Perturbing Practices: A Case Study of the Effects of Virtual Manipulatives as Novel Didactic Objects on Rational Function Instruction

by

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ABSTRACT

The advancement of technology has substantively changed the practices of numerous professions, including teaching. When an instructor first adopts a new technology, established classroom practices are perturbed. These perturbations can have positive and negative, large or small, and long- or short-term effects on instructors' abilities to teach mathematical concepts with the new technology. Therefore, in order to better understand teaching with technology, we need to take a closer look at the adoption of new technology in a mathematics classroom. Using interviews and classroom observations, I explored perturbations in mathematical classroom practices as an instructor implemented virtual manipulatives as novel didactic objects in rational function instruction. In particular, the instructor used didactic objects that were designed to lay the foundation for developing a conceptual understanding of rational functions through the coordination of relative size of the value of the numerator in terms of the value of the denominator. The results are organized according to a taxonomy that captures leader actions, communication, expectations of technology, roles, timing, student engagement, and mathematical conceptions.

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PROBLEM STATEMENT

The addition of anything new or different in a classroom setting inevitably causes perturbations of existing practices. The term perturbation describes a disruption that causes a system to attempt to regain equilibrium. For example, a pebble dropped into a glass of water causes a perturbation since it disrupts the surface of the water but laws of physics dictate that the water will attempt to revert back to its original state prior to the pebble being introduced. This dissertation describes an investigation of the perturbations that occur in a mathematics classroom when novel virtual manipulatives functioning as didactic objects are implemented as part of instruction. These particular virtual manipulatives were designed to help students build a conceptual understanding of rational functions.

Technological advancements significantly change the practices and routines found in numerous professions. When a company adopts a new technology, employees experience perturbations in the existing practices. The perturbations in practice can have positive or negative, large or small, and short- or long-term effects on employees' abilities to accomplish their work. Examples of the perturbations in practice caused by the adoption of new technology are found in emergency rooms where new medical equipment has been implemented (Edmondson, Bohmer, & Pisano, 2001) and on labor floors that have introduced new machines (Pickering, 1995).

Mathematics education is another profession in which new technology is being regularly adopted (Pope, 2013). This new technology comes in many forms such as hardware (e.g. iPads, laptops, interactive whiteboards, document cameras, projectors), software (e. g. Graphing Calculator, Maplesoft, Geometer's Sketchpad), and educational website licenses (e.g. Nearpod, MyBigCampus, Khan Academy, GeoGebra). The goal associated with the implementation of new technology in instruction is to facilitate instruction and improve student achievement and understanding. However, in order to achieve this goal, we need to better understand the process of adopting new technology in instruction. In particular, we need to account for instructors' current mathematical meanings of concepts, the perturbations experienced by instructors when implementing a new technology, and the effect these perturbations have on the instruction of mathematical concepts.

Virtual manipulatives are one type of technology that can be used to restructure student understanding of mathematical concepts, provided they function as didactic objects, i.e., are accompanied by reflective mathematical discourse (Thompson, 2002). In particular, a virtual manipulative that encourages students to conceptualize a rational function by comparing the behavior of the numerator in terms of the denominator can draw on schemes of relative size, along with covariational reasoning, to build an understanding that connects with previous understandings of fractions and division. A meaningful understanding of rational functions, in turn, can be a stepping stone to a deep understanding of quotient of functions and limits (Yerushalmy, 1997).

This dissertation study was designed to provide insight into the perturbations that occur in established mathematical classroom practices when novel didactic objects that use technology are integrated in instruction by a novice instructor. The study was designed to collect information on an instructor's current understandings of rational functions, mathematical classroom practices, and support the instructor in developing meanings for rational functions that are useful for implementing the novel virtual manipulative in instruction as a didactic object. The study then identified perturbations in mathematical classroom practices that occurred when the didactic object was used by a novice instructor for the first time in rational function instruction.

Statement of the Problem

The integration of technology in multiple professions has been problematic (Ertmer, 1999; Iansiti, 1995), and mathematics education is a profession in which new technological devices are being regularly introduced with the goals of improving student engagement, achievement, and understanding. Virtual manipulatives that function as didactic objects are one type of technology that has the potential to promote deeper and more coherent understandings of mathematical concepts. However, in order to harness this potential, we need to explore the relationship between instructors' technological pedagogical content knowledge and their mathematical classroom practices when a novel virtual manipulative is introduced into instruction as a didactic object.

Research Questions

This study identified the perturbations that occur in a novice instructor's mathematical meanings and classroom practices when he attempted to use a new approach to teaching rational functions for which he had already established activity structures (Leinhardt, Weidman, & Hammond, 1987), and the new approach employed technology in a central way. The study answered the following questions:

- In what ways do novel virtual manipulatives that are used as didactic objects perturb a novice instructor's existing mathematical classroom practices?

More specifically,

- What characteristics, other than amount of teaching experience, classify an instructor as a novice? Are there aspects of planning a lesson, teaching a lesson, and reflecting on a lesson that differentiate novice from experienced instructors?
- How does a novice instructor perceive a novel virtual manipulative that functions as a didactic object for teaching rational functions, both mathematically and as an instructional tool?
- What are the differences between a novice instructor's image of the meanings for rational functions that students might develop from the novel didactic object and how the instruction fosters these meanings?

Outline of the Study

In this study I uncovered perturbations in mathematical classroom practices when a novice instructor introduced virtual manipulatives as didactic objects in rational function instruction. The identification of perturbations in mathematical classroom practices assisted me in generating a framework that I hope to develop further in the future. This framework could aid professional development leaders at all levels of education to support mathematics instructors when blending technology with established practices.

In order to identify perturbations in practice, I developed a two-phase study that analyzed mathematical classroom practices both prior to and subsequent to the introduction of virtual manipulatives as novel didactic objects. (For the sake of parsimony, I refer to "virtual manipulatives used as novel didactic objects" as simply "novel didactic objects.") The didactic objects I choose to explore promote a conceptual way of understanding rational functions in pre-calculus instruction. By conducting semi-structured, task-based interviews, observing classroom instruction, and capturing the perspective of the instructors based on journal entries and stimulated recall of their instruction, I identified these instructors' mathematical meanings for rational functions and classroom practices associated with their rational function instruction. I then tracked perturbations in mathematical classroom practices that resulted from the use of the novel didactic objects by the novice instructor when he taught rational functions.

Chapter 2 provides a review of the literature including perturbations of practices found in industry, technological pedagogical content knowledge, as well as technology and mathematical classroom practices. Chapter 3 describes the theoretical perspective I used throughout the study. This chapter contains a conceptual analysis of rational functions built on two major schemes, namely relative size and covariational reasoning. Chapter 4 outlines the phases of the study and details the methods of data collection and data analysis I employed. Chapter 5 characterizes a novice instructor through the presentation of the findings from the first phase of data collection. Chapter 6 presents the perturbations in practice experienced by the novice instructor when implementing the novel virtual manipulatives as didactic objects into his rational function instruction. Chapter 7 provides a discussion of the findings from the data and the implications that these findings have for the mathematics education research community.

CHAPTER 2

LITERATURE REVIEW

In this chapter I provide a review of the literature that explores disruptions of practice found in industry, technology in mathematics education, mathematical classroom practices, potential perturbations in mathematics classrooms, and the Technological Pedagogical Content Knowledge (TPACK) framework.

Perturbations of Practice in Industry

Every profession has practices or agreed upon ways of conducting business, such as achieving goals, communicating, and roles when working together. The introduction of novel technology has the potential to disrupt these practices in big and small ways, for a short term or long term period, and in predictable or unpredictable ways. In industry, disruptions of new technology have been studied in the context of medical procedures in a hospital and production lines on a labor floor.

Edmondson, Bohmer, and Pisano (2001) studied the disruptions in routines that occurred when minimally invasive cardiac surgery (MICS) equipment was introduced into cardiac surgery. The authors identified three activity structures that suffered disruptions, namely authority structure, psychological safety, and team stability. The authority structure delineates the chain of command so that responsibility and accountability are established. Because the authority structure is the underlying frame for activity, disrupting the authority structure has residual effects on other activity structures. Psychological safety refers to the belief that well-intentioned interpersonal risks will not be punished. When technology implementation occurs there is a trial and error period where individuals need to feel comfortable asking questions. Team stability is reflected in the ways in which individual actions and abilities are coordinated so that routines emerge. This stability can be disrupted when new technology is implemented since new individual actions have to be coordinated so that “teamwork” can run as smoothly as previously.

When MICS equipment was introduced into surgery, the authority structure was radically changed; the role of the surgeon changed from order giver to team member and the role of the nurses changed from order takers to information providers as the once quiet operating room now required a constant stream of communication between all team members. In effect, the surgical team went through the process of learning a new kind of teamwork that was significantly different from their established practices for traditional surgery.

Another example of technology-induced disruptions in practice was found on the labor floors at General Electric when numerically controlled machines were introduced. Pickering (1995) described the adoption of numerically controlled (N/C) machine tools by General Electric’s (GE) Aero Engine Group in the early 1960’s, which caused multiple disruptions in established practices on the labor floor and within management. “Instead of requiring detailed human control, N/C equipment was controlled by digital computers executing instructions compiled by programmers; and as corollary, shop-floor labor was reduced to human button pushers. Or so, at least, it worked out in the Servomechanisms Lab at MIT” (Pickering, 1995, p.159). When implementing N/C equipment on the labor floor at GE the disruptions to existing practices were emergent (Pickering, 1995). In other words, none of the disruptions were considered before they occurred. During the original pilot program of N/C equipment, GE management did not take the time to assess the responsibilities of the workers who were manning the equipment. There was no thought to whether the workers were button pushers or responsible for the success of the equipment. Management's lack of forethought on responsibilities of operators led to disruptions within the working arrangements and functions of management.

GE's implementation of N/C equipment on the labor floor changed the roles of management and the roles of N/C operators. GE management's role began as a boss dictating changes and forcing implementation with no consideration to opinions of the shop-floor workers. After the accommodation of job enrichment, management's role changed to a monitor or partner in production practices, valuing the opinions of the N/C operators. Functions of management changed to accommodate N/C operators' opinions on how the N/C equipment was functioning and what changes needed to be made for the equipment to be more effective. This is a stark contrast to the mindless button pushers they originally wanted. GE management did not know all the answers when implementing technology on the labor floor, but they were willing to correct the emergent disruptions that occurred.

In both of these cases, when a novel innovative technology was introduced into established practices, disruptions followed. Table 1 summarizes, describes and provides examples of the aspects of practice that were disrupted when novel technology was introduced in professions outside of mathematics education.

Table 1. *Aspects of Practice Perturbed in Professions outside of Mathematics Education*

Aspects of practice	Description	Example
Leader Actions	Leader's interpretation of the technology and how the leader implements the technology	Edmondson et al. (2001) demonstrated how the surgeon's beliefs in the technology were correlated with how the ER team adapted to the technology.
Communication	The discourse and environment	In Edmondson et al. (2001), the discourse in the ER changed from the surgeon being the only speaker to every member of the team needing to communicate.
Expectations of Technology	Predicted outcomes for the implementation process	In Pickering (1995), prior to implementation GE management expected the technology to increase production.
Roles and Responsibilities	The individual's original responsibilities are altered during the implementation process	In Pickering (1995), the role of workers evolved from button pushers to integral members in the success of the machines.

These cases can be used as a lens for viewing the current technology integration in mathematics classrooms. School districts purchase technology for use in mathematics classrooms without considering the perturbations in established mathematical practices that the mathematics instructors might face when implementing the novel technology.

Technology in Education

The type of technology available and applicable in mathematics instruction has grown significantly in the past two decades (Ozel, Yetkiner, & Capraro, 2008). These new technologies include calculators (graphing calculators especially), interactive whiteboards (SMART, Star boards, etc.), immediate response devices, computers, software, and web-based applications (Kaput, 1992; Pope 2013; Ozel et al., 2008). As such technologies have become more common in classrooms, the role that technology plays in instructor knowledge has also developed.

Technological Pedagogical Content Knowledge

Technology is now part of the framework describing what it takes to teach. The Technological Pedagogical Content Knowledge (TPACK) framework is an extension of a previous framework, that is, as shown in Figure 1, depicted by some researchers as a Venn diagram (Koehler & Mishra, 2009). The previous framework included only content knowledge and pedagogical knowledge (Shulman, 1986; 1987) since technology was not as prevalent three decades ago, and these were thought to be the two major types of knowledge required to support effective teaching.

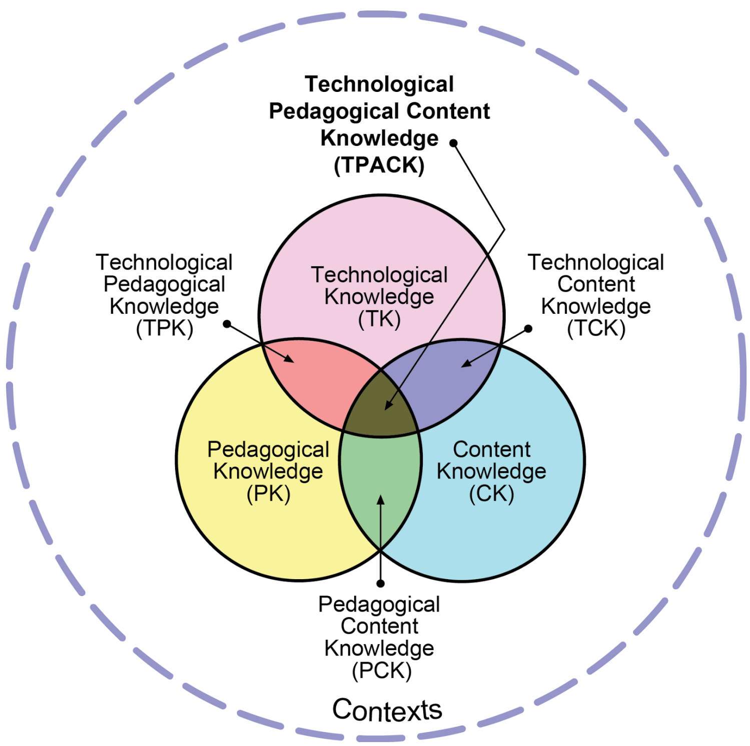

Figure 1. Technological pedagogical content knowledge framework

Content knowledge, exactly as it sounds, is the knowledge of a content area. Instructors are considered content specialists and use their knowledge of a particular subject to educate students. Content knowledge includes knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge (Koehler & Mishra, 2009; Shulman, 1986). For example, a mathematics instructor must have a repertoire of mathematical understandings, as well as the ability to see the interconnectedness of mathematical concepts. Instructors who do not have a comprehensive base of content knowledge can confuse students, cause students to learn content incorrectly, and perhaps even deter students from pursuing a certain content area (Hill, Rowan, & Ball, 2005).

However, although necessary, mathematical content knowledge is not a sufficient condition for being an effective instructor; *pedagogical knowledge* is also required. Pedagogical knowledge refers to understanding how students learn, lesson planning, assessment, and classroom management skills. These aspects of teaching might seem routine, but without pedagogical knowledge instructors cannot impart their content knowledge. An instructor needs to have firm grasp on their students' abilities, on how best to present the content, on the sequencing of the content, and on the current state of their students' understanding. At the same time, an instructor must know how to maintain a safe and productive learning environment so that all students feel comfortable participating.

In addition to content knowledge and pedagogical knowledge, there are aspects of teaching that fall within the intersection of these types of knowledge, known as *pedagogical content knowledge*. Shulman (1986; 1987) describes pedagogical content knowledge as the transformation that occurs when the instructor interprets the subject matter, finds ways to represent the subject matter, and adapts the instructional materials to alternative conceptions and students' prior knowledge. Instructors of mathematics must combine their knowledge of mathematics with their pedagogical knowledge in order to attempt to build mathematical understandings within their students. Mathematical knowledge for teaching is one of the major attempts to expand Shulman's (1986; 1987) types of knowledge for teaching that is tailored to mathematics education.

In order to understand how an instructor envisions presenting a mathematical concept, it is important to look at the knowledge a mathematics instructor employs to build those presentations, namely the blend of mathematical knowledge and pedagogical knowledge. This mathematical knowledge for teaching is an essential part of an instructor's ability to teach effectively and is a source for decisions instructors make in their classroom instruction (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). On a deeper level, instructors are creating classroom practices through their understandings of the mathematics and pedagogical knowledge. These classroom practices are further developed when the instructor begins to form an image of what the lesson will look like when implemented.

"Instructors' mathematical meanings constitute their images of the mathematics they teach and intend that students have" (Thompson, 2016, p.437). An instructor's image of the mathematics has a powerful impact on lesson preparation and instruction of a mathematical concept. Instructors' ways of thinking affects what instructors wish students to learn, what actions instructors take, what instructors teach, and how instructors influence student understandings (Thompson & Thompson, 1996).

Were it not for technology, it would be sufficient for an instructor to have strong content knowledge, pedagogical knowledge, and pedagogical content knowledge. However, the increasing presence of technology in classrooms has changed the responsibilities of instructors. Thus, the TPACK framework includes *technological knowledge*, along with content knowledge and pedagogical knowledge, in order to describe how instructor understanding of educational technologies interacts with these other types of knowledge to support effective teaching (Mishra & Koehler, 2006; Koehler & Mishra, 2008; Koehler & Mishra, 2009).

What is technological knowledge? Defining technological knowledge is difficult since any definition would become outdated with the addition of new technologies. Thus, Koehler & Mishra (2009) define technological knowledge as ways of thinking about and working with technology that can be applied to all technology tools and resources. However, technological knowledge is not independent of the content and pedagogical knowledge necessary for effective teaching.

Instructors must navigate the intersection of technological and content knowledge known as *technological content knowledge*. Technological content knowledge is an understanding of the manner in which technology and content influence and constrain one another. This definition suggests that instructors must have a deep understanding of the manner in which the subject matter, in this case mathematics, can be changed by the application of particular technologies. For instance, Dr. Thompson created the virtual manipulatives used in this study to illustrate the conceptual meanings behind rational functions.

An additional intersection of knowledge exists between technological and pedagogical knowledge known as *technological pedagogical knowledge*. Technological pedagogical knowledge is an understanding of how teaching and learning can change when specific technologies are used in particular ways. Instructors need a deeper understanding of the constraints and affordances of technologies and the pedagogical contexts within which they function. For example, instructors may choose to frequently use overhead projectors based on their belief that it is their job as the teacher to hold students' attention so that they can learn (Stigler & Hiebert, 1999).

The critical intersection of technological knowledge with content knowledge and pedagogical knowledge is *technological pedagogical content knowledge* (TPACK). "TPACK is the basis of effective teaching with technology, requiring an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge to develop new epistemologies or strengthen old ones" (Koehler & Mishra, 2009, p.66). In this way, TPACK reflects all aspects of the expertise need to teach content with technology, something that is relevant for instructors today.

The TPACK framework has encouraged further research in instructor education, instructor professional development, and instructor use of technology (Archambault & Crippen, 2009; Cox & Graham, 2009; Schmidt et al., 2009). This framework allows "instructors, researchers, and instructor educators to move beyond oversimplified approaches that treat technology as an "add-on," instead to focus again, and in a more ecological way, upon the connections among technology, content, and pedagogy as they play out in classroom contexts" (Koehler & Mishra, 2009, p.67). In sum, TPACK illustrates how technology is intended to be an integral, rather than supplemental, part of mathematical classroom practices.

This framework begins a discussion on characterizing teaching expertise. Many studies define an instructor expertise in terms of the years of service in the classroom or student achievement (Borko & Livingston, 1989; Leinhardt & Greeno, 1986). However, years of teaching experience does not capture the characteristics of an expert instructor since time is not the only variable when looking at a classroom environment and student learning (Hogan, Rabinowitz, & Craven, 2003). A more robust measure of expertise for an instructor captures the extent of content and pedagogical knowledge, deep pedagogical content knowledge, and sophisticated technological pedagogical content knowledge.

Mathematical Classroom Practices

Every discipline and profession has practices which are established, maintained, and changed over time. Some researchers use the term community of practice to describe a group of people in which practices evolve naturally over time since the members in the community share a common interest (Lave & Wenger, 1991). Communities of practice rely on a process of sharing with the group to personally and professionally develop the members of the community. There are many different types of communities of practice, both informal and formal, and one type that is of interest to educational researchers exists with the purpose of gaining knowledge related to a specific field, such as mathematics

(Dube, Bourhis, & Jacob, 2005; Kietzmann et al., 2013). Mathematics classrooms can therefore be considered as communities of practice, with influences from both mathematical and classroom practices. In this way, as seen in Figure 2, mathematical classroom practices can be thought of as a blend of mathematical practices (stemming from mathematics as a discipline) and classroom practices (associated with the learning environment).

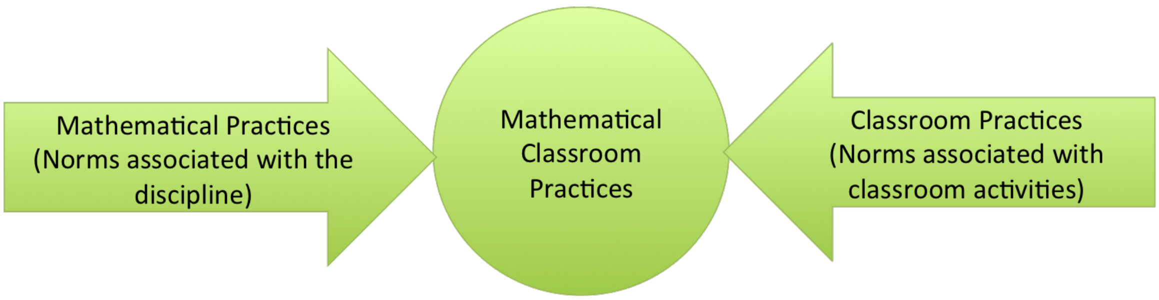

Figure 2. Diagram of mathematical classroom practice

Mathematical practices. In the field of mathematics, there exists a strong community of practice that mathematicians through the centuries have built. This community of practice has shared methods for solving problems and ways of articulating ideas and results to others in their field. The mathematical practices that have evolved in this community of practice have been applied in varying degrees in elementary, secondary, and post-secondary education. For instance, disciplinary engagement refers to the ways in which students in a classroom enact the practices held by the discipline that they are studying (Engle & Conant, 2002).

More formally, mathematical standards of educational policy can stem from practices that characterize accepted mathematical disciplinary practices. Thus, the current Common Core Standards for Mathematical Practice (Table 2) are based on problem solving, reasoning, proof, communication, representation, and connections as a basis for describing how students should engage in mathematical activity (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). These eight core practices represent "varieties of expertise that mathematics educators at all levels should seek to develop in their students" (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). Although these practices are intended for guiding mathematics classroom activity, it is important to note that mathematical practices do not exist solely in mathematics classrooms. For example, during a science lab students might graph the results of an experiment using a Cartesian plane. This would require decisions about how to construct the representation in a way that communicates relevant information to others (e.g., deciding on units and scale). In this example, the science lab is an opportunity for students to utilize mathematical practices outside of a mathematics classroom.

Table 2. *Common core standards of mathematical practices*

Common Core Standards of Mathematical Practice
1. Make sense of problems
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

These eight core practices are broad and describe how the authors of the Common Core Standards want students to engage in mathematical activity. Researchers have also studied the specific practices that mathematicians adopt while solving problems. For example, Carlson and Bloom (2005) investigated mathematicians solving problems and identified a multidimensional framework that describes the interplay between four problem-solving phases (orienting, planning, executing, and checking) and four problem-solving attributes (resources, heuristics, affect, and monitoring). This research suggests that students need a large number of reasoning patterns, knowledge, and behaviors at their disposal to be effective problem solvers. These practices include management of both resources and emotional responses that crop up during the problem-solving process.

Classroom practices. In addition to disciplinary practices, mathematical classroom practices are also shaped by the nature of the activity and the environment in which they take place. Classroom practices consist of the activity structures (Leinhardt, Weidman, & Hammond, 1987), rules of discourse, and management of the classroom. Activity structures are goal-directed segments of instructor and student behavior than involve certain actions. Classroom practices are the configuration of activity structures that are played out by routines that exist in a particular classroom. An example of an activity structure would be an instructor of a course starting every class session with a problem that is designed to review past material or to introduce the day's lesson.

One arena that is of particular interest to educational researchers is the set of classroom practices associated with rules of discourse since speech essentially unites the cognitive and social (Barnes, 1974 as cited in Cazden, 1988). These practices focus on the communication and interactions that occur in a classroom, specifically how participants voice and exchange ideas as they engage in constructing mathematical meanings. S*peaking with meaning* is a prime example of a rule of discourse because it emphasizes the importance of communicating meaningfully with others (Clark, Moore, & Carlson, 2008). This classroom practice requires both the instructor and students to hold one another accountable for articulating ideas and processes in ways that can be understood by others in the classroom. Although developed in the context of a mathematics classroom, speaking with meaning is a practice that also applies to other subject areas and learning environments.

Similarly, Hackenberg (2010) discusses student and instructor interactions as a linked chain of bearable perturbations that she calls mathematical care relationships (MCRs) but that are applicable to other subject areas as well. MCRs measure the quality of the interaction between a student and instructor in both affective and cognitive realms with the goal of increasing mathematical learning (Hackenberg, 2005). MCRs require a two way street of information and participation between the instructor and student. The instructor needs to place herself in the mindset of the student. This act of decentering helps the instructor understand the student's mathematical meanings and prepare discussions (with tasks) to build on those meanings. In turn, the student must feel comfortable to freely communicate with the instructor. Thus, MCRs emerge through the coordination and effort of both parties.

Establishing, Maintaining, and Changing Mathematical Classroom Practices

In order to better understand how students learn mathematics, mathematics education researchers not only study the practices that exist in mathematics classrooms, but also how they develop and change over time (cf. Cobb & Yackel, 1996; Hackenberg, 2005; Hackenberg, 2010). One perspective looks at classroom communities in general so that practices exist prior to and independently of instructors and their students (Cobb, Stephan, McClain, & Gravemeijer, 2011). Accordingly, there is a prescribed way of reasoning and communicating into which instructors and their students are expected to conform (Cobb et al., 2011). However, an alternative to this sociocultural perspective is to look at a particular classroom community and analyze the ways in which the instructor and his or her students construct the mathematical classroom practices through interaction. Cobb and Yackel (1996) developed a sociological perspective on mathematical activity that involved identifying both social norms and socio-mathematical norms within a mathematics classroom. Social norms involve beliefs about the roles and the general nature of mathematical activity in school. Socio-mathematical norms encompass the community's agreed upon process for conducting mathematical activities (e.g. what counts as a different, sophisticated, efficient, and acceptable mathematical solution). In this approach, the authors found that both the social and socio-mathematical norms were created through negotiation between the instructor and students. This negotiation relied on both parties working together to establish norms in the classroom. Thus, mathematical classroom practices can be seen as an emergent phenomenon, resulting from a collective effort on the part of an instructor and students.

Establishing practices. Practices are established when ways of reasoning, arguing, and symbolizing are accepted by the majority (Cobb et al., 2011). It is important to note that the reasoning, arguing, and symbolizing occur within the context and discussion of a particular mathematical idea. These established practices emerge as a reorganization of prior practices held by the instructor and his or her students. In other words, neither instructors nor their students are blank slates (Steffe & Thompson, 2000), so the establishment of practices in a learning environment requires participants to discard, reorganize, and supplement practices that they have previously established.

Maintaining practices. Maintaining mathematical classroom practices relies on consistency and reflection (Tabulawa, 1997). Instructors cannot simply fix their mathematical classroom practices and stick to them throughout the duration of their career (Pickering, 1995). In fact, taking the approach of a local classroom with emergent practices it would be expected that each year a fine-tuning must be made to accommodate the new set of students and the practices that emerge. Continual reinforcement and revision of mathematical classroom practices, by the instructor and her students, through tasks, activities, and discussion is necessary to keep emergent practices honed. In this case, maintaining practices can be seen as a form of housekeeping where one is trying to remove the layer of dust from objects. This maintenance amplifies or bolsters the mathematical classroom practices. In contrast, changing practices requires reorganization. In this case, instead of simply dusting, objects are removed, added, or replaced.

Changing practices. Changing practices requires a modification, if not complete reorganization, of the ways participants engage in activity. Regarding mathematical classroom practices, curricula, resources, and technology can all provoke change (Duffy & Roehler, 1986; Richardson, 1990). Appleton (2008) found instructors given a new curriculum experienced uncomfortable changes in their practices. Some instructors even chose to avoid the new curriculum and just continued to teach the old curriculum. However, the instructors who attempted to adopt the new curriculum had a transformation occur in their practices, prompted by new ways of thinking. Instructors' ways of thinking can be perturbed and modified, and this, in turn, affects their classroom practices.

In addition to definitive events that evoke changes in practices (such as the adoption of a new curriculum), the passage of time can also be thought of as an instigator of continual changes in practices. Pickering (1995) suggests that practices or goals are *temporally emergent* and can be transformed in real-time, which includes encounters with material agency. Temporal emergence refers to the idea that the disruptions caused by material agency, or non-human agents, are never decisively known. This means that there is a constant expectation that practices will have to be changed continuously. Pickering (1995) explains that performativity of new machines must be found out in real time, just as human practices used in conjunction with the machine can only become known as they emerge over time.

Mathematical classroom practices are a blend of the norms associated with the discipline of mathematics and the norms associated with classroom activities and instruction. These mathematical classroom practices are established and maintained over time. However, similar to the impact of new curriculum on mathematical classroom practices, novel technology can perturb and change the established mathematical classroom practices.

Technology and Mathematical Classroom Practices

Technology is an umbrella term that is used to describe or refer to any modern electronic device and engineering innovation. In a mathematics classroom, technology can take the form of calculators, interactive whiteboards, immediate response devices, computers, software, and web-based applications (Kaput, 1992; Ozel et al., 2008; Pope, 2013). Each of these different technologies has the potential to impact learning in different ways.

Regardless of how sophisticated or innovative the technology is, though, its very presence in a mathematics classroom will not necessarily impact mathematical classroom practices. For example, placing a projector into a mathematics classroom where the instructor proceeds to present PowerPoint slides with same information she would have written on the board does not change the mathematical classroom practices of the instructor. Instead, this instructor has experienced a shift in pedagogical knowledge which now incorporates technology. In terms of the TPACK framework, the change is localized to the intersection of just technological and pedagogical knowledge. In order to meaningfully affect instructors' mathematical classroom practices, the novel technology must push the instructors to make changes within their technological pedagogical content knowledge, the very center of the TPACK framework. Didactic objects that incorporate technology are one example of an educational resource that has the potential to impact mathematical classroom practices and prompt instructors to employ and coordinate technological knowledge, pedagogical knowledge, and content knowledge.

Didactic Objects

Thompson (2002) defines *didactic objects* as tools or objects that are created with the intent of supporting reflective mathematical discourse (p.198) and considers them to have two components: first, the object itself, and, second, the classroom discussion that the instructor designs to engage students in constructing shared mathematical understandings. For example, base ten blocks (or the virtual equivalent) together with a plan for guiding a conceptual understanding of place value would constitute a didactic object. It is important to note that designing the plan for guiding a conceptual understanding should include questions that prompt a discussion. These questions that are included in the plan should be conceptual rather than calculational (Thompson, Phillip, Thompson, & Boyd, 1994). Questions that are calculational elicit and solidify answer-getting behaviors and the idea that only one process or answer exists. Conceptual questions that prompt deeper thinking and multiple answers will cultivate stronger meanings, as well as a conceptual understanding. In this way, the goal of didactic objects is to promote ways of thinking that go beyond what is actually present in the discussion surrounding the object. Ideally the discussion that the instructor leads will help students take the conversation and extend the ideas so that, instead of creating islands of knowledge, the instructors are helping students build bridges that support more coherent and connected ways of thinking.

Design of didactic objects. Thompson (2002) points out that objects cannot be didactic in and of themselves. “Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such” (Thompson, 2002, p.198). Thus, the design of didactic objects requires the creator to consider the two components of didactic objects, the object and the discussion, equally.

Design of the object. The object used in conjunction with the discussion to make up the didactic object can be anything. This wide interpretation of “object” gives the creator a limitless perspective on what type of object can be implemented. However, the creator must be aware of why she has selected a specific object and how the object will assist students in developing conceptual understandings of mathematics. For example, if the goal is to develop a stronger understanding of adding polynomials (by stronger, I am referring to more than “combining like terms” when given the two function rules), the object might be designed to display the graphs of two polynomials that are not easily identified, so the students cannot fall back on algebraic manipulation.

Design of the discussion. Designing the discussion to accompany an object is not an easy task. Since the object has such a wide interpretation, it is even more important for the discussion to have a tight focus. For example, there is a big difference between telling students to "graph or represent a situation" that is being depicted (e.g., in an applet) and directing their attention to relevant quantities and covariational relationships. For instance, if an instructor were to use an applet depicting a bottle filling with water, with the intention of having the students construct a height-volume graph of the amount of water in the bottle, the discussion should be designed in such a way as to help students to identify the height of the water in the bottle and the volume of the water in the bottle as quantities that are changing over time and to attend to how the change in these quantities is affected by the shape of the bottle. Virtual manipulatives, such as applets, can be considered as a technologically enhanced form of physical manipulatives, such as algebra tiles or unit cubes (Moyer, Bolyard, & Spikell, 2002). The combination of a thought-provoking discussion and a virtual manipulative can be considered as a didactic object (Thompson, 2002). When used as didactic objects, virtual manipulatives can demonstrate attributes and behaviors of advanced mathematical concepts in readily available and flexible ways.

Virtual Manipulatives

Manipulatives are physical objects or concrete models that can be touched and moved around by the learner (Durmus & Karakirik, 2006; Heddens, 2005). In mathematics instruction, manipulatives afford opportunities for learners to interact with abstract mathematical concepts and procedures through visualization and movement. For example, base ten blocks (cubes and bars representing 1's and 10's place value) allow learners to explore the concept of place value and mathematical operations through activities such as physically configuring the blocks and combining them in various ways (Moyer et al., 2002).

However, we now recognize that the benefits of using manipulatives do not necessarily require the sense of touch, e.g., moving around physical objects, and this has led to the creation of a new class of computer-based manipulatives (Durmus & Karakirik, 2006; Moyer et al., 2002; Moyer-Packenham, Sallkind, & Boylard, 2008). Moyer et al. (2002) define a virtual manipulative as a "web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p.373). Thus, a computer program that allows students to select, configure, and combine representations of base ten blocks using keystrokes or by moving a hand control (i.e. computer mouse) would be considered a virtual manipulative.

Design and impact of virtual manipulatives. Virtual manipulative design is flexible and there are multiple software and web-based programs that can be used to create virtual manipulatives (i.e. Graphing Calculator, GeoGebra, Geometers Sketchpad, etc.). This flexibility allows the creator to design virtual manipulatives with verbal (i.e. letters, numbers) and visual (i.e. pictures, animations) information codes that can be presented separately or simultaneously (Moyer-Packenham et al., 2008). Decisions regarding how best to present information have been informed by theories that are based on information processing models of cognition, such as Dual Coding Theory (Clark & Paivio, 1991). According to this theory, presenting visual and verbal codes in tandem will have a positive effect on student recall since the information can be processed and encoded in two complementary ways. The contiguity principle (which states that presenting words and pictures together temporally or spatially) is now recognized as one of the basic principles of effective multimedia instructional design (Mayer & Anderson, 1992).

However, just as with any other instructional materials, there are factors beyond design considerations that contribute to the effectiveness of virtual manipulatives. The very presence of virtual manipulatives in mathematics instruction does not guarantee that students will develop solid conceptual understandings. For instance, students might fail to link the actions made with a manipulative and the underlying intended mathematical understanding (Clements & McMillen, 1996). Thus, additional support might be necessary to foster these meaningful connections.

Multimedia in the classroom. Although research on instructional multimedia predominantly uses short segments of material or content (e.g. a few minutes or less) (Mayer & Anderson, 1992) and is conducted in a laboratory setting (e.g. using human subjects versus authentic students), there is some research exploring virtual manipulatives in a classroom setting. The results of some such studies indicate that the use of virtual manipulatives in the classroom results in higher learning gains through the additional visualization opportunities (Bolyard, 2006; Moyer et al. 2002; Yerushalmy, 1997). In addition to the gains made by students in mathematics when taught with virtual manipulatives, there appears to be an increase in student engagement, including on task behavior (Drickey, 2000). The use of virtual manipulatives creates a dynamic learning environment (Durmus & Karakirik, 2006; Moyer-Packenham et al., 2008) by allowing educators and students to explore mathematical concepts more deeply. When this happens, the role of the instructor becomes less of a “sage on the stage,” and more of a discussion leader, or “guide on the side” (King, 1993).

Virtual Manipulatives as Didactic Objects

Research on the use of objects (physical or virtual) in mathematics instruction has traditionally focused solely on how the tool itself supports student learning and understanding in terms of cognition (Spicer, 2000). However, there has been a shift to expand the focus beyond the object itself to include the accompanying discussion (Bowers, Bezuk, & Aguilar, 2011). As previously discussed, didactic objects require more than just an object. Didactic objects are the combination of two components, the object and the discussion. The design of both components relies on the creator making the effort to keep the mathematical understanding she would like to co-construct with her students in the forefront.

Virtual manipulatives vs. didactic objects. In sum, the term ‘didactic objects’ refers to a much broader category of instructional resources, and some virtual manipulatives might be used as didactic objects. However, not all virtual manipulatives are didactic objects since virtual manipulatives, in and of themselves, do not require conversation between the students or with the instructor. In fact, many virtual manipulatives can be seen in online education platforms where there are instructional prompts that guide the online learner to see the indicated pieces and features of the manipulative.

Impact of technology on mathematical classroom practices. Instructors have established practices of teaching that they must modify in various way to incorporate technology. Mathematical classroom practices provide the instructor with stability and consistency on a daily basis (Tabulawa, 1997). These established mathematical classroom practices are deeply seeded in instructors, which make altering those practices challenging. However, mathematical classroom practices provide researchers an opportunity to analyze the effect that a new technology, such as virtual manipulatives, has on those practices.

According to Sherman (2014), cognitive technology can serve as either an amplifier or as a reorganizer of mental activity. Cognitive technologies assist in influencing thought and learning by transcending the limitations of the mind in thinking and problem-solving activities. If a technology can more efficiently perform a tedious process that is usually done by hand, then the technology can be labeled as an amplifier. An example of an amplifier technology is a calculator since the technology offers faster computational results that would take longer if done by hand. Technology designed to shift the focus of students' mathematical thinking or activity is classified as a reorganizer. Reorganizing technology can take the form of novel representations, which draw attention to salient aspects of a mathematical concept that might have been difficult to see without the technology. An example of a reorganizer technology can be found when answering questions surrounding the relationship between two quantities such as the height of the water in a bottle and the volume of water in the bottle (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). There are a number of web-based or software based technologies that can produce a graph relating the two quantities and demonstrate the changes in the two quantities as more water is added. Using technological tools to demonstrate and represent the changes in the two quantities, students are able to focus on the interpretation within the context of the problem posed, in this case the relationship that exists between the height of the water in the bottle and the volume of the water in the bottle. Reorganizing technologies of this type allow students to focus on using the representations rather than producing them (Greeno & Hall, 1997).

The addition of reorganizing technology in mathematics education allows for curriculum to be restructured, giving priority to a new set of skills and abilities (Heid, 1988; Palmiter, 1991; Schwarz & Hershkowitz, 1999). Restructuring curriculum, however, is often met with resistance since the change impacts instructors' established mathematical classroom practices. Technology designed as a reorganizer can also affect the structure of the learning environment, such as a flipped classroom model (Gannod, Burge, & Helmick, 2008; Prober & Heath, 2012). In a flipped classroom the instruction is provided in the form of video lectures to be watched outside of class. During class, students work in groups to discuss problems or engage in projects. A flipped classroom models how drastically instructors' mathematical classroom practices can be altered by technology. The impact of technology on mathematical classroom practices requires instructors to rethink, and in some cases abandon, old practices for new ones.

Instructors make pedagogical decisions by drawing on their knowledge of and ideas about mathematics (Thompson, 1984; Silverman & Thompson, 2008). This is why mathematics instructors might have to transform their ways of thinking and teaching to include new resources causing instructors to try a new approach. This new approach to teaching requires additional ways of thinking, such as how to use the technology and what benefit the technology will have on student understanding. The instructor might even need to change the ways of thinking that she has about the mathematical concept that is to be taught with the new resource (Silverman & Thompson, 2008). However, if mathematics instructors do not make the necessary changes to their ways of thinking, integrating the new resource in their mathematical classroom practices could cause additional perturbations.

Potential Perturbations

Instructors need to understand the mathematics they teach differently than they currently understand it in order to use reorganizing technology effectively. Traditionally, a mathematics classroom has an instructor, who leads and develops instruction based on his or her content and pedagogical knowledge. In such a classroom, the instructor, as seen by the students, is the authority in the classroom and the mathematics expert. The role of students is to take notes and complete tasks given by the instructor throughout the lesson. Communication between instructor and student in this type of learning environment is minimal, with only occasional questions that typically follow an Initiation-Response-Evaluation pattern (Cazden, 1988; Mehan, 1979). However, a new approach with reorganizing technology disrupts the traditional mathematics classroom practices by changing lesson planning, structure, and communication.

Mathematics instructors must transform their ways of thinking to include the new approach. A new approach with reorganizing technology requires additional ways of thinking; such as how to use the technology and what benefit will the technology have on student understandings. However, if mathematics instructors do not make the necessary changes to their ways of thinking, integrating technology in their mathematical classroom practice could cause perturbations. “If students are equipped with tools that allow them to make and explore conjectures, then it is likely that they will rapidly press the edges of the instructor's knowledge of the subject" (Schwartz, 1999, p.113). This is a legitimate fear for mathematics instructors because they will need to actively expand their mathematical knowledge. Cooney and Wilson (1995) suggest that instructors make decisions on what resources to use in a lesson by considering how smooth the lesson will run. When students push the boundaries of instructors' mathematical meanings, instructors might become uncomfortable and choose to abandon the approach entirely.

Instructors need to rethink their presentation of the mathematics concepts so that the presentation reflects their new understandings of the mathematics they want students to learn. Based on the instructors' new understandings of the mathematics, classroom structure will need to change from direct instruction and note taking, to discussions on what the technology is demonstrating mathematically. These discussions require instructors to develop ideas of student responses and understandings prior to the implementation of the new approach. Discussions force both instructor and student to participate as equals, listening carefully and interpreting understandings. The new approach with technology changes the roles of the instructor and students. Mathematics instructors are still the authority in the classroom but their roles transform from dictators of instruction to discussion facilitators. The role of the student changes from mindless note taker to discussion participant.

The adoption of new approaches, involving technology, to mathematics teaching and learning creates disruptions within established classroom practices. However, the classroom environment, instructor actions, and stability of the mathematics classroom, between instructor and student, can determine whether the new approach with technology is successful.

Instructors need to rethink the activities they will arrange for students so that they are coherent with the new understandings that they (the instructors) present and are coherent with the new forms of activities that the new technology makes possible. Ozel et al. (2008) believe effective implementation of technology augments the learning of every student by providing diversity in instructional models, developing a student-centered learning environment, and restructuring the teaching and learning process to make it intellectually rigorous (p.82).

Edmondson et al. (2001) found similar characteristics of effective implementation of technology in hospitals. The authors "take the perspective that when new technology disrupts existing work routines, the adopting organization must go through a learning process, making cognitive, interpersonal, and organizational adjustments that allow new routines to become ongoing practice" (Edmondson et al., 2001, p.686).

Similar to the approach of Appleton (2008), mentoring is a viable way to battle the low self-confidence mathematics instructors' experience when integrating technology into current mathematical classroom practices. Mentoring could evoke changes in the mathematical practices of an instructor by assessing the instructor's mathematical meanings. After assessment, mentor and instructor collaborate on a new approach with technology. The mentor guides and discusses the potential uses of the technology while assessing any perturbations that occur in the instructor's mathematical meanings. The mathematics instructor then teaches the new approach with technology to students as the mentor looks for perturbations in existing mathematical classroom practices. A reflection on the new approach with technology is made at the completion of the lesson to discuss the effectiveness of the approach.

It does not make sense to study the intersection without including the mathematical concept and the instructor understanding of the mathematical concept. You cannot study the perturbations in practice that occur when a novel didactic object is integrated in instruction if you do not know the mathematical meanings the instructor possesses for the mathematical concept that is to be taught. In the following chapter, I present my theoretical perspective that includes a conceptual analysis of the mathematical concept I selected, namely rational functions.

Perturbations in practice in industry due to the introductions of novel technology have been researched in fields such as emergency rooms and labor floors. However, there is limited research on the perturbations in practice in mathematics classrooms when novel technology is integrated in instruction. I explored the perturbations in mathematical classroom practices that occurred in the context of rational functions, which represented a concept taught with a conceptual approach using novel didactic objects.

CHAPTER 3

THEORETICAL PERSPECTIVE

This chapter presents the theoretical perspective for the dissertation study I conducted. I start with a description of how rational functions are traditionally taught as the backdrop to an alternative conceptual understanding. Then I provide a brief discussion of the central constructs of Piaget's (1971) genetic epistemology in order to describe how this alternative conceptual understanding of rational functions might develop. Finally, I conclude this chapter with an interpretation of rational functions and their characteristics based on the conceptual analysis I have created.

Traditional Teaching of Rational Functions

Individuals are first introduced to rational functions in Intermediate Algebra, a course that is usually taken in high school. After being taught how to combine functions by adding, subtracting, and taking the product of given polynomials, students are introduced to rational functions as the result of dividing two polynomials. It is in this context that the issue of domain can be problematized since the resulting function can be discontinuous at certain input values. In particular, the graph of a rational function can contain holes (e.g. at input values for which both the function in the numerator and the denominator equal zero), or vertical asymptotes (at input values for which the function in the denominator, but not the numerator, equals zero), or both. For example, as seen in Figure 3, $h(x)=\dfrac{2}{x+2}$ has a vertical asymptote at x = -2.

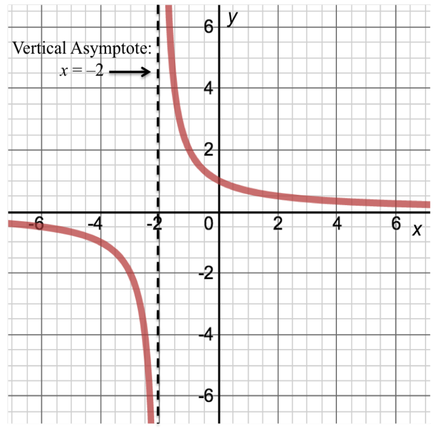

Figure 3. The graph of $h(x) = \dfrac{2}{x+2}$.

Traditionally students are taught how to algebraically simplify the rule of a rational function and then to find the vertical asymptotes by setting the denominator equal to zero. However, this calculational orientation (Thompson et al., 1994) does not provide students with a conceptual understanding of how rational functions behave. In particular, simply setting the denominator equal to zero does not capture the covariational relationship that exists between the two polynomials that make up the rational function.

As the polynomial in the denominator of an algebraically simplified rational function is getting closer to zero for some input value, the entire rational function will increase (or decrease) without bound, resulting in a vertical asymptote. This issue sets the stage for the adoption of a conceptual analysis that encourages individuals to explore rational functions more dynamically and to construct an understanding of the covariational relationship that exists when two functions are combined through division.

Conceptually, rational functions can be considered as the "fractions of algebra." Just as a fraction is the ratio of two integers, a rational function is the ratio of two polynomials (Tussy & Gustafson, 2008). Because the focus of this analysis is a *ratio* relationship between two *functions*, relative size and covariational thinking can be used as foundational unifying constructs. A conceptual analysis considers the ways in which an individual may understand a mathematical concept, such as rational functions (Glasersfeld, 1995; Thompson, 2008). My conceptual analysis describes ways of thinking about relative size, together with covariational reasoning to construct a coherent understanding of rational function. This conceptual analysis generates models of thinking that will lend themselves to the ways of thinking about rational functions that I have identified as being useful. (However, I am not claiming that my conceptual analysis covers every detail involved in an individual understanding rational functions.) In this conceptual analysis I also introduce ways of thinking that will assist individuals in developing a coherent scheme of meanings that would constitute a powerful understanding of rational functions.

Piagetian Constructs

In this study I constructed and analyzed ways of thinking about relative size and covariational reasoning as a bridge to connect understandings of fractions and rational functions. In order to discuss how new understandings develop and relate to previous understandings and to better articulate what it means to have a conceptual understanding of rational functions, I drew on Piaget's (1977) ideas regarding schemes, assimilation, and accommodation (Montangero & Maurice-Naville, 1997).

Schemes. A scheme, which describes how information is encoded and retrieved, can be considered as a mental structure or organization of actions which supports flexible thinking and is repeatable (Piaget, 1954, 1972; Monatangero & Maurice-Naville, 1997).

Here, *action* does not necessarily refer to an experience outside of the body that can be observed by another, but can also extend to the context of the mind, e.g., *mental actions*. Thus, *action* encompasses "all movement, all thought, or all emotions that respond to a need" (Piaget, 1968, p.6 as cited in Thompson, Carlson, Byerley, & Hatfield, 2014, p.10).

Piaget's use of scheme allowed him to address the mental organizations that support flexible thinking and reasoning without emphasis on the contents of the mental organizations (Thompson et al., 2014). Although there is no way of truly knowing what another person's schemes actually consist of or what the person knows, these organizations of mental activity, schemes, can be expressed through behavior; an observer of the behavior can attempt to discern meanings and ways of thinking of the individual, thus supporting a useful model of student understanding. Gaining understanding, or learning, can be thought of as the result of existing schemes being augmented or new schemes developing.

Assimilation and accommodation. In some cases, new information can find a home in an existing scheme. *Assimilation* is the process of thoughts or cognitive information "fitting" into a current scheme. Initially, assimilation occurs when the individual mentally identifies features of a stimulus that might match an existing scheme. However, beyond the process of identification, assimilation involves making inferences. "Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complex system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference" (Piaget, 1958, p.59 cited in Montangero & Maurice-Naville, 1997). Thus, assimilation occurs when the mind recognizes and interprets a stimulus, such as thoughts or cognitive information, and assigns the stimulus to an existing scheme.

Accommodation, on the other hand, is the process that occurs when an individual is unable to assimilate a stimulus, such as thoughts or cognitive information, within a current scheme. “Accommodation is a source of change, whereas assimilations guarantees the conservation of a system” (Montangero & Maurice-Naville, 1997, p.65). An accommodation occurs if a modification is made to an existing scheme, or if a new scheme is created. Another way accommodation can be triggered is when an individual assimilates a stimulus into a scheme but the actual result does not “fit,” or is inconsistent, with the anticipated result.

I used these Piagetian constructs to describe schemes, assimilations, and accommodations that might occur during the interviews I conducted. This Piagetian lens assisted me in identifying generalizations made by the subjects in my study with respect to the ways of thinking about rational functions. In particular, the approach I took with rational functions drew on a scheme of relative size, together with covariational reasoning. At the same time, however, because my subjects were graduate teaching assistants in the school of mathematical and statistical sciences, who had already learned about rational functions, I expected that they would attempt to assimilate the new instructional materials into their existing rational function scheme. I also expected the graduate teaching assistants to experience disequilibrium when they could not assimilate the new instructional materials into their existing schemes for rational functions which might lead to potential accommodations.

Rational Function Scheme

A conceptual understanding of rational functions can be built on existing schemes of relative size, together with covariational thinking. First, relative size lays the groundwork for understanding ratio and division in terms of arithmetic quantities. Covariational thinking then broadens this conceptual understanding so that it can be applied algebraically to rational functions.

Relative size. A major theme in mathematics concerns the relationships between various quantities, where a quantity is the measurable attribute of an object (Carlson, Oehrtman, & Moore, 2015). For example, the size of one quantity can be thought of, or measured, in terms of the size of another quantity. In this case the relationship between the two quantities is one of relative size. For example, when comparing the relative height of an elephant (approximately 120 inches) and a mouse (approximately 1.25 inches), the height of the elephant could be measured in units that correspond to the height of the mouse. So, if a stack of 96 mice reaches to the top of the elephant, then the relative size (in this case, height) of the elephant to a mouse is 96.

In the case of a multiplicative comparison, the resulting quantity is called a *ratio*. If the quantities are being compared according to the same attribute (e.g., height), then the ratio of the two represents the answer to the question: "How many times as large is one quantity in terms of another quantity?" In the previous example, we could say that the ratio of an elephant to a mouse is 96 since an elephant is 96 times as large (tall) as the mouse. The use of the phrase, "times as large," here reflects the multiplicative nature of the comparison. Of course, multiplicative comparisons can also be made using numbers. For example, the multiplicative comparison of 3 to 4 leads to a ratio of ¾, indicating that the measure of 3 in terms of 4 is ¾. The equivalent statement framed multiplicatively would be 3 is ¾ times as large as 4.

Notice that, in both of these cases, we are measuring something, i.e., the size of the elephant or the size of 3. Indeed, the idea of ratio is at the very heart of measurement (Thompson & Saldanha, 2003,p.109). Conceptualizing measurement requires an individual to imagine a ratio relationship (i.e. Quantity A is some number times as large as Quantity B) that is invariant across changes in the unit of measure (Thompson & Saldanha, 2003, p.110*)*. An individual's ability to understand that unit substitution does not change the magnitude of a quantity (amount) is a conceptual breakthrough that Wildi (1991) emphasized by distinguishing between the measure of a quantity and the magnitude of a quantity. Thus, a quantity's magnitude is independent of the unit in which the quantity is measured, even though the quantity's measure will be different (Thompson & Saldanha, 2003). In other words, the choice of a comparison quantity does not change the magnitude of a quantity. In the elephant example, the size of the elephant is still the same, regardless if we compare it with a mouse or a house although it has a different measure in the two cases.

Division is a mathematical operation that is built on the concept of ratio. In this case the two associated quantities are the numerator and the denominator, and the quotient is the result of expressing the numerator in terms of the denominator. Using the vocabulary associated with division, the quotient is therefore the dividend measured in units of the divisor. Conceptualizing division in this way (as opposed to the use of a grouping metaphor) represents an opportunity to think about what it means to express one quantity in terms of another. For example, if 21 is divided by 2, then 21 can be thought of as 21/2 times as large as 2, or, equivalently, it means that 21 measured in units of 2 is 21/2.

This way of thinking focuses attention on the role that units play when constructing and interpreting the result of dividing two quantities. In particular, because the dividend is being measured in units of the divisor, the quotient is measured relative to the divisor (Coughlin, 2010). In the previous example, 21 divided by 2 = 10 ½ means that 10 ½ times the quantity 2 equals 21. What happens, though, if the quantities involved in division are not whole numbers, but fractions instead? In this case, the multiplicative comparison no longer uses whole numbers. However, the dividend is still measured in terms of the divisor. For example, ¾ divided by ½ is 1 ½ meaning ¾ is 1 ½ times as large as ½.

Despite the fact that there is no difference in the relationship between the dividend and the divisor when fractions, instead of whole numbers, are involved, such instances often lead to confusion. Research has shown that individuals have difficulty interpreting the result of fraction division, namely using the unit of the divisor when the result is not a whole number (Coughlin, 2010). One hypothesis is that this issue stems from the fact that the commonly used procedural rule of inverting and multiplying obscures the role of the divisor and, consequently, the meaning of division. In addition to the interpretation of results, the construction of problem situations involving fraction division is also challenging. Ball (1990) found that mathematics majors at the college level struggled when asked to propose situations where dividing by fractions would be appropriate.

Relative size thus lays the arithmetic foundations for understanding rational functions. But, since rational functions are *ratios of polynomial functions*, covariational reasoning is also at play. In particular, the numerator and denominator of a rational function are both functions, and the resulting ratio is also a function. Therefore, the covariational relationship between the input quantity and the output quantity of each of these components must also be considered and coordinated.

Covariational Reasoning

The type of covariational reasoning needed to understand rational functions is layered. Rational functions require an individual to construct an image of the changing value of the polynomial function in the numerator with respect to the varying input value, as well as an image of the changing value of the polynomial function in the denominator with respect to the same varying input value. This imagery is foundational to a conceptual understanding of function (Carlson et al., 2015). As an additional layer, a higher order of thinking must then be applied in order to construct an image of the value of the polynomial function in the numerator in terms of the value of the polynomial function in the denominator as a single quantity. This image is powerful because it allows the individual to construct a rational function without the need for a function rule.

There are two distinct ways of thinking about function, *function as process* and *functions as covariation* (Thompson, 1994). Both ways of thinking, function as process or function as covariation, rely on the individual thinking quantitatively. Reasoning quantitatively about a concept suggests that the individual perceives a situation in terms of quantities (Thompson, 1989), where a quantity is an attribute of an object that is measureable (Carlson et al., 2015).

Function as process suggests that individuals see a function as an input-output process, where, for example, an input value is placed into a function machine and the output value emerges out the other side (Clement, 2001; Eisenberg, 1991). This type of thinking does not support individuals in visualizing a smooth continuous string of input values with a corresponding continuous string of output values. Function as process constructs a chunky image of a set of input values with corresponding set of output values. For example, if an individual views the function $g(x) = 3x - 4$ as a process of substitution of the value of x which will yield a result for $g(x)$, then this individual will most likely not think about how varying x affects the output values.

An individual with function as process ways of thinking can build to thinking about function as covariation. Function as covariation suggests that the individual can imagine the values of two quantities varying in tandem (Saldanha & Thompson, 1998). Covariational reasoning requires an individual to think about how the value represented by a variable changes with respect to the value represented by a second variable. Individuals with covariational ways of thinking have a smoother image of function. Rather than thinking of functions as machines, individuals can coordinate the smooth varying values of one quantity represented by a variable and the corresponding smooth varying values of a second quantity represented by a different variable.

An individual must have an understanding of function notation in order to construct coherent meanings of rational functions. The individual should understand that function notation such as $f(x) = 2x^2$ represents the covariational relationship between the dependent and independent quantities. However, simply being able to rewrite a function with an equal sign and a function rule does not imply that the individual understands function or function notation. There are nuances of function notation that cannot be taken for granted; such as *f* is the name of the function and not an additional variable (Mohr, 2008)

Relative Size Coordinated with Covariational Thinking

A conceptual understanding of rational functions requires an individual to coordinate the co-varying relationship of the polynomial in the numerator (the input quantity and the output quantity of the polynomial) and the polynomial in the denominator (the input quantity and the output quantity of the polynomial). In order to see this relationship the individual must see the value of the numerator and the value of the denominator as two individual quantities while also seeing the *relative size* of the output quantity of the numerator in terms of the output quantity of the denominator as a single quantity. In other words, there are three *covariational relationships* that the individual must coordinate and seamlessly transition between (See Figure 4).

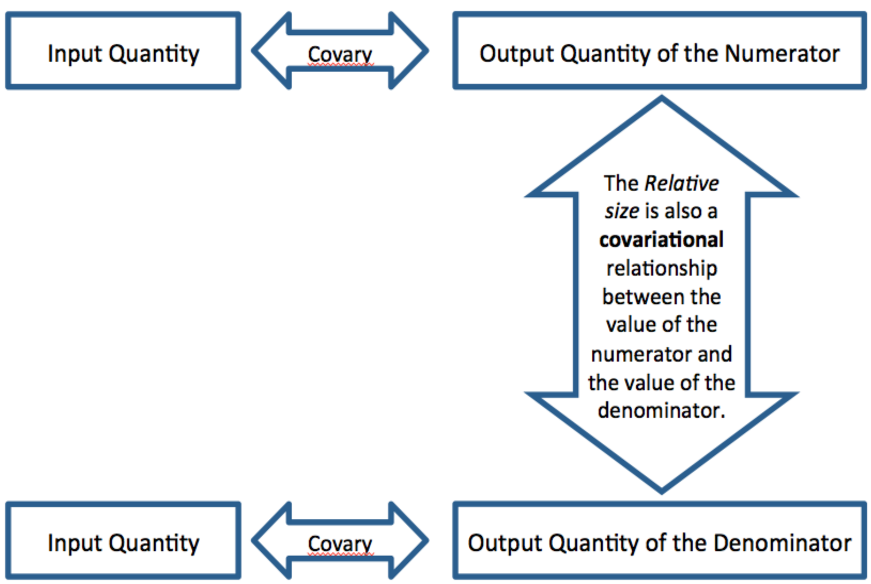

Figure 4. Covariational relationships within a rational function

The relative size of one quantity (Quantity A) to another quantity (Quantity B) is a quantity that is constructed from the ratio of the value of Quantity A in terms of the value of Quantity B. In the context of rational functions, the value of the numerator can be considered as Quantity A and the value of the denominator can be seen as a second quantity, Quantity B, for a given value of the input. The relative size of the value of the numerator of the rational function in terms of the value of the denominator of the rational function can then be constructed as a third quantity (Quantity C) for a given value of the input. Extending this idea to multiple values of the input over the domain of the rational function, this conceptual orientation to rational functions also relies on the coordination of the covariation of the value of the numerator and the value of the denominator as the input value varies. This means that, as the input value changes, the value of the numerator will change, the value of the denominator will change, and the resulting value of the relative size of the value of the numerator measured in terms of the value of the denominator will change.

Understanding rational functions based on a scheme of relative size coordinated with covariational reasoning also leads to specific ways of interpreting features of rational functions, such as asymptotes. By interpreting a rational function as a relationship between quantities, it is also possible to draw conclusions about the behavior of the function and its graphical representation that are not driven by algebraic manipulations.

Conceptual Interpretation of Rational Functions

A conceptual approach to rational functions requires that an individual attend to the behavior of the covariational relationship between the input value and output value of the function in the numerator and denominator, both individually and as a single quantity. This means that vertical asymptotes, holes, and horizontal asymptotes of rational functions can be described by considering the behavior of the ratio of the two polynomials as the input value changes.

Vertical asymptotes and holes. Let us consider an example as the basis of a discussion on the behavior of rational functions around vertical asymptotes and holes. Figure 5 provides the graphical representation of the function in the numerator of a rational function (on the left) and the function in the denominator of a rational function (on the right).

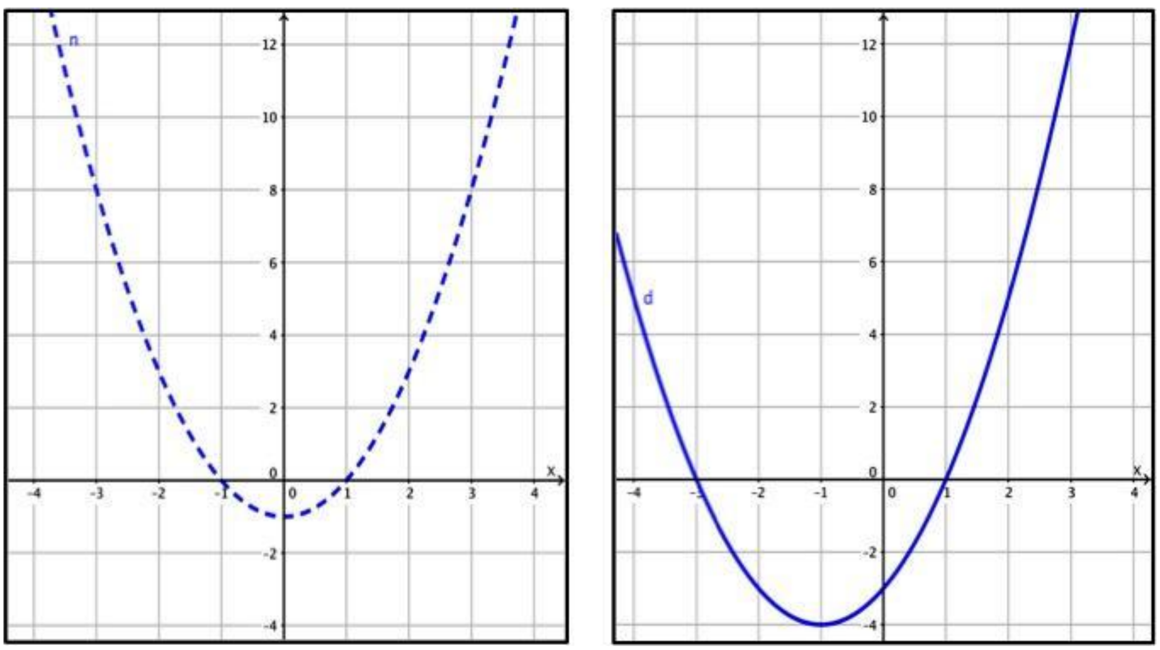

Figure 5. Graph of the Function in the numerator (left) and the denominator (right).

Notice that this example does not provide the rule of the rational function as is usually found in curriculum, so the focus can be on the behavior rather than algebraic manipulation. In particular, we want to analyze the behavior of the two quantities around values of the input for which the denominator approaches zero. Consider the value of the numerator, as the value of the input increases to -3; notice that the output value of the numerator approaches a positive number, approximately 8. By looking at the graph on the right, we see that, as the value of the input increases to -3, the value of the denominator is decreasing to zero. The fact that the value of the numerator approaches a *positive* number and the value of the denominator approaches zero *from above*, means that the relative size of the value of the numerator in terms of the value of the denominator is increasing without bound. This behavior suggests that a vertical asymptote exists at the input value of -3.

We can also analyze the behavior of the rational function as the input value decreases to -3, namely as the value of the input approaches -3 from the right hand side. As shown in Figure 5, when the value of the input decreases to -3, the value of the numerator approaches a positive number, approximately 8, and the value of the denominator increases to zero. The fact that the value of the numerator approaches a *positive* number and the value of the denominator approaches zero *from below*, means that the relative size of the value of the numerator in terms of the value of the denominator is decreasing without bound.

Using a similar analysis of behavior, we can also characterize holes in a rational function. As shown in Figure 5, the denominator approaches zero at the input value of 1. As the value of the input increases to 1, the value of the numerator approaches zero, and the value of the denominator approaches zero, as well. The fact that *both* the value of the numerator and the denominator approach zero as the input value approaches 1 means that the graph of the rational function has a hole at this input value.

Horizontal Asymptotes. Determining horizontal asymptotes requires an analysis of the end behavior of a rational function. In other words, one must consider the value of the numerator and the value of the denominator as the input quantity increases and decreases without bound. In order to do this, an individual must imagine that the static image of the function extends beyond the edge of the depicted graph. What happens to the relative size of these values? If the relative size "stabilizes" to a certain value, then this value represents a horizontal asymptote of the rational function. Otherwise, the rational function does not have a horizontal asymptote.

Notice that this conceptual approach provides a much richer picture of rational functions. Finally, this approach gives meaning behind the value of the horizontal asymptote that goes beyond computing the ratio of the leading coefficients.

Importance of a Conceptual Approach to Rational Functions

Although I have chosen to work with rational functions, the same conceptual understanding built on schemes of relative size, together with covariational reasoning, can be applied to quotients of functions, in general. Individuals still need to coordinate the covariational relationship between the input quantity and the output quantity of the function in the numerator and denominator as well as the relative size of the value of the numerator in terms of the value of the denominator. Individuals who have a conceptual understanding of the behavior of rational functions (e.g. vertical and horizontal asymptotes) have a better platform for calculus concepts, such as limits. The concept of limits requires understandings of many of the same ideas, including infinity, dividing by zero, zero divided by zero, and asymptotes (Hitt & Lara, 1999). Thus, rational functions, if taught conceptually, can help individuals form these understandings and images prior to entering calculus.

In this chapter I presented the theoretical perspective for the dissertation study I conducted. I described how rational functions are traditionally taught as a backdrop to an alternative conceptual understanding. Then I provided a brief discussion of the central constructs of Piaget's (1971) genetic epistemology in order to describe how this alternative conceptual understanding of rational functions might develop. Finally, I concluded this chapter with an interpretation of rational functions that is based on the conceptual analysis I created. In the next chapter, I provide a methodology for this dissertation study that characterized a novice instructor and investigated perturbations in mathematical classroom practices when a novice instructor teaching pre-calculus introduced novel didactic objects into his rational function instruction.

CHAPTER 4

METHODS

In this chapter I describe the methods I used in this study to characterize a novice instructor and to investigate perturbations in mathematical classroom practices when a novice instructor teaching a pre-calculus course introduced novel didactic objects in his rational function instruction. I begin this chapter with an overview of the study I designed, followed by a discussion of the research activities within each phase of the study. Finally, I close this chapter with details on how I analyzed the data.

Overview of Study

In this study I first characterized what it means to be a novice instructor and then investigated the perturbations that occurred when a novice instructor introduced novel didactic objects into his lessons for rational functions. The study consisted of two phases of data collection (See Figure 6). In Phase 1, the pre-intervention semester, a novice instructor and an experienced instructor participated in a pre-interview, classroom observations, and a post-interview. During Phase 2, the intervention semester, the novice instructor participated in two pre-interviews, classroom observations, and a post-interview.

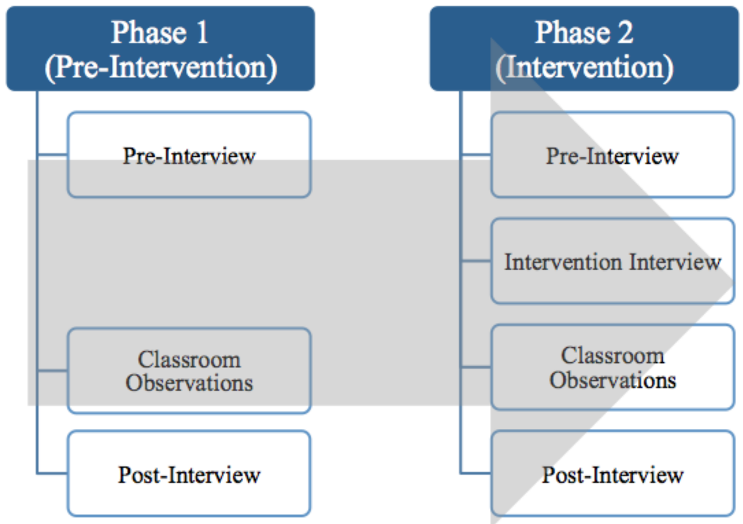

Figure 6. Outline of the Study

The interviews were semi-structured, task-based interviews (Goldin, 2000), approximately two hours in length. The tasks were designed to serve three purposes: first, to characterize the instructor's meanings and ways of thinking about rational functions; second, to give insight into the instructor's current mathematical classroom practices; and third, to introduce an instructor to novel didactic objects that were designed to foster a conceptual understanding of rational functions. All interviews were recorded using a camera focused on the instructor, a second camera focused on the written artifacts produced by the instructor (first pre-interview) and the computer screen on which the novel didactic object (intervention interview) or the video clips (post-interview) were displayed.

The interviews allowed me to differentiate between an experienced and novice instructor and to generate hypotheses regarding the effect of novel didactic objects on the novice instructor's classroom instruction and mathematical meanings for rational functions. Studies with a generative purpose interpret data through a larger lens unlike studies with a convergent interpretation. The generative approach to data analysis allows the researcher to discern new models and theories from the data (Clement, 2000). The goal of this study was to generate theory about perturbations in a novice instructor's mathematical classroom practices that occurred when novel didactic objects are implemented in rational function instruction.

The classroom observations in this study were designed to provide a detailed account of the goings on while the instructor taught rational functions with and without the novel didactic objects. I conducted one set of classroom observations in each phase of the study. Each class session was approximately 50 minutes in length. Every class session in which the instructor taught rational functions was recorded. A video camera was placed in the back of the classroom to record the instructor's discourse and gestures, as well as the projector screen. Recording both discourse and gesture provides a well-rounded, integrated picture of learning as an interactional, versus mental, phenomenon (Koschmann & LeBaron, 2002).

Role of researcher. My role in this study differed depending on the research activity. During some of the interviews, I acted as an investigator by asking the instructor to complete tasks that helped me to identify his mathematical meanings relevant to understanding rational functions and inquiring into his classroom practices. During the intervention interview, I again asked the instructor to complete tasks that gave me insight into the mathematical meanings he had for rational functions, but then I took on more of an instructor role as I introduced the novice instructor to the novel didactic objects that he would later use in his rational function instruction. The didactic objects were novel to the instructor who had no prior knowledge of the didactic objects used in this study. Finally, during the classroom observations, my role was that of observer. The purpose of the observer was "to identify and account for aspects of a culture by analyzing regularities and patterns that arise as, say, a instructor and students interact during mathematics instruction" (Cobb, 1989, p.33).

The participants. The participants in this study were two graduate teaching assistants (instructors) teaching MAT170: Pathways Pre-Calculus (Carlson et al., 2015) at a large public university in the southwest United States in the Fall of 2016 and Spring of 2017. The participants were selected from a pool of volunteers who previously taught the course and a pool of volunteers who were teaching the course for the first time. Both pools of volunteers were enrolled in a teaching seminar that assisted instructors in developing mathematical meanings and instruction. The participants signed a consent form prior to participating in the study (see Appendix A).

Norbert (the novice instructor) was a full-time graduate student that was working as a graduate teaching assistant (GTA) in the fall of 2016 at a large public university in the southwest United States. He was enrolled in a Ph.D. program for pure mathematics. Fall 2016 was Norbert's first semester as a full-time graduate student and first semester teaching Pathways Pre-Calculus. Norbert had no teaching experience prior to this semester.

Edwin (the experienced instructor) was a full-time graduate student that was working as a GTA in the fall of 2016 at a large public university in the southwest United States. He was enrolled in a Ph.D. program for mathematics education. It is important to note that this was his first year as a mathematics education Ph.D. student. In the previous two years he was enrolled in a Ph.D. program for applied mathematics. Fall 2016 was

Edwin's fifth semester as a full-time graduate student and third semester teaching Pathway's Pre-Calculus. Edwin had no teaching experience prior to becoming a graduate student.

Research Phases

There were two phases to this study: a pre-intervention phase and an intervention phase. The pre-intervention phase was designed to showcase the differences between a novice instructor and an experienced instructor as they planned, taught, and reflected on their rational function lessons. The intervention phase contained a series of research activities that allowed me to explore the perturbations in the novice instructor's mathematical classroom practices that occurred when novel didactic objects were introduced into rational function instruction (See Figure 7). Both phases consisted of interviews and classroom observations, but Phase 2 included an additional pre-interview during which the novice instructor was introduced to the novel didactic objects.

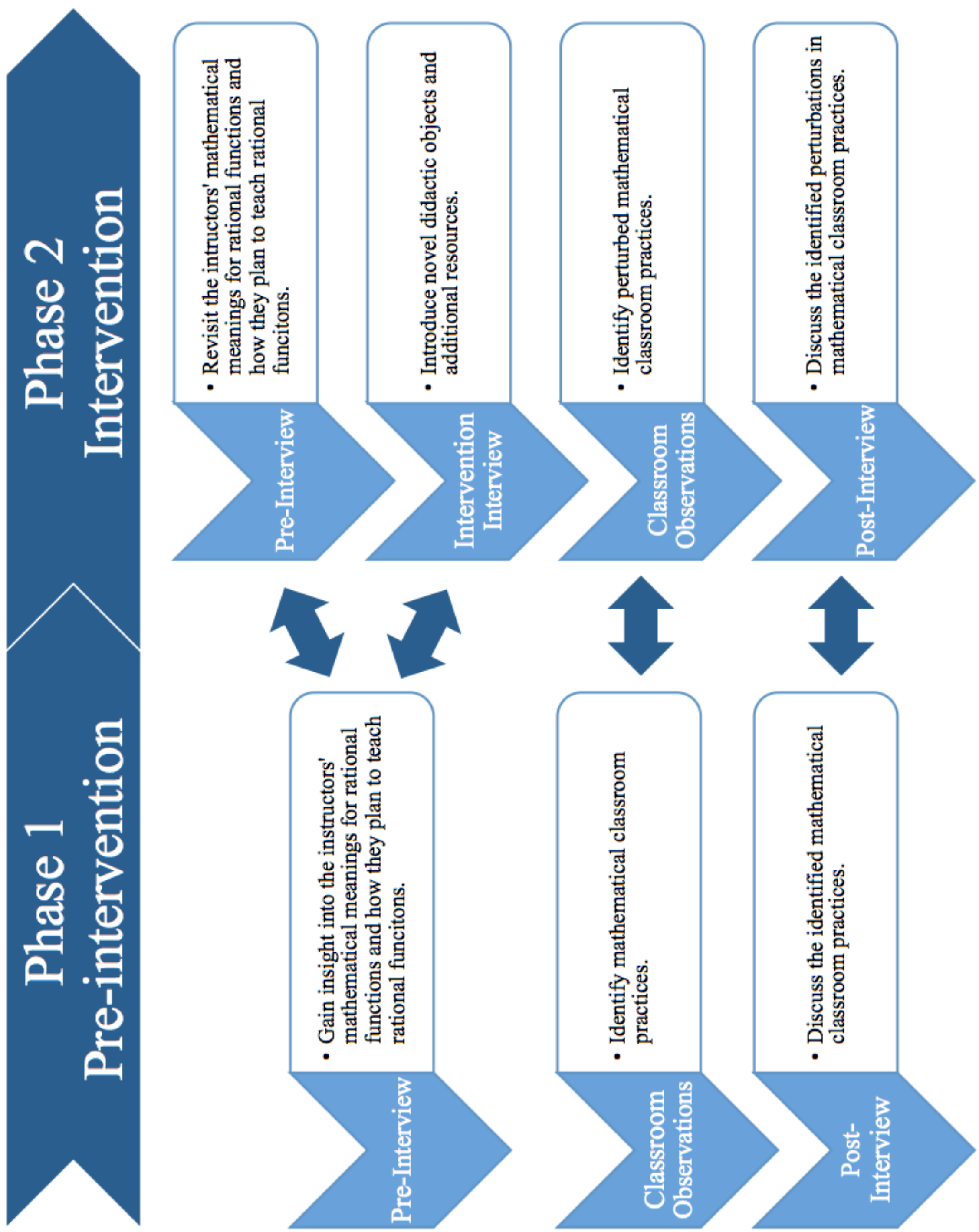

Figure 7. Outline of study, including goals of each activity.

Phase 1 (Pre-Intervention). Phase 1 was designed with the goal of characterizing a novice instructor while also providing a picture of the novice instructors' meanings for rational functions and existing mathematical classroom practices surrounding rational function instruction. First, I needed to characterize what it means to be a novice instructor. I could have used time in the classroom as a measure of the instructor's novice status, but this is not adequate since teaching experience is not sufficient, in and of itself, to capture what it means to be a novice. There are new-to-the-job instructors who have extensive pedagogical content knowledge and there are instructors with many years of teaching experience who have very little pedagogical content knowledge (Berliner, 2001; Glaser, 1987; 1990). Therefore, I chose to conduct interviews with both an instructor who had no previous experience teaching Pathways and a more seasoned instructor to see if I could flesh out what differentiated them in terms of how they thought about rational functions. Toward this end, I conducted a pre-interview, classroom observations, and a post-interview in Phase 1 using two participants (a new-to-Pathways instructor and a seasoned instructor).

Pre-Interview. During the pre-interview, I attempted to gain insight into the instructors' current understandings of rational functions and gather descriptions of the instructors' mathematical classroom practices prior to the introduction to the novel didactic objects. For example, I was interested in whether they have a more calculational or conceptual orientation to instruction (Thompson et al., 1994) on rational functions and, accordingly, how they go about teaching rational functions to their students.

The Phase 1 pre-interview incorporated tasks that targeted the instructors' mathematical meanings of rational functions. For this purpose, I designed interview tasks to probe the instructors' problem-solving approaches and schemes of rational functions. In particular, the tasks were designed to elicit their meanings of relative size, together with covariational reasoning, in the context of rational functions.

In order to elicit thinking about relative size, I designed tasks that were both static and dynamic. Through conversations surrounding these tasks I constructed models of the instructors' understanding of relative size as it pertains to fractions and rational functions. During this process, I made an effort to consider the ways of thinking I wanted the novice instructor to have after the intervention had ended. Appendix B contains the interview protocol.

One of the first tasks was a word problem that allowed me to elicit the instructors' meanings for division and dividing fractions (Figure 8). Research has shown that instructors have difficulty interpreting the result of fraction division, namely using the unit of the divisor when the result is not a whole number (Coughlin, 2010).

Every cheerleader needs 1/5 yard of ribbon to decorate a football player's locker for homecoming. A spool contains 2 ¼ yards of ribbon. How many 1/5 yard pieces of ribbon can be supplied by 1 spool?

Figure 8. Division of fractions task

I designed this task as a way to ease the instructors into thinking about relative size while I assessed their current thinking on division and dividing fractions. My focus was on the treatment of the remainder of ribbon and the unit associated with the remainder. The goal of this task was to find out how instructors think about division of numbers as a precursor to how they might think about rational functions as quotients of functions, and, more specifically, whether they can interpret the value of the numerator as measured in terms of the value of the denominator.

This task afforded me the opportunity to evaluate the instructors' ways of thinking about division with a specific focus on their process of solving. I did not expect the instructors to experience any disequilibrium in the process of dividing fractions. However, I expected that there could be disequilibrium when I ask, "What does your answer mean in the context of the situation?" If an instructor explained that the solution is the number of cheerleaders, which is a possibility based on Coughlin (2010), then he did not attend to the divisor as a measurement unit, or that the instructor equates a $^1\!/_5$ yard piece of ribbon as equivalent to one cheerleader.

The instructors might experience disequilibrium based on the word choice of "how many." Asking a question of "how many" might result in thinking that the answer must be a whole number. This would be a learned assumption since many mathematics textbooks ask how many and the result is a whole number (Yan & Lianghuo, 2006). This means that the instructors might find an answer that is not a whole number to be unusual, which can lead to further disequilibrium.

Additional disequilibrium might occur when the instructor is asked "How much ribbon do you need to supply all 12 cheerleaders?" This question forces the instructors to consider the unit of the divisor in order to correctly answer how much more ribbon is necessary to supply 12 cheerleaders. I expect some disequilibrium in answering this question, if the pre-service instructor did not correctly identify the divisor as the measurement unit. An instructor might also struggle if his scheme for division is limited to an “into” model or measurement model of division (Kouba, 1989; Kouba & Franklin, 1995; Ma, 1999), in which the numerator must be larger than the denominator.

In another task I provided instructors with a sequence of six static images of two bars, one red and the other blue (Figure 9 and Figure 10). Each static bar image was chosen to explore instructors’ thinking on relative size. I asked the instructors to determine the relative size of the red bar in terms of the blue bar for each static bar image. The images were revealed one at a time and instructors were asked to arrange the static images in order of the value of relative size, after every static image we discussed the instructor’s answer. The static bar task was created so the instructors could not find a specific value for the relative size. In fact, values were purposefully absent from the tasks to encourage instructors to reason about the relative size of the red bar in terms of the blue bar instead of trying to calculate an exact numerical answer. I designed these tasks similar to Dr. Patrick Thompson’s work so that the instructors could not rely on their number sense to create an exact or “nice” answer for the task (Thompson & Thompson, 1994; Thompson & Saldanha, 2003).

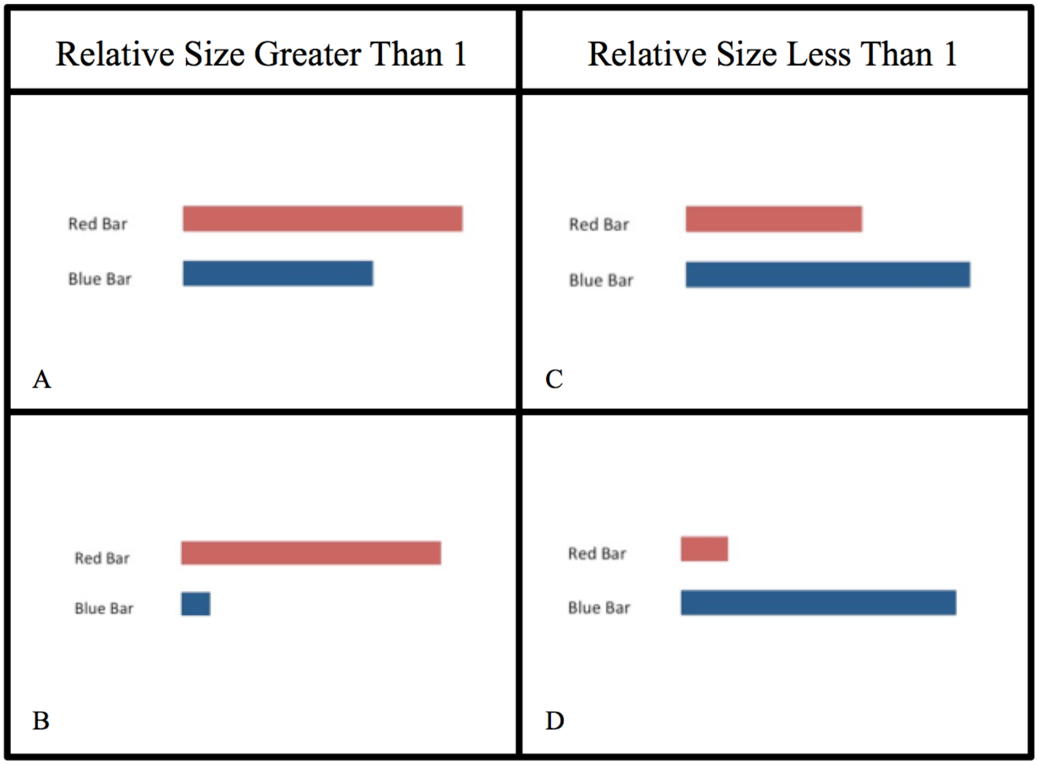

Figure 9. Static bar task where the relative size is greater than or less than 1.

In the first static bar image (Figure 9A) the relative size of the red bar in terms of the blue bar is approximately one and one-third. This task was first since I hypothesized that instructors with an "into" model for measurement (Kouba, 1989; Kouba & Franklin, 1995; Ma, 1999) would be more comfortable in answering the question if the red bar had a greater length than the blue bar.

In the second image (Figure 9B) the length of the blue bar has decreased, resulting in a larger relative size of the red bar in terms of the blue bar. In this case the relative size of the red bar in terms of the blue bar is approximately eight. This task supports instructors who have an "into" model of division, but the image could represent a whole number relative size (versus one and one-third from Figure 2A). Asking the instructors to compare Figure 9A and 9B, in both of which the relative size is greater than one, allowed me to see how the instructors understand change in the relative size as the blue bar changes in length. For example, if the red bar remains the same length but the blue bar decreases in length, the relative size of the red bar in terms of the blue bar increases.

The third static bar image (Figure 9C) has a relative size of approximately two-thirds. This is the first image for which the measuring stick (the blue bar) is larger than the thing to be measured (the red bar). This image was designed to perturb an “into” model of division and require instructors to adopt another method of determining relative size. One of the hypothesized outcomes is that instructors might identify the relative size of the red bar in terms of the blue bar as the difference in the length of the bars.

The relative size represented in Figure 9D is approximately one-fifth. This task continued to perturb instructors understanding of relative size since the length of the red bar is smaller than the length of the blue bar. Asking instructors to compare Figure 9C and 9D in which the relative size of the red bar in terms of the blue bar is less than one allowed me to once again see how the instructors understand change in the relative size, but this time as the red bar changes in length. For example, if the blue bar remains the same and the red bar decreases in length, the relative size of the red bar in terms of the blue bar decreases.

The continuation of the first task included two additional static bar images in which the relative size of the red bar in terms of the blue bar is equal to one (Figure 10). When instructors were asked to compare Figure 10A and 10B, I hypothesized that, if instructors used the length of the bars for quantifying the relative size of the red bar in terms of the blue bar, the instructors would provide a larger value for Figure 10A than for Figure 10B.

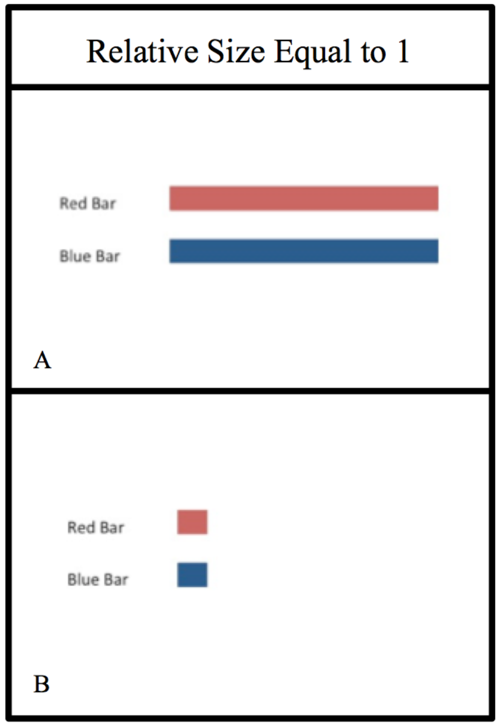

Figure 10. Static bar task where the relative size is equal to 1.

Once all six of the static bar images were revealed to the instructors, I asked them to order the images based on relative size from largest to smallest. The correct order would be Figure 9B, 9A, 10A and 10B, 9C, and lastly 9D. Since the bars depicted in Figure 10A and Figure 10B have the same relative size, these should be placed one on top of the other (see Figure 11).

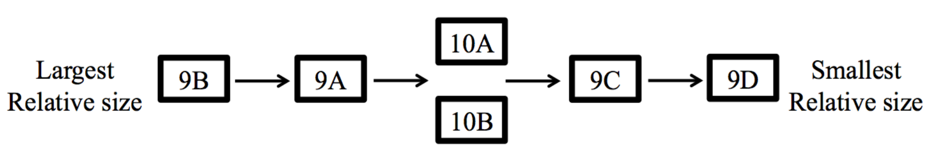

Figure 11. Figures 9A-9D, 10A, and 10B ordered from largest to smallest relative size.

Up until this point in the interview, I only explored schemes of relative size using static images. Such static images provide a solid foundation upon which to build an understanding of how relative size differs according to the lengths of the two bars. However, static images of relative size do not support a smooth understanding of rational functions which requires covariational reasoning. For this reason, a second type of task was needed to explore how static notions of relative size could be coupled with covariational reasoning. After the completion of the static bar activity, I introduced the instructors to a dynamic bar task using a virtual manipulative. In this task, I asked similar questions as in the static bar activity.

The virtual manipulative, created using GeoGebra, produced the same type of images used in the static bar activities; there are two bars on the screen, one red and one blue (Figure 12). When the virtual manipulative plays, the bars change in length depending on the setting of the two sliders in the top left corner. For example, when the one slider is set to three and the other slider is set to two, the red bar decreases in length as the blue bar increases in length. In essence, the dynamic bar task takes an accumulation of static bar images and plays them like a flipbook.

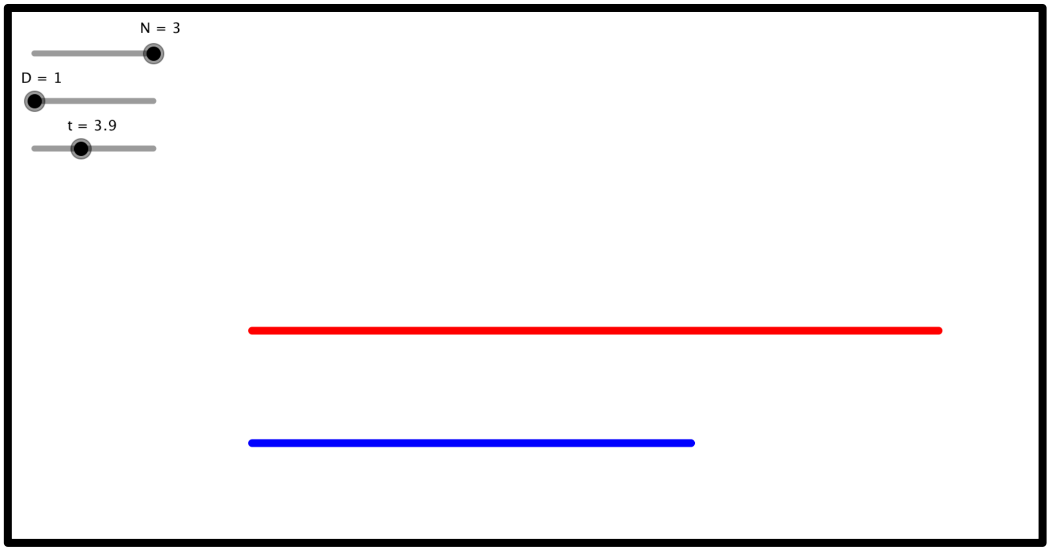

Figure 12. Screen shot of dynamic bar virtual manipulative

During this task instructors were asked to participate in an embodiment activity (Glenberg & Robertson, 2000; Glenberg, Witt, & Metcalfe, 2013) that further explored thinking about relative size and coordinating changes in relative size. Using embodiment activities as a foundation upon which to help students build a conceptual understanding is a hallmark of the Pathways curriculum (Carlson, 1999). The embodiment activity that was used in this study involved an instructor tracking the relative size of the red bar in terms of the blue bar as the distance between his index fingers. This activity revealed thinking about relative size in ways that could not be seen in the static bar activity. For example, when the length of the red bar is decreasing and the length of the blue bar is a constant, an instructor might track the length of the red bar instead of the relative size. This would indicate that the instructor had not internalized relative size but instead focused on the length of a single bar, in this case the length of the red bar.

Upon the completion of the static and dynamic bar tasks, instructors were provided with a graph of the relative size of the red bar in terms of the blue bar (Figure 13). I asked the instructors to describe and draw what the relative size of the red bar in terms of the blue bar is at specific points I indicated on the graph. As an example, at the point (2,1), the relative size of the red bar in terms of the blue bar is one. The depiction of the relative size of the red bar in terms of the blue bar at a value of one should result in an image of two bars of equal length.

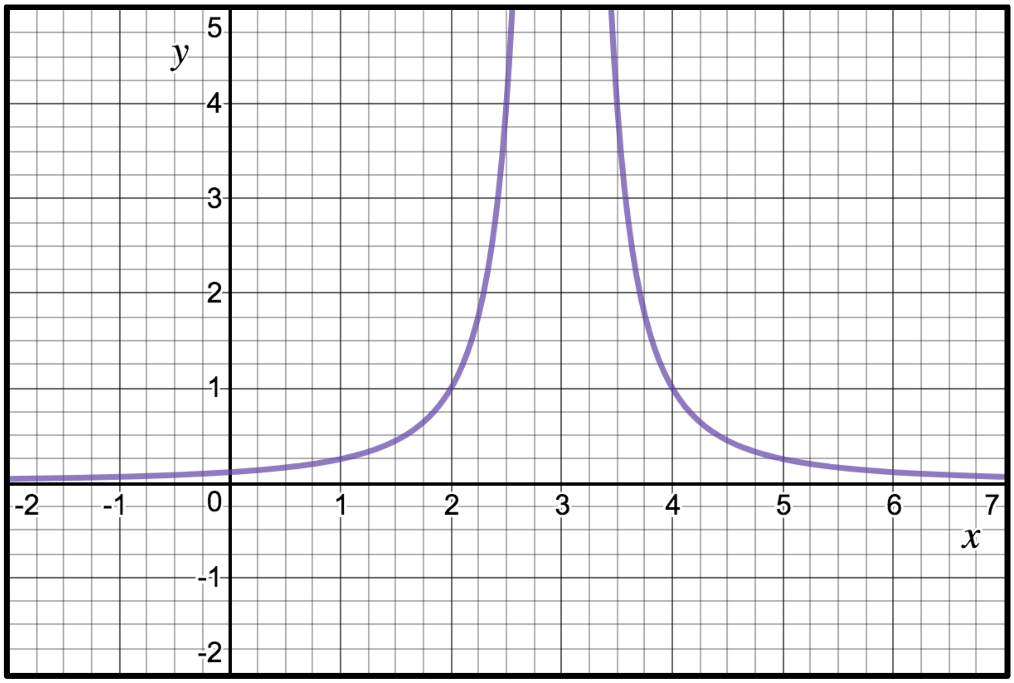

Figure 13. Graph depicting the relative size of the red bar in terms of the blue bar

Once the instructors correctly identified the relative size of the red bar in terms of the blue bar, I asked them to predict what happens to the relative size of the red bar in terms of the blue bar as the input value varies. For example, as the input quantity increases without bound starting at the value of four, how will the relative size change and what could this look like for the red and blue bars? I expected an answer for this might be an explanation that the relative size is decreasing toward zero as the input values increases without bound from four. This could be a result of the red bar decreasing as the blue bar remains constant, the red bar remains constant while the blue bar increases, or as the red bar decreases, the blue bar increases.

In addition, the instructors were asked to describe the behavior of the relative size around the input value of three, in the hope that they might verbalize their thoughts on the relationship between relative size and the existence of a vertical asymptote. I asked the instructors if they had ever seen a function that resembles the behavior found in the provided graph. I asked this question to see if the instructors might recognize the graphed function as a rational function. If the instructor did recognize the graph of the relative size of the red bar in terms of the blue bar as a graph of a rational function or quotient of functions, then the instructor was asked to explain how rational functions and relative size are related. In this context, the instructors could articulate that the red bar represents the output of the function in the numerator while the blue bar represents the output value of the denominator. The ratio of the two output values results in the graph of the rational function.

Once all the tasks were complete, I asked the instructors to describe the rational function instruction they planned to implement (see Appendix B). This was the first time I was introduced to the instructional practices the instructors had planned for rational function instruction. For instance, I asked the instructors to provide a detailed account of how they planned to introduce the concept of rational functions to students, whether technology would be part of the instruction (and, if so, in what manner), and what activities would be used as part of the instruction. I also asked the instructors what mathematical meanings for rational functions they hoped their students would have at the end of the rational function instruction.

Journal and observations. Classroom observations were bookended by video journal entries made by the instructors. Journaling is a methodology that can be used to move back and forth in time and across students and content. This method allows instructors to prepare for future lessons simultaneously with reflecting on what happened the day before (Lampert, 2003). In this study, each of the video journal entries served a different purpose. First, an instructional preparation video journal kept by the instructors allowed me to capture the instructional preparation made after the pre-interview but before the classroom observations. The second time video journal entries were made was in the time period between the classroom observations and the post-interview. The purpose of these entries was to capture the instructor's perspective, i.e., thoughts and reflections, immediately following the lessons on rational functions.

The classroom observations in Phase 1 of the study provided a picture of existing mathematical classroom practices during rational function instruction. I took notes on the instructors' gestures and actions while teaching rational functions. These notes and classroom observation videos allowed me to focus on recurring mathematical classroom practices that the instructors demonstrated when teaching rational functions.

Post-interview. The post-interview in Phase 1 of the study targeted the instructors' mathematical classroom practices prior to the introduction of the novel didactic objects. The interview protocol can be found in Appendix C. This post-interview allowed me to construct a baseline model of the instructors' mathematical classroom practices and better characterize what it means to be a novice instructor. During this interview, I showed the instructors video clips from their rational function lessons and asked them to reflect on their intentions and how they perceived that their instructional decisions and actions (e.g., choice of examples, questioning, and gestures) contributed to student learning. Schoenfeld (1998) used a similar method to study instructors' perception of students' thinking. These mathematical classroom practices that I identified and confirmed with the instructors helped me to detect perturbations that occurred in Phase 2 of the study.

Intervention Activities. The names of the novel didactic objects I introduced to the novice instructor included Moving Vectors, Sum Bar, Rat Bar, and Rat Graph. It is important to note that I use the term didactic object from my perspective rather than that of the novice instructor. To an instructor, these interventions initially are tools or applets (Cobb, Yackel, & McClain, 2014; Dewey, 1944) and may eventually become didactic objects if their value and use for teaching the concept is internalized. I used GeoGebra to create Moving Vectors and Sum Bar as a way to draw attention to the coordination necessary to evaluate the covariational relationship between multiple quantities (See Appendix D for screenshots of each of the virtual manipulatives). Dr. Thompson designed Rat Bar and Rat Graph to foster a conceptual understanding of rational functions. He developed the didactic objects using Graphing Calculator Software to progressively scaffold an understanding of the numerator and denominator of a rational function as single quantities, but also to see the relative size of the numerator measure in terms of the denominator as a separate quantity.

Table 3 contains the supports that were used in this study for implementing the novel didactic objects in rational function instruction. These included teaching guides for every novel didactic object, student activity guides for select didactic objects, and a worksheet that allowed students to further practice with Rat Graph that is completed online (using iMathAS). These supports worked together as a system to serve two purposes: first, to assist the instructors when implementing the novel didactic objects in rational function instruction; and second, to help students construct a conceptual understanding of rational functions.

Table 3. *Supports used in this study for implementing didactic objects*

Virtual Manipulative	Resource	Description	Author	Location
Moving Vectors	Teacher Guide	Contains instructions on how to use the virtual manipulative and discussion prompts to construct the graph of the sum of two functions represented graphically	Krysten Pampel	Appendix E
Sum Bar	Teacher Guide	Contains instructions on how to use the virtual manipulative and discussion prompts for embodied activity to track the sum of two quantities	Krysten Pampel	Appendix E
Rat Bar	Activity Guide	Guide for the four progressive stages of conceptual understanding for rational function	Dr. Patrick Thompson	Appendix F
	Teacher Guide	Contains instructions on how to use the virtual manipulative and discussion prompts for embodied activity to track the relative size of one quantity in terms of the other	Krysten Pampel	Appendix E
Rat Graph	Activity Guide	Guide for the four progressive stages of conceptual understanding for rational function	Dr. Patrick Thompson	Appendix F
	Teacher Guide	Contains instructions on how to use the virtual manipulative and discussion prompts to construct the graph of the relative size of two functions represented graphically	Krysten Pampel	Appendix E
	Online Worksheet	Additional practice for students to construct rational functions using Rat Graph.	Krysten Pampel	Appendix E

In order to assist the instructors with preparation and implementation of the novel didactic objects in the classroom, I created teaching guides for every didactic object. The teaching guides I created are a revision of Thompson's four-stage activity guide, which provides step-by-step instructions that assisted the instructor with setting the didactic objects for each stage, as well as discussion prompts he needed to deliver to students at certain times throughout the instruction of rational functions. In addition to the teaching guides, I created an online worksheet that the instructor assigned to his students. This online worksheet was used as an assessment to test students understanding of relative size. The students had an opportunity to practice graphing rational functions using the conceptual approach demonstrated in-class by the instructors.

In Moving Vectors, two functions are graphed in the same plane. The instructor can check a box in the upper left corner to turn on the vectors, which represent the output value of the respective functions. After having a conversation with the class about how to add the two functions together to make a graph of $f(x) + g(x)$, which is available in the guide that accompanies Moving Vectors, the instructor can move the end of the vectors of $g(x)$ to the point of the vectors of $f(x)$. This process is repeated until all the vectors have been moved. Then the instructor can sketch the graph of $f(x) + g(x)$. The aforementioned guide I authored to assist instructors in using Moving Vectors as part of rational function instruction can be found in Appendix E.

Sum Bar further helps students coordinate the covariational relationship between two quantities by assisting the instructor in leading the students through an embodiment activity. In this activity the students demonstrate, using the distance between their hands, the sum of the red bar and the blue bar depicted in Sum Bar. The Sum Bar guide found in Appendix E, provides the instructor with detailed instruction on how to change the lengths of the red and blue bars as well as questions to ask the students in order to guide the discussion.

As seen in Figure 2, the didactic object (Rat Bar) uses the Scenario slider in a single position and the Display slider in three different positions for labeling the red and blue bars that represent the value of the numerator and the value of the denominator of one specific rational function. In the first display option, the red bar is labeled "Top" representing a single magnitude, and the blue bar is labeled "Bot". The "*a*" located at the bottom of the didactic object represents the parameter values from zero to 6.28. If the play button is pressed, the animation will begin changing the lengths of the red and blue bars so that the instructors and their students can identify the relative size of the numerator measured in terms of the denominator.

Dr. Thompson also created an instructional activity guide (found in Appendix F) that supports four progressive stages of conceptual understanding for rational functions. The first three stages use Rat Bar see Figure 14.

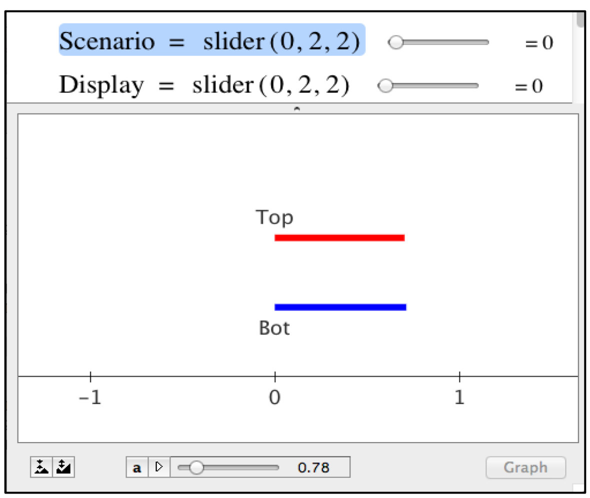

Figure 14. A screen capture of Rat Bar used in the first stage.

The first stage, Stage 1, of the guide assists students in conceptualizing and representing relative size as the quotient of functions. Rat Bar starts out, as seen in Figure 15, with the red and blue bars of equal length. In other words, the measure of the top in units of the bottom is one. During Stage 1, the instructor continues to change the parameter value that alters the length of the red and blue bars and asks students, "What is the relative size of the top measured in units of the bottom?" Students respond by suggesting a value for the relative size of the top measured in terms of the bottom. Once the instructor has completed this with the class numerous times, there is a discussion stemming from the question "How can we represent these relative sizes so that we can differentiate among them?" The intent of this question is to lead the students to use function notation, such as Top(a)/Bot(a).

The second stage, Stage 2, of the guide assists the students in internalizing relative size as a quantity. Rat Bar is altered slightly, as shown in Figure 15, when the Display slider is moved into position one relabeling the top and bottom labels to "Top(a)" and "Bot(a)".

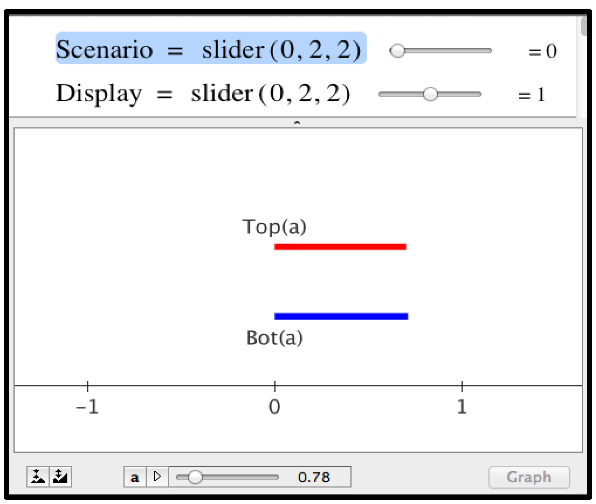

Figure 15. A screen capture of the Rat Bar in Stage 2.

The goal of Stage 2 is for students to no longer just see two *separate quantities* but instead to see the *relative size* of these quantities as its own quantity. This coordination is foundational for students to sketch a graph of the relative size of Top(a)

measured in terms of Bot(a) in relation to the parameter value, a. In this embodied stage, the instructor asks the students to use the distance between their hands to represent the relative size of Top(*a*) measured in terms of Bot(*a*). When changing the parameter values, the students begin to change the distance between their hands. Students are given the opportunity to discuss and practice as they attempt to vary the distance between their hands to represent the relative size of Top(*a*) measured in terms of Bot(*a*) as the parameter value varies. This stage scaffolds to the next by having the students develop an image of the relative size of Top(*a*) in terms of Bot(*a*) that they can coordinate with the parameter value, *a*.

The third stage of the guide, Stage 3, assists students in graphing relative size. In this stage, students use distance between the tabletop and their left index finger to represent the relative size of Top(*a*) in terms of Bot(*a*). In addition, the students use their right index finger sliding along the tabletop to represent the parameter values, a, that are varying. Once the students have tracked the relative size vertically above the table and the parameter values horizontally along the table, the students are instructed to keep their left index finger directly above their right forefinger as they track both quantities' values. After practicing numerous times, a discussion should be prompted by the question, "Does this activity you have just done have anything to do with graphing?" Lastly, the students attempt to sketch a graph of the relative size of Top(a) measured in terms of Bot(a) in relation to the parameter value, a.

The fourth stage, Stage 4, uses a different didactic object (Rat Graph), as shown in Figure 16 and Appendix D. In this stage, the goal is for the students to solidify the ways of thinking that they have just developed to envision graphing, where n(x) represent the function in the numerator and d(x) represent the function in the denominator of the rational function r(x). Rat Graph offers additional practice for the students to envision the relative magnitude of n(x) and d(x) and construct graphs of $r(x) = \frac{n(x)}{d(x)}$. By introducing Rat Graph to the students first, a conversation can be had about the ways of thinking that surround coordinating the numerator and denominator of a rational function. The instructor can then assist students in continuing their thinking to imagine the relationships extending beyond the edges of the applet.

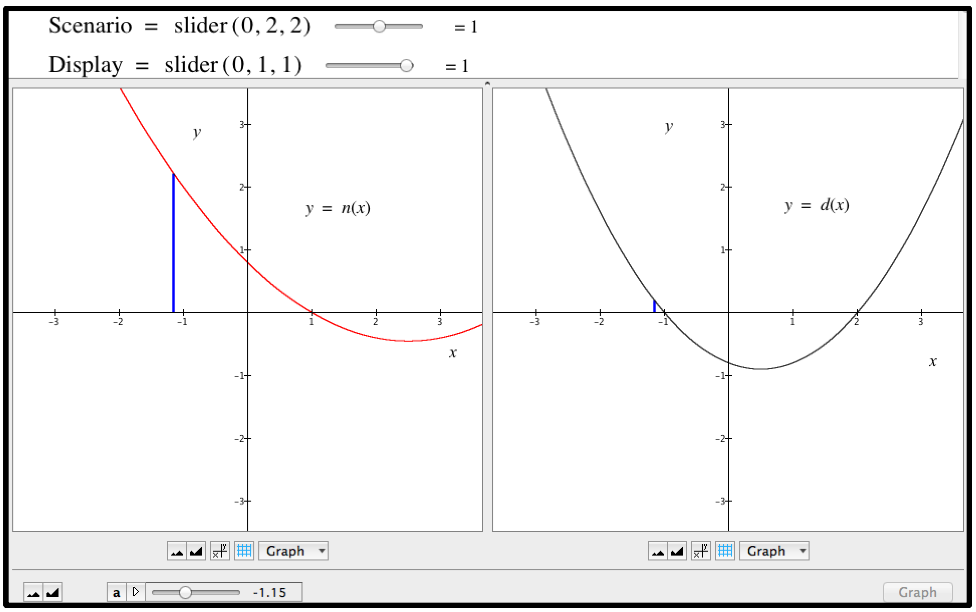

Figure 16. A screen capture of Rat Graph used in Stage 4 of the guide.

Phase 2 (Intervention). The major goal of Phase 2 was to trace the effects of novel didactic objects on a novice instructor's mathematical classroom practices. Toward this end, I conducted a pre-interview, an intervention interview, classroom observations, and a post-interview in Phase 2 using one participant, the novice (new-to-Pathways) instructor.

Pre-Interview. The pre-interview in Phase 2 of the study was similar to the Phase 1 pre-interview and consisted of tasks created to reevaluate the novice instructor's mathematical meanings for rational functions. This pre-interview helped to identify changes that might have occurred in his mathematical meanings and mathematical classroom practices for rational functions since Phase 1.

One task during the pre-interview allowed me to probe the novice instructor's problem-solving approach and mathematical meanings of rational functions. (Appendix B contains the interview protocol and tasks.) In this task, the instructor is given the graph of two functions, $f(x)$ and $g(x)$, and asked to construct the graph of their quotient

$h(x) = \frac{f(x)}{g(x)}$.

Using the graph of $f(x) = -x + 3$ and $g(x) = 2x + 1$, construct a graph of

$$h(x) = \frac{f(x)}{g(x)}$$

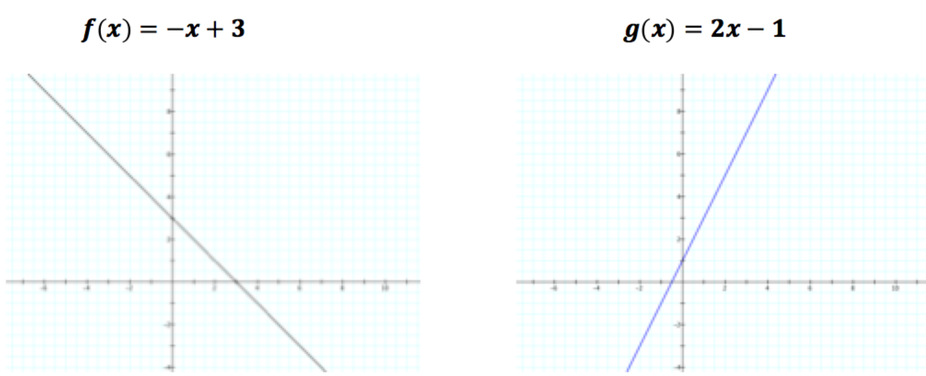

Figure 17. Rational function task used in the pre-interview.

I created this problem, seen in Figure 17, with the intent of gaining insights into the instructor's meaning and ways of thinking about rational functions. Through questioning, I explored whether his approach involved seeing the numerator and the denominator, each as a single quantity, and then the ratio of the two as a third quantity. Additional tasks were constructed based on the outcomes of Phase 1 (see Appendix B).

Intervention Interview. During the intervention interview, I introduced the novice instructor to the novel didactic objects, see Table 3, he later implemented in his rational function instruction, as well as collected his initial thoughts on how he wanted to implement the didactic objects in the classroom. Since these didactic objects were novel to the instructors, I acted as an instructor having him complete the tasks as if he were a student. After we worked through all of the didactic objects, I asked questions on how he planned to implement the didactic objects in his rational function instruction. I also asked the instructor to describe stumbling blocks he expected students would face during the lessons.

After guiding the instructor through the novel didactic objects and the additional resources, I began to ask questions regarding the implementation of the novel didactic objects and the instructor's initial perception of the novel didactic objects. I also asked the instructor where he might expect students to struggle during the implementation of the novel didactic objects. For instance, he might focus on stumbling blocks associated with the technology instead of focusing on obstacles stemming from the mathematical conceptions, or vice-versa.

Journal and observations. As in Phase 1 of the study, classroom observations were once again bookended by video journal entries made by the novice instructor, each of which will serve a different purpose. First, in the time between the intervention interview and their lessons, the instructor voiced his thoughts on the instruction of rational functions with the addition of the novel didactic objects. These thoughts elaborated on how the instructor anticipated implementing the didactic object in his instruction on rational functions.

I intended for the video journaling activity to serve as an occasion for the instructor to further reflect on the implications of teaching rational function with novel didactic objects that promote a conceptual understanding. My hope for the instructional preparation video journal was to obtain further insight into the instructor's lesson planning process, as well as an account for changes in his perception of the novel didactic objects. In these ways, the video journal was a valuable method that allowed me access to the instructor's thoughts during the gap between pre-interviews and classroom observation. The second time journal entries were made was in the time period between the classroom observations and the post-interview. The purpose of these entries was to capture the instructor's perspective, i.e., his thoughts and reflections, immediately following his use of the novel didactic objects as part of his lessons on rational functions.

The classroom observations in Phase 2 of the study required a slightly different focus than those in Phase 1. I still looked at the instructor's mathematical classroom practices but, rather than identifying recurring practices, I focused on finding perturbations that occurred while teaching rational functions when implementing the novel didactic objects. My notes and the use of Studio Code helped me to identify video clips for the stimulated recall activity in the post-interview.

Post-interview. The post-interview in Phase 2 of the study targeted the perturbations that occurred when the novice instructor taught rational functions using the novel didactic objects introduced to him in the intervention interview. I employed stimulated recall as a method of data collection to serve two purposes: first, I had access to the instructors' gestures, activities, and answer to questions without interrupting the overall classroom dynamics, and second, I ensured the analysis of the data reflected the instructor's perspectives in addition to my own. The interview protocol can be found in Appendix C.

Methods of Data Analysis

This section presents the methods I used for data analysis in this study, which covers how I utilized each method, where in the study I used the methods, and an explanation of why the methods were appropriate.

Overview. The data analysis methods I used in this study are consistent with the grounded theory approach (Glaser & Strauss, 1967, Strauss & Corbin, 1990). In general, the approach I used in this study to analyze the data follows a three-step cyclical process:

1. Analyze data in order to identify instances that provided insight into the perturbations in mathematical classroom practices experienced by the novice instructor when introducing the novel didactic objects in the classroom, and formulate initial hypotheses based on these instances.
2. Analyze the data for a second time searching for evidence to support or contradict my initial impressions I formulated in the first analysis of the data.
3. Evaluate my initial hypotheses, accept, reject, or revise, with evidence gathered in the second step.

Part 1: Reduction of data. This study generated an estimated 30 hours of video recordings. I analyzed each video session with Studio Code as soon as possible following the recording. The timely analysis was necessary in identifying the perturbations in mathematical classroom practices during the classroom observations, as well as in pinpointing the video clips used in the post-interview.

The analysis of the interviews and classroom observations exposed possible perturbations in mathematical classroom practices experienced by the novice instructor when he implemented novel didactic objects in his rational function instruction. The theories I discerned from the data used a generative approach (Clement, 2000). The generative approach to data analysis helped me to generate hypotheses regarding the effect of novel didactic objects on the novice instructor's classroom instruction. These hypotheses I identified within the videos formed the basis of the second stage of data analysis.

Part 2: Analysis of perturbations. I utilized grounded theory (Glaser & Strauss, 1967, Strauss & Corbin, 1990) to infer possible perturbations in mathematical classroom practices that occurred throughout the study. After analyzing the data, the results were examined using hypothesized perturbations I identified in part 1 of the data analysis (after the classroom observations) and the retrospective self-report given by the instructors during the post-interview. I used stimulated recall and gesture analysis to draw conjectures on the types of perturbations in mathematical classroom practices that are evoked within the novice instructor when introducing novel didactic objects in his rational function instruction.

Stimulated recall procedures are used in situations where think aloud protocols would interfere with the performance of the task being examined (Stough, 2001). The stimulated recall methodology involves videotaping an activity with the intent to display video to a participant at a later time. While watching the video, the participant is asked to retrospectively self-report on her actions, gestures, and discourse. Although, stimulated recall procedures are valuable, there are limitations associated with this methodology. In particular, the self-reflection might not truly convey the reasoning behind the participant's actions, gestures, or discourse during the original recorded activity. However, participants generally recall more when they have prompts than they would otherwise. In my study, stimulated recall was used to prompt the instructor to reflect on his mathematical classroom practices when teaching rational functions.

There is more to discourse than just verbal utterances. Gestures can function as co-participants with speech as two parts of a single utterance (Koschmann & Lebaron, 2002, p. 252). A gesture is considered to be a movement of the body that is clearly part of an individual's acknowledged intention to convey meaning (Kendon, 1987). Gestural performance is sensitive both to the composition of the audience and to prior interaction while showing evidence of the consequentiality of gesture for the development of subsequent understanding (Koschmann & Lebaron, 2002, p. 270). Gestures are interactive phenomena in as much as they serve to regulate co-presence, affect the actions of others, accomplish something in the social world, and so contribute to the partially constitute social actions (Moerman 1990, p. 17). Treating gestures as the embodiment of thinking allowed me to analyze the gestures of the instructor as material signs that embody the knowledge being articulated while simultaneously shaping and lending structure to social interactions of the classroom.

The task of journaling I gave the instructors required them to immediately, upon completion of each lesson on rational functions, capture their thoughts on the lesson using guiding questions found in Appendix G. For example: Overall how do you feel the lesson went? And were there any notable disruptions or hiccups in the lesson? The act of journaling "offers a rich means for describing practice; for recording and examining beliefs, assumptions, questions, and challenges; and for expressing feelings and identifying problems" (Pine, 2008, p.194). For this reason, journaling is used as a method of data collection in educational research (Epp, 2008). The instructors' video journals provided me tangible evidence of their processes, struggles, and reasons for actions taken throughout the lesson. These video journals allowed me to further identify perturbations in their mathematics classroom practices that can be addressed and confirmed in the post-interview.

When analyzing the responses prompted by the stimulated recall and journaling activities, I used two general principles to accept, reject, or revise hypothesized perturbations in mathematical classroom practices experienced by the instructor. First, I did not infer more information from the instructors' gestures, video journal entries, and discourse beyond what is necessary to explain the instructors' actions. This ensured that I avoided making unsubstantiated claims. Second, I required substantial evidence to support a hypothesis before I accepted the hypothesis. For example, I considered evidence to be stronger when both the instructors and I concurred that a perturbation was experienced during the classroom observations. However, in the cases where I was unable to find enough evidence to accept or reject a hypothesis, I revised the hypothesis.

Synthesis of Analysis. First I coded the data using the initial perturbations-in-practice taxonomy shown in Table 4. This taxonomy, based on the perturbations in practice found in professions outside of education (Table 1), is effective for seeing similarities between perturbations in industry and mathematics classrooms. In particular, the taxonomy contains leader actions, communication, expectation of technology, and roles and responsibilities as aspects of practice that are perturbed when novel technology is introduced.

Table 4. *Aspects of Practice Perturbed by Novel Technology*

Aspects of practice	Description	Example
Leader Actions	How instructor perceives novel didactic object and how the instructor uses the technology in the classroom	The instructor's introduction of the didactic object demonstrates his uneasy feeling toward trying something new.
Communication	Classroom discourse surrounding the novel didactic object	The instructor's students no longer rely on exact answers but instead they explain the behavior of the function.
Expectations of Technology	What understandings the instructor expects students to develop	The instructor expects the novel didactic object to assist student in forming an image of the mathematics.
Roles and Responsibilities	Responsibilities of the instructor and students when using the didactic objects	The instructor's role is altered from lecturer to discussion facilitator.
Student Engagement	Student participation while the didactic object is being used	The instructor's students are more attentive in the lesson through the activities that accompany the didactic objects.
Mathematical Conceptions	How students perceive the mathematics addressed by the novel didactic object	The instructor struggles to understand students' mathematical conception of the concept while teaching with didactic objects.

This taxonomy was constructed in a pilot study I conducted using two novel didactic objects in a mathematics classroom. The findings from this pilot study led me to add two aspects of practice that can be perturbed when novel didactic objects are introduced in an instructional setting. First, student engagement refers to how students perceive and act when a novel didactic object is used. Second, students' mathematical conceptions can be perturbed, which will consequently perturb mathematical classroom practices. I used this tailored taxonomy as a starting point for categorizing aspects of practice that are perturbed when novel didactic objects are introduced into rational function instruction. However, I was not able to fully explain the perturbations that the instructors experience when implementing the novel didactic objects. Therefore I used the stimulated recall post-interview session together with my own observations to further revise the taxonomy.

CHAPTER 5

RESULTS OF PHASE 1: CHARACTERIZING A NOVICE INSTRUCTOR

This chapter presents selected data and results from Phase 1 of this study. This chapter does not include a full analysis of every task and observation in the interest of readability. Instead, the focus of the chapter is on an analysis of presented episodes that were representative of the distinctions between the two instructors, one new-to-Pathways and the other a seasoned instructor.

The results presented in this chapter were obtained through the interview transcriptions and video analysis of data collected in Phase 1 of the study. In the following sections, I present results that pertain to the distinctions between the two instructors and provide a basic model of the instructors' meanings for rational functions and existing mathematical classroom practices surrounding their rational function instruction. This chapter is broken into three sections: division, relative size, and covariation; approach to promoting discussion of rational functions; and rational function lesson reflections, these sections represent the similarities and differences found in Phase 1 between the two instructors. In each of the sections I present results on the new-to-Pathways instructor, Norbert, and the experienced instructor, Edwin (These pseudonyms were chosen to help the reader associate Norbert with the novice instructor and Edwin with the experienced instructor).

Division, Relative Size, and Covariation

In the pre-interview of Phase 1, Norbert and Edwin demonstrated their thinking about division, relative size, and covariation to discuss rational functions. The results found in the pre-interview of Phase 1 hinted toward Norbert and Edwin having similar understandings for division, relative size, and covariation to discuss and describe rational functions in a conceptual manner. Norbert and Edwin completed tasks designed to probe their understanding of rational functions through division, relative size, and covariation. The first task of the pre-interview (Figure 18) asked for the instructors to evaluate the following.

Evaluate the following:

1.) $\frac{6}{1/5}$

Figure 18: *First task of Phase 1 Pre-interview.*

While answering this task (Figure 18), Edwin described how he was thinking about the problem (Excerpt 1).

Excerpt 1

Edwin: OK. So I think of this as... First I want to think about what this is, so one fifth. I think of this as, if you take one whole thing whatever your unit is it doesn't really matter and split up into five equal sections and you take one copy of those and... and then you're wanting to see how many of these you can fit into something of size 6. So we know something six times as large as our unit now and that's going to be very large. So we're asking how many of these lengths (points to one-fifth) can we fit into this (points to the six).

Edwin stated that we want to see how many of the one-fifths can fit into something of size 6 (lines 4-5). He went on to describe that the result of the task should be very large using a multiplicative comparison of the result and the unit one-fifth (lines

5-6). Edwin then restated that we are trying to find how many of these length of one-fifth can fit into 6 (lines 6-8). Edwin solved the problem by drawing a picture (Figure 19).

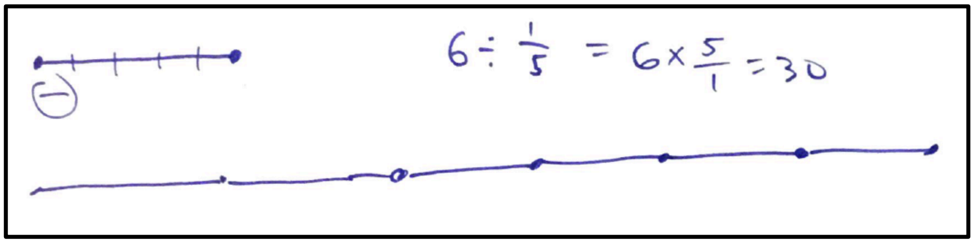

Figure 19: *Edwin's work from task 1.*

Norbert described his thinking of the first task in a similar way to Edwin (Excerpt 2). However, Norbert solved the problem in a procedural way. Only after being probed to find an alternative way to demonstrate how to find the answer of the task, did Norbert decide to draw an image.

Excerpt 2

1	Norbert:	Ok number one. Evaluate six divided by one fifth. Ok so for division I
2		try and measure how many times my denominator goes into my
3		numerator. So here I'm going to see how many copies of one fifth go
4		into my numerator. So in order to do so I'm going to clear my
5		denominator by multiplying essentially by 1 or 5 over 5. So
6		multiplying by 1 doesn't change the value I'm allowed to do that and by
7		multiplying by 1 in the form of 5 over 5 I can get my denominator to
8		look nice in terms of just being over 1. Then I also have the
9		multiplication on top which will give me 30 over one. So completely
10		simplified six over one fifth will give me 30. So if I'm looking at how
11		many times one fifth goes into six, it goes in 30 times.

Norbert outlined his thinking about division as how many times the value in the denominator goes into the value of the numerator (lines 3-4). He continued to solve the problem using the traditional flip and multiply process (lines 5-11). Norbert finished this task by restating that we are looking for how many times one-fifth goes into six, which in this case is 30 (lines 9-10). After some additional questions on how Norbert could represent his result using a drawing, Norbert provided an image of six circles that were divided into one-fifth sections (Figure 20).

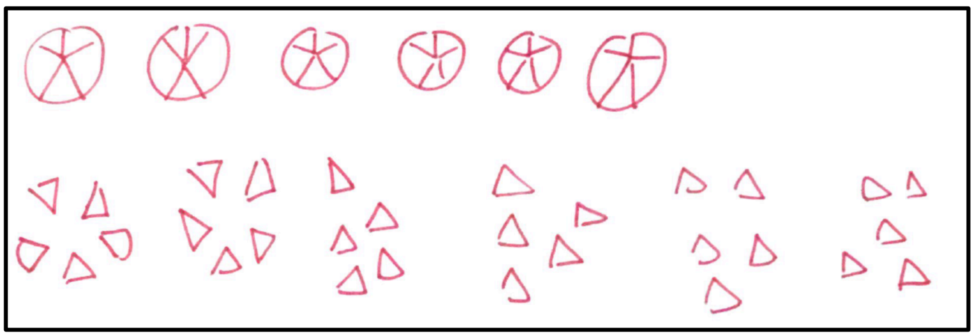

Figure 20: *Norbert's work for task 1.*

Edwin and Norbert demonstrated a quotitive interpretation of division (Correa, Nunes, & Bryant, 1998; Fischbein, Deri, Nello, & Marino, 1985; Greer, 1992) where they both are thinking, how many times does one-fifth go into six. This quotitive interpretation of division has a strong correlation to the concept of ratio. In this case the two associated quantities are the numerator (six) and the denominator (one-fifth), and the quotient is the result of expressing the numerator in terms of the denominator. Conceptualizing division in this way (as opposed to the use of a grouping metaphor or partitive interpretation of division) represents an opportunity to think about what it means to express one quantity in terms of another. Edwin and Norbert demonstrated this way of thinking multiple times throughout the Phase 1 pre-interview even when numerical values were not present.

The static bars task was designed intentionally without values in order to test if the instructors would persist in using their model of division and to explore instructors' thinking on relative size (See Figure 9 in Methods). The values were purposefully absent from the tasks to encourage instructors to reason about the relative size of the red bar in terms of the blue bar instead of trying to calculate an exact numerical answer.

In the pre-interview I asked the instructors to determine the relative size of the red bar in terms of the blue bar for each static bar image. The images were revealed one at a time and instructors were asked to arrange the static images in order of the value of relative size, after every static image the instructors answered questions about how they came up with their answer. Edwin answered the first static bar image (See Figure 9A in Methods) similarly to the first task in the pre-interview (Excerpt 3, the notation of KP in the Excerpt refers to the researcher and author of this dissertation).

Excerpt 3

KP:	What I'm going to be looking for is the relative size of the red bar in terms of the blue bar. OK. So what is the relative size of the red bar in terms of the blue bar?
Edwin:	Relative size of the red bar in terms of the blue bar. So it's like one value? Can I write on this?
KP:	Yes please do.
Edwin:	(Marks on the static image) I would say one and a half.
KP:	OK. So what is one and a half?
Edwin:	So one and a half blue bars go into the red bar.
KP:	OK. So the relative size of the red bar in terms of the blue bar is?

11	Edwin:	Is one half… is one and a half.

Edwin answered the first static bar image by writing tick marks on the image (Figure 21). Edwin's first mark occurred at the end of the length of the blue bar and extended through the red bar above. The second mark was made at the end of the length of the red bar and the final mark was made about halfway through the length of the blue bar. After making all of the marks on the static bar image Edwin stated that the answer would be one and a half (line 7). He rearticulated his answer as one and a half blue bars go into the red bar (line 9). He responded in the same manner as the answer of the first task, which demonstrated cohesion in his thinking about division. In response to the original question of the static bar task being asked a second time, Edwin answered by stating one and a half (line 11).

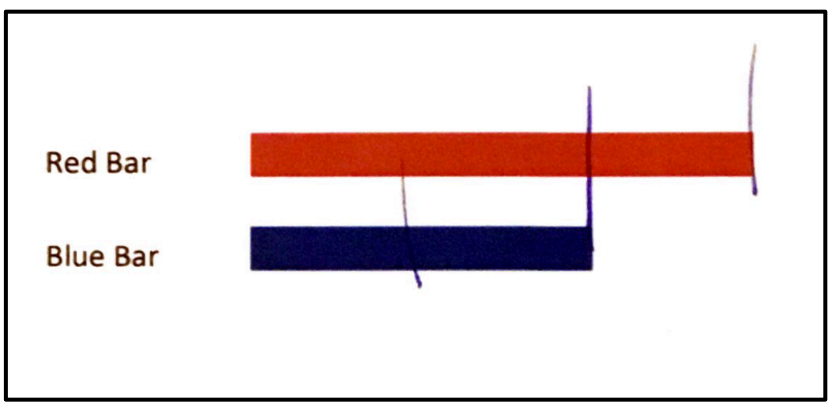

Figure 21: *Edwin's written work of static bar image 9A.*

As the static bar task progressed, Edwin's answers became more confident. When the third static bar image (Figure 9C) was revealed Edwin responded in multiple ways that indicated the use of his understanding of division and correlation to relative size (Excerpt 4).

Excerpt 4

KP:	All right. Let's try another one. So what is the relative size of the red bar terms of the blue bar?
Edwin:	Two Thirds.
KP:	So when you say two thirds what are... what is two thirds representing to you?
Edwin:	The number... the number of blue bars that you can fit into the red bar. So you can fit two thirds of this ruler (points to blue bar) into that (points to red bar).
KP:	OK so the relative size of the red bar in terms of the blue bar is?
Edwin:	Two thirds.
KP:	Two thirds.
Edwin:	Yes.
KP:	OK. Is there another way we could say that and say that the relative size of the red bar in terms of the blue bar is two thirds?
Edwin:	The red bar is two thirds as large as the blue bar.

Edwin answered two-thirds shortly after the image was flipped over but was probed further to describe what the answer of two-thirds represented (lines 4-5). Edwin restated his answer of two-thirds as the number of blue bars that can fit into the red bar (lines 6-8). His answer stated in this manner still demonstrates his understanding of division. Edwin was again asked to think of an additional way to describe the relative size of the red bar in terms of the blue bar. He responded by saying that the red bar is two thirds as large as the blue bar (line 15).

When faced with the static bar task Norbert exhibited his understanding of division once again by comparing one quantity in terms of another quantity (Excerpt 5). In the case of the first static bar image (See Figure 9A in Methods), Norbert measured the length of the red bar in terms of the length of the blue bar.

Excerpt 5

KP: I would like to know what is the relative size of the red bar in terms of that blue bar?

Norbert: Relative bar or relative size in terms... of the red bar in terms of the blue bar. So I'm looking at the red bar and the red bar is longer than the blue bar so I'm thinking in terms of the blue bar I'm going to have more than one copy of the blue bar in there and it looks like if I divide my red bar half that I have a little over one half of the... little over.... So one half of one half of the red bar is a little less than the blue bar. So I look at the piece that's hanging over from the blue bar and I identify one half and it looks like it's about one half of the remaining... remaining distance. So if I want to measure the red bar in terms of the blue bar I would say that the red bar is three halves times the blue bar. I can fit one entire copy of the blue bar into the red bar plus an additional half copy of the blue bar into the red bar.

KP: Oh ok so three... three halves.

Norbert: Three halves. Yes.

KP: OK. So the relative size of the red bar in terms of the blue bar is what?

Norbert: Three halves. The red bar is three halves times the blue bar or one and

	one half times the blue bar.

Norbert displayed his understanding of division by fitting the length of the blue bar into the length of the red bar (lines 3-14). He determined that the relative size of the red bar in terms of the blue bar is three halves. When asked the original question for the static bar task a second time, Norbert rephrased his answer to that of a multiplicative comparison stating that the length of the red bar is one and one half times the length of the blue bar (lines 17-19).

When the third static bar image (Figure 9C) was revealed, Norbert's response indicated the use of his understanding of division and correlation to relative size (Excerpt 6).

Excerpt 6

KP:	OK. All right. So we are just going to put these off to the side we've got more. (flips over next image) So we're still looking for the relative size of the red bar in terms of the blue bar.
Norbert:	Absolutely. So here you're throwing a curveball at me, we have the red bar is a little bit smaller than the blue bar. So I'm going to look and see OK I can fit an entire copy of the blue bar into the red bar so I know that my relative size is going to be less than one. So if I look at this it looks like I can fit... So just eyeballing it looks like it's about two thirds of the blue bar covers the red bar. So in terms of relative size the amount of blue bar that I can fit into the red bar, the relative size of the red bar to the blue bar would be two thirds.

KP:	OK
Norbert:	Because in the other examples I was able to fit more than one copy here in this example I'm only able to fit two thirds of the entire blue bar into the red bar.

Norbert explained the static bar image (Figure 9C) was a curve ball since this was the first image that the length of the red bar was smaller than the length of the blue bar (lines 4-5 and 13-15). However, even with this curve ball Norbert persisted in using his understanding of division to construct the relative size of the red bar in terms of the blue bar (lines 7-11).

Edwin and Norbert continued to exhibit a quotitive interpretation of division through out the static bar tasks and began to leverage this interpretation of division to identify the relative size of the red bar in terms of the blue bar. Thus far Edwin and Norbert had not been exposed to simultaneously varying the lengths of the red and blue bars. The task leading up to dynamic bars had single values or static images that would result in a single answer.

The dynamic bars tasks (See Figure 12 in Methods) required Edwin and Norbert to coordinate the changing lengths of the red and blue bar while utilizing their understanding of division. This coordination of the changing lengths relied on the instructors' ability to use covariational reasoning in tandem with division to find the relative size of the red bar in terms of the blue bar.

During the dynamic bars task the instructors participated in an embodiment activity that required them to demonstrate the relative size of the red bar in terms of the blue bar as the distance between their hands. This physical motion allowed me to see how the instructors thought the relative size of the red bar in terms of the blue bar changed as the length of the red and blue bars varied.

While participating in the dynamic bars task, Edwin and Norbert exhibited the ability to coordinate the relative size of the red bar in terms of the blue bar between their hands and articulate their coordination all of the varying quantities i.e., the length of the red bar, the length of the blue bar, and the relative size of the red bar in terms of the blue bar (Excerpt 7 and 8).

Excerpt 7

KP:	So in this scenario right.
Edwin:	Yeah.
KP:	Relative size starts where? So the relative size of the red bar in terms of the blue bar starts where in this scenario?
Edwin:	Really large so like a thousand something.
KP:	OK and then what's ending up happening, as the red bar decreases and as the blue bar increases?
Edwin:	That number is decreasing.
KP:	To where?
Edwin:	Really fast to 1.
KP:	To 1?
Edwin:	To 1 yeah. It's decreasing to one and then after that it's also still decreasing past that and... and decreasing to zero.
KP:	To zero because...

Edwin: Yeah I see two... like I imagine this is two scenarios because like I split it... it's weird to think... like imagine two... like a stopping point when they're at the same size and I have to ask myself OK what happens past that. And then the ruler of the red line is still decreasing and blue line is still increasing. So I mean I guess it doesn't even matter if you do the stopping point, you can just... the red lines decreasing and the blue line is increasing so the relative size is always going to be decreasing.

In Excerpt 7, Edwin is tracking the relative size of the red bar in terms of the blue bar between his index fingers as the red bar decreases in length and the blue bar increases in length. Edwin identified the relative size of the red bar in terms of the blue bar at the beginning of the scenario as really large l (line 5). However, Edwin articulated the change in the relative size of the red bar in terms of the blue bar as being split into two parts in his mind (lines 12-13).

The first part of Edwin’s image consisted of the red bar having a length that was greater than the blue bar to the point of the length of the red bar and the blue bar being the same (lines 16-17). The second part of his image of the scenario consisted of what happens after the length of red bar and the length of the blue bar are the same (lines 18-19). In other words, Edwin imagined a division at the point where the red and blue bars were the same length. After this point, Edwin used a second image where the length of the red bar is smaller than the length of the blue bar (lines 18-19). He convinced himself almost immediately after describing the two images that they are not necessary. Instead, he stated that we can think as the red bar decreases and the blue bar increases that the relative size of the red bar in terms of the blue bar is decreasing (lines 18-21). Edwin demonstrated his ability to think about the covarying lengths of the bars and the effect of the covariation on the relative size of the red bar in terms of the blue bar. This thinking persisted into the next task associated to the graph of the relative size of the red bar in terms of the blue bar.

Norbert, like Edwin, experienced little trouble with utilizing his understanding of division with the covariation of the length of the red and blue bars (Excerpt 8).

Excerpt 8

KP: So let's talk about what's happening in this scenario. So I'm going to... I'm just going to have this play so we can talk about it. So what's happening in this scenario?

Norbert: So right now my red bar is decreasing in magnitude and my blue bar is increasing in magnitude. So in the case where the red bar is decreasing I know that when my red bar decreases, I have a relative size going down. I also know that when my blue bar increases my relative size is also going down. So the combination of both those will cause relative size to decrease.

KP: OK.

Norbert: And once... So I hit that point I have a certain rate of decrease, it's happening pretty fast before I... before I hit the point one, and then that rate of decrease of relative size sort of slows down a bit as I... As I'm... as my red bar is going towards zero and my blue bar is going past one.

In this excerpt, Norbert articulated that relative size of the red bar in terms of the blue bar is decreasing as the red bar decreases in length and the blue bar increases in length (lines 4-9). Norbert also identified the a point, just like Edwin, after the length of the red and blue bars are the same length. Norbert explained this point in terms of the rate of decrease of the relative size of the red bar in terms of the blue bar (lines 11-14).

During the dynamic bar task, Edwin and Norbert exhibited the ability to coordinate the relative size of the red bar in terms of the blue bar between their hands and articulate their coordination all of the varying quantities i.e. the length of the red bar, the length of the blue bar, and the relative size of the red bar in terms of the blue bar. This coordination of the covarying quantities in conjunction with their understanding of division became even more apparent when completing the next task.

The task following dynamic bars consisted of a graph of the relative size of the red bar in terms of the blue bar (See Figure 12 in Methods). This task required the instructors to describe and draw images of the relative size of the red bar in terms of the blue bar at specified points.

Once the instructors identified the relative size of the red bar in terms of the blue bar, they were asked to predict what happens to the relative size of the red bar in terms of the blue bar as the input value varies. The predictions made by the instructors highlighted their ability to construct a conceptual understanding of rational functions using their understanding of division in conjunction with covariational reasoning.

Edwin responded easily to the task by making multiple predictions about all of the scenarios of the changing red and blue bars for the points specified on the graph. It is important to note that upon first seeing the task, Edwin stated that the graph looked similar to a rational function. This conversation of rational functions was revisited after all the predictions were made (Excerpt 9).

Excerpt 9

KP:	OK. So because you said this is a rational function. How would that relate back to like the red and blue bars for think about a rational function?
Edwin:	So I'm imagining these bars changing with respect to a varying quantity x.
KP:	So that's (slides finger along x-axis) like the input?
Edwin:	Yeah. Well each of these are represented by some polynomial (points to red and blue bars). And so the values... the relative sizes are dependent upon the output of this polynomial expression.
KP:	OK. So the output of the function of the rational function?
Edwin:	Of Each of these sizes. And then… I imagine the rational function... the output of the rational function to represent the relative size of the values that each of those polynomials has.
KP:	Ok so the top bar here would represent what?
Edwin:	The output of a polynomial.
KP:	Of any polynomial?
Edwin:	Of the... the numerator of the polynomial of the rational function pre-supposing that it's a rational function.
KP:	OK… So if we're talking about a rational function then what would the blue bar or the bottom bar represent?

Edwin:	The length of the line or the output of a polynomial as well.
KP:	That is...
Edwin:	In the denominator.
KP:	That it is denominator and then the relative size of the red bar in terms of blue bar would represent?
	Relative size starts where? So the relative size of the red bar in terms of the blue bar starts where in this scenario?
Edwin:	The output of rational function.

Edwin articulated why he thought the graph of the relative size of the red bar in terms of the blue bar was similar to a rational function by him imagining the bars changing with respect to the input quantity (line 4-5). Edwin continued his explanation by stating that the bars represented polynomial functions and the output values seen on the graph were dependent on the output values of the polynomial functions (lines 7-9). After being probed further about his statement, Edwin identified all of the quantities that he was coordinating in order to construct the graph of the relative size of the red bar in terms of the blue bar that he related to rational functions (lines 14-28). Edwin articulated the covariational relationship that exists within a rational function using his understanding of division, relative size, and covariation.

When faced with the same task Norbert provided similar results through his explanation of the specified points and articulated that the graph of the relative size of the red bar in terms of the blue bar could be similar to a rational function by writing a possible function for the depicted graph (Excerpt 10).

Excerpt 10

KP:	So we want the relative size of the red bar in terms of blue bar as the input value decreases to three.
Norbert:	Ok. As the input value decreases to three, so as we're decreasing to three our relative size is increasing so decreasing to three our relative size is increasing. So that would correspond to the same... same three scenarios we had earlier.
KP:	OK
Norbert:	Where either the red bar is increasing and blue bar is decreasing; blue bar is fixed red bar is increasing; or red bar is fixed and blue bar is decrease.
KP:	Okay great. Now what if we say we wanted to figure out the... we want the relative size right. We're going to describe the behavior the relative size of the red bar in terms of the blue bar as the input value increases from three. So we can say increases without bound if it's more comfortable.
Norbert:	So increasing from three, we see that our real size is decreasing. So we can have the possible scenarios where a blue bar is fixed and our red bar is decreasing, getting smaller and smaller, the scenario where a red bar is fixed or a blue bar is increasing... red bar is fixed and our blue bar is increasing without bound, or the scenario where both our red bar is decreasing and our blue bar is increasing without bound.
KP:	OK. All right. What's the relative size at 3?

Norbert: So the relative size at three is... if you want to think about in practicality would be undefined it doesn't exist. But looking at the graph the limit as the... the limit of the relative size as we approach three would be getting arbitrarily large or approaching infinity.

KP: So have you ever seen a graph like this before?

Norbert: I have seen a graph with this before.

KP: What kind of graph comes to mind?

Norbert: So the example I think of is one over x squared. So I guess one over, X minus three squared would be.

KP: You can write it down. If you don't want to try to keep it inside your brain.

Norbert: Ok so 1 over X minus three squared.

KP: OK

Norbert: So as I kind of... I go towards the tails of my graph. I'm getting smaller and smaller approaching zero but I'm never getting negative because it's a squared term. And as I approach three my denominator gets arbitrarily small so then my function values get arbitrarily large.

KP: So what would... say... So if we what... can we call this a function? Is that okay with you if we put an f(x) there?

Norbert: Sure.

KP: So that would be okay with you?

Norbert: That's fine with me.

KP:	All right. So this function here is (points to the written function) the function in the numerator, in this case, would represent what?
Norbert:	Would represent the magnitude of the... magnitude of the red bar.
KP:	OK. And then the denominator would be the...
Norbert:	Magnitude of the blue bar.

In this excerpt, Norbert described the behavior of the red and blue bars that would result in the behavior of the relative size of the red bar in terms of the blue bar (lines 3-10). When asked if he had seen any graphs that reminded him of the graph of the relative size of the red bar in terms of the blue bar, Norbert decided to write out an expression that resembled a rational function (lines 30-39). Norbert articulated that the numerator of the function that he had written represented the length of red bar and the denominator represented the length of the blue bar. In this task Norbert used covariational reasoning with his understanding of division to articulate the relative size of red bar in terms of blue bar.

Throughout the entire pre-interview Edwin and Norbert paralleled one another's understandings and responses to the tasks provided. The lack of discrepancies in Edwin and Norbert's mathematical thinking provided an even platform for identifying differences in their rational function instruction without having to consider their mathematical knowledge as being significantly different.

Approach to Promoting Discussion of Rational Functions

Differences between Edwin and Norbert were uncovered during classroom observation data collection. These differences were significantly based on Edwin and

Norbert's focus and approach to teaching rational functions. Edwin approached rational functions with more of a conceptual orientation (Thompson et al., 1994) that promoted discussions and conversations within his rational function lessons. This included asking open-ended questions that allowed for students to explain their thinking. Norbert approached rational functions with a calculational orientation (Thompson et al., 1994), which is more aligned with the traditional approach to teaching rational functions. Norbert was more of a lecturer and asked questions that represented small tests of student knowledge (such as rules or properties) but provided little opportunities for students to articulate their thinking (Mehan, 1979).

In preparing to teach module 6 on rational functions Edwin stated how he wanted his students to build an understanding of division and to a greater extent relative size (Excerpt 11) in his video journal.

Excerpt 11

Edwin: I want to think about a division of two numbers as a measurement scheme so... maybe I used scheme in a very wrong sense but I want them to think about it as a measuring process. So the denominator is, like your ruler, and the numerator is the thing you're measuring. So I'll start off probably with asking them simple numerical questions about division. So very similar to module three I think we should revisit this actually. Sorry module four, beginning of module four, starts with how many times does this compare to this and they use percentages instead of like relative size.

Edwin's repeated references to schemes (lines 1-2), a process (lines 3-4) , and division (lines 4-6) show that he planned to have his students construct an understanding of rational functions based on relative size. Also, he saw how this concept ties in with previously learned material and consequently planned to build on and revisit (lines 5-9) these previous understandings, thus creating a coherent curriculum (Thompson, 1994, undergraduate curriculum).

Norbert revealed in his video journal that he would construct his rational function lessons in more of a traditional manner. He concocted a plan where he laid out the formal definition of rational function and leveraged the students' knowledge of polynomials to find key features of the rational function (Excerpt 12).

Excerpt 12

Norbert: The topic of this module investigation is the introduction of rational functions and looking at the behavior of rational functions not quite end behavior but like the internal behavior of the rational functions. Primarily when approaching either singularities or vertical asymptotes. So the plan of attack is to first introduce what a rational function is and define it. And. I'm going to change up the order of the problems in the book a little bit. So I'd start off with either using Problem three problem three looks at. Arbitrary rational function and asks what values make the rational function undefined. So we're looking for values where the denominator function is zero. And what values make ah... the rational function equal to zero so what are the roots. And those would be ah… the numerator function. And the reason I want to do that

	first is because primarily ah... when looking at functions I feel you want to look for, I don't know, roots of the function, where the function equals zero. That seems like a good starting point and it will be a nice transition considering we just learned roots and umm things of that nature in the last module with polynomial functions.

Norbert chose a traditional approach to rational functions, where he planned to instruct students to find the values of the function that are undefined by setting the denominator of the function equal to zero (lines 8-10). Further in the excerpt, Norbert explained that the approach to finding the roots of the rational function was similar to polynomial functions, a previously covered concept in the course (lines 12-17).

The dichotomy between Edwin and Norbert’s approach to teaching rational functions persisted when they taught the first lesson. Edwin implemented his plan to draw attention to the division of quantities and leveraged student understanding of division to construct an understanding of rational functions (Excerpt 13).

Excerpt 13

Edwin:	All right so rational functions have a lot to do with measuring stuff. So I'm assuming you guys have an understanding of measuring things. Let's give a... let's start with some appropriate measurements that we might use to measure different things like for example, if I wanted to measure my height, what are some appropriate units of measure that you would want to measure my height in?
Student:	Feet.

Student: Centimeters.

Edwin: So I heard a couple answers, feet, centimeters, meters. Why are these relatively good unit of measures to measure my height with?

Student: Keeps our numbers like with in a certain range. It's always like one to a 100, it's not like one to 400ths of a mile.

Edwin: OK, yeah. So you... you wouldn't use miles to measure me, right? I'm assuming, why?

Student: Too large.

Student: It's too big.

Edwin: Why? What's too big, be specific?

Student: The mile.

Student: The thing you are measuring.

Edwin: The thing you're measuring with is really big compared to the thing you're measuring, right?

Student: (As a class answer) Yes.

Edwin: So for that reason not only is it difficult to see what the corresponding measurement is but you end up getting a number that you might feel very comfortable with, right now. Today though we're going to be talking about, well what if we do measure things with really big... measuring sticks and what if we measure things that are really big compared to the measuring stick? What does that do? What happens to the output? Maybe you guys... how big... how tall do you think I am?

Student: (Multiple students respond with varying heights around 5 feet)

Edwin: 5'8"? Did you just say 5'1"? 5'1" so I noticed you guys are using... first of all are you using two different measuring sticks to measure me. That's kind of unique, in my opinion. What if you were to measure me with one? So you are using feet and you are using inches to measure me. Let's start... let's stick with just one measuring stick. How tall do you think I am in terms of one measuring stick?

Student: 60 inches

Student: 68 inches

Edwin: 68 inches. What about feet?

Student: 5 feet.

Student: 5 and 8/12ths.

Edwin: 5 and 8/12ths. Why do you say 8/12ths?

Student: No that was a lie.

Student: 8/12ths of a foot.

Student: Oh yeah that’s true.

Edwin: 8/12ths of a foot so that would represent five feet and eight inches.

Student: (Multiple students respond) Yes.

Edwin: So five feet and 8/12ths is five feet and eight inches. All right. So I want you to get used to measuring because that is all we're going to be doing in all of this module. We are going to be measuring. We are going to talk about what happens if your measuring stick is really small? What happens if your measuring stick is really large? What happens if the thing you are measuring is really large? And what happens if you are measuring... the thing you are measuring with is really small? So let's give an example, I have this whiteboard right here in front of me and let's say I want to measure it in terms of this marker. How many markers do you think I could fit into this whiteboard?

In this excerpt Edwin asked questions that had multiple answers and pushed the students to discuss measurement when comparing two quantities (lines 2-6, 13-14, 32-36). This discussion allowed Edwin to build a connection between the students meaning of measurement and division which allowed him to set up for a later discussion of rational functions. Norbert implemented his plan to draw on the similarities of polynomial functions and rational functions (Excerpt 14).

Excerpt 14

Norbert: So today at this point we really have only had exposure to these types of functions (points to the list of functions on the board). Really we have only had exposure to exponential and polynomial functions and some special cases of polynomial functions. So we're going to build a new type of function from some of the functions that we have already seen. So I'm going to define a rational function. And what a rational function is... is it's a function? Obviously. That doesn't make poor decisions. It's a function that is a ratio of polynomial functions. So q(x) is a polynomial, p(x) is a polynomial. So my first question is, should we be worried that taking a ratio of polynomials is actually a function? So what makes a function, a function?

Student: For every input there is an output.

Norbert: So for every input there is an output that is true, but how many outputs for every...

Student: One?

Edwin: There is exactly one output for every input. (writes on the board) One output for every input. So q(x) is a polynomial, p(x) is a polynomial. That means I plug in an input, right? So I plug in an input. This is going to give me... q(x) is going to give me one output, p(x) is going to give me one output, so I'm only going to get one ratio, one fraction for a value of x. Does that make sense what I am saying here? (mumbles of agreement) So because q(x) and p(x) are functions, if I plug something into h(x), I'm only going to get one number for q(x), one number for p(x). So I'm only going to get one fraction at the end of the day. So we're ok here? No function illegality happening. So the first thing I want to look at is domain for rational functions. So what do I always say? What number do we not want to divide by?

Student: Zero.

Norbert: Zero because we will rip a whole in the universe and kill everyone. So for this general form h(x) is equal to q(x) over p(x), where both of those polynomials, where are the cases where I am dividing by zero?

Student: When p(x) equals zero?

Norbert: When p(x) equals zero, right? So p(x) equals zero, great. Now what if p(x) equals zero? What does... What does that mean? What are the values of x so that p(x) equals zero? Pretend I'm a gardener. What am I going to be working with? (Makes pulling action with hands) My roots.

Student: Oh. (laughs from students) What were you doing?

Norbert: I was... I was pulling weeds or carrots of something. So I'm going to be looking at the values of X so that p(x) equals zero or the roots of p(x). So we can say that our domain is all values of X, real numbers, so that p(x) is different from zero. Where we will run into trouble, are the points where p(x) equals zero for the roots of our bottom function. So does that make sense? What we are doing here, any issues? So for example, if I had h(x) equals one over one minus x, right. So one is a polynomial, one minus x is a polynomial we have a rational function here. Can someone tell me what my domain of my rational function is?

Student: Ah negative infinity to 1.

Norbert: OK.

Student: And then 1 to infinity.

Norbert: Right. So we look at the roots of one minus x, it's simply just one. So we have to remove one from our domain. We start off with all the reals and we take out the point so that my bottom function is equal to zero. My bottom function is equal to zero when x equals one so we're

	removing that point from our domain. So any issues here?

In this excerpt Norbert started his lesson exactly as he had planned, as a review of the functions covered in previous sections (lines 1-11). Throughout this lesson Norbert asked questions that required the students to give short calculational responses (lines 12, 15, 28). Norbert provided generalized forms of rational functions (lines 16-27) and gave students the properties and characteristics of rational functions (lines 38-46). This was a common theme that saturated all aspects of Norbert's instruction of rational functions. He acted as the "sage on the stage" where Edwin took on a role that emulated the "guide on the side" (King, 1993).

In the post interview of phase one, Edwin and Norbert were asked if they would teach rational functions in the same way as they did this semester (Excerpt 15 and Excerpt 16).

Excerpt 15

KP:	All right . So why did you decide to start teaching rational functions in this way?
Edwin:	So I feel like a lot of students have this preconceived notion that like if I want to talk about measurement... measuring things like I feel like these students have these preconceived notions that the thing you're measuring has to be bigger than the thing you're measuring with and it has to be like bigger for a certain amount like it can be huge it has to be like a sweet spot like... like what this guy here said. Like the numbers got to be around 1 to 100. And like I wanted to break that way of

thinking like right off the bat so I want to just think about OK I know you feel comfortable with this. These units of measure, but in this lecture we're going to be talking about you know measuring things where it really doesn't matter what we what the size of the thing you're measuring with and really doesn't matter what the size of the thing you're measuring.

KP: Ok would you start the lesson the same... Like this whole entire rational function lessons would you start this the same way? If given the chance.

Edwin: Probably.

KP: Yeah?

Edwin: I would say so.

KP: OK. Have you started rational function instruction like this in the past?

Edwin: No.

KP: No. First time?

Edwin: First time.

KP: OK. Why did you want to start rational functions in this way?

Edwin: Again like I felt like I had this new powerful notion of like measurement... measurement and I wanted to convey that to... to the students. And like I never did that before like I would normally just like I initially just like go out and say this is a rational function. This is a vertical asymptote. This is a horizontal asymptote like a traditional

	ways of thinking about it and that's kind of what I did.

In this excerpt Edwin expressed his concern that students usually think of fraction in the sense of a smaller quantity measuring a bigger quantity which results in a number greater than 1 (lines 3-15). Edwin stated that he wants students to move away from and expand their understanding to include fractional results of comparing two quantities as well. Edwin explained that this was the first time that he taught rational functions in this manner (line 25) and that he would probably teach it the same way again (line 19), if given the chance. He elicited his feeling about teaching rational functions in a conceptual manner as a powerful notion rather than his previous ways of teaching rational functions (lines 27-32).

Excerpt 16

KP:	So it is going to be a little awkward on that. So we're going to go ahead and start with your first day. (Shows first clip. You can hear the video playing.) All right. So my question is why did you decide to start rational functions in this manner?
Norbert:	So we're just trying to... The idea was to create a bridge between what we've already... what we've already learned and really I think any good math class is, the more you can set up a math class like a story the better... I think the better the understanding comes and that's just the real... the real reason is, hey we really only looked at these types of functions. These aren't the only ones out there. But here's how we can take the ones we've already looked at and start creating new ones.

KP: So who does this start up help? Like does it help you? does it help your students? does it help the overall like structure pathways? When you started the lesson in this way, what did you view it as being the most beneficial to?

Norbert: I do that as being the most beneficial to the students in the sense that so I think because I've noticed with a lot of stuff like when we went from linear to exponential functions it was kind of once we started the exponential functions nothing is ever going to be linear again.

KP: Oh.

Norbert: In the sense like there's the kind of like when you're learning topics their to sep... separate entities and....

KP: Like it doesn't exist anymore?

Norbert: Exactly like everything prior doesn't it. So that's.. that was kind of the main idea was, hey like this stuff is still... still existing and it's actually has a big role in what we are doing today.

KP: Would you still start your lesson off this way? Would you start rational functions off this way again?

Norbert: I would. Given the chance again, I would make the connection to prior stuff.

Norbert clearly explained that he introduced rational functions in this manner because he felt like he was making a bridge from the students' prior knowledge (lines 5-11, 16-19). Norbert elaborated on his reasoning for wanting to connect the students prior knowledge by explaining his feels that students seem to think everything in a prior section of the curriculum doesn't exist as soon as a new section has begun (lines 24-26). Norbert stated that if given the chance he would start his rational function lessons in the same manner (line 29-30).

Throughout these excerpts Edwin demonstrated a conceptual approach to teaching rational functions whereas Norbert took a calculational approach. Edwin's conceptual approach to rational functions allowed for students to be more involved in their learning. Edwin was able to become a guide in the students' learning allowing him to ask questions that caused the students to articulate their thinking and encouraged class participation. Norbert's approach to rational functions was calculational and allowed for some class participation. What is most striking between Edwin and Norbert was the approach to teaching rational functions. Edwin willingly changed his approach to teaching rational functions by creating conceptual lessons that stimulated his students and diverged from his previous calculational approach. Norbert stuck to the curriculum and played things safe throughout the rational function instruction.

Rational Function Lesson Reflections

When reflecting on the rational function lessons Edwin and Norbert differed on their focus. Edwin reflected on his actions and the actions and behaviors of the students. Norbert reflected on his teaching and what he wanted to accomplish in the lessons. This difference is focus during the reflection process gives insight into what the instructor felt was valuable to them (Lampert, 2003).

After teaching the first lesson on rational functions Edwin reflected on his preparation for the lesson and how this impacted his students (Excerpt 17 and 18).

Excerpt 17

Edwin:	I could of done way better I could have been more organized. In general I am little lazy when it comes to preparing. This is why I should prepare. I should be prepared for one of the student’s questions that asked, with regards to relative size in terms of what so I think this is a problem with...this is his problem. His problem was, well my problem that I didn't develop in his head I couldn't... I didn't give him the right environment for him to develop the right meanings. He was asking relative size, what is the unit of the relative size.

Edwin explained that he should have been more organized and prepared (lines 1-2) since he felt that his lack of preparation for the lesson affected his ability to answer his students’ questions (lines 2-5). Edwin admitted that his lack of preparation did not provide his students with an environment to help in developing the mathematical meanings that he wanted them to create (lines 5-8). As Edwin continued his reflection of the first lesson on rational functions he stated that the number of problems that are covered in class does not correlate to the students building meaning related to the mathematics (Excerpt 18)

Excerpt 18

Edwin:	I think it was a really good representation of an average day because I didn't really prepare so much. I should prepare a lot more. I need to work on that. But I have these meanings and I wanted students to develop these meanings so I slowly gradually work with that. I think I

did a decent job of that. So to give specific details as to why this was an average day in class. Like I don't use pathways very much. I try to but I really try to like take a few pathways problems. Very few, because there's like 10 pathways problems per investigation. And let's be honest, can students develop meanings from 10 problems? No they can develop meanings with like one problem I think. So I work with one problem but I work with it very long. And so we did problem number one and problem number two for our entire 50 minutes. And I think that did better for my students than just going through it really quickly and like that would have been... I think that would be detrimental to start talking about like notations that they'd never seen before and just like flood them with problems all over the place. I don't think that's really good.

Edwin articulated for a second time during this reflection of the first rational function lesson that he could have prepared more but that this lesson did resemble an average day in this classroom (lines 1-2). In his reflection Edwin stated he has meanings for rational functions he wanted to develop in his students (lines 2-4) and these meanings do not need to be developed using numerous problems (lines 6-10). Edwin continued to state that he felt that completing too many problems could be detrimental to his students (lines 14-17). Even though Edwin reflected on his actions, his major focus was on his students and how his actions affected his students. Norbert's reflection of the first lesson on rational functions focused predominantly on a mistake that he made on a single problem covered in the lesson (Excerpt 19).

Excerpt 19

Norbert: The topic was rational functions, vertical asymptotes in particular ah so I felt I did a reasonably good job of motivating the topics and we want to examine more types of functions up until this point we've only really looked at exponential and polynomial functions in the special cases linear and quadratic. The motivation was to build the functions from polynomial functions that we already know really well and tie in that really a lot of the a lot of the analysis is the same as the analysis of polynomials. So I think I did a pretty good job of making that point clear. We went through some problems mostly looking at started off looking at where rational functions are undefined. So looking for roots of the denominator function and where the roots of the rational function are and those occur when the roots of the numerator function so at the roots of the numerator function with a with a minor caveat that it can't be a root if it's not my domain. So if it's a root that shared between both the numerator and denominator function we can't have it. And we did problem, Problem 2. You kind of look at identify some of the behavior and function and with problem 2 it went pretty smoothly until I made a mistake and said that some said that some portion of the functions should be decreasing when it should be increasing and that caused some trouble. And we tried to graph the function. And because

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	of that, really lost a good deal of time of me trying to regroup and cover myself so that that was not good at all not good on my part. But the majority of the information that I wanted to cover it covered it I'm going to have to go over review a little bit of stuff for the next lesson.

When reflecting on his first rational function lesson Norbert focused on his performance by stating that he did a relatively good job (lines 1-5). He justified his motivation in connecting polynomial functions to rational functions (lines 5-9). Norbert articulated the problems that he covered in the class (lines 9-20) to include the mistake he made in the second problem (lines 16-22). Throughout the remainder of his reflection on the first rational function lesson Norbert continued to bring up his mistake from the second problem.

The reflection of the first rational function lesson for Edwin and Norbert differed in focus significantly. Edwin discussed his actions and the effects of those actions on his student educational environment. Norbert had a more egocentric reflection that focused on what transpired in the lesson but dwelt heavily on his mistake when completing a problem with his students. This difference in the reflective focus of Edwin and Norbert continued into the following rational function lesson reflection.

After teaching the second day of rational functions Edwin and Norbert once again reflected using their video journal. Edwin articulated that his lesson did not go well based on the behavior of his students (Excerpt 20).

Excerpt 20

Edwin:	I am slightly concerned because there was a lot of silence. There were a

lot of people trying to figure out what's going on. I can tell people were not enthusiastic about the material. I asked them to d... draw some tables, I could tell maybe one or two people actually did. So people weren't following directions very much. I could tell people weren't enthusiastic and so building meanings in that kind of environment seems to be a little difficult. So I could tell that there were a lot of points people where people were just like huh? And I couldn't understand some of the question still... I still can't.

Edwin expressed his concern about the second rational function lesson not going well by stating that there was silence (line 1) and that his students did not appear engaged in his lesson (lines 2-5). He explained that this unenthusiastic environment was not conducive to building meanings (lines 5-7) and that he could not at the time nor still understand some of the questions that the students were asking (lines 7-9). Edwin assessed the success of his lesson based on the student feedback that he was receiving through questions and overall demeanor of his students.

As Norbert reflected on his second rational function lesson he expressed his need to fill in gaps from the previous lesson and what he covered in the lesson (Excerpt 21)

Excerpt 21

Norbert: I started off the lesson with the review of module six, investigation one trying to fill in some of the gaps that I needed to cover on Wednesday. And that review went pretty well I think I covered all the holes that I missed and my students seemed to understand the topics that were taught on Wednesday. I wanted to motivate end behavior of rational functions by... This is a continuation in the sense of end behavior of polynomial functions because I built my rational function as a combination of polynomial functions. So we started off looking at a case from the book where the degrees of my numerator and denominator functions are equal. In this case we approach the ratio of the leading coefficients of my polynomial functions I make up my rational function. I finagled the book problem a little bit just to give coefficients different from one for both of my polynomials in my combination in my rational function just to see that we're actually approaching a ratio of the leading coefficients. I really wanted to stress that even with rational functions it is very similar to just looking at regular polynomials. We are looking at really the leading coeff... the leading terms of my polynomials in my rational function to tell us the whole story.

Norbert focused his reflection on the events that occurred in the lesson and his motivation for those events. Norbert started his reflection with a discussion of the review and continuation of Wednesday’s material that was not covered (lines 1-5). Norbert reflected on the events of his second rational function lesson through the description of his focus of end behavior of rational functions and the connections that can be made from end behavior of polynomials (lines 5-8, 15-19).

The reflection of the second rational function lesson for Edwin and Norbert differed once again in the focus of their reflection. Edwin discussed how his lesson was less than ideal because of the behavior that he saw in his students throughout the lesson. Norbert reflected on the events of his lesson discussing the connection that he wanted to make between the end behavior of polynomial function and the end behavior of rational functions.

The completion of the final rational function lesson ended with a reflection on the lesson by both instructors. Edwin reflected on his lesson with an emphasis on the meaning behind the definitions of rational functions (Excerpt 22).

Excerpt 22

Edwin: I think I embedded the meanings that I wanted inside of all of those definitions. That's basically what we did... what I just covered definitions that these students need to really learn but I really wanted to emphasize meaning behind those definitions and I think the textbook didn't really help me out with that so much. So I went on my own with that. They were able to solve all the problems. They were able to understand what was going on, I think at least. But they kept going back to the high school things that they learned and there was no meaning behind them they kept asking... like the one girl, she asked like isn't like a horizontal asymptote like the thing you get really close to but you never touch. I was just like huh that is not the meaning that we want in fact this is why we don't draw out the dotted lines and instructors like if you think about a dotted line and think about what

	that graph looks like it hugs (air quotes) the vertical line. Oh I don't like that. It's just so geometric.

Edwin stated that he thinks that he was able to embed all the meanings into the definitions for vertical and horizontal asymptotes of rational functions even though the textbook, in his opinion, did not set up the definitions the way he wanted (lines 1-5). Edwin expressed his irritation with the students reverting back to their previous knowledge of rational functions (lines 5-15) instead of applying the meanings to the behavior of rational functions that he was trying to develop with them for the previous two lessons.

After teaching his final lesson on rational functions Norbert reflected on his performance and his concern about covering too much material in his lesson (Excerpt 23)

Excerpt 23

Norbert:	I probably went over a little bit more about limits than was required just because I felt that that would be good for the students when they go along, moving on towards calculus in the next, next years. And the one thing I definitely could have improved on, I was a little bit informal with the explanation for limits, certain limits and what that means. I wanted to...so we have certain cases when we approach horiz... a vertical asymptote where we both shoot off to infinity both shoot off to negative infinity and so on and so forth. And I wanted to make the point that if they both shoot off at the same direction the behavior is the same then the limit still does exist even though it's an

11	infinite limit. And that might have been a little bit confusing I think I'm
12	going to review a little bit and kind of go over the topics some more to
13	clarify some things.

In this reflection Norbert expressed his concern that he covered too much material (lines 1-3) and might have confused his students with an informal definition of limits (lines 3-11). Norbert reflected that he would need to review limits with his students in order to clarify some of the misunderstandings that might have happened in the final lesson of rational functions (lines 11-13).

During Phase 1 data collection Edwin and Norbert's reflections on their rational functions lessons differed due to their focus. In his reflection, Edwin utilized his students as a way to measure the success of his lessons, which led him to identify areas for him to improve as an instructor. In contrast, Norbert reflected on his actions and performance as an instructor and used himself (rather than his students) as the measure of his success.

Summary

The results found in Phase 1 suggested that similarities and differences existed between Norbert and Edwin. The pre-interview of Phase 1 revealed that Norbert and Edwin had similar ways of thinking about measurement, division, relative size, and covariation in a conceptual manner. The lack of discrepancies in Norbert and Edwin's mathematical thinking provided an even platform for identifying differences in the novice and experienced instructors' instruction on rational functions without having to consider their mathematical knowledge as being significantly different. In other words, both Norbert and Edwin demonstrated that they possessed the schemes that would support a conceptual approach to rational function instruction. After the pre-interview of Phase 1 the differences between Norbert and Edwin became more apparent in the video journal, classroom observations, and post-interview.

Norbert and Edwin planned and executed different approaches to teaching rational functions in Phase 1. Norbert approached rational functions in a calculational way by presenting definitions and rules found in the curriculum. The questions Norbert asked of his students during the classroom observations required short numerical responses. Edwin approached rational function in a conceptual way by leading a discussion on measurement, division, relative size and covariation. The questions asked by Edwin in the classroom observations gave students the opportunity to discuss and describe their thinking of the mathematics. The reflections completed after the classroom observations revealed additional differences between Norbert and Edwin.

In the video journal reflections of the Phase 1 data collection Norbert and Edwin reflected on their teaching of rational functions using a different focus. Norbert primarily reflected on his actions and performance as an instructor using himself as the measure of his success. Edwin, on the other hand, utilized his students as a way to measure the success of his rational function lesson and this focus led him to identifying areas of improvement that he could make as an instructor.

Additional differences that were found in Phase 1 involved the amount of time spent on lesson preparation and student involvement. Norbert planned a lesson no later than 18 hours before teaching the lesson. Edwin prepared lessons approximately 30 minutes before teaching the lesson. Norbert and Edwin also structured their daily rational function lessons different. For example, during the classroom observation of Phase 1, Norbert was at the board instructing 90% of the class while the other 10% was dedicated to asking students questions and waiting on student responses to questions. Edwin was instructing the class 50% of the class while 50% of the class time was dedicated to asking students questions, letting students work on problems, and discussing the mathematics as a class.

The differences found in Phase 1 between Norbert and Edwin pointed toward a distinction between the two instructors. Norbert exhibited characteristics of a novice instructor through his ability to articulate his mathematical content knowledge, choice of his teaching approach to rational functions, and his lesson reflection focus after teaching rational functions.

After characterizing the differences between the novice and experienced instructors, the next phase of the study focused on how the practices of the novice instructor were perturbed by the addition of a virtual manipulative serving as a didactic object.

CHAPTER 6

RESULTS OF PHASE 2: PERTURBATIONS IN NOVICE INSTRUCTOR'S CLASSROOM PRACTICES

This chapter presents selected data and results from Phase 2 of this study. This chapter does not include a full analysis of every task and observation, in the interest of readability. Instead the focus of the chapter is on an analysis of presented episodes that were representative of the perturbations the novice instructor, Norbert, experienced while teaching rational functions with novel virtual manipulatives serving as didactic objects. It is important to note that I am not saying that Norbert thought of the virtual manipulatives as didactic objects. Indeed, initially Norbert viewed the intervention activities simply as applets and it is precisely the trajectory along which he internalized them as didactic objects that was the focus of this research.

The results presented in this chapter were obtained through the interview transcriptions and video analysis of data collected in Phase 2 of the study. In each of the following sections, I present results that pertain to the effects of novel didactic objects on a novice instructor's mathematical classroom practices and changes that occurred in the novice instructor's mathematical meanings and mathematical classroom practices for rational functions. This chapter is broken into sections that correlate to the taxonomy of aspects of practices perturbed by the novel didactic objects of this study (Table 5).

Table 5. *Aspects of Practice Perturbed by the Novel Didactic Objects of this Study*

Aspects of practice	Description
Leader Actions	How instructor perceives Moving Vectors, Sum Bar, Rat Bar, and Rat Graph and how the instructor uses these virtual manipulatives in the classroom
Communication	Classroom discourse on rational functions surrounding Moving Vectors, Sum Bar, Rat Bar, and Rat Graph.

Expectations of Technology	What understandings about rational functions does the instructor expect students to develop
Roles and Responsibilities	Responsibilities of the instructor and students when using these virtual manipulatives
Student Engagement	Student participation while these virtual manipulatives are implemented in rational function instruction.
Mathematical Conceptions	How students perceive rational functions when these virtual manipulatives are implemented in instruction.

Leader Actions

Leader actions, in a mathematics classroom, refer to the instructor perception of the novel didactic objects and how the instructor uses the novel didactic objects in his teaching. Once exposed to all of the virtual manipulatives, from the intervention interview of Phase 2 data collection, Norbert explained how he saw the virtual manipulatives working with and becoming part of his rational function instruction (Excerpt 24).

Excerpt 24

KP:	All right. So let's talk about just a couple more questions and then I'll let you go ponder for a while. So how would you want to build these applets into your rational function instruction currently? Just spit balling ideas at this point.
Norbert:	So the as of right now my plan is... the most organic way to fit in there would be to have them as so... To use them as many examples, in the sense of like I usually kind of go through theory first and then provide examples of like let's take what we just discussed and do it that way. So maybe like having... having it pulled off to the side just ready so I can... I get to that example just make a couple of clicks and up on the board

not wasting any time.

Norbert explained the easiest way to incorporate the novel didactic objects into his rational function instruction was to make the virtual manipulatives into examples that would be covered in his lesson (lines 5-8). He hinted toward the necessity of the didactic objects being readily available for him to quickly have the didactic objects displayed (lines 8-11). This initial perception of the didactic objects suggested Norbert's willingness to use the didactic objects in his rational function instruction with the caveat that the didactic objects caused as little disruption as possible within his lessons on rational functions. Norbert continued to articulate his perception of the didactic objects as the intervention interview continued (Excerpt 25).

Excerpt 25

KP:	So how do you see this fitting into like the last semester stuff that you did?
Norbert:	So with the last semester's... (Long pause)
KP:	It's been a while. I know it's OK.
Norbert:	The last... I don't... I think I would need to do a little more... a little more surgery with my lessons because kind of the context in which I set... I think the context in which I set it up last semester is not really... at least right now at this point in time I don't see the... I don't see how I can like smoothly put these guys in there because I think the way that I set it up was more as this is just an extension of polynomial functions not so much as this is.

When asked about fitting the novel didactic objects into his rational function lessons from the previous semester, Norbert described the need to alter his lessons (lines 5-7) since his set up to rational functions in the previous semester was an extension of polynomial functions (lines 8-11). This was the first indication Norbert articulated in Phase 2 data collection that suggested the task of implementing the didactic objects would not be as easy as making the didactic objects examples to be covered as a class. Near the end of the intervention interview of Phase 2 Norbert explained that he saw value in the didactic objects but was concerned that his own shortcomings and inabilities would take away from the benefits students could get from the didactic objects.

As Norbert conducted his planning session for the first rational function lesson where the didactic objects would be implemented, he once again articulated his hesitant willingness to use the didactic objects (Excerpt 26).

Excerpt 26

Norbert: So I'm going to, for the first time, incorporate some applets into my lesson. I normally don't use technology but in this case I will…. So I'm planning on using the applets for this first lesson when introducing… I'm just going to be using the... the vert... the applet that.. What's it called? The moving vectors applet. So we have two graphs of functions and we identify the values of the function with the vectors from the… emanating from the x axis. So my plan in using this is... last time I taught rational functions I really looked at teaching rational functions from very algebraic perspective in the sense of just looking at the operations where we are doing operations on the output in an algebraic sense. So what that kind of misses is that we have a lot of questions and a lot of problems we do where we are either given graphs or tables and we don't really have a rule, a function rule at our hands in certain situations. So incorporating this applet this will kind of... This will provide a way to do these sorts of things with the rational functions even if we don't have a rule at hand and say we just have a graph for example.

Norbert explained that he would be incorporating technology into his lesson for the first time (lines 1-2). Norbert described his lesson in the previous semester as very algebraic (lines 7-11) and missed the connection to problems where the function rule was not provided (lines 11-14). He perceived the didactic object, Moving Vectors, as a way to motivate a connection to problems given without function rules (lines 14-17). Further into his planning of the first rational function lesson with the didactic objects, Norbert described his reservations for teaching with the didactic objects (Excerpt 27).

Excerpt 27

Norbert: I have some slight reservations about using the applets not... not because of the merit of the applets I think they are great but I just... I'm not the best when it comes to technology and I haven't really incorporated technology in my lessons so I'm just a little bit nervous about that, how it's going to transition. Just the smoothness of that, but I have... I have the idea that my students are going to like this a lot more having... because my drawings aren't really the best so having an actual nice computer simulation are probably very beneficial. And I think my students are really going to... So when I implement these applets I think they're going to like it a lot more than what I normally do. Maybe I might have a change of heart and use technology more in the future. In terms of my planning it's actually made the planning process a little bit less time consuming because I have something that I can put my hands on... and put my lesson around this. So I'd say it's actually easier using these applets and... I really... I am excited about using the Moving Vectors applet.

In this excerpt, Norbert described his hesitation in using the didactic objects (lines 1-2) because of his limited experience integrating technology into his lessons (lines 2-6). But then he began reflecting on the possible positive contributions that didactic object could make for rational function instruction. He predicted that his students would enjoy the didactic objects because they look better than his drawings (lines 6-11). He reflects further stating that this experience might have him incorporating technology into his lesson in the future (lines 11-12). Norbert also perceived the didactic objects as making his lesson planning easier (lines 12-15) and he was excited to use the didactic object, Moving Vectors, in his upcoming lesson (lines 15-16). These sentiments of the didactic objects making the lesson planning for rational functions easier and the overall excitement in using the didactic objects in the rational function lesson were a common theme throughout Norbert's planning sessions from his video journaling. In terms of leader actions, Norbert experienced a perturbation that had a positive effect on his lesson planning for rational functions even though he remained hesitant in integrating the didactic objects into his lessons.

During the first lesson Norbert defined rational functions in a different way than in the previous semester (Excerpt 28).

Excerpt 28

Norbert:	Rational functions are functions that measure the relative size of one polynomials output with respect to another polynomials output. So I have used the term relative size frequently in this class. When I say relative size what idea are we thinking about? So if I want to... Suppose I had a stake with length 5 inches. Stick A has length 5 inches and Stick B has length 3 inches. If I want the relative size of Stick A with respect to Stick B, How do I compute that?
Student:	Divide?
Norbert:	Divide. Which one do I divide by which?
Student:	Five by three.
Norbert:	I divide the relative size of Stick A by the relative size of Stick B. Outstanding. So suppose I give you two polynomial functions, p(x) and r(x), and I wanted to find a function h(x), this is going to be a rational function that measures the relative size of the output of p(x) with respect to the output of r(x). How might I set that guy up? Similar to the relative size example we just did with the lengths of sticks.
Student:	p(x) over r(x).

Norbert: p(x) over r(x), outstanding. p(x) divided by r(x), okay? And algebraically... so if we are given nice formulas for our p(x) and r(x) polynomials, computing the value of h(x) is very straight forward.

Instead of starting off his lesson with a review of polynomial functions, as in the previous semester, Norbert framed rational functions more conceptually, introducing them as the relative size of one polynomial with respect to another polynomial (lines 1-2). He provided the students an example with stick lengths so that they could form an analogy between relative size of sticks with outputs of polynomials (lines 4-11) and related the stick example back to the construction of a rational function using two polynomial functions (lines 12-20). When asked in the post-interview about his approach to teaching rational functions Norbert explained that he wanted to connect rational functions to the comparison of two objects (Excerpt 29).

Excerpt 29

Norbert: So the... the connection that I wanted to make was... so... we are starting off with a definition and our idea is that we want to look at two polynomials and measure the relative size of one with respect to another. So I could have... I could have made more clear why we might want to do that if we had a polynomial function that measures something and we wanted to compare these two objects that are being measured by these functions. So that was the big idea, which is just that it might be useful at some point in a practical application to measure relative sizes of polynomials. So then we've looked at relative sizes for in terms of just actual values because output values are going to be just actual values. Looking at relative size of two polynomial outputs there's really no difference in how we think about these things. Or work with these rational functions. So that was the big idea, just connecting this to previous material looking at... relative size. What does that even mean when we were talking about that past and how does that relate to this new function we're defining.

Norbert articulated his desire to have students see the big idea of rational functions connecting to relative size (lines 7-9) for the purpose of a practical application and that this connection could be made by having students relate previous material to the definition of rational functions (lines 12-16). Norbert’s leadership exhibited two distinct perturbations when introducing rational functions with didactic objects. The first perturbation is related to how he defined rational functions as the relative size of one polynomial with respect to another polynomial.

The second perturbation was highlighted by Norbert’s word choice when he discussed the algebraic form of rational functions. In Excerpt 28, lines 17-19, Norbert explained that if given “nice formulas” calculating the value of the rational function is straightforward. The term “nice formulas” was used two additional times in the first rational function lesson when Norbert introduced the didactic object Moving Vectors. When asked about what he meant by “nice formulas” he responded that his students feel more comfortable with algebraic forms (Excerpt 30).

Excerpt 30

KP:	So in this clip you say nice formulas. What did you mean by given nice formulas?
Norbert:	Really just given a formula at all. So I've noticed that with my class, my students really like the formula stuff better. Those type questions on homework and exams. And the ones where they're given graphs or tables not so much which I think it has something to say with like I kind of focus more on the algebraic stuff than tables and graphs and that sort of stuff.
KP:	OK.
Norbert:	So that's all I meant was just really I guess in this sense every formula would be nice but I mean these like these nice closed form sort of things that always seem to work out nicely when do our problems.
KP:	So then when given graphs, are they not nice?
Norbert:	So yes. At this point in time, I think my students would say that they're not nice because it's harder to tell what's going on. Well in terms of doing arithmetic... arithmetic operations it's harder to tell what's going on.

Norbert described “nice formulas” as any formulas that are provided (line 3) especially the ones that work out nicely (lines 10-12). When pushed further on whether tables and graphs could be considered “not nice,” Norbert reported that these representations are not nice in the eyes of his students (lines 14-17). Norbert hypothesized that his students might prefer algebraic forms over other representations because of his emphasis in the classroom (lines 6-8).

Prior to teaching rational functions with didactic objects Norbert had not made the distinction that one representation of rational functions was nicer than another. Norbert's distinction between the two representations in Phase 2 reflected what he thought his students found to be nice about mathematics. The distinction made by Norbert suggested that his practices associated with leader actions were perturbed. Norbert changed his word choice based on his perception of his students and how they would react when given a graphical representation of a rational function, which led to his actions changing in the classroom.

Norbert identified another perturbation during the post-interview of Phase 2 when participating in stimulated recall where he watched clips of his teaching of rational functions from the recorded classroom observations. When discussing changes that he made to his teaching of rational functions Norbert explained that he had come early to class each day that he taught rational functions (Excerpt 31).

Excerpt 31

KP:	You had made mention that you came early to set up. So how early do you usually show up to class?
Norbert:	So I usually... four or five minutes before the class starts in when I get there.
KP:	So how much earlier... how much earlier did you feel like you were getting?
Norbert:	About I'd say 10 minutes earlier to get myself... I tried to get there at

8		45. Give myself fifteen minutes before class started.
9	KP:	Have you continued to show up that early to class?
10	Norbert:	No not any more.

Norbert remarked when teaching topics outside of rational functions he would arrive four to five minutes before class began (line 3-4), but, when teaching rational functions, Norbert explained that he arrived about ten to fifteen minutes before the start of the class (lines 7-8). However, Norbert no longer attempted to show up to class as early once the rational function lessons were completed (line 10). This finding suggested that the addition of the didactic objects to the rational function instruction perturbed his actions as a leader. Instead of arriving five minutes to the start of class, Norbert made sure that he was in class approximately three times earlier. This finding might seem mundane in comparison to other perturbations mentioned previously in this section. However, this finding demonstrates Norbert’s attempt to reestablish equilibrium related to his established comfort level of preparedness immediately before class began. Norbert spent the additional time before class checking the projector and making sure that the didactic objects were readily available on the computer in the classroom.

Communication

Communication refers to the discourse surrounding the implementation of the novel didactic objects in rational function lessons. In general, communication can be considered challenging since conversations are analogous to nonlinear, chaotic systems (Thompson & Thompson, 1994). Communication in a classroom presents a new level of difficulty since this setting requires upwards of thirty-five individuals to articulate their thinking, attempt to understand one another, and build additional ways of thinking. Introducing novel didactic objects to the environment further increases complexity and potential perturbations through additional cognitive and attitudinal constraints.

During Norbert's rational function lesson with novel didactic objects he exhibited perturbations associated with the discourse surrounding the novel didactic object. The first instance of the identified perturbations was in Norbert's reflection on the first rational function lesson where he introduced the novel didactic object, Moving Vectors (Excerpt 32).

Excerpt 32

Norbert:	Overall I think the lesson went pretty well. I covered the majority stuff that I wanted to cover. My students seemed pretty responsive and I felt... felt pretty good about teaching with the applets. It went smoother than I thought it would. Going in before class and preparing everything really did help. And I think my students did pretty… pretty well during this lesson. They were responsive, about as responsive as they normally are. It's kind of hard to get them to answer questions sometimes they don't feel like doing that but for the most part I think it went pretty well. Today's lesson is a pretty average representation of what's... what happens in class. I have a couple of students who are pretty good at answering questions and the rest are more... I wouldn't say they're not participating it just they don't... they don't like to talk as much which I can't... I can't fault them for that. Not everyone needs to talk every lesson, every day.

In this excerpt Norbert discussed how he thought the lesson went (lines 1-2) even with the addition of the novel didactic objects (lines 2-6) and described that his students were as responsive as usual (lines 6-9). Norbert explained that the students who do not answer questions could not be classified as not participating (lines 10-14) and not all students need to talk during the lesson (line 13-14). Norbert exhibited a difference in his reflection of the Phase 2 lesson with the novel didactic objects since he focused on his students' discourse in the classroom setting rather than on his own actions and performance as he had done in previous reflections.

When asked in the post-interview, prior to the stimulated recall segment, if he liked the level of participation he received from his students, Norbert discussed his acceptance of the level of participation he received (Excerpt 33).

Excerpt 33

KP:	So do you like the level of participation that you receive from your class?
Norbert:	Yes I do like it. I would always, I mean, if possible I would always like to increase the level of participation but a part of me is like you know they're paying for this class, they're here every day, you know how much really can I ask. But if given the choice that I would always want to increase the level of participation but with what I currently have, I'm happy.

Norbert quickly responded that he did like the amount of participation he received from his students (line 3) and explained that he would like the level of participation to increase (lines 3-4). However, Norbert articulated that he felt as if he could not really ask for more from his students since they were paying and attending to the lesson (lines 4-5). This excerpt confirmed Norbert's sentiments in the first reflection of the rational function lesson in Phase 2. Norbert struggled with wanting his students to be more responsive but believing that the level of participation he received was sufficient since he could not ask for more.

During the third rational function lesson, Norbert introduced the novel didactic object, Rat Bar, with a conversation about the graphs associated with the output value of a function (Excerpt 34) which he referred to as a "geometric interpretation (lines 5-6)."

Excerpt 34

Norbert: Earlier we made the connection that we are looking at functions, function graphs (Turns off board lights). We can identify the values of the function with a certain geometric interpretation (Turns on projector and brings down projector screen). So given a function, suppose this is what my graph looks like (Draws graph on the board). What geometric interpretation can I give to this point? …this function value? (points to the point made on the drawn graph) Let's suppose the value of that input is *a*, here is *f*(*a*). Does anybody remember the geometric object we attributed to that function value? (displays Rat Bar)

After Norbert asked his initial question about the geometric interpretation that can be given to a point, he received no response from the students (lines 2-6) so he decided to draw a graph of a function on the board and place a point onto the graph of the function. When the students remained silent (perhaps confused by what was being asked), Norbert added that the input value of the point is *a* and the output value of the point is *f*(*a*) (lines 7-8). Norbert rephrased his initial question by asking what geometric object could the students attribute to that function value (lines 8-9). This question was met with another long and non-responsive pause from his students.

During the stimulated recall segment of the post-interview Norbert viewed this classroom observation clip and was asked how he wanted his students to respond to his question leading up to the use of Rat Bar (Excerpt 35).

Excerpt 35

KP:	OK so this is where you start Rat Bar. (Plays the next clip) So how were you hoping for them to respond? Because they don't respond the way you want them to... no one responds... So what were you hoping that they were going to give you?
Norbert:	Because like what I was hoping was… I mean I'm pretty sure I said this exact same line of dialogue two days ago. I was hoping that someone would just be like, “Oh he wants an arrow. This is exactly what he asked us the other day.” That's... That's what I was looking for was put a frikkin vector there.
KP:	OK. What quantity…
Norbert:	But I'm not bitter or anything.
KP:	…Obviously, what quantity are you really wanting them to really attend to?
Norbert:	So I want them to focus on the output value of this... at this input value x and that that corresponds to a vector emanating from the x-axis.

KP:	All right.
Norbert:	Because it's like… I kind of like... the idea is just like the... It's... I always come back just beating them over the head with stuff that's like I'm not going to let you guys not understand what this is. I want this to be at the point where it's second nature.

Norbert expressed that he wanted his students to realize an arrow should be placed on the graph to represent the output value of the function (lines 5-8). Norbert expressed his frustration that students did not pick up on his reference to the earlier lesson that included Moving Vectors (line 9), but reflected that he was not “bitter” about this failed moment during his rational function lesson (line 11). He further explained his desire to have the students see the output value of the function as represented by a vector emanating from the x-axis (lines 14-15) and how this representation should be second nature to the students (lines 19-20). Norbert exhibited perturbation around the communication, in this case lack of response from his students, surrounding the introduction of the novel didactic object, Rat Bar. The fact that his frustration was still so vivid during the post interview that occurred a week after he completed all of the lessons on rational functions indicated a lasting perturbation surrounding the communication involved in teaching rational functions with the didactic objects. This was in stark contrast to the uneasy contentedness that Norbert previously associated with his students’ level of responsiveness and communication in the classroom.

As the stimulated recall segment of the post interview of Phase 2 continued, Norbert watched an additional clip of his third rational function lesson when implementing the novel didactic object, Rat Graph (Excerpt 36).

Excerpt 36

Norbert:	What does that mean about the magnitude of the output? Is it increasing or decreasing? (Checks watch)
Student:	Increasing.
Norbert:	Increasing. Outstanding. So as X approaches zero from the left, the magnitude is increasing. OK. (Looks at watch) So now we need to find which way are our ultimate vector is pointing. So let's look at this one more time. (Looks at watch) So as we move towards zero, which way are our numerator function vectors pointing? Up or down? (Looks at watch.)

In this clip, Norbert exhibited behaviors that had not been seen in previous lessons, such as frequently checking his watch (lines 1-2, 4-9). Later in the clip Norbert began jogging in place as he waited for his students to respond to a question he posed. After asking questions Norbert expressed unrest by checking his watch multiple times and jogging in place. After watching the clip of his behavior in the stimulated recall segment of the post interview for Phase 2, Norbert articulated his reasons for his behavior (Excerpt 37).

Excerpt 37

KP:	(Plays the video clip) So you asked a question. So you jog in place and

2		it's the oddest thing. I'm trying to figure out, why?
3	Norbert:	So if I remember this correctly I just... I think it was just like someone
4		for the love of God answer me; we don't have time for this.

Norbert explained his behavior as an outward display of unrest since he felt that there was no time for waiting on students to answer his questions (lines 3-4). This display and embodiment of unrest exhibited by Norbert indicated an additional perturbation surrounding the discourse associated with the novel didactic objects. The novel didactic objects required Norbert to rely on his students to respond to his questions, which took additional time from the overall class.

Expectations of Technology

An instructor's expectation of technology refers to the mathematical understandings the instructor expects students to develop when implementing novel didactic objects into the mathematics classroom. In a mathematics classroom the expectation of technology exhibited by an instructor includes the proposed impact the technology will have on the students' mathematical understanding of the concept being presented, either as an amplifier or as a reorganizer (Sherman, 2014). Once exposed to all of the virtual manipulatives, from the intervention interview of Phase 2 data collection, Norbert explained what mathematical understandings that he wanted his students to develop by the end of the rational function lessons with the novel didactic objects (Excerpt 38).

Excerpt 38

1	KP:	So what mathematical understandings from module 6, do you want

your students to kind of walk away with? What understandings are you wanting them to have?

Norbert: Right. So the understanding and I would want them to have is... So I kind of... I kind of like to break down the rational functions into... Break it down into the three like three types... So really like the way that I view it is that three types like the classes are the types of end behavior you exhibit. Where there's the one where... So I'm going to zero in where really in the long run I think of OK... in the long run I'm one over some polynomial.

KP: OK.

Norbert: Or not even a polynomial... well it's still a polynomial but one over some power of x. Is what that... that's essentially happening in the long run. And then some other stuff is happening in between. And we have the means to find out what that stuff is happening in between the techniques just from regular polynomials, reduces to finding roots and so on and so forth but. So there's that class. There's the one where you approach some other asymptote where long run I'm just ending up like a constant function and then some stuff happening in between. And I think so now that I'm thinking about it. The last one is shooting off, some how. Where in the long run I'm looking like just a regular old polynomial.

KP: Oh a slant asymptote?

Norbert: Yes.

KP: Yeah.

Norbert: In the long run like that. So the way that I'm thinking about like this is I still keep it... keep it in those categories.

KP: OK.

Norbert: But now for the finding out what's in between I think that this will be a much better... much better way to do so with looking at the relative size of the vectors and then deciding, OK which way am I going to be pointing in the long run.

KP: OK.

Norbert: So the ultimate main thing is just that really mod... modulo like the combinations of the ideas, nothing we're doing is new. In the sense that we're just doing polynomials then we're just adding division, that's all it is. That's the main goal, this is not... this is not as scary as it looks.

In this excerpt Norbert explained his desire to have students think about the three cases of end behavior, namely a horizontal asymptote at zero, an asymptote at a non-zero constant, or the presence of a slant asymptote (lines 4-10, 12-17). Norbert discussed the same procedural understanding of rational functions in Phase 1 of data collection. However, in this expert Norbert continued his explanation of the understandings that he wanted his students to develop with more of a behavioral expectation. Norbert expressed how looking at relative size (lines 29-32) could help his students understand the behavior of the rational function outside of end behavior. The mathematical understanding of rational functions Norbert wanted his students to develop was altered by the prospect of teaching with the novel didactic objects. Prior to the Phase 2 intervention interview, Norbert described the understanding of rational functions he wanted students to develop in a procedural way. Once exposed to the novel didactic objects in the intervention interview, Norbert hinted at a change in his understanding for students by discussing the behavior of the rational function. This slight sway in focus from procedural to discussing behavior indicated the existence of a possible perturbation in Norbert's classroom mathematical practices with respect to the expectation of technology.

Norbert's video journal entry for planning the first rational function lesson further confirmed the existence of a perturbation (Excerpt 39).

Excerpt 39

Norbert: So in terms of key ideas and understanding I want my students to have, the big plan is I'm going to first introduce rational function so what is the definition. And so looking at ratios of polynomials we're looking at the relative size of one polynomials output with respect to another polynomial and start off with the lesson like looking at... OK let's do things algebraically suppose we have rules for these polynomials where I create my rational function and I have an example here where we're going to look at just OK we look at the value at the numerator function and then the denominator function and then look at the relative size of those guys. And the story that I've set up with rational functions is the... well not with rational functions... with functions in general kind of the prototype that I have had this year is that we say okay we introduce a new function and we look at vertical intercepts, x intercepts, or roots. How the function behaves whether it's invertible properties like that which we will look at first in an algebraic sense with rational functions. So this is more geared to be... I assume that my students have familiarity with rational functions or just like basic operations of functions. So this is going to be a good refresher of the algebraic... algebraic notions. In that sense looking at... still staying in the... in the land of algebraicness. We will then introduce the domain of rational functions. So we look at the values where our rational functions undefined and we get to the point where we say oh that's where our denominator function is zero because really the only bad thing that can happen is dividing by zero. So then we look at the two cases and what happens when we divide, that is when we have an undefined point. Those cases being holes where... so here we're still algebraically, we have a root of both the numerator and denominator so we can factor out a term x. Minus that root in our function, it identically looks like something else except at that point we just removed. Then we are done. And then algebraically looking at asymptotes we have that… we have a root of the denominator function. So our numerator function approaches a fixed value whereas our denominator function approaches zero. So we have something dividing by something smaller and smaller. It's going to blow up in some fashion, really not concerned at this point which way it blows up, either increase or decrease without bound but just the idea that it blows up. So next what we'll look at is... algebraically with asymptotes... is how do we know when it's going to…. which way it's going to blow up. Here we just kind of calculate some values approaching the asymptote and then also get to discuss the notation x approaching *a* from the left, approaching *a* from the right, x arrow *a* minus, x arrow *a* plus, so on and so forth. And I also have an example here where we're going to look at a function with an asymptote and get a rough, it's a very simple example, will get a rough graph of what this looks like just to kind of look at like what happens near an asymptote. And then after that I'll get the actual definition we have in the book where it's saying an asymptote occurs when we approach from one side of the asymptote we blow up either positive or negative and likewise from the other side blow up either positive or negative. So at this point I will not have used the applets yet this is going to be a refresher of the algebraic properties. For me particularly I like algebraic stuff when it comes to like functions and function rules more than graphical means. Which is kind of ironic because I want to be a geometer but in terms of like function and rules, I like the algebraic stuff better. So after doing all that at this point is when I'm going to introduce the applets. So I'm going to say like OK guys we've had situations where we don't have a rule at hand. This is not something new to my students and I'm pretty sure they don't like the examples where we're given graphs and have to do stuff so I'm going to be like, OK we have these examples, how could we do the same sort of analysis with say a graph, for example.

In this excerpt Norbert defined a rational function as a ratio of two polynomials and the relative size of the output of one polynomial with respect to the output of the other polynomial (lines 2-5) is the output of the rational function. After articulating this conceptual definition of rational functions, Norbert explained the first lesson for rational functions would start with an introduction to the algebraic form of rational functions (lines 16-20). He detailed the properties of rational functions to include asymptotes (lines 21-36) and the book definition of when an asymptote occurred (lines 36-49). Towards the end of Norbert's planning session for the first lesson he discussed the addition of the technology as a way to talk about the graphical representation of rational functions (lines 54-60). Norbert explained that this would occur when a rule for the rational function is not provided.

The sentiments shared by Norbert in the planning session for the first lesson indicate a perturbation in his expectation of technology. Norbert defined rational functions in a different way than found in Phase 1. He emphasized the relative size of the output values of the two polynomials that make up the rational function rather than defining the rational function in a traditional manner. Even though Norbert planned an algebraic lesson for rational functions, his description of rational functions was altered.

This once again hinted toward a perturbation experienced by Norbert implementing the novel didactic objects in his rational function instruction.

During the post-interview of Phase 2 data collection the existence of a perturbation became clear when Norbert was asked if he would have changed anything from his Phase 2 rational function lessons (Excerpt 40).

Excerpt 40

KP: OK. Would you change anything from your lessons?

Norbert: Yes I would change my lesson. So the first couple of... the first two investigations I did algebraic interpretation first and then geometric interpretation. The algebraic interpretation in both situations took a lot longer than I expected. So what I would do was flip those guys where I would do geometric interpretations first and then algebraic interpretations because the students are, I think more familiar with the algebraic interpretation and I think I took more time... spent more time on what they were already familiar with as opposed to the sort of new thing... new things and I wanted to introduce. So that's the biggest thing that I would change is just geometry first and then algebra.

Norbert explained how he would change his first two lessons on rational functions to emphasize the graphs more, which he referred to as the "geometric interpretation" (lines 4-7) rather than the algebraic representations that he had started with in Phase 2. He reasoned that his students seemed to be familiar with the algebraic interpretation of rational functions (lines 7-8) so he would have rather spent more time in the context of graphs (lines 8-10). Shortly after this moment in the post-interview, Norbert was asked if he still believed that teaching the algebraic properties of rational functions was necessary before introducing the didactic objects (Excerpt 41).

Excerpt 41

Norbert: I still think... I still think yes it is. Actually no I don't even believe that answer that I just said. I think OK so I change my answer. I think you can get away from teaching the... so you can teach the algebraic stuff second but I really think you need to hammer home the connection between what this vector in my graph represents with respect to a function of value and really... really get that connection because once you have that the algebraic stuff just kind of falls out afterwards from what you've done.

Norbert answered the question first with a yes he believes that the algebraic rules should be taught first (line 1) but quickly changed his answer to no (lines 1-2). He stated that we could move away from teaching the algebraic rules first (lines 2-3) and that the algebraic rules fall out after a connection is made between the output values of the functions and the vector emanating from the x-axis in Rat Graph (lines 4-8).

Roles and Responsibilities

Roles and responsibilities, an aspect of mathematical classroom practice, refers to the responsibilities of the instructor and students when the novel didactic objects are implemented in mathematics instruction. For example, if the instructor experienced a shift in positioning (for example positioning students as co-constructors of understanding versus recipients of knowledge) when implementing a new technology in a mathematical lesson, then it might be said that the instructor exhibited signs of perturbations with respect to roles and responsibilities. There was evidence that Norbert experienced some perturbations related to roles and responsibilities in the planning, teaching, and reflecting on his rational function lessons.

The multiple planning entries in Norbert's video journal indicated that he found the novel didactic objects as a useful tool in planning his lessons on rational functions. During the post-interview Norbert was asked questions to elicit more information about the impact of the novel didactic objects on his lesson planning (Excerpt 42).

Excerpt 42

KP:	OK. So during the video journal planning session that you had you get stating that the applets helped you plan your lessons. So could you explain how the applets helped you plan your lessons?
Norbert:	So the biggest... the biggest thing was just having something tangible that I can... So just a point where so I have this applet. And then I have something around which I can base... base my lessons. Kind of gives me a nice rock foundation of I'm going to introduce this thing and from this my lesson is going to be kind of emanating from this applet. So it really just gave me like kind of what's... what's the word when you're climbing a rock where you put your foot... not a foot hold. I don't know.... it just gave... gave me a couple like foundation points just it made the... so that the skeleton of my plan was just kind of already given to me and then fill in the meat and bones and everything. Bones are already there. Meat and organs. It was a lot easier.
KP:	OK. How...

Norbert: So it kind of, sorry to interrupt, it'd be kind of like... drawing a picture and then coloring a picture versus here's something for you to color. That's kind of the way that I was thinking about it.

KP: OK. So how is planning with the applets compared to planning with just the curriculum? So last semester you only had the curriculum to kind of run off of so what is the comparison there?

Norbert: So the big difference is that drawing... so building that skeleton for my lesson was all up to me in the sense of like I... so I completely missed an entire interpretation of these rational functions that being the geometric interpretation that we've gone over a couple of times. And so just the biggest... the biggest thing is that preparing in the first semester there were so many things, one big thing, but so many things within that big thing that I didn't even consider to go over which was that whole geometric flavor to the to the subject.

KP: How much did you practice with the applets before teaching them?

Norbert: So I went over for the rat... which one was it? Sum bar and sum graph, I didn't do any practice with those ones because I thought that those were going to be pretty straightforward. We saw that that bit me... bit me on the tail when I tried to move the unmoveable vectors. It was the ones, the blue ones you could move and the orange ones you couldn't or something along that line. So the reason there was just like I... I mean I didn't even use sum bar because I kind of consolidate sum bar

38	and sum graph into one thing. But those ones I didn't practice at all.
39	And looking back I should've. It would have saved me a little bit of
40	time. But the rat... rat bar and rat graph I think I practiced four times
41	total before my lesson.

Norbert used two metaphors to describe how the didactic objects helped him plan his lessons, namely as having a skeleton (lines 4-14) or a coloring page (lines 16-18). He articulated the benefit of the novel didactic objects providing him with a structure or outline to construct the rest of his lesson around. When asked to compare planning with the curriculum versus the novel didactic objects, Norbert responded that he had more responsibility to construct his own lesson with just the curriculum (lines 22-25). He explained that without the novel didactic objects he missed the entire graphical interpretation of rational functions (lines 25-29). Norbert was asked about how much he practiced with the novel didactic objects before teaching. He responded that he did not think it was necessary to practice with all the novel didactic objects before teaching (lines 31-33) and expressed that this lack of preparation had side effects during the lesson (lines 33-36). Norbert regretted not practicing more with the first two didactic objects (lines 38-39) but did explain that he practiced the last two novel didactic objects at least four times before teaching (lines 39-41).

Norbert's answers to the probing questions in the post-interview indicated two possible perturbations associated with Norbert's responsibilities as the instructor. The first perturbation is indicated by Norbert's explanation that planning with the novel didactic objects provided him with additional assistance in constructing his rational function lessons. He articulated that, with only the curriculum, he had missed the graphical interpretation of rational functions, which included the behavior of the rational functions when a function rule was not presented. The second perturbation is identified in the preparation required to teach with the novel didactic objects. Norbert expressed how he wished he had practiced the first two didactic objects since the lack of practice effected his implementation during the lesson. The need for additional practice suggested that implementing the novel didactic object into the rational function lessons requires the instructor to practice with the novel didactic objects multiple times before teaching.

An additional perturbation associated with roles and responsibilities was identified during a reflection entry in Norbert's video journal (Excerpt 43).

Excerpt 43

Norbert:	So all in all I think the lesson went very well I just. Would have liked to have... So I didn't factor in properly the fact that I need some time for my students to think before answering questions and also that most of them don't like to answer questions. I think that dragged me back a little bit. But. All in all I think am very well and should be able to make up the time Monday.

During Norbert's reflection on his first lesson on rational functions, he explained that he thought the lesson went well (line 1) but realized that he did not factor in time for the students to think before answering questions (lines 1-4). Even though Norbert's reflection just touched on the time needed for students to answer questions, this perturbed Norbert's practices associated with the roles and responsibilities. The addition of the novel didactic object in Norbert's rational function lesson required more student participation, a fact that Norbert had not considered. The time necessary for students to think and answer questions associated with the novel didactic object was an indication of a two-part perturbation. The first part resting on Norbert's need to plan more time for students to think and answer questions associated with the novel didactic object. The second part required students to take more of a responsibility in their learning of the mathematics.

Student Engagement

Student engagement, in a mathematics classroom, refers to student participation while the novel didactic object is being used in a lesson. For example, the instructor's students might be less attentive as the didactic object is presented or decide not to participate because the discussion surrounding the didactic object is uncomfortable compared to an average lesson. After teaching his third rational function lesson Norbert reflected on his students' engagement during the lesson when the novel didactic objects, Rat Bar and Rat Graph, were implemented (Excerpt 44).

Excerpt 44

Norbert: Overall I feel the lesson went very well with the minor caveat you know just didn't get to cover everything I wanted. I remember in my planning I said I hope I'll cover everything and famous last words. I still don't have a feel for like how quickly my students are going to answer questions. I felt great teaching with the applets, the Rat Graph and Rat Bar applets. My students really seemed to understand the concepts I was trying to hammer home, that we're measuring the relative size of these vectors. When we looked at relative size of

9	polynomial outputs and they were able to pick up ideas that I wanted.
10	That we can just forget about the sign, look at relative size and then
11	notice where the vectors are pointing in order to get our ultimate sign.
12	And I think my students did very well during this lesson, even more so
13	than normal. They answer my questions quite quicker than normal.
14	Even though it still took them a little bit of time to answer them and
15	they seemed to have a good grasp on the things that I was trying to...
16	Trying to put forth.

Norbert expressed his dismay at not covering more of the problems that he had planned (lines 1-3) and remarked that he needed to have a better feel for how long students would take answering questions (lines 3-5). Norbert reflected on the use of the didactic objects in his lesson stating that he felt great teaching with Rat Bar and Rat Graph (lines 5-6). He seemed to believe that his students understood the concepts that he wanted (lines 6-11) and explained that the students did very well in the lesson - even better than normal (lines 12-16).

Norbert's reflection in the third rational function lesson suggested a change in his students' behavior while the novel didactic objects were used. He expressed how the students performed better than usual in the rational function lesson. Norbert's remark on the students' behavior hinted toward a possible perturbation in student engagement. In order to identify the perturbation Norbert was asked probing questions during the post-interview about changes to his classroom while teaching rational functions with the novel didactic objects (Excerpt 45).

Excerpt 45

KP: All right. So did you notice any changes in your classroom while teaching rational functions this semester?

Norbert: So I noticed one big difference, I noticed that the engagement of the students was a lot better so that they... I noticed that my students were very, even more so than they normally are, more engaged in the lesson. And I think they just had a lot to do with... we had nice shiny applets up on the projector so it was kind of like, oh what's going on here? And they were more inclined to pay attention to what was going on.

Norbert described the engagement of the students as a big difference in his classroom when teaching rational function with the didactic objects (lines 3-4). He remarked how students were more engaged than normal in the lesson (lines 4-5). He equated this change in the students' level of engagement to the novel didactic objects (lines 6-8). Norbert discussed shortly after this moment in the post-interview that more of the students had their heads up and their attention was directed to the didactic object on the screen. Norbert's response to the probing questions asked in the post-interview confirmed the existence of a perturbation in the students' level of engagement when the novel didactic objects were used in the rational function lessons.

Mathematical Conceptions

Mathematical conception, as an aspect of practice, refers to how students perceive the mathematics addressed by the implementation of a novel didactic object. For example, when an instructor implements a novel didactic object in a mathematics lesson there is no guarantee that the students will interpret the mathematical concept being presented in the manner the instructor intended. This might lead to the instructor struggling to understand students' mathematical conception of the concept while teaching with the novel didactic object. During the classroom observation of the third rational function lesson, Norbert introduced the didactic object, Rat Graph, through questions related to the characteristics of a function (Excerpt 46).

Excerpt 46

Norbert: We're going to look at this guy on the left, your left, is going to be my numerator function. And the guy on the right is going to be my denominator function. And we are going to see if we can use this idea of looking at relative size of vectors in order to compute a good approximation of what our graph should look like. Okay. So first things first let's look at... intercepts. So we have the red guy as our numerator function and what is the behavior of this numerator function? So let's roll this guy for a second. As I let x increase, what happens to the value of my numerator function? The guy on the left. (Long pause) So better yet using our identification, how does this vector change, as x is increasing?

Student: It stays constant.

Norbert: It stays constant. So if that vector doesn't change what does that mean about the function value?

Student: It doesn't change.

Norbert: Doesn't change. So the numerator function is a constant function. Okay. Now if a numerator function is a constant function and it looks like this constant value is happening... I don't care it happening somewhere not zero. So it is happening somewhere that is not zero. Is this numerator function ever going to be equal to zero? No. The numerator function will never be zero. So what does that tell us about intercepts...let's say, vertical intercepts of our rational function? No horizontal intercepts of our rational function.

Student: There are none.

Norbert: There won't be any because the roots or the horizontal intercepts happen when p(x) is equal to zero and that is never going to happen in this case. So we are not going to have any vertical intercepts. OK. So now what about horizontal intercepts, for any function how do I determine a horizontal intercept? So it's where I'm going to be intercepting the graph somewhere here. Right. What does that mean about my value of x?

Student: It's zero.

Norbert: Zero. So I look for the value of my rational function at x equals zero. So I look for the relative size of these vectors, when I'm at the value x equals zero. So looking at this guy we have that the relative size... sorry the vector for the numerator function is going to be two and then what about the vector for the denominator function? What is that going to be? So how long is this vector happening at zero?

Student: It's (0,0).

Norbert:	It's zero. So then this would be two divided by zero, oh god, I don't want to tear a hole in the universe.

Norbert introduced the Rat Graph didactic object to his students (lines 1-5) and started a conversation about the intercepts of the graph that would help in creating the graph of the rational function (lines 5-11). During this excerpt Norbert relied on students to answer questions using the didactic object and their previous knowledge regarding intercepts of a function (lines 12-32). At the end of this excerpt Norbert asked when the input is at the value of zero, how long is the vector for the denominator function (lines 33-38)? Norbert's student responded with an ordered pair at the origin. In the moment of teaching, Norbert dismissed the student's answer by taking zero as the answer and moving the class forward to the next characteristic of the graph of the rational function. This moment in the classroom observation of the third rational function lesson taught by Norbert was flagged as a possible perturbation (namely hearing what he wanted to hear) that was then confirmed in the post-interview.

During the stimulated recall segment of the post-interview in Phase 2 of data collection Norbert watched this clip and was asked if he believed that the student thought the length of the vector was zero (Excerpt 47).

Excerpt 47

KP:	Do you believe that the student thinks that the vector has a length of zero?
Norbert:	Umm from... I think he does. I think he was answering... So this is a lot of extrapolation on my part.

KP:	That's OK.
Norbert:	I think he does know that the length of the vector is zero. But I think that he was giving me like the... I think what was going on in his head was the point (0,0), it's not like it's got a length but that was essentially what he was going for. If that makes sense, like... the position of... the position vector is... (0,0) essentially.

Norbert believed this student understood that the length of the vector was zero but, instead of providing the length as a solution, gave the position of the vector instead. When asked what made him so confident in the student’s response Norbert said that this was just his interpretation of what the student was thinking. The student’s mathematical conception when introduced to the novel didactic object, Rat Graph, caused him some confusion about position and length of a vector, but Norbert experienced no change or discomfort in the student’s response.

Another instance in the classroom observation of the third rational function lesson was flagged as a potential perturbation when Norbert asked the class about the end behavior of the rational function formed by the didactic object, Rat Graph (Excerpt 48).

Excerpt 48

Norbert:	Okay so we are going to fit this in for the last minute, end behavior. So now we are looking at if we keep increasing x or we keep decreasing x, what happens to the output of our vectors?
Student:	It approaches zero.
Norbert:	Ah it approaches zero, so here [name] I'm going to run through this

animation and I want you to quickly tell me why it is going to approach zero. So what's happening?

Student: Which one? The right one or the left one?

Norbert: So you are going to need both of them if you are going to tell me why it approaches zero. So we want to measure the relative size of this red vector with respect to the blue vector, right?

Student: Uh huh.

Norbert: The red vector is getting larger or smaller?

Student: It's staying constant.

Norbert: Staying constant but the blue vector is...?

Student: Constantly moving like it is increasing.

Norbert: It is increasing getting larger and larger and larger.

Student: So those numbers would be getting smaller and smaller and smaller approaching zero.

Norbert: Yes. So the output will be getting smaller and smaller and smaller. In which our vectors will look like (draw on the board decreasing lengths of vectors approaching zero) and will eventually level off to zero. Now lets look at the opposite direction, if we decrease without bound. So our red vector is staying constant and how is the length of the blue vector changing?

Student: Constantly going... constantly decreasing.

Norbert: Constantly decreasing. What's important is the magnitude is getting larger. So then the output is going to be approaching what? It stays constant. So if that vector doesn't change what does that mean about the function value?

Student: Zero.

Norbert: It's going to be approaching zero, outstanding.

In this excerpt Norbert asked a question about the behavior of the output value of the rational function as the input increased or decreased without bound (lines 1-3). A student responded quickly that the output values would approach zero (line 4). Norbert required the student to explain why the output values of the rational function would approach zero using the didactic object, Rat Graph (lines 5-7). The student exhibited confusion about which part of Rat Graph would help him explain why the output value of the rational function would approach zero as the input value increased or decreased without bound (line 8). Norbert helped the student discover how to use the didactic object through probing questions (lines 9-26). This moment in the classroom observation of the third rational function taught by Norbert was flagged because of the student’s quick answer to Norbert’s question but lack of confidence in using the didactic object to explain his thinking. During the stimulated recall segment of the post-interview Norbert viewed this clip and was asked questions associated with the student’s thinking in the moment (Excerpt 49).

Excerpt 49

KP: So my question is what do you think he actually understands in the moment?

Norbert:	So I think he was just looking at this picture I have here (points to image on the board) and that's what he was… like he kind of cheated, for lack of better words. Oh I see those vectors are going to get smaller and smaller they are going to zero.
KP:	OK.
Norbert:	So that's what I think happened. That's kind of why I probed a little bit more because I was like you just looked at the board. I mean like I was happy that he was paying attention to what was on the board. I was happy in that sense but I kind of just wanted to probe a little more and be like, did you just look at the board or did you get that from the applet?

Norbert believed that the student used the graph of the rational function that was on the board to answer the question about end behavior (lines 3-6). Norbert proceeded to explain that he chose to ask more questions (lines 8-9) because he felt as if the student did not use the didactic object to construct his answer (lines 9-13). In this moment, Norbert probed a student's thinking as a way to test if the student had made a lucky guess when answering his original question. This was the first time that Norbert pushed a student to articulate their thinking about the mathematics surrounding the novel didactic object. Based on the finding in the classroom observation and the stimulated recall segment of the post-interview, Norbert experienced a perturbation in his practice associated to mathematical conceptions.

Summary

In this chapter I presented results obtained through the interview transcriptions and video analysis of data collected in Phase 2 of the study. These results pertained to the effects of novel didactic objects on a novice instructor's mathematical classroom practices and changes that occurred in the novice instructor's mathematical meanings and mathematical classroom practices for rational functions. Within the framework (Table 5) describing aspects of practice perturbed by the introduction of the novel didactic objects, the results are summarized below:

Leader Actions. Norbert exhibited the effects of perturbations associated with leader actions in planning and teaching rational functions with novel didactic objects. The earliest perturbation Norbert experienced was in the planning stage. He articulated that the didactic objects helped him formulate his lessons with more ease even though he still had reservations about teaching with the didactic objects. When teaching rational functions with didactic objects Norbert experienced perturbations associated with how he introduced rational functions and his word choice when discussing algebraic and graphical representations of rational functions. An additional perturbation was revealed in the post-interview when Norbert explained that he was showing up to class approximately ten minutes earlier to teach his rational function lessons. In these ways, the introduction of novel didactic objects perturbed Norbert's actions as a leader and resulted in changes to classroom practices.

Communication. Norbert experienced perturbations associated with communication when teaching rational functions with novel didactic objects. Norbert expressed the first perturbation during his reflection of his first rational function lesson where he implemented the novel didactic object, Moving Vectors. He articulated that he wished that his students would be more responsive in class but felt as if he could not ask more of his students. When teaching his third rational function lesson, Norbert experienced a perturbation surrounding the discourse of the introduction of the novel didactic object, Rat Bar. He asked his students to explain what geometric object is associated with the output of a graph of a function and received no response. Norbert expressed his frustration that his students were unable to give him his expected response of an arrow, which he felt should have been second nature. An additional perturbation was revealed in the stimulated recall segment of the post-interview when Norbert explained that he was continually checking his watch and jogging in place after asking the students a question because they did not have time to waste. In these ways, the introduction of the novel didactic objects perturbed the communication of Norbert's classroom and resulted in changes to classroom practices.

Expectation of Technology. After teaching rational functions with the novel didactic objects, Norbert changed his opinion on which representation of rational functions should be taught first. This change indicates that a perturbation occurred in Norbert's practices associated with the expectation of technology. Before being introduced to the novel didactic objects Norbert's conception of a rational function was limited to the ratio of two polynomials. After the novel didactic objects were introduced, Norbert changed his conception of rational functions to include the ratio of two polynomial functions where the ratio represented the relative size of the output of one polynomial function in terms of the output of the other polynomial function. The instruction of the novel didactic objects perturbed Norbert's practices associated with the expectation of technology so much that in the post-interview he articulated that he would change his lessons on rational functions. These changes consisted of leading with and emphasizing the graphical representation of rational functions rather than the algebraic.

Roles and Responsibilities. Norbert experienced perturbations associated with roles and responsibilities when planning, teaching, and reflecting on his rational function lesson with novel didactic objects. Norbert expressed the first perturbation during the post-interview when he was probed to explain how the novel didactic objects affected his planning for the rational function lessons. He articulated that the novel didactic objects allowed for him to discuss the graphical representations of rational functions, which was missing from his previous semester's lessons on rational functions. In planning to teach with the novel didactic objects Norbert experienced another perturbation surrounding his responsibilities as the instructor when implementing the novel didactic objects. He expressed the need for practicing with the novel didactic objects multiple times before teaching. An additional perturbation was revealed in a video journal entry on the reflection of the first lesson on rational functions where the didactic object, Moving Vectors, was introduced. Norbert explained that he did not plan for the additional time students would require to think and answer his questions surrounding the novel didactic object. This perturbation affected both Norbert's need to plan for additional time for questioning and the students' need to take more responsibility in their mathematical education. These perturbations affected Norbert's practices associated with roles and responsibilities, the responsibilities of the instructor and students when the novel didactic objects are implemented in mathematics instruction.

Student Engagement. After teaching the third rational function lesson with the novel didactic objects, Rat Bar and Rat Graph, Norbert noticed a change in his students' level of engagement. This change indicated that a perturbation might have occurred in the practices associated with the student engagement. Based on Norbert's responses to the probing questions asked during the post-interview, the perturbation to student engagement was confirmed. Norbert expressed that the students' level of engagement in the lesson was a big difference to the classroom environment and the rational function lesson. Norbert attributed the difference in the engagement to the novel didactic objects that were used in the rational function lessons.

Mathematical Conceptions. After teaching the third rational function lesson with the novel didactic object, Rat Graph, Norbert probed student thinking to further identify how the student answered his original question. Norbert's decision to probe the student further about his thinking suggests that there was a perturbation in Norbert's practice. Based on Norbert's response to questions asked in the post-interview about this moment during the classroom observation, the perturbation was confirmed. Norbert engaged in additional questioning and required his student to describe his thinking using the didactic object. This change indicated that a perturbation occurred in Norbert's practices associated with mathematical conceptions. Norbert perceived the students answer as being a possible guess and changed his questioning to force the student to reveal his thinking using the novel didactic object.

CHAPTER 7

DISCUSSION AND CONCLUSIONS

In this chapter, I will highlight the study's key findings with respect to the perturbations in practice that occur when novel virtual manipulatives that are used as didactic objects are implemented by a novice instructor's rational function instruction. I discuss the results of this study relative to the research questions presented in Chapter 1 as well as contributions, implications, and limitations.

- In what ways do novel virtual manipulatives that are used as didactic objects perturb a novice instructor's existing mathematical classroom practices?

More specifically,

- What characteristics, other than amount of teaching experience, classify an instructor as a novice? Are there aspects of planning a lesson, teaching a lesson, and reflecting on a lesson that differentiate novice from experienced instructors?
- How does a novice instructor perceive a novel virtual manipulative that functions as a didactic object, both mathematically and as an instructional tool?
- What are the differences between a novice instructor's image of the meanings students might develop from the novel didactic object and how the instruction fosters these meanings?

This chapter provides suggestions for curriculum and instruction surrounding rational functions and the use of novel virtual manipulatives as didactic objects in a mathematics classroom. The conclusion of this chapter addresses the limitations of this study as well as possible directions for future mathematics education research.

Key Findings

The key findings of this study included characteristics of a novice instructor and perturbations in practice. I found multiple instances in the first phase of data collection that classified Norbert as a novice instructor. The data collected in the first phase of this study set the stage for exploring the perturbations that occur when a novice instructor uses the novel virtual manipulatives as didactic objects in rational function instruction. I found multiple instances during the interviews, classroom observations, and video journaling of how the implementation of the novel virtual manipulatives as didactic objects changed Norbert's mathematical classroom practices. I saw perturbations characteristic of leader actions, communication, expectations of technology, responsibilities, student engagement, and mathematical conceptions.

Characteristics of a Novice Instructor

In order to maximize the emergence of perturbations for the study, I chose to focus on the instruction of a novice instructor. I therefore had to first verify that Norbert was a novice instructor using a different measure than just his time teaching in a mathematics classroom. I compared Norbert, the instructor I hypothesized was a novice, and Edwin, the instructor I hypothesized to be experienced, as they planned, taught, and reflected on lessons regarding rational functions. During the Phase 1 data collection of this study, I validated my original hypothesis of Norbert displaying the characteristics of a novice instructor.

Using the data collected in Phase 1 of the study and a comparison with Edwin's mathematics instruction, I identified examples in Norbert's mathematics instruction that could be classified as novice. There are three aspects in which I identified Norbert as a novice instructor; mathematical understanding of rational functions, classroom discourse, and lesson reflection.

The pre-interview during Phase 1 of data collection revealed that Norbert and Edwin had similar ways of thinking about division, relative size, and covariation with respect to a conceptual understanding of rational functions. Edwin and Norbert demonstrated a quotitive interpretation of division (Correa et al., 1998; Fischbein et al., 1985; Greer, 1992) where they both were thinking, how many times does the value, one-fifth fit into the value, six (Excerpts 1-2). Conceptualizing division in this way (as opposed to the use of a grouping metaphor or partitive interpretation of division) represents an opportunity to think about what it means to express one quantity in terms of another. When asked to find the relative size of the red bar in terms of the blue bar with both static and dynamic bars, Edwin and Norbert leveraged their quotitive interpretation of division to demonstrate in an embodiment activity the relative size of the red bar in terms of the blue bar (Excerpts 3-8). When given the graph of the relative size of the red bar in terms of the blue bar, Edwin and Norbert predicted how the red and blue bars would change in length with respect to the input quantity (Excerpts 9-10). These predictions highlighted their ability to construct a conceptual understanding of rational functions using their understanding of division in conjunction with covariational reasoning. The lack of discrepancies in Norbert's and Edwin's mathematical thinking provided an even platform for identifying differences in the novice and experienced instructor instruction on rational functions without having to consider their mathematical knowledge as significantly different.

After the pre-interview of Phase 1 the differences between Norbert and Edwin became more apparent in the video journal, classroom observations, and post-interview. Norbert and Edwin planned and executed different approaches to teaching rational functions in Phase 1. Edwin planned lessons surrounding rational functions that drew attention to the division of quantities and leveraged students understanding of division to construct a conceptual understanding of rational functions (Excerpts 11 and 13). Edwin approached rational function in a conceptual way by leading a discussion on division, relative size and covariation. The questions asked by Edwin in the classroom observations gave students the opportunity to discuss and describe their thinking of the mathematics. Norbert approached rational functions in a calculational way by presenting definitions and rules found in the curriculum associated with rational functions (Excerpt 12 and 14). The questions Norbert asked of his students during the classroom observations required mainly short numerical responses.

When reflecting on the lessons on rational functions, Edwin and Norbert focused on different aspects of instruction. For example, when Edwin reflected on his second rational function lesson, he stated that his students did not seem engaged in the lesson which was not conducive to students building the meanings for rational functions he wanted (Excerpt 20). Edwin focused on his students as a way to measure the success of his rational function lesson and this focus led him to identifying areas of improvement that he could make as an instructor (Excerpts 17, 18, and 22). In contrast, Norbert primarily reflected on his actions and performance as an instructor using himself as the measure of his success (Excerpts 19, 21, and 23). For instance, in the first rational function lesson reflection Norbert primarily discussed a mistake that he made in one of the problems he presented to the students (Excerpt 19). Norbert articulated what transpired in the lesson but dwelled on his mistake. This egocentric type of reflection was common for all of Norbert's reflections in Phase 1 of data collection.

Additional differences between the two instructors were related to the preparation time and amount of time using direct instruction. Norbert planned a lesson no later than 18 hours before teaching the lesson. Edwin prepared lessons approximately 30 minutes before teaching the lesson. Norbert and Edwin structured their daily rational function lessons differently. For example, during the classroom observation of Phase 1, Norbert taught using direct instruction 90% of the time while the other 10% was dedicated to asking students questions and waiting on student responses to questions. Edwin directly instructed his class only 50% of the time, with the rest of the class time dedicated to asking students questions, letting students work on problems, and discussing the mathematics as a class.

Perturbations in Practice

In order to validate the perturbations in practices I had identified in the classroom observations and video journal activity, I asked Norbert to retrospectively analyze the instances I had identified. During this stimulated recall segment of the post-interview session of Phase 2 of data collection, Norbert reaffirmed my original hypotheses of perturbations he experienced in the classroom while implementing the novel virtual manipulatives as didactic objects.

Using the framework (Table 3), I identified examples in Norbert's mathematics instruction where perturbations had occurred when he implemented the novel virtual manipulatives as didactic objects in his rational function instruction. There are six aspects in which the introduction of novel virtual manipulatives as didactic objects can impact practice; leader actions, communication, expectation of technology, roles/responsibilities, student engagement, and mathematical conceptions.

First, the impact of the novel virtual manipulatives as didactic objects needs to be viewed with a focus on the person who is positioned as the leader in the classroom, i.e. Norbert in this study. I identified instances where Norbert's interpretation of the novel virtual manipulatives as didactic objects and his actions as he implemented the novel virtual manipulatives as didactic objects in rational function instruction perturbed his actions as a leader. For instance, when Norbert introduced the rational function with the novel didactic objects he defined rational function as the relative size of one polynomial with respect to another polynomial, which was a different approach than the previous semester where he defined a rational function as the ratio of two polynomials (Excerpt 29). An additional perturbation was highlighted when Norbert discussed "nice formulas" referring to the algebraic representation of rational functions (Excerpt 28). Norbert experienced perturbations in his practices due to his perception of the didactic objects and his actions when teaching rational functions with the didactic objects (Excerpts 24-31).

Second, communication practices surrounding the novel virtual manipulatives as didactic objects were disrupted. Norbert's once comfortable environment, where direct instruction was routine and well established, was altered by the implementation of the novel didactic objects. I identified instances of discourse surrounding the novel didactic objects changing. For example, when reflecting on his first lesson with the novel didactic object, Moving Vectors, Norbert expressed his desire for his students to be more responsive in class (Excerpt 32). When implementing the novel didactic object, Rat

Graph, Norbert exhibited unique behaviors that included frequently checking his watch and jogging in place after posing a question for his students to answer (Excerpt 36). Norbert articulated during the post-interview that waiting on his students to answer questions was maddening and led to his need to release the frustration of waiting. Norbert experienced perturbations surrounding the discourse of the introduction of the novel didactic object in his desire to have students be more responsive, his frustration when students were unable to answer the questions he posed, and his mentality that there was no time to waste waiting for his students to answer his questions (Excerpts 32-37).

Third, implementing the novel didactic objects in instruction comes with certain expectations that cause perturbations if the expectations are not met. For instance, Norbert expressed that teaching rational functions using algebraic rules and properties was necessary (Excerpt 38). However, during the post-interview Norbert articulated that he believed that he could move away from teaching the algebraic rules and instead start with the graphical representations used in the novel didactic objects (Excerpt 41). After teaching rational functions with the novel didactic objects Norbert's practices associated with the expectation of technology were perturbed so much that in the post-interview he articulated that he would change his lessons on rational functions (Excerpt 38- 41).

Fourth, the impact of the novel didactic objects affects the roles and responsibilities of the individuals who will be exposed to and use the innovation, i.e. Norbert and his students. The implementation of the novel didactic objects forced Norbert to see the value in practicing with the novel didactic objects prior to teaching a lesson where the novel didactic object would be included (Excerpt 42). Norbert identified the need to factor in more time for student to answer questions when teaching with novel didactic objects (Excerpt 43). This perturbation affected both Norbert needing to plan for additional time for questioning and the students need to take more responsibility in their mathematical education.

Fifth, implementing novel didactic object in instruction can cause perturbations in student engagement. During the post-interview Norbert discussed the engagement of his students seemed to increase when he taught with the novel didactic objects, Rat Bar and Rat Graph (Excerpt 45). Norbert identified that more students had their heads up and were looking at the didactic object being displayed on the projector screen. Based on Norbert's responses to the probing questions asked during the post-interview, the perturbation to student engagement was confirmed. Norbert expressed that the students' level of engagement in the lesson was a big difference to the classroom environment and the rational function lesson. Norbert attributed the difference in the engagement as a result of the novel didactic objects that were used in the rational function lessons (Excerpt 44-45).

Finally, novel didactic objects impact the mathematical conceptions that emerged in the classroom. For instance, after teaching the third rational function lesson with the novel didactic object, Rat Graph, Norbert probed a student's thinking to further identify how the student answered his original question (Excerpt 48). Norbert's decision to probe the student further about his thinking suggests that there was a perturbation in Norbert's practice. Based on Norbert's response to questions asked in the post-interview about this moment during the classroom observation, the perturbation was confirmed (Excerpt 49). Norbert perceived the student's answer as being a possible guess and changed his questioning to force the student to reveal his thinking by appealing to the novel didactic object. This change indicated that a perturbation occurred in Norbert's practices associated with mathematical conceptions that emerged in the classroom (Excerpt 46-49).

Summary of Findings

In this section, I address the specific research questions that this study attempted to answer:

Research Question #1: In what ways do novel virtual manipulatives that are used as didactic objects perturb a novice instructor's existing mathematical classroom practices?

In this study, the novel virtual manipulatives that Norbert used as didactic objects in his rational function instruction did perturb the existing mathematical classroom practices. Perturbations, both positive and negative, occurred in Norbert's classroom practices with regard to leader actions, communication, expectation of technology, roles/responsibilities, student engagement, and mathematical conceptions. These perturbations stemmed from Norbert's perception of the novel didactic objects mathematically and as an instructional tool.

Research Question #2: What characteristics, other than amount of teaching experience, classify an instructor as a novice? Are there aspects of planning a lesson, teaching a lesson, and reflecting on a lesson that differentiate novice from experienced instructors?

Expert and novice instructors can differ based on mathematical content knowledge, ability to articulate mathematical content knowledge, choice of teaching approach, and lesson reflection focus. Although in this study, the novice and expert instructors shared similar relevant mathematical content knowledge, they differed according to their planning, comfort with a student-centered classroom, and measure of instructional success.

Research Question #3: How does a novice instructor perceive a novel virtual manipulative that functions as a didactic object for teaching rational functions, both mathematically and as an instructional tool?

A virtual manipulative used as a didactic object can initiate a shift in an instructor's mathematical meanings, but may not necessarily be valued as an instructional tool. Norbert's perception of the novel didactic objects mathematically and as an instructional tool had a significant impact on the perturbations that occurred in his mathematical classroom practices. Norbert initially viewed rational functions symbolically with little to no emphasis on the covariational relationship that exists between the input quantity of the rational function, the value of the numerator, the value of the denominator, and the value of the relative size of the numerator in terms of the denominator. In Phase 1 of data collection, Norbert taught rational functions in a traditional manner by discussing the properties and characteristics of rational function using algebraic representations. After being exposed to the novel didactic objects, Norbert's perception changed to include the covariational relationship that exists within a rational function. However, Norbert's new perception of the novel didactic objects mathematically was not strong enough to influence his rational function instruction to foster a deeper meaning of rational functions within his students.

Norbert's perception of the virtual manipulatives as novel didactic objects as an instructional tool, at first, was that of confusion. During the intervention interview in Phase 2 of data collection, Norbert articulated the need for time to think about how he would use the novel didactic objects as an instructional tool. Norbert said that he was interested in using the novel didactic objects in his classroom and explained that he did see value in his students learning rational functions with these instructional tools. During the video journal lesson planning entries, Norbert explained that teaching with the didactic objects made planning his lesson easier since he had a solid outline to follow. Norbert's actions and discourse while teaching with the novel didactic objects in his rational function instruction told a slightly different story. Norbert saw the virtual manipulatives just as applets throughout his teaching of rational functions. However, in the post-interview of Phase 2, Norbert expressed a desire to teach rational functions using the applets first rather than the algebraic interpretations. This suggested that Norbert might be moving from seeing the virtual manipulative just as applets to seeing the virtual manipulatives as didactic objects. With a shift in the order of activities that prioritized the interventions, he demonstrated that they have a legitimate place in instruction.

Research Question #4: What are the differences between a novice instructor's image of the meanings for rational functions that students might develop from the novel didactic object and how the instruction fosters these meanings?

Norbert's image of how the meanings of rational functions students should develop changed throughout the study as he taught the curriculum for the first time and was introduced to the novel didactic objects that he later implemented in his rational function instruction. After being exposed to the novel didactic objects Norbert attempted to foster his image of the meanings student should develop from the novel didactic objects in every rational function lesson. However, the study revealed differences between Norbert's image of the meanings students should develop from the novel didactic objects and how the instruction Norbert led fostered these meanings.

During Phase 1 of data collection, prior to the introduction of the novel didactic objects, Norbert taught rational functions in a traditional manner by emphasizing the properties of rational functions using algebraic representations and procedural methods. Once exposed to the didactic objects in Phase 2 of data collection, Norbert began to articulate the value of a student determining the location of the asymptotes of rational function by assessing the behavior of the function, namely the relative size of the numerator in terms of the denominator. During the video journal planning session entries, Norbert continued to express the value of students thinking about the behavior of rational functions rather than the procedures and his excitement in teaching with the novel didactic objects. However, during the classroom observations, Norbert did not stay true to having the students think about the behavior of rational functions when locating asymptotes. Instead, Norbert led his class in a procedural lesson that looked at the properties of rational functions. When Norbert did introduce the novel didactic objects into his instruction of rational functions, he still fell back into giving students specific values to construct graphs of the rational functions. Thus, even though Norbert saw the value in students understanding the covariational relationship of rational functions, he was unable to break away from his comfortable routine to cultivate that meaning of rational functions within his students. During the post-interview, after teaching rational functions with the novel didactic objects, Norbert articulated that he would change how he taught rational functions for the next semester. He explained how he would want to teach with the novel didactic objects first and then discuss the properties of rational functions. Norbert's post-interview discussion suggested that he still sees the value of the instructional tools and is looking to the future when he can implement the novel didactic objects at the onset of rational function instruction.

Contributions to the Literature

This study explored how a novice instructor implemented virtual manipulatives as didactic objects in rational function instruction. In the TPACK framework, it therefore sits squarely within the intersection of technology knowledge, content knowledge, pedagogical knowledge, and pedagogical content knowledge. In various degrees, it contributes to our understanding of teaching expertise, use of technology in the classroom, didactic objects, and a conceptual understanding of rational functions.

Teaching Expertise. Contrasting experts and novices sheds light on possible paths along a trajectory of increased fluency and expertise (Carlson & Bloom, 2005; Chi, Feltovich & Glaser, 1981; Schoenfeld, 1992; Shepherd & van de Sande, 2014). In the context of teaching, expertise has been linked to many characteristics, both teacher- and student-focused. For instance, time in the classroom, a structure of mathematical knowledge that supports student understanding, pattern recognition in the moment of teaching, a flexible automaticity in actions and behaviors to allow conscious processing of information, and student achievement on assessments have all been used to characterize teaching expertise (Berliner, 2001; Glaser, 1987; 1990). Expertise, then, roughly corresponds to how a teacher scores on such a "pedagogical report card."

Instructors who are novices in the sense of a pedagogical report card (e.g., time in classroom, structured mathematical knowledge, pattern recognition in the moment, and flexible automaticity in their actions and behaviors) can vary considerably with respect to adapting to and growing from trying new approaches and resources in the classroom

(Rich, 1993). Without support and training, novice instructors demonstrate extremely impoverished pedagogical and mathematical content knowledge (Musgrave & Carlson, 2017). Given training and mentoring, however, some novice instructors can respond to unfamiliar instructional approaches (such as facilitating cooperative learning) by modifying their teaching based on an understanding of their students' needs, thereby moving along a trajectory toward expertise.

For this reason, much research has focused on how to design, implement, and evaluate effective mentoring for instructors (Appleton, 2008; Fluckiger, McGlamery, & Edick, 2006; Zimpher & Rieger, 1988). One key ingredient to the successful mentoring of instructors involves a system of shared beliefs, e.g. teacher buy-in (Turnbull, 2002). In particular, the participants have to reach a mutual understanding of the intent, the implementation, and the benefits of the approach being supported by the mentorship. If an instructor does not believe in how the model or approach is supposed to improve student understanding, then the benefits of mentoring (even if intensive) are significantly hindered (Thompson & Thompson, 1994; 1996).

In my study, however, Norbert did buy into the conceptual analysis of rational functions. He understood and valued the mathematical meanings that could be constructed from interacting with the virtual didactic objects. Even though Norbert's classroom implementation of the virtual didactic objects was unpolished and clumsy, he began to shift to a more student-centered analysis of his teaching and was enthusiastic about finding alternative ways to implement the virtual didactic objects in the future. Therefore, virtual didactic objects have the potential to help novice instructors take baby steps towards becoming experts at helping students construct meaningful mathematics.

Use of Technology in the Classroom. Instructors play a critical role in the successful integration of technology in the classroom (Bitner & Bitner, 2002; Loveless, DeVoogd & Bohlin, 2001, Romano, 2003) since the direct determination of the instruction is in the hands of the instructor rather than the external educational requirements (Chen, 2008). In mathematics education, the general message is that electronic technologies ought to be used to enhance student learning of mathematics but there is a lack of research on the specifics of how to accomplish this. A popular lens for understanding the role of technology in student learning is based on the assumption that the technology will have a unidirectional impact, *input-output* approach, where the technology is injected into classrooms and the learning effects result from this addition (Lynch, 2003; 2006). Research studies that use a input-output approach measure technology usage rather than generating new knowledge about education (Wagner, 1993). Studies with this approach are usually large-scale quasi-scientific studies that generalize the effects of technology on learning through the use of surveys (Angrist & Lavy, 2002) but fail to answer questions on how to improve educational practice (Lynch, 2006).

In my study, however, the use of technology in the classroom used an approach that sought to find the perturbations that occurred in the mathematical classroom practices of an instructor when implementing technology. As a comparison to the input-output approach that usually involves large-scale survey instruments, my study was small-scale and had an interview and classroom observation driven approach. This approach allowed for aspects of the instructor's practices to be assessed for changes when implementing technology in instruction. In this way, this study provides further insight into ways in which educational practices dealing with technology can be improved.

Didactic Objects. Didactic objects are objects, i.e. images, documents, technology, etc., in conjunction with a conversation or discussion that can support students' construction of mathematical ideas (Thompson, 2002). Research studies that incorporate didactic objects as defined by Dr. Thompson have focused on the effects of the didactic objects on student understanding of mathematics in traditional and online classroom environments (Bowers et al., 2010; Poddiakov, 2001) as well as teacher professional development (Lima, McClain, Castillo-Garsow, & Thompson, 2009).

In my study, I focused on the effect of virtual manipulatives as novel didactic objects on an instructor's rational function instruction. This focus is unique as it uses Dr. Thompson's definition of didactic objects but, instead of assessing the effect on the students, it looks at how the didactic objects are used by the instructor. This shift in focus allows for additional conversation around the implementation of didactic objects by instructors and ties to the professional development of instructors (Lima et al, 2009) so that instructors can be better prepared to orchestrate reflective discourse (Thompson, 2002).

Conceptual Understanding of Rational Functions. Typically, rational function instruction is approached as an extension of the mathematics of polynomials (such as finding roots). However, this study took a novel approach to rational function instruction by introducing a conceptual analysis of the topic that was based on a connected web of schemes, including measurement, division, covariation, etc. By taking this approach, the goal was to add meaningful coherence to the pre-calculus curriculum (Thompson, 1994). Indeed, in the short duration of the study, Norbert shifted his notion of coherence from a procedural focus to this conceptually oriented approach that supports students as they reason quantitatively and embark on a transition to a deeper understanding of limits.

Limitations of the Study

Although this study begins to identify the perturbations in classroom practices experienced by a novice mathematics instructor, it has several limitations in terms of the participants and the choice of perturbing technology. The instructors that participated in this study were graduate students at a large southwestern university. First, the participants were chosen from a very restricted pool of instructors. Edwin was a graduate student in the mathematics education doctoral program who had transferred from the pure mathematics doctorate program. Norbert was a graduate student in the pure mathematics doctorate program. Edwin was in his third semester of teaching pre-calculus in Phase 1 of data collection and had taken multiple courses focusing on research in undergraduate mathematics education. Norbert was in his first semester of teaching pre-calculus during Phase 1 of data collection, and in his second semester during Phase 2 of data collection. Edwin was not chosen at random since he was the only male graduate student that fit the criteria of an experienced pre-calculus instructor. Norbert was chosen from the Teaching Undergraduate Mathematics Education (TUME) seminar since he was the first to volunteer to participate in the study. In addition, since only one experienced and novice instructor was studied, it is difficult to make any overarching claims that would be applicable to every mathematics instructor implementing novel didactic objects in instruction. In particular, the study only focused on graduate students teaching introductory mathematics courses at a university.

Another limitation arises from the decision to use novel didactic objects as the representative perturbing technology. The study did not explore perturbations from other technologies but instead used novel didactic objects that were intentionally selected to elicit perturbations in classroom practices. Other technologies might have elicited different perturbations than those that were found in this study. This study deliberately assessed the effects of novel didactic objects in a mathematics classroom on the concept of rational functions. The didactic objects I used in this study were designed by Dr. Thompson to foster a conceptual understanding of rational functions. This concept was well suited for the study because rational functions are predominantly taught from a procedural orientation and therefore fostering a conceptual orientation afforded additional opportunities for perturbations.

This study begins to identify the perturbations in classroom practices experienced by a mathematics educator. However, this study does not fully explain all types or forms of perturbations a mathematics instructor might experience when introducing novel didactic objects in instruction. Additional research is needed before a more comprehensive list can be constructed. For this study, I purposefully choose to use a broad brush to paint picture of the perturbations that occur when novel didactic objects are added to instruction. This broad brush provides a backdrop against which researchers can delve more deeply into the nuances that accompany perturbations in classroom practices and explore their duration and resolution.

Directions for Future Research and Development

This study produced several lines of possible future research and curricular development. For example, additional research is needed to understand the connection of instructors' mathematical meanings to the instructional practices they employ in instruction with or without the implementation of technology. The expansion of technology in learning environments suggests that curriculum and instruction will need to change to encompass the new technological resources. For instance professional development opportunities to create applets might help instructors revisit and reconceptualize their mathematical meanings. Another change might need to come from the curriculum companies putting more effort into creating applets that accompany the curricula. However, the creation of applets, whether by instructors or professionals, is not enough; guides must be made to help instructors effectively implement the applets as didactic objects. If these guides are forgotten or pushed aside as a waste of time, the applets could become glorified pictures or videos with little to no impact on the classroom environments and student mathematical understandings.

Future research is needed to understand the relationship between the mathematical meanings of an instructor and the resulting perturbations in classroom practices when the instructor implements novel didactic objects in his or her instruction. For instance, little is understood about the technological pedagogical content knowledge that allows an instructor to effectively foster student understanding. This study sets the stage for such research by demonstrating the promise and potential of virtual manipulatives used as didactic objects to promote curricular coherence (Thompson, 1994) and extend instructors' mathematical meanings (Musgrave & Carlson, 2017) of rational functions.

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APPENDIX A

CONSENT FORM

Title of research study: Investigating Perturbations in Mathematical Classroom Practices Using Virtual Manipulatives as Novel Didactic Objects

Investigators: Carla Van De Sande & Krysten Pampel

Why am I being invited to take part in a research study?
We invite you to take part in a research study because you currently teach a section of MAT 170 and participating in TUME.

Why is this research being done?
This study will attempt to characterize the perturbations that occur when virtual manipulatives as novel didactic objects are implemented into rational function instruction.

How long will the research last?
We expect that individuals who elect to participate will teach MAT170 in both Fall 2016 and Spring 2017. In other words there will be two semesters of data collection on your course. The first semester will consist of 2 interviews (about 1 hour per interview) and classroom observations (up to 5 class days). The second semester will consist of 3 interviews (about 1 hour per interview) and classroom observations (up to 5 class days).

How many people will be studied?
We expect at least two graduate students teaching MAT 170 to participate in this research study.

What happens if I say yes, I want to be in this research?
If you agree to be apart of this research study you will take part in the first phase of data collection this Fall 2016 by participating in interviews (2) and classroom observation (up to 5). The first interview will occur a week prior to the first classroom observation and will cover tasks related to the rational functions. The classroom observations will consist of the research sitting in the back of the room video taping you and your projector screen while teaching rational functions. Please note none of your students will be captured in the video. No more than a week after the final classroom observation there will be a second interview. In this interview we will discuss moments in your lesson that were of interest. During this phase of data collection you will also be asked to keep an instructional journal (provided to you by the researcher) that will be used with question prompts to help you reflect on your lessons.

The second phase of the data collection does not begin until Spring 2017. This phase will consist of interviews (3) and classroom observations (up to 5). The first interview will happen a week before the classroom observations and you will be asked to complete tasks related to rational functions. The second interview will be conducted roughly 2 days before the classroom observations. In this interview you will be introduced to didactic objects and resources that will be used to teach rational functions. This will include questions about how you would implement the didactic objects into your rational function instruction for MAT 170. The classroom observations will consist of the research sitting in the back of the room video taping you and your projector screen while teaching rational functions. Once again, none of your students will be captured in the video. In these classroom observations you will implement the didactic objects that were introduced in the second interview. No more than a week after the final classroom observation there will be a third interview. In this interview we will discuss moments in your lesson that were of interest. You will also be asked to reflect on the observations and answers questions about the didactic objects and resources. During this phase of data collection you will once again be asked to keep an instructional journal (provided to you by the researcher) that will be used with question prompts to help you reflect on your lessons.

What happens if I say yes, but I change my mind later?
You can leave the research at any time and it will not be held against you.

Is there any way being in this study could be bad for me?
Your participation will entail completing mathematical tasks while describing aloud how you are thinking. All efforts will be made to conceal your identity from individuals outside of the research community. It may take a minute or two to overcome initial discomfort related to being recorded and talking about your instruction.

Will being in this study help me in any way?
We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include improving mathematical classroom practices in the classroom and learn new ways to integrate technology in your classroom.

What happens to the information collected for the research?
We cannot promise complete confidentiality. The results of this study might be used in reports, presentations or publications but your name will not be used. Copies of your work and recorded interviews will be stored on a password protected hard drive in the researcher's office, which remains locked whenever she is not present.

Who can I talk to?
If you have questions, concerns, or complaints, talk to the research team by emailing Krysten Pampel at krysten.pampel@asu.edu or Carla Van De Sande at carla.vandesande@asu.edu

This research has been reviewed and approved by the Social Behavioral IRB. You may talk to them at (480) 965-6788 or by email at research.integrity@asu.edu if:

- The research team is not answering your questions, concerns, or complaints.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You have questions about your rights as a research participant.
- You want to get information or provide input about this research.

Your signature documents your permission to take part in this research, which includes agreeing to be videotaped in interviews and classroom observations.

Signature of Participant: ____________________________________ Date: __________

Printed Name of Participant: ____

APPENDIX B

PRE-INTERVIEW PROTOCOL AND TASKS

Phase 1: Pre-interview

Task 1

Evaluate the following:

1.) $\frac{6}{\frac{1}{5}}$

2.) $\frac{\frac{5}{6}}{\frac{1}{4}}$

- Please make sure to think out loud so that I can understand what you are thinking when completing the task
- 6 divided by 1/5
  - What does your answer mean about the relationship between __6___ and ___1/5___?
  - Could you use your answer to compare the sizes?
    - "Number of __1/5ths__ that fit into __6___"
  - Is there anyway that you could represent your thinking about the result of your evaluation?
    - Since flip and multiply does not provide a model could you make one? "Circles or bars"
- 5/6 divided by 1/4
  - What does your answer mean about the relationship between __5/6___ and ___1/4___?
  - Could you use your answer to compare the sizes?
    - "Number of __1/4ths__ that fit into __5/6___"
  - Is there anyway that you could represent your thinking about the result of your evaluation?
    - Since flip and multiply does not provide a model could you make one? "Circles or bars"

Task 2

Every cheerleader needs 1/5 yard of ribbon to decorate a football player's locker for homecoming. A spool contains 2 ¼ yards of ribbon. How many 1/5 yard pieces of ribbon can be supplied by 1 spool?

- Please make sure to think out loud so that I can follow along with your thinking and process.
- *How many 1/5 yard pieces of ribbon can be supplied by 1 spool?*
- How many cheerleaders can be supplied by 1 spool of ribbon?
  - Can you explain to me what you are thinking about?
  - Why did you decide to…?
  - What does your answer mean in the context of the situation?
  - How much more ribbon do you need to supply all 12 cheerleaders?

Task 2 Option B *(Note: I never had to use this task.)*
A seamstress must cut 1/3-yard pieces of fabric from material that is 1 ¾ yards long. How many 1/3 pieces can be cut?

Student 1: The problem is asking how many 1/3-yard pieces are in 1 and ¾ yards of fabric. This is a division problem: 1 ¾ divided by 1/3 equals 7/4 * 3/1= 21/4= 5 ¼. However, 1/12< 1/3 therefore the remaining fabric is not big enough to be considered.

Student 2: Seven fourths divided by one third become 7/4 multiplied by 3/1 through the process of dividing fractions. The answer is 21/4; or 5 and ¼ 1/3 yard pieces. So the correct answer is 5 pieces with ¼-yard left over.

- Given this contextual problem, here are two student answers and explanations for the problem. Please read the solutions and explanations out load.
- What do you think about the two student responses?
- Would you give the same score to both of the students? Why or why not?
- Is one of the explanations better than the other?
- Which has correct answer
    - Have EXP describe which explanation is better

Task 3: Static Bar Examples given on index cards.

i. R>B
    1. What is the size of the red bar (relative to) in terms of the size of the blue bar?
    2. >1
        a. I noticed your answer is bigger than 1 why is that?
    3. =1
        a. I noticed that your answer is 1; can you explain how you got that?
    4. <1
        a. I noticed that your answer is less than 1
            i. What is the size of the blue bar relative to the red bar?
            ii. Why do you think the relative size is
            iii. How did you come up with the relative size?

ii. B>R
    1. What is the size of the red bar (relative to) in terms of the size of the blue bar?
    2. >1
        a. I noticed your answer is bigger than 1 why is that?
    3. =1
        a. I noticed that your answer is 1; can you explain how you got that?
    4. <1
        a. I noticed that your answer is less than 1
            i. What is the size of the blue bar relative to the red bar?
            ii. Why do you think the relative size is
    - If wrong talk about easier cases
        - Such as same size, twice as large

iii. Ask about R=B
    1. Can you
    2. How would you draw two bars

iv. Comparative static images
    1. In which case, is the size of the red bar relative to the size of the blue bar larger? (*Have the students order the static bars from largest to smallest.)*

v. Which mathematical operation or operations would you associated with the activity?
    1. Give options of addition, sub, multi, division

- If I multiply by 2.5 to the red and blue bars what would our relative size be?
- If I added 2 to the red bar and the blue bar what would happen to our relative size?

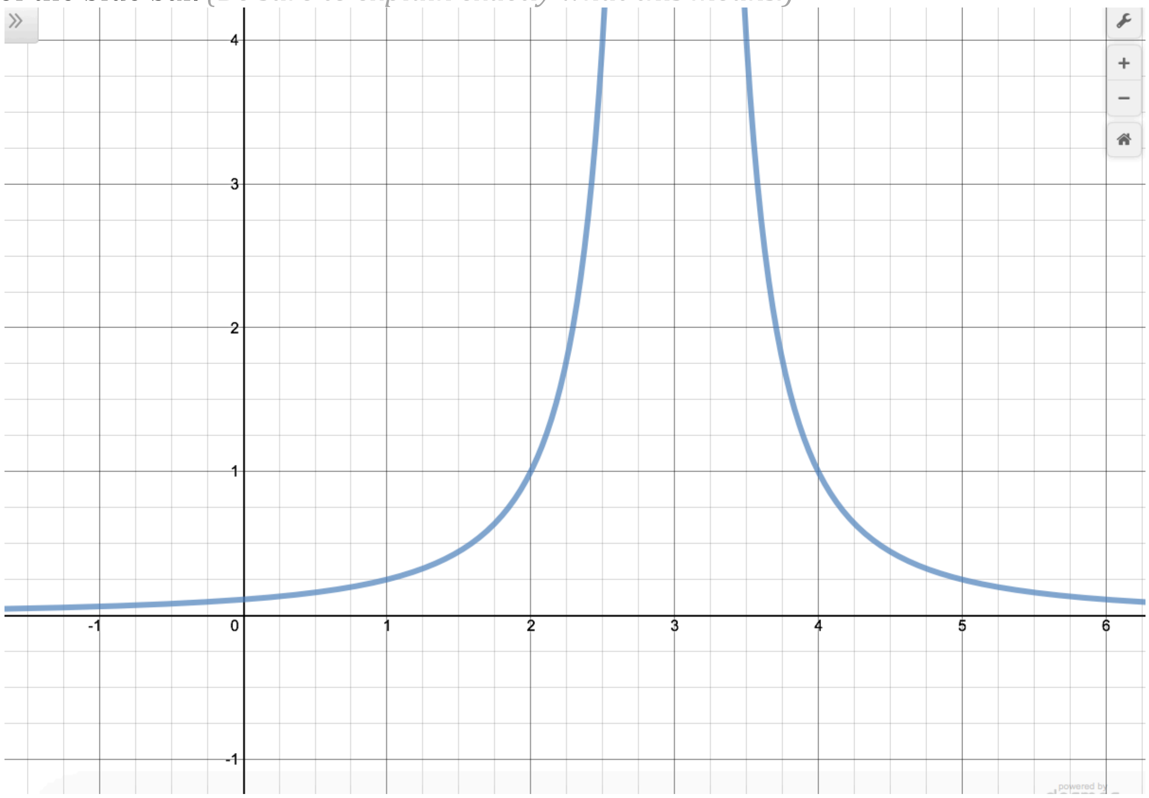
Task 4: Conversation about Task 1 and Task 3
Numbers and comparison of two numbers as ratio

- Do you see any similarities in the tasks you have completed so far?
    - Make sure to see if the student addresses the ways of thinking needed to understand the tasks.
- Do you see a relationship, what if any, between the bar activity and the first two tasks?
- Could you make a model to help illustrate this relationship?
- Could you draw a picture of the red and blue bars that would give you the same solution as you found in the first task?

Task 5: Dynamic Bar Examples

- Animations (RB is constant and BB decreases; RB constant and BB increasing; RB increasing and BB decreasing)
    - What is the relative size of the red bar in terms of the blue bar when both bars are gone?
    - How is the relative size of the red bar in terms of the blue bar changing as the blue bar decreases in length?
    - If the blue bar were to no longer be seen (and the red bar is the same), what would the relative size of the red bar in terms of the blue bar be?
    - Using the zoom feature to better understand the students thinking:
        - If the student thinks about the size of the bars
            - When zooming out the student will think that the relative size is affected.
        - If the student thinks about the relative size of the red bar in terms of the blue bar.
            - When zooming out the student will think that the relative size is the same regardless of the size of he bars.
    - N=1 D=1 (0→ t → 10)
    - N=1 D=1 (10→ t → 0)
    - N=3 D=1 (0→ t → 10)
    - N=3 D=3 (0→ t → 10)
    - N=3 D=3 (10→ t → 0)
    - N=3 D=2 (0→ t → 10)
    - N=3 D=2 (10→ t → 0)

Task 6: This graph below depicts the relative size of the red bar in terms of the size of the blue bar. *(Be sure to explain exactly what this means.)*



- Select random points
    - What could the relative size of the RB in terms of the BB be?
    - Can you draw a possible picture to represent the relative size?
    - Can you demonstrate with your hands how the relative size is changing as the input
        - Increases to 0?
        - Decreases to 3?
- Behavior as the input changes
    - What would happen to your image as the values of the input approached the value ________?
        - Would the RB increase or decrease?
        - Would the BB increase or decrease?
        - What would happen to the relative size of the RB in terms of the BB?
- Rational Function
    - Can you construct the rule of the function graphed in this image?
    - How do the function of the numerator and the function of the denominator effect the construction of the rational function?
    - Add arrows to see if students change opinion about the graph.
- In your own words, what is relative size?
    - Do we have to have a RB and BB in order to find/have a relative size?
    - Could you give me a different scenario where we could use relative size?

Task 7
The following is a rational function:

$$h(x) = \frac{x+3}{x^2+4x-1}$$

- Starting with a fixed value of 2
    - What would the output of this function be if the input is 2?
    - Does your answer for the output of the function tell us anything about the value of numerator?
    - Does your answer for the output of the function tell us anything about the value of the denominator?
    - What does your answer mean about the value of the numerator and the value of the denominator?
    - In terms of the bars what would they look like when the input value of the function is 2?
    - Could you make a table of values for this function?
- What would happen to the output value of the function as the input value increased from 2?
    - What does your answer indicate about the value of the numerator as the input value increases?
    - What does your answer indicate about the value of the denominator as the input value increases?
    - What would happen to the lengths of the bars based on your answer?
- What would happen to the output value of the function as the input value decreased from 2?
    - What does your answer indicate about the value of the numerator as the input value increases?
    - What does your answer indicate about the value of the denominator as the input value increases?
    - What would happen to the lengths of the bars based on your answer?

Phase 2: Pre-interview

Personal Background

- What type of degrees do you have?
- What degree are you working on now?
- Did you have any previous teaching experience before ASU?
- How many semesters have you taught MAT170: Pathways Pre-Calculus?
- What made you interested in becoming a teaching assistant? (Motivation)
- Do you plan to teach after getting your degree?
- Does your instruction differ from the instruction you received or still receive? If so, in what ways? (Technology, practices, strategies, classroom environment, expectations, etc.)
- Do you use anything that you learned in the TA Seminar led by Katie?
- Do you use anything from the TUME Seminar led by Marilyn?
    - o Compare the TA and TUME

Read the Scenarios

- Which scenario do you identify with more and in what ways?
    - o (Two scenarios dump truck (discovery based learning) and bulldozer (direct instruction).)
- In your own words, how do you prepare a general lesson for your course?
    - o TUME seminar, key points for lesson, side notes, etc. (Use of Technology)
- How do you decide what tasks to use in your lessons?
- What mathematical meanings for rational functions do you want students to have at the end of your lessons?
- Do you remember any thing specific about rational functions that students struggled with last semester?
    - o What were those stumbling blocks?
- Have you used technology (simulations, animations, manipulative software) to teach so far this semester?
    - o If so, what types of technology did you use?
    - o How did you implement the technology?

Tasks

Evaluate the following:

1.) $\frac{3/4}{2/3}$

- Please make sure to think out loud so that I can understand what you are thinking when completing the task
- 3/4 divided by 2/3
    - What does your answer mean about the relationship between __3/4___ and ___2/3___?
    - Could you use your answer to compare the sizes?
        - o "Number of __2/3rds__ that fit into __3/4ths___"

- Is there anyway that you could represent your thinking about the result of your evaluation?
    - Since flip and multiply does not provide a model could you make one? "Circles or bars"

2.) 6/7 of what number is 5?

- 6/7 of what number is 5?
  - What does your answer mean about the relationship between __6/7___ and ___5___?
  - Could you use your answer to compare the sizes?
    - "__5___ is _______ times as large as ____6/7ths____"
  - Is there anyway that you could represent your thinking about the result of your evaluation?
    - Since flip and multiply does not provide a model could you make one? "Circles or bars"

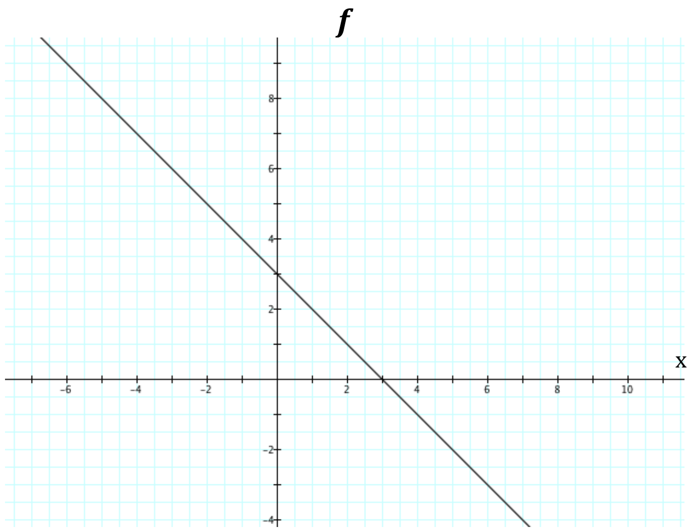
3.) ¼ of what number is 5/6?

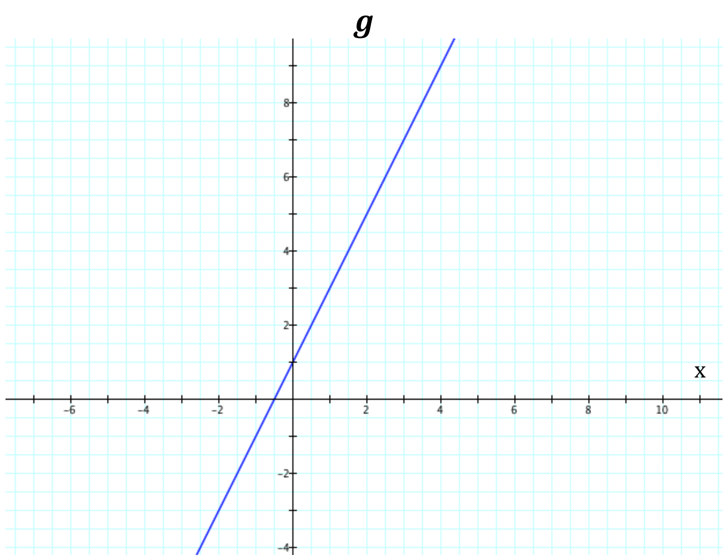
- ¾ of what number is 5/6?
  - What does your answer mean about the relationship between __3/4___ and ___5/6___?
  - Could you use your answer to compare the sizes?
    - "__5/6ths___ is _______ times as large as ____6/7ths____"
  - Is there anyway that you could represent your thinking about the result of your evaluation?
    - Since flip and multiply does not provide a model could you make one? "Circles or bars"

Dynamic Bars D=2 N=2

- Statics instances of dynamic bars (6 instances)
  - What is the relative size of the red bar in terms of the blue bar?
  - Discuss the changes that occur when the bars change length.
- Dynamic bars
  - Can you draw the graph of the relative size?
  - Please draw the graph of the relative size to the best of your ability

Using the graph of $f(x)$ and $g(x)$, construct a graph of $h(x)=\frac{f(x)}{g(x)}$





- Draw the graph
    - Look to see if he is creating the functions of the two graphs in order to graph.
    - How did you create the graph?
    - What were you looking at in order to graph h?

Dynamic Graphing

- Draw the graph using the animation to help
- N=1 D=1
    - Look to see if he is creating the functions of the two graphs in order to graph.
    - How did you create the graph?
    - What were you looking at in order to graph the relative size?
- N=2 D=2
    - Look to see if he is creating the functions of the two graphs in order to graph.
    - How did you create the graph?
    - What were you looking at in order to graph the relative size?

Construct a graph of the following rational function.

$$k(x) = \frac{4 - x}{28 - 3x - x^2}$$

- Draw the graph
  - How did you create the graph?
  - What were you looking at in order to graph the rational function?
  - Do you see any relationship between graphing rational functions and the tasks that I had you complete?

Phase 2: Intervention Interview

Introduction of Tool/Supports

- Do you have any experience with Graphing Calculator? If so, in what context did you use it? (Explain/discuss Graphing Calculator with short demo)

Now we are going to go through the applets that you will use in your classroom. As we go through the applets, I would like for you to take on the role of the student and I will take on the role of the instructor. Please feel free to stop me at any time during the demonstration of the activity and applets to ask clarifying questions!

- Moving vectors
- Sum Bar
- Rat Bar
- Rat Graph

The questions I am going to ask you will involve instructional decisions for your upcoming lessons on rational function. I understand that you will not have solid answers for these questions and that your answers may change over the next couple of days. That is why, I would like for you to again keep a video journal of instructional decisions that you make over the next couple of days. Here is a notebook that will be used as a back up plan just in case something does not work with the video journal. <u>I also have questions for you to answer as you are completing your video journal.</u> Please make sure to voice any questions or thoughts you may have about the applets, teaching rational functions, or anything else related to this study.

- How would you build these applets into your rational function instruction?
  - How do you see this fitting in with your current materials/lessons?
  - Where would you add these applets into your instruction?
- What mathematical understandings do you want your students to have after your rational function lessons with these applets?
- Can you please identify what part of your lesson will build the students understanding of this concept?
- How does this approach differ from your past rational function instruction?
- What stumbling blocks do you think students will have during this lesson?
  - How do you anticipate handling the stumbling blocks?

APPENDIX C

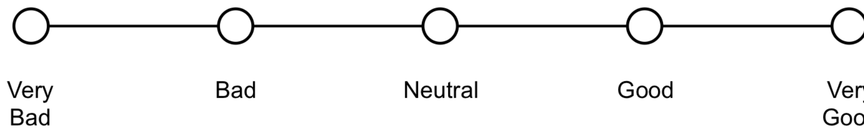
POST-INTERVIEW PROTOCOL AND TASKS

Phase 1: Post-interview

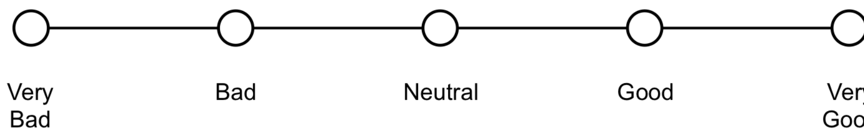
Discussion of Observations

Now that you have completed the lessons on rational functions, I would like to get your thoughts about how the lessons went.

- Overall how do you feel the rational function lessons went?



- How do you think your students did during the rational function lessons?



- Do you believe that your rational function lessons are a good representation of an average day in class?
    - If yes, give specific details as to why these lessons represent an average day in class?
    - If not, give details on what made these lessons not an average day in class?
- Did you notice any changes in your classroom while teaching rational functions?
- What was the overall learning goal of the rational function lessons? *What understandings of rational functions did you want your students to come away with?*
- Do you think your students demonstrated the understanding(s) that you were striving for in these lessons? *Do you believe that the students have those understandings? What leads you to your conclusion on the students understanding?*
    - If yes, please give an example from class.
    - If no, please give an example from class.
- Were there any notable disruptions in the rational function lessons? Did you experience any moments while teaching rational functions that was awkward or unusual?
    - Why do you believe these disruptions might have occurred?
    - Could you have done anything differently to avoid the disruption?
- Did you cover everything you wanted to cover in the rational function lessons? *(This includes tasks, discussions, and activities)*
    - If yes, why?
    - If no, what tasks, discussions, and activities would you have liked to cover?
- Would you change anything from your lesson?
    - If so, would you modify anything? If not, why?
    - Time, order of lessons, etc.
- Did your original timeline for the rational function lessons change?

  - If yes, in what ways?
  - If no, what do you think helped you in constructing a timeline that was successful?
- Did you have any difficulty in selecting the tasks/activities for these rational function lessons?
  - If yes, why?
  - If no, why not?
- Have you ever considered adding technology into your lessons?
  - If so, what type of technology would you want to add?
  - What would the technology look like?
  - How would the technology help?
  - Who would the technology help?
- Have you used technology in other lessons?
- Are there other concepts that you would want to add technology to?
  - If yes, what type of technology would you want to add?
  - What would the technology look like?
  - How would the technology help?
  - Who would the technology help?
- After teaching rational functions this semester, did your own understandings of rational functions change?
  - If so, how?
- Why do you think your understandings changed?
- Are you planning on giving an assessment on rational functions to your students?
  - If yes, what type of assessment?
  - When would you be giving the assessment?
  - Can I get copies of your assessment? (*I do not need any identifying information about the students. Only the scores and the questions.*)
- Since MAT170 has a common final, I would like permission to gather information about the Mod 6 items that are on the exam. This entails only the answers from those items with no student information.

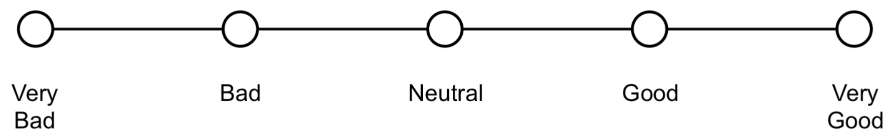
Now I am going to show you short video clips from the observations. My goal in doing this is to help you recall events from the lesson. We will watch the video clips one at a time and I will ask you to describe your thoughts and emotions at the time to the best of your ability.

Phase 2: Post-interview

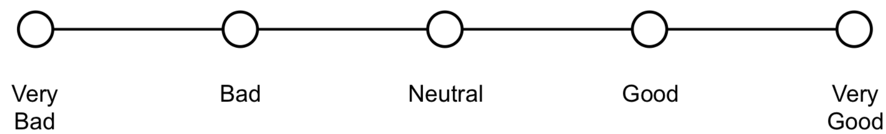
<u>Discussion of Observations</u>

Now that you have completed the lessons on rational functions, I would like to get your thoughts about how the lessons went.

- Overall how do you feel the rational function lessons went?

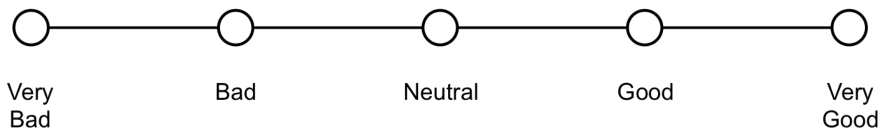


- How do you think your students did during the rational function lessons?



- Do you believe that your rational function lessons are a good representation of an average day in class?
    - If yes, give specific details as to why these lessons represent an average day in class?
    - If not, give details on what made these lessons not an average day in class?
- Did you notice any changes in your classroom while teaching rational functions?
- What was the overall learning goal of the rational function lessons? *What understandings of rational functions did you want your students to come away with?*
- Do you think your students demonstrated the understanding(s) that you were striving for in these lessons? *Do you believe that the students have those understandings? What leads you to your conclusion on the students understanding?*
    - If yes, please give an example from class.
    - If no, please give an example from class.
- Were there any notable disruptions in the rational function lessons? Did you experience any moments while teaching rational functions that was awkward or unusual?
    - Why do you believe these disruptions might have occurred?
    - Could you have done anything differently to avoid the disruption?
- Did you cover everything you wanted to cover in the rational function lessons? *(This includes tasks, discussions, and activities)*
    - If yes, why?
    - If no, what tasks, discussions, and activities would you have liked to cover?
- Would you change anything from your lesson?
    - If so, would you modify anything? If not, why?
    - Time, order of lessons, etc.

- Did your original timeline for the rational function lessons change?
    - If yes, in what ways?
    - If no, what do you think helped you in constructing a timeline that was successful?
- Did you have any difficulty in selecting the tasks/activities for these rational function lessons?
    - If yes, why?
    - If no, why not?
- How do you think the implementation of the applets went in your class?



- Did you experience any moments during your planning for the applets that was awkward or unusual? If so what were they?
- Did you consider the assessments that you will give your students when planning?
- During your video journal planning sessions, you kept stating that the applets helped you plan your lessons. Can you please explain how the applets helped you plan your lessons?
    - Do you think the curriculum does a good job of helping you plan your rational function lessons?
    - How is planning with the applets compare to planning with the curriculum?
- How much did you practice with the applets before teaching with them?
    - Did you have the guides out at the same time?
- Did you experience any moments during the implementation of the applets that was awkward or unusual?
    - If so what/when did they occur?
    - Do you still have questions about the applets or applet guides?
- Did you experience or see a change in your classroom structure/environment when the applets were introduced?
    - Were the discussions in your classroom affected? More or less?
- Would you use this package of materials again?
    - If so, would you modify anything? If not, why?
        - Time, order of lessons, etc.
    - Do you still believe that it is important to teach the algebraic rules of rational functions before using the applets? Why?
- If you were to teach with these applets again would you teach them in the same manner?
- Did these applets change your understanding of rational function in any way?
    - If so, how?
- Do you believe that the students developed the mathematical understandings of rational functions that you designed your lessons for?
    - What leads you to your conclusion?

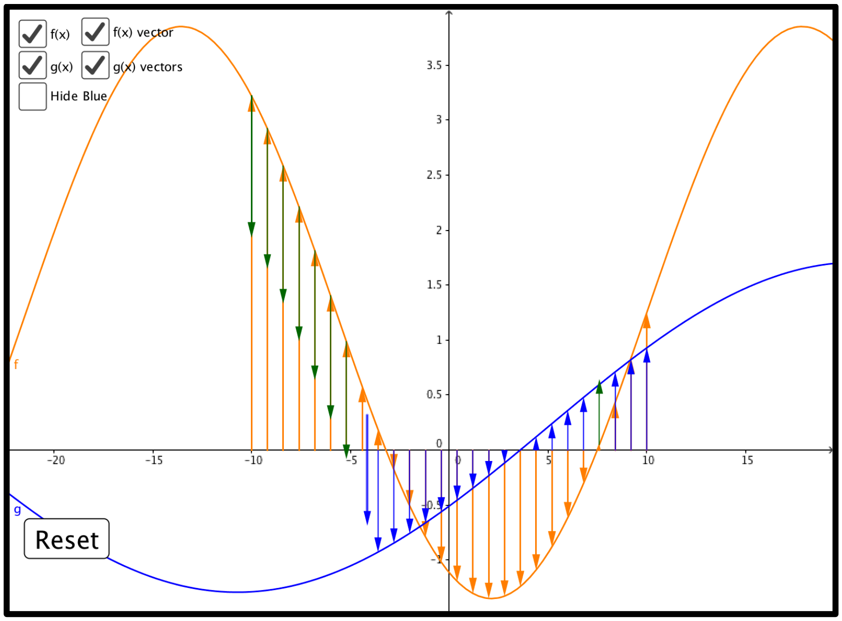
- Did you change mathematical understandings for students from the first semester to the second semester based on planning using the applets?
    - Would you add any additional mathematical understandings for rational functions after teaching in this semester with the applets?
- Do you think your understanding of rational functions has changed over the two semesters? If yes, in what ways? If no, why not?
- Do you think your instruction for rational functions has changed over the course of the two semesters?
- Are you planning on giving an assessment on rational functions to your students?
    - If yes, what type of assessment?
    - When would you be giving the assessment?
    - Can I get copies of your assessment? (*I do not need any identifying information about the students. Only the scores and the questions.*)
- Since MAT170 has a common final, I would like permission to gather information about the Mod 6 items that are on the exam. This entails only the answers from those items with no student information.

Now I am going to show you short video clips from the observation. My goal in doing this is to help you recall events from the lesson. We will watch the video clips one at a time and I will ask you to describe your thoughts and emotions at the time to the best of your ability.

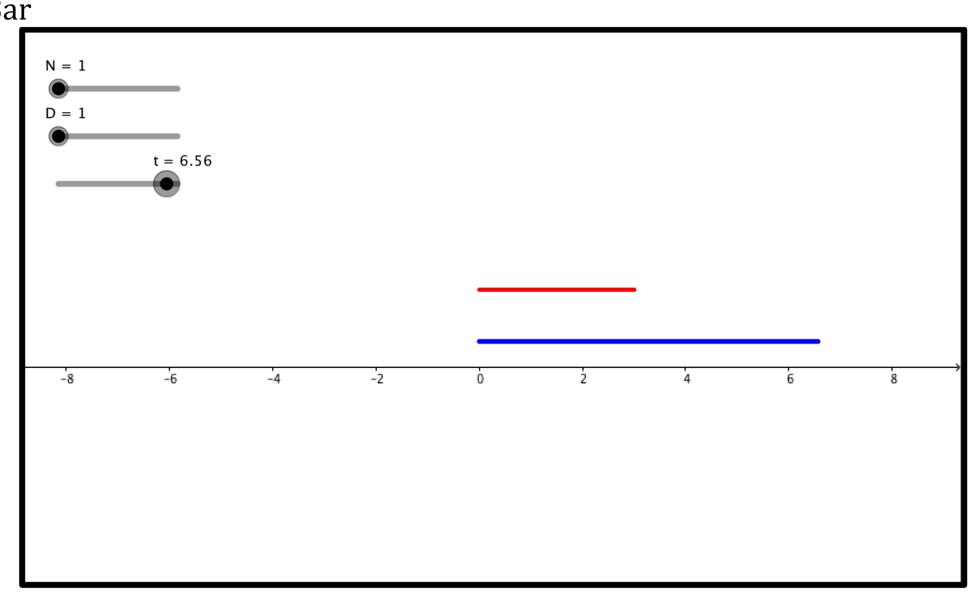
APPENDIX D

DIDACTIC OBJECTS

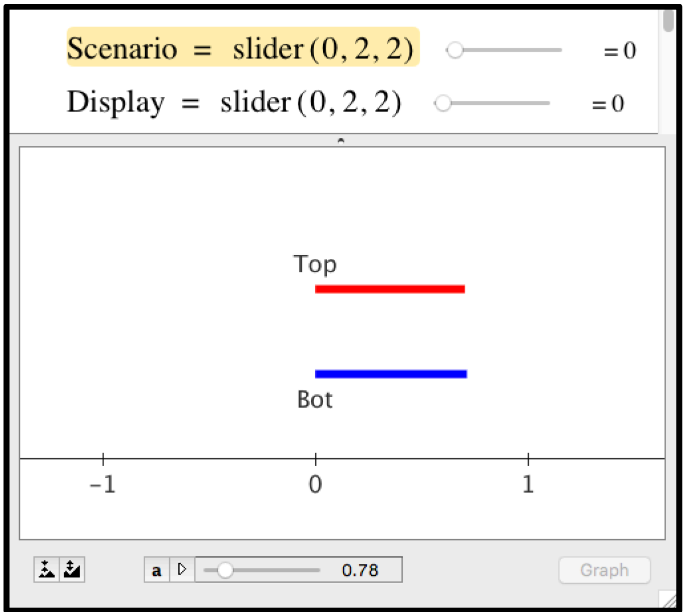
Moving Vectors



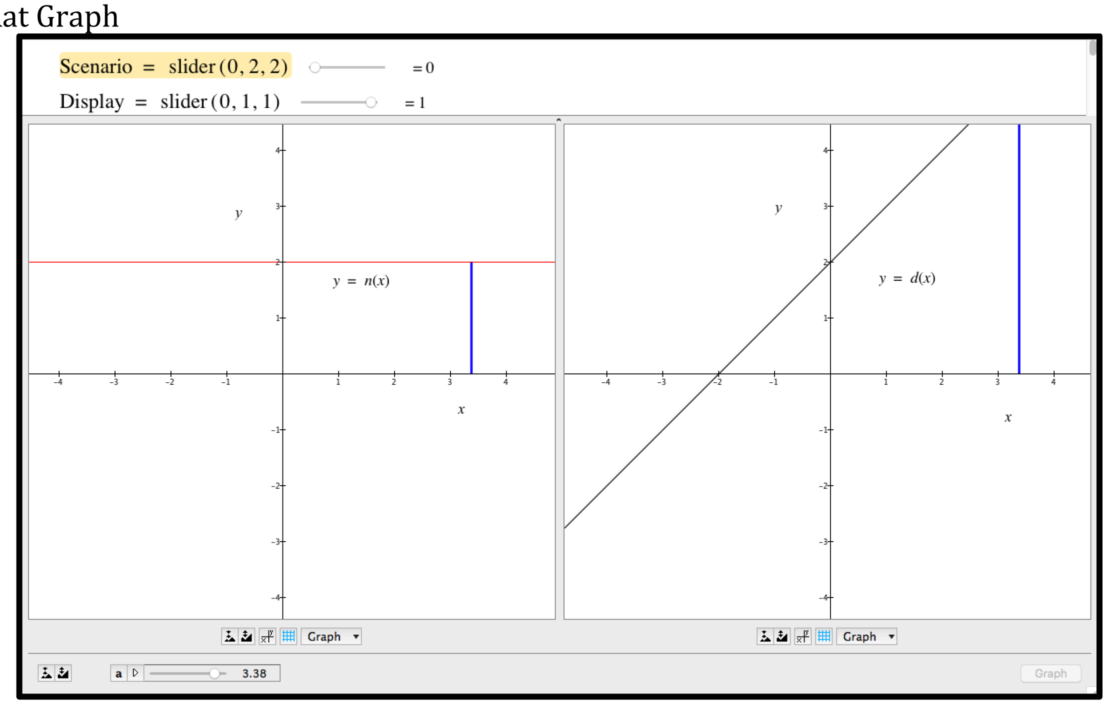
Sum Bar



Rat Bar



Rat Graph



APPENDIX E

ADDITIONAL DIDACTIC OBJECT SUPPORTS

Open Moving Vectors.ggb

<u>Object setting:</u>
The f(x) and g(x) buttons are check.

<u>Discussion:</u>
We have already discussed algebraic procedures to find f(x) + g(x) but now we do not know the rule of the function. Instead, we have the graph of both f(x) and g(x). How can we determine f(x) + g(x) without knowing the rule of the functions? (*Let students have a discussion. Once the student groups or pairs have made some sort of agreement move to next discussion.*)

<u>Discussion:</u>
What do we have to envision if we were to find the f(x)+g(x) graph? What does f(x) represent? What does g(x) represent? What does f(x)+g(x) represent? (*Hopefully students will explain that they need to envision taking the vertical distance from the horizontal axis to the graph of f(x) and add the vertical distance from the horizontal axis to the graph of g(x).)*

<u>Object setting:</u>
Check the f(x) vector and g(x) vector after the students explain what f(x) and g(x) represent.

<u>Object setting:</u>
Show students the capabilities of the GeoGebra file such as moving the g(x) vectors as well as the reset button.

<u>Discussion:</u>
Now that we have discussed what we have to envision to find f(x) + g(x), use the graph and the magnitude of f(x) and g(x) to construct the graph of f(x) + g(x). (*In order for students to graph a sketch of the function they will need to uncheck the f(x) and g(x) vectors and use the ABC drop down menu to select the pen tool. Allow students to explore the abilities of the GeoGebra file.*)

Note: As you walk the room make sure that the students are not only moving the g(x) vectors and trying to trace f(x) + g(x). You can remind students to use the reset button if they get stuck or make a mistake. Also suggest placing one arrow at the beginning of the other arrow.

<u>Discussion:</u>
Have students re-explain what they had to do to find the graph of f(x) + g(x).
Note: You may even want students to write down what f(x), g(x) and f(x) + g(x) represent.

<u>Object setting:</u>
Show sample of traced f(x) + g(x) graph. (*You can type into the input bar h(x)=f(x)+g(x), this will give the exact graph of f(x)+g(x).*)

Open the file Sum Bar.ggb

Object setting:
N = 1 and D = 1

Discussion: What is the sum of Top and Bot? (*Repeat this question for each of the object settings below*.)

Object setting:
1.) Set t =1.12, Top + Bot = 4
2.) Set t = 3.04, Top + Bot = 6
3.) Set t = 5.92, Top + Bot = 9
4.) Set t = -0.96, Top + Bot = 2

Discussion: How could we represent the value of the sum of Top and Bot? Is there a notation that would help us distinguish between the values of the sum of Top and Bot?

Note: If students do not suggest function notation urge them in that direction

Object setting:
N = 1, D =1, t = -0.96

Discussion: Ask students to use the distance between their index fingers to represent the value of the sum of Top(t) and Bot(t).

Note: As you slide the t value scale from -0.96, to 1.12, 3.04, and 5.92, you should notice the students hands growing further and further apart. (*You are trying to get students to see the sum of Top(t) and Bot(t) as a quantity*)

Open the file Rat Bar.gcf.

Object setting:
Scenario = 0 and Display = 0

Discussion: What is the relative magnitude of the Top and Bot? In other words, what is the measure of the Top in units of the Bottom? (*Repeat this question for each of the object settings below*.)

Object setting:
1.) Set a =0.78, Top/Bot = 1
2.) Set a = 1.12, Top/Bot = 2
3.) Set a = 1.46, Top/Bot = 8

Discussion: How can we represent these relative magnitudes to differentiate among them? Is there a notation that would help us make a distinction between each of these relative magnitudes?

Note: If students do not suggest function notation urge them in that direction

Object setting:
Scenario = 0, Display =1, a = 0.78

Discussion: Ask students to use the distance between their index fingers to represent the relative magnitude of Top(a) and Bot(a).
Note: As you slide the a value scale from 0.78 to 1.12, 1.38, 1.46, and 1.54, you should notice the students hands growing further and further apart. (*You are trying to get students to see the relative magnitude as a quantity*)

Object setting:
Scenario = 0, Display = 1, a = 0
Press the play button on the a slider
Discussion: Explain to the students that they should still use the distance between their index fingers to represent the relative magnitude.
Note: As the slider for a is playing the students demonstrate relative magnitude as increasing and decreasing. *Let the students practice and discuss until majority of students seem satisfied with their representation.*

Object setting:
Pause the a slider until you explain the discussion below then begin the slider again.
Discussion: Tell students to use the distance between the tabletop and their left hand to represent the relative magnitude.
Note: Look for students to demonstrate relative magnitude using tabletop and their left hand.

Object setting:
Pause the a slider until you explain the discussion below then begin the slider again.
Discussion: Point out the value of the a slider has been varying from 0 to 3.14. Tell students for this next part they are still going to track the relative magnitude with the table-top and their left hand but they will now track the value of a by sliding their right index finger along the table top.
Note: Be Patient and let the students struggle. Students will need a significant amount of time to coordinate both the relative magnitude and the a value. (*Perfection is not a priority*)

Object setting:
Pause the a slider until you explain the discussion below then begin the slider again.
Discussion: Tell students to still track both the relative magnitude and the a value, but this time they will need to keep their left hand directly above their right index finger.
Note: Be Patient and let the students struggle. Students will need a significant amount of time to coordinate both the relative magnitude and the a value. (*Perfection is not a priority*)

Discussion: When students can coordinate both quantities' values ask: Has what we have just done have anything to do with graphing? What was your left hand sketching in the air?

Note: You can always go back to the pervious discussion for additional practice if students seem lost. Have the students pay attention to what their left hand if sketching.

Discussion: Tell students to sketch a graph of the relative magnitude of Top(a) to Bot(a) in relation to the value of a on graph paper. Encourage students to discuss one another's graphs and answer the questions on the worksheet.

Open file Rat Graph.gcf

Discussion: Explain to the students that these are graphs of two functions n and d. The left graph is n(x) and the right graph is d(x). Can we use the same way of thinking we just developed in the first activity to envision and construct a graph of h(x) = n(x)/d(x)?

Object setting:
Scenario = 0, Display = 1
Discussion: Ask again, can we use the same way of thinking we just developed in the first activity to envision and construct a graph of n(x)/d(x)? Additionally, ask students to describe the behavior of h(x), as x increases/decreases without bound. Also make sure to ask questions about the behavior of h(x) when n(x) and/or d(x) equals zero. Have students construct the sketch of the graph h(x) = n(x)/d(x) on graph paper.

Object setting:
Scenario = 1, Display = 1
Discussion: Have students construct the sketch of the graph h(x) = n(x)/d(x) on graph paper. Once the students have began to construct the sketch of the graph of h makes sure to discuss the behavior of h(x) similarly to the pervious object setting.

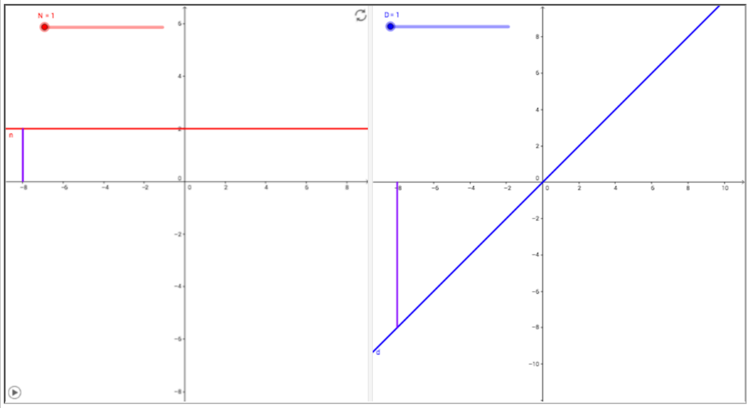
Object setting:
Scenario = 2, Display = 1
Discussion: Have students construct the sketch of the graph h(x) = n(x)/d(x) on graph paper. Once the students have began to construct the sketch of the graph of h makes sure to discuss the behavior of h(x) similarly to the pervious object setting.

iMathAS Online Worksheet

This was covered in 02-LessonLogic the online worksheet in MathAS would be a perfect summative assessment for this lesson on rational functions.

Page 1 of iMathAS Worksheet

Make sure that the settings on animation below are N=1 and D=1 in order to answer the questions on this page.



Use the graphs of n and d above. Suppose the rational function h is defined by $h(x) = \frac{n(x)}{d(x)}$.

What happens to h(x) as $x \to 0^-$?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

Submit

What happens to h(x) as $x \to 0^+$?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

Submit

What happens to h(x) as x increases without bound?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

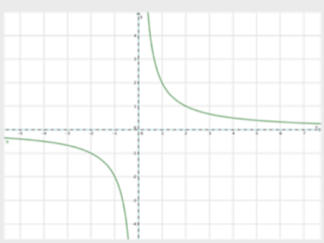
Submit

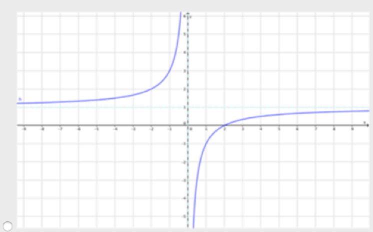
What happens to h(x) as x decreases without bound?

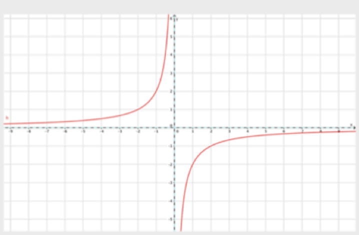
- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

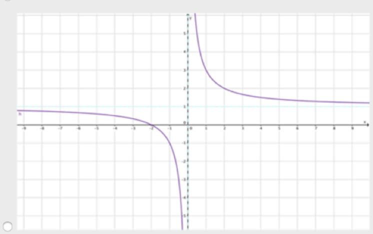
Submit

Select the graph of *h* , when N=1 and D=1.



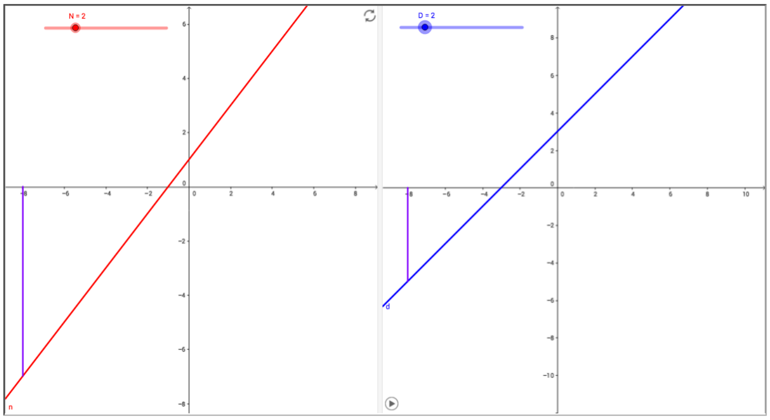






Page 2 of iMathAS Worksheet

Make sure that the settings on animation below are N=2 and D=2 in order to answer the questions on this page.



Use the graphs of n and d above. Suppose the rational function h is defined by $h(x) = \dfrac{n(x)}{d(x)}$.

What happens to h(x) as $x \to -1^-$?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

Submit

What happens to h(x) as $x \to -1^+$?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

Submit

What happens to h(x) as x increases without bound?

- h(x) approaches 1
- h(x) decreases without bound
- h(x) increases without bound

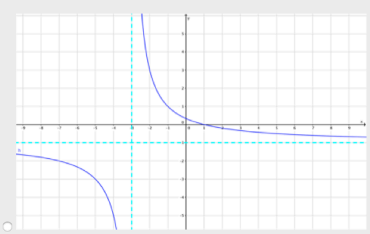
Submit

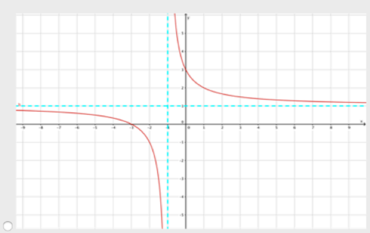
What happens to h(x) as x decreases without bound?

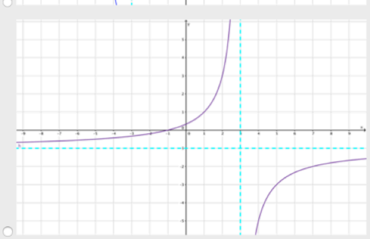
- h(x) approaches 1
- h(x) decreases without bound
- h(x) increases without bound

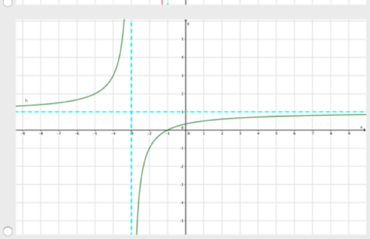
Submit

Select the graph of h , when N=2 and D=2.









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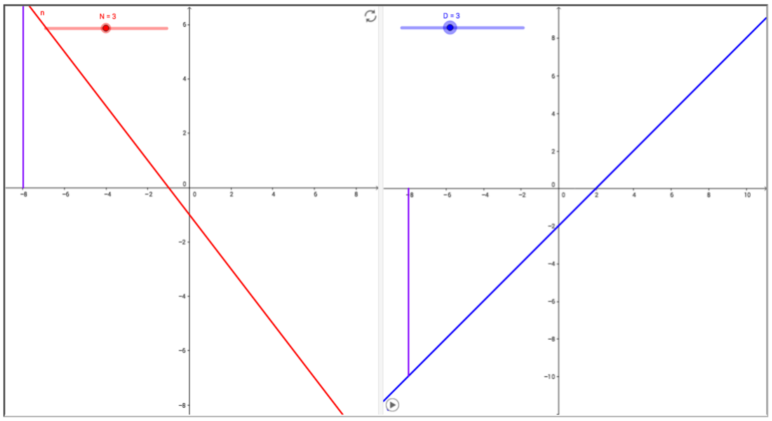
Page 3 of iMathAS Worksheet

Make sure that the settings on animation below are N=3 and D=3 in order to answer the questions on this page.



Use the graphs of n and d above. Suppose the rational function h is defined by $h(x) = \frac{n(x)}{d(x)}$.

What happens to h(x) as $x \to 2^-$?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

Submit

What happens to h(x) as $x \to 2^+$?

- h(x) approaches 0
- h(x) decreases without bound
- h(x) increases without bound

Submit

What happens to h(x) as x increases without bound?

- h(x) approaches -1
- h(x) decreases without bound
- h(x) increases without bound

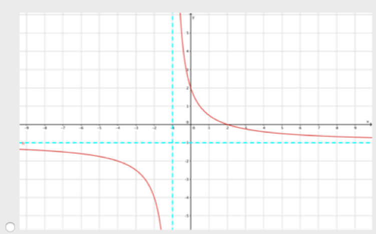
Submit

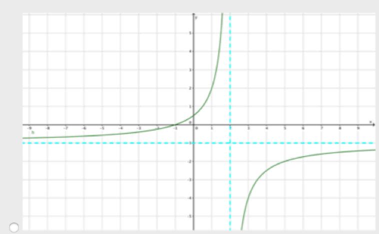
What happens to h(x) as x decreases without bound?

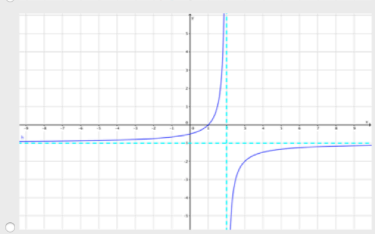
- h(x) approaches -1
- h(x) decreases without bound
- h(x) increases without bound

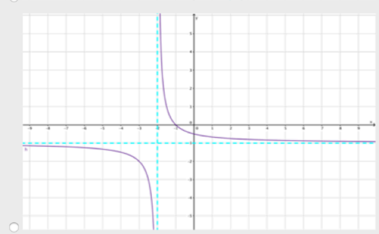
Submit

Select the graph of h , when N=3 and D=3.









Submit

APPENDIX F

DR THOMPSON'S ACTIVITY GUIDE

Rational Function Activity

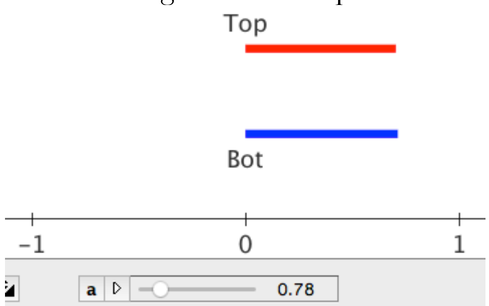
Pat Thompson

Open the file *Rational Function #1.gcf.*

Phase 1: Conceptualizing and representing relative magnitude as a quotient of functions

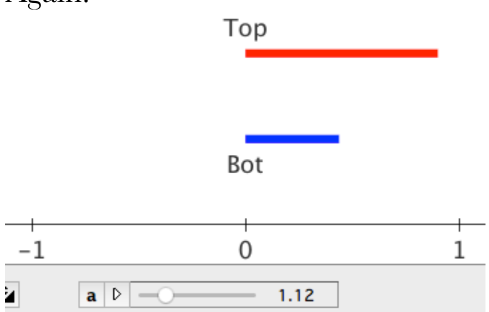
Scenario = 0, Display = 0

1. Relative magnitude of Top and Bottom: Measure Top in units of Bottom



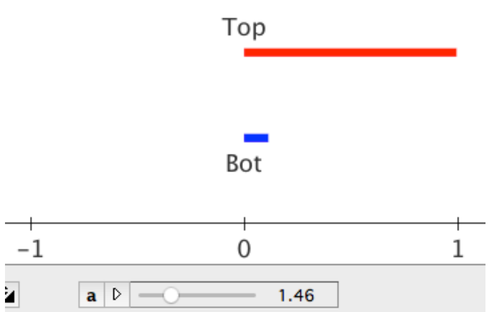
Relative magnitude of Top and Bottom is 1.
Represent that: Top/Bot = 1

2. Again:



Represent this: Top/Bot = 2 (approximately)

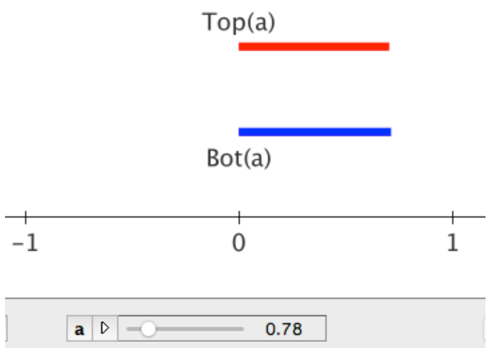
3. Again:



Represent this: Top/Bot = 8 (approximately)

4. Problem: We said "Top/Bot" equals 3 different numbers. How can we represent these relative magnitudes to differentiate among them?

   Students often have suggested using subscripts or other similar device:
   $\text{Top}_1/\text{Bot}_1=1$, $\text{Top}_2/\text{Bot}_2=2$, $\text{Top}_3/\text{Bot}_3=8$
   Accept this, then ask, "How could we represent these relative magnitudes so that we can re-create them?" Point out that we used the *a* slider to change magnitudes.
5. Lead the conversation toward the use of function notation, e.g. Top(0.78)/Bot(0.78) = 1. Introduce the idea yourself if it does not arise from the students. Recreate the above displays, and represent the relative magnitude using function notation and values of *a*.
   Scenario = 0, Display = 1



Phase 2: Internalizing Relative Magnitude as a Quantity

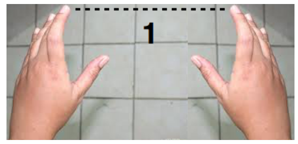
All values of a *are approximate. Don't get hung up on getting them exact while working with students. But practice setting the value of* a *to these different settings before using the tool with students.*

Scenario = 0, Display = 1, a = 0.78

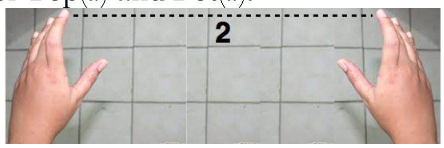
The aim in this phase is that students cease seeing just two magnitudes separately and instead begin to see relative magnitude of these magnitudes as a quantity in its own right *in addition* to seeing the two magnitudes separately. This phase is designed to that students develop an embodied image of relative magnitude of Top and Bot that they can subsequently (in the next phase) coordinate with the value of *a* (actually, with experiential time). Their ability to coordinate relative magnitude with experiential time will be foundational for their ability to sketch a graph of the relative magnitude of Top(*a*) and Bot(*a*) in relation to the value of *a*.

Ask students to hold their hands like below. The initial distance between their index fingers will represent the relative magnitude of 1 (alert them that they might want to think ahead about how large they want a distance of 1 to be).

6. Slide *a* to a value of 0.78. Ask students to use the distance between their index fingers to represent the relative magnitude of Top(*a*) and Bot(*a*).



7. Slide *a* to a value of 1.12. Ask students to use their index fingers to represent the relative magnitude of Top(*a*) and Bot(*a*).



8. Slide *a* to a value of 1.38. Ask students to use their index fingers to represent the relative magnitude of Top(*a*) and Bot(*a*).
9. Slide *a* to a value of 1.46. Ask students to use their index fingers to represent the relative magnitude of Top(*a*) and Bot(*a*). Deal playfully with their complaints that the distance is too large or that their arms are not long enough.
10. Slide *a* to a value of 1.59. Ask students to use their index fingers to represent the relative magnitude of Top(*a*) and Bot(*a*). Deal playfully with their complaints that the distance is too large or that their arms are not long enough.
11. Repeat Steps 6-10, starting with 0.78, but moving the slider to any successive values of *a*. Repeat until students can represent various relative magnitudes with ease.

    Scenario = 0, Display = 1, a = 0.

Tell students that the value of *a* can vary automatically (demonstrate by clicking the *a* slider's play button).

Reset *a* to 0. Tell students to use their index fingers to track the relative magnitude as the bars vary.

Click the play button on the *a* slider. *DO NOT PAUSE IT—REGARDLESS OF HOW MANY STUDENTS ASK YOU TO PAUSE IT.*

Let students practice, discuss, or anything else they find helpful as they attempt to vary the distance between their index fingers to represent the relative magnitude of Top(*a*) and Bot(*a*) as it varies. Continue until all students are satisfied that they are representing reasonably well the relative magnitude as it varies.

Phase 3: Graphing Relative Magnitude

12. Tell students to rotate their left hand 90° clockwise and to use the table top as their right hand, and to use their index finger as their measuring point.
13. Reset the *a* slider to 0.
14. Play the *a* slider and let students represent the relative magnitude as it varies. Stop when students are satisfied with their attempts to represent the varying relative magnitude.
15. Stop when students are satisfied with their attempts to represent the varying relative magnitude.
16. Point out that the value of the *a* slider has been varying the whole time that the were tracking the relative magnitude of Top(a) and Bot(a), and that its value varies from 0 to 3.14.
17. Tell students to continue using their left hand to track the relative magnitude, but in addition to slide their right forefinger along the table to represent the value of *a*. *STUDENTS WILL TAKE A WIDE VARIETY OF TIMES TO ATTAIN PROFICIENCY. DO NOT MAKE EXACTNESS A PRIORITY.*
18. Tell students to keep their left hand directly above their right forefinger as they track both quantities' values. *STUDENTS WILL TAKE A WIDE VARIETY OF TIMES TO ATTAIN PROFICIENCY. DO NOT MAKE EXACTNESS A PRIORITY.*
19. When students can coordinate both quantities' values satisfactorily, ask them if what they have just done has anything to do with graphs. If necessary, ask them if their left index finger traced a graph "in the air". If necessary, have students repeat #18 again, this time paying attention to what they are sketching "in the air".
20. Tell students to sketch the graph of the relative magnitude of Top(a) to Bot(a) in relation to the value of *a*.
21. Discuss their graphs. Why do they have the shape they do? What does the graph say about the relative magnitude of Top(a) and Bot(a) for values of *a* between 0 and 1? Between 1 and 2? Between 2 and 3?
22. Write $f(x)$ = Top(x)/Bot(x). Ask what they think the graph of $y = f(x)$ will look like. **Scenario = 0, Display = 2**
23. Point out that these are the functions they were working with. That the value of the top magnitude was $|\sin(a)|$ for each value of *a* and that the value of the bottom magnitude was $|\cos(a)|$ for each value of *a*.
24. Ask students what the graph of $y = |\sin(x)|/|\cos(x)|$ might look like.

Phase 4: Doing the same thing, with graphs

Open the file *Rational Function #2.gcf*

25. Tell students that these are the graphs of two functions *n* and *d*. The left graph is $y = n(x)$ and the right graph is $y = d(x)$.

26. Ask students if they can use the way of thinking that they just developed to envision the graph of $y = n(x)/d(x)$. Discuss.
    Scenario = 0, Display = 1

27. Ask students again if they can use the way of thinking that they just developed to envision the graph of $y = n(x)/d(x)$. Discuss as needed.
28. Tell students to construct the graph of $y = n(x)/d(x)$. Discuss as needed.
    Scenario = 1, Display = 1

29. Tell students to construct the graph of $y = n(x)/d(x)$. Discuss as needed.
    Scenario = 2, Display = 1

30. Tell students to construct the graph of $y = n(x)/d(x)$. Discuss as needed. Note that the two functions approach 0 simultaneously at several values of x. Go into this issue as deeply as you dare.

APPENDIX G

JOURNALING PROMPTS

These will be the questions provided to the participants in conjunction with the journal that they will use to answer these questions before they teach and after each lesson on rational functions. This journal will be given back to the researcher after the final observation but before the post-interview.

Phase 1: Journal Prompts

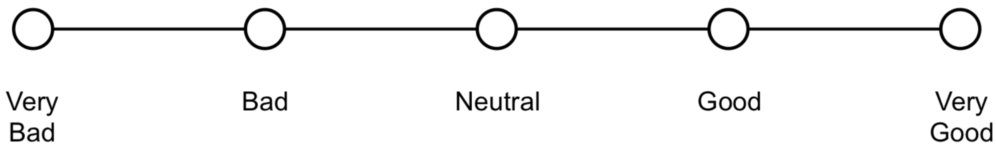
Please provide as much detail as possible when answering the questions. Remember I am trying to better understand your planning and teaching process the more information you can provide me the better.

Answer these questions before teaching rational functions, preferably when you are planning your lessons on rational functions:

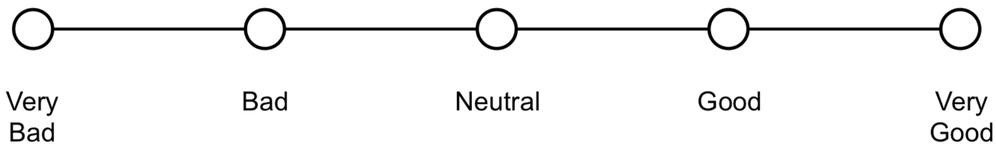
- How are you planning on teaching rational functions?
- Do you have a time line of how many days this lesson will take? If yes, please give detail about time line.
- What key ideas/understandings do you want students to walk away with?
- What activities or tasks will you use to assist students in building those understandings?
    - Please be specific and explain which task/activity goes with the key understandings you selected from above.
- Do you plan to use any resources outside of the Pathways Pre-Calculus curriculum during these lessons?
    - If yes, why?
- Did you have any difficulty in selecting the tasks/activities for these rational function lessons?
    - If yes, why?
    - If no, why not?
- Do you believe that your students will have previous experience with rational functions?
    - If yes, what type of experiences?
        - How will there previous experiences affect the rational function lessons you have planned?
    - If no, what leads you to believe that your students have not been exposed to rational functions until this point?

After each lesson on rational functions answer the following questions:

- Overall how do you feel the lesson went?



- How do you think your students did during the lesson?



- Was today’s lesson a good representation of an average day in class?
    - If yes, give specific details as to why this day was an average day in class?
    - If not, give details on what made this not an average day in class?
- What was the overall learning goal of this lesson?
    - Do you think your students demonstrated the understanding(s) that you were striving for in this lesson?
        - If yes, please give an example from class.
        - If no, please give an example form class.
- Were there any notable disruptions in the lesson?
    - Why do you believe these disruptions might have occurred?
    - Could you have done anything differently to avoid the disruption?
- Did you cover everything you wanted to cover in this lesson? *(This includes tasks, discussions, and activities)*
    - If yes, why?
    - If no, what tasks, discussions, and activities would you have liked to cover?
- After today’s lesson has your original timeline for the rational function lessons changed?
    - If yes, in what ways?

Phase 2: Journal Prompts

Please provide as much detail as possible when answering the questions. Remember I am trying to better understand your planning and teaching process the more information you can provide me the better.

Answer these questions before teaching rational functions, preferably when you are planning your lessons on rational functions:

- How are you planning on teaching rational functions with the applets?
- Do you have a time line of how many days this lesson will take? If yes, please give detail about time line.
- What key ideas/understandings do you want students to walk away with?
- What activities or tasks, including the applets, will you use to assist students in building those understandings?
    - Please be specific and explain which task/activity/applet goes with the key understandings you selected from above.
- Do you have any reservations about using the applets in your rational function instruction? Please give specific details.
- How do you think your students will have to adapt when you implement the applets?
- Has planning with the applets changed the way you usually plan your lessons?
    - Is the planning process more time consuming?
    - Was it easier to plan your lesson using the applets?
    - Are you experiencing any frustration or excitement in planning to teach with the applets?

After each lesson on rational functions answer the following questions:

- Overall how do you feel the lesson went?

Very Bad	Bad	Neutral	Good	Very Good
○	○	○	○	○

- How did you feel while teaching with the applets?

Very Bad	Bad	Neutral	Good	Very Good
○	○	○	○	○

- How do you think your students did during the lesson?

Very Bad	Bad	Neutral	Good	Very Good
○	○	○	○	○

- Was today's lesson a good representation of an average day in class?
    - If yes, give specific details as to why this day was an average day in class?
    - If not, give details on what made this not an average day in class?
- Were there any notable disruptions in the lesson?
    - Why do you believe these disruptions might have occurred?
    - Could you have done anything differently to avoid the disruption?
- What was the overall learning goal of this lesson?
    - Do you think your students demonstrated the understanding(s) that you were striving for in this lesson?
        - If yes, please give an example from class.
        - If no, please give an example from class.
- Did you cover everything you wanted to cover in this lesson? *(This includes tasks, discussions, activities, and applets)*
    - If yes, why?
    - If no, what tasks, discussions, and activities would you have liked to cover?
- After today's lesson has your original timeline for the rational function lessons changed?
    - If yes, in what ways?
    - If no, what makes you think that you are still aligned with your original timeline?