On the Uncrossing Partial Order on Matchings

by

Younghwan Kim

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Graduate Supervisory Committee:

Susanna Fishel, Chair
Andrew Bremner
Andrzej Czygrinow
Henry Kierstead
Julien Paupert

ARIZONA STATE UNIVERSITY

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ABSTRACT

The uncrossing partially ordered set P_n is defined on the set of matchings on $2n$ points on a circle represented with wires. The order relation is $\tau' \leq \tau$ in P_n if and only if τ' is obtained by resolving a crossing of τ. I identify elements in P_n with affine permutations of type $(0, 2n)$. Using this identification, I adapt a technique in Reading for finding recursions for the cd-indices of intervals in Bruhat order of Coxeter groups to the uncrossing poset P_n. As a result, I produce recursions for the cd-indices of intervals in the uncrossing poset P_n. I also obtain a recursion for the ab-indices of intervals in the poset $\hat{P}_n$, the poset P_n with a unique minimum $\hat{0}$ adjoined.

Reiner-Stanton-White defined the cyclic sieving phenomenon (CSP) associated to a finite cyclic group action on a finite set and a polynomial. Sagan observed the CSP on the set of non-crossing matchings with the q-Catalan polynomial. Bowling-Liang presented similar results on the set of k-crossing matchings for $1 \leq k \leq 3$. In this dissertation, I focus on the set of all matchings on $[2n] := \{1, 2, \ldots, 2n\}$. I find the number of matchings fixed by $\frac{2\pi}{d}$ rotations for $d|2n$. I then find the polynomial $X_n(q)$ such that the set of matchings together with $X_n(q)$ and the cyclic group of order $2n$ exhibits the CSP.

To my parents

To my wife

To my son

ACKNOWLEDGMENTS

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INTRODUCTION

1.1 Matchings and the Uncrossing Partial Order on Matchings

A (complete) matching on $[2n]$ is a partition of $[2n]$ of type $(2, 2, 2, \ldots, 2)$. We represent a matching by listing its n blocks, as $\{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ where $i_r < j_r$ for all $r \in [n]$. Two blocks (i_r, j_r) and (i_s, j_s) form a *crossing* if $i_r < i_s < j_r < j_s$. We also represent a matching by wires in a circle as follows. Label $1, 2, \ldots, 2n$ in cyclic order on a circle. For each block (i_r, j_r) in a matching, connect the boundary points i_r and j_r with a wire. For example, Figure 1.1 shows a wire diagram for a 4-crossing matching $\{(1, 4), (2, 6), (3, 8), (5, 7)\}$.

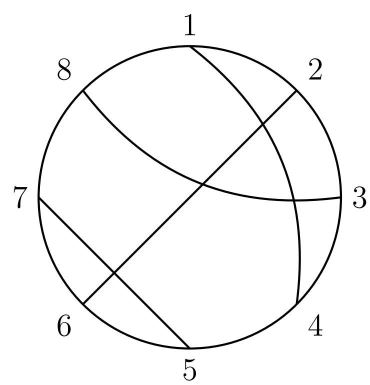


Figure 1.1: A Wire Diagram of a 4-crossing Matching $\{(1, 4), (2, 6), (3, 8), (5, 7)\}$

Since there are $\prod_{i=0}^{n-1} \binom{2n-2i}{2} = (2n)!/2^n$ ways to make n blocks if the order of the blocks matters, the number of matchings on $[2n]$ is $(2n)!/(n!2^n)$. On the other hand, there are $2n - 1$ choices for a partner of 1, say $\tau(1)$. There are $2n - 3$ choices for a partner of the smallest number in $[2n] \backslash \{1, \tau(1)\}$, and so forth. Thus, the number of matchings on $[2n]$ is also expressed as $\prod_{i=1}^{n} (2i - 1)$.

The uncrossing partially ordered set P_n on matchings is defined on the set of matchings on $2n$ points on a circle represented with wires, with an order relation: $\tau' \leq \tau$ in P_n if and only if τ' is obtained by resolving a crossing of τ. For example, if $\tau = \{(1,3),(2,4)\}, \tau' = \{(1,2),(3,4)\}$ and $\tau'' = \{(1,4),(2,3)\}$ in P_2 then $\tau' \leq \tau$ and $\tau'' \leq \tau$. Figure 1.2 shows the Hasse diagram of P_2. Figure 2.13 in Chapter 2 shows the Hasse diagram of P_3.

Let $c(\tau)$ be the number of crossings of the matching τ. The poset P_n is graded of rank $\binom{n}{2}$ with rank function given by $c(\tau)$. The poset P_n has a unique maximum element, namely $\{(1,n{+}1),(2,n{+}2),\ldots,(n,2n)\}$. There are Catalan number $\frac{1}{n+1}\binom{2n}{n}$ of minimal elements, which are the noncrossing matchings.

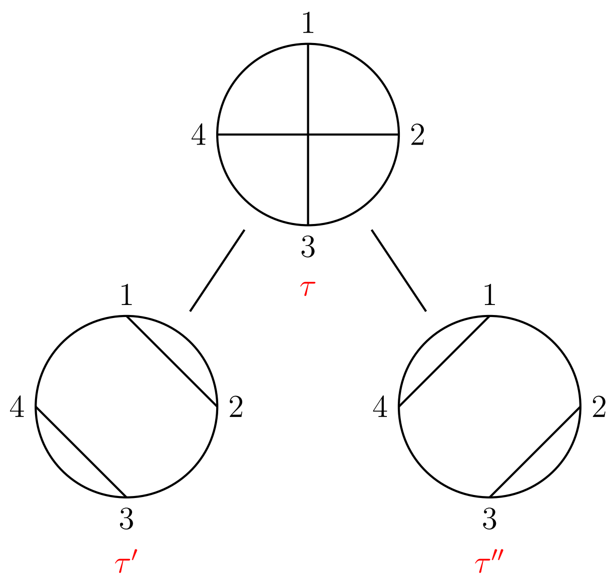


Figure 1.2: The Hasse Diagram of P_2

The remaining sections in this chapter are brief summaries of Chapter 3 and Chapter 4.

1.2 Summary of Chapter 3

A graded poset P with a unique maximum and a unique minimum is *Eulerian* if, for every non-trivial interval $[x, y]$ in P, the number of elements of odd rank in $[x, y]$ is equal to the number of elements of even rank in $[x, y]$. One nice property of Eulerian posets is their very simple Möbius functions $\mu_P(s, t) = (-1)^{\ell(s,t)}$. Another nice property of Eulerian posets is that they have a cd-index. A cd-index is a non-commutative generating function, which is an efficient way to encode the flag enumeration of Eulerian posets. The cd-index arose from the work of Bayer and Billera (1984) on flag f-vectors and flag h-vectors of Eulerian posets. Bayer and Klapper (1991) and Stanley (1994) proved the existence of the cd-indices of Eulerian posets. Ehrenborg and Readdy (1998) showed the way to obtain the cd-index of some operations, for example, the cd-index of *pyramid* of P, the product of a poset P with a chain of length one. Reading (2004) presented a recursive formula for the cd-indices of intervals in the Bruhat order on a Coxeter group.

In Chapter 3 we first define an induced subposet $\mathcal{MP}_n$ of affine permutations $\tilde{S}_{2n}$ of type $(0, 2n)$ in Definition 3.2 and Definition 3.5 in Chapter 3. Then we construct an order-reversing bijection in Theorem 3.9 between P_n and $\mathcal{MP}_n$, and thus Theorem 3.9 describes the elements in P_n with the elements in the induced subposet $\mathcal{MP}_n$. Lam (2014) introduced a representation of matchings in P_n: to a matching $\tau \in P_n$, associate $g_\tau : \mathbb{Z} \to \mathbb{Z}$ by

$$g_\tau(i) = \begin{cases} \tau(i) & \text{if } i < \tau(i) \\ \tau(i) + 2n & \text{if } i > \tau(i). \end{cases} \tag{1.1}$$

Lam showed that this map $\tau \mapsto g_\tau$ identifies P_n with an induced subposet of dual Bruhat order of affine permutations of type $(n, 2n)$ (see Lam (2014), Theorem 4.16). However, we need a map to affine permutations of type $(0, 2n)$ in order to apply a technique in Reading (2004) for finding recursions for the cd-indices of intervals in Bruhat order on Coxeter groups. We slightly modify Lam's map $\tau \mapsto g_\tau$ to adapt Reading's ideas to our situation.

The main result in Chapter 3 is recursive formulas in Theorem 3.17 for the cd-indices of intervals in the uncrossing poset P_n.

Theorem 1.1 (Theorem 3.17)**.** *Let $u, w \in \mathcal{MP}_n$ and let $s = s_i$ and $s' = s_{i+n}$ for some $0 \leq i < 2n$. Let $u < sus', w < sws'$ and $u \leq w$.*

(1) If $sus' \notin [u,w]$, then $\Phi_{[sus',sws']} = \Phi_{[u,w]}$, and

$$\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{Pyr([u,w])} \\
&= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]}\right).
\end{aligned}$$

(2) If $sus' \in [u,w]$, then

$$\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{Pyr([u,w])} - \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w \\ svs'<v}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \\
&= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w}} \sigma_s(v)\Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]}\right).
\end{aligned}$$

Finally, using the cd-indices of intervals in P_n, we present a recursion in Theorem 3.29 for the ab-indices of intervals in the poset $\hat{P}_n$ where $\hat{P}_n$ is a poset P_n with a unique minimum element $\hat{0}$ adjoined.

Theorem 1.2 (Theorem 3.29)**.** *The ab-index of $\hat{P}_n$ is recursively given by*

$$\Psi_{\hat{P}_n}(a,b) = (a-b)^{\binom{n}{2}} + \sum_{i=0}^{\binom{n}{2}-1} (a-b)^i b \sum_{\ell(x)=2\binom{n}{2}-2i} \Psi_{[e,x]}(a,b)$$

where $x \in \mathcal{MP}_n$. Let $\tau \in P_n$ such that $c(\tau) = k \leq \binom{n}{2}$ and $\phi(\tau) = w \in \mathcal{MP}_n$. The ab-index of the interval $[\hat{0}, \tau] \subset \hat{P}_n$ is recursively given by

$$\Psi_{[\hat{0},\tau]}(a,b) = (a-b)^k + \sum_{i=0}^{k-1} (a-b)^i b \sum_{\substack{x : x > w \\ \ell(x) = 2\binom{n}{2} - 2i}} \Psi_{[w,x]}(a,b).$$

1.3 Summary of Chapter 4

Reiner *et al.* (2004) defined the *cyclic sieving phenomenon*, CSP for short. To define the CSP, we need: (1) a finite set X, (2) a cyclic group C generated by an element $c \in C$ of order n acting on X, (3) a polynomial $X(q)$ with integer coefficients in a variable q. We say the triple $(X, X(q), C)$ exhibits the CSP if for all integers d,

$$|\{x \in X : c^d(x) = x\}| = X(\zeta^d)$$

where $\zeta = e^{\frac{2\pi i}{n}}$ is a n^{th}-root of unity.

In many cases, we can take $X(q)$ as the q-analog of the cardinality of the set X. Let $[m]!_q := [m]_q[m-1]_q \ldots [2]_q[1]_q$ denote the q-binomial coefficient, and let $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$ denote the q-analog of m. We present a couple of examples.

Example 1.3. Let $\binom{[4]}{2}$ be the set of 2-elements subsets of $\{1, 2, 3, 4\}$. Let $C_4 = \langle (1\ 2\ 3\ 4) \rangle$ and let $X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$. Figure 1.3 shows the C_4-orbits in $\binom{[4]}{2}$. For $\zeta = e^{\frac{2\pi i}{4}} = i$, we see that $X(\zeta^0) = 6$ and $X(\zeta^2) = 2$ and $X(\zeta) = X(\zeta^3) = 0$. Thus the triple $\left(\binom{[4]}{2}, X(q), C_4\right)$ exhibits the CSP.

Example 1.4. Let NCM_3 be the set of non-crossing matchings on $[6]$. Let $C_6 = \langle (1\ 2\ 3\ 4\ 5\ 6) \rangle$ and let $Cat_3(q) = \frac{1}{[4]_q} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6$. Figure 1.4 shows the C_6-orbits in NCM_3. For $\zeta = e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}i}{2}$, we see that $Cat_3(\zeta^0) = 5$ and

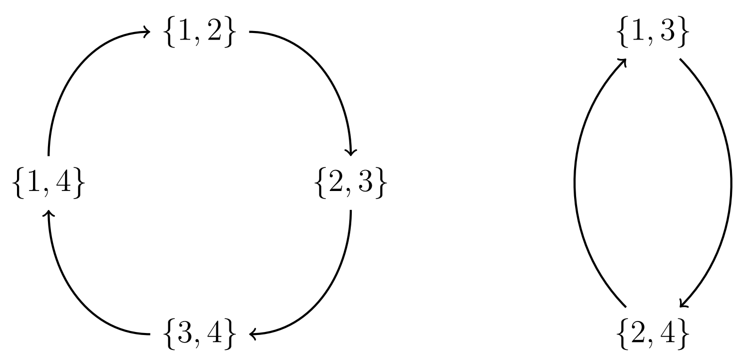


Figure 1.3: The C_4-orbits in $\binom{[4]}{2}$

$Cat_3(\zeta^2) = Cat_3(\zeta^4) = 2$ and $Cat_3(\zeta^3) = 3$ and $Cat_3(\zeta) = Cat_3(\zeta^5) = 0$. Thus the triple $(NCM_3, Cat_3(q), C_6)$ exhibits the CSP.

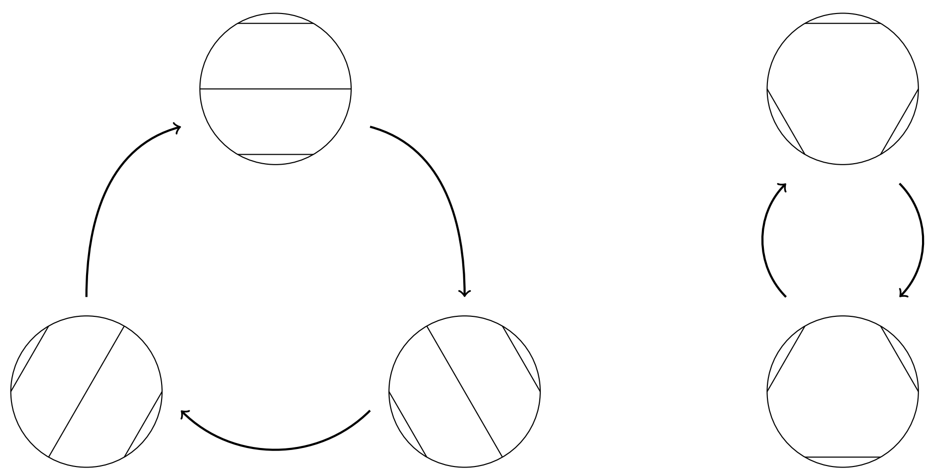


Figure 1.4: The C_6-orbits in NCM_3

We mention that $X(q) = \begin{bmatrix}4\\2\end{bmatrix}_q$ in Example 1.3 is the q-analog of the cardinality $\binom{4}{2}$ of the set $\binom{[4]}{2}$, and $Cat_3(q) = \frac{1}{[4]_q}\begin{bmatrix}6\\3\end{bmatrix}_q$ in Example 1.4 is the q-analog of the cardinality of the set NCM_3.

Sagan (2011) observed the CSP on non-crossing matchings. Bowling and Liang (2017) observed the CSP on the set of k-crossing matchings for $k = 1, 2, 3$. The CSP polynomials in Bowling and Liang (2017) are the q-analog of the cardinality of the set of k-crossing matchings.

In Chapter 4, we mainly focus on the CSP on the whole set of matchings on $[2n]$ instead of the set of matchings of a certain number of crossing. We conjecture that there exist CSP polynomials $X_n(q)$ for the set of matchings on $[2n]$. Because the number of matchings on $[2n]$ is $(2n)!/(2^n \cdot n!)$ or $\prod_{i=1}^{n}(2i-1) = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, the possible candidates for CSP polynomials $X_n(q)$ could be the q-analog of them:

$$Y_n(q) = \frac{[2n]!_q}{([2]_q)^n [n]_q!} \quad \text{or} \quad Z_n(q) = [1]_q[3]_q \cdots [2n-1]_q.$$

Notice that $Y_2(q) = \frac{[4]_q}{[2]_q^2[2]_q!} = \frac{(q^2+1)(q^2+q+1)}{q+1}$ which is not a polynomial. Note also that $Z_2(q) = [3]_q = q^2+q+1 \not\equiv q^2+2 \mod q^4-1$, and thus $Z_2(q)$ is not a CSP polynomial by Example 4.8 in Chapter 4.

Thus we think of a different way to find the polynomials $X_n(q)$. An equivalent condition to the definition of the CSP is presented in Proposition 2.1 in Reiner *et al.* (2004). We first prove that for given set X and a cyclic group C of order N, there is a way, which is equivalent to Proposition 2.1 in Reiner *et al.* (2004), to construct a polynomial $f(q)$ for which the triple $(X, f(q), C)$ exhibits the CSP in Proposition 4.17 and Proposition 4.19 in Chapter 4.

Proposition 1.5 (Proposition 4.17)**.** *Let X be a finite set. Let a cyclic group $C = \langle c \rangle$ act on X where $|c| = N$. Let a_d be the number of elements of X fixed by $c^{N/d}$ for $d|N$. Define b_d by the equation*

$$a_d = \sum_{d|r} \frac{N}{r} b_r. \tag{1.2}$$

Let $f(q)$ be

$$\begin{aligned} f(q) &= \sum_{d|N} b_d(q^{N-d} + q^{N-2d} + \cdots + q^d + 1) \\ &= \sum_{d|N} b_d \frac{[N]_q}{[d]_q}. \end{aligned}$$

Then, the evaluation $f(\zeta^d)$ is equal to the number of elements of X fixed by c^d.

Proposition 1.6 (Proposition 4.19)**.** *The polynomial $f(q)$ constructed in Proposition 4.17 together with X and C exhibits the CSP.*

We then find the number $a_{d,n}$ of matchings on $[2n]$ fixed by the action $c^{2n/d}$ for divisors d of $2n$ in Proposition 4.28.

Proposition 1.7 (Proposition 4.28)**.** *If 2 divides d, then*

$$a_{d,n} = 1 + n\sum_{i\geq 0} \frac{(2i+1)!}{(i+1)!}\binom{\frac{2n}{d}-1}{2i+1}\left(\frac{d}{2}\right)^i. \tag{1.3}$$

If d is not divisible by 2, then

$$a_{d,n} = \prod_{i=1}^{n/d}(2i-1)d. \tag{1.4}$$

Finally we find the polynomials $X_n(q)$ such that P_n together with C_{2n}, the cyclic group of order $2n$, exhibits the CSP in Theorem 4.29.

Theorem 1.8 (Theorem 4.29)**.** *Let $C_{2n} = \langle c\rangle$ where $c = (1\ 2\ \ldots\ 2n)$. Let $X_n(q)$ be*

$$\begin{aligned} X_n(q) &= \sum_{d|2n} b_{d,n}\frac{[2n]_q}{[d]_q} && (1.5)\\ &= \sum_{d|2n} b_{d,n}(q^{2n-d}+q^{2n-2d}+\cdots+q^{2d}+q^d+1) && (1.6)\end{aligned}$$

where the coefficients $b_{d,n}$ satisfy

$$a_{d,n} = \sum_{d|r}\frac{2n}{r}b_{r,n}. \tag{1.7}$$

Then, the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP.

Chapter 2

PRELIMINARIES

In this chapter we give background information on partially ordered sets, the cd-index, affine permutations and Bruhat order. In Section 2.4, we introduce the uncrossing posets. The definitions and notations in this chapter are mainly obtained from Stanley (2012), Björner and Brenti (2005) and Lam (2014). Relatively less well-known materials on these topics are presented in Chapter 3 and 4.

2.1 Partially Ordered Sets

A *partially ordered set* P, or for short *poset*, is a set together with a binary relation denoted $\leq$, satisfying the following three axioms:

1. For all $x \in P$, $x \leq x$ (*reflexivity*),
2. $x \leq y$ and $y \leq x$ implies $x = y$ (*antisymmetry*),
3. $x \leq y$ and $y \leq z$ implies $x \leq z$ (*transitivity*).

If $x \leq y$ and $x \neq y$ then we write $x < y$. We say that y *covers* x, equivalently x is *covered by* y, and denoted $x \lessdot y$, if $x < y$ and there is no z such that $x < z < y$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $x \lessdot y$ then y is drawn above x. Figure 2.1 is the Hasse diagram of the poset of subsets of $\{a, b, c\}$ ordered by inclusion.

For two posets P and Q, a map $\phi : P \to Q$ is called an *order-preserving bijection* if $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q. Two posets P and Q are *isomorphic*, denoted $P \cong Q$, if there exists an order-preserving bijection $\phi : P \to Q$. The *dual* poset P^* of a poset P is the poset on the same set of elements P such that for all x and

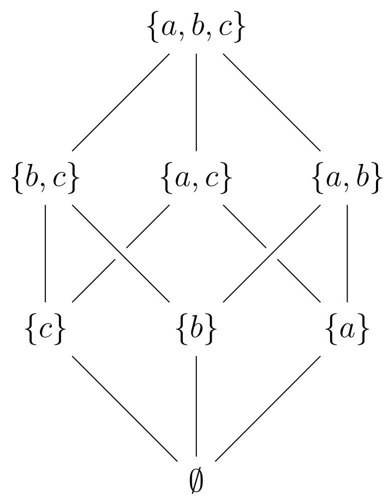


Figure 2.1: The Hasse Diagram of the Poset of Subsets of $\{a, b, c\}$

y, $x \leq y$ in P if and only if $y \leq x$ in P^*. A map ϕ is called an *order-reversing bijection* if ϕ is an order-preserving bijection between P and Q^*. By an *induced subposet*, or for short *subposet*, of P, we mean a subset Q of P and the partial ordering on Q such that for $x, y \in Q$ we have $x \leq y$ in Q if and only if $x \leq y$ in P. For $x \leq y$ in P, a (closed) *interval* $[x, y] = \{z \in P : x \leq z \leq y\}$ is a subposet of P. Given two posets P and Q, form their *product* $P \times Q$ on the set $\{(x, y) : x \in P, y \in Q\}$ such that $(x_1, y_1) \leq (x_2, y_2)$ in $P \times Q$ if $x_1 \leq x_2$ in P and $y_1 \leq y_2$ in Q. The *join* $x \vee y$ of two elements x and y is the unique minimal element in $\{z : x \leq z, y \leq z\}$ if it exists. The *meet* $x \wedge y$ is the unique maximal element in $\{z : z \leq x, z \leq y\}$ if it exists. A *lattice* is a poset P such that every pair $x, y \in P$ has a meet and a join.

We say P has a $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$. Similarly, we say P has a $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for all $x \in P$. We call the elements $\hat{0}$ and $\hat{1}$, if exist, the *minimum* and the *maximum* elements of P respectively. A subset $C = \{x_1, x_2, \ldots, x_n\}$ of P is called a *chain* if $x_1 < x_2 < \cdots < x_n$. A chain is called *maximal* if it is not contained in a larger chain of P. The *length* $\ell(C)$ of a finite chain is defined by $\ell(C) = |C| - 1$. If P is a poset such that all maximal chains have the same length n, then we say that P is *graded of rank* n. In this case there is a unique *rank function* $\rho : P \to \{0, 1, \dots, n\}$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if $x \lessdot y$ in P. If P is graded of rank n and has p_i elements of rank i, then the polynomial

$$F(P, q) = \sum_{i=0}^{n} p_i q^i = \sum_{x \in P} q^{\rho(x)}$$

is called the *rank-generating function* of P. A graded poset P with a unique maximum and a unique minimum is *Eulerian* if, for every interval $[x, y]$ in P where $x < y$, the number of elements of odd rank in $[x, y]$ is equal to the number of elements of even rank in $[x, y]$.

2.2 Affine Permutatons and Bruhat Order

Let $\tilde{S}_n^k, n \geq 2$ be the group of all bijections w of $\mathbb{Z}$ in itself such that

1. $w(x + n) = w(x) + n$ for all $x \in \mathbb{Z}$ and
2. $\sum_{x=1}^{n} w(x) - (1 + 2 + \cdots + n) = nk$,

with composition as group operation. This is the set of affine permutations of type (k, n). We abbreviate $\tilde{S}_n^0$ by $\tilde{S}_n$. The *window notation* of $w \in \tilde{S}_n$ is $w = [a_1, \dots, a_n]$ if $w(i) = a_i$ for $i \in [n]$. As a set of generators for $\tilde{S}_n$ we take the set of periodic adjacent transpositions $\tilde{S} = \{s_0, s_1, \dots, s_{n-1}\}$ where

$$\begin{aligned} s_i &:= [1, 2, \dots, i-1, i+1, i, i+2, \dots, n] \text{ for } 1 \leq i \leq n-1, \\ s_0 &:= [0, 2, 3, \dots, n-1, n+1]. \end{aligned}$$

Then, the pair $(\tilde{S}_n, S)$ is a Coxeter system and $\tilde{S}_n$ is the Coxeter group of *affine permutations* of the integers.

There are several partial orders defined on $\tilde{S}_n$. We need (strong) Bruhat order and from now on, $\tilde{S}_n$ denotes the affine permutations, together with this order. For $w \in$

$\tilde{S}_n$, a decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ with letters in S is called a *reduced decomposition* for w if k is minimal. The word $i_1 i_2 \ldots i_k$ is called a *reduced word* for w. We say k is the *length* of w and denote it by $\ell(w)$. Fix a reduced decomposition for $w = t_1 t_2 \ldots t_k$ where $t_i \in S$ for all $1 \leq i \leq k$. The Bruhat order is defined by $v \leq_B w$ if and only if there is a reduced subword $t_{i_1} t_{i_2} \ldots t_{i_j}$ of $t_1 t_2 \ldots t_k$ for v such that $1 \leq i_1 < i_2 < \cdots < i_j \leq k$. We will write $v \leq w$ for $v \leq_B w$ if there is no possibility of confusion. Figure 2.2 is an example of an interval in the Bruhat order on $\tilde{S}_2$.

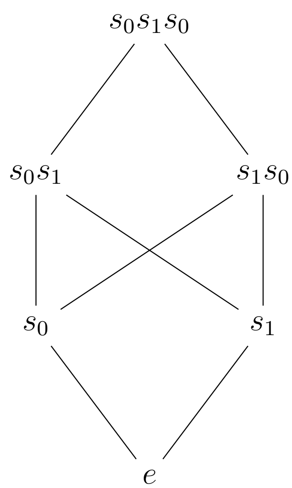


Figure 2.2: The Hasse Diagram of the Interval $[e, s_0 s_1 s_0]$ in $\tilde{S}_2$

2.3 The cd-index

The cd-index arises from the work of Bayer and Billera (1984) on flag f-vectors of Eulerian posets. They extended the f-vector and the h-vector of a polytope Δ to the flag f-vector and the flag h-vector of the order complex $\Delta(P)$ of a finite graded poset P. A *polytope* is the convex hull of finitely many points in $\mathbb{R}^d$. A *hyperplane* of $\mathbb{R}^d$ is an affine subspace of dimension $d - 1$. A *half-space* is either of the two parts into which a hyperplane divides an affine space. A *supporting hyperplane* of a polytope is a hyperplane such that the polytope is contained in one of half-spaces. A *face* of a polytope is any intersection of a supporting hyperplane with the polytope. A *simplex* is the convex hull of affinely independent points. A polytope is called *simplicial* if each of its faces, except possibly the polytope itself, is a simplex.

Let Δ be a finite $(d-1)$-dimensional simplicial complex with f_i i-dimensional faces. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of Δ. The vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$, called the *$h$-vector*, is defined by the relation

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$$

where $f_{-1} = 1$ unless $\Delta = \emptyset$.

Let P be a graded poset, rank n, with a $\hat{0}$ and a $\hat{1}$. The *order complex* $\Delta(P)$ of P is defined as follows: the vertices of $\Delta(P)$ are the elements of $P - \{\hat{0}, \hat{1}\}$, and the faces of $\Delta(P)$ are the chains of $P - \{\hat{0}, \hat{1}\}$. Then, $\Delta(P)$ is a simplicial complex, and Bayer and Billera (1984) enumerated faces of this order complex $\Delta(P)$ as follows. For a chain C in $P - \{\hat{0}, \hat{1}\}$, define $\rho(C) = \{\rho(x) : x \in C\}$. Let $[n] := \{1, 2, \ldots, n\}$ and let $2^{[n]}$ be the set of all subsets of $[n]$. For any $S \subseteq [n-1]$, define

$$\alpha_P(S) = \left|\{C \subseteq P : C \text{ is a chain such that } \rho(C) = S\}\right|.$$

We call the function $\alpha_P : 2^{[n-1]} \to \mathbb{N}$ the *flag f-vector*. We define

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

The function $\beta_P : 2^{[n-1]} \to \mathbb{N}$ is called the *flag h-vector* of P. Then we can check for the order complex $\Delta = \Delta(P)$

$$\begin{aligned} f_i(\Delta) &= \sum_{|S|=i+1} \alpha_P(S) \\ h_i(\Delta) &= \sum_{|S|=i} \beta_P(S). \end{aligned}$$

Thus the flag f-vector α_P of the order complex $\Delta(P)$ counts flags (or chains) of $P - \{\hat{0}, \hat{1}\}$ by length, which are faces of $\Delta(P)$ by dimension, as the f-vector counts faces of a polytope by dimension. The flag h-vector β_P plays a role as the h-vector of a polytope.

Now we define ab-index of a poset and cd-index of an Eulerian poset. Define the *characteristic monomial* u_S of $S \subseteq [n-1]$ by $u_S = e_1 e_2 \ldots e_{n-1}$, where

$$e_i = \begin{cases} b & \text{if } i \in S \\ a & \text{if } i \notin S. \end{cases}$$

Define a noncommutative polynomial $\Psi_P(a,b)$, called the *ab-index* of P, by

$$\Psi_P(a,b) = \sum_{S \subseteq [n-1]} \beta_P(S) u_S.$$

Thus $\Psi_P(a,b)$ is a noncommutative generating function for the flag h-vector β_P. By definition of flag h-vector $\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T)$, it is true that

$$\Psi_P(a+b,b) = \sum_{S \subseteq [n-1]} \alpha_P(S) u_S$$

which is a generating function for the flag f-vector α_P. By Bayer and Klapper (1991), if P is an Eulerian poset of rank n, then there exists a polynomial $\Phi_P(c,d)$ in the noncommutative variables c and d such that $\Psi_P(a,b) = \Phi_P(a+b, ab+ba)$. The polynomial $\Phi_P(c,d)$ is called the *cd-index* of P.

Example 2.1. Let us compute the cd-index of the interval $[e, s_0 s_1 s_0]$ in $\tilde{S}_2$ presented in Figure 2.2. There are 2 chains containing an element of rank 1 but no element of rank 2, so the coefficient of ba is 2 in $\Psi_{[e,s_0 s_1 s_0]}(a+b,b)$. There are 4 chains containing an element of rank 1 and an element of rank 2, so the coefficient of b^2 is 4 in $\Psi_{[e,s_0 s_1 s_0]}(a+b,b)$. Similarly, the coefficients of a^2 and ab are 1 and 2 respectively. Thus we have

$$\Psi_{[e,s_0 s_1 s_0]}(a+b,b) = a^2 + 2ab + 2ba + 4b^2.$$

By replacing a by $a-b$, we get

$$\begin{aligned}\Psi_{[e,s_0s_1s_0]}(a,b) &= (a-b)^2+2(a-b)b+2b(a-b)+4b^2\\ &= a^2+ab+ba+b^2.\end{aligned}$$

By substituting $c=a+b$ and $d=ab+ba$, we have the cd-index $\Phi_{[e,s_0s_1s_0]}(c,d)=c^2$.

2.4 The Uncrossing Posets

In this section we introduce the circular planar electrical networks and the uncrossing posets. The definitions and notations are mainly obtained from Lam (2014) and Lam (2015). For more details for circular electrical planar networks, we refer to Curtis *et al.* (1998) and Colin de Verdière *et al.* (1996). For more details for uncrossing posets, we refer to Alman *et al.* (2015), Kenyon (2012), and Huang *et al.* (2014).

2.4.1 The Circular Planar Electrical Networks

A *circular planar electrical network*, or for short *an electrical network*, is a finite weighted undirected graph Γ embedded into a disk, with *boundary* vertices and *interior* vertices. Each edge represents a resistor and the weight of the edge represents the conductance of the resistor. The electrical properties of Γ are encoded in a $n\times n$ *response matrix* $\Lambda(\Gamma)$ which sends a vector of voltages at the n boundary vertices $\{v_1,v_2,\ldots,v_n\}$, to the vector of currents induced on the same vertices. Two electrical networks Γ_1 and Γ_2 are electrically equivalent if they have the same response matrix. For example, electrical networks Γ_1 and Γ_2 have the same response matrix

$$\Lambda(\Gamma_1)=\Lambda(\Gamma_2)=\begin{bmatrix}1/2 & -1/2\\ 1/2 & -1/2\end{bmatrix}.$$

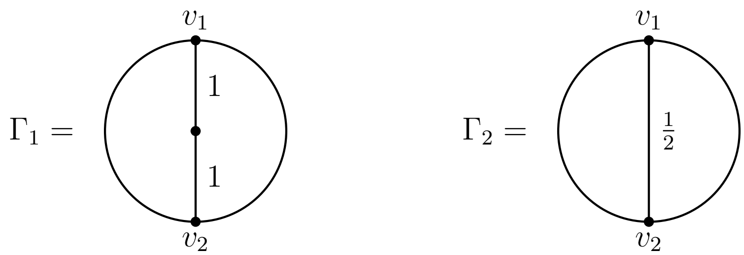


Figure 2.3: Equivalent Electrical Networks

In Colin de Verdière *et al.* (1996), four well-known electrically equivalent transformations are introduced:

1. series and parallel transformations,
2. removing interior degree 1 vertices (pendant removal),
3. removing loops,
4. $Y - \Delta$ transformation.

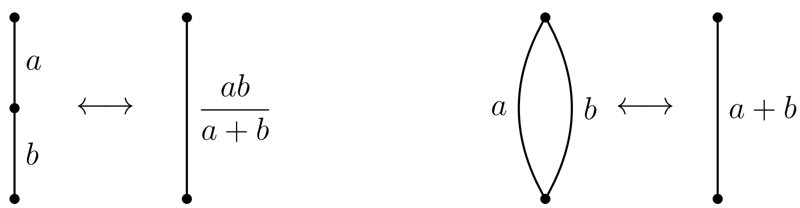


Figure 2.4: Series and Parallel Transformations

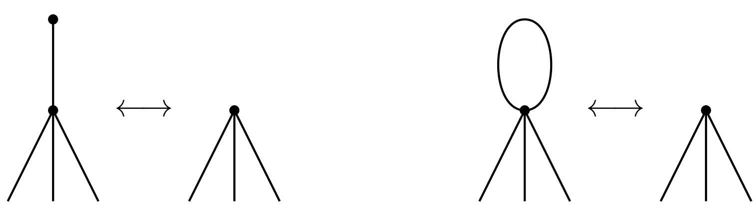


Figure 2.5: Removing a Pendant and a Loop

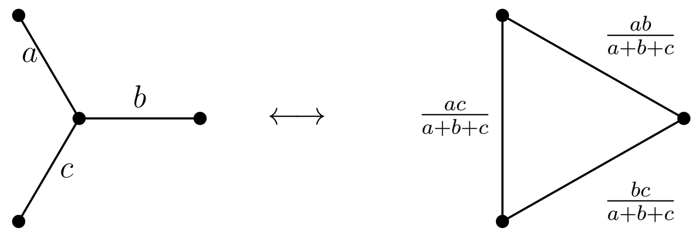


Figure 2.6: $Y - \Delta$ Transformation

Colin de Verdière *et al.* (1996) showed that for any two circular planar electrical networks Γ_1 and Γ_2 having the same response matrix, there is a sequence of electrically equivalent transformations $T_1, \ldots, T_k$ such that $\Gamma_2 = T_k \ldots T_1 \Gamma_1$.

2.4.2 Medial Graphs

Let Γ be a unweighted electrical network. The *medial graph* $M(\Gamma)$ of Γ is defined as follows. Let $v_1, \ldots, v_n$ be the boundary vertices in a circular order. First for each v_i, place a vertex t_{2i-1} on left side of v_i and another vertex t_{2i} on the right side of v_i on the circle. Next add a vertex t_e for each edge $e \in E(\Gamma)$. Join t_e with $t_{e'}$ if e and e' share a vertex and border the same face. Here, a face means a polygon enclosed by edges. Since the networks are planar graphs, faces are well-defined. For boundary vertices t_{2i-1} or t_{2i}, we draw an edge to t_e if v_i is an endpoint of e. If a vertex v_i is isolated, then join t_{2i-1} and t_{2i} by an edge. Observe that each vertex t_e is of degree 4, and each vertex t_i on the circle has degree 1. Finally, we form *wires* in $M(\Gamma)$ in the following way. Starting at each t_i, draw a path $t_i t_{e_1} t_{e_2} \ldots t_{e_k} t_j$ so that the edges $t_{e_{i-1}} t_{e_i}$ and $t_{e_i} t_{e_{i+1}}$ separate the other two edges incident to t_{e_i}.

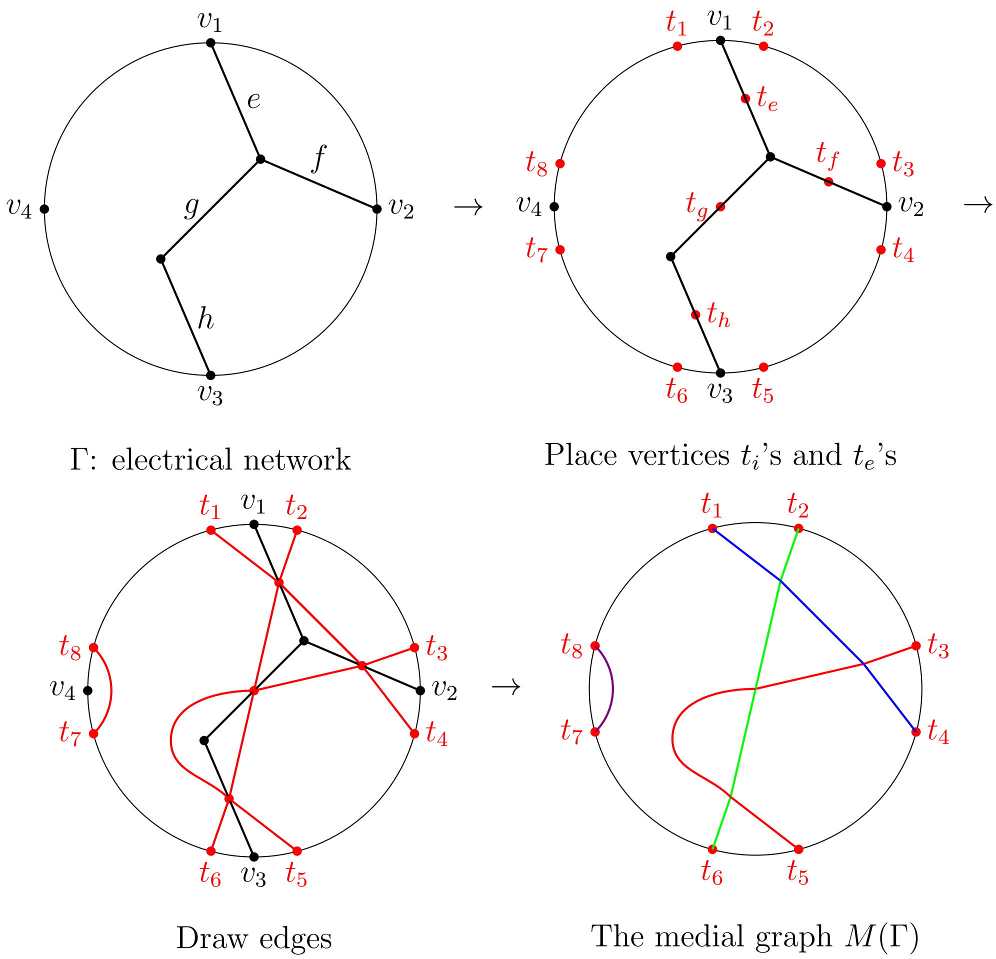


Figure 2.7: The Medial Graph $M(\Gamma)$ of an Electrical Network Γ

Wires in medial graphs either join boundary vertices to boundary vertices or form cycles inside the circle. A medial graph is called *lensless* if it satisfies the following three conditions:

1. every wire begins and ends on the circle,
2. any two wires intersect at most once,
3. no wire has a self intersection.

A medial graph can be reduced to a lensless medial graph by removing bubbles and loops.

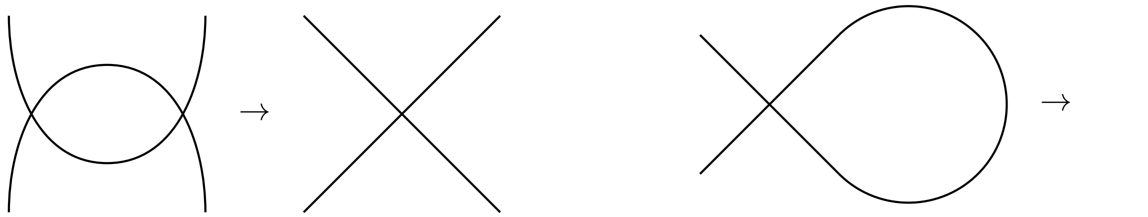
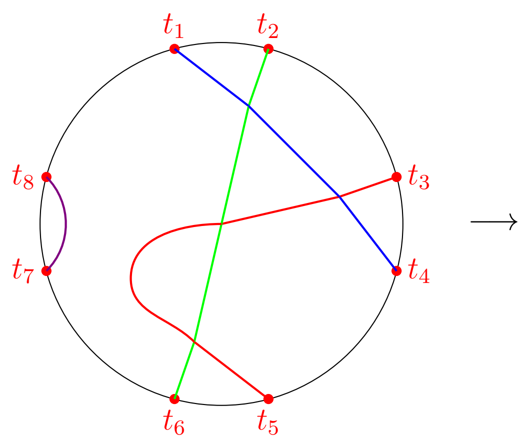
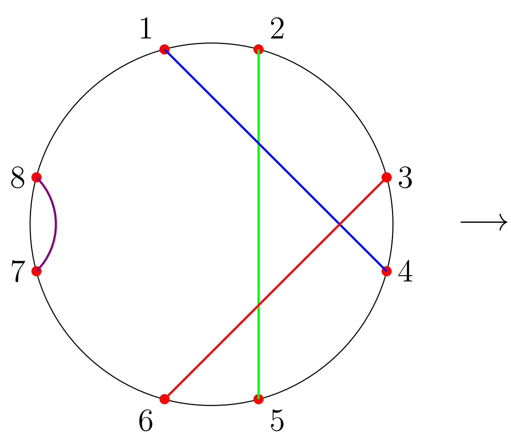


Figure 2.8: Removal of a Bubble and a Loop

A lensless medial graph induces a matching on the set $[2n]$.



A medial graph $M(\Gamma)$ with a bubble



$\longrightarrow \qquad \{(1,4),(2,5),(3,6),(7,8)\}$

A lensless medial graph

Figure 2.9: A Matching Obtained from a Lensless Medial Graph

One may want to do electrically equivalent transformations first, then obtain the same matching.

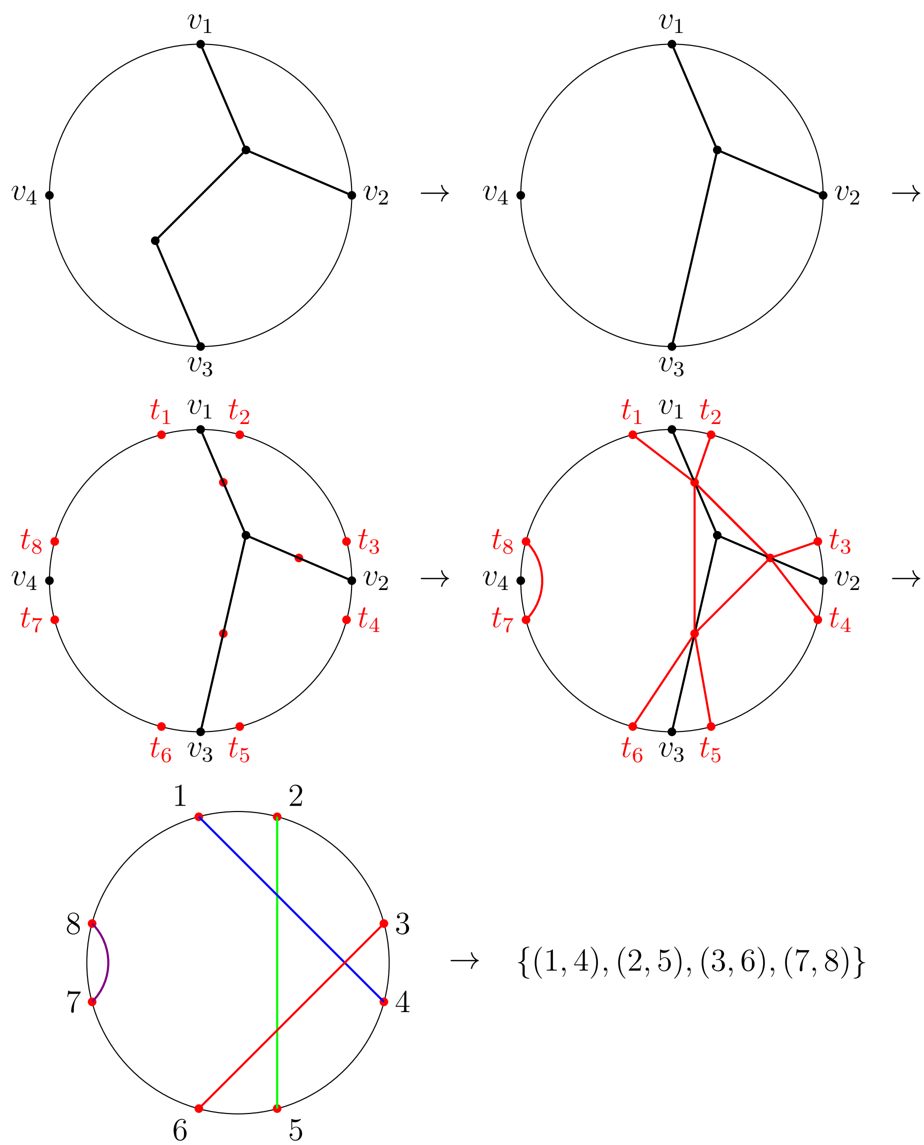


Figure 2.10: The Same Matching Obtained from an Equivalent Network

Conversely, the electrical network Γ can be recovered from the medial graph $M(\Gamma)$. Note, however, that not all matchings on $[2n]$ can arise from electrical networks. For example, there is no electrical network Γ such that the medial graph $\Lambda(\Gamma)$ induces $\{(1,2),(3,8),(4,7),(5,6)\}$.

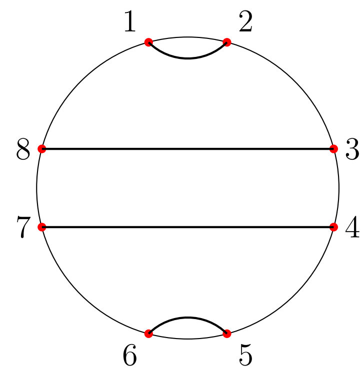


Figure 2.11: A Matching Which Can't Be Obtained from Any Electrical Network

Lam (2014) showed that by compactification of the space of electrical networks all matchings on $[2n]$ are obtained.

2.4.3 The Uncrossing Posets

We have seen that a matching τ on $[2n]$ can be represented by a lensless medial graph $M(\tau)$. Now resolve any crossing in $M(\tau)$ in either of two ways:

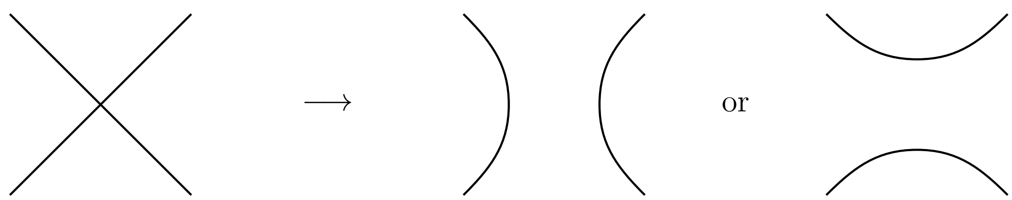


Figure 2.12: Two Ways to Resolve a Crossing

This gives a new lensless medial graph $M' = M(\tau')$ for some matching τ' on $[2n]$. Then we define the partial order $\tau' \leq \tau$ on the set of matchings on $[2n]$ if the lensless medial graph $M(\tau')$ is obtained by resolving a crossing of the lensless medial graph $M(\tau)$. Let $c(\tau)$ be the number of crossings of a lensless medial graph for τ. From now let P_n denote the set of matchings on $[2n]$ together with this order. We call P_n the *uncrossing partial order on matchings* on $[2n]$, the *uncrossing poset* for short. The poset P_n is graded of rank $\binom{n}{2}$ with rank function given by $c(\tau)$. The poset P_n has a unique maximum element, namely the matching $\{(1, n+1), (2, n+2), \ldots, (n, 2n)\}$, and Catalan number $\frac{1}{n+1}\binom{2n}{n}$ of minimal elements, which are non-crossing matchings. Figure 2.13 shows the Hasse diagram of P_3.

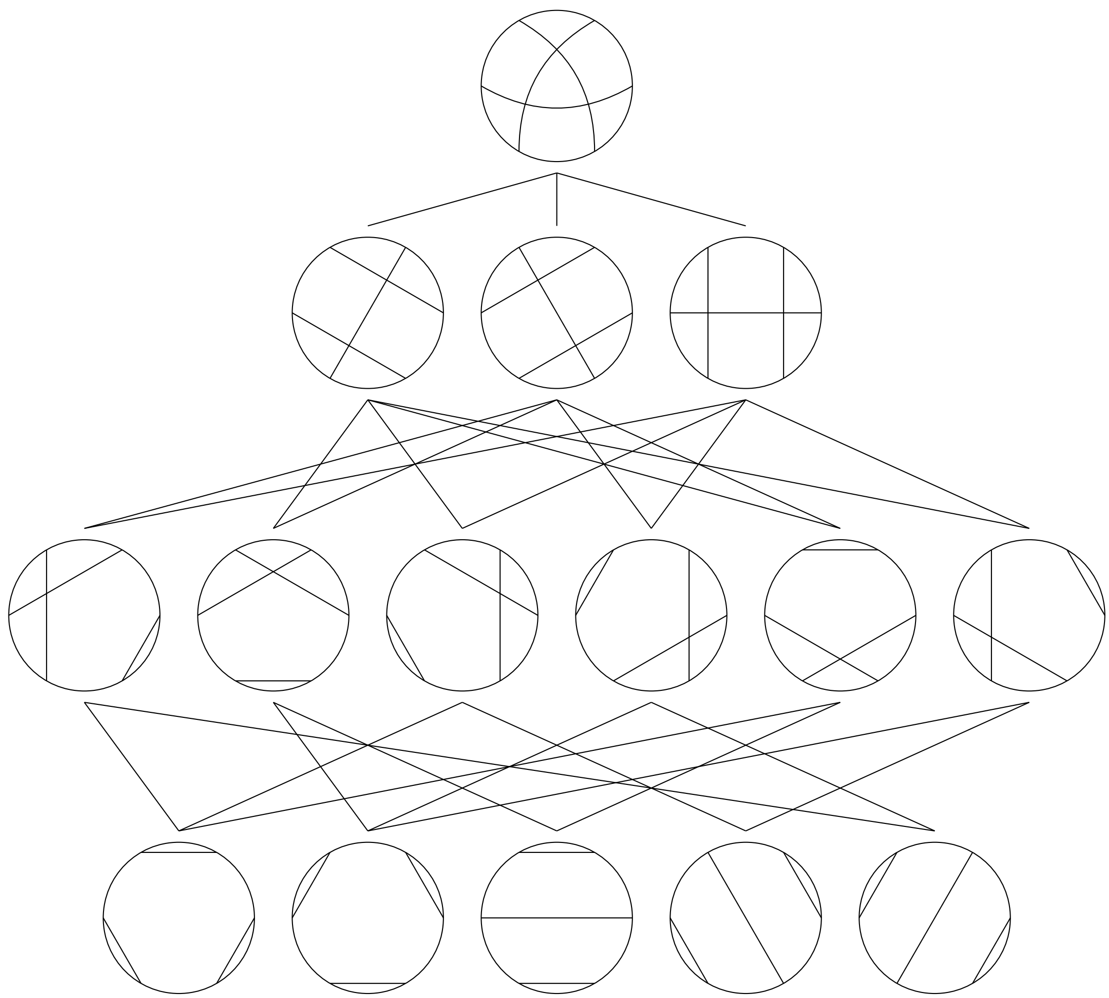


Figure 2.13: The Hasse Diagram of P_3 (Courtesy of Thomas Lam, Used with Permission)

This partial order has been studied by Alman *et al.* (2015), Huang *et al.* (2014), Kenyon (2012), Lam (2014) and Lam (2015). Let $\hat{P}_n$ denote P_n with a unique minimum element $\hat{0}$ adjoined, where we let $c(\hat{0}) = -1$. The poset $\hat{P}_n$ was conjectured to be Eulerian by Alman *et al.* (2015), Huang *et al.* (2014), and Kim and Lee (2014), and is proved to be so by Lam (2015).

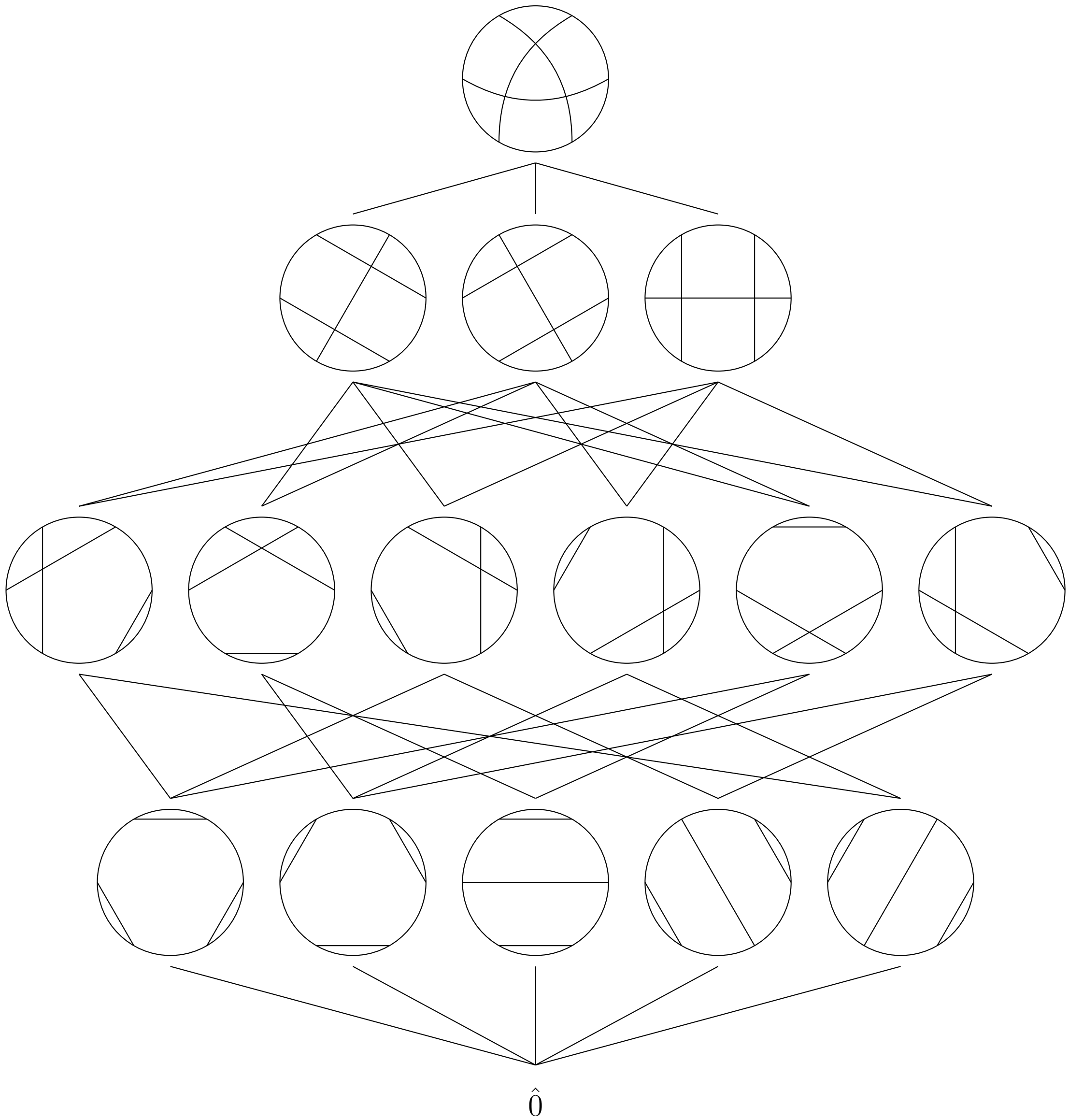


Figure 2.14: The Hasse Diagram of the Eulerian Poset $\hat{P}_3$

Theorem 2.2 (Lam (2015), Theorem 1)**.** *$\hat{P}_n$ is an Eulerian poset.*

Lam (2015) used the map $\tau \mapsto g_\tau$ (see (1.1) in Chapter 1, Lam (2014)) as the main tool for proving Theorem 2.2. Using this map, Lam showed that the number of odd elements equals the number of even rank elements in intervals in $\hat{P}_n$. To be specific, he first showed that the number of odd rank elements equals the number of even rank elements in any interval $[\tau, \eta] \subset P_n$ by descending induction on $c(\tau) + c(\eta)$. Then he showed the number of odd rank elements equals the number of even rank elements in any interval $[\hat{0}, \eta] \subset \hat{P}_n$ by establishing an involution $\sigma \mapsto s_i \cdot \sigma$ on the set $\{\sigma \in (\hat{0}, \eta] | s_i \cdot \sigma \neq \sigma\}$.

The rank-generating function of P_n was discovered by Touchard (1950) and Riordan (1975),

Theorem 2.3 (Touchard (1950), Riordan (1975))**.** *The rank-generating function $F(P_n, q)$ is*

$$F(P_n, q) = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} (-1)^k q^{k(k-1)/2} \binom{2n}{n+k}.$$

Chapter 3

THE CD-INDICES OF INTERVALS IN THE UNCROSSING PARTIAL ORDER ON MATCHINGS

In this chapter, we present a set of results on flag enumeration in intervals in the uncrossing partial order on matchings. We produce recursions for the cd-indices of intervals in the uncrossing poset P_n. In Section 3.1 we define an induced subposet $\mathcal{MP}_n$ of Bruhat order on affine permutations $\tilde{S}_{2n}$ of type $(0, 2n)$. We prove that there is an order-reversing bijection between P_n and $\mathcal{MP}_n$. With this bijection we are able to explicitly describe the elements in P_n. In Section 3.2, we recall the recursive formulas for the cd-indices of intervals in Bruhat order of a Coxeter group studied by Reading (2004). In Section 3.3, we prove that there are recursive formulas for the cd-indices of intervals in P_n. Furthermore, we present a recursive formula for the ab-indices of intervals in the poset $\hat{P}_n$.

3.1 Modular Palindromic Permutations

Lam (2014) maps matchings $\tau \in P_n$ to affine permutations g_τ of type $(n, 2n)$ as (1.1) in Chapter 1. Lam showed that this map $\tau \mapsto g_\tau$ identifies P_n with an induced subposet of dual Bruhat order of affine permutations of type $(n, 2n)$ (Theorem 4.16 in Lam (2014)), through Theorem 8.3.7 in Björner and Brenti (2005) which characterizes affine Bruhat order in terms of a matrix which tracks inversions.

We notice that if we slightly modify Lam's map it is possible to show that P_n is identified with an induced subposet of dual Bruhat order of affine permutations $\tilde{S}_{2n}$ of type $(0, 2n)$ without Theorem 8.3.7 in Björner and Brenti (2005). We believe that this identification is necessary to apply a technique in Reading (2004) for finding recursions for the cd-indices of intervals in Bruhat order on Coxeter groups. Moreover, we are able to explicitly describe the elements in the induced subposet of affine permutations as modular palindromic permutations (see Theorem 3.9).

For these reasons, we slightly modify Lam's map $\tau \mapsto g_\tau$ to define an injective map $\phi : P_n \to \tilde{S}_{2n}$ as follows. For a matching $\tau \in P_n$, define $h_\tau : [2n] \to \mathbb{Z}$ by

$$h_\tau(i) = \begin{cases} \tau(i) - n & \text{if } i < \tau(i) \\ \tau(i) + n & \text{if } i > \tau(i). \end{cases}$$

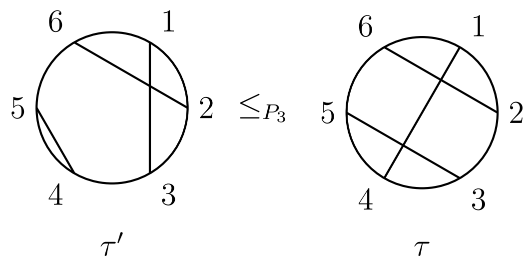
Now define $\phi : P_n \to \tilde{S}_{2n}$ by $\phi(\tau) = [h_\tau(1), h_\tau(2), \dots, h_\tau(2n)]$ in the window notation of the affine permutation group $\tilde{S}_{2n}$.

Example 3.1. Let $\tau = \{(1,4),(2,6),(3,5)\}$ and $\tau' = \{(1,3),(2,6),(4,5)\}$ in P_3. The images of τ and τ' under Lam's map are $g_\tau = [g_\tau(i)]_{i=1}^6 = [4,6,5,7,9,8]$ and $g_{\tau'} = [g_{\tau'}(i)]_{i=1}^6 = [3,6,7,5,10,8]$ in the window notation. On the other hands, the images under the map ϕ are $\phi(\tau) = [1,3,2,4,6,5] = s_2 s_5 \in \tilde{S}_6$ and $\phi(\tau') = [0,3,4,2,7,5] = s_2 s_0 s_3 s_5 \in \tilde{S}_6$.

With the map ϕ, we explicitly describe the subposet of $\tilde{S}_{2n}$ which is isomorphic to the dual of P_n. First, we define a *modular palindromic* permutation and a subset $\mathcal{MP}_n$ of the affine permutation group $\tilde{S}_{2n}$.

Definition 3.2. An affine permutation $w \in \tilde{S}_{2n}$ is *modular palindromic* if w has a reduced decomposition $w = s_{i_1} s_{i_2} \dots s_{i_{2k}}$ with $|i_{2k+1-r} - i_r| = n$ for all $r \in [k]$.

Example 3.3. Let $\tau = \{(1,4),(2,6),(3,5)\}$ and $\tau' = \{(1,3),(2,6),(4,5)\}$ in P_3 as in Example 3.1. Both $\phi(\tau) = s_2 s_5$ and $\phi(\tau') = s_2 s_0 s_3 s_5$ are modular palindromic because $|2-5| = |0-3| = 3$. We also observe that ϕ reverses the order in P_3; in other words, $\tau' \leq \tau$ in P_3 but $\phi(\tau') \geq \phi(\tau)$ in Bruhat order.



$$\longleftrightarrow \quad \phi(\tau') = s_2 s_0 s_3 s_5 \ge_B s_2 s_5 = \phi(\tau)$$

Figure 3.1: Matchings in P_3 and Their Images under the Map ϕ

Example 3.4. Let $n = 3$. Let $w = s_0 s_5 s_1 s_4 s_2 s_3 \in \tilde{S}_6$. Notice that w is a modular palindromic permutation because $|0-3| = |5-2| = |1-4| = 3$. The window notation of the permutation is $w = [2, 3, 7, 0, 4, 5]$. Assume $w = \phi(\tau)$ for some $\tau \in P_3$. Since $w(3) = 7$, we must have $h_\tau(3) = 7$, but it is impossible because $1 \le h_\tau(3) \le 5$. Thus $w \notin \phi(P_3)$.

The previous example shows that not all modular palindromic permutations are images of matchings under the map ϕ. Which conditions are needed for modular palindromic permutations to be in the image of the map ϕ? We need the following definition.

Definition 3.5. The subset $\mathcal{MP}_n$ of $\tilde{S}_{2n}$ is the set of all modular palindromic permutations $w \in \tilde{S}_{2n}$ such that no reduced word of w contains a subword of n consecutive integers.

Remark 3.6. The consecutive integers in Definition 3.5 need not be in adjacent positions in the reduced word.

Now we use $\mathcal{MP}_n$ as a poset, equipped with Bruhat order.

Example 3.7. Let $n = 2$. We see that $\mathcal{MP}_2$ contains $e, s_0 s_2$ and $s_1 s_3$. Notice that all length four modular palindromic permutations, whose reduced words are 1023, 0132, 2130 and 3201 respectively, have a subword of 2-consecutive integers, and thus

$\mathcal{MP}_2 = \{e, s_0s_2, s_1s_3\}$. Figure 3.2 shows the Hasse diagrams of P_2 and the dual of $\mathcal{MP}_2$, respectively.

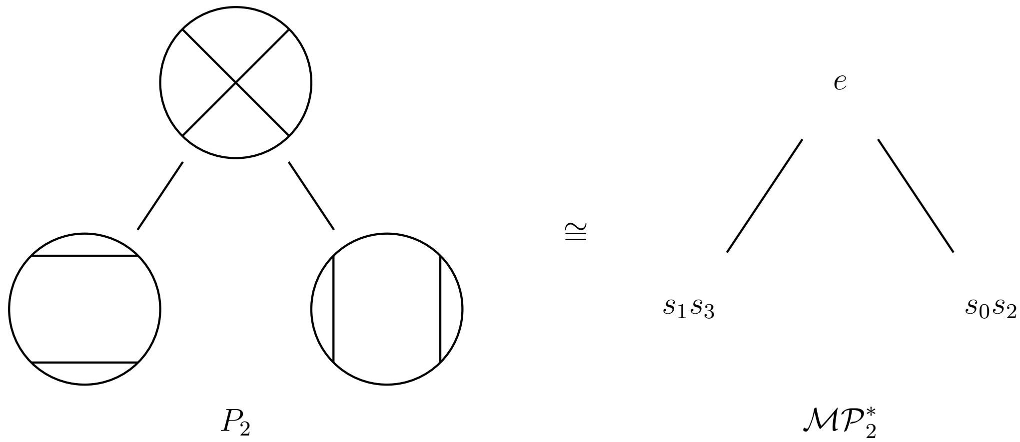


Figure 3.2: The Hasse Diagrams of P_2 and $\mathcal{MP}_2^*$

Example 3.8 (revisited)**.** Let $n = 3$. Let $w = s_0s_5s_1s_4s_2s_3 \in \tilde{S}_6$ with a reduced word 051423. Notice that w is a modular palindromic permutation, which has a subword 543, a 3 consecutive integers. Thus $w \notin \mathcal{MP}_3$.

The following theorem is one of the main results in this dissertation.

Theorem 3.9. *The map ϕ is an order-reversing bijection between P_n and $\mathcal{MP}_n$.*

To prove this theorem, we first state and prove a lemma.

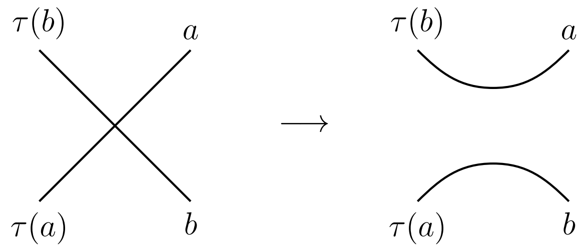
Lemma 3.10. *The image of P_n under the map ϕ is contained in $\mathcal{MP}_n$. In other words, $\phi(P_n) \subseteq \mathcal{MP}_n$.*

Proof. Let $\tau \in P_n$. We must show that $\phi(\tau)$ is modular palindromic and no reduced word for $\phi(\tau)$ contains a subword of n consecutive integers. First, we prove that $\phi(\tau)$ is modular palindromic using decreasing induction on the ranks of $\tau \in P_n$.

(i) Base case: the unique maximum element $\hat{1} = \{(1, n+1), (2, n+2), \ldots, (n, 2n)\}$. Notice that $\phi(\hat{1}) = e \in \mathcal{MP}_n$ as required.

(ii) Induction step: suppose $\tau \leq \hat{1}$ and $\phi(\tau) \in \mathcal{MP}_n$. Choose a crossing generated by a pair of wires $(a, \tau(a))$ and $(b, \tau(b))$ such that $a < b < \tau(a) < \tau(b)$. We can resolve the crossing in two ways.

Case 1: from $\{(a, \tau(a)), (b, \tau(b))\}$ to $\{(a, \tau(b)), (b, \tau(a))\}$. Let $\tau' \in \mathcal{MP}_n$ be the matching obtained by resolving the crossing this way.

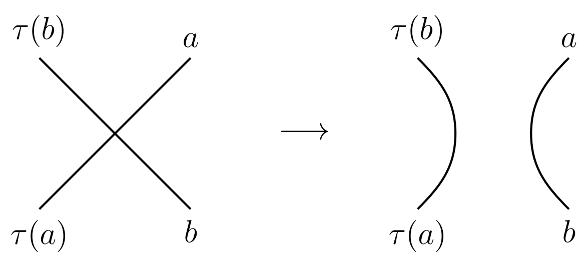


In window notation, we see that

$$\begin{array}{rcl} & & \ldots, \quad a \quad, \ldots, \quad b \quad, \ldots, \ \tau(a)\ , \ldots, \ \tau(b)\ , \ldots \\ \phi(\tau) & = & [\ldots, \tau(a) - n, \ldots, \tau(b) - n, \ldots, a + n, \ldots, b + n, \ldots] \\ \phi(\tau') & = & [\ldots, \tau(b) - n, \ldots, \tau(a) - n, \ldots, b + n, \ldots, a + n, \ldots]. \end{array}$$

Observe that $\phi(\tau')$ is obtained from $\phi(\tau)$ by swapping the numbers in the a-th spot and b-th spot and swapping the numbers $a+n$ and $b+n$. Thus, $\phi(\tau') = t_{a+n,b+n}\phi(\tau)t_{a,b}$ where $t_{a,b} = s_a s_{a+1} \ldots s_{b-2} s_{b-1} s_{b-2} \ldots s_{a+1} s_a$ is the periodic transposition of a and b. Therefore, $\phi(\tau')$ is modular palindromic.

Case 2: from $\{(a, \tau(a)), (b, \tau(b))\}$ to $\{(a, b), (\tau(a), \tau(b))\}$.



$\phi(\tau)$ $\qquad$ $\phi(\tau'')$

In window notation, we see that

$$\begin{array}{rcl} & & \ldots, \quad a \quad, \ldots, \quad b \quad, \ldots, \quad \tau(a) \quad, \ldots, \quad \tau(b) \quad, \ldots \\ \phi(\tau) & = & [\ldots, \tau(a)-n, \ldots, \tau(b)-n, \ldots, \quad a+n \quad, \ldots, \quad b+n \quad, \ldots] \\ \phi(\tau'') & = & [\ldots, \quad b-n \quad, \ldots, \quad a+n \quad, \ldots, \tau(b)-n, \ldots, \tau(a)+n, \ldots]. \end{array}$$

Observe that $\phi(\tau'')$ is obtained from $\phi(\tau)$ by swapping the numbers in the b-th spot and $\tau(a)$-th spot and swapping the numbers $b-n$ and $\tau(a)-n$, which is equivalent to changing the numbers $b+n$ and $\tau(a)+n$ by periodicity of $2n$. Thus, $\phi(\tau'') = t_{b+n,\tau(a)+n}\phi(\tau)t_{b,\tau(a)}$, and it is modular palindromic.

Secondly, we prove that there is no reduced word for $\phi(\tau)$ which contains a subword of n consecutive integers. For the sake of contradiction, suppose there is a reduced word for $\phi(\tau)$ containing a subword of n consecutive integers. Without loss of generality, assume the consecutive integers are increasing. Take a maximal length subword $s_{a+1}s_{a+2}\ldots s_{a+m}$ of $\phi(\tau)$ where s_{a+i}'s are periodic adjacent transpositions and $m \geq n$ and the indices are taken modulo $2n$ if necessary. Then, observe that $\phi(\tau)(a+m+1) = a+1$, which contradicts that $i-n < \phi(\tau)(i) < i+n$ by the definition of ϕ for all i. $\square$

Corollary 3.11. *The map ϕ is order-reversing. In other words, $\tau' \leq \tau$ in P_n implies $\phi(\tau') \geq \phi(\tau)$ in $\mathcal{MP}_n$ or in $\tilde{S}_{2n}$.*

Proof. By the proof of the previous lemma, we see that if a matching τ' is obtained by resolving a crossing of a matching τ, then $\phi(\tau') = t_{a,b}\phi(\tau)t_{a+n,b+n}$ for some $a, b \in [2n]$. Since a reduced word of $\phi(\tau)$ is a subword of a reduced word of $\phi(\tau')$, we conclude that $\phi(\tau) \leq \phi(\tau')$ as required. $\square$

The following lemma will be used when we prove Theorem 3.9; in particular, we will use the lemma to show that the map ϕ is surjective.

Lemma 3.12. *Let $p, q \in \mathcal{MP}_n$ such that $q = s_a p s_{a+n}$ and $\ell(q) = \ell(p) + 2$. Let $x, y \in [2n]$ such that $p(x) \equiv a \pmod{2n}$ and $p(y) \equiv a + 1 \pmod{2n}$. Then,*

(1) The numbers $x, y, a+n$ and $a+n+1$ are distinct.

(2) $q(i) = p(i)$ for all $i \in [2n] \backslash \{a+n, a+n+1, x, y\}$.

(3) $q(a+n) = p(a+n+1)$, $q(a+n+1) = p(a+n)$, $q(x) \equiv a+1 \pmod{2n}$, and $q(y) \equiv a \pmod{2n}$.

Proof. (1) It is clear that $x \neq y$ and $a+n \neq a+n+1$. Since $p \in \mathcal{MP}_n$, we know $a < p(a+n) < a+2n$ and $a+1 < p(a+n+1) < a+2n+1$, and thus $a+n \neq x$ and $a+n+1 \neq y$. Asume $y = a+n$, so that $p(a+n) \equiv a+1 \pmod{2n}$. Then any reduced decomposition for p must contain a subword of the form $s_{a+1}s_{a+2}\dots s_{a+n-2}s_{a+n-1}$. Then any reduced decomposition for $q = s_a p s_{a+n}$ must contain a subword of n consecutive integers, which contradicts $q \in \mathcal{MP}_n$. Assume $x = a+n+1$, so that $p(a+n+1) \equiv a \pmod{2n}$. Then any reduced decomposition for p must contain a subword of the form $s_{a+2n+1}s_{a+2n}\dots s_{a+n+2}s_{a+n+1}$. Then any reduced decomposition for $q = s_a p s_{a+n}$ must contain a subword of n consecutive integers, which contradicts $q \in \mathcal{MP}_n$.

(2) Let $i \in [2n]\backslash\{a+n, a+n+1, x, y\}$. Then it is clear that $s_{a+n}(i) = i$. Note that $p(i) \notin \{a, a+1\}$ since $i \notin \{x, y\}$, and thus $s_a(p(i)) = p(i)$. Then we have

$$q(i) = (s_a p s_{a+n})(i) = s_a(p(i)) = p(i).$$

(3) Since s_{a+n} acts on the right of p by swapping numbers in the $(a+n)$-th spot and $(a+n+1)$-th spot and s_a acts on the left of p by swapping numbers a and $a+1$,

we have

$$\begin{aligned}
q(a+n+1) &= s_a p s_{a+n}(a+n+1) \\
&= s_a(p(a+n)) \\
&= p(a+n).
\end{aligned}$$

Note that $(a+n+1)-n < p(a+n+1) < (a+n+1)+n$ and $a+n-n < q(a+n) < a+n+n$ since $p, q \in \mathcal{MP}_n$, and equivalently $a+2 \leq p(a+n+1) \leq a+2n$ and $a+1 \leq q(a+n) \leq a+2n-1$, which forces

$$\begin{aligned}
q(a+n) &= s_a p s_{a+n}(a+n) \\
&= s_a(p(a+n+1)) \\
&= p(a+n+1).
\end{aligned}$$

Note that $x-n < p(x) = a < x+n$ and $x-n < q(x) < x+n$ since $p, q \in \mathcal{MP}_n$, and equivalently $a-n+1 \leq x \leq a+n-1$, which forces $s_{a+n}(x) = x$. Then observe that

$$\begin{aligned}
q(x) &= (s_a p s_{a+n})(x) \\
&= s_a(p(x)) \\
&= s_a(a) \\
&= a+1 = p(y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
q(y) &= (s_a p s_{a+n})(y) \\
&= s_a(p(y)) \\
&= s_a(a+1) \\
&= a = p(x).
\end{aligned}$$

$\square$

Proof of Theorem 3.9. By Lemma 3.10 and Corollary 3.11, we need to show that the map ϕ is surjective to complete the proof. Let $p \in \mathcal{MP}_n$. By definition, no reduced decomposition of p contains a subword of n consecutive integers, and thus $i - n < p(i) < i + n$ for all $i \in [2n]$. Define τ_p by

$$\tau_p(i) \quad = \quad \begin{cases} p(i) + n & \text{if} \quad p(i) \leq n \\ p(i) - n & \text{if} \quad p(i) \geq n + 1 \end{cases}$$

for $i \in [2n]$. We claim that τ_p is a matching on $[2n]$. To show τ_p is a matching, we need to show that τ_p is an involution with no fixed point. In other words, we must show that $\tau_p(i) \neq i$ and $\tau_p^2(i) = i$ for all $i \in [2n]$. Since $i - n < p(i) < i + n$, either $i < \tau_p(i) < i + 2n$ or $i - 2n < \tau_p(i) < i$, and thus $\tau_p(i) \neq i$. Observe that

$$\tau_p^2(i) \quad = \quad \begin{cases} p(p(i) + n) + n & \text{if} \quad p(i) \leq n \text{ and } p(p(i) + n) \leq n \\ p(p(i) + n) - n & \text{if} \quad p(i) \leq n \text{ and } p(p(i) + n) \geq n + 1 \\ p(p(i) - n) + n & \text{if} \quad p(i) \geq n + 1 \text{ and } p(p(i) - n) \leq n \\ p(p(i) - n) - n & \text{if} \quad p(i) \geq n + 1 \text{ and } p(p(i) - n) \geq n + 1. \end{cases} \tag{3.1}$$

Since p is an affine permutation in $\tilde{S}_{2n}$, we see that

$$p(p(i) - n) + n = p(p(i) + n - 2n) + n = p(p(i) + n) - n$$

and

$$p(p(i) - n) - n = p(p(i) + n - 2n) - n = p(p(i) + n) - 3n.$$

By this observation, we have $\tau_p^2(i) \equiv p(p(i) + n) + n \pmod{2n}$. Since $1 \leq \tau_p(i) \leq 2n$, we can simplify (3.1) as follows.

$$\tau_p^2(i) \equiv p(p(i) + n) + n \pmod{2n} \text{ and } 1 \leq \tau_p^2(i) \leq 2n. \tag{3.2}$$

Now we prove that τ_p is an involution by induction on $\ell(p)/2$.

Base case: $p = e$. Observe that

$$e(e(i)+n)+n = e(i+n)+n = (i+n)+n = i+2n,$$

and $\tau_e^2(i) = i$ for all $i \in [2n]$ as desired.

Induction step: Suppose $\tau_p^2(i) = i$ for all $i \in [2n]$ for $p \in \mathcal{MP}_n$ with $\ell(p) < 2\binom{n}{2}$. Then $p(p(i)+n)+n \equiv i \pmod{2n}$, or equivalently, $p(p(i)+n) \equiv i+n \pmod{2n}$. Let $q = s_a p s_{a+n} \in \mathcal{MP}_n$ with $\ell(q) = \ell(p)+2$. Let $x, y \in [2n]$ such that $p(x) \equiv a \pmod{2n}$ and $p(y) \equiv a+1 \pmod{2n}$. Let $i \in [2n] \backslash \{a+n, a+n+1, x, y\}$. By Lemma 3.12, we see $p(i)+n \not\equiv a+n \pmod{2n}$ and $p(i)+n \not\equiv a+n+1 \pmod{2n}$, and thus

$$\begin{aligned} q(q(i)+n)+n &\equiv q(p(i)+n)+n \\ &\equiv (s_a p s_{a+n})(p(i)+n)+n \\ &\equiv s_a(p(p(i)+n)+n \\ &\equiv s_a(i+n)+n \\ &\equiv i+2n, \end{aligned}$$

where the last equality is due to $i \notin \{a+n, a+n+1\}$, and hence $\tau_q^2(i) = i$ for $i \in [2n] \backslash \{a+n, a+n+1, x, y\}$. We calculate $q(q(i)+n)+n$ for $i \in \{a+n, a+n+1, x, y\}$ as follows. Since $a+1 < p(a+n+1) < a+2n+1$ and $p(a+n+1) \not\equiv a \pmod{2n}$, we see that

$$\begin{aligned} q(q(a+n)+n)+n &\equiv q(p(a+n+1)+n)+n \\ &\equiv (s_a p s_{a+n})(p(a+n+1)+n)+n \\ &\equiv (s_a p)(p(a+n+1)+n)+n \\ &\equiv s_a(p(p(a+n+1)+n))+n \\ &\equiv s_a(a+n+1+n)+n \\ &\equiv a+3n, \end{aligned}$$

and hence $\tau_q^2(a+n) = a+n$. Since $a < p(a+n) < a+2n$ and $p(a+n) \not\equiv a+1 \pmod{2n}$, we see that

$$\begin{aligned}
q(q(a+n+1)+n)+n &\equiv q(p(a+n)+n)+n \\
&\equiv (s_a p s_{a+n})(p(a+n)+n)+n \\
&\equiv (s_a p)(p(a+n)+n)+n \\
&\equiv s_a(p(p(a+n)+n))+n \\
&\equiv s_a(a+n+n)+n \\
&\equiv a+3n+1,
\end{aligned}$$

and hence we have $\tau_q^2(a+n+1) = a+n+1$. Observe that

$$\begin{aligned}
q(q(x)+n)+n &\equiv q(a+1+n)+n \\
&\equiv p(a+n)+n \\
&\equiv p(p(x)+n)+n\} \\
&\equiv x+n+n \\
&\equiv x+2n,
\end{aligned}$$

and thus we have $\tau_q^2(x) = x$. Observe that

$$\begin{aligned}
q(q(y)+n)+n &\equiv q(a+n)+n \\
&\equiv p(a+1+n)+n \\
&\equiv p(p(y)+n)+n \\
&\equiv y+n+n \\
&\equiv y+2n,
\end{aligned}$$

and thus we have $\tau_q^2(y) = y$. Hence by induction, τ_p is an involution and therefore τ_p

is a matching on $[2n]$. By the construction of τ_p, observe that

$$\phi(\tau_p)(i) = h_{\tau_p}(i) \quad = \quad \begin{cases} \tau_p(i) - n = (p(i)+n) - n & \text{if} \quad i < \tau_p(i) \Leftrightarrow p(i) \leq n \\ \tau_p(i) + n = (p(i)-n) + n & \text{if} \quad i > \tau_p(i) \Leftrightarrow p(i) \geq n+1 \end{cases}$$

and thus $\phi(\tau_p)(i) = p(i)$ for all $i \in [2n]$, which shows that the map ϕ is surjective, and the proof follows. □

3.2 The cd-index of the Posets P_n

In the previous section, we saw that the poset P_n, without $\hat{0}$ adjoined, is isomorphic to a subposet of the dual Bruhat order of affine permutations. Reading (2004) provided recursive formulas for the cd-indices of intervals in the Bruhat order on a Coxeter group, and it looks promising to examine the recursions to compute the cd-index of P_n.

The product of a poset P with a chain of length one is called the *pyramid* of P, and denoted by $Pyr(P)$. We will use the proposition from Ehrenborg and Readdy (1998) which produces the cd-index of $Pyr(P)$ from the cd-index of P.

Proposition 3.13 (Ehrenborg and Readdy (1998), Proposition 4.2)**.** *Let P be an Eulerian poset. Then $Pyr(P)$ is also Eulerian, and moreover the cd-index of $Pyr(P)$ is given by*

$$\Phi_{Pyr(P)} = \frac{1}{2}\left(\Phi_p \cdot c + c \cdot \Phi_P + \sum_{x:\hat{0}<x<\hat{1}} \Phi_{[\hat{0},x]} \cdot d \cdot \Phi_{[x,\hat{1}]}\right).$$

A *zipper* in a poset P is a triple of distinct elements $x, y, z \in P$ such that $\{w : w < x\} = \{w : w < y\}$ and $z = x \vee y$ covers x and y but covers no other element.

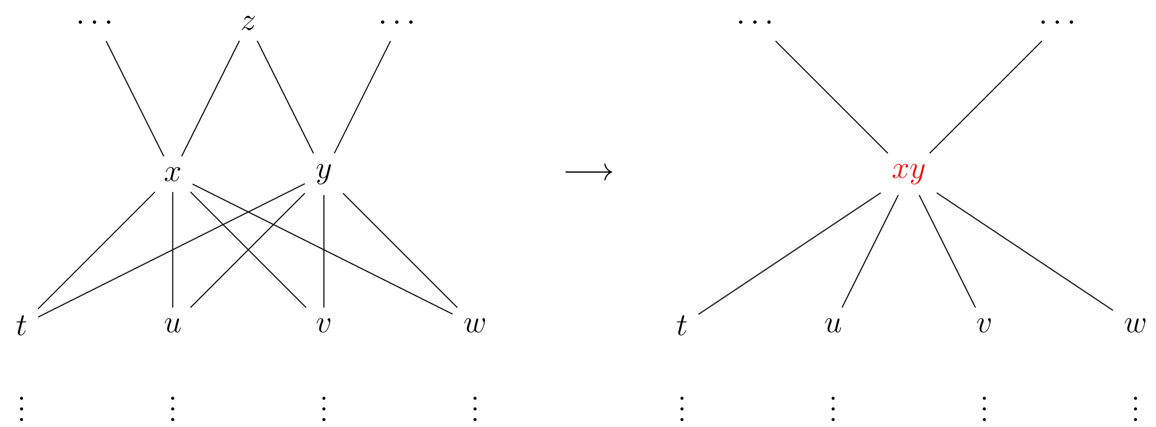


Figure 3.3: A Zipper $\{x, y, z\}$ and Zipping the Zipper $\{x, y, z\}$

The operation *zip* of a zipper $\{x, y, z\} \subset P$ is defined as follows. Let xy be a single new element not in P, and define $P' = (P - \{x, y, z\}) \cup \{xy\}$, with a binary relation $\preceq$ on P', given by:

$$\begin{aligned} a \preceq b \qquad & \text{if } a \leq b \text{ in } P - \{x, y, z\} \\ xy \preceq a \qquad & \text{if } x \leq a \text{ or if } y \leq a \text{ in } P - \{x, y, z\} \\ a \preceq xy \qquad & \text{if } a \leq x \text{ or (equivalently) if } a \leq y \text{ in } P - \{x, y, z\} \\ xy \preceq xy. & \end{aligned}$$

Figure 3.3 shows a zipper and the operation zip of the zipper. We can think of the operation zip of a zipper $\{x, y, z\}$ as deleting z and identifying x with y. According to Reading (2004), the zip operation of a zipper produces a new poset and the zip operation of an Eulerian poset is also Eulerian. Furthermore, the following theorem provides a formula for the cd-index of the resulting poset in terms of the cd-index of the initial Eulerian poset.

Theorem 3.14 (Reading (2004), Theorem 4.6)**.** *Let P' be the result of the operation zip of a zipper in P. Then P' is a poset under the partial order $\preceq$. If P is Eulerian* *then so is P'. Moreover, P has a cd-index Φ_P if and only if P' has a cd-index $\Phi_{P'}$, and*

$$\Phi_{P'} = \Phi_P - \Phi_{[\hat{0},x]} \cdot d \cdot \Phi_{[z,\hat{1}]}.$$

The following theorem is a structural recursion for Bruhat intervals.

Theorem 3.15 (Reading (2004), Theorem 5.5)**.** *Let (W,S) be a Coxeter system. Let $w,u \in W$ and $s \in S$. Let $ws > w, us > u$ and $u \leq w$.*

(1) If $us \notin [u,w]$ then $[u,ws] \cong [u,w] \times [1,s]$ and $[us,ws] \cong [u,w]$.

(2) If $us \in [u,w]$, then $[u,ws]$ can be obtained from $[u,w] \times [1,s]$ by a sequence of zippings.

From these two theorems, Reading (2004) produced recursions for the cd-indices of Bruhat intervals. For $v \in W$ and $s \in S$, define $\sigma_s(v) := \ell(vs) - \ell(v) \in \{-1,1\}$.

Theorem 3.16 (Reading (2004), Theorem 6.1)**.** *Let (W,S) be a Coxeter system. Let $w,u \in W$ and $s \in S$. Let $u < us, w < ws$ and $u \leq w$.*

(1) If $us \notin [u,w]$, then $\Phi_{[us,ws]} = \Phi_{[u,w]}$, and

$$\begin{aligned} \Phi_{[u,ws]} &= \Phi_{Pyr([u,w])} \\ &= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{v:u<v<w} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]}\right). \end{aligned}$$

(2) If $us \in [u,w]$, then

$$\begin{aligned} \Phi_{[u,ws]} &= \Phi_{Pyr([u,w])} - \sum_{\substack{v:u<v<w \\ vs<v}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \\ &= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{v:u<v<w} \sigma_s(v)\Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]}\right). \end{aligned}$$

Even though $\mathcal{MP}_n$ is not a Coxeter group, the order relation in $\mathcal{MP}_n$ is Bruhat order. Thus we consider an analogue of Theorem 3.16 for $\mathcal{MP}_n$. We find that there are recurrence relations for the cd-indices of intervals in $\mathcal{MP}_n$, and also in P_n. For $s = s_i \in S$ define $s' = s_{i+n} \in S$. For $v \in \mathcal{MP}_n$ and $s \in S$, define $\sigma_s(v) := \frac{1}{2}[\ell(svs') - \ell(v)] \in \{-1, 1\}$.

Theorem 3.17. *Let $u, w \in \mathcal{MP}_n$ and let $s = s_i$ and $s' = s_{i+n}$ for some $0 \leq i < 2n$. Let $u < sus', w < sws'$ and $u \leq w$.*

(1) If $sus' \notin [u, w]$, then $\Phi_{[sus', sws']} = \Phi_{[u,w]}$, and

$$\begin{aligned}\Phi_{[u,sws']} &= \Phi_{Pyr([u,w])} \\ &= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]}\right).\end{aligned}$$

(2) If $sus' \in [u, w]$, then

$$\begin{aligned}\Phi_{[u,sws']} &= \Phi_{Pyr([u,w])} - \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w \\ svs'<v}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \\ &= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w}} \sigma_s(v)\Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]}\right).\end{aligned}$$

Example 3.18. Let $u = e$ and $w = s_1s_2s_5s_4$ in $\mathcal{MP}_3$. Figure 3.4 shows the Hasse diagrams for intervals $[u, w], [s_3us_0, s_3ws_0], [u, s_3ws_0]$ and $[u, s_2ws_5]$.

(1) Let $s = s_3$ and $s' = s_0$. Observe that $u < sus'$, $w < sws'$, $u \leq w$ and $sus' \notin [u, w]$. We see that $\Phi_{[sus',sws']} = \Phi_{[s_3s_0,s_3s_1s_2s_5s_4s_0]} = c = \Phi_{[e,s_1s_2s_5s_4]} = \Phi_{[u,w]}$. Notice that the intervals $[u, w]$ and $[s_3us_0, s_3ws_0]$ are isomorphic to Boolean

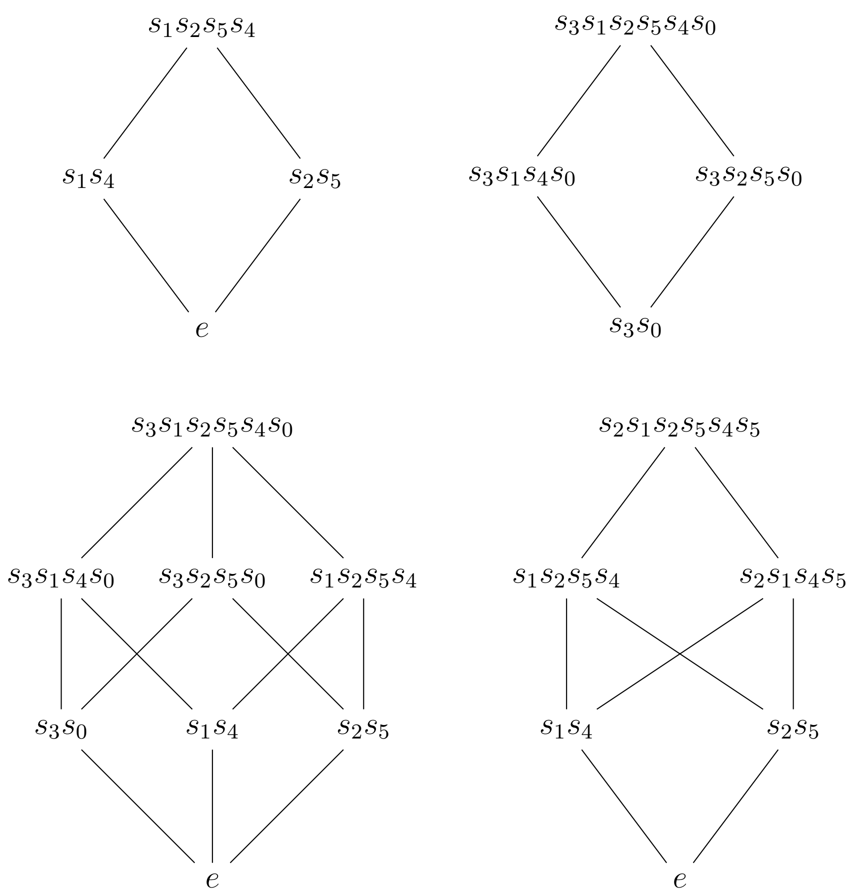


Figure 3.4: The Intervals $[u, w], [s_3us_0, s_3ws_0], [u, s_3ws_0]$ and $[u, s_2ws_5]$ in $\mathcal{MP}_3$

lattice B_2. The cd-index of B_2 is $\Phi_{B_2}(c, d) = c$. We also have

$$\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{[e,s_3s_1s_2s_5s_4s_0]} \\
&= \frac{1}{2}\left(\Phi_{[e,s_1s_2s_5s_4]} \cdot c + c \cdot \Phi_{[e,s_1s_2s_5s_4]} + \sum_{e<v<s_1s_2s_5s_4} \Phi_{[e,v]} \cdot d \cdot \Phi_{[v,s_1s_2s_5s_4]}\right) \\
&= \frac{1}{2}\left(c \cdot c + c \cdot c + \Phi_{[e,s_1s_4]} \cdot d \cdot \Phi_{[s_1s_4,s_1s_2s_5s_4]} + \Phi_{[e,s_2s_5]} \cdot d \cdot \Phi_{[s_2s_5,s_1s_2s_5s_4]}\right) \\
&= \frac{1}{2}(c^2 + c^2 + d + d) = c^2 + d.
\end{aligned}$$

Note that the interval $[u, s_3ws_0]$ is isomorphic to Boolean lattice B_3. The cd-index of B_3 is $\Phi_{B_3}(c, d) = c^2 + d$. Observe that the number of maximal chains in the interval $[u, s_3ws_0]$ is $\Phi_{B_3}(2, 2) = 2^2 + 2 = 6$.

(2) Let $s = s_2$ and $s' = s_5$. Observe that $u < sus'$, $w < sws'$, $u \leq w$ and $sus' \in [u, w]$. Note that $\sigma_{s_2}(s_1 s_4) = \frac{1}{2}(4-2) = 1$ and $\sigma_{s_2}(s_2 s_5) = \frac{1}{2}(0-2) = -1$. We see that

$$\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{[e,s_2 s_1 s_2 s_5 s_4 s_5]} \\
&= \frac{1}{2}\left(\Phi_{[e,s_1 s_2 s_5 s_4]} \cdot c + c \cdot \Phi_{[e,s_1 s_2 s_5 s_4]} + \sum_{e<v<w} \sigma_{s_2}(v)\Phi_{[e,v]} \cdot d \cdot \Phi_{[v,s_1 s_2 s_5 s_4]}\right) \\
&= \frac{1}{2}\left(c \cdot c + c \cdot c + \Phi_{[e,s_1 s_4]} \cdot d \cdot \Phi_{[s_1 s_4, s_1 s_2 s_5 s_4]} - \Phi_{[e,s_2 s_5]} \cdot d \cdot \Phi_{[s_2 s_5, s_1 s_2 s_5 s_4]}\right) \\
&= \frac{1}{2}(c^2 + c^2 + d - d) = c^2.
\end{aligned}$$

Observe that the interval $[u, s_2 w s_5]$ is isomorphic to Bruhat order of the symmetric group S_3. The cd-index of Bruhat order of S_3 is $\Phi_{S_3}(c, d) = c^2$. Observe that the number of maximal chains in the interval $[u, s_2 w s_5]$ is $\Phi_{S_3}(2, 2) = 2^2 = 4$.

3.3 Proof of Theorem 3.17

In this section we let $u, w \in \mathcal{MP}_n$ and let $s = s_i \in \tilde{S}$ and let $s' = s_{i+n} \in \tilde{S}$. All intervals in this section are induced subposets in $\mathcal{MP}_n$. Suppose $u < sus'$ and $w < sws'$. We employ the map η in Reading (2004) to our situation. Define a map $\eta : [u, w] \times [e, ss'] \to [u, sws']$, as follows:

$$\begin{aligned}
\eta(v, e) &= v \\
\eta(v, ss') &= \begin{cases} svs' & \text{if } svs' > v \\ v & \text{if } svs' < v \end{cases}
\end{aligned}$$

The following proposition is known as Lifting Property of Bruhat order.

Proposition 3.19 (Björner and Brenti (2005), Proposition 1.2 (Lifting Property of Bruhat order))**.** *Let (W, S) be a Coxeter system. Let $u, w \in W$ and $s \in S$. If $w > ws$ and $us > u$, then the following are equivalent:*

(i) $w > u$

(ii) $ws > u$

(iii) $w > us$.

We will need the following proposition, which is the analogue of Proposition 3.19 for $\mathcal{MP}_n$.

Proposition 3.20 (Lifting Property of $\mathcal{MP}_n$)**.** *Let $u, w \in \mathcal{MP}_n$ and let $s = s_i \in \tilde{S}$. Let $s' = s_{i+n} \in \tilde{S}$. If $w > sws'$ and $sus' > u$, then the following are equivalent:*

(i) $w > u$

(ii) $sws' > u$

(iii) $w > sus'$.

Proof. Since $w > sws'$, by transitivity (ii) implies (i). Since $sus' > u$, by transitivity (iii) implies (i). So assume (i) $w > u$. Choose a reduced decomposition for $sws' = t_1 t_2 \dots t_{2q}$ where $t_i \in \tilde{S}$ for all $i \in [2q]$ such that the reduced word is modular palindromic. Then $w = st_1 t_2 \dots t_{2q} s'$ is also reduced and modular palindromic. There is a reduced decomposition for u

$$u = t_{i_1} t_{i_2} \dots t_{i_{2r}}$$

which is a subword of $w = st_1 t_2 \dots t_{2q} s'$. Since $sus' > u$, we have $t_{i_1} \neq s$ and $t_{i_{2r}} \neq s'$, and thus both (ii) $sws' > u$ and (iii) $w > sus'$ hold as desired. □

We claim that the map η is well-defined. To show this, let $v \in [u, w]$. Because we have assumed $w < sws'$ we know that $v \in [u, w] \subset [u, sws']$, so we may assume that $\eta(v, ss') = svs'$. In this case, $v < svs'$, thus we see $u \leq v < svs' < sws'$ where the last inequality is due to the lifting property, and therefore $\eta(v, ss') \in [u, sws']$. The following proposition and Proposition 5.1 in Reading (2004) are the same statement on different posets: Bruhat order of Coxeter groups and the induced subposet $\mathcal{MP}_n$. Here, we show the order-preserving part of the statement on the induced subposet $\mathcal{MP}_n$. For the proof of surjective part, see Proposition 5.1 in Reading (2004).

Proposition 3.21. *If $u < sus'$ and $w < sws'$, then $\eta : [u, w] \times [e, ss'] \to [u, sws']$ is an surjective order-preserving map.*

Proof. Suppose $(v_1, a_1) \leq (v_2, a_2)$ in $[u, w] \times [e, ss']$. Since $e \leq a_1 \leq a_2 \leq ss'$, we break into three cases.

Case 1: $a_1 = e$ and $a_2 = e$. Then $\eta(v_1, a_1) = v_1 \leq v_2 = \eta(v_2, a_2)$.

Case 2: $a_1 = e$ and $a_2 = ss'$. Then either $\eta(v_2, a_2) = v_2$ with $v_2 > sv_2s'$ or $\eta(v_2, a_2) = sv_2s'$ with $sv_2s' > v_2$. In either case, we see that $\eta(v_2, a_2) \geq v_1$.

Case 3: $a_1 = ss'$ and $a_2 = ss'$. Then either $\eta(v_1, a_1) = v_1$ with $v_1 > sv_1s'$ or $\eta(v_1, a_1) = sv_1s'$ with $sv_1s' > v_1$. We also have either $\eta(v_2, a_2) = v_2$ with $v_2 > sv_2s'$ or $\eta(v_2, a_2) = sv_2s'$ with $sv_2s' > v_2$. We break up this case into four subcases.

Subcase 3-1: $\eta(v_1, a_1) = v_1$ and $\eta(v_2, a_2) = v_2$. Then, $\eta(v_1, a_1) \leq \eta(v_2, a_2)$.

Subcase 3-2: $\eta(v_1, a_1) = v_1$ and $\eta(v_2, a_2) = sv_2s'$. Then, $\eta(v_1, a_1) = v_1 \leq v_2 < sv_2s' = \eta(v_2, a_2)$.

Subcase 3-3: $\eta(v_1, a_1) = sv_1s'$ and $\eta(v_2, a_2) = v_2$. Then, $\eta(v_1, a_1) = sv_1s' \leq v_2 = \eta(v_2, a_2)$ by the lifting property.

Subcase 3-4: $\eta(v_1, a_1) = sv_1s'$ and $\eta(v_2, a_2) = sv_2s'$. Then, $\eta(v_1, a_1) = sv_1s' \leq sv_2s' = \eta(v_2, a_2)$ by the lifting property. $\square$

For every $v \in [u, w]$ with $svs' < v$, notice that the image of the elements (v, e), (svs', ss'), (v, ss') under the map η is the single element v. From this observation, we let $v_1, v_2, \ldots, v_k$ be a linear ordering of the elements of the set $Z = \{v : u < v < w, svs' < v\}$ such that the ranks (or lengths) of elements are weakly increasing, in other words, $\ell(v_1) \leq \ell(v_2) \leq \cdots \leq \ell(v_k)$. Define posets Q_i for $0 \leq i \leq k$ recursively as follows. Let $Q_0 = [u, w] \times [e, ss']$. Let Q_i to be the poset obtained by zipping $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ in Q_{i-1}. In the following proposition, we show that this is indeed a proper zipping.

Proposition 3.22. *The triples* $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ *in* Q_{i-1} *for* $1 \leq i \leq k$ *are zippers.*

Proof. First, we claim that $(v_i, e), (sv_is', ss')$ and (v_i, ss') are elements of Q_{i-1}. The element (v_i, ss') has not been deleted yet, and we have not identified (v_i, ss') with any element because it is at a rank higher than we have yet made identifications. The only elements ever deleted are of the form (x, ss') where $x > sxs'$, so (v_i, e) and (sv_is', ss') have not been deleted. The only identification one could make involving (v_i, e) and (sv_is, ss') is to identify them to each other, and that has not happened yet, and thus the claim is proved.

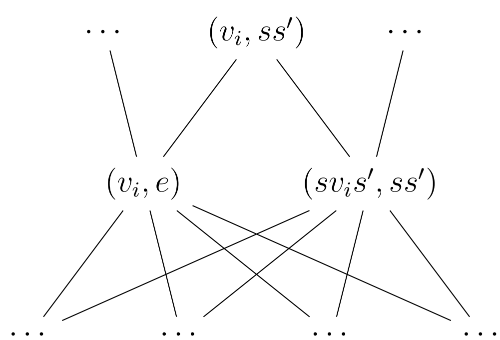


Figure 3.5: The Triple $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ in Q_{i-1}

Second, we check the conditions in the definition of a zipper. Suppose $(x, a) < (v_i, e)$, so that $x < v_i$ and $a = e$. Then $x < sxs'$; otherwise the triple $\{(sxs', e), (x, ss'), (x, e)\}$ is a zipper in Q_{i-1} with $\ell(x) < \ell(v_i)$ which is impossible. Then we have $x \leq sv_is'$ by lifting property and thus $(x, a) < (sv_is', ss')$. Now suppose $(x, a) < (sv_is', ss')$, so that either $a = e$ or $a = ss'$. If $a = e$ then $(x, a) < (v_i, e)$ since $x \leq sv_is' < v_i$. Assume $a = ss'$. Then $x < sv_is'$ which implies $\ell(x) < \ell(sv_is')$ so x is not v_r for any $r \leq i$, and then $x < sxs'$. Then the triple $\{(x, ss'), (sxs', e), (sxs', ss')\}$ is a zipper with $\ell(sxs') < \ell(v_i)$ in Q_{i-1} which is impossible. Hence the triple satisfies the first condition $\{(x,a) : (x,a) < (v_i, e)\} = \{(x,a) : (x,a) < (sv_is', ss')\}$. The second condition is obvious because $(v_i, e) \lessdot (v_i, ss')$ and $(sv_is', ss') \lessdot (v_i, ss')$. Assume (x,a) is covered by (v_i, ss'). If $a = e$ then $x = v_i$, so $(x,a) = (v_i, e)$. If $a = ss'$ then $x \lessdot v_i$. Since $sv_is' < v_i$ and $x < sxs'$, by lifting property $x \leq sv_is'$ which implies $x = sv_is'$. Hence (v_i, ss') covers no other element than (v_i, e) and (sv_is', ss'), and thus the third condition holds. Therefore, the triple $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ is a zipper in Q_{i-1}. $\square$

In terms of zipper and operation zip, we can think of the image of $[u, w] \times [e, ss']$ under the map η as a sequence of zipping operations of the triples $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ for $v_i \in Z$.

Example 3.23. Let $u = e, w = s_1s_3s_2s_6s_7s_5$ in $\mathcal{MP}_4$ and let $s = s_2$ and $s' = s_6$. Then, $u < sus', w < sws'$ and $u < w$. Figure 3.6 shows the Hasse diagram of $[u, w] \times [e, ss']$. We consider the map $\eta : [u, w] \times [e, ss'] \to [u, sws']$. We get the result of zipping the zipper $\{(s_2s_6, e),\ (e, s_2s_6),\ (s_2s_6, s_2s_6)\}$ as in Figure 3.7.

The following proposition is an analogue of Proposition 5.2 in Reading (2004).

Proposition 3.24. *Let $u < sus'$ and $w < sws'$ and $sus' \not\leq w$. Then $svs' > v$ for all $v \in [u, w]$, and η is an order-preserving bijection.*

Proof. Suppose that there is a $v \in [u, w]$ with $svs' < v$. Since $u < sus'$ and $u \leq v$, by lifting property, $sus' \leq v$, and thus $sus' \leq w$, which is a contradiction. Therefore, $v < svs'$ for all $v \in [u, w]$.

In the Proposition 3.21, it is proved that the map η is order-preserving. Observe

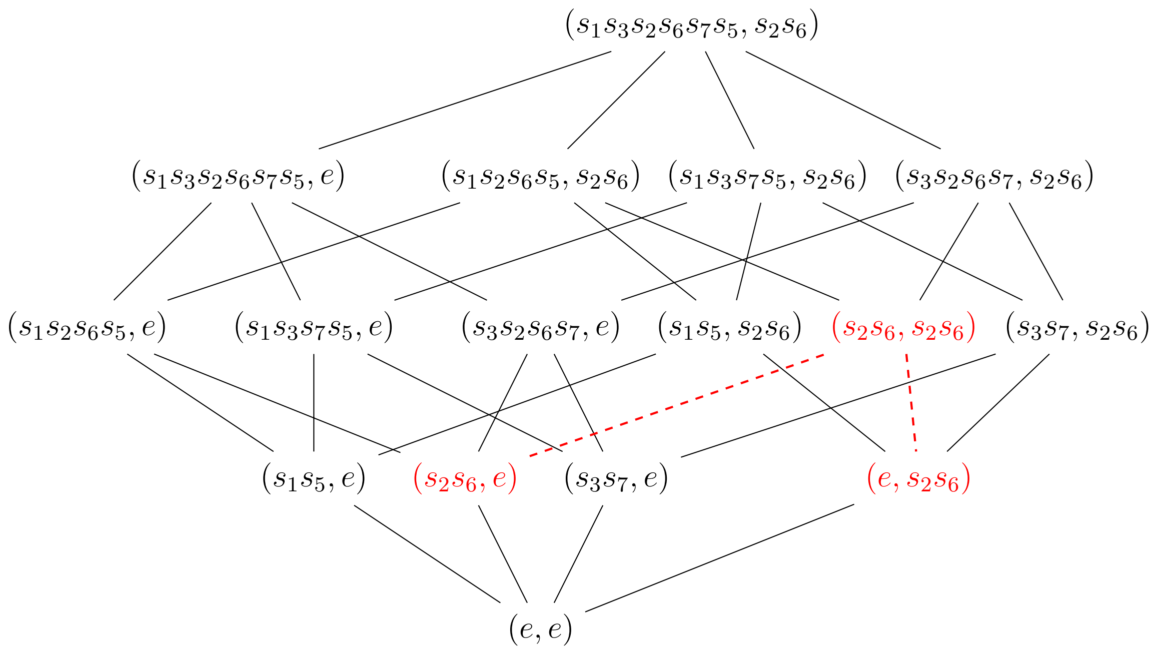


Figure 3.6: The Interval $[u, w] \times [e, ss']$

that the inverse image of $v \in [u, sws']$ under the map η is

$$\eta^{-1}(v) = \begin{cases} \{(v, e)\} & \text{if } v \in [u, w] \\ \{(svs', ss')\} & \text{if } v \in [u, sws'] \backslash [u, w], \end{cases}$$

and the inverse map η^{-1} is well-defined. Now suppose $v_1, v_2 \in [u, sws']$ with $v_1 \leq v_2$. We consider the following four cases:

Case 1: $v_1 \in [u, w]$ and $v_2 \in [u, w]$. Then $\eta^{-1}(v_1) = (v_1, e)$ and $\eta^{-1}(v_2) = (v_2, e)$, thus $\eta^{-1}(v_1) = (v_1, e) \leq (v_2, e) = \eta^{-1}(v_2)$.

Case 2: $v_1 \in [u, w]$ and $v_2 \in [u, sws'] \backslash [u, w]$. Then $\eta^{-1}(v_1) = (v_1, e)$ and $\eta^{-1}(v_2) = (sv_2s', ss')$, hence $v_1 < sv_1s'$ and $v_2 > sv_2s'$. Thus $\eta^{-1}(v_1) = (v_1, e) \leq (sv_2s', ss') = \eta^{-1}(v_2)$ by lifting property.

Case 3: $v_1 \in [u, sws'] \backslash [u, w]$ and $v_2 \in [u, sws'] \backslash [u, w]$. Then $\eta^{-1}(v_1) = (sv_1s', ss')$ and $\eta^{-1}(v_2) = (sv_2s', ss')$, hence $v_1 > sv_1s'$ and $v_2 > sv_2s'$. Thus $\eta^{-1}(v_1) =$

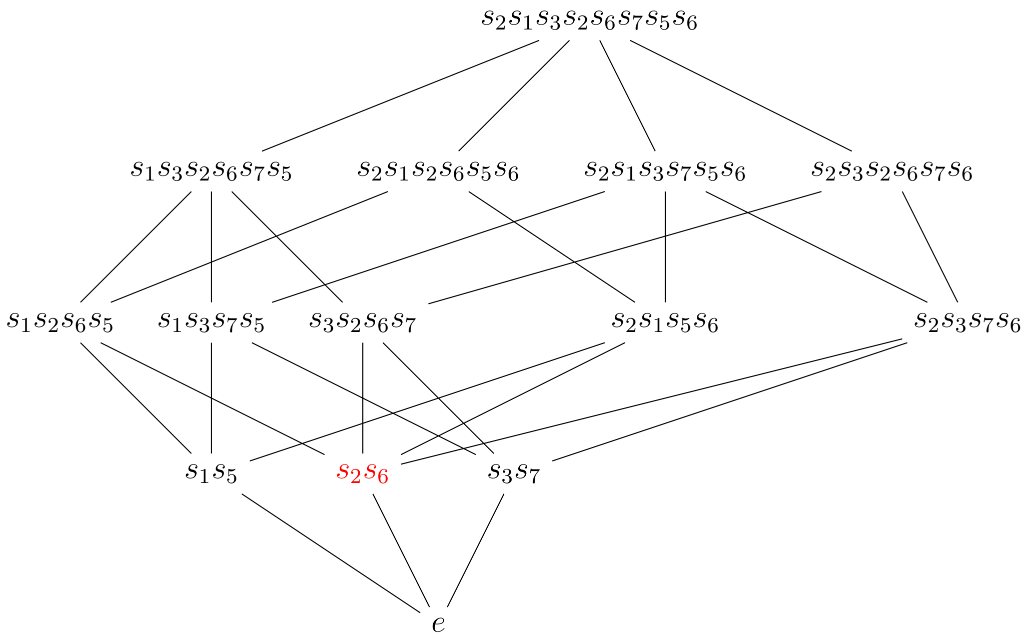


Figure 3.7: The Result of Zipping the Zipper $\{(s_2s_6, e),\, (e, s_2s_6),\, (s_2s_6, s_2s_6)\}$

$(sv_1s', ss') \leq (sv_2s', ss') = \eta^{-1}(v_2)$ by lifting property.

Case 4: $v_1 \in [u, sws']\backslash[u, w]$ and $v_2 \in [u, w]$. This case is impossible because $v_1 \leq v_2$. $\square$

Proposition 3.24 directily implies the following corollary.

Corollary 3.25. *Let $u < sus'$ and $w < sws'$ and $sus' \not\leq w$. Then the map θ : $[u, w] \to [sus', sws']$ with $\theta(v) = svs'$ is an order-preserving bijection.*

In other words, if the condition in Corollary 3.25 holds, then the map θ makes a copy $[sus', sws']$ of the interval $[u, w]$.

We have proven the following theorem which is an analogue of Theorem 5.5 in Reading (2004).

Theorem 3.26. *Let $w < sws'$ and $u < sus'$ and $u \leq w$. If $sus' \notin [u, w]$ then $[u, sws'] \cong [u, w] \times [1, ss']$ and $[sus', sws'] \cong [u, w]$. If $sus' \in [u, w]$, then $[u, sws']$ can be obtained from $[u, w] \times [1, ss']$ by a sequence of zippings.*

Now we are ready to prove Theorem 3.17. This proof is essentially the same as the proof of Theorem 6.1 in Reading (2004). Here we reproduce this proof for completeness.

Proof of Theorem 3.17. Part (1) follows from Proposition 3.24 and Corollary 3.25. So we focus on part (2), in the case of $sus' \in [u, w]$. Define the posets Q_i as in Proposition 3.22. By Theorem 3.14,

$$\Phi_{Q_{i-1}} - \Phi_{Q_i} = \Phi_{[(u,e),(v_i,e)]} \cdot d \cdot \Phi_{[(v_i,ss'),(w,ss')]}.$$

where intervals in the right hand side are in Q_{i-1}. Since $Q_k = [u, sws']$, sum from $i = 1$ to $i = k$ to obtain

$$\Phi_{[u,sws']} = \Phi_{Q_0} - \sum_{j=1}^{k} \Phi_{[(u,e),(v_j,e)]} \cdot d \cdot \Phi_{[(v_j,ss'),(w,ss')]}.$$

where intervals in right hand side are in Q_{j-1}. Notice that the interval $[(u, e), (v_j, e)]$ in Q_{j-1} is isomorphic to the interval $[(u, e), (v_j, e)]$ in Q_0 which is also isomorphic to $[u, v_j]$. Similarly, the interval $[(v_j, ss'), (w, ss')]$ in Q_{j-1} is isomorphic to the interval $[(v_j, ss'), (w, ss')]$ in Q_0 which is isomorphic to $[v_j, w]$, and the second part of the theorem is proved. $\square$

Therefore, we have recurrence relations for the cd-indices $\Phi(c, d)$ of intervals in P_n, or equivalently, $\mathcal{MP}_n$. Then we also have recurrence relations for the ab-indices $\Psi(a, b)$ of the intervals by the relation $\Psi(a, b) = \Phi(a+b, ab+ba)$. Recall that poset $\hat{P}_n$ is P_n with a unique minimum element $\hat{0}$ adjoined. We prove the following proposition which helps us compute ab-indices of intervals $[\hat{0}, \tau]$ in $\hat{P}_n$ recursively.

Proposition 3.27. *Let P be a graded poset of rank n with a unique maximum element $\hat{1}$ and multiple minimal elements such that every interval in P is Eulerian. Let $\hat{P}$ be* *the poset P with a unique minimum element $\hat{0}$ adjoined, where $\rho(\hat{0}) = -1$. If $\hat{P}$ is Eulerian, then the ab-index of $\hat{P}$ is given by*

$$\Psi_{\hat{P}}(a,b) = (a-b)^n + \sum_{i=0}^{n-1}(a-b)^i b \sum_{\rho(x)=i} \Psi_{[x,\hat{1}]}(a,b). \tag{3.3}$$

Proof. By the definition of ab-index and the definition of flag f-vector and h-vector,

$$\begin{aligned}\Psi_{\hat{P}}(a+b,b) &= \sum_{S\subseteq[n-1]\cup\{0\}} \alpha_P(S)u_S \\ &= \sum_{\hat{0}<t_1<\cdots<t_{k-1}<\hat{1}} a^{\rho(\hat{0},t_1)-1}ba^{\rho(t_1,t_2)-1}b\ldots ba^{\rho(t_{k-1},\hat{1})-1},\end{aligned}$$

and thus

$$\Psi_{\hat{P}}(a,b) = \sum_{\hat{0}<t_1<\cdots<t_{k-1}<\hat{1}} (a-b)^{\rho(\hat{0},t_1)-1}b(a-b)^{\rho(t_1,t_2)-1}b\ldots b(a-b)^{\rho(t_{k-1},\hat{1})-1}. \tag{3.4}$$

We rearrange this summation in terms of t_1, the lowest element in the chain except $\hat{0}$. If there is no t_1 in the chain, then the summand will be $(a-b)^n$. If $\rho(t_1) = i \in \{0,1,,\ldots,n-1\}$, then the summand will be $(a-b)^i b[(a-b)^{\rho(t_1,t_2)-1}b\ldots b(a-b)^{\rho(t_{k-1},\hat{1})-1}]$. By this observation, we can write (3.4) as

$$\begin{aligned}\Psi_{\hat{P}}(a,b) &= (a-b)^n + \sum_{i=0}^{n-1}(a-b)^i b \sum_{t_1<\cdots<t_{k-1}<\hat{1}} (a-b)^{\rho(t_1,t_2)-1}b\ldots b(a-b)^{\rho(t_{k-1},\hat{1})-1} \\ &= (a-b)^n + \sum_{i=0}^{n-1}(a-b)^i b \sum_{\rho(t_1)=i} \Psi_{[t_1,\hat{1}]}(a,b)\end{aligned}$$

as desired. □

Remark 3.28. If we are only interested in the number of maximal chains in $\hat{P}_n$, then we plug in $a = b = 1$ in (3.3),

$$\Psi_{\hat{P}}(1,1) = \sum_{\rho(x)=0} \Psi_{[x,\hat{1}]}(1,1).$$

Proposition 3.27 directly implies the following theorem. We have a recursion for the ab-indices of the poset $\hat{P}_n$ and its intervals $[\hat{0},\tau]$.

Theorem 3.29. *The ab-index of $\hat{P}_n$ is recursively given by*

$$\Psi_{\hat{P}_n}(a,b) = (a-b)^{\binom{n}{2}} + \sum_{i=0}^{\binom{n}{2}-1} (a-b)^i b \sum_{\ell(x)=2\binom{n}{2}-2i} \Psi_{[e,x]}(a,b)$$

where $x \in \mathcal{MP}_n$. Let $\tau \in P_n$ such that $c(\tau) = k \leq \binom{n}{2}$ and $\phi(\tau) = w \in \mathcal{MP}_n$. The ab-index of the interval $[\hat{0}, \tau] \subset \hat{P}_n$ is recursively given by

$$\Psi_{[\hat{0},\tau]}(a,b) = (a-b)^k + \sum_{i=0}^{k-1} (a-b)^i b \sum_{\substack{x:x>w \\ \ell(x)=2\binom{n}{2}-2i}} \Psi_{[w,x]}(a,b).$$

Example 3.30. Figure 3.8 shows the Hasse diagram of the dual of $\mathcal{MP}_3$. There are three $x \in \mathcal{MP}_3$ with $\ell(x) = 2$ such that the ab-index $\Psi_{[e,x]} = 1$ because every interval $[e,x]$ is a length 1 chain. There are six $x \in \mathcal{MP}_3$ with $\ell(x) = 4$ such that the ab-index $\Psi_{[e,x]} = a+b$ because every interval $[e,x]$ is isomorphic to Boolean lattice B_2. There are five $x \in \mathcal{MP}_3$ with $\ell(x) = 6$. For three of them, we see that the ab-index $\Psi_{[e,x]} = (a+b)^2$ because the intervals are isomorphic to Bruhat order S_3. For two of them, we see that the ab-index $\Psi_{[e,x]} = (a+b)^2 + (ab+ba)$ because the intervals are isomorphic to Boolean lattice B_3. By Theorem 3.29 we compute the ab-index of $\hat{P}_3$

$$\begin{aligned}
\Psi_{\hat{P}_3} &= (a-b)^3 + \sum_{i=0}^{2} (a-b)^i b \sum_{\ell(x)=6-2i} \Psi_{[e,x]} \\
&= (a-b)^3 + b \sum_{\ell(x)=6} \Psi_{[e,x]} + (a-b)b \sum_{\ell(x)=4} \Psi_{[e,x]} + (a-b)^2 b \sum_{\ell(x)=2} \Psi_{[e,x]} \\
&= (a-b)^3 + b[5(a+b)^2 + 2(ab+ba)] + (a-b)b[6(a+b)] + (a-b)^2 b[3 \cdot 1] \\
&= a^3 + 2a^2b + 5aba + 4ab^2 + 4ba^2 + 5bab + 2b^2a + b^3 \\
&= (a+b)^3 + (a+b)(ab+ba) + 3(ab+ba)(a+b).
\end{aligned}$$

Thus, the cd-index of $\hat{P}_3$ is

$$\Phi_{\hat{P}_3} = c^3 + cd + 3dc.$$

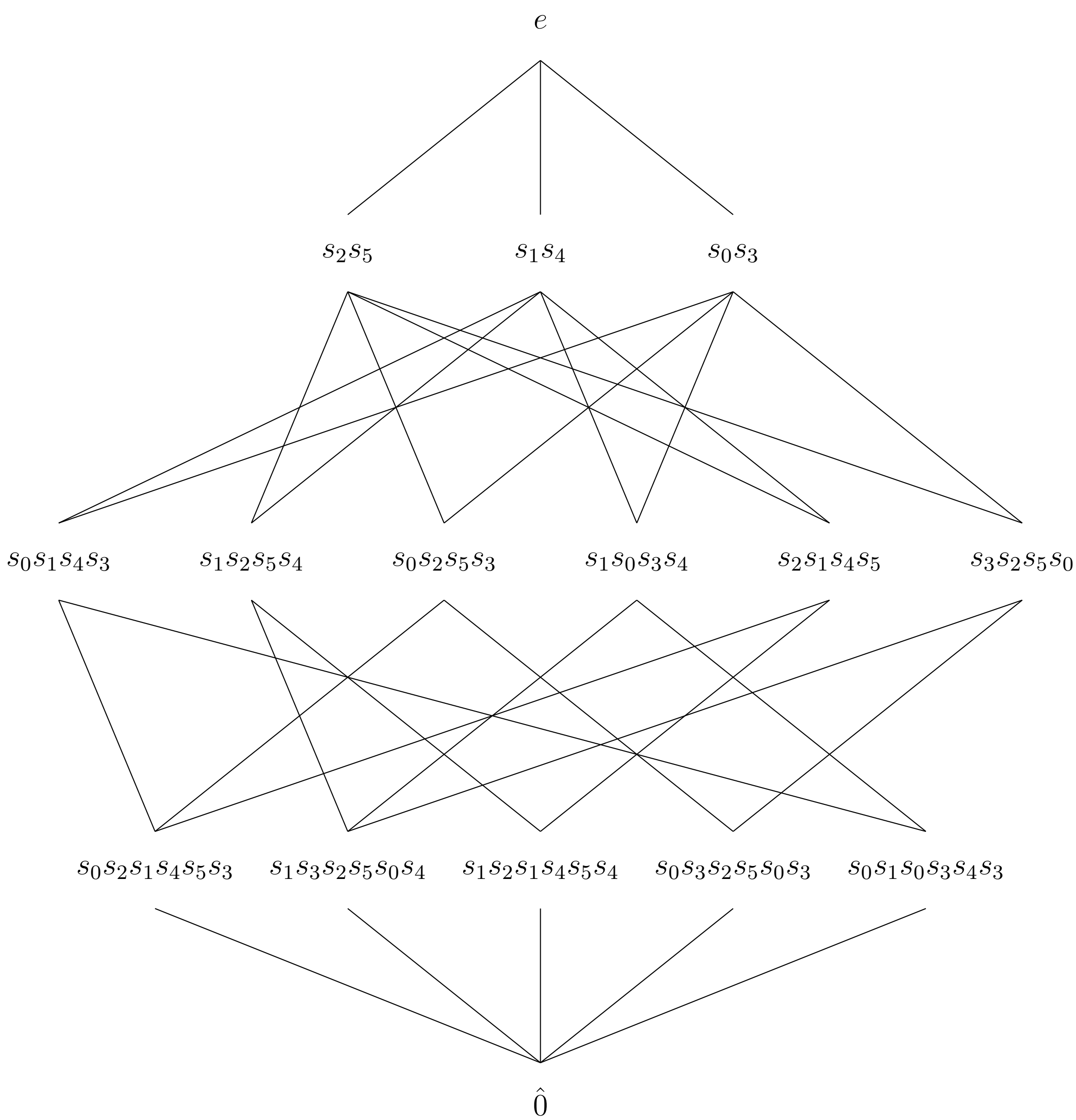


Figure 3.8: The Hasse Diagram of the Dual of $\mathcal{MP}_3$

Moreover, we see that there are $2^3 + 2 \cdot 2 + 3 \cdot 2 \cdot 2 = 24$ maximal chains. Or by Remark 3.28,

$$\begin{aligned}\Psi_{\hat{P}}(1,1) &= \sum_{\ell(x)=6} \Psi_{[e,x]}(1,1) \\ &= 5(1+1)^2 + 2(1+1) = 24.\end{aligned}$$

3.4 Future Work

Since we describe the elements in P_n with modular palindromic permutations in $\mathcal{MP}_n \subset \tilde{S}_{2n}$, we believe that it could be easier to see the structure of the poset P_n. Then we could examine other properties of the poset P_n by exploring properties of the poset $\mathcal{MP}_n$.

Problem 3.31. Enumerate objects in the poset P_n or equivalently in $\mathcal{MP}_n$, such as intervals with respect to lengths, intervals isomorphic to particular posets, maximal chains, and so forth.

The main results in this chapter are recurrence relations for the cd-indices and the ab-indices. We produce recursions for the cd-indices of intervals in the poset P_n and the ab-indices of intervals of the form $[\hat{0}, \tau]$ in the poset $\hat{P}_n$. Thus the most natural problem would be the following.

Problem 3.32. Find a recursion for the cd-indices of intervals of the form $[\hat{0}, \tau]$ in the poset $\hat{P}_n$. Find a closed formula for the cd-indices of intervals in the posets P_n and $\hat{P}_n$.

We may pay attention to lower bounds on the coefficients of the cd-index of an interval in $\mathcal{MP}_n$. The following problem is a special case of a conjecture by Stanley (1994).

Problem 3.33. For any $u \leq w$ in $\mathcal{MP}_n$, prove that the coefficients of $\Phi_{[u,w]}$ are non-negative.

Chapter 4

THE CYCLIC SIEVING PHENOMENON ON MATCHINGS

In this chapter, we present a set of results on the cyclic sieving phenomenon, CSP for short, on matchings. For given set X and a cyclic group C of order N which acts on X, we present a way to construct a polynomial $f(q)$ for which the triple $(X, f(q), C)$ exhibits the CSP. From now we will abuse P_n to denote the set of matchings on $[2n]$ if there is no possibility of confusion. We find the number of elements in P_n fixed by the action $c^{2n/d}$ for divisors d of $2n$ where c is the cyclic shift of order $2n$. Then we find the polynomials $X_n(q)$ such that P_n together with C_{2n}, the cyclic group of order $2n$, exhibits the CSP. In Section 4.1, we review definitions, examples and previous results on the cyclic sieving phenomenon on matchings. In Section 4.2, we find the polynomials $X_n(q)$ for any prime n. In Section 4.3, we discuss a general way to construct a CSP polynomial. In Section 4.4, we find the polynomials $X_n(q)$ for any $n \in \mathbb{N}$. Since Section 4.2 is a special case of Section 4.4, readers may skip Section 4.2. We keep Section 4.2 because we believe that it would be helpful to understand Section 4.4.

4.1 Cyclic Sieving Phenomenon

The cyclic sieving phenomenon was defined by Reiner *et al.* (2004). Let X be a finite set. Let C be a cyclic group generated by an element $c \in C$ of order n acting on X. Let $X(q)$ be a polynomial with integer coefficients in a variable q.

Definition 4.1 (Reiner *et al.* (2004))**.** The triple $(X, X(q), C)$ exhibits the *cyclic sieving phenomenon* (CSP) if for all integers d, the number of elements of X fixed by c^d equals the evaluation $X(\zeta^d)$ where $\zeta = e^{\frac{2\pi i}{n}}$ is a n^{th}-root of unity.

In particular, since $X(1) = |X|$, the polynomial $X(q)$ can be thought of as a generating function for the set X. Reiner-Stanton-White (Reiner *et al.* (2004), Theorem 1.1) proved that $(X, X(q), C_n)$ exhibits the CSP with the collection X of all k-elements subsets of $[n] := \{1, 2, \ldots, n\}$, the cyclic group $C_n = \langle c \rangle$ where $c = (1\ 2\ \ldots\ n)$, and q-binomial coefficients

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q}$$

where $[m]!_q := [m]_q[m-1]_q \ldots [2]_q[1]_q$ and $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$.

Example 4.2. Take $n = 4$ and $k = 2$, and let $C_4 = \langle c \rangle$ where $c = (1\ 2\ 3\ 4)$ and let $X(q) = \left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_q = 1 + q + 2q^2 + q^3 + q^4$. For $\zeta = e^{\frac{2\pi i}{4}} = i$, we see that $X(\zeta^0) = 6$, and it means all 2-subsets are fixed by the identity. We also check that $X(\zeta^2) = 2$, and we interpret two 2-subsets, namely $\{1,3\}, \{2,4\}$, are fixed by $c^2 = (1\ 3)(2\ 4)$, and $X(\zeta) = X(\zeta^3) = 0$ which means no 2-subset is fixed by $c = (1\ 2\ 3\ 4)$ or $c^3 = (1\ 4\ 3\ 2)$.

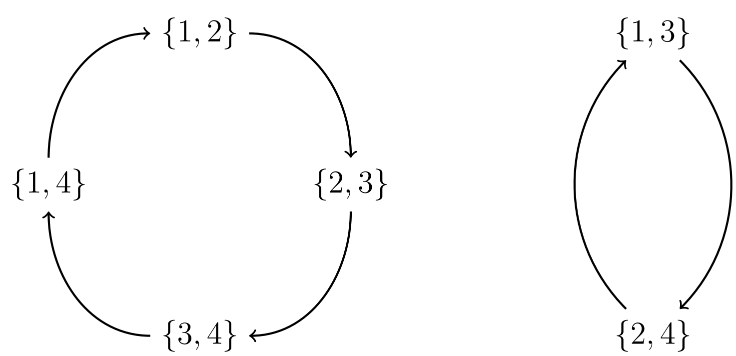


Figure 4.1: The C_4-orbits in $\binom{[4]}{2}$

The CSP is observed on the set of non-crossing matchings on $[2n]$.

Theorem 4.3 (Sagan (2011), Theorem 8.1)**.** *Let X be the collection of non-crossing matchings on $[2n]$. Let $C_{2n} = \langle c \rangle$ where $c = (1\ 2\ \ldots\ 2n)$. Let $Cat_n(q)$ be the n-th q-Catalan number,*

$$Cat_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

Then the triple $(X, Cat_n(q), C_{2n})$ exhibits the CSP.

Example 4.4. Let NCM_3 be the set of non-crossing matchings on $[6]$. Let $C_6 = \langle(1\ 2\ 3\ 4\ 5\ 6)\rangle$ and let $Cat_3(q) = \frac{1}{[4]_q}\begin{bmatrix}6\\3\end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6$. Figure 1.4 shows the C_6-orbits in NCM_3. For $\zeta = e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}i}{2}$, we see that $Cat_3(\zeta^0) = 5$ and $Cat_3(\zeta^2) = Cat_3(\zeta^4) = 2$ and $Cat_3(\zeta^3) = 3$ and $Cat_3(\zeta) = Cat_3(\zeta^5) = 0$. Thus the triple $(NCM_3, Cat_3(q), C_6)$ exhibits the CSP.

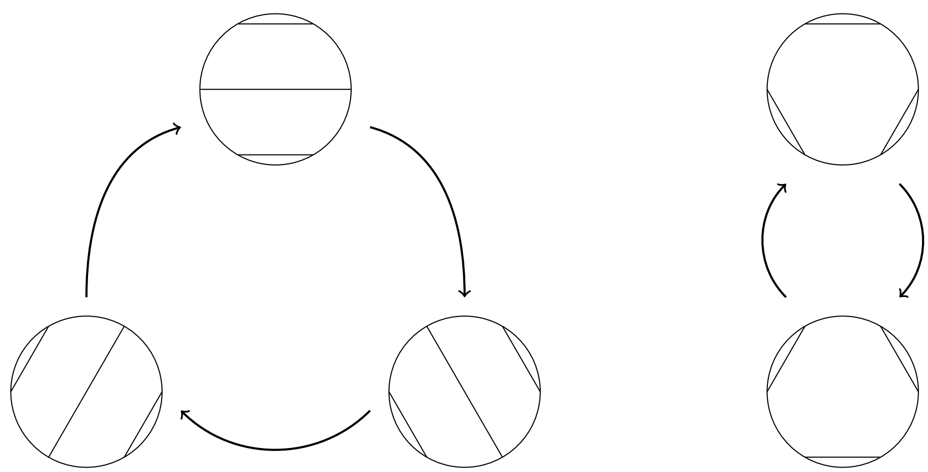


Figure 4.2: The C_6-orbits in NCM_3

This result is found in a survey of CSP by Sagan (2011). This is a special case, when $m = 2$, of Theorem 1.3 in Rhoades (2010).

Theorem 4.5 (Rhoades (2010), Theorem 1.3)**.** *Let $\lambda = (n, n, \ldots, n) \vdash mn$ be a rectangular partition and let $X = SYT(\lambda)$ be the set of standard Young tableaux of shape λ. Let $C = \mathbb{Z}/mn\mathbb{Z}$ act on X by jeu-de-taquin promotion. Then the triple $(X, C, X(q))$ exhibits the CSP, where $X(q)$ is the q-analog of the hook length formula*

$$X(q) = f^\lambda(q) := \frac{[mn]!_q}{\prod_{(i,j)\in\lambda}[h_{ij}]_q}.$$

There is a bijection between standard Young tableaux in $SYT(n, n)$ and non-crossing matchings. For $T \in SYT(n, n)$, form a corresponding sequence of parentheses by placing a left parenthesis under each number in the first row and a right parenthesis under each number in the second row. Then match the parentheses.

Example 4.6. Let $n = 3$ and $T \in SYT(3,3)$ as in Figure 4.3.

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \mapsto \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ (&) & (& (&) &) \end{array} \mapsto \{(1,2),(3,6),(4,5)\}.$$

Figure 4.3: An Example of the Bijection Between $SYT(n,n)$ and Matchings on $[2n]$

Applying this bijection to Theorem 4.5 gives the matching description of jeu-de-taquin promotion. As stated in Petersen *et al.* (2009), Dennis White discovered this interpretation of promotion, although never published. We see that the hook length formula is reduced to the q-Catalan number when $\lambda = (n,n)$. Thus the bijection by Dennis White and Theorem 4.5 directly imply Theorem 4.3.

Motivated by the previous result, the CSP on the non-crossing matchings, Bowling and Liang (2017) showed that the CSP is observed on the set of k-crossing matchings for $k = 1,2,3$ in P_n. Bowing-Liang first found that the number of one-crossing, two-crossing, and three-crossing matchings on $[2n]$ are $\binom{2n}{n-2}$, $\frac{n+3}{2}\binom{2n}{n-3}$, and $\frac{1}{3}\binom{n+5}{2}\binom{2n}{n-4} + \binom{2n}{n-3}$ respectively. Then they set $f_k(q)$ to be the q-analog of the number of k-crossing matchings. Next they enumerated the number of k-crossing matchings fixed by c^d for $k = 1,2,3$. They finished their proof by showing that the evaluation $f_k(\zeta^d)$ is the number of k-crossing matchings fixed by c^d for $k = 1,2,3$.

Theorem 4.7 (Bowling and Liang (2017), Theorem 2, Theorem 3, Theorem 4)**.** *Let X_k be the collection of k-crossing matchings on $[2n]$ for $k = 1,2,3$. Let $C_{2n} = \langle c \rangle$*

where $c = (1\ 2\ \dots\ 2n)$*. Let*

$$\begin{aligned}
f_1(q) &= \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q \\
f_2(q) &= \frac{[n+3]_q}{[2]_q} \begin{bmatrix} 2n \\ n-3 \end{bmatrix}_q \\
f_3(q) &= \frac{1}{[3]_q} \begin{bmatrix} n+5 \\ 2 \end{bmatrix}_q \begin{bmatrix} 2n \\ n-4 \end{bmatrix}_q + \begin{bmatrix} 2n \\ n-3 \end{bmatrix}_q
\end{aligned}$$

Then the triple $(X_k, f_k(q), C_{2n})$ *exhibits the CSP for* $k = 1, 2, 3$.

Both Theorem 4.3 and Theorem 4.7 establish the CSP on the set of matchings of a particular number of crossings. One could keep working on the set of elements of a certain number of crossings, but we change our focus to the whole set P_n of matchings on $[2n]$. After writing code in SAGE (Stein *et al.* (2016)) to compute the number of elements of P_n fixed by c^d with respect to the number of crossings (see Appendix), we found some patterns. We conjectured that there exist polynomials $X_n(q)$ for $X = P_n$ and $C_{2n} = \langle c \rangle$ where $c = (1\ 2\ \dots\ 2n)$ such that the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP. Because $|P_n| = (2n)!/(2^n \cdot n!)$ or $|P_n| = 1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)$, the first possible candidates for $X_n(q)$ could be the q-analog of $|P_n|$:

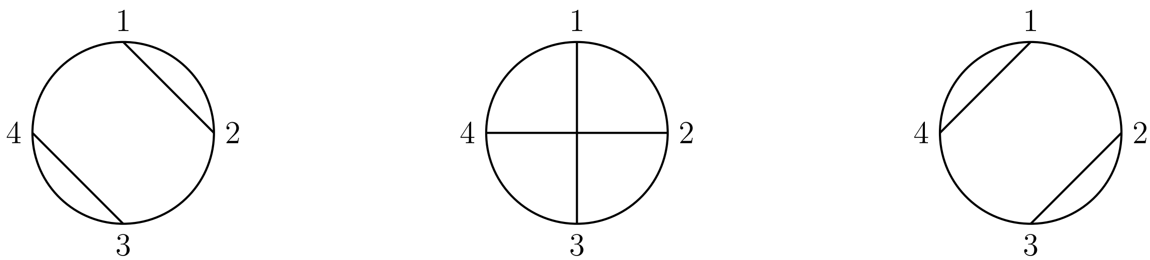
$$X_n'(q) = \frac{[2n]!_q}{([2]_q)^n [n]_q!} \quad \text{or} \quad X_n''(q) = [1]_q [3]_q \cdots [2n-1]_q.$$

Notice that $X_2'(q) = \frac{[4]_q}{[2]_q^2 [2]_q!} = \frac{(q^2+1)(q^2+q+1)}{q+1}$ which is not a polynomial. Note also that $X_2''(q) = [3]_q = q^2 + q + 1 \not\equiv q^2 + 2 \mod q^4 - 1$, and thus $X_2''(q)$ is not a CSP polynomial by Example 4.8.

4.2 The CSP on P_n for a Prime n

In the previous section, we see that our candidates for $X_n(q)$ do not work. Thus, we try to construct a CSP polynomial, and first we assume n is a prime number because it could be a good point to start. We first examine when $n = 2$.

Example 4.8. Let $X_2(q) = Cat_2(q) + f_1(q) = q^2 + 2$ by Theorem 4.3 and Theorem 4.7. Let $C_4 = \langle c \rangle$ where $c = (1\ 2\ 3\ 4)$. Then we claim that the triple $(P_2, X_2(q), C_4)$ exhibits the CSP. Observe that there is one element in P_2 fixed by c and c^3, which is $\{(1,3),(2,4)\}$. Observe also that all three elements in P_2 are fixed by c^0 and c^2. Thus if we let $\zeta = e^{\frac{2\pi i}{4}} = i$ be a fourth-root of unity, then we see that $X_2(1) = X_2(-1) = 3$ and $X_2(i) = X_2(-i) = 1$, and this shows the triple $(P_2, X_2(q), C_4)$ exhibits the CSP.



$\{(1,2),(3,4)\}$ $\{(1,3),(2,4)\}$ $\{(1,4),(2,3)\}$

Figure 4.4: Elements in P_2

Let us see one more example.

Example 4.9. Let $X_3(q) = Cat_3(q) + f_1(q) + f_2(q) + f_3(q) = q^6 + q^5 + 3q^4 + 2q^3 + 3q^2 + q + 4$. Let $C_6 = \langle c \rangle$ where $c = (1\ 2\ 3\ 4\ 5\ 6)$. Then the triple $(P_3, X_3(q), C_6)$ exhibits the CSP. To check this, first observe that the number of elements in P_3 fixed by c^d for $d = 0, 1, 2, 3, 4,$ and 5 is the row sum of $(d+1)$-th row of A_3 in Appendix B, namely 15,1,3,7,3, and 1 respectively. One could check this in Figure 2.13 in Chapter 2. Thus if we let $\zeta = e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ be a sixth-root of unity, then we see that

$$\begin{aligned} X_3(1) &= 15 \\ X_3(-1) &= 7 \\ X_3(\zeta) &= X_3(\zeta^5) = 1 \\ X_3(\zeta^2) &= X_3(\zeta^4) = 3, \end{aligned}$$

and this shows the triple $(P_3, X_3(q), C_6)$ exhibits the CSP.

Wire diagrams on $2n$ points on a circle are a good visualization of matchings on $[2n]$. If d divides $2n$, then the action of $c^{2n/d}$ on matchings can be interpreted as the $2\pi/d$ rotation of corresponding wire diagrams. We state and prove following lemmas.

Lemma 4.10. *Define a sequence t_n $(n \geq 1)$ by $t_n = t_{n-1} + (2n-2)t_{n-2}$ and $t_1 = 1$, $t_2 = 3$. Then the number of elements in P_n fixed by $c^n = (1\ 2\ \ldots\ 2n)^n = (1\ n+1)(2\ n+2)\ldots(n\ 2n)$, the 180 degree rotation, is t_n.*

Proof. We will count the number of ways to construct $\tau \in P_n$ which is fixed by c^n. Let $\tau(n)$ denote the partner of n in τ, then $\tau(n) \in [2n]\backslash\{n\}$. Assume $\tau(n) = m \neq 2n$. Since τ is fixed by 180 degree rotation, $\tau(2n) = m + n$ where modulo $2n$ is taken if necessary. Thus τ contains the pairs (n, m) and $(2n, m+n)$. There are $(2n-2)$ choices for the value of m. Once the value of m is determined, the number of unpaired numbers in τ is $|[2n]\backslash\{n, 2n, m, m+n\}| = 2n-4$, and there are t_{n-2} ways to construct τ. Now assume $\tau(n) = 2n$. Then τ contains the pair $(n, 2n)$. Since there are $|[2n]\backslash\{n, 2n\}| = 2n-2$ unpaired numbers in τ, there are t_{n-1} ways to construct τ. Thus, t_n satisfies the recursion $t_n = t_{n-1} + (2n-2)t_{n-2}$. It is clear that $t_1 = 1$ and $t_2 = 3$, and the proof follows. $\square$

Remark 4.11. The sequence t_n is A047974 in the OEIS (Sloane (2003)).

Remark 4.12. Lemma 4.10 is true for not only primes but also for any natural numbers, but the following lemma is only true for odd primes.

Lemma 4.13. *For a prime $n \geq 3$, the number of elements in P_n fixed by c^d for $d \in \{1, 3, 5, \ldots, 2n-1\}\backslash\{n\}$ is 1. The number of elements in P_n fixed by c^d for $d \in \{2, 4, 6, \ldots, 2n-2\}$ is n.*

Proof. Suppose $d \in \{1, 3, 5, \ldots, 2n-1\} \backslash \{n\}$. Since d and $2n$ are relatively prime, c^d is a generator of the cyclic group C_{2n}. Let $\tau \in P_n$ be fixed by c^d. Then τ is fixed by c. Thus the matching τ is determined by $\tau(1)$, the partner of 1 in τ, because all pairs in τ will be of the form $(1+x, \tau(1)+x)$ for all $0 \leq x \leq 2n-1$. Since τ is a matching, we have $\tau(1) = n+1$. Thus τ is the matching $\{(1, n+1), (2, n+2), \ldots, (n, 2n)\}$, and the number of elements fixed by c^d for $d \in \{1, 3, 5, \ldots, 2n-1\} \backslash \{n\}$ is 1. Now suppose $d \in \{2, 4, 6, \ldots, 2n-2\}$. Let $\tau \in P_n$ be fixed by c^d. Since $\gcd(d, 2n) = 2$ where gcd stands for the greatest common divisor, τ is fixed by c^2. Hence the matching τ contains the pairs $(1, \tau(1)), (3, \tau(1)+2), (5, \tau(1)+4), \ldots, (2n-1, \tau(1)+2n-2)$, and thus $\tau(1) = m$ for some $m \in \{2, 4, \ldots, 2n\}$, and the value of m determines τ. Thus the number of elements fixed by c^d for $d \in \{2, 4, 6, \ldots, 2n-2\}$ is n. □

The following lemma will be used to prove that the coefficients of $X_n(q)$ are integers in Theorem 4.15.

Lemma 4.14. *Let* $\alpha_n = \frac{1}{2n}\left(\frac{(2n)!}{2^n \cdot n!} - t_n - n + 1\right)$ *and* $\beta_n = \frac{t_n - 1}{n}$. *For any prime* $n \geq 3$, *both* α_n *and* β_n *are integers.*

Proof. We will show that $t_n = a_{2,n} = 1 + n\sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!}\binom{n-1}{2i+1}$ in (4.7) in Proposition 4.28. Then we see that $\beta_n = (t_n - 1)/n = \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!}\binom{n-1}{2i+1}$ is an integer. Notice that $\frac{(2n)!}{2^n \cdot n!} = 1 \cdot 3 \cdot \cdots \cdot (2n-1)$ is the product of first n odd numbers, and it is divisible by n. Thus we see that $\frac{(2n)!}{2^n \cdot n!} - n$ is divisible by $2n$. We also observe that $t_n - 1 = n\sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!}\binom{n-1}{2i+1}$ is divisible by $2n$ because $\binom{n-1}{2i+1}$ is even for all $i \geq 0$. Thus α_n is an integer. □

We also prove Lemma 4.14 by an exponential generating function argument.

Proof of Lemma 4.14 using EGF. First, we show the function $f(x) = e^{x+x^2}$ is an exponential generating function for the sequence t_n. Define $T(x)$ by

$$\begin{aligned}
T(x) &= \sum_{n\geq 0} \frac{t_n x^n}{n!} \\
&= 1 + \frac{t_1 x}{1!} + \frac{t_2 x^2}{2!} + \sum_{n\geq 0} \frac{[t_{n-1} + 2(n-1)t_{n-2}]x^n}{n!}.
\end{aligned}$$

Differentiate both sides, to see that

$$\begin{aligned}
T'(x) &= 1 + 3x + \sum_{n\geq 3} \frac{t_{n-1}x^{n-1}}{(n-1)!} + 2x \sum_{n\geq 3} \frac{t_{n-2}x^{n-2}}{(n-2)!} \\
&= 1 + 3x + [T(x) - 1 - x] + 2x[T(x) - 1] \\
&= (1+2x)T(x).
\end{aligned}$$

Solving this separable differential equation with initial condition $T(0) = 1$, we have $T(x) = e^{x+x^2}$. Second, we prove α_n and β_n are integers. Consider a function $B(x) = e^x(e^{x^2} - 1)$. We see that

$$\begin{aligned}
B(x) &= e^{x^2+x} - e^x \\
&= \sum_{n\geq 0} \frac{(t_n - 1)x^n}{n!} \\
&= \sum_{n\geq 0} \frac{n\beta_n x^n}{n!}.
\end{aligned}$$

Thus $n\beta_n$, the coefficient of $x^n/n!$ in $B(x) = e^x(e^{x^2}-1) = \left(\sum_{n\geq 0} x^n/n!\right)\left(\sum_{n\geq 1} x^{2n}/n!\right)$, is

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{n!}{(2k-1)!\left(\frac{n-2k+1}{2}\right)!}.$$

We see that

$$\begin{aligned}
\beta_n &= \sum_{k=1}^{\frac{n-1}{2}} \frac{(n-1)!}{(2k-1)!\left(\frac{n-2k+1}{2}\right)!} \\
&= (n-1)\sum_{k=1}^{\frac{n-1}{2}} \frac{(n-2)!}{(2k-1)!\left(\frac{n-2k+1}{2}\right)!} \\
&= (n-1)\left[\sum_{k=1}^{\frac{n-1}{2}-1} \frac{(n-2)!}{(2k-1)!\left(\frac{n-2k+1}{2}\right)!} + 1\right]
\end{aligned}$$

is an even integer because $(n-1)$ is even and $(2k-1)+\left(\frac{n-2k+1}{2}\right)=\frac{n-1}{2}+k\leq n-2$ for $1\leq k\leq\frac{n-1}{2}-1$. Notice that $\frac{(2n)!}{2^n\cdot n!}=1\cdot 3\cdot\cdots\cdot(2n-1)$ is the product of first n odd numbers, and it is divisible by n. Since $t_n-1=n\beta_n$ which is divisible by $2n$, we have $\alpha_n=\frac{1}{2n}\left(\frac{(2n)!}{2^n\cdot n!}-n\beta_n-n\right)$ is an integer. $\square$

In the following theorem, we present the polynomials $X_n(q)$ for any odd prime n.

Theorem 4.15. *Let $n\geq 3$ be a prime number. Let $C_{2n}=\langle c\rangle$ where $c=(1\ 2\ \ldots\ 2n)$. Let $X_n(q)$ be the polynomial*

$$\begin{aligned}X_n(q)&=\alpha_n[2n]_q+\beta_n\cdot\frac{[2n]_q}{[2]_q}+\frac{n-1}{2}\cdot\frac{[2n]_q}{[n]_q}+1\\&=\alpha_n\sum_{i=0}^{2n-1}q^i+\beta_n\sum_{i=0}^{n-1}q^{2i}+\frac{n-1}{2}(1+q^n)+1\end{aligned}$$

where $\alpha_n=\frac{1}{2n}\left(\frac{(2n)!}{2^n\cdot n!}-t_n-n+1\right)$ and $\beta_n=\frac{t_n-1}{n}$. Then, the triple $(P_n,X_n(q),C_{2n})$ exhibits the CSP.

Proof. By Lemma 4.14, the polynomial $X_n(q)$ is with integral coefficients. Let $\zeta=e^{2\pi i/2n}$ be a $(2n)$-th root of unity. By Lemma 4.10 and Lemma 4.13, we need to check:

$$\begin{aligned}&X_n(\zeta^d)=1\qquad\text{for } d\in\{1,3,5,\ldots,2n-1\}\backslash\{n\}\\&X_n(\zeta^d)=n\qquad\text{for } d\in\{2,4,6,\ldots,2n-2\}\\&X_n(\zeta^0)=X_n(1)=|P_n|=\frac{(2n)!}{2^n\cdot n!}\\&X_n(\zeta^n)=X_n(-1)=t_n.\end{aligned}$$

For $d\in\{1,3,5,\ldots,2n-1\}\backslash\{n\}$, since $\zeta^d\neq 1$ and $(\zeta^d)^2\neq 1$ and $(\zeta^d)^n=-1$, thus we have $X_n(\zeta^d)=1$. For $d\in\{2,4,6,\ldots,2n-2\}$, since $\zeta^d\neq 1$ and $(\zeta^d)^2\neq 1$ and $(\zeta^d)^n=1$, thus we have $X_n(\zeta^d)=n$. We compute that

$$\begin{aligned}X_n(1)&=\alpha_n\cdot 2n+\frac{t_n-1}{n}\cdot n+\frac{n-1}{2}\cdot 2+1=\frac{(2n)!}{2^n\cdot n!}\\X_n(-1)&=\frac{t_n-1}{n}\cdot n+1=t_n\end{aligned}$$

as desired. $\square$

Example 4.16 (revisited)**.** Let $n = 3$. By Theorem 4.15,

$$\begin{aligned} X_3(q) &= [6]_q + 2\frac{[6]_q}{[2]_q} + \frac{[6]_q}{[3]_q} + 1 \\ &= (q^5 + q^4 + q^3 + q^2 + q + 1) + 2(q^4 + q^2 + 1) + (q^3 + 1) + 1 \end{aligned}$$

since $t_3 = t_2 + 4t_1 = 7$ and $\alpha_3 = \frac{1}{6}(15 - t_3 - 3 + 1) = 1$. Thus, we have $X_3(q) = q^5 + 3q^4 + 2q^3 + 3q^2 + q + 5$ which is congruent to $X_3(q) = q^6 + q^5 + 3q^4 + 2q^3 + 3q^2 + q + 4$ in Example 4.9 under modulo $q^6 - 1$.

4.3 Construction of a CSP Polynomial in General

Inspired by the work presented in the previous section, we continue to try to construct a CSP polynomial in a general setting. In this section, all propositions hold for not only a set of matchings but also any finite sets. We state and prove the following proposition.

Proposition 4.17. *Let X be a finite set. Let a cyclic group $C = \langle c \rangle$ act on X where $|c| = N$. Let a_d be the number of elements of X fixed by $c^{N/d}$ for $d|N$. Define b_d by the equation*

$$a_d = \sum_{d|r} \frac{N}{r} b_r. \tag{4.1}$$

Let $f(q)$ be

$$\begin{aligned} f(q) &= \sum_{d|N} b_d(q^{N-d} + q^{N-2d} + \cdots + q^d + 1) \\ &= \sum_{d|N} b_d \frac{[N]_q}{[d]_q}. \end{aligned}$$

Then, the evaluation $f(\zeta^d)$ is equal to the number of elements of X fixed by c^d.

Proof. First, we prove that for $k|N$ the evaluation $f(\zeta^k)$ is equal to $a_{N/k}$, the number of elements of X fixed by c^k. Note that

$$q^{N-r} + q^{N-2r} + \cdots + q^r + 1\Big|_{q=\zeta^k} = \frac{q^N - 1}{q^r - 1}\bigg|_{q=\zeta^k} = 0$$

unless $N|kr$. If $N|kr$ or equivalently $\frac{N}{k}|r$, we have

$$b_r(q^{N-r} + q^{N-2r} + \cdots + q^r + 1)\big|_{q=\zeta^k} = b_r(1 + 1 + \cdots + 1) = b_r\frac{N}{r}.$$

Thus we see that

$$f(\zeta^k) = \sum_{\frac{N}{k}|r} \frac{N}{r} b_r = a_{N/k}.$$

Next, we prove that for $j \nmid N$ the evaluation $f(\zeta^j)$ is equal to the number of elements of X fixed by c^j. We claim that for $j \nmid N$ the number of elements of X fixed by c^j is equal to the number of elements of X fixed by c^k if $\gcd(N, j) = k$. Assume $x \in X$ is fixed by c^j. By the Euclidean algorithm there are $a, b \in \mathbb{Z}$ such that $aN + bj = k$. Then we see that x is fixed by c^k since $c^k \cdot x = c^{aN+bj} \cdot x = c^{aN}(c^{bj} \cdot x) = e \cdot x = x$. Now assume x is fixed by c^k. Since j is a multiple of k, the element x is fixed by c^j. Thus our claim is proved. Hence we must show that the evaluation $f(\zeta^j)$ is equal to $f(\zeta^k) = a_{N/k}$. Notice that

$$q^{N-r} + q^{N-2r} + \cdots + q^r + 1\big|_{q=\zeta^j} = \left.\frac{q^N - 1}{q^r - 1}\right|_{q=\zeta^j} = 0$$

unless $N|jr$. Suppose $N|jr$, then $\frac{N}{k}|r$ because $\gcd(N, j) = k$. We see that

$$b_r(q^{N-r} + q^{N-2r} + \cdots + q^r + 1)\big|_{q=\zeta^j} = b_r(1 + 1 + \cdots + 1) = b_r\frac{N}{r}.$$

Hence we have

$$f(\zeta^j) = \sum_{\frac{N}{k}|r} \frac{N}{r} b_r = a_{N/k},$$

as required. $\square$

Moreover, we claim that the triple $(X, f(q), C)$ exhibits the CSP. Since elements within a C-orbit share the same stabilizer subgroup, whose cardinality we call the stabilizer-order for the orbit. Recall Proposition 2.1 in Reiner *et al.* (2004).

Proposition 4.18 (Proposition 2.1 in Reiner *et al.* (2004))**.** *Let X be a finite set. Let a cyclic group $C = \langle c \rangle$ act on X where $|c| = N$. Let $X(q)$ be a polylnomial with nonnegative coefficients. Then the following are equivalent conditions for a triple $(X, X(q), C)$.*

1. *For every $0 \le d \le N-1$,*

$$[X(q)]_{q=\zeta^d} = |\{x \in X : c^d(x) = x\}|.$$

2. *The coefficient α_ℓ defined uniquely by the expansion*

$$X(q) \equiv \sum_{\ell=0}^{N-1} \alpha_\ell q^\ell \mod q^N - 1$$

has the following interpretation: α_ℓ counts the number of C-orbits on X for which the stabilizer-order divides ℓ. In particular, α_0 counts the total number of C-orbits on X, and α_1 counts the number of free C-orbits on X.

By the second part of Proposition 4.18, for given set X and cyclic group C there is always a unique polynomial $X(q)$ of degree less than n making the triple $(X, X(q), C)$ the CSP triple. This observation implies the following proposition.

Proposition 4.19. *Let X be a finite set. Let a cyclic group $C = \langle c \rangle$ act on X where $|c| = N$. Then the polynomial $f(q)$ constructed in Proposition 4.17 together with X and C exhibits the CSP.*

Proof. By Proposition 4.18, there exists a unique polynomial $X(q)$ of degree at most $N-1$ such that $(X, X(q), C)$ exhibits the CSP. Then $X(\zeta^d)$ counts the number of elements of X fixed by c^d for all $0 \le d \le N-1$. Let $F(q) := f(q) - X(q)$, then we notice that $F(q)$ is a polynomial in $\mathbb{Q}[q]$ of degree at most $N-1$. By construction of $f(q)$, for a N-th root of unity ζ we have $F(\zeta^d) = f(\zeta^d) - X(\zeta^d) = 0$ for all $0 \le d \le N-1$. Thus $F(q)$ should have factors $q - \zeta^d$ for $0 \le d \le N-1$. Then

$F(q)$ is divisible by $\prod_{d=0}^{N-1}(q-\zeta^d) = q^N - 1$, which forces $F(q)$ to be identically zero. Therefore, we conclude that $f(q) = X(q)$, and the triple $(X, f(q), C)$ exhibits the CSP. $\square$

Remark 4.20. Proposition 4.19 (together with Proposition 4.17) is equivalent to the second condition in Proposition 4.18. We use Proposition 4.17 to construct a CSP polynomial.

We have seen that the coefficients a_d's and b_d's are related by $a_d = \sum_{d|r} \frac{N}{r} b_r$ in (4.1). The following proposition shows how to express the coefficients b_d's in terms of a_d's.

Proposition 4.21. *Suppose the coefficients a_d's and b_d's are as in Proposition 4.17. Let $N = |c|$. Let $P = \{p_1, p_2, \ldots, p_m\}$ be the set of primes that divide N/d. Then*

$$\begin{aligned} b_d &= \frac{d}{N} \sum_{S \subseteq P} (-1)^{|S|} a_{d \prod_{p \in S} p} \\ &= \frac{d}{N} \left[a_d - (a_{dp_1} + \cdots + a_{dp_m}) + \cdots + (-1)^m a_{dp_1 p_2 \ldots p_m} \right]. \end{aligned} \tag{4.2}$$

Proof. Let D_N be the poset on the set of all divisors of N partially ordered by $i \leq j$ if and only if j is divisible by i. Let $g(d) = a_d$ and $f(d) = \frac{N}{d} b_d$. The equation (4.1) can be written as $g(d) = \sum_{r: d \leq r} f(r)$. By the dual form of Möbius inversion formula (Proposition 3.7.2 in Stanley (2012)), we have $f(d) = \sum_{d \leq r} \mu(d,r) g(r)$. Since

$$\mu(d,r) = \begin{cases} (-1)^t & \text{if } r/d \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

(see Example 3.8.4 in Stanley (2012)) and the proof follows. $\square$

The following proposition explains how the coefficients α_ℓ's and b_d's are related to each other.

Proposition 4.22. *The coefficients α_ℓ's and b_d's are related by*

$$\alpha_\ell = \sum_{d|\ell} b_d, \tag{4.3}$$

or equivalently for the set $Q := \{p_1, \ldots, p_l\}$ of prime divisors of d,

$$\begin{aligned} b_d &= \sum_{S \subseteq Q} (-1)^{|S|} \alpha_{d/\prod_{p\in S} p} \\ &= \alpha_d - (\alpha_{d/p_1} + \cdots + \alpha_{d/p_l}) + \cdots + (-1)^m \alpha_{d/(p_1 p_2 \ldots p_l)}. \end{aligned} \tag{4.4}$$

Proof. Since α_ℓ is the coefficient of q^ℓ, for divisors d of N such that $N - kd = \ell$ for some k or equivalently $d|\ell$ we have $\alpha_\ell = \sum_{d|\ell} b_d$. By using Möbius inversion formula, the proof follows. $\square$

Remark 4.23. The coefficient b_d counts the number of C-orbits on X for which the stabilizer-order is exactly d. Or equivalently, the coefficient b_d count the number of C-orbits on X of order N/d.

The previous remark implies the following corollary.

Corollary 4.24. *The coefficients b_d's are nonnegative integers.*

4.4 The CSP on P_n for Any $n \in \mathbb{N}$

In this section we use the results presented in the previous section to construct a CSP polynomial $X_n(q)$ on P_n. We first examine when $n = 6$.

Example 4.25. Let $X_6(q) = 837q^{11} + 889q^{10} + 842q^9 + 893q^8 + 837q^7 + 897q^6 + 837q^5 + 893q^4 + 842q^3 + 889q^2 + 837q + 902$. Let $C_{12} = \langle c \rangle$ where $c = (1\ 2\ \ldots 11\ 12)$. Then the triple $(P_6, X_6(q), C_{12})$ exhibits the CSP. To check this, first observe that the number of elements in P_6 fixed by c^d for $d = 0, 1, 2, \ldots, 11$ is the row sum of $(d+1)$-th row of A_6 in Appendix B. Thus if we let $\zeta = e^{\frac{2\pi i}{12}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ be a 12th-root of unity, then we see that

$$\begin{aligned}
X_6(1) &= 10395 \\
X_6(-1) &= 331 \\
X_6(\zeta) &= X_6(\zeta^5) = X_6(\zeta^7) = X_6(\zeta^{11}) = 1 \\
X_6(\zeta^4) &= X_6(\zeta^8) = 27 \\
X_6(\zeta^2) &= X_6(\zeta^{10}) = 7 \\
X_6(\zeta^3) &= X_6(\zeta^9) = 13,
\end{aligned}$$

and this shows the triple $(P_6, X_6(q), C_{12})$ exhibits the CSP.

For all $n \geq 1$, we want to find the polynomials $X_n(q)$ for which the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP. Suppose d divides $2n$. First, we want to count the number of elements in P_n fixed by $2\pi/d$ rotation. Let $a_{d,n}$ be the number of such elements in P_n. For example, Lemma 4.10 tells us that $a_{2,n} = a_{2,n-1} + (2n-2)a_{2,n-2}$ with $a_{2,1} = 1$ and $a_{2,2} = 3$.

Proposition 4.26. *Suppose $2n = dk$ for some $k \in \mathbb{Z}$. Then the sequence $a_{d,n}$ satisfies the following recurrence relations depending on the parity of d.*

1. *If 2 divides d, then*

$$a_{d,n} = a_{d,n-\frac{d}{2}} + (2n-d)a_{d,n-d} \tag{4.5}$$

   with the initial condition $a_{d,\frac{d}{2}} = 1$ and $a_{d,d} = d+1$.

2. *If d is not divisible by 2, then*

$$a_{d,n} = (2n-d)a_{d,n-d} \tag{4.6}$$

   with the initial condition $a_{d,d} = d$.

Proof. We first show the recurrence relations.

(2|d) Suppose 2 divides d. We will count the number of ways to construct $\tau \in P_n$ which is fixed by $2\pi/d$ rotation. First assume $\tau(2n) = n$. Then τ should contain the pairs $(k, n+k), (2k, n+2k), \ldots, (n-k, 2n-k), (n, 2n)$ because τ is fixed by $2\pi/d$ rotation. We see that the number of unpaired numbers in τ is $2n - |\{k, 2k, \ldots, 2n-k, 2n\}| = 2n - d$. There is a bijection between ways to pair up $2n-d$ unpaired numbers in τ onto matchings fixed by $2\pi/d$ rotation in $P_{n-\frac{d}{2}}$. For instance, see Example 4.27. Thus there are $a_{d,n-\frac{d}{2}}$ ways to construct τ. Now assume $\tau(2n) \neq n$. Observe that $\tau(2n) \neq k$, otherwise both pairs $(2n, k)$ and $(k, 2k)$ are in τ, which contradicts τ is a matching. For the same reason, it is obvious that $\tau(2n) \notin \{k, 2k, \ldots, 2n-k\}$. Thus $\tau(n) = m$ for some $m \in [2n]\backslash\{k, 2k, \ldots, 2n-k\}$, and there are $2n-d$ choices for the value of m. Once the value of m is determined, τ should contains the pairs $(k, m+k), (2k, m+2k), \ldots, (2n, m)$ where modulo $2n$ is taken if necessary. Since the number of unpaired numbers in τ is $2n - |(\{k, 2k, \ldots, 2n-k, 2n\} \cup \{m, m+k, \ldots, m+2n-k\})| = 2n-2d$, there are $(2n-d)\cdot a_{d,n-d}$ ways to construct τ. Hence, we have the recurrence relation $a_{d,n} = a_{d,n-\frac{d}{2}} + (2n-d)a_{d,n-d}$.

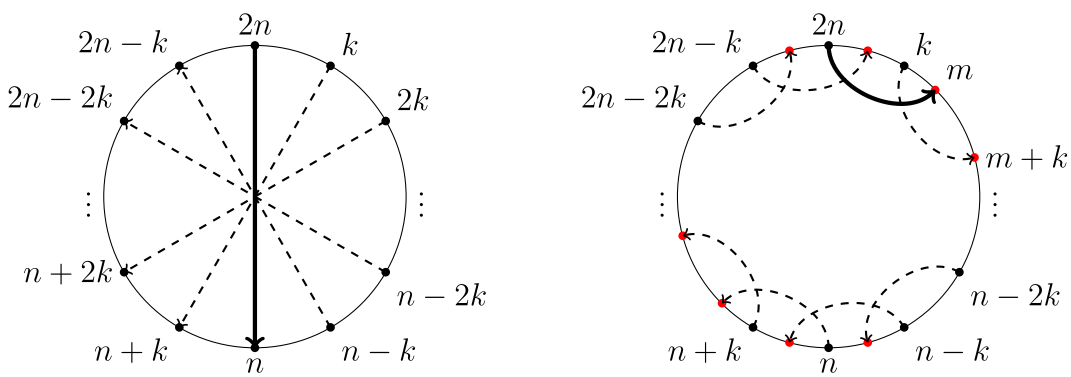


Figure 4.5: The Case of $2|d$

($2 \nmid d$) Suppose d is not divisible by 2. Then k is divisible by 2. First assume $\tau(2n) = n$.

Note that $n \notin k\mathbb{Z}$ and $n + \frac{k}{2} \in k\mathbb{Z}$ and $n - \frac{k}{2} \in k\mathbb{Z}$ since $n = \frac{dk}{2}$ and d is odd. Then τ should contain the pairs $(\frac{k}{2}, n + \frac{k}{2}), (k, n + k), (\frac{3k}{2}, n + \frac{3k}{2}), \dots, (n - \frac{k}{2}, 2n - \frac{k}{2}), (n, 2n)$. Note that the number of unpaired numbers in τ is $2n - |\{\frac{k}{2}, k, \dots, 2n - \frac{k}{2}, 2n\}| = 2n - 2d$, and there are $a_{d,n-d}$ ways to construct τ when $\tau(2n) = n$. Now, assume $\tau(2n) \neq n$. Then $\tau(2n) = m$ for some $m \in [2n] \backslash (\{k, 2k, \dots, 2n - k, 2n\} \cup \{n\})$, and we have $2n - d - 1$ choices for the value of m. Once the value of m is determined, τ should contain the pairs $(k, m+k), (2k, m+2k), \dots, (2n-k, m+2n-k), (2n, m)$ where modulo $2n$ is taken if necessary. Since the number of unpaired numbers in τ is $2n - |\{k, 2k, \dots, 2n - k\} \cup \{m, m + k, \dots, m + 2n - k\}| = 2n - 2d$, there are $(2n - d - 1) \cdot a_{d,n-d}$ ways to construct τ when $\tau(2n) \neq n$. Hence, we have the recurrence relation $a_{d,n} = a_{d,n-d} + (2n - d - 1)a_{d,n-d} = (2n - d)a_{d,n-d}$.

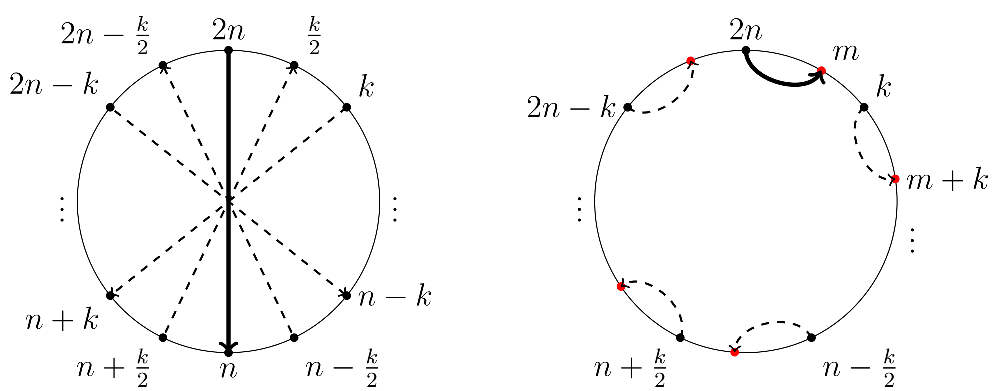


Figure 4.6: The Case of $2 \nmid d$

Next we show the initial conditions.

$(2|d)$ Suppose 2 divides d. Assume $2n = d$. Let $\tau \in P_n = P_{\frac{d}{2}}$ be an element fixed by $2\pi/d$ rotation. Since there are d points on the circle, the matching τ should be $\tau = \{(1, n + 1), (2, n + 2), \dots, (n, 2n)\}$. Thus $a_{d,\frac{d}{2}} = 1$. Now assume $n = d$. Let $\sigma \in P_n = P_d$ be an element fixed by $2\pi/d$ rotation. Then $\sigma(1) = m$ for some

$m \in \{2, 4, \ldots, 2d\} \cup \{d+1\}$. Since $\sigma(1)$ determines the matching σ, we have $a_{d,d} = d + 1$ as desired.

$(2 \nmid d)$ Suppose d is not divisible by 2. Let $n = d$. Let $\tau \in P_n = P_d$ be an element fixed by $2\pi/d$ rotation. Then $\tau(1) = m$ for some $m \in \{2, 4, \ldots, 2d\}$. Notice that $d + 1$ is even, so it is in the set $\{2, 4, \ldots, 2d\}$. Since $\tau(1)$ determines the matching τ, we have $a_{d,d} = d$ as desired.

$\square$

Example 4.27. Let $n = 10$ and $d = 4$, then $2n = 20$ and $k = 2n/d = 5$. Let $\tau \in P_{10}$ be fixed by $2\pi/d = \pi/2$ rotation. Suppose $\tau(20) = 10$, then it forces $\tau(5) = 15$. So, there are $2n - d = 16$ unpaired numbers in τ. Let T be the set of unpaired numbers in τ, namely $T = [20]\backslash\{5, 10, 15, 20\}$. Consider a map $T \to [16]$ by

$$\begin{cases} i \mapsto i & \text{if } 1 \leq i \leq 4 \\ i \mapsto i-1 & \text{if } 6 \leq i \leq 9 \\ i \mapsto i-2 & \text{if } 11 \leq i \leq 14 \\ i \mapsto i-3 & \text{if } 16 \leq i \leq 19. \end{cases}$$

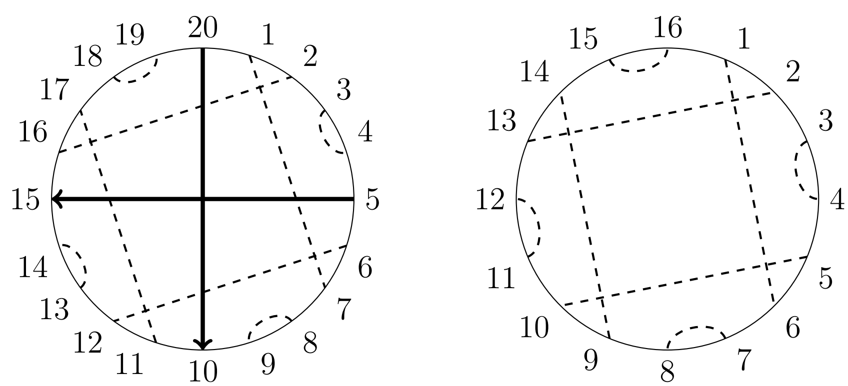


Figure 4.7: A Way to Pair up Elements in T and a Matching in P_8

The image of a way to pair up elements in T under this map is a matching fixed by $\pi/2$ rotation in P_8.

We find a formula for $a_{d,n}$ in the following proposition.

Proposition 4.28. *If 2 divides d, then*

$$a_{d,n} = 1 + n \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \binom{\frac{2n}{d} - 1}{2i+1} \left(\frac{d}{2}\right)^i. \tag{4.7}$$

If d is not divisible by 2, then

$$a_{d,n} = \prod_{i=1}^{n/d} (2i-1)d. \tag{4.8}$$

Proof. Suppose 2 divides d. Let $u_n = 1 + n \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \binom{\frac{2n}{d}-1}{2i+1} \left(\frac{d}{2}\right)^i$. Observe that $u_{d/2} = 1$ and $u_d = d + 1$. Then we calculate that

$$\begin{aligned}
&u_{n-\frac{d}{2}} + (2n-d)u_{n-d} \\
&= 1 + \left(n - \frac{d}{2}\right) \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \binom{\frac{2n}{d} - 2}{2i+1} \left(\frac{d}{2}\right)^i \\
&\qquad + (2n-d) \left[1 + (n-d) \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \binom{\frac{2n}{d} - 3}{2i+1} \left(\frac{d}{2}\right)^i\right] \\
&= 1 + \left(n - \frac{d}{2}\right) \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \binom{\frac{2n}{d} - 2}{2i+1} \left(\frac{d}{2}\right)^i \\
&\qquad + (2n-d) \left[1 + (n-d) \sum_{i \geq 1} \frac{(2i-1)!}{i!} \binom{\frac{2n}{d} - 3}{2i-1} \left(\frac{d}{2}\right)^{i-1}\right] \\
&= 1 + \left(n - \frac{d}{2}\right) \left(\frac{2n}{d} - 2\right) + (2n-d) \\
&\qquad + \sum_{i \geq 1} \left[\frac{(2i+1)!}{(i+1)!} \left(n - \frac{d}{2}\right) \binom{\frac{2n}{d} - 2}{2i+1} \left(\frac{d}{2}\right)^i \right. \\
&\qquad\qquad \left. + (2n-d)(n-d) \frac{(2i-1)!}{i!} \binom{\frac{2n}{d} - 3}{2i-1} \left(\frac{d}{2}\right)^{i-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \frac{2n}{d}\left(\frac{2n}{d} - 1\right)\left(\frac{d}{2}\right) \\
&\qquad + \sum_{i\geq 1}\left[\left(\frac{d}{2}\right)\frac{(\frac{2n}{d}-1)!}{(i+1)!(\frac{2n}{d}-2i-3)!}\left(\frac{d}{2}\right)^{i} + 2\left(\frac{d}{2}\right)^{2}\frac{(\frac{2n}{d}-1)!}{i!(\frac{2n}{d}-2i-2)!}\left(\frac{d}{2}\right)^{i-1}\right] \\
&= 1 + n\left(\frac{2n}{d} - 1\right) + \sum_{i\geq 1}\frac{(\frac{2n}{d}-1)!}{i!(\frac{2n}{d}-2i-3)!}\left[\frac{1}{i+1} + \frac{2}{\frac{2n}{d}-2i-2}\right]\left(\frac{d}{2}\right)^{i+1} \\
&= 1 + n\left(\frac{2n}{d} - 1\right) + \sum_{i\geq 1}\frac{\frac{2n}{d}!}{(i+1)!(\frac{2n}{d}-2i-2)!}\left(\frac{d}{2}\right)^{i+1} \\
&= 1 + n\left(\frac{2n}{d} - 1\right) + n\sum_{i\geq 1}\frac{(2i+1)!}{(i+1)!}\binom{\frac{2n}{d}-1}{2i+1}\left(\frac{d}{2}\right)^{i} \\
&= 1 + n\sum_{i\geq 0}\frac{(2i+1)!}{(i+1)!}\binom{\frac{2n}{d}-1}{2i+1}\left(\frac{d}{2}\right)^{i} = u_n.
\end{aligned}
$$

Thus if $2|d$, we have $a_{d,n} = u_n$ as required.

Now suppose d is not divisible by 2. Let $v_n = \prod_{i=1}^{n/d}(2i-1)d$. Notice that $v_d = d$. We compute that

$$
\begin{aligned}
(2n-d)v_{n-d} &= (2n-d)\prod_{i=1}^{n/d-1}(2i-1)d \\
&= \prod_{i=1}^{n/d}(2i-1)d = v_n.
\end{aligned}
$$

Thus if $2 \nmid d$, we have $a_{d,n} = v_n$ as required. $\square$

We also prove the case of $2|d$ by using exponential generating functions.

proof of (4.7) *in Proposition 4.28 using EGF.* For a fixed even number d, let $c_i = a_{d,\frac{d}{2}(i+1)}$. Then c_i satisfies $c_k = c_{k-1} + dkc_{k-2}$ with initial conditions $c_0 = 1$ and $c_1 = d+1$. Let $y = \sum_{k\geq 0}\frac{c_k x^k}{k!}$ be an exponential generating function for c_k and let

$Y = \sum_{k\geq 0} \frac{c_k x^{k+1}}{(k+1)!}$ be an antiderivative of y. By usual EGF methods, we see that

$$\begin{aligned}
y &= \sum_{k\geq 0} \frac{c_k x^k}{k!} \\
&= 1 + (d+1)x + \sum_{k\geq 2} \frac{c_{k-1}x^k}{k!} + dx \sum_{k\geq 2} \frac{c_{k-2}x^{k-1}}{(k-1)!} \\
&= 1 + (d+1)x + [Y - x + C_1] + dx[Y + C_2] \\
&= 1 + C_1 + (1 + C_2)dx + (1 + dx)Y.
\end{aligned}$$

By comparing the constant term and x term in both sides, we have $C_1 = C_2 = 0$, and thus $y = 1 + dx + (1 + dx)Y$. By solving this differential equation, we get

$$y = (1 + dx)e^{x+\frac{dx^2}{2}}.$$

Hence, c_k is the coefficient of $x^k/k!$ in $(1 + dx)e^{x+\frac{dx^2}{2}}$, which is

$$c_k = \sum_{i+2j=k} \frac{k!}{i!j!} \left(\frac{d}{2}\right)^j + d \sum_{i+2j=k-1} \frac{k!}{i!j!} \left(\frac{d}{2}\right)^j.$$

Suppose k is even. Then we see that

$$\begin{aligned}
c_k &= \sum_{0\leq j\leq \frac{k}{2}} \frac{k!}{(k-2j)!j!} \left(\frac{d}{2}\right)^j + \sum_{0\leq j\leq \frac{k}{2}-1} \frac{k!}{(k-2j-1)!j!} \left(\frac{d}{2}\right)^{j+1} \cdot 2 \\
&= 1 + \sum_{1\leq j\leq \frac{k}{2}} \frac{k!}{(k-2j)!j!} \left(\frac{d}{2}\right)^j + \sum_{0\leq j\leq \frac{k}{2}-1} \frac{k!}{(k-2j-1)!j!} \left(\frac{d}{2}\right)^{j+1} \cdot 2 \\
&= 1 + \sum_{0\leq j\leq \frac{k}{2}-1} \frac{k!}{(k-2j-2)!(j+1)!} \left(\frac{d}{2}\right)^{j+1} + \sum_{0\leq j\leq \frac{k}{2}-1} \frac{k!}{(k-2j-1)!j!} \left(\frac{d}{2}\right)^{j+1} \cdot 2
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{0 \le j \le \frac{k}{2}-1} \frac{k!}{(k-2j-2)!j!} \left(\frac{d}{2}\right)^{j+1} \left(\frac{1}{j+1} + \frac{2}{k-2j-1}\right) \\
&= 1 + \sum_{0 \le j \le \frac{k}{2}-1} \frac{k!}{(k-2j-2)!j!} \left(\frac{d}{2}\right)^{j+1} \frac{k+1}{(k-2j-1)(j+1)} \\
&= 1 + \sum_{0 \le j \le \frac{k}{2}-1} \frac{(k+1)\cdot k!}{(k-2j-1)!(j+1)!} \left(\frac{d}{2}\right)^{j+1} \\
&= 1 + \sum_{0 \le j \le \frac{k}{2}-1} \frac{(k+1)\cdot k!}{(k-2j-1)!(j+1)!} \frac{(2j+1)!}{(2j+1)!} \left(\frac{d}{2}\right)^{j+1} \\
&= 1 + \sum_{0 \le j \le \frac{k}{2}-1} \frac{(k+1)\cdot (2j+1)!}{(j+1)!} \binom{k}{2j+1} \left(\frac{d}{2}\right)^{j+1}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
a_{d,n} &= c_{\frac{2n}{d}-1} \\
&= 1 + \sum_{0 \le j \le \frac{n}{d}-\frac{3}{2}} \frac{\frac{2n}{d}\cdot(2j+1)!}{(j+1)!} \binom{\frac{2n}{d}-1}{2j+1} \left(\frac{d}{2}\right)^{j+1} \\
&= 1 + n \sum_{j \ge 0} \frac{(2j+1)!}{(j+1)!} \binom{\frac{2n}{d}-1}{2j+1} \left(\frac{d}{2}\right)^{j}.
\end{aligned}
$$

Now suppose k is odd. Then we see that

$$
\begin{aligned}
c_k &= \sum_{0 \le j \le \frac{k-1}{2}} \frac{k!}{(k-2j)!j!} \left(\frac{d}{2}\right)^{j} + \sum_{0 \le j \le \frac{k-1}{2}} \frac{k!}{(k-2j-1)!j!} \left(\frac{d}{2}\right)^{j+1} \cdot 2 \\
&= 1 + \sum_{1 \le j \le \frac{k-1}{2}} \frac{k!}{(k-2j)!j!} \left(\frac{d}{2}\right)^{j} + \sum_{0 \le j \le \frac{k-1}{2}} \frac{k!}{(k-2j-1)!j!} \left(\frac{d}{2}\right)^{j+1} \cdot 2 \\
&= 1 + \sum_{1 \le j \le \frac{k-3}{2}} \frac{k!}{(k-2j-2)!(j+1)!} \left(\frac{d}{2}\right)^{j+1} + \sum_{0 \le j \le \frac{k-1}{2}} \frac{k!}{(k-2j-1)!j!} \left(\frac{d}{2}\right)^{j+1} \cdot 2
\end{aligned}
$$

$$= 1 + \sum_{0 \le j \le \frac{k-3}{2}} \frac{k!}{(k-2j-2)!j!} \left(\frac{d}{2}\right)^{j+1} \left(\frac{1}{j+1} + \frac{2}{k-2j-1}\right) + \frac{k!}{(\frac{k-1}{2})!} \left(\frac{d}{2}\right)^{\frac{k+1}{2}} \cdot 2$$

$$= 1 + \sum_{0 \le j \le \frac{k-3}{2}} \frac{(k+1)\cdot(2j+1)!}{(j+1)!} \binom{k}{2j+1} \left(\frac{d}{2}\right)^{j+1} + \frac{k!}{(\frac{k-1}{2})!} \left(\frac{d}{2}\right)^{\frac{k+1}{2}} \cdot 2$$

$$= 1 + \sum_{0 \le j \le \frac{k-1}{2}} \frac{(k+1)\cdot(2j+1)!}{(j+1)!} \binom{k}{2j+1} \left(\frac{d}{2}\right)^{j+1}.$$

Thus, we have

$$\begin{aligned} a_{d,n} &= c_{\frac{2n}{d}-1} \\ &= 1 + \sum_{0 \le j \le \frac{n}{d}-1} \frac{\frac{2n}{d}\cdot(2j+1)!}{(j+1)!} \binom{\frac{2n}{d}-1}{2j+1} \left(\frac{d}{2}\right)^{j+1} \\ &= 1 + n \sum_{j \ge 0} \frac{(2j+1)!}{(j+1)!} \binom{\frac{2n}{d}-1}{2j+1} \left(\frac{d}{2}\right)^{j}. \end{aligned}$$

□

Now we are ready to find the polynomials $X_n(q)$ such that the set P_n of matchings on $[2n]$ with $X_n(q)$ and the cyclic group C_{2n} exhibits the CSP.

Theorem 4.29. *Let $C_{2n} = \langle c \rangle$ where $c = (1\ 2\ \ldots\ 2n)$. Let $X_n(q)$ be*

$$X_n(q) = \sum_{d|2n} b_{d,n} \frac{[2n]_q}{[d]_q} \tag{4.9}$$

$$= \sum_{d|2n} b_{d,n}(q^{2n-d} + q^{2n-2d} + \cdots + q^{2d} + q^d + 1) \tag{4.10}$$

where the coefficients $b_{d,n}$ satisfy

$$a_{d,n} = \sum_{d|r} \frac{2n}{r} b_{r,n}. \tag{4.11}$$

Then, the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP.

Proof. By Proposition 4.17 and Proposition 4.19 and Proposition 4.28 the proof follows. □

Example 4.30 (revisited)**.** Let $n = 6$. The cyclic sieving phenomenon polynomial is

$$X_6(q) = b_{1,6}[12]_q + \frac{b_{2,6}[12]_q}{[2]_q} + \frac{b_{3,6}[12]_q}{[3]_q} + \frac{b_{4,6}[12]_q}{[4]_q} + \frac{b_{6,6}[12]_q}{[6]_q} + \frac{b_{12,6}[12]_q}{[12]_q}.$$

By (4.2), we get

$$\begin{bmatrix} 12b_{1,6} \\ 6b_{2,6} \\ 4b_{3,6} \\ 3b_{4,6} \\ 2b_{6,6} \\ b_{12,6} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1,6} \\ a_{2,6} \\ a_{3,6} \\ a_{4,6} \\ a_{6,6} \\ a_{12,6} \end{bmatrix} = \begin{bmatrix} 10395 - 331 - 27 + 7 \\ 331 - 13 - 7 + 1 \\ 27 - 7 \\ 13 - 1 \\ 7 - 1 \\ 1 \end{bmatrix}$$

Here we get the last equality by Proposition 4.26, and we get the polynomial

$$\begin{aligned} X_6(q) = X_6(q) &= 837[12]_q + \frac{52[12]_q}{[2]_q} + \frac{5[12]_q}{[3]_q} + \frac{4[12]_q}{[4]_q} + \frac{3[12]_q}{[6]_q} + \frac{1[12]_q}{[12]_q}. \\ &= 837q^{11} + 889q^{10} + \cdots + 837q + 902 \end{aligned}$$

as in Example 4.25.

4.5 Future Work

In this chapter, we constructed CSP polynomials for the set of matchings on $[2n]$ by counting matchings fixed by $c^{2n/d}$. In this way our CSP polynomials are expressed as a linear combination of $[2n]_q/[d]_q$ for $d|2n$, which are polynomials of degree less than $2n$. Even though we constructed CSP polynomials, we count directly matchings fixed by $c^{2n/d}$. If we find CSP polynomials without direct counting, it would be much more interesting.

Problem 4.31. Find an expression of the cardinality of the set of matchings on $[2n]$ such that the q-analog of the expression are CSP polynomials.

The CSP is observed on the set of non-crossing matchings in Sagan (2011) and the set of one-, two- and three-crossing matchings in Bowling and Liang (2017). We tried to construct a polynomial for four-crossing matchings in the same way we use in Section 4.4. The way we obtain a recurrence relation in Section 4.4 is by breaking cases down into (1) $\tau(2n) = n$ or (2) $\tau(2n) = m \neq n$. It is not easy to find a recurrence relation for the number of four-crossing matchings fixed by $c^{2n/d}$. This is because in either case when we pair up the unpaired numbers it is difficult to control the number of crossings. Thus we will work on the set of k-crossing matchings.

Problem 4.32. For $3 < k < \binom{n}{2}$ find polynomials $X_{n,k}(q)$ such that the set of k-crossing matchings together with $X_{n,k}(q)$ and C_{2n} exhibits the CSP.

The set $\mathcal{C}_n$ of chains in the uncrossing poset P_n could be an interesting object that we may observe the CSP.

Problem 4.33. Let $\mathcal{C}_n$ be the set of chains in the uncrossing poset P_n. Find a polynomial $f_n(q)$ such that $\mathcal{C}_n$ together with $f_n(q)$ and C_{2n} exhibits the CSP.

We may also investigate the *homomesy* phenomenon defined by Propp and Roby (2015), which is another interesting example of dynamical algebraic combinatorics. See also the surveys by Roby (2016) and by Striker (2017) for more recent works and examples.

Definition 4.34 (Propp and Roby (2015))**.** A group action on a set of combinatorial objects, along with a statistic, exhibits homomesy if the average of the statistic on each orbit is the same as the average of the statistic over the whole set.

Problem 4.35. For a group action on the set P_n of matchings, not necessarily cyclic shifts, find a statistic such that the average value of the statistic on every orbit is the same as the average of the statistic over P_n.

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APPENDIX A

SAGE CODE FOR THE CSP ON MATCHINGS

We wrote code in SAGE (Stein *et al.* (2016)) to compute the number of elements of P_n fixed by c^d with respect to the number of crossings. The following is the code for $n = 7$.

```
n=7
a=range(2*n)
b=range(2*n)
c=range(2*n)
d=range(2*n)
D=matrix(2*n, binomial(n,2)+1)
M=PerfectMatchings(2*n)
m=factorial(2*n)/(factorial(n)*2^n)
for i in range(2*n):
    d[i]=0
for i in range(2*n):
    if i==2*n-1:
        a[i]=1
    else:
        a[i]=i+2
for i in range(2*n):
    if i==0:
        b[i]=Permutation(a)
    else:
        b[i]=b[i-1].left_action_product(b[0])
for i in range(2*n):
    if i==0:
        c[i]=b[2*n-1]
    else:
        c[i]=b[i-1]
for i in range(2*n):
    for k in range(m):
      if M[k].conjugate_by_permutation(c[i])==M[k]:
        D[i,M[k].number_of_crossings()]=D[i,M[k].number_of_cro
         ssings()]+1
D
```

APPENDIX B

THE SAGE DATA

We present the data we obtained. Let A_n be the $2n \times (\binom{n}{2} + 1)$ matrix whose (i, j) entry $A_n(i, j)$ is the number of $(j-1)$-crossing elements in P_n fixed by c^{i-1}. For $2 \leq n \leq 6$ we present the matrices A_n.

$$A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 5 & 6 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 3 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 14 & 28 & 28 & 20 & 10 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 4 & 4 & 4 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 42 & 120 & 180 & 195 & 165 & 117 & 70 & 35 & 15 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 10 & 0 & 20 & 5 & 15 & 5 & 10 & 5 & 5 & 5 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 132 & 495 & 990 & 1430 & 1650 & 1617 & 1386 & 1056 & 726 & 451 & 252 & 126 & 56 & 21 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 1 \\ 6 & 0 & 0 & 8 & 0 & 0 & 6 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 20 & 15 & 30 & 30 & 30 & 45 & 26 & 36 & 18 & 27 & 12 & 18 & 8 & 9 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 8 & 0 & 0 & 6 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$