

Rigorous Proofs of Old Conjectures
and New Results
for Stochastic Spatial Models in Econophysics
by
Stephanie Jo Reed

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Graduate Supervisory Committee:

Nicolas Lanchier, Chair
Erika Camacho
Abba Gumel
Sebastien Motsch
Hal Smith

ARIZONA STATE UNIVERSITY

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ABSTRACT

This dissertation examines six different models in the field of econophysics using interacting particle systems as the basis of exploration. In each model examined, the underlying structure is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each $x \in \mathcal{V}$ represents an individual who is characterized by the number of coins in her possession at time t . At each time step t , an edge $(x, y) \in \mathcal{E}$ is chosen at random, resulting in an exchange of coins between individuals x and y according to the rules of the model. Random variables ξ_t , and $\xi_t(x)$ keep track of the current configuration and number of coins individual x has at time t respectively. Of particular interest is the distribution of coins in the long run. Considered first are the uniform reshuffling model, immediate exchange model and model with saving propensity. For each of these models, the number of coins an individual can have is nonnegative and the total number of coins in the system is conserved for all time. It is shown here that the distribution of coins converges to the exponential distribution, gamma distribution and a pseudo gamma distribution respectively. The next two models introduce debt, however, the total number of coins again remains fixed. It is shown here that when there is an individual debt limit, the number of coins per individual converges to a shifted exponential distribution. Alternatively, when a collective debt limit is imposed on the whole population, a heuristic argument is given supporting the conjecture that the distribution of coins converges to an asymmetric Laplace distribution. The final model considered focuses on the effect of cooperation on a population. Unlike the previous models discussed here, the total number of coins in the system at any given time is not bounded and the process evolves in continuous time rather than in discrete time. For this model, death of an individual will occur if they run out of coins. It is shown here that the survival probability for the population is impacted by the level of cooperation along with how productive the population is as whole.

DEDICATION

To my son Michael, this is for you.

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Chapter 1

INTRODUCTION

1.1 Econophysics

Plutarch once said “An imbalance between rich and poor is the oldest and most fatal ailment of all republics [4].” This imbalance between the rich and the poor is still as ever-present today as it has been throughout history [6]. In most societies that ever existed, wealth classes form over time even when each individual in the society begins with roughly the same amount of wealth. It is unclear why this occurs, and getting at the root cause of this has proven to be quite difficult in part because there isn’t much data available on individual wealth. Consider specifically the distribution of wealth in the United States; although each individual is required to report their yearly earnings for tax purposes, it would be highly unusual for one to report on the status of their total wealth at all let alone report this information on a regular basis. As such, using traditional statistical methods to examine the distribution of wealth across a population over time is seemingly untenable. The field of econophysics tries to solve problems such as this and provides tools to examine the distribution of wealth by modeling monetary interactions between individuals on a microscopic level. As its name implies, econophysics is the combining of economics and physics. Specifically, econophysics is the field of research where known problem solving techniques and principles in physics are interpreted for and applied to problems in economics. Appearing first in [17], the term econophysics was described well in this paper asserting that “behavior of large numbers of humans as measured, e.g., by economic indices might conform to analogs of the scaling laws that have proved useful in de-

scribing systems composed of large numbers of inanimate objects.” It was with this notion in mind that the research presented in this dissertation was conducted. Many models describing the exchange of money between individuals have been looked at by [3, 5, 6, 9, 11] among others. It is intuitive that examining the way money is exchanged between individuals on a granular scale could help to answer questions about the distribution of wealth in a population. This intuition is the motivation behind choosing the framework of interacting particle systems to model wealth flows in a population.

1.2 Interacting Particle Systems

In the fields of physics, biology, sociology, economics and others, the mathematical models used to describe the phenomena that happen within these respective fields are often lacking in helping to build an understanding of the role that spatial structure and/or social network (represented by a connected graph) plays in how the overall process will evolve in the long run. It is natural to make the assertion that this spatial component taking the form of local interactions is very important in many of these processes and that in fact, the spatial component can have a large impact on the evolution of such processes. For example, two groups of the same species who have become separated by some physical barrier will no longer be able to interact with one another and will evolve independently of one another. Interacting particle systems are mathematical models that take into account the spatial structure of the underlying process as well as the rules that govern the interactions between single particles, where particles are interpreted as a single unit in the population being modeled, i.e. a person, animal, cell, etc. Looking at the spatially explicit versions of these models allows one to deduce the overall effects that single interactions between particles can have on the process as a whole. The general framework for interacting

particle systems has each particle located on a vertex of a given graph. The graph provides the spatial structure for the process as only those particles connected by an edge can interact with one another and it is assumed that the underlying graph remains constant throughout the process. The process will evolve in time by allowing adjacent particles to interact with one another. Each interaction will result in a changing of properties of the participating particles according to the rules of the process. More rigorously, given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an interacting particle system is defined as a continuous-time or a discrete-time Markov chain where the state at time t is a configuration:

$$\xi_t : \mathcal{V} \rightarrow \mathcal{K}$$

where each $x \in \mathcal{V}$ is a spatial location for a particle in the process, and each $i \in \mathcal{K}$ corresponds to a type. It is said that $\xi_t(x)$ is the type of vertex x at time t . For each vertex $x \in \mathcal{V}$, the so-called interaction neighborhood:

$$N_x = \{y \in \mathcal{V} : (x, y) \in \mathcal{E}\}$$

is the set of vertices (particles) with whom x can interact.

The focus of this dissertation is to solve existing problems in the field of econophysics utilizing the framework of interacting particle systems. The uniform reshuffling model, immediate exchange model, model with saving propensity, and the models with debt are all well known in the field of econophysics. Discussed in chapters 2 and 3 are discretized versions of these five important models. In chapter 4, another model is introduced where the effect of cooperation between individuals is studied. In each of the models discussed in this dissertation, the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ gives the underlying spatial structure and each vertex x in the set \mathcal{V} represents an individual who is characterized by the amount of coins in his/her possession at time t . The goal of chapter 2, is to find the stationary distribution for the number of coins per individ-

ual for the uniform reshuffling model, immediate exchange model, and model with saving propensity. Similarly, the goal of chapter 3, is to find the stationary distribution for the number of coins per individual for the model with individual debt limit and the model with collective debt limit. For the five models discussed in chapter 2 and chapter 3, the underlying graph \mathcal{G} is a finite connected graph and remains fixed throughout the process. The goal of the chapter 4 is to determine the effect that cooperation has on the probability of survival of a population of individuals where death of an individual will occur if that individual needs to spend a coin but has no coin to spend. Unlike the other models discussed here, \mathcal{G} can be either finite or infinite and although \mathcal{G} is initially connected, it is possible for \mathcal{G} to become disconnected if a cut vertex is removed from the graph via death of the individual located there.

1.3 Models on Connected Graphs

Chapter 2 of this dissertation is concerned with three models: the uniform reshuffling model, the immediate exchange model and the model with saving propensity. Simulations of the uniform reshuffling model, which was first studied in [6], suggest the conjecture that when the number of individuals and the average number of coins per individual is large, the stationary distribution for the number of coins per individual is the exponential distribution. Simulations of the immediate exchange model done in [8] suggest a similar conjecture that the stationary distribution for the number of coins per individual is the gamma distribution. Simulations in [16] of the model with saving propensity suggest the conjecture that the stationary distribution for the number of coins per individual is a gamma distribution. In this dissertation, interacting particle systems are used to look at discrete versions of these three models, and for each, the underlying graph is finite and connected. Further, by design the processes are conservative in that at any given time t , there are exactly M coins in circulation.

Let $N = |\mathcal{V}|$ be finite and denote the number of individuals in the processes. For each process, the set of types is

$$\mathcal{K} = \{0, 1, \dots, M\}$$

so that no individual has a negative amount of coins at any time, and so that each individual always has an integer valued amount of coins.

1.3.1 *Uniform Reshuffling Model*

The simplest of the three models is the uniform reshuffling model. The dynamics of the process are as follows: at each time step t , an edge $(x, y) \in \mathcal{E}$ is chosen at random, the total amount of coins the two individuals x and y possess at that time are pooled together and then redistributed among the two individuals uniformly. The simulations performed in [6] for this model under the assumption that all the individuals are equally likely to interact suggest that, as the population size approaches infinity, the distribution of money converges to an exponential distribution.

1.3.2 *Immediate Exchange Model*

The dynamics of the immediate exchange model are as follows: at each time step t , an edge $(x, y) \in \mathcal{E}$ is chosen at random, then the interacting individuals x and y each choose a random number of coins to give to the other from their own set of coins uniformly. They then exchange the coins they have chosen to give. For this model, it was conjectured based on numerical simulations in [8] that, when the individuals are equally likely to interact, the distribution of money converges to a gamma distribution. This result has been proved analytically in [10] for an infinite-population version of the model.

1.3.3 Model with Saving Propensity

The dynamics of the model with saving are as follows: at each time step t , an edge $(x, y) \in \mathcal{E}$ is chosen at random, then the interacting individuals x and y each choose a random amount of their own coins to save for themselves. At this time remaining coins are redistributed according to the uniform reshuffling model. The numerical results in [4] show that, when all the individuals are equally likely to interact, the limiting distribution of money is similar but not exactly equal to a gamma distribution.

Chapter 2 of this dissertation not only gives rigorous proofs of the results in [4, 6, 8, 10], but also extends those results to any connected graph \mathcal{G} . Even though the local evolution rules for those three models are simplistic and do not take into account all the complexity of financial interactions in the real world, numerical simulations performed by other scientists show that the limiting behavior of those models matches well with real world data for the distribution of money.

1.4 Models with Debt

Chapter 3 of this dissertation is concerned with the two models introduced by [6, 18] where debt is included: the model with individual debt limit L_i and the model with collective debt limit L_c . These models have also been studied by others via simulations. These simulations suggest the conjectures that when the number of individuals and average number of coins per individual is large, the stationary distribution for the number of coins per individual is a shifted exponential distribution for the model with individual debt limit L_i , and an asymmetric Laplace distribution for the model with collective debt limit L_c . Here interacting particle systems are used to study discrete versions of these models where the underlying graph \mathcal{G} is once again

assumed to be finite and connected. For simplicity, \mathcal{G} will be a directed multigraph such that $(x, y) \in \mathcal{E}$ if and only if $(y, x) \in \mathcal{E}$. Once again both models with debt are conservative in that at any given time, there are exactly M coins in circulation. For both models, each individual always has an integer valued amount of coins.

1.4.1 Model with Individual Debt Limit L_i

The model with individual debt limit has the set of types

$$\mathcal{K} = \{-L_i, -L_i + 1, \dots, 0, \dots, M + (N - 1)L_i\}$$

which means that each individual can have no more than $M + (N - 1)L_i$ coins and no less than $-L_i$ coins at any time. The dynamics of this process are as follows: at each time step t , an edge $(x, y) \in \mathcal{E}$ is chosen at random, and if person x has more than $-L_i$ coins he gives one coin to person y , otherwise the exchange is cancelled.

1.4.2 Model with Collective Debt Limit L_c

In the model with collective debt limit, let $\mathcal{V}^* = \mathcal{V} \cup \{\star\}$, where \star represents the location of a bank. The bank loans out at most L_c coins in total and once those L_c coins are borrowed, it will not loan out more coins regardless of the distribution of debt among the individuals. This model has the set of types

$$\mathcal{K} = \{-L_c, -L_c + 1, \dots, 0, \dots, M + L_c\}$$

so that each individual can have no more than $M + L_c$ (all coins in circulation plus all coins from the bank) and no less than $-L_c$ (bank is empty and only one individual is in debt) coins at any time. The dynamics of this process are as follows: at each time step t , an edge $(x, y) \in \mathcal{E}$ is chosen at random and there are four possibilities for the interaction between x and y :

Case 1: ($\xi_t(x) > 0$ and $\xi_t(y) \geq 0$) In this case person x gives one coin to person y .

Case 2: ($\xi_t(x) \leq 0$ and $\xi_t(y) \geq 0$) In this case, if the bank has at least one coin left to loan, person x borrows a coin from the bank and gives it to person y , otherwise the exchange is cancelled.

Case 3: ($\xi_t(x) > 0$ and $\xi_t(y) < 0$) In this case, person x gives one coin to person y who then immediately gives this coin to the bank to pay off some of her debt.

Case 4: ($\xi_t(x) \leq 0$ and $\xi_t(y) < 0$) In this case, if the bank has at least one coin left to loan, person x borrows a coin from the bank and gives it to person y who then immediately gives this coin to the bank to pay off some of her debt. If the bank has already loaned out all L_c of its coins, the exchange is cancelled.

Previous conjectures by those who have worked on the discrete model with individual debt limit L_i are proved in chapter 3. Also contained in chapter 3 is an expression for the stationary distribution for the model with collective debt limit L_c . Thus far, the expression for the stationary distribution for the model with collective debt limit L_c is too complicated to simplify to prove the conjecture made by others that the stationary distribution for the number of coins per individual is Laplace. However, a computer was used to plot the Laplace distribution alongside the actual stationary distribution which yielded an almost perfect fit giving more evidence that the conjecture is in fact true. Furthermore, it is shown in chapter 3 that the process $\xi_t(\star)$ is a supermartingale. This insight is the basis for a heuristic argument in favor of the conjecture that the stationary distribution for the number of coins per individual

is asymmetric Laplace.

1.5 Model with Cooperation

Chapter 4 of this dissertation is concerned with processes where cooperation plays a role in the dynamics of the system. In this model, the underlying graph \mathcal{G} is a locally finite connected graph which can be either finite or infinite in order, and the process is a continuous-time Markov chain with set of types $\mathbb{Z}^+ \cup \{-1, 0\}$.

The dynamics of this process is as follows: Each agent x in the system spends one coin at rate one and earns one coin at rate ϕ_x chosen from a fixed distribution ϕ . To incorporate the concept of cooperation in the system, fix a nonnegative constant μ as the rate of cooperation. For any pair of neighbors on the graph, the “richer” of the two x gives the “poorer” of the two y one coin at rate μ assuming this interaction will not result in making x more poor than y . If at any time during the process an agent with zero coin needs to spend a coin, she dies and is assigned type -1 for the remainder of the process. Any agent who has died can no longer interact, spend or earn coins.

The reason why a continuous-time Markov chain is used, as opposed to a discrete-time Markov chain as with the first five models, is because the graph \mathcal{G} can be infinite. Indeed, when the number of individuals is countably infinite, choosing a pair of individuals uniformly at random like for the first five models is not well defined. Instead, it is necessary to use independent Poisson processes to describe the times at which pairs of neighbors interact, which results in a continuous-time Markov chain.

1.5.1 Model with Cooperation Where \mathcal{G} Is Finite

First studied are the simple cases on a finite connected graph where there is either perfect cooperation ($\mu = \infty$) or no cooperation ($\mu = 0$). It is shown that in the

case that in average the population spends more money than they need to survive, perfect cooperation results in less individuals expected to survive than if there were no cooperation. On the other hand, when in average the population earns more money than they need to survive, perfect cooperation results in more individuals expected to survive than if there were no cooperation.

1.5.2 Model with Cooperation Where \mathcal{G} Is the One Dimensional Integer Lattice

Finally studied is the system where \mathcal{G} is the one dimensional integer lattice. In the case where the population earns less than they need to survive in average, it is shown that the proportion of individuals who eventually die has a positive lower bound that does not depend on the number of coins each agent starts with or the value of μ .

Chapter 2

RIGOROUS RESULTS FOR THE DISTRIBUTION OF MONEY ON CONNECTED GRAPHS

2.1 Abstract

This paper is concerned with general spatially explicit versions of three stochastic models for the dynamics of money that have been introduced and studied numerically by statistical physicists: the uniform reshuffling model, the immediate exchange model and the model with saving propensity. All three models consist of systems of economical agents that consecutively engage in pairwise monetary transactions. Computer simulations performed in the physics literature suggest that, when the number of agents and the average amount of money per agent are large, the limiting distribution of money as time goes to infinity approaches the exponential distribution for the first model, the gamma distribution with shape parameter two for the second model and a distribution similar but not exactly equal to a gamma distribution whose shape parameter depends on the saving propensity for the third model. The main objective of this paper is to give rigorous proofs of these conjectures and also extend these conjectures to generalizations of the first two models and a variant of the third model that include local rather than global interactions, i.e., instead of choosing the two interacting agents uniformly at random from the system, the agents are located on the vertex set of a general connected graph and can only interact with their neighbors.

2.2 Introduction

The objective of this paper is to give rigorous proofs of various conjectures (as well as extensions of these conjectures) about general spatially explicit versions of models in econophysics describing the dynamics of money within a population of economical agents. The term econophysics was coined by physicist Eugene Stanley to refer to the subfield of statistical physics that applies concepts from traditional physics to economics. The terminology is motivated by the idea that molecules can be viewed as individuals, energy as money, and collisions between two molecules as exchanges of money between two individuals. The models we consider in this paper are known in the mathematics literature as interacting particle systems [8] and are inspired from models for the dynamics of money reviewed in [10] that consist of large systems of N economic agents that interact to engage in pairwise monetary transactions. The models in [10] are examples of discrete-time Markov chains where, at each time step, two agents are selected uniformly at random to interact, which results in an exchange of money between the two agents in an overall conservative system, meaning that the total amount of money in the system, say M dollars, remains constant. By analogy with the temperature in physics, the average amount of money per agent $T = M/N$ is called the money temperature. The main problem about these models is to find the limiting distribution of money, which is mathematically defined as the limit as time goes to infinity of the probability that a given agent has a given amount of money.

The first paper introducing such models is [3] where several rules for the exchange of money are considered. In the most natural version, called the uniform reshuffling model, the total amount of money the two interacting agents possess before the interaction is uniformly redistributed between the two agents after the interaction. More precisely, using the same notation as in the review [10] and letting m_i and m'_i

be the amount of money agent i has before and after the interaction, respectively, an interaction between agents i and j results in the update

$$\begin{aligned} m_i &\rightarrow m'_i = \epsilon(m_i + m_j) \\ m_j &\rightarrow m'_j = (1 - \epsilon)(m_i + m_j) \end{aligned} \quad \text{where } \epsilon = \text{Uniform}(0, 1). \quad (2.1)$$

The computer agent-based simulations performed in [3] suggest that, for all the rules under consideration including (2.1), the limiting distribution of money approaches an exponential distribution with mean T when both the population size N and the money temperature T are large, i.e., the probability that a given individual has m dollars approaches

$$P(m) = \frac{1}{T} e^{-m/T} \quad \text{where } T = M/N.$$

Strictly speaking, since the independent uniform random variables ϵ used at each update are continuous, the probability of having exactly m dollars is equal to zero, so the limit above given in the physics literature has to be understood as follows: As time goes to infinity, the probability that a given individual has at least m dollars approaches

$$\int_m^\infty \frac{1}{T} e^{-x/T} dx = e^{-m/T}.$$

One of the models in [3] assumes that one of the two interacting agents chosen at random gives one dollar to the other agent if she indeed has at least one dollar. The simulations in [3] suggest that the limiting distribution of money for this model also approaches the exponential distribution, which has been proved analytically and extended to general spatial models in [7].

The second model we consider in this paper is inspired from the so-called immediate exchange model introduced and studied numerically in [4]. In this model, two agents are again chosen uniformly at random at each time step, but we now assume that each of the two interacting agents gives a random fraction of her fortune to the

other agent. More precisely, an interaction between agents i and j results in the update

$$\begin{aligned} m_i &\rightarrow m'_i = (1 - \epsilon_i) m_i + \epsilon_j m_j \\ m_j &\rightarrow m'_j = (1 - \epsilon_j) m_j + \epsilon_i m_i \end{aligned} \quad \text{where } \epsilon_i, \epsilon_j = \text{Uniform}(0, 1) \quad (2.2)$$

are independent. Note that the uniform reshuffling model (2.1) can be obtained from the immediate exchange model (2.2) by assuming that the two uniform random variables used at each update are not independent but instead satisfy $\epsilon_i + \epsilon_j = 1$. Interestingly, this slight change in the interaction rules creates a new behavior. Indeed, the numerical simulations in [4] suggest that the limiting distribution of money now approaches a gamma distribution with mean T and shape parameter two when the population size and the money temperature are large:

$$P(m) = \frac{4m}{T^2} e^{-2m/T} \quad \text{where } T = M/N.$$

As previously, this limit has to be understood as follows: As time goes to infinity, the probability that a given individual has at least m dollars approaches

$$\int_m^\infty \frac{4x}{T^2} e^{-2x/T} dx = \left(1 + \frac{2m}{T}\right) e^{-2m/T}.$$

Shortly after the publication of [4], the convergence to the gamma distribution has been proved analytically in [5] for an infinite-population version of the immediate exchange model.

The third and last model we consider in this paper is inspired from another generalization of the uniform reshuffling model that includes saving propensity [2]. The two interacting agents now save a fixed fraction λ of their fortune and only the combined remaining fortune is reshuffled randomly between the two agents, which makes the uniform reshuffling model the particular case $\lambda = 0$. In equations, an interaction

between agents i and j results in the update

$$\begin{aligned} m_i &\rightarrow m'_i = \lambda m_i + \epsilon(1 - \lambda)(m_i + m_j) \\ m_j &\rightarrow m'_j = \lambda m_j + (1 - \epsilon)(1 - \lambda)(m_i + m_j) \end{aligned} \quad \text{where } \epsilon = \text{Uniform}(0, 1). \quad (2.3)$$

The computer simulations performed in [9] suggest that the limiting distribution of money converges to a gamma distribution. However, the results in [1] show that the limiting distribution is similar but not exactly equal to a gamma distribution with mean T and shape parameter r that depends on the saving propensity as follows:

$$P(m) = \frac{1}{\Gamma(r)} \left(\frac{r}{T}\right)^r m^{r-1} e^{-rm/T} \quad \text{where } T = M/N \quad \text{and} \quad r = \frac{1 + 2\lambda}{1 - \lambda}.$$

This again has to be understood as follows: As time goes to infinity, the probability that a given individual has at least m dollars is close to

$$\int_m^\infty \frac{1}{\Gamma(r)} \left(\frac{r}{T}\right)^r x^{r-1} e^{-rx/T} dx \quad \text{where } r = \frac{1 + 2\lambda}{1 - \lambda}.$$

Note that the distribution above reduces to the exponential distribution when $\lambda = 0$, in accordance with the numerical results for the uniform reshuffling model in [3]. Note also that the gamma distribution with shape parameter $r = 2$, which approximates the limiting distribution of the immediate exchange model, is obtained by setting $\lambda = 1/4$.

2.3 Model Description

The models we study in this paper are discrete-state versions of the models (2.1)–(2.3) that also include a spatial structure in the form of local interactions.

- Discrete-state versions means that we assume that there is a total of M coins in the system, where M is a nonnegative integer, and that individuals are characterized by the number of coins they possess, which we again assume to be a nonnegative integer. In particular, the fortune of each individual is a discrete

quantity rather than a continuous one, and each exchange of money can only result in a finite number of outcomes.

- Local interactions, as opposed to global interactions where any two individuals in the system may interact at each time step, means that individuals are located on the set of vertices \mathcal{V} of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that we assume to be connected, and that only neighbors, i.e., individuals connected by an edge $e \in \mathcal{E}$, can interact to exchange coins. The graph \mathcal{G} has to be thought of as representing a social network where only individuals connected by an edge (friends, business partners, etc.) can interact to exchange money.

As previously, we let $N = \text{card}(\mathcal{V})$ be the population size. Each of the three models is again a discrete-time Markov chain but the state at time $t \in \mathbb{N}$ is now a spatial configuration

$$\xi : \mathcal{V} \rightarrow \mathbb{N} \quad \text{where} \quad \xi(x) = \text{number of coins at vertex } x.$$

In addition to the fact that the amount of money individuals possess is discrete rather than continuous, the main difference with the non-spatial models described in the previous section is that, at each time step, the interacting pair is not selected by choosing a pair uniformly at random but by choosing an edge $e \in \mathcal{E}$ uniformly at random. Note that the non-spatial models in the previous section can be viewed as particular cases where \mathcal{G} is the complete graph with N vertices.

Uniform reshuffling model. The version of the uniform reshuffling model (X_t) we consider evolves in discrete time as follows. At each time step, say t , an edge (x, y) is chosen uniformly at random from the edge set \mathcal{E} , which results in an interaction between the economical agents at vertex x and at vertex y . Following [3], we assume that the total amount of coins both agents have at time t is uniformly redistributed

between the two agents at time $t + 1$. Since each coin is treated as an indivisible unit, the number of outcomes is finite. More precisely, we let

$$U = \text{Uniform} \{0, 1, \dots, X_t(x) + X_t(y)\} \quad (2.4)$$

and update the configuration by setting

$$X_{t+1}(x) = U \quad \text{and} \quad X_{t+1}(y) = X_t(x) + X_t(y) - U \quad (2.5)$$

while $X_{t+1} \equiv X_t$ on the set $\mathcal{V} - \{x, y\}$. Note that

$$X_t(x) + X_t(y) - U = \text{Uniform} \{0, 1, \dots, X_t(x) + X_t(y)\} \quad \text{in distribution,}$$

indicating that, though (2.5) is not symmetric in x and y , the numbers of coins the agents at x and y receive from the interaction are indeed equal in distribution.

Immediate exchange model. In our version of the immediate exchange model (Y_t), we again choose an edge (x, y) uniformly at random at each time step, which results in an interaction between the two agents incident to the edge. Following [4], we now assume that the two agents give a (uniform) random number of their coins to the other agent. More precisely, we let

$$U_1 = \text{Uniform} \{0, 1, \dots, Y_t(x)\} \quad \text{and} \quad U_2 = \text{Uniform} \{0, 1, \dots, Y_t(y)\} \quad (2.6)$$

be independent, and update the configuration by setting

$$Y_{t+1}(x) = Y_t(x) - U_1 + U_2 \quad \text{and} \quad Y_{t+1}(y) = Y_t(y) - U_2 + U_1 \quad (2.7)$$

while $Y_{t+1} \equiv Y_t$ on the set $\mathcal{V} - \{x, y\}$.

Uniform saving model. As previously, an edge (x, y) is chosen uniformly at random at each time step, which results in an interaction between the two agents incident to the edge. In the original model with saving introduced in [2], each agent saves a

fixed (deterministic) fraction of their fortune and the combined remaining amount of money is uniformly redistributed between the two agents. In contrast, we add more randomness to the process by assuming that the number of coins each agent saves is also random. This results in a model (Z_t) that combines the previous two types of interactions: random exchange and uniform reshuffling. More precisely, we let

$$U_1 = \text{Uniform}\{0, 1, \dots, Z_t(x)\} \quad \text{and} \quad U_2 = \text{Uniform}\{0, 1, \dots, Z_t(y)\} \quad (2.8)$$

be independent. These are the random numbers of coins vertex x and vertex y save before the exchange. Then, given that $U_1 = c_x$ and $U_2 = c_y$, we let

$$U = \text{Uniform}\{0, 1, \dots, Z_t(x) + Z_t(y) - c_x - c_y\} \quad (2.9)$$

be the random number of coins vertex x gets after uniform reshuffling. In particular, the new configuration is obtained by setting

$$Z_{t+1}(x) = c_x + U \quad \text{and} \quad Z_{t+1}(y) = Z_t(x) + Z_t(y) - c_x - U \quad (2.10)$$

while $Z_{t+1} \equiv Z_t$ on the set $\mathcal{V} - \{x, y\}$. Note that the number of coins at vertex y after the interaction can be written and interpreted in the following manner:

$$Z_{t+1}(y) = Z_t(x) + Z_t(y) - c_x - U = c_y + (Z_t(x) + Z_t(y) - c_x - c_y - U)$$

which is the number of coins vertex y saves before the interaction plus the number of coins vertex y gets after uniform reshuffling of the coins involved in the exchange.

2.4 Main Results

Numerical simulations of the uniform reshuffling model (2.4)–(2.5) on the complete graph suggest that the limiting distribution of money approaches the exponential distribution

$$\frac{1}{T} e^{-c/T} \quad \text{for all } c = 0, 1, \dots, M,$$

shown in Figure 2.1 when the number of vertices and the money temperature are large. This is in agreement with the numerical results found for the continuous counterpart (2.1). Similarly, numerical simulations of the immediate exchange model (2.6)–(2.7) on the complete graph are in agreement with the numerical results found for the continuous counterpart (2.2), suggesting again that the limiting distribution of money approaches in this case the gamma distribution

$$\frac{4c}{T^2} e^{-2c/T} \quad \text{for all } c = 0, 1, \dots, M,$$

shown in Figure 2.2 when the number of vertices and the money temperature are large. These results are expected since our versions of the uniform reshuffling and immediate exchange models are good approximations of models (2.1) and (2.2) when the number of coins is large. Now, in contrast with model (2.3), our version of the uniform saving model (2.8)–(2.10) does not include any parameter measuring the saving propensity. As for the immediate exchange model, simulations of the uniform saving model suggest convergence to the gamma distribution with shape parameter two, which is close to the limit of model (2.3) with saving propensity $\lambda = 1/4$.

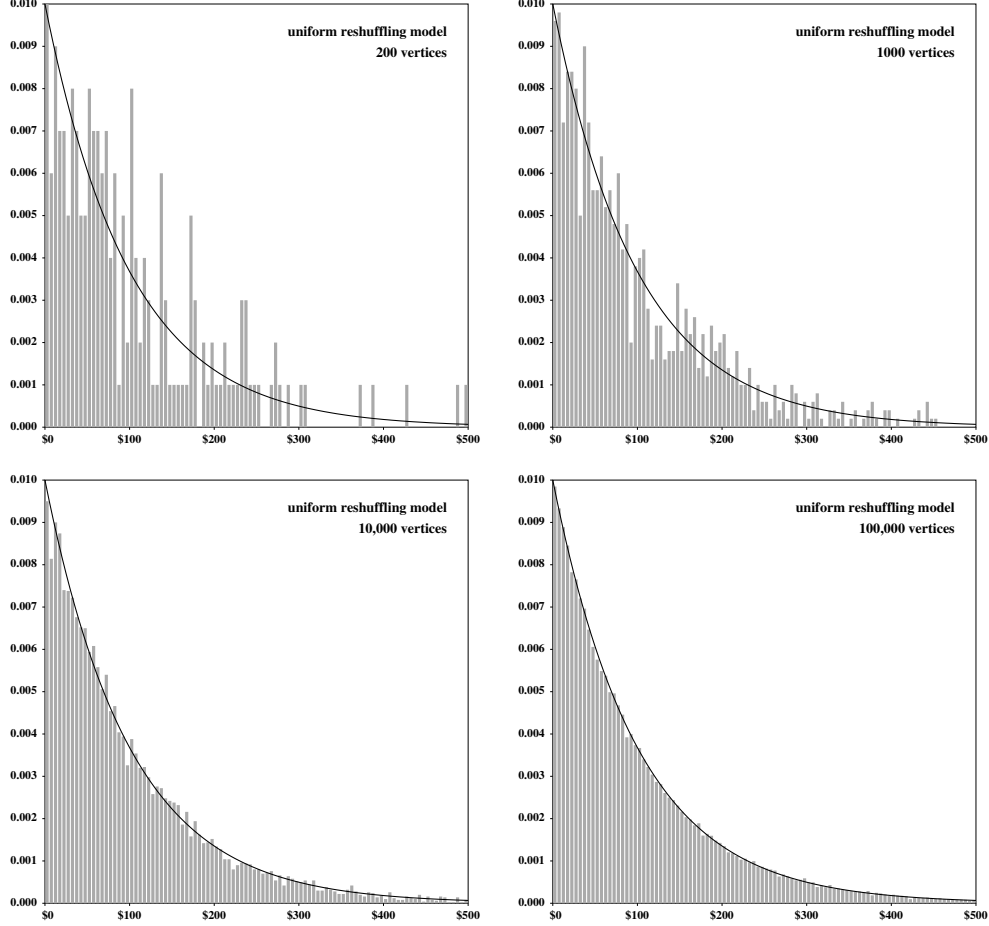


Figure 2.1: Simulation Results for a Single Realization of the Uniform Reshuffling Model (2.4)–(2.5) on the Complete Graph. The Number of Vertices Used in Each Simulation Is Indicated at the Top Right of the Pictures. For Each of the Four Simulations, All the Vertices Start with \$100. The Gray Histograms Represent the Distribution of Money After 10^6 Updates While the Black Solid Curve Is the Exponential Distribution with Mean $T = 100$.

Our analytical results for the three models (2.4)–(2.10) not only give rigorous proofs of the three conjectures above when the number of vertices and the money temperature are large, they also extend these conjectures in several directions:

1. The convergence to a distribution of money that approaches the exponential distribution or the gamma distribution holds regardless of the initial configuration of the system while the numerical results in [3, 4, 9] assume that each agent starts with T dollars.

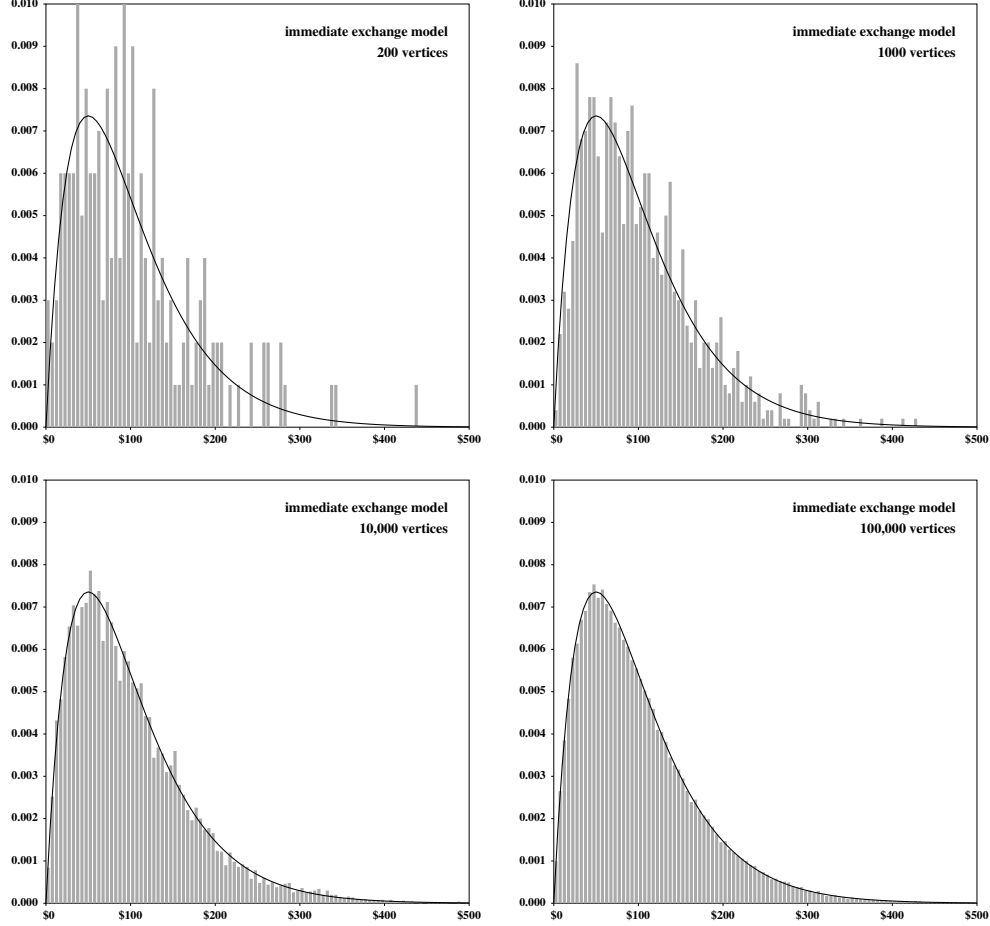


Figure 2.2: Simulation Results for a Single Realization of the Unmmiedate Rexchange Model (2.6)–(2.7) on the Complete Graph. The Number of Vertices Used in Each Simulation Is Indicated at the Top Right of the Pictures. For Each of the Four Simulations, All the Vertices Start with \$100. The Gray Histograms Represent the Distribution of Money After 10^6 Updates While the Black Solid Curve Is the Gamma Distribution with Mean $T = 100$ and Shape Parameter Two.

2. The convergence to a distribution of money that approaches the exponential distribution or the gamma distribution holds for the general spatial models on any connected graphs while the numerical results in [3, 4, 9] focus on the complete graph only.

3. The results in 1 and 2 appear as particular cases of more general results that give the exact expression of the limiting distribution of money for all possible values of the population size and the money temperature while the conjectures in [3, 4, 9] are only true under the assumption that these two quantities are large.

The level of generality of our results is a good illustration of the advantage of using mathematical tools as opposed to computer simulations that cannot be performed for all possible connected graphs with all possible number of vertices containing all possible number of coins starting from all possible initial configurations.

We now state our results and briefly sketch their proofs. For all three models, there is a positive probability that an interaction between x and y results in the same number of coins moving from $x \rightarrow y$ and from $y \rightarrow x$ and therefore no change after the update. This shows that the processes are aperiodic. It can also be proved that the three processes are irreducible, which is an intrinsic consequence of the connectedness of the network of interactions. These two ingredients together with finiteness of the state space imply that, for each of the three models, there is a unique stationary distribution to which the process converges regardless of the initial configuration.

For the model with uniform reshuffling, the symmetry of the evolution rules can be used to prove that the probability of a transition from $\xi \rightarrow \eta$ is equal to the probability of a transition from $\eta \rightarrow \xi$ for any two configurations ξ and η . This implies that the process is doubly stochastic from which it follows that the unique stationary distribution is uniform on the set of all possible configurations, i.e., under the stationary distribution, all the configurations are equally likely. This can be used to obtain an explicit expression of the limiting distribution of money. Some basic algebra also implies that this distribution approaches the exponential distribution with mean T when N and T are both large, in agreement with the conjecture in [3].

Theorem 2.1 (uniform reshuffling) – *For all connected graph \mathcal{G} with N vertices, regardless of the number M of coins and the initial configuration,*

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) = \binom{M - c + N - 2}{N - 2} \bigg/ \binom{M + N - 1}{N - 1}.$$

In particular, when N and $T = M/N$ are large,

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) \approx \frac{1}{T} e^{-c/T}.$$

In contrast, the immediate exchange model is not doubly stochastic. However, one can use reversibility to have an implicit expression of the unique stationary distribution. Some combinatorial techniques lead to an explicit expression while some basic algebra implies that the limiting distribution of money approaches the gamma distribution with mean T and shape parameter two when N and T are both large, in agreement with the conjecture in [4].

Theorem 2.2 (immediate exchange) – *For all connected graph \mathcal{G} with N vertices, regardless of the number M of coins and the initial configuration,*

$$\lim_{t \rightarrow \infty} P(Y_t(x) = c) = (c + 1) \binom{M - c + 2N - 3}{2N - 3} \bigg/ \binom{M + 2N - 1}{2N - 1}.$$

In particular, when N and $T = M/N$ are large,

$$\lim_{t \rightarrow \infty} P(Y_t(x) = c) \approx \frac{4c}{T^2} e^{-2c/T}.$$

Turning to the uniform saving model, though its evolution rules are somewhat different from the evolution rules of the immediate exchange model, it can be proved that their respective stationary distributions satisfy the same detailed balance equation and therefore are equal. In particular, in contrast with the model with saving introduced in [2] that converges to a distribution similar but not exactly equal to a gamma distribution, the limiting distribution of money for our discrete version of the model

with saving converges exactly to the gamma distribution with shape parameter two in the large population and large temperature limit.

Theorem 2.3 (uniform saving) – *For all connected graph \mathcal{G} with N vertices, regardless of the number M of coins and the initial configuration,*

$$\lim_{t \rightarrow \infty} P(Z_t(x) = c) = (c + 1) \binom{M - c + 2N - 3}{2N - 3} \bigg/ \binom{M + 2N - 1}{2N - 1}.$$

In particular, when N and $T = M/N$ are large,

$$\lim_{t \rightarrow \infty} P(Z_t(x) = c) \approx \frac{4c}{T^2} e^{-2c/T}.$$

The rest of this paper is devoted to proofs. Theorem 2.1 is proved in Section 2.5. Sections 2.6 and 2.7 focus on the reversibility of the immediate exchange model and the uniform saving model, respectively, and give the corresponding detailed balance equations. Section 2.8 gives the common final step to complete the proof of Theorems 2.2 and 2.3.

2.5 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1 about the limiting distribution of money for the uniform reshuffling model (2.4)–(2.5). The proof relies on the following two key ingredients:

1. There exists a unique stationary distribution π_X to which the uniform reshuffling model converges starting from any initial configuration.
2. The uniform distribution on the set of all possible configurations is stationary.

With these two preliminary results in hand, the theorem follows using some basic combinatorics and some basic algebra. From now on, we let

$$\mathcal{C}_{N,M} = \{\xi : \mathcal{V} \rightarrow \mathbb{N} \text{ such that } \sum_{x \in \mathcal{V}} \xi(x) = M\}$$

be the set of all possible configurations with exactly M coins. Also, for every vertex $x \in \mathcal{V}$ and every configuration $\xi : \mathcal{V} \rightarrow \mathbb{N}$, we let

$$\xi^x(z) = \xi(z) + \mathbb{1}\{z = x\} \quad \text{for all } z \in \mathcal{V}$$

be the configuration obtained from ξ by adding one coin at vertex x . We now prove existence and uniqueness of the stationary distribution.

Lemma 2.4 – *There is a unique stationary distribution π_X and*

$$\lim_{t \rightarrow \infty} P_\eta(X_t = \xi) = \pi_X(\xi) \quad \text{for all } \xi, \eta \in \mathcal{C}_{N,M}.$$

Proof. According to [6, Theorem 7.7], it suffices to prove that the process is finite, irreducible and aperiodic. Finiteness is obvious while aperiodicity follows from the fact that

$$\begin{aligned} P(X_{t+1} = \xi \mid X_t = \xi) &= \frac{1}{\text{card}(\mathcal{E})} \sum_{(x,y) \in \mathcal{E}} P(\text{Uniform}\{0, 1, \dots, \xi(x) + \xi(y)\} = \xi(x)) \\ &= \frac{1}{\text{card}(\mathcal{E})} \sum_{(x,y) \in \mathcal{E}} \left(\frac{1}{\xi(x) + \xi(y) + 1} \right) \geq \left(\frac{1}{M+1} \right) > 0 \end{aligned}$$

for every configuration $\xi \in \mathcal{C}_{N,M}$. To prove that the process is also irreducible, let $x, y \in \mathcal{V}$. Since the graph is connected, there exists a path

$$(x_0, x_1, \dots, x_t) \subset \mathcal{V} \quad \text{such that} \quad x_0 = x, \quad x_t = y \quad \text{and} \quad t < N.$$

In particular, for all $\xi \in \mathcal{C}_{N,M-1}$,

$$\begin{aligned} P(X_t = \xi^y \mid X_0 = \xi^x) &\geq \prod_{s=0}^{t-1} P(X_{s+1} = \xi^{x_{s+1}} \mid X_s = \xi^{x_s}) \\ &= \prod_{s=0}^{t-1} \left(\frac{1}{\text{card}(\mathcal{E})} \right) \left(\frac{1}{\xi(x_s) + \xi(x_{s+1}) + 2} \right) \geq \left(\frac{1}{N^2(M+1)} \right)^N > 0. \end{aligned} \tag{2.11}$$

Let $m \leq M$. We deduce from (2.11) by induction that, for all

$$\xi \in \mathcal{C}_{N,M-m} \quad \text{and} \quad x, y \in \mathcal{V}^m$$

there exists $t < mN$ such that

$$P(X_t = (\dots (\xi^{y_1})^{y_2} \dots)^{y_m} \mid X_0 = (\dots (\xi^{x_1})^{x_2} \dots)^{x_m}) \geq \left(\frac{1}{N^2(M+1)} \right)^{mN}. \quad (2.12)$$

Since any $\xi, \eta \in \mathcal{C}_{N,M}$ can be obtained from the configuration with zero coin by adding M coins at the appropriate vertices, it follows from (2.12) with $m = M$ that

$$P(X_t = \eta \mid X_0 = \xi) \geq \left(\frac{1}{N^2(M+1)} \right)^{MN} > 0 \quad \text{for some } t < MN.$$

This shows that the process is irreducible and completes the proof. \square

The next lemma shows that the uniform reshuffling model is doubly stochastic, from which we deduce that the unique stationary distribution π_X from the previous lemma is the uniform distribution on the set of configurations $\mathcal{C}_{N,M}$.

Lemma 2.5 – *The unique stationary distribution is $\pi_X = \text{Uniform}(\mathcal{C}_{N,M})$.*

Proof. Let $\xi, \eta \in \mathcal{C}_{N,M}$. Note that $P(X_{t+1} = \eta \mid X_t = \xi) > 0$ if and only if

$$\xi \equiv \eta \text{ on } \mathcal{V} - \{x, y\} \quad \text{and} \quad \xi(x) + \xi(y) = \eta(x) + \eta(y)$$

for some $(x, y) \in \mathcal{E}$, in which case we have

$$\begin{aligned} P(X_{t+1} = \eta \mid X_t = \xi) &= \frac{1}{\text{card}(\mathcal{E})} P(\text{Uniform} \{0, 1, \dots, \xi(x) + \xi(y)\} = \eta(x)) \\ &= \frac{1}{\text{card}(\mathcal{E})} \frac{1}{\xi(x) + \xi(y) + 1}. \end{aligned}$$

In particular, either $P(X_{t+1} = \eta \mid X_t = \xi) = P(X_{t+1} = \xi \mid X_t = \eta) = 0$ or

$$\begin{aligned} P(X_{t+1} = \eta \mid X_t = \xi) &= \frac{1}{\text{card}(\mathcal{E})} \frac{1}{\xi(x) + \xi(y) + 1} \\ &= \frac{1}{\text{card}(\mathcal{E})} \frac{1}{\eta(x) + \eta(y) + 1} = P(X_{t+1} = \xi \mid X_t = \eta). \end{aligned}$$

This shows that the transition matrix of the process is symmetric and so doubly stochastic. Therefore, it follows from [6, Section 7.3] that the uniform distribution on

the set of configurations is stationary. By the uniqueness of the stationary distribution π_X established in the previous lemma, we conclude that $\pi_X = \text{Uniform}(\mathcal{C}_{N,M})$.

□

With Lemmas 2.4 and 2.5 in hand, we are now ready to prove the theorem.

Proof of Theorem 2.1. This is similar to the proofs of Lemma 4 and Theorem 1 in [7] that we briefly recall. First, we note that

$$\text{card}(\mathcal{C}_{N,M}) = \text{card} \{c \in \mathbb{N}^N : c_1 + \dots + c_N = M\} = \binom{M+N-1}{N-1}.$$

Since in addition all the configurations are equally likely under π_X according to Lemma 2.5, and since there are $\text{card}(\mathcal{C}_{N-1,M-c})$ configurations with exactly c coins at vertex x ,

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) = \frac{\text{card}(\mathcal{C}_{N-1,M-c})}{\text{card}(\mathcal{C}_{N,M})} = \binom{M-c+N-2}{N-2} \bigg/ \binom{M+N-1}{N-1}.$$

This shows the first part of the theorem. In addition, when N and T are large,

$$\begin{aligned} \lim_{t \rightarrow \infty} P(X_t(x) = c) &= \frac{(M-c+N-2) \cdots (M-c+1)}{(M+N-1) \cdots (M+1)} \frac{(N-1)!}{(N-2)!} \\ &= \frac{(M-c+N-2) \cdots (M-c+1)}{(M+N-2) \cdots (M+1)} \frac{(N-1)}{(M+N-1)} \\ &\approx \left(\frac{N}{NT} \right) \left(1 - \frac{c}{NT} \right)^N \approx \frac{1}{T} e^{-c/T}. \end{aligned}$$

This shows the second part of the theorem. □

2.6 Reversibility of the Immediate Exchange Model

This section collects preliminary results about the immediate exchange model (2.6)–(2.7) that will be useful to prove Theorem 2.2. As for the uniform reshuffling model, the first step is to show that there exists a unique stationary distribution π_Y to which

the immediate exchange model converges starting from any initial configuration. Contrary to the uniform reshuffling model, the process is not doubly stochastic and so the uniform distribution is no longer stationary. However, an implicit expression of the (unique) stationary distribution can be found using reversibility.

Lemma 2.6 – *There is a unique stationary distribution π_Y and*

$$\lim_{t \rightarrow \infty} P_\eta(Y_t = \xi) = \pi_Y(\xi) \quad \text{for all } \xi, \eta \in \mathcal{C}_{N,M}.$$

Proof. As for the uniform reshuffling model, it suffices to establish finiteness, irreducibility and aperiodicity. Finiteness is again obvious. Letting

$$U(z, \xi) = \text{Uniform} \{0, 1, \dots, \xi(z)\} \quad \text{for all } z \in \mathcal{V}$$

be independent, aperiodicity follows from the fact that

$$\begin{aligned} P(Y_{t+1} = \xi \mid Y_t = \xi) &= \frac{1}{\text{card}(\mathcal{E})} \sum_{(x,y) \in \mathcal{E}} P(U(x, \xi) = U(y, \xi)) \\ &= \frac{1}{\text{card}(\mathcal{E})} \sum_{(x,y) \in \mathcal{E}} \frac{\min(\xi(x) + 1, \xi(y) + 1)}{(\xi(x) + 1)(\xi(y) + 1)} \geq \left(\frac{1}{M+1} \right) > 0 \end{aligned}$$

for every $\xi \in \mathcal{C}_{N,M}$. Also, letting $(x, y) \in \mathcal{E}$ and $\xi \in \mathcal{C}_{N,M-1}$,

$$\begin{aligned} P(Y_{t+1} = \xi^y \mid Y_t = \xi^x) &= \frac{P(\text{Uniform} \{0, 1, \dots, \xi(x) + 1\} = U(y, \xi) + 1)}{\text{card}(\mathcal{E})} \\ &= \frac{1}{\text{card}(\mathcal{E})} \frac{\min(\xi(x) + 1, \xi(y) + 1)}{(\xi(x) + 2)(\xi(y) + 1)} \geq \frac{1}{N^2(2M+2)} > 0. \end{aligned}$$

Repeating the proof of Lemma 2.4, we deduce that, for all $\xi, \eta \in \mathcal{C}_{N,M}$,

$$P(Y_t = \eta \mid Y_0 = \xi) = \left(\frac{1}{N^2(2M+2)} \right)^{MN} > 0 \quad \text{for some } t < MN,$$

which shows irreducibility. \square

We now give an implicit expression of π_Y using reversibility.

Lemma 2.7 – *The distribution π_Y is reversible and*

$$\pi_Y(\xi) = \frac{\mu(\xi)}{\sum_{\eta \in \mathcal{C}_{N,M}} \mu(\eta)} \quad \text{where} \quad \mu(\xi) = \prod_{z \in \mathcal{V}} (\xi(z) + 1). \quad (2.13)$$

Proof. Let $\xi \neq \eta$ in $\mathcal{C}_{N,M}$ and assume that, for some $(x, y) \in \mathcal{E}$,

$$\xi \equiv \eta \text{ on } \mathcal{V} - \{x, y\} \quad \text{and} \quad \xi(x) + \xi(y) = \eta(x) + \eta(y). \quad (2.14)$$

Letting $U(z, \xi) = \text{Uniform}\{0, 1, \dots, \xi(z)\}$ be independent, we have

$$\begin{aligned} P(Y_{t+1} = \eta \mid Y_t = \xi) &= \frac{P(\xi(x) + U(y, \xi) - U(x, \xi) = \eta(x))}{\text{card}(\mathcal{E})} \\ &= \frac{1}{\text{card}(\mathcal{E})} \sum_{c_x=0}^{\xi(x)} \sum_{c_y=0}^{\xi(y)} \frac{\mathbb{1}\{c_x = \xi(x) - \eta(x) + c_y\}}{(\xi(x) + 1)(\xi(y) + 1)} \\ &= \frac{1}{\text{card}(\mathcal{E})} \sum_{c_x=0}^{\xi(x)} \frac{\mathbb{1}\{\xi(x) - \eta(x) \leq c_x \leq \xi(x) - \eta(x) + \xi(y)\}}{(\xi(x) + 1)(\xi(y) + 1)}. \end{aligned}$$

Since $\xi(x) - \eta(x) + \xi(y) = \eta(y)$, we get

$$\begin{aligned} Q_{x,y}(\xi, \eta) &= \text{card}(\mathcal{E}) (\xi(x) + 1)(\xi(y) + 1) P(Y_{t+1} = \eta \mid Y_t = \xi) \\ &= \min(\xi(x), \xi(x) - \eta(x) + \xi(y)) - \max(0, \xi(x) - \eta(x)) + 1 \\ &= \min(\xi(x), \eta(y)) + \min(\xi(x), \eta(x)) - \xi(x) + 1. \end{aligned}$$

Using also that $\eta(y) - \xi(x) = \xi(y) - \eta(x)$,

$$\begin{aligned} Q_{x,y}(\xi, \eta) &= \min(\xi(x), \eta(y)) + \min(\xi(x), \eta(x)) - \xi(x) + 1 \\ &= \min(0, \eta(y) - \xi(x)) + \min(\xi(x), \eta(x)) + 1 \\ &= \min(0, \xi(y) - \eta(x)) + \min(\xi(x), \eta(x)) + 1 \\ &= \min(\eta(x), \xi(y)) + \min(\eta(x), \xi(x)) - \eta(x) + 1 = Q_{x,y}(\eta, \xi) \end{aligned}$$

from which it follows that

$$\begin{aligned} & (\xi(x) + 1)(\xi(y) + 1) P(Y_{t+1} = \eta | Y_t = \xi) \\ &= (\eta(x) + 1)(\eta(y) + 1) P(Y_{t+1} = \xi | Y_t = \eta). \end{aligned} \tag{2.15}$$

If on the contrary condition (2.14) is not satisfied, since there are only two neighbors exchanging money at each time step, we must have

$$P(Y_{t+1} = \eta | Y_t = \xi) = P(Y_{t+1} = \xi | Y_t = \eta) = 0. \tag{2.16}$$

Combining (2.15) and (2.16), we conclude that, in any case,

$$\mu(\xi) P(Y_{t+1} = \eta | Y_t = \xi) = \mu(\eta) P(Y_{t+1} = \xi | Y_t = \eta) \quad \text{where} \quad \mu(\xi) = \prod_{z \in \mathcal{V}} (\xi(z) + 1).$$

By uniqueness, this implies that π_Y is reversible and satisfies (2.13). \square

2.7 Reversibility of the Uniform Saving Model

The objective of this section is to prove that Lemmas 2.6 and 2.7 in the previous section also hold for the uniform saving model (2.8)–(2.10). The main ideas behind the proofs are the same as for the immediate exchange model but the technical details are somewhat different.

Lemma 2.8 – *There is a unique stationary distribution π_Z and*

$$\lim_{t \rightarrow \infty} P_\eta(Z_t = \xi) = \pi_Z(\xi) \quad \text{for all } \xi, \eta \in \mathcal{C}_{N,M}.$$

Proof. Let $\xi \in \mathcal{C}_{N,M}$ and let

$$U(z, \xi) = \text{Uniform} \{0, 1, \dots, \xi(z)\} \quad \text{and} \quad U_c = \text{Uniform} \{0, 1, \dots, c\}$$

be independent for all $z \in \mathcal{V}$ and $c \in \mathbb{N}$. Then,

$$\begin{aligned} P(Z_{t+1} = \xi | Z_t = \xi) &\geq \frac{1}{\text{card}(\mathcal{E})} \sum_{(x,y) \in \mathcal{E}} P(U(x, \xi) = \xi(x), U(y, \xi) = \xi(y)) \\ &= \frac{1}{\text{card}(\mathcal{E})} \sum_{(x,y) \in \mathcal{E}} \frac{1}{(\xi(x) + 1)(\xi(y) + 1)} \geq \left(\frac{1}{M+1} \right)^2 > 0 \end{aligned}$$

so the process is aperiodic. Also, letting $(x, y) \in \mathcal{E}$ and $\xi \in \mathcal{C}_{N, M-1}$,

$$\begin{aligned} P(Z_{t+1} = \xi^y \mid Z_t = \xi^x) &\geq \frac{P(\text{Uniform}\{0, 1, \dots, \xi(x) + 1\} = \xi(x), U(y, \xi) = \xi(y), U_1 = 0)}{\text{card}(\mathcal{E})} \\ &= \frac{1}{\text{card}(\mathcal{E})} \frac{1}{2(\xi(x) + 2)(\xi(y) + 1)} \geq \left(\frac{1}{N(M + 2)} \right)^2 > 0. \end{aligned}$$

Repeating the proof of Lemma 2.4, we deduce that, for all $\xi, \eta \in \mathcal{C}_{N, M}$,

$$P(Z_t = \eta \mid Z_0 = \xi) = \left(\frac{1}{N(M + 2)} \right)^{2MN} > 0 \quad \text{for some } t < MN,$$

so the process is irreducible. As previously, convergence to a unique stationary distribution follows from the fact that the process is finite, irreducible and aperiodic.

□

Lemma 2.9 – *The distribution π_Z is reversible and*

$$\pi_Z(\xi) = \frac{\mu(\xi)}{\sum_{\eta \in \mathcal{C}_{N, M}} \mu(\eta)} \quad \text{where} \quad \mu(\xi) = \prod_{z \in \mathcal{V}} (\xi(z) + 1). \quad (2.17)$$

Proof. Let $\xi \neq \eta$ in $\mathcal{C}_{N, M}$ be two configurations. As for the uniform reshuffling and immediate exchange models, when condition (2.14) is not satisfied,

$$P(Z_{t+1} = \eta \mid Z_t = \xi) = P(Z_{t+1} = \xi \mid Z_t = \eta) = 0. \quad (2.18)$$

To study the transition probability when (2.14) holds, let

$$U(z, \xi) = \text{Uniform}\{0, 1, \dots, \xi(z)\} \quad \text{and} \quad U_c = \text{Uniform}\{0, 1, \dots, c\}$$

be independent for all $z \in \mathcal{V}$ and $c \in \mathbb{N}$. By conditioning on all the possible values of $U(x, \xi)$ and $U(y, \xi)$ and using independence, we get

$$P(Z_{t+1} = \eta \mid Z_t = \xi) = \frac{1}{\text{card}(\mathcal{E})} \sum_{c_x=0}^{\xi(x)} \sum_{c_y=0}^{\xi(y)} \frac{P(c_x + U_{\xi(x)+\xi(y)-c_x-c_y} = \eta(x))}{(\xi(x) + 1)(\xi(y) + 1)}. \quad (2.19)$$

By conditioning on all the possible values of $U_{\xi(x)+\xi(y)-c_x-c_y}$ and using again independence, the numerator in the sum above can be written as

$$\begin{aligned} P(c_x + U_{\xi(x)+\xi(y)-c_x-c_y} = \eta(x)) &= \frac{\mathbb{1}\{c_x \leq \eta(x) \leq \xi(x) + \xi(y) - c_y\}}{\xi(x) + \xi(y) - c_x - c_y + 1} \\ &= \frac{\mathbb{1}\{c_x \leq \eta(x) \leq \eta(x) + \eta(y) - c_y\}}{\xi(x) + \xi(y) - c_x - c_y + 1} = \frac{\mathbb{1}\{c_x \leq \eta(x)\} \mathbb{1}\{c_y \leq \eta(y)\}}{\xi(x) + \xi(y) - c_x - c_y + 1}. \end{aligned} \quad (2.20)$$

Combining (2.19) and (2.20), we obtain that

$$Q_{x,y}(\xi, \eta) = \text{card}(\mathcal{E}) (\xi(x) + 1)(\xi(y) + 1) P(Z_{t+1} = \eta \mid Z_t = \xi)$$

can be written using symmetry as

$$\begin{aligned} Q_{x,y}(\xi, \eta) &= \sum_{c_x=0}^{\xi(x)} \sum_{c_y=0}^{\xi(y)} \frac{\mathbb{1}\{c_x \leq \eta(x)\} \mathbb{1}\{c_y \leq \eta(y)\}}{\xi(x) + \xi(y) - c_x - c_y + 1} \\ &= \sum_{c_x=0}^{\xi(x) \wedge \eta(x)} \sum_{c_y=0}^{\xi(y) \wedge \eta(y)} \left(\frac{1}{\xi(x) + \xi(y) - c_x - c_y + 1} \right) \\ &= \sum_{c_x=0}^{\eta(x) \wedge \xi(x)} \sum_{c_y=0}^{\eta(y) \wedge \xi(y)} \left(\frac{1}{\eta(x) + \eta(y) - c_x - c_y + 1} \right) = Q_{x,y}(\eta, \xi) \end{aligned}$$

from which it follows that

$$\begin{aligned} (\xi(x) + 1)(\xi(y) + 1) P(Z_{t+1} = \eta \mid Z_t = \xi) \\ = (\eta(x) + 1)(\eta(y) + 1) P(Z_{t+1} = \xi \mid Z_t = \eta). \end{aligned} \quad (2.21)$$

Combining (2.18) and (2.21), we conclude that, in any case,

$$\mu(\xi) P(Z_{t+1} = \eta \mid Z_t = \xi) = \mu(\eta) P(Z_{t+1} = \xi \mid Z_t = \eta) \quad \text{where} \quad \mu(\xi) = \prod_{z \in \mathcal{V}} (\xi(z) + 1),$$

showing that π_Z is reversible and satisfies (2.17). \square

2.8 Proof of Theorems 2.2 and 2.3

Lemmas 2.6–2.9 in the previous two sections imply that, though the evolution rules of the immediate exchange model and of the uniform saving model are different,

both processes converge to the same stationary distribution $\pi = \pi_Y = \pi_Z$ which is characterized by

$$\pi(\xi) = \frac{\mu(\xi)}{\sum_{\eta \in \mathcal{C}_{N,M}} \mu(\eta)} \quad \text{where} \quad \mu(\xi) = \prod_{z \in \mathcal{V}} (\xi(z) + 1).$$

To complete the proof of Theorems 2.2 and 2.3, the last step is to find a more explicit expression of the stationary distribution by computing the denominator

$$\Lambda(N, M) = \sum_{\xi \in \mathcal{C}_{N,M}} \prod_{z \in \mathcal{V}} (\xi(z) + 1) = \sum_{c_1 + \dots + c_N = M} (c_1 + 1)(c_2 + 1) \cdots (c_N + 1).$$

To compute $\Lambda(N, M)$, we start with the following technical lemma.

Lemma 2.10 – For all $M, K \in \mathbb{N}$,

$$S(M, K) = \sum_{c=0}^M (c+1) \binom{M-c+K}{K} = \binom{M+K+2}{K+2}.$$

Proof. We prove the result by induction on $M + K$. The fact that

$$\begin{aligned} S(0, K) &= \sum_{c=0}^0 (c+1) \binom{0-c+K}{K} = \binom{K}{K} = 1 = \binom{0+K+2}{K+2} \\ S(M, 0) &= \sum_{c=0}^M (c+1) \binom{M-c+0}{0} = \sum_{c=0}^M (c+1) = \frac{(M+1)(M+2)}{2} = \binom{M+0+2}{0+2} \end{aligned}$$

shows that the result holds when $M = 0$ or $K = 0$. Now, let $m \in \mathbb{N}^*$ and assume that the result holds whenever $M + K < m$. Using the well-known identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{for all } 1 \leq k < n$$

consecutively in the following two cases

$$n = M - c + K \quad \text{and} \quad k = K \quad \text{with} \quad K \geq 1 \quad \text{and} \quad M > c$$

$$n = M + K + 2 \quad \text{and} \quad k = K + 2 \quad \text{with} \quad K \geq 1 \quad \text{and} \quad M \geq 1$$

and assuming that $M + K = m$ with $M, K \geq 1$, we get

$$\begin{aligned}
S(M, K) &= \sum_{c=0}^{M-1} (c+1) \left[\binom{(M-1)-c+K}{K} + \binom{M-c+(K-1)}{K-1} \right] + (M+1) \\
&= S(M-1, K) + \sum_{c=0}^M (c+1) \binom{M-c+(K-1)}{K-1} = S(M-1, K) + S(M, K-1) \\
&= \binom{M+K+2-1}{K+2} + \binom{M+K+2-1}{K+2-1} = \binom{M+K+2}{K+2}.
\end{aligned}$$

This completes the proof. \square

Using Lemma 2.10, we can now compute $\Lambda(N, M)$.

Lemma 2.11 – *For all $N, M \geq 1$, we have*

$$\Lambda(N, M) = \sum_{c_1+\dots+c_N=M} (c_1+1)(c_2+1)\dots(c_N+1) = \binom{M+2N-1}{2N-1}.$$

Proof. We prove the result by induction on N . Observing that

$$\Lambda(1, M) = \sum_{c_1=M} (c_1+1) = (M+1) = \binom{M+2-1}{2-1}$$

shows that the result holds for $N = 1$. Now, fix $N \geq 2$ and assume that the result holds for $N - 1$ vertices. Decomposing according to the possible values of c_N , we get

$$\begin{aligned}
\Lambda(N, M) &= \sum_{c_N=0}^M (c_N+1) \sum_{c_1+\dots+c_{N-1}=M-c_N} (c_1+1)(c_2+1)\dots(c_{N-1}+1) \\
&= \sum_{c=0}^M (c+1) \Lambda(N-1, M-c) = \sum_{c=0}^M (c+1) \binom{M-c+2N-3}{2N-3}.
\end{aligned}$$

Finally, applying Lemma 2.10, we obtain

$$\Lambda(N, M) = S(M, 2N-3) = \binom{M+2N-3+2}{2N-3+2} = \binom{M+2N-1}{2N-1},$$

which completes the proof. \square

We are now ready to prove the theorems.

Proof of Theorems 2.2 and 2.3. Combining Lemmas 2.6 and 2.7, we obtain that, regardless of the initial configuration and regardless of the choice of vertex $x \in \mathcal{V}$,

$$\begin{aligned} \lim_{t \rightarrow \infty} P(Y_t(x) = c) &= \sum_{\xi: \xi(x)=c} \pi(\xi) \\ &= \sum_{c_1 + \dots + c_{N-1} = M-c} \frac{(c_1 + 1) \cdots (c_{N-1} + 1)(c + 1)}{\Lambda(N, M)} = \frac{(c + 1) \Lambda(N - 1, M - c)}{\Lambda(N, M)}. \end{aligned}$$

This, together with Lemma 2.11, implies that

$$\lim_{t \rightarrow \infty} P(Y_t(x) = c) = (c + 1) \binom{M - c + 2N - 3}{2N - 3} \bigg/ \binom{M + 2N - 1}{2N - 1}.$$

This proves the first part of Theorem 2.2. Now, observe that

$$\begin{aligned} &(c + 1) \binom{M - c + 2N - 3}{2N - 3} \bigg/ \binom{M + 2N - 1}{2N - 1} \\ &= (c + 1) \frac{(2N - 1)(2N - 2)}{(M + 2N - 1)(M + 2N - 2)} \frac{(M - c + 2N - 3) \cdots (M - c + 1)}{(M + 2N - 3) \cdots (M + 1)}. \end{aligned}$$

In particular, when N and T are large, this is approximately

$$(c + 1) \left(\frac{2N}{M} \right)^2 \left(1 - \frac{c}{M} \right)^{2N} \approx \frac{4c}{T^2} \left(1 - \frac{c}{NT} \right)^{2N} \approx \frac{4c}{T^2} e^{-2c/T}.$$

This completes the proof of Theorem 2.2. The proof of Theorem 2.3 is exactly the same since both models converge to the same stationary distribution $\pi = \pi_Y = \pi_Z$.

□

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Chapter 3

RIGOROUS RESULTS FOR THE DISTRIBUTION OF MONEY ON CONNECTED GRAPHS (MODELS WITH DEBTS)

3.1 Abstract

In this paper, we continue our analysis of spatial versions of agent-based models for the dynamics of money that have been introduced in the statistical physics literature, focusing on two models with debts. Both models consist of systems of economical agents located on a finite connected graph representing a social network. Each agent is characterized by the number of coins she has, which can be negative in case she is in debt, and each monetary transaction consists in one coin moving from one agent to one of her neighbors. In the first model, that we name the model with individual debt limit, the agents are allowed to individually borrow up to a fixed number of coins. In the second model, that we name the model with collective debt limit, agents can borrow coins from a central bank as long as the bank is not empty, with reimbursements occurring each time an agent in debt receives a coin. Based on numerical simulations of the models on complete graphs, it was conjectured that, in the large population/temperature limits, the distribution of money converges to a shifted exponential distribution for the model with individual debt limit, and to an asymmetric Laplace distribution for the model with collective debt limit. In this paper, we prove exact formulas for the distribution of money that are valid for all possible social networks. Taking the large population/temperature limits in the formula found for the model with individual debt limit, we prove convergence to the shifted exponential distribution, thus establishing the first conjecture. Simplifying

the formula found for the model with collective debt limit is more complicated, but using a computer to plot this formula shows an almost perfect fit with the Laplace distribution, which strongly supports the second conjecture.

3.2 Introduction

The main objective of this paper is to continue the mathematical analysis of economical models for the dynamics of money initiated by the authors in [6, 7]. These models consist of (typically) large systems of economical agents, where each agent is characterized by the amount of money, or number of coins, she has at a given time. The processes evolve in discrete time and, at each time step, two agents are selected at random from the entire population to engage in a monetary transaction. The main problem about these models is to find the limiting distribution of money, i.e., letting $\xi_t(x)$ be the number of coins agent x has at time t , the objective is to find

$$\lim_{t \rightarrow \infty} P(\xi_t(x) = c)$$

the probability that agent x has c coins at equilibrium. Physicists have introduced a number of economical models and, relying on numerical simulations, were able to derive interesting conjectures about the distribution of money in the limit as the number of individuals, and the average number of coins per individual called the money temperature, both tend to infinity. The simplest such system, which we call the **one-coin model**, was introduced in [2]. In this model, after two agents have been selected to engage in a monetary transaction, one of the two agents chosen at random gives one of her coins to the other agent (if she has at least one). The authors of [2] conjectured that, for this simple model, the distribution of money converges to the exponential distribution with mean the money temperature in the large population/temperature limits.

More realistic variants of the one-coin model where monetary transactions involve multiple coins were also introduced in the statistical physics literature, such as the three models studied analytically by the authors in [7] that can be described as follows.

- In the **uniform reshuffling model** also introduced in [2], all the coins of the two agents selected to interact are uniformly redistributed between the two agents. The simulations performed in [2] suggest that, for this type of monetary transaction, the distribution of money at equilibrium converges to the exponential distribution with mean the money temperature in the large population/temperature limits, just like in the one-coin model.
- In the **immediate exchange model** introduced in [3], each of the two interacting agents chooses independently and uniformly at random a number of her coins that she gives to the other interacting agent. Results from [3, 4] suggest that, in this case, the distribution of money converges to a gamma distribution with mean the money temperature and shape parameter two in the large population/temperature limits.
- In the **model with saving propensity** introduced in [1], the two interacting agents independently save some of their coins, and the remaining coins are uniformly redistributed between the two agents, just like in the uniform reshuffling model. The computer simulations performed in [8] suggest that, like in the immediate exchange model, the limiting distribution of money converges to a gamma distribution with mean the money temperature and shape parameter two in the large population/temperature limits.

All the conjectures above have been recently proved analytically by the authors in [6] for the one-coin model, and in [7] for the other three models. In addition, we were

able to extend these results to more general models where the economical agents are located on a finite connected graph and can only interact with their neighbors. Thinking of the graph as a social network, this means that individuals can only exchange money with their friends or business partners, which results in a more realistic model. Interestingly, we were able to prove that the distribution of the number of coins a given agent has at equilibrium does not depend on the number of connections of this agent, in particular the distribution is the same for agents with a large number of connections and for agents with only one neighbor.

In this paper, we continue our analysis, looking now at two models with debts introduced in the physics literature [2, 9]. The inclusion of debts is modeled by the fact that the agents can now have a negative number of coins. The evolution rules at each transaction are the same as in the idealized one-coin model, but the models also differ qualitatively in that [2] assumes that the agents have the same individual debt limit whereas [9] assumes that there is a collective limit.

- We call model with **individual debt limit** the model with debts introduced in [2]. In this model, agents are allowed to individually borrow L_i coins, and the numerical simulations performed in [2] suggest that the distribution of money at equilibrium now converges to a shifted exponential distribution in the large population/temperature limits.
- We call model with **collective debt limit** the model in [9]. In this model, agents can borrow coins from a bank that starts with L_c coins as long as the bank is not empty, with reimbursements occurring each time an agent in debt receives a coin. It was conjectured in [9] that the distribution of money at equilibrium converges to an asymmetric Laplace distribution in the large population/temperature limits.

As in [6, 7], we study spatial generalizations of these economical models where agents are located on a social network. Following an approach similar to [6], we give a complete proof of (and extend) the conjecture in [2] about the model with individual debt limit. The model with collective debt limit is more challenging. Our main result gives an exact expression of the distribution of money at equilibrium for all possible number of individuals and coins, but we were not able to simplify this expression in the large population/temperature limits to prove the conjecture in [9]. However, using a computer to plot the exact distribution found analytically shows an almost perfect fit with the Laplace distribution found numerically in [9], which strongly supports their conjecture.

In the next two sections, we give a rigorous definition of the spatial versions of the two models with debts introduced in [2, 9], and state our main results about the distribution of money at equilibrium. The other sections are devoted to proofs.

3.3 Model Description

In contrast with [2, 9] that rely on numerical simulations restricted to models where all pairs of individuals are equally likely to interact at each time step, our analysis is general enough to account for local interactions and network structure. This means that the individuals are more realistically located on the set of vertices of a general finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ representing a social network. The structure of the network is incorporated in the dynamics by assuming that only individuals located on vertices connected by an edge of the graph, that can be thought of as friends or business partners, may interact to exchange money. Note that the models in [2, 9] can be viewed as the particular cases where \mathcal{G} is a complete graph.

The two models with debts studied in this paper are spatially explicit variants of the basic one-coin model introduced in [2], and studied analytically on general

connected graphs in [6]. Having a general finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the (spatial) one-coin model is a discrete-time Markov chain in which the state at time t is a configuration

$$\xi_t : \mathcal{V} \rightarrow \mathbb{N} \quad \text{where} \quad \xi_t(x) = \text{number of coins agent } x \text{ has at time } t.$$

In order to describe the dynamics, because the flow of money at each transaction is oriented from one vertex to another, it is convenient to define the set of directed edges

$$\vec{\mathcal{E}} = \{(x, y), (y, x) : \{x, y\} \in \mathcal{E}\}.$$

At each time step, a directed edge, say $(x, y) \in \vec{\mathcal{E}}$, is chosen uniformly at random, which results in the transfer of one coin from vertex x to vertex y if and only if there is a least one coin at x before the interaction. The restriction on the transfer reflects the fact that individuals cannot have debts. In contrast, the models introduced in [2, 9] allow the individuals to go into debt and have a negative number of coins. In particular, the state at time t is now

$$\xi_t : \mathcal{V} \rightarrow \mathbb{Z} \quad \text{where} \quad \xi_t(x) = \begin{cases} + \text{ number of coins } x \text{ has when } \xi_t(x) \geq 0 \\ - \text{ number of coins } x \text{ borrowed when } \xi_t(x) < 0. \end{cases}$$

As in the basic one-coin model, a directed edge $(x, y) \in \vec{\mathcal{E}}$ representing the flow of money (one coin) is chosen uniformly at random at each time step. The conditions under which the transaction indeed occurs are however different.

Model with individual debt limit. In the model with debts introduced in [2], agents are allowed to borrow individually up to L_i coins, which is modeled by assuming that the transaction indeed occurs if and only if the state at vertex x exceeds $-L_i$. In equations, letting (x, y) be the selected edge and ξ the configuration before the

interaction, the configuration after the interaction is

$$(\sigma_{x,y}\xi)(z) = \xi(z) + \mathbb{1}\{\xi(x) > -L_i \text{ and } z = y\} \\ - \mathbb{1}\{\xi(x) > -L_i \text{ and } z = x\} \quad \text{for all } z \in \mathcal{V},$$

obtained by moving one coin from x to y if and only if the state at x exceeds $-L_i$. Recalling that, at each time step, a directed edge is chosen uniformly at random, and using that the total number of directed edges is twice the number of edges in \mathcal{E} , the model with individual debt limit is the discrete-time Markov chain (X_t) with transition probabilities

$$P(X_{t+1} = \sigma_{x,y}\xi \mid X_t = \xi) = \frac{1}{2 \text{card}(\mathcal{E})} \quad \text{for all } (x,y) \in \vec{\mathcal{E}}.$$

Model with collective debt limit. The model with collective debt limit introduced in [9] is more complicated. The money can be borrowed from a central bank that we represent by adding a vertex \star to the vertex set \mathcal{V} , so the vertex set becomes

$$\mathcal{V}^\star = \mathcal{V} \cup \{\star\} \quad \text{where } \star = \text{location of the central bank.}$$

Though there is some flow of money between the bank and the individuals, the edge set is unchanged, i.e., there is no edge between the central bank and the individuals. As previously, a directed edge, say (x,y) , is chosen at each time step. From the point of view of x ,

- In case x has at least one coin, one coin moves from vertex x to vertex y , which results in the state at vertex x to decrease by one.
- In case x has zero coin or is in debt, and there is at least one coin in the bank, x takes one coin from the bank to give to y , which results in the state at x to decrease by one.
- In case x has zero coin or is in debt, and the bank has no coin, nothing happens.

From the point of view of y ,

- In case y is not in debt before the interaction and indeed receives one coin, the state of the bank does not change further and the state at vertex y increases by one.
- In case y is in debt before the interaction and indeed receives one coin, y gives one coin to the bank to reimburse part of her debt, so the state of the bank increases by one and the state at vertex y increases by one.

Note in particular that, in case both x and y are in debt and there is at least one coin at the bank before the interaction, the bank gives one coin to x , and y immediately gives it back to the bank so the state of the bank does not change. In equations, letting ξ be the configuration before the interaction, the configuration after the interaction is

$$\begin{aligned}
(\tau_{x,y}\xi)(z) &= \xi(z) + \mathbb{1}\{\max(\xi(x), \xi(\star)) > 0 \text{ and } z = y\} \\
&\quad - \mathbb{1}\{\max(\xi(x), \xi(\star)) > 0 \text{ and } z = x\} \quad \text{for all } z \in \mathcal{V} \\
(\tau_{x,y}\xi)(\star) &= \xi(\star) + \mathbb{1}\{\max(\xi(x), \xi(\star)) > 0 \text{ and } \xi(x) > 0 \text{ and } \xi(y) < 0\} \\
&\quad - \mathbb{1}\{\max(\xi(x), \xi(\star)) > 0 \text{ and } \xi(x) \leq 0 \text{ and } \xi(y) \geq 0\}.
\end{aligned}$$

In particular, using again that there are $2 \text{card}(\mathcal{E})$ directed edges, the model with collective debt limit is the Markov chain (Y_t) with transition probabilities

$$P(Y_{t+1} = \tau_{x,y}\xi \mid Y_t = \xi) = \frac{1}{2 \text{card}(\mathcal{E})} \quad \text{for all } (x, y) \in \vec{\mathcal{E}}.$$

From now on, we let $N = \text{card}(\mathcal{V})$ be the number of vertices, which is also the number of individuals in the system. Note that, each time a transaction indeed occurs, the state of one vertex decreases by one, the state of another vertex increases by one, and the state of all the other vertices does not change. In particular, letting M be the

initial number of coins in the population,

$$\sum_{z \in \mathcal{V}} X_t(z) = \sum_{z \in \mathcal{V}} Y_t(z) = M \quad \text{for all } t > 0.$$

The model with individual debt limit is thus characterized by the structure of the network, and the two parameters M and L_i . Similarly, letting L_c be the initial number of coins in the central bank, the model with collective debt limit is characterized by the structure of the network, and the two parameters M and L_c . Finally, the average number of coins per individual is denoted by $T = M/N$, and called the money temperature by analogy with the notion of temperature in physics.

3.4 Main Results

As previously mentioned, the numerical simulations of the model with individual debt limit on the complete graph performed in [2], i.e., the model in which all pairs of individuals are equally likely to be selected, suggest that the limiting distribution of money approaches a shifted exponential distribution in the large population/temperature limits. The gray histogram in Figure 3.1 shows the distribution of money obtained from numerical simulations that we have reproduced. More precisely, the conjecture in [2] states that, when the number of individuals N and the money temperature $T = M/N$ are large, we have the approximation

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) \approx f_X(c) = \frac{1}{T + L_i} \exp\left(-\frac{c + L_i}{T + L_i}\right). \quad (3.1)$$

The black curve in Figure 3.1 shows the graph of the density function f_X . Note that this distribution is similar to the distribution of money in the one-coin model except that the money temperature T is replaced by $T + L_i$, and the range is shifted by $-L_i$. This shows that the model with individual debt limit behaves essentially like the one-coin model in which the average number of coins per individual is now $T + L_i$, due

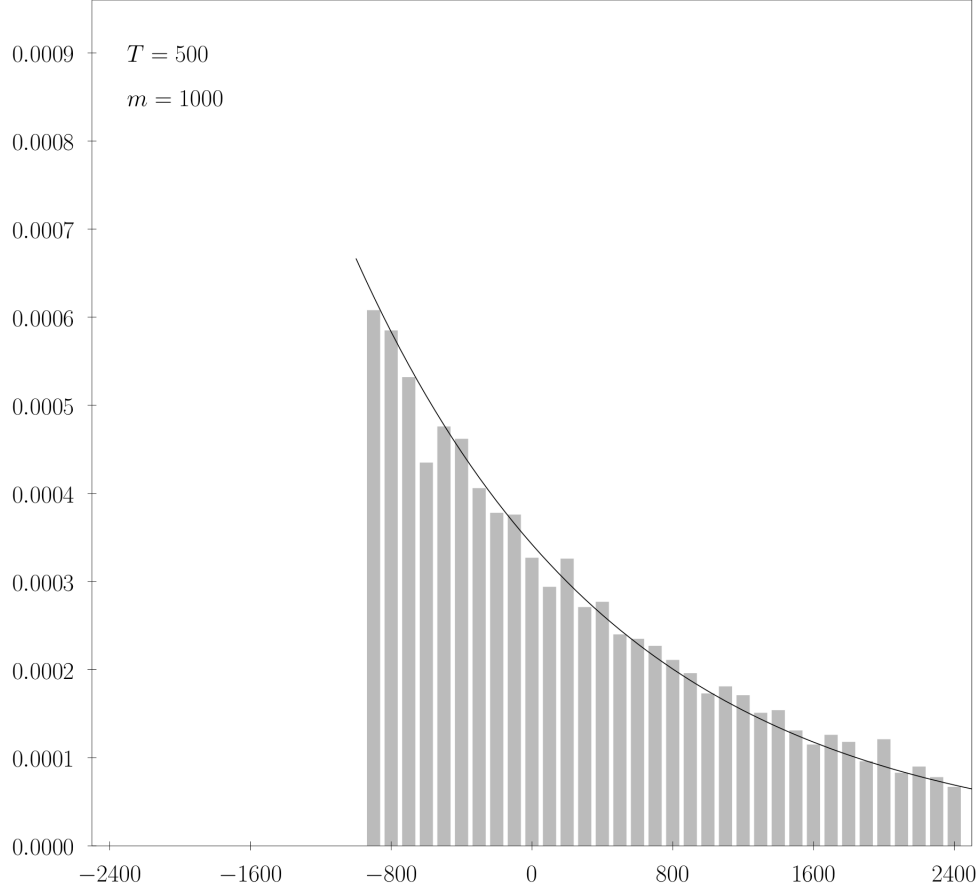


Figure 3.1: Distribution of Money for the Model with Individual Debt Limit $L_i = 1000$ and Temperature $T = 500$. The Solid Curve Corresponds to the Graph of the Density Function (3.1) While the Gray Histogram Gives the Distribution of Money Obtained From 5×10^{10} Iterations of the Process on the Complete Graph with $N = 10,000$ Vertices.

to the fact that each individual can use L_i additional coins. Using reversibility of the stochastic process to identify the stationary distribution and some basic combinatorics to count the total number of admissible configurations, we get the following theorem.

Theorem 3.1 (individual debt limit) – *For all connected graph \mathcal{G} with N vertices, regardless of the number M of coins, the limit L_i and the initial configuration,*

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) = \frac{\Lambda_X(N-1, M-c, L_i)}{\Lambda_X(N, M, L_i)} \quad \text{for all} \quad -L_i \leq c \leq M + L_i(N-1)$$

where the function Λ_X is given by

$$\Lambda_X(N, M, L_i) = \binom{M + L_i N + N - 1}{N - 1} \quad \text{for all } N, M, L_i.$$

In particular, when N and $T = M/N$ are large,

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) \approx f_X(c) \quad \text{for all } -L_i \leq c \leq M + L_i(N - 1).$$

The first part of the theorem gives an exact expression of the distribution of money for all possible parameters of the system, and for all possible connected graphs. The second part shows that taking the limit as N and T tend to infinity in this exact expression implies the conjecture (3.1).

We now look at the model with collective debt limit. Recall that the numerical simulations of the model on the complete graph in [9] suggest that the distribution of money at equilibrium now approaches an asymmetric Laplace distribution in the large population/temperature limits. The gray histogram in Figure 3.2 shows the distribution of money obtained from numerical simulations that we have reproduced. More precisely, the conjecture in [9] states that, in the large population limit, and when M/N and L_c/N are large, we have the approximation

$$\lim_{t \rightarrow \infty} P(Y_t(x) = c) \approx f_Y(c) = \begin{cases} K e^{-ac} & \text{for } c \geq 0 \\ K e^{+bc} & \text{for } c \leq 0 \end{cases} \quad (3.2)$$

where, letting $\rho = L_c/M$,

$$K \sim \frac{1}{T} \left(\sqrt{1 + \rho} - \sqrt{\rho} \right)^2 \quad a \sim \frac{1}{T} \left(1 - \sqrt{\frac{\rho}{1 + \rho}} \right) \quad b \sim \frac{1}{T} \left(\sqrt{\frac{1 + \rho}{\rho}} - 1 \right). \quad (3.3)$$

The black curve in Figure 3.2 shows the graph of the density function f_Y . As for the model with individual debt limit, using reversibility to identify the stationary distribution and some combinatorics to count the total number of admissible configurations leads to an exact expression of the distribution of money for all possible

connected graphs and all possible M and L_c . The combinatorial analysis, however, is more complicated for the model with collective debt limit.

Theorem 3.2 (collective debt limit) – *For all connected graph \mathcal{G} with N vertices, regardless of the number M of coins, the limit L_c and the initial configuration,*

$$\lim_{t \rightarrow \infty} P(Y_t(x) = c) = \begin{cases} \frac{\Lambda_Y(N-1, M-c, L_c)}{\Lambda_Y(N, M, L_c)} & \text{for all } 0 \leq c \leq M + L_c \\ \frac{\Lambda_Y(N-1, M-c, L_c + c)}{\Lambda_Y(N, M, L_c)} & \text{for all } -L_c \leq c \leq 0 \end{cases}$$

where the function Λ_Y is given by

$$\Lambda_Y(N, M, L_c) = \sum_{a=0}^{L_c} \sum_{b=0}^N \binom{N}{b} \binom{a-1}{b-1} \binom{M+a+N-b-1}{N-b-1}.$$

In contrast with Theorem 3.1, we were not able to simplify the expression of Λ_Y sufficiently to prove that the distribution of money indeed converges to (3.2)–(3.3) in the large population/temperature limits. However, using a computer program to plot the exact expression of the distribution of money found in the theorem (the black squares in Figure 3.2) shows an almost perfect fit with the Laplace distribution (the solid curve in Figure 3.2) found via numerical simulations in [9], which strongly supports their conjecture. To further support the conjecture, we also give a partial proof of (3.3), assuming that (3.2) holds, in the last section of this paper. More precisely, we prove that, at equilibrium, the number of coins in the bank is a strict supermartingale, which suggests that the number of coins in the bank divided by the collective debt limit L_c converges to zero as $L_c \rightarrow \infty$. Assuming that this is indeed the case and that the distribution of money is indeed described by an asymmetric Laplace distribution (3.2), we prove (3.3) rigorously.

The rest of the paper is devoted to proofs and organized as follows. In the next section, we collect preliminary results for both models, showing the existence and uniqueness of their stationary distribution. We also use reversibility to prove that, at

equilibrium, all the configurations are equally likely. In the following two sections, we use some combinatorics to obtain an explicit expression of the stationary distribution, from which Theorems 3.1 and 3.2 can be easily deduced. Finally, the last section gives the partial proof of (3.3) under the assumption that (3.2) holds.

3.5 Ergodicity and Reversibility

This section collects some preliminary results about the models with individual debt limit and collective debt limit that will be useful later to prove our theorems. More precisely, we prove that both processes are irreducible, aperiodic and reversible. From now on, the set of all possible configurations of the model with individual debt limit is denoted by

$$\mathcal{C}_{N,M,L_i} = \{\xi : \mathcal{V} \rightarrow \mathbb{Z} : \sum_{x \in \mathcal{V}} \xi(x) = M \text{ and } \xi(x) \geq -L_i \text{ for all } x \in \mathcal{V}\}$$

while the set of configurations of the model with collective debt limit is

$$\mathcal{D}_{N,M,L_c} = \{\xi : \mathcal{V} \rightarrow \mathbb{Z} : \sum_{x \in \mathcal{V}} \xi(x) = M \text{ and } \sum_{x \in \mathcal{V}} (-\xi(x)) \mathbf{1}\{\xi(x) < 0\} \leq L_c\}.$$

Because the proofs of irreducibility, aperiodicity, and reversibility are similar for both models, instead of studying both processes separately, we give the details of the proofs for the process with individual debt limit and then briefly explain how the proof can be adapted to show the analog for the process with collective debt limit. We start with irreducibility. Intuitively, the reason why the two processes are irreducible is because the graph is connected, which allows us to move coins from any vertex to any other vertices.

Lemma 3.3 (irreducibility) – *The process (X_t) is irreducible.*

Proof. Letting $\xi \neq \xi'$, the objective is to show that

$$P(X_t = \xi' \mid X_0 = \xi) > 0 \quad \text{for some } t \in \mathbb{N}. \quad (3.4)$$

The key is to use the following metric d on the set of configurations:

$$d(\xi, \xi') = \sum_{z \in \mathcal{V}} |\xi(z) - \xi'(z)| \quad \text{for all } \xi, \xi' : \mathcal{V} \rightarrow \mathbb{Z}.$$

Because $\xi \neq \xi'$, there exist $x, y \in \mathcal{V}$ such that

$$\xi(x) > \xi'(x) \quad \text{and} \quad \xi(y) < \xi'(y).$$

Then, define the configuration $\eta \in \mathcal{C}_{N, M, L_i}$ as

$$\eta(x) = \xi(x) - 1, \quad \eta(y) = \xi(y) + 1, \quad \eta(z) = \xi(z) \text{ for all } z \neq x, y$$

and let $\Gamma = (x, z_1, z_2, \dots, z_s, y)$ be a self-avoiding path of the graph \mathcal{G} connecting x and y . Note that such a path exists because the graph is connected. Note also that, because Γ is a self-avoiding path, the integer s is less than the diameter of the graph. Now, define recursively

$$\xi_1 = \tau_{x, z_1} \xi, \quad \xi_2 = \tau_{z_1, z_2} \xi_1, \dots, \quad \xi_{i+1} = \tau_{z_i, z_{i+1}} \xi_i, \dots, \quad \xi_{s+1} = \tau_{z_s, y} \xi_s.$$

Because $\xi(x) > \xi'(x) \geq -L_i$, we can move a coin from x to z_1 , which gives $\xi_1(z_1) > -L_i$. By using a simple induction, we deduce that $\xi_i(z_i) > -L_i$ for all i . This also implies that we can move a coin $s+1$ times to bring it from x to y therefore $\xi_{s+1} = \eta$ and

$$P(X_{s+1} = \eta \mid X_0 = \xi) \geq P_\xi(X_1 = \xi_1) \prod_{i=1}^s P_{\xi_i}(X_{i+1} = \xi_{i+1}) = \left(\frac{1}{2 \text{card}(\mathcal{E})} \right)^{s+1}. \quad (3.5)$$

In addition, because $\xi(x) > \xi'(x)$, we must have

$$\begin{aligned} |\eta(x) - \xi'(x)| &= \eta(x) - \xi'(x) = (\xi(x) - 1) - \xi'(x) \\ &= (\xi(x) - \xi'(x)) - 1 = |\xi(x) - \xi'(x)| - 1. \end{aligned} \quad (3.6)$$

Similarly, because $\xi(y) < \xi'(y)$, we have

$$\begin{aligned} |\eta(y) - \xi'(y)| &= \xi'(y) - \eta(y) = \xi'(y) - (\xi(y) + 1) \\ &= (\xi'(y) - \xi(y)) - 1 = |\xi(y) - \xi'(y)| - 1. \end{aligned} \quad (3.7)$$

Combining (3.6) and (3.7), we deduce that

$$\begin{aligned}
d(\eta, \xi') &= \sum_{z \in \mathcal{V}} |\eta(z) - \xi'(z)| \\
&= \sum_{z \neq x, y} |\eta(z) - \xi'(z)| + |\eta(x) - \xi'(x)| + |\eta(y) - \xi'(y)| \\
&= \sum_{z \neq x, y} |\eta(z) - \xi'(z)| + (|\xi(x) - \xi'(x)| - 1) + (|\xi(y) - \xi'(y)| - 1) \\
&= \sum_{z \in \mathcal{V}} |\xi(z) - \xi'(z)| - 2 = d(\xi, \xi') - 2 < d(\xi, \xi').
\end{aligned} \tag{3.8}$$

Observing also that $-L_i \leq \xi(z) \leq M + L_i(N - 1)$ for all $z \in \mathcal{V}$, we have

$$d(\xi, \xi') = \sum_{z \in \mathcal{V}} |\xi(z) - \xi'(z)| \leq \sum_{z \in \mathcal{V}} (M + NL_i) \leq N(M + NL_i) \tag{3.9}$$

for all $\xi, \xi' \in \mathcal{C}_{N, M, L_i}$. Combining (3.5), (3.8) and (3.9), we conclude that

$$P(X_t = \xi' \mid X_0 = \xi) \geq \left(\frac{1}{2 \text{card}(\mathcal{E})} \right)^{DN(M + NL_i)}$$

for some $t \leq DN(M + NL_i)$, where D is the diameter of the graph \mathcal{G} . This shows that (3.4) holds so the process is irreducible and the proof is done. \square

Lemma 3.4 (irreducibility) – *The process (Y_t) is irreducible.*

Proof. The proof is identical to the proof of Lemma 3.3 with only exception:

$$-L_c \leq \xi(z) \leq M + L_c \quad \text{for all } \xi \in \mathcal{D}_{N, M, L_c} \quad \text{and } z \in \mathcal{V}.$$

This implies that, for all $\xi, \xi' \in \mathcal{D}_{N, M, L_c}$,

$$d(\xi, \xi') = \sum_{z \in \mathcal{V}} |\xi(z) - \xi'(z)| \leq \sum_{z \in \mathcal{V}} (M + 2L_c) \leq N(M + 2L_c)$$

so the same argument as in the proof of Lemma 3.3 gives

$$P(X_t = \xi' \mid X_0 = \xi) \geq \left(\frac{1}{2 \text{card}(\mathcal{E})} \right)^{DN(M + 2L_c)}$$

for some $t \leq DN(M + 2L_c)$, where D is the diameter of the graph \mathcal{G} . \square

In view of irreducibility, in order to prove that the two processes are aperiodic, it suffices to identify a configuration that has period one, which is done in the next two lemmas.

Lemma 3.5 (aperiodicity) – *The process (X_t) is aperiodic.*

Proof. Let $\xi \in \mathcal{C}_{N,M,L_i}$ such that $\xi(x) = -L_i$ for some $x \in \mathcal{V}$. Because the graph \mathcal{G} is connected, vertex x has at least one neighbor $y \in \mathcal{V}$. Also, given that the directed edge (x, y) is the one chosen at the next time step, because vertex x has reached its debt limit, it cannot give any coin to y so the exchange of money is canceled. In particular,

$$P(X_{t+1} = \xi \mid X_t = \xi) \geq P(\text{edge } (x, y) \in \vec{\mathcal{E}} \text{ is selected}) = \frac{1}{2 \text{card}(\vec{\mathcal{E}})} > 0.$$

This shows that configuration ξ has period one. Because the process is irreducible, all the configurations must have the same period, therefore the process is aperiodic.

\square

Lemma 3.6 (aperiodicity) – *The process (Y_t) is aperiodic.*

Proof. Starting with a configuration $\xi \in \mathcal{D}_{N,M,L_c}$ such that $\xi(x) = -L_c$ for some $x \in \mathcal{V}$, and following the exact same reasoning as in the proof of Lemma 3.5, give the result. \square

Irreducibility and aperiodicity, together with the fact that the number of configurations is finite, imply that, for both models, there exists a unique stationary distribution to which the process converges starting from any initial configuration. The next natural step is to find this stationary distribution for each process. In view of

the large number of configurations and more importantly the fact that the agents are located on a general finite connected graph rather than a complete graph, writing the transition matrix in order to compute the stationary distribution looks impossible. However, one can easily find the stationary distribution observing that the processes are reversible and using the corresponding detailed balance equations.

Lemma 3.7 (reversibility) – *The process (X_t) is reversible and*

$$P(X_{t+1} = \xi' \mid X_t = \xi) = P(X_{t+1} = \xi \mid X_t = \xi') \quad \text{for all } \xi, \xi' \in \mathcal{C}_{N,M,L_i}.$$

Proof. The equations to be proved are obvious when $\xi = \xi'$. To deal with the non-trivial case where the two configurations are different, we distinguish two scenarios:

- (a) $P(X_{t+1} = \xi' \mid X_t = \xi) > 0$ and $\xi \neq \xi'$
- (b) $P(X_{t+1} = \xi' \mid X_t = \xi) = 0$ and $\xi \neq \xi'$.

Because the configurations that can be reached from ξ in one step are of the form $\sigma_{x,y}\xi$, in the context of scenario (a), we have $\xi(x) > -L_i$ and

$$\xi'(x) = \xi(x) - 1, \quad \xi'(y) = \xi(y) + 1, \quad \xi'(z) = \xi(z) \text{ for all } z \neq x, y \quad (3.10)$$

for some $(x, y) \in \vec{\mathcal{E}}$. In particular,

$$\xi'(y) = \xi(y) + 1 \geq -L_i + 1 > -L_i$$

which, together with (3.10), implies that $\sigma_{y,x}\xi' = \xi$. Therefore,

$$\begin{aligned} P(X_{t+1} = \xi' \mid X_t = \xi) &= P(X_{t+1} = \sigma_{x,y}\xi \mid X_t = \xi) = 1/(2 \text{card}(\mathcal{E})) \\ &= P(X_{t+1} = \sigma_{y,x}\xi' \mid X_t = \xi') = P(X_{t+1} = \xi \mid X_t = \xi') \end{aligned}$$

because the probabilities of choosing directed edge (x, y) and directed edge (y, x) (out of all the possible $2 \text{card}(\mathcal{E})$ directed edges) are equal. Now, in the context of scenario (b), condition (3.10) does not hold for any of the directed edges. Equivalently,

$$\xi(y) = \xi'(y) - 1, \quad \xi(x) = \xi'(x) + 1, \quad \xi(z) = \xi'(z) \text{ for all } z \neq x, y$$

does not hold for any of the directed edge $(y, x) \in \vec{\mathcal{E}}$ from which it follows that configuration ξ cannot be reached from ξ' in one step. In conclusion, in the context of scenario (b),

$$P(X_{t+1} = \xi' \mid X_t = \xi) = P(X_{t+1} = \xi \mid X_t = \xi') = 0.$$

In either case, $P(X_{t+1} = \xi' \mid X_t = \xi) = P(X_{t+1} = \xi \mid X_t = \xi')$. \square

Lemma 3.8 (reversibility) – *The process (Y_t) is reversible and*

$$P(Y_{t+1} = \xi' \mid Y_t = \xi) = P(Y_{t+1} = \xi \mid Y_t = \xi') \quad \text{for all } \xi, \xi' \in \mathcal{D}_{N,M,L_c}.$$

Proof. As for irreducibility and aperiodicity, the proof of reversibility is quite similar for both processes, so we only focus on the differences. As previously, the detailed balanced equations are obvious when $\xi = \xi'$ and the idea is again to distinguish between the two scenarios

$$(a) \quad P(Y_{t+1} = \xi' \mid Y_t = \xi) > 0 \quad \text{and} \quad \xi \neq \xi'$$

$$(b) \quad P(Y_{t+1} = \xi' \mid Y_t = \xi) = 0 \quad \text{and} \quad \xi \neq \xi'.$$

In the context of scenario (b), the same argument as in the proof of Lemma 3.7 gives

$$P(Y_{t+1} = \xi' \mid Y_t = \xi) = P(Y_{t+1} = \xi \mid Y_t = \xi') = 0.$$

In the context of scenario (a), because the configurations that can be reached from ξ in one step are of the form $\tau_{x,y}\xi$, we now have $\max(\xi(x), \xi(\star)) > 0$ and

$$\xi'(x) = \xi(x) - 1, \quad \xi'(y) = \xi(y) + 1, \quad \xi'(z) = \xi(z) \quad \text{for all } z \neq x, y \quad (3.11)$$

for some $(x, y) \in \vec{\mathcal{E}}$. Then, we have the following three implications:

$$\xi(x) > 0 \quad \text{and} \quad \xi(\star) > 0 \quad \text{imply that} \quad \xi'(\star) \geq \xi(\star) > 0$$

$$\xi(x) \leq 0 \quad \text{and} \quad \xi(\star) > 0 \quad \text{imply that} \quad \xi'(y) > 0 \quad \text{or} \quad \xi'(\star) = \xi(\star) > 0$$

$$\xi(x) > 0 \quad \text{and} \quad \xi(\star) = 0 \quad \text{imply that} \quad \xi'(y) > 0 \quad \text{or} \quad \xi'(\star) = 1.$$

In either case, $\max(\xi'(y), \xi'(\star)) > 0$ which, together with (3.11), implies that $\tau_{y,x} \xi' = \xi$. In particular, the two transition probabilities are equal:

$$\begin{aligned} P(Y_{t+1} = \xi' \mid Y_t = \xi) &= P(Y_{t+1} = \tau_{x,y} \xi \mid Y_t = \xi) = 1/(2 \operatorname{card}(\mathcal{E})) \\ &= P(Y_{t+1} = \tau_{y,x} \xi' \mid Y_t = \xi') = P(Y_{t+1} = \xi \mid Y_t = \xi'). \end{aligned}$$

This completes the proof. \square

3.6 Proof of Theorem 3.1

In this section, we prove the explicit expression for the distribution of money given in Theorem 3.1 for the model with individual debt limit, and take the limit as N and T both tend to infinity to deduce and extend conjecture (3.1) to general finite connected graphs. As previously mentioned, irreducibility and aperiodicity of the model with individual debt limit proved in Lemmas 3.3 and 3.5 and the fact that the number of configurations is finite imply that the process (X_t) has a unique stationary distribution, say π_X , to which it converges starting from any initial configuration:

$$\lim_{t \rightarrow \infty} P_\eta(X_t = \xi) = \pi_X(\xi) \quad \text{for all } \xi, \eta \in \mathcal{C}_{N,M,L_i}. \quad (3.12)$$

In addition, the detailed balanced equations in Lemma 3.7 imply that, under the stationary distribution, all the configurations are equally likely. Indeed, the lemma shows that the transition matrix of the process is symmetric, and therefore doubly stochastic, from which it follows that

$$\pi_X = \text{Uniform}(\mathcal{C}_{N,M,L_i}). \quad (3.13)$$

Letting $\Lambda_X(N, M, L_i) = \operatorname{card}(\mathcal{C}_{N,M,L_i})$ be the number of admissible configurations for the process with individual debt limit, it directly follows from (3.12) and (3.13) that, regardless of the initial configuration of the system and the choice of $x \in \mathcal{V}$,

$$\lim_{t \rightarrow \infty} P(X_t(x) = c) = \frac{\Lambda_X(N-1, M-c, L_i)}{\Lambda_X(N, M, L_i)} \quad (3.14)$$

for all $c = -L_i, \dots, M + L_i(N - 1)$, because the numerator is equal to the number of configurations such that there are c coins at vertex x , while the denominator is equal to the total number of configurations. In particular, to complete the proof of theorem, the next step is to count the number of configurations, which is done in the next lemma.

Lemma 3.9 – *For all N, M and L_i , we have*

$$\Lambda_X(N, M, L_i) = \binom{M + L_i N + N - 1}{N - 1}.$$

Proof. By definition, $\Lambda_X(N, M, L_i)$ is the number of integer solutions to

$$\xi(x_1) + \xi(x_2) + \dots + \xi(x_N) = M \quad \text{with} \quad \xi(x_i) \geq -L_i \text{ for all } i.$$

Letting $\eta(x_i) = \xi(x_i) + L_i + 1$, this is the number of integer solutions to

$$\eta(x_1) + \eta(x_2) + \dots + \eta(x_N) = M + N(L_i + 1) \quad \text{with} \quad \eta(x_i) \geq 1 \text{ for all } i$$

which is known to be the binomial coefficient

$$\binom{M + L_i N + N - 1}{N - 1}.$$

See for instance [5, Figure 1.3]. \square

Using (3.14) and Lemma 3.9, we are now ready to complete the proof of Theorem 3.1, which not only shows conjecture (3.1) but also extends the conjecture to all finite connected graphs.

Proof of Theorem 3.1. According to Lemma 3.9,

$$\Lambda_X(N, M, L_i) = \frac{1}{(N - 1)!} \prod_{k=1}^{N-1} (M + L_i N + k).$$

Similarly, we have

$$\Lambda_X(N-1, M-c, L_i) = \frac{1}{(N-2)!} \prod_{k=1}^{N-2} (M-c + L_i N - L_i + k).$$

Taking the ratio, we get

$$\begin{aligned} \frac{\Lambda_X(N-1, M-c, L_i)}{\Lambda_X(N, M, L_i)} &= \frac{(N-1)!}{(M + L_i N + N - 1)(N-2)!} \prod_{k=1}^{N-2} \left(\frac{M-c + L_i N - L_i + k}{M + L_i N + k} \right) \\ &= \frac{N-1}{M + L_i N + N - 1} \prod_{k=1}^{N-2} \left(\frac{M + L_i N + k - (c + L_i)}{M + L_i N + k} \right) \\ &= \frac{N-1}{M + L_i N + N - 1} \prod_{k=1}^{N-2} \left(1 - \frac{c + L_i}{M + L_i N + k} \right). \end{aligned}$$

Recalling that $T = M/N$, in the limit as $N, T \rightarrow \infty$,

$$\begin{aligned} \frac{\Lambda_X(N-1, M-c, L_i)}{\Lambda_X(N, M, L_i)} &\sim \left(\frac{N}{N(T + L_i)} \right) \left(1 - \frac{c + L_i}{N(T + L_i)} \right)^N \\ &\sim \left(\frac{1}{T + L_i} \right) \exp \left(- \frac{c + L_i}{T + L_i} \right). \end{aligned}$$

This, together with (3.14), completes the proof of the theorem. \square

3.7 Proof of Theorem 3.2

This section is devoted to the proof of Theorem 3.2 that describes the limiting behavior of the model with collective debt limit. Following the same argument as in Section 3.6, but using Lemmas 3.4, 3.6 and 3.8 instead of Lemmas 3.3, 3.5 and 3.7, we obtain that

$$\lim_{t \rightarrow \infty} P_\eta(Y_t = \xi) = \pi_Y(\xi) \quad \text{for all } \xi, \eta \in \mathcal{D}_{N, M, L_c} \quad (3.15)$$

where the (unique) stationary distribution π_Y is

$$\pi_Y = \text{Uniform}(\mathcal{D}_{N, M, L_c}). \quad (3.16)$$

Now, given that there are c coins at vertex x , there is a total of $M-c$ coins distributed across the rest of the graph. In addition, the number of coins from the bank all the

agents excluding x can borrow is L_c when $c \geq 0$ but $L_c + c$ when $c < 0$. Hence, letting $\Lambda_Y(N, M, L_c) = \text{card}(\mathcal{D}_{N,M,L_c})$ and applying (3.15) and (3.16), we deduce that, regardless of the initial configuration,

$$\lim_{t \rightarrow \infty} P(Y_t(x) = c) = \begin{cases} \frac{\Lambda_Y(N-1, M-c, L_c)}{\Lambda_Y(N, M, L_c)} & \text{for all } 0 \leq c \leq M + L_c \\ \frac{\Lambda_Y(N-1, M-c, L_c + c)}{\Lambda_Y(N, M, L_c)} & \text{for all } -L_c \leq c \leq 0. \end{cases}$$

In particular, to complete the proof of Theorem 3.2, it suffices to compute the number of configurations $\Lambda_Y(N, M, L_c)$, which is done in the next lemma.

Lemma 3.10 – *For all N, M and L_c ,*

$$\Lambda_Y(N, M, L_c) = \sum_{a=0}^{L_c} \sum_{b=0}^N \binom{N}{b} \binom{a-1}{b-1} \binom{M+a+N-b-1}{N-b-1}.$$

Proof. Introducing the set

$$\mathcal{D}_{N,M,L_c}^+ = \{\xi \in \mathcal{D}_{N,M,L_c} : \xi(x) \geq 0 \text{ for all } x \in \mathcal{V}\}$$

and its complement $\mathcal{D}_{N,M,L_c}^- = \mathcal{D}_{N,M,L_c} \setminus \mathcal{D}_{N,M,L_c}^+$, we have

$$\Lambda_Y(N, M, L_c) = \text{card}(\mathcal{D}_{N,M,L_c}) = \text{card}(\mathcal{D}_{N,M,L_c}^+) + \text{card}(\mathcal{D}_{N,M,L_c}^-). \quad (3.17)$$

Because \mathcal{D}_{N,M,L_c}^+ is the set of configurations with nobody in debt, $\mathcal{D}_{N,M,L_c} = \mathcal{C}_{N,M,0}$.

In particular, a direct application of Lemma 3.9 implies that

$$\text{card}(\mathcal{D}_{N,M,L_c}^+) = \text{card}(\mathcal{C}_{N,M,0}) = \Lambda_X(N, M, 0) = \binom{M+N-1}{N-1}. \quad (3.18)$$

To count the number of configurations in \mathcal{D}_{N,M,L_c}^- , we let

$\phi(N, M, L_c, a, b) =$ number of configurations in \mathcal{D}_{N,M,L_c} with a coins
borrowed from the bank and where the debt
is shared among b individuals.

Then, we have the following decomposition:

$$\text{card}(\mathcal{D}_{N,M,L_c}^-) = \sum_{a=1}^{L_c} \sum_{b=1}^{a \wedge (N-1)} \phi(N, M, L_c, a, b). \quad (3.19)$$

Now, observe that

- There are N choose b ways to choose the individuals in debt.
- Following the same reasoning as in the proof of Lemma 3.9, there are $a - 1$ choose $b - 1$ ways to distribute the debt among those individuals,
- Similarly, there are $(M + a) + (N - b) - 1$ choose $N - b - 1$ ways to distribute the $M + a$ coins among the remaining $N - b$ individuals.

This implies that

$$\phi(N, M, L_c, a, b) = \binom{N}{b} \binom{a-1}{b-1} \binom{M+a+N-b-1}{N-b-1}. \quad (3.20)$$

Finally, combining (3.17)–(3.20) and using the convention

$$\binom{-1}{-1} = 1 \quad \text{and} \quad \binom{n}{k} = 0 \quad \text{when} \quad n < k \quad \text{or} \quad k < 0 \leq n$$

we conclude that

$$\begin{aligned} \Lambda_Y(N, M, L_c) &= \binom{M+N-1}{N-1} + \sum_{a=1}^{L_c} \sum_{b=1}^{a \wedge (N-1)} \phi(N, M, L_c, a, b) \\ &= \binom{M+N-1}{N-1} + \sum_{a=1}^{L_c} \sum_{b=1}^{a \wedge (N-1)} \binom{N}{b} \binom{a-1}{b-1} \binom{M+a+N-b-1}{N-b-1} \\ &= \sum_{a=0}^{L_c} \sum_{b=0}^N \binom{N}{b} \binom{a-1}{b-1} \binom{M+a+N-b-1}{N-b-1}. \end{aligned}$$

This completes the proof. \square

3.8 Partial Proof of the Conjecture (3.3)

As previously mentioned, the plot (using a computer) of the explicit expression for the distribution of money proved in Theorem 3.2 strongly suggests convergence to the asymmetric Laplace distribution conjectured in [9]. In this section, we give more evidence that this conjecture is true. More precisely, we assume convergence to an asymmetric Laplace distribution

$$f_Y(c) = \begin{cases} K e^{-ac} & \text{for } c \geq 0 \\ K e^{+bc} & \text{for } c \leq 0 \end{cases} \quad (3.21)$$

and argue that the three parameters K , a and b are indeed given by (3.3). To compute these three parameters, the basic idea is to derive a system of three equations involving these parameters. The next lemma gives two such equations.

Lemma 3.11 – *For all N, M and L_c , we have*

$$\frac{K}{a} + \frac{K}{b} = 1 \quad \text{and} \quad \frac{K}{a^2} - \frac{K}{b^2} = \frac{M}{N}.$$

Proof. The first equation directly follows from the fact that, because f_Y is a density function, its integral must be equal to one, while computing this integral gives

$$\int_{\mathbb{R}} f_Y(x) dx = \int_{\mathbb{R}_-} K e^{+bx} dx + \int_{\mathbb{R}_+} K e^{-ax} dx = \frac{K}{a} + \frac{K}{b}.$$

The second equation follows from looking at the mean number of coins per individual. Because each interaction has either no effect or moves one coin from a vertex x to a vertex y , which results in the state at x to decrease by one and the state at y to decrease by one, we have

$$\sum_{z \in \mathcal{V}} Y_t(z) = \sum_{z \in \mathcal{V}} Y_0(z) = M \quad \text{for all } t \in \mathbb{R}_+. \quad (3.22)$$

But our assumption that the distribution of money converges to the asymmetric Laplace distribution given in (3.21) also implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} E\left(\frac{1}{N} \sum_{z \in \mathcal{V}} Y_t(z)\right) &= \lim_{t \rightarrow \infty} E(Y_t(x)) = \int_{\mathbb{R}} x f_Y(x) dx \\ &= \int_{\mathbb{R}_-} K x e^{+bx} dx + \int_{\mathbb{R}_+} K x e^{-ax} dx = \frac{K}{a^2} - \frac{K}{b^2}. \end{aligned} \quad (3.23)$$

Combining (3.22) and (3.23) gives the second equation. \square

To find a third equation involving the parameters K, a and b , the next step is to argue that the number of coins in the bank, when rescaled by its initial value L_c , tends to zero as time goes to infinity. More precisely, we believe that

$$\lim_{t \rightarrow \infty} \frac{Y_t(\star)}{L_c} \rightarrow 0 \quad \text{as} \quad N, M/N, L_c/N \rightarrow \infty. \quad (3.24)$$

The next lemma shows that, as long as there is at least one coin in the bank, the number of coins in the bank behaves like a supermartingale, which suggests that (3.24) is true.

Lemma 3.12 – *For all N, M, L_c , we have*

$$\lim_{t \rightarrow \infty} E(Y_{t+1}(\star) - Y_t(\star) \mid Y_t(\star) > 0) < 0.$$

Proof. Letting $(x, y) \in \vec{\mathcal{E}}$ be the oriented edge selected at time t , we define

$$J_t = (\text{sign}(Y_t(x)), \text{sign}(Y_t(y))) = \text{type of interaction at time } t.$$

In words, the random variable J_t keeps track of whether the agent x selected to give a coin and the agent y selected to receive a coin are in debt (state $-$), have zero coin (state 0), or have a surplus of coins (state $+$). There are $3^2 = 9$ types of interactions and, as long as there is at least one coin in the bank, the type of interaction determines

whether the number of coins in the bank decreases, stays the same, or increases. More precisely,

$$\begin{aligned}
Y_{t+1}(\star) &= Y_t(\star) - 1 \quad \text{when } J_t = (-, 0), (-, +), (0, 0), (0, +) \\
Y_{t+1}(\star) &= Y_t(\star) \quad \text{when } J_t = (-, -), (0, -), (+, 0), (+, +) \\
Y_{t+1}(\star) &= Y_t(\star) + 1 \quad \text{when } J_t = (+, -).
\end{aligned} \tag{3.25}$$

Now, define the corresponding conditional probabilities

$$p(\epsilon_1, \epsilon_2) = \lim_{t \rightarrow \infty} P(J_t = (\epsilon_1, \epsilon_2) \mid Y_t > 0) \quad \text{for all } \epsilon_1, \epsilon_2 \in \{-, 0, +\}.$$

Because edges (x, y) and (y, x) are equally likely to be chosen,

$$p(\epsilon_1, \epsilon_2) = p(\epsilon_2, \epsilon_1) \quad \text{for all } \epsilon_1, \epsilon_2 \in \{-, 0, +\}. \tag{3.26}$$

Also, using convergence to $\pi_Y = \text{Uniform}(\mathcal{D}_{N,M,L_c})$ and that, regardless of the number of coins in the bank, there is a positive fraction of configurations with zero coin at a given vertex,

$$p(0, -) + p(0, 0) + p(0, +) = \lim_{t \rightarrow \infty} P(Y_t(x) = 0 \mid Y_t(\star) > 0) > 0. \tag{3.27}$$

Combining (3.25)–(3.27), we deduce that

$$\begin{aligned}
&\lim_{t \rightarrow \infty} E(Y_{t+1}(\star) - Y_t(\star) \mid Y_t(\star) > 0) \\
&= p(+, -) - p(-, 0) - p(-, +) - p(0, 0) - p(0, +) \\
&= - (p(-, 0) + p(0, 0) + p(0, +)) = - \lim_{t \rightarrow \infty} P(Y_t(x) = 0 \mid Y_t(\star) > 0) < 0.
\end{aligned}$$

This completes the proof. \square

The previous lemma shows that the process $(Y_t(\star))$ stopped at the time it reaches zero is a supermartingale. Because the inequality in the lemma is strict, this suggests that (3.24) holds. To make the proof perfectly rigorous, we would need to prove a

slightly stronger result, namely that the conditional expectation is less than a negative constant that does not depend on the parameters of the system. Numerical simulations of the model with collective debt limit support this result, revealing that the number of coins in the bank drops quickly and then fluctuates around values that are negligible compared to L_c . Moving forward with our heuristic argument, we now assume that (3.24) holds in order to derive a third equation involving the parameters K, a and b .

Lemma 3.13 – *Assume that (3.24) holds. Then,*

$$\frac{K}{a^2} \sim \frac{M + L_c}{N} \quad \text{as } N, M/N, L_c/N \rightarrow \infty.$$

Proof. The trick to establish the lemma is now to look at the mean number of coins among the individuals who have no debt. Due to (3.24), the set of all individuals with no debt share a total of about $M + L_c$ coins at equilibrium. In equation, this can be written as

$$\lim_{t \rightarrow \infty} \sum_{z \in \mathcal{V}} Y_t(z) \mathbf{1}\{Y_t(z) \geq 0\} \sim M + L_c. \quad (3.28)$$

In other respects, our assumption that the distribution of money converges to the Laplace distribution given in (3.21) implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} E \left(\frac{1}{N} \sum_{z \in \mathcal{V}} Y_t(z) \mathbf{1}\{Y_t(z) \geq 0\} \right) \\ = \int_{\mathbb{R}} x f_Y(x) \mathbf{1}\{x \geq 0\} dx = \int_{\mathbb{R}_+} K x e^{-ax} dx = \frac{K}{a^2}. \end{aligned} \quad (3.29)$$

Combining (3.28) and (3.29) proves the lemma. \square

Using Lemmas 3.11 and 3.13, we can now prove (3.3) that we recall in the next lemma.

Lemma 3.14 – *Let $\rho = L_c/M$. Then, as $N, M/N, L_c/N \rightarrow \infty$,*

$$K \sim \frac{1}{T} \left(\sqrt{1 + \rho} - \sqrt{\rho} \right)^2 \quad a \sim \frac{1}{T} \left(1 - \sqrt{\frac{\rho}{1 + \rho}} \right) \quad b \sim \frac{1}{T} \left(\sqrt{\frac{1 + \rho}{\rho}} - 1 \right).$$

Proof. Lemma 3.13 and the second equation in Lemma 3.11 imply that

$$\frac{K}{a^2} \sim \frac{M + L_c}{N} \quad \text{and} \quad \frac{K}{b^2} = \frac{K}{a^2} - \left(\frac{K}{a^2} - \frac{K}{b^2} \right) \sim \frac{M + L_c}{N} - \frac{M}{N} = \frac{L_c}{N}.$$

In particular, some basic algebra gives

$$a \sim \sqrt{\frac{KN}{M + L_c}} \quad \text{and} \quad b \sim \sqrt{\frac{KN}{L_c}} \quad (3.30)$$

which, together with the first equation in Lemma 3.11, implies that

$$\frac{1}{K} = \frac{1}{a} + \frac{1}{b} \sim \sqrt{\frac{M + L_c}{KN}} + \sqrt{\frac{L_c}{KN}} \quad \text{and} \quad K \sim \left(\frac{\sqrt{N}}{\sqrt{M + L_c} + \sqrt{L_c}} \right)^2. \quad (3.31)$$

Combining (3.30) and (3.31), we deduce that

$$\begin{aligned} a &\sim \left(\frac{\sqrt{N}}{\sqrt{M + L_c} + \sqrt{L_c}} \right) \sqrt{\frac{N}{M + L_c}} = \frac{1}{\sqrt{M + L_c}} \left(\frac{N}{\sqrt{M + L_c} + \sqrt{L_c}} \right) \\ b &\sim \left(\frac{\sqrt{N}}{\sqrt{M + L_c} + \sqrt{L_c}} \right) \sqrt{\frac{N}{L_c}} = \frac{1}{\sqrt{L_c}} \left(\frac{N}{\sqrt{M + L_c} + \sqrt{L_c}} \right). \end{aligned} \quad (3.32)$$

Recalling $T = M/N$ and $\rho = L_c/M$, and using (3.31) and (3.32), we get

$$\begin{aligned} K &\sim \frac{1}{T} \left(\frac{1}{\sqrt{1 + \rho} + \sqrt{\rho}} \right)^2 = \frac{1}{T} \left(\sqrt{1 + \rho} - \sqrt{\rho} \right)^2 \\ a &\sim \frac{1}{T} \sqrt{\frac{1}{1 + \rho}} \left(\frac{1}{\sqrt{1 + \rho} + \sqrt{\rho}} \right) = \frac{1}{T} \left(1 - \sqrt{\frac{\rho}{1 + \rho}} \right) \\ b &\sim \frac{1}{T} \sqrt{\frac{1}{\rho}} \left(\frac{1}{\sqrt{1 + \rho} + \sqrt{\rho}} \right) = \frac{1}{T} \left(\sqrt{\frac{1 + \rho}{\rho}} - 1 \right). \end{aligned}$$

This completes the proof. \square

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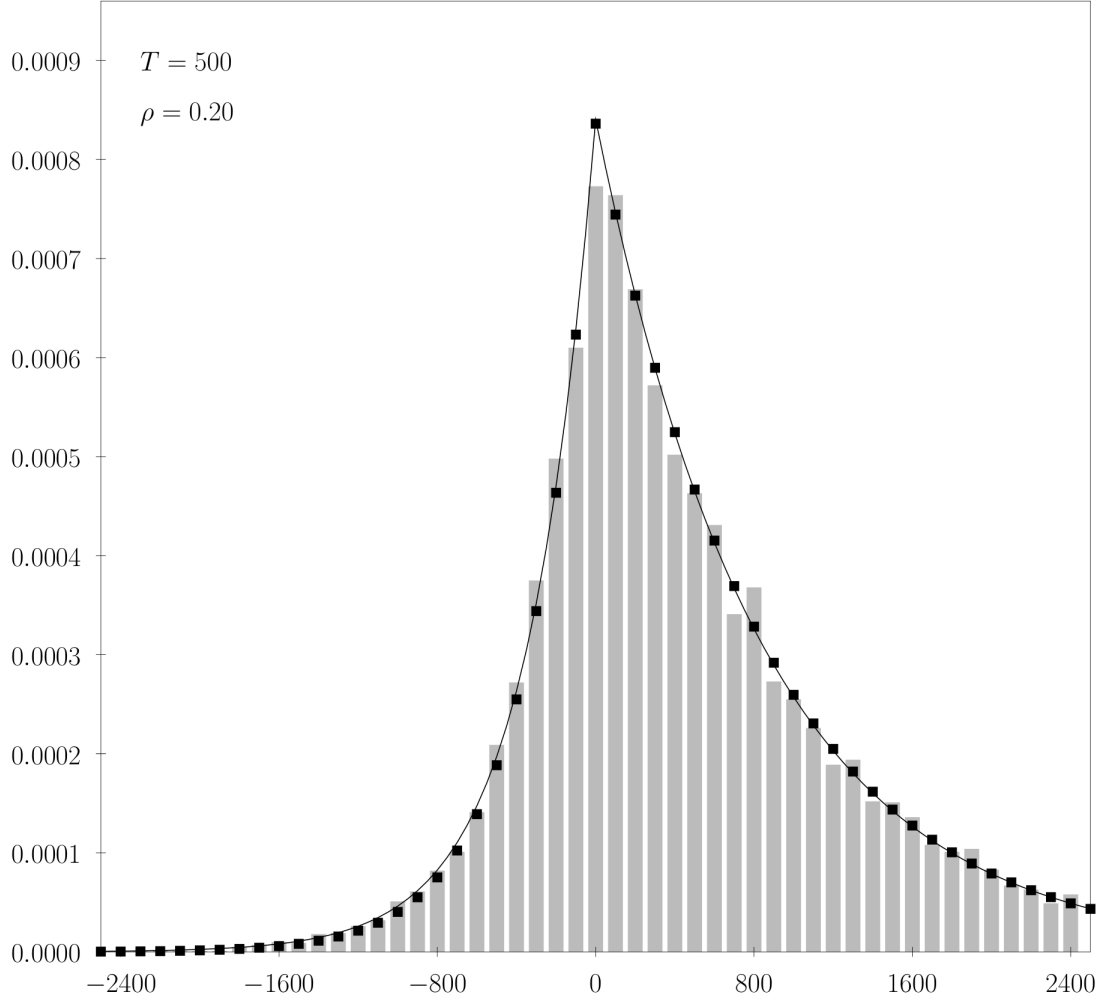


Figure 3.2: Distribution of Money for the Model with Collective Debt Limit with Money Temperature $T = 500$ and Parameter $\rho = 0.20$. The Solid Curve Corresponds to the Graph of the Density Function (3.2)–(3.3) While the Black Squares Correspond to the Distribution of Money Obtained From the Exact Expression in Theorem 3.2 with $N = 100$. The Gray Histogram Gives the Distribution of Money Obtained From 5×10^{10} Iterations of the Stochastic Process on the Complete Graph with $N = 10,000$ Vertices.

Chapter 4

THE ROLE OF COOPERATION IN SPATIALLY EXPLICIT ECONOMICAL SYSTEMS

4.1 Abstract

This paper is concerned with a model in econophysics, the subfield of statistical physics that applies concepts from traditional physics to economics. In our model, economical agents are represented by the vertices of a connected graph and are characterized by the number of coins they possess. Agents independently spend one coin at rate one for their basic need, earn one coin at a rate chosen independently from a fixed distribution ϕ and exchange money at rate μ with one of their nearest neighbors, with the richest neighbor giving one coin to the other neighbor. If an agent needs to spend one coin when her fortune is at zero, she dies, i.e., the corresponding vertex is removed from the graph. Our first results focus on the two extreme cases of lack of cooperation $\mu = 0$ and perfect cooperation $\mu = \infty$ for finite connected graphs. These results suggest that, when overall the agents earn more than they spend, cooperation is beneficial for the survival of the population, whereas when overall the agents earn less than they spend, cooperation becomes detrimental. The infinite one-dimensional system is also studied. In this case, when the agents earn less than they spend in average, the density of agents that die eventually is bounded from below by a positive constant that does not depend on the initial number of coins per agent or the level of cooperation.

4.2 Introduction

Models in econophysics typically consist of large systems of economical agents who earn, spend and exchange money. For a review of such models, we refer the reader to [8]. These models so far have mainly been studied by statistical physicists. From a mathematical point of view, they fall into the category of stochastic processes known as interacting particle systems [4, 7]. The most basic model in econophysics has been studied in [3] based on numerical simulations but was also considered earlier in [1, 2]. This model consists of a system of N interacting economical agents that are characterized by the number of dollars they possess, and evolves as follows: at each time step, an agent chosen uniformly at random gives one dollar to another agent again chosen uniformly at random, unless the first agent has no money in which case nothing happens. The main conjecture about this model is that, when the number of agents and the money temperature, defined as the average amount of money per agent, are both large, the limiting distribution of money is well approximated by the exponential distribution with parameter the money temperature.

Spatially explicit versions of this model where agents are located on the vertices of a finite connected graph and can only exchange money with their nearest neighbors have been recently introduced and studied analytically in [6]. The non-spatial model considered in [3] is simply obtained by assuming that the underlying graph is the complete graph with N vertices. It is proved in [6] that the conjecture in [3] is indeed correct and in fact holds for all spatially explicit versions, not only the process on the complete graph.

In this paper, we study variants of the spatially explicit models [6] where agents also earn money, spend money and die if they run out of money. In addition, we assume that the exchange of money occurs in a cooperative setting, meaning that

the flow of money is exclusively directed from “rich” agents to “poor” agents. We also follow the framework of interacting particle systems [7] by assuming that the process evolves in continuous rather than in discrete time. This approach will allow us to define the system on infinite graphs using an idea of Harris [4] that consists in constructing the process from a collection of independent Poisson processes.

4.3 Model Description

To define our spatial model formally, we let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite or infinite locally finite connected graph. Each vertex represents an economical agent who is either alive and characterized by the amount of money she possesses, or dead. To fix the ideas, we assume that the amount of money agents who are alive possess is a nonnegative integer representing a number of credits or coins, while we use the state -1 for dead agents. In particular, the state of the system at time t is a spatial configuration

$$\xi_t : \mathcal{V} \longrightarrow \{-1, 0, 1, 2, \dots\}$$

with the value of $\xi_t(x)$ indicating that agent x is dead or representing the number of coins this agent possesses when she is alive. To define the evolution rules, we attach to each vertex $x \in \mathcal{V}$ a random variable ϕ_x chosen independently from a fixed distribution ϕ . The individual at vertex x earns one coin at rate ϕ_x and, to ensure her survival, spends one coin at rate one. The population is also characterized by its level of cooperation which is measured using a nonnegative parameter μ as follows: nearest neighbors that are alive interact at rate μ and, in case one neighbor has at least two more coins than the other neighbor, she gives one coin to the other neighbor. In particular, the “richest” agent before the interaction does not give any coin if this makes her “poorer” than her neighbor. Finally, if an individual has zero coin at the

time she needs to spend one coin then she dies and the corresponding vertex is removed from the graph. To describe the dynamics formally, for each spatial configuration ξ , we let

$$\textbf{spending} \quad \xi_x^-(z) = \xi(z) - \mathbb{1}\{z = x\} \quad \text{for all } z \in \mathcal{V}$$

$$\textbf{earning} \quad \xi_x^+(z) = \xi(z) + \mathbb{1}\{z = x\} \quad \text{for all } z \in \mathcal{V}$$

be the configurations obtained respectively by removing/adding one coin at vertex x .

Also, for each edge $(x, y) \in \mathcal{E}$ of the network of interactions, we let

$$\begin{aligned} \textbf{cooperation} \quad \xi_{(x,y)}(z) = & \xi(z) + \mathbb{1}\{\xi(x) < \xi(y) - 1\}(\mathbb{1}\{z = x\} - \mathbb{1}\{z = y\}) \\ & + \mathbb{1}\{\xi(y) < \xi(x) - 1\}(\mathbb{1}\{z = y\} - \mathbb{1}\{z = x\}) \end{aligned}$$

be the configuration obtained by moving one coin from the richer to the poorer vertex if the two vertices are at least two coins apart. The dynamics of the system is then described by the Markov generator L defined on the set of cylinder functions by

$$\begin{aligned} Lf(\xi) = & \sum_{x \in \mathcal{V}} (f(\xi_x^-) - f(\xi)) \mathbb{1}\{\xi(x) \neq -1\} \\ & + \sum_{x \in \mathcal{V}} \phi_x (f(\xi_x^+) - f(\xi)) \mathbb{1}\{\xi(x) \neq -1\} \\ & + \sum_{(x,y) \in \mathcal{E}} \mu (f(\xi_{(x,y)}) - f(\xi)) \mathbb{1}\{\xi(x) \neq -1, \xi(y) \neq -1\}. \end{aligned}$$

The first sum describes the rate at which vertices spend one coin, the second sum the rate at which they earn one coin, and the third sum the rate at which neighbors exchange one coin. As previously mentioned, the process is well defined on locally finite graphs, including infinite graphs, and can be constructed from a collection of independent Poisson processes. More precisely,

- for all $x \in \mathcal{V}$, let $N_t^-(x)$ be a Poisson process with intensity one,
- for all $x \in \mathcal{V}$, let $N_t^+(x)$ be a Poisson process with intensity ϕ_x ,
- for all $(x, y) \in \mathcal{E}$, let $N_t(x, y)$ be a Poisson process with intensity μ .

We further assume that these processes are independent. This implies that, with probability one, the arrival times are all distinct. A general result due to Harris [4] then shows that the process can be constructed using the following rules:

- At the arrival times of the Poisson process $N_t^-(x)$, we take one coin from the individual at vertex x if this individual is still alive.
- At the arrival times of the Poisson process $N_t^+(x)$, we give one coin to the individual at vertex x if this individual is still alive.
- At the arrival times of $N_t^+(x, y)$, we move one coin from x to y if x has at least two more coins than y or one coin from y to x if y has at least two more coins than x .

4.4 Main Results

To begin with, we compare the two processes with the same earning rates ϕ_z in the absence of cooperation $\mu = 0$ and in the presence of perfect cooperation $\mu = \infty$ on finite connected graphs to understand whether cooperation is beneficial or detrimental for survival. Our first results look at the probability of global survival that we define as

$$p_\mu(c, (\phi_z)) = P(\xi_t(z) \neq -1 \text{ for all } (z, t) \in \mathcal{V} \times \mathbb{R}_+ \mid \xi_0 \equiv c)$$

where c refers to the common initial number of coins per agent and where the earning rates ϕ_z are independent realizations of the distribution ϕ for all $z \in \mathcal{V}$. Estimates for the probability of global survival can be expressed in terms of the two key quantities

$$\mathcal{D} = \max_{x \in \mathcal{V}} \sum_{z \in \mathcal{V}} d(x, z) \quad \text{and} \quad \Phi = \frac{1}{N} \sum_{z \in \mathcal{V}} \phi_z \quad (4.1)$$

where d refers to the graph distance and N to the population size. Using that, as long as all the agents are alive, the total number of coins on the graph behaves like a random walk that increases at rate $N\Phi$ and decreases at rate N together with the fact that nearest neighbors are at most one coin apart in the presence of perfect cooperation, we get the following theorem.

Theorem 4.1 – *In the presence of perfect cooperation $\mu = \infty$,*

$$p_\infty(c, (\phi_z)) \geq \max(0, 1 - \Phi^{-(Nc - \mathcal{D} + 1)}).$$

The proof relies, among other things, on an application of the optional stopping theorem for martingales. The inequality in the statement turns out to be an equality when $N = 1$. In particular, since the system in the absence of cooperation behaves like N independent copies of a one-person system, the theorem also gives the probability of global survival when $\mu = 0$. Using this and some basic algebra, it can be proved that, when $\Phi > 1$ and c is large, the probability of global survival is larger in the presence of perfect cooperation than in the absence of cooperation.

Theorem 4.2 – *Assume that $\Phi > 1$. Then, there exists $c_0 < \infty$ that depends on N such that*

$$p_0(c, (\phi_z)) = \prod_{z \in \mathcal{V}} \max(0, 1 - \phi_z^{-(c+1)}) \leq \max(0, 1 - \Phi^{-(Nc - \mathcal{D} + 1)}) \leq p_\infty(c, (\phi_z))$$

for all $c \geq c_0$.

More generally, we conjecture that, when $\Phi > 1$, i.e., when overall the agents earn more than they spend, the probability of global survival is larger in the presence of perfect cooperation than in the absence of cooperation regardless of the initial value c .

We now focus on the two-person system: we set $\mathcal{V} = \{x, y\}$ and assume that

vertices x and y are connected by an edge. In this case, Theorem 4.1 implies that when

$$\Phi = \frac{\phi_x + \phi_y}{2} > 1 \quad \text{and} \quad \phi_x < 1 < \phi_y$$

global survival is possible in the presence of perfect cooperation whereas individual x dies almost surely in the absence of cooperation, showing again that cooperation is beneficial. Cooperation, however, becomes detrimental when

$$\Phi = \frac{\phi_x + \phi_y}{2} < 1 \quad \text{and} \quad \phi_x < 1 < \phi_y.$$

In this case, regardless of the level of cooperation μ , global survival is not possible so, to measure the effect of cooperation, we study instead

$$E_\mu(c, (\phi_z)) = E(\text{card}\{z \in \mathcal{V} : \xi_t(z) \neq -1 \text{ for all } t \in \mathbb{R}_+\} \mid \xi_0 \equiv c),$$

the expected number of individuals that live forever. Due to perfect cooperation and the fact that individual x dies almost surely, it can be proved that the last time both individuals each have one coin is almost surely finite and that, between this time and the first time one of the two individuals dies, the process behaves according to a certain seven-state Markov chain. Using a first-step analysis to study this Markov chain and part of the proof of Theorem 4.1, the expected value of the number of individuals that live forever can be computed explicitly.

Theorem 4.3 – Assume that $\mathcal{V} = \mathcal{E} = \{x, y\}$ and that

$$\Phi = \frac{\phi_x + \phi_y}{2} < 1 \quad \text{and} \quad \phi_x < 1 < \phi_y.$$

Then, letting $\Psi = 8 + 2\phi_x + 2\phi_y$, for all $c \geq 1$,

$$\begin{aligned} E_\infty(c, \phi_x, \phi_y) &= \left(\frac{2}{\Psi}\right) \left(1 - \frac{1}{\phi_y}\right) + \left(\frac{\phi_y}{\Psi} + \frac{1}{4}\right) \left(1 - \left(\frac{1}{\phi_y}\right)^2\right) \\ &< 1 - \left(\frac{1}{\phi_y}\right)^{c+1} = E_0(c, \phi_x, \phi_y). \end{aligned}$$

Our approach to prove this result works in theory for all complete graphs, but becomes computationally intractable even with only three vertices. More generally, we conjecture that, at least on the complete graph and when $\Phi < 1$, i.e., when overall the agents earn less than they spend, the expected number of individuals that live forever is larger in the absence of cooperation than in the presence of perfect cooperation. In a nutshell, we conjecture that cooperation is beneficial for populations that are “productive” but detrimental for populations that are not.

Finally, we look at the infinite system in one dimension: the underlying graph is represented by the integers with each integer being connected to its predecessor and to its successor. In this case, the process is more difficult to study because the graph is infinite. The next result shows that, when the expected value of ϕ is less than one, the density of individuals who die eventually in the infinite one-dimensional system is bounded from below by a positive constant that does not depend on the level of cooperation or on the initial number of coins per agent.

Theorem 4.4 – *Assume that $E(\phi) < 1$. Then,*

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbb{1}\{\xi_t(z) = -1 \text{ for some } t\} = l$$

where $l > 0$ does not depend on μ or on the initial fortune c per vertex.

To prove this result, we first identify a collection of events that ensure that a given agent dies before time one. This, together with the ergodic theorem, implies that the density of agents that die before time one is positive. This density, however, depends *a priori* on the initial fortune. Then, we define a sink as a vertex such that the agents in any finite interval that contains this vertex earn overall less than they spend. The law of large numbers implies that the density of sinks is bounded from below by a constant that does not depend on the initial fortune. Using finally that, at time one, each sink is located between two agents who already died, we use a

recursive argument to prove that each sink dies eventually. In conclusion, the density of individuals who die eventually is bounded from below by the density of sinks which, in turn, is bounded from below by a positive constant that does not depend on the initial fortune. This gives the result.

The proof of Theorem 4.4 also suggests that, when the expected value of ϕ is larger than one, the density of agents who live forever can be made arbitrarily close to one by choosing the initial fortune c large enough. The proof of this result, however, requires additional arguments that we were not able to make rigorous.

4.5 Proof of Theorems 4.1 and 4.2

In this section, we start by collecting some preliminary results about martingales that will be used later to prove the first two theorems. The first step is to estimate probabilities related to the continuous-time Markov chain (W_t) with transition rates

$$\begin{aligned}\lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(W_{t+\epsilon} = W_t + 1) &= \sum_{z \in \mathcal{V}} \phi_z \\ \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(W_{t+\epsilon} = W_t - 1) &= \text{card}(\mathcal{V}) = N.\end{aligned}\tag{4.2}$$

Recall from (4.1) that $\Phi = (1/N) \sum_{z \in \mathcal{V}} \phi_z$. To state our next results, we also define

$$T_i = \inf \{t : W_t = i\} \quad \text{for all } i \in \mathbb{Z}.$$

Lemma 4.5 – *Assume that $K \leq Nc \leq M$ and $\Phi \neq 1$. Then,*

$$p(K, M) = P(T_M < T_K \mid W_0 = Nc) = \frac{1 - \Phi^{-(Nc-K)}}{1 - \Phi^{-(M-K)}}.$$

Proof. This follows from the optional stopping theorem applied to the martingale (Φ^{-W_t}) stopped at time $T = \min(T_K, T_M)$. See [5, Example 5.1] for a proof.

□

Lemma 4.6 – *For all $M \leq Nc$ and all $\Phi > 0$,*

$$q(M) = P(T_M = \infty \mid W_0 = Nc) = \max(0, 1 - \Phi^{-(Nc-M)}).$$

Proof. We distinguish three cases depending on the value of Φ .

- When $\Phi = 1$, the process (W_t) is the one-dimensional symmetric random walk which is known to be recurrent. This gives the probability $q(M) = 0$.
- When $\Phi < 1$, the law of large numbers implies that $W_t \rightarrow -\infty$ almost surely. In particular, the stopping time T_M is again almost surely finite and the probability $q(M) = 0$.
- When $\Phi > 1$, the law of large numbers now gives $W_t \rightarrow \infty$ so

$$\{T_M = \infty\} = \{T_K < T_M \text{ for all } K \geq Nc\} \quad \text{almost surely.}$$

Since in addition we have the inclusions

$$\{T_{K+1} < T_M\} \subset \{T_K < T_M\} \quad \text{for all } K \geq Nc,$$

by continuity from above and Lemma 4.5, we get

$$\begin{aligned} q(M) &= P(T_K < T_M \text{ for all } K \geq Nc \mid W_0 = Nc) \\ &= P(\lim_{K \rightarrow \infty} \{T_K < T_M\} \mid W_0 = Nc) \\ &= \lim_{K \rightarrow \infty} P(T_K < T_M \mid W_0 = Nc) = 1 - \Phi^{-(Nc-M)}. \end{aligned}$$

Observing also that $1 - \Phi^{-(Nc-M)} \leq 0$ if and only if $\Phi \leq 1$ gives the result. \square

Lemma 4.6 is the main ingredient to prove Theorem 4.1. To see the connection between the previous martingale results and the economical system, define

$$\tau = \inf \{t : \xi_t(x) = -1 \text{ for some } x \in \mathcal{V}\} \quad \text{and} \quad Z_t = \sum_{z \in \mathcal{V}} \xi_t(z)$$

and observe that, before time τ , the individual at z is alive, earns one coin at rate ϕ_z and spends one coin at rate one, therefore

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(Z_{t+\epsilon} = Z_t + 1 \mid \tau > t) &= \sum_{z \in \mathcal{V}} \phi_z \\ \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(Z_{t+\epsilon} = Z_t - 1 \mid \tau > t) &= \text{card}(\mathcal{V}) = N. \end{aligned}$$

In other words, by time τ , the total number of coins behaves like the Markov chain (W_t) . Using this and the previous lemma, we can now prove the theorem.

Proof of Theorem 4.1. In the limiting case $\mu = \infty$ and as long as all the individuals are alive, each time an individual has at least two more coins than one of her neighbors, this individual instantaneously gives a coin to one of her poorest neighbors, therefore

$$|\xi_t(x) - \xi_t(y)| \leq 1 \quad \text{for all } (x, y) \in \mathcal{E} \text{ and } t < \tau.$$

Now, letting $x, y \in \mathcal{V}$ be arbitrary, there exist

$$z_0 = x, z_1, \dots, z_d = y \in \mathcal{V} \quad \text{such that} \quad (z_i, z_{i+1}) \in \mathcal{E} \quad \text{for all } i = 0, 1, \dots, d-1$$

where $d = d(x, y)$. In particular, the triangle inequality implies that

$$\begin{aligned} |\xi_t(x) - \xi_t(y)| &\leq |\xi_t(z_0) - \xi_t(z_1)| + \dots + |\xi_t(z_{d-1}) - \xi_t(z_d)| \\ &\leq d = d(x, y) \end{aligned} \tag{4.3}$$

for all $t < \tau$. Now, on the event that $\tau < \infty$, just before that time, there is at least one vertex, say x , with zero coin, while the other vertices have a positive fortune. This, together with (4.3), implies that the total number of coins satisfies

$$Z_{\tau-} = \sum_{z \in \mathcal{V}} \xi_{\tau-}(z) = \sum_{z \in \mathcal{V}} |\xi_{\tau-}(x) - \xi_{\tau-}(z)| \leq \sum_{z \in \mathcal{V}} d(x, z).$$

Taking the maximum over all possible configurations gives

$$Z_{\tau-} \leq \max_{x \in \mathcal{V}} \sum_{z \in \mathcal{V}} d(x, z) = \mathcal{D}.$$

Finally, using Lemma 4.6 and observing that all the individuals survive if and only if $\tau = \infty$ gives the following lower bound for the probability of global survival

$$\begin{aligned}
p_\infty(c, (\phi_z)) &= P(\tau = \infty \mid \xi_0(z) = c \text{ for all } z \in \mathcal{V}) \\
&\geq P(Z_t \geq \mathcal{D} \text{ for all } t \mid \xi_0(z) = c \text{ for all } z \in \mathcal{V}) \\
&= P(W_t > \mathcal{D} - 1 \text{ for all } t \mid W_0 = Nc) \\
&= P(T_{\mathcal{D}-1} = \infty \mid W_0 = Nc) = q(\mathcal{D} - 1) \\
&= \max(0, 1 - \Phi^{-(Nc - \mathcal{D} + 1)}).
\end{aligned}$$

This completes the proof of the theorem. \square

Using Lemma 4.6 and Theorem 4.1, we can now prove Theorem 4.2.

Proof of Theorem 4.2. It follows from Lemma 4.6 that, in the presence of only one vertex, say x , the probability of survival is given by

$$p_0(c, \phi_x) = q(-1) = \max(0, 1 - \phi_x^{-(c+1)}).$$

Since in the absence of cooperation $\mu = 0$, the system with N individuals consists of N independent copies of a one-person system, we get

$$p_0(c, (\phi_z)) = \prod_{z \in \mathcal{V}} p_0(c, \phi_z) = \prod_{z \in \mathcal{V}} \max(0, 1 - \phi_z^{-(c+1)}).$$

It directly follows that

$$p_0(c, (\phi_z)) = 0 \quad \text{when} \quad \phi_z \leq 1 \quad \text{for some} \quad z \in \mathcal{V}$$

so the inequality to be proved is obvious in this case. Assume now that $\phi_z > 1$ for all $z \in \mathcal{V}$. In this case, we have the following inequalities:

$$\begin{aligned}
\log(p_0(c, (\phi_z))) &= \sum_{z \in \mathcal{V}} \log(1 - \phi_z^{-(c+1)}) \leq - \sum_{z \in \mathcal{V}} \phi_z^{-(c+1)} \\
\log(p_\infty(c, (\phi_z))) &\geq \log(1 - \Phi^{-(Nc - \mathcal{D} + 1)}) \geq - \frac{\Phi^{-(Nc - \mathcal{D} + 1)}}{1 - \Phi^{-(Nc - \mathcal{D} + 1)}}.
\end{aligned}$$

In particular, since $\Phi > 1$, for all $N \geq 2$ and c sufficiently large,

$$\begin{aligned} \log(p_\infty(c, (\phi_z))) &\geq -\frac{\Phi^{-(Nc-\mathcal{D}+1)}}{1-\Phi^{-(Nc-\mathcal{D}+1)}} \geq -2\Phi^{-(Nc-\mathcal{D}+1)} \geq -2\left(\min_{z \in \mathcal{V}} \phi_z\right)^{-(Nc-\mathcal{D}+1)} \\ &\geq -\left(\min_{z \in \mathcal{V}} \phi_z\right)^{-(c+1)} \geq -\sum_{z \in \mathcal{V}} \phi_z^{-(c+1)} \geq \log(p_0(c, (\phi_z))). \end{aligned}$$

This completes the proof of the theorem. \square

4.6 Proof of Theorem 4.3

As stated in the introduction, the two-person system is simple enough that we may calculate certain probabilities by hand. Since there are only two vertices, we will call them x and y and the rates at which they earn a coin ϕ_x and ϕ_y , respectively. To simplify the notation, write

$$X_t = \xi_t(x) \quad \text{and} \quad Y_t = \xi_t(y) \quad \text{for all } t \geq 0.$$

Letting $T_- = \inf\{t : \min(X_t, Y_t) = -1\}$, the process

$$\Phi^{-(X_{t \wedge T_-} + Y_{t \wedge T_-})} = \left(\frac{2}{\phi_x + \phi_y}\right)^{X_{t \wedge T_-} + Y_{t \wedge T_-}}$$

is again a martingale. Using that the individuals' fortunes differ by at most one coin in the presence of perfect cooperation, and repeating the proofs of Lemmas 4.5 and 4.6, we easily show that, when both individuals start with c coins, the probability of global survival satisfies

$$\begin{aligned} p_\infty(c, \phi_x, \phi_y) &= P(\min(X_t, Y_t) \geq 0 \text{ for all } t \mid X_0 = Y_0 = c) \\ &\geq P(X_t + Y_t > 0 \text{ for all } t \mid X_0 = Y_0 = c) \\ &= \max(0, 1 - (2/(\phi_x + \phi_y))^{2c}) \end{aligned}$$

in the case of perfect cooperation. In particular, when

$$\phi_x + \phi_y > 2 \quad \text{and} \quad \phi_x < 1 < \phi_y,$$

while individual x dies almost surely in the absence of cooperation, global survival is possible in the presence of perfect cooperation, showing that cooperation is beneficial in this case. We now focus on the parameter region

$$\phi_x + \phi_y < 2 \quad \text{and} \quad \phi_x < 1 < \phi_y \quad (4.4)$$

and show that, in this case, cooperation is detrimental: individual x again dies almost surely while individual y is more likely to live forever in the absence of cooperation than in the presence of perfect cooperation. The probability of survival can be computed explicitly.

Using again that the individuals' fortunes differ by at most one coin in the presence of perfect cooperation, together with the fact that global survival is not possible when (4.4) holds, implies that the stopping time T_- is almost surely finite and that

$$(X_{T_-}, Y_{T_-}) \in \{(-1, 0), (-1, 1), (0, -1), (1, -1)\}.$$

To simplify the notation, we rename these four states as well as the three adjacent states as shown in Figure 4.1 and define the stopping times and corresponding probabilities

$$\tau_i = \inf \{t : (X_t, Y_t) = S_i\} \quad \text{and} \quad p_i = P(T_- = \tau_i) \quad \text{for } i = 1, 2, 3, 4.$$

The probabilities p_i are computed explicitly in the next lemma.

Lemma 4.7 – *Assume (4.4) and perfect cooperation. Then,*

$$p_1 = p_2 = \frac{2}{\Psi} \quad p_3 = \frac{\phi_x}{\Psi} + \frac{1}{4} \quad p_4 = \frac{\phi_y}{\Psi} + \frac{1}{4}$$

where $\Psi = 8 + 2\phi_x + 2\phi_y$.

Proof. Observe that T_- is almost surely finite when (4.4) holds. Since in addition the individuals' fortunes differ by at most one coin before time T_- ,

$$T_+ = \sup \{t : X_t = Y_t = 1\} < \infty \quad \text{almost surely.}$$

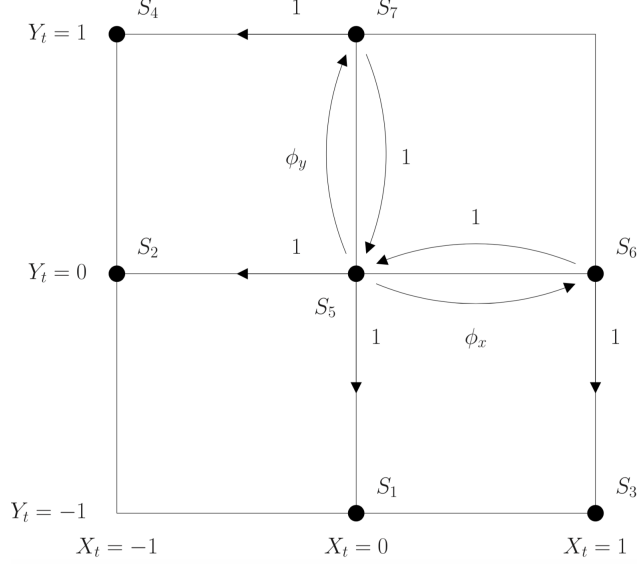


Figure 4.1: The Seven States and Transition Rates Between Times T_+ and T_- .

Also, between time T_+ and time T_- , the process consists of the seven-state continuous-time Markov chain whose transition rates are indicated in Figure 4.1. Referring again to the picture for the name of the states, we define the conditional probabilities

$$p_{ij} = P(T_- = \tau_i | (X_0, Y_0) = S_j) \quad \text{for all } (i, j) \in \{1, 2, 3, 4\} \times \{5, 6, 7\}.$$

Using a first-step analysis and looking at the probabilities at which the process starting from state S_5 jumps to each of the four adjacent states, we get

$$p_{15} = \frac{1}{2 + \phi_x + \phi_y} + \frac{\phi_x p_{16}}{2 + \phi_x + \phi_y} + \frac{\phi_y p_{17}}{2 + \phi_x + \phi_y}.$$

The same idea gives $p_{16} = p_{17} = (1/2)p_{15}$. Solving the system, we get

$$p_{15} = \frac{2}{4 + \phi_x + \phi_y} \quad \text{and} \quad p_{16} = p_{17} = \frac{1}{4 + \phi_x + \phi_y}.$$

Since in addition the first state visited after time T_+ is equally likely to be S_6 and S_7 , we conclude that the probability p_1 is given by

$$p_1 = \frac{p_{16} + p_{17}}{2} = \frac{1}{4 + \phi_x + \phi_y} = \frac{2}{\Psi}$$

which, by symmetry, is also the value of p_2 . To compute p_3 , we again use a first-step analysis to obtain a system involving the three conditional probabilities:

$$p_{35} = \frac{\phi_x p_{36}}{2 + \phi_x + \phi_y} + \frac{\phi_y p_{37}}{2 + \phi_x + \phi_y} \quad p_{36} = \frac{1}{2} + \frac{p_{35}}{2} \quad p_{37} = \frac{p_{35}}{2}.$$

Solving the system gives

$$p_{35} = \frac{\phi_x}{4 + \phi_x + \phi_y} \quad p_{36} = \frac{1}{2} + \frac{\phi_x}{8 + 2\phi_x + 2\phi_y} \quad p_{37} = \frac{\phi_x}{8 + 2\phi_x + 2\phi_y}$$

from which it follows as before that

$$p_3 = \frac{p_{36} + p_{37}}{2} = \frac{\phi_x}{8 + 2\phi_x + 2\phi_y} + \frac{1}{4} = \frac{\phi_x}{\Psi} + \frac{1}{4}.$$

By symmetry, the value of p_4 is obtained by exchanging the role of ϕ_x and ϕ_y in the previous expression, which completes the proof. \square

Using the previous lemma as well as Lemma 4.6 and conditioning on the first boundary state visited, we deduce that the expected number of individuals that survive in the presence of perfect cooperation, which is also the probability that y survives, is given by

$$E_\infty(c, \phi_x, \phi_y) = p_2 p_0(0, \phi_y) + p_4 p_0(1, \phi_y) = \left(\frac{2}{\Psi}\right) \left(1 - \frac{1}{\phi_y}\right) + \left(\frac{\phi_y}{\Psi} + \frac{1}{4}\right) \left(1 - \left(\frac{1}{\phi_y}\right)^2\right).$$

Since in addition

$$1 - \frac{1}{\phi_y} < 1 - \left(\frac{1}{\phi_y}\right)^2 \leq 1 - \left(\frac{1}{\phi_y}\right)^{c+1}$$

for all $\phi_y > 1$ and $c \geq 1$, and since

$$\left(\frac{2}{\Psi}\right) + \left(\frac{\phi_y}{\Psi} + \frac{1}{4}\right) = P(T_- = \tau_2 \text{ or } T_- = \tau_4) \leq 1,$$

we conclude that

$$E_\infty(c, \phi_x, \phi_y) < 1 - \left(\frac{1}{\phi_y}\right)^{c+1} = E_0(c, \phi_x, \phi_y).$$

This completes the proof of Theorem 4.3.

4.7 Proof of Theorem 4.4

As explained in the introduction, the first step to prove Theorem 4.4 is to identify a collection of events that simultaneously occur with positive probability and ensure that a given vertex, say the origin, dies before time one. These events are defined from the collection of independent Poisson processes introduced at the end of the model description as follows:

$$A_1 = \{N_1^+(0) = 0 \text{ and } N_1^-(0) \geq (c+1)^2\}$$

$$A_2 = \{N_1^+(z) = N_1^-(z) = 0 \text{ for all } z \in \mathbb{Z} \text{ such that } 0 < |z| \leq c+1\}$$

$$A_3 = \{N_1^+(c+2) + \cdots + N_1^+(c+n+1) \leq n \text{ for all } n > 0\}$$

$$A_4 = \{N_1^+(-(c+2)) + \cdots + N_1^+(-(c+n+1)) \leq n \text{ for all } n > 0\}.$$

The times at which neighbors exchange a coin are unimportant in the proof of the theorem. Let A be the event that consists of the intersection of these four events.

Lemma 4.8 – *For all $\mu \in [0, \infty]$, we have $P(\xi_1(0) = -1 \mid A) = 1$.*

Proof. To begin with, we ignore the exchange of money between $c+1$ and its right neighbor and between $-(c+1)$ and its left neighbor. Recalling that an agent can receive one coin from a neighbor only if this neighbor has at least two more coins, on the event $A_1 \cap A_2$,

$$\xi_1(0) = -1 \quad \text{and} \quad c \geq \xi_t(z) \geq |z| - 1 \quad \text{for all } 0 < |z| \leq c+1 \text{ and } t \in (0, 1). \quad (4.5)$$

Note that the second inequality above becomes an equality when $\mu = \infty$. In this case, the total loss of coins among the $2c+3$ vertices around zero is given by

$$(c+1) + 2c + 2(c-1) + \cdots + 2 \times 1 + 2 \times 0 = (c+1)^2,$$

which explains our definition of the event A_1 . Observe that (4.5) implies that there are exactly c coins at vertex $c+1$ until time one. In particular, looking at the full

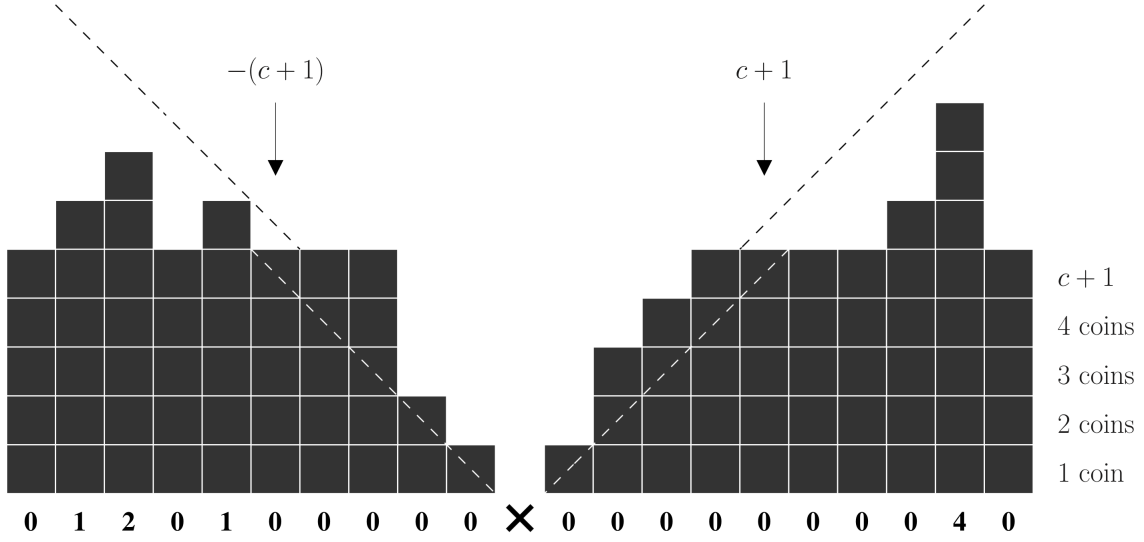


Figure 4.2: Typical Configuration at Time One When A Occurs: The Agent at 0 Is Dead and the Fortune of the Agents at Distance at Least $c+2$ From the Origin Is Below the Black Dashed Line. The Numbers at the Bottom of the Picture Give the Number of Coins These Agents Earned by Time One. In the Picture, We Assume That These Agents Do Not Spend Any Coin, in Which Case The Fortune of the Agents Within Distance $c+1$ of the Origin Is Above the White Dashed Line.

system and allowing the exchange of money between $c+1$ and its right neighbor, on the event A_3 ,

$$\begin{aligned} & \text{number of coins traveling } c+1 \rightarrow c+2 \text{ by time one} \\ & \geq \text{number of coins traveling } c+2 \rightarrow c+1 \text{ by time one.} \end{aligned} \tag{4.6}$$

By symmetry, on the event A_4 ,

$$\begin{aligned} & \text{number of coins traveling } -(c+1) \rightarrow -(c+2) \text{ by time one} \\ & \geq \text{number of coins traveling } -(c+2) \rightarrow -(c+1) \text{ by time one.} \end{aligned} \tag{4.7}$$

Combining (4.5)–(4.7), we deduce that given the event A we must have $\xi_1(0) = -1$.

□

To prove that the event A has a positive probability, we let

$$\epsilon = -\frac{E(\phi) - 1}{2} > 0 \quad \text{so that} \quad E(\phi) = 1 - 2\epsilon$$

and call vertex $z \in \mathbb{Z}$

a right ϵ -sink when $\phi_z + \phi_{z+1} + \cdots + \phi_{z+n} \leq (n+1)(1-\epsilon)$ for all $n \in \mathbb{N}$

a left ϵ -sink when $\phi_z + \phi_{z-1} + \cdots + \phi_{z-n} \leq (n+1)(1-\epsilon)$ for all $n \in \mathbb{N}$.

Then, we have the following result.

Lemma 4.9 – *We have $P(z \text{ is a left } \epsilon\text{-sink}) = P(z \text{ is a right } \epsilon\text{-sink}) = a > 0$.*

Proof. Define the process

$$X_n = X_n(z) = \phi_z + \phi_{z+1} + \cdots + \phi_{z+n} - (n+1)(1-\epsilon) \quad \text{for all } n \in \mathbb{N}.$$

Since the random variables $\phi_z, \phi_{z+1}, \dots, \phi_{z+n}$ are independent and identically distributed, it follows from the strong law of large numbers that

$$\lim_{n \rightarrow \infty} \frac{X_n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n (\phi_{z+i} - (1-\epsilon)) = E(\phi) - (1-\epsilon) = -\epsilon < 0.$$

In particular, there exists N , fixed from now on, such that

$$P(X_n \leq 0 \text{ for all } n \geq N) = P\left(\sum_{i=1}^n (\phi_{z+i} - (1-\epsilon)) \leq 0 \text{ for all } n \geq N\right) \geq 1/2. \quad (4.8)$$

In addition, since $E(\phi) < 1 - \epsilon$, we have $p = P(\phi \leq 1 - \epsilon) > 0$ so

$$P(X_n \leq 0 \text{ for all } n < N) \geq P(\phi_{z+i} \leq 1 - \epsilon \text{ for all } i < N) = p^N > 0. \quad (4.9)$$

Finally, combining (4.8) and (4.9) and using that the events $\{X_n \leq 0\}$ for different values of $n \in \mathbb{N}$ are positively correlated, we conclude that

$$\begin{aligned} P(z \text{ is a right } \epsilon\text{-sink}) &= P(X_n \leq 0 \text{ for all } n \geq 0) \\ &= P(X_n \leq 0 \text{ for all } n \geq N \mid X_n \leq 0 \text{ for all } n < N) P(X_n \leq 0 \text{ for all } n < N) \\ &\geq P(X_n \leq 0 \text{ for all } n \geq N) P(X_n \leq 0 \text{ for all } n < N) \geq (1/2) p^N > 0. \end{aligned}$$

It also follows from obvious symmetry that the probability that z is a left ϵ -sink is equal to the probability that it is a right ϵ -sink. This completes the proof. \square

Using the previous lemma, we can now prove that the event A has positive probability.

Lemma 4.10 – *We have $P(A) > 0$.*

Proof. Since the Poisson processes in the graphical representation are independent

$$P(A) = P(A_1) P(A_2) P(A_3) P(A_4).$$

In addition, for any given c finite, the first two events have positive probability while, by symmetry, the last two events have the same probability, i.e.,

$$P(A_1) P(A_2) > 0 \quad \text{and} \quad P(A_3) = P(A_4). \quad (4.10)$$

In particular, to conclude, it suffices to prove that the event A_3 has a positive probability. By conditioning on the event that vertex $c + 2$ is a right ϵ -sink, we get

$$\begin{aligned} P(A_3) &\geq P(A_3 \mid c + 2 \text{ is a right } \epsilon\text{-sink}) P(c + 2 \text{ is a right } \epsilon\text{-sink}) \\ &= a P(A_3 \mid c + 2 \text{ is a right } \epsilon\text{-sink}) \end{aligned} \quad (4.11)$$

where $a > 0$ according to Lemma 4.9. Now, let

$$Y_n = \text{Poisson}(n(1 - \epsilon)) \text{ be independent for all } n > 0.$$

Using that the events that define the event A_3 are positively correlated and recalling the definition of right ϵ -sink, we deduce that

$$P(A_3 \mid c + 2 \text{ is a right } \epsilon\text{-sink}) \geq P(Y_n \leq n \text{ for all } n > 0) = \prod_{n>0} P(Y_n \leq n). \quad (4.12)$$

In other respects,

$$\begin{aligned} \prod_{n>0} P(Y_n \leq n) > 0 \quad &\text{if and only if} \quad \sum_{n>0} -\log(1 - P(Y_n > n)) < \infty \\ &\text{if and only if} \quad \sum_{n>0} P(Y_n > n) < \infty \end{aligned} \tag{4.13}$$

which follows from standard large deviations estimates for the Poisson distribution. Combining (4.11)–(4.13), we deduce that $P(A_3) > 0$ which, together with (4.10), gives the lemma. \square

Since the random variables ϕ_z are independent and identically distributed, we may apply the ergodic theorem together with Lemmas 4.8 and 4.10 to deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbf{1}\{\xi_1(z) = -1\} \geq P(A) > 0. \tag{4.14}$$

Note however that this does not imply our theorem since the probability of $A_1 \cap A_2$, and therefore the lower bound $P(A)$, depends on c , the initial number of coins per vertex.

The second step of the proof is to identify an infinite collection of vertices, that we call ϵ -sinks, that are removed eventually. The density of such vertices is bounded from below by a positive constant that does not depend on c . More precisely, we call vertex $z \in \mathbb{Z}$ an ϵ -sink if

$$\phi_{z-m} + \phi_{z-m+1} + \cdots + \phi_{z+n} \leq (m+n+1)(1-\epsilon) \quad \text{for all } m, n \in \mathbb{N}. \tag{4.15}$$

Lemma 4.11 – *We have $P(z \text{ is an } \epsilon\text{-sink}) \geq a^2 > 0$.*

Proof. Let $A_{m,n}$ be the event in (4.15) and observe that

$$A_{m,0} \cap A_{0,n} \subset A_{m,n} \quad \text{for all } m, n \in \mathbb{N}.$$

In particular, the event that z is an ϵ -sink is

$$\bigcap_{m,n} A_{m,n} = \bigcap_{m,n} (A_{m,0} \cap A_{0,n}) = \left(\bigcap_m A_{m,0} \right) \cap \left(\bigcap_n A_{0,n} \right). \quad (4.16)$$

Using that $A_{0,n} = \{X_n \leq 0\}$ where the process (X_n) has been defined in the proof of Lemma 4.9 and obvious symmetry, we also have

$$P\left(\bigcap_m A_{m,0}\right) = P\left(\bigcap_n A_{0,n}\right) = P(X_n \leq 0 \text{ for all } n \geq 0) = a > 0 \quad (4.17)$$

according to Lemma 4.9. Combining (4.16) and (4.17), and using that the events $A_{m,0}$ and $A_{0,n}$ are positively correlated, we conclude that

$$P(z \text{ is an } \epsilon\text{-sink}) = P\left(\bigcap_{m,n} A_{m,n}\right) \geq P\left(\bigcap_m A_{m,0}\right) P\left(\bigcap_n A_{0,n}\right) = a^2 > 0.$$

This completes the proof. \square

To complete the proof of the theorem, the last step is to show that all the ϵ -sinks die eventually with probability one, which is done in the following lemma.

Lemma 4.12 – *Assume that $x \in \mathbb{Z}$ is an ϵ -sink. Then $\xi_t(x) = -1$ for some t .*

Proof. For all times t , we define

$$z_t^- = \sup \{z \leq x : \xi_t(z) = -1\} \quad \text{and} \quad z_t^+ = \inf \{z \geq x : \xi_t(z) = -1\}.$$

In view of (4.14) and since -1 is an absorbing state for each vertex,

$$I_t = (z_t^-, z_t^+) \text{ is bounded at time } t = 1 \text{ and nonincreasing in } t$$

for the inclusion. Now, set $T_0 = 1$ and define recursively

$$\begin{aligned} T_i &= \inf \{t > T_{i-1} : I_t \neq I_{t-}\} \quad \text{when } T_{i-1} < \infty \\ &= \infty \quad \text{when } T_{i-1} = \infty \end{aligned}$$

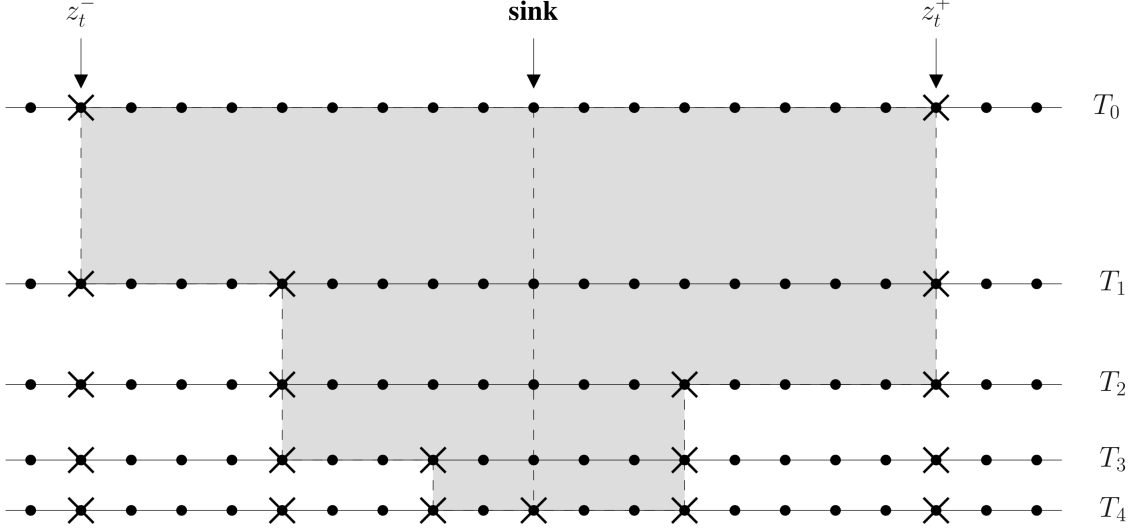


Figure 4.3: Picture of the Construction in Lemma 4.12 with the Sequence of Stopping Times T_i . The Crosses \times Represent the Agents That are Dead. The Gray Region Shows the Interval I_t From Time $T_0 = 1$ Until the Sink Dies. In Our Example, It Takes Four Steps to Kill the Sink Located at the Center of the Picture.

for all $i > 0$. See Figure 4.3 for a picture. Given that time T_i is finite and that the interval I_{T_i} is nonempty, by the definition of ϵ -sink, between time T_i and time T_{i+1} , the process

$$Z_t = \xi_t(z_{T_i}^- + 1) + \xi_t(z_{T_i}^- + 2) + \cdots + \xi_t(z_{T_i}^+ - 1)$$

is dominated stochastically by a one-dimensional random walk with a negative drift. This implies that the expected number of coins in the interval I_t is decreasing, therefore one of the vertices in the interval must reach state -1 in a finite time and

$$P(T_{i+1} < \infty \mid T_i < \infty \text{ and } I_{T_i} \neq \emptyset) = 1.$$

Recall also that the interval is bounded at time one and observe that, by definition of the stopping times, the length of the interval decreases by at least one at each step, i.e.,

$$|I_{T_0}| < \infty \quad \text{and} \quad |I_{T_{i+1}}| \leq |I_{T_i}| - 1 \quad \text{when} \quad T_i < T_{i+1} < \infty.$$

In summary, it takes only a finite number steps for I_t to become empty and the duration of each step is almost surely finite. Since in addition the sink dies at the time I_t becomes empty,

$$\inf \{t : \xi_t(x) = -1\} = \inf \{t : I_t = \emptyset\} < \infty$$

with probability one. This completes the proof. \square

As previously, since the random variables ϕ_z are independent and identically distributed, we may apply the ergodic theorem which, together with Lemmas 4.11 and 4.12, implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbb{1}\{\xi_t(z) = -1 \text{ for some } t\} \\ \geq \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbb{1}\{z \text{ is an } \epsilon\text{-sink}\} \geq a^2 > 0. \end{aligned}$$

Since a does not depend on c , this proves Theorem 4.4.

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CONCLUSION

5.1 Summary of Findings for Conservative Models on Connected Graphs

One purpose of this dissertation was to prove and extend many conjectures about a few fundamental models in the field of econophysics. The uniform reshuffling model, the immediate exchange model, the model with saving propensity and the models with individual and collective debt limit are all models well known in the fields of econophysics. Much research has been done about these models regarding the limiting distribution of wealth in a population of individuals whose interactions are governed by the rules of such processes. However, in each case, only conjectures about the limiting distribution of assets had been made up until the writing of chapters 2 and 3 of this dissertation. Many people looked at these models from a computational point of view, using simulations as a main basis for conjecture about the limiting behavior of such models with little or no analytical results to back up their assertions. Another area where insight into these models was lacking is the role that the spatial structure played. As the number of possible graphs with N vertices grows quickly with N , to simplify things, all simulations of these five models done previously by researchers were on the complete graph with N individuals. Looking only at the complete graph makes simulating these processes much less computationally expensive but doing so is at the expense of eliminating the spatial component representing the structure of the social network. Looking at these models on the complete graph gives no insight into the role that microscopic interactions has on the limiting behavior of the process. As such, all conjectures that were made about these five models based

on numerical results are only applicable to the specific case where the individuals are interacting on the complete graph. For obvious reasons, this is not a scenario likely to happen in the real world as not every individual in a population knows every other individual. Rather, people only tend to interact with those that they know. This shortcoming is largely overcome in chapters 2 and 3 of this text as excepting the model with collective debt limit, all of the previous numerical results are proven analytically and even extended to any connected graph with N vertices. As for the model with collective debt limit, an expression for the limiting distribution of wealth is found in chapter 3 analytically, however, this expression is not in a closed form thus making it difficult to verify the numerical results of others. Instead, a heuristic argument is given in chapter 3 further giving support to the previous numerical results regarding this model. Although the conjecture that the limiting distribution of wealth for the model with collective debt limit has yet to be verified, insight about this model was still gained. It is shown in chapter 3 that there does exist a unique stationary distribution of wealth and in accordance with the first four models discussed in this text, this stationary distribution is the same regardless of the underlying structure of the graph \mathcal{G} . Using a computer to plot the asymmetric Laplace distribution alongside an approximation of the expression found for $\lim_{t \rightarrow \infty} P(Y_t(x) = c)$ where $x \in \mathcal{V}$ is arbitrary gives strong visual evidence that the true distribution of wealth is asymmetric Laplace.

5.2 Summary of Findings for Model with Cooperation

The second purpose of this dissertation was to examine the effect of cooperation on the distribution of wealth in a population over time.

5.2.1 *Summary of Findings for Model with Cooperation on a Finite Connected Graph*

In the case where \mathcal{G} is finite, an interesting result is shown in chapter 4. This result being that depending on the overall productiveness of the population as a whole, cooperation will either improve the chance of overall survival or guarantee that overall survival is impossible. Specifically, when the population on average produces more resources than they use in average, it is shown that cooperation increases the probability of overall survival. However, the opposite is true when the population produces less than is needed for survival on average. When productiveness is too low, adding the element of cooperation by setting $\mu > 0$ ensures that survival of the population as a whole is impossible. It should be noted that as long as there are individuals who are more productive than is needed, there is a positive probability that some individuals will survive. It is likely that the nonproductive individuals will die first allowing the overall productiveness of the remaining population to increase. If this production level increases to the point that the remaining population produces more resource on average than is needed, there would then be a positive probability that all remaining individuals will survive and having $\mu > 0$ improves this probability for the remaining individuals.

5.2.2 *Summary of Findings for the Model with Cooperation on the One Dimensional Integer Lattice*

In the case where $\mathcal{G} = \mathbb{Z}$, total survival of the population is impossible regardless of the level of cooperation. However, it is shown in chapter 4 that when the average rate of production of the population is less than is needed for survival, the proportion of individuals who eventually die is bounded from below by a positive constant that

is independent of both μ and the initial number of coins per individual.

5.3 Final Thoughts

This research was motivated by the need to understand what causes the state of wealth distribution in the real world. One reason why it is necessary to examine these real world phenomena from an analytical point of view is the lack of data regarding wealth of individuals. The six models looked at here are clearly well suited for looking at wealth distributions as was the purpose of this dissertation, however, the beauty in interacting particle systems is that many models in this field are often found to be analogous to phenomena in other fields, and thus these models can often be reinterpreted for other contexts while the analytic results found regarding these models will still hold. This gives a level of profoundness to proving analytic results about interacting particle systems in that these results can have consequences in the future which at the present have yet to be conceived. For example, conic sections were first defined in ancient Greece and at that time, there was no apparent use for such an abstract notion. It wasn't until Johannes Kepler came along centuries later and used ellipses to describe the motion of celestial bodies that the conic sections had an application. Even though the first five models are rather simple and far from accounting for all the complexity of real world monetary transactions, statistical physicists showed that their numerical results fit well the few data available. In particular, the main objective of this dissertation is not pursue their work and validate their models but instead to rely on a rigorous analytical treatment to prove that their conjectures are indeed correct.

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APPENDIX A
CO-AUTHOR PERMISSIONS

I certify that my co-author, Dr. Nicolas Lanchier has given me permission, to include all material in my PhD thesis for Chapters 2, 3 and 4.