Sparky the Saguaro:

Teaching Experiments Examining Students' Development of the Idea of Logarithm

by

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ABSTRACT

There have been a number of studies that have examined students' difficulties in understanding the idea of logarithm and the effectiveness of non-traditional interventions. However, there have been few studies that have examined the understandings students develop and need to develop when completing conceptually oriented logarithmic lessons. In this document, I present the three papers of my dissertation study. The first paper examines two students' development of concepts foundational to the idea of logarithm. This paper discusses two essential understandings that were revealed to be problematic and essential for students' development of productive meanings for exponents, logarithms and logarithmic properties. The findings of this study informed my later work to support students in understanding logarithms, their properties and logarithmic functions. The second paper examines two students' development of the idea of logarithm. This paper describes the reasoning abilities two students exhibited as they engaged with tasks designed to foster their construction of more productive meanings for the idea of logarithm. The findings of this study provide novel insights for supporting students in understanding the idea of logarithm meaningfully. Finally, the third paper begins with an examination of the historical development of the idea of logarithm. I then leveraged the insights of this literature review and the first two papers to perform a conceptual analysis of what is involved in learning and understanding the idea of logarithm. The literature review and conceptual analysis contributes novel and useful information for curriculum developers, instructors, and other researchers studying student learning of this idea.

DEDICATION

I would like to dedicate this dissertation to all unborn children – especially my current unborn child and future children.

it came to my many iterations of defining the idea of logarithm. I will never forget the

times spent together, and I hope to stay connected and collaborate for many years to

come. You all have helped me in more ways than you may realize, and I am truly grateful

for your friendships.

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INTRODUCTION

The idea of logarithm is useful both in mathematics (e.g., number theory –
primes, statistics – non-linear regression, chaos theory – fractal dimension, calculus –
differential equations) and in modeling real-world relationships (e.g., Richter scale,
Decibel scale, population growth, radioactive decay). Therefore, a goal for mathematics
educators should be to assist students in developing coherent meanings for the idea of
logarithms. How does one achieve this goal? One approach is to research the aspects of
the idea of logarithm students have difficulties with. In particular, studies have shown
that students have difficulty with logarithmic notation, logarithmic properties and
logarithmic functions (Weber, 2002; Kenney, 2005; Strom, 2006; Gol Tabaghi, 2007).
Another approach is to develop and test the efficiency of interventions relative to
standard curriculum (Weber, 2002; Panagiotou, 2010). Although these methods may shed
light on epistemological obstacles students encounter and/or the effectiveness of a non-
traditional intervention, neither approach examines the reasoning abilities needed to
coherently understand and use the idea of logarithm. In fact, relatively few studies have
examined what meanings students have for the idea of logarithm (Kenney, 2005; Gol
Tabaghi, 2007), and fewer have examined how students come to conceptualize the idea
of logarithm (Kastberg, 2002).

This study investigates three undergraduate precalculus students' understandings
of the idea of logarithm and concepts foundational to the idea of logarithm as they work
through an exploratory lesson on exponential and logarithmic functions. The findings of
this study may reveal essential components that students must conceptualize in order to
hold a productive meaning for the idea of logarithm. For example, in order to reason

through tasks involving logarithmic expressions, logarithmic properties, and logarithmic functions in a way that builds off prior meanings and serves to be useful for more complex tasks, students may find it helpful to conceptualize that multiplying by A and then multiplying by B has the overall effect of multiplying by AB, and conceptualize that an exponent on a value b represents the number of b-tupling[1] periods that have elapsed. In this study, I model the students' thinking as they participate in an exponential and logarithmic sequence designed to assist students in developing coherent meanings for the idea of logarithm. I also discuss the importance of conceptualizing the essential components in the context of the lesson.

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[1] A b-tupling occurs when a quantity becomes b times as large. Therefore, a b-tupling period is the amount

STATEMENT OF PROBLEM

Teachers and researchers have recognized that students face challenges when introduced to logarithms and logarithmic functions. In an effort to lighten the burden on students, some teachers have tried incorporating the history of logarithms into their lessons (Panagiotou, 2011), changing the notation (Hammack & Lyons, 1995), and approximating logarithms with repeated division (Vos & Espedal, 2016), yet research continues to report that many students struggle to develop coherent understandings for logarithmic notation, properties and function (Weber, 2002; Kenney, 2005; Strom, 2006; Gol Tabaghi, 2007). Adding to the problem, standard curriculum often fails to present the material in a meaningful and coherent way. A review of 5 precalculus and calculus texts[2] revealed that $y = \log_b(x)$ was introduced as the inverse to $y = b^x$, with the properties of logarithms stated shortly after. It seems necessary to first understand the ways in which students develop productive meanings for the idea of logarithm if we wish to improve the curriculum. Unfortunately, little research has been conducted on the understandings students develop during an instructional sequence on exponential and logarithmic functions. Additionally, little is reported on the ways in which students develop coherent understandings of the idea of logarithm (i.e. logarithmic notation, logarithmic properties, the logarithmic function).

The difficulties students have with developing coherent understandings of the idea of logarithm is likely multidimensional. In a typical precalculus course, logarithmic functions are the first function family introduced that does not specify a function rule,

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[2] (1) Spiegler, Adam, "Functions Modeling Change: A Preparation for Calculus" (2011). *Faculty Books*. 92. (2&3) Stewart, J. (2010). *Calculus: early transcendentals*. Cengage Learning. [2nd and 6th editions] (4) Anton, H. Calculus with Analytic Geometry, 1988. (5) Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*.

leaving students with no direction on how to determine the value of $\log_b(m)$ given values of *b* and *m*. Instead, students are expected to either apply their understandings of the idea of logarithm, exponents and powers to approximate the value of a logarithm for some input value, or, more commonly, use technology to calculate its value. In fact, the Common Core State Standards (CCSS) for mathematics have as one of the goals for high school students that they be able to write the corresponding logarithmic equation given an exponential equation, and calculate the value using technology (only for bases 2, 10 and *e*). Logarithmic functions are also the first function family that students encounter in which the function name is not a single letter. This may introduce an added complexity for students who already struggle in using function notation (Thompson, 2013; Musgrave & Thompson, 2014). Additionally, aspects of logarithmic notation have a dual nature to them (Kenney, 2005). For example, in $\log_b(x) = y$, *b*, *x*, and *y* take on a variety of meanings – *b* often takes on the form of a parameter (staying consistent within the context of a problem, but varying from problem to problem), *x* serves as the input variable to the logarithmic function and is a tupling, and *y* serves as the output variable to the logarithmic function and is the number of *b*-tupling periods needed to result in an *x*-tupling.

In addition to these unavoidable complexities, studies have shown that students struggle to understand, explain and apply the three properties of logarithms (Gol Tabaghi, 2007; Weber, 2002). In an exploratory study, I found students also have difficulties interpreting the expression $\log_b(x)$ in a coherent and meaningful way. Some students even claimed that in order for the expression to have meaning, one would need to know

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what the expression was equal to (so that the equation could be rewritten in exponential form). Therefore, it seems reasonable to assume that if students continue to have difficulties in understanding the idea of logarithm (i.e. logarithmic notation, logarithmic properties, the logarithmic function), they must still need to develop some understanding(s) foundational to the topic. This investigation intends to discover understandings that are foundational to understanding the idea of logarithm and research how students come to understand the idea of logarithm in hopes of contributing to current research in this area. An additional goal of this study is to inform curriculum so that students can build more coherent understandings of logarithms.

The primary questions motivating this investigation are:

– What understandings are foundational to understanding the idea of logarithm?

– What understandings of the idea of logarithm do students develop during an exponential and logarithmic instructional sequence that emphasizes quantitative and covariational reasoning?

While examining research on student understandings of exponential and logarithmic functions, I was inspired by the conceptually-based exponential situation involving Ellis' et al. (2012, 2015) Jactus the Cactus. The instructional sequence designed for this study was created to support the subjects in learning the foundational ideas of exponential functions. The activities in this study were also designed to promote a contextual interpretation of the idea of logarithm before introducing a generalized form. This investigation seeks to offer new research on the understandings foundational to the idea of logarithm, as well as the understandings of the idea of logarithm that

undergraduate precalculus students develop during an exponential and logarithmic instructional sequence that emphasizes quantitative and covariational reasoning. Furthermore, this research may inform future work to support students in understanding logarithms, their properties and logarithmic functions.

LITERATURE REVIEW

This study will investigate the understandings foundational to understanding the idea of logarithm and the understandings of the idea of logarithm students develop during an exponential and logarithmic instructional sequence that emphasizes quantitative and covariational reasoning. Thus, I will organize the relevant literature into three categories:

1. Background for Investigation, Quantitative Reasoning and Covariational Reasoning

2. Research literature on students' understandings of exponents and the exponential function

3. Research literature on students' understandings of the idea of logarithm and the logarithmic function

Background for Investigation, Quantitative Reasoning and Covariational Reasoning

For the last 35 years, textbooks have introduced the idea of logarithm by presenting some version of Euler's definition for logarithm – usually in the form of a biconditional statement relating an exponential equation to its equivalent logarithmic equation (Panagiotou, 2011). A review of 5 precalculus and calculus texts[3] revealed that examples and exercises on writing the equivalent form of exponential and logarithmic equations often follow the presentation of the definition. Then, without much development, anywhere from 3-6 logarithmic properties are stated and more examples and exercises ensue. There is no explanation of how to think about a logarithm or what the input quantity or the output quantity of a logarithmic function represents. Weber's (2002) pilot study revealed that most students who were taught using this traditional approach to teaching the idea of logarithm were often unable to recall or justify properties

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[3] Previously listed

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correctly. For example, none of the students in the control group in his study correctly determined the value of $\log_x(x)$. Kenney's (2005) study further revealed that students often applied incorrect procedures to isolate x when presented with logarithmic equations involving more than one logarithmic expression, such as $\log_5(x) + \log_5(x+4) = 1$. To justify their work, the students in her study stated their applied method was a logarithmic property. These "properties" often included eliminating the logarithmic notation, so long as the base value was the same for both logarithmic expressions, and rewriting the remaining numerals and symbols in an alternate form. For example, for the previous example, one of the students simplified the equation as $x + (x+4) = 1$ and solved for x. Kenney also observed that students, when presented with an equation involving only one logarithmic expression, rewrote the logarithmic expression in the equivalent exponential form. In addition, while I conducted research for my block grant, I noticed a tendency in the students in my study to rewrite logarithmic expressions in exponential form as a way of coping with the task of graphing $y = \log_{10}(x)$. I hypothesized that the students did not have a productive way to think about logarithms and therefore relied on rewriting the equation in exponential form to make sense of the task. I decided to test to see if more students struggled to give meaning to logarithmic expressions (not equations where students could rewrite the equation in exponential form). Four of my colleagues and I decided to add a question addressing the meaning of a logarithmic expression (Figure 0.1) on a Pathways precalculus exam.

Which of the following best describes what $\log_{3.45}(65.2)$ represents[4]?

    A. The number of factors of 65.2 there are in 3.45

    B. The number of factors of 3.45 there are in 65.2

    C. There are 65.2 factors of 3.45

    D. There are 3.45 factors of 65.2

    E. Not enough information – you need what $\log_{3.45}(65.2)$ is equal to.

Figure 0.1. Exam Question Addressing a Logarithmic Expression

Across all five classes, "E" was the most frequently chosen distractor. The results from these and other studies (Kastberg, 2002; Strom, 2006; Gol Tabaghi, 2007; Kenney & Kastberg, 2013) suggest students struggle with the idea of logarithm, logarithmic notation and the logarithmic function. Smith and Thompson (2007) argue that if students are to utilize algebraic notation to assist them in representing ideas and reasoning productively, then their ideas and reasoning must become sophisticated enough to justify the use of the notation in the first place. I argue that the same is true for the idea of logarithm and logarithmic notation. That is, before students begin using logarithmic notation and the logarithmic properties to represent their ideas and reasoning, their reasoning must identify a need for such tools. How does one develop such sophisticated reasoning? Smith and Thomson (2007) claim that it is through years of developing quantitative reasoning that make algebraic knowledge more meaningful and productive (pg. 10). In the paragraphs that follow, I describe quantitative reasoning and discuss its relevance in learning and understanding the idea of logarithm.

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[4] The answer choices were designed using Weber's (2002) definition of logarithm. I have since modified my definition of logarithm.

A quantity is a mental construction of a measurable attribute of an object (Thompson, 1990, 1993, 1994, 2011). That is, quantities do not exist out in the world; they are created in the mind of an individual when she conceptualizes measuring a quality of an object (Thompson, 2011). For example, suppose a saguaro cactus was purchased on January 1st of this year. When one imagines measuring the height (attribute) of the cactus (object), or measuring the elapsed (attribute) time (object) since the cactus' purchase, we say she has conceptualized a quantity. Furthermore, one is said to participate in the act of quantification when, after conceptualizing a quantity, she conceptualizes the attribute's unit of measure such that the attribute's measure is proportional to its unit (Thompson, 2011). For example, to engage in the act of quantification, one could imagine the cactus (object) and the cactus' height (attribute), and determine that the height of the cactus was 5.4 feet (where the attribute's measure, 5.4 feet, is 5.4 times as large as the unit of measure, 1 foot). In this example, we refer to 5.4 – the numerical measurement that a quantity may assume – as a value. When the measurable attribute of an object doesn't change throughout a situation, we call it a constant or fixed quantity. For example, the price paid for the cactus on the first of January would be considered a constant quantity. On the other hand, if the value of a quantity changes throughout a situation, we call it a varying quantity.

Mathematics is often used to model and describe how two or more quantities relate. A quantitative operation occurs in the mind of an individual and is when "one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 2011, pg. 9). For example, one could conceive of the height of the cactus on January 1st and the height of the cactus on February 1st as two individual quantities. Next,

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he could conceptualize how many times as large the February 1st cactus is compared to the January 1st cactus as a new quantity by multiplicatively comparing the two preconceived quantities by means of a ratio. When one conceives of three quantities related by means of a quantitative operation, we say he has conceptualized a quantitative relationship. Changing which quantity is determined by the quantitative operation changes the quantitative relationship (Thompson, 1990). In the previous example, the "growth factor" comparing how many times as large the February 1st cactus is compared to the January 1st cactus is a ratio; however, if one wished to determine the height of the cactus on February 1st given that the saguaro grew by a factor of 3 over the month of January, she would need to re-conceive the "growth factor" as representing a 3-tupling[5] (i.e. tripling). When one analyzes a situation and assigns his observations (i.e. quantities, quantitative relationships) to a network of quantities and quantitative relationships, called a quantitative structure, he is said to engage in quantitative reasoning (Thompson, 1988, 1990, 1993, 1994, 2011).

When a student engages in the essential constructs of quantitative reasoning she may end up developing a need for logarithmic notation on her own – possibly making the notation more meaningful to her. For example, suppose Mary purchased a cactus on January 1st and noticed the cactus was growing in a peculiar way. Mary might conceptualize the cactus' height as a quantity and decide to measure the cactus' height using the cactus' height at different moments. Suppose she initially documented the cactus' height on a wall and concluded that the cactus is one cactus tall on the first of January. One week later, Mary documented the cactus' new height on the wall, measured

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[5] I define an *m*-tupling as the event in which a quantity becomes *m* times as large.

its current height using its initial height as the unit of measure, and concluded that the cactus one week later had a measure of 2 (in units of the initial cactus) – therefore participating in the act of quantification. Suppose she then concluded that in that one-week's time, the cactus' height 2-tupled (doubled). If Mary conceptualized the factor by which the cactus may grow (the tupling value) as a quantity, resulting from multiplicatively comparing the two heights, she engaged in a quantitative operation. If, after documenting the cactus' growth over a long period of time, Mary concludes that the 2-tupling (doubling) period is one week, she may be curious to determine how many 2-tupling (doubling) periods need to elapse for the initial cactus to 9-tuple in height (to determine how long she has until she needs to take the cactus outside). Mary could then use logarithmic notation to represent the value of that particular quantity – specifically, $\log_2(9)$. In general, I define $\log_b(m)$ to represent the number of b-tupling periods[6] necessary to result in an m-tupling. The steps used to solve for the inverse relationship to the general representation of an exponential relationship, $y = a(b)^x$, informed this decision. For example, when solving for x applying Euler's definition, we get $x = \log_b(y / a)$, therefore indicating that the argument to the logarithmic function is a y/a-tupling. That is, in order for the initial value of the exponential relationship to be equal to y, the initial value must y/a-tuple or become y/a times as large.

To illustrate the difference between algebraic reasoning and quantitative reasoning in an exponential situation, consider the following task: *Suppose cactus A was 14 feet tall on January 1ˢᵗ and doubles (2-tuples) in height each week and suppose cactus*

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[6] Recall, a *b*-tupling occurs when a quantity becomes *b* times as large. Therefore, a *b*-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become *b* times as large. We say that the second quantity has *b-tupled* over some interval of change of the first quantity.

B is 5 feet tall on January 1ˢᵗ and triples (3-tuples) in height each week. After how many

weeks will the two cacti be the same height? A typical algebraic solution to this problem

involves defining variables, developing expressions that represent the heights of the cacti,

setting those expressions equal to one another, and solving for the unknown value. If x

represents the number of weeks since January 1ˢᵗ, then $14(2)^x$ represents the height of

cactus A x weeks after January 1ˢᵗ, and $5(3)^x$ represents the height of cactus B x weeks

after January 1ˢᵗ. We wish to solve $14(2)^x = 5(3)^x$ for x. Although algebraic solutions

may vary, a typical solution follows the form of the solution in Figure 0.2.

$$14(2)^x = 5(3)^x$$
$$\frac{14}{5} = \frac{3^x}{2^x}$$
$$\frac{14}{5} = \left(\frac{3}{2}\right)^x$$
$$\ln\left(\frac{14}{5}\right) = \ln\left(\left(\frac{3}{2}\right)^x\right)$$

$$\ln\left(\frac{14}{5}\right) = x\ln\left(\frac{3}{2}\right)$$

$$x = \frac{\ln\left(\frac{14}{5}\right)}{\ln\left(\frac{3}{2}\right)}$$

Figure 0.2. A Typical Algebraic Response

On the other hand, a response that utilizes quantitative reasoning does not require

the use of symbols to represent relationships, but rather deals with the relationships

themselves. Here is one example of such reasoning: Initially, cactus A's height is 14/5

times as tall as cactus B's height. Therefore, cactus B's height needs to 14/5-tuple as well

as 2-tuple as many times as cactus A's height did over the entire interval. For any one-

week change, the height of cactus B 3-tuples – this is equivalent to the height of the

cactus experiencing a 2-tupling and then immediately experiencing a 1.5-tupling. That is,

the 2-tupled height becomes 1.5 times as large for an overall 3-tuple in height. So, from

the start, any time that cactus B triples (3-tuples), the necessary doubling is taken into

account. In Figure 0.3, the height of cactus B is documented at different moments of a one-week period, specifically demonstrating a doubling (2-tupling) and then immediately a 1.5-tupling. It is worth noting that the 2-tupling and 1.5-tupling periods for cactus B are less than one week long and remain constant throughout this situation (with the 2-tupling period longer than the 1.5-tupling period). Also, for any portion of a week, say w weeks (where $0 < w < 1$), cactus A will grow by a factor of 2^w and cactus B will grow by a factor of 3^w, or $2^w 1.5^w$. That is, if w of a 3-tupling period has elapsed, then w of the corresponding cactus' 2-tupling period will have elapsed and w of that same cactus' 1.5-tupling period will have elapsed. Therefore, what remains to be determined is how many of these 1-week periods need to elapse for the accumulated 1.5-tuplings to result in a 14/5-tupling. The expression $\log_{1.5}(14/5)$ represents this specific value.
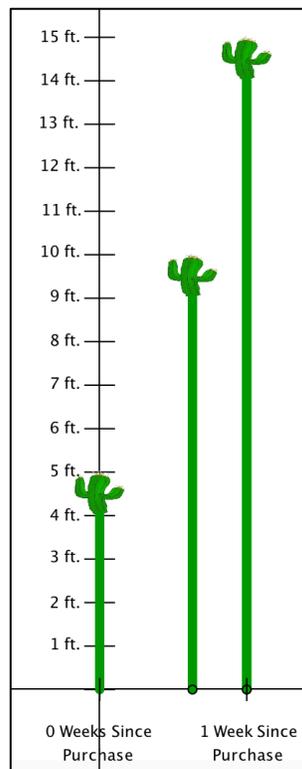
Figure 0.3. Cactus B at Different Moments Throughout the First Week

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When a student engages in the essential constructs of quantitative reasoning she may also end up constructing the logarithmic properties on her own – possibly making them more meaningful to her. For example, suppose a saguaro's height doubles (2-tuples) and subsequently triples (3-tuples). Overall, the saguaro's height will grow by a factor of 6 (i.e. experience a 6-tupling) (Figure 0.4). To conceive of this new tupling is an example of a quantitative operation. Then, the number of weeks needed for the cactus to 2-tuple (become 2 times as large) and then 3-tuple (become 3 times as large) will be the same as the number of weeks needed for the cactus to 6-tuple (become 6 times as large). This is a specific case of one of the logarithmic properties. If the 2-tupling period is one week, then symbolically we write $\log_2(2) + \log_2(3) = \log_2(6) = \log_2(2 \cdot 3)$. A more detailed explanation of the other logarithmic properties can be found in my conceptual analysis.
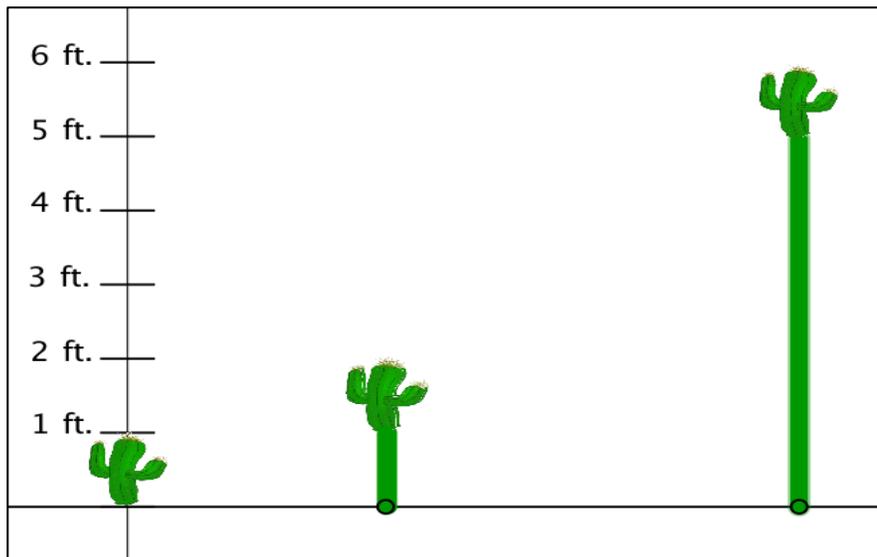
Figure 0.4. A Cactus' Height Doubling and Then Successively Tripling

For one to understand $\log_b(x)$ as representing a functional relationship in Thompson and Carlson's (2017) sense, he must rely on the essential constructs of

quantitative reasoning. That is, he must first conceive of the quantities represented by b, x and $\log_b(x)$. Recall, using my definition, b and x both represent tuplings and $\log_b(x)$ represents the number of b-tupling periods needed to result in an x-tupling. It is also worth noting here that the role of b is that of a parameter – staying consistent within a particular situation, but able to differ across situations. He must then conceive of the two quantities, x and $\log_b(x)$, as "varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33). This quantitative relationship can be modeled in two ways – $x = b^y$ or $\log_b(x) = y$, where y also represents the number of b-tupling periods needed to result in an x-tupling. Thus, if we know the value of y, we can determine the corresponding value of x using the first equation, and if we know the value for x, we can determine the corresponding value of y using the second equation. In addition to quantitative reasoning, covariational reasoning also plays a role in developing students' meanings for the logarithmic function.

Before I describe covariational reasoning in detail and discuss its role in the context of the logarithmic function, I will briefly describe its predecessor, variational reasoning. When one engages in variational reasoning, he is conceptualizing the value of a varying quantity. How he conceptualizes the value of a varying quantity may differ from another's conception, however. Thompson and Carlson (2017) generated a framework that summarized six levels of variational reasoning. At the lowest levels of the framework, no variation in the quantity's values is considered – either a variable is

viewed as a symbol that fails to take on any value, or it is viewed as an unknown. Students reasoning at the next two levels of the framework either only consider specific values that the varying quantity may assume, such as only considering the values presented in a table, or they imagine the quantity increasing or decreasing without considering any specific values whatsoever. A student reasoning at the fifth level of the framework, chunky continuous variation, conceptualizes the varying values of a quantity as changing by fixed intervals – similar to laying (possibly different-sized) rulers along a number line. In this case, the values within the interval hold a different meaning to the student than the values at the endpoints of the intervals, in the sense that the values within the interval "come along" with the interval. This way of thinking may be troublesome for students when working with exponential growth. For example, suppose the 4-tupling period of an exponential function is one week. A student reasoning at the chunky continuous level may struggle to imagine or identify the 3-day growth factor if their "interval rulers" are one week long. On the other hand, a student reasoning at the sixth level of the framework, smooth continuous variation, may also conceptualize the varying values of a quantity as changing by intervals, but the values at the endpoints of the interval and values within the interval hold the same meaning. In some sense, the student can recursively consider, or anticipate smaller intervals whose values (both endpoints and values within the interval) also vary in a similar, smooth and continuous, manner. These constructs are used in understanding Thompson and Carlson's (2017) covariational framework.

When a student conceptualizes two quantities' values varying in tandem, he engages in covariational reasoning. Thompson and Carlson (2017) argued that

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covariational reasoning is essential for students' mathematical development. The authors

also presented studies whose results suggested students experienced difficulties with

functional relationships when they did not appear to engage in covariational reasoning

and showed signs of improvement while engaging in covariational reasoning. Like

variational reasoning, students can reason at a variety of levels. Thompson and Carlson

(2017) generated a framework that summarized six levels of covariational reasoning. I

describe each level and hypothesize how a student at that level may reason about a

logarithmic function (Table 0.1).

Table 0.1

An Overview of the Covariation Framework with Logarithmic Examples

Level	Description	Example of student reasoning using $y = \log_2(x)$
No coordination	The student focuses on one variable's variation without conceptualizing simultaneous variation in the other variable	The student may attend only to the x-tupling. For example, the student may claim that $y = \log_2(16)$ represents a 16-tupling.
Precoordination of values	The student conceptualizes two quantities' values as varying, but not simultaneously. He imagines changes in one variable, followed by changes in the next.	Assuming the 2-tupling period is one week, the student may conclude that if the x-tupling value changes from 2 to 16, then the number of weeks must change from 1 to 4.
Gross coordination of values	The student conceptualizes two quantities varying simultaneously, but in a gross variation manner – not considering specific values, but coordinating whether or not the quantities are increasing or decreasing.	Assuming the 2-tupling period is one week, then as the overall x-tupling value increases, the number of weeks needed to grow by that increasing factor must also increase.

Coordination of values	The student coordinates one quantity's values with the second quantity's corresponding values. The student also anticipates forming an ordered pair with both values.	Assuming the 2-tupling period is one week, then when the value of x is 16, or represents a 16-tupling, $\log_2(16) = 4$ means that 4 weeks, (i.e. four 2-tupling periods) have passed. This corresponds with the point (16,4).
Chunky continuous covariation	The student conceptualizes two quantities varying simultaneously, both in a chunky continuous manner.	The student may envision the value of x varying in a chunky continuous manner with the intervals having endpoints that are whole number powers of two, corresponding with whole number values for $\log_2(x)$. Values of x and $\log_2(x)$ within the respective intervals do not hold the same meanings as those at the endpoints.
Smooth continuous covariation	The student conceptualizes two quantities varying simultaneously, both in a smooth and continuous manner.	The student may envision the value of x varying in a smooth continuous manner with the intervals having endpoints that are whole number powers of two, corresponding with whole number values for $\log_2(x)$. Values of x and $\log_2(x)$ within the respective intervals hold the same meanings as those at the endpoints. For example, the 3 in $\log_2(3) \approx 1.585$ would represent a 3-tupling (tripling) and the 1.585 would suggest about 1.585 2-tupling (doubling) periods passed.

When one reasons covariationally, she is consciously aware of two quantities' values varying in tandem (Carlson, Jacobs, Coe, Larsen, Hsu, 2002; Saldanha & Thompson, 1998) and may visualize this covariation by coupling the two quantities' values in her mind as a new conceptual object. Thompson and Carlson (2017) refer to this coupling as a multiplicative object. A student who reasons in this way with the

logarithmic function may begin to develop connections across representational contexts. For example, he may view the values on the horizontal and vertical axes of the Cartesian plane as representing the varying values of x-tuplings and the number of b-tuplings needed to result in an x-tupling, respectively. He may then conclude that the graph of $y = \log_b(x)$ consists of infinitely many points whose ordered pairs represent the coupling of the two quantities' corresponding values. In other words, each point making up the graph of $y = \log_b(x)$ is a visual representation of the student's conceptualized multiplicative object. When this student observes a table of values relating x-tuplings with the number of b-tuplings needed to result in an x-tupling, he may also view each of the rows as representing the coupling of the two quantities' values. It is worth noting that using tables as a representational tool for continuous functions may be limiting to students because tables do not take into account what happens in between the entries. Nevertheless, students who reason at the smooth continuous covariation level should be able to imagine both of the quantities' values varying within the intervals presented by the table.

A student who has conceptualized the coupling of two quantities' values in her mind as a new conceptual object may also develop an intellectual need for logarithmic (function) notation. That is, she may desire to represent her conceptualized multiplicative object in a way that does not require a table or graph. She may define x to be a tupling that can vary smoothly and continuously, but lack the tools needed to represent the number of b-tupling periods needed to result in an x-tupling. I hypothesize that a student in this state is likely to find $\log_b(x)$ to be more meaningful and may visualize the notation

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in the following way: $\underbrace{\log_b \overbrace{(x)}^{\text{varying}}}_{\text{varying}}$, or in the form of a formula, $\overset{\text{varying}}{y} = \underbrace{\log_b \overbrace{(x)}^{\text{varying}}}_{\text{varying}}$.

Research Literature on Students' Understandings of Exponents and the Exponential Function

Viewing exponentiation as repeated multiplication is a primitive, yet insufficient interpretation. When the value of the exponent is a natural number, this conception is adequate. However, when the value is a non-natural real number, say $-\pi$, how might we interpret the exponent in this case? The interpretation of exponentiation as repeated multiplication fails to describe this case. While some researchers advocate a repeated multiplication approach (e.g. Goldin & Herscovics, 1991; Weber, 2002), others believe this approach limits students (e.g. Ellis, Ozgur, Kulow, Williams & Amidon, 2015; Davis, 2009; Confrey & Smith, 1995). In particular, Confrey and Smith (1995) argue that the standard way of teaching multiplication through repeated addition is inadequate for describing a variety of situations such as magnification, multiplicative parts (i.e. finding a fraction of a split), reinitializing and creating an array. Weber (2002) proposed that students first understand exponentiation as a process (in terms of APOS theory) before viewing exponential and logarithmic expressions as the result of applying the process. Once this reasoning ability is achieved, the student should be able to generalize the understanding to cases in which the exponent is a non-natural number. Specifically, Weber stressed to his students that "b^x represents the number that is the product of x many factors of b" and that "$\log_b(m)$ is the number of factors of b there are in m."

Using this language conveys a more productive meaning[7] of an exponent than merely viewing an exponent as repeated multiplication. For example, we can now describe $9^{2.5}$ to be the number that is the product of two and a half factors of 9, while under the view of repeated multiplication, a student might write " $9 \cdot 9 \cdot ?$ ". If a coherent understanding of exponential functions (and later logarithmic functions) is desired of our students, it is imperative that they have productive meanings for exponents.

Confrey and Smith (1995) presented a theoretical approach for understanding exponential functions emphasizing the use of two constructs: splitting and covariation. The construct of splitting is "a primitive model…that provides an operational basis for multiplication and division" (Confrey & Smith, 1995). Direction in the splitting structure suggests either multiplication or division (doubling vs. halving, etc.). The authors provide empirical evidence (students utilize the idea of halving to determine the area per child on a playground) that they claim suggests that splitting is an intuitive construct for multiplication and division. Confrey and Smith describe, compare and contrast two "worlds" of mathematics: the counting (additive) world, and the splitting (multiplicative) world. They briefly examine the history of Napier's continuous approach for examining arithmetic and geometric sequences and note that it is important to identify the isomorphic attributes of the splitting world when one makes a discovery in the counting world. For example, the identity in the counting world is 0, while in the splitting world the identity is 1. They also note that a link between the two worlds is present – particularly that the counting world numbers are often used as index numbers for the

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[7] There are issues with Weber's (2002) definition. For example, the phrase "factors of *b*" may invite the students to consider the prime factorization of *b*. Also, the phrase "in *m*" is unclear. These issues led me to develop my modified definition of logarithm.

splitting world and that often counting numbers are used to represent the results of splitting (although not all the time – i.e. Richter scale). The origin of the counting world is 0 (acting as a boundary for positive and negative numbers) while the origin of the splitting world is 1 (acting as a boundary "between whole numbers and fraction values" (pgs. 76-77) when discussing powers). Often, with exponential functions, we have an initial value that is not 1 (our origin) – however, we can think of that initial value as being a whole (or 1) of something.

Confrey and Smith (1995) compare and contrast the covariation approach to functions with the correspondence approach to functions. They claim that the correspondence approach is the approach that dominates curriculum – where the set theory definition of function is utilized, algebraic rules are emphasized and a directionality from x to $f(x)$ is implied. On the other hand, they describe the covariation approach as considering two sets of data and the relationship between the sets. That is, this approach encourages the description of how one quantity varies in relation to another and allows for the discussion of rates of change, differences, and accumulation. In particular, exponential functions can be characterized as having constant multiplicative rates of change (Ellis et al., 2015). Confrey and Smith described how to produce exponential functions using splitting and covariation and conclude that the use of covariation, splitting and the idea of the isomorphism between the two worlds helps avoid concealing the relevant splitting unit/base that relates to the functional situation and helps avoid an overreliance on algebraic representation.

Ellis et al. (2015) conducted a small-scale teaching experiment with three middle school students that examined continuously covarying quantities. The students were

asked to consider a scenario of a cactus named Jactus whose height doubled every week. Eventually, the initial height, weekly growth factor and amount of time needed to double were altered to provide variety. The authors noticed three significant shifts in the students' thinking over the course of the study. At first, the students attended only to Jactus' height and concluded he grew by means of repeated multiplication. Eventually, the students began to coordinate this repeated multiplication with the corresponding changes in the amount of time that elapsed. The second shift consisted of students determining the factor by which Jactus' height grew for varying changes in the number of weeks by means of calculating the ratio of two heights. Finally, the third shift involved the students generalizing the reasoning noted in the second shift to include non-natural exponents (i.e. to determine the 1-day growth factor). The authors noted that a student's ability to coordinate the growth factor (or ratio of height values) with the changes in elapsed time contributed to the student successfully defining the relationship between the elapsed time and Jactus' height. This study leveraged findings from Ellis et al.'s study of Jactus the Cactus to promote more meaningful discussions on logarithms.

Research Literature on Students' Understandings of the Idea of Logarithm and the Logarithmic Function

The topics of logarithmic notation and logarithmic functions often pose a variety of challenges to students (Kenney, 2005; Weber, 2002). Similar to the complexities present in function notation, logarithmic notation consists of multiple parts each with their own dual nature (Kenney, 2005). As stated previously, in the equation $y = \log_b(x)$, b, x, and y take on a variety of meanings – b often takes on the form of a parameter (staying consistent within the context of a problem, but varying from problem to

problem), *x* serves as the input variable to the logarithmic function and is a tupling, and *y*

serves as the output variable to the logarithmic function and is the number of *b*-tupling

periods needed to result in an *x*-tupling. Kenney (2005) noted that because function

names are often one letter, students do not naturally view $\log(x)$ as representing an

output to a function. Weber (2002) recognized these and other obstacles students

encounter and conducted a pilot study that compared a traditional approach to teaching

logarithmic functions with a more conceptual approach using technology (MAPLE) that

introduced $\log_b(m)$ as the number of factors of *b* there are in *m*. Weber's way of

discussing the meaning of a logarithmic expression more clearly describes what the

multiple parts of the notation represent - therefore addressing the issues Kenney observed

in her study. However, Weber's definition of logarithm may introduce other problems.

For example, the phrase "factors of *b*" may invite the students to consider the prime

factorization of *b*. Also, the phrase "in *m*" is unclear. These issues led me to develop my

modified definition of logarithm.

     In addition to these unavoidable complexities, Kenney's (2005) study uncovered

other difficulties students have in understanding logarithmic notation. Kenny investigated

students' understandings of logarithmic notation in two phases (questionnaire and student

interviews). The data revealed that students displayed mixed understandings of the bases

in the expressions. For example, the students appeared to think that different bases

always meant the logarithmic expressions were not equivalent (with the inputs being the

same). However, when the expression involved the sum of logarithms, some students

claimed equivalence because the bases would cancel out. Students also claimed that *ln*

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was equivalent to $\log_{10}$. One possible reason for this misconception is that both of these logarithmic bases appear on graphing calculators and are used when solving for the input to an exponential function. The study also revealed that students would disregard or "cancel out" the word "log" when simplifying equations involving logarithms and solving for *x*. Despite the aforementioned difficulties, a few of the students were successful in arriving at the correct answer. However, Weber (2002) found that this was an unlikely result of traditionally taught students.

Weber's (2002) pilot study examined the effects of non-traditional instruction of exponents and logarithms. The participants of the study were college students from two different college algebra and trigonometry classes at a university in the southern region of the United States. 15 students from each class voluntarily participated in the study. The first group of 15 students made up the control group and experienced traditional instruction on exponents and logarithms while the second group of 15 students participated in a more conceptually taught lesson lead by the author which incorporated the use of the program MAPLE. Students were taught a basic loop that used repeated addition to perform multiplications of integers and were later asked to write a similar program for exponentiation. Each class spent approximately the same amount of time covering the topics. Three weeks after instruction, students from each class were individually interviewed and asked a series of questions involving exponents, logarithmic expressions, logarithmic properties, and equations involving logarithmic expressions. While students in both groups were able to evaluate simple calculations, students in the experimental group were able to recall more properties of exponents and logarithms than

the control group. These students were also able to provide justifications for the properties - unlike the students in the control group. Weber found that the students who received more conceptually based instruction were more likely to catch their mistakes when it came to identifying and justifying properties of logarithms and exponents.

This data emphasizes the importance and need for more coherent and conceptually taught lessons for exponents, logarithmic expressions and logarithmic functions. We are doing our students a disservice when we simply present them with a list of rules to memorize and expect them to remember everything at face value.

THEORETICAL PERSPECTIVE

This chapter presents the theoretical perspective for this study. I begin by presenting my conceptual analysis for the idea of logarithm and conclude by discussing the theoretical perspective that informs my methods for my study.

Conceptual Analysis

Exponential and logarithmic relationships are two sides of a coin – when one discusses elements of one relationship, he is, in some form or another, discussing components of the other relationship as well. In this conceptual analysis, I examine a variety of aspects often categorized under exponential relationships because I see them as being important for one to come to understand the idea of logarithm and the logarithmic function. In particular, I develop the ideas of growth factor, the exponential relationship, tuplings and tupling periods, exponent, growth factor conversions, the exponential function, logarithmic notation, logarithmic properties and the logarithmic function. I also briefly examine a few prerequisite understandings one must have to make sense of these listed ideas. At the end of this section, I develop a hypothetical learning trajectory informed by my conceptual analysis.

Division as Measurement

Students must understand the construct of division as measurement. That is, to measure Quantity A in terms of Quantity B, we write $\dfrac{\text{Quantity A}}{\text{Quantity B}}$. If $\dfrac{\text{Quantity A}}{\text{Quantity B}} = m$, we say Quantity A is m times as large as Quantity B. As long as Quantity A and Quantity B are measured using the same unit, this ratio will remain constant.

Multiplying by *A* and Then Multiplying by *B* Has the Same Overall Effect as Multiplying by *AB* ($\times A \times B = \times AB$)

Students must have the understanding that multiplying by *A* and then multiplying by *B* is equivalent to multiplying by *AB*. For example, multiplying some value by 2 and then by 3 is equivalent to multiplying the value by 6. Therefore, if a value *A*-tuples (becomes *A* times as large) and then *B*-tuples (becomes *B* times as large), overall the value will *AB*-tuple (become *AB* times as large) (this claim is informed by my research in RUME IV).

Growth Factor / The Exponential Relationship

When comparing two values of the same quantity (say value A and value B), we can determine how many times as large one value is than another by calculating a quotient to evaluate a ratio$\left(\dfrac{\text{value A}}{\text{value B}} \right)$. If value B is *m* times as large as value A, then by convention we say the quantity grew by a factor of *m*, or became *m* times as large. In the future, I will refer to this as an *m*-tuple. Note: this is not to be confused with the definition of *m*-tuple as an ordered set of *m* numbers (in the *m*-dimensional Cartesian plane). Similarly, an *m*-tupling occurs when a quantity becomes *m* times as large.

When one attends to the values of two varying quantities, Quantity A and Quantity B, and notices that for equal changes in the Quantity A, Quantity B grows by a constant factor, then there exists a geometric relationship between the two quantities. In the continuous case, we more specifically refer to the relationship between the two quantities as exponential. For the rest of this proposal, I assume continuity unless stated otherwise.

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Tuples, Tuplings & Tupling Periods / Exponents / Growth Factor

Conversions

In this section, I discuss concepts foundational to exponential functions. I begin by developing a necessity for exponential notation and then argue how my definition for exponent is useful for converting from one growth factor to another growth factor (often called partial or *n*-unit growth factors).

Recall that for two exponentially related quantities, for equal changes in one quantity, the other quantity grows by a constant factor. That is, for example, for any change of *n* in Quantity A, Quantity B will become *b* times as large (or *b*-tuples). By convention, we say the *n*-unit growth factor is *b*. However, we can also say that *n* is the *b*-tupling period, the amount/value of change of our input to our exponential function necessary for our output to *b*-tuple, or become *b* times as large. If *m* *b*-tupling periods have elapsed (that is, *nm* units of the input quantity), then by convention, we write b^m to represent the factor by which the quantity grows in that period. It is worth noting that this interpretation for exponents differs from the repeated multiplication approach because it takes into account all real values of *m*. For example, suppose the 4-tupling (or quadrupling) period for a population is one week and suppose 1.5 weeks elapse, then the factor by which the population grew over the course of the 1.5 weeks can be expressed as $4^{1.5}$ (which is equivalent to 8). Or, suppose that for every 1 radian a dial rotates, the amount of frozen yogurt dispensed from a machine 1.5-tuples. Then if the dial rotates an angle of π radians, the amount of frozen yogurt dispensed from a machine will 1.5^π - tuple.

Ellis and colleagues (2015) found that before students were able to reason with non-natural number exponents, they first had to reason with natural number exponents. Therefore, as students are beginning to conceptualize the idea of exponent, it may be necessary to present students with cases where *m*, the number of elapsed *b*-tupling periods, is a natural number. For example, suppose the 3-tupling (or tripling) period for a population is 1 week and suppose 2 weeks (two 3-tupling periods) have elapsed, then the factor by which the population grows over the 2 weeks is $3 \times 3 = 9$. To represent the case where two 3-tupling periods have elapsed we can also write 3^2. In this instance, it is easy to calculate the 2-week growth factor – however, this is not always the case.

Still assuming the 1-week growth factor is 3, suppose we now wish to represent the 1-year, or 52 week growth factor. We need a way to represent the growth factor that corresponds to the case where 52 3-tupling periods have elapsed; specifically, we write 3^{52}. Similar reasoning can be employed to determine the 1-day, or 1/7th week growth factor. To represent the case where 1/7th of a 3-tupling period has elapsed, we write $3^{1/7}$. In both of these cases, we let the exponent on 3 to represent the number of elapsed 3-tupling periods (weeks). This reasoning remains consistent for exponents less than or equal to zero, too. For example, in the case where no time has elapsed, the population would not change (i.e. grow by a factor of 1); this corresponds with the equation $3^0 = 1$. If the change in the number of weeks is -3 (i.e. we are looking "back in time" for a total of 3 weeks), then the -3 week growth factor is 3^{-3} or 1/27 (since over the 3 weeks prior to when 0 weeks have elapsed, the population would both become 1 and would increase by a factor of 27). In general, if we let *x* represent the number of elapsed 3-tupling periods

(weeks), then 3^x represents the *x*-week growth factor.

The Exponential Function

In this section, I describe how one might come to define an exponential function. To productively discuss the ideas in this section, students must have an understanding for division as measurement and growth factors, they must conceptualize exponents to represent a number of elapsed tupling periods, understand how to represent changes in quantities' values, and recognize that for exponential relationships between two quantities, for equal changes in one quantity, the other quantity grows by a constant factor.

Suppose (x_1, y_1) and (x, y) are points that satisfy an exponential relationship. Since *y* is y/y_1 times as large as y_1, and since the relationship is exponential, then for any change of x-x_1 in the input quantity, the output quantity will become y/y_1 times as large. Similarly, if the 1-unit growth factor is *b*, then for any change of x-x_1 in the input quantity, the corresponding growth factor will be b^{x-x_1}. Therefore, we can conclude

$\dfrac{y}{y_1} = b^{x-x_1}$ (the two different expressions representing the same growth factor are equivalent), or $y = y_1 b^{x-x_1}$ (*y* is b^{x-x_1} times as large as y_1). In the case where (x_1, y_1) is the vertical intercept, say $(0, a)$, we have $y = ab^x$. Therefore, if $f(x) = y$, then $f(x) = ab^x$, where *a* is the initial value of the output quantity and *b* is the 1-unit growth factor. Consider Sparky, a saguaro cactus whose height is growing exponentially. If Sparky was 5 feet tall when he was purchased and 10 feet tall one week later, then in one week, he became 2 times as large. Thus, the one-week growth factor is 2. If we wish to define the

relationship relating the number of weeks since Sparky's purchase, x, and his height in feet, y, we can use the reasoning described above to conclude $\frac{y}{5} = 2^{x-0}$ or $y = 5(2)^x$.

Logarithmic Notation

Recall exponential functions have the quality that, for equal changes in the input quantity, the output quantity grows by a constant factor. That is, for any change of n in the input quantity, the output quantity will b-tuple, or become b times as large. By convention, we say the n-unit growth factor is b. However, we can also say that n is the b-tupling period, the amount/value of change of our input to our exponential function necessary for our output to become b times as large.

Often, when working with exponential functions, students are given explicit information about only one growth factor. This may be the one-year growth factor, the three-day growth factor, etc. This information also informs the student of a tupling period. For example, if the one-week growth factor is 2, then the 2-tupling period is one week. However, in a situation where the 2-tupling period is one week, a student may be interested in determining the number of weeks necessary to 10-tuple, or become 10 times as large (based on information presented in the task at hand). In this case, the 10-tupling period will be longer than the 2-tupling period (1 week), but can still be measured using a one-week unit of measure (or the 2-tupling period). However, since 10 is not a power of 2, this value can be difficult to calculate. Moreover, in general, determining the change in the input of an exponential function necessary for the initial value of the function to m-tuple, or become m times as large, is not a trivial task. That is, there is no simple rule that provides instructions on how to calculate the m-tupling period. However, with the use of

modern technology, these calculations are possible. The 10-tupling period and the *e*-tupling period are the most common units used to measure all other tupling periods. However, any tupling period can be used to measure the change in input necessary for the initial value of the function to *m*-tuple. For example, if the 3-tupling period is one day, we can use it to measure the 27-tupling period (3 days). In general, we write $\log_b(m)$ to represent the number of *b*-tupling periods it takes the initial value of our exponential function to result in an *m*-tupling.

Logarithmic Properties

We start with the meaning of "$\log_b(x)$" being "the number of *b*-tupling periods needed to result in an *x*-tupling". After being presented with logarithmic notation, students are often asked to manipulate logarithmic expressions or equations using one or more of the following logarithmic properties:

1. $\log_b(X) + \log_b(Y) = \log_b(XY)$

2. $\log_b(X) - \log_b(Y) = \log_b(X/Y)$

3. $\log_b(X^y) = y \log_b(X)$

4. $\log_b(X) = \dfrac{\log_c(X)}{\log_c(b)}$ (or more accurately, $\dfrac{\log_b(X)}{\log_b(Y)} = \dfrac{\log_c(X)}{\log_c(Y)}$)

5. $\log_b(b^x) = x$

6. $b^{\log_b(x)} = x$

The understanding that multiplying by *X* and then multiplying by *Y* is equivalent to multiplying by *XY* is foundational to understanding the first logarithmic property. Therefore, if a value experiences an *X*-tupling and then experiences a *Y*-tupling, overall

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the value will experience an *XY*-tupling. If we let T_X represent the *X*-tupling period, T_Y represent the *Y*-tupling period, and T_{XY} represent the *XY*-tupling period (each not yet measured in a specified unit), then $T_X + T_Y = T_{XY}$. Therefore, the number of *b*-tupling periods needed to result in an *XY*-tupling is equal to the number of *b*-tupling periods needed to result in an *X*-tupling plus the number of *b*-tupling periods needed to result in a *Y*-tupling, or $\log_b(X) + \log_b(Y) = \log_b(XY)$. If we consider a mystical cactus named Sparky whose height 2-tuples each week, and suppose his height experiences a 2-tupling and suppose his height then experiences an 8-tupling after the 2-tupling. His 2-tupled height will become 8 times as large. His height will have become 16 times as large as it was before it 2-tupled, for an overall 16-tuple in height. The number of weeks (2-tupling periods) needed to result in a 2-tupling (1 week) followed by the number of 2-tupling periods to result in an 8-tupling (3 weeks) will be the number of 2-tupling periods needed to result in a 16-tupling (4 weeks). Symbolically, we represent this case as $\log_2(2) + \log_2(8) = \log_2(16)$.

To understand the second logarithmic property, one can build off the first logarithmic property and the understanding that *X* is *X/Y* times as large as *Y*. That is, if a value experiences a *Y*-tupling and then experiences an *X/Y*-tupling after the *Y*-tupling, the *Y*-tupled value will become *X/Y* times as large. Therefore, the value will have become *X* times as large as it was before it *Y*-tupled, for an overall *X*-tuple. If we let T_X represent the *X*-tupling period, T_Y represent the *Y*-tupling period, and $T_{X/Y}$ represent the *X/Y*-tupling period (each not yet measured in a specified unit), then $T_{X/Y} + T_Y = T_X$. Therefore, the number of *b*-tupling periods needed to result in an *X*-tupling is equal to the number of *b*-

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tupling periods needed to result in an *X/Y*-tupling plus the number of *b*-tupling periods

needed to result in an *Y*-tupling, or $\log_b(X) = \log_b(X/Y) + \log_b(Y)$. Alternatively,

$\log_b(X/Y) = \log_b(X) - \log_b(Y)$. Considering the same example used for the first

logarithmic property, we can calculate the number of weeks needed for Sparky's height

to experience an 8-tupling by subtracting the number of weeks (2-tupling periods) needed

for Sparky's height to experience a 2-tupling from the number of weeks (2-tupling

periods) needed for Sparky's height to experience a 16-tupling.

The understanding that an exponent on a value, *X*, to represent the number of *X*-

tupling periods that have elapsed is foundational to understanding the third logarithmic

property. That is, if a value experiences *y X*-tupling periods, then overall the value will

experience an X^y-tupling. If we let T_X represent the *X*-tupling period and T_{X^y} represent the

X^y-tupling period (both not yet measured in a specified unit), then $T_{X^y} = yT_X$. Therefore,

the number of *b*-tupling periods needed to experience an X^y-tupling is *y* times as large as

the number of *b*-tupling periods needed to experience an *X*-tupling, symbolically

$\log_b(X^y) = y\log_b(X)$. The number of weeks (2-tupling periods) needed to result in a 2-

tupling 5 times ($\log_2(2^5)$) is 5 times as large as the number of 2-tupling periods needed

to result in a 2-tupling ($5\log_2(2)$).

A less discussed, but useful property of logarithms is the change of base relation.

This property is used to rewrite logarithmic expressions using a different base value,

often as an alternative way of calculating the exact value, and is frequently presented as

$\log_b(X) = \dfrac{\log_c(X)}{\log_c(b)}$. To understand this property, students must have the understanding

that *A* is *A/B* times as large as *B*. Therefore, if we let T_X represent the *X*-tupling period

and T_Y represent the *Y*-tupling period (each not yet measured in a specified unit), then T_X

is T_X/T_Y times as large as T_Y. This relationship will not change based on the units used to

measure either tupling period. That is, if we suppose *b*>0 and use the *b*-tupling period to

measure the *X*- and *Y*- tupling periods, then the *X*-tupling period will always be $\dfrac{\log_b(X)}{\log_b(Y)}$

times as large as a *Y*-tupling period. Put another way, $\dfrac{\log_b(X)}{\log_b(Y)} = \dfrac{\log_c(X)}{\log_c(Y)}$. Notice, if we

let *Y* = *b*, then $\log_b(X) = \dfrac{\log_b(X)}{1} = \dfrac{\log_b(X)}{\log_b(b)} = \dfrac{\log_c(X)}{\log_c(b)}$. Considering the same example

used for the previous logarithmic properties, the 3-tupling (tripling) period measured in

weeks is about 1.585 and the 2-tupling (doubling) period measured in weeks is 1.

Therefore, the number of weeks needed to 3-tuple (1.585 weeks) is 1.585/1 times as large

as the number of weeks needed to 2-tuple (1 week). Alternatively, since the number of

days will always be 7 times as large as the number of weeks, then the 3-tupling (tripling)

period measured in days is $1.585(7) = 11.095$ and the 2-tupling (doubling) period

measured in days is 1(7)=7. Thus, the number of days needed to 3-tuple (triple) will be

$11.095 / 7 = 1.585$ times as large as the number of days needed to 2-tuple (double). In

general, the 3-tupling (tripling) period will always be approximately 1.585 times as large

as the 2-tupling (doubling) period. If we were to measure these periods in weeks, days,

years, or any other appropriate measurement the relationship would still be true. That is,

$$\frac{\log_2(3)}{\log_2(2)} = \frac{\log_{1.1}(3)}{\log_{1.1}(2)} = \frac{\log_{2^{52}}(3)}{\log_{2^{52}}(2)} = \frac{\log_c(3)}{\log_c(2)} \approx 1.585 .$$

The understanding that the exponent on a value, *b*, represents the number of *b*-tupling periods that have elapsed is foundational to understanding the last two logarithmic properties. Therefore, to represent that *x* *b*-tupling periods have elapsed, one writes b^x. Students must also understand that b^x may also represent a b^x-tupling. Additionally, the understanding that $\log_b(m)$ represents the number of *b*-tupling periods needed to result in an *m*-tupling is also foundational to understanding the last two logarithmic properties. Therefore, since b^x conveys that *x* *b*-tupling periods have elapsed and also conveys a b^x-tupling, then the number of *b*-tupling periods needed to result in a b^x-tupling is *x*. Symbolically, we write $\log_b(b^x) = x$. On the other hand, if a value *b*-tuples $\log_b(x)$ many times, the number of *b*-tupling periods needed to result in an *x*-tupling, overall the value will *x*-tuple. Symbolically, we write $b^{\log_b(x)} = x$.

The Logarithmic Function

To conceptualize the logarithmic function in Thomspon and Carlson's (2017) sense, one must first understand *b* and *x* to represent tuplings and $\log_b(x)$ as representing the number of *b*-tupling periods needed to experience an *x*-tupling. He must then conceive of the x-tupling and the number of *b*-tupling periods needed to experience the *x*-tupling as "varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33). In particular, if we know the value for *x*, we can determine the corresponding value of $\log_b(x)$, given a value for *b*. That is, for any given tupling, there will be exactly one number of *b*-tupling periods that are needed to achieve the same growth.

The following taxonomy (Table 0.2) summarizes the components to understanding the idea of logarithm along with the final understandings students should hold for each one.

Table 0.2

Taxonomy of the Idea of Logarithm

Component of the idea of Logarithm	Desired understanding
Division as measurement	To measure Quantity A in terms of Quantity B, we write $\dfrac{\text{Quantity A}}{\text{Quantity B}}$. If $\dfrac{\text{Quantity A}}{\text{Quantity B}} = m$, we say Quantity A is m times as large as Quantity B.
Multiplying by A and then multiplying by B has the same overall effect as multiplying by AB. $\times A \times B = \times AB$	If a value A-tuples (becomes A times as large) and then B-tuples (becomes B times as large), overall the value will AB-tuple (become AB times as large).
Growth Factor	When coordinating the values of two quantities, if the value of the first quantity increases by n-units while the next value of the second quantity is m times as large as its current value, then the n-unit growth factor is m.
The Exponential Relationship	When relating two continuous quantities, Quantity A and Quantity B, if for equal changes in Quantity A, Quantity B grows by a constant factor, then the two quantities have an exponential relationship.
Tuples (VERB)	If the value of a quantity becomes m times as large, we say the quantity's value m-tuples.
Tuplings (NOUN)	An m-tupling is the event in which the value of a quantity becomes m times as large.
Tupling period	An m-tupling period is the amount of change in the independent quantity needed for the dependent quantity to become m times as large.
Exponent (on a value, b)	The number of elapsed b-tupling periods. Symbolically, this value is written superscript to b.
Growth Factor Conversion	The factor by which a quantity will grow over x, b-tupling periods is represented as b^x. If $c^1 = b^x$, then one c-tupling period is the same as x b-tupling periods.

The Exponential Function	The function $f(x) = ab^x$ relates two quantities "varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33) where x represents the varying values of the first quantity, a represents the initial value of the second quantity, and b represents the 1-unit (of the first quantity) growth factor.
Logarithmic Notation	$\log_b(X)$ represents the number of b-tupling periods it takes (the initial value of an exponential function) to result in an X-tupling.
LP1: $\log_b(X) + \log_b(Y) = \log_b(XY)$	The number of b-tupling periods needed to result in an XY-tupling is the same as the number of b-tupling periods needed to result in an X-tupling plus the number of b-tupling periods needed to result in a Y-tupling.
LP2: $\log_b(X) - \log_b(Y) = \log_b(X/Y)$	The number of b-tupling periods needed to result in an X/Y-tupling is the same as the number of b-tupling periods needed to result in an X-tupling minus the number of b-tupling periods needed to result in a Y-tupling.
LP3: $\log_b(X^y) = y\log_b(X)$	The number of b-tupling periods needed to experience an X^y-tupling is y times as large as the number of b-tupling periods needed to experience an X-tupling.
LP4: $\log_b(X) = \dfrac{\log_b(X)}{\log_b(b)} = \dfrac{\log_c(X)}{\log_c(b)}$	The X-tupling period will always be k times as large as the b-tupling period (this value does not depend on the unit chosen to measure both the X- and b-tupling periods).
LP5: $\log_b(b^x) = x$	The number of b-tupling periods needed to experience a b-tupling, x times, is x.
LP6: $b^{\log_b(x)} = x$	If a value b-tuples $\log_b(x)$ times, the number of b-tupling periods needed to result in an x-tupling, overall the value will x-tuple.
The Logarithmic Function	A covarying relationship between an x-tupling and the number of b-tupling periods needed to experience an x-tupling ($\log_b(x)$). These two quantities vary in such a way that every value of the x-tupling determines exactly one value of the number of b-tupling periods needed to experience an x-tupling.

Theoretical Perspective

I begin this section with a brief discussion of radical constructivism (Glasersfeld, 1995), the theoretical perspective that informs the teaching experiment methodology. A

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central claim of radical constructivism is that knowledge is constructed in the mind of an individual and therefore cannot be directly accessed by anyone else. Steffe and Thompson (2000a) refer to an individual's mathematical reality as "student's mathematics." If students' mathematics were accessible to researchers, there would be little need for mathematics education research. Therefore, at best, researchers can attempt to form a model of students' thinking, referred to as "mathematics of students." A model is considered reliable when the student's utterances, written work, and movements are in alignment with the model. However, to say one has developed the mathematics of a student is not the same as stating that a model directly represents a student's mathematics (this is an impossible goal).

In a study of student learning, one must decide on a theory of knowledge and a theory of learning to inform her hypotheses, data collection and analysis. Researchers who choose different theories to inform their studies will usually have different conjectures, methods and findings. Under the theoretical perspective of radical constructivism, where individuals construct their own knowledge, the researcher will design her study to center around the constructions made by the subject. If the subject responds in a way that is mathematically incorrect, the researcher will be interested to explore the ways in which the student was thinking for his claims to make sense to him. Under a different theoretical approach, perhaps where knowledge is something to be found, the researcher might disregard the student's incorrect claims and simply conclude that the student has not yet made the necessary mathematical discoveries. Piaget's genetic epistemology (2001) is a theoretical framework that works well with the teaching experiment methodology used in this study. The framework focuses on both "what

knowledge consists of [cognitive structures - schemes] and the ways in which knowledge develops [how those structures come into existence]" (Piaget, 2001, p. 2). When a researcher develops the mathematics of a student, she is trying to model the student's cognitive structures that comprise knowledge, known as schemes. These structures are organizations of mental actions or mental operations (reversible actions) and may even be complex and contain other schemes (Piaget, 2001). An action is "all movement, all thought, or all emotion – [that] responds to a need" (Piaget, 1967, p. 6). Researchers rely on a student's observable actions when attempting to form models of his schemes, such as utterances, written work, movements or body language.

Researchers who are interested in how a student comes to learn a particular idea must also try and model what the student does with his schemes. When a student applies a scheme to a particular environment and achieves outcomes that do not conflict with his anticipated results, he assimilates the environment to the scheme. In cases of assimilation, no noteworthy learning takes place because the student remains in a state of equilibrium. However, if the student achieves outcomes that conflict with his anticipated results, the assimilation is unsuccessful and he will be in a state of disequilibrium. To cope with his unrest, the student may modify the scheme he originally applied to the environment or he may create a new scheme altogether (Piaget, 2001). Learning takes place when either of these accommodations occurs. Piaget went on to develop more constructs to discuss the development of knowledge.

Piaget specified that students abstract knowledge in a variety of ways. For instance, when a student makes an empirical abstraction, he abstracts knowledge from an object (mentally constructed in his mind) such as color, size, weight, etc. (Thompson,

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1985). When a student makes a pseudo empirical abstraction, he attends only to the

results of actions he has performed. For example, when solving the equation

$234 = 54(1.5)^x$ for x, a student may arrive at the answer $x = \dfrac{\ln(234/54)}{\ln(1.5)}$ and conclude

that all solutions to problems beginning as $y = a(b)^x$ will be of the form $x = \dfrac{\ln(y/a)}{\ln(b)}$. On

the other hand, when the student distinguishes an action from the initial and resulting

stages (differentiation), creates an action to represent this differentiated action

(projection), or coordinates these new actions together (coordination/integration), he has

made a reflective abstraction (Piaget, 2001). In the previous example, if the student had

instead reflected on what each step of his solution represented and why the actions

involved calculated those values, his abstraction would likely have been classified as a

reflective abstraction. Piaget (2001) labeled this level of thinking to be more advanced

than thinking involved in making empirical and pseudo-empirical abstractions.

Furthermore, if the student then chose to reflect on his reflective abstraction to form a

generalization he will have made a reflected abstraction. Finally, thematization occurs

when the subject is able to provide an outline of an activity without needing to provide

the minor details. Researchers who are interested in using the teaching experiment

methodology to model student learning (i.e. cognitive structuring and restructuring)

should make sure to provide students with opportunities for reflection (Derry, 1996).

Throughout my study, I will attempt to model the participants' knowledge development

of the idea of logarithm. I will focus on the abstractions I believe must be made in order

to develop a coherent model of logarithmic functions. I will then argue why and at what

points I believe the participants of my study experience different types of abstraction as a means of explaining their paths of knowledge development.

METHODOLOGY

Teaching Experiment

Two of the overall goals for researchers using teaching experiments are to develop models of students' mathematics and to understand the progress students make over an extended period of time (Steffe & Thompson, 2000a). This is a step up from Piaget's clinical interview, where researchers only attempt to model the student's current thinking. Developing the mathematics of students throughout a teaching experiment is a demanding task involving much scrutiny. Hypotheses of student thinking must be developed, tested, revised, and tested again until a reliable model is formed. This process is never quite finished during teaching experiments because the researcher is interested in the development of the student's thinking over the course of the experiment. Therefore, once a reliable model has been formed, the researcher can form a new hypothesis for how the student will act in a different mathematical scenario. While the researcher interacts with the student, she should expect to encounter a few constraints. For example, the language and actions exhibited by the students may perplex the researcher. However, this constraint is valuable when trying to develop the mathematics of students because the language and actions of the students are informed by the students' mathematics. Researchers should also expect to encounter moments when students "hit a wall" in their thinking. When a student can't seem to move forward in his thinking, despite the assistance of the researcher or other medium, he makes what are considered to be essential mistakes. Essential mistakes made by the student can serve to assist the researcher in identifying the boundaries of the mathematics of the student (Steffe & Thompson, 2000a). When a student makes what appears to be an essential mistake during

a teaching experiment, the researcher's goal should be to try and understand and model what the student can do and how the student must be thinking for his actions to make sense to him.

Before conducting a teaching experiment, it is recommended that researchers conduct exploratory teaching (Steffe & Thompson, 2000a). One of the purposes of exploratory teaching is to form a reliable model of the student's current thinking; this phase will inform the researcher of the student's initial way of thinking and will help the researcher develop hypotheses to test throughout the teaching experiment. When using this methodology in a study of student learning of ideas of logarithm, properties of logarithms, logarithmic growth, and logarithmic functions, the exploratory teaching stage may let the researcher explore students' understandings of prerequisites to the idea of logarithm, such as division as measurement or that multiplying by A and then multiplying by B has the same overall effect as multiplying by AB. During this stage, the researcher may find that the student makes essential mistakes with these or other topics and may hypothesize ways in which the student may respond to logarithmic tasks throughout the upcoming episodes in the teaching experiment.

Teaching experiments are made up of a series of recorded teaching episodes, a teaching agent, one or more students, a witness and designated time in between episodes to conduct retrospective analyses (Steffe & Thompson, 2000a). The role of the teaching agent is multifaceted. Prior to each teaching episode, the teaching agent should have hypotheses about the student's thinking and how that thinking will affect the student's utterances, movements, and written work. The teaching agent should also develop a series of tasks designed to test her hypotheses. However, during the teaching episode, the

teaching agent needs to temporarily set her hypotheses aside so that she can focus on what is actually happening. She must also be prepared to go down paths in which she was unprepared for – for if she knew where everything was going to lead, there would be no point to doing the research. When the teaching agent goes along with a student's claim without knowing where it is headed or tries to decenter (Piaget, 1955; Steffe & Thompson, 2000b; Carlson, Bowling, Moore & Ortiz, 2007) in an effort to explore the student's reasoning, she is engaging in what is called a responsive and intuitive interaction. On the other hand, when the teaching agent initiates discussions with the student to compare the student's response to a hypothesized action, she is engaging in what is called an analytical interaction. After the student responds, the teaching agent may have to modify her hypotheses and introduce new situations so that she can continue to model the student's thinking (either on the fly or during the next session).

The teaching agent should try to engender generalizing assimilations, functional accommodations, or developmental accommodations in the student throughout the teaching episodes. An assimilation is generalizing when the scheme involved is used in situations that the student would deem familiar, but include novel (from the researcher's perspective) elements. In cases of generalizing assimilation, students may have to make minor accommodations to their schemes. Functional accommodations occur when the student uses his scheme in a new way or develops new operations. The teaching agent may engender developmental accommodations when she presents a student with a task that, in the researcher's perspective, contain elements that are beyond the scope of the student's schemes. In order for the student to solve the tasks, he would need to conduct a major reorganization of his schemes. Such tasks can also be used to analyze the

developmental stage of the student (Steffe & Thompson, 2000a).

Steffe and Thompson (2000a) share a couple techniques used by teaching agents during teaching episodes to check in with the student's progress. One technique is to share with the student what "another student" (real or imaginary) claimed; this may evoke some level of doubt within the student and may help the researcher in identifying boundaries of the mathematics of the student. Another technique is to ask the student to anticipate what will happen after a certain operation is performed. For example, I've programmed my Sparky the Saguaro Geogebra applet to display two cacti some number of weeks apart. If "1" is typed in the "Weeks Before" box, the screen will display two cacti named "Sparky" and "Weeks Before" that are one week apart. Because the 2-tupling period is set to be one week, the "Sparky" cactus will be 2 times as tall as the "Weeks Before" cactus. When the animation is running, the horizontal distance between the two cacti will remain constant (1), and the "Sparky" cactus will always be 2 times as tall as the "Weeks Before" cactus. Therefore, I could ask the interviewee how he thinks the animation will change when we type in $\log_2(3)$ in the "Weeks Before" box. How the student responds may shed light on his interpretation of logarithmic notation. In this case, the animation will display two cacti, about 1.585 weeks apart and the "Sparky" cactus will be 3 times as tall as the "Weeks Before" cactus because $\log_2(3)$ represents the number of 2-tupling periods (weeks) needed for the height to experience a 3-tupling (tripling).

After each teaching episode, the teaching agent conducts a retrospective analysis. This stage in the teaching experiment is particularly important and must be adequately

planned for (Steffe & Thompson, 2000a). During the retrospective analysis, the teaching agent reviews the recordings of the previous teaching episode(s) and analyzes the student's utterances, written work, and movements, to develop a model of the student's mathematics. The teaching agent develops mental records of the interactions made with the student during the episodes, but may not remember every detail. When the teaching agent conducts a retrospective analysis, she may recall a realization she made during the episode regarding the student's thinking that she otherwise may have forgotten. During this stage, the teaching agent revisits the hypotheses she set aside (modifying them as needed), creates new tasks to test her models in future episodes and develops hypotheses of how the student will respond to such tasks. If the teaching agent believes she has developed a reliable model of the student's thinking, she might design tasks that present the student with opportunities to reexamine and modify his thinking. During these stages of the teaching experiment, the teaching agent can also rely on the witness for additional assistance and outside opinions.

During a teaching experiment, the witness observes the interactions between the teaching agent and the student. When the teaching agent is engaged in responsive and intuitive interactions with the student, the witness is able to analyze the interactions from an outside perspective. During the retrospective analysis stage, the witness may offer insight that the teaching agent may have otherwise missed. The teaching agent may also call on the witness during the teaching episode for additional questioning and assistance. For example, if the student makes what appears to be an essential mistake and can't seem to move forward, despite the assistance of the teaching agent, the witness can be asked to intervene and ask questions that address the student's thinking from a different angle.

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In a study of student learning at the undergraduate level of the idea of logarithm, properties of logarithms, logarithmic growth, and logarithmic functions, the researcher must be prepared to encounter students who have already developed schemes for exponent, the idea of logarithm, etc. For example, during the exploratory teaching stage of the experiment, the teaching agent may find that the student views exponents as representing repeated multiplication; this interpretation can lead to confusion for the students when the exponent is not a natural number (Confrey & Smith, 1995; Weber, 2002; Davis, 2009; Ellis, Ozgur, Kulow, Williams & Amidon, 2015). In this case, the teaching agent can develop tasks to provide the student opportunities to make an accommodation to his scheme for exponent. If the student has been previously introduced to Euler's definition of logarithm in a previous course, he may be inclined to rewrite logarithmic equations using exponential equations to eliminate the logarithmic notation (Kenney, 2005). Researchers can counter this inclination by presenting a situation to the student where using exponential notation does not simplify the task.

If the researcher decides to incorporate the use of technology in her teaching experiment, such as a Geogebra applet, it will serve her well to be as fluent with the technology as possible. During her interactions with the student, she may see a need to use the technology to visually represent how she is interpreting the student's descriptions. For example, last year I conducted some exploratory teaching using my Sparky the Saguaro Geogebra applet with a student I will call Mike. Mike was discussing the percent change in Sparky's height from one moment to the next and was referring to some measuring lines I had previously programmed in the applet. Based on Mike's utterances, I hypothesized that he was envisioning more measuring lines during his discussion of

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percent change. I decided to program what I believed he was imagining to check my hypothesis. This skill may also be useful if the teaching agent wishes to test or challenge the student's thinking, or create a new situation "on the fly."

Researchers using the teaching experiment methodology must also take into consideration what activities the student will be engaged in between teaching episodes. If the student will be working on homework assignments between meetings, he may make accommodations to his schemes or make abstractions not witnessed by the teaching agent (as seen in Thompson, 1994). However, if the researcher does not offer homework in between episodes, it will not guarantee that the student will not advance his thinking before the next session. Therefore, the teaching agent must accept that she may not be present when the student makes important abstractions or accommodations.

Hypothetical Learning Trajectory

The construct of hypothetical learning trajectory (HLT) (Simon, 1995; Simon & Tzur, 2004) was originally developed to assist educators in planning mathematics lessons and eventually was modified to include a framework for the learning process. HLTs consist of a list of learning goals for students, tasks intended to promote such learning goals, and hypotheses about student learning within the mathematical context. Simon & Tzur (2004) were inspired by Piaget's construct of reflective abstraction and refined the construct of HLT to include an outline of the learning process. The authors emphasized that a student learns (develops new ways of thinking) when she reflects on her actions when completing tasks and the effects of those actions. In Table 0.3, I organized my HLT by presenting a task, the goal(s) for student learning that the task promotes, and a discussion on the role of the task in Simon & Tzur's (2004) framework for learning. It is

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worth noting that this HLT was developed using my conceptual analysis. The HLT

outlines one possible trajectory students might take while developing an understanding of

the idea of logarithm. In a later section, I discuss how during a teaching experiment

researchers must be open to modifying their tasks and trajectory based on the student's

responses.

Table 0.3

HLT for the Idea of Logarithm and the Logarithmic Function

Task

i. Cactus C (A, D) is how many times as tall as Cactus B?

ii. Cactus B is how many times as tall as Cactus C (A, D)?

iii. Given any two cacti, describe how you determine how many times as tall one is than the other?

iv. Draw Cactus E given Cactus E is 5.5 times as tall as Cactus B.

v. Draw Cactus F given Cactus C is 3 times as tall as Cactus F.

Cactus A   Cactus B   Cactus C   Cactus D

vi. If Cactus B is 8 inches tall, how tall are Cacti A, C, D and E?

vii. Cactus H is how many times as tall as Cactus G if Cactus G is 34 inches tall and Cactus H is 102 inches tall?

viii. Cactus I is how many times as tall as Cactus J if Cactus J is x inches tall and Cactus I is y inches tall?

ix. How would you describe the cactus' growth in the diagram to the right given that the cactus on the left grew to be the cactus on the right?

x. If a cactus was 23 inches tall when it was purchased and grew to be 156 inches tall, by what factor did the cactus grow?

xi. If a cactus was m inches tall when it was purchased and grew to be k inches tall, by what factor did the cactus grow?

Learning Goal(s)	Discussion
Division as Measurement: Student will understand that to measure Quantity A in terms of Quantity B, we write $\dfrac{\text{Quantity A}}{\text{Quantity B}}$. If $\dfrac{\text{Quantity A}}{\text{Quantity B}} = m$, we say Quantity A is m times as large as Quantity B. As long as Quantity A and Quantity B are measured using the same unit, this ratio will remain constant. *Growth Factor:* Student will understand that to determine how many times as large one value of a quantity is than another, she can calculate the ratio $\dfrac{\text{value B}}{\text{value A}}$. If value B is m times as large as value A, then by convention we say the quantity grew by a factor of m. m is a growth factor. *Tuplings:* (Chance to intro) Student will understand that the phrase, "grow by a factor of b" can also be expressed as "the *quantity* b-tuples" or "the *quantity* experienced a *b*-tupling."	i. Measuring different cacti with the same ruler. ii. Measuring the same cactus with different cacti as the ruler. Addresses reciprocal relationship. iii. Provide the student a chance to reflect on their actions in (i) and (ii). iv. Given ruler-cactus's height and growth factor, determine new cactus's height. v. Given final-cactus's height and growth factor, determine ruler-cactus's height. vi. Problems (i)-(iv) addressed the relationships between Cactus B and the other cacti. This problem introduces a new unit to measure the other cacti with. vii. Specific case using division as measurement. viii. Generalized case using division as measurement. ix. Addresses two instances of a single quantity. Can be used to lead into a discussion of the term "growth factor" and even "tuplings." x. Addresses two values of a single quantity. Can be used to continue a discussion of the term "growth factor" and even "tuplings." xi. Generalized case for division as measurement and growth factor.

Task
i.

(A) At some point in time, Sparky the cactus was this tall.	(B) After some time, Sparky's height doubled (becomes 2 times as large). Draw the resulting Sparky.	(C) After some more time, Sparky's height then quadrupled (becomes 4 times as large) from point (B). Draw the resulting Sparky.

ii. By what overall factor did Sparky grow from point (A) to point (C)?
In other words, overall Sparky's height experienced a _____-tupling.

iii. If Sparky's height becomes 3 times as large and *then* 5 times as large, overall his height will experience a _____-tupling.

iv. If Sparky's height becomes 34 times as large and *then* 57 times as large, overall his height will experience a _____-tupling.

v. If Sparky's height becomes *X* times as large and *then* *Y* times as large, overall his height will experience a_____-tupling.

Learning Goal(s)	Discussion
$\times A \times B = \times AB$: Student will understand that multiplying by *A* and then multiplying by *B* has the same overall effect as multiplying by *AB*. *Tuplings:* Student will understand that the phrase, "grow by a factor of *b*" can also be expressed as "the *quantity* b-tuples" or "the *quantity* experienced a *b*-tupling."	i. Review tupling language, growth factors. Addresses single tuplings. ii. Considers overall effect of two successive tuplings. iii. Considers overall effect of two successive tuplings. The student could draw a picture if they need it. iv. Considers overall effect of two successive tuplings that is large enough where the student will not want to draw a picture and will have to reflect on how she solved the first three tasks. v. Generalizes the overall effect of two successive tuplings.

Task (This task requires the use of the attached Geogebra Applet)
 i. Emily purchased the mystical cactus shown in the video (Geogebra Applet) on Sunday, January 1st and named the saguaro Sparky. She decided to record the displayed time-lapse video of Sparky's growth and noticed he was growing in a peculiar way. Watch the video and discuss what you observe.
 ii. Document and observe Sparky's height every: week (2 weeks, 1/7 week (day), 1.585 weeks, etc.) What changes? What stays consistent?
iii. If Emily's friend Morgan visited every Tuesday (every other Tuesday, every day, every third Tuesday, etc.) to document Sparky's growth, would she make the same claims?
iv. If Emily's friend Kevin visited every Friday (every other Friday, every day, every third Friday, etc.) to document Sparky's growth, would he make the same claims?
 v. What is the 1-week (2-week, 1/7th-week, 1.585-week, etc.) growth factor?
vi. What is the 2-tupling (4-tupling, 1.1-tupling, 3-tupling, etc.) period? In other words, how long does it take Sparky's height to become 2 (4, 1.1, 1.585, etc.) times as large?

Learning Goal(s)	Discussion
The Exponential Relationship: Student will understand that for equal changes in Quantity A, Quantity B will grow by a constant factor.	Ellis and colleagues (2012) found that students attended just to the growth factors before attending to the covarying quantities.
Tuplings and Tupling Periods: Student will understand that if the *n*-unit growth factor is *b*, or a quantity *b*-tuples in *n*-units, then the *b*-tupling period is *n* units.	Before answering the questions, students will be introduced to the Geogebra applet. During this introduction, students are encouraged to reflect on the quantities individually and then together by adjusting the viewing settings.

Task
Recall the 1-week growth factor is 2, and thus the 2-tupling period is 1 week.
  i. By what factor does Sparky grow every two (three, six) weeks? *Two (Three, Six) weeks is equivalent to how many 2-tupling periods? How else might we represent this growth factor so that we convey that two (three, six) 2-tupling periods have elapsed?*
 ii. By what factor does Sparky grow every 52 weeks (1 year)? *52 weeks is equivalent to how many 2-tupling periods? How else might we represent this growth factor so that we convey that fifty-two 2-tupling periods have elapsed?*
iii. By what factor does Sparky grow every day (1/7th of a week)? *One day is equivalent to how many 2-tupling periods? How else might we represent this growth factor so that we convey that 1/7th 2-tupling periods have elapsed?*
iv. By what factor does Sparky grow every -1 weeks? *-1 weeks is equivalent to how many 2-tupling periods? How else might we represent this growth factor so that we convey that -1 2-tupling periods have elapsed?*
 v. By what factor does Sparky grow if no time has elapsed (0 weeks)? *Zero weeks is equivalent to how many 2-tupling periods? How else might we represent this growth factor so that we convey that zero 2-tupling periods have elapsed?*
vi. By what factor does Sparky grow by every *x* weeks? x *weeks is equivalent to how many 2-tupling periods? How else might we represent this growth factor so that we convey that* x *2-tupling periods have elapsed?*
vii. Suppose a different cactus' height 17-tuples every year. By what factor will this cactus grow every week?

Learning Goal(s)	Discussion
Tuplings and Tupling Periods: (Chance to review) Student will understand that if the *n*-unit growth factor is *b*, or a quantity *b*-tuples in *n*-units, then the *b*-tupling period is *n*. *Exponent:* Student will understand that the exponent, *x*, on a number, *b*, represents the number of *b*-tupling periods. *Growth Factor Conversions:* The student will understand that the factor by which a quantity will grow over *x* *b*-tupling periods is represented as b^x .	i. This task encourages the student to think of multiple representations for values she can easily calculate. ii. This task encourages the student to see the *usefulness* for exponential notation (not necessarily a *need* because the calculation can still be done as in part (i)). iii. This task encourages the student to see the need for exponential notation (here I say *need* because the value cannot be calculated in the same way as the previous tasks). iv. This task encourages the student to develop a meaning for negative exponents. v. This task encourages the student to develop a meaning for an exponent of 0. vi. This task is meant to help the student develop the understanding that the factor by which a quantity will grow over *x* 2-tupling periods is represented as 2^x . vii. This task is meant to encourage the student to reflect on her previous work and make a growth factor conversion in a different context.

Task
Recall the 1-week growth factor is 2, and thus the 2-tupling period is 1 week. Also recall that initially (week 0) Sparky is 1 foot tall. Suppose that after *x* weeks, Sparky is *y* feet tall.
 i.   Fill in the blank: After *x* weeks, Sparky's height is ___ times as large as his height at week 0.
 ii. Use the 1-week growth factor to represent this same growth factor.
iii. Given any number of weeks, *x*, write an equation that determines the corresponding height of Sparky, *y*. *Hint: write an equation relating your answers to (i) and (ii).*

iv.   Now, suppose initially (week 0) Sparky was 3 feet tall and still doubled in size each week. Write an equation that determines *y*, Sparky's height in feet, given *x*, the number of weeks since Sparky's purchase.

 v.   Suppose a pool is being filled with water so that the volume of water in the pool 1.5-tuples every hour. At 9am, there were 15 gallons of water in the pool. Write an equation that determines the number of gallons of water in the pool, *g*, in terms of the number of hours since 9am, *h*.

Learning Goal(s)	Discussion
The Exponential Function: The function $f(x) = ab^x$ relates two quantities "varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33) where *x* represents the varying values of the first quantity, *a* represents the initial value of the second quantity, and *b* represents the 1-unit (of the first quantity) growth factor.	i. This task is meant to guide the student to represent the *x* week growth factor using Sparky's height values and division as measurement. ii. This task is meant to help the student represent the same growth factor using the number of weeks and the fact that every week Sparky's height 2-tuples. iii. This task is meant to help the student develop the equation of the exponential function for the Sparky situation. iv. This task is meant to evaluate whether or not the student has reflected on the previous three tasks and present a situation where the initial value is not 1. v. This task is meant to see if the student can apply the reasoning presented in (i)-(iv) in a new situation.

Task

   i. How many 2-tupling periods (weeks) does it take for Sparky's height to result in a 2-tupling (4-tupling, 8-tupling)?
  ii. How many 2-tupling periods (weeks) does it take for Sparky's height to result in a 3-tupling (5-tupling, 7-tupling)?
 iii. In general, $\log_b(m)$ represents the number of *b*-tupling periods needed to result in an *m*-tupling. Use this notation to represent your answers to parts (i) and (ii). Verify your answers with the applet.

Learning Goal(s)	Discussion
Logarithmic Notation: Student will understand that $\log_b(m)$ represents the number of *b*-tupling periods needed to result in an *m*-tupling	i. This task is meant to guide the student to see that the number of 2-tupling periods needed to result in an *m*-tupling is something that can be measured. ii. This task is meant to help the student begin to see the need for a way to represent the exact value. This task can also be used to analyze how the student views the relationships between tuplings. iii. Introduces the notation and has the student practice utilizing the notation. Using the applet, the student can see the usefulness of the notation. Before verifying answer using the applet, the researcher can ask the student to anticipate what she will observe when she hits enter. This will inform the researcher of how the student is viewing the notation.

Task

i.

(A) At some point in time, Sparky the cactus was this tall.	(B) After 1 week, Sparky's height doubled (2-tupled, became 2 times as large). Draw the resulting Sparky.	(C) After about 1.585 weeks, Sparky's height then tripled (3-tupled, became 3 times as large). Draw the resulting Sparky.

ii. By what factor did Sparky grow from point (A) to point (C)? How long did it take to grow by this factor?

In other words, overall Sparky's height will experience a _____-tupling in _____ weeks.

iii. If Sparky's height 3-tuples *then* 5-tuples, overall his height will experience a _____-tupling.

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a 3-tupling, the number of 2-tupling periods (weeks) needed to result in a 5-tupling, and the number of 2-tupling periods (weeks) needed to result in a 15-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____ weeks to 3-tuple and _____ weeks to 5-tuple, then it will take _____ weeks to 15-tuple.

iv. If Sparky's height 34-tuples *then* 57-tuples, overall his height will experience a _____-tupling.

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to 34-tuple, the number of 2-tupling periods (weeks) needed to 57-tuple, and the number of 2-tupling periods (weeks) needed to 1938-tuple. Write an equation representing the relationship between these three values.

In other words, if it takes _____ weeks to 34-tuple and _____ weeks to 57-tuple, then it will take _____ weeks to 1938-tuple.

v. If Sparky's height *X*-tuples *then* *Y*-tuples, overall his height will experience a _____-tupling.

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a *X*-tupling, the number of 2-tupling periods (weeks) needed to result in a *Y*-tupling, and the number of 2-tupling periods (weeks) needed to result in a *XY*-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____ weeks to *X*-tuple and _____ weeks to *Y*-tuple, then it will take _____ weeks to *XY*-tuple.

vi. Now, discuss how your equations would change had you measured in days instead of weeks.

Learning Goal(s)	Discussion
Logarithmic Property #1: $\log_b(X) + \log_b(Y) = \log_b(XY)$ The student will understand that the number of b-tupling periods needed to result in an XY-tupling is equal to the number of b-tupling periods needed to result in an X-tupling plus the number of b-tupling periods needed to result in a Y-tupling	i. Revisiting prerequisite $\times A \times B = \times AB$ ii. Attending to both covarying quantities. This task was designed to help the student recognize that multiplying the growth factors corresponds with adding the corresponding tupling periods. iii. Same as (ii), but without requiring the student to draw pictures. However, the numbers are small enough where if the student needs to reason with pictures, he can. iv. Same as (ii), but the student must reflect on previous tasks and the relationships because the numbers are large enough where the student will not want to draw a picture. v. Requires the student to generalize. vi. The overall relationship will remain the same, but the base value will change (how we measure the tupling periods).

Task

i.

(A) At some point in time, Sparky the cactus was this tall.

(B) After some time, Sparky's height 5-tupled in size. Draw the resulting Sparky.

(C) After 1 week, Sparky's height then 2-tupled in size from point (B). Draw the resulting Sparky.

ii. By what factor did Sparky grow from point (A) to point (C)? If it took Sparky approximately 3.3219 weeks to grow by this factor, how long did it take Sparky to 5-tuple?

iii. If it takes Sparky's height 3.585 weeks to experience a 12-tupling and 2 weeks to experience a 4-tupling, how long does it take for Sparky's height to experience a 3-tupling?

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a 12-tupling, the number of 2-tupling periods (weeks) needed to result in a 4-tupling, and the number of 2-tupling periods (weeks) needed to result in a 3-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____weeks to 12-tuple and _____weeks to 4-tuple, then it will take _____weeks to 3-tuple.

iv. Describe how you would determine the 17-tupling period given that the 34-tupling period is approximately 5.087 weeks

v. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in an X-tupling, the number of 2-tupling periods (weeks) needed to result in a Y-tupling, and the number of 2-tupling periods (weeks) needed to result in an X/Y-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____weeks to X-tuple and _____weeks to Y-tuple, then it will take _____weeks to X/Y-tuple.

vi. Now, discuss how your equations would change had you measured in days instead of weeks.

Learning Goal(s)	Discussion
Logarithmic Property #2: $\log_b(X) - \log_b(Y) = \log_b(X/Y)$ The student will understand that the number of *b*-tupling periods needed to result in a *X/Y*-tupling is equal to the number of *b*-tupling periods needed to result in a *X*-tupling minus the number of *b*-tupling periods needed to result in a *Y*-tupling	i. Revisiting prerequisite $\times A \times B = \times AB$ ii. Attending to both covarying quantities. This task was designed to help the student recognize that if the larger tupling period is known along with a smaller tupling period, the remaining tupling period can be determined using subtraction. iii. Same as (ii), but without requiring the student to draw pictures. However, the numbers are small enough where if the student needs to reason with pictures, he can. iv. Same as (ii), but the student must reflect on previous tasks and the relationships because the numbers are large enough where the student will not want to draw a picture. The student is also required to calculate the ratio to determine the missing tupling. v. Requires the student to reflect on his previous work and make a generalization. vi. The overall relationship will remain the same, but the base value will change (how we measure the tupling periods).

Task

Recall that the 2-tupling period is 1 week.

i. Determine the $2^4 = 16$-tupling period. *What does the 4 as the exponent represent?*

ii. The 16-tupling period is how many times as large as the 2-tupling period?

iii. Given that the quadrupling or 4-tupling period is 2 weeks, describe how you would determine the 4^{50}-tupling period. *What does the 50 as the exponent represent?*

iv. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in an *X*-tupling and the number of 2-tupling periods (weeks) needed to result in an X^y-tupling. Write an equation representing the relationship between these two values.

v. Now, discuss how your equations would change had you measured in days instead of weeks.

Learning Goal(s)	Discussion
Logarithmic Property #3: $\log_b(X^y) = y\log_b(X)$ The student will understand that the number of *b*-tupling periods needed to result in an X^y-tupling is *y* times as large as the number of *b*-tupling periods needed to result in an *X*-tupling	i. Review. *Recall, the exponent of 4 represents the number of 2-tupling periods that have elapsed.* ii. Has the student multiplicatively compare the two values. The values are small enough that the student could draw a picture if he needed to. iii. Requires that the student reflects on his work in (ii) iv. Requires the student to make a generalization. v. The overall relationship will remain the same, but the base value will change (how we measure the tupling periods).

Task

The 10-tupling period is about 3.3 weeks and the 15-tupling period is about 3.9 weeks.

  i. The 15-tupling period is how many times as large as the 10-tupling period?

  ii. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to 10-tuple and the number of 2-tupling periods (weeks) needed to 15-tuple. Write an equation representing the relationship between these two values.

  iii. How would your answer to (i) change if the two periods been measured in days? In years? How would your answer to (i) remain the same if the two periods been measured in days? In years? Explain.

  iv. Use logarithmic notation to represent the number of 1.104-tupling periods (days) needed to 10-tuple and the number of 1.104-tupling periods (days) needed to 15-tuple. Write an equation representing the relationship between these two values.

  v. Compare your answers in (ii) and (iv).

  vi. Develop an equation relating $\log_b(X)$, $\log_b(Y)$, $\log_c(X)$, and $\log_c(Y)$ (for $b,c,X,Y>0$)

Learning Goal(s)	Discussion

Change of Base Rule (Property 4):
$$\frac{\log_b(X)}{\log_b(Y)} = \frac{\log_c(X)}{\log_c(Y)}$$
The student will understand that the *X*-tupling period will always be $\dfrac{\log_b(X)}{\log_b(Y)}$ times as large as the *Y*-tupling period, for any value *b*.
	i. Review on division as measurement.
	  ii. Representing work in (i) using notation.
	  iii. Student must reflect on role of units in measuring relative size. As long as the units used to measure each quantity are the same, the ratio will be constant.
	  iv. Builds off reasoning in (iii).
	  v. Reflect on relationship between (ii) and (iv)
	  vi. Requires the student to generalize.

Task

i. What does *y* represent in the expression 2^y?

ii. Represent the number of 2-tupling periods needed to result in a 2^y-tupling using logarithmic notation.

iii. Represent the number of 2-tupling periods needed to result in a 2^y-tupling without using logarithmic notation.

iv. Write an equation relating your answers in (ii) and (iii).

v. Simplify $\log_b(b^x)$

vi. What does *y* represent in the expression $2^y = x$?

vii. Represent the number of 2-tupling periods needed to result in an *x*-tupling using logarithmic notation.

viii. Simplify $2^{\log_2(x)}$

ix. Simplify $b^{\log_b(x)}$

Learning Goal(s)	Discussion
Logarithmic Property 5: $\log_b(b^x) = x$ The number of *b*-tupling periods needed to result in an *b*-tupling *x* times is *x*. *Logarithmic Property 6:* $b^{\log_b(x)} = x$ *b*-tupling the number of *b*-tupling periods needed to result in an *x*-tupling is equivalent to an *x*-tupling.	i. Review meaning for exponents. ii. Review logarithmic notation. iii. Review meaning for exponents. iv. Special case of property 5: $\log_2(2^y) = y$ v. Generalize from (i-iv). vi. Review meaning for exponents. vii. Review logarithmic notation. viii. Special case of property 6: $2^{\log_2(x)} = x$ xi. Generalize from (vi-viii)

Task

Recall $\log_b(x)$ represents the number of *b*-tupling periods needed to result in an *x*-tupling.

  i. Describe how $\log_2(x)$ ($\log_{1/2}(x)$) varies as *x* varies.

  ii. Graph the relationship of $\log_2(x)$ ($\log_{1/2}(x)$) with respect to *x*. If necessary, create a table of values.

  iii. T/F: Every value of *x* determines exactly one value of $\log_2(x)$. Explain your answer.

Learning Goal(s)	Discussion
The Logarithmic Function: The student will understand that as the x-tupling value and the number of *b*-tupling periods needed to result in an *x*-tupling vary simultaneously, "there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33).	i. Addresses covariation ii. Function in graphical (and table) representation iii. Addresses functional relationship (Note: usually by this lesson, students have had a few weeks with functions, function notation, etc.).

THREE PAPERS

In this section, I present the three papers of my dissertation study. I began my work with the idea of logarithm by first examining its historical development. I then leveraged the insights of this literature review to perform a conceptual analysis for the idea of logarithm based on quantitative and covariational reasoning. This conceptual analysis informed the hypothetical learning trajectories and task design used in two consecutive studies and evolved based on the findings of each study. The first study examines two students' development of concepts foundational to the idea of logarithm. This paper discusses two essential understandings that were revealed to be problematic and essential for students' development of productive meanings for exponents, logarithms and logarithmic properties. The findings of this study informed my later work to support students in understanding logarithms, their properties and logarithmic functions. The second study examines two students' development of the idea of logarithm. This paper describes the reasoning abilities two students exhibited as they engaged with tasks designed to foster their construction of more productive meanings for the idea of logarithm. The findings of this study provide novel insights for supporting students in understanding the idea of logarithm meaningfully. Finally, I conclude this section with my current conceptual analysis of what is involved in learning and understanding the idea of logarithm. The literature review and conceptual analysis contributes novel and useful information for curriculum developers, instructors, and other researchers studying student learning of this idea.

PAPER 1:

SPARKY THE SAGUARO: TEACHING EXPERIMENTS EXAMINING STUDENTS'

DEVELOPMENT OF CONCEPTS FOUNDATIONAL TO THE IDEA OF

LOGARITHM

ABSTRACT

There have been a number of studies that have examined students' difficulties in understanding the idea of logarithm and the effectiveness of non-traditional interventions. However, few studies have focused on the understandings students develop while participating in conceptually oriented exponential and logarithmic lessons. Furthermore, little information has been reported about the understandings foundational to the idea of logarithm that students need for constructing a robust meaning for logarithm. This study explores two undergraduate precalculus students' understandings of concepts foundational to the idea of logarithm as they individually completed an exploratory lesson on exponential and logarithmic functions. Over several weeks, the students participated in individual teaching experiments that focused on Sparky – a mystical saguaro that doubled in height every week. The exponential lesson was centered on growth factors and tupling (e.g., doubling, tripling) periods in an effort to support the students in developing the understandings necessary to discuss logarithms and logarithmic properties meaningfully. This paper discusses two essential understandings that were revealed to be problematic and essential for students' development of productive meanings for exponents, logarithms and logarithmic properties. The findings of this study may inform future work to support students in understanding logarithms,

their properties and logarithmic functions.

INTRODUCTION

Studies have revealed that most students have weak conceptions of the idea of logarithm even after experiencing instruction aimed at teaching this idea (Weber, 2002; Kenney, 2005). In an effort to support student learning of the idea of logarithm, some teachers have tried incorporating the history of logarithms into their lessons (Panagiotou, 2011), changing the notation (Hammack & Lyons, 1995), and approximating logarithms with repeated division (Vos & Espedal, 2016), yet researchers continue to report that many students struggle to develop coherent understandings for logarithmic notation, properties and the logarithmic function (Kenney, 2005; Strom, 2006; Weber, 2002; Gol Tabaghi, 2007). Adding to the problem, standard curriculum provides little support for helping students (or teachers) construct a strong meaning for what a logarithm represents. A review of 5 precalculus and calculus texts[8] revealed that $y = \log_b(x)$ was introduced as the inverse to $y = b^x$, with the properties of logarithms simply stated shortly after. This top down approach of beginning with a formal definition of logarithm has not been effective. In response, I propose to investigate strategies for helping students develop productive meanings for exponents, exponential functions and other concepts

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[8] (1) Spiegler, Adam, "Functions Modeling Change: A Preparation for Calculus" (2011). *Faculty Books*. 92. (2&3) Stewart, J. (2010). *Calculus: early transcendentals*. Cengage Learning. [2nd and 6th editions] (4) Anton, H. Calculus with Analytic Geometry, 1988. (5) Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*.

foundational to the idea of logarithm that might support students in understanding logarithms and their properties.

I argue that understanding the idea of logarithm requires more than just memorizing and applying Euler's definition. To understand the idea of logarithm meaningfully, one must first conceptualize tuplings[9] and their corresponding tupling periods in exponential situations. That is, one must attend to the multiplicative growth of the output quantity of an exponential function while also attending to the corresponding changes in the input quantity of an exponential function. After conceptualizing these quantities, one must attend to how they vary together and imagine one tupling period relative to another. Therefore, I claim that it is necessary for students to engage in quantitative reasoning and covariational reasoning to understand the idea of logarithm coherently. It is well documented that students who engage in quantitative reasoning are more likely to reason productively while working on conceptually challenging tasks (Castillo-Garsow, 2010; Ellis, 2007; Hackenberg, 2010; Moore, 2010; Moore, K. C., & Carlson, M. P., 2012; Saldanha & Thompson, 1998; Thompson, 1993, 1994b). Furthermore, Thompson and Carlson (2017) have argued that covariational reasoning is an essential way of thinking for constructing meaningful function formulas and graphs. Therefore, if a goal for students is for them to utilize the idea of logarithm as they work through conceptually challenging tasks, then it would follow that they should develop an understanding of the idea of logarithm that is based on their conceptualizing and representing quantities, while also attending to how the quantities' values vary in tandem.

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[9] A *b*-tupling occurs when a quantity becomes *b* times as large. Therefore, a *b*-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become *b* times as large. We say that the second quantity has *b-tupled* over some interval of change of the first quantity.

This paper reports on two undergraduate precalculus students' understandings of concepts foundational to the idea of logarithm as they individually worked through an exploratory lesson on exponential and logarithmic functions. The findings of this study revealed two essential understandings that students must conceptualize in order to hold a productive meaning for the idea of logarithm. That is, students must first conceptualize that multiplying by A, then multiplying the resulting value by B has the same effect as multiplying the initial value by AB, and second that an exponent on a value b represents the number of elapsed b-tupling periods. When discussing the results I illustrate the role of these ideas in constructing logarithmic expressions, logarithmic properties, and logarithmic functions. I conclude by discussing the importance of conceptualizing these two essential understandings in the context of the lesson.

RESEARCH QUESTION

The primary question motivating this investigation is:

- What understandings foundational to the idea of logarithm must students develop during an exponential and logarithmic instructional sequence aimed at supporting students in acquiring a strong meaning for the idea of logarithm?

LITERATURE REVIEW

Quantitative Reasoning

Smith and Thompson (2007) argue that if students are to utilize algebraic notation to assist them in representing ideas and reasoning productively, then their ideas and reasoning must become sophisticated enough to justify the use of the notation. It thus seems reasonable that logarithmic notation and properties should be introduced as a way

to represent an idea that students have first conceptualized. If a goal for students is for them to utilize the idea of logarithm as they work through conceptually challenging tasks, then it would follow that they should develop an understanding of the idea of logarithm that is based on their conceptualizing and representing quantities. In this section, I elaborate my perspective on what is involved in conceptualizing and reasoning with quantities.

A quantity is a mental construction of a measurable attribute of an object (Thompson, 1990, 1993, 1994a, 2011). That is, quantities do not exist out in the world; they are created in the mind of an individual when she conceptualizes measuring some quality of an object, such as a person's height or the person's distance from home as she drives to work (Thompson, 2011). One is said to participate in the act of quantification when, after conceptualizing a quantity, she conceptualizes the attribute's unit of measure such that the attribute's measure is proportional to its unit (Thompson, 2011); one's distance from home is some number of times as large as one foot. The numerical measurement that a quantity assumes is referred to as a value. When the measurable attribute of an object doesn't change throughout a situation, it is called a constant or fixed quantity. On the other hand, if the value of a quantity changes throughout a situation, we call it a varying quantity.

Mathematics is often used to model and describe how two or more quantities relate. A quantitative operation occurs in the mind of an individual and is when "one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 2011, pg. 9). When one conceives of three quantities related by means of a quantitative operation, he has conceptualized a quantitative relationship. Changing which

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quantity is determined by the quantitative operation changes the quantitative relationship

(Thompson, 1990). When one analyzes a situation and assigns his observations (i.e.

quantities, quantitative relationships) to a network of quantities and quantitative

relationships, called a quantitative structure, he is said to engage in quantitative reasoning

(Thompson, 1988, 1990, 1993, 1994a, 2011).

Research Literature on Students' Understandings of Exponents and Exponential Functions

A student who conceptualizes exponentiation only as repeated multiplication will

likely be limited to interpreting natural number exponents. In cases when an exponent is a

non-natural real number, say $-\pi$, the interpretation of exponentiation as repeated

multiplication is ineffective. While some researchers advocate a repeated multiplication

approach (e.g. Goldin & Herscovics, 1991; Weber, 2002), others believe this approach

limits students (e.g. Ellis, Ozgur, Kulow, Williams & Amidon, 2015; Davis, 2009;

Confrey & Smith, 1995). In particular, Confrey and Smith (1995) argue that the standard

way of teaching multiplication through repeated addition is inadequate for describing a

variety of situations such as magnification, multiplicative parts (i.e. finding a fraction of a

split), reinitializing and creating an array. Weber (2002) proposed that students first

understand exponentiation as a process (in terms of APOS theory) before viewing

exponential and logarithmic expressions as the result of applying the process. A student

with a process conception of exponent will be able to generalize her understanding to

cases in which the exponent is a non-natural number. Specifically, Weber stressed to his

students that "b^x represents the number that is the product of x many factors of b." With

this conception, we can describe $9^{2.5}$ to be the number that is the product of two and a

half factors of 9, while under the view of repeated multiplication, a student might write "

$9 \cdot 9 \cdot ?$ ". If a coherent understanding of exponential functions (and later logarithmic

functions) is desired of our students, it is imperative that they have productive meanings

for exponents.

Confrey and Smith (1995) claimed that exponential function learning involves

mental actions of splitting and covariation. The authors describe splitting as a

multiplicative operation different from repeated addition that arises in situations

involving magnification, similarity and sharing, for example. Direction in the splitting

structure suggests either multiplication or division (doubling vs. halving, etc.). The

authors provided empirical evidence (students utilize the idea of halving to determine the

area per child on a playground) that they claim suggests that splitting is an intuitive

construct for multiplication and division. Confrey and Smith described the covariation

approach as considering two sets of data and the relationship between the sets. That is,

this approach encourages the description of how one quantity varies in relation to another

and allows for the discussion of rates of change, differences, and accumulation. In

particular, exponential functions can be characterized as having constant multiplicative

rates of change (Ellis et al., 2015). Confrey and Smith described how to produce

exponential functions using splitting and covariation and concluded that the use of

covariation, splitting and the idea of the isomorphism between the additive and

multiplicative worlds helps avoid concealing the relevant splitting unit/base that relates to

the functional situation and helps avoid an overreliance on algebraic representation.

This study's intervention expanded on Ellis et al.'s (2015) small-scale teaching

experiment that examined continuous quantities covarying exponentially. The three middle school participants were asked to consider a scenario of a cactus named Jactus whose height doubled every week. Eventually, the initial height, weekly growth factor and amount of time needed to grow by the provided factor were altered to provide variety. The authors noticed three significant shifts in the students' thinking over the course of the study. At first, the students attended only to Jactus' height and concluded he grew by means of repeated multiplication. Eventually, the students began to coordinate this repeated multiplication with the corresponding changes in the amount of time that elapsed. The second shift consisted of students determining the factor by which Jactus' height grew for varying changes in the number of weeks by means of calculating the ratio of two heights. Finally, the third shift involved the students generalizing the reasoning noted in the second shift to include non-natural exponents (i.e. to determine the 1-day growth factor). The authors noted that a student's ability to coordinate the growth factor (or ratio of height values) with the changes in elapsed time contributed to the student successfully defining the relationship between the elapsed time and Jactus' height. This study leveraged findings from Ellis et al.'s study of Jactus the Cactus to promote more meaningful discussions on logarithms.

Research Literature on Students' Understandings of Logarithms

There have been a number of studies that have examined students' difficulties in understanding the idea of logarithm (i.e. logarithmic notation, logarithmic properties, the logarithmic function) and the effectiveness of non-traditional interventions. However, few studies have focused on the understandings students develop while participating in conceptually oriented exponential and logarithmic lessons. Furthermore, little

information has been reported about the understandings foundational to the idea of logarithm students must develop to understand the idea of logarithm well.

The difficulties students have with developing coherent understandings of the idea of logarithm is likely multidimensional. In a typical precalculus course, logarithmic functions are the first function family introduced that does not specify a function rule, leaving students with no direction on how to determine the value of $\log_b(m)$ given values of b and m. Instead, students are expected to either apply their understandings of the idea of logarithm, exponents and powers to approximate the value of a logarithm for some input value, or, more commonly, use technology to calculate its value. In fact, the Common Core State Standards (CCSS) for mathematics have as one of the goals for high school students that they are able to write the corresponding logarithmic equation given an exponential equation, and calculate the value using technology (only for bases 2, 10 and e). Logarithmic functions are also the first function family that students encounter in which the function name is not a single letter. This may introduce an added complexity for students who already struggle in using function notation (Thompson, 2013; Musgrave & Thompson, 2014). Additionally, aspects of logarithmic notation have a dual nature to them (Kenney, 2005). For example, in $y = \log_b(x)$, b, x, and y may take on a variety of meanings to an individual – b often takes on the form of a parameter (staying consistent within the context of a problem, but varying from problem to problem), x serves as the input variable to the logarithmic function and is a tupling[10], and y serves as the output

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[10] Recall: A b-tupling occurs when a quantity becomes b times as large. Therefore, a b-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become b times as large. We say that the second quantity has b-*tupled* over some interval of change of the first quantity.

variable to the logarithmic function and is the number of *b*-tupling periods needed to *x*-tuple.

In addition to these unavoidable complexities, studies have shown that students struggle to understand, explain and apply the properties of logarithms (Weber, 2002; Kenney, 2005; Gol Tabaghi, 2007). Some students in Kenney's (2005) study experienced difficulties simplifying equations involving at least two logarithmic expressions shortly after successfully applying Euler's definition with single logarithmic expressions. This suggests that the standard approach to introduce students to logarithms by giving them the statement that $\log_b(x) = y \leftrightarrow b^y = x$ is an insufficient foundation for students trying to develop an understanding of the properties of logarithms. Therefore, it seems reasonable to consider that if students continue to have difficulties in understanding the idea of logarithm (i.e., logarithmic notation, logarithmic properties, the logarithmic function), they may still need to develop some understanding(s) foundational to the topic. This study was designed to shed light on conceptions that build the foundation for the idea of logarithm. An additional goal of this study was to inform curriculum so that students can build more coherent understandings of the idea of logarithm.

CONCEPTUAL ANALYSIS

In this section, I present the conceptual analysis that guided the design of my intervention and goals for student learning of the idea of logarithm. In general, conceptual analysis is used to describe the mental operations that might explain why people think the way that they do (Glasersfeld, 1995). In this conceptual analysis, I convey my understanding of the idea of logarithm. In doing so, I focus on major

constructions that need to be made as one develops the idea of logarithm for themselves.

For example, I defined $\log_b(m)$ to represent the number of b-tupling periods it takes to

result in an m-tupling. To illustrate the usefulness of this definition, consider a task and

solution (Figure 1.1).

The starfish population in Hawaii has increased 20% per year since 1990 and is modeled by the function $f(t) = 1500(1.2)^t$, with t representing the number of years since 1990. Determine how long it will take for the population to reach 3480 starfish.

$$(1) \quad f(t) = 1500(1.2)^t$$

$$(2) \quad 3480 = 1500(1.2)^t$$

$$(3) \quad \frac{3480}{1500} = (1.2)^t$$

$$(4) \quad 2.32 = (1.2)^t$$

$$(5) \quad t = \log_{1.2}(2.32)$$

$$(6) \quad t \approx 4.6 \text{ years}$$

Figure 1.1. A Solution to an Exponential Function

In line (3), we see the ratio $\frac{3480}{1500}$. This calculates the factor by which the initial

value of the exponential function grows. In particular, in the unspecified amount of time,

the population of starfish grows by a factor of 2.32, or 2.32-tuples. Therefore, to

determine precisely how long it takes for the population to 2.32-tuple, we must utilize the

fact that the population of starfish 1.2-tuples every year, and ask the question, "How

many years (1.2-tupling periods) does it take to 2.32-tuple?" Using logarithmic notation,

we can represent this exact value as $\log_{1.2}(2.32)$. Then, with the use of technology, we

can determine $\log_{1.2}(2.32) \approx 4.6$, and conclude that after approximately 4.6 1.2-tupling

periods, or years, the starfish population will reach 3480 starfish. This definition for logarithm relies on the understanding that a designated tupling-period can be used to measure a different tupling-period. Of course, in order to discuss these ideas in a meaningful manner, the student must also develop a meaning for division as measurement, growth factors, tuplings and tupling-periods, and logarithmic notation as determining how many *base*-tupling periods are needed to grow by another factor.

The meanings I hypothesize to be critical for understanding exponential and logarithmic ideas are further clarified in the following Taxonomy (Table 1.1). The table provides a more detailed description of the specific ways of thinking and understandings that are productive for students to construct in the process of learning about logarithms and logarithmic functions. This paper describes two conceptions that assist students in developing a number of these desired understandings (by means of more fine grained constructions).

Table 1.1

Taxonomy of the Idea of Logarithm

Conceptions related to the idea of logarithm	Desired understanding
Division as measurement	To measure a value of Quantity A in terms of a value of Quantity B, we write $\dfrac{\text{Value of Quantity A}}{\text{Value of Quantity B}}$. If $\dfrac{\text{Value of Quantity A}}{\text{Value of Quantity B}} = m$, we say Quantity A is m times as large as Quantity B.
Growth Factor	When coordinating the values of two quantities, if the value of the first quantity increases by n-units while the next value of the second quantity is m times as large as its current value, then the n-unit growth factor is m.

Exponential growth	When relating two continuous quantities, Quantity A and Quantity B, if for equal changes in Quantity A, Quantity B grows by a constant factor, then the two quantities have an exponential relationship.
Tuples (VERB)	If the value of a quantity becomes m times as large, we say the quantity's value m-tuples.
Tuplings (NOUN)	An m-tupling is the event in which the value of a quantity becomes m times as large.
Tupling period	An m-tupling period is the amount of change in the independent quantity needed for the dependent quantity to become m times as large.
Exponent (on a value, b)	The number of elapsed b-tupling periods. Written where x is the number of elapsed b-tupling periods.
Growth Factor Conversion	The factor by which a quantity will grow over x, b-tupling periods is represented as b^x. If $c^1 = b^x$, then one c-tupling period is the same as x b-tupling periods.
The Exponential Function	The function $f(x) = ab^x$ relates two quantities "varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33) where x represents the varying values of the first quantity, a represents the initial value of the second quantity, and b represents the 1-unit (of the first quantity) growth factor.
Logarithmic Notation	$\log_b(X)$ represents the number of b-tupling periods it takes (the initial value of an exponential function) to result in an X-tupling.
LP1: $\log_b(X) + \log_b(Y) = \log_b(XY)$	The number of b-tupling periods needed to result in an XY-tupling is the same as the number of b-tupling periods needed to result in an X-tupling plus the number of b-tupling periods needed to result in a Y-tupling.
LP2: $\log_b(X) - \log_b(Y) = \log_b(X/Y)$	The number of b-tupling periods needed to result in an X/Y-tupling is the same as the number of b-tupling periods needed to result in an X-tupling minus the number of b-tupling periods needed to result in a Y-tupling.

LP3: $\log_b(X^y) = y\log_b(X)$	The number of b-tupling periods needed to experience an X^y-tupling is y times as large as the number of b-tupling periods needed to experience an X-tupling.
LP4: $$\log_b(X) = \frac{\log_b(X)}{\log_b(b)} = \frac{\log_c(X)}{\log_c(b)}$$	The X-tupling period will always be k times as large as the b-tupling period (this value does not depend on the unit chosen to measure both the X- and b-tupling periods).
LP5: $\log_b(b^x) = x$	The number of b-tupling periods needed to experience a b-tupling, x times, is x.
LP6: $b^{\log_b(x)} = x$	If a value b-tuples $\log_b(x)$ times, the number of b-tupling periods needed to result in an x-tupling, the value will x-tuple.
The Logarithmic Function	A covarying relationship between an x-tupling and the number of b-tupling periods needed to experience an x-tupling ($\log_b(x)$). These two quantities vary in such a way that every value of the x-tupling determines exactly one value of the number of b-tupling periods needed to experience an x-tupling.

This Taxonomy highlights the reasoning abilities and understandings that are included in my hypothetical learning trajectory (HLT) (Simon, 1995; Simon & Tzur, 2004) for learning the idea of logarithm. My HLTs consisted of a list of learning goals for students, tasks intended to promote such learning goals, and hypotheses about student learning within the mathematical context. The task associated with each learning goal typically progressed through four stages based on my hypotheses of student learning: (1) Activity Problem – offers a starting point for students, (2) Optional Activity Problem – encourages student to consider relationships between quantities and effects of previous actions but can still be verified by engaging with the activity, (3) Non-activity Problem – encourages student to reflect on his thinking as he engaged with the previous problems while considering relationships between quantities, (4) Abstract Problem – encourages

student to generalize through reflection on activity-effect relationships. This progression was specifically designed to provide the student opportunities to advance and strengthen her thinking while reflecting on the preceding questions. This decision was guided by Simon and Tzur's (2004) emphasis that a student learns (develops new ways of thinking) when she reflects on her actions and their effects when completing tasks and was inspired by their specific task sequence on equivalent fractions.

THEORETICAL PERSPECTIVE

This study investigated the ways of thinking that are needed when learning the idea of logarithm. The intention is not to classify how every student will come to learn the idea of logarithm, but rather to model the mathematical realities of individual students. Doing so will initiate a conversation of epistemic students one might encounter while teaching the idea of logarithm. The theoretical perspective used for this study is radical constructivism (Glasersfeld, 1995), which proposes that knowledge is constructed in the mind of an individual and therefore cannot be directly accessed by anyone else. Under this perspective, researchers can, at best, attempt to form a model of students' thinking (Steffe & Thompson 2000a). A model is considered reliable when the student's utterances, written work, and movements are in alignment with the model and does not necessarily have to be mathematically correct. That is, if the subject responds in a way that is mathematically incorrect, the researcher will be interested in modeling how the student was thinking for his claims to make sense to him. When a researcher develops such models, she is trying to model the student's cognitive structures that comprise knowledge, known as schemes. These structures are organizations of mental actions or mental operations (reversible actions) and may even be complex and contain other

schemes (Piaget, 2001). An action is "all movement, all thought, or all emotion – [that] responds to a need" (Piaget, 1967, p. 6). Researchers rely on a student's observable actions when attempting to form models of his schemes, such as utterances, written work, movements or body language.

Researchers who are interested in using the teaching experiment methodology to model student learning (i.e. cognitive structuring and restructuring) should make sure to provide students with opportunities for reflection (Derry, 1996). The goal of my study was to model my subjects' knowledge development of concepts foundational to the idea of logarithm as each subject completed lessons in a teaching experiment designed to advance her meanings. My data collection and analysis focused on understanding and characterizing the meanings the students constructed as they engaged in tasks and responded to questions that provided opportunities for reflection.

METHODOLOGY

For this study, I conducted two consecutive teaching experiments (Steffe & Thompson, 2000a) that focused on advancing and characterizing students' ways of thinking as they completed lessons that were designed to support their understanding of concepts foundational to the idea of logarithm. I recruited two precalculus undergraduate students, Lexi and Aaliyah (both pseudonyms), to participate in the teaching experiments. Lexi participated in four 1.5-hour teaching episodes over the course of a three-week period as a substitute for attending class on exponential and logarithmic ideas. Her grade in the class at the start of the interviews was a D-. I selected Lexi because she was motivated and was verbal, and I was interested in identifying foundational ways of thinking that might benefit all students in learning the idea of logarithm. Aaliyah, on the

other hand, participated in the teaching experiment after attending her classes on the two topics. We met 7 times over the course of a 3.5-week period for approximately 1.5 hours each session. Her grade in the class at the start of the interviews was an A.

Prior to the start of each teaching experiment, I updated my hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004) for the idea of logarithm. I referred to these hypothetical learning trajectories as I developed and upgraded the progression of tasks used for each teaching experiment. The instructional sequence designed as the focus for this study evolved from the conceptually-based exponential situation designed by Ellis et al. (2012, 2015) entitled Jactus the Cactus – which examined a mystical cactus whose height doubled in size each week. I also provided the students with tasks unrelated to the Sparky situation in order to supplement the instructional sequence and provide opportunities for the students to continue to advance their thinking with alternative scenarios. All of the tasks used in this study were designed to support the subjects in learning the foundational ideas of exponential functions and to promote a contextual interpretation of the idea of logarithm before introducing a generalized form. Figure 1.2 provides an overview of the flow of both teaching experiments. My breakdown of what it means to learn the idea of logarithm is centered on the ideas of tuplings and tupling periods. These ideas stem from a discussion on growth factors. In exponential lessons, however, information about the growth factor is often provided through a discussion on a set percent change, or by representing the percent a new output value is of a current output value in an exponential function. Therefore, I began the teaching experiments with a brief discussion on determining percentages of values and progressed to determine the corresponding growth factor given a quantity's growth expressed as a percent. Due to the

adaptive nature of teaching experiments, I do not examine each and every task prepared

for and/or used in this study in this section (see Appendix for planned tasks). However,

throughout the presentation of results, I describe tasks that best reveal the student

learning that the tasks were designed to address.
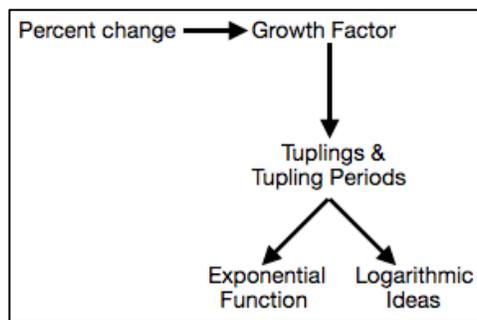
Figure 1.2. General Flow of the Teaching Experiments

       Throughout the teaching experiment, I often prompted the students to compare

Sparky's height at two different instances of time. To accompany the Sparky the Saguaro

tasks, I designed a Geogebra applet that displayed a dynamic image of Sparky's height

and time elapsed since January 1st. The applet was designed to provide a variety of

viewing options. The students could view Sparky grow as if watching a time-lapse video,

observe his height above the corresponding elapsed time since his purchase, document

his height every m weeks, and document his height for any m-week change with the

additional option of displaying "measuring lines" to help determine Sparky's current

height in terms of "how many times as tall Sparky is as compared to his previous height"

(Figure 1.3). Throughout the teaching experiments I used these displays to explore and

advance my subject's understanding of tuplings and tupling periods.
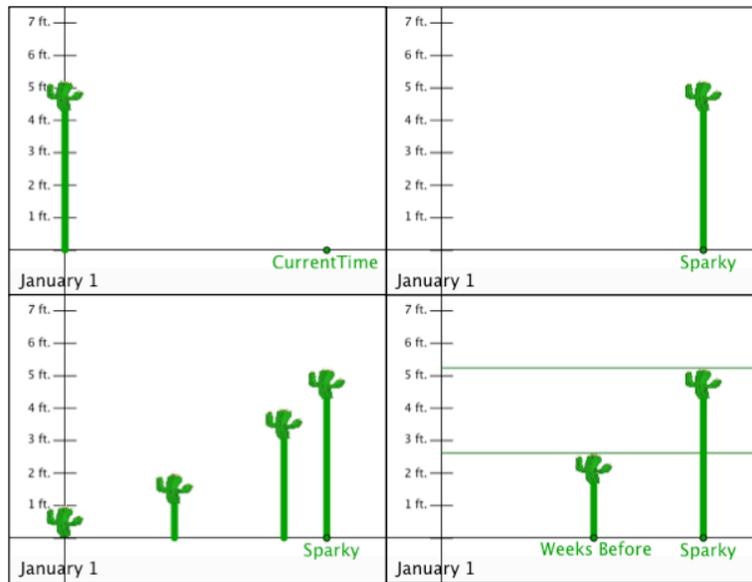
Figure 1.3. Different Display Options in Sparky the Saguaro Geogebra Applet

Following each teaching episode, I conducted a retrospective analysis (Steffe &

Thompson, 2000a) and analyzed the students' actions (verbal, written, and motions)

following an open, axial and selective coding approach (Strauss & Corbin, 1998) in an

attempt to develop models of student thinking and to inform future sessions. As an

example, I considered the students' use and explanation of the Geogebra applet images in

the context of their solutions to gain insights into their conceptions of the covarying

quantities in the situation. During this analysis stage, I watched the recordings of each

interview and made note of shifts in the student's thinking or moments when the student

made an essential mistake (Steffe & Thompson, 2000a). In the subsequent episodes I

tested my hypotheses, modified my claims as needed, and asked questions I thought

would support my subject in both confronting problematic conceptions and developing

desirable conceptions and ways of thinking (as described in my conceptual analysis).

Following the teaching experiments, I revisited every episode again to refine my

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categorizations. The subjects were not asked to complete assignments between teaching episodes. The results describe the thinking that my subjects revealed as they engaged with tasks designed to support their learning the idea of logarithm.

RESULTS

This section presents results from analyzing video data of Lexi and Aaliyah as they independently completed tasks in the exponential lesson.

Foundational Understanding #1: Multiplying by *A*, then multiplying the resulting value by *B*, has the same effect as multiplying by *AB* ($\times A \times B = \times AB$)

This section examines clips from the teaching episodes that suggest both Lexi and Aaliyah experienced difficulties in viewing the overall effect of multiplying by *A* and then by *B* as being the same as multiplying by *AB*. I also report on the students' thinking as they completed tasks that I designed to support them in constructing a productive meaning for multiplicative growth.

Teaching Experiment #1: Lexi's Experiences Involving Foundational Understanding #1

Recall that my teaching experiment began by prompting the students to compare Sparky's height at different moments and to describe a new height as a percent of an old height. During the first teaching episode I noticed that when Lexi was asked to determine n% of a value, she didn't directly multiply the value by $n/100^{\text{ths}}$, but rather determined either 1% or 10% of the reference value and then scaled up the resulting value accordingly. For example, during a supplementary task, I asked Lexi to determine 73% of $27. Lexi began by dividing $27 by 100 to determine 1% of $27, and then multiplied the resulting value by 73 to determine $19.71 was 73% of $27. There was no evidence to

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suggest that Lexi viewed the resulting value ($19.71) as being $73/100^{ths}$ (or 0.73) of the

initial value ($27). Lexi's dominant meaning for percentages allowed her to answer the

questions I posed. However, I hypothesized that her strong calculational orientation

(Thompson, Philipp, Thompson, & Boyd, 1994) and the weaknesses in her meaning for

percent and what is involved in determining a percent of a number would make it

difficult for her to determine the corresponding growth factor in the situation.

To better understand and advance her meaning for percent during the second

teaching episode, I presented Lexi with the following two questions:

1. Suppose the division button on your calculator wasn't working. How would you
   determine 1% of $45.67?
2. Suppose the division button on your calculator wasn't working. How would you
   determine 73% of $45.67?

The purpose of this task was to support Lexi in conceptualizing what it means to

determine n% of a number. In particular I hoped that she would see that n% of a number

is $n/100^{ths}$ of the number being referenced. She responded to the first question by stating

that she could divide $45.67 by 100 to calculate 1% of $45.67. I then reminded her that

she should assume the division button on the calculator was broken and that she needed

to come up with a different way to calculate 1% of $45.67. Lexi's next response was to

multiply $45.67 by 1/100 by entering 1/100 into the calculator, again making use of the

division button. I followed by asking her, "What is another way to represent 1/100?" and

she responded, "0.2? 0.1? 0.01?" – eventually settling on 0.01. Lexi's statement suggests

that she was uncertain about using 0.01 to represent $1/100^{th}$. When attempting the second

question of the task that prompted her to determine a larger percentage of $45.67, Lexi

stated, "Don't we just do the same thing?" She followed by saying that she could

determine 73% of $45.67 by multiplying $45.67 by 0.73. Lexi's attention to the results of her actions for the first problem suggests that she did not consider what the 0.73 represented in the situation. I then asked Lexi how she might calculate the same value by using her answer in part (1). She explained that she would just have to multiply the 1% value by 73 to calculate 73% of $45.67. I attempted to draw Lexi's attention to the actions she performed in hopes that she would see that multiplying by 0.73 has the same effect as multiplying by 0.01 and then by 73. That is, multiplying a value by 0.73 produces 73 1/100ths of that value. Instead, Lexi claimed that one method (the first) uses the 1% and the other (multiplying by 0.73) doesn't "necessarily need the 1% to find (the output)." Lexi's description of the two methods suggests that she did not view these two approaches as equivalent. In other words, Lexi's actions suggest she viewed multiplying by 0.01 and then by 73 as being quantitatively different than multiplying by 0.73.

During the remaining time in the second teaching episode, Lexi worked on a task that prompted her to determine different growth factors to represent Sparky the Saguaro's growth[11] over different periods of elapsed time. In an attempt to determine the 3-week growth factor, Lexi began by noting Sparky's initial height of one foot at week zero and then claimed, "three time(s)– no, every week it's doubling, or times two for the height. So to get to week three, you'd say it's like, you wouldn't say 6 times as large – that wouldn't make sense. I feel like you would say 3 times as large – that doesn't make sense either." This response suggests that Lexi first considered multiplying the 1-week growth factor (2) by the number of elapsed weeks (3) to calculate the 3-week growth factor. However, she quickly ruled out that option and looked to other values appearing in the

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[11] Recall Sparky the Saguaro is the mystical cactus whose height doubles in size each week.

situation. Lexi then appeared to observe the height of the cactus three weeks after its purchase and eventually concluded that at the end of week 3 Sparky would be 8 times as large as the initial Sparky. However, there was no evidence to suggest that Lexi had contemplated the relationship between the 1-week growth factor (2) and the number of weeks elapsed (3), as a means to obtain the 3-week growth factor (8). In particular, although Lexi noted that Sparky was doubling in height every week, her responses and attention to the heights of the cacti suggest that she had not yet conceptualized that if Sparky doubles in height three weeks in a row, that will have the same effect as growing by a factor of 2^3, or 8.

During the third lesson, Lexi and I discussed the biconditional nature between statements involving growth factors and tupling periods. For example, I conveyed that we say the n-unit growth factor is b if and only if the b-tupling period is n-units. In the Sparky context, since the 1-week growth factor is 2, the 2-tupling (or doubling) period is 1 week. Lexi correctly determined the 2- and 4-tupling periods while observing Sparky's growth each week. However, she struggled to explain n-tupling periods when n was not a power of 2. For example, when I asked Lexi to approximate the 3-tupling (or tripling) period, she claimed that it would be 1.5 weeks (so that the three foot Sparky would lie halfway between the 2 foot and 4 foot Sparky). Under the assumption that Sparky was three feet tall after 1.5 weeks, I asked Lexi to determine the number of weeks it would take Sparky to 9-tuple (or to determine the total amount of elapsed time if Sparky 3-tupled in height again). Lexi's first response did not build off her answer to the 3-tupling period. Instead, Lexi treated the task as a new problem and claimed the 9-tupling period would be 3.5 weeks and then modified her response to be 3.25 weeks (so that the 9 foot

tall Sparky would lie closer to the 8 foot tall Sparky). Lexi's response suggests she did not use the understanding that if Sparky's height 3-tupled (or tripled) two times in a row, his resulting height would be 9 times as large as his initial height. Furthermore, the 9-tupling period would be (1.5x2) 3 weeks (based on her first response). However, this is impossible because Sparky becomes 8 times as large during a 3-week period. After I explained why this was an incorrect amount of time, Lexi stated that the 3-tupling period would have to be less than 1.5 weeks. Again, if this was the case, the 9 foot tall Sparky would appear before the 8 foot tall Sparky – a contradiction! Lexi's second incorrect response suggests that she had still not distinguished the relationship between the 3-tupling and the 9-tupling periods (specifically that 3-tupling twice is equivalent to 9-tupling once). For the remaining portion of the teaching session, Lexi continued to struggle with the idea that if Sparky's height first m-tupled and then n-tupled, we could describe his total growth as growing by a factor of mn.

During the retrospective analysis of the third teaching episode, I hypothesized that Lexi's difficulties with the aforementioned ideas were due to her not understanding that A-tupling then B-tupling had the same overall effect as AB-tupling. As a result I designed a task (Figure 1.4) aimed at supporting her in conceptualizing this foundational understanding. I anticipated that if Lexi engaged with this task she would begin to develop this foundational way of thinking.

(A) At some point in time, Sparky was this tall.	(B) After some time, Sparky 2-tupled in height. Draw the resulting Sparky.	(C) After some more time, Sparky then 4-tupled in height. Draw the resulting Sparky.
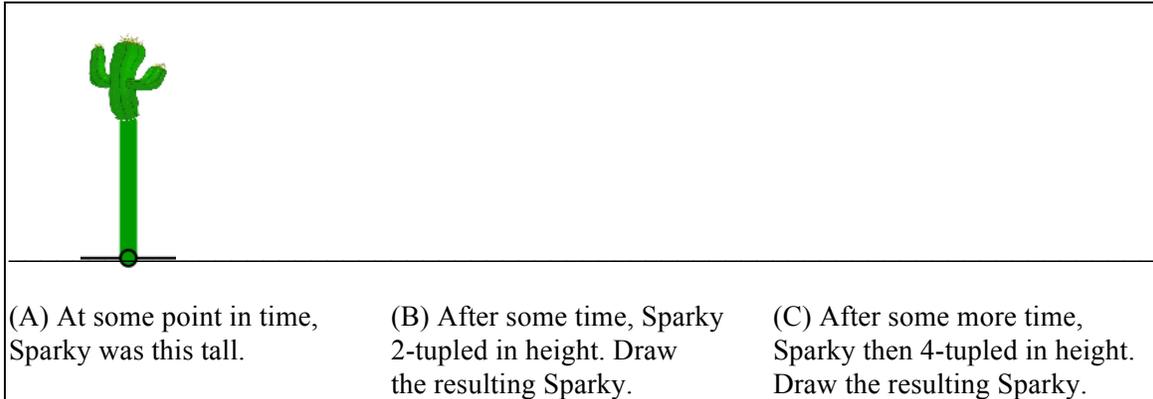
Figure 1.4. Task to Address Foundational Understanding #1

Lexi drew Sparky (B) with ease. Using a straightedge, she marked Sparky (A)'s height and drew a new Sparky that was 2 times as tall as the first. However, Lexi was unable to construct Sparky (C)'s height accurately. At first, she drew a cactus that was 2 times as tall as Sparky (B). It appeared as though Lexi interpreted the tupling language to mean doubling. After I clarified that Sparky (C) should be 4 times as tall as Sparky (B), Lexi drew the correct Sparky (C) by using her straightedge, documenting Sparky (B)'s height and constructing a length that is 4 times as tall as Sparky (B)'s height. Afterwards, Lexi and I had the following discussion:

> Emily: Sparky (C) is how many times as large as Sparky (A)?
> Lexi: Um, wouldn't it be like 6 times as large?
> Emily: OK, can you verify that?
> Lexi: Sure (reaching for straightedge)
> Emily: And as you are marking that off, can you explain how you concluded it should be 6?
> Lexi: Um, well I figured that it would be 6 times as tall because right here this is two times so then that 2 plus that 4 would be 6. (Uses the straightedge to measure how many Sparky (A)'s fit into Sparky (C)) Oh so maybe I was wrong. OK, wait, so it's 8 because is it because it's 4 times 2? Would you multiply those instead of adding them?
> Emily: Mhmm
> Lexi: OK
> Emily: But can you, can you think about, um, instead of just saying "We're going to multiply instead of add," can you think about why it is multiplication?

Lexi: Um, I guess that would make sense because right here, if you're like doubling it in height, you're multiplying it by two. And then if you're 4-tupling it I guess you are going to increase it by like another factor of 4. So instead of adding the factors you would need to multiply them.

Following this first activity, Lexi correctly completed and interpreted two similar tasks – one where Sparky tripled and then doubled in height, and another where Sparky tripled twice in a row. During the remaining time in this teaching episode, Lexi consistently applied similar reasoning, with one exception. In this instance she failed to make sense of the quantities in the situation and expressed that she felt lost in the numbers. However, as soon as I helped her refocus her attention on the relevant quantities, she began to reason in a productive manner. In the Discussion section of this paper I elaborate how the development of this understanding was essential for Lexi as she attempted questions involving the first logarithmic property. In particular, I discuss the importance of first conceptualizing the relationship between the tuplings (i.e., Foundational Understanding #1) before discussing the relationship between their corresponding tupling periods (i.e., the first logarithmic property).

Teaching Experiment #2: Aaliyah's Experiences Involving Foundational Understanding #1

During her exploratory interview, I presented Aaliyah with a variety of tasks aimed at revealing her conceptions and ways of thinking before beginning the teaching sessions. Two of the tasks (Figure 1.5) focused on the understanding that multiplying a value by A, and then multiplying the resulting value by B has the same effect as multiplying the starting value by AB. Both tasks examine the height of a cactus experiencing two multiplicative growth spurts. However, the first task provides an initial

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height and the second task does not.

1. Suppose you purchased a saguaro cactus that was 2.5 feet tall. If the measure of a saguaro cactus's height doubles (grows by a factor of 2) and *then immediately* triples (grows by a factor of 3), by what overall factor did the cactus grow?
A. 5
B. 6
C. 12.5
D. 15
E. The answer would vary based on the units chosen to measure initial height of the cactus (e.g. inches, feet, meters, etc.).

2. If the measure of a saguaro cactus's height quadruples (grows by a factor of 4) and *then immediately* triples (grows by a factor of 3), by what overall factor did the cactus grow?
A. 3.5
B. 7
C. 12
D. There is not enough information – you need to know the initial height of the cactus.
E. The answer would vary based on the units chosen to measure initial height of the cactus (e.g. inches, feet, meters, etc.).

Figure 1.5. Two Tasks Examining the First Foundational Understanding

Initially, Aaliyah approached the first task by multiplying the initial height by 2, and then she took the resulting value, 5, and multiplied it by 3 to arrive at 15. I asked if this was what we were trying to find and she replied, "Yes." I then asked if there was a difference between, "by what overall factor did the cactus grow" and "how tall is the resulting cactus" and she said the two phrases mean the same thing to her. I decided to discuss Aaliyah's thinking regarding the second question before trying to advance her thinking on the overall idea. For the second task, Aaliyah claimed that there was not enough information to answer the question because the initial value was not provided. To challenge her thinking, I drew an initial cactus with no specified height and asked if we could identify the height of the cactus after it quadruples in size. The following dialogue is what ensued.

Aaliyah: It's possible but you wouldn't have a number estimate of what it would

be.

Emily: OK, but even if we don't know the number, there would still be a height right?

Aaliyah: Mhm.

Emily: Could you go ahead and um at least identify…how tall the resulting cactus would be after it quadruples in height?

Aaliyah: *Draws a tick mark at the top of the initial cactus' height and measures the cactus' height using two fingers. She keeps her fingers spaced that far apart and marks off four total tick marks spaced apart by that length creating an invisible segment that is four times as large as the initial cactus.*

Emily: So this is like the resulting spot right here?

Aaliyah: Yeah

Emily: Can you describe what you were doing with your fingers as you were trying to measure that out?

Aaliyah: Um, basically I took the initial height and then I doubled it and made a mark and then I tripled it and made a mark and so forth.

Emily: Ok, so you have four copies of the initial cactus?

Aaliyah: Yes.

This except reveals that Aaliyah was able to represent the height of a new cactus in terms of a cactus with an arbitrary height. I then asked Aaliyah to determine the height of the resulting cactus if the cactus she just finished drawing tripled in height. Aaliyah then went through the same process as she had before and drew tick marks that were equally spaced according to the height of the quadrupled cactus – ending with a cactus that was three times as tall as the quadrupled cactus' height. With each stage of the cactus' growth documented, I then asked Aaliyah, "So, what if I asked you, 'How many times as large is this final cactus…compared to the starting cactus?' Would you be able to answer that question?" The following conversation ensued.

Aaliyah: Sort of, kind of. I feel like if we're comparing the big one to the small one it would be…for this one three times as large (*pointing to the middle stage*).

Emily: And how did you get three?

Aaliyah: Because I remember from the initial question that you asked me to make it three times as large, so it's basically taking…um…the biggest size and you kinda divide it by the smaller portion to get an answer.

Emily: What about this cactus (points to middle), how many times as large is

this cactus compared to the initial one?

Aaliyah: Four times as large.

Emily: And, so um, if the biggest cactus is three times as large as the middle one, and the middle one is four times as large as the smallest one, um, did you say that we could determine that the biggest cactus is some number times as large as the smallest one?

Aaliyah: It's possible, yeah.

Emily: And are you saying that it's possible only if we know the numbers, for the like how many feet tall they are?

Aaliyah: Not necessarily because if you wanted to you could take this cactus (middle) and this one (largest) and you can divide it by fours within each section (each third)

Emily: And if you did that, how many would you get?

Aaliyah: That would be 12 times as large.

Emily: Ok, so this final cactus…should be 12 times as large as this (initial) cactus?

Aaliyah: Mhm.

This excerpt supports that my intervention was successful in supporting Aaliyah to conceptualize the effects of growing by a factor of four and then three. However this exchange does not reveal Aaliyah's conception of how the measure of the initial height of the cactus affects the cactus' growth. I concluded our discussion of this task by asking Aaliyah if her answer would change if the initial height was different. She replied, "Possibly, I feel like it probably would be. But then again it might not. See here's why I think yes and no: I say no because even if you did have an initial height for the small cactus, you're still times, you know still going by times 12. So, it's technically the same thing but with numbers. And then….I think that was my yes reason. And the other reason was because maybe…what if for the smaller cactus, um, no I'm going to stick with the first theory." Therefore, Aaliyah concluded that the initial height of the cactus would not affect her answer, suggesting that she had conceptualized the final height of the cactus in terms of an arbitrary initial height – in particular, that given any height for the initial value of the cactus, the final cactus' height would be 12 times as large.

Despite the success of the previous intervention, I was unsure if Aaliyah had considered how the three factors (3, 4, and 12) in the second task were related. I suggested we revisit the first task to see if her thinking had changed. After looking over the problem again, Aaliyah decided she wanted to change her answer:

Aaliyah: So it's growing by a factor of 2 and then 3 so if I wanted to add the two together it's basically growing by 5. So if they just wanted the factor, then it would be A.

Emily: And is that consistent with how you thought about number [two] with this drawing out stuff?

Aaliyah: Right, now that I'm looking at it visually, and then reading it, it matches up better.

Emily: So how is that reasoning, now that you've had that little aha moment, how is that similar to the stuff that we talked about in number [two]?

Aaliyah: Because with number [two], that's basically saying you quadrupled it and then from there you tripled it. So if you wanted to take the 2.5 and double it, which was 5, and then triple it, which was 15, is basically the same thing. So, if you wanted to know how much it grew, that's saying, "Oh it grew by this, like times this many from the initial height." So that's how I came up with five. Because you doubled it and then you ended up tripling it and then I kinda added the two together rather than timesing it because if you times two by three then it would be six and then that's a completely different answer. So I felt like adding two and three together sounded more logical.

Emily: Ok, can you explain why it sounded more logical?

Aaliyah: Because (reaches for calculator) so say if I did 2.5 times 6…(*sees calculation results in 15*)…hmm…and then 2.5 times 5…OOHHH…it's 6! Wait, wait, wait wait. I'm sorry I keep changing my answer!

After some vacillation, Aaliyah concluded that the final cactus would be six times as tall as the initial cactus if the initial cactus' height doubled and then tripled in size. She also verified her answer by drawing a picture that illustrated her reasoning. To challenge her thinking further I asked, "So what if I had a cactus that doubled in height, then tripled in height, then quadrupled in height?" Aaliyah replied, "That would be 2 times 3 which is 6 times 4 which would be 24," and concluded, "That would be 24 times the initial height…24 would be the growth factor from the initial height because we don't know

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what it is." Aaliyah's response and attention to her actions suggests that she had conceptualized determining the overall growth factor as multiplying the individual factors together. However, later in the teaching experiment we revisited this foundational understanding and Aaliyah provided stronger reasoning to justify her actions. In our discussion I asked Aaliyah what the overall growth factor would be if a cactus' height 3-tupled (tripled) and then from that point 5-tupled in size. Aaliyah responded, "I want to say 15, take 3 times 5, no, mm, wait. If he becomes three times as large – whatever height that may be, and then you take that 3 times as large and you make it into five of them, it's 15, no, it's three times as large, so it's three within each five, it's fifteen." Aaliyah's attention to each of her actions between the initial and resulting stages of the task suggests that she conceptualized the roles of the individual and overall factors in the situation and the relationship between them (either during the interview or some time between the exploratory interview and that teaching session).

Lexi and Aaliyah were not alone in their difficulties with this foundational understanding. When I recruited subjects for the second teaching experiment, I gave the previously mentioned tasks (Figure 3) to 124 students. Only 39.5% answered the first question correctly and 52.4% answered the second question correctly. This data provides evidence that the foundational understanding of multiplying by *A,* then multiplying the resulting value by *B* has the same effect as multiplying by *AB* needs to be addressed prior to introducing lessons on exponential and logarithmic growth.

Foundational Understanding #2: The exponent on a growth factor, *b*, represents the number of elapsed *b*-tupling periods

Teaching Experiment #2: Lexi's Experiences Involving Foundational

Understanding #2

In this section, I present and discuss clips from the teaching episodes that suggest Lexi did not view the exponent on a growth factor, *b*, as representing the number of elapsed *b*-tupling periods. This understanding, or lack thereof, was less prevalent throughout the teaching experiment, particularly because I readjusted the third and fourth episode dialogue to focus more on discussing the quantities (i.e. growth factors, tupling-periods) instead of representing them using exponential notation. It is unclear as to whether or not Lexi had constructed the desired understandings during the course of the teaching experiment. I conclude this section by discussing how the intervention may have impacted Lexi's thinking.

Although Lexi had claimed during the first interview that any one-week change in the number of weeks would result in Sparky growing by a factor of 2, she did not appear to understand that this 1-week growth factor could be used to represent cases where a doubling in height occurred more than once (i.e., using exponents). For example, as Lexi examined Table 1.2, which she constructed during the first interview, she noted that it was easier to observe that Sparky's height was doubling each week by attending to the values in the Decimal Notation column and made no claims about how the doubling was represented in the Exponential Notation column.

Table 1.2

Differentiating Between Product, Exponential, and Decimal Notation

	Product Notation	Exponential Notation	Decimal Notation
Height at purchase	1	1	1
Height after 1 week	1(2)	$1(2)^1$	2
Height after 2 weeks	1(2)(2)	$1(2)^2$	4
Height after 3 weeks	1(2)(2)(2)	$1(2)^3$	8
Height after 4 weeks	1(2)(2)(2)(2)	$1(2)^4$	16

As a result, I decided to reintroduce the idea that the exponents on the 2's represent the number of doubling periods (weeks) that have elapsed since Sparky's purchase. However, despite our brief discussion, Lexi did not appear to understand that the exponent on a value, b, represents the number of b-tupling periods. This was apparent as Lexi tried to determine the 1-day, 4-day and 8-day growth factors.

Lexi's initial attempt to determine the 1-day growth factor involved her dividing the 1-week growth factor (2) by the number of days in a week – arriving at 2/7. However, when Lexi calculated 2/7ths in her calculator and observed a value less than 1, she claimed her method would not work because multiplying by 2/7 will "make the value smaller." Had Lexi conceptualized that the exponent on 2 represented the number of doubling periods, or weeks, that have elapsed and that the entire expression represented the growth factor for that specified period of time, I hypothesize that she would have been able to conclude that $2^{1/7}$ was the 1-day growth factor because one day is $1/7^{\text{th}}$ of a 2-tupling period. Although Lexi was confident that a 7 must be involved in calculating the 1-day growth factor using the 1-week growth factor, she did not know how to proceed. Trying a different method, I used the applet to calculate the 1-day growth factor and asked Lexi to define a function to determine Sparky's height in feet in terms of the number of days since his purchase. I then asked Lexi to compare the two function definitions (one in terms of weeks and the other in terms of days since Sparky's purchase) taking into account the relationship between weeks and days ($d=7w$). Unfortunately, our conversation did not appear to have any lasting effect on Lexi's thinking. For example, when Lexi was asked to determine the 4-day growth factor, she said, "So wouldn't we have to just do the same thing, but with a 4 in it?" To represent her

answer, she wrote the 1-day growth factor (approximately 1.1) obtained from the applet with an exponent of 4. Lexi's attention to the results of her actions suggests that she viewed finding a new growth factor as a process of plugging in the value representing the designated amount of time as the exponent to the appropriate known growth factor. Similarly, she referred to the expression $2^{1/7}$ from our previous discussion when asked to express the 4-day growth factor in terms of the 1-week growth factor, and wrote $2^{4/7}$. It was unclear as to whether or not Lexi viewed $2^{1/7}$ or $2^{4/7}$ as being equivalent to the 1-day and 4-day growth factors respectively. It appeared that Lexi viewed the numerator of the fraction in the exponent of 2 as representing a number of days and simply replaced the 1 with a 4 instead of reasoning that a 4-day period is $4/7^{\text{ths}}$ of a 1-week period.

Finally, I asked Lexi to determine the 8-day growth factor given the 4-day growth factor. Lexi referred to her previous answer of 1.1^4 and replaced the 4 with an 8, claiming that the 8-day growth factor was 1.1^8. I then reposed the question and said, "If I said, '1.485 is the 4-day growth factor' how would you find the 8-day growth factor?" Lexi immediately responded that she would multiply the value by two. Had Lexi conceptualized that the exponent on a value, b, represents the number of b-tupling periods, she should have been able to conclude that the 8-day growth factor is because there are two four-day periods that make up an eight-day period. Lexi experienced similar difficulties at the beginning of our third lesson.

In the final teaching episode, exponential notation was not used until we began discussing the major ideas behind the third logarithmic property ($\log_b(X^y) = y\log_b(X)$). The following dialogue demonstrates Lexi's struggle in viewing the exponent on a

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growth factor, 4, as representing the number of 4-tupling periods that have elapsed.

> Emily: Given that the quadrupling or 4-tupling period is 2 weeks, describe how you would determine the 4 to the 50th -tupling period. Whatever that is. It is pretty big – I don't want to punch it into my calculator.
> Lexi: This comes out to be two weeks – this beginning part – doesn't it? Um. I don't know what to do with the 50.
> Emily: Ok. What does this exponent on the 4 represent?
> Lexi: The number of tupling periods it takes to get there.
> Emily: So the number of 4-tupling periods that have passed. How long is a 4-tupling period?
> Lexi: I don't even know. I'm like, I don't remember.
> Emily: So read the statement again.
> Lexi: Oh, it's two weeks.
> Emily: So a 4-tupling period is 2 weeks and the exponent on 4 represents the number of 4-tupling periods have passed. So how long is the 4 to the 50th -tupling period? How can we calculate that?
> Lexi: I don't know what you do with the 50. I understand what it is, I don't know what to do with it though to get this extra value that it's giving us.

At first, it appears as though Lexi viewed the exponent, 50, as representing the

number of 4-tupling periods it takes to grow by a factor of 4^{50}. However, she was unable

to immediately conclude that if there are 50 4-tupling periods and each 4-tupling period is

2 weeks long, then 100 weeks have elapsed. This suggests she did not conceptualize the

multiplicative relationship between the number of elapsed 4-tupling periods and the

number of weeks in a 4-tupling period. In an effort to help her conceptualize that the   -

tupling period will be $2n$ weeks, I presented Lexi with related examples where the

exponent was a smaller number (e.g. 2, 5, 10). After working through a few of these

smaller-exponent examples, Lexi arrived at the correct answer, but seemed to get there by

attending to the results of her previous actions rather than conceptualizing what the

exponent represented and conveyed in the situation. This data suggests that her difficulty

was with her understanding of the meaning of the exponent, and possibly with the tupling

language. Either way, this dialogue suggests Lexi began to view the value of an exponent as representing a number of tuplings throughout the course of the instructional sequence on tupling periods.

Teaching Experiment #2: Aaliyah's Experiences Involving Foundational Understanding #2

During her exploratory interview, I presented Aaliyah with a task (Figure 1.6) that focused on the understanding that the exponent on a value, *b*, represents the number of elapsed *b*-tupling periods. Initially, Aaliyah chose answer choice A stating, "it made me think of the equation, I don't remember what it's called, but when they give you how many times it triples, per hour with your initial number, it's written out *a* times *b* to the whatever power it may be. Oh, and usually when it's to the power, it typically conveys time." Aaliyah's conception that the exponent represented time worked for her when the *base*-tupling period[12] was 1-hour, 1-minute, or 1-second, etc. However, as in the case of this task, Aaliyah experienced difficulties interpreting the meaning of exponents when the provided *base*-tupling period was not a typical unit of measure (e.g. 2-hours, 6 minutes, etc.).

On Saturday morning, Mary made a batch of bread dough and set the dough in a warm place to rise. Suppose the volume of the dough triples every 2 hours and suppose the starting volume of the dough was 45 cm^3. Which of the following statements best describes what information the 5.5 conveys in the expression: $45(3)^{5.5}$?
 A. 5.5 hours have elapsed
 B. 5.5 two-hour (tripling) periods have elapsed
 C. 3 gets multiplied by itself 5.5 times
 D. This expression doesn't make sense – you can't multiply a number by itself 5.5 times.
 E. None of the above

Figure 1.6. Task Examining the Second Foundational Understanding

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[12] Recall an *m*-tupling period is the amount of change in the independent quantity needed for the dependent quantity to become *m* times as large.

Throughout the teaching sessions that followed, Aaliyah demonstrated consistent thinking when it came to determining a specific growth factor provided an amount of time. For example, when asked to determine the 6-hour growth factor for the previous task, Aaliyah would take the number of elapsed hours, divide by the number of hours it took to triple, and use the resulting fraction as the exponent to 3. In general, if the *b*-tupling period was *n* units, and *m* units elapsed, Aaliyah would claim that the *m*-unit growth factor was $b^{m/n}$. Although her approach determines the correct answer, Aaliyah experienced difficulty when asked to describe what the value of the exponent represented. Instead of attending to the entire value of the exponent, she would often focus just on the numerator of her fraction in the exponent and claim that that many units elapsed. This technique proved troublesome when the exponent was simplified or written as a fraction where the value of the denominator was not the same as the number of units needed for the output quantity's value to *base*-tuple.

During the fourth teaching episode, I asked Aaliyah to determine different growth factors for the following situation: On Saturday morning, Mary made a batch of bread dough and set the dough in a warm place to rise. Initially, at 9am, the dough's volume was 45cm^3. Suppose the volume of the dough 3-tuples (triples) every 2 hours. Aaliyah employed her fractional approach to determine the 1-hour, 2-hour, 25-hour, ½-hour and *H*-hour growth factors. However, when I asked her to interpret the simplified value of the exponent she determined for the ½-hour growth factor, she quickly resorted to discussing the values of her original fractional exponent as seen in the following dialogue:

Emily: What about if we wanted to find the ½ hour growth factor, or we could say 30 minutes.
Aaliyah: Then you could do three to the power 30 over 120 if you wanted to make

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the two hours into minutes.

Emily: Ok, so you're, are you saying like 30 over 120 is the 30 minutes that we're talking about over the 120 minutes in two hours?

Aaliyah: Yes.

Emily: That's how you're getting that fraction?

Aaliyah: Mhm

Emily: OK. Um, so 30 divided by 120 simplifies to a fourth, or .25. What does that .25, or a fourth represent in that context then?

Aaliyah: The growth factor for a 30-minute time change

Emily: So this, this .25 is the growth factor?

Aaliyah: Yes. For the 30 minutes, well no because you don't have the three. So the .25 is another way of saying um, is basically representing the 30 minutes. But, yeah, representing 30 minutes within two hours by itself, without the growth factor.

Later in that same interview we revisited this part of the task and Aaliyah stated, "I would take the three to the power of .25 divided by two to give me that many times the three can 3-tuple." Her suggestion that we rewrite the growth factor as   was an (incorrect) attempt to make the denominator of the exponent be the number of hours it took for the dough's volume to 3-tuple (triple). She made a similar attempt in the fifth teaching episode when trying to interpret the growth factor $4^{3/4}$ in the Sparky situation. Aaliyah rewrote the growth factor to be $4^{0.75/2}$ stating, "I wrote the divided by two because it 4-tuples every 2 weeks and … and then with three fourths after I turned it into a decimal, I was basing the decimal after how many times will it 4-tuple in 2 weeks…but, since that's not the case, … it's probably just .75 by itself. But then you don't know, well I don't know if um…if it's the final answer for how many times it can 4-tuple or something else." We discussed why the rewritten growth factor was not equivalent to the original problem and concluded that instead we could have written $4^{1.5/2}$. I then asked Aaliyah to interpret the growth factor and she said, "It basically means since, so because um, the cactus 4-tuples every two weeks, we want to figure out how many times it'll 4-

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tuple within a 1.5 week period." Aaliyah concluded that in 1.5 weeks, the cactus would 4-tuple .75 times.

Aaliyah's difficulties with interpreting the value of an exponent suggest she struggled to conceive the number of elapsed *base*-tupling periods as a new quantity. Therefore when the quantitative relationship changed (i.e., when she was provided with the value of an exponent and asked to find which growth factor was being represented), Aaliyah was forced to make changes to her thinking in order to make sense of the values presented. Both Lexi and Aaliyah were not alone in their difficulties with this foundational understanding. In fact, as I recruited for the second teaching experiment, I gave the previously mentioned task (Figure 1.6) to the same 124 students. Only 43.5% answered the question correctly. This evidence suggests that the foundational understanding that the exponent on a growth factor, *b*, represents the number of elapsed *b*-tupling periods is worth discussing in detail prior to lessons on exponential and logarithmic growth.

DISCUSSION

The understanding that multiplying by *A* and then multiplying by *B* has the same effect as multiplying by *AB* is a critical understanding that must be applied throughout a lesson on exponential and logarithmic functions. Types of problems that involve such reasoning include: calculating percentages of values, determining partial growth factors, representing, interpreting and calculating logarithmic values, and working with and explaining logarithmic properties. A student who does not hold this understanding can be successful in answering questions to determine percentages of values, as reported when Lexi first calculated 1% of a value and then scaled her answer to find a different percent.

However, if it is our goal that students develop coherent understandings of exponential and logarithmic functions and other related topics, then as instructors we must ensure that this foundational understanding is also developed. In the following paragraphs, I describe how this crucial understanding is found in the types of problems listed above and I discuss possible ramifications of not having this understanding.

Calculating Percents (x 1/100 x *n* = x *n* /100): When determining *n*% of a value, a student may first determine 1% of the reference value, either by dividing the reference value by 100 or multiplying the reference value by 0.01, and then scale the resulting value by a factor of *n*. Or, a student may multiply the initial value by *n*/100 (either in fractional or decimal form). While calculating a percentage of a value in two steps is a mathematically correct method, it is not the most productive method. A student who relies on the two-step method and has difficulties seeing multiplying a value by *n*/100 to determine *n*% of the reference value as being equivalent may also experience difficulties determining the percent change from growth factors in exponential situations. For example, if a student with this understanding is informed that every year the money in his bank account will grow by a factor of 1.08, he might find it difficult to conclude that the amount in his account a year later will be 108/100 or 108% of his current amount. Furthermore, he may struggle to conclude that the one-year percent change was 8%.

Determining (Partial) Growth Factors: In the case of Sparky, to determine the 1-day growth factor, we wish to find the number such that when we multiply by this factor 7 times it will have the same effect as multiplying by the 1-week growth factor, 2. Symbolically we write $b^7 = 2$ and then solve for *b* to find the 1-day growth factor.

However, a student that does not hold the first foundational understanding may have difficulties setting up this equation and not be able to see why

$b \times b \times b \times b \times b \times b \times b = b^7$ or why $b^7 = 2$. When Lexi was presented with the task of determining the 1-day growth factor, she appeared troubled and decided to divide the 1-week growth factor by 7. However, she quickly concluded that her attempt was incorrect after observing that the growth factor was less than 1. Lexi experienced similar struggles when trying to determine the 3-week growth factor. Had Lexi developed this foundational understanding, she should have been able to conclude that when Sparky doubles in height three weeks in a row ($\times 2 \times 2 \times 2$), that will have the same effect as growing by a factor of 2^3, or 8. However, at the time, Lexi had not yet developed the foundational understanding that multiplying by A and then by B has the same overall effect as multiplying by AB, and had to resort to other measures in order to arrive at an answer that made sense to her.

Logarithms: Recall $\log_b(m)$ represents the number of b-tupling periods needed to m-tuple (or grow by a factor of m). In other words, this expression represents the number of times a value must b-tuple in order to have the same effect as m-tupling. In regards to the foundational understanding, m takes the role of $\times AB$ and the b takes the role of the individual factors. If students do not hold the foundational understanding, they may struggle to envision the relationship between b and m.

Logarithmic Properties: This foundational understanding is most clearly present in the first logarithmic property, $\log_b(XY) = \log_b(X) + \log_b(Y)$, which is interpreted as the number of b-tupling periods needed to XY-tuple is equal to the number of b-tupling

periods needed to *X*-tuple plus the number of *b*-tupling periods needed to *Y*-tuple. To correctly reason through problems involving this property, a student must first be able to conclude that multiplying by *X* and then multiplying by *Y* has the same effect as growing by a factor of *XY*. After developing this understanding of how the tuplings relate, he may consider the relationship between the corresponding tupling periods. Specifically, he may conclude that the *XY*-tupling period will be the same as the sum of the *X*-tupling period and the *Y*-tupling period. From this point, the student might be able conclude that this relationship will stay consistent as long as the tupling periods are measured using the same unit. Therefore, without this first foundational understanding, it is practically impossible to then reason about the first logarithmic property. Before the intervention in the fourth episode, Lexi struggled to identify the number of weeks it would take for Sparky to grow by a factor of 10 given Sparky's 2-tupling period and its 5-tupling period. However, as Lexi completed the intervention in the fourth episode and attempted a similar task, she conceptualized and related relevant quantities, resulting in her reasoning that if it took Sparky one week to 2-tuple and approximately 1.58 weeks to 3-tuple, then it should take 1+1.58=2.58 weeks to 6-tuple. In other words, the number of 2-tupling periods (weeks) needed to 2-tuple plus the number of 2-tupling periods (weeks) needed to 3-tuple is equal to the number of 2-tupling periods (weeks) needed to 6-tuple. Symbolically, $\log_2(2) + \log_2(3) = \log_2(6)$ - a specific case of the first logarithmic property!

The second foundational understanding students must develop for a coherent understanding of exponential and logarithmic functions is that the exponent on a growth

factor, b, represents the number of b-tupling periods that have elapsed. Students who hold a repeated multiplication view of exponentiation may struggle in interpreting the expression $2^{1.5}$ in the Sparky context. However, a student who views the exponent on 2 as the number of doubling periods that have elapsed may interpret this expression to be representing the factor by which Sparky grows over a 1.5 week period. Similarly, such a student should be able to generalize this statement for any number of weeks and therefore be able to meaningfully define the exponential function relating Sparky's height with the number of weeks that have passed since January 1st. Students who have developed this foundational understanding may find it easier to determine and interpret growth factors. For example, if the exponent on 2 represents an elapsed number of weeks and we wish to determine the 1-day growth factor, then since one day is $1/7^{th}$ of a week, the factor by which Sparky will grow over the course of one day is $2^{1/7}$. Similarly, provided $2^{3/4}$ is a growth factor in the Sparky situation, one may interpret this to be the 3/4-week growth factor. With Aaliyah, we observed this understanding was not entirely necessary for her to be able to determine growth factors. However, when she was asked to interpret the amount of time it would take for the output value to grow by a specific factor, she experienced difficulties when she did not interpret the exponent to be the number of elapsed $base$-tupling periods. This understanding is also foundational for understanding logarithms and logarithmic properties. Even if a student understands $\log_b(m)$ to be the number of b-tupling periods needed to m-tuple (or grow by a factor of m), he may not be able to correctly apply this understanding to solve for the input to an exponential function if he does not see the exponent on b as representing a number of b-tuplings. To

compound this issue, exponential notation is utilized in the third logarithmic property, $\log_b(X^y) = y\log_b(X)$, which can be interpreted as the number of b-tupling periods needed to X-tuple y times is y times as large as the number of b-tupling periods needed to X-tuple once. In order for this property to make sense to the student, it is crucial that he develops the understanding that the exponent on a growth factor, b, represents the number of b-tupling periods that have elapsed (among other understandings).

<center>CONCLUSION</center>

Many studies have examined aspects of logarithms that present difficulties for students, while others have investigated the effectiveness of interventions. In this study, however, I examined the subjects' thinking as they participated in a conceptually based lesson on exponential and logarithmic functions in an effort to determine the understandings foundational to the idea of logarithm students must develop. My findings revealed two understandings foundational to learning logarithms; first that multiplying by A and then multiplying the resulting value by B has the same effect as multiplying the initial value by AB, and second that the exponent on a number b represents the number of elapsed b-tupling periods. The results of this study also suggest that the meaning for exponents may be strengthened as the student discusses tupling periods throughout the instructional intervention preparing for logarithmic notation. These findings will be used to improve the Sparky the Saguaro lesson for future research in an effort to provide students more opportunities to develop these foundational understandings at the beginning of the intervention. The Geogebra applet utilized in this study can be requested at egkuper@asu.edu.

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APPENDIX

Task 0:
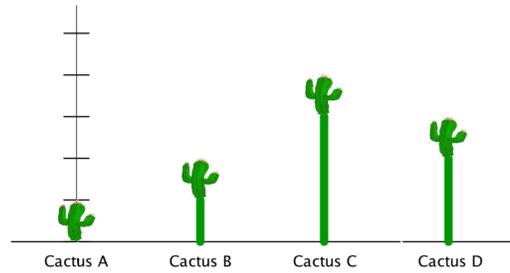
i. Represent 1% of Cactus C's height. Is your representation an approximation? Is there exactly one height that corresponds to 1% of Cactus C's height? What would you need to do to find the exact height corresponding to 1% of Cactus C's height?

ii. What would you need to do to find the exact height corresponding to 27% of cactus A's height?

iii. The height of Cactus C is how many times as large as the height of Cactus B? What height corresponds to 1% of Cactus B's height? Using this measurement as a unit of measure, how tall is Cactus C?

iv. The height of Cactus C is what percent of the height of Cactus B?

v. How many feet taller is Cactus C than Cactus B? This difference of Cactus C's and Cactus B's height is how many times as large as Cactus B's height? This difference in height is what percent of Cactus B's height?

vi. What is the relationship between the height of Cactus C as a percent of the height of Cactus B, and the difference in their heights as a percent of Cactus B's height?

Task 1: (using the same picture above)

i. Cactus C (A, D) is how many times as tall as Cactus B?

ii. Cactus B is how many times as tall as Cactus C (A, D)?

iii. Given any two cacti, describe how you determine how many times as tall one is than the other?

iv. Draw Cactus E given Cactus E is 5.5 times as tall as Cactus B.

v. Draw Cactus F given Cactus C is 3 times as tall as Cactus F.

vi. If Cactus B is 8 inches tall, how tall are Cacti A, C, D and E?

vii. Cactus H is how many times as tall as Cactus G if Cactus G is 34 inches tall and Cactus H is 102 inches tall?

viii. Cactus I is how many times as tall as Cactus J if Cactus J is x inches tall and Cactus I is y inches tall?

ix. How would you describe the cactus' growth in the diagram to the right given that the cactus on the left grew to be the cactus on the right?

x. If a cactus was 23 inches tall when it was purchased and grew to be 156 inches tall, by what factor did the cactus grow?

xi. If a cactus was m inches tall when it was purchased and grew to be k inches tall, by what factor did the cactus grow?

Task 2

i.

| (A) At some point in time, Sparky the cactus was this tall. | (B) After some time, Sparky's height doubled (becomes 2 times as. large). Draw the resulting Sparky. | (C) After some more time, Sparky's height then quadrupled (becomes 4 times as large) from point (B). Draw the resulting Sparky. |

ii. By what overall factor did Sparky grow from point (A) to point (C)?
     In other words, overall Sparky's height experienced a _____-tupling.

iii. If Sparky's height becomes 3 times as large and *then* 5 times as large, overall his height will experience a ____-tupling.

iv. If Sparky's height becomes 34 times as large and *then* 57 times as large, overall his

     height will experience a ____-tupling.

v.   If Sparky's height becomes *X* times as large and *then* *Y* times as large, overall his

     height will experience a____-tupling.

Task 3 (This task requires the use of the Geogebra Applet)

i.   Emily purchased the mystical cactus shown in the video (Geogebra Applet) on Sunday, January 1st and named the saguaro Sparky. She decided to record the displayed time-lapse video of Sparky's growth and noticed he was growing in a peculiar way. Watch the video and discuss what you observe.

ii. Document and observe Sparky's height every: week (2 weeks, 1/7 week (day), 1.585 weeks, etc.) What changes? What stays consistent?

iii. If Emily's friend Morgan visited every Tuesday (every other Tuesday, every day, every third Tuesday, etc.) to document Sparky's growth, would she make the same claims?

iv. If Emily's friend Kevin visited every Friday (every other Friday, every day, every third Friday, etc.) to document Sparky's growth, would he make the same claims?

v.   What is the 1-week (2-week, 1/7th-week, 1.585-week, etc.) growth factor?

vi. What is the 2-tupling (4-tupling, 1.1-tupling, 3-tupling, etc.) period? In other words, how long does it take Sparky's height to become 2 (4, 1.1, 1.585, etc.) times as large?

Task 4
Recall the 1-week growth factor is 2, and thus the 2-tupling period is 1 week.

   i.   By what factor does Sparky grow every two (three, six) weeks?
   ii.   By what factor does Sparky grow every 52 weeks (1 year)?
   iii.   By what factor does Sparky grow every day ($1/7^{th}$ of a week)?
   iv.   By what factor does Sparky grow every -1 weeks?
   v.   By what factor does Sparky grow if no time has elapsed (0 weeks)?
   vi.   By what factor does Sparky grow by every x weeks?
   vii. Suppose a different cactus' height 17-tuples every year. By what factor will this cactus grow every week?

Task 5
Recall the 1-week growth factor is 2, and thus the 2-tupling period is 1 week. Also recall that initially (week 0) Sparky is 1 foot tall. Suppose that after x weeks, Sparky is y feet tall.

   i.   Fill in the blank: After x weeks, Sparky's height is ___ times as large as his height at week 0.
   ii.   Use the 1-week growth factor to represent this same growth factor.
   iii.   Given any number of weeks, x, write an equation that determines the corresponding height of Sparky, y.
   iv.   Now, suppose initially (week 0) Sparky was 3 feet tall and still doubled in size each week. Write an equation that determines y, Sparky's height in feet, given x, the number of weeks since Sparky's purchase.
   v.   Suppose a pool is being filled with water so that the volume of water in the pool 1.5-tuples every hour. At 9am, there were 15 gallons of water in the pool. Write an equation that determines the number of gallons of water in the pool, g, in terms of the number of hours since 9am, h.

Task 6

   i.   How many 2-tupling periods (weeks) does it take for Sparky's height to result in a 2-tupling (4-tupling, 8-tupling)?
   ii.   How many 2-tupling periods (weeks) does it take for Sparky's height to result in a 3-tupling (5-tupling, 7-tupling)?
   iii.   In general, $\log_b(m)$ represents the number of b-tupling periods needed to result in an m-tupling. Use this notation to represent your answers to parts (i) and (ii). Verify your answers with the applet.

Task 7

i.

(A) At some point in time, Sparky the cactus was this tall.	(B) After 1 week, Sparky's height doubled (2-tupled, became 2 times as large). Draw the resulting Sparky.	(C) After about 1.585 weeks, Sparky's height then tripled (3-tupled, became 3 times as large). Draw the resulting Sparky.

ii. By what factor did Sparky grow from point (A) to point (C)? How long did it take to grow by this factor?
   In other words, overall Sparky's height will experience a _____-tupling in _____ weeks.

iii. If Sparky's height 3-tuples *then* 5-tuples, overall his height will experience a _____-tupling.
   Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a 3-tupling, the number of 2-tupling periods (weeks) needed to result in a 5-tupling, and the number of 2-tupling periods (weeks) needed to result in a 15-tupling. Write an equation representing the relationship between these three values.
   In other words, if it takes _____ weeks to 3-tuple and _____ weeks to 5-tuple, then it will take _____ weeks to 15-tuple.

iv. If Sparky's height 34-tuples *then* 57-tuples, overall his height will experience a _____-tupling.
   Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to 34-tuple, the number of 2-tupling periods (weeks) needed to 57-tuple, and the number of 2-tupling periods (weeks) needed to 1938-tuple. Write an equation representing the relationship between these three values.
   In other words, if it takes _____ weeks to 34-tuple and _____ weeks to 57-tuple, then it will take _____ weeks to 1938-tuple.

v. If Sparky's height *X*-tuples *then* *Y*-tuples, overall his height will experience a _____-tupling.
   Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a *X*-tupling, the number of 2-tupling periods (weeks) needed to result in a *Y*-tupling, and the number of 2-tupling periods (weeks) needed to result in a *XY*-tupling. Write an equation representing the relationship between these three values.
   In other words, if it takes _____ weeks to *X*-tuple and _____ weeks to *Y*-tuple, then it will take _____ weeks to *XY*-tuple.

vi. Now, discuss how your equations would change had you measured in days instead of weeks.

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Task 8

i.

| (A) At some point in time, Sparky the cactus was this tall. | (B) After some time, Sparky's height 5-tupled in size. Draw the resulting Sparky. | (C) After 1 week, Sparky's height then 2-tupled in size from point (B). Draw the resulting Sparky. |

ii. By what factor did Sparky grow from point (A) to point (C)? If it took Sparky approximately 3.3219 weeks to grow by this factor, how long did it take Sparky to 5-tuple?

iii. If it takes Sparky's height 3.585 weeks to experience a 12-tupling and 2 weeks to experience a 4-tupling, how long does it take for Sparky's height to experience a 3-tupling?
Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a 12-tupling, the number of 2-tupling periods (weeks) needed to result in a 4-tupling, and the number of 2-tupling periods (weeks) needed to result in a 3-tupling. Write an equation representing the relationship between these three values.
In other words, if it takes _____ weeks to 12-tuple and _____ weeks to 4-tuple, then it will take _____ weeks to 3-tuple.

iv. Describe how you would determine the 17-tupling period given that the 34-tupling period is approximately 5.087 weeks

v. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in an X-tupling, the number of 2-tupling periods (weeks) needed to result in a Y-tupling, and the number of 2-tupling periods (weeks) needed to result in an X/Y-tupling. Write an equation representing the relationship between these three values.
In other words, if it takes _____ weeks to X-tuple and _____ weeks to Y-tuple, then it will take _____ weeks to X/Y-tuple.

vi. Now, discuss how your equations would change had you measured in days instead of weeks.

Task 9
Recall that the 2-tupling period is 1 week.

i. Determine the $2^4 = 16$-tupling period.

ii. The 16-tupling period is how many times as large as the 2-tupling period?

iii. Given that the quadrupling or 4-tupling period is 2 weeks, describe how you would determine the 4^{50}-tupling period.

iv. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in an X-tupling and the number of 2-tupling periods (weeks) needed to result in an X^y-tupling. Write an equation representing the relationship between these two values.

v. Now, discuss how your equations would change had you measured in days instead of weeks.

Task 10
The 10-tupling period is about 3.3 weeks and the 15-tupling period is about 3.9 weeks.

i.   The 15-tupling period is how many times as large as the 10-tupling period?
ii. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to 10-tuple and the number of 2-tupling periods (weeks) needed to 15-tuple. Write an equation representing the relationship between these two values.
iii. How would your answer to (i) change if the two periods been measured in days? In years? How would your answer to (i) remain the same if the two periods been measured in days? In years? Explain.
iv. Use logarithmic notation to represent the number of 1.104-tupling periods (days) needed to 10-tuple and the number of 1.104-tupling periods (days) needed to 15-tuple. Write an equation representing the relationship between these two values.
v.   Compare your answers in (ii) and (iv).
vi. Develop an equation relating $\log_b(X)$, $\log_b(Y)$, $\log_c(X)$, and $\log_c(Y)$ (for $b,c,X,Y > 0$)

Task 11

i.   What does y represent in the expression 2^y?
ii.   Represent the number of 2-tupling periods needed to result in a 2^y-tupling using logarithmic notation.
iii. Represent the number of 2-tupling periods needed to result in a 2^y-tupling without using logarithmic notation.
iv.   Write an equation relating your answers in (ii) and (iii).
v.   Simplify $\log_b(b^x)$.
vi.   What does y represent in the expression $2^y = x$?
vii. Represent the number of 2-tupling periods needed to result in an x-tupling using logarithmic notation.
viii. Simplify $2^{\log_2(x)}$.
ix.   Simplify $b^{\log_b(x)}$.

Task 12
Recall $\log_b(x)$ represents the number of b-tupling periods needed to result in an x-tupling.

i.   Describe how $\log_2(x)$ varies as x varies.
ii. Graph the relationship of $\log_2(x)$ with respect to x. If necessary, create a table of values.
     T/F: Every value of x determines exactly one value of $\log_2(x)$. Explain your answer.

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PAPER 2:

SPARKY THE SAGUARO: TEACHING EXPERIMENTS EXAMINING STUDENTS'

DEVELOPMENT OF THE CONCEPT OF LOGARITHM

ABSTRACT

Studies have documented student difficulties in understanding and learning the
idea of logarithm. However, few studies have examined student reasoning as they
complete tasks designed to support them in acquiring strong meanings for this idea. This
study investigated two undergraduate precalculus students' ways of thinking and
understandings of exponential and logarithmic functions as they examined growth
patterns of Sparky – a mystical saguaro that doubled in height every week. The lessons
were designed to support students in understanding foundational ideas for understanding
and using the idea of logarithm and logarithmic properties meaningfully, including the
ideas of growth factor and tupling (e.g., doubling, tripling) periods. This paper describes
the reasoning abilities two students exhibited as they engaged with tasks designed to
foster their construction of more productive meanings for the idea of logarithm. The
findings of this study provide novel insights for supporting students in understanding the
idea of logarithm meaningfully.

INTRODUCTION

The idea of logarithm is useful both in mathematics (e.g., number theory –
primes, statistics – non-linear regression, chaos theory – fractal dimension, calculus –

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differential equations) and in modeling real-world relationships (e.g., Richter scale, Decibel scale, population growth, radioactive decay). Therefore, a goal for mathematics educators should be to assist students in developing coherent meanings for the idea of logarithm. How does one achieve this goal? One approach is to research the aspects of the idea of logarithm students have difficulties with. In particular, studies have shown that students have difficulty with logarithmic notation, logarithmic properties and logarithmic functions (Weber, 2002; Kenney, 2005; Strom, 2006; Gol Tabaghi, 2007). Another approach is to develop and test the efficiency of interventions relative to standard curriculum (Weber, 2002; Panagiotou, 2010). Although these methods may shed light on epistemological obstacles students encounter and/or the effectiveness of a non-traditional intervention, neither approach examines the reasoning abilities needed to coherently understand and use the idea of logarithm. In fact, relatively few studies have examined what meanings students have for the idea of logarithm (Kenney, 2005; Gol Tabaghi, 2007), and fewer have examined how students come to conceptualize the idea of logarithm (Kastberg, 2002). In response, I propose to investigate students' thinking and developing understandings as they work through a conceptually oriented exponential lesson designed to foster students' construction of productive meanings for the idea of logarithm.

I argue that a productive understanding of the idea of logarithm requires more than just memorizing and applying Euler's definition after completing an exponential lesson. Rather, to understand the idea of logarithm meaningfully, one must first

conceptualize tuplings[13] and their corresponding tupling periods in exponential situations. That is, one must attend to the multiplicative growth of the output quantity of an exponential function while also attending to the corresponding changes in the input quantity of an exponential function. After conceptualizing these quantities, one must attend to how they vary together and imagine one tupling period relative to another. Therefore, I claim that it is necessary for students to engage in quantitative reasoning and covariational reasoning to understand the idea of logarithm coherently. It is well documented that students who engage in quantitative reasoning are more likely to reason productively while working on conceptually challenging tasks (Thompson, 1993, 1994b; Carlson, 1998; Saldanha & Thompson, 1998; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Ellis, 2007; Castillo-Garsow, 2010; Hackenberg, 2010; Moore, 2010; Moore & Carlson, 2012). Furthermore, Carlson et al. (2002) and Thompson and Carlson (2017) have argued that covariational reasoning is an essential way of thinking for constructing meaningful function formulas and graphs. Therefore, if a goal for students is for them to utilize the idea of logarithm as they work through conceptually challenging tasks, then it would follow that they should develop an understanding of the idea of logarithm that is based on their conceptualizing and representing quantities, while also attending to how the quantities' values vary in tandem.

This study investigated two undergraduate precalculus students' understandings of the idea of logarithm as they each worked through an exploratory lesson on exponential and logarithmic functions. The findings of this study revealed an essential

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[13] A *b*-tupling occurs when a quantity becomes *b* times as large. Therefore, a *b*-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become *b* times as large. We say that the second quantity has *b-tupled* over some interval of change of the first quantity.

conceptualization that students must construct in order to hold a productive meaning for many of the components of the idea of logarithm. That is, in order to reason through tasks involving logarithmic expressions, logarithmic properties, and logarithmic functions in a way that builds off prior meanings and serves to be useful for more complex tasks, students must conceptualize *base*-tupling periods as a multiplicative object. Specifically, students must conceptualize a *b*-tupling period as providing information about a specific change in the input quantity necessary to result in the output quantity growing by a factor of *b*. In this study, I modeled both students' thinking as they participated in an exponential and logarithmic sequence designed to assist students in developing coherent meanings for the idea of logarithm. I also discussed the importance of conceptualizing this essential component in the context of the lesson.

Research Question

The primary question motivating this investigation is:

- What understandings of the idea of logarithm do students develop during an exponential and logarithmic instructional sequence aimed at supporting students in acquiring a strong meaning for the idea of logarithm?

LITERATURE REVIEW

Quantitative Reasoning

Smith and Thompson (2007) argue that if students are to utilize algebraic notation to assist them in representing ideas and reason productively, then their ideas and reasoning must become sophisticated enough to justify the use of the notation in the first place. It thus seems reasonable that logarithmic notation and properties should be introduced so that the notation represents measurable attributes of objects that students

have conceptualized. This approach has been referred to as quantitative reasoning (Thompson, 1990, 1993, 1994a, 2011) and describes the mental processes involved in conceptualizing quantities and the relationships between quantities. If a goal for students is for them to utilize the idea of logarithm as they work through conceptually challenging tasks, then it would follow that they should develop an understanding of the idea of logarithm that is built on quantitative reasoning. In this section, I briefly describe the components of quantitative reasoning.

A quantity is a mental construction of a measurable attribute of an object (Thompson, 1990, 1993, 1994a, 2011). That is, quantities do not exist out in the world; they are created in the mind of an individual when she conceptualizes measuring a quality of an object (Thompson, 2011). Furthermore, one is said to participate in the act of quantification when, after conceptualizing a quantity, she conceptualizes the attribute's unit of measure such that the attribute's measure is proportional to its unit (Thompson, 2011). The numerical measurement that a quantity may assume is referred to as a value. When the measurable attribute of an object doesn't change throughout a situation, it is called a constant or fixed quantity. On the other hand, if the value of a quantity changes throughout a situation, we call it a varying quantity.

Mathematics formulas and graphs are often used to model and describe how two or more quantities relate and change together. A quantitative operation occurs in the mind of an individual when "one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 2011, pg. 9). When one conceives of three quantities related by means of a quantitative operation, he has conceptualized a quantitative relationship. Changing which quantity is determined by the quantitative operation

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changes the quantitative relationship (Thompson, 1990). When one analyzes a situation

and assigns his observations (i.e. quantities, quantitative relationships) to a network of

quantities and quantitative relationships, called a quantitative structure, he is said to

engage in quantitative reasoning (Thompson, 1988, 1990, 1993, 1994a, 2011).

Multiplicative Object

When a student engages in covariational reasoning and conceptualizes two

quantities' values varying in tandem, she will likely encounter opportunities to

conceptualize the coupling of the values simultaneously and use notation, a point on a

graph or some other means to represent her conception. Researchers (Thompson &

Saldhanha, 2003; Thompson & Carlson, 2017) refer to this conceptualization of the

coupling of two quantities' values as a multiplicative object. Consider, for example, the

expression $\log_2(8)$. I define $\log_2(8)$ as the number of 2-tupling periods needed to result

in an 8-tupling. This expression conveys information about two different tupling periods

while simultaneously representing the number of the 2-tupling periods necessary to result

in an 8-tupling (specifically this number is 3). I anticipate that students' abilities to

conceptualize the values of two (or more) quantities simultaneously as a new conceptual

object will assist students in developing robust meanings for the idea of logarithm.

Research Literature on Students' Understandings of Exponents and Exponential Functions

A student who conceptualizes exponentiation only as repeated multiplication will

likely be limited to interpreting natural number exponents. In cases when an exponent is a

non-natural real number, say $-\pi$, we need a way of thinking that will allow us to interpret

the exponent. The interpretation of exponentiation as repeated multiplication is not

helpful here. While some researchers advocate a repeated multiplication approach (e.g. Goldin & Herscovics, 1991; Weber, 2002), others believe this approach limits students (e.g. Ellis, Ozgur, Kulow, Williams & Amidon, 2015; Davis, 2009; Confrey & Smith, 1995). In particular, Confrey and Smith (1995) argue that the standard way of teaching multiplication through repeated addition is inadequate for describing a variety of situations such as magnification, multiplicative parts (i.e. finding a fraction of a split), reinitializing and creating an array. Weber (2002) proposed that students first understand exponentiation as a process (in terms of APOS theory) before viewing exponential and logarithmic expressions as the result of applying the process. A student with a process conception of exponent will be able to generalize her understanding to cases in which the exponent is a non-natural number. Specifically, Weber stressed to his students that "b^x represents the number that is the product of x many factors of b." With this conception, we can describe $9^{2.5}$ to be the number that is the product of two and a half factors of 9, while under the view of repeated multiplication, a student might write " $9 \cdot 9 \cdot ?$ ". If a coherent understanding of exponential functions (and later logarithmic functions) is desired of our students, it is imperative that they have productive meanings for exponents.

Ellis et al. (2015) conducted a small-scale teaching experiment with three middle school students that examined continuously covarying quantities. The students were asked to consider a scenario of a cactus named Jactus whose height doubled every week. Eventually, the initial height, weekly growth factor and amount of time needed to double were altered to provide variety. The authors noticed three significant shifts in the

students' thinking over the course of the study. At first, the students attended only to Jactus' height and concluded he grew by means of repeated multiplication. Eventually, the students began to coordinate this repeated multiplication with the corresponding changes in the amount of elapsed time. The second shift involved students determining the factor by which Jactus' height grew, for varying changes in the number of weeks, by means of calculating the ratio of two heights. During the third shift students generalized the reasoning described in the second shift to include non-natural exponents (i.e., to determine the 1-day growth factor). The authors noted that a student's ability to coordinate the growth factor (or ratio of height values) with the changes in elapsed time contributed to the student successfully defining the relationship between the elapsed time and Jactus' height. This study leveraged findings from Ellis et al.'s study of Jactus the Cactus.

Research Literature on Students' Understandings of Logarithms

The topics of logarithmic notation and logarithmic functions often pose a variety of challenges to students (Kenney, 2005; Weber, 2002). Similar to the complexities present in function notation, logarithmic notation consists of multiple parts each with their own dual nature (Kenney, 2005). In the equation $\log_b(x) = y$, b, x, and y can take on a variety of meanings to an individual – b often takes on the form of a parameter (staying consistent within the context of a problem, but varying from problem to problem), x serves as the input variable to the logarithmic function and is a tupling, and y serves as the output variable to the logarithmic function and is the number of b-tupling

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periods[14] needed to *x*-tuple. Kenney (2005) noted that because function names are often one letter, students do not naturally view log(*x*) as representing an output to a function. Weber (2002) recognized these and other obstacles students encounter and conducted a pilot study that compared a traditional approach to teaching logarithmic functions with a conceptual approach that introduced $\log_b(m)$ as the number of factors of *b* there are in *m*. Weber's way of discussing the meaning of a logarithmic expression more clearly describes what the multiple parts of the notation represent - therefore addressing the issues Kenney observed in her study.

In addition to these unavoidable complexities, Kenney's (2005) study uncovered other difficulties students have in understanding logarithmic notation. Kenny investigated students' understandings of logarithmic notation in two phases. The data revealed that students displayed mixed understandings of the bases in the expressions. For example, the students appeared to think that different bases always meant the logarithmic expressions were not equivalent (with the inputs being the same). However, about 32% of the students in her study claimed $\log_3(x) + \log_3(x+1) = \log_5(x) + \log_5(x+1)$ because the bases would cancel out. Students also claimed that the notation for the natural logarithm (ln) and the common logarithm (log) were equivalent. One possible reason for this misconception is that both of these logarithmic bases appear on graphing calculators and are often used when finding an input to an exponential function for a specific output value. The study also revealed that students would disregard or "cancel out" the word "log" when simplifying equations involving logarithms and solving for *x*. Despite the

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[14] A *b*-tupling occurs when a quantity becomes *b* times as large. Therefore, a *b*-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become *b* times as large. We say that the second quantity has *b-tupled* over some interval of change of the first quantity.

aforementioned difficulties, a few of the students were successful in arriving at the correct answer. However, Weber (2002) found that this was an unlikely result of traditionally taught students.

Weber's (2002) pilot study examined the effects of non-traditional instruction of exponents and logarithms. The participants of the study were college students from two different college algebra and trigonometry classes at a university in the southern region of the United States. 15 students from each class voluntarily participated in the study. The first group of 15 students made up the control group and experienced traditional instruction on exponents and logarithms while the second group of 15 students participated in more conceptually taught lessons led by the author which incorporated the use of the program MAPLE. Students were taught a basic loop that used repeated addition to perform multiplications of integers and were later asked to write a similar program for exponentiation. Each class spent approximately the same amount of time covering the topics. Three weeks after instruction, students from each class were individually interviewed and asked a series of questions involving exponents, logarithmic expressions, logarithmic properties, and equations involving logarithmic expressions. While students in both groups were able to evaluate simple calculations, students in the experimental group were able to recall more properties of exponents and logarithms than the control group. These students were also able to provide justifications for the properties - unlike the students in the control group. Weber also reported that students in the experimental group were more likely to catch their mistakes when it came to identifying and justifying properties of logarithms and exponents. This data highlights the importance of and need for more coherent and conceptually taught lessons for exponents,

logarithmic expressions and logarithmic functions. This finding provides a compelling

argument for the benefits of curriculum and instruction that is more conceptually focused.

CONCEPTUAL ANALYSIS

In this section, I present the conceptual analysis that guided the design of my

intervention and goals for student learning of the idea of logarithm. In general,

conceptual analysis is used to describe the mental operations that might explain why

people think the way that they do (Glasersfeld, 1995). In this conceptual analysis, I

convey my understanding of the idea of logarithm. In doing so, I focus on major

constructions that need to be made as one develops the idea of logarithm for themselves.

For example, I defined $\log_b(m)$ to represent the number of b-tupling periods it takes to

result in an m-tupling. To illustrate the usefulness of this definition, consider a task and

solution (Figure 2.1).

The starfish population in Hawaii has increased 20% per year since 1990 and is

modeled by the function $f(t) = 1500(1.2)^t$, with t representing the number of years since

1990. Determine how long it will take for the population to reach 3480 starfish.

$$(1) \quad f(t) = 1500(1.2)^t$$
$$(2) \quad 3480 = 1500(1.2)^t$$
$$(3) \quad \frac{3480}{1500} = (1.2)^t$$
$$(4) \quad 2.32 = (1.2)^t$$
$$(5) \quad t = \log_{1.2}(2.32)$$
$$(6) \quad t \approx 4.6 \text{ years}$$

Figure 2.1. A Solution to an Exponential Function

In line (3), we see the ratio $\frac{3480}{1500}$. This calculates the factor by which the initial value of the exponential function grows. In particular, in the unspecified amount of time, the population of starfish grows by a factor of 2.32, or 2.32-tuples. Therefore, to determine precisely how long it takes for the population to 2.32-tuple, we must utilize the fact that the population of starfish 1.2-tuples every year, and ask the question, "How many years (1.2-tupling periods) does it take to 2.32-tuple?" Using logarithmic notation, we can represent this exact value as $\log_{1.2}(2.32)$. Then, with the use of technology, we can determine $\log_{1.2}(2.32) \approx 4.6$, and can conclude that after approximately 4.6 1.2-tupling periods, or years, the starfish population will reach 3480 starfish. This definition for logarithm relies on the understanding that a designated tupling-period can be used to measure a different tupling-period. Of course, in order to discuss these ideas in a meaningful manner, the student must also develop a meaning for division as measurement, growth factors, tuplings and tupling-periods, and logarithmic notation as determining how many *base*-tupling periods are needed to grow by another factor.

The meanings I hypothesize to be critical for understanding exponential and logarithmic ideas are further clarified in the following Taxonomy (Table 2.1). The table provides a more detailed description of the specific ways of thinking and understandings that are productive for students to construct in the process of learning about logarithms and logarithmic functions. This paper describes one conception that assists students in developing a number of these desired understandings (by means of more fine grained constructions).

Table 2.1

Taxonomy of the Idea of Logarithm

Conceptions related to the idea of logarithm	Desired understanding
Division as measurement	To measure a value of Quantity A in terms of a value of Quantity B, we write $\dfrac{\text{Value of Quantity A}}{\text{Value of Quantity B}}$. If $\dfrac{\text{Value of Quantity A}}{\text{Value of Quantity B}} = m$, we say Quantity A is m times as large as Quantity B.
Multiplying by A and then multiplying by B has the same overall effect as multiplying by AB. ($\times A \times B = \times AB$)	If a value A-tuples (becomes A times as large) and then the A-tupled value B-tuples (becomes B times as large), overall the initial value will AB-tuple (become AB times as large).
Growth Factor	When coordinating the values of two quantities, if the value of the first quantity increases by n-units while the next value of the second quantity is m times as large as its current value, then the n-unit growth factor is m.
The Exponential Relationship	When relating two continuous quantities, Quantity A and Quantity B, if for equal changes in Quantity A, Quantity B grows by a constant factor, then the two quantities have an exponential relationship.
Tuples (VERB)	If the value of a quantity becomes m times as large, we say the quantity's value m-tuples.
Tuplings (NOUN)	An m-tupling is the event in which the value of a quantity becomes m times as large.
Tupling period	An m-tupling period is the amount of change in the independent quantity needed for the dependent quantity to become m times as large.
Exponent (on a value, b)	The number of elapsed b-tupling periods. Written where x is the number of elapsed b-tupling periods.
Growth Factor Conversion	The factor by which a quantity will grow over x, b-tupling periods is represented as b^x . If $c^1 = b^x$, then one c-tupling period is the same as x b-tupling periods.
LP3: $\log_b(X^y) = y \log_b(X)$	The number of b-tupling periods needed to experience an X^y-tupling is y times as large as the number of b-tupling periods needed to experience an X-tupling.

LP4: $\log_b(X) = \dfrac{\log_b(X)}{\log_b(b)} = \dfrac{\log_c(X)}{\log_c(b)}$	The X-tupling period will always be k times as large as the b-tupling period (this value does not depend on the unit chosen to measure both the X- and b-tupling periods).
LP5: $\log_b(b^x) = x$	The number of b-tupling periods needed to experience a b-tupling, x times, is x.
LP6: $b^{\log_b(x)} = x$	If a value b-tuples $\log_b(x)$ times, the number of b-tupling periods needed to result in an x-tupling, the value will x-tuple.
The Logarithmic Function	A covarying relationship between an x-tupling and the number of b-tupling periods needed to experience an x-tupling ($\log_b(x)$). These two quantities vary in such a way that every value of the x-tupling determines exactly one value of the number of b-tupling periods needed to experience an x-tupling.

This Taxonomy highlights the reasoning abilities and understandings that are included in my hypothetical learning trajectory (HLT) (Simon, 1995; Simon & Tzur, 2004) for learning the idea of logarithm. My HLTs consisted of a list of learning goals for students, tasks intended to promote such learning goals, and hypotheses about student learning within the mathematical context. The task associated with each learning goal typically progressed through four stages based on my hypotheses of student learning: (1) Activity Problem – offers a starting point for students, (2) Optional Activity Problem – encourages student to consider relationships between quantities and effects of previous actions but can still be verified by engaging with the activity, (3) Non-activity Problem – encourages student to reflect on his thinking as he engaged with the previous problems while considering relationships between quantities, (4) Abstract Problem – encourages student to generalize through reflection on activity-effect relationships. For example, the task designed to support students in developing an understanding of the first logarithmic property (Figure 2.5 in Results section) began with an activity that had the student draw

Sparky at different moments along a timeline. The student was provided information about Sparky's initial height (in the form of a picture), the factors that Sparky's height grew by, and the corresponding number of weeks it took to grow by the provided factors. The student could rely on her drawing to make conclusions about the overall growth factor after two consecutive tuplings and to determine the overall-tupling period. The remaining questions did not require the student to draw a picture documenting Sparky's height, but still provided information about the consecutive tuplings that occurred and asked the student to determine the overall tupling and its corresponding tupling period. The second question involved growth factors that were small enough that if the student wished to document Sparky's height with a diagram, she could. However the third question involved growth factors that were too large and the fourth question generalized the growth factors as variables – therefore requiring the student to think about how she could determine the overall growth factor and its corresponding tupling period in order to answer the questions rather than relying on the drawing activity. This progression was specifically designed to provide the student opportunities to advance and strengthen her thinking, while reflecting on the preceding questions. This progression was inspired by Simon and Tzur's (2004) claim that a student learns (develops new ways of thinking) when she reflects on her actions and their effects when completing a task. The task design was further informed by their specific task sequence on equivalent fractions.

THEORETICAL PERSPECTIVE

This study proposes ways of thinking that are productive for learning and using the idea of logarithm. The intention is not to classify how every student will come to learn the idea of logarithm, but rather to model the mathematical realities of individual

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students. This information should provide insights about the mental constructions (ways of thinking) that are critical for developing in all students. The theoretical perspective that guided the design of this study is radical constructivism (Glasersfeld, 1995). This theory proposes that knowledge is constructed in the mind of an individual and therefore cannot be directly accessed by anyone else. Under this perspective, researchers can, at best, attempt to form a model of students' thinking (Steffe & Thompson 2000). A model is considered reliable when the student's utterances, written work, and movements are in alignment with the model and does not necessarily have to be mathematically correct. That is, if the subject responds in a way that is mathematically incorrect, the researcher will be interested in modeling how the student was thinking for his claims to make sense to him. When a researcher develops such models, she is trying to model the student's cognitive structures that comprise knowledge, known as schemes. These structures are organizations of mental actions or mental operations (reversible actions) and may even be complex and contain other schemes (Piaget, 2001). An action is "all movement, all thought, or all emotion – [that] responds to a need" (Piaget, 1967, p. 6). Researchers rely on a student's observable actions when attempting to form models of his schemes, such as utterances, written work, movements or body language.

Researchers who are interested in using the teaching experiment methodology to model student learning (i.e. cognitive structuring and restructuring) should make sure to provide students with opportunities for reflection (Derry, 1996). The goal of my study was to model my subjects' knowledge development of concepts foundational to the idea of logarithm as they completed lessons in a teaching experiment designed to advance their meanings. My data collection and analysis focused on understanding and

characterizing the meanings the students constructed as they engaged in tasks and responded to questions that provided opportunities for reflection.

METHODOLOGY

For this study, I conducted two teaching experiments (Steffe & Thompson, 2000) that focused on advancing and characterizing students' ways of thinking and meanings as they completed lessons that were designed to support their understanding of the idea of logarithm. I administered a pre-study survey to 124 students in four precalculus sections and recruited two precalculus undergraduate students, Abigail and Aaliyah (both pseudonyms), to participate in the teaching experiments. Abigail was chosen to participate in the teaching experiment because her responses to the pre-study survey suggested she had already developed the prerequisite understandings for learning the idea of logarithm and she explicitly stated that she did not know anything about logarithms. Abigail participated in six 2-hour teaching episodes over the course of a five-week period as a substitute for attending class on logarithmic ideas. Prior to the teaching sessions, Abigail attended classes on exponential functions. Her grade in the class at the start of the interviews was an A. Aaliyah, on the other hand, participated in the teaching experiment after attending her classes on the topic. Her pre-study survey responses suggested she had already worked with the idea of logarithm, but may have developed unproductive understandings. Her responses also revealed that she still needed to develop a few of the prerequisite understandings to the idea of logarithm. We met 7 times over the course of a 3.5-week period for approximately 1.5 hours each session. Her grade in the class at the start of the interviews was an A.

Prior to the start of each teaching experiment, I updated my hypothetical learning

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trajectories (Simon, 1995; Simon & Tzur, 2004) for the idea of logarithm. I referred to these hypothetical learning trajectories as I developed and updated the progression of tasks used for each teaching experiment. The instructional sequence designed for this study evolved from the conceptually-based exponential situation Ellis et al. (2012, 2015) created focusing on Jactus the Cactus – the mystical cactus whose height grew exponentially with respect to time. To supplement the instructional sequence involving Sparky the Saguaro, a cactus whose height doubled in size each week, the students worked with two additional exponential situations. The first focused on Mary – a woman who made a batch of bread dough and set the dough in a warm place to rise. Mary noticed that the bread dough tripled in size every two hours. The second exponential situation involved filling a pool with water – specifically the volume of the water in the pool 1.5-tupled each hour. The tasks used in this study were designed to support the subjects in learning the foundational ideas of exponential functions and to promote a contextual interpretation of the idea of logarithm before introducing a generalized form. The subjects were not asked to complete assignments between teaching episodes. I discuss the repercussions of this decision in the Discussion section.

Following each teaching episode, I conducted a retrospective analysis (Steffe & Thompson, 2000) and analyzed the students' actions (verbal, written, and motions) following an open, axial and selective coding approach (Strauss & Corbin, 1998) in an attempt to develop models of student thinking and to inform future sessions. As an example, I considered the students' use and explanation of the Geogebra applet images in the context of their solutions as a way to gain insights into their conceptions of the covarying quantities in the situation. During the analysis stage I watched the recordings

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of each interview and made note of shifts in the student's thinking and identified moments when the student made an essential mistake (Steffe & Thompson, 2000). In the subsequent episodes I tested my hypotheses, modifying my claims as needed, and asked questions I thought would support my subject in confronting problematic conceptions and develop desirable conceptions and ways of thinking (as described in my conceptual analysis). Following the teaching experiments, I revisited every episode again to refine my categorizations. The presentation of students' thinking on each task in the teaching experiment is beyond the scope of this paper. My results describe two students' development of essential meanings and ideas that are described in my conceptual analysis of logarithm, and revealed to be critical for constructing a strong meaning for the idea of logarithm.

RESULTS

In the following sections I report my analysis of the discussions between me and the subjects as they responded to tasks designed to advance their understanding of the idea of logarithm. In my analysis of these discussions I use my theoretical constructs to characterize progress in student reasoning and understanding.

Tuples, Tuplings, Tupling Periods

Introducing the language

During their exploratory interviews I probed Abigail's and Aaliyah's meanings for colloquial terms such as doubles, tripled or quadruple. Their responses support that both subjects viewed each of these terms as describing multiplicative growth. When I prompted Abigail to explain what it meant for a cactus to double in height she replied by saying that the cactus' height would become 2 times as tall. When prompting each

subject to describe how she would determine the cactus' new height if his height doubled or tripled in size, both students stated that she would multiply the height of a cactus by 2 or 3, respectively. Both students further acknowledged that there was no similar colloquial term to describe a cactus' growth when its height grew by a factor of about 6.78. I used this task as a segue for introducing the tupling language. I introduced the tupling language by discussing how instead of saying the cactus' height became 6.78 times as large, we can simply say the cactus' height 6.78-tupled. I then explained that the term 2-tuple could be used to replace double. Subsequent to this discussion, each student attempted to use the tupling language to describe the growth of a cactus that began at m inches tall and grew to be k inches tall. For example, Aaliyah said, "you take k and divide it by m to say that it grew to be whatever-the-number-may-be-tupled." Abigail responded by saying, "so it grew by k, m (*wrote k/m*)-tupled." It is noteworthy that even though the students' phrasing relative to tupling was sometimes lacking in precision, they both provided responses that suggested that they associated the use of the term tupled with multiplicative growth.

Following the teaching episode where the new language was introduced, Abigail consistently used the tupling language correctly – including correct tense (e.g., 2-tuple for double, 2-tupled for doubled, 2-tupling for doubling.). There were a few times when Aaliyah's word choice suggested that she viewed a "tuple" as a new unit of measure. For example, early on when asked to describe the overall growth of a cactus that X-tupled and then Y-tupled in height, Aaliyah stated, "you would take X and times it by Y to give you your amount of tuplings." However, in the few instances when similar descriptions arose later in the teaching experiment, Aaliyah appeared to drop the preceding number out of

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carelessness and was more specific when probed.

Conceptualizing the base-tupling period as a multiplicative object

Over the course of both teaching experiments, Abigail and Aaliyah's use of the tupling period language improved in precision and accuracy as a result of repeated requests to use and interpret the new language. At first, both students either exclusively described a *b*-tupling period in terms of a specific amount of elapsed time, or as the number of times a quantity *b*-tupled in size. For example, when I asked Abigail to describe what a 3-tupling period meant in the Mary's bread dough context[15], she said, "Um, each time the dough is three times as large from when it started." It is noteworthy that Abigail did not mention the amount of time needed for the dough to 3-tuple in size; her focus was on the multiplicative growth of the bread dough's size. However, when I asked Abigail how long a 3-tupling period was, she responded immediately, "two hours." These excerpts reveal that Abigail was capable of interpreting a 3-tupling period as representing the specific amount of elapsed time necessary for the bread dough to 3-tuple. It is noteworthy that Abigail did not spontaneously coordinate the elapsed time with the description of multiplicative growth. I had similar conversations with Aaliyah. When I asked her what the 5 in 3^5 represented, she replied, "All I'm thinking of right now is how it can be 5 times the bread can 3-tuple in size." However, when I asked how many 2-hour periods elapsed in the same case, Aaliyah replied, "Five. Oh, it could also represent how many times the two-hour time frame has um, elapsed, or how many of them they were." Like Abigail, Aaliyah first attended to the multiplicative growth of the bread dough and then attended to the corresponding elapsed time. There was no evidence to suggest that

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[15] Mary is a woman who made a batch of bread dough and set the dough in a warm place to rise. Mary noticed that the bread dough tripled in size every two hours.

she simultaneously coordinated these two quantities on her own. However, both of these conversations were encouraging because they suggested that both students were on their way to conceptualizing a *b*-tupling period as a multiplicative object. Specifically, coordinating that a *b*-tupling period is a change in the input quantity corresponding to the event in which the output quantity *b*-tuples.

The term "*b*-tupling period" allows one to simultaneously describe changes in the input quantity of an exponential relationship with the multiplicative growth in the output quantity. On the other hand, just discussing the change in the input quantity places a burden on the students to recall how the output quantity changes in tandem. In fact, both Abigail and Aaliyah experienced difficulties reasoning with multiplicative growth when they were only provided information about the elapsed time. For example, Abigail's pre-study survey responses, her responses during the exploratory interview, and her expressed meanings in the first half of her first teaching episode suggest that she understood that if a quantity's value *A*-tupled and then the *A*-tupled value *B*-tupled, then overall the initial value would have experienced an *AB*-tupling. However, in the second half of the first teaching episode, after becoming familiar with the Sparky situation and shortly after justifying why an *X*-tupling followed by a *Y*-tupling corresponds to an overall *XY*-tupling, Abigail surprisingly claimed that the 3-week growth factor in the Sparky situation would be 6 "because it will be three periods times two, cause it's doubling." The introduction of a linear quantity such as elapsed time altered Abigail's approach to multiplicative growth. Using the Geogebra applet, Abigail and I viewed Sparky's growth for any three-week change and Abigail observed that at the end of the 3-week interval, Sparky's height was 8 times as large as the Sparky's height at the start of

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the 3-week interval. When asked why she thought that, she replied, "Three week period. Well it would have to be…Oh! So it's just two (*feet*) times two times two times two. So it would be instead of um, three times two, which is what I was thinking as the factor, it would be 2 to the third. Two times two times two. Two to the third." Both students exhibited similar difficulties when they confounded tupling periods with tuplings.

During each teaching experiment, Abigail and Aaliyah sometimes referred to a *b*-tupling period as a *b*-tupling (or vice versa). This conception created issues for the students as they worked on a variety of tasks, particularly because the term *b*-tupling period provides information about the input quantity to an exponential relationship and the term *b*-tupling provides information about how the output quantity to an exponential relationship is growing. For example, toward the end of Aaliyah's teaching experiment, I asked her to interpret and approximate the value of $\log_2(3)$ after being provided arrows representing both the 2-tupling and 3-tupling periods. She first described $\log_2(3)$ to be "how many 2-tupling periods will it take to 3-tuple," and then stated, "We can take your three tuplings and you can see how many times a 2-tupling period will fit into a 3-tuple. So that would be one, a little bit over a half, well technically 3 over 2 is one and a half." Aaliyah's first approximation was based on how many of the 2-tupling period arrows were needed to make up or "fit into" the 3-tupling period arrow (a correct approach). However, she also multiplicatively compared the tuplings themselves to conclude that 3 is 1.5 times as large as 2. Aaliyah went on to state that the length of the provided arrows must have been incorrect since the 3-tupling period arrow was not exactly one and a half 2-tupling period arrows long. In this example Aaliyah did not appear to distinguish

between her two methods for answering the question. While tuplings and tupling periods are obviously related, it is important that students develop individual understandings for both phrases – particularly because the idea of logarithm relies both on an understanding of tuplings and tupling periods.

The Logarithmic Definition

Before Introducing Logarithmic Notation

During both teaching experiments, I engaged Abigail and Aaliyah with tasks (Figure 2.2) designed to assist them in conceptualizing the quantities that logarithmic expressions are used to represent. The purpose of the first task was to get the students to identify the overall tupling experienced by the volume of the water in the pool (essentially determining what would be the desired argument to a logarithmic expression). The second task was designed to assist the students in conceptualizing the quantity represented by a logarithmic expression.

1. Suppose a pool is being filled with water so that the volume of water in the pool 1.5-tuples every hour. At 9am, there were 15 gallons of water in the pool. Over some amount of time (since 9am), the volume of the pool reached 123 gallons. Therefore, in this unknown amount of time, the volume of water in the pool:

a.   15-tupled in size, or became 15 times as large

b.   15/123-tupled in size, or became 15/123 times as large

c.   123/15-tupled in size, or became 123/15 times as large

d.   123-tupled in size

e.   None of the above.

2. Suppose a pool is being filled with water so that the volume of water in the pool 1.5-tuples every hour. At 9am, there were 15 gallons of water in the pool. Over some amount of time (since 9am), the volume of the pool reached 123 gallons. If we wish to determine the number of hours it takes for the volume to reach 123 gallons, we wish to determine:

A.   The number of 123/15-tupling periods it takes to 1.5-tuple

B.   The number of 1.5-tupling periods it takes to 123/15-tuple

C.   The number of 1.5-tupling periods it takes to 123-tuple

D.   The number of 123-tupling periods it takes to 1.5-tuple

E.   None of the above.

Figure 2.2. Tasks Motivating the Idea of Logarithm

In response to the first task, both students claimed that the growth factor could be

determined by dividing the final output value by the initial output value, therefore

choosing answer choice C. After reading through the possible answer choices to the

second question, Abigail settled on answer choice B but was unable to explain why she

knew that was the correct answer saying, "It's hard to word. It's just hard to- I don't

know how to." However, when I asked her if there was anything about the other answer

choices that made her worried that she picked the wrong one, Abigail confidently replied,

"No." I hypothesize that this was because she was beginning to conceptualize the number

of 1.5-tupling periods as also representing the number of elapsed hours. That is, I

hypothesize Abigail may have interpreted answer choice B as also stating: The number of

hours it takes to 123/15-tuple. On the other hand, Aaliyah, who had already been

introduced to logarithms in her class, acknowledged that to find the desired number of

hours, she would solve $123 = 15(1.5)^x$ for x by "put[ing] log in." In particular, she wanted

to evaluate $\dfrac{\log(123/15)}{\log(1.5)}$ to determine the desired number of hours, but expressed that she

didn't know why her method worked other than it "gets x by itself." I suggested we

return to the task at hand in hopes of eventually understanding what is meant by the

expression $\dfrac{\log(123/15)}{\log(1.5)}$. After Aaliyah and I reexamined what was meant by 1.5-tupling

period, she also settled on answer choice B stating, "I mean it looks like it matches up

because 1.5 is replacing the x amount of hours, so for the most part it all looks right."

Both students acknowledged that problems like those in Figure 2.2 were essentially

asking for the value of a specific exponent, however they each made an interesting

attempt to determine the exact value.

When Abigail and Aaliyah worked with problems of the form $\dfrac{a}{b} = c^x$ with x

unknown and difficult to estimate, they both resorted to evaluating $\dfrac{a}{b}$ and claimed that

the value of the exponent was either equal to or could be determined from the resulting

value. For example, when Abigail attempted to solve for h in the equation $\dfrac{200}{45} = 3^h$ she

appeared to resort back to applying a familiar procedure as evidenced by her saying,

"because h isn't by itself." She then determined 200/45 was approximately 4.4 and

claimed that 4.4 "time periods" had elapsed. Like Abigail, Aaliyah attempted to use a growth factor to determine the number of 3-tupling periods when working with the equation $200 = 45(3)^{x/2}$ stating, "Wouldn't we take the 200 divided by 45 and then that answer divided by 2?...Because then that'll give you the number of times three can 3-tuple or triple per say." These excerpts suggest that both Abigail and Aaliyah were attempting to solve for the unknown in the exponent – despite the fact that I did not ask them to determine this value. In an effort to determine the value of the unknown in the exponent, the students performed the only calculation available to them in an attempt to determine the (logarithmic) value. At this point I reminded the students that I was not asking them to determine the specific value, but rather to interpret what quantity's value we would be finding if we did determine the exponent. I also informed the students that in some cases, these (soon-to-be-called logarithmic) values were easier to determine.

After Introducing Logarithmic Notation

I introduced logarithmic notation to both students after prompting them to evaluate or approximate the value of a logarithm for various numeric values. That is, for example, I first asked both of the students to determine the number of 2-tupling periods needed for Sparky's height to experience a 4-tupling before introducing them to the expression $\log_2(4)$. Abigail's initial attempts to represent the number of 2-tupling periods needed to grow by a specific factor sometimes involved the value of the answer itself. For example, she had already determined that the number of 2-tupling periods needed to 2-tuple was one. However, instead of representing this number as $\log_2(2)$, she wrote $\log_2(1)$ - writing the answer as the argument to the logarithmic function. After I

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interpreted what she wrote down and gave a specific example of my own, Abigail correctly represented the specific logarithmic values – representing the number of 2-tupling periods needed to 4-tuple as $\log_2(4)$ and the number of 2-tupling periods needed to 5-tuple as $\log_2(5)$, for example. Aaliyah also correctly represented the specific logarithmic values after an instructor example. I decided to evaluate the students' understandings of logarithmic expressions using the Geogebra applet designed to examine Sparky's growth.

One of the viewing options in the Sparky Geogebra applet allows the user to view a dynamic image of Sparky at its current height simultaneously with a dynamic image of Sparky some number of "weeks before" (entered by the user). To evaluate the students' understandings of logarithmic expressions, I asked each subject to explain what she would see if I entered expressions such as $\log_2(6)$ in the "Weeks Before" box. In general, Abigail's response involved approximating the number of weeks elapsed from when Sparky was the height of the Previous Sparky to when Sparky was the height of current Sparky. For example, when we entered $\log_2(6)$ in the "Weeks Before" box, she anticipated that the two cacti would be a little over 2 weeks apart. She also stated how many times as large the current Sparky's height would be than the Previous Sparky's height. That is, when we entered $\log_2(6)$ in the "Weeks Before" box, she stated that the current Sparky would "be 6 times larger" than the previous Sparky. Abigail's descriptions of her anticipations suggest she coordinated both the amount of elapsed time represented by the logarithmic expression as well as the growth factor conveyed in the argument to the logarithm. On the other hand, Aaliyah did not spontaneously attend to the

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elapsed time represented by the logarithmic expressions, but only attended to how the cacti's heights would (multiplicatively) compare. For example, when we entered $\log_2(3.75)$ in the "Weeks Before" box, she only stated that the current Sparky would be 3.75 times as large as the previous Sparky. However, when probed, Aaliyah also discussed the amount of time that would separate the Previous and Current Sparkies.

In these teaching experiments, the 2-tupling period was the most common tupling period used to measure all other tupling periods. In other words, in most of the logarithmic expressions discussed throughout the experiments, 2 was used as the base value. However, I also made sure to provide the students opportunities to share how they were thinking about logarithmic expressions and equations that involved other base values. For example, I asked the students to compare the expressions $\log_2(8)$ and $\log_4(8)$. Abigail and Aaliyah each stated that both expressions were measuring the same overall growth (the 8-tupling) in two different ways – the first using the 2-tupling or one-week period, and the second using the 4-tupling or two-week period as the unit of measure. Abigail went on to say that $\log_2(8)$ should be twice as large as $\log_4(8)$ because the first expression measured the 8-tupling period with a one-week period and the second expression measured the 8-tupling period with a 2-week period. Aaliyah came to a similar realization after comparing the specific values of $\log_2(8)$ with $\log_4(8)$, $\log_2(16)$ with $\log_4(16)$, and $\log_2(64)$ with $\log_4(64)$, stating that the "4-tupling period will always be two times as large as the 2-tupling period" and concluded $\frac{1}{2}\log_2(64) = \log_4(64)$. The students' responses to this task suggest that they each began

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to develop an understanding that the argument to the logarithmic expression was the tupling period being measured and that the base value in the expression represented the specific tupling period that was being used as the unit of measure. The students displayed similar ways of thinking when working with logarithmic equations involving more than one logarithmic expression. For example, when Abigail was asked in a later teaching episode how the logarithmic properties would change if instead of measuring everything in weeks we measured using different amounts of elapsed time, she stated, "I'm pretty sure the bases would just change." She continued to think out loud in an attempt to verify her claim saying, "Ok, if it was a 6-day period. Because 2-tupling is a one week period. So six days would be um, yeah so just the bases would change." Abigail's attention to the dual nature of the tupling periods represented as base values in the logarithmic expressions suggests that Abigail viewed the subscript of logarithmic notation as also conveying an elapsed amount of time. However, Abigail knew to write the growth factor corresponding to the given amount of time as the base value in the logarithmic notation. For example, in the task Abigail was referring to, she concluded that changing the measuring stick from a 2-tupling period to a 6-day period would result in a new base of $2^{6/7}$ - the 6-day growth factor. This example again suggests Abigail viewed the arguments to the logarithmic expressions as the tupling periods being measured (therefore left unchanged if measured using a different ruler) and that the base value in the expression represented the specific tupling period that was being used as the unit of measure.

Imagery

Abigail: Before introducing Abigail to logarithmic notation, I asked her to determine the number of 2-tupling periods needed in order for Sparky to grow by different factors (specifically powers of 2). Abigail appeared to visualize the equivalent exponential equation and used repeated multiplication to determine and verify the specific number of 2-tupling periods (i.e., the exponent on 2) needed to result in the specific growth factor. For example, when Abigail was asked to determine the number of 2-tupling periods needed to 2-tuple and 4-tuple, she gave the answers 1 and 2 respectively and stated, "I'm seeing this as 2 to the 1 equals 2. So this [*the exponent*] is the number that we're looking for. So 2 to the 2 equals 4." In cases where the answers weren't whole numbers she applied this same approach and estimated the whole number values the answer would fall between. This method appeared to be how Abigail typically visualized the number of b-tupling periods needed to m-tuple throughout the teaching experiment – also after logarithmic notation was introduced. For example, toward the end of the teaching experiment, Abigail was asked to justify the first logarithmic property. In response, she wrote exponential equations for each logarithmic expression, determined the value of the exponent, and verified that the equality held (Figure 2.3).

Figure 2.3. Abigail's Work to Justify First Logarithmic Property

Abigail expressed a desire to verify the second logarithmic property in a similar way, but I suggested she try to explain why the property is true without using specific values for *b*, *X* and *Y*. She proceeded to discuss increments of *b*-tupling periods, but did not attend to how the specific tuplings (*X*, *Y* and *X/Y*) were related, or what role they played in the property; in this case, we subtract one number of *b*-tupling periods from the other. I suggested we try imagining the three tupling periods as being represented using arrows. Abigail and I set up the diagram demonstrating the relationship between the tupling periods (Figure 2.4) and reasoned that *Y*-tupling and then *X/Y*-tupling will be equivalent to *X*-tupling. Afterwards, I asked her to explain the equation for the second property that she came up with (i.e. $\log_b(X / Y) = \log_b(X) - \log_b(Y)$). She stated, "OK, ok! The number of times we b-tuple for something to *X*-tuple minus the number of *b*-tuplings it takes for us to *Y*-tuple something, would equal, aha ok, would equal the number of *b*-tuplings it takes for something to *X*, *Y* (*X/Y*)-tuple." It appeared as though Abigail's engagement in this task helped her to begin to "see" the relationship between the three tupling periods. While Abigail still had a tendency to imagine the equivalent exponential notation throughout the experiment, this conversation opened a door to imagining the relationships between tupling periods differently.
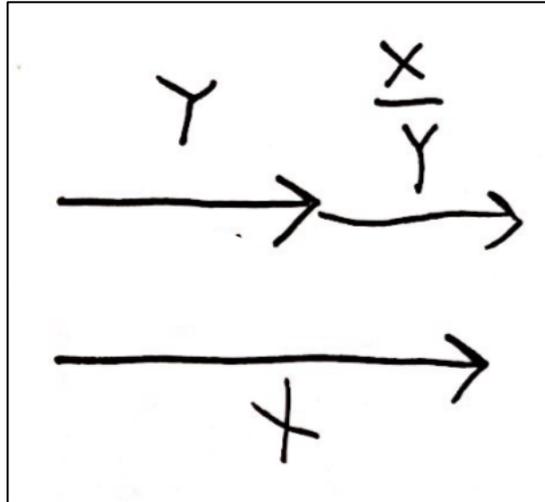
Figure 2.4. Diagram Used to Represent the Relationship Between Tupling Periods in Second Logarithmic Property

Aaliyah: During my interactions with Aaliyah, we spent extra time representing tupling periods using horizontal arrows. As a result, it appeared Aaliyah often imagined measuring one tupling period using another when solving tasks that were logarithmic in nature. For example, when trying to determine the number of 4-tupling periods needed to 8-tuple in the Sparky context, Aaliyah stated, "Because without doing anything, 1-week, 2-tupling period, 2-weeks, 4-tupling period, 3-weeks, 8-tupling periods. We're looking at a 4-tupling period to find a 8-tupling period. And so that means you're taking two weeks. So that's basically saying oh how many times can we take the 4-tupling period and try to find the 8 and it's one and a half because the four-tupling period won't fit twice. So only half of it will." After I introduced logarithmic notation, Aaliyah demonstrated an understanding that logarithmic equations could be written as exponential equations. However, this understanding appeared to be the result of a previous experience with Euler's definition. Aaliyah stated she could "switch the log conversion so then it doesn't have the log in it." As the teaching experiment progressed, Aaliyah did not always

represent different tupling periods using arrows (unless encouraged to do so). During

some of these conversations, there was not enough evidence to suggest that she applied

the same measuring process to determine logarithmic values.

Logarithmic Property #1: $\log_b(X) + \log_b(Y) = \log_b(XY)$

The Task (Answers listed in Appendix)
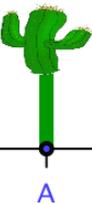
1.

A   B   C

(A) At some point in time, Sparky the cactus was this tall.

(B) After 1 week, Sparky's height doubled (2-tupled, became 2 times as large). Draw the resulting Sparky.

(C) After about 1.585 weeks, Sparky's height then tripled (3-tupled, became 3 times as large) from point (B). Draw the resulting Sparky.

- By what factor did Sparky grow from point (A) to point (C)? How long did it take to grow by this factor?
- In other words, overall Sparky's height will experience a [6]-tupling in ___weeks.
2. If Sparky's height 3-tuples *then* 5-tuples, overall his height will experience a ___-tupling.
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to 3-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to 5-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to [15]-tuple?
- Write an equation using logarithmic notation representing the relationship between these three values.
3. If Sparky's height 34-tuples *then* 57-tuples, overall his height will experience a ___-tupling.
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to 34-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to 57-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to [34*57]-tuple?
- Write an equation using logarithmic notation representing the relationship between these three values.
4. If Sparky's height *X*-tuples *then* *Y*-tuples, overall his height will experience a ___-tupling.
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to *X*-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to *Y*-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to [*XY*]-tuple?
- Write an equation using logarithmic notation representing the relationship between these three values.
5. Now, discuss how your equations would change had you measured in: 2-week periods, 1-day periods, 1-year periods, and *b*-tupling periods.

Figure 2.5. Task Designed to Assist Students in Developing an Understanding of the First Logarithmic Property

This task was designed to assist the students in developing an understanding of

the first logarithmic property. Specifically, the questions provided information of Sparky's growth over two consecutive periods of elapsed time (i.e., two sub-tuplings and their corresponding tupling periods). The students were guided to determine the overall-tupling and then asked to determine its corresponding tupling period. Lastly, the students were then asked to make generalizations about how all three tupling periods were related.

Abigail

For the first question, Abigail initially determined the overall tupling claiming that from point A to point C, Sparky's height would experience a 6-tupling. She appeared to rely on her understanding that an A-tupling immediately followed by a B-tupling results in an overall AB-tupling to make this conclusion. Abigail then relied on the timeline in the diagram to determine that the 6-tupling period would be 2.585 weeks. For the subsequent questions (2, 3, 4, & 5), a diagram of the situation was neither provided nor required of the students. Without drawing a diagram of the situation, Abigail determined that if Sparky 3-tupled and then immediately 5-tupled in height, overall his height would experience a 15-tupling. She then determined the number of weeks it would take to experience the 3- and 5-tuplings by evaluating $\log_2(3)$ and $\log_2(5)$, respectively, and noted that we could also determine the number of weeks needed for Sparky's height to 15-tuple by evaluating $\log_2(15)$. However, before we evaluated $\log_2(15)$, I asked Abigail if we could determine the number of 2-tupling periods needed for Sparky to 15-tuple using the information she had already determined (the number of 2-tupling periods needed for Sparky's height to 3-tuple and the number of 2-tupling periods needed for Sparky's height to 5-tuple). Abigail looked back at the first task and said, "3-tuples then

5-tuples. Oh! So if [he] 3-tuples and then he 5-tuples, ooh, OK I'm going to say I would say that we would add these ($\log_2(3)$ and $\log_2(5)$) together." This excerpt suggests that Abigail reflected on the effects of the actions she performed in the first task to conclude that in order to determine an overall-tupling period, she could add the two consecutive sub-tupling periods. Abigail completed the remaining tasks (questions 3 and 4) using this same reasoning – that, to determine the number of weeks it would take to *AB*-tuple, one could add the number of weeks needed to *A*-tupled and the number of weeks needed to *B*-tuple. In addition, she completed the remaining questions by developing the relationship $\log_2(X) + \log_2(Y) = \log_2(XY)$ and concluded that measuring the tupling periods using a different measurement would require a change in the base value.

Aaliyah

Aaliyah approached this set of questions in a similar manner to Abigail. One difference was that during the second question, before being asked to revisit her thinking in the first question, Aaliyah performed a variety of calculations with the values of $\log_2(3)$ and $\log_2(5)$ in an effort to determine the number of weeks it would take Sparky to 15-tuple. For example, she first attempted to multiply 1.585 and 2.322 (the approximate values of $\log_2(3)$ and $\log_2(5)$, respectively). She also suggested that she could multiply 1.585 weeks by 5 stating, "So then you want to take that many weeks [*1.585*] and you want to 5-tuple it." This excerpt suggests Aaliyah did not distinguish the differences between tuplings and the corresponding tupling periods. However, after looking back at her work for the first question, she concluded that the 15-tupling period could be determined by adding the values of $\log_2(3)$ and $\log_2(5)$.

Logarithmic Property #2: $\log_b(X) - \log_b(Y) = \log_b(X/Y)$

The Task (Answers listed in Appendix)

1.

A	B	C

(A) At some point in time, Sparky the cactus was this tall.

(B) After some time, Sparky's height grew by some factor. {2^{nd}:} Draw the resulting Sparky. From (B) to (C) he 2-tupled over 1 week.

(C) After a <u>total</u> of 3.3219 weeks, Sparky's height 10-tupled in size from point (A). {1^{st}:} Draw the resulting Sparky.

- By what factor did Sparky grow from point (A) to point (B)? How long did it take Sparky to 5-tuple?

2. Suppose in some unknown amount of time, Sparky's height ___-tuples. 2 weeks (2-tupling periods) later, Sparky's height 4-tupled in size. If overall, Sparky's height 12-tupled in size over a 3.585 week (2-tupling) period, how long does it take for Sparky's height to experience the ___-tupling?

- How many 2-tupling (1-week) periods need to elapse for Sparky's height to 12-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to 4-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to [3]-tuple?
- Write an equation using logarithmic notation representing the relationship between these three values.

3. If Sparky's height Y-tuples *then* ___-tuples, overall his height will experience an X-tupling.

- How many 2-tupling (1-week) periods need to elapse for Sparky's height to X-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to Y-tuple?
- How many 2-tupling (1-week) periods need to elapse for Sparky's height to [X/Y]-tuple?
- Write an equation using logarithmic notation representing the relationship between these three values.

Figure 2.6. Task Designed to Assist Students in Developing an Understanding of the Second Logarithmic Property

This task was designed to assist the students in developing the second logarithmic property. Specifically, the questions provided information of Sparky's overall tupling during a specific period of time as well as information of a sub-tupling and its corresponding tupling period. The students were guided to determine the remaining sub-

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tupling and then asked to determine its corresponding tupling period. Lastly, the students were then asked to make generalizations about how all three tupling periods were related.

Abigail

Abigail read through the first question and immediately concluded that since Sparky 2-tupled in height from point B to point C, he would have to be half of Sparky's height at point C and therefore 5-times as tall as the Sparky at point A. She concluded the question stating that it would take Sparky 2.3219 weeks to experience the 5-tupling by subtracting the 1 week that it took to double from the 3.3219 weeks it took to 10-tuple. For the remaining questions, a diagram of the situation was neither provided nor required of the students. Abigail chose to not draw Sparky's height at the various stages of his growth, but rather drew a timeline labeling both the tuplings and corresponding tupling periods as she read the question (Figure 2.7): "OK, so suppose in some unknown amount of time, Sparky's height blank tuples. OK, so… some unknown amount of time, Sparky's height blank tuples. Two weeks later, OK so we don't, we don't know the starting point. But this is the unknown tupling, then two weeks later, Sparky's height 4-tupled in size, so times four. If overall Sparky's height 12-tupled in size, oh my goodness, OK. Overall Sparky's height 12-tupled in size, how long does it take for Sparky's height to experience the blank tupling?"
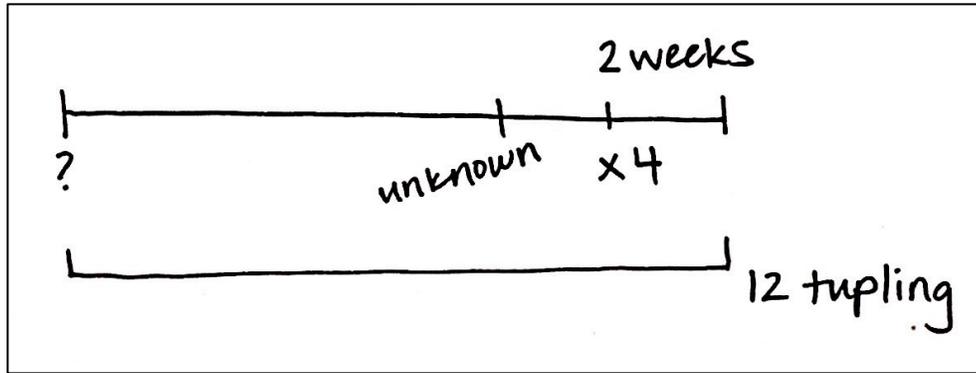
Figure 2.7. Abigail's Drawing for the Second Problem in the Set of Tasks

Referring to her timeline, Abigail stated, "OK. I'm just trying to figure out if I should subtract this [*12-4*]? Yeah, I think I, I think that's what it would be. So at this unknown time, I'm going to say he 8-tupled." In an effort to help Abigail catch her mistake, I initiated the following conversation:

Emily: So he 8-tuples in size and then 4-tuples in size

Abigail: OK

Emily: What's the overall tupling?

Abigail: 32.

Emily: Ok, so is an 8-tuple what we're looking for?

Abigail: No. So ok, I see. So 4 times 3 equals 12. So he 3-tupled. At this point he
          had 3-tupled. That's why I was like, "I don't know if I should subtract,"
          OK, two weeks later, if overall Sparky's height 12-tuples in 3.585 weeks,
          how long does it take for Sparky's height to experience the 3-tupling. OK
          so if this total is 3.585 minus 2 is 1.585 weeks.

Abigail's response revealed that she developed the understanding that she needed to find

the number that when multiplied by 4 results in 12 in order to find the missing sub-tupling value after reflecting on the relationship between the individual and overall growth factors. After determining this value, Abigail recognized that all she had left to do was find the difference between the corresponding tupling periods. In the generalized case, the students were told that Sparky's height Y-tupled, then grew by an unknown factor, overall resulting in an X-tupling. Abigail struggled to determine the sub-tupling of X/Y. Abigail proposed Sparky would X/Y-tuple, but didn't express confidence in her answer. She verified her thinking stating, "OK, well so if Y times something equals X, and X over Y, the Y's would cross- cancel out." This excerpt reveals that Abigail determined the sub-tupling value by choosing a hypothetical sub-tupling value, multiplying it with the provided sub-tupling value, and verifying that the provided overall-tupling value was the result.

Abigail experienced fewer difficulties when determining the missing sub-tupling period. When she worked with specific values, Abigail subtracted the sub-tupling period that was given from the overall tupling period to arrive at the correct answer. When working with logarithmic notation, Abigail relied on her understanding of the first logarithmic property to initially conclude that $\log_b(Y) + \log_b(X/Y) = \log_b(X)$ prior to her using algebra to construct the equation $\log_b(X/Y) = \log_b(X) - \log_b(Y)$. However, despite her conclusion, there was no evidence to suggest that she understood that to find the X/Y-tupling period, she could subtract the Y-tupling period from the X-tupling period. Furthermore, there was no evidence to suggest that she understood that the number of b-tupling periods needed to X/Y-tuple was the same as the difference between the number

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of *b*-tupling periods needed to *X*-tuple and the number of *b*-tupling periods needed to *Y*-tuple. In a later episode, as previously discussed in the Imagery section, there was evidence to suggest that Abigail's use of arrows to represent the individual tupling periods helped her see that finding the *X*/*Y*-tupling period could be achieved by subtracting the *Y*-tupling period from the *X*-tupling period.

Aaliyah

Aaliyah approached the first question in a similar way to Abigail. She first determined that from point A to point B, Sparky would 5-tuple in height and concluded that it would take 2.3219 weeks to experience the 5-tupling by subtracting the 1 week that it took to double from the 3.3219 weeks it took to 10-tuple. After reading the second question, Aaliyah expressed that she needed to draw a picture in order to answer the questions in the set. This suggests that Aaliyah did not reflect on the effects of her previous actions in the first task or at least did not feel comfortable relying on her reflection to answer the second question. Using her drawing, Aaliyah completed the second problem just as she did the first task. In the third question, Aaliyah struggled to determine the sub-tupling of *X*/*Y*, like Abigail. After she revisited her thinking on the two previous questions to see how she used the provided tuplings, she determined the missing sub-tupling would be *X*/*Y*. As Aaliyah attempted to write a generalization to determine the *X*/*Y*-tupling period using logarithms, she wrote the correct statement but claimed she felt she did something wrong. Aaliyah quickly changed her mind stating, "Nevermind, because it follows the log function. Cause if it's something minus something, then it's usually in the division form," apparently relying on something she learned prior to the teaching experiment.

It is worth noting that this was the last of the logarithmic properties covered with Aaliyah during her teaching experiment. We were unable to discuss the remaining properties because of the prolonged time spent focusing on a number of prerequisite understandings to the idea of logarithm (Kuper, 2018a). Therefore, for the remainder of the paper, I will be focusing on the understandings Abigail developed.

Logarithmic Property #3: $\log_b(X^y) = y\log_b(X)$

The Task (Answers listed in Appendix)

1. Suppose we observed Sparky's height 4-tuple in size three times in a row. On the paper provided, document Sparky's height at these moments.
   - Represent and determine the overall growth factor for this situation.
   - How many 2-tupling (1-week) periods need to elapse for Sparky's height to 4-tuple?
   - How many 2-tupling (1-week) periods need to elapse for Sparky's height to [64]-tuple?
   - Write an equation using logarithmic notation representing the relationship between these two values.
2. If fifty 4-tupling periods elapse, overall Sparky will ___-tuple.
   - How many 2-tupling (1-week) periods need to elapse for Sparky's height to 4-tuple?
   - How many 2-tupling (1-week) periods need to elapse for Sparky's height to [4^{50}]-tuple?
   - Write an equation using logarithmic notation representing the relationship between these two values.
3. If y X-tupling periods elapse, overall Sparky will ___-tuple.
   - How many 2-tupling (1-week) periods need to elapse for Sparky's height to X-tuple?
   - How many 2-tupling (1-week) periods need to elapse for Sparky's height to [X^y]-tuple?
   - Write an equation using logarithmic notation representing the relationship between these two values.

Figure 2.8. Task Designed to Assist Students in Developing an Understanding of the Third Logarithmic Property

The purpose of this task was to assist the students in developing the understanding that if y X-tupling periods elapse, an overall X^y-tupling would occur. Furthermore, the X^y-tupling period would be y times as large as the X-tupling period.

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Abigail

As Abigail began to work on the first question, she appeared to view tupling periods as happening at an instance instead of over some interval of the independent variable. This was revealed by her drawing the initial height of Sparky at week 0, then a 4-foot tall Sparky at week 2 and when pointing to the initial cactus she said, "Are we including this initial one?" as if to say the two cacti she drew represented two of the three 4-tuplings. I responded, "Can you use your fingers to demonstrate one 4-tupling period?" Abigail then moved her finger from the initial height to the 4-foot tall Sparky. She then proceeded to draw a 16-foot tall cactus and noted that after one more 4-tupling period Sparky would be 64 feet tall. I asked Abigail to think about the relationship between $\log_2(4)$ and $\log_2(64)$. As Abigail began discussing the corresponding tupling periods, I encouraged her to draw line segments to represent the tupling periods she was referring to. Immediately following that suggestion, we had the following conversation:

Abigail: OK. So this [*4-tupling period*] would be log base 2 of 4, and this [*64-tupling period*] is log base 2 of 64. So it could be 3 on the other side here, so 3 log base 2 of 4 equals log base 2 of 64.

Emily: And how did you decide to take three and multiply by log base 2 of 4?

Abigail: Um, because if this is two weeks, and this is 6 weeks, 6 divided by 2 is 3.

Emily: And where do we see that three popping up?

Abigail: Um, three times in a row

Abigail's interpretation of $3\log_2(4)$ made it appear as though she viewed the expression as representing three 4-tupling periods.

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When asked to consider the situation where fifty 4-tupling periods elapse (question 2), Abigail stated, "OK. If fifty 4-tupling periods elapse, overall Sparky will … ok … so 50 log base 2 of 4 equals…um, so that would be log base 2 of 4 to the 50?" I asked her to explain her thinking and she replied, "Um, so just comparing it to this one [*question #1*], um, they're both 4-tupling, 4-tupling periods…ok, yes, cause this [$\log_2(4)$] is a 4-tupling period and 50 of those have happened. And if these two equations, or these two expressions are equal to each other, um I, I don't know if it works for all of them, but if we just move this [*the 50*] inside the parentheses." Abigail appeared to be relying on her previous conclusions, so I encouraged her to make more sense of the relationship than just relying on "moving numbers around." This suggestion resulted in the following conversation:

Abigail: OK. So this is saying…this is a two, this is a two-week period (*points to*

$\log_2(4)$ *in expression* $50\log_2(4)$). OK. So 50 times…if 50 4-tupling

periods elapse, overall Sparky will 4 to the 50-tuple. Well yeah, I guess that

makes sense.

Emily: And why?

Abigail: Um, because if log base 2 of 4 is a two week period, and which is a 4-

tupling period, and 50 of those happen, yeah, that that would make sense to

me.

Emily: Ok, what would we expect this value ($\log_2(4^{50})$) to be when we punch it

into the calculator?

Abigail: Mmm…Should we know this?

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Emily: Well, think out loud. Think about the relationships that you've been talking

           about.

Abigail: Uh, I would expect it to be, ah! Ok, so (*referring to previous work*) if that's

           two weeks times three of those, so in this case it would be 50 times 2 would

           be 100. I would expect to get 100.

Again, Abigail appeared to interpret expressions in the form $n\log_b(m)$ as representing

the case where *n m*-tupling periods elapse. She also relied on her previous work to

confirm her thinking - a tendency of hers as she worked on developing the generalized

form of the logarithmic property as well (question 3).

Logarithmic Property #4 (AKA Change of Base): $\log_b(X) = \dfrac{\log_b(X)}{\log_b(b)} = \dfrac{\log_c(X)}{\log_c(b)}$

   The Task (Answers listed in Appendix)

1. Suppose we observed Sparky's height 2-tuple in size five times in a row.
   - On the paper provided, document Sparky's height at these moments.
   - Determine the overall growth factor for this situation.
   - On the paper provided, identify the 2-tupling period and the [32]-tupling period.
   - The [32]-tupling period is how many times as large as the 2-tupling period.
   What unit are you using to measure the tupling periods in this situation? Will this affect your answer?
2. How many 4-tupling (2-week) periods need to elapse for Sparky's height to 2-tuple? How many 4-tupling (2-week) periods need to elapse for Sparky's height to [32]-tuple?
   - Using this information, determine how many times as large the [32]-tupling period is compared to the 2-tupling period.
3. How many 10-tupling periods need to elapse for Sparky's height to 2-tuple? How many 10-tupling periods need to elapse for Sparky's height to [32]-tuple?
   - Using this information, determine how many times as large the [32]-tupling period is compared to the 2-tupling period.
4. The 15-tupling period is how many times as large as the 10-tupling period?
5. In general, the *X*-tupling period is ___times as large as the *Y*-tupling period.
   - Develop an equation relating $\log_b(X)$, $\log_b(Y)$, $\log_c(X)$, and $\log_c(Y)$ (for $b,c,X,Y > 0$)

Figure 2.9. Task Designed to Assist Students in Developing an Understanding of the Fourth Logarithmic Property

The purpose of this task was to direct the student's attention to the relative size of two tupling periods. Additionally, this task encourages the student to determine the relative size of two tupling periods using different units of measure (i.e., other tupling periods). The fourth question provides the student an opportunity to reflect on her thinking in the first three questions but also has the added complexity of not suggesting a unit to measure the 10- and 15-tupling periods.

Abigail

Abigail began the first question by drawing Sparky's height after each 2-tupling. She identified the 32-tupling period and the 2-tupling period, and concluded that the 32-tupling period was "5 times as large" as the 2-tupling period because the doubling "occurred five times." She identified that she used the "2-tupling, 1-week unit" to

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measure both of the tupling periods. I asked Abigail if her final answer would change if she had used a different ruler to measure the tupling periods and at first, she said that her answer would stay the same "as long as we're consistent" but then changed her mind and said the answer would also change. I asked her to go ahead and determine how many times as large the 32-tupling period was compared to the 2-tupling period using the 4-tupling period as her unit of measure (question 2). At first, she wanted to find the difference between her measurements but then recalled she needed to divide her measurements in order to find how many times as large the 32-tupling period is compared to the 2-tupling period. After observing that 2.5/0.5 was also equal to 5, Abigail concluded that the tupling period used to measure did not affect her answer. We finished this task using the 10-tupling period as the measuring stick (question 3). Abigail represented how many times as large the 32-tupling period was compared to the 2-tupling

period by writing $\dfrac{\log_{10}(32)}{\log_{10}(2)}$ and anticipated that when entered in a calculator, the value

would be 5. Abigail also concluded that $\dfrac{\log_{2}(32)}{\log_{2}(2)} = \dfrac{\log_{4}(32)}{\log_{4}(2)} = \dfrac{\log_{10}(32)}{\log_{10}(2)} = 5$ after

reviewing her previous work.

The fourth question asked Abigail to multiplicatively compare the 15-tupling period with the 10-tupling period. She began her work by stating, "It is 15 over 10 which is 1.5 times as large as the 10-tupling period." Despite the confidence in her tone, Abigail wanted to also see what answer she would get if she "used logs." She determined that $\log_{2}(15) \approx 3.906$ stating the value represented "the number of 2-tupling periods for something to 15-tuple" and determined $\log_{2}(10) \approx 3.32$. After comparing her two

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methods, she recognized that calculating 15/10 was not right because she was comparing "the tupling size." Abigail concluded that the 15-tupling period would be 3.91/3.32 or $\frac{\log_2(15)}{\log_2(10)}$ times as large as the 10-tupling period.

As Abigail attempted to make a generalization, she began stating the X-tupling period would be "X over Y times as large as the Y-tupling period." When I asked what quantities she was comparing when she wrote X/Y, she recognized that she had compared the tuplings and not the tupling periods. She then decided that she would need to use logarithmic notation to represent the tupling periods, but had a difficult time because she was not provided which tupling period to use to measure (i.e., the base value). Despite Abigail's claims that changing the base didn't matter in the previous equations, it seemed as though she still hadn't made the realization that she could use any base and the relationship would still hold. I suggested we use the 2-tupling, 1-week period to measure both the X- and Y-tupling periods and Abigail concluded that $\frac{\log_2(X)}{\log_2(Y)}$ determined how many times as large the X-tupling period was compared to the Y-tupling period. After making this claim, she was asked to relate $\log_b(X)$, $\log_b(Y)$, $\log_c(X)$, and $\log_c(Y)$ and she concluded $\frac{\log_b(X)}{\log_b(Y)} = \frac{\log_c(X)}{\log_c(Y)}$.

Abigail's difficulty distinguishing between tuplings and tupling periods affected her development of understanding the fourth logarithmic property. In response to both the fourth and fifth questions, Abigail compared the tuplings themselves and not the tupling periods. Recall she first claimed that the 15-tupling period was 1.5 times as large as the

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10-tupling period in the fourth task before incorporating logarithms into her work and realizing that calculating 15/10 was not the correct calculation because she was comparing the tuplings and not the tupling periods. Similarly in the fifth task, Abigail initially concluded that the *X*-tupling period was *X/Y* times as large as the *Y*-tupling period before deciding she needed to use logarithmic notation to correctly represent the tupling *periods*. This data demonstrates the importance of distinguishing between quantities (specifically tuplings and tupling periods) while working on logarithmic tasks.

To assist Abigail in developing the understanding typically presented as the change of base formula, that $\log_b(X) = \dfrac{\log_c(X)}{\log_c(b)}$, I suggested we revisit the first scenario where Abigail concluded $\dfrac{\log_2(32)}{\log_2(2)} = \dfrac{\log_4(32)}{\log_4(2)} = \dfrac{\log_{10}(32)}{\log_{10}(2)} = 5$. She noted that the denominator of the first fraction was equal to one, therefore implying that $\log_2(32) = \dfrac{\log_2(32)}{\log_2(2)} = 5$. Abigail applied this same approach in subsequent tasks involving logarithmic values with bases other than 10 or *e*. For example, I asked Abigail to determine the amount of time it would take for Sparky to become 100 feet tall (given that he started at a height of 1 foot tall and doubled in size each week). Abigail first set up the equation $2^x = 100$ and solved for *x* writing $\dfrac{\log_2(100)}{\log_2(2)} = \dfrac{\log_{10}(100)}{\log_{10}(2)}$ concluding the answer was 6.64 weeks – suggesting she realized that when given a logarithmic expression she is unable to enter in her calculator, she must first set up a quotient of the original logarithmic expression and a logarithmic expression in which the original base is

both in the base and the argument, before changing the base to something she can

calculate. In the future, I plan on developing a way to assist students in constructing the

understanding that determining the value of $\log_b(m)$ also calculates how many times as

large the *m*-tupling period is than the *b*-tupling period. Doing so may help students who

are struggling to understand why $\log_b(m) = \dfrac{\log_b(m)}{\log_b(b)}$, or furthermore why

$$\log_b(m) = \dfrac{\log_c(m)}{\log_c(b)} \, .$$

Logarithmic Property #5: $\log_b(b^x) = x$

   The Task (Answers listed in Appendix)

1. Simplify: $\log_b(b^x)$ 2. Evaluate: $\log_2(2^{52})$

Figure 2.10. Task Designed to Assist Students in Developing an Understanding of the Fifth Logarithmic Property

Unlike a number of the previous tasks, I began this task by presenting Abigail

with the generalized version of the 5th logarithmic property for the purpose of evaluating

her meaning for the idea of logarithm. In the case that her meaning for logarithm was

insufficient to answer the first question, I was prepared to engage her in a task (question

2) to advance her understanding.

   Abigail

Abigail answered the first question quickly and concluded that the expression

would simplify to *x* by stating $\log_b(b^x) = x\log_b(b) = x \cdot 1 = x$. It appeared as though

Abigail applied her understanding for the third logarithmic property to simplify this

generalized case. For the second question, she immediately stated the answer would be

52 stating, "The number of two-tuplings it takes to 2-tuple 52 times is 52." Abigail did

not appear to develop or need to develop any additional ways of thinking about the idea

of logarithm in order to answer these questions.

Logarithmic Property #6: $b^{\log_b(x)} = x$

The Task (Answers listed in Appendix)

1. Simplify: $b^{\log_b(x)}$ 2. Evaluate: $2^{\log_2(17)}$

Figure 2.11. Task Designed to Assist Students in Developing an Understanding of the Sixth Logarithmic Property

Unlike a number of the previous tasks, I presented Abigail with the generalized

version of the 6th logarithmic property first to evaluate the effectiveness of her meaning

for the idea of logarithm. In the case that her meaning for logarithm was insufficient to

answer the first question, I was ready to provide a specific example (question 2) to

provide opportunities to advance her thinking.

Abigail

While working on the first question, Abigail said that the logarithmic expression

in the exponent was throwing her off. I asked her to interpret the meaning of the exponent

and she stated, "the number of b-tuplings for something to x-tuple." I then asked her what

an exponent on a value b typically represented and she said, "The number of b-tuplings -

this b is tupled log base b of x number of times." Despite her correct statements, Abigail

did not appear to realize that if something b-tuples the number of times needed to overall

x-tuple, then the value will x-tuple. I then decided to present her with the second question

and asked her to evaluate the expression. She initially determined the value of $\log_2(17)$,

wrote this value as an exponent to the number 2, and then anticipated that she would get

17 when she plugged the expression into a calculator, stating, "Ok, well first we figured

out what log base 2 of 17 was. And then we multiplied 2 times that number of times. I

guess it's just that, it's just that – if you're going to find the number of times you multiply

it to 17-tuple, and then you're going to 2-tuple that many times, you're going to get this

(17) number." Following this explanation, she returned to the general expression and

simplified it to be *x*.

In the next interview, one week later, I presented Abigail with a new task (Figure

2.12). I hypothesized that if I changed the quantitative relationship, Abigail would be

more likely to view a logarithmic expression as the value of an exponent.

Fill in the exponent/box to make the statement true (no calculator): $7.2^{\square} = 64.3$

Figure 2.12. Additional Task Examining Abigail's Understanding of Sixth Logarithmic
Property

Abigail and I engaged in the following conversation as she worked through the task:

Abigail: Fill in the exponent slash box to make the statement true, no calculator.

7.2 to what equals 64.3?

Emily: And if the box is too small you can just write your answer off to the side.

Abigail: Ok. I can't use a calculator at all?

Emily: Nope.

Abigail: Um, statement true…

Emily: Can you describe what you would want to punch into a calculator if you

had one?

Abigail: Um, I would do log base 7.2 of 64.3.

Emily: And what would that represent?

Abigail: Um, this would represent the number of times something is 7.2-tupled to

have an overall growth of 64.3.

Emily: And what does the exponent on 7.2 represent in that case?

Abigail: The number of 7.2-tupling periods for something to 64.3-tuple.

Emily: And how is what you just said different from how you interpreted the

log statement?

Abigail: How's it different?

Emily: Mhm

Abigal: I don't know I think they're the same, I mean I thought I said them the

same way. They are the same.

Emily: So could we write that expression in the box and be happy, or is there

something else that we need to do to make the statement true?

Abigail: If we wrote this in the box,…yeah, we could write that in the box.

Emily: Let's just kind of rewrite the statement, so 7.2 raised to this equals 64.3?

You're happy with that statement?

Abigail: Oh, yes. 7.2 to the log base 7.2 of 64.3 equals 64.3.

It seemed as though this task made it easier for Abigail to see a logarithmic expression as

the value of an exponent. I hypothesize that had I introduced this question prior to the

generalized property a week earlier, Abigail would have experienced fewer difficulties.

DISCUSSION & CONCLUSION

Many studies have examined aspects of logarithms that present difficulties for

students, while others have investigated the effectiveness of interventions. In this study I

examined two students' thinking as they participated in a conceptually based lesson on exponential and logarithmic functions. Recall that the purpose of this study was to model the mathematical realities of individual students for the purpose of illustrating how particular students might reason when experiencing instruction aimed at teaching the idea of logarithm meaningfully. My findings revealed the importance of conceptualizing a *b*-tupling period as a multiplicative object. That is, conceptualizing that a *b*-tupling period is a change in the input quantity corresponding to an event in which the output quantity grows by a factor of *b* may assist students in working with logarithms when comparing growth factors and the corresponding input intervals. When Abigail and Aaliyah only acknowledged the tupling or only acknowledged the elapsed time, their reasoning often led to unproductive conclusions. For example, recall Abigail's initial struggles to conceptualize the 3-week growth factor in the Sparky situation. Despite having developed the understanding that *A*-tupling and then *B*-tupling results overall in an *AB*-tupling, Abigail stated that the 3-week growth factor would be 6 (not 8). On the other hand, recall Abigail's work with the third logarithmic property – when Abigail coordinated both the tupling and the elapsed time when discussing the 4-tupling period, she experienced clarity in her thinking and concluded that $\log_2(4^{50})$ would be 100 because 50 4-tupling, 2-week periods elapsed.

The results of this study also suggest that the imagery associated with logarithms may vary between students. Students may imagine the equivalent exponential form or think about different magnitudes of the input quantity in relation to the corresponding multiplicative growth. In addition, it may be beneficial to assist students in imagining

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alternative representations of components of the idea of logarithm. Consider Abigail's experiences with representing tupling periods with arrows when trying to justify the second logarithmic property. She appeared to make more sense of the relationships between tupling periods when she was able to "see" how the tupling periods related.

Additionally, providing opportunities for the students to reflect on their previous thinking helped Abigail and Aaliyah strengthen their foundational understandings and advance their meaning for the idea of logarithm. Recall how both students realized that they could add the corresponding tupling periods as they looked back at their diagrams drawn in the first logarithmic property. Without this opportunity, the students might have just looked at how the numbers were related rather than thinking about how the referenced quantities were represented and related in the moment. As I stated earlier, the students in this study were not asked to complete any assignments between the teaching episodes. As a result, the students were not provided opportunities to engage in repeated reasoning of the idea of logarithm outside of the teaching episodes. This proved to be slightly troublesome when the students were asked to apply ideas they had previously encountered in a new context. For example, during the discussion focused on interpreting exponents as a number of elapsed *base*-tupling periods, Abigail interpreted an exponent on 8 as "the number of 8-tupling periods that have elapsed." However, two episodes later (about 1.5-2 weeks later) Abigail appeared to regress back to expressing her prior meaning for exponent as repeated multiplication as evidenced by her describing the exponent on *b* as "the number of times you multiply *b* times itself." This finding highlights the need for students to be engaged in repeatedly applying newly learned (and more productive) ways of thinking. It is also noteworthy that both students had a

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tendency to revert back to prior ways of thinking that were productive for them in other contexts.

It is highly unlikely that this study's models of student thinking summarize all possible ways of thinking students have when participating in exponential and logarithmic lessons. Therefore, more research into students' developing conceptions and reasoning abilities when working through lessons on the idea of logarithm (and exponential functions) must be conducted. As more research is conducted, we can continue to improve the design of logarithmic curriculum and provide professional development for teachers so they are better equipped to support students in developing productive understandings of the idea of logarithm.

Finally, it is my hope that curriculum developers and researchers will have greater clarity about critical reasoning abilities and understandings that students need to acquire about the idea of logarithm. In particular, my descriptions of my tasks and my characterization of my subjects' thinking as they engaged in these tasks should be useful in designing instruction to advance students' meanings, in addition to helping other researchers' focus their investigations.

The Geogebra applet and tasks utilized in this study can be requested at egkuper@asu.edu.

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Answers to Tasks designed to assist students in developing an understanding of the logarithmic properties	

Figure 2.5 Answers	1. Sparky grew by a factor of 6 in 2.585 weeks. In other words, overall Sparky's height will experience a 6-tupling in 2.585 weeks. 2. If Sparky's height 3-tuples *then* 5-tuples, overall his height will experience a 15-tupling. $\quad$ - $\log_2(3) \approx 1.585$ weeks $\quad$ - $\log_2(5) \approx 2.322$ weeks $\quad$ - $\log_2(15) = \log_2(3) + \log_2(5) \approx 3.907$ weeks $\quad$ - $\log_2(15) = \log_2(3) + \log_2(5)$ 3. If Sparky's height 34-tuples *then* 57-tuples, overall his height will experience a 1938-tupling. $\quad$ - $\log_2(34) \approx 5.087$ weeks $\quad$ - $\log_2(57) \approx 5.833$ weeks $\quad$ - $\log_2(1938) = \log_2(34 \cdot 57) = \log_2(34) + \log_2(57) \approx 10.920$ weeks $\quad$ - $\log_2(1938) = \log_2(34 \cdot 57) = \log_2(34) + \log_2(57)$ 4. If Sparky's height *X*-tuples *then* *Y*-tuples, overall his height will experience a *XY*-tupling. $\quad$ - $\log_2(X)$ weeks $\quad$ - $\log_2(Y)$ weeks $\quad$ - $\log_2(XY)$ weeks $\quad$ - $\log_2(XY) = \log_2(X) + \log_2(Y)$ 5. The bases of each of the logarithmic expressions would change to: 4, $2^{1/7}$, 2^{52}, b
Figure 2.6 Answers	1. Sparky grew by a factor of 5 in 2.322 weeks. 2. In this unknown amount of time, Sparky's height 3-tuples. $\quad$ - $\log_2(12) \approx 3.585$ weeks $\quad$ - $\log_2(4) = 2$ weeks $\quad$ - $\log_2(3) = \log_2(12) - \log_2(4) \approx 1.585$ weeks $\quad$ - $\log_2(3) = \log_2(12) - \log_2(4)$ 3. If Sparky's height *Y*-tuples *then* *X/Y*-tuples, overall his height will experience a *X*-tupling. $\quad$ - $\log_2(X)$ weeks $\quad$ - $\log_2(Y)$ weeks $\quad$ - $\log_2(X/Y)$ weeks $\quad$ - $\log_2(X/Y) = \log_2(X) - \log_2(Y)$

Figure 2.8
 	
Answers
	1. The overall growth factor is $4^3 = 64$
	
	  - $\log_2(4) = 2$ weeks
	
	  - $\log_2(4^3) = \log_2(64) = 6$ weeks
	
	  - $\log_2(4^3) = \log_2(64) = 3\log_2(4)$
	
	2. If fifty 4-tupling periods elapse, overall Sparky will 4^{50}-tuple
	
	  - $\log_2(4) = 2$ weeks
	
	  - $\log_2(4^{50}) = 100$ weeks
	
	  - $\log_2(4^{50}) = 50\log_2(4)$
	
	3. If y X-tupling periods elapse, overall Sparky will X^y-tuple
	
	  - $\log_2(X)$ weeks
	
	  - $\log_2(X^y)$ weeks
	
	  - $\log_2(X^y) = y\log_2(X)$

Figure 2.9
 	
Answers
	1. The overall growth factor is 32 or 2^5
	  - The 32-tupling period is 5 weeks.
	  - The 32-tupling period is 5 times as large as the 2-tupling period.
	
	2. $\log_4(2) = 0.5$ 4-tupling periods are needed for Sparky's height to 2-tuple.
	
	  $\log_4(32) = 2.5$ 4-tupling periods are needed for Sparky's height to 32-tuple. The 32-tupling period is 2.5/0.5=5 times as large as the 2-tupling period.
	
	3. $\log_{10}(2) \approx 0.301$ 10-tupling periods are needed for Sparky's height to 2-tuple.
	
	  $\log_{10}(32) \approx 1.505$ 10-tupling periods are needed for Sparky's height to 32-tuple.
	
	  The 32-tupling period is $\dfrac{\log_{10}(32)}{\log_{10}(2)} = 5$ times as large as the 2-tupling period.
	
	4. *Students can use any tupling period to measure the 10- and 15-tupling periods. Since the 2-tupling period is commonly used throughout the Sparky situation, this may be the most used option.* The 15-tupling period is $\dfrac{\log_2(15)}{\log_2(10)} \approx 1.176$ times as large as the 10-tupling period.
	
	5. $\dfrac{\log_b(X)}{\log_b(Y)} = \dfrac{\log_c(X)}{\log_c(Y)}$

Figure 2.10 Answers	1. $\log_b(b^x) = x$ 2. $\log_2(2^{52}) = 52$
Figure 2.11 Answers	1. $b^{\log_b(x)} = x$ 2. $2^{\log_2(17)} = 17$

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PAPER 3:

THE IDEA OF LOGARITHM

INTRODUCTION

The idea of logarithm has many practical and theoretical uses (Vagliardo, 2004).

Despite its functionality, the idea of logarithm is treated superficially in most precalculus

textbooks. Definitions and properties are presented as statements of fact with little

attention given to help students understand the quantities they relate. While a number of

studies have examined the effectiveness of non-traditional interventions to teach

logarithms (Hammack & Lyons, 1995; Weber, 2002; Panagiotou, 2011; Vos & Espedal,

2016), research continues to report that students struggle to understand logarithmic

notation, the logarithmic properties, and the logarithmic function (Kenney, 2005; Strom,

2006; Weber, 2002; Gol Tabaghi, 2007). My work to support students in developing

strong and more coherent meanings for the idea of logarithm began with an examination

of the historical development of the idea of logarithm. I then leveraged the insights of this

literature review to perform a conceptual analysis of what is involved in learning and

understanding the idea of logarithm. The literature review and conceptual analysis

contributes novel and useful information for curriculum developers, instructors, and other

researchers studying student learning of this idea.

HISTORICAL DEVELOPMENT OF THE IDEA OF LOGARITHM

The idea of logarithm was introduced by John Napier in the year 1614 to make

mathematical calculations more manageable (Stoll, 2006; Villarreal-Calderon, 2008;

Panagiotou, 2011). Astrological work at the time involved the multiplication and division

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of very large numbers – a time-consuming process. Leading up to the seventeenth-century, mathematicians developed a variety of practices for simplifying such calculations to cut down the computation time. One common technique used by mathematicians, referred to as prosthaphaeresis, utilized trigonometric identities, such as $\sin(\alpha) \cdot \sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$, to convert multiplication and division problems to problems involving only addition and subtraction (Villarreal-Calderon, 2008). With today's technology, this method seems overly tedious. However, at the time of Napier, prosthaphaeresis helped astronomers save time and reduce errors in their calculations and estimations (Villarreal-Calderon, 2008).

Another common method for simplifying calculations involving multiplication and division incorporated the use of arithmetic and geometric sequences. In 1544, a German mathematician, Michael Stifel considered the relationship between the arithmetic sequence $\{1, 2, 3, 4, \ldots, n\}$ and the geometric sequence $\{2, 4, 8, 16, \ldots, 2^n\}$. He noted that multiplying terms in the geometric sequence correlated with adding the corresponding terms in the arithmetic sequence (Katz, 2004; Villarreal-Calderon, 2008; Panagiotou, 2011). For example, to determine $4 \cdot 32$ (the product of the second and fifth entries in the geometric sequence), one could add the corresponding terms in the arithmetic sequence, $2 + 5$, and refer to that entry in the geometric sequence, 128 (the seventh term in the geometric sequence). This approach is limited to determining products of numbers that are powers of the same number. That is, using this method it is impossible to calculate $5 \cdot 1370$. This limitation inspired Napier to develop a method that could be used to calculate the product of any two numbers.

Napier considered a situation (Figure 3.1) that examined two points, *P* and *Q* –

with *P* traveling along a ray (CD) and *Q* traveling along a line segment (AB), with

segment AB having a length 10^7. The point *Q* (traveling along the line segment) began

at one extreme end (point A) and moved toward the opposite end (point B), traveling at a

speed proportional to the remaining distance needed to travel along the segment (QB).

That is, the point *Q* travels geometrically (Ayoub, 1993; Villarreal-Calderon, 2008;

Panagiotou, 2011). On the other hand, the point *P* (traveling along ray CD) began at the

endpoint of the ray (point C) and moved along the ray at a constant speed equal to the

starting speed of the point *Q*. That is, the point P travels arithmetically. Using this model,

Napier concluded that at any given moment, the distance traveled by the point on the ray

(CP) was defined as the logarithm of the distance remaining for the point to travel on the

line segment (QB) (Cajori, 1893; Confrey & Smith, 1995; Katz, 2004; Villarreal-

Calderon, 2008; Panagiotou, 2011). This definition of logarithm is quite different from

the standard definition of logarithm used today. One major difference is that the notion of

a base with Napier's logarithm is inapplicable. Also, Napier's decision to define the

length of AB as 10^7 meant that the Napier logarithm of one was not zero. Consequently,

the logarithmic properties of logarithms today do not hold for Napier logarithms (Ayoub,

1993; Panagiotou, 2011). Despite these major differences, Napier's logarithms did what

they were intended to do: they allowed one to multiply any two numbers using addition

(Katz, 2004; Villarreal-Calderon, 2008).
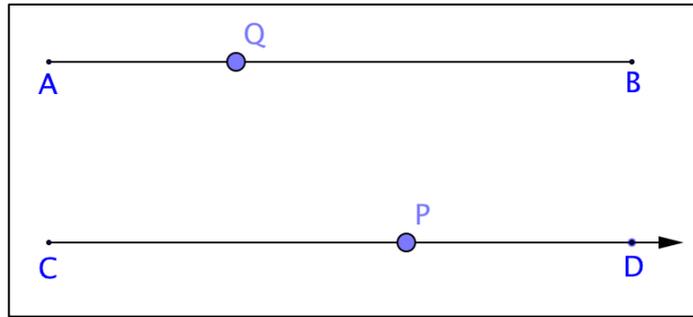
Figure 3.1: Napier's Logarithmic Situation

Over the next 20 years, Napier created a table of logarithmic values. Subsequently, Napier and Henry Briggs, an English mathematician, decided that calculations would be easier to perform if the logarithm of 10 was 1, instead of 10^7 (Villarreal-Calderon, 2008). Thus, the definition of the common logarithm was born. After Napier's death in 1617, Briggs set out to determine the logarithms of prime numbers. Then, using his calculations, Briggs proceeded to calculate the logarithms of all natural numbers up to 20,000 and from 90,000 to 100,000 to as many as 14 decimal places, organizing the values in a table. Dutchman Adrian Vlacq set out to complete the table, and in 1628 he published the logarithms of all natural numbers between 1 and 100,000 (Cajori, 1893; Villarreal-Calderon, 2008). Over the years that followed, people began to use the idea of logarithm in new ways. As a result, the definition of idea of logarithm also evolved.

At the beginning of the seventeenth-century, mathematicians recognized the functional relationship between the values in the table and began representing the relationship graphically. Eventually, the logarithmic function began appearing in calculus and the logarithmic series $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} = \ldots$ was derived (Gol Tabaghi,

2007). Mathematicians began questioning whether the argument of the logarithmic function could be a negative number. In fact, Jean Bernoulli believed $\log(-x) = \log(x)$. However, Leonhard Euler, one of Bernoulli's students eventually proved that logarithms of negative numbers are not real (Boyer, 1968). Euler continued working with the logarithmic function and in the year 1770, he developed a new definition for the logarithmic function – one that described logarithms in terms of exponents. That is, "if x > 0, the logarithm of x to base a ($a > 0$, $a \neq 1$), is the real number y such that $a^y = x$ and is symbolized with $y = \log_a(x)$" (Euler 1770/1984: 63-64). Euler's definition has been widely accepted throughout the mathematical community and variations of his definition are often found in most of today's curricula (Panagiotou, 2011).

The logarithmic function, like the trigonometric functions, is special in the sense that there is no simple rule for calculating the function's values. Mathematicians relied on the use of tables to determine the values of logarithmic functions. However, just a few years after Napier published his definition, William Oughtred developed the slide rule – a tool to replace a book containing logarithmic tables. Over the years, improvements were made to the slide rule so that one could calculate fractional powers and roots, such as 25.4 to the 7.1th power, and even trigonometric values (Stoll, 2006). The slide rule served to be very useful for centuries until it was used to create a computing machine that eventually made the use of the slide rule obsolete. In the year 1972, Hewlett-Packard came out with the HP-35 – the first pocket scientific calculator that exceeded the computational power of the slide rule (Stoll, 2006). With technology on the rise, improvements to calculators are still being made. In recent years, graphing calculators

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have been programmed to calculate logarithmic values where the base can be a value

other than 10 or *e*. Prior to this update, the user needed to know the change of base

formula in order to calculate the value of $\log_{1.2}(3.7)$, for example. What began as a

search for a way to simplify calculations eventually led to the creation of one of the most

useful devices in mathematics.

The idea of logarithm did not vanish with the advent of the calculator. Practical

uses of the idea of logarithm persisted, including simplifying calculations by turning

multiplication problems into addition problems and division problems into subtraction

problems (Vagliardo, 2004). These uses are taught in standard precalculus level

curriculum, frequently presented in the form of logarithmic properties that are used to

simplify or expand logarithmic expressions, and later to determine complicated

derivatives. Euler's definition of logarithm provided a way to determine the inverse

function for an exponential function. That is, the idea of logarithm can be used to undo

exponentiation and can be applied to solve for the independent variable of an exponential

function, thus expressing the independent variable of an exponential function in terms of

its dependent variable.

The usefulness of the idea of logarithm extends to advanced areas of mathematics

(Vagliardo, 2004). For example, the idea of logarithm is used to locate primes in number

theory, to describe natural growth and decay in biology, to formulate non-linear

regression in statistics, to model the laws of motion in physics, in calculating fractal

dimension in chaos theory, in interpreting the Richter scale in geology, and in calculating

Ph in chemistry (Vagliardo, 2004). If a goal for students is that they develop rich

understandings of the relationships modeled in the aforementioned mathematics or sciences courses, it will benefit them to acquire a strong understanding of logarithm notation, logarithmic properties, and the logarithmic function.

A typical precalculus course introduces a variety of functions including linear, quadratic, exponential, polynomial, rational, trigonometric, their properties (domain, range, roots, concavity, initial values, over what intervals the function is increasing or decreasing, asymptotes, long-run behavior, etc.), and their inverses. Since the idea of logarithm falls under the topics covered in a typical precalculus course and is necessary for courses following precalculus, it seems reasonable to assume that the logarithmic function would be analyzed with as much scrutiny as other functions in precalculus. Unfortunately, however, this is not always the case. A review of 5 precalculus and calculus texts[16] revealed that a typical section on the idea of logarithms first introduced Euler's definition of logarithm and often presented $y = \log_b(x)$ as the inverse to $y = b^x$. Shortly after stating the definition, the texts typically list the logarithmic properties and occasionally include a reference to the properties of exponents to justify the statements. The sections conclude with exercises that provide students practice for using the idea of logarithm to simplify and expand logarithmic expressions, solve for the input to an exponential function, etc. This approach leaves students with an impoverished image of the idea of logarithm and promotes an image that doing mathematics is about applying meaningless rules mindlessly. In addition, students who are introduced to Euler's definition may view logarithmic notation as an instruction to rewrite the term using an

[16] (1) Spiegler, Adam, "Functions Modeling Change: A Preparation for Calculus" (2011). *Faculty Books*. 92. (2&3) Stewart, J. (2010). *Calculus: early transcendentals*. Cengage Learning. [2nd and 6th editions] (4) Anton, H. Calculus with Analytic Geometry, 1988. (5) Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*.

exponential equation in order to eliminate the logarithmic notation (Kenney, 2005). This may create further issues for students attempting to develop understandings of logarithmic expressions – where the student is not informed of what the expression is equal to, or the logarithmic properties – where more than one logarithmic expression is involved. In contrast, Weber's (2002) approach to introduce $\log_b(m)$ as the number of factors of *b* in *m* led to more students (compared to a control group) being able to recall and apply properties of exponents and logarithms. The results of Weber's study suggest that students might benefit from a more coherent and conceptually focused curricula for introducing and teaching the idea of logarithm.

BACKGROUND FOR CONCEPTUAL ANALYSIS

In this section, I present my conceptual analysis of the idea of logarithm. I briefly examine the theoretical perspective and theoretical framework that informed the process of performing a conceptual analysis. I also examine Weber's (2002) non-traditional definition of logarithm and provide a rationale for the definition I propose. Finally, I conclude with a discussion of the role of quantitative reasoning in my description of what is involved in understanding and learning idea of logarithm.

Theoretical Perspective and Framework

The purpose of conceptual analysis is to describe the mental operations that might explain why people think the way that they do (Glasersfeld, 1995). The idea of conceptual analysis stems from the theoretical framework of Piaget's (2001) genetic epistemology and the theoretical perspective of radical constructivism. Piaget's genetic epistemology focuses on both "what knowledge consists of [cognitive structures -

schemes] and the ways in which knowledge develops [what those structures do]" (Piaget, 2001, p. 2). These cognitive structures, or schemes, are organizations of mental actions or mental operations (reversible actions) (Piaget, 2001). An action is "all movement, all thought, or all emotion – [that] responds to a need" (Piaget, 1967, p. 6). In a discussion of what it means to understand the idea of logarithm we must consider the individual's schemes. For example, if a person has a meaning for the idea of logarithm, he has a scheme for the idea of logarithm. If he engages with a situation and associates the situation as involving the idea of logarithm, he has assimilated the situation to his scheme for the idea of logarithm. In cases of assimilation, no noteworthy learning takes place because the person remains in a state of equilibrium. Thompson and Saldanha describe a person's understanding as "assimilation to a scheme" (2003). On the other hand, if the person engages with a situation and achieves outcomes that conflict with his anticipated results, the assimilation is unsuccessful and he will be in a state of disequilibrium. To cope with his unrest, he may modify his meanings or he may develop a new meaning altogether (Piaget, 2001). Learning takes place when either of these accommodations occurs.

A central claim of radical constructivism is that knowledge is constructed in the mind of an individual and therefore cannot be directly accessed by anyone else. Therefore, while I am unable to access anyone else's understanding of the idea of logarithm, I can do my best to convey my understanding of the idea of logarithm through this conceptual analysis. In doing so, I focus on major constructions that need to be made as one develops the idea of logarithm for herself. How people come to develop such constructions (possibly by means of "smaller" constructions) is beyond the scope of this

paper.

Logarithm as a Number of Factors

Weber (2002) defined $\log_b(m)$ as the number of factors of b in m. Using his

definition, $\log_5(125)$ is described as the number of factors of 5 in 125. The equation

$\log_5(125) = 3$ is a statement that there are 3 factors of 5 in 125. While his definition

presents logarithms more conceptually than Euler's definition, the phrasing can be

slightly misleading. For example, the phrase "factors of b" in the definition of $\log_b(m)$

may influence students to think of the prime factorization of b. Also, the phrase "in m" is

vague. What does it mean for one number to be *in* another number? I found myself

unable to reconcile these issues and decided to construct a definition for logarithm

grounded in quantitative reasoning.

Quantitative Reasoning

Smith and Thompson (2007) argue that if students are to utilize algebraic notation

to assist them in representing ideas and reasoning productively, then their ideas and

reasoning must become sophisticated enough to justify the use of the notation in the first

place. I argue that the same is true for the idea of logarithm. That is, before students begin

using logarithmic notation and the logarithmic properties to represent their ideas and

reasoning, their reasoning must identify a need for such tools. How does one develop

such sophisticated reasoning? Smith and Thompson (2007) claim that it is through years

of developing and using quantitative reasoning that one's algebraic knowledge becomes

meaningful and productive (pg. 10) for representing quantitative relationships. It is well

documented that students who engage in quantitative reasoning are more likely to reason

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productively while working on conceptually challenging tasks (Castillo-Garsow, 2010; Ellis, 2007; Hackenberg, 2010; Moore, 2010; Moore, K. C., & Carlson, M. P., 2012; Saldanha & Thompson, 1998; Thompson, 1993, 1994b). If a goal for students is that they utilize the idea of logarithm as they work through conceptually challenging tasks, then it would follow that they should develop an understanding of the idea of logarithm that is attentive to what quantities the logarithmic function relates. In this section, I describe quantitative reasoning and discuss its relevance in learning and understanding the idea of logarithm.

A quantity is a mental construction of a measurable attribute of an object (Thompson, 1990, 1993, 1994a, 2011). That is, quantities do not exist out in the world; rather, they are created in the mind of an individual when she conceptualizes measuring a quality of an object (Thompson, 2011). Furthermore, one is said to participate in the act of quantification when, after conceptualizing a quantity, she conceptualizes the attribute's unit of measure such that the attribute's measure is proportional to its unit (Thompson, 2011). The numerical measurement that a quantity may assume is referred to as a value. When the measurable attribute of an object doesn't change throughout a situation, it is called a constant or fixed quantity. On the other hand, if the value of a quantity changes throughout a situation, we call it a varying quantity.

Mathematics is often used to model and describe how two or more quantities relate. A quantitative operation occurs in the mind of an individual and is when "one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 2011, pg. 9). When one conceives of three quantities related by means of a quantitative operation, he has conceptualized a quantitative relationship. Changing which

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quantity is determined by the quantitative operation changes the quantitative relationship (Thompson, 1990). When one analyzes a situation and assigns his observations (i.e. quantities, quantitative relationships) to a network of quantities and quantitative relationships, called a quantitative structure, he is said to engage in quantitative reasoning (Thompson, 1988, 1990, 1993, 1994a, 2011).

When a student engages in the essential constructs of quantitative reasoning she may end up developing a need for logarithmic notation on her own – possibly making the notation more meaningful to her. Consider the following example: Mary purchased a cactus on January 1st of this year and noticed the cactus was growing in a peculiar way. Mary might conceptualize the cactus' (object's) height (attribute) or elapsed (attribute) time (object) as quantities and decide to measure the cactus' height using the cactus' initial height at different moments since January 1st. Suppose she initially documented the cactus' height on a wall and concluded that the cactus is one cactus tall on the first of January. One week later, Mary documented the cactus' new height on the wall, measured its current height using its initial height as the unit of measure, and concluded that the cactus one week later had a measure of 2 (in units of the height of the initial cactus) – therefore participating in the act of quantification. Suppose she then concluded that in that one-week's time, the cactus' height 2-tupled (doubled). If Mary conceptualized the factor by which the cactus may grow (the tupling value) as a quantity, resulting from multiplicatively comparing the two heights, she engaged in a quantitative operation. If, after documenting the cactus' growth over a long period of time, Mary concludes that the 2-tupling (doubling) period is one week, she may be curious to determine how many 2-tupling (doubling) periods need to elapse for the initial cactus to 9-tuple in height (to

determine how long she has until she needs to take the cactus outside). Mary could then use logarithmic notation to represent the value of that particular quantity – specifically, $\log_2(9)$.

In general, I define $\log_b(m)$ to represent the number of b-tupling periods[17] necessary to result in an m-tupling. The steps used to solve for the inverse relationship to the general representation of an exponential relationship, $y = a(b)^x$, informed this decision. For example, when solving for x applying Euler's definition, we get $x = \log_b(y/a)$, therefore indicating that the argument to the logarithmic function is a y/a-tupling. That is, in order for the initial value of the exponential relationship to become y, the initial value must y/a-tuple or become y/a times as large. My conceptual analysis for the idea of logarithm expanded from this definition and examines components of exponential and logarithmic situations similarly. Following my conceptual analysis of the idea of logarithm, I illustrate the difference between algebraic reasoning and quantitative reasoning in an exponential and logarithmic setting.

CONCEPTUAL ANALYSIS OF THE IDEA OF LOGARITHM

Exponential and logarithmic relationships are two sides of a coin – when one discusses elements of one relationship, he is, in some form or another, discussing components of the other relationship as well. In this conceptual analysis, I examine a variety of aspects often categorized under exponential relationships because I see them as being important for one to come to understand the idea of logarithm. In particular, I develop the ideas of growth factor, the exponential relationship, tuplings and tupling

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[17] A b-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become b times as large. We say that the second quantity has b-tupled over some interval of change of the first quantity.

periods, exponent, growth factor conversions, the exponential function, logarithmic

notation, logarithmic properties and the logarithmic function. I also briefly examine a few

prerequisite understandings one must have to make sense of these listed ideas.

Division as Measurement (Prerequisite)

Students must understand the construct of division as measurement. That is, to

measure a value of Quantity A in terms of a value of Quantity B, measured in the same

unit, we can calculate the quotient $\frac{\text{Value of Quantity A}}{\text{Value of Quantity B}}$. If $\frac{\text{Value of Quantity A}}{\text{Value of Quantity B}} = m$,

we say the value of Quantity A is m times as large as the value of Quantity B. As long as

Quantity A and Quantity B are measured using the same unit, this ratio will remain

constant.

**Multiplying by *A* and then multiplying by *B* has the same overall effect as
multiplying by *AB* ($\times A \times B = \times AB$) (Prerequisite)**

Students must have the understanding that multiplying by A and then multiplying

the resulting value by B is equivalent to multiplying the original value by AB. For

example, multiplying some value by 2 and then the resulting value by 3 is equivalent to

multiplying the original value by 6. Therefore, if a value A-tuples (becomes A times as

large) and then the A-tupled value B-tuples (becomes B times as large), overall the

starting value will AB-tuple (become AB times as large).

Growth Factor / The Exponential Relationship

When comparing two values of the same quantity (say value A and value B), we

can determine how many times as large one value is than another by calculating a

quotient to evaluate a ratio $\left(\frac{\text{value B}}{\text{value A}} \right)$. If value B is m times as large as value A, then by

convention we say the quantity's value grew by a factor of *m,* or became *m* times as large. If this particular (multiplicative) growth corresponds with a (additive) change of n in another quantity, then by convention we say the *n*-unit growth factor is *m*.

When one attends to the values of two varying quantities, Quantity A and Quantity B, and notices that for equal changes in the value of Quantity A, the value of Quantity B grows by a constant factor, then there exists a geometric relationship between the two quantities. In the continuous case, we more specifically refer to the relationship between the two quantities as exponential. For the rest of this paper, I assume continuity unless stated otherwise.

Tuples, Tuplings & Tupling Periods / Exponents / Growth Factor Conversions

In this section, I discuss concepts foundational to exponential functions. I begin by justifying and defining a few terms I use in my conceptual analysis. I then discuss how I motivate exponential notation and argue how my definition for exponent is useful for converting from one growth factor to another growth factor (often called partial or *n*-unit growth factors).

It has been my observation that students are comfortable in using colloquial terms such as doubles or triples to describe a quantity's values becoming two or three times as large, respectively. Because the idea of logarithm builds off of growth factors, I wanted to introduce similar language that students could use to describe the case where a quantity's values become 1.5 times as large, for example. Therefore, I say that to *b*-tuple (verb) is to become *b* times as large. Note: this is not to be confused with the definition of *m*-tuple as an ordered set of *m* numbers (in the *m*-dimensional Cartesian plane). So instead of using the term "double", I would say 2-tuple. I use the phrase *b*-tupling (noun)

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to represent the instance where a quantity's value becomes *b* times as large, or the event in which a quantity's value *b*-tuples. Finally, a *b*-tupling period (noun) is the amount of change in the input quantity of an exponential function needed for the output quantity of the exponential function to become *b* times as large. Typically, this first quantity is time, but it doesn't have to be.

Recall that for two exponentially related quantities, equal changes in the value of one quantity imply that the value of the other quantity grows by a constant factor. That is, for any change of *n* in the value of Quantity A, the value of Quantity B will become *b* times as large (or *b*-tuples). By convention, we say the *n*-unit growth factor is *b*. However, we can also say that *n*-units is the *b*-tupling period. Recall the *b*-tupling period is the amount of change of our exponential function input value needed for the output value to *b*-tuple, or become *b* times as large. If *m* *b*-tupling periods have elapsed (that is, *nm* units of the input quantity), then the value of the output will grow by a different factor. By convention, we write b^m to represent the factor by which the output quantity grows when *m* *b*-tupling periods elapse. It is worth noting that this interpretation for exponents differs from the repeated multiplication approach because it takes into account all real values of *m*. For example, suppose the 4-tupling (or quadrupling) period for a population is one week and suppose 1.5 weeks elapse, then the factor by which the population grew over the course of the 1.5 weeks can be expressed as $4^{1.5}$ (which is equivalent to 8). Or, suppose that for every 1 radian a dial rotates, the amount of frozen yogurt dispensed from a machine 1.5-tuples. Then if the dial rotates an angle of π radians, the amount of frozen yogurt dispensed from a machine will 1.5^π-tuple.

Ellis and colleagues (2015) found that before students were able to reason with non-natural number exponents, they first had to reason with natural number exponents. Therefore, as students are beginning to conceptualize the idea of exponent, it may be necessary to present students with cases where *m*, the number of elapsed *b*-tupling periods, is a natural number. For example, suppose the 3-tupling (or tripling) period for a population is 1 week and suppose 2 weeks (two 3-tupling periods) have elapsed, then the factor by which the population grows over the 2 weeks is $3 \times 3 = 9$. To represent this factor, we can also write 3^2, communicating that two 3-tupling periods have elapsed. In this instance, it is easy to calculate the 2-week growth factor – however, this is not always the case.

Still assuming the 1-week growth factor is 3, suppose we now wish to represent the 1-year, or 52 week growth factor. We need a way to represent the growth factor that corresponds to the case where 52 3-tupling periods have elapsed; specifically, we write 3^{52}. Similar reasoning can be employed to determine the 1-day, or 1/7th week growth factor. To represent the growth factor in the case where 1/7th of a 3-tupling period has elapsed, we write $3^{1/7}$. In both of these cases, we let the exponent on 3 represent the number of elapsed 3-tupling periods (1-week periods). This reasoning remains consistent for exponents less than or equal to zero, too. For example, in the case where no time has elapsed, the population would not change (i.e. grow by a factor of 1); this corresponds with the equation $3^0 = 1$. If the change in the number of weeks is -3 (i.e., we are looking "back in time" for a total of 3 weeks), then the -3 week growth factor is 3^{-3} or 1/27 (since over the 3 weeks prior to when 0 weeks have elapsed, the population would both

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become 1 and would increase by a factor of 27). In general, if we let x represent the number of elapsed 3-tupling periods (1-week periods), then 3^x represents the x-week growth factor and can therefore be used to determine any other growth factor.

The Exponential Function

In this section, I describe how one might come to define an exponential function. To meaningfully discuss the ideas in this section, students must have developed the understandings outlined at the beginning of this conceptual analysis of division as measurement and growth factors. Students must also conceptualize exponents to represent the number of elapsed *base*-tupling periods, understand how to represent changes in quantities' values, and recognize that for exponential relationships between two quantities, for equal changes in the input quantity, the output quantity grows by a constant factor.

Suppose (x_1, y_1) and (x, y) are points that satisfy an exponential relationship. Since y is y/y_1 times as large as y_1, and since the relationship is exponential, then for any change of $x - x_1$ in the input quantity, the output quantity will become y/y_1 times as large. Similarly, if we suppose the 1-unit growth factor is b, then for any change of $x - x_1$ units in the input quantity, the corresponding growth factor will be b^{x-x_1} (because 1-unit or b-tupling periods elapsed). Therefore, the two different expressions representing the same growth factor are equivalent, $y/y_1 = b^{x-x_1}$. We can also conclude that y is b^{x-x_1} times as large as y_1 ($y = y_1 b^{x-x_1}$). In the case where (x_1, y_1) is the vertical intercept, say $(0, a)$, we have $y = ab^x$. Therefore, if $f(x) = y$, then $f(x) = ab^x$, where a is the initial

value of the output quantity and *b* is the 1-unit growth factor. Consider Sparky, a saguaro cactus whose height is growing exponentially. If Sparky was 5 feet tall when he was purchased and 10 feet tall one week later, then in one week, he became $\frac{10}{5} = 2$ times as large, or 2-tupled. Thus, the one-week growth factor is 2. If we wish to define the formula relating the number of weeks since Sparky's purchase, *x*, and his height in feet, *y*, we can use the reasoning described above to conclude $\frac{y}{5} = 2^{x-0}$ or $y = 5(2)^x$.

  I recognize that this method of relating equivalent growth factors is not the only way to define exponential functions, nor is it the typical approach found in most curricula. However, ironically, students are often expected to use this equality when they solve for the inverse relationship of an exponential function. For example, when solving $y = ab^x$ for *x*, students are often taught to divide both sides of the equation by *a*, before applying Euler's definition for logarithm. I am unaware of any studies that have examined students' understandings for this operation, although I would hypothesize that students would view this step as just another procedure to follow to "get the answer." In this conceptual analysis, I place an emphasis on the exponential growth and encourage students to see a formula as emerging from conceptualizing and then representing new quantities in a situation. This is similar to an approach for the development of linear functions that begins by first conceptualizing how the two varying quantities are changing together, and then constructing a formula to represent this relationship.

Logarithmic Notation

  Recall exponential functions have the quality that, for equal changes in the input

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quantity, the output quantity grows by a constant factor. That is, for any change of *n* in the input quantity, the output quantity will *b*-tuple, or become *b* times as large. By convention, we say the *n*-unit growth factor is *b*. However, we can also say that *n* is the *b*-tupling period, the amount/value of change of our input to our exponential function necessary for our output to become *b* times as large.

Often, when working with exponential functions, students are given explicit information about only one growth factor. This may be the one-year growth factor, the three-day growth factor, etc. This information also informs the student of a tupling period. For example, if the one-week growth factor is 2, then the 2-tupling period is one week. However, in a situation where the 2-tupling period is one week, a student may be interested in determining the number of weeks necessary to 10-tuple, or become 10 times as large (based on information presented in the task at hand). In this case, the 10-tupling period will be longer than the 2-tupling period (1 week), but can still be measured using a one-week unit of measure (or the 2-tupling period). However, since 10 is not a power of 2, this value can be difficult to calculate. Moreover, in general, determining the change in the input of an exponential function necessary for the initial value of the function to *m*-tuple, or become *m* times as large, is not a trivial task. That is, there is no simple rule that provides instructions on how to calculate the *m*-tupling period. However, with the use of modern technology, these calculations are possible. The 10-tupling period and the *e*-tupling period are the most common units used to measure all other tupling periods. However, any tupling period can be used to measure the change in input necessary for the initial value of the function to *m*-tuple. For example, if the 3-tupling period is one day, we can use it to measure the 27-tupling period (3 days). To represent the number of 3-

tupling periods needed to elapse in order for a 27-tupling to occur, we write $\log_3(27)$. In general, we write $\log_b(m)$ to represent the number of b-tupling periods it takes a value of our exponential function to experience an m-tupling.

Logarithmic Properties

I start with the meaning of "$\log_b(x)$" as "the number of b-tupling periods needed to result in an x-tupling". After being introduced to and practice using logarithmic notation, students are often asked to manipulate logarithmic expressions or equations using one or more of the following logarithmic properties:

1. $\log_b(X) + \log_b(Y) = \log_b(XY)$

2. $\log_b(X) - \log_b(Y) = \log_b(X/Y)$

3. $\log_b(X^y) = y\log_b(X)$

4. $\log_b(X) = \dfrac{\log_c(X)}{\log_c(b)}$ (or more accurately, $\dfrac{\log_b(X)}{\log_b(b)} = \dfrac{\log_c(X)}{\log_c(b)}$)

5. $\log_b(b^x) = x$

6. $b^{\log_b(x)} = x$

The understanding that multiplying by X and then multiplying by Y is equivalent to multiplying by XY is foundational to understanding the first logarithmic property. Therefore, if a value experiences an X-tupling and then experiences a Y-tupling, overall the initial value will experience an XY-tupling. If we let T_X represent the X-tupling period, T_Y represent the Y-tupling period, and T_{XY} represent the XY-tupling period (each not yet measured in a specified unit), then the sum of the X-tupling period and the Y-

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tupling period should be the same as the *XY*-tupling period ($T_X + T_Y = T_{XY}$). Therefore, now measuring each of these tupling periods using the same unit, the number of *b*-tupling periods needed to result in an *XY*-tupling is the same as the number of *b*-tupling periods needed to result in an *X*-tupling plus the number of *b*-tupling periods needed to result in a *Y*-tupling, or $\log_b(X) + \log_b(Y) = \log_b(XY)$. If we consider a mystical cactus named Sparky whose height 2-tuples each week, and suppose his height experiences a 2-tupling and suppose his height then experiences an 8-tupling after the 2-tupling. His 2-tupled height will become 8 times as large. His height will have become 16 times as large as it was before it 2-tupled, for an overall 16-tuple in height. The number of weeks (2-tupling periods) needed to result in a 2-tupling (1 week) followed by the number of 2-tupling periods to result in an 8-tupling (3 weeks) will be the same as the number of 2-tupling periods needed to result in a 16-tupling (4 weeks). Symbolically, we represent this case as $\log_2(2) + \log_2(8) = \log_2(16)$.

To understand the second logarithmic property, one can build off the first logarithmic property and the understanding that *X* is *X/Y* times as large as *Y*. That is, if a value experiences a *Y*-tupling and then experiences an *X/Y*-tupling after the *Y*-tupling, the *Y*-tupled value will become *X/Y* times as large. Therefore, the value will have become *X* times as large as it was before it *Y*-tupled, for an overall *X*-tuple. If we let T_X represent the *X*-tupling period, T_Y represent the *Y*-tupling period, and T_{XY} represent the *X/Y*-tupling period (each not yet measured in a specified unit), then $T_{X/Y} + T_Y = T_X$. Therefore, now measuring each of these tupling periods using the same unit, the number of *b*-tupling

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periods needed to result in an X-tupling is the same as the number of b-tupling periods needed to result in an X/Y-tupling plus the number of b-tupling periods needed to result in an Y-tupling, or $\log_b(X) = \log_b(Y) + \log_b(X/Y)$. Alternatively, $\log_b(X) - \log_b(Y) = \log_b(X/Y)$. Considering the same example used for the first logarithmic property, we can calculate the number of weeks needed for Sparky's height to experience an 8-tupling by subtracting the number of weeks (2-tupling periods) needed for Sparky's height to experience a 2-tupling from the number of weeks (2-tupling periods) needed for Sparky's height to experience a 16-tupling.

The understanding that an exponent on a value, X, represents the number of X-tupling periods that have elapsed is foundational to understanding the third logarithmic property. That is, if a value experiences y X-tupling periods, then overall the value will experience an X^y-tupling. If we let T_X represent the X-tupling period and T_{X^y} represent the X^y-tupling period (both not yet measured in a specified unit), then the X^y-tupling period will be y times as large as the X-tupling period, $T_{X^y} = yT_X$. Therefore, now measuring each of these tupling periods using the same unit, the number of b-tupling periods needed to experience an X^y-tupling is y times as large as the number of b-tupling periods needed to experience an X-tupling, symbolically $\log_b(X^y) = y\log_b(X)$. Therefore, the number of weeks (2-tupling periods) needed to result in a 2^5-tupling ($\log_2(2^5)$) is 5 times as large as the number of 2-tupling periods needed to result in a 2-tupling ($5\log_2(2)$).

A less discussed, but useful property of logarithms is the change of base formula.

This property is used to rewrite logarithmic expressions using a different base value, often as an alternative way of calculating the exact value, and is frequently presented as

$\log_b(X) = \dfrac{\log_c(X)}{\log_c(b)}$. To understand this property, students must have the understanding

that A is A/B times as large as B. Therefore, if we let T_X represent the X-tupling period and T_Y represent the Y-tupling period (each not yet measured in a specified unit), then T_X is T_X / T_Y times as large as T_Y. This relationship will not change based on the units used to measure either tupling period. That is, if we suppose $b > 0$, $b \neq 1$ and use the b-tupling period to measure the X- and Y- tupling periods, then the X-tupling period will always be

$\dfrac{\log_b(X)}{\log_b(Y)}$ times as large as a Y-tupling period. Put another way, $\dfrac{\log_b(X)}{\log_b(Y)} = \dfrac{\log_c(X)}{\log_c(Y)}$ for

$b,c > 0$, $b,c \neq 1$. Notice, if we let $Y = b$, then $\log_b(X) = \dfrac{\log_b(X)}{1} = \dfrac{\log_b(X)}{\log_b(b)} = \dfrac{\log_c(X)}{\log_c(b)}$.

Considering the same example used for the previous logarithmic properties, the 3-tupling (tripling) period measured in weeks is about 1.585 and the 2-tupling (doubling) period measured in weeks is 1. Therefore, the number of weeks needed to 3-tuple (1.585 weeks) is 1.585/1 times as large as the number of weeks needed to 2-tuple (1 week). Alternatively, since the number of days will always be 7 times as large as the number of weeks, then the 3-tupling (tripling) period measured in days is $1.585(7) = 11.095$ and the 2-tupling (doubling) period measured in days is $1(7) = 7$. Thus, the number of days needed to 3-tuple (triple) will be $11.095 / 7 = 1.585$ times as large as the number of days needed to 2-tuple (double). In general, the 3-tupling (tripling) period will always be

approximately 1.585 times as large as the 2-tupling (doubling) period. If we were to

measure these periods in weeks (2-tupling periods), days (about 1.1-tupling periods),

years (2^{52}-tupling periods), or any other appropriate measurement the relationship would

still be true. That is, $\dfrac{\log_2(3)}{\log_2(2)} = \dfrac{\log_{1.1}(3)}{\log_{1.1}(2)} = \dfrac{\log_{2^{52}}(3)}{\log_{2^{52}}(2)} = \dfrac{\log_c(3)}{\log_c(2)} \approx 1.585$.

   The understanding that the exponent on a value, b, represents the number of b-

tupling periods that have elapsed is foundational to understanding the last two

logarithmic properties. Therefore, to represent that x b-tupling periods have elapsed, one

writes b^x. Students must also understand that b^x may also represent a b^x-tupling.

Additionally, the understanding that $\log_b(m)$ represents the number of b-tupling periods

needed to result in an m-tupling is also foundational to understanding the last two

logarithmic properties. Therefore, since b^x conveys that x b-tupling periods have elapsed

and also conveys a b^x-tupling, then the number of b-tupling periods needed to result in a

b^x-tupling is x. Symbolically, we write $\log_b(b^x) = x$. On the other hand, if the number of

elapsed b-tupling periods is $\log_b(b)$, the number of b-tupling periods needed to result in

an x-tupling, an x-tupling will occur. Symbolically, we write $b^{\log_b(x)} = x$.

The Logarithmic Function

   To conceptualize the logarithmic function in Thomspon and Carlson's (2017)

sense, one must first understand b and x to represent tuplings and $\log_b(x)$ as representing

the number of b-tupling periods needed to experience an x-tupling. He must then

conceive of the x-tupling and the number of b-tupling periods needed to experience the x-

tupling as "varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, pg. 33). In particular, if we know the value for *x*, we can determine the corresponding value of $\log_b(x)$, given a value for *b*. That is, for any given tupling, there will be exactly one number of *b*-tupling periods that are needed to achieve the same growth.

THE IDEA OF LOGARITHM

A goal for student learning of the idea of logarithm should include the aforementioned ways of thinking and understandings. As students become more fluent in using these ways of thinking I conjecture that students will be better equipped to use the idea of logarithm meaningfully to model quantitative relationships involving exponential growth. Repeated efforts to conceptualize and represent quantitative relationships that are related "logarithmically" should also strengthen students' understandings of exponential growth and exponential functions. How to support students in developing such understandings is beyond the scope of this paper.

My conceptual analysis calls for the idea of logarithm to be presented in a way that supports students in reasoning quantitatively through conceptually rich exponential and logarithmic tasks. To illustrate the difference between algebraic reasoning and quantitative reasoning in an exponential situation, consider the following task: *Suppose cactus A was 14 feet tall on January 1*[st] *and doubles (2-tuples) in height each week and suppose cactus B is 5 feet tall on January 1*[st] *and triples (3-tuples) in height each week. After how many weeks will the two cacti be the same height?* A typical algebraic solution

to this problem involves defining variables, developing expressions that represent the heights of the cacti, setting those expressions equal to one another, and solving for the unknown value. If x represents the number of weeks since January 1$^{\text{st}}$, then $14(2)^x$ represents the height of cactus A x weeks after January 1$^{\text{st}}$, and $5(3)^x$ represents the height of cactus B x weeks after January 1$^{\text{st}}$. We wish to solve $14(2)^x = 5(3)^x$ for x.

Although algebraic solutions may vary, a typical solution follows the form of the solution in Figure 3.2.

$$14(2)^x = 5(3)^x$$

$$\frac{14}{5} = \frac{3^x}{2^x}$$

$$\frac{14}{5} = \left(\frac{3}{2}\right)^x$$

$$\ln\left(\frac{14}{5}\right) = \ln\left(\left(\frac{3}{2}\right)^x\right)$$

$$\ln\left(\frac{14}{5}\right) = x\ln\left(\frac{3}{2}\right)$$

$$x = \frac{\ln\left(\frac{14}{5}\right)}{\ln\left(\frac{3}{2}\right)}$$

Figure 3.2. A Typical Algebraic Response

On the other hand, a response that utilizes quantitative reasoning does not require the use of symbols to represent relationships, but rather deals with the relationships themselves. Here is one example of such reasoning: Initially, cactus A's height is 14/5 times as tall as cactus B's height. Therefore, cactus B's height needs to 14/5-tuple as well as 2-tuple as many times as cactus A's height did over the entire interval. For any one-

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week change, the height of cactus B 3-tuples – this is equivalent to the height of the cactus experiencing a 2-tupling and then immediately experiencing a 1.5-tupling. That is, the 2-tupled height becomes 1.5 times as large for an overall 3-tuple in height. So, from the start, any time that cactus B triples (3-tuples), the necessary doubling is taken into account. In Figure 3.3, the height of cactus B is documented at different moments of a one-week period, specifically demonstrating a doubling (2-tupling) and then immediately a 1.5-tupling. It is worth noting that the 2-tupling and 1.5-tupling periods for cactus B are less than one week long and remain constant throughout this situation (with the 2-tupling period longer than the 1.5-tupling period) (see Figure 3). Also, for any portion of a week, say w weeks (where $0 < w < 1$), cactus A will grow by a factor of 2^w and cactus B will grow by a factor of 3^w, or $2^w 1.5^w$. That is, if w of a 3-tupling period has elapsed, then w of the corresponding cactus' 2-tupling period will have elapsed and w of that same cactus' 1.5-tupling period will have elapsed. Therefore, what remains to be determined is how many of these 1-week periods need to elapse for the accumulated 1.5-tuplings to result in a 14/5-tupling. The expression $\log_{1.5}(14/5)$ represents this specific value.
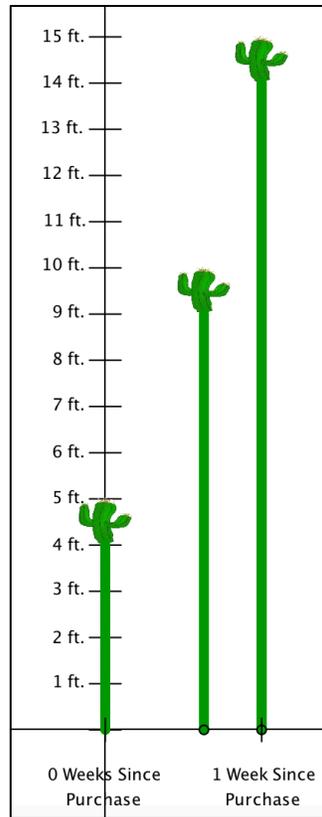
Figure 3.3. Cactus B at Different Moments Throughout the First Week

CONCLUSION

In this paper, I discussed the historical development of the idea of logarithm and described the reasoning abilities and understandings possessed by a student who has a strong understanding of the idea of logarithm. In doing so, I've elaborated the role of quantitative reasoning in conceptualizing a logarithmic expression and logarithmic function. My conceptual analysis of the idea of logarithm should be useful to other researchers studying student learning of logarithm and to curriculum developers who wish to support students in developing strong meanings for logarithm. I did not explore how students come to develop these understandings, nor have I examined how to conclude that students have developed such understandings. However, it is my hope that readers view my characterizations of specific ways of thinking and understandings that

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contribute to a productive meaning for logarithm as a useful theoretical grounding for

designing curriculum and instruction to improve student learning of this idea.

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CONCLUSION

Many studies have examined aspects of logarithms that present difficulties for students, while others have investigated the effectiveness of interventions. In this dissertation study, however, I examined students' thinking as they individually participated in a conceptually based lesson on exponential and logarithmic functions. In doing so, I gained insight on two understandings foundational to the idea of logarithm students must develop. In particular, students must develop the understanding that multiplying by A and then multiplying the resulting value by B has the same effect as multiplying the initial value by AB. This understanding is a critical and must be applied throughout a lesson on exponential and logarithmic functions. Types of problems that involve such reasoning include: calculating percentages of values, determining partial growth factors, representing, interpreting and calculating logarithmic values, and working with and explaining logarithmic properties. Additionally, I found that the understanding that the exponent on a number b represents the number of elapsed b-tupling periods is not necessary for determining growth factors. However, this understanding was necessary for one student, Aaliyah, to develop in order to determine an amount of elapsed time when provided a growth factor. Furthermore, even if a student understands $\log_b(m)$ to be the number of b-tupling periods needed to m-tuple (or grow by a factor of m), he may not be able to correctly apply this understanding to solve for the input to an exponential function if he does not see the exponent on b as representing a number of b-tuplings.

The findings of my dissertation study also revealed the importance of students conceptualizing a b-tupling period as a multiplicative object. That is, conceptualizing that

a *b*-tupling period is a change in the input quantity corresponding to an event in which the output quantity grows by a factor of *b* may assist students in working with logarithms when comparing growth factors and the corresponding input intervals. I also found that the imagery associated with logarithms may vary between students and that students may benefit from imagining alternative representations of components of the idea of logarithm. For example, students may imagine the equivalent exponential form of a logarithmic equation or think about different magnitudes of the input quantity in relation to the corresponding multiplicative growth. Additionally, providing opportunities for the students to reflect on their previous thinking helped the students strengthen their foundational understandings and advance their meaning for the idea of logarithm. My findings also highlighted the need for students to be engaged in repeatedly applying newly learned (and more productive) ways of thinking.

It is highly unlikely that this dissertation study's models of student thinking summarize all possible ways of thinking students have when participating in exponential and logarithmic lessons. Therefore, more research into students' developing conceptions and reasoning abilities when working through lessons on the idea of logarithm (and exponential functions) must be conducted. As more research is conducted, we can continue to improve the design of logarithmic curriculum and provide professional development for teachers so they are better equipped to support students in developing productive understandings of the idea of logarithm. It is my hope that curriculum developers and researchers will have greater clarity about critical reasoning abilities and understandings that students need to acquire about the idea of logarithm. In particular, my conceptual analysis, descriptions of my tasks, and my characterization of my subjects'

thinking as they engaged in these tasks should be useful in designing instruction to advance students' meanings, in addition to helping other researchers' focus their investigations.

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APPENDIX A

TEACHING EXPERIMENT TASKS

Task 1

i.   Cactus C (A, D) is how many times as tall as Cactus B?

Cactus A   Cactus B   Cactus C   Cactus D

ii. Cactus B is how many times as tall as Cactus C (A, D)?

iii. Given any two cacti, describe how you determine how many times as tall one is than the other?

iv. Draw Cactus E given Cactus E is 5.5 times as tall as Cactus B.

v.   Draw Cactus F given Cactus C is 3 times as tall as Cactus F.

vi. If Cactus B is 8 inches tall, how tall are Cacti A, C, D and E?

vii. Cactus H is how many times as tall as Cactus G if Cactus G is 34 inches tall and Cactus H is 102 inches tall?

viii. Cactus I is how many times as tall as Cactus J if Cactus J is x inches tall and Cactus I is y inches tall?

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ix. How would you describe the cactus' growth in the diagram to the right given that the cactus on the left grew to be the cactus on the right?

x. If a cactus was 23 inches tall when it was purchased and grew to be 156 inches tall, by what factor did the cactus grow?

xi. If a cactus was m inches tall when it was purchased and grew to be k inches tall, by what factor did the cactus grow?

Task 2
i.

| (A) At some point in time, Sparky the cactus was this tall. | (B) After some time, Sparky's height doubled (becomes 2 times as. large). Draw the resulting Sparky. | (C) After some more time, Sparky's height then quadrupled (becomes 4 times as large) from point (B). Draw the resulting Sparky. |

ii. By what overall factor did Sparky grow from point (A) to point (C)?

In other words, overall Sparky's height experienced a _____-tupling.

iii. If Sparky's height becomes 3 times as large and *then* 5 times as large, overall his

height will experience a _____-tupling.

iv. If Sparky's height becomes 34 times as large and *then* 57 times as large, overall his

height will experience a _____-tupling.

v. If Sparky's height becomes *X* times as large and *then* *Y* times as large, overall his

height will experience a_____-tupling.

Task 3 (This task requires the use of the attached Geogebra Applet)

 i. Emily purchased the mystical cactus shown in the video (Geogebra Applet) on Sunday, January 1st and named the saguaro Sparky. She decided to record the displayed time-lapse video of Sparky's growth and noticed he was growing in a peculiar way. Watch the video and discuss what you observe.

 ii. Document and observe Sparky's height every: week (2 weeks, 1/7 week (day), 1.585 weeks, etc.) What changes? What stays consistent?

 iii. If Emily's friend Morgan visited every Tuesday (every other Tuesday, every day, every third Tuesday, etc.) to document Sparky's growth, would she make the same claims?

 iv. If Emily's friend Kevin visited every Friday (every other Friday, every day, every third Friday, etc.) to document Sparky's growth, would he make the same claims?

 v. What is the 1-week (2-week, 1/7th-week, 1.585-week, etc.) growth factor?

 vi. What is the 2-tupling (4-tupling, 1.1-tupling, 3-tupling, etc.) period? In other words, how long does it take Sparky's height to become 2 (4, 1.1, 1.585, etc.) times as large?

Task 4
Recall the 1-week growth factor is 2, and thus the 2-tupling period is 1 week.

    i.   By what factor does Sparky grow every two (three, six) weeks?

    ii.   By what factor does Sparky grow every 52 weeks (1 year)?

    iii.   By what factor does Sparky grow every day ($1/7^{th}$ of a week)?

    iv.   By what factor does Sparky grow every -1 weeks?

    v.   By what factor does Sparky grow if no time has elapsed (0 weeks)?

    vi.   By what factor does Sparky grow by every x weeks?

    vii.   Suppose a different cactus' height 17-tuples every year. By what factor will this cactus grow every week?

Task 5
Recall the 1-week growth factor is 2, and thus the 2-tupling period is 1 week. Also recall that initially (week 0) Sparky is 1 foot tall. Suppose that after x weeks, Sparky is y feet tall.

i. Fill in the blank: After x weeks, Sparky's height is ___ times as large as his height at week 0.

ii. Use the 1-week growth factor to represent this same growth factor.

iii. Given any number of weeks, x, write an equation that determines the corresponding height of Sparky, y.

iv. Now, suppose initially (week 0) Sparky was 3 feet tall and still doubled in size each week. Write an equation that determines y, Sparky's height in feet, given x, the number of weeks since Sparky's purchase.

v. Suppose a pool is being filled with water so that the volume of water in the pool 1.5-tuples every hour. At 9am, there were 15 gallons of water in the pool. Write an equation that determines the number of gallons of water in the pool, g, in terms of the number of hours since 9am, h.

Task 6

   i. How many 2-tupling periods (weeks) does it take for Sparky's height to result in a 2-tupling (4-tupling, 8-tupling)?

  ii. How many 2-tupling periods (weeks) does it take for Sparky's height to result in a 3-tupling (5-tupling, 7-tupling)?

iii. In general, $\log_b(m)$ represents the number of b-tupling periods needed to result in an m-tupling. Use this notation to represent your answers to parts (i) and (ii). Verify your answers with the applet.

Task 7
i.

| (A) At some point in time, Sparky the cactus was this tall. | (B) After 1 week, Sparky's height doubled (2-tupled, became 2 times as large). Draw the resulting Sparky. | (C) After about 1.585 weeks, Sparky's height then tripled (3-tupled, became 3 times as large). Draw the resulting Sparky. |

ii. By what factor did Sparky grow from point (A) to point (C)? How long did it take to grow by this factor?

In other words, overall Sparky's height will experience a _____-tupling in _____ weeks.

iii. If Sparky's height 3-tuples *then* 5-tuples, overall his height will experience a _____-tupling.

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a 3-tupling, the number of 2-tupling periods (weeks) needed to result in a 5-tupling, and the number of 2-tupling periods (weeks) needed to result in a 15-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____weeks to 3-tuple and _____weeks to 5-tuple, then it will take _____weeks to 15-tuple.

iv. If Sparky's height 34-tuples *then* 57-tuples, overall his height will experience a _____-tupling.

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to 34-tuple, the number of 2-tupling periods (weeks) needed to 57-tuple, and the number of 2-tupling periods (weeks) needed to 1938-tuple. Write an equation representing the relationship between these three values.

In other words, if it takes _____weeks to 34-tuple and _____weeks to 57-tuple, then it will take _____weeks to 1938-tuple.

v. If Sparky's height *X*-tuples *then* *Y*-tuples, overall his height will experience a _____-tupling.

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a *X*-tupling, the number of 2-tupling periods (weeks) needed to result in a *Y*-tupling, and the number of 2-tupling periods (weeks) needed to result in a *XY*-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____ weeks to *X*-tuple and _____ weeks to *Y*-tuple, then it will take _____ weeks to *XY*-tuple.

iv. Now, discuss how your equations would change had you measured in days instead of weeks.

Task 8

i.

(A) At some point in time, Sparky the cactus was this tall.

(B) After some time, Sparky's height 5-tupled in size. Draw the resulting Sparky.

(C) After 1 week, Sparky's height then 2-tupled in size from point (B). Draw the resulting Sparky.

ii. By what factor did Sparky grow from point (A) to point (C)? If it took Sparky approximately 3.3219 weeks to grow by this factor, how long did it take Sparky to 5-tuple?

iii. If it takes Sparky's height 3.585 weeks to experience a 12-tupling and 2 weeks to experience a 4-tupling, how long does it take for Sparky's height to experience a 3-tupling?

Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in a 12-tupling, the number of 2-tupling periods (weeks) needed to result in a 4-tupling, and the number of 2-tupling periods (weeks) needed to result in a 3-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____weeks to 12-tuple and _____weeks to 4-tuple, then it will take _____weeks to 3-tuple.

iv. Describe how you would determine the 17-tupling period given that the 34-tupling period is approximately 5.087 weeks

v. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in an X-tupling, the number of 2-tupling periods (weeks) needed to result in a Y-tupling, and the number of 2-tupling periods (weeks) needed to result in an X/Y-tupling. Write an equation representing the relationship between these three values.

In other words, if it takes _____weeks to X-tuple and _____weeks to Y-tuple, then it will take _____weeks to X/Y-tuple.

vi. Now, discuss how your equations would change had you measured in days instead of weeks.

Task 9
Recall that the 2-tupling period is 1 week.

  i. Determine the $2^4 = 16$ -tupling period.

  ii. The 16-tupling period is how many times as large as the 2-tupling period?

  iii. Given that the quadrupling or 4-tupling period is 2 weeks, describe how you would determine the 4^{50} -tupling period.

  iv. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to result in an X-tupling and the number of 2-tupling periods (weeks) needed to result in an X^y -tupling. Write an equation representing the relationship between these two values.

  v. Now, discuss how your equations would change had you measured in days instead of weeks.

Task 10
The 10-tupling period is about 3.3 weeks and the 15-tupling period is about 3.9 weeks.

i. The 15-tupling period is how many times as large as the 10-tupling period?

ii. Use logarithmic notation to represent the number of 2-tupling periods (weeks) needed to 10-tuple and the number of 2-tupling periods (weeks) needed to 15-tuple. Write an equation representing the relationship between these two values.

iii. How would your answer to (i) change if the two periods been measured in days? In years? How would your answer to (i) remain the same if the two periods been measured in days? In years? Explain.

iv. Use logarithmic notation to represent the number of 1.104-tupling periods (days) needed to 10-tuple and the number of 1.104-tupling periods (days) needed to 15-tuple. Write an equation representing the relationship between these two values.

v. Compare your answers in (ii) and (iv).

vi. Develop an equation relating $\log_b(X)$, $\log_b(Y)$, $\log_c(X)$, and $\log_c(Y)$ (for $b,c,X,Y > 0$)

Task 11

i. What does y represent in the expression 2^y ?

ii. Represent the number of 2-tupling periods needed to result in a 2^y-tupling using logarithmic notation.

iii. Represent the number of 2-tupling periods needed to result in a 2^y-tupling without using logarithmic notation.

iv. Write an equation relating your answers in (ii) and (iii).

v. Simplify $\log_b(b^x)$

vi. What does y represent in the expression $2^y = x$?

vii. Represent the number of 2-tupling periods needed to result in an x-tupling using logarithmic notation.

viii. Simplify $2^{\log_2(x)}$

ix. Simplify $b^{\log_b(x)}$

Task 12

Recall $\log_b(x)$ represents the number of b-tupling periods needed to result in an x-tupling.

iii. Describe how $\log_2(x)$ varies as x varies.

iv. Graph the relationship of $\log_2(x)$ with respect to x. If necessary, create a table of values.

T/F: Every value of x determines exactly one value of $\log_2(x)$. Explain your answer.