓ. $\square$

Theorem 1.3.6 (Czygrinow, Molla, Nagle, & Oursler)**.** *For every $\ell \geq 4$, there exists n_0 such that for every $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and $\delta^+(G) \geq \frac{n+1}{3}$, then G contains a directed ℓ-cycle.*

Proof. Let $\ell \geq 4$. Fix $\lambda > 0$ and n_0 such that Lemmas 4.1.5, 4.1.6, and 4.1.7 apply in the following argument. Let G be an oriented graph on $n > n_0$ vertices such that $\delta^+(G) \geq \frac{n+1}{3}$. Assume for contradiction that G does not contain a directed C_ℓ. Let

$$U := \{u \in V(G) : d^-_G(u) = 0\}.$$

Note that $G - U$ does not contain a directed C_ℓ and that the minimum out-degree of $G - U$ is equal to $\delta^+(G)$. We also have that

$$\binom{|G-U|}{2} \geq |E(G-U)| \geq |G-U|\delta^+(G),$$

so, $(|G - U| - 1) \geq 2\delta^+(G) \geq \frac{2n}{3}$, and $|G - U| > \frac{2n}{3}$.

When $\ell = 5$, Lemma 4.1.5 directly implies that $G - U$ is λ-extremal. When $\ell \neq 5$, we have that $G - U$ is λ-extremal by Lemma 4.1.6.

In either case we have that $G - U$ is λ-extremal. Because $\delta^-(G - U) \geq 1$, if (G', c) is the simple edge-colored graph determined by $G - U$, then

$$d^c(G') = \delta^+(G - U) + 1 = \delta^+(G) + 1 \geq \frac{n+4}{3} \geq \frac{(|G'| + 4)}{3}.$$

Therefore Lemma 4.1.7 implies that (G', c) contains a properly colored C_ℓ. By Fact 1.3.10, such a C_ℓ corresponds to a directed C_ℓ in $G - U \subseteq G$, which is a contradiction. $\square$

4.2 Digraph and Rainbow Subgraph Relationship

In this section we give proofs of the results mentioned in Subsection 1.3.1, that the minimum degree bound in Theorem 1.3.5 is tight when 3 does not divide ℓ, and that if

(G, c) is an edge colored graph, G' an directed graph associated with G with a significant number of 1-in directions of a graph F, then G contains a rainbow F.

Proposition 1.3.11. *Theorem 1.3.5 is the best possible for sufficiently large n when 3 does not divide ℓ.*

Proof. Let G' be the n-vertex blowup of a directed C_3 on $[n]$ and (G, c) the simple edge colored graph determined by G'. Then G does not contain a rainbow ℓ-cycle, $\delta^c(G) \geq \lfloor \frac{n}{3} \rfloor + 1$, and when $\ell \equiv 1 \pmod 3$ and $n \pmod 3 \in \{0, 1\}$ this provides a sharp bound for Theorem 1.3.5. When $\ell \equiv 2 \pmod 3$ we can modify G to create another sharpness example for Theorem 1.3.5. This example is created by adding new edges, each of which are colored with the color $n + 1$ which is distinct from any previous edge color on G. The new edges are added inside each of the three parts so that every vertex is incident to at least one new edge. Then the minimum color degree is $\lfloor \frac{n}{3} \rfloor + 2 = \lceil \frac{n+4}{3} \rceil$, but because $\ell \equiv 2 \pmod 3$ and at most one new edge can appear in a rainbow subgraph, there does not exist a rainbow ℓ-cycle.

The sharpness example for Theorem 1.3.5 when $\ell \equiv 1$ and $n \equiv 2 \pmod 3$ also starts with (G, c). Let $m := \lfloor \frac{n}{3} \rfloor$, so $n = 3m + 2$ and label the parts V_1, V_2 and V_3 so that edges in G' go from V_i to V_{i+1} for $i \in [3]$. We can assume that $V_1 = [m + 1]$ and $V_2 = \{i \in \mathbb{N} : m + 2 \leq i \leq 2m + 2\}$. Note that the minimum color degree is $m + 1$, as witnessed by vertices in V_2, but, for a sharpness example, we want the minimum color degree to be $m + 2 = \lfloor \frac{n+6}{3} \rfloor = \lceil \frac{n+4}{3} \rceil$. We modify the coloring c to achieve this in the following way: for every $i, j \in [m + 1]$, we let

$$c(\{j, m + 1 + i\}) = \begin{cases} n + 1 & \text{if } i = j \\ i & \text{otherwise,} \end{cases}$$

and we leave the color on all other edges unchanged. Now every vertex has color degree $m + 2$, and we have not created a rainbow ℓ-cycle. To see this, assume, for a contradiction, that $C = u_1, \dots, u_\ell$ is such an ℓ-cycle. Because ℓ is not divisible by 3, without loss of generality we can assume that there exists $j \in [3]$ and $i \in [\ell]$ such that $u_i \in V_{j-1}$, $u_{i+1} \in V_j$ and $u_{i+2} \in V_{j-1}$, i.e., the cycle must change direction at least once and we can assume, by potentially reversing the labeling of C, that this reversal goes from the forward direction to the backward direction. Furthermore, the coloring and the fact that C is rainbow imply that $j = 2$. Without loss of generality we can assume that $i = 1$ and that $u_1 = 2$, $u_2 = (m+1)+1$ and $u_3 = 1$, so $c(u_1u_2) = 1$ and $c(u_2u_3) = n + 1$. Because for every $u \in N(u_3, V_3)$, we have that $c(uu_3) = 1$ and $c(u_1u_2) = 1$, it must be that $u_4 \in V_2$. Now, because $\ell \equiv 1 \pmod 3$, and $u_1 \in V_1$ and $u_4 \in V_2 = V_{4-2}$, there must exist an index $4 \le i \le \ell$ such that $u_i \in V_{i-2}$ and $u_{i+1} \in V_{i-3}$, i.e., we must move in the backward direction at least one more time (we could potentially have $u_\ell \in V_2$ and $u_{\ell+1} = u_1 \in V_1$). Let i be the smallest such index, so we have that $u_{i-1} \in V_{i-3}$, $u_i \in V_{i-2}$, and $u_{i+1} \in V_{i-3}$. Then either the edge $u_{i-1}u_i$ or u_iv_{i+1} must be given the color $n+1$, a contradiction to the fact that $c(u_2u_3) = n+1$. □

Proposition 1.3.12. *For every graph F and $\alpha > 0$, there exists n_0 such that for every $n \ge n_0$ the following holds. Let G be a graph on n vertices, let c be an edge-coloring of G and let G' be a directed graph associated with (G, c). If F' is a 1-in direction of F and G' contains at least $\alpha n^{|F|}$ copies of F', then G contains a rainbow F.*

Proof. Let $\ell = |F|$. We can assume that $n_0 > \frac{\ell^4}{\alpha}$, so

$$\ell^4 < \alpha n_0 \le \alpha n. \tag{4.1}$$

Let $\Psi \subseteq V(G')^\ell$ be the set of ℓ-tuples $(v_1, \dots, v_\ell) \in V(G')^\ell$ such that $\{v_1, \dots, v_\ell\}$ contains F' so that for some $2 \le i \le \ell - 1$, we have that $c(\{v_1, v_2\}) = c(\{v_i, v_{i+1}\})$ and v_iv_{i+1} is a directed edge in G'. Call an element $(v_1, \dots, v_\ell) \in \Psi$ an *i-repeat* if $c(\{v_1, v_2\}) = c(\{v_i, v_{i+1}\})$, so every element in Ψ is an i-repeat for some $2 \le i \le \ell - 1$. If we assume for a contradiction that G has no rainbow F, then we can map every copy of F' in G' to an element in Ψ. To see why, let F' be on vertices $\{v_1, \dots, v_\ell\}$ and since G does not contain a rainbow

F, there exist edges $\{u_1, u_2\}$ and $\{u_3, u_4\}$ with the same color. If $\{u_1, u_2\} \cap \{u_3, u_4\} = \emptyset$, then we can obviously order the vertices so that u_1u_2 and u_3u_4 are directed edges in G. Otherwise, without loss of generality assume that $u_1 = u_4$. Note that if there exists a directed path $u_3u_1u_2$ or $u_2u_1u_3$ in G', then we can order the vertices so that u_1u_2 and u_3u_4 are directed edges in G as well. Otherwise no such path exists. Note that u_1u_2 and u_1u_3 cannot both be directed edges in G' as the directed edges leaving u_1 are rainbow in G. If both u_2u_1 and u_3u_1 are directed edges in G', since F' is a 1-in direction the directed edge u_1u_2 or u_1u_3 must exist. But then F' contains a directed path $u_3u_1u_2$ or $u_2u_1u_3$, contradicting the assumption that no such path exists. Thus we can always order the vertices (after possibly relabeling) so that u_1u_2 and u_3u_4 are directed edges in G. Therefore we can map F' to an element in Ψ. On the other hand there are at most $\ell!$ copies of F' on the vertices $\{v_1, \ldots, v_\ell\}$, and if we associate each copy of F' with a possible starting edge v_1v_2, there are at least $(\ell - 3)!$ ways to map F' into Ψ. Thus

$$|\Psi| \geq \frac{\alpha}{\ell^3} n^\ell. \tag{4.2}$$

To get an upper bound on $|\Psi|$, observe that we can generate every element in Ψ with the following procedure. First pick $2 \leq i \leq \ell - 1$ such that there exists an i-repeat in Ψ. Then for j from 1 to ℓ, pick a vertex v_j so that $v_1, \ldots, v_j$ are the initial j vertices of some i-repeat in Ψ. We clearly have at most n choices for each selection v_j. Crucially when $j = i + 1$, we have exactly one choice for v_j, because there is only one vertex $u \in N^+(v_i)$ such that $c(\{v_i, u\}) = c(\{v_1, v_2\})$. Therefore, by (4.1),

$$|\Psi| \leq (\ell - 2) n^{\ell - 1} < \frac{\alpha}{\ell^3} n^\ell,$$

which contradicts (4.2). $\square$

4.3 Non-extremal Case

We use the following corollary to the main result of Ji, Wu, & Song in [15] (Corollary 1.5).

Corollary 4.3.1 (Ji, Wu, & Song 2018 [15]). *For every $n \in \mathbb{N}$, $\varepsilon < 0.6976$ and ℓ such that $4 \leq \ell \leq 1.4334 \cdot \varepsilon n + 2$, the following holds. If G is an oriented graph on n vertices that does not contain a directed triangle and $\delta^0(G) \geq (0.3024 + \varepsilon)n$, then for every $u \in V(G)$, there exists a directed C_ℓ which contains u.*

We use the following fact several times throughout this section.

Fact 4.3.2. *If G is an oriented graph that contains a vertex x such that x is in a directed triangle and a directed C_4, then x is in a closed ℓ-walk for every $\ell \geq 3$ such that $\ell \neq 5$.*

We now collect a few simple facts which aid in identifying λ-extremal oriented graphs in what follows.

Proposition 4.3.3. *For every $\lambda > 0$, there exists n_0 and $\alpha > 0$ such that for every $n \geq n_0$ and every oriented graph G on n vertices the following holds:*

(1) If $G' \subseteq G$, $|G'| \geq (1-\alpha)n$, and G' is $(\lambda - \alpha)$-extremal, then G is λ-extremal.

(2) If $|E(G)| \geq (\frac{1}{3} - \alpha)n^2$ and G has no transitive triangles, then G is λ-extremal.

(3) If $\delta^0(G) \geq (\frac{1}{3} - \alpha)n$ and $V_1, V_2 \subseteq V(G)$ are disjoint sets each of order at least $(\frac{1}{3} - \alpha)n$ such that $|E(V_1)|$, $|E(V_2)|$, and $|E(V_2, V_1)|$ are each at most αn^2, then G is λ-extremal.

Proof. To see (1), note that if $\{V_1', V_2', V_3'\}$ is a $(\lambda - \alpha)$-extremal partition of G', then if we define $V_1 := V_1' \cup V(G - G')$, $V_2 := V_2'$, and $V_3 := V_3'$ we have that, for every $i \in [3]$,

$$|E(V_i, V_{i+1})| \geq |E(V_i', V_{i+1}')| \geq \frac{(n-\alpha)^2}{9} - (\lambda - \alpha)n^2 \geq \frac{n^2}{9} - \lambda n^2,$$

so $\{V_1, V_2, V_3\}$ is a λ-extremal partition of G.

To see (2), first note that, because G has no transitive triangles, the graph underlying G is K_4-free. Therefore by the Erdős & Simonovits stability theorem, there exists a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that $|V_i| \in \{\lfloor \frac{n}{3} \rfloor, \lceil \frac{n}{3} \rceil\}$ for $i \in [3]$, and, for some $\alpha \ll \beta \ll \lambda$, there are at least $\frac{n^2}{9} - \beta^2 n^2$ edges between V_i and V_{i+1} for every $i \in [3]$. Call a vertex $v \in V_i$, *i-typical* if it is adjacent to all but at most βn vertices in V_{i-1} and all but at most βn vertices in V_{i+1}. Because, for every $i \in [3]$,

$$|V_i||V_{i+1}| - (\frac{n^2}{9} - \beta^2 n^2) < \beta n \cdot 2\beta n,$$

there are fewer than $4\beta n$ vertices in V_i that are not i-typical. Therefore we can assume, by possibly changing the labeling of V_1, V_2 and V_3, that there exists a directed triangle $v_1 v_2 v_3$ such that v_i is i-typical for every $i \in [3]$. For $i \in [3]$, let

$$U_i := \{u \in V_i : u \text{ is } i\text{-typical and } u \text{ is adjacent to both } v_{i-1} \text{ and } v_{i+1}\}.$$

Note that $|U_i| \geq |V_i| - 6\beta n \geq (\frac{1}{3} - 7\beta)n$, and that, because there are no transitive triangles, every edge between a vertex $u_i \in U_i$ and $u_{i+1} \in U_{i+1}$ must be directed from u_i to u_{i+1}. Therefore

$$|E(V_i, V_{i+1})| \geq |E(U_i, U_{i+1})| \geq \sum_{u \in U_i} d^+(u, U_{i+1}) \geq |U_i|(|U_{i+1}| - \beta n) \geq \frac{n^2}{9} - \lambda n^2.$$

To see (3), let $V_3 := V(G) \setminus (V_1 \cup V_2)$. We have that, by the minimum semidegree condition,

$$|E(V_2, V_3)| \geq \sum_{v \in V_2} d^+(v) - |E(V_2)| - |E(V_2, V_1)| \geq \frac{n^2}{9} - 4\alpha n^2,$$

and similarly $|E(V_3, V_1)| \geq \frac{n^2}{9} - 4\alpha n^2$. Since $|V_2| + |V_3| = n - |V_1| \leq (\frac{2}{3} + \alpha)n$, we have that

$$|V_2||V_3| \leq \left((\frac{1}{3} + \frac{\alpha}{2})n\right)^2 \leq \frac{n^2}{9} + \frac{\alpha n^2}{3} + \frac{\alpha^2 n^2}{4}.$$

so

$$|E(V_3, V_2)| \leq |V_2||V_3| - |E(V_2, V_3)| \leq 5\alpha n^2.$$

and, by the minimum semidegree condition,

$$|E(V_1, V_2)| \geq \sum_{v \in V_2} d^-(v) - |E(V_2)| - |E(V_3, V_2)| \geq \frac{n^2}{9} - \lambda n^2. \qquad \square$$

The following lemma allows us to convert statements involving minimum semidegree to analogous statements involving minimum out-degree.

Lemma 4.3.4. *For every $\ell \in \mathbb{N} \setminus \{1, 2, 3, 5\}$ and $\beta > 0$, there exists $\alpha > 0$ and n_0 such that for every $n \geq n_0$ and $\xi \geq \frac{1}{3} - \alpha$ the following holds, and when $\ell = 5$ and $\beta > 0$ there exists n_0 such that for every $\xi \geq \frac{(n+1)}{3n}$ the following holds. If G is an oriented graph on n vertices that does not contain a closed ℓ-walk and $\delta^+(G) = \xi n$, then there exists $G' \subseteq G$ such that $|G'| \geq (1 - \beta)n$ and $\delta^0(G') \geq (\xi - \beta)|G'|$.*

Proof. Let n_0, α, and γ be such that

$$0 < \frac{1}{n_0} \ll \alpha \ll \gamma \ll \beta, \frac{1}{\ell},$$

and assume that G is an n-vertex counterexample for some $n \geq n_0$. Let $x \in V(G)$ be such that it maximizes $d^-(x)$, and define $\eta := \frac{d^-(x)}{n}$. Then

$$d^-(v) \leq \eta n \text{ for every } v \in V(G). \tag{4.3}$$

Claim 4.3.5. $\eta > (\xi + \gamma)$.

Proof. Assume for contradiction that

$$\eta \leq \xi + \gamma, \tag{4.4}$$

and let

$$V' := \{v \in V(G) : d^-(v) < (\xi - \beta^2)n\}.$$

Then, by (4.3) and (4.4),

$$n \cdot \xi n \leq \sum_{v \in V(G)} d^-(v) \leq (n - |V'|) \cdot (\xi + \gamma) n + |V'| \cdot (\xi - \beta^2) n = (\xi + \gamma) n^2 - |V'|(\gamma + \beta^2) n,$$

which implies that

$$|V'| \leq \frac{\gamma n^2}{(\gamma + \beta^2) n} \leq \frac{\frac{\beta^3}{2}}{\beta^2} n = \beta \frac{n}{2}.$$

If we let $G' := G - V'$, then we have that

$$\delta^-(G') \geq (\xi - \beta^2) n - |V'| \geq (\xi - \beta)|G'| \qquad \text{and} \qquad \delta^+(G') \geq \xi n - |V'| \geq (\xi - \beta)|G'|,$$

which contradicts the assumption that G is a counterexample. $\square$

Claim 4.3.6. *If U and W are disjoint subsets of $V(G)$ such that $|U| \geq \eta n$ and $|W| \geq \xi n$, then there exists a path wvu such that $w \in W$ and $u \in U$.*

We defer the proof of Claim 4.3.6 so that we can first show how it and Claim 4.3.5 together imply a contradiction. To this end, assume that Claim 4.3.6 holds and note that Claim 4.3.6, with $N^-(x)$ and $N^+(x)$ playing the roles of U and W, respectively, implies that there exists a closed 4-walk containing x. Therefore we can assume that $\ell \geq 5$.

First assume $\ell \geq 6$. If there exists a vertex $y \in N^+(x)$ such that there is a vertex $z \in N^+(y) \cap N^-(x)$, then $xyzx$ is a directed triangle containing x. But by Fact 4.3.2, there exists a closed ℓ-walk in G, a contradiction. Therefore for every $y \in N^+(x)$, we can assume that $d^+(y, N^-(x)) = 0$, so by (4.3) and Claim 4.3.5,

$$d^+(y, N^+(x)) \geq d^+(y) + |N^+(x)| - (n - |N^-(x)|) \geq 2\xi n + (\xi + \gamma) n - n \geq (\gamma - 3\alpha) n \geq \ell.$$

Therefore there exists a path $y_1, \ldots, y_{\ell-2}$ of length $(\ell - 2)$ in $N^+(x)$. Since there is no closed ℓ-walk, we have that $N^+(y_{\ell-2})$ is disjoint from $N^-(x)$, so Claim 4.3.6, with $N^-(x)$ and $N^+(y_{\ell-2})$ playing the roles of U and W, respectively, implies that there exists $w, v, u \in V(G)$ such that $y_{\ell-2}wvux$ is a path in G. We then have that $xy_3 \ldots y_{\ell-2}wvux$ is a closed ℓ-walk, which is a contradiction.

Now assume $\ell = 5$. In this case, $\delta^+(G) \geq \frac{(n+1)}{3}$, so $d^-(x) \geq \frac{(n+1)}{3}$. In fact we get that $d^-(x) > \frac{(n+1)}{3}$, as otherwise, by the maximality of $d^-(x)$, $\Delta^-(x) = \delta^+(G)$ implying $d^+(v) = d^-(v) = \frac{(n+1)}{3}$ for all vertices and satisfying the lemma. Assume there exists an x, x'-path on 3 vertices and an x, x'-path on 4 vertices. Then $N^-(x)$ and $N^+(x')$ are disjoint, but $d^-(x) + d^+(x) + d^+(x') > n$. Then there exists $y \in N^+(x', N^+(x))$. Because there exists an x, x'-path on 3 vertices, there exists an x, y-path on 4 vertices. But then $N^-(x)$ and $N^+(y)$ must be disjoint, and Claim 4.3.6, with $N^-(x)$ and $N^+(y)$ playing the roles of U and W respectively, implies that there exists $w \in N^+(y)$, $v \in V(G)$, and $u \in N^-(x)$ such that $xywvux$ is a directed C_5, a contradiction. Thus there exists no vertex x' in an x, x'-path on both 3-vertices and 4-vertices.

Therefore $N^+(x)$ is an independent set, because if there exists yz in $E(G[N^+(x)])$, then for every vertex $x' \in N^+(z)$ we have the paths xzx' and $xyzx'$, a contradiction. So for every $y \in N^+(x)$, we have that $N^+(y)$ is disjoint from $N^+(x)$. We also have that $N^+(y)$ is an independent set, because if there exists zx' in $E(G[N^+(y)])$, then we have the paths $xyzx'$ and xyx', a contradiction. Because $\delta^+(G) \geq \frac{(n+1)}{3}$, $N^+(x)$ and $N^+(y)$ are disjoint, and $N^+(y)$ is an independent set, we can conclude that for every $z \in N^+(y)$ there exists $w \in N^+(z, N^+(x))$. But now since $\delta^+(G) \geq \frac{(n+1)}{3}$, $N^+(x)$ is an independent set, and $d^-(x) > \frac{(n+1)}{3}$, there exists $u \in N^+(w, N^-(x))$. But then $xyzwux$ is a directed C_5, a contradiction.

Proof of Claim 4.3.6. Assume for contradiction that such a path does not exist. Let $X_1 \subseteq W$ be such that $|X_1| = \xi n$, and let $X_2 := N^+(X_1)$. By our assumption, $N^+(X_2)$ is disjoint from U. If we let $X_3 = N^+(X_2) \setminus X_1$, then the sets X_3, X_1, and U are pairwise disjoint, so

$$\frac{|X_3|}{n} + \xi + \eta - 1 \leq 0. \tag{4.5}$$

With (4.3) we have that

$$|E(X_2, X_3)| \leq \eta n \cdot |X_3|. \tag{4.6}$$

Since $|X_1| = \xi n$, we also get that $|E(X_1, X_2)| \geq |X_1|\delta^+(G) = (\xi n)^2$. Therefore

$$|E(X_2, X_1)| \leq |X_1||X_2| - |E(X_1, X_2)| \leq \xi n \cdot |X_2| - (\xi n)^2,$$

so

$$|E(X_2, X_3)| \geq \delta^+(G) \cdot |X_2| - |E(X_2, X_1)| \geq (\xi n)^2. \tag{4.7}$$

Together (4.6) and (4.7) imply that $\eta n \cdot |X_3| \geq |E(X_2, X_3)| \geq \xi^2 n^2$, so

$$\frac{|X_3|}{n} \geq \frac{\xi^2}{\eta},$$

and, with (4.5) and the fact that $\xi \geq (\frac{1}{3} - \alpha)$, we have that

$$0 \geq \frac{\xi^2}{\eta} + \xi + \eta - 1 \geq \frac{(\frac{1}{3} - \alpha)^2}{\eta} + (\frac{1}{3} - \alpha) + \eta - 1.$$

This implies that $\eta^2 - (\frac{2}{3} + \alpha)\eta + (\frac{1}{3} - \alpha)^2 \leq 0$. Solving yields

$$\eta \leq \frac{\frac{2}{3} + \alpha + \sqrt{4\alpha - 3\alpha^2}}{2} \leq (\frac{1}{3} - \alpha) + \gamma \leq \xi + \gamma,$$

a contradiction to Claim 4.3.5. $\square$

$\square$

The following is a corollary to Theorem 1.3.7 and Lemma 4.3.4.

Corollary 4.3.7. *For every $\ell \geq 4$ and $\alpha > 0$, there exists n_0 such that for every $n \geq n_0$ the following holds. If G is an oriented graph on n vertices that does not contain a closed ℓ-walk, then there exists $x, y \in V(G)$ such that $d^+(x) < (\frac{1}{3} + \alpha)n$ and $d^-(y) < (\frac{1}{3} + \alpha)n$.*

Proof. Let n_0, β and α be such that

$$0 < \frac{1}{n_0} \ll \beta \ll \alpha \ll \frac{1}{\ell}.$$

Assume for contradiction that $\delta^+(G) \geq (\frac{1}{3}+\alpha)n$, then Lemma 4.3.4 implies that there exists a subgraph G' of G such that $|G'| \geq (1-\beta)n$ and $\delta^0(G') \geq (\frac{1}{3}+\alpha-\beta)|G'| \geq (\frac{1}{3}+\frac{\alpha}{2})|G'|$. By Theorem 1.3.7, $|G'|$ must contain a closed ℓ-walk, a contradiction.

By reversing the orientation of the edges in G, the previous argument implies that there exists $y \in V(G)$ such that $d^-(y) < (\frac{1}{3}+\alpha)n$ as well. □

Lemma 4.3.8. *For every $\ell \geq 4$ and $\lambda > 0$, there exists $\alpha > 0$ and n_0 such that for every $n \geq n_0$ the following holds. If G is an oriented graph on n vertices that does not contain a closed ℓ-walk and $\delta^0(G) \geq (\frac{1}{3}-\alpha)n$, then G is λ-extremal.*

Proof. We start the proof with two claims.

Claim 4.3.9. *Suppose that $X^+, X^- \subseteq V(G)$ such that $|X^+|, |X^-| \geq (\frac{1}{3}-\alpha)n$ and such that $|X^+ \cap X^-| \leq (\frac{1}{3}-21\alpha n)$. If there does not exist a path x^+yx^- with $x^+ \in X^+$ and $x^- \in X^-$, then, for $\sigma \in \{-,+\}$, there exists $Y^\sigma \subseteq X^\sigma \setminus X^{-\sigma}$ such that $|Y^\sigma| \geq |X^\sigma \setminus X^{-\sigma}| - 7\alpha n$ and Y^σ is independent.*

Proof. Call a path x^+yx^- with $x^+ \in X^+$ and $x^- \in X^-$ a *forbidden path*, and note that we are assuming that there are no forbidden paths. For $\sigma \in \{-,+\}$, let

$$U^\sigma = \{u \in X^\sigma \setminus X^{-\sigma} : d^{-\sigma}(u, X^\sigma \setminus X^{-\sigma}) > 0\},$$

let $X := X^+ \cup X^-$, and let $Y^\sigma := X^\sigma \setminus (X^{-\sigma} \cup U^\sigma)$. Then Y^σ is independent, so we prove the claim if we show that $|U^\sigma| \leq 7\alpha n$.

Define $W^\sigma := N^\sigma(U^\sigma) \setminus X$. Since there are no forbidden paths, $W^- \cap W^+ = \emptyset$, so

$$|X| + |W^+| + |W^-| \leq n. \tag{4.8}$$

We first prove the following implication:

$$|U^\sigma| \geq \alpha n \Rightarrow |W^\sigma| \geq \delta^0(G) - (\frac{1}{3}+\alpha)|U^\sigma|. \tag{4.9}$$

To see that (4.9) holds, assume that $|U^\sigma| \geq \alpha n$. Because $G[U^\sigma]$ has no closed ℓ-walk and αn is sufficiently large, Corollary 4.3.7 implies that there exists $u \in U^\sigma$ such that

$$d^\sigma(u, U^\sigma) \leq (\frac{1}{3}+\alpha)|U^\sigma|. \tag{4.10}$$

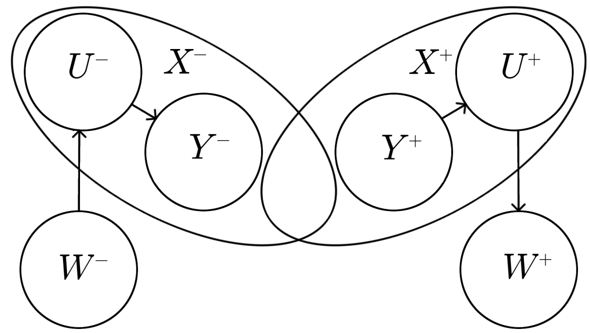


Figure 4.1: Relationship Between X^+, X^-, U^+, U^-, Y^+, Y^-, W^+, and W^-

By the definition of U^σ, there exists $w \in N^{-\sigma}(u, X^\sigma)$, so because there are no forbidden paths, we have that $d^\sigma(u, X^{-\sigma}) = 0$. Furthermore we have that $d^\sigma(u, U^\sigma) = d^\sigma(u, X^\sigma)$, implying $d^\sigma(u, X) = d^\sigma(u, U^\sigma)$. By the definition of W^σ,

$$d^\sigma(u) = d^\sigma(u, W^\sigma) + d^\sigma(u, X) = d^\sigma(u, W^\sigma) + d^\sigma(u, U^\sigma).$$

and this with (4.10) gives us that

$$|W^\sigma| \geq d^\sigma(u, W^\sigma) \geq d^\sigma(u) - d^\sigma(u, U^\sigma) \geq \delta^0(G) - (\frac{1}{3} + \alpha)|U^\sigma|,$$

proving (4.9).

We now use (4.9) to complete the proof of this claim by showing that $|U^\sigma| < 7\alpha n$. Assume $|U^\sigma| \geq \alpha n$ and, for convenience, define $\Gamma := (\frac{1}{3} - \alpha)n$.

If $|U^{-\sigma}| \geq \alpha n$, then by (4.8), (4.9), $U^- \cup U^+ \subseteq X \setminus (X^+ \cap X^-)$, $|X| \geq 2\Gamma - |X^+ \cap X^-|$, and $|X^+ \cap X^-| \leq \Gamma - 20\alpha n$, we have that

$$\begin{aligned}
n &\geq |X| + 2\Gamma - (\frac{1}{3} + \alpha)(|U^-| + |U^+|) \geq |X| + 2\Gamma - (\frac{1}{3} + \alpha)(|X| - |X^+ \cap X^-|) \\
&= (\frac{2}{3} - \alpha)|X| + 2\Gamma + (\frac{1}{3} + \alpha)|X^+ \cap X^-| \\
&\geq (\frac{10}{3} - 2\alpha)\Gamma - (\frac{1}{3} - 2\alpha)|X^+ \cap X^-| \geq 3\Gamma + (\frac{1}{3} - 2\alpha)20\alpha n,
\end{aligned}$$

a contradiction.

Otherwise $|U^{-\sigma}| < \alpha n$. Then there exists $v \in X^{-\sigma} \setminus U^{-\sigma}$ with $d^{-\sigma}(v, X^{-\sigma} \setminus U^{-\sigma}) = 0$. To see this, assume the contrary and let $v_1 \in Y^{-\sigma}$. Then there exists $v_2 \in N^-(v_1, X^{-\sigma} \setminus U^{-\sigma})$. As $Y^{-\sigma}$ is an independent set, we have that $v_2 \in X^+ \cap X^-$. There also exists $v_3 \in N^-(v_2, X^{-\sigma} \setminus U^{-\sigma})$. Since there are no forbidden paths, $v_3 \notin X^\sigma \cap X^{-\sigma}$, so $v_3 \in Y^{-\sigma}$. Similar to v_1, there exists $v_4 \in X^\sigma \cap X^{-\sigma}$ in $N^-(v_3, X^{-\sigma} \setminus U^{-\sigma})$, but $v_2 v_4 v_3$ is a forbidden path, a contradiction.

So there exists $v \in X^{-\sigma} \setminus U^{-\sigma}$ such that $d^{-\sigma}(v, X^{-\sigma} \setminus U^{-\sigma}) = 0$. But then the sets $N^{-\sigma}(v)$, $X^{-\sigma} \setminus U^{-\sigma}$, W^σ, and U^σ are pairwise disjoint since there are no forbidden paths. This with (4.9) implies that

$$|U^\sigma| \leq n - (d^{-\sigma}(v) + (|X^{-\sigma}| - |U^{-\sigma}|) + |W^\sigma|) \leq n - (3\Gamma - \alpha n - (\frac{1}{3} + \alpha)|U^\sigma|),$$

so $(\frac{2}{3} - \alpha)|U^\sigma| < n - (3\Gamma - \alpha n) \leq 4\alpha n$. Therefore $|U^\sigma| < 7\alpha n$. □

Note that for any $v \in V(G)$ that is not in a directed C_4, Claim 4.3.9, with $N^-(v)$ and $N^+(v)$ playing the roles of X^- and X^+ respectively, implies that both the out-neighborhood and the in-neighborhood of v contain large independent sets.

Claim 4.3.10. *Suppose xyz is a directed triangle and x and y are not in a directed C_4, then*

$$|N^-(x) \cap N^+(y)| \geq (\frac{1}{3} - 18\alpha)n. \tag{4.11}$$

Proof. First note that for every vertex v that is not in a directed C_4, Claim 4.3.9, with $N^-(v)$ and $N^+(v)$ playing the roles of X^- and X^+ respectively, implies that there are independent subsets of $N^-(v)$ and $N^+(v)$ of order at least

$$\delta^0(G) - 7\alpha n \geq (\frac{1}{3} - 8\alpha)n.$$

Therefore there exists $U \subseteq N^-(x)$ and $W \subseteq N^+(y)$ such that

$$|U|, |W| \geq \delta^0(G) - 7\alpha n \geq (\frac{1}{3} - 8\alpha)n \tag{4.12}$$

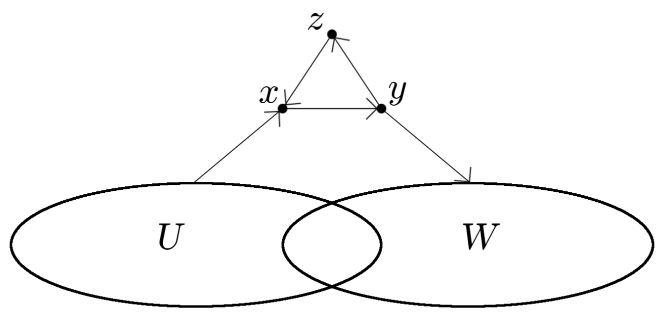


Figure 4.2: Relationship Between x, y, z, U, and W

and U and W are independent sets. Suppose for contradiction that

$$|U \cap W| \leq |N^-(x) \cap N^+(y)| < (\frac{1}{3} - 18\alpha)n. \tag{4.13}$$

With (4.12) and (4.13), we have that

$$|U \cup W| = |U| + |W| - |U \cap W| \geq 2 \cdot (\frac{1}{3} - 8\alpha)n - |U \cap W| > n - 2\delta^0(G). \tag{4.14}$$

Then $U \cap W = \emptyset$ since for a vertex $v \in U \cap W$, $N^-(v)$, $N^+(v)$, and $U \cup W$ are pairwise disjoint because U and W are independent sets. But then

$$n \geq |N^-(v)| + |N^+(v)| + |U \cup W| > \delta^0(G) + \delta^0(G) + n - 2\delta^0(G) = n.$$

Thus we have that

$$|U \cup W| \geq (\frac{2}{3} - 16\alpha)n. \tag{4.15}$$

Since there are no directed C_4 that contain the edge xy, there are no edges from W to U. This with (4.15) and the fact U and W are independent sets implies that for every $u \in U$ and $w \in W$,

$$|N^-(u) \cap N^+(w)| \geq 2 \cdot \delta^0(G) - (n - |U \cup W|) \geq (\frac{1}{3} - 18\alpha)n.$$

Therefore if $u \in U$ and $w \in N^+(x, W)$, there exists $v \in N^-(u) \cap N^+(w)$ so that $xwvux$ is a directed C_4, a contradiction. Thus $d^+(x, W) = 0$, and, by a similar argument, $d^-(y, U) = 0$. But then $N^+(x) \cup N^-(y) \subseteq V(G) \setminus (U \cup W)$, so with (4.15) we have

$$|N^+(x) \cap N^-(y)| \geq 2 \cdot \delta^0(G) - (n - |U \cup W|) \geq (\frac{1}{3} - 18\alpha)n.$$

But for every $v \in N^+(x) \cap N^-(y)$, $yzxvy$ is a directed C_4, a contradiction. $\square$

Case 1: $\ell \neq 5$**.**

We can assume that $\alpha < 10^{-5} \cdot \lambda$, that there exists a triangle $v_1v_2v_3$ in G, and that v_i is not in a directed C_4 for $i \in [3]$ by Corollary 4.3.1 and Fact 4.3.2.

Let $i \in [3]$ and let $U_i := N^+(v_{i-1}) \cap N^-(v_{i+1})$, From Claim 4.3.10, we have that

$$|U_i| \geq (\frac{1}{3} - 0.001\lambda)n. \tag{4.16}$$

Because the sets $N^+(v_i)$ and $N^-(v_i)$ are disjoint for every $i \in [3]$, the sets U_1, U_2, U_3 are pairwise disjoint. Then (4.16) implies that

$$|V(G) \setminus (U_1 \cup U_2 \cup U_3)| \leq 0.003\lambda n.$$

To prove that G is λ-extremal, it suffices to show that for every $i \in [3]$ and $u \in U_i$, we have $d^+(u, U_{i+1}) \geq (\frac{1}{3} - 0.01\lambda)n$ by Proposition 4.3.3(1).

Suppose $d^-(v_{i-1}) > (\frac{1}{3} + 0.001\lambda)n$. Then Claim 4.3.9 and the fact that v_{i-1} is not in a directed C_4 imply that there exists an independent set $Y \subseteq N^-(v_{i-1})$ with $|Y| \geq n - 2\delta^0(G)$, which is a contradiction. Therefore $d^-(v_{i-1}) \leq (\frac{1}{3} + 0.001\lambda)n$, and, because $U_{i+1} \subseteq N^-(v_{i+2}) = N^-(v_{i-1})$, this and (4.16) imply that

$$|N^-(v_{i-1}) \setminus U_{i+1}| = d^-(v_{i-1}) - |U_{i+1}| \leq 0.002\lambda n. \tag{4.17}$$

But u is in the triangle $v_{i-1}uv_{i+1}$ for all $u \in U_{i+1}$. Therefore u is not in a C_4 and Claim 4.3.10 implies that $d^+(u, N^-(v_{i-1})) \geq (\frac{1}{3} - 0.001\lambda)n$. With (4.17), we have that

$$d^+(u, U_{i+1}) = d^+(u, N^-(v_{i-1})) - d^+(u, N^-(v_{i-1}) \setminus U_{i+1}) \geq (\frac{1}{3} - 0.01\lambda)n,$$

which is what we wanted to show. Therefore G is λ-extremal completing this case.

Case 2: $\ell = 5$**.**

We can assume that there exists a transitive triangle since Proposition 4.3.3(2) and the minimum semidegree condition imply G is λ-extremal otherwise. Therefore there exists

$x \in V(G)$ with $uw \in E(N^+(x))$. Let $Z := N^-(x) \cap N^+(w)$. Because there are no directed C_5 containing the path xuw, Z is an independent set. We will show that

$$|Z| \geq (\frac{1}{3} - 21\alpha)n. \tag{4.18}$$

Suppose that $|Z| < (\frac{1}{3} - 21\alpha)n$, then by Claim 4.3.9 with $N^-(x)$ and $N^+(w)$ playing the roles of X^- and X^+ respectively, there exists $Y^- \subseteq N^-(x) \setminus Z$ and $Y^+ \subseteq N^+(w) \setminus Z$ such that Y^- and Y^+ are independent sets that have order at least $\delta^0(G) - |Z| - 7\alpha n$. For every $y^- \in Y^-$ and $y^+ \in Y^+$, we have that $N^-(y^-)$ and $N^+(y^+)$ are disjoint, since there are no directed C_5 containing the path y^-xwy^+. We also have that $N^-(y^-)$ does not intersect Y^-, because Y^- is an independent set, and does not intersect $Z \cup Y^+$, because $Z \cup Y^+ \subseteq N^+(w)$ and there are no directed C_5 containing the path y^-xuw. By a similar argument, $N^+(y^+)$ does not intersect $Y^+ \cup Z \cup Y^-$. Therefore the sets $N^-(y^-)$, $N^+(y^+)$, Y^-, Y^+, and Z are pairwise disjoint, implying that

$$n \geq |Z| + 2\delta^0(G) + 2(\delta^0(G) - |Z| - 7\alpha n) = 4\delta^0(G) - |Z| - 14\alpha n,$$

and contradicting the assumption that $|Z| < (\frac{1}{3} - 21\alpha)n$.

Let $z \in Z$. Note that,

$$d^+(a, Z) = 0 \text{ for every } a \in N^+(z), \tag{4.19}$$

because there are no directed C_5 containing the path $xwza$ and $Z \subseteq N^-(x)$. To complete the proof, we only need to show that there exists an independent set $B \subseteq N^+(z)$ such that $|B| \geq (\frac{1}{3} - 24\alpha)n$. This is because $|E(B, Z)| = 0$, so by Proposition 4.3.3(3) with Z, B, 100α playing the roles of V_1, V_2, and α, respectively, G is λ-extremal. Therefore we may assume that $N^+(z)$ is not independent and by a similar argument,

$$\text{if } B \subseteq N^-(z) \text{ and } |B| \geq (\frac{1}{3} - 24\alpha)n, \text{ then } B \text{ is not independent.} \tag{4.20}$$

Suppose there exists $ab \in E(G[N^+(z)])$ such that $d^+(b, N^+(z)) = 0$. Then with (4.19), we have that b has no out-neighbors in $Z \cup N^+(z)$, so, with (4.18) and the fact that Z is independent, if we define $B := N^+(b, N^-(z))$ we have that

$$|B| \geq d^+(b) + d^-(z) - (n - |Z| - d^+(z)) \geq (3\delta^0(G) - n) + |Z| \geq (\frac{1}{3} - 24\alpha)n.$$

By (4.20), there must exist an edge $cd \in E(G[B])$, but then we have a directed C_5 $abcdza$, a contradiction. If there is no such edge $ab \in E(G[N^+(z)])$, then the set

$$C := \{c \in N^+(z) : \text{ there exists a path } abc \text{ in } G[N^+(z)]\}$$

is not empty. By the fact that there is no directed C_5 in $G[N^+(z)]$, Corollary 4.3.7, and (4.18), we have that there exists $c \in C$ such that

$$d^+(c, N^+(z)) = d^+(c, C) \leq (\frac{1}{3} + \alpha)|N^+(z)| < |Z| - 3\alpha n.$$

This with (4.19) and the fact that Z is independent imply that

$$\begin{aligned} d^+(c, N^-(z)) &\geq d^+(c, V(G) \setminus N^+(z)) + d^-(z) - (n - |Z| - d^+(z)) \\ &= d^+(c) - d^+(c, N^+(z)) + d^-(z) - (n - |Z| - d^+(z)) \\ &\geq (3\delta^0(G) - n) + (|Z| - d^+(c, N^+(z)) > 0. \end{aligned}$$

But by the definition of C, there exists a path abc in $G[N^+(z)]$ for every $d \in N^+(c, N^-(z))$. Thus $zabcdz$ is a directed C_5, a contradiction. $\square$

With the proof above completed, we are now ready to prove the main lemma of this section, restated below.

Lemma 4.1.5 (Non-extremal lemma)**.** *Suppose $\lambda > 0$. For every $\ell \in \mathbb{N} \setminus \{1, 2, 3, 5\}$ there exists $\alpha > 0$ and n_0 such that every oriented graph G on $n \geq n_0$ vertices that does not contain a closed ℓ-walk and such that $\delta^+(G) \geq (\frac{1}{3} - \alpha)n$ is λ-extremal. Furthermore, when $\ell = 5$ and $\lambda > 0$, there exists n_0 such that every oriented graph G on $n \geq n_0$ vertices that does not contain a closed ℓ-walk and such that $\delta^+(G) \geq \frac{n+1}{3}$ is λ-extremal.*

Proof. Assume that G does not have a closed ℓ-walk, and let λ', β, α and n_0 be such that

$$0 < \frac{1}{n_0} \ll \alpha \ll \beta \ll \lambda' \ll \lambda.$$

If $\ell \neq 5$, then Lemma 4.3.4 with 0.9β playing the role of β implies that there exists $G' \subseteq G$ such that $n' := |G'| \geq (1 - 0.9\beta)n \geq (1 - \beta)n$ and

$$\delta^0(G') \geq (\frac{1}{3} - \alpha - 0.9\beta)|G'| \geq (\frac{1}{3} - \beta)n'.$$

If $\ell = 5$, then we have that $\delta^+(G) \geq \frac{(n+1)}{3}$, and Lemma 4.3.4 implies that there exist $G' \subseteq G$ such that $n' := |G'| \geq (1 - \beta)n$ and

$$\delta^0(G') \geq (\frac{(n+1)}{3n} - \beta)n' \geq (\frac{1}{3} - \beta)n'.$$

Lemma 4.3.8, with β and λ' playing the roles of α and λ, respectively, implies that G' is λ'-extremal. By Proposition 4.3.3(1) with $\min\{\lambda - \lambda', \beta\}$ playing the role of α implies that G is λ-extremal. □

4.4 Extremal Case

In this section we prove the following lemma from Section 4.1.

Lemma 4.1.7 (Extremal lemma)**.** *For every $\ell \geq 4$ that is not divisible by 3, there exists n_0 and $\lambda > 0$ such that for every $n \geq n_0$ the following holds. Suppose that G is a graph on n vertices and c is an edge-coloring of G such that (G, c) is λ-extremal. If $\ell \neq 5$ and* (1.1) *holds, i.e.,*

$$\delta^c(G) \geq \begin{cases} \frac{n+5}{3} & \textit{if } \ell = 1 \pmod 3 \\ \frac{n+7}{3} & \textit{if } \ell = 2 \pmod 3, \end{cases}$$

then G contains a rainbow C_ℓ. Furthermore, if $\ell \equiv 1 \pmod 3$ and $\delta^c(G) \geq \frac{n+4}{3}$, then G contains a properly colored C_ℓ. Finally, if $\ell \equiv 2 \pmod 3$, $\delta^c(G) \geq \frac{n+4}{3}$, and there exists an oriented graph G' such that (G, c) is the simple edge-colored graph determined by G', then G contains a properly colored C_ℓ.

Proof. To make the proof easier to digest, it is broken into a number of claims. For contradiction, assume that (G, c) an edge-minimal counterexample.

Claim 4.4.1. *At least one of the following conditions hold:*

(I) $\delta^c(G) = \frac{(n+5)}{n}$, $\ell \equiv 1 \pmod 3$ *and G does not have a rainbow C_ℓ;*

(II) $\delta^c(G) = \frac{(n+4)}{n}$, $\ell \equiv 1 \pmod 3$ *and G does not have a properly colored C_ℓ; or*

(III) $\delta^c(G) = \frac{(n+7)}{n}$, $\ell \equiv 2 \pmod 3$ *and G does not have a rainbow C_ℓ;*

(IV) $\delta^c(G) = \frac{(n+4)}{n}$, $\ell \equiv 2 \pmod 3$, *G does not have a properly colored C_ℓ, and there exists an oriented graph G' such that (G, c) is the simple edge-colored graph determined by G'.*

Furthermore, the following condition always holds

(V) $\delta^c(G - e) < \delta^c(G)$ *for every $e \in E(G)$.*

Note that Claim 4.4.1(V) implies that G does not contain a monochromatic path on four vertices, a fact that we use multiple times.

Let n_0, λ, β and γ be such that

$$0 < \frac{1}{n_0} \ll \lambda \ll \beta \ll \gamma \ll \frac{1}{\ell}. \tag{4.21}$$

Let $m := \left\lfloor \frac{n}{3} \right\rfloor$, and note that Claim 4.4.1 implies that the following inequality holds since $\delta^c(G)$ is an integer:

$$\delta^c(G) \geq \left\lceil \frac{(n+4)}{3} \right\rceil = m + 2. \tag{4.22}$$

Let $\{Y_1, Y_2, Y_3\}$ be an λ-extremal partition of (G, c). For every $i \in [3]$, call $x \in Y_i$ an *i-good vertex* if

$$d^c(x, Y_{i+1}) \geq |Y_{i+1}| - \lambda^{\frac{1}{2}} n \qquad \text{and} \qquad d(x, Y_{i-1}) \geq |Y_{i-1}| - \lambda^{\frac{1}{2}} n,$$

and let $\widetilde{X}_i$ be the set of i-good vertices. Let $\widetilde{X} := \widetilde{X}_1 \cup \widetilde{X}_2 \cup \widetilde{X}_3$ and let $Q := V(G) \setminus \widetilde{X}$. Vertices in $\widetilde{X}$ are called *good vertices*. Partition Q into $\{Q_1, Q_2, Q_3\}$ (with some parts potentially empty), so that, for every $i \in [3]$ and every $x \in Q_i$,

$$d^c(x, \widetilde{X}_{i+1}) = \max\{d^c(x, \widetilde{X}_1), d^c(x, \widetilde{X}_2), d^c(x, \widetilde{X}_3)\},$$

with ties broken arbitrarily. Finally, for each $i \in [3]$, let $X_i := \widetilde{X}_i \cup Q_i$, let

$$X_i'' := \{x \in Q_i : d^c(x, X_i) \geq 3\},$$

let $X_i' := Q_i \setminus X_i''$, let $\widehat{X}_i := \widetilde{X}_i \cup X_i'$, and let $p_i := m - |\widehat{X}_i|$.

Claim 4.4.2. *We have that $|Q| \leq 0.5\beta^2 n$. In particular, this implies that, for every $i \in [3]$, every $x \in \widetilde{X}_i$, and every $z \in X_i$, we have that*

(A) $(\frac{1}{3} - \beta^2)n \leq |X_i| \leq (\frac{1}{3} + \beta^2)n$,

(B) $|\widetilde{X}_i| \geq |X_i| - \beta^2 n \geq (\frac{1}{3} - 2\beta^2)n$,

(C) $d^c(z, X_{i+1}) \geq (\frac{1}{9} - \beta^2)n$,

(D) $d^c(x, X_{i+1}) \geq |X_{i+1}| - \beta^2 n$,

(E) $d(x, X_{i-1}) \geq |X_{i-1}| - \beta^2 n$,

(F) $|p_1| + |p_2| + |p_3| \leq \beta^2 n$, *and*

(G) $\sum_{i \in [3]}(|X_i'| + |X_i''|) \leq \beta^2 n$.

Proof. This claim follows from the definition of an λ-extremal partition, the preceding definitions, (4.21) and (4.22). The details are omitted. □

For $1 \leq k < k'$, let $P = v_1 \dots v_k$ be a path and let $Q = v_1 \dots v_k v_{k+1} \dots v_{k'}$ be a path that begins with the same vertices as P and is such that $F := E(Q) \setminus E(P)$ is rainbow and the colors used on F are disjoint from the colors used on the edges of P. In particular, if P is rainbow, then Q is rainbow and if P is properly colored than Q is properly colored. We say that Q *is an extension of P in the forward direction* or that Q *is constructed from*

P *in the forward direction* if, for every $k \leq j \leq k' - 1$, we have that $v_j \in X_i$ implies that $v_{j+1} \in X_{i+1}$. Similarly, we say that Q *is an extension of* P *in the backward direction* or that Q *is constructed from* P *in the backward direction* if $v_j \in X_i$ implies that $v_{j+1} \in X_{i-1}$ for every $k \leq j \leq k' - 1$. By Claim 4.4.2(C), when $k < k' \leq \ell$, we can always construct a k'-vertex path Q from P in the forward direction. With Claim 4.4.2(B), we can also assume that all of the vertices in $V(Q) \setminus V(P)$ are good vertices, or that the colors of edges in $E(Q) \setminus E(P)$ avoid a set C of at most γn colors.

For every $i \in [3]$ and $x \in \widetilde{X}_i$, let c_x be a color that appears most often on the edge set $E(x, X_{i-1})$ where ties are broken arbitrarily. We say that c_x is the *primary color* of x. Let $\{S, T\}$ be a partition (with S potentially empty) of the edge set $\bigcup_{i \in [3]} E(X_{i-1}, \widetilde{X}_i)$ where

$$T := \bigcup_{i \in [3]} \{yx \in E(X_{i-1}, \widetilde{X}_i) : y \in X_{i-1}, x \in \widetilde{X}_i, \text{ and } c(yx) = c_x\}.$$

The edges in T are *typical edges* and the edges in S are *special edges*. Let G_T and G_S be the spanning subgraphs of G with edge sets T and S respectively. For all $U \subseteq V(G)$ and $v \in V(G)$, let $N^t(v, U) := N_{G_T}(v, U)$ and let $N^s(v, U) := N_{G_S}(v, U)$ be the set of *typical neighbors* and *special neighbors* of v in U respectively. Let $d^t(v, U) := |N^t(v, U)|$ and $d^s(v, U) := |N^s(v, U)|$. For every $W \subseteq V(G)$, let $e^s(W, U) := \sum_{v \in W} d^s(v, U)$ and $e^t(W, U) := \sum_{v \in W} d^t(v, U)$. Furthermore, let $d^{ct}(v, U) := d^c(v, N^t(v, U))$ and let $d^{cs}(v, U) := d^c(v, N^s(v, U))$ be the number of colors used on edges from v to its typical neighbors in U and special neighbors in U respectively. Note that for every $x \in \widetilde{X}_i$, by the definition of T, $d^{ct}(x, X_{i-1}) = 1$.

Claim 4.4.3. *For every $i \in [3]$ and $x \in \widetilde{X}_i$, we have that $d^t(x, X_{i-1}) \geq |X_{i-1}| - \beta \frac{n}{2}$, i.e., for all but at most $\beta \frac{n}{2}$ vertices $y \in X_{i-1}$, we have that $xy \in E(G)$ and $c(xy) = c_x$.*

Proof. Suppose for contradiction that $d^t(x, X_{i-1}) < |X_{i-1}| - \beta \frac{n}{2}$ for some $x \in \widetilde{X}_i$. Note that $d^t(x, X_{i-1}) + d^s(x, X_{i-1}) = d(x, X_{i-1})$, so by Claim 4.4.2(B) and (E),

$$d^s(x, \widetilde{X}_{i-1}) \geq d^s(x, X_{i-1}) - 0.1\beta n = d(x, X_{i-1}) - d^t(x, X_{i-1}) - 0.1\beta n > 0.3\beta n. \quad (4.23)$$

The argument is broken into two cases depending on the number of colors on edges from x to $\widetilde{X}_{i-1}$. In both cases the outline of the argument is the same, we construct a rainbow cycle C_ℓ on vertices $xv_2 \dots v_\ell$ by first constructing a rainbow path P from x in the backward direction. If $\ell = 2 \pmod 3$, then $v_2 \in N^s(x, \widetilde{X}_{i-1})$ and P is the path xv_2. If $\ell = 1 \pmod 3$, then $v_2 \in \widetilde{X}_{i-1}$, $v_3 \in \widetilde{X}_{i-2}$, and $P = xv_2v_3$ is a rainbow path. We extend P to a rainbow $(\ell-1)$-path Q in the forward direction so that its final vertex $v_{\ell-1} \in \widetilde{X}_{i-2}$. The construction of P and Q will be such that we can find a vertex in $v_\ell \in \widetilde{X}_i$ to complete a rainbow C_ℓ.

Case 1: $d^c(x, \widetilde{X}_{i-1}) > 0.01\beta n$

Claim 4.4.2(D) implies that $d^c(x, X_{i+1}) \geq (\frac{1}{3} - \beta^2)n$, so

$$d^c(x) \geq (\frac{1}{3} - \beta^2)n + 0.01\beta n \geq \frac{n}{3} + 4.$$

By the edge-minimality of (G, c) (Claim 4.4.1(V)), every 3-vertex path that has x as an endpoint is rainbow. If we suppose zyx is a path where $c(zy) = c(yx)$, then in $G - yx$, the color degree of y is $d^c_G(y)$ and the color degree of x is still at least $m + 3$, a contradiction.

By (4.23), there exists $v_2 \in N^s(x, \widetilde{X}_{i-1})$. If $\ell = 2 \pmod 3$, then let P be the 2-vertex path xv_2. If $\ell = 1 \pmod 3$, then, by Claim 4.4.2(B) and (E), there exist $v_3 \in N(v_2, \widetilde{X}_{i-2})$ and let P be a rainbow 3-vertex path xv_2v_3. We can extend P in the forward direction to a rainbow path $Q = xv_2 \dots v_{\ell-1}$ such that $v_{\ell-1} \in \widetilde{X}_{i-2}$. Because $d^c(x, \widetilde{X}_{i-1}) > 0.01\beta n$, there exists $Y \subseteq N(x, X_{i-1})$ such that for every $y \in Y$, we have that $c(xy) \notin c(E(Q))$ and

$$|Y| \geq 0.01\beta n - (\ell - 1) \geq 0.005\beta n. \quad (4.24)$$

Because $v_{\ell-1} \in \widetilde{X}_{i-2}$ and $Y \subseteq X_{i-1}$, Claim 4.4.2(D) and (4.24) imply that

$$d^c(v_{\ell-1}, Y) \geq |Y| - \beta^2 n \geq 0.001\beta n > \ell,$$

so there exists $v_\ell \in N(v_{\ell-1}, Y)$ such that $xv_2 \ldots v_{\ell-1}v_\ell$ is a rainbow path. By the edge-minimality of (G, c) (Claim 4.4.1(V)) and the selection of Y, the path $v_{\ell-1}v_\ell x$ is rainbow path that avoids colors in $c(E(Q))$. Therefore $xv_2 \ldots v_\ell x$ is a rainbow C_ℓ.

Case 2: $d^c(x, \widetilde{X}_{i-1}) \leq 0.01\beta n$

First assume that $\ell \equiv 2 \pmod 3$. By (4.23), there exists $v_2 \in N^s(x, \widetilde{X}_{i-1})$. Let P be the 2-vertex path xv_2. Now assume that $\ell \equiv 1 \pmod 3$. By (4.23) and the case, there exist at least two distinct vertices $y_1, y_2 \in N^s(x, \widetilde{X}_{i-1})$ such that both of the edges xy_1 and xy_2 are assigned the same color ϕ by c. Note that $\phi \neq c_x$ because xy_1 and xy_2 are special edges. Since $y_1, y_2 \in \widetilde{X}_{i-1}$, Claim 4.4.2(B) and (E) implies that there exist two distinct vertices z_1 and z_2 in $N(y_1, \widetilde{X}_{i-2}) \cap N(y_2, \widetilde{X}_{i-2})$. For every $j, k \in [2]$, by the edge-minimality of (G, c) (Claim 4.4.1(V)) the path $z_j y_k x y_{3-k}$ is not monochromatic, so $c(z_j y_k) \neq \phi$. Furthermore, again because there does not exists a monochromatic path on 4 vertices, there exist $j, k \in [2]$ such that that $c(z_j y_k) \neq c_x$. If we let $v_2 := y_k$ and $v_3 := z_j$, we then have that $P := xv_2v_3$ is a monochromatic path that avoids the color c_x.

Let

$$Y := \{y \in N(x, \widetilde{X}_{i-1}) : c(xy) \notin c(E(P))\}.$$

Let $C := c(E(x, Y))$ be the set of colors used on the edges from x to Y. Because $c_x \notin c(E(P))$, we have that $c_x \in C$. Since $|E(P)| \leq 2$, with Claim 4.4.2(B) and (E), we have that

$$|Y| \geq \frac{d(x, \widetilde{X}_{i-1})}{3} > 0.1n. \tag{4.25}$$

By the case,

$$|C| = d^c(x, Y) \leq d^c(x, \widetilde{X}_{i-1}) \leq 0.01\beta n, \tag{4.26}$$

so, in the forward direction, we can extend P to a rainbow path $Q = xv_2 \ldots v_{\ell-1}$ such that $v_{\ell-1} \in \widetilde{X}_{i-2}$ that avoids the colors in C. Because, $v_{\ell-1}$ is $(i-2)$-good and $Y \subseteq \widetilde{X}_{i-1}$, Claim 4.4.2(D), (4.25) and (4.26) imply that $d^c(v_{\ell-1}, Y) \geq |Y| - \beta^2 n \geq |C| + \ell$. Therefore there exists $v_\ell \in N(v_{\ell-1}, Y)$ such that $xv_2v_3\ldots v_\ell$ is a rainbow path that avoids the colors in C. By the definitions of Y and C, we have that $xv_\ell \in E(G)$ and $c(xv_\ell) \in C$, so $xv_2v_3\ldots v_\ell$ is a rainbow C_ℓ. $\square$

Claim 4.4.4. *For every $i \in [3]$ and distinct vertices $x, x' \in \widetilde{X}_i$, we have that $c_x \neq c_{x'}$. In particular for every $y \in X_{i-1}$, we have that $d^{\mathrm{t}}(y, \widetilde{X}_i) = d^{\mathrm{ct}}(y, \widetilde{X}_i)$ as the typical edges from y to $\widetilde{X}_i$ are each given a distinct color.*

Proof. By Claim 4.4.3, there exist two distinct vertices $y, y' \in N^t(x, X_{i-1}) \cap N^t(x', X_{i-1})$. The edge-minimality of (G, c) (Claim 4.4.1(V)) implies that $c_x \neq c_{x'}$. $\square$

Call a rainbow (respectively, properly colored) C_k on vertices $v_1 \ldots v_k$ a *strong (respectively, properly colored) C_k* if for some $i \in [3]$, $v_1 \in \widetilde{X}_i$ and $v_k \in N^t(v_1, X_{i-1})$.

Claim 4.4.5. *Suppose that $1 \leq k < k' \leq \ell$, $x \in \widetilde{X}_i$, and $y \in X_j$. If P is a rainbow x, y-path on k vertices that avoids the color c_x and $k' - k \equiv (i-1) - j \pmod 3$, then there exists a strong rainbow $C_{k'}$. Similarly, if P is a properly colored x, y-path on k vertices such that c_x is not used on the edge in P that is incident to x and $k' - k \equiv (i-1) - j \pmod 3$, then there exists a strong properly colored $C_{k'}$.*

In particular, if there exists a strong rainbow (respectively, properly colored) C_k, then there exists a strong (respectively, properly colored) rainbow $C_{k'}$ whenever $k' - k$ is divisible by 3.

Proof. If P is a properly colored x, y-path such that c_x is not used on the edge incident to x, then we can extend P in the forward direction to a properly colored x, z-path Q on $k' - 1$ vertices without using the color c_x on the new edges. If P is a rainbow x, y-path that avoids the color c_x, then we can extend P in the forward direction to a rainbow x, z-path Q on $k' - 1$ vertices that avoids the color c_x. Let $x \in X_i$, and as

$$j + (k' - 1) - k \equiv j - 1 + (k' - k) \equiv i - 2 \pmod 3,$$

we get that $z \in X_{i-2}$. By Claims 4.4.2(C) and 4.4.3,

$$|d^c(z, N^t(x, X_{i-1}))| \geq (\frac{1}{9} - \beta^2)n - \frac{\beta n}{2} > \ell,$$

so there exists $w \in N(z, X_{i-1}) \cap N^t(x, X_{i-1})$ so that $c(zw) \notin c(E(Q)) \cup \{c_x\}$. Then $xQzwx$ is the desired strong properly colored or rainbow $C_{k'}$.

To see the final implications, suppose that $v_1 \dots v_k v_1$ is a strong rainbow or properly colored C_k with $v_1 \in \widetilde{X}_i$ and $v_k \in N^t(v_1, X_{i-1})$. Then apply the first part of the lemma with k, k', i, $i-1$, v_1, v_k and the path $v_1 \dots v_k$ playing the roles of k, k', i, j, x, y and P, respectively. $\square$

Claim 4.4.6. *We have that* $\ell \equiv 1 \pmod 3$.

Proof. Assume for contradiction that $\ell \equiv 2 \pmod 3$, so either conditions Claim 4.4.1(III) or (IV) holds. We can assume that G has no rainbow C_ℓ. Let $\Phi := \delta^c(G) - (m+2)$. Then

$$\delta^c(G) = m + 2 + \Phi, \tag{4.27}$$

and $\Phi \geq 0$ when $\delta^c(G) \geq \frac{(n+4)}{3}$, and $\Phi \geq 1$ when $\delta^c(G) \geq \frac{(n+7)}{3}$. We can assume that X_1, X_2 and X_3 are labeled so that $|X_3| \leq m$, and, subject to this, $|X_2| + |X_3|$ is as small as possible. Therefore

$$|X_3| \leq m \qquad \text{and} \qquad |X_2| + |X_3| \leq 2m + 1. \tag{4.28}$$

as otherwise we get that $|X_1| \leq n - |X_2| + |X_3| \leq m$ and the set X_1 would have been fixed as X_3 instead as $|X_3| + |X_1| < |X_2| + |X_3|$.

Let $i \in [3]$, then the following claims hold.

(a) There does not exists a 2-vertex rainbow x, y-path that avoids c_x with $x \in \widetilde{X}_i$ and $y \in X_{i-1}$, i.e., for every $x \in \widetilde{X}_i$ we have that $d^s(x, X_{i-1}) = 0$.

(b) If xzy is a 3-vertex rainbow path with $x \in \widetilde{X}_i$ and $y \in X_i$, then $c_x \in \{c(xz), c(zy)\}$. Furthermore, if G does not contain a properly colored C_ℓ, then $c(xz) = c_x$.

(c) If $\ell \neq 5$ (so $\ell \geq 8$), then there does not exist a pair of disjoint edges xu and zy in $G[X_i]$ such that $x, z \in \widetilde{X}_i$ and c_x, c_z, $c(xu)$, and $c(zy)$ are pairwise disjoint.

The first two claims follow directly from Claim 4.4.5. The third claim also follows from Claim 4.4.5. To see this, note that if there exists such a pair of disjoint edges, then using Claim 4.4.2(C) and Claim 4.4.3, we can find a rainbow x, y-path xuv_3v_4zy on 6-vertices that avoids c_x by picking $v_3 \in N(u, X_{i+1})$ and $v_4 \in N(v_3, X_{i-1}) \cap N^t(z, X_{i-1})$.

For every $i \in [3]$ and $x \in \widetilde{X}_i$, using (a) and (4.27), we can compute that the number of colors other than c_x that are used on edges incident to x in $E(G[X_i])$ is at least

$$\delta^c(G) - d^c(x, X_{i-1}) - d^c(x, X_{i+1}) \geq (m + 2 + \Phi) - 1 - |X_{i+1}| \tag{4.29}$$

We will now deduce a contradiction.

Case 1: Condition Claim 4.4.1(III) holds. Then G has no rainbow C_ℓ and $\Phi = 1$. By (4.28) and (4.29), we have that

$$\forall x \in \widetilde{X}_2, \exists x', x'' \in N(x, X_2) \text{ such that } c(xx'), c(xx'') \text{ and } c_x \text{ are pairwise distinct.} \tag{4.30}$$

Now fix $x \in \widetilde{X}_2$. By (4.30), there exist $u_1, u_2 \in N(x, X_2)$ such that the colors $c(xu_1)$, $c(xu_2)$ and c_x are pairwise distinct. By Claim 4.4.2(B) and Claim 4.4.4, there exists $z \in \widetilde{X}_2 \setminus \{x, u_1, u_2\}$ such that $c_z \notin \{c(xu_1), c(xu_2), c_x\}$. By (4.30) again, there exist $y_1, y_2 \in N(z, X_2)$ such that the colors $c(zy_1)$, $c(zy_2)$ and c_z are pairwise distinct. If $\{x, u_1, u_2\}$ and $\{z, y_1, y_2\}$ are disjoint sets, then we can pick $i \in [2]$ such that $c(zy_i) \neq c_x$ and then pick $j \in [2]$ so that $c(xu_j) \neq c(zy_i)$. The pair of disjoint edges zy_i and xu_j contradicts (c). If there exists $i \in [2]$ such that $y_i = x$, then we can pick $j \in [2]$ so that $c(xu_j) \neq c(zx)$. Recall that z was selected so that $c_z \notin \{c(xu_1), c(xu_2), c_x\}$, so we have that zxu_j is a rainbow path that avoids c_z, which contradicts (b) (with z, x and u_j playing the roles of x, z and y, respectively). Because we selected z so that $z \notin \{x, u_1, u_2\}$, the final case is when there exists $i \in [2]$ and $j \in [2]$ such that $u_i = y_j$. Without loss of generality assume that $i = j = 1$. If $c(xu_1) \neq c(zy_1)$, then $xu_1z = xy_1z$ is a rainbow path on 3-vertices that does not use c_z, which contradictions (b). If $c(xu_1) = c(zy_1)$, then the disjoint pair of edges xu_2 and zy_1 contradicts (c), because, by the selection of z, $c(xu_2) \neq c_z$, and we also have that $c(xu_2) \neq c(xu_1) = c(zy_1)$ and $c(zy_1) = c(xu_1) \neq c_x$.

Case 2: Condition Claim 4.4.1(IV) holds. In this case, G has no properly colored C_ℓ and $\Phi = 0$. Let $y \in \widetilde{X}_1$. Suppose that there exists $y' \in N(y, X_1))$ such that $c(yy') \neq c_y$. Then, by Claim 4.4.2(B) and (C), there exists $x \in N(y', \widetilde{X}_2)$ such that $c(y'x) \neq c(yy')$. By (a), we can assume that $c(y'x) = c_x$. Since $|X_3| \leq m$ by (4.28), (4.29) implies that there exists $x' \in N(x, X_2)$ such that $c(xx') \neq c_x$. Note that $yy'xx'$ is a properly colored path and $c(yy') \neq c_y$, so Claim 4.4.5 implies that there exists a properly colored C_ℓ, a contradiction. Therefore, with (a), we have that the only color used on the edges in $E(y, X_3 \cup X_1)$ is c_y. Define

$$A := \{x' \in N(y, X_2) : c(yx') \neq c_y\}, \tag{4.31}$$

then $|A| \geq d^c(y) - 1 \geq m + 1$. Let $x \in \widetilde{X}_2$, and define

$$B := \{x' \in N(x, X_2) : c(xx') \neq c_x\}. \tag{4.32}$$

By (4.29), $|B| \geq m + 1 - |X_3|$. Then by (4.28),

$$|A \cap B| \geq (m + 1) + (m + 1 - |X_3|) - |X_2| = 2m + 2 - (|X_2| + |X_3|) > 0,$$

so there exists $x' \in A \cap B$. If there exists $y' \in N(x', X_1)$ such that $c(y'x') \neq c(yx')$, then $yx'y'$ is rainbow and $c(yx') \neq c_y$, which violates (b). Therefore every edge from x' to X_1 is colored $c(yx')$, and, using (4.28), the number of neighbors of x' in X_2 that are not colored $c(yx')$ is at least

$$d^c(x') - 1 - |X_3| \geq m + 1 - |X_3| \geq 1. \tag{4.33}$$

Since Claim 4.4.1(IV) holds, we can assume that there exists G' such that (G, c) is the simple edge-colored graph associated with G'. (Because we have now introduced the directed graph G', we will use set notation for edges in G for the remainder of this proof to avoid any possible confusion). Therefore for every edge $\{u, v\}$ in $E(G)$, we have that $c(\{u, v\}) \in \{u, v\}$. Therefore if $\{u, v\}$ and $\{u', v\}$ are two distinct edges incident to a vertex $v \in V(G)$ and $c(\{u, v\}) = c(\{u', v\})$, then $c(\{u, v\}) = c(\{u', v\}) = v$. In particular, $c_x = x$ and $c_y = y$. Because $c(\{x, x'\}) \neq c_x = x$ and $c(\{y, x'\}) \neq c_y = y$, we have that $c(\{y, x'\}) = c(\{x, x'\}) = x'$. This, with (4.33), implies that there exists $x'' \in N_G(x', X_2)$ such that $c(\{x', x''\}) = x''$. But then the path $xx'x''$ violates (b). This contradiction completes the proof of this claim. $\square$

Claim 4.4.7. *The following hold:*

(i) For every $x \in \widetilde{X}_i$ and $y \in N(x, X_i \cup X''_{i+1})$, we have that $c(xy) = c_x$.

(ii) For every $x \in \widetilde{X}_i$, we have that $d^{cs}(x, X_{i-1}) \geq d^c(x) - 1 - |\widehat{X}_{i+1}| \geq p_{i+1} + 1$.

(iii) If $y \in X''_{i-1}$ and $d^s(y, \widetilde{X}_i) \geq 1$, then $d^c(y, X_i) \geq (\frac{1}{6} - \beta^2)n$.

(iv) If $y \in X'_{i-1}$ and $d^s(y, \widetilde{X}_i) \geq 1$, then $d^c(y, X_i) \geq d^c(y) - 3$.

(v) If $y \in \widetilde{X}_{i-1}$ and $d^s(y, \widehat{X}_i) \geq 1$, then $d^c(y, \widehat{X}_i) \geq d^c(y) - 1$.

Proof. Because G has no rainbow C_ℓ, Claims 4.4.5 and 4.4.6 imply that

$$\forall x \in \widetilde{X}_i \text{ and } y \in N(x) \text{ such that } c(xy) \neq c_x,\ c(E(y, X_{i+1})) \subseteq \{c_x, c(xy)\}. \qquad (4.34)$$

Note that for every $x \in \widetilde{X}_i$ and $y \in N(x, X_i \cup X''_{i+1})$, by the definition of X''_{i+1} and Claim 4.4.2(C), we have that $d^c(y, X_{i+1}) \geq 3$. Thus $c(E(y, X_{i+1})) \not\subseteq \{c_x, c(xy)\}$, and by (4.34), $c(xy) = c_x$. Thus (i) holds. Furthermore,

$$d^{cs}(x, X_{i-1}) = d^c(x) - 1 - d^c(x, \widehat{X}_{i+1}) \geq d^c(x) - 1 - |\widehat{X}_{i+1}| \geq (m - |\widehat{X}_{i+1}|) + 1,$$

so (ii) holds.

For the remaining implications assume $y \in X_{i-1}$ and that there exists $x \in N^s(y, \widetilde{X}_i)$. By (4.34), the only colors that appear on edges in $E(y, X_{i+1})$ are $c(yx)$ and c_x. This implies that

$$d^c(y, X_i \cup X_{i-1}) \geq d^c(y) - 1. \tag{4.35}$$

Now suppose that $y \in X''_{i-1}$. By construction of X_i, $d^c(y, \widetilde{X}_i) \geq d^c(y, \widetilde{X}_{i-1})$. Therefore Claim 4.4.2(B) and (4.35), imply that

$$d^c(y, \widetilde{X}_i) \geq \frac{d^c(y, \widetilde{X}_i \cup \widetilde{X}_{i+1})}{2} \geq \frac{(d^c(y, X_i \cup X_{i+1}) - 2\beta^2 n)}{2} \geq \frac{(d^c(y) - 1 - 2\beta^2 n)}{2}.$$

With (4.22), we have (iii). To see (iv) recall that if $y \in X'_{i-1}$, then $d^c(y, X_{i-1}) \leq 2$. So with (4.35), we have that $d^c(y, X_i) \geq d^c(y) - 3$.

To prove (v), suppose that $y \in \widetilde{X}_{i-1}$. By (4.34), for every $w \in N(y, X_{i+1})$ we have that $c(yw) \in \{c_x, c(xy)\}$. Since there exists $w \in N^t(y, X_{i+1})$, we have that $c_y = c(yw) \in \{c_x, c(xy)\}$. Furthermore, for every $z \in N(y, X_{i-1} \cup X''_i)$, by (i) with $i-1$, y and z playing the roles of i, x and y, respectively, we have that $c(yz) = c_y$. Therefore

$$c(E(y, X_{i-1} \cup X''_i \cup X_{i+1})) \subseteq \{c_y, c_x, c(xy)\} = \{c_x, c(xy)\}.$$

Since $c(xy) \in c(E(y, \widehat{X}_i))$ and $V(G) \setminus \widehat{X}_i = X_{i-1} \cup X''_i \cup X_{i+1}$, this implies that

$$c(E(y)) \setminus c(E(y, \widehat{X}_i)) \subseteq \{c_x\},$$

and we have that $d^c(y, \widehat{X}_i) \geq d^c(y) - 1$. $\square$

Claim 4.4.8. *For every $y \in X_{i-1}$, if $d^s(y, \widetilde{X}_i) \geq 4$, there exists $x \in N^s(y, \widetilde{X}_i)$ such that $d^{cs}(y, \widetilde{X}_i - x) = 1$. This implies that for every $y \in X_{i-1}$, we have that $d^{cs}(y, \widetilde{X}_i) \leq 3$, so $d^t(y, \widetilde{X}_i) = d^c(y, \widetilde{X}_i) - d^{cs}(y, \widetilde{X}_i) \geq d^c(y, \widetilde{X}_i) - 3$. This means that for every $y \in X_{i-1}$, we have that $d^t(y, \widetilde{X}_i) \geq (\frac{1}{9} - \beta)n$, and if $d^s(y, \widetilde{X}_i) \geq 1$, then $d^t(y, \widetilde{X}_i) \geq (\frac{1}{6} - \beta)n$.*

Proof. Let $x, x' \in N^s(y, \widetilde{X}_i)$ be distinct vertices. We say that (x, x') is a *y-pair* if the colors $c(yx)$, $c(yx')$, c_x and $c_{x'}$ are distinct. There are no y-pairs, because if (x, x') is a y-pair, then by Claim 4.4.3, there exists $z \in N^t(x, X_i) \cap N^t(x', X_i)$. For every such z, the cycle $xyx'z$ is a strong rainbow C_4, a contradiction by Claim 4.4.5.

For contradiction, assume that $d^s(y, \widetilde{X}_i) \geq 4$ and that for every $x \in N^s(y, \widetilde{X}_i)$, we have that $d^{cs}(y, \widetilde{X}_i - x) \geq 2$. If every special edge from y to $\widetilde{X}_i$ is given a unique color, let $\{x_1, x_2, x_3, x_4\}$ be an arbitrarily selected set of 4 vertices in $N^s(y, \widetilde{X}_i)$. Otherwise, there exists $x_2, x_3 \in N^s(y, \widetilde{X}_i)$ such that $c(yx_2) = c(yx_3)$. Since $d^{cs}(y, \widetilde{X}_i) \geq 2$, there exists $x_1 \in N^s(y, \widetilde{X}_i)$ such that $c(yx_1) \neq c(yx_2) = c(yx_3)$. Because $d^{cs}(y, \widetilde{X}_i - x_1) \geq 2$, there exists $x_4 \in N^s(y, \widetilde{X}_i - x_1)$ such that $c(yx_4) \neq c(yx_2) = c(yx_3)$.

Define $a := c(yx_1)$. In all cases, we have that $c(yx_1) = a$, $c(yx_2) \neq a$, $c(yx_3) \neq a$, and $c(yx_4) \notin \{c(yx_2), c(yx_3)\}$. In what follows, we use Claim 4.4.4 implicitly. Since $c_{x_2} \neq c_{x_3}$ and neither (x_1, x_2) nor (x_1, x_3) is a y-pair, one of c_{x_2} or c_{x_3} must be a. We can assume that $c_{x_2} = a$. Let $b := c(yx_3)$ and note that $c_{x_1} = b$, since $c_{x_3} \neq c_{x_2} = a$, $c(yx_1) = a \neq c(yx_3)$ and (x_1, x_3) is not a y-pair. Furthermore, $c(yx_4) = a$, because (x_1, x_4) is not a y-pair, $c_{x_1} = b$, $c(yx_4) \neq c(yx_3) = b$, and $c_{x_4} \neq c_{x_2} = a$. But then $c_{x_4} \neq c_{x_1} = b = c(yx_3)$ and $c_{x_3} \neq c_{x_2} = a = c(yx_4)$, so (x_3, x_4) is a y-pair, a contradiction.

For the remaining implications, the first statement implies that if $d^s(y, \widetilde{X}_i) \geq 4$, then $d^{cs}(y, \widetilde{X}_i) \leq 2$, and, clearly, if $d^s(y, \widetilde{X}_i) \leq 3$, we have that $d^{cs}(y, \widetilde{X}_i) \leq 3$, so

$$d^t(y, \widetilde{X}_i) = d^{ct}(y, \widetilde{X}_i) = d^c(y, \widetilde{X}_i) - d^{cs}(y, \widetilde{X}_i) \geq d^c(y, \widetilde{X}_i) - 3.$$

The remaining implications follow from Claims 4.4.2(C) and 4.4.7(iii),(iv) and (v). $\square$

Call a C_4 $xyx'y'$ a *special* C_4 if there exists i such that $x, x' \in \widetilde{X}_i$, $y, y' \in X_{i-1}$, the edges xy and $x'y'$ are special edges, and the edges xy' and $x'y$ are typical edges.

Claim 4.4.9. *Exactly three colors are used on the edges of every special C_4 and the same color is used on the two special edges. In particular, every special C_4 is a strong properly colored C_4.*

Proof. Suppose that $xyx'y'$ is a special C_4 with $x, x' \in \widetilde{X}_i$ and $y, y' \in X_{i-1}$ for some $i \in [3]$. Assume that xy and $x'y'$ are the special edges.

We will first show that the color $c(xy')$ is used exactly once on the cycle. By the definition of typical and special edges, we have that $c(xy') = c_x \neq c(xy)$, and, with Claim 4.4.4, we have $c(xy') = c_x \neq c_{x'} = c(x'y)$. If $c(xy') = c(x'y')$, then the color degree of both x and y' is the same in $G - xy'$ as it is in G, and this contradicts the edge-minimality of (G, c) (Claim 4.4.1(V)). Indeed, this is clearly true for y' and is true for x because, by Claim 4.4.3, x has typical neighbors in X_{i-1} other than y'.

By symmetry, $c(x'y)$ is used exactly once on the cycle as well. As $xyx'y'$ is not a strong C_4 by Claim 4.4.6, $c(xy) = c(x'y')$. □

Claim 4.4.10. *For every $i \in [3]$ and every pair of vertices $y, y' \in X_{i-1}$ the following holds. For any color a, if $Z := \{x \in N^s(y, X_i) : c(xy) = a\}$ and $Z' := \{x' \in N^s(y', X_i) : c(x'y') \neq a\}$, then*

$$|Z \cup Z'| < (\frac{1}{6} + \gamma)n,$$

Proof. Assume for contradiction that $|Z \cup Z'| \geq (\frac{1}{6} + \gamma)n$. We can assume that one of Z or Z', say Z, is non-empty. This and Claim 4.4.8 imply that $d^t(y, \widetilde{X}_i) \geq (\frac{1}{6} - \beta)n$, so, by Claim 4.4.2(A),

$$|Z| \leq |\widetilde{X}_i| - d^t(y, \widetilde{X}_i) \leq (\frac{1}{3} + \beta^2)n - (\frac{1}{6} - \beta)n < (1/6 + \gamma)n \leq |Z \cup Z'|,$$

so $|Z'| > 0$. Therefore by Claim 4.4.8, $d^t(y', \widetilde{X}_i) \geq (\frac{1}{6} - \beta)n$ as well. With Claim 4.4.2(A), there exists $x' \in N^t(y, \widetilde{X}_i) \cap (Z \cup Z')$ and $x \in N^t(y', \widetilde{X}_i) \cap (Z \cup Z')$. Therefore $xyx'y'$ is a special C_4 with special edges xy and $x'y'$ such that and $c(xy) = a \neq c(x'y')$. Claim 4.4.9 implies that $xyx'y'$ is a strong rainbow C_4, a contradiction. □

We now label X_1, X_2, and X_3 in a careful way to make the rest of the proof proceed more smoothly.

Claim 4.4.11. *We can assume that $p_3 \geq 0$, $|X''_1| \leq 2p_3$ and $|X_2| \leq m + 2p_3 + 2$.*

Proof. We need to prove that there exists $i \in [3]$ such that $p_i \geq 0$, $|X''_{i+1}| \leq 2p_i$ and $|X_{i-1}| \leq m + 2p_i + 2$. First note that for $i \in [3]$, because

$$|X_{i-1}| - m = |X''_{i-1}| + |\widehat{X}_{i-1}| - m = |X''_{i-1}| - p_{i-1},$$

the inequality $|X_{i-1}| \leq m + 2p_i + 2$, is equivalent to

$$|X''_{i-1}| - p_{i-1} \leq 2p_i + 2. \tag{4.36}$$

Also note that

$$\sum_{j\in[3]} |X''_j| = n - \sum_{j\in[3]} |\widehat{X}_j| = p_1 + p_2 + p_3 + (n - 3m) \leq p_1 + p_2 + p_3 + 2. \tag{4.37}$$

For $i \in [3]$,

$$p_i = \max_{j\in[3]}\{p_j\} \Rightarrow |X''_{i+1}| \geq 2p_i + 1, \tag{4.38}$$

because then $p_i \geq 0$, and by (4.37), we have that

$$|X''_{i-1}| - p_{i-1} \leq p_{i+1} + p_i + 2 \leq 2p_i + 2,$$

so (4.36) holds. If $|X''_{i+1}| \leq 2p_i$, then we are done, so assume that $|X''_{i+1}| > 2p_i + 1$.

Assume that $p_3 = \max_{i\in[3]}\{p_i\}$, so we have that $p_3 \geq 0$ and (4.38) gives us that

$$|X''_1| \geq 2p_3 + 1. \tag{4.39}$$

This with (4.37) implies that

$$0 \leq |X''_3| \leq p_1 + p_2 + p_3 + 2 - |X''_1| \leq p_1 + (p_2 - p_3) + 1. \tag{4.40}$$

We have that

$$p_1 \geq 0, \tag{4.41}$$

because if $p_1 < 0$, then, because $p_2 \leq p_3$, (4.40) implies that $p_1 = -1$, $p_2 = p_3$ and $|X_3''| = 0$. But this contradicts (4.38) (with 2 playing the role of i). Since $p_3 \geq \max\{0, p_2\}$, (4.40) and (4.41) give us that $|X_3''| - p_3 \leq |X_3''| \leq p_1 + 1 < 2p_1 + 2$, so (4.36) is satisfied with $i = 1$. This with (4.41) implies that

$$|X_2''| \geq 2p_1 + 1. \tag{4.42}$$

By (4.37), (4.39), and (4.42), we have that

$$2p_1 + 2p_3 + 2 + |X_3''| \leq \sum_{i \in [3]} |X_i''| \leq p_1 + p_2 + p_3 + 2,$$

so $0 \leq |X_3''| \leq (p_2 - p_3) - p_1$. With (4.41), we have that $p_2 = p_3$ and $|X_3''| = 0$. This contradicts (4.38) (again with 2 playing the role of i). □

Note that for every $i \in [3]$ such that $p_i \geq 0$, $|X_{i+1}''| \leq 2p_i$ and $|X_{i-1}| \leq m + 2p_i + 2$. All of the following claims are valid with the indices $i-1$, i and $i+1$ playing the roles of 2, 3, and 1, respectively.

One of our main goals is to show that that there must exist a special edge between $\widetilde{X}_1$ and $\widetilde{X}_2$, which we prove in Claim 4.4.15 To do this, we use Claim 4.4.12 to bound the number of special edges from $\widetilde{X}_2$ to X_1' and then Claim 4.4.14 provides a bound on the number of special edges from $\widetilde{X}_2$ to X_1''.

Claim 4.4.12. *If $y \in \widehat{X}_1$, then $d^s(y, \widetilde{X}_2) \leq 2p_3 + 5$.*

Proof. Assume for contradiction that there exists $y \in \widehat{X}_1$ such that

$$d^s(y, \widetilde{X}_2) \geq 2p_3 + 6. \tag{4.43}$$

By Claim 4.4.11, $p_3 \geq 0$, so we can assume $d^s(y, \widetilde{X}_2) \geq 6$ which with Claim 4.4.8 implies that

$$d^{cs}(y, \widetilde{X}_2) \leq 2. \tag{4.44}$$

With Claim 4.4.7(iv) and (v) we have that

$$d^c(y, \widetilde{X}_2) \geq d^c(y) - 3. \tag{4.45}$$

So (4.43), (4.44), and (4.45) imply that

$$|X_2| \geq d(y, X_2) \geq d^c(y, X_2) - d^{cs}(y, \widetilde{X}_2) + d^s(y, \widetilde{X}_2) \geq d^c(y) - 5 + (2p_3 + 6) \geq m + 2p_3 + 3,$$

which contradicts Claim 4.4.11. □

Claim 4.4.13. *For every $y \in X_1''$, we have that $d^s(y, \widetilde{X}_2) \leq \frac{n}{10}$.*

Proof. Suppose for a contradiction that there exists $y \in X_1''$ such that $d^s(y, \widetilde{X}_2) > \frac{n}{10}$. By Claim 4.4.8, there exists a color a such that if $Z := \{x \in N^s(y, \widetilde{X}_2) : c(xy) = a\}$, then

$$|Z| \geq \frac{n}{10} - 1 \geq 0.09n. \tag{4.46}$$

Let $U := N^t(y, \widetilde{X}_2)$, and note that, by Claim 4.4.8,

$$|U| \geq (\frac{1}{6} - \beta)n \geq 0.16n. \tag{4.47}$$

Let $u \in U$ and suppose that there exist $w \in N^s(u, \widehat{X}_1)$ such that $c(uw) \neq a$. Then, by Claims 4.4.2(A), 4.4.7((iv),(v)) and 4.4.8, and (4.46),

$$d^t(w, Z) \geq |Z| + d^t(w, \widetilde{X}_2) - |\widetilde{X}_2| > 0,$$

so there exists $x \in N^t(w, Z)$, which implies $uwxy$ is a special C_4. This contradicts Claim 4.4.9, as the special edges, uw and xy, are assigned distinct colors. Therefore using Claims 4.4.7(ii) and 4.4.11, we have that for every $u \in U$, the number of colors other than a that are used on special edges from u to vertices in X_1'' is at least

$$d^{cs}(u, X_1) - 1 \geq p_3 \geq \frac{|X_1''|}{2}.$$

By averaging, there exists $y' \in X_1'' - y$ such that if we let

$$Z' := \{x' \in N^s(y', U) : c(x'y') \neq a\},$$

then $|Z'| \geq \frac{|U|}{2}$. With (4.46) and (4.47), we have that

$$|Z \cup Z'| \geq |Z| + \frac{|U|}{2} \geq 0.17n \geq (\frac{1}{6} + \gamma)n,$$

which contradicts Claim 4.4.10. $\square$

Claim 4.4.14. *For any $p \geq p_3$ such that $p \geq 1$ the following holds. If $U \subseteq X_1$ such that $|U| \leq 0.01n$, then $e^s(U, \widetilde{X}_2) \leq 0.3pn$.*

Proof. Suppose

$$e^s(U, \widetilde{X}_2) > 0.3pn.$$

By Claim 4.4.8, for every vertex in $u \in U \setminus X_1''$, we have that $d^s(u, \widetilde{X}_2) \leq 2p_3 + 5 \leq 2p + 5$. Since $p \geq 1$ and $|U| \leq 0.01n$ we have that

$$e^s(U \cap X_1'', \widetilde{X}_2) = e^s(U, \widetilde{X}_2) - e^s(U \setminus X_1'', \widetilde{X}_2) \geq 0.3pn - (2p + 5)0.01n > 0.2pn.$$

By Claim 4.4.11, we have that $|U \cap X_1''| \leq |X_1''| \leq 2p_3 \leq 2p$. By averaging, there exists $y \in U \cap X_1''$ such that $d^s(y, \widetilde{X}_2) > \frac{n}{10}$, a contradiction to Claim 4.4.13. $\square$

Claim 4.4.15. *We have $p_2 \leq -1$, $p_3 = 0$, and n is not congruent to 2 modulo 3.*

Proof. Let $U := X_1'' \cup X_1'$. By Claim 4.4.2(G) we have that $|U| \leq 0.01n$. By Claim 4.4.7(ii), every $x \in \widetilde{X}_2$ sends at least $p_3 + 1$ special edges to X_1. By Claim 4.4.2(B), $|\widetilde{X}_2| \geq 0.3n$, Claim 4.4.14 implies that there exists a special edges yx with $y \in X_1 \setminus U = \widetilde{X}_1$ and $x \in \widetilde{X}_2$. Note that Claim 4.4.7(v) implies that

$$|\widehat{X}_2| \geq d^c(y, \widehat{X}_2) - 1 \geq m + 1, \tag{4.48}$$

so $p_2 \leq -1$. Let $a := c(yx)$.

Now redefine $U := X_1 \setminus N^t(x, X_1)$. By Claim 4.4.3, we have that $|U| \leq 0.01n$. Let $W := N^t(y, \widetilde{X}_2)$. Note that for every special edge $y'x'$ with $y' \in X_1 \setminus U = N^t(x, X_1)$ and $x' \in W$, we have the special C_4 $xyx'y'$. By Claim 4.4.7(v) and Claim 4.4.8, we have that $|W| \geq d^c(y) - 4 \geq 0.3n$. Again, by 4.4.7(ii), for every $w \in W$ we have that $d^{cs}(w, X_1) \geq p_3 + 1$. Therefore Claim 4.4.14 implies that there exists a special C_4, and Claims 4.4.5 and 4.4.9 imply that G has a properly colored C_ℓ. Therefore, with Claim 4.4.6, we can assume condition Claim 4.4.1(I) holds, so $\delta^c(G) = \frac{(n+5)}{3}$.

If $n \equiv 2 \pmod 3$, then $\delta^c(G) = m + 3$, and Claim 4.4.7(ii) implies that for every $w \in W$ we have $d^{cs}(w, X_1) \geq p_3 + 2$. Therefore, whenever $p_3 \geq 1$ or $n \equiv 2 \pmod 3$, there exists $p \geq p_3$ such that $p \geq 1$ and every vertex in W send at least p special edges to X_1 that are not colored a. Claim 4.4.14 implies that there exists an edge $y'x'$ such that $y' \in X_1 \setminus U = N^t(y, \widetilde{X}_2)$, $x' \in W = N^t(y, \widetilde{X}_2)$ and $c(y'x') \neq a$, which contradicts Claim 4.4.9. Therefore $p_3 = 0$ and n is not congruent to 2 modulo 3. $\square$

By Claim 4.4.15, $p_2 \leq -1$, $p_3 = 0$ and n is not congruent to 2 modulo 3. Therefore $n \leq 3m + 1$, $|\widehat{X}_2| \geq m + 1$ and $|\widehat{X}_3| = m$. Thus $|\widehat{X}_1| \leq m$, so $p_1 \geq 0$, and that

$$|X_2''| + |X_3''| \leq |X_1''| + |X_2''| + |X_3''| = n - \sum_{j \in [3]} |\widehat{X}_j| \leq p_1 + p_2 + p_3 + 1 \leq p_1.$$

Therefore $p_1 \geq 0$, $|X_2''| \leq 2p_1$, and $|X_3| = m - p_3 + |X_3''| \leq m + 2p_1 + 2$, so Claim 4.4.15 is valid with the indices 2, 3 and 1 playing the roles of the indices 1, 2, and 3, respectively (see the text after Claim 4.4.11). This implies that $p_3 \leq -1$, which contradicts Claim 4.4.15. This contradiction proves Lemma 4.1.7. $\square$

REFERENCES

[1] N. Alon, A. Pokrovskiy, and B. Sudakov. Random subgraphs of properly edge-coloured complete graphs and long rainbow cycles. *Israel Journal of Mathematics*, 222(1):317–331, Oct 2017.

[2] N. Alon and A. Shapira. Testing subgraphs in directed graphs. *Journal of Computer and System Sciences*, 69:354–382, 01 2003.

[3] L. Caccetta and R. Häggkvist. On minimal digraphs with given girth. *Congressus numerantium*, XXI, 1978.

[4] K. Corrádi and A. Hajnal. On the maximal number of independent circuits in graph. *Acta Mathematica Academiae Scientiarum Hungarica*, 14, 09 1963.

[5] A. Czygrinow. Tight co-degree condition for packing of loose cycles in 3-graphs. *Journal of Graph Theory*, 83(4):317–333, 2016.

[6] A. Czygrinow, L. DeBiasio, and B. Nagle. Tiling 3-uniform hypergraphs with $K_3^4 - 2e$. *Journal of Graph Theory*, 75:124–136, 2014.

[7] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, s3-2(1):69–81, 1952.

[8] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.

[9] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel Journal of Mathematics*, 2:183–190, 09 1964.

[10] S. Glock and F. Joos. A rainbow blow-up lemma, 2018. Preprint.

[11] P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, s1-10(1):26–30, 1935.

[12] J. Han, C. Zang, and Y. Zhao. Minimum vertex degree thresholds for tiling complete 3-partite 3-graphs. *Journal of Combinatorial Theory, Series A*, 149:115 – 147, 2017.

[13] J. Han and Y. Zhao. Minimum vertex degree threshold for C_4^3□tiling. *Journal of Graph Theory*, 79(4):300–317, 2015.

[14] J. Hladký, D. Král', and S. Norine. Counting flags in triangle-free digraphs. *Combinatorica*, 37(1):49–76, June 2016.

[15] Y. Ji, S. Wu, and H. Song. On short cycles in triangle-free oriented graphs. *Czechoslovak Mathematical Journal*, 68(1):67–75, Mar 2018.

[16] P. Keevash, D. Mubayi, B. Sudakov, and J. Verstraëte. Rainbow turán problems. *Combinatorics, Probability and Computing*, 16(1):109–126, 2007.

[17] L. Kelly, D. Kühn, and D. Osthus. A dirac-type result on hamilton cycles in oriented graphs. *Combinatorics, Probability and Computing*, 17(5):689–709, 2008.

[18] L. Kelly, D. Kühn, and D. Osthus. Cycles of given length in oriented graphs. *Journal of Combinatorial Theory, Series B*, 100(3):251 – 264, 2010.

[19] J. Kim, D. Kühn, A. Kupavskii, and D. Osthus. Rainbow structures in locally bounded colourings of graphs, 2018. Preprint.

[20] D. Kirkpatrick and P. Hell. On the complexity of general graph factor problems. *SIAM J. Comput.*, 12:601–609, 08 1983.

[21] D. Kühn and D. Osthus. Loose hamilton cycles in 3-uniform hypergraphs of high minimum degree. *Journal of Combinatorial Theory, Series B*, 96(6):767 – 821, 2006.

[22] Daniela Kühn, Deryk Osthus, and Diana Piguet. Embedding cycles of given length in oriented graphs. *European Journal of Combinatorics*, 34(2):495 – 501, 2013.

[23] B. Li, B. Ning, C. Xu, and S. Zhang. Rainbow triangles in edge-colored graphs. *European Journal of Combinatorics*, 36:453 – 459, 2014.

[24] H. Li. Rainbow C_3's and C_4's in edge-colored graphs. *Discrete Mathematics*, 313(19):1893 – 1896, 2013. Cycles and Colourings 2011.

[25] R. Mycroft. Packing k-partite k-uniform hypergraphs. *Journal of Combinatorial Theory, Series A*, 138:60 – 132, 2016.

[26] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *Journal of Combinatorial Theory, Series A*, 116(3):613 – 636, 2009.

[27] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs. *Journal of Combinatorial Theory, Series A*, 119(7):1500 – 1522, 2012.

[28] P. Turán. Eine extremalaufgabe aus der graphentheorie. *Matematikai és Fizikai Lapok*, 48:436–452, 1941.

[29] W. T. Tutte. The factorization of linear graphs. *Journal of the London Mathematical Society*, s1-22(2):107–111, 1947.

[30] R. Čada, A. Kaneko, Z. Ryjáček, Y. Zdeněk, and K. Yoshimoto. Rainbow cycles in edge-colored graphs. *Discrete Math.*, 339(4):1387–1392, April 2016.