

Hybrid Subgroups of Complex Hyperbolic Lattices

by

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A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved April 2019 by the
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May 2019

ABSTRACT

In the 1980's, Gromov and Piatetski-Shapiro introduced a technique called "hybridization" which allowed them to produce non-arithmetic hyperbolic lattices from two non-commensurable arithmetic lattices. It has been asked whether an analogous hybridization technique exists for complex hyperbolic lattices, because certain geometric obstructions make it unclear how to adapt this technique. This thesis explores one possible construction (originally due to Hunt) in depth and uses it to produce arithmetic lattices, non-arithmetic lattices, and thin subgroups in $SU(2, 1)$.

ACKNOWLEDGMENTS

First and foremost, to my advisor Julien Paupert, I thank you for all of your guidance over the years and for introducing me to an area of mathematics that I've grown to love. I appreciate your patience and your prompt, respectful responses to even the silliest of math questions I've asked. You have been a constant source of inspiration, both as a mathematician and as a person.

To my supervisory committee Nancy Childress, Susanna Fishel, Matthias Kawski, and Brett Kotschwar, I thank each and every one of you for the support you've provided during my journey, for the insights you've shared (both mathematical and about life in general), and for the the many impromptu hallway conversations that I will always remember fondly.

To my cohorts Donnie Adams, Mary Cook, Lauren Crider, Zach Harrison, David Polletta, and the countless other people I've met along the way, I thank you all for the many casual mathematical discussions and general comeraderie over the years.

To my wife Jen, I thank you for the constant love, praise, and encouragement you've shown me throughout the challenging graduate experience and throughout our entire relationship. You've been my rock, and I could not have done this without you.

To my family Mike, Rebel, Ayla, Deborah, Jonny, Vanessa, Azuriah, Jaidyn, and Emma, I thank you all for the love and support you've provided me along the way, even though you don't understand my research.

Last, and certainly not least, if you are reading this, I both acknowledge you and appreciate you. I regret to inform you that there is no money to be found tucked in between the printed pages of this document.

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Chapter 1

INTRODUCTION

This thesis largely aims to explore a certain class of discrete groups, which we call *hybrid subgroups*, of the group of complex hyperbolic isometries $\mathrm{PU}(n, 1)$.

The theory of discrete subgroups of (semisimple) Lie groups is very rich (see [35] or [53] for some standard texts). To a semisimple Lie group G , we can associate a *symmetric space* X (a smooth manifold with the property that, for every $p \in X$, there is an involutive isometry of X with p as an isolated fixed point) on which G acts both transitively and by isometries. For a torsion-free discrete subgroup Γ of G , the quotient space $\Gamma \backslash X$ is again a smooth manifold and the geometric structure of this quotient is inherited from the structure determined by G and X . In this way, there is a natural correspondence between the algebraic structure of discrete subgroups of G and the possible manifolds that can be endowed with a given geometric structure (determined by G and X). When Γ has torsion, the quotient is no longer a manifold (instead, we call it an *orbifold*), but for the most part this is only a minor technicality.

One may coarsely sort discrete subgroups by covolume (that is, volume of the coset space G/Γ with respect to the Haar measure, or equivalently, the volume of $\Gamma \backslash X$). For infinite covolume discrete groups, there may be deformations – continuous families of discrete faithful representations $\rho_t : \Gamma \rightarrow G$ that are pairwise non-conjugate – and the deformation theory is very well-developed. For example, see [30] and [31] for deformations of Fuchsian groups, the survey [1] for deformations of Kleinian groups, and [27] for deformations of (G, X) structures.

Beyond the deformation theory, there has also been renewed interest in a particular family of infinite covolume discrete groups called *thin groups* (these are contained

a finite covolume discrete subgroups and are Zariski-dense in G) thanks to Sarnak [54] for use in “superstrong approximation theory” which is a far-reaching generalization of the Chinese Remainder Theorem in the context of Cayley graphs. From the perspective of deformations, we expect that thin groups are, in some sense, ubiquitous, although codifying this notion precisely is still unclear (see [22]) and constructing examples has proven challenging. Remarkably, there has been considerable success in producing examples of thin subgroups using techniques from hyperbolic geometry (see, for example, [32] or discussions in [42]), and the hybrid technique introduced in Chapter 3 successfully produce an example as well, although it is an open question as to the necessary conditions for producing more thin hybrids.

In the case that Γ has finite covolume, we say Γ is a *lattice*, and the deformation story is quite different. When G is a simple Lie group not locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$, the celebrated rigidity theorems due to Mostow [39] (in the the case where $\Gamma \backslash G$ is compact) and Prasad [52] (the non-cocompact case) tell us that for a lattice $\Gamma < G$, all discrete faithful representations $\rho : \Gamma \rightarrow G$ with finite covolume image are conjugate. In particular, for the case of real hyperbolic manifolds of dimension $n > 2$, these results imply that the geometric structure of the hyperbolic manifold is uniquely determined by the volume. As a result of this rigidity, it is a challenge to produce interesting lattices.

In [4] Borel and Harish-Chandra prove that every simple Lie group G contains a lattice, and they give a construction of such a lattices that generalizes the standard example of $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$. Lattices that arise in this way are called *arithmetic*, and due to Margulis’ arithmeticity theorem [34], there is a strong sense in which most lattices are arithmetic. Specifically, this theorem says that if G has real rank at least 2 (that is, the associated symmetric space contains an isometrically embedded copy of \mathbb{R}^n for $n \geq 2$), then all irreducible lattices are arithmetic. As such, up to

a compact factor, the only simple Lie groups which can contain a non-arithmetic lattice are $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$, $\mathrm{Sp}(n, 1)$, and F_4 . In fact, Gromov and Schoen [25] showed that $\mathrm{Sp}(n, 1)$ contains only arithmetic lattices, and Corlette [10] showed that F_4 also contains only arithmetic lattices. Due to their exceptional nature, an even bigger challenge is to find these non-arithmetic lattices in $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$.

In $\mathrm{SO}(n, 1)$, the corresponding symmetric space is real hyperbolic space $\mathbf{H}_{\mathbb{R}}^n$, and some of the earliest examples of nonarithmetic lattices in $\mathrm{SO}(n, 1)$ were produced by Vinberg in [57] (see also [42]), and the results extended into higher dimensions in [58]. These groups are generated by reflections in the walls (that is, totally geodesic real hyperplanes) of hyperbolic polytopes. Takeuchi [56] later classified all possible triangle groups (discrete groups generated by reflections in the sides of a hyperbolic triangle) by arithmeticity. More generally, Gromov and Piatetski-Shapiro [24] introduced a technique called “interbreeding” or “hybridization” for producing non-arithmetic lattices in every dimension n from two non-commensurable arithmetic groups in $\mathrm{SO}(n, 1)$ (two groups $\Gamma_1, \Gamma_2 < G$ are *commensurable* if there exists a $g \in G$ for which $\Gamma_1 \cap g^{-1}\Gamma_2g$ has finite index in both Γ_1 and $g\Gamma_2g^{-1}$). In a sense, all known nonarithmetic lattices in $\mathrm{SO}(n, 1)$ are produced using geometric techniques.

In $\mathrm{SU}(n, 1)$, the associated symmetric space is complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^n$, and in a stark contrast with the real hyperbolic setting, very little is known about nonarithmetic lattices (see [46] for a survey of complex hyperbolic lattices). One reason for this is that the geometric techniques from real hyperbolic space do not carry over naturally to complex hyperbolic space: $\mathbf{H}_{\mathbb{C}}^n$ does not contain any totally geodesic real hypersurfaces, and so techniques involving reflections in or gluing along these hypersurfaces have no natural analog (in particular, this means there is no obvious analog to the Gromov–Piatetski-Shapiro construction).

The first examples of nonarithmetic lattices in $\mathrm{SU}(2, 1)$ were produced by Picard

[51] among a list of 27 lattices constructed from monodromy groups of hypergeometric functions. Although arithmeticity was not a consideration in Picard’s time, 7 of these lattices are non-arithmetic and their arithmeticity was determined by Deligne and Mostow [12]. In [40], Mostow produced 7 new examples of non-arithmetic lattices in $SU(2, 1)$ by considering groups generated by three complex reflections (these are analogous to triangle groups in $SO(2, 1)$), and Deraux, Falbel, and Paupert in [15] re-examined these groups to find simpler fundamental domains. In a construction similar to that of Mostow, Deraux–Parker–Paupert [16] and [17] produced several new examples of non-arithmetic lattices in $SU(2, 1)$. As well, they examined the commensurability of all new and previously known non-arithmetic lattices in $SU(2, 1)$, ultimately determining that they fall into 22 commensurability classes. In $SU(3, 1)$, Deligne–Mostow [12] produce a single non-arithmetic lattice amidst their treatment of Picard’s construction, and recently Deraux [13] found a single non-arithmetic lattice in $SU(3, 1)$ among lattices produced by Couwenberg–Heckman–Looijenga [11] (which generalizes the earlier work of Deligne–Mostow) and determined that it was not commensurable to the Deligne–Mostow lattice. There are no known examples of non-arithmetic lattices in $SU(n, 1)$ for $n > 3$, and it’s unknown if they can exist at all. One possible approach to resolving this question is to find a suitable hybrid construction in the spirit of Gromov–Piatetski-Shapiro, and indeed this is the main aim of this writing.

This thesis is organized as follows:

Chapter 2 provides a background of the relevant information from complex hyperbolic geometry and arithmetic groups. The information contained in this chapter is completely standard, and one can see [23] or [44] for a more complete treatment of complex hyperbolic geometry and [38] for arithmetic groups. We also summarize the strategy of Gromov and Piatetski-Shapiro to motivate a candidate hybrid con-

struction, originally due to Hunt (see the references in [48]) for $SU(n, 1)$ and explore it further in the subsequent chapters.

Chapter 3 explores hybrid subgroups in some of the Picard modular groups, $SU(2, 1; \mathcal{O}_d)$ (where \mathcal{O}_d is the ring of integers for the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$) which are among the simplest arithmetic lattices in $SU(2, 1)$. The main result of this chapter is that both thin subgroups and lattices can arise as the product of his hybrid construction. Because it is joint work with Julien Paupert and has been submitted for review as of the time of this writing, this chapter is a faithful reproduction of that paper (with only minor formatting changes to adhere to guidelines).

Chapter 4 gives an introduction to Mostow’s lattice construction and further explores hybrid subgroups within a certain subset of these lattices (specifically, within Mostow’s lattices of “small phase shift”, which is defined precisely later). The main result in this chapter is that all of Mostow’s small phase shift lattices can be recovered via a hybrid construction. Surprisingly, some of these non-arithmetic lattices arise as hybrids of non-commensurable arithmetic groups.

Chapter 5 proposes some future directions and demonstrates partial progress made towards these goals. In particular, we examine hybrids in other non-arithmetic lattices in $SU(2, 1)$, namely those of Deraux–Parker–Paupert. We also discuss the hybrid construction in higher dimensions, where the “intersection condition” of the construction becomes nontrivial. In particular, we demonstrate the possible necessity of this condition by using the Gauss–Picard and Eisenstein–Picard lattices to produce a non-discrete subgroup in $SU(3, 1)$ when the condition is relaxed.

Chapter 2

BACKGROUND

2.1 Complex Hyperbolic Geometry

The material contained within this section is completely standard; see for example [23] or [44].

2.1.1 Complex Hyperbolic Space

Let H be a Hermitian matrix of signature $(n, 1)$ and let $\mathbb{C}^{n,1}$ denote \mathbb{C}^{n+1} endowed with the Hermitian form $\langle \cdot, \cdot \rangle_H$ given by

$$\langle x, y \rangle_H = y^* H x.$$

(we will omit the subscript when the Hermitian form is clear from context). Let V_- denote the set of points $z \in \mathbb{C}^{n,1}$ for which $\langle z, z \rangle < 0$, and let V_0 denote the set of points for which $\langle z, z \rangle = 0$ (visually, V_0 forms the *light cone* in Figure 2.1, and V_- is the interior of this cone).

Given the usual projectivization map $\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \rightarrow \mathbb{C}\mathbb{P}^n$, *complex hyperbolic*

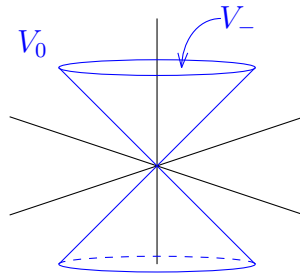


Figure 2.1: Visualization of Subsets V_- and V_0 in $\mathbb{C}^{n,1}$

n -space, denoted $\mathbf{H}_{\mathbb{C}}^n$, is $\mathbb{P}(V_-)$ with distance d coming from the Bergman metric

$$\cosh^2\left(\frac{1}{2}d(\mathbb{P}(x), \mathbb{P}(y))\right) = \frac{\langle x, y \rangle \langle y, x \rangle}{\langle x, x \rangle \langle y, y \rangle} \quad (2.1)$$

The ideal boundary $\partial_{\infty}\mathbf{H}_{\mathbb{C}}^n$ is then identified with $\mathbb{P}(V_0)$.

The choice of Hermitian form gives rise to different geometric models. When

$$H = \begin{pmatrix} I_{n \times n} & 0 \\ 0 & -1 \end{pmatrix}$$

we obtain the *ball model*. By normalizing the last coordinate of a point $z = [z_1, \dots, z_n, 1] \in \mathbf{H}_{\mathbb{C}}^n$, we have

$$\langle z, z \rangle < 0 \Leftrightarrow |z_1|^2 + \dots + |z_n|^2 < 1$$

and thus we can identify $\mathbf{H}_{\mathbb{C}}^n$ with the open unit ball in \mathbb{C}^n and $\partial_{\infty}\mathbf{H}_{\mathbb{C}}^n$ with the boundary $(2n - 1)$ -sphere. When

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{(n-1) \times (n-1)} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

one obtains the *Siegel model*. By normalizing the last coordinate of a point $z = [z_1, \dots, z_n, 1]$ in $\mathbf{H}_{\mathbb{C}}^n$, we have

$$\langle z, z \rangle < 0 \Leftrightarrow 2 \operatorname{Re}(z_1) + |z_2|^2 + \dots + |z_n|^2 < 0$$

and thus we identify $\mathbf{H}_{\mathbb{C}}^n$ with the convex domain in the interior of the paraboloid given by $2 \operatorname{Re}(z_1) + |z_2|^2 + \dots + |z_n|^2 = 0$, and $\partial_{\infty}\mathbf{H}_{\mathbb{C}}^n$ is identified with the boundary paraboloid along with a distinguished point at infinity, $p_{\infty} = [1, 0, 0] \in \mathbb{C}\mathbb{P}^n$. For a given point z in the Siegel model of $\mathbf{H}_{\mathbb{C}}^n$, let $t \in \mathbb{R}$ and $u \in \mathbb{R}^+$ such that

$$z = \left[\frac{-|z_2|^2 - \dots - |z_n|^2 - u + it}{2}, z_2, \dots, z_n, 1 \right].$$

In this way we can identify $\mathbf{H}_{\mathbb{C}}^n$ with $\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$ and coordinates (z_2, \dots, z_n, t, u) ; the boundary $\partial_{\infty} \mathbf{H}_{\mathbb{C}}^n$ is identified with $(\mathbb{C}^{n-1} \times \mathbb{R} \times \{0\}) \cup \{p_{\infty}\}$. These coordinates, called *horospherical coordinates*, provide us with a natural analog of the half space model of real hyperbolic space.

2.1.2 Complex Hyperbolic Isometries

Just as in real hyperbolic space, any holomorphic isometry of $\mathbf{H}_{\mathbb{C}}^n$ can be classified as one of the following three types depending on its fixed point(s). Specifically, a holomorphic isometry is:

- *elliptic* if it has a fixed point in $\mathbf{H}_{\mathbb{C}}^n$.
- *parabolic* if it has exactly one fixed point in the boundary (and no fixed points in $\mathbf{H}_{\mathbb{C}}^n$).
- *loxodromic* if it has exactly two fixed points in the boundary (and no fixed points in $\mathbf{H}_{\mathbb{C}}^n$).

Given a vector $v \in \mathbb{C}^{n,1}$ with $\langle v, v \rangle = 1$ and complex number ζ with unit modulus, the linear map

$$R_{v,\zeta}(x) : x \mapsto x + (\zeta - 1)\langle x, v \rangle v$$

descends to an elliptic isometry of $\mathbf{H}_{\mathbb{C}}^n$ called a *complex reflection*. We note that, unlike reflections in real hyperbolic geometry, the choice of ζ introduces an added degree of freedom in complex reflections, and as such complex reflections may take on any order.

Let $U(n, 1)$ denote the group of unitary matrices preserving the Hermitian form on $\mathbb{C}^{n,1}$, that is

$$U(n, 1) := \{A \in \text{GL}(n+1, \mathbb{C}) : \forall x, y \in \mathbb{C}^{n,1}, \langle Ax, Ay \rangle = \langle x, y \rangle\}.$$

It is clear from the definition of the metric in Equation 2.1 that $U(n, 1)$ acts on $\mathbf{H}_{\mathbb{C}}^n$ by holomorphic isometries and that scalar matrices act trivially. It is a fact that the holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^n$ is exactly $\text{PU}(n, 1) := U(n, 1)/U(1)$, and the full isometry group is generated by $\text{PU}(n, 1)$ and the antiholomorphic involution $z \mapsto \bar{z}$ (conjugation after lifting to a vector in $\mathbb{C}^{n,1}$). Given a matrix $A \in U(n, 1)$ representing a complex hyperbolic isometry, Chen and Greenberg [9] gave a classification of the isometries:

Theorem 1 (Theorem 3.4.1 of [9]). *Let g be an isometry of $\mathbf{H}_{\mathbb{C}}^n$ and let A_g be a lift of g into $U(n, 1)$.*

1. *g is elliptic if and only if A_g is semisimple and all of its eigenvalues have norm 1.*
2. *g is parabolic if and only if A_g is not semisimple and all of its eigenvalues have norm 1.*
3. *g is loxodromic if and only if A_g is semisimple and has exactly $n - 1$ many eigenvalues of norm 1.*

We also say that g as above is a *hyperbolic* isometry if it is loxodromic and there exists a lift A_g in $U(n, 1)$ for which all eigenvalues of A_g are real.

2.1.3 Totally Geodesic Subspaces

A \mathbb{C}^k -plane (or a \mathbb{C} -line when $k = 1$) is projective k -dimensional subspace of $\mathbb{C}\mathbb{P}^n$ that intersects $\mathbf{H}_{\mathbb{C}}^n$, and this intersection is isometrically a copy of $\mathbf{H}_{\mathbb{C}}^k \subset \mathbf{H}_{\mathbb{C}}^n$. A vector $v \in \mathbb{C}^{n,1}$ orthogonal to a \mathbb{C}^{n-1} -plane is called a *polar vector*, and each \mathbb{C}^{n-1} -plane is the fixed point locus of a complex reflection $R_{v,\zeta}$ for some ζ . An \mathbb{R}^k -plane is a projective totally real subspace of $\mathbb{C}\mathbb{P}^n$ that intersects $\mathbf{H}_{\mathbb{C}}^n$. Every \mathbb{R}^n -plane is the fixed point set of a (unique) anti-holomorphic involution called a *real reflection*.

Theorem 2 (Section 3.1.11 in [23]). *Every totally geodesic submanifold in $\mathbf{H}_{\mathbb{C}}^n$ is either a \mathbb{C}^k plane or an \mathbb{R}^k -plane, for $0 \leq k \leq n$.*

This implies that $\mathbf{H}_{\mathbb{C}}^n$ has no totally geodesic real hypersurfaces. In particular, the boundary of a complex hyperbolic manifold is not itself a totally geodesic complex hyperbolic manifold.

2.2 Discrete Subgroups of Lie Groups

For a connected semisimple Lie group G , one can take a maximal compact subgroup K to construct the associated *symmetric space* $X = G/K$ on which G acts transitively and by isometries (see [26] for a standard reference on symmetric spaces). A subgroup Γ of G is *discrete* if it inherits the discrete topology, and such a discrete group is a *lattice* if $\Gamma \backslash X$ has finite volume. Following Sarnak [54], a discrete subgroup Δ of a lattice Γ is *thin* if $\Delta \backslash X$ has infinite volume but Δ and Γ have the same Zariski-closure in G . We say that Γ is *cocompact* if $\Gamma \backslash X$ is compact, and Γ is *non-cocompact* otherwise. Two subgroups $\Gamma_1, \Gamma_2 < G$ are called *commensurable* if there exists a $g \in G$ for which $\Gamma_1 \cap g\Gamma_2g^{-1}$ has finite index in both Γ_1 and $g\Gamma_2g^{-1}$ (in some literature, this is referred to as *commensurable in the wide sense*, but we will have no need for such a distinction). Commensurability is an equivalence relation and cocompactness is a commensurability-invariant.

One key property in the study of lattices is that of arithmeticity (see [38] for a standard reference), which is also a commensurability-invariant. Before defining arithmeticity, we give an intermediate definition:

Definition. Let H be a closed subgroup of $\mathrm{SL}(n, \mathbb{R})$ with only finitely-many components and let H_{\circ} denote the identity component. If there exists $\mathcal{Q} \subset \mathbb{Q}[x_{1,1}, \dots, x_{n,n}]$ such that (1) the variety $\mathrm{Var}(\mathcal{Q})$ is a subgroup of $\mathrm{SL}(n, \mathbb{R})$ and (2) $H_{\circ} = \mathrm{Var}(\mathcal{Q})_{\circ}$,

then we say that H is an *algebraic group defined over \mathbb{Q}* .

Definition. Γ is an *arithmetic subgroup of G* if and only if there exist

- a closed, connected, semisimple subgroup G' of some $\mathrm{SL}(n, \mathbb{R})$ that is defined over \mathbb{Q} ,
- compact normal subgroups K and K' of G_\circ and G' , respectively,
- an isomorphism $\varphi : G_\circ/K \rightarrow G'/K'$,

such that $\varphi(\overline{\Gamma \cap G_\circ}) \cap \overline{G' \cap \mathrm{SL}(n, \mathbb{Z})}$ has finite index in both $\varphi(\overline{\Gamma \cap G_\circ})$ and $\overline{G' \cap \mathrm{SL}(n, \mathbb{Z})}$ (here an overline denotes the images of the respective groups in the appropriate quotient groups).

Example 3. It is clear from the definition that $\mathrm{SL}(n, \mathbb{Z})$ is an arithmetic lattice in $\mathrm{SL}(n, \mathbb{R})$. In particular, when $n = 2$, we obtain the *modular group*, which acts on the upper half-plane model of $\mathbf{H}_{\mathbb{R}}^2$ by Möbius transformations. It is easily seen that any unipotent upper triangular matrix fixes the point at infinity, and so choosing a fundamental domain that includes this point (as in Figure 2.2), one readily sees that the quotient is non-compact (hence $\mathrm{SL}(2, \mathbb{Z})$ is a non-cocompact lattice).

In fact, the existence of unipotent elements is exactly a test for cocompactness of a lattice:

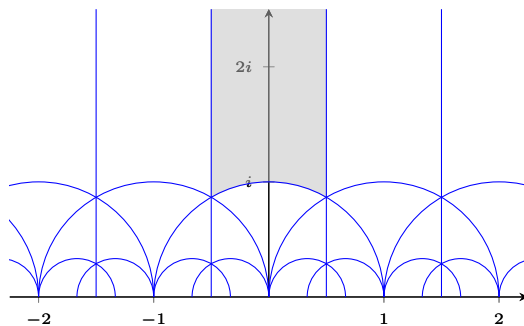


Figure 2.2: Fundamental Domain for the Action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbf{H}_{\mathbb{R}}^2$.

Theorem 4 (Godement’s compactness criterion, Prop 5.3.1 in [38]). *Let $G < \mathrm{SL}(n, \mathbb{R})$ be defined over \mathbb{Q} . Then $G \cap \mathrm{SL}(n, \mathbb{Z})$ is non-cocompact if and only if $G \cap \mathrm{SL}(n, \mathbb{Z})$ contains a (nontrivial) unipotent element.*

Similar to the example of $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$, Borel and Harish-Chandra further showed that arithmetic lattices exist in all (real) linear algebraic groups.

Theorem 5 (Theorem 1 of [4]). *Let $G < \mathrm{GL}(n, \mathbb{C})$ be defined over \mathbb{Q} . Then $G \cap \mathrm{GL}(n, \mathbb{Z})$ is a lattice in $G \cap \mathrm{GL}(n, \mathbb{R})$.*

Example 6. Given \mathcal{O}_d , the ring of integers for the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$, the *Picard modular groups* $\mathrm{SU}(n, 1; \mathcal{O}_d)$ are arithmetic lattices in $\mathrm{SU}(n, 1)$. Let $\{1, \tau\}$ be a \mathbb{Z} -basis for \mathcal{O}_d and consider the following \mathbb{R} -algebra embedding of \mathbb{C} into $M(2, \mathbb{R})$:

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} : \mathbb{C} \rightarrow M(2, \mathbb{R})$$

$$(x + iy) \mapsto \begin{pmatrix} 1 & \frac{1}{2}(\tau + \bar{\tau}) \\ 0 & \frac{1}{2}(\tau - \bar{\tau}) \end{pmatrix}^{-1} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}(\tau + \bar{\tau}) \\ 0 & \frac{1}{2}(\tau - \bar{\tau}) \end{pmatrix}$$

In this way, we exactly have that $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathcal{O}_d) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) \cap M(2, \mathbb{Z})$. This map extends to a map from $\mathrm{GL}(n, \mathbb{C})$ into $\mathrm{GL}(2n, \mathbb{R})$ by applying $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ at each matrix entry, and thus

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SU}(n, 1; \mathcal{O}_d)) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SU}(n, 1)) \cap \mathrm{GL}(2n, \mathbb{Z}).$$

Since $\mathrm{SU}(n, 1)$ is a linear algebraic group defined over \mathbb{Q} , the result follows from the previous theorem of Borel–Harish-Chandra. These lattices also contain non-identity unipotent elements, and are non-cocompact by Godement’s compactness criterion.

Checking arithmeticity in practice often requires the use of some commensurability invariants. One such invariant is the *adjoint trace field*, $\mathbb{Q}(\mathrm{Tr} \mathrm{Ad} \Gamma)$, where $\mathrm{Ad} : \Gamma \rightarrow$

$\mathrm{GL}(\mathfrak{g})$ is the adjoint representation and $\mathrm{Tr Ad} \Gamma = \{\mathrm{Tr Ad}(\gamma) \ : \ \gamma \in \Gamma\}$. The following lemma provides a criterion that is well-suited for checking arithmeticity of lattices in $\mathrm{SU}(n, 1)$.

Lemma 7 (Lemma 4.1 in [40]). *Let E/F be an imaginary quadratic extension of a totally real number field, and let $H = (h_{ij})$ be a Hermitian matrix of signature $(n, 1)$ with coefficients in E . A lattice $\Gamma < \mathrm{SU}(H, \mathcal{O}_E)$ is arithmetic if and only if, for every $\varphi \in \mathrm{Gal}(E/\mathbb{Q})$:*

1. $\varphi(\overline{h_{ij}}) = \overline{\varphi(h_{ij})}$, and
2. if φ does not restrict to the identity on $\mathbb{Q}(\mathrm{Tr Ad} \Gamma)$, then ${}^\varphi H = (\varphi(h_{ij}))$ has signature $(n + 1, 0)$ or $(0, n + 1)$.

In a very strong sense, most lattices are arithmetic. Given a simple Lie group G , Margulis [34] showed that non-arithmetic lattices can only exist when G has real rank 1; that is, up to finite index, $G = \mathrm{Isom}(X)$ for a hyperbolic space X . Moreover, the combined work of Gromov–Schoen [25] and Corlette [10] implies that non-arithmetic lattices can only exist when $X = \mathbf{H}_{\mathbb{R}}^n$ or $X = \mathbf{H}_{\mathbb{C}}^n$. Due to their exceptional nature, it is a major challenge to find non-arithmetic lattices in real and complex hyperbolic isometry groups.

Real hyperbolic lattices and their arithmetic properties have been very extensively studied; see [33] and the references therein for a rather comprehensive resource. Notably, Gromov and Piatetski-Shapiro in [24] showed the existence of non-arithmetic lattices in every dimension via a particular construction that they call “interbreeding” (also called “hybridization”).

To contrast, very little is known about complex hyperbolic lattices, especially non-arithmetic lattices. At present, non-arithmetic examples are only known in di-

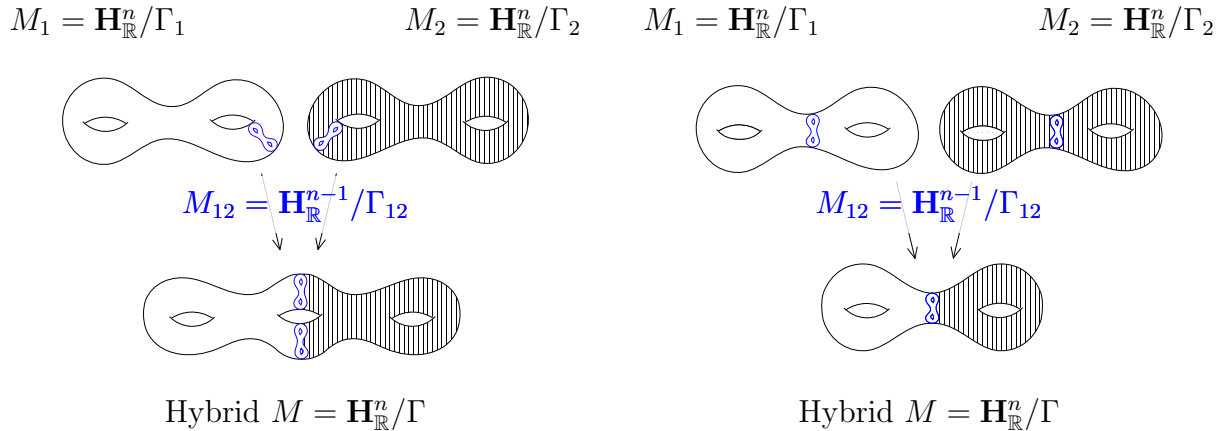


Figure 2.3: M_{12} Does Not Separate

Figure 2.4: M_{12} Separates

mensions 2 and 3 (see [46], [13]), and it is an open question as to whether or not they exist at all in higher dimensions.

2.3 Gromov–Piatetski-Shapiro Hybrids

The Gromov–Piatetski-Shapiro (GPS) hybridization construction introduced in [24] is as follows: Begin with two arithmetic lattices Γ_1, Γ_2 in $\text{PO}(n, 1)$ with common sublattice $\Gamma_{12} \leq \text{PO}(n - 1, 1)$ (for simplicity in exposition, we’ll assume these lattices are all torsion-free). Geometrically, this yields two finite-volume hyperbolic manifolds $M_i = \mathbf{H}_{\mathbb{R}}^n / \Gamma_i$ each containing totally geodesic submanifolds isometric to $M_{12} = \mathbf{H}_{\mathbb{R}}^{n-1} / \Gamma_{12}$. Cut and glue (a single connected component of) $M_1 - M_{12}$ and $M_2 - M_{12}$ along M_{12} to obtain the *hybrid manifold* $M = M_1 \sqcup_{M_{12}} M_2$. Since M_{12} is totally geodesic, the metric is still well-defined (and hyperbolic) at the glue locus, so $M = \mathbf{H}_{\mathbb{R}}^n / \Gamma$ is a finite-volume hyperbolic manifold, and the resulting lattice Γ is the *hybrid lattice*. Algebraically, Γ is the amalgamated free product $\Gamma_1 *_{\Gamma_{12}} \Gamma_2$, and in particular, Γ is generated by Γ_1 and Γ_2 . Figures 2.3 and 2.4 provide a geometric visualization for the construction.

2.4 Complex Hyperbolic Hybrid Construction

The lack of totally geodesic real hypersurfaces in $\mathbf{H}_{\mathbb{C}}^n$ poses an issue to find a suitable complex-hyperbolic analog to the GPS construction. Hunt proposed one possible analog (see the references contained within [48]) where one starts with two arithmetic lattices Γ_1, Γ_2 in $\mathrm{PU}(n, 1)$, embeddings $\iota_i : \mathrm{PU}(n, 1) \hookrightarrow \mathrm{PU}(n + 1, 1)$, and totally geodesic hypersurfaces Σ_1 and Σ_2 such that

1. Σ_1 and Σ_2 are orthogonal,
2. $\iota_j(\Gamma_j)$ stabilizes Σ_j , and
3. $\iota_1(\Gamma_1) \cap \iota_2(\Gamma_2)$ is a lattice in $\mathrm{PU}(n - 1, 1)$.

The resulting hybrid subgroup is then $H(\Gamma_1, \Gamma_2) := \langle \iota_1(\Gamma_1), \iota_2(\Gamma_2) \rangle$.

As Paupert [48] showed, hybrids in $\mathrm{PU}(2, 1)$ may be non-discrete. As further explored by Paupert and the author in Chapter 3 ([49]), even accounting for discreteness does not guarantee that the resulting subgroup is a lattice or an amalgamated free product as in the GPS construction.

With this in mind, we present a slightly more general and flexible notion of a hybrid.

Definition. Let $\Gamma_1, \Gamma_2 < \mathrm{PU}(n, 1)$ be lattices. We define a *hybrid of Γ_1, Γ_2* to be any group $H(\Gamma_1, \Gamma_2)$ generated by discrete subgroups $\Lambda_1, \Lambda_2 < \mathrm{PU}(n + 1, 1)$ stabilizing totally geodesic hypersurfaces Σ_1, Σ_2 (respectively) such that

1. Σ_1 and Σ_2 are orthogonal,
2. $\Gamma_i = \Lambda_i|_{\Sigma_i}$, and
3. $\Lambda_1 \cap \Lambda_2$ is a lattice in $\mathrm{PU}(n - 1, 1)$.

Remark. With this new definition, the resulting hybrid is not unique. In particular, given any Λ_i as above and any non-identity $A \in \text{PU}(n+1, 1) - \Lambda_i$ fixing Σ_i pointwise, we have $\Gamma_i = \Lambda_i|_{\Sigma_i} = \langle \Lambda_i, A \rangle|_{\Sigma_i}$.

Remark. The hybrids explored in [48] and [49] are still hybrids in this new sense as well, taking $\Lambda_j = \iota_j(\Gamma_j)$.

Chapter 3

HYBRID LATTICES AND THIN SUBGROUPS OF PICARD MODULAR GROUPS

The following chapter explores hybrid subgroups in the context of some well-known arithmetic lattices in $\mathrm{PU}(2, 1)$, the *Picard modular groups*. This chapter is joint work with Julien Paupert and, at present, is in review for publication. With his permission, it has been faithfully reproduced with only minor modifications to meet formatting guidelines. It should also be noted that the presentation attributed to Mark–Paupert [36] in Section 3.5 for $\mathrm{PU}(2, 1; \mathcal{O}_7)$ is missing a relation; our result is unaffected by this fact.

Abstract

We consider a certain hybridization construction which produces a subgroup of $\mathrm{PU}(n, 1)$ from a pair of lattices in $\mathrm{PU}(n - 1, 1)$. Among the Picard modular groups $\mathrm{PU}(2, 1, \mathcal{O}_d)$, we show that the hybrid of pairs of Fuchsian subgroups $\mathrm{PU}(1, 1, \mathcal{O}_d)$ is a lattice when $d = 1$ and $d = 7$, and a geometrically infinite thin subgroup when $d = 3$, that is an infinite-index subgroup with the same Zariski-closure as the full lattice.

3.1 Introduction

Lattices in rank 1 real (semi)simple Lie groups are still far from understood. A key notion is that of *arithmetic lattice* which we will not define properly here but note that by a famous result of Margulis a lattice in such a Lie group is arithmetic if and only if it has infinite index in its commensurator.

Margulis' celebrated arithmeticity theorem states that every lattice of a simple real Lie group G is arithmetic whenever the real rank of G is at least two. Thus

non-arithmetic lattices can only exist in real rank one, that is when the associated symmetric space is a hyperbolic space. In real hyperbolic space, where the Lie group is $\mathrm{PO}(n, 1)$, Gromov and Piatetski-Shapiro produced in [24] a construction yielding non-arithmetic lattices in $\mathrm{PO}(n, 1)$ for all $n \geq 2$ (see below for more details), in fact producing in each dimension infinitely many non-commensurable lattices, both cocompact and non-cocompact. In quaternionic hyperbolic spaces (and the Cayley octave plane), work of Corlette and Gromov-Schoen implies as in the higher rank case that all lattices are arithmetic.

The case of complex hyperbolic spaces, where the associated Lie group is $\mathrm{PU}(n, 1)$, is much less understood. Non-arithmetic lattices in $\mathrm{PU}(2, 1)$ were first constructed by Mostow in 1980 in [40], and subsequently by Deligne-Mostow and Mostow as monodromy groups of certain hypergeometric functions in [12] and [41], following pioneering work of Picard. More recently, Deraux, Parker and the first author constructed new families of non-arithmetic lattices in $\mathrm{PU}(2, 1)$ by considering groups generated by certain triples of complex reflections (see [16], [17]). Taken together, these constructions yield 22 commensurability classes of non-arithmetic lattices in $\mathrm{PU}(2, 1)$, and only 2 commensurability classes in $\mathrm{PU}(3, 1)$. The latter two are noncocompact; one is a Deligne-Mostow lattice and the other was constructed by Couwenberg-Heckman-Looijenga in 2005 and recently found to be non-arithmetic by Deraux, [13]. Major open questions in this area remain the existence of non-arithmetic lattices in $\mathrm{PU}(n, 1)$ for $n \geq 4$, as well as the number (or finiteness thereof) of commensurability classes in each dimension.

The Gromov–Piatetski-Shapiro construction, which they call interbreeding of 2 arithmetic lattices (now often referred to as hybridization), produces a lattice $\Gamma < \mathrm{PO}(n, 1)$ from 2 lattices Γ_1 and Γ_2 in $\mathrm{PO}(n, 1)$ which have a common sublattice $\Gamma_{12} < \mathrm{PO}(n - 1, 1)$. Geometrically, this provides two hyperbolic n -manifolds $V_1 =$

$\Gamma_1 \backslash H_{\mathbb{R}}^n$ and $V_2 = \Gamma_2 \backslash H_{\mathbb{R}}^n$ with a hyperbolic $(n-1)$ -manifold V_{12} which is isometrically embedded in V_1 and V_2 as a totally geodesic hypersurface. This allows one to produce the hybrid manifold V by gluing $V_1 - V_{12}$ and $V_2 - V_{12}$ along V_{12} (more precisely, in case V_{12} separates V_1 and V_2 , by gluing $V_1^+ - V_{12}$ and $V_2^+ - V_{12}$ along V_{12} , with V_i^+ a connected component of $V_i - V_{12}$). The resulting manifold is also hyperbolic because the gluing took place along a totally geodesic hypersurface, and its fundamental group Γ is therefore a lattice in $\text{PO}(n, 1)$. The main point is then that if Γ_1 and Γ_2 are both arithmetic but non-commensurable, their hybrid Γ is non-arithmetic. Note that the resulting hybrid Γ is algebraically an amalgamated free product of Γ_1 and Γ_2 over Γ_{12} (say, in the case where V_{12} separates both V_1 and V_2), and in all cases is generated by its sublattices Γ_1 and Γ_2 .

It is not straightforward to adapt this construction to construct lattices in $\text{PU}(n, 1)$, the main difficulty being that there do not exist in complex hyperbolic space any totally geodesic real hypersurfaces. In fact, it has been a famous open question since the work of Gromov–Piatetski-Shapiro to find some analogous construction in $\text{PU}(n, 1)$. Hunt proposed the following construction (see [48] and references therein). Start with 2 arithmetic lattices Γ_1 and Γ_2 in $\text{PU}(n, 1)$, and suppose that one can embed them in $\text{PU}(n+1, 1)$ in such a way that (a) each stabilizes a totally geodesic $H_{\mathbb{C}}^n \subset H_{\mathbb{C}}^{n+1}$ (b) these 2 complex hypersurfaces are orthogonal, and (c) the intersection of the embedded Γ_i is a lattice in the corresponding $\text{PU}(n-1, 1)$. The resulting hybrid $\Gamma = H(\Gamma_1, \Gamma_2)$ is then defined as the subgroup of $\text{PU}(n+1, 1)$ generated by the images of Γ_1 and Γ_2 . (See the end of Section 3.2 for a more detailed and concrete description when $n = 2$).

It is not clear when, if ever, such a group has any nice properties. One expects in general the hybrid group to be non-discrete, and in fact the first author showed in [48] that this happens infinitely often among hybrids in $\text{PU}(2, 1)$ of pairs of Fuchsian

triangle subgroups of $\mathrm{PU}(1, 1)$. It was observed there that one can easily arrange for the hybrid to be discrete by arranging for the two subgroups Γ_1, Γ_2 to already belong to a known lattice. But even in the simplest case of arithmetic cusped lattices (where the matrix entries are all in \mathcal{O}_d , the ring of integers of $\mathbb{Q}[i\sqrt{d}]$ for some squarefree $d \geq 1$), it was not known whether the discrete hybrid Γ could ever be a sublattice of the corresponding Picard modular group $\Gamma(d) = \mathrm{PU}(2, 1, \mathcal{O}_d)$, as opposed to an infinite-index (discrete) subgroup of $\Gamma(d)$. Following Sarnak ([54]) we will call *thin subgroup* of a lattice Γ any infinite-index subgroup having the same Zariski-closure as Γ .

In this note we show that in fact both behaviors can occur, even among this simplest class of hybrids of sublattices of the Picard modular groups $\Gamma(d)$. More precisely, we consider for $d = 3, 1, 7$ the hybrid subgroup $H(d)$ defined as the hybrid of two copies of $\mathrm{SU}(1, 1, \mathcal{O}_d)$ inside the Picard modular group $\mathrm{PU}(2, 1, \mathcal{O}_d)$ (when $d = 7$ we consider in fact for simplicity the hybrid of two copies of $\mathrm{U}(1, 1, \mathcal{O}_7)$). These specific values of d are those for which a presentation of $\mathrm{PU}(2, 1, \mathcal{O}_d)$ is known (by [20], [19] and [36]). Our main results can be summarized as follows (combining Theorems 12, 23, and 26 and Propositions 28 and 29).

- Theorem 8.**
1. *The hybrid $H(3)$ is a thin subgroup of the Eisenstein-Picard lattice $\mathrm{PU}(2, 1, \mathcal{O}_3)$. It has full limit set $\partial_\infty \mathbb{H}_\mathbb{C}^2 \simeq S^3$ and is therefore geometrically infinite.*
 2. *The hybrid $H(1)$ has index 2 in the Gauss-Picard lattice $\mathrm{PU}(2, 1, \mathcal{O}_1)$.*
 3. *The hybrid $H(7)$ is the full Picard lattice $\mathrm{PU}(2, 1, \mathcal{O}_7)$.*

Remark. (a) We also give analogous results for two related hybrids $H'(3)$ and $H'(1)$ in Corollaries 20 and 24. In terms of Fuchsian triangle groups these groups

are defined as the hybrids of two copies of the (orientation-preserving) triangle groups $(2, 6, \infty)$ and $(2, 4, \infty)$ respectively, as opposed to $(3, \infty, \infty) \simeq \text{PU}(1, 1, \mathcal{O}_3)$ and $(2, \infty, \infty) \simeq \text{PU}(1, 1, \mathcal{O}_1)$ (so, replacing the elliptic generator by one of its square roots). An interesting feature of $H'(3)$ is that it has infinite index in its normal closure in $\Gamma(3)$, whereas all other hybrids we consider are normal in $\Gamma(d)$.

- (b) In all cases we also show that the hybrid Γ is not an amalgamated free product of Γ_1 and Γ_2 over their intersection. In case Γ is itself a lattice this follows from general considerations of cohomological dimension, and for $H(3)$ and $H'(3)$ we show this by finding sufficiently many relations among the generators for Γ , see Corollary 15.
- (c) One of the main geometric difficulties in analyzing these groups is understanding the parabolic subgroups. By construction the generators contain a pair of (opposite) parabolic isometries (as well as an elliptic isometry when $d = 3$, two elliptic isometries when $d = 1$, and two elliptic and two loxodromic isometries when $d = 7$), however it seems hard in general to determine the rank of the parabolic subgroups of the hybrid. In the cases where the hybrid is a lattice we obtain indirectly that the parabolic subgroups must have full rank, but in the thin subgroup case we do not know what this rank is.
- (d) The parabolic isometries appearing in the generators for our hybrids are by construction vertical Heisenberg translations, since they preserve a complex line (see Section 3.2). It turns out that Falbel ([18]) and Falbel-Wang ([21]) studied a group formally similar to our hybrid $H(3)$, obtained by completely different methods, namely by finding all irreducible representations of the figure-eight knot group Γ_8 into $\text{PU}(2, 1)$ with unipotent boundary holonomy. Falbel showed

in [18] that there are exactly 3 such representations, one of which has image contained in $\Gamma(3) = \text{PU}(2, 1, \mathcal{O}_3)$ and the two others in $\Gamma(7) = \text{PU}(2, 1, \mathcal{O}_7)$. These are all generated by a pair of opposite horizontal Heisenberg translations. The image of the former representation is shown in [18] and [21] to be, like our hybrids $H(3)$ and $H'(3)$, a thin subgroup of $\Gamma(3)$ with full limit set, whereas the images of the latter two representations are shown in [14] to have non-empty domain of discontinuity (and hence have infinite index in $\Gamma(7)$). We were inspired by some of the arguments of [18] and [21].

- (e) Kapovich found in [28] the first examples of infinite-index normal subgroups of lattices in $\text{PU}(2, 1)$, among a family of four lattices first constructed by Livné in his thesis (and predating the term *thin subgroup*). Parker showed in [45] (sections 6 and 7) that this description could be extended to the Eisenstein-Picard modular group $\text{PU}(2, 1, \mathcal{O}_3)$, and that some of Kapovich's results extended to that case as well. It was shown to us by an anonymous referee that our hybrid $H(3)$ is in fact commensurable to the infinite-index normal subgroup of $\text{PU}(2, 1, \mathcal{O}_3)$ obtained in this way.
- (f) Discrete groups generated by opposite parabolic subgroups have been studied in higher rank by Oh, Benoist-Oh and others. A conjecture of Margulis states that if G is a semisimple real algebraic group of rank at least 2 and Γ a discrete Zariski-dense subgroup containing irreducible lattices in two opposite horospherical subgroups, then Γ is an arithmetic lattice in G . Oh showed in [43] that this holds when G is a split real Lie group, Benoist-Oh extended this in [3] to the case of $G = \text{SL}(3, \mathbb{R})$, and very recently Benoist-Miquel treated the general case in [2].

The paper is organized as follows. In section 2 we review basic facts about complex

hyperbolic space, its isometries, subspaces and boundary at infinity. In Sections 3,4,5 we consider each of the hybrids $H(3)$, $H(1)$ and $H(7)$ respectively. In section 6 we review and apply basic facts about limit sets and geometrical finiteness to the non-lattice hybrid $H(3)$. We would like to thank Elisha Falbel for pointing out a simplification of the proof of Theorem 12, and an anonymous referee for several useful comments.

3.2 Complex Hyperbolic Space, Isometries and Boundary at Infinity

We give a brief summary of basic definitions and facts about complex hyperbolic geometry, and refer the reader to [23], [9] or [44] for more details.

3.2.1 Projective Models of $\mathbf{H}_{\mathbb{C}}^n$

Denote $\mathbb{C}^{n,1}$ the vector space \mathbb{C}^{n+1} endowed with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n, 1)$. Define $V^- = \{Z \in \mathbb{C}^{n,1} | \langle Z, Z \rangle < 0\}$ and $V^0 = \{Z \in \mathbb{C}^{n,1} | \langle Z, Z \rangle = 0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}P^n$ denote projectivization. One may then define complex hyperbolic n -space $\mathbf{H}_{\mathbb{C}}^n$ as $\pi(V^-) \subset \mathbb{C}P^n$, with the distance d (corresponding to the Bergman metric) given by:

$$\cosh^2 \frac{1}{2} d(\pi(X), \pi(Y)) = \frac{|\langle X, Y \rangle|^2}{\langle X, X \rangle \langle Y, Y \rangle} \quad (3.1)$$

The boundary at infinity $\partial \mathbf{H}_{\mathbb{C}}^n$ is then naturally identified with $\pi(V_0)$. Different Hermitian forms of signature $(n, 1)$ give rise to different models of $\mathbf{H}_{\mathbb{C}}^n$. Two of the most common choices are the Hermitian forms corresponding to the Hermitian matrices $H_1 = \text{Diag}(1, \dots, 1, -1)$ and:

$$H_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (3.2)$$

In the first case, $\pi(V^-) \subset \mathbb{C}P^n$ is the unit ball of \mathbb{C}^n , seen in the affine chart $\{z_{n+1} = 1\}$ of $\mathbb{C}P^n$, hence the model is called the *ball model* of $\mathbf{H}_{\mathbb{C}}^n$. In the second case, we obtain the *Siegel model* of $\mathbf{H}_{\mathbb{C}}^n$, which is analogous to the upper-half space model of $\mathbf{H}_{\mathbb{R}}^n$ and is likewise well-adapted to parabolic isometries fixing a specific boundary point. We will mostly use the Siegel model in this paper and will give a bit more details about it below. We will use the following *Cayley transform* J to pass from the ball model to the Siegel model (see [47]); a key point for us is that $J \in \mathrm{GL}(3, \mathbb{Z})$, hence conjugating by J preserves integrality of matrix entries.

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \quad (3.3)$$

3.2.2 Isometries

It is clear from (3.1) that $\mathrm{PU}(n, 1)$ acts by isometries on $\mathbf{H}_{\mathbb{C}}^n$, denoting $\mathrm{U}(n, 1)$ the subgroup of $\mathrm{GL}(n+1, \mathbb{C})$ preserving the Hermitian form, and $\mathrm{PU}(n, 1)$ its image in $\mathrm{PGL}(n+1, \mathbb{C})$. It turns out that $\mathrm{PU}(n, 1)$ is the group of holomorphic isometries of $\mathbf{H}_{\mathbb{C}}^n$, and the full group of isometries is $\mathrm{PU}(n, 1) \times \mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor corresponds to a real reflection (see below). A holomorphic isometry of $\mathbf{H}_{\mathbb{C}}^n$ is of one of the following three types:

- *elliptic* if it has a fixed point in $\mathbf{H}_{\mathbb{C}}^n$
- *parabolic* if it has (no fixed point in $\mathbf{H}_{\mathbb{C}}^n$ and) exactly one fixed point in $\partial\mathbf{H}_{\mathbb{C}}^n$

- *loxodromic*: if it has (no fixed point in $\mathbb{H}_{\mathbb{C}}^n$ and) exactly two fixed points in $\partial\mathbb{H}_{\mathbb{C}}^n$

3.2.3 Totally Geodesic Subspaces

A *complex k -plane* is a projective k -dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ non-trivially (so, it is an isometrically embedded copy of $\mathbb{H}_{\mathbb{C}}^k \subset \mathbb{H}_{\mathbb{C}}^n$). Complex 1-planes are usually called *complex lines*. If $L = \pi(\tilde{L})$ is a complex $(n-1)$ -plane, any $v \in \mathbb{C}^{n+1} - \{0\}$ orthogonal to \tilde{L} is called a *polar vector* for L .

A *real k -plane* is the projective image of a totally real $(k+1)$ -subspace W of \mathbb{C}^{n+1} , i. e. a $(k+1)$ -dimensional real linear subspace such that $\langle v, w \rangle \in \mathbb{R}$ for all $v, w \in W$. We will usually call real 2-planes simply real planes, or \mathbb{R} -planes. Every real n -plane in $\mathbb{H}_{\mathbb{C}}^n$ is the fixed-point set of an antiholomorphic isometry of order 2 called a *real reflection* or \mathbb{R} -reflection. The prototype of such an isometry is the map given in affine coordinates by $(z_1, \dots, z_n) \mapsto (\bar{z}_1, \dots, \bar{z}_n)$; this is an isometry provided that the Hermitian form has real coefficients.

We will need to distinguish between the following types of parabolic isometries. A parabolic isometry is called *unipotent* if it has a unipotent lift to $U(n, 1)$. In dimensions $n > 1$, unipotent isometries are either *2-step* (also called *vertical*) or *3-step* (also called *horizontal*), according to whether the minimal polynomial of their unipotent lift is $(X-1)^2$ or $(X-1)^3$ (see section 3.4 of [9]). Another way to distinguish these two types is that 2-step unipotent isometries preserve a complex line (in fact, any complex line through their fixed point) but no real plane, whereas 3-step unipotent isometries preserve a real plane (in fact, an entire *fan* of these, see section 2.3 of [50]) but no complex line.

3.2.4 Boundary at Infinity and Heisenberg Group

In the Siegel model associated to the Hermitian form given by the matrix H_2 in (3.2), $\mathbb{H}_{\mathbb{C}}^n$ can be parametrized by $\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$ as follows, denoting as before by π the projectivization map: $\mathbb{H}_{\mathbb{C}}^n = \{\pi(\psi(z, t, u)) \mid z \in \mathbb{C}^{n-1}, t \in \mathbb{R}, u \in \mathbb{R}^+\}$, where:

$$\psi(z, t, u) = \begin{pmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{pmatrix} \quad (3.4)$$

With this parametrization the boundary at infinity $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$ corresponds to the one-point compactification:

$$\{\pi(\psi(z, t, 0)) \mid z \in \mathbb{C}^{n-1}, t \in \mathbb{R}\} \cup \{\infty\}$$

where $\infty = \pi((1, 0, \dots, 0)^T)$. The coordinates $(z, t, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$ are called the *horospherical coordinates* of the point $\pi(\psi(z, t, u)) \in \mathbb{H}_{\mathbb{C}}^n$.

The punctured boundary $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n - \{\infty\}$ is then naturally identified to the *generalized Heisenberg group* $\text{Heis}(\mathbb{C}, n)$, defined as the set $\mathbb{C}^{n-1} \times \mathbb{R}$ equipped with the group law:

$$(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\text{Im}(z_1 \cdot \bar{z}_2))$$

where \cdot denotes the usual Euclidean dot-product on \mathbb{C}^{n-1} . This is the classical 3-dimensional Heisenberg group when $n = 2$. The identification of $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n - \{\infty\}$ with $\text{Heis}(\mathbb{C}, n)$ is given by the simply-transitive action of $\text{Heis}(\mathbb{C}, n)$ on $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n - \{\infty\}$, where the element $(z_1, t_1) \in \text{Heis}(\mathbb{C}, n)$ acts on the vector $\psi(z_2, t_2, 0)$ by left-multiplication by the following *Heisenberg translation* matrix in $U(n, 1)$:

$$T_{(z_1, t_1)} = \begin{pmatrix} 1 & -z_1^* & (-|z_1|^2 + it_1)/2 \\ 0 & I_{n-1} & z_1 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.5)$$

In other words: $T_{(z_1, t_1)}\psi(z_2, t_2, 0) = \psi(z_1 + z_2, t_1 + t_2 + 2\text{Im}(z_1 \cdot \bar{z}_2), 0)$.

In the above terminology, the unipotent isometry (given by the projective action of) $T_{(z_1, t_1)}$ is 2-step (or vertical) if $z_1 = 0$ and 3-step (horizontal) otherwise.

3.2.5 The Hybridization Construction

We will first embed the pair of Fuchsian groups into $\text{SU}(2, 1)$ in the ball model of $H_{\mathbb{C}}^2$; there, two preferred orthogonal complex lines L_1 and L_2 are given by (the coordinate axes in the standard affine chart) $L_1 = \pi(\text{Span}(e_1, e_3))$ and $L_2 = \pi(\text{Span}(e_2, e_3))$, where (e_1, e_2, e_3) denotes the canonical basis of \mathbb{C}^3 and $\pi : \mathbb{C}^3 - \{0\} \rightarrow \mathbb{C}P^2$ the projectivization map. These intersect at the origin $O = \pi(e_3)$.

We will embed $\text{SU}(1, 1)$ in the stabilizer of each of these complex lines in the obvious block matrix form, namely via the injective homomorphisms:

$$\begin{aligned} \iota_1 : \text{SU}(1, 1) &\longrightarrow \text{SU}(2, 1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \iota_2 : \text{SU}(1, 1) &\longrightarrow \text{SU}(2, 1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \end{aligned} \tag{3.7}$$

In the notation from the introduction, given two lattices Γ_1, Γ_2 in $\text{SU}(1, 1)$, we consider the hybrid $H(\Gamma_1, \Gamma_2) = \langle \iota_1(\Gamma_1), \iota_2(\Gamma_2) \rangle < \text{PU}(2, 1)$.

3.3 A Hybrid Subgroup of the Eisenstein-Picard Modular Group $\mathrm{PU}(2, 1, \mathcal{O}_3)$

Denoting $\omega = \frac{-1+i\sqrt{3}}{2}$, the following matrices generate $\mathrm{SU}(1, 1; \mathcal{O}_3)$ in the disk model of $\mathbb{H}_{\mathbb{C}}^1$:

$$E = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad U = \begin{pmatrix} 1+i\sqrt{3} & -i\sqrt{3} \\ i\sqrt{3} & 1-i\sqrt{3} \end{pmatrix}, \quad I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that $\mathrm{PSU}(1, 1; \mathcal{O}_3)$ acts on the unit disk as (the orientation-preserving subgroup of) a $(3, \infty, \infty)$ triangle group. (An anonymous referee pointed out that we had omitted I in a previous version of the paper).

We consider the hybrid group $H(3) = H(\mathrm{SU}(1, 1; \mathcal{O}_3), \mathrm{SU}(1, 1; \mathcal{O}_3))$, which by definition is generated by $\iota_j(E), \iota_j(U)$ and $\iota_j(I)$ for $j = 1, 2$. By the following observation it will suffice for our purposes to study the subgroup $\tilde{H}(3) = \langle \iota_1(E), \iota_2(E), \iota_1(U), \iota_2(U) \rangle$, which is a hybrid of two copies of $\mathrm{PSU}(1, 1; \mathcal{O}_3)$.

Lemma 9. *$\tilde{H}(3)$ is normal in $H(3)$, with index dividing 4.*

Proof. Since the $\iota_j(E), \iota_j(I)$ are diagonal they all commute; moreover for each $j = 1, 2$ $\iota_j(I)$ commutes with $\iota_j(E), \iota_j(U)$. It suffices therefore to show that $\iota_j(I)$ conjugates $\iota_{3-j}(U)$ into $\tilde{H}(3)$ for $j = 1, 2$; a straightforward computation gives: $\iota_j(I)\iota_{3-j}(U)\iota_j(I) = \iota_{3-j}((EUE)^{-1})$. Therefore $\tilde{H}(3)$ is normal in $H(3)$; the quotient group is a quotient of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as it is generated by the images of $\iota_1(-I)$ and $\iota_2(-I)$ which have order 2 and commute. \square

It will be more convenient for us to work in the Siegel model, in other words to conjugate by the Cayley transform J given in (3.3); we will abuse notation slightly by denoting again $H(3), \tilde{H}(3)$ the conjugates $J^{-1}H(3)J, J^{-1}\tilde{H}(3)J$. Concretely, we

consider $\tilde{H}(3) = \langle E_1, U_1, E_2, U_2 \rangle$, with:

$$E_1 = J^{-1}\iota_1(E)J = \begin{pmatrix} \omega^2 & \omega^2 - 1 & \omega + 2 \\ i\sqrt{3} & 1 + i\sqrt{3} & \omega^2 - 1 \\ i\sqrt{3} & i\sqrt{3} & \omega^2 \end{pmatrix}, \quad U_1 = J^{-1}\iota_1(U)J = \begin{pmatrix} 1 & 0 & i\sqrt{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_2 = J^{-1}\iota_2(E)J = \begin{pmatrix} \omega^2 & -i\sqrt{3} & i\sqrt{3} \\ \omega + 2 & 1 + i\sqrt{3} & -i\sqrt{3} \\ \omega + 2 & \omega + 2 & \omega^2 \end{pmatrix}, \quad U_2 = J^{-1}\iota_2(U)J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i\sqrt{3} & 0 & 1 \end{pmatrix}.$$

Remark. Since E_1, E_2 are both regular elliptic of order 3 with the same eigenspaces, they are either equal or inverse of each other. It turns out that $E_2 = E_1^{-1}$ in $\text{PU}(2, 1)$ (the matrices satisfy $E_2 = \omega E_1^{-1}$). We will therefore omit the generator E_2 from now on.

In [20] the authors determine that the Eisenstein-Picard modular group $\text{PU}(2, 1; \mathcal{O}_3)$ has presentation:

$$\text{PU}(2, 1; \mathcal{O}_3) = \langle P, Q, R \mid R^2, (QP^{-1})^6, PQ^{-1}RQP^{-1}R, P^3Q^{-2}, (RP)^3 \rangle, \quad \text{where}$$

$$P = \begin{pmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

A straightforward computation gives the following:

Lemma 10. *The generators for the hybrid $\tilde{H}(3)$ can be expressed in terms of the*

Falbel-Parker generators for $\mathrm{PU}(2, 1; \mathcal{O}_3)$ as follows:

$$\begin{aligned} U_1 &= Q^2, \\ U_2 &= RQ^2R, \\ E_1 &= P^2(RQ^2)^2P^{-2}. \end{aligned}$$

Lemma 11. *The hybrid $\tilde{H}(3)$ is a normal subgroup of $\mathrm{PU}(2, 1; \mathcal{O}_3)$.*

Proof. It suffices to check that generators of $\mathrm{PU}(2, 1; \mathcal{O}_3)$ conjugate generators of $\tilde{H}(3)$ into $\tilde{H}(3)$. Straightforward computations give the following:

$$\begin{aligned} P^{-1}U_1P &= U_1 & P^{-1}U_2P &= U_1^{-1}E_1 & P^{-1}E_1P &= U_2^{-1}E_1^{-1}U_1 \\ Q^{-1}U_1Q &= U_1 & Q^{-1}U_2Q &= U_1^{-1}E_1 & Q^{-1}E_1Q &= U_2U_1 \\ R^{-1}U_1R &= U_2 & R^{-1}U_2R &= U_1 & R^{-1}E_1R &= E_1^{-1} \end{aligned}$$

□

We can then form the quotient group $G = \mathrm{PU}(2, 1; \mathcal{O}_3)/\tilde{H}(3)$, which by Lemma 10 has presentation:

$$G = \mathrm{PU}(2, 1; \mathcal{O}_3)/\tilde{H}(3) = \left\langle P, Q, R \left| \begin{array}{l} R^2, (QP^{-1})^6, PQ^{-1}RQP^{-1}R, \\ P^3Q^{-2}, (RP)^3, Q^2 \end{array} \right. \right\rangle.$$

(Note that the relation Q^2 makes the other three relators corresponding to the generators of $\tilde{H}(3)$ superfluous). The Tietze transformation $a = PQ^{-1}$, $b = Q$, $c = R$, yields the following presentation for G :

$$G = \langle a, b, c \mid c^2, a^6, [a, c], (ab)^3, (cab)^3, b^2 \rangle$$

Note that this is a quotient of an extension of the $(2, 3, 6)$ triangle group:

Theorem 12. *The hybrid $H(3)$ has infinite index in $\mathrm{PU}(2, 1, \mathcal{O}_3)$.*

Proof. Note that $G/\langle\langle c \rangle\rangle$ is the $(2, 3, 6)$ -triangle group, hence infinite. Therefore G is also infinite, in other words $\tilde{H}(3)$ has infinite index in $\mathrm{PU}(2, 1, \mathcal{O}_3)$. By Lemma 9, $H(3)$ also has infinite index in $\mathrm{PU}(2, 1, \mathcal{O}_3)$. \square

Corollary 13. *The hybrid $H(3)$ is a thin subgroup of $\mathrm{PU}(2, 1, \mathcal{O}_3)$.*

Proof. The only additional statement is that $H(3)$ is Zariski-dense in $\mathrm{PU}(2, 1)$, which is simple to see in rank 1, as it reduces essentially to irreducibility. Indeed, by [9] if a discrete subgroup Γ is not Zariski-dense then it preserves a strict subspace of $\mathbb{H}_{\mathbb{C}}^2$ or it fixes a point on $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^2$. This is easily seen not to be the case, as E_1 does not preserve the unique complex line preserved by both U_1 and U_2 . (This also follows from the fact that $H(3)$ has full limit set). \square

We conclude this section with a few remarks about the algebraic structure of the hybrid $H(3)$. We do not know a complete presentation for $H(3)$, in fact it may be non-finitely presented as far as we know (see [29] and Proposition 4.2 of [21]). The following observations are obtained by direct computation using the generators in matrix form.

Lemma 14. *The following relations hold between the generators E_1, U_1, U_2 for $H(3)$:*

$$E_1^3 = (U_1 U_2)^3 = (E_1 U_1^{-1} U_2)^3 = (E_1 U_2 U_1^{-1})^3 = (E_1^{-1} U_1 U_2^{-1})^3 = (E_1^{-1} U_2^{-1} U_1)^3 = 1.$$

Corollary 15. *The hybrid $H(3)$ has finite abelianization; in particular it is not isomorphic to the amalgamated product of $i_1(\mathrm{SU}(1, 1, \mathcal{O}_3))$ and $i_2(\mathrm{SU}(1, 1, \mathcal{O}_3))$ over their intersection.*

Proof. Observe that by Lemma 14, the following relations hold in the abelianization $H(3)^{ab}$ (we slightly abuse notation by using the same symbol for elements of $H(3)$)

and their image in $H(3)^{ab}$: $E_1^3 = U_1^6 = 1$, $U_1^3 = U_2^3$. Therefore $H(3)^{ab}$ is a quotient of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The second statement follows by observing that the abelianization of $SU(1, 1, \mathcal{O}_3)$ is \mathbb{Z} , as the former is a $(3, \infty, \infty)$ triangle group. \square

It is interesting to note that this also tells us the behavior of a related hybrid group, namely the hybrid of two $(2, 6, \infty)$ triangle groups, rather than $(3, \infty, \infty)$ (which each $(2, 6, \infty)$ group contains with index 2). A simple way to view this new hybrid $H'(3)$ as a subgroup of $\Gamma(3) = PU(2, 1, \mathcal{O}_3)$ containing the previous hybrid $H(3)$ is to take the obvious square root of the previous generator E_1 in terms of the Falbel-Parker generators, in other words to take $H(3)$ to be generated by $E'_1 = P^2(RQ^2)P^{-2}$, and $U_1 = Q^2, U_2 = RQ^2R$ unchanged.

Lemma 16. *The hybrid $H'(3)$ is contained in $[\Gamma(3), \Gamma(3)]$.*

Proof. From the Falbel-Parker presentation for Γ_3 we get (abusing notation slightly again by using the same symbol for elements of $\Gamma(3)$ and their image in $\Gamma(3)^{ab}$):

$$\Gamma(3)^{ab} = \Gamma(3)/[\Gamma(3), \Gamma(3)] = \langle P, Q, R \mid R = P^3 = Q^2 = [P, Q] = 1 \rangle.$$

The result then follows by noting that the generators listed above for $H'(3)$ all become trivial in the abelianization. \square

The following is Lemma 6 of [21].

Lemma 17. *The commutator subgroup $[\Gamma(3), \Gamma(3)]$ has abelianization $\mathbb{Z} \oplus \mathbb{Z}$.*

Lemma 18. *The hybrid $H'(3)$ has finite abelianization.*

Proof. This follows from the relations given in Lemma 14 by noting that $H'(3)$ is generated by E'_1, U_1, U_2 with $(E'_1)^2 = E_1$. \square

The following is well known but we include it for completeness:

Lemma 19. *If $K_1 < K_2$ are two groups with $[K_2 : K_1]$ and K_1^{ab} finite, then K_2^{ab} is finite.*

Proof. Denote i the inclusion map from K_1 into K_2 , and $\pi_i : K_i \rightarrow K_i^{ab}$ the quotient maps for $i = 1, 2$. Then $\pi_2 \circ i$ is a homomorphism from K_1 to an abelian group, so by the universal property of abelianizations $\pi_2 \circ i$ factors through K_1^{ab} , i.e. there is a homomorphism $i_* : K_1^{ab} \rightarrow K_2^{ab}$ such that $i_* \circ \pi_1 = \pi_2 \circ i$. Since $K_1 = i(K_1)$ has finite index in K_2 by assumption and π_2 is surjective, $\pi_2(K_1) = i_*(\pi_1(K_1)) = i_*(K_1^{ab})$ has finite index in K_2^{ab} . The result follows since K_1^{ab} is finite. \square

Combining Lemmas 16, 17, 18 and 19 gives the following:

Corollary 20. *The hybrid $H'(3)$ has infinite index in $[\Gamma(3), \Gamma(3)]$, hence also in $\Gamma(3)$.*

It is interesting to note that, in contrast with the previous hybrid $H(3)$ which was normal in $\Gamma(3)$, $H'(3)$ now has infinite index in its normal closure $\langle\langle H'(3) \rangle\rangle = \Gamma(3)$ in $\Gamma(3)$ (the presentation of $\Gamma(3)/\langle\langle H'(3) \rangle\rangle$ obtained by adding the generators of $H'(3)$ to the presentation for $\Gamma(3)$ now gives the trivial group).

3.4 A Hybrid Subgroup of the Gauss-Picard Modular Group $\mathrm{PU}(2, 1, \mathcal{O}_1)$

The following matrices generate $\mathrm{SU}(1, 1; \mathcal{O}_1)$ in the ball model of $\mathbb{H}_{\mathbb{C}}^1$:

$$E = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U = \begin{pmatrix} 1+i & -i \\ i & 1-i \end{pmatrix}.$$

We now consider the hybrid group $H(\mathrm{SU}(1, 1; \mathcal{O}_1), \mathrm{SU}(1, 1; \mathcal{O}_1))$, which by definition is generated by $\iota_1(E)$, $\iota_1(U)$, $\iota_2(E)$ and $\iota_2(U)$. It will be again more convenient for us to work in the Siegel model, in other words to conjugate by the Cayley transform J given in (3.3). We thus consider the group $H(1) = \langle E_1, U_1, E_2, U_2 \rangle$, where:

$$E_1 = J^{-1}\iota_1(E)J = \begin{pmatrix} i & -1+i & 1-i \\ -2i & 1-2i & -1+i \\ -2i & -2i & i \end{pmatrix}, \quad U_1 = J^{-1}\iota_1(U)J = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_2 = J^{-1}\iota_2(E)J = \begin{pmatrix} i & 2i & -2i \\ 1-i & 1-2i & 2i \\ 1-i & 1-i & i \end{pmatrix}, \quad U_2 = J^{-1}\iota_2(U)J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}.$$

A presentation for the Gauss-Picard lattice $\text{PU}(2, 1; \mathcal{O}_1)$ was first found in [19], however for our purposes it is more convenient to use the following presentation given in [36]:

$$\begin{aligned} [T_\tau, T_2] &= T_v^4, [T_v, T_2], [T_v, T_\tau], [T_v, R], R^4, I^2, [R, I], \\ RT_2R^{-1} &= T_\tau^2T_2^{-1}T_v^4, RT_\tau R^{-1} = T_\tau T_2^{-1}T_v^2, \\ \text{PU}(2, 1; \mathcal{O}_1) &= \left\langle T_2, T_\tau, T_v, R, I \mid [I, T_2]^2, (IT_v)^3 = R, [I, T_\tau] = T_\tau IR^2, (T_v IR^{-1} T_v^2 I)^2, \right. \\ &\quad IT_v^{-1} T_\tau I R T_2^{-1} T_v^{-1} = T_2 T_\tau^{-1} I T_\tau R^2 T_v I, \\ &\quad \left. (IT_v^{-1} T_\tau I R T_2^{-1} T_v^{-1})^2 = R^{-1} T_2^{-1} T_\tau T_v^{-3} \right\rangle \end{aligned}$$

where

$$T_2 = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_\tau = \begin{pmatrix} 1 & -1+i & -1 \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix},$$

$$T_v = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

A straightforward computation gives the following:

Lemma 21. *The generators for the hybrid $H(1)$ can be expressed in terms of the Mark-Paupert generators for $\text{PU}(2, 1; \mathcal{O}_1)$ as follows:*

$$\begin{aligned} U_1 &= T_v, \\ U_2 &= IT_v I, \\ E_1 &= T_v^{-1} T_\tau I R T_2^{-1} I, \\ E_2 &= IT_v^{-1} T_\tau I R T_2^{-1}. \end{aligned}$$

Lemma 22. *The hybrid $H(1)$ is a normal subgroup of $\text{PU}(2, 1; \mathcal{O}_1)$.*

Proof. It suffices to check that generators of $\text{PU}(2, 1; \mathcal{O}_1)$ conjugate generators of $H(1)$ into $H(1)$. Note that there is nothing to check for $T_v = U_1$ as it is a generator for both groups; also note that $R^2 = (U_1 U_2)^3 \in H(1)$. Straightforward computations give the following relations:

$$\begin{aligned} T_2^{-1} U_1 T_2 &= U_1 & T_2^{-1} U_2 T_2 &= R^2 E_2^{-1} U_2 E_2 R^2 \\ T_\tau^{-1} U_1 T_\tau &= U_1 & T_\tau^{-1} U_2 T_\tau &= (R^2 U_1 E_1) U_2 (R^2 U_1 E_1)^{-1} \\ R^{-1} U_1 R &= U_1 & R^{-1} U_2 R &= U_2 \\ I^{-1} U_1 I &= U_2 & I^{-1} U_2 I &= U_1 \\ \\ T_2^{-1} E_1 T_2 &= R^2 U_1^{-1} E_2 U_1^{-1} E_2^{-1} R^2 & T_2^{-1} E_2 T_2 &= R^2 U_2^{-1} E_1 U_2^{-1} E_1^{-1} R^2 \\ T_\tau^{-1} E_1 T_\tau &= (R^2 U_2 U_1) E_2 (R^2 U_2 U_1)^{-1} & T_\tau^{-1} E_2 T_\tau &= (R^2 U_2 U_1) E_1 (R^2 U_2 U_1)^{-1} \\ R^{-1} E_1 R &= (U_1 U_2 U_1)^{-1} E_2 (U_1 U_2 U_1) & R^{-1} E_2 R &= (U_2 U_1 U_2)^{-1} E_1 (U_2 U_1 U_2) \\ I^{-1} E_1 I &= E_2 & I^{-1} E_2 I &= E_1 \end{aligned}$$

□

Theorem 23. *The hybrid $H(1)$ has index 2 in the full Gauss-Picard lattice $\text{PU}(2, 1; \mathcal{O}_1)$.*

Proof. A presentation for the quotient $\text{PU}(2, 1; \mathcal{O}_1)/H(1)$ is obtained from the presentation for $\text{PU}(2, 1; \mathcal{O}_1)$, to which we add as relations the generators of the subgroup

$H(1)$ written as words in the generators for $\text{PU}(2, 1; \mathcal{O}_1)$ as in Lemma 21.

$$\begin{aligned}
[T_\tau, T_2] &= T_v^4, [T_v, T_2], [T_v, T_\tau], [T_v, R], R^4, I^2, [R, I], \\
RT_2R^{-1} &= T_\tau^2T_2^{-1}T_v^4, RT_\tau R^{-1} = T_\tau T_2^{-1}T_v^2, \\
\text{PU}(2, 1; \mathcal{O}_1)/H(1) &= \left\langle T_2, T_\tau, T_v, R, I \mid \begin{array}{l} [I, T_2]^2, (IT_v)^3 = R, [I, T_\tau] = T_\tau IR^2, (T_v IR^{-1} T_v^2 I)^2, \\ IT_v^{-1} T_\tau I R T_2^{-1} T_v^{-1} = T_2 T_\tau^{-1} I T_\tau R^2 T_v I, \\ (IT_v^{-1} T_\tau I R T_2^{-1} T_v^{-1})^2 = R^{-1} T_2^{-1} T_\tau T_v^{-3}, \\ T_v, IT_v I, T_v^{-1} T_\tau I R T_2^{-1} I, IT_v^{-1} T_\tau I R T_2^{-1} \end{array} \right\rangle
\end{aligned}$$

Since $T_v = 1$ in the quotient, the relation $(IT_v)^3 = R$ implies $I = R$. The relation coming from E_1 implies that $I = T_\tau T_2^{-1}$, and substituting into the relation on the fourth line above yields $I = T_\tau$. With this, T_1 and I commute, and the relation on the fifth line above yields $T_2 = 1$. Thus the presentation above simplifies to

$$\text{PU}(2, 1; \mathcal{O}_1)/H(1) = \langle T_2, T_\tau, T_v, R, I \mid I = R = T_\tau, T_2 = T_v = I^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z}$$

□

We now consider the related hybrid $H'(1)$ as in the case of $d = 3$, namely taking $H'(1)$ to be the hybrid of two copies of the Fuchsian triangle group $(2, 4, \infty)$, rather than $(2, \infty, \infty) \simeq \text{SU}(1, 1, \mathcal{O}_1)$. We immediately get the following result by noting that $H'(1)$ contains $H(1)$, which has index 2 in the full lattice $\Gamma(1)$, as well as a new element of order 4 not belonging to $H(1)$.

Corollary 24. *The hybrid $H'(1)$ is equal to the full lattice $\Gamma(1) = \text{PU}(2, 1; \mathcal{O}_1)$.*

3.5 A Hybrid Subgroup of the Picard Modular Group $\text{PU}(2, 1, \mathcal{O}_7)$

The following matrices generate $\text{U}(1, 1; \mathcal{O}_7)$ in the ball model of $\mathbb{H}_{\mathbb{C}}^1$:

$$U = \begin{pmatrix} 1 + i\sqrt{7} & -i\sqrt{7} \\ i\sqrt{7} & 1 - i\sqrt{7} \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{7}}{2} & 1 \\ -1 & \frac{1}{2} + i\frac{\sqrt{7}}{2} \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the Siegel model, the corresponding hybrid $H(7) = H(\mathrm{U}(1, 1; \mathcal{O}_7), \mathrm{U}(1, 1; \mathcal{O}_7))$ has the following generators:

$$\begin{aligned}
U_1 = J^{-1}\iota_1(U)J &= \begin{pmatrix} 1 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & U_2 = J^{-1}\iota_2(U)J &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & 1 \end{pmatrix}, \\
A_1 = J^{-1}\iota_1(A)J &= \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{7}}{2} & -\frac{3}{2} + i\frac{\sqrt{7}}{2} & \frac{1}{2} - i\frac{\sqrt{7}}{2} \\ 1 & 2 & -\frac{3}{2} + i\frac{\sqrt{7}}{2} \\ 1 & 1 & -\frac{1}{2} + i\frac{\sqrt{7}}{2} \end{pmatrix}, & B_1 = J^{-1}\iota_1(B)J &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ -2 & -2 & 1 \end{pmatrix}, \\
A_2 = J^{-1}\iota_2(A)J &= \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{7}}{2} & -1 & 1 \\ \frac{3}{2} - i\frac{\sqrt{7}}{2} & 2 & -1 \\ \frac{1}{2} - i\frac{\sqrt{7}}{2} & \frac{3}{2} - i\frac{\sqrt{7}}{2} & -\frac{1}{2} + i\frac{\sqrt{7}}{2} \end{pmatrix}, & B_2 = J^{-1}\iota_2(U)J &= \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

In [36] the authors determine that $\mathrm{PU}(2, 1; \mathcal{O}_7)$ has presentation

$$\begin{aligned}
& [T_\tau, T_1] = T_v, [T_v, T_1], [T_v, T_\tau], [T_v, R], R^2, (RT_\tau)^2, \\
& (RT_1)^2 = T_v, I_0^2, I_1^2, [R, I_0], [R, I_1 I_0 T_1^{-1} T_\tau]^2, \\
& [R, I_1 I_0 T_1^{-1} T_\tau] = T_v I_0 I_1 T_\tau T_1^{-1} I_1 I_0 T_\tau T_1^{-2} T_v, \\
& [R, I_1 I_0 T_1^{-1} T_\tau] = T_v T_1^{-1} I_0 T_1 I_0 T_\tau^{-1} I_1 R I_0 T_v^{-1}, \\
& [I_0, T_v^{-1} T_\tau T_1] = T_1 I_0 I_1 I_0 T_1^{-1}, \\
& R[R, I_1 I_0 T_1^{-1} T_\tau] = T_1 I_0 T_v T_1^{-2} I_0 T_1 T_v^{-1} R, \\
& I_1 = T_1^2 T_\tau R T_1^2 I_0 T_1^{-1} I_0 T_1 I_0
\end{aligned}
\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \mathrm{PU}(2, 1; \mathcal{O}_7) = \left\langle T_1, T_\tau, T_v, R, I_0, I_1 \mid \right.$$

(3.8)

where

$$\begin{aligned}
T_1 &= \begin{pmatrix} 1 & -1 & -\frac{1}{2} + i\frac{\sqrt{7}}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & R &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
T_\tau &= \begin{pmatrix} 1 & -\frac{1}{2} + i\frac{\sqrt{7}}{2} & -1 \\ 0 & 1 & \frac{1}{2} + i\frac{\sqrt{7}}{2} \\ 0 & 0 & 1 \end{pmatrix}, & I_0 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
T_v &= \begin{pmatrix} 1 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & I_1 &= \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{7}}{2} & \frac{1}{2} + i\frac{\sqrt{7}}{2} & 2 \\ \frac{1}{2} + i\frac{\sqrt{7}}{2} & 2 & \frac{1}{2} - i\frac{\sqrt{7}}{2} \\ 2 & \frac{1}{2} - i\frac{\sqrt{7}}{2} & -\frac{1}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix}.
\end{aligned}$$

In terms of these generators, the generators for $H(7)$ can be written as follows:

$$\begin{aligned}
U_1 &= T_v, \\
U_2 &= I_0 U_1 I_0, \\
A_1 &= T_1 I_0 T_1 R, \\
A_2 &= I_0 A_1 I_0, \\
B_1 &= (I_0 T_1) R (I_0 T_1)^{-1}, \\
B_2 &= I_0 B_1 I_0.
\end{aligned}$$

Lemma 25. *The hybrid $H(7)$ is a normal subgroup of $\text{PU}(2, 1; \mathcal{O}_7)$.*

Proof. Since we have that

$$\begin{aligned}
R &= (A_1 A_2 B_1 A_1 B_2)^{-1} B_1 (A_1 A_2 B_1 A_1 B_2) \in H(7), \\
T_v &= U_1 \in H(7),
\end{aligned}$$

and $I_0H(7)I_0 \subseteq H(7)$, it suffices to check conjugation by T_1 and T_τ :

$$T_1^{-1}A_1T_1 = (A_1A_2^{-1}B_2A_2^{-1}A_1)^{-1}A_2(A_1A_2^{-1}B_2A_2^{-1}A_1)$$

$$T_\tau^{-1}A_1T_\tau = (A_1^{-1}A_2U_1)^{-1}A_2(A_1^{-1}A_2U_1)$$

$$T_1^{-1}A_2T_1 = (B_2A_2A_1^{-1}A_2^{-1}B_1)^{-1}A_2(B_2A_2A_1^{-1}A_2^{-1}B_1)$$

$$T_\tau^{-1}A_2T_\tau = (B_1A_1^{-1}A_2U_1)^{-1}A_1(B_1A_1^{-1}A_2U_1)$$

$$T_1^{-1}B_1T_1 = (A_1^{-1}A_2^{-1}B_1)^{-1}B_1(A_1^{-1}A_2^{-1}B_1)$$

$$T_\tau^{-1}B_1T_\tau = (A_2U_1)^{-1}B_2(A_2U_1)$$

$$T_1^{-1}B_2T_1 = R$$

$$T_\tau^{-1}B_2T_\tau = (A_1^{-1}A_2)^{-1}B_1(A_1^{-1}A_2)$$

$$T_1^{-1}U_1T_1 = U_1$$

$$T_\tau^{-1}U_1T_\tau = U_1$$

$$T_1^{-1}U_2T_1 = (A_1^2A_2^{-1}B_2A_2^{-1}A_1)^{-1}U_2(A_1^2A_2^{-1}B_2A_2^{-1}A_1)$$

$$T_\tau^{-1}U_2T_\tau = (U_2B_1A_1^{-1}A_2)^{-1}U_1(U_2B_1A_1^{-1}A_2)$$

□

Theorem 26. *The hybrid $H(7)$ is the full lattice $\text{PU}(2, 1; \mathcal{O}_7)$.*

Proof. We consider the quotient

$$\mathrm{PU}(2, 1; \mathcal{O}_7)/H(7)$$

The relators coming from the generators U_1, B_1 and A_1 of $H(7)$ immediately imply that, in the quotient, $T_v = R = 1$ and $T_1^2 = I_0$, respectively. Moreover, the relation $(RT_1)^2 = T_v$ implies that $T_1^2 = I_0 = 1$, and the relation defining I_1 implies that $I_1 = T_\tau$, whence $T_\tau^2 = 1$. Substituting this into the relations on the third and fourth lines of the presentation (3.8), we get that $T_1 = 1$ and $T_\tau = 1$, respectively. \square

3.6 Limit Sets and Geometrical Finiteness

3.6.1 Limit Sets

We first briefly recall the definition and two classical facts about limit sets of discrete groups of isometries. The space we consider in this paper is the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$, but these definitions and facts hold more generally in any negatively curved symmetric space (so, hyperbolic space of any dimension over the real or complex numbers or quaternions, or hyperbolic plane over the octonions).

Definition. Let X be a negatively curved symmetric space, $\partial_\infty X$ its boundary at infinity (or visual, or Gromov boundary), and Γ a discrete subgroup of $\mathrm{Isom}(X)$. The *limit set* $\Lambda(\Gamma)$ of Γ is defined as the set of accumulation points in $\partial_\infty X$ of the orbit Γx_0 for any choice of $x_0 \in X$; this does not depend on the choice of x_0 .

A basic property of $\Lambda(\Gamma)$ is that it is the minimal (nonempty) closed Γ -invariant subset of $\partial_\infty X$, in fact the orbit Γp_∞ is dense in $\Lambda(\Gamma)$ for any $p_\infty \in \Lambda(\Gamma)$. We will use the following two classical properties of limit sets; recall that a discrete subgroup Γ of $\mathrm{Isom}(X)$ is called *non-elementary* if $\Lambda(\Gamma)$ contains more than two points.

Proposition 27. *Let X be a negatively curved symmetric space and Γ a discrete subgroup of $\text{Isom}(X)$.*

(a) *If Γ is a lattice in $\text{Isom}(X)$ then $\Lambda(\Gamma) = \partial_\infty X$.*

(b) *If Γ' is a nonelementary normal subgroup of Γ then $\Lambda(\Gamma') = \Lambda(\Gamma)$.*

The following result is an immediate consequence of this and Lemmas 11, 22 (or Theorem 23).

Proposition 28. *For $d = 1, 3$ the hybrid $H(d)$ has full limit set: $\Lambda(H(d)) = \partial_\infty \mathbb{H}_\mathbb{C}^2 \simeq S^3$.*

3.6.2 Geometrical Finiteness

The original notion of geometrical finiteness for a Kleinian group $\Gamma < \text{Isom}(\mathbb{H}_\mathbb{R}^3)$ was to admit a finite-sided polyhedral fundamental domain. This was later shown to admit several equivalent formulations, then systematically studied by Bowditch in higher-dimensional real hyperbolic spaces in [7], and more generally in pinched Hadamard manifolds in [8]. In [7], Bowditch labelled the five equivalent formulations of the definition of geometrical finiteness (GF1)-(GF5), with (GF3) corresponding to the original notion. He then showed in [8] that the four other formulations, now labelled F1, F2, F4, and F5, remain equivalent in the more general setting (but not the original one). The most convenient for our purposes will be condition F5, which we now recall.

Let as above X be a negatively curved symmetric space and Γ a discrete subgroup of $\text{Isom}(X)$. The *convex hull* $\text{Hull}(\Gamma)$ of Γ in X is the convex hull of the limit set $\Lambda(\Gamma)$, more precisely the smallest convex subset of X whose closure in $\overline{X} = X \cup \partial_\infty X$ contains $\Lambda(\Gamma)$. This is invariant under the action of Γ , and the *convex core* $\text{Core}(\Gamma)$ of Γ in X is defined as the quotient of $\text{Hull}(\Gamma)$ under the action of Γ .

Definition. We say that Γ satisfies condition F5 if (a) for some $\varepsilon > 0$, the tubular neighborhood $N_\varepsilon(\text{Core}(\Gamma))$ in X/Γ has finite volume, and (b) there is a bound on the orders of the finite subgroups of Γ .

Proposition 29. *The hybrid $H(3) < \text{Isom}(\mathbb{H}_\mathbb{C}^2)$ is geometrically infinite.*

Proof. We show that $H(3)$ does not satisfy condition F5. By Proposition 28, $\Lambda(H(3)) = \partial_\infty \mathbb{H}_\mathbb{C}^2$, hence $\text{Hull}(H(3)) = \mathbb{H}_\mathbb{C}^2$. Now by Theorem 12, $H(3)$ has infinite index in a lattice, therefore it acts on $\mathbb{H}_\mathbb{C}^2$ with infinite covolume, in other words $\text{Core}(H(3))$ has infinite volume hence so does any of its tubular neighborhoods. \square

Chapter 4

NONARITHMETIC HYBRIDS IN $SU(2, 1)$

A *triangle group* is one generated by reflections in the sides of a triangle. Because of this very simple and very geometric description, these groups have been very well-studied. In particular, in 1977, Takeuchi [56] classified all arithmetic and non-arithmetic triangle groups in $SL(2, \mathbb{R})$. In 1980, Mostow [40] explored a similar family of groups in $SU(2, 1)$ which are generated by complex reflections in lines with order-3 symmetry (this is analogous to a triangle group where the triangle is equilateral). Remarkably, many of these groups end up being non-arithmetic, and this provided some of the earliest known examples of non-arithmetic lattices in $SU(2, 1)$. In this section we explore hybrid subgroups in some of Mostow's lattices, and the main result of this section is the following

Theorem 1. *Among Mostow's non-arithmetic lattices $\Gamma(p, t) < PU(2, 1)$, the groups $\Gamma(4, 1/12)$ and $\Gamma(5, 1/5)$ arise as hybrids of non-commensurable arithmetic lattices in $PU(1, 1)$.*

4.1 Mostow's Lattices

Following along with [40] and [15], $p \geq 3$ is an integer, t is a real number satisfying $|t| < 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $\alpha = \frac{1}{2} \csc(\pi/p)$, $\varphi = e^{\pi it/3}$, and $\eta = e^{\pi i/p}$. The Hermitian form is given by $\langle x, y \rangle = x^T H \bar{y}$ where

$$H = \begin{pmatrix} 1 & -\alpha\varphi & -\alpha\bar{\varphi} \\ -\alpha\bar{\varphi} & 1 & -\alpha\varphi \\ -\alpha\varphi & -\alpha\bar{\varphi} & 1 \end{pmatrix}.$$

With p, t as above, the reflection groups to consider are $\Gamma(p, t) = \langle R_1, R_2, R_3 \rangle$ where

$$R_1 = \begin{pmatrix} \eta^2 & -i\eta\bar{\varphi} & -i\eta\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -i\eta\varphi & \eta^2 & -i\eta\bar{\varphi} \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i\eta\bar{\varphi} & -i\eta\varphi & \eta^2 \end{pmatrix}.$$

When $|t| < \frac{1}{2} - \frac{1}{p}$, Mostow refers to these groups as having “small phase shift.”

Following the notation in [15], we study closely related groups $\tilde{\Gamma}(p, t) = \langle R_1, J \rangle$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

J has order 3 and $R_i = JR_{i+3}J^{-1}$ (where indices are taken modulo 3). It is sufficient to study these groups $\tilde{\Gamma}(p, t)$ due to the following result:

Proposition 30 (Lemma 16.1 in [40], Prop 2.11 in [15]). *$\Gamma(p, t)$ has index dividing 3 in $\tilde{\Gamma}(p, t)$.*

Remark (Tables 1 and 2 in [40], Remark 5.3 in [15]). Given $p = 3, 4, 5$, there are only finitely-many values of t for which $\Gamma(p, t)$ is discrete. They are given in Table 4.1.

p	$ t < \frac{1}{2} - \frac{1}{p}$	$ t \geq \frac{1}{2} - \frac{1}{p}$
3	$0, \frac{1}{30}, \frac{1}{18}, \frac{1}{12}, \frac{5}{42}$	$\frac{1}{6}, \frac{7}{30}, \frac{1}{3}$
4	$0, \frac{1}{12}, \frac{3}{20}$	$\frac{1}{4}, \frac{5}{12}$
5	$\frac{1}{10}, \frac{1}{5}$	$\frac{11}{30}, \frac{7}{10}$

Table 4.1: Values of p and t for Which $\Gamma(p, t)$ is Discrete

Theorem 31 (Theorem 17.3 in [40]). *The following lattices $\Gamma(p, t) \leq \text{PU}(2, 1)$ are nonarithmetic: $\Gamma(3, 5/42)$, $\Gamma(3, 1/12)$, $\Gamma(3, 1/30)$, $\Gamma(4, 3/20)$, $\Gamma(4, 1/12)$, $\Gamma(5, 1/5)$, $\Gamma(5, 11/30)$. The non-cocompact lattices $\Gamma(p, t)$ are arithmetic.*

4.2 Hybrids in Mostow's Lattices

When $\Gamma(p, t)$ has small phase shift, the fundamental domain for each of Mostow's groups is built by coning over two polytopes that intersect in a right-angled hexagon (see Figure 4.1 for a topological picture, or Figure 1 on Page 16 of [15] for a geometric picture in coordinates) which becomes degenerate for larger t -values. Taking lifts to $\mathbb{C}^{2,1}$, one sees that non-adjacent sides for each hexagon intersect in positive vectors, which are given explicitly below:

$$\begin{aligned}
 v_{123} &= \begin{pmatrix} -i\eta\bar{\varphi} \\ 1 \\ i\eta\varphi \end{pmatrix}, & v_{231} &= \begin{pmatrix} i\bar{\eta}\varphi \\ -i\eta\bar{\varphi} \\ 1 \end{pmatrix}, & v_{312} &= \begin{pmatrix} 1 \\ i\bar{\eta}\varphi \\ -i\eta\bar{\varphi} \end{pmatrix}, \\
 v_{321} &= \begin{pmatrix} i\bar{\eta}\varphi \\ 1 \\ -i\eta\varphi \end{pmatrix}, & v_{132} &= \begin{pmatrix} -i\eta\varphi \\ i\bar{\eta}\bar{\varphi} \\ 1 \end{pmatrix}, & v_{213} &= \begin{pmatrix} 1 \\ -i\eta\varphi \\ i\bar{\eta}\bar{\varphi} \end{pmatrix}.
 \end{aligned}$$

Geometrically, v_{ijk} is the polar vector to the mirror for the complex reflection

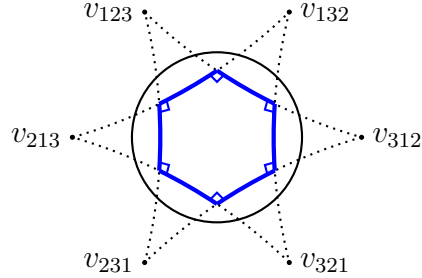


Figure 4.1: Core Right-Angled Hexagon

$J^{\pm 1}R_jR_k$ (for $k = j \pm 1$). What's more,

Proposition 32 (Proposition 2.13(3) in [15]). $v_{ijk} \perp v_{jik}$ and $v_{ijk} \perp v_{ikj}$.

For the hybrid construction, we use the subspaces (considered as projective subspaces of $\mathbf{H}_{\mathbb{C}}^2$) corresponding to v_{ijk}^{\perp} . Since $Jv_{ijk} = v_{kij}$, it suffices to look only at v_{312}^{\perp}

and v_{312}^\perp as the remaining subspaces are obtained by successive applications of J . In homogeneous coordinates, one sees that

$$\begin{aligned} v_{312}^\perp &= \{[z, i\bar{\eta}\bar{\varphi}, 1]^T : z \in \mathbb{C}\} \\ v_{321}^\perp &= \{[i\bar{\eta}\bar{\varphi}, z, 1]^T : z \in \mathbb{C}\} \end{aligned}$$

Let $\Lambda_{ijk} \leq \Gamma(p, t)$ be the stabilizer subgroup of v_{ijk}^\perp , which is naturally identified with a subgroup of $\text{PU}(1, 1)$, and let Γ_{ijk} be a lift of this group to $\text{SU}(1, 1)$.

Proposition 33. Γ_{312} is a cocompact lattice in $\text{SU}(1, 1)$. It is arithmetic for all pairs (p, t) appearing in Table 4.1 except $(3, 1/30)$, $(3, 1/12)$, $(3, 5/42)$, and $(4, 3/20)$.

Proof. R_1 and R_3J both stabilize v_{312}^\perp :

$$\begin{aligned} R_1 : [z, i\bar{\eta}\bar{\varphi}, 1]^T &\mapsto [\eta^2 z + \bar{\varphi}^2 - i\eta\varphi, i\bar{\eta}\bar{\varphi}, 1]^T \\ R_3J : [z, i\bar{\eta}\bar{\varphi}, 1]^T &\mapsto [i\bar{\eta}\bar{\varphi}/z, i\bar{\eta}\bar{\varphi}, 1]^T \end{aligned}$$

Let A and B be the following elements in $\text{SU}(1, 1)$ corresponding to the actions of R_1 and R_3J on v_{312}^\perp , respectively.

$$A = \frac{1}{\eta} \begin{pmatrix} \eta^2 & \bar{\varphi}^2 - i\eta\varphi \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{-i\bar{\eta}\bar{\varphi}}} \begin{pmatrix} 0 & i\bar{\eta}\bar{\varphi} \\ 1 & 0 \end{pmatrix}, \quad A^{-1}B = \frac{1}{\sqrt{-i\bar{\eta}\bar{\varphi}}} \begin{pmatrix} -\bar{\varphi}^2 + i\eta\varphi & i\bar{\eta}\bar{\varphi} \\ \eta^2 & 0 \end{pmatrix}.$$

One then sees that

$$\begin{aligned} |\text{Tr}(A)| &= |e^{i\pi/p} + e^{-i\pi/p}|, \\ |\text{Tr}(B)| &= 0, \\ |\text{Tr}(A^{-1}B)| &= |e^{i\pi(-1/2+1/p+t/3)} + e^{-2\pi it/3}|. \end{aligned}$$

All of these values are less than 2 when $p \geq 3$ and $|t| \neq \frac{1}{2} - \frac{1}{p}$ and so the elements are elliptic. Thus $\langle A, B \rangle$ is a cocompact triangle group (and therefore Γ_{312}

is a cocompact lattice). By computing orders of these elements for (p, t) values in Table 4.1, one obtains Table 4.2 showing the corresponding triangle groups, and arithmeticity(A)/non-arithmeticity(NA) of each can be checked by comparing with the main theorem of [56]. \square

Proposition 34. Γ_{321} is a cocompact lattice in $SU(1, 1)$. It is arithmetic for all pairs (p, t) appearing in Table 4.1 except $(3, -5/42)$, $(3, -1/12)$, $(3, -1/30)$, and $(4, -3/20)$.

Proof. R_2 and JR_3^{-1} both stabilize v_{321}^\perp :

$$R_2 : [i\bar{\eta}\varphi, z, 1]^T \mapsto [i\bar{\eta}\varphi, \eta^2 z + \varphi^2 - i\eta\bar{\varphi}, 1]^T$$

$$JR_3^{-1} : [i\bar{\eta}\varphi, z, 1]^T \mapsto [i\bar{\eta}\varphi, i\bar{\eta}\varphi/z, 1]^T$$

Let A and B be the following elements in $SU(1, 1)$ corresponding to the actions of R_2 and JR_3^{-1} on v_{321}^\perp , respectively.

$$A = \frac{1}{\eta} \begin{pmatrix} \eta^2 & \varphi^2 - i\eta\bar{\varphi} \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{-i\bar{\eta}\varphi}} \begin{pmatrix} 0 & i\bar{\eta}\varphi \\ 1 & 0 \end{pmatrix}, \quad A^{-1}B = \frac{1}{\sqrt{-i\eta\bar{\varphi}}} \begin{pmatrix} i\eta\bar{\varphi} - \varphi^2 & i\bar{\eta}\varphi \\ \eta^2 & 0 \end{pmatrix}$$

One can check that

$$|\mathrm{Tr}(A)| = |e^{i\pi/p} + e^{-i\pi/p}|,$$

$$|\mathrm{Tr}(B)| = 0,$$

$$|\mathrm{Tr}(A^{-1}B)| = |e^{i\pi(1/2+1/p-t/3)} - e^{2\pi it/3}|.$$

All of these values are less than 2 when $p \geq 3$ and $|t| \neq \frac{1}{2} - \frac{1}{p}$ and so the elements are elliptic. Thus $\langle A, B \rangle$ is a cocompact triangle group (and therefore Γ_{321} is a cocompact lattice). By computing orders of these elements for (p, t) values in Table 4.1, one obtains Table 4.3 showing the corresponding triangle groups, and arithmeticity(A)/non-arithmeticity(NA) of each can be checked by comparing with the main theorem of [56]. \square

(p, t)	$\Delta(x, y, z)$	A/NA	(p, t)	$\Delta(x, y, z)$	A/NA
$(3, -5/42)$	$\Delta(2, 3, 7)$	A	$(4, -3/20)$	$\Delta(2, 4, 5)$	A
$(3, -1/12)$	$\Delta(2, 3, 8)$	A	$(4, -1/12)$	$\Delta(2, 4, 6)$	A
$(3, -1/18)$	$\Delta(2, 3, 9)$	A	$(4, 0)$	$\Delta(2, 4, 8)$	A
$(3, -1/30)$	$\Delta(2, 3, 10)$	A	$(4, 1/12)$	$\Delta(2, 4, 12)$	A
$(3, 0)$	$\Delta(2, 3, 12)$	A	$(4, 3/20)$	$\Delta(2, 4, 20)$	NA
$(3, 1/30)$	$\Delta(2, 3, 15)$	NA	$(5, -1/5)$	$\Delta(2, 4, 5)$	A
$(3, 1/18)$	$\Delta(2, 3, 18)$	A	$(5, -1/10)$	$\Delta(2, 5, 5)$	A
$(3, 1/12)$	$\Delta(2, 3, 24)$	A	$(5, 1/10)$	$\Delta(2, 5, 10)$	A
$(3, 5/42)$	$\Delta(2, 3, 42)$	NA	$(5, 1/5)$	$\Delta(2, 5, 20)$	A

Table 4.2: Properties of Γ_{312}

(p, t)	$\Delta(x, y, z)$	A/NA	(p, t)	$\Delta(x, y, z)$	A/NA
$(3, -5/42)$	$\Delta(2, 3, 42)$	NA	$(4, -3/20)$	$\Delta(2, 4, 20)$	NA
$(3, -1/12)$	$\Delta(2, 3, 24)$	A	$(4, -1/12)$	$\Delta(2, 4, 12)$	A
$(3, -1/18)$	$\Delta(2, 3, 18)$	A	$(4, 0)$	$\Delta(2, 4, 8)$	A
$(3, -1/30)$	$\Delta(2, 3, 15)$	NA	$(4, 1/12)$	$\Delta(2, 4, 6)$	A
$(3, 0)$	$\Delta(2, 3, 12)$	A	$(4, 3/20)$	$\Delta(2, 4, 5)$	A
$(3, 1/30)$	$\Delta(2, 3, 10)$	A	$(5, -1/5)$	$\Delta(2, 5, 20)$	A
$(3, 1/18)$	$\Delta(2, 3, 9)$	A	$(5, -1/10)$	$\Delta(2, 5, 10)$	A
$(3, 1/12)$	$\Delta(2, 3, 8)$	A	$(5, 1/10)$	$\Delta(2, 5, 5)$	A
$(3, 5/42)$	$\Delta(2, 3, 7)$	A	$(5, 1/5)$	$\Delta(2, 4, 5)$	A

Table 4.3: Properties of Γ_{321}

Theorem 35. For $|t| < \frac{1}{2} - \frac{1}{p}$, the hybrid $H(\Gamma_{312}, \Gamma_{321})$ is the full lattice $\tilde{\Gamma}(p, t)$.

Proof. The group $K = \langle R_1, R_3J, R_2, JR_3^{-1} \rangle$ is a subgroup of $H(\Gamma_{312}, \Gamma_{321})$. Since $J = (R_3J)^{-1}(JR_3^{-1})^{-1}$, $K = \langle R_1, J \rangle = \tilde{\Gamma}(p, t)$. \square

By comparing with the table on Page 418 of [33], one sees that Γ_{312} and Γ_{321} are both arithmetic and noncommensurable in the cases where $(p, t) = (4, 1/12)$ and $(5, 1/5)$. This means that

Theorem 36. $\Gamma(4, 1/12)$ and $\Gamma(5, 1/5)$ are nonarithmetic lattices obtained by interbreeding two noncommensurable arithmetic lattices.

FUTURE DIRECTIONS

5.1 Hybrids in Deraux–Parker–Paupert Lattices

In [16] the authors examine a new family of subgroups also generated by 3 complex reflections with threefold symmetry. Following along with [16], $p \geq 3$ is an integer, $\tau = -\frac{1+i\sqrt{7}}{2}$, $a = e^{2\pi i/p}$, $\alpha = 2 - a^3 - \bar{\alpha}^3$, $\beta = (\bar{a}^2 - a)\tau$. The Hermitian form is given by $\langle x, y \rangle = x^*Hy$ where

$$H = \begin{pmatrix} \alpha & \beta & \bar{\beta} \\ \bar{\beta} & \alpha & \beta \\ \beta & \bar{\beta} & \alpha \end{pmatrix}$$

With p, τ as above, the groups to consider are $\Gamma(p, \tau) = \langle J, R_1, R_2, R_3 \rangle$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} a^2 & \tau & -a\bar{\tau} \\ 0 & \bar{a} & 0 \\ 0 & 0 & \bar{a} \end{pmatrix}.$$

and $R_2 = JR_1J^{-1}$, $R_3 = J^{-1}R_1J$.

Theorem 37 (Theorem 1.1 in [16]). $\Gamma(p, \tau)$ is a lattice precisely when $p = 3, 4, 5, 6, 8, 12$. Moreover, $\Gamma(p, \tau)$ is cocompact if and only if $p = 3, 5, 8, 12$, and $\Gamma(p, \tau)$ is arithmetic if and only if $p = 3$.

Let m_i be the mirror for the complex reflection R_i . A relatively straightforward computation show that points in m_1 have the form $[1, (-\bar{\beta}z - \alpha)/\beta, z]^T$ for a complex parameter z , and applications of J give the parameterizations for points in m_2 and m_3 . For simplicity, we'll define $M_1(z) = [1, (-\bar{\beta}z - \alpha)/\beta, z]^T$.

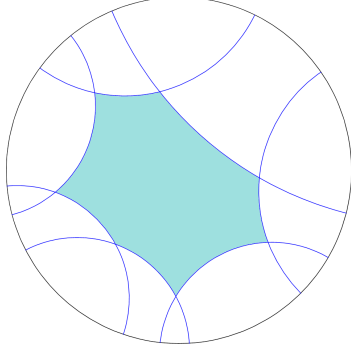


Figure 5.1: $p = 3$

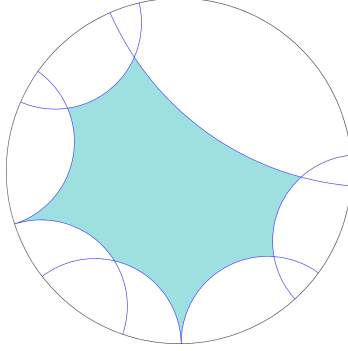


Figure 5.2: $p = 4$

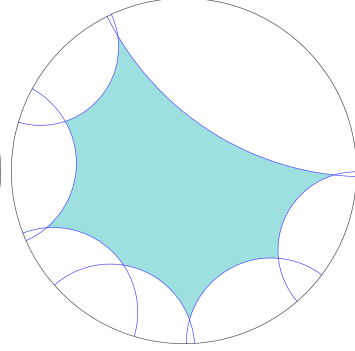


Figure 5.3: $p = 5$

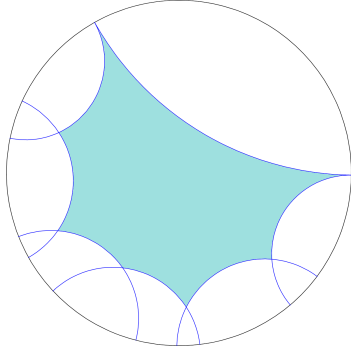


Figure 5.4: $p = 6$

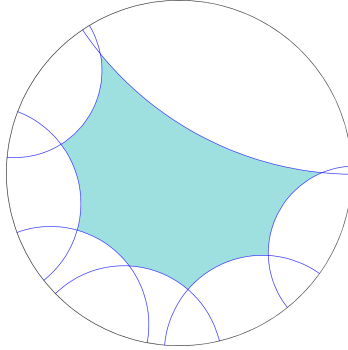


Figure 5.5: $p = 8$

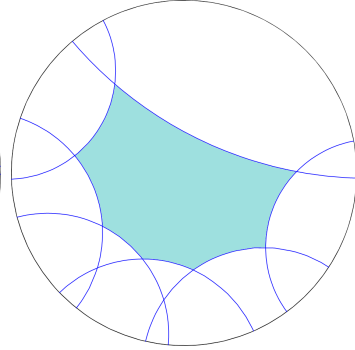


Figure 5.6: $p = 12$

Proposition 38. *Let $\Lambda_1 = \langle (R_1R_2)^2, (R_1R_3)^2, (R_1R_2R_3R_2^{-1})^3, (R_1R_3^{-1}R_2R_3)^3 \rangle$. Then $\Lambda_1 \leq \text{Stab}(m_1)$ and $\Gamma_1 = \Lambda_1|_{m_1}$ is (isomorphic to) a lattice in $\text{PU}(1,1)$.*

Proof. That Λ_1 stabilizes m_1 is a straightforward check that the polar vector to m_1 is an eigenvector of each of the generators. One can see that the restriction to m_1 is a lattice in $\text{PU}(1,1)$ by looking at the Dirichlet domain based at $M_1 \left(\frac{\alpha\bar{\beta}^2 - \alpha^2\beta}{4\alpha^2 - \beta^3 - \bar{\beta}^3} \right)$. In particular, the bisectors corresponding to the actions of elements $(R_1R_2)^2$, $(R_1R_2)^{-2}$, $(R_1R_3)^2$, $(R_1R_3)^{-2}$, $(R_1R_2R_3R_2^{-1})^{-3}$, $(R_1R_3^{-1}R_2R_3)^3$, and $(R_1R_2)^2(R_1R_3)^2(R_1R_3^{-1}R_2R_3)^3$ form a finite-area heptagon that (possibly strictly) contains a fundamental domain for Λ_1 . See Figures 5.1, 5.2, 5.3, 5.4, 5.5, 5.6. \square

Question 39. Is Γ_1 arithmetic?

The action of $(R_1R_2)^2$ on m_1 gives rise to an element of $\text{PU}(1,1)$ which we can lift to an element of $\text{SU}(1,1)$, call this A . Similarly, let B correspond to $(R_1R_3)^2$, C

correspond to $(R_1 R_2 R_3 R_2^{-1})^3$, and D correspond to $(R_1 R_3^{-1} R_2 R_3)^3$. As such, $\Gamma_1 = \langle A, B, C, D \rangle$, where the generators are given explicitly by

$$\begin{aligned}
A &= \begin{pmatrix} -i\bar{a}^3 & 0 \\ ia - ia^2\bar{\tau} & ia^3 \end{pmatrix} \\
B &= -\frac{i}{\tau} \begin{pmatrix} -a^3 - \bar{\tau} & -\bar{\tau}a^4 - \tau a - \bar{a}^2 \\ 2\bar{a} - a^2\bar{\tau} & -\tau a^3 + a^3 + \tau\bar{a}^3 + \bar{\tau} \end{pmatrix} \\
C &= -\frac{1}{\tau\sqrt{\bar{a}^3}} \begin{pmatrix} -i + i\tau\bar{a}^3 - i\tau a^3 - i\bar{\tau}\bar{a}^6 & ia + i\bar{\tau}\bar{a}^2 - i\bar{a}^5 - i\bar{\tau}\bar{a}^8 \\ i\bar{\tau}\bar{a} - i\tau a^2 - i\bar{a}^4 & i - i\tau\bar{a}^3 - i\bar{a}^6 \end{pmatrix} \\
D &= -\frac{1}{\tau\sqrt{\bar{a}^3}} \begin{pmatrix} -\bar{\tau}i\bar{a}^3 + i\bar{\tau}a^3 - i + i\tau + i\bar{a}^6 & 2ia + 2i\bar{\tau}\bar{a}^2 - i\bar{\tau}a^4 - 2i\bar{a}^5 - i\bar{\tau}\bar{a}^8 \\ i\bar{\tau}\bar{a} - ia^2 - i\bar{\tau}\bar{a}^4 & i\bar{\tau}\bar{a}^3 + ia^3 - i\bar{a}^6 + i\tau\bar{a}^6 + i - i\tau \end{pmatrix}
\end{aligned}$$

In order to determine arithmeticity, it should be straightforward to determine the appropriate Hermitian form for Γ_1 and the adjoint trace field to then apply Lemma 7.

The arrangement of the mirrors have different configurations for $p = 3$ (where they intersect in $\mathbf{H}_{\mathbb{C}}^2$), for $p = 4$ (where they intersect in $\partial_{\infty}\mathbf{H}_{\mathbb{C}}^2$), and for $p \geq 5$ (where they are ultraparallel). As such, it may be useful to explore each of these cases separately.

5.1.1 The Case $p \geq 5$

Since the m_i are all ultraparallel, there is a unique perpendicular passing through each pair. We follow with [16] and let m_{ij} denote the unique complex line perpendicular to both m_i and m_j . In particular, m_{12} is stabilized by reflections R_1 and R_2 , so we can find elements X_1, X_2 in $SU(1, 1)$ corresponding to their actions, respectively.

$$X_1 = \frac{1}{\sqrt{a^3}} \begin{pmatrix} a^3 & a\tau \\ 0 & 1 \end{pmatrix} \quad X_2 = \frac{1}{\sqrt{a}} \begin{pmatrix} \bar{a} & 0 \\ -\bar{\tau}a & a^2 \end{pmatrix}$$

Lemma 40. *Let $\Lambda_2 = \langle R_1, R_2 \rangle$. Then $\Gamma_2 := \Lambda_2|_{m_{12}}$ is a cocompact arithmetic lattice in $\mathrm{PU}(1, 1)$*

Proof. X_1 and X_2 both act on m_{12} by order- p rotations about distinct points, and their product is elliptic of order 2, whence Γ_2 is a $(2, p, p)$ -triangle group. Comparing with the classification of arithmetic triangle groups in [56], we see that Γ_3 is arithmetic for $p = 5, 6, 8, 12$. \square

Question 41. For $p \geq 5$, is $H(\Gamma_1, \Gamma_2) = \langle \Lambda_1, \Lambda_2 \rangle$ a lattice in $\Gamma(p, \tau)$?

The computer algebra system Magma [6] [5] is unable to compute the index of this subgroup, suggesting the answer to this question is likely false. It is presently unclear how to approach, however, as the techniques used in [49] do not seem to carry over in this case.

5.2 Hybrids in $\mathrm{PU}(n, 1)$ for $n \geq 3$

Given the success of the hybrid construction producing arithmetic lattices in $\mathrm{PU}(2, 1)$, the next obvious challenge is to try it in $\mathrm{PU}(3, 1)$ (where, up to commensurability, we only have two examples of non-arithmetic lattices, see [13] and references therein). For $n \geq 3$, the third condition of the construction (that $\Lambda_1 \cap \Lambda_2$ be a lattice in $\mathrm{PU}(n - 1, 1)$) is nontrivial. It is presently unclear if the assumption is strictly necessary, but certainly without it, it is not too difficult to construct a non-discrete hybrid, as the following example shows.

Example 42. Consider the ball model of $\mathbf{H}_{\mathbb{C}}^3$. Let $\Gamma_1 = \mathrm{SU}(2, 1; \mathcal{O}_1)$, $\Gamma_2 = \mathrm{SU}(2, 1; \mathcal{O}_3)$, and $\Lambda_j = \iota_j$ where ι_j are the following embeddings of $\mathrm{SU}(2, 1)$ into $\mathrm{SU}(3, 1)$:

$$\iota_1 : \begin{pmatrix} A_{1 \times 1} & A_{1 \times 2} \\ A_{2 \times 1} & A_{2 \times 2} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_{1 \times 1} & A_{1 \times 2} \\ 0 & A_{2 \times 1} & A_{2 \times 2} \end{pmatrix} \quad \iota_2 : \begin{pmatrix} A_{1 \times 1} & A_{1 \times 2} \\ A_{2 \times 1} & A_{2 \times 2} \end{pmatrix} \mapsto \begin{pmatrix} A_{1 \times 1} & 0 & A_{1 \times 2} \\ 0 & 1 & 0 \\ A_{2 \times 1} & 0 & A_{2 \times 2} \end{pmatrix}$$

The intersection of these subgroups is precisely the set of block-diagonal matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_{2 \times 2} \end{pmatrix}$$

where the submatrix $A_{2 \times 2} \in \text{SU}(1, 1)$ has coefficients in $\mathcal{O}_1 \cap \mathcal{O}_3 = \mathbb{Z}$. It is an elementary exercise to see that any element of $\text{SU}(1, 1)$ has the form

$$\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$$

from which it follows that $\text{SU}(1, 1; \mathbb{Z})$ is finite.

Consider the following element in $H(\Gamma_1, \Gamma_2)$:

$$A = \iota_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \cdot \iota_2 \begin{pmatrix} 1 + i\sqrt{3} & 0 & -i\sqrt{3} \\ 0 & 1 & 0 \\ i\sqrt{3} & 0 & 1 - i\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 + i\sqrt{3} & 0 & 0 & -i\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ \sqrt{3} & 0 & 0 & -i - \sqrt{3} \end{pmatrix}$$

S is elliptic with fixed point

$$\left[\left(-\frac{1}{6} - \frac{i}{6} \right) \left(-\sqrt{3} + \sqrt{6(\sqrt{3} - 1)} - 3 \right), 0, 0, 1 \right]$$

and all eigenvalues of S have absolute value 1. To have finite order, it must be that all of A 's eigenvalues are roots of unity. A has characteristic polynomial

$$(x - i)(x - 1) \left(x^2 + (1 - i)\sqrt{d}x - (1 - i)x - i \right)$$

and this divides the following polynomial in $\mathbb{Z}[x]$

$$p = (x - 1)(x^2 + 1)(x^8 - 4x^7 + 8x^6 + 12x^5 + 2x^4 + 12x^3 + 8x^2 - 4x + 1)$$

There are only finitely-many cyclotomic polynomials with degree at most 8, (and with the aid of a computer, one can determine these 18 polynomials explicitly). It is

straightforward to verify that p is divisible by exactly 2 cyclotomic polynomials, $x - 1$ and $x^2 + 1$, (with no multiplicity). It follows thusly that A can have at most 3 eigenvalues which are roots of unity, hence A is an elliptic with infinite order and therefore $H(\Gamma_1, \Gamma_2)$ is not discrete.

Remark. We remark that the only known nonarithmetic lattices in $SU(3, 1)$ (see [13]) are defined over $\mathbb{Q}(\zeta_{12})$, which is the composite of $\mathbb{Q}(i)$ and $\mathbb{Q}(i\sqrt{3})$.

One may ask whether there exist hybrids at all that do satisfy this intersection condition. When n is odd, indeed there is a fairly straightforward construction (somewhat similar to that used in [24]). Let E/F be an imaginary quadratic extension of a totally real number field, and let $\alpha_1, \dots, \alpha_{n+2} \in F$ such that $\alpha_1, \dots, \alpha_{n+1} > 0$ and $\alpha_{n+2} < 0$. Define the following Hermitian matrices:

$$H_{12} = \text{diag}\{\alpha_3, \alpha_4, \alpha_5, \dots, \alpha_{n+2}\}$$

$$H_1 = \text{diag}\{\alpha_2, \alpha_3, \alpha_4, \dots, \alpha_{n+2}\}$$

$$H_2 = \text{diag}\{\alpha_1, \alpha_3, \alpha_4, \dots, \alpha_{n+2}\}$$

$$H = \text{diag}\{\alpha_1, \dots, \alpha_{n+2}\}$$

Suppose further that α_1, α_2 are in two different classes of $F^\times/N_{E/F}(E^\times)$. By the work of Borel–Harish-Chandra, it follows that $\Gamma_1 := SU(H_1, \mathcal{O}_E)$ is an arithmetic lattice in $SU(H_1)$ and $\Gamma_2 := SU(H_2, \mathcal{O}_E)$ is an arithmetic lattice in $SU(H_2)$ (Γ_1 and Γ_2 are *arithmetic lattices of the first type*, see Chapter 5 in [37]), and the Γ_i are not commensurable (see Theorem 6.11 of [37] or Chapter 10 of [55]). We can then consider the following maps from $GL(n + 1, \mathbb{C})$ into $GL(n + 2, \mathbb{C})$

$$\begin{aligned} \iota_1 : \begin{pmatrix} A_{1 \times 1} & A_{1 \times n} \\ A_{n \times 1} & A_{n \times n} \end{pmatrix} &\mapsto \begin{pmatrix} 1 & & \\ & A_{1 \times 1} & A_{1 \times n} \\ & A_{n \times 1} & A_{n \times n} \end{pmatrix} \\ \iota_2 : \begin{pmatrix} A_{1 \times 1} & A_{1 \times n} \\ A_{n \times 1} & A_{n \times n} \end{pmatrix} &\mapsto \begin{pmatrix} A_{1 \times 1} & & A_{1 \times n} \\ & 1 & \\ A_{n \times 1} & & A_{n \times n} \end{pmatrix} \end{aligned}$$

In this way, ι_1 embeds $\mathrm{SU}(H_1)$ into $\mathrm{SU}(H)$ and ι_2 embeds $\mathrm{SU}(H_2)$ into $\mathrm{SU}(H)$ so that their intersection is precisely $\mathrm{SU}(H_{12})$. $\Lambda_1 := \iota_1(\Gamma_1)$ stabilizes $\Sigma_1 = (1, 0, \dots, 0)^\perp$, $\Lambda_2 := \iota_2(\Gamma_2)$ stabilizes $\Sigma_2 = (0, 1, 0, \dots, 0)^\perp$, and the Σ_i are orthogonal. Moreover, $\Lambda_1 \cap \Lambda_2 \cong \mathrm{SU}(H_{12}, \mathcal{O}_E)$, an arithmetic lattice in $\mathrm{SU}(H_2)$, so indeed $\Lambda = \langle \Lambda_1, \Lambda_2 \rangle$ is a hybridization of Γ_1 and Γ_2 , although it is unclear if Λ is a lattice or arithmetic.

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