Examining the Development of Students' Covariational

Reasoning in the Context of Graphing

by

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ABSTRACT

Researchers have documented the importance of seeing a graph as an emergent trace of how two quantities' values vary simultaneously in order to reason about the graph in terms of quantitative relationships. If a student does not see a graph as a representation of how quantities change together then the student is limited to reasoning about perceptual features of the shape of the graph.

This dissertation reports results of an investigation into the ways of thinking that support and inhibit students from constructing and reasoning about graphs in terms of covarying quantities. I collected data by engaging three university precalculus students in asynchronous teaching experiments. I designed the instructional sequence to support students in making three constructions: first imagine representing quantities' magnitudes along the axes, then simultaneously represent these magnitudes with a correspondence point in the plane, and finally anticipate tracking the correspondence point to track how the two quantities' attributes change simultaneously.

Findings from this investigation provide insights into how students come to engage in covariational reasoning and re-present their imagery in their graphing actions. The data presented here suggests that it is nontrivial for students to coordinate their images of two varying quantities. This is significant because without a way to coordinate two quantities' variation the student is limited to engaging in static shape thinking.

I describe three types of imagery: a correspondence point, Tinker Bell and her pixie dust, and an actor taking baby steps, that supported students in developing ways to coordinate quantities' variation. I discuss the figurative aspects of the students' coordination in order to account for the difficulties students had (1) constructing a

multiplicative object that persisted under variation, (2) reconstructing their acts of covariation in other graphing tasks, and (3) generalizing these acts of covariation to reason about formulas in terms of covarying quantities.

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CHAPTER 1
PROBLEM STATEMENT

Conceptualizing formulas and graphs is an important part of secondary and undergraduate mathematics. Nearly every mathematics problem in the secondary and undergraduate curricula asks students to create or interpret one of these mathematical objects. Yet the research community continues to document the difficulties students experience constructing, interpreting, and reasoning about formulas and graphs (e.g., Carlson, 1998; Monk, 1992; Moore & Carlson, 2012; Moore & Thompson, 2015). This suggests that the meanings mathematics education researchers hold for these mathematical objects are not aligned with the meanings students construct in the classroom.

As Leinhardt, Zaslavsky, and Stein (1990) described, individuals construct graphs and formulas in order to organize mathematical ideas that simultaneously focus on issues of relation as well as entity (p. 3). I interpret this to mean that, for the expert, graphs and formulas represent relationships between pairs of numbers as well as relationships between these pairs. For example one might understand a formula, such as $y = 45x + 12$, as a description of how a value of x is related to a value of y as well as how values of x and values of y change together. In terms of graphing, this involves understanding a graph both as a collection of points but also, as Bell and Janvier (1981) explained, as a representation that "exposes features of the situation not immediately obvious from the numerical data" (p. 34).

Many students do not have an opportunity to construct these meanings. School mathematics focuses on supporting students in constructing static relationships between two things, call them *x* and *y*. For example, students construct a graph only to read off a single point or they construct formulas to repeatedly plug in numbers. As a result, students often conceptualize variables as placeholders for specific values and do not imagine variables varying (White & Mitchelmore, 1996). As Thompson and Carlson (2017) explained, if the student views variables statically, and thus does not conceptualize variables varying, then the student cannot imagine expressions as representations of relationships among varying quantities. As a result, the student focuses on symbolic manipulations as opposed to anything those manipulations might represent. In the context of graphing, the student focuses on attributes of the graph such as intersection points, locations of peaks and valleys of the curve, and the slantiness of the curve. With this conception of graphs, the student will not be equipped to reason about the meaning of these features in terms of relevant quantities in the situation (e.g., Bell & Janvier, 1981; Carlson, 1998; McDermott, Rosenquist, & van Zee, 1987).

This focus on static relationships is especially problematic for two reasons. First, focusing on static relationships creates a disconnect between students' daily experiences – where they imagine and experience objects and people moving continuously, and school mathematics – where they are asked to think about point-wise associations between values. Second, there is a growing body of research that documents the importance of imagining quantities' values varying when conceptualizing rates (Johnson, 2015; Thompson, 1994a; Thompson & Thompson, 1992), functions (Castillo-Garsow, 2010; Moore, 2010; Thompson, 1994c), and graphs (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, Paoletti, Stevens, & Hobson, 2016). After high school, reasoning about variation is essential to understand derivatives (Zandieh, 2000), accumulation functions (Thompson & Silverman, 2008), the Fundamental Theorem of Calculus (Thompson, 1994b), differential equations (Rasmussen, 2001), and continuous functions (Roh & Lee, 2011).

To understand the limitations of thinking about static relationships consider a student entering calculus, a course focused on studying how variables change together. If the student does not conceptualize variables varying, then there is nothing for the student to imagine changing. As a result, the student must memorize procedures for taking the derivative, interpret meanings for derivative in terms of pictorial attributes of a graph (i.e., slope – or for the student, the slantiness of the tangent line), and think about integration as a static area under a curve. In order for students to understand many mathematical ideas and to relate their daily experiences with school mathematics, it is essential that students have the opportunity to engage in variational and covariational reasoning. Additionally, educators must provide opportunities for students to conceptualize formulas and graphs as representations of how varying values of two quantities change together.

Simply stated, covariational reasoning entails thinking about how two quantities' values change together. However, there is no single understanding of what ways of thinking constitute covariational reasoning. Confrey (1988) and Confrey and Smith (1995) described a notion of covariation where students coordinate a completed change in the value of x with a completed change in the value of y. Thompson and Thompson (1992) and Thompson (1994a) described a notion of covariation where students track two quantities' varying values simultaneously. Finally, Carlson et al. (2002) described a developmental notion of covariation where students begin by coordinating directional changes in the values of two quantities and eventually coordinate continuous change in one quantity with the instantaneous rate of change of another quantity.

While there are applications in which each of these notions of covariation are productive for analyzing students' covariational reasoning, Castillo-Garsow (2010, 2012) and Castillo-Garsow, Johnson, and Moore (2013) convincingly argued that a student must conceptualize smooth variation to understand exponential functions, rates, and trigonometric functions. Since Thompson's notion of covariation is based on images of smooth variation, Castillo-Garsow and his colleague's work suggests that the research community needs to better understand the ways of thinking involved in Thompson's conception of covariational reasoning.

Saldanha and Thompson (1998) elaborated Thompson's notion of covariation. They explained that their notion of covariation entails a student coupling two quantities' values in such a way that as the student imagines time varying continuously she imagines both quantities' values varying together. They go on to explain that with this image of covariation, the student imagines *how* quantities' values vary within an interval as opposed to coordinating completed changes in quantities' values.

Thompson (2011b) explained that in order for students to engage in this type of covariational reasoning the student must (1) construct two quantities, (2) imagine the measures of these quantities varying smoothly, and (3) unite the measures of two quantities by constructing a multiplicative object that simultaneously represents the two measures. Ideally, students will develop these ways of thinking throughout their grade school education. While this is a worthwhile goal for the future of school mathematics, it will not solve our immediate problem; students who have spent years focusing on static relationships between letters must learn to reason covariationally in order to develop meaningful conceptions of formulas, graphs, functions, rate of change, derivative, etc. It is imperative that we as a research community understand how to support individuals accustomed to focusing on static relationships in engaging in covariational reasoning.

Many studies that investigate how students engage in covariational reasoning are situated in the context of a student's graphing activity (e.g., Carlson et al., 2002; Castillo-Garsow, 2010; Moore et al., 2016; Saldanha & Thompson, 1998; Whitmire, 2014). This suggests an underlying theoretical hypothesis: by studying how students come to construct graphical representations one can understand the nuances of the mental actions involved in covariational reasoning. However, the research community does not yet understand how students operationalize the ways of thinking they construct in their graphing activity when reasoning about formulas and tables. In this dissertation I addressed this hypothesis by studying what aspects of the constructions students make in their graphing activity are operative and independent of the representation system.

More specifically, this dissertation was designed to address the following research questions:

1. What ways of thinking do students engage in when conceptualizing and representing how two quantities change together? How do students construct these ways of thinking? And, what ways of thinking support/inhibit students from reasoning about how two quantities change together?
2. How do students operationalize their scheme for covariational reasoning across problem types including graphs and formulas?

CHAPTER 2
LITERATURE REVIEW

Covariational reasoning, reasoning about how two quantities' values change together, is an essential part of mathematical thinking. This way of thinking can be traced back to the 1700s when Euler first conceptualized functions as covariational relationships. Although reasoning about how quantities' values vary together is no longer part of the contemporary conception of function, researchers have identified the importance of covariational reasoning in conceptualizing various mathematical ideas such as rates (Johnson, 2015; Thompson, 1994a; Thompson & Thompson, 1992), the behavior of exponential and trigonometric functions (Castillo-Garsow, 2010; Moore, 2010; Thompson, 1994c), graphical representations of relationships between quantities' values (Carlson et al., 2002; Moore & Thompson, 2015), derivatives (Zandieh, 2000), accumulation functions (Thompson & Silverman, 2008), the Fundamental Theorem of Calculus (Thompson, 1994b), differential equations (Rasmussen, 2001), and continuity (Roh & Lee, 2011).

Through my review of the literature I identified three different conceptions of covariational reasoning. Confrey and Smith (1994, 1995) described a conception of covariational reasoning based in coordinating successive values of two number sequences. Thompson (Saldanha & Thompson, 1998; Thompson, 1994b, 2011b; Thompson & Thompson, 1992) described a conception of covariational reasoning based in his theory of quantitative reasoning. He focused on supporting students in conceptualizing smooth variation of a quantity's value and constructing a multiplicative object that unites two varying quantities' values. Finally, Carlson et al. (2002) described a developmental conception of covariational reasoning where students begin by coordinating the value of one quantity with changes in the other and eventually coordinate the instantaneous rate of change of the function with continuous changes in the input variable.

In the following sections I provide more detail on these three conceptions of covariational reasoning and the mental activities associated with each conception. I will examine the similarities and differences among these three conceptions of covariational reasoning and will use this analysis to explicate the conception of covariational reasoning upon which I based this study.

The Role of Covariational Reasoning in the Evolution of Mathematics

The late 15th and early 16th centuries were a critical time in the development of mathematics. There were three significant advancements: (1) mathematicians such as Bombelli and Stifel expanded the concept of number to embrace all real numbers, (2) mathematicians and scientists such as Kepler and Galileo studied motion as a central problem of science, and (3) mathematicians such as Viète and Descartes developed a symbolic algebra. As a result of these advancements, mathematicians began to shift their thinking about relationships from static and discrete relationships between numbers to dynamic and continuous relationships between quantities (Kleiner, 1989). This change in thinking marks the beginning of covariational reasoning, reasoning about how quantities' values change together, as a critical way of thinking in mathematics.

Scientists and mathematicians spent much of the 17th and 18th centuries thinking about relationships between variables and expressing these relationships through equations and curves. These relationships became known as functions. Euler provided one of the earliest conceptions of function in 1755 when he explained,

> If however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others (quoted in Kleiner, 1989, p. 288).

Thinking about dynamic and continuous relationships between variables remained a prominent way of thinking about mathematics until the mid 20th century. In the 1930s Nicolas Bourbaki, a collective pseudonym for a group of mathematicians, published a series of books formalizing an abstract and self-contained mathematics. As a result of these publications the mathematics community became focused on developing and teaching a formal and rigorous mathematics. Since mathematicians considered reasoning about how variables changed together to be an intuitive understanding of function, and as Poincare described, "intuition can not give us rigor, nor even certainty", conceptualizing functions as relationships between variables was no longer sufficient in the mathematics community (Poincare, 1969, p. 207). As a result, mathematicians adopted a new meaning for function based on Dirichlet's conception of function: "y is a function of x, for a given domain of values of x, whenever a precise law of correspondence between x and y can be stated clearly" (quoted in Boyer, 1946, p. 13).

The mathematics community's adoption of Dirichlet's definition of function was significant because, as Kleiner (1989) explained, Dirichlet described the concept of function as an *arbitrary* correspondence. With Dirichlet's definition of function, mathematicians could no longer think about functions as analytic expressions or curves let alone relationships between continuously changing variables (Kleiner, 1989, p. 292).

Three Meanings for Covariational Reasoning

After Dirichlet formalized the concept of function, covariational reasoning, an intuitive way of thinking, disappeared from the mathematics and math education community and remained absent for nearly fifty years. Notions of covariation reemerged in the 1980s when researchers began focusing on the importance of covariational reasoning in constructing meaningful conceptions of graphs (Bell & Janvier, 1981; McDermott et al., 1987), exponential functions (Confrey, 1988; Rizzuti, 1991), and rate of change (Thompson, 1990).

The research community continues to highlight the importance of covariational reasoning in various mathematical disciplines. However, there is no single understanding of what ways of thinking constitute covariational reasoning. In the following section I elaborate three meanings of covariational reasoning based on Confrey & Smith, Thompson, and Carlson's work.

Confrey & Smith's Conception of Covariational Reasoning

Confrey (1988) and Confrey & Smith (1994, 1995) described covariational reasoning as a process of coordinating successive values of two variables. They focused on tabular representations to support students in coordinating the change in the value of y from y_m to y_{m+1} with the change in the value of x from x_m to x_{m+1}. This conception of covariational reasoning entails a student identifying patterns of change in the value of x, patterns of change in the value of y, and then coordinating these patterns of change to answer questions about a situation. When engaged in this activity, students focus on the repeated action of conceptualizing the change in the value of *x* and change in the value of *y*. Confrey and Smith (1995) explained that this repeated action is the basis of generating operations that enable the student to define the relation between values of *x* and values of *y* (p. 79).

For example, consider the relationship represented in Figure 1. A student might construct the pattern that in each subsequent row, the value of *x* always increases by 3 and the value of *y* increases by 8. She can coordinate these changes and conceptualize that if the value of *x* increases by 1.5, ½ of 3, then the value of *x* must increase by 4, ½ of 8. Notice that this student is not thinking about *how* the value of *y* changes as the value of *x* increases from 1 to 2.5. Instead, she determined a new pair of values that satisfies the relationship between *x* and *y* by generalizing the relationship she constructed between the change in the value of *x* and the change in the value of *y*.

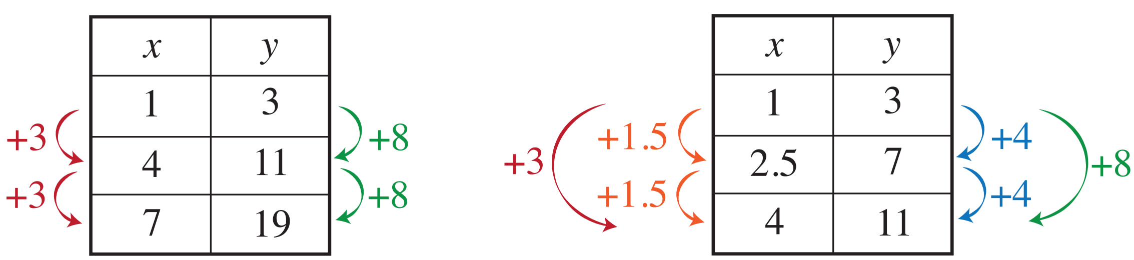

Figure 1: Constructing patterns in the value of *x* and the value of *y*.

Confrey & Smith (1995) explain that constructing a pattern of change in the value of *x* gives students the opportunity to construct the variable, *x*. They explain, “Systematically selecting data values, ordering one’s data, and examining them for patterns are all a part of this process [of constructing a variable]" (p. 78). This suggests that for Confrey & Smith, variables are lists of possible values a quantity can assume.

Confrey & Smith argued that their conception of covariational reasoning supports students in understanding *a covariation approach to function* where the student conceptualizes a function as "a juxtaposition of two sequences, each of which is generated independently through a pattern of data values" (Confrey & Smith, 1995, p. 67). In the *covariation approach to function*, students focus on patterns between changes in two quantities' values. Confrey & Smith reported that students found the *covariation approach to function* more intuitive than the arbitrary correspondence meaning for function where students focus on functions as algebraic rules and directional mappings from x to $f(x)$ (p. 79).

Confrey and Smith's conception of covariational reasoning and the *covariation approach to function* emerged from their research on how students conceptualize and reason about exponential relationships (e.g., Confrey & Smith, 1994, 1995). In this study the researchers supported students in constructing two number worlds: one based on an additive conception of counting and the other on a multiplicative conception of splitting. Confrey and Smith proposed that once a student constructs these number worlds, she could conceptualize exponential functions by coordinating additive changes in the value of x with multiplicative changes in the value of y.

As with all research endeavors, one's research questions and methodology gives the researcher a lens to understand his data. But this lens can also blind the researcher as one often sees only what she is looking for. Confrey and Smith's (1995) empirical background focused them on ways of thinking about exponential functions and other functional relationships that related two structured worlds (p. 79). However, this way of thinking does not extend to reasoning about relationships that are governed by an unknown pattern of change. More specifically, this conception of covariational reasoning would not support students in reasoning about relationships between quantities whose values cannot be represented by polynomial or exponential relationships.

Thompson's Conception of Covariational Reasoning

Thompson's conception of covariational reasoning is closely related to his conception of quantitative reasoning. When a student engages in quantitative reasoning she conceptualizes a situation by "reason[ing] about quantities, their magnitudes, and their relationships to other quantities" (Thompson, 1988, p. 164). As Thompson (2011b) explained, a learner constructs a quantitative relationship among quantities in a static situation. When the student introduces an image of variation to the situation and imagines two quantities' values varying together so that the quantitative relationship remains unchanged, she is engaging in covariational reasoning. In this section I will describe Thompson's theory of quantitative reasoning and elaborate how this extends to a conception of covariational reasoning.

Thompson (1988, 1990, 1994a, 2008, 2011b) outlined a theory of quantitative reasoning and proposed meanings for quantity, quantitative operation, and quantitative structure that provide a foundation for thinking about algebraic reasoning and covariational reasoning. Thompson (1990, 2011b) explained that a quantity is a mental construction of a quality of an object that one can imagine measuring. Students construct quantities by conceptualizing an attribute to be measured and the way in which they would measure it. Thompson emphasized that quantities exist in the mind. A teacher might reference an area or volume but this is only a quantity for the student if he conceptualizes area/volume as the result of measuring some attribute of an object. For example, one can construct Bob's height as an attribute of Bob, an object, to measure. One can conceive of measuring Bob's height by determining the relative size of the distance between the floor and the top of Bob's head with a piece of string that is one foot long. While the student must imagine a way to measure Bob's height, the student does not have to physically engage in this activity in order to conceptualize Bob's height as a quantity.

Thompson explained that a second aspect of quantitative reasoning is constructing relationships between quantities through quantitative operations. Thompson (1988) described four quantitative operations: combining quantities additively, comparing quantities additively, combining quantities multiplicatively, and comparing quantities multiplicatively (see Figure 2).

Operation	*Example*
Combine quantities additively	*Unite two sets; consider two regions as one.*
Compare quantities additively	*"How much more of this is there than that"*
Combine quantities multiplicatively	*Combine distance and force to get torque; combine linear dimensions to get regions*
Compare quantities multiplicatively	*"How many times as large is this than that"*

Figure 2: Four quantitative operations (Thompson, 1988, p. 164).

Quantitative operations are different than the numerical operations of addition, subtraction, multiplication, etc. As Thompson (1994a) explained, quantitative operations are non-numerical and are used to create quantities whereas numerical operations are used to evaluate quantities (p. 13). It is important that a student be able to differentiate between a quantitative operation and the corresponding numerical operation. For example, suppose a student wants to determine how much taller Bob is than Sue. He must first construct Bob's height and Sue's height as quantities and then differentiate the necessary quantitative operation (additive comparison) from the numerical operation (subtraction). As Thompson (1993) described, differentiating between quantitative and numerical operations is often challenging for students since there is no way to represent quantitative operations; mathematicians use arithmetic to represent both quantitative and numerical operations.

The last aspect of quantitative reasoning is constructing a quantitative relationship. Once a student has constructed quantities and conceptualized a quantitative operation between these quantities, he can construct a quantitative relationship between the quantities operated on and the resulting quantity (Thompson, 1990, p. 13). In the example above, when a student additively compares the quantities Bob's height and Sue's height he has constructed the quantity the difference between Bob's height and Sue's height. Conceptualized in this way, these three quantities in relation to one another form a quantitative relationship. There is a subtle distinction between conceptualizing quantitative operations and quantitative relationships. If the student focuses on the activity of operating on two quantities then he is conceptualizing a quantitative operation. If he focuses on the result of operating as well as the relationship between the result and the operands then the student is conceptualizing a quantitative relationship.

Thompson (2011b) explained that one constructs a quantitative structure while conceptualizing a static situation. For example, to construct the quantity the difference between Bob's height and Sue' height one must imagine Bob's height and Sue's height at a given moment in time and then additively compare these quantities' values. To construct this quantitative relationship, a student must have a static conception of the situation. Thompson goes on to explain, "imagining those relationships to remain the same under changed circumstances is tightly related to covariational reasoning" (p. 46). Returning to our example, this means that a student begins to engage in covariational reasoning by introducing an image of variation to the situation and imagining how the difference between Bob's height and Sue's height varies as time elapses. As the student imagines time varying the quantitative relationship between Bob's height, Sue's height, and the difference between Bob's height and Sue's height is invariant; the quantitative relationship between the three quantities does not change as the student imagines time elapsing. What changes is the result of the numerical operation that assigns a value to the quantity the difference between Bob's height and Sue's height. Note that a student must differentiate the numerical operation of subtraction from the quantitative operation of additive comparison before he can conceptualize a situation in such a way that the quantitative relationship remains invariant while the numerical relationship varies.

Variational reasoning. In order for a student to imagine a quantitative relationship to remain constant while a numerical relationship varies, she must imagine an attribute's value varying as some object in the situation moves/changes. Thus, as Thompson (2011b) described, a student's construction of an invariant relationship is closely tied with her construction of varying quantities and variables (p. 46). Thompson has attended to students' conceptualizations of varying quantities since the early 1990s. For example, Thompson (e.g., Thompson, 1994a; Thompson & Thompson, 1992) explored the difference between a student's conceptualization of a ratio and a rate. He explained that a student conceptualizes a ratio when she constructs a multiplicative comparison of two non-varying quantities. On the other hand, a student conceptualizes a rate when she attends to both quantities accruing simultaneously and continuously so that the total accumulation of the accruals remain in constant ratio. Thus, students construct ratios from static and discrete conceptualizations of situations and students construct rates from dynamic and continuous conceptualizations of situations.

For example, consider a car traveling at a constant speed of 70 miles per hour. A student who conceptualizes speed as a ratio might understand the car traveled 70 miles every hour – a static comparison of 70 miles and 1 hour. This student is not imagining the number of hours elapsed varying continuously from 0 to 1 hour. Instead, she focuses on what happened after traveling for an hour – the car traveled 70 miles. If the student conceptualizes speed as a rate she might conceptualize how the distance traveled varies as the number of hours elapsed varies continuously from 0 to 1 hour. She might understand the number of miles the car traveled is always 70 times as large as the number of hours elapsed since the car started traveling at 70 miles per hour. A key difference between these conceptualizations is how the student imagines the quantities, distance traveled and time elapsed, varying.

Saldanha and Thompson (1998) coordinated conceptions of variation and covariation and explained,

> In our theory, images of covariation are developmental. In early development one coordinates two quantities' values – think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values

> persist. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values (Saldanha & Thompson, 1998, p. 2).

This description of covariation highlights the importance of imagining a quantity's value varying continuously when engaging in covariational reasoning. Additionally, Thompson and Saldanha described a developmental conception of covariational reasoning where students go from thinking about one quantity then the other to imagining the quantities' values varying together continuously. Unlike Confrey and Smith (1994, 1995), Saldanha and Thompson (1998) do not believe that coordinating completed changes in the quantities' values is part of engaging in covariational reasoning and constructing variables. Instead, Saldanha and Thompson suggested that a student constructs a variable in such a way that he anticipates the quantity's value *always* varying.

Multiplicative objects. Constructing invariant quantitative relationships and conceptualizing continuous variation are only part of Thompson's conception of covariational reasoning. These mental actions enable a student to imagine two quantities varying together so that they satisfy some invariant relationship. However, in order to reason about and re-present one's conception of how the quantities change together the student must coordinate two images of variation and construct what Thompson calls a multiplicative object. Thompson introduced this idea in 1998 when he and Saldanha explained their meaning for covariation:

> Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value (Saldanha & Thompson, 1998, pp. 1-2).

For Saldanha and Thompson, multiplicative objects do not necessarily involve the numerical operation of multiplication or the quantitative operation of multiplicative comparison. Instead, Thompson and Saldanha extend the work of Inhelder and Piaget (1964) and conceptualize multiplicative objects as mental constructions an individual makes when uniting two or more quantities simultaneously (Thompson, 2011b, p. 47).

According to Inhelder and Piaget (1964), multiplicative relationships are schemas an individual constructs that can be described by the word "simultaneous" (p. 182). Thus, multiplicative relationships are more closely related to the logical conjunction ($A \wedge B$) than the numerical operation of multiplication. An individual constructs a multiplicative object through multiple classification; the student constructs the object from its attributes. As Thompson described,

> A person creates a multiplicative object when he takes two attributes of already-conceptualized quantities as one property of a newly conceptualized object. For example, when a person takes measures *x* and *y* of two quantities and

> conceptualizes the *pair* (x,y)—meaning that the pair of measures constitutes one thing (e.g. a relationship between x and y), the person has conceptualized a multiplicative object (personal communication September 29, 2015).

In the following paragraphs I will elaborate Thompson's conception of multiplicative objects.

Typically, individuals conceptualize objects and then identify attributes of the object. For example, suppose one conceptualizes the following object (see Figure 3). After conceptualizing the object the individual might abstract the properties red and circle from this object.

Figure 3: An object that is both red and circular.

On the other hand, suppose an individual constructs an object that is both red and circular. Here, the individual must construct an object out of its properties. One would need to construct a single object that is a red circle (Figure 4). In this example, the object the individual constructs simultaneously has both the attribute red and the attribute circle. The simultaneity of these attributes is what makes the construction multiplicative.

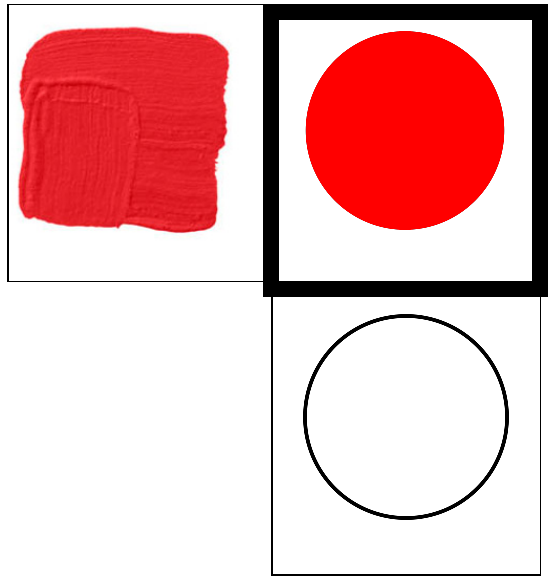

Figure 4: Construct an object that has both the attributes red and circle.

When an individual conceptualizes two attributes simultaneously he has the opportunity to construct a relationship between the two attributes. This relationship is a third attribute of the multiplicative object. Returning to the red circle example, consider one is classifying the objects in Figure 5. The individual could identify all of the objects that are red, all of the objects that are circles, and all of the objects that are red circles. Thus the individual can construct a red circle as a single attribute based on the relation red *AND* circle. Conceptualizing this third attribute, the relation red *AND* circle, is essential when constructing the multiplicative object.

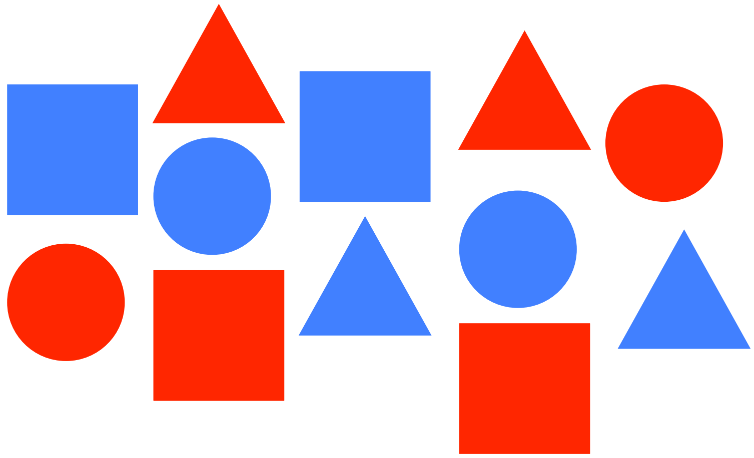

Figure 5: A collection of blue and red shapes.

Thompson (2011b) provides the following examples of multiplicative objects:

- A student can construct a rectangle's area as a multiplicative object that unites the rectangle's length and width.
- A student can construct a point in the Cartesian plane as a multiplicative object that unites the distance of the point from the horizontal axis with the distance of the point from the vertical axis (p. 47).

As Saldanha and Thompson (1998) explained, when a student constructs a multiplicative object in the context of covariational reasoning he is organizing his thoughts about how two quantities' values vary together so that whenever he imagines variation of one quantity he necessarily imagines variation in the other. For example, if the student has constructed the point (x, y) in the Cartesian coordinate system as a multiplicative object then as he imagines the value of x varying continuously he understands that the value of y necessarily varies as well. With this conception, the student can conceptualize graphs as an emergent representation of how quantities' values change together.

Not all conceptions of a point in the Cartesian plane are multiplicative objects. For example, if a student conceptualizes the point (3, 5) as a command to act by going over 3 units and up 5 units, then this student has not constructed a multiplicative object. Instead, he is enacting a procedure to determine an appropriate location to place the point (3, 5). For the student's conception of the point to be a multiplicative object the student must first conceptualize the attributes of x and y independently. One way to do this is to imagine their values represented along the horizontal and vertical axes, respectively. Then the student can imagine extending these measures into the plane as if there were dotted lines extending from the ends of each measure on the axes. Next the student can construct a point (x, y) at the intersection of these extended measures in the plane. By focusing on the intersection of the two extended measures the student can conceptualize the point as conveying two attributes simultaneously. With this understanding the researcher can say the student has constructed an object (the correspondence point) from its properties (two quantities' measures) and thus the student has constructed a multiplicative object.

An individual can also conceptualize functions as multiplicative objects by uniting the measures of two quantities and attending to the relationship created in one's mind by simultaneously attending to two varying quantities. It is likely that students come to construct functions as multiplicative objects by operationalizing their conception of a point as a multiplicative object so that it is no longer dependent on a graphical context. Then, as the individual imagines the value of x varying, the student has the opportunity to construct a function as a multiplicative object.

Just as not all conceptions of points in the Cartesian plane are multiplicative objects, not all conceptions of functions are multiplicative objects. For example, according to the APOS literature, many students imagine functions as activities that transform values of x into values of y. As Breidenbach, Dubinsky, Hawks, and Nichols (1992) described, a student might believe "a function is an equation in which a variable is manipulated so that an answer is calculated using numbers in place of that variable" (p. 252). It would cause cognitive conflict for this student to think of the value of x and y simultaneously since he imagines the value of x causes the value of y. This is true even if the student has a process conception of function and "can think of a function in terms of accepting inputs, manipulating them in some way, and producing outputs without the need to make explicit calculations" (Arnon et al., 2014, p. 30). This student still thinks of the value of x causing the value of y. Thus, a point-wise conception of function does not support students in conceptualizing functions (or points) as multiplicative objects.

Thompson (2011b) summarized his conception of covariational reasoning and the relationship between the mental acts of conceptualizing continuous variation of a quantity's value, constructing multiplicative objects, and conceptualizing invariant relationships. He said:

> In summary, there are two considerations in examining students' construction of quantitative covariation. The first is conceiving the quantities themselves and images of them that entail their values varying. The second is to conceptualize the multiplicative object made by uniting those quantities in thought and maintaining that unit while also maintaining a dynamic image of the situation in which it is embedded. This act, of uniting two quantities conceptually within an image of a

situation that changes while staying the same, is nontrivial. Yet it is at the heart of using mathematics to model dynamic situations (Thompson, 2011b, p. 48).

Carlson's Conception of Covariational Reasoning

Carlson et al. (2002) defined *covariational reasoning* "to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Carlson (1998) identified behaviors undergraduate students engaged in when they interpreted and represented dynamic situations. Carlson et al. (2002) used these behaviors to develop a Covariation Framework that consists of two parts: five mental actions of covariational reasoning (see Table 1) and five corresponding levels of covariational reasoning. For Carlson et al., these mental actions are developmental ranging from coordinating values of x with values of y to coordinating the instantaneous rate of change of a function with continuous changes in the input of the function. Like Confrey and Smith (1995), the mental actions Carlson et al. (2002) described involve coordinating changes in the values of the inputs and output. Students progress to more sophisticated conceptions of covariational reasoning by imagining a smaller and smaller interval of the input until the student conceptualizes the instantaneous rate of change of the function.

These five mental actions provide a framework for researchers to classify students' behaviors as they engage in tasks and covariational reasoning. Researchers can then use the mental actions a student exhibits to classify the student's level of covariational reasoning. Carlson et al. explained that a student is exhibiting a given level of covariational reasoning when the student's thinking consists of the mental actions associated with that level *and* the actions associated with all lower developmental levels.

For example, for a researcher to classify a student's thinking as level 4 covariational reasoning then the researcher should have evidence that the student can consistently engage in mental actions 1, 2, 3 and 4.

Table 1

Mental Actions of Carlson et al.'s Covariation Framework (2002, p. 357)

Mental action	**Description of mental action**	**Behaviors**
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	Labeling the axes with verbal indications of coordinating the two variables (e.g., *y* changes with changes in *x*)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Constructing an increasing straight line Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/constructing secant lines Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable	Constructing contiguous secant lines for the domain Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Carlson et al. described the Covariation Framework as an analytical tool that gives researchers a nuanced way to evaluate students' covariational reasoning as well as a common language for classifying students' thinking in the context of a specific problem. Since the authors conceptualized their framework as an analytical tool, as opposed to a theory of learning, they do not describe a process of reasoning by which a student who engages in mental action 3 might come to engage in mental action 4.

Synthesis: Choosing a Framework for the Teaching Experiment

Carlson et al. (2002) acknowledged the role that Thompson and Confrey & Smith's work played in their thinking about covariation. Thus, it is not surprising that there are similarities between the mental actions Carlson et al. described in their Covariation Framework and the mental actions that Confrey and Smith and Thompson described. For example, Carlson et al. (2002) defined Mental Action 3 as "coordinating the amount of change of one variable with changes in the other variable" (p. 357). This mental action is analogous to Confrey & Smith's (1994, 1995) conception of covariational reasoning as coordinating successive values of two variables. Carlson et al. (2002) defined Mental Action 5 as "Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function" (p. 357). This mental action aligns with Saldanha and Thompson's (1998) thinking that covariational reasoning entails conceptualizing continuous covariation.

It seems that Carlson et al.'s (2002) Covariation Framework combines Confrey and Smith's conception of covariational reasoning with Thompson's conception of covariational reasoning. However, as Castillo-Garsow (2010) explained these two conceptions are not compatible.

Castillo-Garsow (2010, 2012) and Castillo-Garsow et al. (2013) described that there are two ways for a student to imagine a quantity's value changing. In the first image the student imagines the change has already happened so he coordinates two completed changes. This is what Castillo-Garsow calls chunky reasoning. Castillo-Garsow et al. (2013) explained that when a student engages in chunky reasoning he imagines that "nothing of importance happens within the chunk because the entire-chunk is imagined all at once" (p. 11). Thus, when a student engages in chunky thinking he focuses on discrete values at the end of an interval(s). The student does not attend to the values the quantity assumes in between these points. Castillo-Garsow et al. (2013) described the space between these discrete points as holes. Note that no matter how small the chunk size, from the researcher's perspective, there will always be a hole in the student's images of the quantity's variation.

The student could also imagine the quantity varying by imagining change in progress and keeping track of the two quantities' magnitudes as he attends to them in his experiential time. This student would imagine the magnitude of each quantity passing through all possible measures between the initial and final value. This way of thinking is what Castillo-Garsow calls smooth thinking. When a student engages in smooth thinking he imagines change in progress. This means that a student engaging in smooth thinking imagines a beginning point but no endpoint. As soon as the student conceptualizes an endpoint the student is no longer imagining change in progress. This can be problematic if the student wants to determine a numerical value. As soon as a student engaging in smooth thinking slows down to compute a value of y, the value of x has changed.

As Castillo-Garsow (2012) explained, the way a student conceptualizes how a quantity's value varies is not dependent on the problem situation. For example, a student can use smooth thinking to construct a graph of a non-continuous function (e.g., step function). So long as the student is imagining change in progress he is engaging in smooth thinking.

Castillo-Garsow's conception of smooth and chunky thinking provides a way to think about Confrey and Smith, Thompson, and Carlson et al.'s conception of covariational reasoning. Confrey and Smith's conception of covariation is based in chunky thinking and Thompson's conception of covariation is based in smooth thinking. It is a little more difficult to classify Carlson et al.'s conception of covariational reasoning. While Carlson et al. recognized the importance of conceptualizing continuous variation they proposed that students come to conceptualize continuous variation by imagining smaller and smaller changes in the value of x. However, Castillo-Garsow et al. (2013) convincingly argued that no matter how small one imagines the chunk, chunky thinking can never become the basis for smooth thinking. This implies that Carlson et al.'s (2002) conception of covariation is based in chunky thinking and is more closely related to Confrey and Smith's conception of covariational reasoning than to Thompson's conception of covariational reasoning.

While Confrey and Smith and Carlson et al. both described conceptions of covariational reasoning based in chunky thinking, there are two main differences between Confrey and Smith's conception of covariational reasoning and Carlson et al.'s conception of covariational reasoning. First, Carlson et al.'s Covariation Framework provides a lens researchers can use to analyze how students conceptualize various function types, including but not limited to polynomial and exponential functions. Additionally, Carlson et al.'s framework suggests a developmental trajectory for engaging in more sophisticated forms of covariational reasoning.

Before I design a study to understand how students develop schemes to reason about relationships between covarying quantities, I must clearly articulate what I mean by covariational reasoning. Castillo Garsow (2010, 2012) and Castillo-Garsow et al. (2013) convincingly argued that a student must conceptualize smooth variation to understand exponential functions, rates, and trigonometric functions. Since Thompson's conception of covariation is the only one that is based on smooth thinking from the earliest developmental conceptions of covariational reasoning, I will base my study on his meaning for covariational reasoning. In particular, I will focus on the three mental actions that Thompson explains constitute covariational reasoning: To engage in covariational reasoning the student must (1) construct invariant relationships, (2) conceptualize quantities' values varying continuously, and (3) construct a multiplicative object to unite two continuously varying quantities.

Extending and Elaborating Castillo-Garsow's Conception of Variation

Castillo-Garsow (2010) introduced the constructs of smooth and chunky thinking. However, he was not the first researcher to discuss the importance of conceptualizing a quantity's value varying. Thompson and Carlson (2017) synthesized prior research on variation (e.g., Carlson et al., 2002; Castillo-Garsow, 2010; Castillo-Garsow et al., 2013; Saldanha & Thompson, 1998; Thompson, 1994a, 2011b; Thompson & Thompson, 1992) and proposed six meanings of variation: a variable is a letter, no variation, discrete variation, gross variation, chunky continuous variation, and smooth continuous variation. Thompson and Carlson organized their meanings for variation from most productive for engaging in covariational reasoning (top) to least productive for engaging in covariational reasoning (bottom) (see Table 2).

These meanings for variation are not developmental. It might seem promising that a student could construct a smooth continuous image of variation by refining their chunky continuous image of variation and imagining smaller and smaller chunks. However, Castillo-Garsow (2012) and Castillo-Garsow et al. (2013) argue that this does not work. They claim that thinking in chunks is chunky thinking no matter what chunk size the student imagines. They argue that thinking about smooth continuous variation necessarily involves imagining something moving. Thus, I do not see these five meanings as a developmental trajectory.

Table 2

Thompson and Carlson's Major Levels of Variational Reasoning, highest to lowest (Thompson & Carlson, 2017, p. 34)

Major Levels of Variational Reasoning	
Level	*Description*
Smooth Continuous Variation	The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-sized intervals of variation, but not necessarily.
Chunky Continuous Variation	The person thinks of variation of a variable's value as changing by intervals of a fixed size. The intervals might be same sized, but not necessarily. The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, etc. "come along" by virtue of each being part of a chunk – like numbers on a ruler, but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, etc. as values. Chunky continuous variation is *not* just thinking that changes happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that the entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on.
Gross Variation	The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing.
Discrete Variation	The person envisions a variable as taking specific values. The person sees the variable's value changing from a to b by taking values a_1, a_2, …, a_n, but does not envision the variable taking any value between a_i and a_{i+1}
No Variation	The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.
Variable as Letter	A variable is a symbol. It has nothing to do with variation.

To elaborate the differences between these six meanings for variation consider a car traveling along a highway and the varying quantity *the number of miles the car has traveled*. Suppose d represents the varying values this quantity assumes. Imagine explaining this situation to a kindergarten student who is just learning the alphabet. This student is likely to imagine d as a letter and does not conceptualize d as a representation of anything in the situation. This student has not coordinated her understanding of letters with her understanding of things changing.

If the student conceptualizes this situation with no image of variation then she might imagine $d = 4$ to be the number of miles car A has traveled and $d = 5$ to be the number of miles car B has traveled. This student imagines each value of d to be associated with a different car. The student is said to have no image of variation because she imagines the same attribute of multiple objects where each object's attribute has a different measure as opposed to conceptualizing an attribute *of a single object* whose magnitude varies.

If the student conceptualizes this situation with a discrete image of variation she might imagine the car is at mile marker 4 then magically the car is at mile marker 5. This is as if the student closed her eyes while the car was traveling and only attended to the locations when her eyes were open. Unlike the student with no image of variation, the student with a discrete image of variation conceptualizes an attribute of a single object varying. However, this student does not attend to the quantity's magnitude within the chunk. Thus, from the researcher's perspective there is a hole in the student's conception of the variable's value.

If the student conceptualizes the situation with gross variation then she may attend to whether his distance from home increases or decreases, but does not attend to the values this distance takes on as she travels. If the student conceptualizes this situation with a chunky continuous image of variation the student will focus on the ends of the chunk, say the miles indicated by the mile marker posts. Although the student might be aware that the car covered all distances between the mile marker posts, the distances highlighted by the mile marker posts have more significance in her thinking than the distances between the mile marker posts. Thus, this student does not attend to *how* the quantity's measure varies within a chunk.

Finally, if the student conceptualizes the situation with a smooth continuous image of variation then she is imagining change in progress and she is keeping track of the distance the car traveled as she imagines the car traveling. This student might also use the mile markers as landmarks to keep track of the distance the car has traveled, but she imagines these distances to be no more significant than the distances between the mile markers. The student conceptualizes these landmark points as a means to keep track of the measure of the varying quantity.

A student must construct an image of variation in each situation she encounters. Thus, a student might construct no image of variation in problem A but construct an image of discrete variation in problem B. This suggests that educators must carefully design tasks in order to support students in constructing images of variation that are productive for that problem. To do this, Thompson and Carlson (2017) recommend leveraging students' use of fictive motion – using a motion verb when the subject is not actually moving – to support students in imagining change in progress. For example, in the phrase "the value of x goes from 1 to 4" the value of x is not moving but we talk as if it is. As Thompson and Carlson explain, cognitive linguists, such as Matlock (2001, 2004), convincingly argue that when a student engages in fictive motion he is actively imagining something moving. This way of thinking is consistent with Castillo-Garsow and colleagues' (2013) conceptualization of smooth thinking as imagining change in progress. Fictive motion is also necessary when constructing variables as representations of varying values of a quantity because, as Lakoff and Núñez (2000) explained, fictive motion enables one to go between static and dynamic conceptualizations of the value of x.

Empirical Support for Constructing Multiplicative Objects

Researchers, such as Benjamin Whitmire (2014), Heather Johnson (2015), Kevin Moore (Moore, Paoletti, & Musgrave, 2013), and Patrick Thompson (Thompson, Joshua, Yoon, Byerley, & Hatfield, in review) have recently provided empirical evidence to support Thomson's conception of covariational reasoning. In particular, results from these studies suggest the importance of constructing invariant relationships and multiplicative objects as part of engaging in covariational reasoning.

Moore and colleagues (Moore et al., 2013) studied how two undergraduate pre-service teachers reasoned about graphs in the Cartesian Coordinate System (CCS) and the Polar Coordinate System (PCS). They asked the students to graph functions, such as $f(\theta) = 2\theta + 1$, in both the CCS and PCS. The students began by plotting discrete points and then considered how the quantities' values changed together. For example, one of the participants related the graph of $f(\theta) = 2\theta + 1$ in the CCS and PCS by explaining, "I'm relating the slope here (*pointing to the CCS graph*), to the difference in the radius of two each time (*tapping along the PCS graph*). Like [the radius is] one, three, five, seven, nine, eleven (*pointing to the corresponding points on the polar graph*), [the radius] increases by two" (p. 466). The authors explained that by reasoning about how two quantities' changed together the student was able to conceptualize *something* stayed the same and thus was able to reason that two graphs that looked different represented the same thing – the same covariational relationship. The results from this study suggest that engaging in covariational reasoning and constructing invariant relationships happen simultaneously in the student's thinking. Students are able to construct invariant relationships by engaging in covariational reasoning.

Although Whitmire (2014) and Johnson (2015) did not describe their findings in terms of multiplicative objects, these authors described the importance of conceptualizing quantities' values simultaneously when reasoning about graphical representations and rate, respectively. As Inhelder and Piaget (1964) explained, "as soon as we have a schema which can be described by the word "simultaneous" we have some sort of multiplicative relationship" (p. 182). Thus, Whitmire (2014) and Johnson's (2015) results provide evidence for the importance of a learner constructing multiplicative objects and uniting two quantities in thought while engaging in covariational reasoning.

Whitmire (2014) conducted one-on-one teaching interviews with university precalculus and first semester calculus students. His interviews centered around "the Homer Task" (this task has also been used by Saldanha and Thompson (1998) and Silverman (2005)). Whitmire (2014) began by presenting students with a computer animation of a person (Homer) driving at a constant speed along a straight road between two cities Shelbyville and Springfield (see Figure 6). He asked the students to construct a graph of Homer's distance from Springfield in terms of his distance from Shelbyville.

In his analysis Whitmire focused on the students' ways of thinking about graphs and how the quantities' values changed. He used his analysis to assess the propitiousness of these ways of thinking for engaging in covariational reasoning. He found that when students plotted discrete points, reasoned about a graph by attending to its shape, and/or reasoned about only one quantity's value varying with respect to time elapsed, the student was not likely to reason about how the two distances changed together. On the other hand, when a student conceptualized both quantities simultaneously and always talked about one quantity in relation to the other he was likely to reason about how the two quantities' magnitudes changed together and was likely to conceptualize a graph in terms of changing magnitudes. Although Whitmire does not discuss his findings in terms of multiplicative objects, attending to two quantities' magnitudes simultaneously entails constructing a multiplicative object that unites the two quantities. Additionally, his data suggests that imagining a point as a representation of both a value of x and a value of y, that is constructing the point (x, y) as a multiplicative object, is essential in order to conceptualize a graph as a representation of how x and y change together.

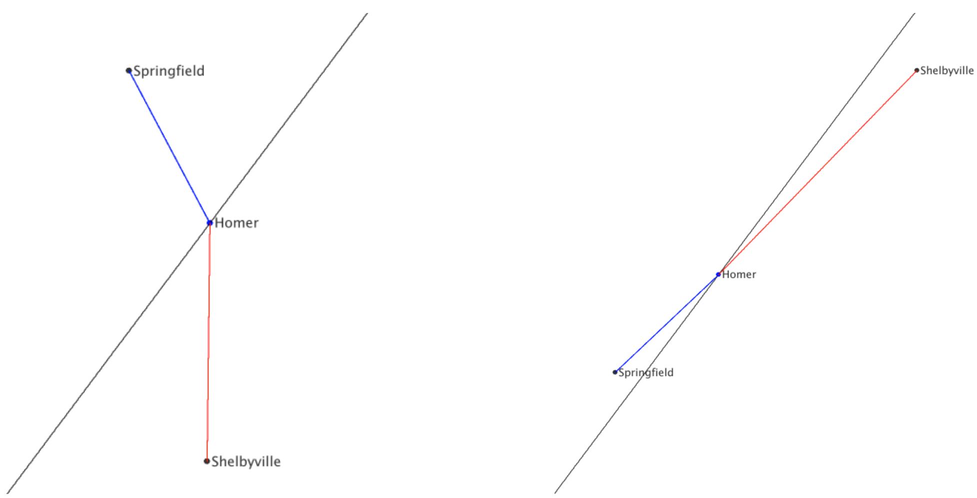

Figure 6: Screen shots from Homer Task where students are asked to reason about Homer's distance from Springfield in terms of his distance from Shelbyville.

Johnson (2015) conducted task-based clinical interviews with secondary students. Each student participated in a series of five interviews where they were asked to complete tasks that enabled the researcher to glean insights to the students' conceptualizations of ratio and rate. In particular, Johnson focused on whether the student compared or coordinated the two quantities' values. She related these constructs to Castillo-Garsow's (2012) conception of chunky and smooth thinking. Johnson (2015) explained,

> The operation of comparison involves chunky images of change and products of the operation of comparison include associations of amounts of change in quantities (e.g., height change more than volume in an interval). In contrast, the operation of coordination involves smooth images of change, and products of the operation include relationships between changing quantities such that change in one quantity would depend on concurrent, continuing change in another quantity (e.g., as height increases, volume continually increases) (Johnson, 2015, p. 70).

Johnson's analysis suggests that in order to conceptualize rates, students need to construct an intensive quantity by "coordinating variation in the intensity of change in one quantity with continuing change in another quantity" (Johnson, 2015, p. 84). She explained that a student constructs this intensive quantity by imagining the variation of one quantity happening simultaneously with continuing change in another quantity. Although Johnson does not interpret her finding in terms of multiplicative objects, her claim suggests that constructing a multiplicative object that binds that two quantities' variation together is essential when conceptualizing a rate.

In another study, Thompson et al. (in review) studied in-service mathematics teachers' meanings for teaching. The authors displayed an image of two bars, a red bar on the horizontal axis and a blue bar on the vertical axis of the Cartesian coordinate system. As the animation played the lengths of the bars varied together, each keeping one end fixed at the origin. The researchers provided each teacher with a response sheet on which the teacher was asked to sketch a graph of how the values of the two quantities were related (Figure 7).

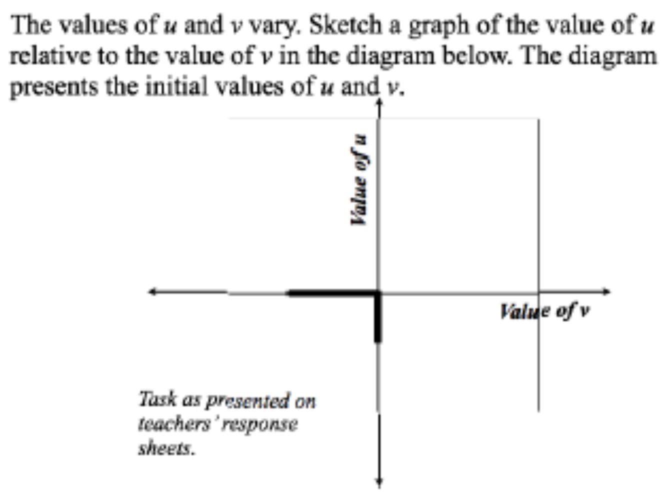

Figure 7: Teachers' response sheet for item assessing teachers' covariational reasoning (in Thompson et al., in review).

Thompson et al. scored the teachers' placement of the initial point separately from the shape of the graph the teacher drew in order to study whether the teacher realized that any point on his/her graph simultaneously represented two values. Since the response sheet included the initial lengths of both bars, if the teacher conceptualized a point on a graph as simultaneously representing two values, then the teacher would have to use the initial lengths of the bars in order to place an initial point on the graph. The authors found that only 18% of South Korean teachers' graphs that had a badly misplaced initial point also had accurate or semi-accurate shape, while 67% of South Korean teachers' graphs with a well-placed initial point also had an accurate or semi-accurate shape. Only 12% of American teachers' graphs that had a badly misplaced initial point also had an accurate or semi-accurate shape, while 52% of American teachers' graphs with a well-placed initial point also had an accurate or semi-accurate shape. These results provide compelling evidence that conceptualizing a point as a multiplicative object is essential in order to conceptualize a graph as an emergent trace of how two quantities' measures change together.

The results described above suggest that the way a student engages in covariational reasoning is dependent upon the multiplicative object the student constructs and the student's conception of variation. For example, one must construct the point (x,y) as a multiplicative object and imagine the value of x (or y) varying smoothly in order to conceptualize a graph as an emergent trace of how two quantities change together. However, this construction is not necessary in order to give a gross description of how two quantities change together – also a type of covariational reasoning. Conceptualizing a student's engagement in covariational reasoning in terms of the multiplicative objects she constructs and the ways she imagines quantities varying is consistent with the levels of covariational reasoning Thompson and Carlson (2017) proposed (see Table 3).

They explained that these levels of covariation "retain emphases on quantitative reasoning and multiplicative objects (Thompson), coordination in quantities' values (Confrey, Carlson), and add ways in which an individual conceives quantities to vary (Castillo-Garsow)" (*ibid,* p. 21). They go on to explain that each level is intended to both describe a class of behaviors and to characterize an individual's capacity to engage in covariational reasoning.

Table 3

Thompson and Carlson's Major Levels of Covariational Reasoning, highest to lowest (Thompson and Carlson, 2017, p. 23)

Major Levels of Covariational Reasoning	
Level	Description
Smooth Continuous Covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and they envision both variables varying smoothly and continuously.
Chunky Continuous Covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of Values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross Coordination of Values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases". The person does not envision that individual values of quantities go together. Instead the person envisions a loose, non-multiplicative link between the overall changes in two quantities' values.
Pre-coordination of Values	The person envisions two variables' values varying, but asynchronously, one variable changes, then the second variable changes, then the first, etc. The person does not anticipate creating pairs of values as multiplicative objects.
No Coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Thompson and Carlson emphasize that classifying a student's engagement in covariational reasoning just in terms of these levels is not sufficient. If a researcher uses these levels of covariational reasoning as a guiding framework for his analysis, then the researcher must also be mindful to model the ways in which a student conceptualizes quantities' values, how the student conceptualizes these values varying, and how the student unites two quantities' values in both thought and representation.

Graphing and Covariational Reasoning

Many studies investigating how students engage in covariational reasoning are situated in the context of a student's graphing activity (e.g., Carlson et al., 2002; Castillo-

Garsow, 2010; Moore et al., 2016; Saldanha & Thompson, 1998; Whitmire, 2014). This suggests an underlying theoretical hypothesis: by studying how students come to construct graphs one can understand the nuances of the mental actions involved in covariational reasoning. In the following section I will discuss the literature on students' graphing activity to illustrate this hypothesis.

Researchers have documented students' difficulty interpreting and constructing graphs (e.g., Bell & Janvier, 1981; Carlson, 1998; McDermott et al., 1987; Monk, 1992). Specifically, researchers suggest that students do not typically think about graphs as representations of how two quantities' values change together (e.g., Dubinsky & Wilson, 2013; Thompson, 1994c). Instead, many students reason based on their perception of the shape of the graph and often conflate visual attributes of a situation (such as the shape of a hill) with the shape of a graph.

For example, Bell and Janvier (1981) found that students were likely to experience situational and pictorial distractions when reasoning about graphs. They explained that situational distractions occur when the student's experience of the situation interferes with his/her ability to attend to the meanings of the features of the graphs and pictorial distractions occur when the student confuses the aspects of the situation. To illustrate these types of distractions, consider the racetrack problem (p. 39). Students experiencing a pictorial distraction would select racetrack G because the shape of the track matches the shape of the graph. Students experiencing a situational distraction attended to the number of bends in the track by thinking about the speed of the car decreasing around a curve, but this student would not attend to the location of the curves relative to the starting position or the depths of the dips in the graph when determining which 3-bend shape to select.

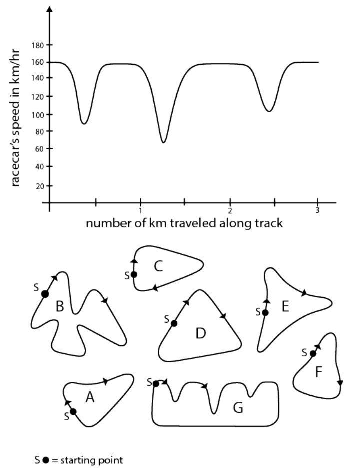

Figure 8: Racetrack problem from Bell and Janvier (1981, p. 39).

Carlson (1998) studied how students who at various points in their mathematics career reasoned about the position of two cars after one hour given graphs of each car's velocity with respect to time (Figure 9). Carlson found that 88% of students who recently completed college algebra with an A and 29% of students who recently completed second semester calculus with an A interpreted the graphs as pictures of the paths of the cars. As a result these students concluded that after one hour the cars were in the same position (because their paths are intersecting) or that Car B was passing Car A (because lines are moving away from one another at t = 1 hour).

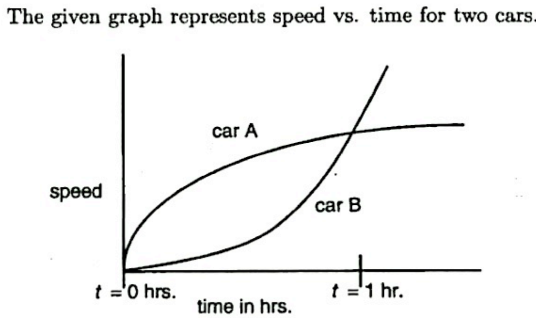

Figure 9: Car A and Car B task from Monk (1992).

Bell and Janvier (1981) and Carlson's (1998) findings provide evidence that students often reason about graphs based on their perception of the shape and pictorial attributes of the graph. As a result, students often confound the shape of the graph with pictures of physical situations. Moore and Thompson (2015) called this *static shape thinking* and explained that a student who engages in static shape thinking might, for example, understand slope as the property of the line that determines whether the line falls or rises as it goes from left to right.

An alternative way of thinking about graphs is what Moore and Thompson (2015) called *emergent shape thinking*. They explained,

> Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation). As opposed to assimilating a graph as a static object, emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities (p. 4).

Central to this conception of graphs is an understanding that a point in the

Cartesian plane is a multiplicative object that unites two quantities' values whose measures are represented on the axes (Figure 10). This intersection point in the plane is the object the student then imagines tracing while engaging in emergent shape thinking.

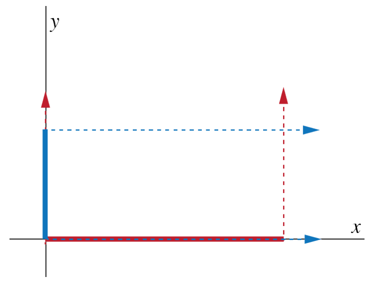

Figure 10: A point as the intersection of two quantities' values extended from the axes.

Researchers are learning that students' tendency to engage in static shape thinking inhibits their ability to engage in emergent shape thinking (e.g., Frank, 2016; Moore et al., 2016). For example, Moore et al. (2016) reported that pre-service teachers have constructed graphing habits that cause them to experience cognitive conflict when trying to reason about the graph as an emergent trace. The authors found that these students were often able to engage in gross coordination of values and describe situations in terms of how quantities varied together. However, these students' graphing habits - such as starting a graph by plotting a point on the vertical axis, reading graphs from left to right, and believing that graphs must pass the 'vertical line test' - inhibited their ability to graphically represent relationships between varying quantities. In addition to documenting the difficulties students encounter when trying to engage in emergent shape thinking, Moore et al. (2016) explained that these findings support Moore and Thompson's (2015) conjecture that conceptualizing graphs as emergent traces is a productive way of thinking about novel phenomena. Whereas ways of thinking that focus on recalling shapes and properties of shapes are limited to phenomena that are compatible with these shapes.

One final note about graphing: math educators and math education researchers might conceptualize graphs as representations of how quantities change together so that the pictorial graph and the individual's conceptualization of how quantities change together are interconnected. As a result, the individual can then use her graph as the basis for further reasoning. However, the studies above provide evidence that this is not how many students conceptualize graphs. Thus, researchers who might conceptualize graphs as representations of how quantities change together must be cautious when ascribing meaning to students' graphing activity. Even though a student constructs a graph by engaging in covariational reasoning, one should not take this as evidence that the student then conceptualizes the completed graph as a representation of that thinking. Research about students' graphing activity must go one step beyond understanding what supports/inhibits students from constructing a graph as an emergent trace. Researchers must then ask students to reason from the products of their graphing activity to understand the meanings students construct from their graphing activity.

CHAPTER 3
THEORETICAL PERSPECTIVE

The purpose of this chapter, my theoretical perspective, is to make explicit to the reader and myself the assumptions I make about knowing and learning. As Thompson (2002) explained, the purpose of these assumptions is "to constrain the types of explanations we [the researcher] give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (p. 192). My theoretical perspective is grounded in my interpretations of Piaget's (1967, 1985) genetic epistemology and von Glasersfeld's (1995) radical constructivism. It is based on the following assumptions:

1. The knower constructs his knowledge through his experiences.
2. There is no universal reality; an individual's reality is the product of his experiences.
3. An individual is a biological creature who continuously adapts his reality in order to make sense of his experiences.

These ideas have informed the design of this study as well as my implementation and analysis of this investigation. In this chapter I describe my theoretical perspective by elaborating my interpretation of Piaget's genetic epistemology focusing on his Theory of Reflective Abstraction. I also address the implications of Piaget's work on teaching and learning mathematics.

Piaget's Genetic Epistemology

For nearly sixty years Piaget created and elaborated a developmental theory of human knowledge, a genetic epistemology. His theory focused primarily on two things:

(1) what knowledge is and (2) how individuals come to know what it is that they know. At the heart of his theory was that an individual comes to know and understand through action. This implies that there is no way for one to transfer knowledge to another individual. Instead, each person must build up – or construct – her knowledge through her actions. For Piaget, actions are more than behaviors; Piaget considered actions to include all thought, movement, and emotion satisfying a need (Piaget, 1967, p. 6). Thus, for Piaget, actions include behavior as well as acts of reasoning and judgment.

Individuals organize their actions into schemes. These schemes include when to apply the action, an anticipation of the result of acting, how these actions work together, and eventually how these actions can chain together. An individual's schemes constitute his *operative knowledge*, knowledge about how to act on an object under certain circumstances or knowledge about what the object will do under different circumstances.

Piaget believed that an individual's reality is not innate. He considered "the ultimate nature of reality to be in continual construction instead of consisting of an accumulation of ready-made structures" (quoted in von Glasersfeld, 1995, p. 57). This implies reality is *always* relative to the individual and an individual's reality is *her* understanding of the world. The individual's reality changes and develops as she experiences discrepancies between what he knows and what she discovers by using that knowledge. As a result, one's knowledge about the world is adaptive—an individual is constantly seeking equilibrium between the understandings she has constructed and the results of using those understandings to make sense of her experiences.

This equilibrium develops through two fundamental processes, assimilation and accommodation. Piaget described assimilation as the incorporation of a stimulus into an internal cognitive structure, a scheme (Bringuier, 1980). For this conception of assimilation to be coherent in Piaget's genetic epistemology we must not think about assimilation as the individual incorporating something *from reality* into his cognitive structure. From Piaget's perspective there are no external objects to assimilate; objects are always an individual's construction and thus one cannot speak of objects without speaking of the individual who has constructed the object. This implies that the individual can only assimilate his own experiences and constructions to his existing schemes.

As von Glasersfeld (1995) described, assimilation is closely related to perception. "The cognitive organism perceives (assimilates) only what it can fit into the structures it already has. … when an organism assimilates, it remains unaware of, or disregards, whatever does not fit into the conceptual structures it possesses" (von Glasersfeld, 1995, p. 63). For example, consider a child's drawing of an object. The child's drawing is not an exact replica of the observer's understanding of the object. Instead the drawing represents the child's interpretation and understanding of the object (see Figure 11). The child's drawing represents his/her assimilation (Piaget & Goretta, 1977). This suggests one's understandings are her interpretations of what she observes and thus knowledge is the result of assimilating to a scheme.

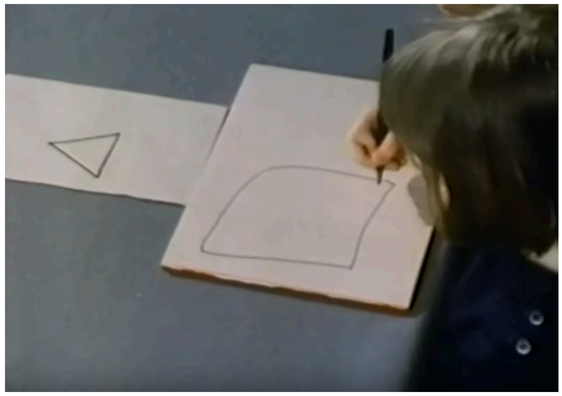

Figure 11: Child draws her assimilation of the shape in front of her (from Piaget and Goretta, 1977).

Understanding one's assimilations does not tell the entire story of her knowledge construction. As Piaget explained, "The scheme of assimilation is general, and as soon as it's applied to a particular situation, it must be modified according to the particular circumstances of the situation" (quoted in Bringuier, 1980, p. 43) These modifications are what Piaget called accommodations. When an individual assimilates an experience to a scheme, she can anticipate the result of acting. When her anticipation of the result of acting does not match her experience she must modify her internal structure to fit her experience. This will cause the individual to experience perturbation and make an accommodation, an adjustment to her cognitive structure(s) in order to fit the particular experience. If her accommodation is successful then she has engaged in assimilation (Piaget, 1985, p. 5). This suggests that there is a reflexive relationship between assimilation and accommodation. Piaget summarized this relationship between assimilation and accommodation when he said,

> Just as there is no accommodation without assimilation – because it is always accommodation to something being assimilated to one scheme or another –

> similarly, there can be no assimilation without accommodation, because the assimilatory scheme is general and must always be accommodated to the particular situation (quoted in Bringuier, 1980, p. 43).

An example might be useful to illustrate this relationship between assimilation and accommodation. Thompson (2011b) described the work of an elementary school student, JJ, who constructed a scheme to determine the amount of time needed to cover a given distance by measuring the distance in units of a speed length (more details in Thompson, 1994a). After repeating this activity numerous times, Thompson asked JJ to determine the speed necessary to cover some distance in a given amount of time. From the researcher's perspective, these two tasks do not require the same ways of thinking about measurement. However, JJ assimilated this new task to her scheme for measuring distances in speed lengths and approached the question by guessing a speed and then using her conception of speed length to see how well the speed she picked worked.

It is important to note that when JJ constructed her speed length scheme, she was always given a speed and a distance and asked to figure out how long it took to cover that distance at the given speed. The new task did not provide this same information. Thus, in order for JJ to assimilate her interpretation of the new situation to her scheme of speed lengths, JJ had to make an accommodation to her speed length scheme to be able to assimilate problems that did not give her a speed. This suggests that for JJ to assimilate, she first needed to make an accommodation to her scheme. Once JJ assimilated the new task to her speed length scheme she needed to make an accommodation to her conceptualization of the problem so that the problem fit her scheme that required she think about measuring distances with a known speed length. She accommodated the situation by guessing speed lengths and then using these speed lengths to measure the distance and finally checking if she got the right time. Thus, JJ made an accommodation to her interpretation of the task because of the assimilation she made.

While accommodation and assimilation are the mechanisms of equilibration, it is still necessary to understand abstraction—the mechanism of accommodation.

Abstraction

Piaget distinguished between two types of abstraction, empirical abstraction and reflective abstraction. An individual engages in empirical abstraction when he abstracts properties from objects he has constructed. Empirical abstractions are not merely observations. Instead, an empirical abstraction is the product of an individual assimilating his environment to a scheme he previously constructed (Piaget, 2001, p. 30). To differentiate between observations and empirical abstractions consider the following examples. Observations include the blue stick is shorter than the red stick or $y = 2x$ is defined to be a linear relationship. Examples of empirical abstractions include the blue stick**s** are shorter than the red stick**s** or $y = \mathrm{m}x$ is called linear since $y = 2x$ is called linear and $y = 4.7x$ is called linear. These examples suggest that empirical abstractions are closely related to generalizations where the individual has relaxed some constraint in his construction of an object.

Psuedo-empirical abstraction is a type of empirical abstraction where the individual abstracts from the products of his actions. From the perspective of the observer, an individual's pseudo-empirical abstraction involves actions that have the potential to be reflected upon. For example, when describing the behavior of the function f, where $f(x) = 3x + 1$, a student might begin by producing a set of input-output pairs such as (0, 1), (1, 4), (2, 7), etc. The student might use these input-output pairs to observe that as the input increases by 1 the output increases by 3. This is an example of pseudo-empirical abstraction because the student is using the product of acting, the (x, y) pairs, to coordinate the values of x and y.

Reflective abstraction, on the other hand, is an abstraction from actions where actions include behavior as well as interpreting, judging, predicting, and reasoning. By engaging in reflective abstraction, an individual is able to systematize his actions. Piaget identified two main phases of a reflective abstraction. In the first phase, réfléchissement, the individual differentiates an action from the product of acting and then projects this action to a scheme where the action becomes a transformation – an object of thought. In the second phase, reflection, the individual coordinates this transformation with his existing schemes. I elaborate these two phases in the following paragraphs.

When an individual engages in reflective abstraction, he differentiates an action (or characteristic of an action) from its consequence. Once the individual differentiates the action from its consequence, which might require numerous attempts at differentiation, he can begin to reflect on the action. As von Glasersfeld (1995) described, reflection is,

> The mysterious capability that allows us to step out of the stream of direct experience, to re-present a chunk of it, and to look at it as though it were direct experience, while remaining aware of the fact that it is not. (p. 90)

In order to re-present the action, the student must re-play and re-construct the experience in which he constructed the action. This involves deferred imitation; the individual must imagine the action occurring without actually engaging in that action and without the presence of the perceptual situation that initially led to the individual's construction of the action. This process enables the individual to project the action or characteristic of action from scheme A where it was used implicitly, or simply implied, and transform it into an object of thought in scheme B. Operating with scheme B the student can imagine performing the action without the presence of specific conditions. For example, operating with scheme A one might need two specific collections in order to imagine joining these collections. With scheme B one can now think about joining two collections without the presence of two specific collections because the individual can think about the activity of joining instead of being limited to thinking about the result of joining two collections; the individual has constructed joining as a transformation.

Once the individual has re-presented the action as an object of thought in scheme B, the student can engage in the second phase of reflective abstraction - reflection. When the student projects the action from scheme A to scheme B the student has introduced a new object(s) of thought into scheme B. This necessitates that the student reorganize scheme B so that the imitation of the action from scheme A is integrated with the other actions and operations in scheme B. This integration occurs as the student coordinates the imitation of the action from scheme A with the elements in scheme B. The student might have to engage in coordination many times before he successfully integrates the imitation with scheme B (Piaget, 2001, p. 53).

From the perspective of learning, the activity of engaging in reflective abstraction is just as important as the product of this activity. Engaging in reflective abstraction might take anywhere from a few moments to a few years. This activity requires the individual to repeatedly construct an understanding by differentiating the action from its consequence, re-present this action without external stimuli, and then reconstruct his conception in order to be able to represent and coordinate his understanding in a new scheme. The individual will likely need to go through this entire cycle, or parts of this cycle, numerous times in order to construct a scheme that the individual can assimilate to in future experiences.

Two Types of Reflective Abstraction

Piaget identified two types of reflective abstraction – reflecting and reflected abstraction. The difference between these types of abstraction is whether the individual is conscious of the knowledge he developed through abstraction. If the individual is not conscious of the knowledge he developed through abstraction, then she engaged in what Piaget called a reflecting abstraction. If the individual is conscious of the result of her abstraction, then she engaged in a reflected abstraction.

Piaget (2001) considered reflecting abstraction to be a constructive process where the individual differentiates an action from its product, imitates this action, projects the imitation to a higher level of thinking, and then integrates this imitation with the other actions and objects at this new level of thinking. I must emphasize that reflecting abstraction does not happen in a moment. The individual might have to differentiate or imitate the action numerous times before he successfully projects the imitation. Then, the individual might have to coordinate this imitation at the new level of thinking many times before she successfully integrates this imitation at the higher level of thinking. As a result, a reflecting abstraction might take an individual anywhere from a couple of minutes to multiple years to complete.

Reflected abstraction involves a retroactive thematization where the individual reflects on the result of his reflecting abstraction so that he becomes conscious of this result. Thus, when an individual engages in reflected abstraction the integrated imitation of an action becomes a conscious object of thought. As a result, an individual is conscious of the product of a reflected abstraction and is able to verbalize his newly constructed knowledge.

Images & Abstraction

Imagination is a critical part of reflecting abstraction; by imagining an action on an object (e.g., imagine *rotating* an object a quarter turn to the right) the student can develop a representation of the action and thus differentiate the action from its consequence. This suggests that students' images are an essential part of their abstractions; in fact the nature of the student's image develops in parallel with his abstractions.

My conception of images entails more than mental pictures. As Thompson (1996) described, an image is,

> Constituted by experiential fragments from kinesthesis, proprioception, smell, touch, taste, vision, or hearing. It seems essential also to include the possibility that images can entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining (Thompson, 1996, p. 267).

Thompson's meaning for image suggests that an individual's image depends on his past experiences and thus an individual's image will be unique to himself. This implies that it is not productive to speak of the images associated with linear relationships without discussing the individual who is conceptualizing and constructing images of linear relationships. Additionally, an individual constructs images in the moment of acting. These images are closely tied to the schemes that are governing his thinking in the moment and the nature of the objects and actions organized by those schemes.

Piaget hypothesized that there are three types of images. The primary difference between these images is how dependent the image is on actions it entails. The following quotations summarize Piaget's three types of images:

> 1. An "internalized act of imitation … the motor response required to bring action to bear on an object … a schema of action."
>
> 2. "In place of merely representing the object itself, independently of its transformations, this image expresses a phase or an outcome of the action performed on the object. … [but] the image cannot keep pace with the actions because, unlike operations, such actions are not coordinated one with the other."
>
> 3. "[An image] that is dynamic and mobile in character … entirely concerned with the transformations of the object. … [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination." (quoted in Thompson, 1994b, pp. 183-184)

I will call these initial images, images of directed actions, and images of transformations respectively.

I will use the following example to differentiate between these three images. Consider the division of the positive integer *a* by the positive integer *b*. When an individual first conceptualizes $\frac{a}{b}$, the individual might compare the relative size of *a* and *b* by either measuring or imagining how many copies of *b* fit into *a*. This individual must have or imagine having specific values of *a* and *b* in order to reason about the relative size of *a* and *b*. In this image of division, the context in which the student is reasoning, the reasoning he engaged in, and the product of this reasoning are all interconnected and part of the same cognitive entity – the individual has yet to differentiate one part from another. He likely has to imitate this set of actions in order to develop an initial image of division. This image is highly figurative and entirely dependent on the context in which he is reasoning. If any part of this image (the context, the act of reasoning, and the product of reasoning) is removed, his image of division will fall apart. This image is an example of an initial image – an image that contains the context the individual constructed, the actions he imposed in that context, and the result of those actions. For the individual, these three aspects are interconnected as a single object of thought.

After the individual repeatedly re-constructs his initial image, he will start to attend to the sequence of actions he enacts. He will focus on $\frac{a}{b}$ as the relative size of *a* and *b*. He no longer needs to imitate each action from the initial image. For example, he no longer needs to imagine determining how many copies of *b* fit into *a*. However, his image of division is still dependent on imagining specific values of *a* and *b*. This image is an example of an image of directed action. The individual attends to the action that connected the context and the outcome – the relation between *a* and *b*, but the context and the outcome of his actions are still part of this image.

Eventually the individual will be able to think about a relation between the relative size of two numbers. The individual does not need to imagine two specific values, he does not need to imagine representing them in the form of a fraction, nor does he need to imagine determining how many copies of one number fit into the other. The individual has constructed the *relation* of relative size as something to reason about. He has constructed relative size as a transformation – an abstracted action. This is an example of an image of a transformation. This image is operative in the sense that it does not depend on a specific context in which to reason. Instead, the individual has abstracted the action of relating the magnitude of two values. This implies that images of transformations are the product of reflecting abstractions where the individual has constructed a representation of an abstracted action – a transformation.

It is important to note that as a researcher, once you have constructed an image (say an image of a transformation), it is hard to imagine what it is like to only have an image of directed action. This is important because it suggests a challenge researchers face when modeling someone else's thinking. If the researcher is not conscious of the images he has constructed then it is likely that he will wrongfully impose his images on someone else's thinking.

Extending Piaget's Theory of Abstraction

Images are not necessarily stable constructions; an individual might have a momentary understanding of relative size as a transformation but then loose that image when he goes to reason about the relative size. As Thompson and Harel (in Thompson, Carlson, Byerley, & Hatfield, 2014, p. 13) explained, an individual can construct understandings in the moment that are easily lost once the individual's attention moves on or the individual can construct stable understandings that are part of the individual's schemes (see Table 4).

Table 4

Thompson and Harel's definitions of understanding, meaning, and ways of thinking (quoted in Thompson et al., 2014)

Construct	Definition
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning. What Harel previously called Way of Understanding
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning.

This suggests that reflecting abstractions do not always result in *stable understandings*. Steffe (1991) elaborated two types of accommodations to help researchers differentiate between reflecting abstractions that result in *understandings in the moment* - what Steffe called functional accommodations, and reflecting abstractions that result in *stable understandings* – what Steffe called metamorphic accommodations.

According to Steffe (1991) a functional accommodations is "an accommodation of a scheme that occurs in the context of using the scheme" (p. 37). I interpret this to mean that in the context of acting, the student has coordinated her existing schemes, at least momentarily, in a new way. This coordination is not a permanent modification to the student's schemes. Instead, this coordination results in a successful, momentary, assimilation. This assimilation is what Thompson and Harel call an *understanding in the* *moment.* From a researcher's perspective, the construct of a functional accommodation is extremely useful. A researcher can use functional accommodations to help explain why a student might be able to solve a given problem in one moment but cannot apply the same type of reasoning to the next problem.

When a student engages in a metamorphic accommodation she reorganizes her scheme(s) and as a result has a new way of thinking, she has constructed a *stable understanding*. She is not necessarily conscious of this new understanding but this understanding is now part of a scheme that the student can then assimilate to future experiences. Steffe relates metamorphic accommodations to the reflection aspect of reflecting abstraction where the individual integrates an imitation of an action with his existing schemes(s).

Steffe's constructs of functional and metamorphic accommodations provide a way for researchers to think about the mechanisms of constructing understandings in the moment and stable understandings. Equally important to theorizing how a student constructs an understanding is to model the space of implications for that understanding. This is what Thompson and Harel (in Thompson et al., 2014) call a meaning. As Thompson et al. (2014) explained, "the *meaning of an understanding* is the space of implications that the current understanding mobilizes – actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort" (p. 12). The authors go on to explain that an individual's scheme constitutes the space of implications anytime the individual assimilates to that scheme. Thus, when a researcher attends to the meanings individuals construct he is able to study the schemes that individuals have constructed.

APOS: A Potential Theoretical Framework

Given my focus on how students conceptualize relationships between quantities' values, sometimes called functional reasoning (e.g., Breslich, 1928), and my theoretical perspective focused on reflective abstraction, one might assume that APOS Theory is a good theoretical fit for my dissertation study. However, as I will describe, the underlying assumptions of APOS Theory make it an inappropriate theoretical framework for the design and analysis of this dissertation study.

In the late 1980s Dubinsky hypothesized that a person's mathematical knowledge "consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them in schemas to make sense of the situations and solve the problems" (Dubinsky & McDonald, 2001, p. 276). He formalized this hypothesis into APOS Theory, which suggests that for each mathematical idea there are three stages of understanding: action, process, and object conceptions.

While Dubinsky intended APOS Theory to be applicable to any mathematical concept, much of the research using this theory focuses on how students understand functions. APOS researchers say that a student has an action conception of function when the student thinks about a function as a particular rule or formula to carry out (Breidenbach et al., 1992). Consider the statement "Function f is defined by $f(x)=3x+1$". A student with an action conception of function will be limited to envisioning the evaluation of f as substituting a specific value for x and then multiplying it by 3 and adding 1. He must repeat this set of actions for any value of x that he is given.

APOS researchers call the second developmental stage of understanding a process conception. They say a student has a process conception of function when he can think of a function as receiving one or more inputs and returning the results, the outputs, without having to go through each step of the transformation (Asiala et al., 1996). Consider again the statement "Function f is defined by $f(x) = 3x + 1$"; a student with a process conception of function can envision substituting a value of x and getting a corresponding output without having to actually compute or know the value of the output.

APOS researchers say a student has an object conception of function, the final stage of understanding, when he can think about functions as objects on which to act. For example thinking about the sum of two functions as a single function necessitates that the student see each of the two original functions as objects that can be terms in the summation (Dubinsky, 1991a).

According to Dubinsky (1991b), students develop these structures (actions, processes, objects) by engaging in reflective abstraction by first projecting existing knowledge onto a higher plane of thought and then reorganizing that knowledge in the higher plane of thought. According to APOS Theory, students must engage in interiorization, coordination, encapsulation, and generalization in order to engage in reflective abstraction and reconstruct existing knowledge in a new way in the higher plane of thought.

At the surface it seems that Dubinsky's APOS Theory is aligned with Piaget's thinking. For example, one might see Dubinsky's constructs of action, process, and object conceptions as aligned with the three types of images Piaget described and one might believe Dubinsky's use of reflective abstraction is consistent with Piaget's writings. However, as I describe in the coming paragraphs, these are only surface level similarities.

APOS Theory does not equip the researcher with the same theoretical tools as Piaget's genetic epistemology. In particular, APOS Theory is not intended to support researchers in conducting a nuanced analysis of student thinking.

One could argue that Dubinsky's constructs of action, process, and object conceptions are aligned with my conception of Piaget's three images, which I have called initial images, images of directed action, and images of transformations. For example, in an action conception the individual is focused on carrying out a series of actions on a previously conceived object. The student needs to explicitly perform these actions and must perform each action – none can be skipped (Arnon et al., 2014, p. 19). This is consistent with my conception of Piaget's notion of initial images where the individual performs some action in such a way that the context of reasoning, the acts of reasoning, and the products of reasoning are all interconnected. If any piece of the image is removed, the image falls apart. The similarity extends one step further: both images are intended to describe understandings in the moment. As Breidenbach et al. (1992) described, a student's thinking might be between an action and process conception suggesting that images of actions, processes, and objects might only be understandings in the moment. However, there is a major difference between Dubinsky's and Piaget's conception of images: Dubinsky's action, process, and object conception are used to describe images that the observer finds propitious. These conceptions cannot be used to describe students' images that, from the observer's perspective, are inaccurate. Thus, APOS researchers are not attending to all images a student constructs. Piaget's images have no preference for what the observer deems correct or incorrect. Instead, Piaget studied the nuances of each student's thinking and modeled the images the student constructed, not the images Piaget wanted the student to construct.

Additionally, APOS researchers do not attend to the nature of the objects students construct. In Dubinsky's early writing he specifies that objects are always cognitive constructions a subject makes at some point in his development in order to deal with and make sense of his perceptions, activity, and thought (Dubinsky, 1991b). However, most research using APOS Theory ignores the origin of the objects that the individual is "transforming".

From my perspective this is problematic because the construction of an object is not a trivial activity for the student; constructing an object is an intellectual achievement. Assuming the student has constructed an object implies that the student has previously engaged in cognitively demanding activity. Additionally, both physical and mental objects can exist at multiple levels of sophistication. For example, a student can conceptualize the physical object "rock" as a solid 3-dimensional shape with jagged edges or as a solid aggregate of minerals that can be classified into igneous, sedimentary, and metamorphic. In the context of graphing, a student might conceptualize a graph as a wire-like-shape or the student could conceptualize a graph as an emergent trace. The actions the student can perform on his mentally constructed object depend on the level of sophistication of this object. For example, a student who has constructed the graph as a wire-like-shape might engage in the activity of a horizontal translation by sliding the graph - an image - to the left or right by some amount. On the other hand, a student who has constructed the graph as an emergent trace representing a dynamic relationship between two quantities might engage in the activity of a horizontal translation by first constructing a new relationship between two varying quantities and then representing this relationship. Both of these hypothetical students might draw the same translated graph, however, the actions each student engaged in and the objects he transformed were very different. Thus, in my opinion, it is essential for the researcher to have a model of the object the student is transforming and the level of sophistication of this object.

Note these are my conceptions of an object. Researchers using APOS Theory assume that a subject has some collection of objects to utilize when he begins engaging in mathematical activity. These researchers ignore what these objects are, how the student constructed these objects, and the level of sophistication of these objects. For APOS researchers, once they attribute an object to a student's thinking, the inner workings of that construction are forgotten.

Dubinsky's APOS Theory also seems to be aligned with Piaget's notion of reflective abstraction. However, there is one significant difference; the APOS constructs of interiorization, coordination, encapsulation, and generalization are used to describe global changes in the student's cognitive structure and are not intended to describe the nuances of the student's activities while engaging with a single problem. Throughout *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*, APOS developer's comprehensive book on the development of and current state of APOS Theory, the authors provide examples of reflective abstraction that involve the student reconstructing complex abstract concepts that are often the focus of an entire mathematics course, such as function or integer. For example,

> (Functions) are first constructed as operations that transform elements in a set, called the domain, into elements in a set, called the range. Then, at a higher stage,

as elements of a function space, functions become content on which new operations are constructed (Arnon et al., 2014, p. 6).

The authors do not discuss how a student might engage in reflective abstraction at the scale of a single problem. This suggests the developers of APOS Theory are more concerned with classifying students' understandings of complex abstract ideas than they are with understanding nuances in students' thinking when engaging in particular types of problems or learning to solve new types of problems. From a researcher's perspective this is problematic because as Steffe (1991) articulated, when a student makes an accommodation to his scheme in the moment of acting, she is not constructing a stable understanding. Stable understandings come from metamorphic accommodations that rarely happen in the moment of acting and are nearly impossible for a researcher to witness in the context of a teaching experiment or even a semester long case study. It is essential for the researcher to be able to describe the nuances of the student's thinking in the context of their study. This requires theoretical constructs, such as differentiation and functional accommodation, which are intended to describe the nuances in a student's thinking while engaging in mathematical activities.

Additionally, Arnon et al.'s (2014) conception of reflection limits the explanatory power of APOS Theory. According to Arnon et al., reflection involves "awareness and contemplative thought, about what Piaget called content and operations on the content" (Arnon et al., 2014, p. 6). This suggests that, for APOS theorists, reflection is a conscious activity. Requiring the student to be conscious of his activity in order to construct new mathematical knowledge limits the interpretative power of APOS Theory. Piaget (1976) discussed the role of consciousness in thought. He described how individuals are often able to complete tasks by engaging in reflecting abstraction *before* they are conscious of the products of these abstractions. For example, Piaget (2001) explains that toddlers engage in reflecting abstraction in order to learn to push a rotating bar away from them in order to bring the desired object toward them. However, Piaget does not assert that the toddler is conscious of this understanding. Piaget's insights about consciousness suggest that students are likely able to coordinate their actions long before they are able to consciously describe what they have done. Thus, it is essential for the researcher to be able to differentiate between reflective abstractions that result in consciousness of thought and those that do not.

In addition to the theoretical discrepancies between Piaget & Dubinsky's conceptions of reflective abstraction, APOS Theory is not situated to help me study how students reason the ways two quantities' values change together. Although some would consider this reasoning to be synonymous with functional reasoning, covariational reasoning is not part of the contemporary mathematical conception of function. In the case of function, the mathematics community has accepted Dirichlet's definition of function, "y is a function of x, for a given domain of values of x, whenever a precise law of correspondence between x and y can be stated clearly", as *the* meaning of function (quoted in Boyer, 1946, p. 13). This is significant because researchers use APOS Theory to classify students' understandings by categorizing the students' conception of a concept. For example, an APOS researcher might say that a student has an action conception of the concept of function. According to Arnon et al. (2014), a concept is an understanding that is agreed upon by mathematicians whereas a conception is an individual's understanding (Arnon et al., 2014, p. 18). If APOS researchers are only interested in classifying student's understandings relative to *the concept,* the mathematician's understanding, then the researcher's classification of the student's understanding does not address any meanings that are not propitious to conceptualizing what the mathematics community has identified as *the concept*. Thus, in the case of function, if researchers are only interested in classifying students' understandings relative to *the concept of function* then the researcher is not addressing covariational reasoning in his characterization of students' function conceptions.

One could argue that APOS Theory can be applied to any mathematical idea and thus one can extend APOS Theory to classify students' engagement with covariational reasoning. However, since there is no commonly accepted conception of covariational reasoning - there is no "concept" of covariational reasoning for researchers to use as a measure by which to classify a student's thinking. As a result, it would not make sense to talk about a student's action conception of covariational reasoning.

Given the prominence of APOS Theory throughout the function literature it is necessary to discuss the limitations of this theory. APOS Theory can be useful in providing an initial interpretation of students' thinking relative to their point-wise meanings for function. But, researchers need to be aware of the theoretical limitations of APOS Theory. In particular, APOS Theory cannot help a researcher think about a student's engagement in covariational reasoning. Additionally, APOS theoretical constructs, such as interiorization and encapsulation, are not intended to describe the nuances of a student's mathematical activity. Instead, these constructs are intended to describe metamorphic changes in a student's mathematical understandings. In order to focus on the nuances of a student's thinking I will ground my dissertation study and analysis in my own interpretation of Piaget's genetic epistemology. This will enable me to describe and attend to the nuances in my participants' thinking.

Implications of Piaget's Genetic Epistemology in Math Education

Adopting Piaget's genetic epistemology as a framework of learning has significant implications in the classroom. If one assumes, as I have, that the individual constructs her knowledge through her experiences, then in the classroom one must assume that students learn from their activity as opposed to learning what the teacher tells them. For example, a teacher might share with the class a procedure for solving equations of the form $ax^2+bx+c=0$. The student can only interpret the procedure in terms of her previous experiences and understandings. As a result, the student might not assimilate the procedure as the teacher intended. Not only does this challenge the typical image of a teacher in the classroom, this means that in a classroom of thirty-five students, each student will have to construct his own knowledge and each student's construction will be different than his peer's construction. Thus, each student will construct her own interpretation of the mathematics the teacher communicates and her own understandings from classroom experiences.

With this perspective, the teacher's role is no longer to disseminate his knowledge to the students. Instead, the teacher's role is to develop and enable experiences for the student. As Thompson (1991) explained,

> It sounds quite non-constructivist to say that, as mathematics educators, what we try to do is shape students' mathematical experiences. Yet, that is what mathematics educators working within a constructivist framework try to do. We

> attempt to provide occasions where students' experiences will be propitious for expanding and generalizing their mathematical knowledge. Not just any experience is satisfactory (pp. 260-261).

By adopting a theoretical perspective grounded in Piaget's genetic epistemology, this suggests that when an educator designs an experience for students, she should design tasks that support students in engaging in abstraction and reflection. This means that the purpose of classroom activities should be to provide opportunities for students to engage in actions and reflect on these actions to construct and coordinate mathematical structures. One should note that reflecting on one's own thinking is not a natural action. As a result, the instructor must work diligently to design situations where the student reflects on his actions. Since imagination and imitation are acts of reflecting, tasks that encourage students to anticipate "What would happen if…?" likely support students in reflecting on their actions. von Glasersfeld (1995) called such tasks thought experiments and hypothesized "thought experiments constitute what is perhaps the most powerful learning procedure in the cognitive domain" (p. 69).

Piaget's genetic epistemology does not suggest an educational environment without teachers. Instead, Piaget's work questions the nature of the teacher in the American classroom. His work suggests that the teacher's primary role should be to construct models of what students know, what the teacher wants students to know, and how students come to know. The teacher can then use these models to design activities that she anticipates will support students in interpreting the tasks in ways that the teacher anticipates so that the student has the opportunity to construct understandings in ways the teacher anticipates will be propitious (Thompson, 2000, p. 427).

This conception of teaching also extends to how I will design and conduct this dissertation study. In addition to supporting students in constructing new understandings, math education researchers seek to understand a student's thinking by building models of his/her thinking. Since Piaget's genetic epistemology suggests that students' understandings are organized in schemes, to model a student's thinking is to build a model of the schemes that the student has constructed. In addition to designing tasks that support reflecting abstractions researchers also need to design tasks that help them understand the bounds of the student's schemes. One way to do this is to engage students in a pre-test, intervention, post-test. The student's activity on the post-test, namely where and how the student engages in generalizing assimilations, will provide insights into the nature of the schemes the student constructed during the intervention. As I will describe in the next chapter, this pre-test, intervention, post-test model will serve as the general structure of my dissertation study.

CHAPTER 4
METHODOLOGY

The purpose of this chapter is to outline the methods I used in order to investigate my research questions:

1. What ways of thinking do students engage in when conceptualizing and representing how two quantities change together? How do students construct these ways of thinking? And, what ways of thinking support/inhibit students from reasoning about how two quantities change together?
2. How do students operationalize their scheme for covariational reasoning across problem types?

In particular, in this chapter I discuss the context of the study and the three phases of my experimental methodology: one-on-one preliminary task-based clinical interviews, teaching experiments, and post teaching experiment task-based clinical interviews. Then I describe the three phases of my analytical methodology: (1) preliminary analysis, (2) ongoing analysis of open and axial coding, and (3) retrospective analysis. As I describe my methodology I will focus on how these methods supported me in addressing my research questions.

Experimental Methodology

There are three components to this dissertation study; I wanted to understand (1) how students develop understandings about relationships between varying quantities', (2) what ways of thinking support/inhibit students from making these constructions, and (3) how students operationalize these newly constructed understandings in problem contexts that, from my perspective, ask students to model or represent relationships between varying quantities. To address these questions I had to construct a three-part model of each student's thinking where I sought to understand the student's mathematical reality before, during, and after instruction. More specifically, before I tried to support the student in developing new understandings, I needed to construct a model of his/her existing ways of thinking about graphical, tabular, and symbolic representations of relationships between varying quantities. This model enabled me to conjecture what ways of thinking support/inhibit students from developing new conceptions of covariational reasoning during instruction. After constructing this model I engaged the student in teaching sessions with the hope that the student would construct new understandings that enabled him/her to conceptualize graphs as emergent relationships. I used the student's mathematical activity during the teaching sessions to model the student's schemes for covariational reasoning focusing on covariational reasoning in the context of graphing. Additionally, in the context of teaching I was able to document how the student's schemes develop and change over the course of multiple teaching sessions. Finally, to gather empirical evidence for how students operationalize newly constructed ways of engaging in covariational reasoning across problem types, I modeled the space of implications for the understandings the student constructed in the teaching sessions.

I employed a three-phase methodology to construct this three-part model. In the first phase I conducted one-on-one task based clinical interviews with each participant to model each student's existing understandings of graphical, tabular, and symbolic representations of how quantities change together. In Phase II I conducted one-on-one teaching experiments to investigate how students construct ways of engaging in covariational reasoning. Finally, in Phase III I conducted a post teaching experiment task based clinical interview to study how students operationalize the understandings they constructed in the teaching experiment across problem types. In the following sections I describe the context of my study and the three phases of my experimental methodology.

Context of the Study

The participants of my study were three undergraduate students who were either enrolled in or had just completed a precalculus course at a large southwestern university. I selected this population because Steffe and Thompson (2000) recommended that a researcher should have "a history of interactions with students similar to the students involved in the teaching experiment" (p. 283). They go on to explain that this history of observing student and activity and participating in interactions with these students can give the researcher confidence that communication is being established between the student and the teacher/researcher. My three years of teaching precalculus and four years working with university precalculus instructors gave me an opportunity to frequently interact with this population and think deeply about the mathematical activity of university precalculus students.

Recruitment and Selection

In May 2016 I recruited ten students who were either currently enrolled in summer semester precalculus (n=7) or had just completed spring semester precalculus (n=3)[1]. The sample consisted of 3 females and 7 males with 4 declared liberal arts majors and 6 declared STEM majors. All ten students completed precalculus in high school, two previously took precalculus in college, and two took calculus in high school (see Table

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[1] I was never an instructor for any of these participants.

5). All participants were given financial compensation and students who were currently enrolled in precalculus also received extra credit towards their homework grade. Each student participated in a four question clinical interview to support me in understanding his/her graphing scheme[2].

Table 5

Description of Recruited Participants

Student	*Gender*	*Major*	*Highest math course taken at time of recruitment interview*
AV	Male	Arts	Calculus in High School
Ali	Female	Arts	Precalculus in High School
TB	Female	Arts	Precalculus in High School
Sue	Female	STEM	Precalculus at University
GR	Male	Arts	Precalculus at University
Bryan	Male	STEM	Precalculus in High School
JG	Male	STEM	Calculus in High School
SR	Male	STEM	Precalculus in High School
NP	Male	STEM	Precalculus is High School
MA	Male	STEM	Calculus in High School

As I analyzed the students' mathematical activity I identified six meanings for graphs (Table 6). While two students demonstrated only one meaning for graphs, eight of the students demonstrated more than one meaning for graphs over the course of the recruitment clinical interview (see Figure 12).

[2] See Appendix A for protocol for recruitment interview

Table 6

Six Meanings for Graphs

Meaning	***Description***
Graphs as pictures	A graph shows the motion of an object over the course of an event. This is consistent with Monk's (1992) notion of *iconic translations*.
Graphs as shapes	A graph is a shape. Once one has the shape she can use it to reason about features of the shape. This is consistent with Moore and Thompson's (2015) notion of *static shape thinking.*
Graphs of one quantity	A graph tracks one quantity's smooth variation in the context of one's image of a changing phenomenon.
Graphs have a few points	The student coordinates two images of changing quantities to conceptualize graphs as a way to show how two quantities change between a few key points.
Graphs of infinite points	The graph is an infinite collection of isolated points such that each conveys a pair of measures at a given moment in time.
Graphs as product of emergent trace	The graph is the result of simultaneously tracking two quantities' magnitudes as one imagines the event occurring.

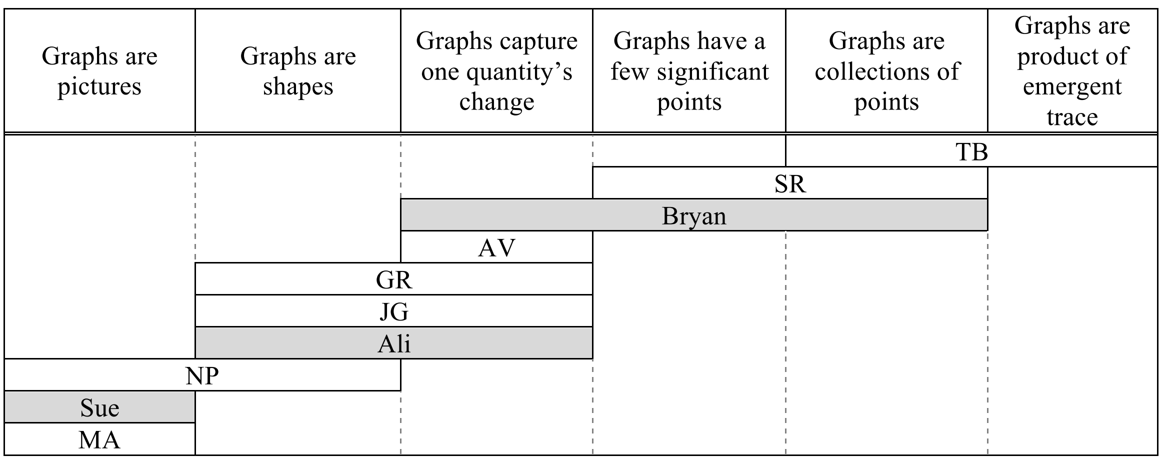


Figure 12: Characterization of meanings recruited participants represented in their graphing activity. Highlighted students were selected to participate in entirety of study.

I used my characterization of each student's meaning for graphs to select three students to participate in the entirety of the study. I decided to select students that demonstrated a variety of meanings for graphs. Thus I selected students who demonstrated a variety of meanings in their own reasoning and I selected students in so that the three students I selected demonstrated five of the six meanings for graphs I identified. I anticipated that selecting students that demonstrated a variety of meanings would support me in addressing my research questions and understanding which meanings and ways of thinking support and inhibit students from engaging in emergent shape thinking.

I selected Sue to understand how one's scheme for quantitative reasoning informs her scheme for covariational reasoning. While both MA and Sue demonstrated thinking about graphs as images of an object's motion, MA was not fluent in the English language. I selected Ali to understand how students who conceptualize graphs as static shapes might come to reason about graphs as emergent representations. I selected Ali over JG and GR because Ali was very reflective about her mathematical activity during the recruitment interview. Finally, I selected Bryan because he was the only student to demonstrate three different meanings for graphs. I expected his engagement in the teaching experiment would provide insights into the ways he coordinated these meanings.

After selecting the students to participate in the teaching experiment, I determined the order in which to conduct the asynchronous teaching experiments. I decided to start with the student with the least propitious meaning for graphs – graphs are pictures – because I thought that understanding what makes emergent shape thinking difficult might support me in seeing how students with more robust reasoning were able to be successful. Thus I conducted the first teaching experiment with Sue in June 2016, the second teaching experiment with Ali in July 2016, and the third teaching experiment with Bryan in August 2016.

In Chapters 6, 7, and 8 I describe each student's engagement in a teaching experiment.

Logistics and Procedures

After selecting the three participants for the study, I engaged each student in one-on-one clinical interviews and teaching sessions. Since covariational reasoning is highly imagistic, I conducted one-on-one interviews and teaching episodes so that I had the opportunity to understand the nuances of each student's imagery.

I conducted all interviews and teaching sessions in a conference room with a lap-top computer. Each session was videotaped to capture the interaction between the student and the interviewer including all gestures. I also used Quicktime® to create a screen capture to record the computer animations and to sync the animations with the participant's voice. Since all recordings included the audio of the interview I used Studiocode® to merge and sync the two video files.

In addition to the video-recorded clinical interviews and teaching sessions, I collected and scanned all of the students' written work. I made sure that the student and I wrote in different color pens to ensure that I could easily differentiate what the student wrote from what I wrote.

Phase I: Pre-Teaching Experiment Clinical Interview

The first phase of my study was a one-on-one task based clinical interview (Clement, 2000; Hunting, 1997) with each student. As Clement (2000) described, a clinical interview allows the researcher to "collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures in the subject's thinking that could not be detected by less open-ended techniques" (p. 341). Thus, at the core of a clinical interview is the researcher's goal to model an individual's hidden mental processes and their organization – his schemes. As Clement explained, a researcher can construct these models of an individual's conceptual understanding by attending to the individual's oral and graphical explanations and asking clarifying questions when necessary.

In this study, I conducted clinical interviews in order to construct models of each student's ways of thinking about graphs, tables, and formulas prior to my teaching experiment and to understand the extent to which the student conceptualized these as representations of how quantities vary together. Additionally, these interviews provided an opportunity for me to model the student's existing ways of engaging in covariational reasoning. The models I constructed were not exact replicas of the students' thinking. Instead, as Clement (2000) described, my goal in modeling was to develop viable characterizations of the mental actions each student engaged in that would account for the student's language and observable actions.

In these interviews I asked students to respond to a set of predetermined questions (see Appendix B). From my perspective, these questions all necessitated the student reason about how two quantities change together. The questions were of varied problem types (e.g., tables, graphs, formulas, word problems) and had different levels of instructional support. Where possible, I selected tasks that are documented in the literature. This allowed me to situate my findings in a larger body of research and understand how the ways of thinking my participants demonstrated were similar to/different from other students at the same/different academic level.

To ensure that my interview methodology did not influence each student's mathematical activity in different ways, I followed the same protocol for each student. I only deviated from the protocol in order to probe the students until I felt confident that I generated enough data to support me in modeling the student's mathematical understandings. Since I tried to understand each student's existing ways of thinking, I was cautious not to ask leading questions or offer guidance during these clinical interviews.

The model I constructed during this phase of the interview was useful in my analysis for two reasons. First, by looking for relationships in how I made sense of the student's thinking when constructing this model and the model I constructed in the teaching experiment I was able to hypothesize what ways of thinking might support/inhibit students from engaging in the teaching experiment. Second, by looking for relationships in how I made sense of the student's thinking when constructing this model and the one I constructed after the teaching experiment, I was able to better understand the schemes the students constructed during the teaching experiment and how the student operationalized those schemes in the post-teaching experiment clinical interview.

Phase II: The Teaching Experiment

In Phase II I conducted one-on-one teaching experiments to study how undergraduate students construct meanings for graphs that enable them to engage in emergent shape thinking, a robust form of covariational reasoning. The underlying hypothesis of any teaching experiment is that when one teaches another individual, the teacher has the opportunity to learn about the student's current state of knowledge and establish bounds on the student's thinking. The teacher/researcher can then use the knowledge she constructs from her teaching practices to inform her theory of knowledge development. This implies that theory can emerge from teaching episodes while also informing pedagogical changes.

While education researchers agree that the teaching experiment methodology should support a reflexive relationship between teaching and researching, researchers adopt different theoretical perspectives and thus conceptualize different methods for achieving this reflexive relationship. For example, Cobb (2000) adopted an emergent theoretical perspective and claimed that a student's cognitive activity is situated in the classroom in which he/she participates. Cobb argued that the classroom culture influences students' beliefs about what it means to learn and understand mathematics. Thus, for Cobb, both the classroom and classroom teacher are essential aspects of a teaching experiment. With this perspective, Cobb conceptualized teaching experiments in which the researchers attend to both the student's individual cognitive activity as well as how the student develops new understandings in the context of classroom activities.

Steffe and Thompson (2000) situated their conception of a teaching experiment within radical constructivism and adopted the belief that mathematics is a product of human intelligence. Thus, for Steffe and Thompson, when a researcher implements a teaching experiment the researcher should focus on understanding the student's mathematics by constructing models of each student's mathematical reality – realities distinct from one another and distinct from the researcher's mathematical reality. In order to develop these models the researcher acts as the teaching agent and works with a small group of students in order to understand each student's mathematical meanings and the progress this student makes in coming to understand a mathematical idea over a period of time. While Steffe and Thompson acknowledge construction is often the result of an interaction, they believe that, ultimately, the construction is an individual accomplishment. The “same” interaction with different students will lead to different student constructions.

Lesh and Kelly (2000) described a different approach to the teaching experiment methodology. Instead of situating their teaching experiment within a specific constructivist perspective, as both Cobb and Steffe and Thompson did, Lesh and Kelly developed a teaching experiment methodology that spans multiple research groups and thus multiple constructivist perspectives. In this type of multi-tiered teaching experiment a teaching experiment for students is used as the context for a teaching experiment with teachers, which is then used as the context for a teaching experiment with researchers. Lesh and Kelly (2000) explained that their conception of a teaching experiment “focus[es] on the nature of developing *ideas* … regardless of whether the relevant development occurs in individuals or groups” (p. 200). By focusing on development of ideas, regardless of individually or in groups, Lesh and Kelly argued that the research program can simultaneously address issues related to what it means for a student to have a deep understanding, what kinds of activities are most productive for developing such an understanding, and how information about the ways students develop this understanding can be used to influence how teachers teach.

Each of these conceptions of the teaching experiment methodology enables the researcher to model student’s mathematical understandings and how the student develops these understandings. However, the nuances of each methodology are influenced by the researcher’s theoretical perspective. This means that one’s theoretical perspective not only influences the nature of her research question but also impacts her choice of experimental methodology. By adopting a theoretical perspective grounded in Piaget's genetic epistemology I am focused on modeling student's schemes and mathematical realities. As a result, I must implement an experimental methodology aligned with my research goals and my assumptions about how students construct knowledge. Thus, a teaching experiment in the sense of Steffe and Thompson (2000) was best suited for my experimental methodology. In the remainder of this section I will describe the details of Steffe and Thompson's teaching experiment methodology. I will focus on how this methodology supported me in both modeling students' schemes and understanding the mechanisms through which an individual constructs new schemes.

Details of Steffe and Thompson's teaching experiment methodology. The teaching experiment in this study was based on the teaching experiment methodology of Steffe and Thompson (2000). Before describing the details of this methodology, it is necessary to first understand Steffe and Thompson's theoretical assumptions. The authors adopt a radical constructivist perspective and emphasize that the student's mathematical reality is distinct from the researcher's mathematical reality. Additionally, the student's mathematical reality is fundamentally unknowable to the researcher. This introduces a theoretical question – if the researcher's goal is to understand a student's mathematical reality, then how can the researcher understand something that is fundamentally unknowable? Steffe and Thompson claim that a researcher can understand the student's mathematical reality by constructing models of the student's reality. They use the phrases *student's mathematics* and *mathematics of students* to differentiate between the student's mathematical reality and the researcher's interpretation of the student's mathematical reality, respectively.

It is imperative to recognize that when engaging in a teaching experiment the researcher is always constructing models of the student's mathematical realities. These models are not direct representations of the student's mathematics; instead they are characterizations of ways of thinking that if the student were to possess them would account for his observable actions including his gestures, written responses, and oral explanations. Thus, the goal of the teaching experiment in this study was to construct models of the student's mathematics that were consistent with his/her behaviors. Note that since the researcher constructs models from the student's observable actions, the researcher's models are constrained by the student's language and behavior during the teaching sessions. As a result, during a teaching session, the teacher/researcher must do everything in her power to understand the student's thinking.

In addition to modeling the student's current mathematical reality, in a teaching experiment the researcher also works to understand how the student's mathematical reality changes and how the student develops new ways of thinking. This is what differentiates a teaching experiment from Piaget's clinical interviews (described in Clement, 2000; Hunting, 1997). While the intent of a clinical interview is to understand the student's current knowledge, the teaching experiment provides an opportunity for the researcher to act as a teacher and investigate how students modify their existing schemes over the course of multiple teaching episodes. More specifically, as Steffe and Thompson (2000) explained, "the interest [of teaching experiments] is in understanding the students' assimilating schemes and how these schemes might change as a result of their mathematical activity" (p. 288). In the context of this study, I investigated how students' conceptualizations of covarying quantities changed over the course of numerous teaching sessions.

In order to study how a student's schemes change as a result of his mathematical activity the researcher must try and teach the student while simultaneously constructing and testing hypotheses about the nature of the student's mathematical reality. According to Steffe and Thompson (2000) this requires a teaching agent, students to participate in the teaching episodes, a witness to the teaching episodes, and a record of what transpired in each teaching episode. The teaching agent is typically the researcher. This enables the teacher/researcher to simultaneously act as both a teacher and a researcher so that she can construct and test hypotheses during a single teaching episode. I served as the teacher/researcher in the teaching experiments for this dissertation study.

In addition to documenting the activity of the teaching episodes (i.e., the student and teacher's written work, conversations, and gestures), as the teacher/researcher I must also document my hypotheses about what is involved in developing the targeted mathematical understanding, a robust conception of covariational reasoning grounded in images of variation, invariant relationships, and multiplicative objects. My pre-dissertation study conceptions are documented in a hypothetical learning trajectory (Simon & Tzur, 2004), described in Chapter 5. In addition to documenting my initial hypothesis about the mental constructions necessary to engage in covariational reasoning, I kept a journal where I documented how my hypothesis changed over the course of this dissertation study.

After I documented my initial hypotheses about the mental actions necessary to engage in emergent shape thinking, I designed tasks that I anticipated would support a student in making these constructions. My initial task designs are described in Chapter 5.

It is important to note that per Steffe and Thompson's (2000) recommendation, I designed tasks with the anticipation that students would experience difficulty answering the questions. As Steffe and Thompson described, this is a good thing; if I appropriately designed the task then the nature of the student's difficulty would reveal something about how the student is thinking about the problem. When I conjectured that the student exhausted his current ways of thinking then I intervened in ways that I hoped would either (1) enable me to better understand the nature of the student's difficulty, or (2) determine if the student could overcome that difficulty. If the student was able to overcome the difficulty, then I had new information about the student's understandings before and after the task and how the student modified his scheme as a result of the teaching activity. If the student did not overcome the difficulty then I had to consider whether this difficulty was an essential mistake, a nonproductive way of thinking that persisted despite my best efforts to eliminate it (Steffe & Thompson, 2000, p. 277). If, on the other hand, the student did not experience difficulty solving the tasks then I had to make a conscious effort not to impose my thinking on the student. Instead I took advantage of all opportunities to ask the student probing questions to reveal the nuances in the student's way of thinking.

Steffe and Thompson (2000) explained that even though the researcher establishes major hypotheses at the beginning of the experiment, the researcher must do his/her best to forget about these hypotheses during the experiment and instead focus on the student's mathematical activity and what actually happens in the teaching episodes. As Steffe and Thompson warn, it is possible that students will engage in the tasks in such unexpected ways that as the researcher, I must abandon my main hypothesis and focus on modeling the student's mathematics. When this happened I had to develop sub-hypotheses that explained the student's language and actions including explanations for the student's difficulties and successes.

As the teacher/researcher, I developed these sub-hypotheses both during the teaching session and between sessions. In between each teaching session I engaged in coding by reviewing records from previous teaching sessions and developing tentative models of the student's mathematics. I met with a witness – Patrick Thompson – to discuss these tentative models. As Steffe and Thompson (2000) explained, having a witness is an essential part of the teaching experiment because the witness provides an outside perspective to the teaching episodes. Serving as both a teacher and a researcher means that in order to model the interaction between the teacher and student I needed to step outside of the interaction in the teaching episode, reflect on this interaction, and then act based on those reflections. In the context of teaching, this reflection occurs in real time. As Steffe and Thompson described, the witness is always outside of the interaction and thus as he observes the interaction between the teacher/researcher and student, the witness might be able to observe aspects of the interaction that I missed acting as both teacher and researcher.

Additionally, since the witness has different mathematical reality, his assimilations of the student's mathematical activity may result in alternative models of the student's mathematics or provide a way of thinking about the student's behaviors that force me to challenge my tentative models. In order for the witness to be able to construct alternative models to the student's mathematics, the witness must have his/her own understanding of what is involved in engaging in covariational reasoning as well as what is involved in modeling student's mathematics. Thus, the choice of a witness is essential for the success of a teaching experiment. Patrick Thompson served as the witness for all three of my teaching experiments. His experience thinking about the mechanisms of covariational reasoning and modeling students' mathematics provided an invaluable perspective while conducting this study and engaging in analysis.

While the teaching experiment methodology provides an opportunity for the student to develop new ways of thinking, this is not the main goal of a teaching experiment. Instead, the purpose of a teaching experiment is to develop nuanced models of the student's mathematics including models about how the student's mathematics might change as a result of the student's mathematical activity and to test the viability of these models. Thus as the teacher/researcher, I must constantly balance my efforts to support the student in developing new ways of thinking with my efforts to model the student's mathematics in the moment.

This balance was particularly important in my dissertation study because my research questions are highly aligned with instructional goals: I wanted to understand how students develop ways of engaging in covariational reasoning and I wanted to understand how students operationalize these ways of thinking. In order to understand how students operationalize these ways of thinking I need to support students in engaging in covariational reasoning. However, my findings will only be meaningful if I am able to understand how the student was able to construct and then operationalize these ways of thinking.

To understand how the student constructed her understandings I must model the interaction between the student and myself, the teacher, during the teaching sessions. The purpose of studying this interaction is not to focus on learning as a social construction. Instead, I studied this interaction in order to understand the experiences that caused both the student and myself, as the teacher, to engage in the interaction in new ways. As Thompson (2013) explained, when two people attempt to engage in a meaningful conversation, the conversation can reach a level of intersubjectivity where each participant has no reason to believe he/she was misunderstood. For the conversation to be in a state of intersubjectivity, both participants must speak and listen with the intent of trying to understand the meanings the other person might have intended. This requires both people to continuously negotiate their understanding of the other person. As Thompson described, "The negotiations that happen involve each person monitoring the other's responses, comparing them to the responses anticipated, and then adjusting his model for the other to make better decisions about how to act and what to expect in the future" (*ibid*, p. 64).

This suggests that during the teaching experiment both the student and myself, as the teacher, continuously negotiate a model of the other person. For example, suppose I have some thought I want to convey to the student. I decide how to express this thought by considering how the student will assimilate what I say. I choose words/tasks/gestures that I anticipate will support the student in interpreting my action as I intend. As Thompson (2013) explained, my actions are towards my *image* of the student. I am purposefully trying to make sense of the student's actions by interpreting them through my model of the student's mathematics. Thus, I am acting upon what Steffe and Thompson (2000) called a *second-order model*. If the student is also trying to engage in a meaningful conversation, then the student will hear my words based on what she thinks I am trying to convey. She too has constructed a model of me, the teacher, which she acts with. However, she likely does not differentiate between her own schemes and the schemes she attributes to myself, the teacher. As a result, she is likely acting with what Steffe and Thompson (2000) called a *first-order model.*

As a researcher, I must look at this interaction as if I was an observer. In doing so, I differentiated between myself as the teacher in the interaction and myself as the researcher observing the interaction. This enabled me to construct a model of the interaction that includes (1) the student's model of the teacher and how this model changes over the course of the teaching sessions, (2) the model of the student I constructed and operated with during the context of the teaching session, and (3) the actions and mathematical experiences that caused each of these models to change. By focusing on the actions and experiences that cause these two models to change, I was able to hypothesize what actions caused the student and me, the teacher, to negotiate our meanings. This will allow me to conjecture what experiences supported/inhibited the student in constructing new understandings.

Phase III: A Post-Teaching Experiment Clinical Interview

After the teaching experiment I engaged each student in a final one-on-one clinical interview. The purpose of this interview was to study how students operationalize ways of engaging in covariational reasoning. In particular, I wanted to see if and how students operationalize their thinking about smooth variation, multiplicative objects, and invariant relationships in non-graphing situations. I hypothesized that if the student constructed emergent shape thinking as a scheme of operations, then the student could generalize the idea of constructing representations that coordinate static conceptions of situations and dynamic images of change to other representations, such as formulas and tables. In order to compare the student's thinking pre and post teaching experiment, this interview followed the exact same structure as the clinical interview in Phase I; I used the same tasks and the same protocol.

These post-teaching experiment clinical interviews concluded my experimental methodology.

Analytical Methodology

The purpose of an analytical methodology was to build and test models of students' mathematics. My analytical methodology is based in grounded theory where the researcher uses his/her understanding of the student's mathematical activity to support and refute his/her conjectures about the schemes and understandings the student constructs.

Preliminary Analysis

The first phase of analysis occurred during and immediately after each interview session. I kept a written journal where I documented my hypotheses about the student's thinking as well as moments in the interviews that did not fit with my working model of the student's thinking. In order to accommodate these moments, I made modifications to my tentative model of the student's thinking. I documented these modifications in this journal. Finally, I recorded any on the spot teaching decisions – including modifications to tasks – as well as a justification for why I chose to make that teaching move at the moment I did. These notes allowed me to retrospectively study and analyze my thinking throughout the study. Additionally, these notes documented how I constructed an initial model of each student's mathematical understandings. Writing these journal entries constituted my *preliminary analysis*.

Ongoing Analysis

Throughout the interview process and teaching sessions I engaged in open and axial coding to construct a grounded theory (Strauss & Corbin, 1990). As Strauss and Corbin (1990) explained, a grounded theory is "discovered, developed, and provisionally verified through systematic data collection and analysis of data pertaining to that phenomenon. Therefore, data collection, analysis, and theory stand in a reciprocal relationship with each other" (p. 23). The grounded theories I built during ongoing analysis were models of my research participants' thinking.

Researchers construct grounded theories through coding where "coding represents the operations by which data are broken down, conceptualized, and put back together in new ways" (Strauss & Corbin, 1990, p. 57). Engaging in coding allows the researcher to better understand what is happening in her data and to make sense of what it could mean (Charmaz, 2006, p. 46). This necessitates the researcher consider the participant's language and behaviors as problematic so there is something for the researcher to analyze. Otherwise, the researcher might unconsciously apply her own meanings to the participant's actions. As the researcher defines and revises her codes she is actively trying to understand the participant's views, actions, and tacit meanings.

In the context of this study this meant that both during the interviews and during the coding process I had to ask a lot of questions about the events in order to better understand the student's experience and not impose my own thinking on the data. For example, when a student used terms/phrases such as graph, function, point, line, variable, change, formula, etc., I had to be cautious not to impose my own meaning for these words on the student's thinking, but instead try to understand what the student meant by using that word. I anticipated that often the student's meanings would not be aligned with my own meaning for the same term. By being sensitive to my own word choice and the student's word choice, I was mindful to ask the participants to explain their meanings throughout the interview process.

Rigorous coding is critical in qualitative data analysis because it is through a researcher's coding activity that she constructs data from her video records and journal entries. Events, such as interviews and teaching sessions, cannot be considered data until the researcher has imposed some type of interpretation on the event; this suggests that videos and journal entries are not data until someone has made sense of them. Thus, the activity of coding produces data to analyze. In the following sections I describe open and axial coding (Strauss & Corbin, 1990), the analytical methodology I used to code my video records and journal entries in order to produce data and a grounded theory.

Open coding. I conducted at least three clinical interviews and three teaching sessions with each research participant. Between each session I engaged in open coding. Strauss and Corbin (1990) define open coding to be "the process of breaking down, examining, comparing, conceptualizing, and categorizing data" (p. 61). The first step in open coding is to impose meaning on participants' activity. One can do this by breaking the phenomena into short exchanges and events and creating labels for each phenomenon. It is important to emphasize that when a researcher applies a code to phenomena, she is doing more than applying a word. Instead, a code represents a meaning the researcher is ascribing to the phenomena. Thus, it was essential that as I engaged in coding, that I also documented the meanings I was conveying with my codes and how that meaning changed as I continued to engage in coding. Figure 13 represents the product of this initial coding: a description of the event and an associated label. In addition to creating a written description of the events and associated label, I also used Studiocode® to label episodes in the video record so that I could easily return to that part of the video during future analysis.

Description of Event	***Label***
I presented Sue with a table of values and asked her to graph the girl's distance from home in terms of elapsed time. Sue drew a house at the origin and explained her graph in terms of the girl getting closer and further from her house, which Sue imagined as the horizontal axis.	Graph as diagram

Meaning of Codes	
Graph as diagram	Student attends to graph as picture of the phenomenon. This might involve drawing objects from the phenomenon along the axes or imagining the curve as a picture of everywhere the object had been.

Figure 13: Example of initial open coding where I assigned labels to phenomena and documented meanings for these labels.

To the best of my ability, I tried to let the labels I used emerge from my data, but my understanding of the literature influenced the labels that I used. For example, labeling phenomena with codes aligned with Thompson and Carlson's (2017) levels of covariational reasoning and levels of variational reasoning were useful in making sense of the students' mathematical activity. I used a single Studiocode code file in order to keep track of the labels I created while coding and the meanings I ascribed to these codes so that I could label similar events with the same name. (See Figure 14 for final code window). After reviewing each interview session, I reviewed all of the interview sessions again using my final code file. This enabled me to apply codes that I created in later videos to events in earlier interview sessions.

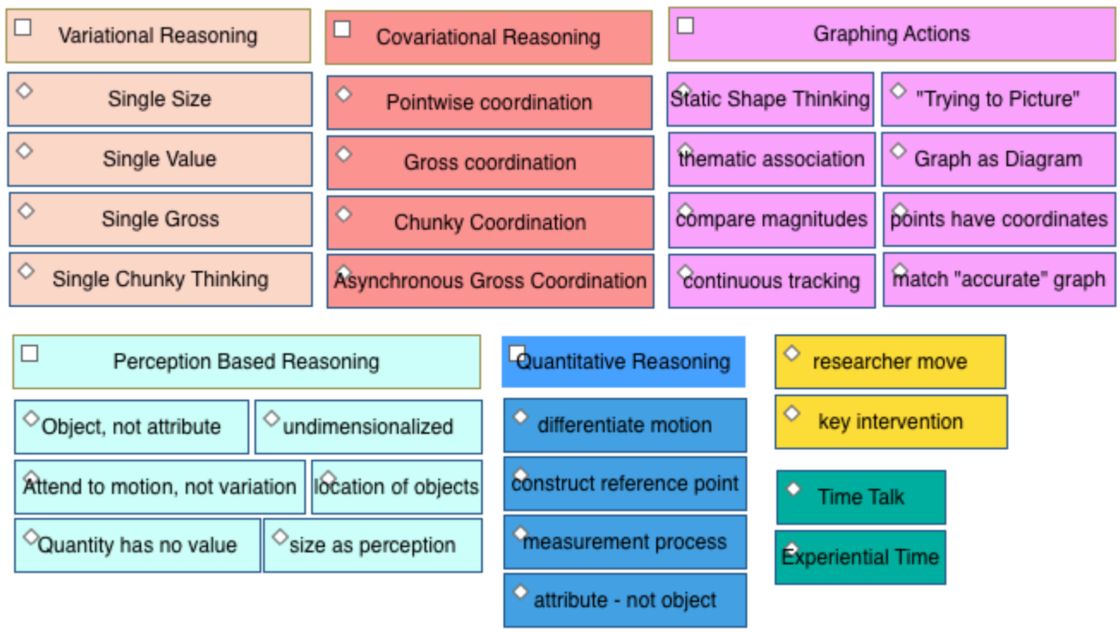


Figure 14: Final code window

Axial coding. After constructing categories through open coding, I began to develop relationships between these categories. This is what Strauss and Corbin (1990) call axial coding (p. 97). They explain that the purpose of axial coding is to put back together the data that one broke apart during open coding. While the authors describe axial coding after open coding, they emphasize that there is not a fine line between these types of coding and researchers might go back and forth between open and axial coding without realizing it.

It is through axial coding that one formalizes and tests his/her models of student's mathematics. One can do this by constructing what Strauss and Corbin (1990) called the *paradigm model.* One constructs a paradigm model by specifying and relating six components; causal conditions, phenomenon, context, intervening conditions, action/interaction strategies, and consequences. Although Strauss and Corbin do not describe the paradigm model in terms of modeling student's mathematics, I believe these six aspects are aligned with aspects of models of student's thinking. Thus, I will interpret each aspect of Strauss and Corbin's paradigm model in terms of modeling student's mathematics.

The first aspect of the paradigm model is *causal conditions*: the "events, incidents, happenings that lead to the occurrence or development of a phenomenon" (p. 96). In the context of modeling student's mathematics, I interpret the causal conditions as the student's existing understandings that he/she brings to bear in the moment of assimilating a task to his/her scheme(s). This also includes my actions as the researcher, such as the way I verbalized a task or how I presented an animation, which might cause the student to assimilate my actions in such a way that she makes a modification, at least momentary, to his existing scheme(s). Thus, my actions might influence the way the student assimilates the task to his/her schemes.

The second aspect, *phenomenon*, is "the central idea, event, happening, incident about which a set of actions or interactions are directed at managing, handling, or to which the set of actions is related" (ibid). In this study, the phenomenon is the way in which the student engages in a particular task.

The third aspect, *context*, is "the specific set of properties that pertain to a phenomenon" (ibid). When studying student thinking, the context includes more than the task as it is written on the paper. The context also includes the student's interpretation of the task, the researcher's questioning, and the student's understanding of the researcher's questions.

The fourth aspect, *intervening conditions*, is "the structural conditions bearing on action/interactional strategies that pertain to a phenomenon. They facilitate or constrain the strategies taken with a specific context" (*ibid*). In the context of this study, I interpret the intervening conditions as the student's existing schemes that are brought to bear by the student's assimilation of the problem/task. While the causal conditions are specific understandings brought to bear by the phenomenon, the intervening conditions are the space of implications of those understandings.

The fifth aspect, *action/interaction strategies*, is the "strategies devised to manage, handle, carry out, respond to a phenomenon under a specific set of perceived conditions" (Strauss & Corbin, 1990, p. 97). In this study the action/interaction strategies are the understandings the student constructs in the moment of acting – the student's functional accommodations.

The final aspect of the model, *consequences*, is the "outcomes or results of action and interaction" (ibid). If one interprets the action and interaction as an understanding, than Harel and Thompson's (in Thompson et al., 2014) conception of meaning and understanding suggests that the consequences are the meanings – the space of implications from the newly constructed understanding. Thus, as Strauss and Corbin (1990) suggested, the consequences of one phenomenon might become the conditions of another (p. 106). In other words, the meanings a student constructs through one experience might be brought to bear as understandings in another experience.

Strauss and Corbin (1990) explained that in order for a researcher to build relationships between the categories established in open coding, thus constructing a tentative model of the student's mathematics, the researcher must constantly move between inductive and deductive thinking. This means the researcher must continuously engage in proposing and checking his/her model against the data. This is the central component of grounding one's model in his/her data.

During axial coding I worked to build a 3-part model of each student's thinking. Each model included a characterization of the student's thinking pre-teaching experiment, during teaching experiment, and post-teaching experiment. I engaged in multiple passes of axial coding both during the interview process and after I completed data collection in order to build a viable model of each student's thinking. These models are presented in Chapters 6, 7, and 8.

Retrospective Analysis

After I finished data collection, I engaged in *retrospective analysis* to formalize a model of each student's thinking and also to identify relationships within the models I constructed for an individual student and across models of multiple students.

Although constructing viable models of each student's thinking is an essential part of this dissertation study, these models alone do not answer my research questions. I still needed to determine what ways of thinking support/inhibit students from engaging in covariational reasoning and to understand how students operationalize ways of engaging in covariational reasoning. This required I abstract relationships within an individual student's thinking and across multiple students' thinking. To determine these relationships I engaged in open, axial, and selective coding again. However, this time I used my activity of constructing a model of each student's thinking as my data source.

After constructing a 3-part model of each student's thinking, I looked for relationships in how I made sense of the student's thinking when constructing each part of the model. Specifically, to study the schemes the student constructed over the course of the teaching session I looked for patterns in the ways of thinking I attributed to the student throughout all three phases of the experimental methodology.

Additionally, to investigate what ways of thinking supported/inhibited the student from constructing new understandings, I looked for relationships between the ways of thinking the student demonstrated in the initial clinical interview and the student's thinking throughout the teaching experiment. I worked to discern ways of thinking that I attributed to the student in the initial clinical interview that influenced the ways she engaged in an intersubjective conversation during the teaching sessions. In doing so, I established ways of thinking that influenced the student's assimilations and supported/inhibited the student from making new constructions

Finally, to study how students do/do not operationalize newly constructed understandings, I identified understandings the student demonstrated throughout the teaching experiment that he/she did or did not demonstrate in the post teaching experiment. Figure 15 depicts the relationships I sought to determine within each student's thinking.

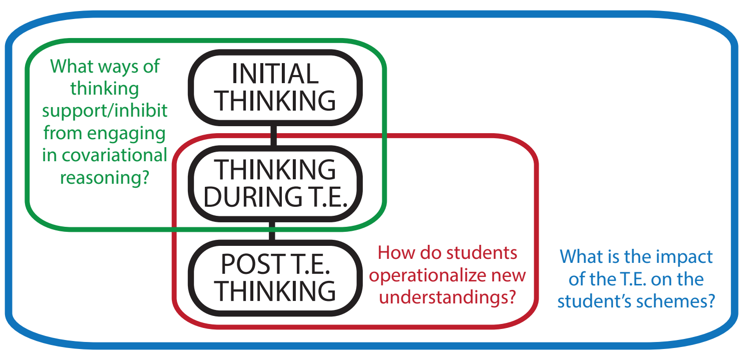


Figure 15: Constant Comparative Analysis - Within Each Student

After I constructed a more global understanding of each student's thinking, I looked across the models I created for each student and looked for patterns in how I made sense of each student's mathematical activity. More specifically, I looked for patterns in how I organized and related constructs to describe the students' thinking. For example, I found that when coding students' variational and covariational reasoning during their graphing activity I often applied two codes: one for the student's activity making the graph and one for the student's activity reasoning from his graph. By studying the ways I made sense of the students' thinking I was able to identify two distinct experiences students have in their graphing activity (discussed in detail in Chapter 9).

Finally, as the methodology above suggests, the success of this study did not dependent on the participants successfully constructing new understandings in the teaching experiment and then operationalizing these understandings in the post-teaching experiment clinical interview. The goal of this study was to understand how students construct and operationalize understandings of covariation in order to refine my hypothesis of what it means to engage in covariational reasoning and how students come to engage in covariational reasoning in the context of graphing.

CHAPTER 5
HYPOTHETICAL LEARNING TRAJECTORY

One purpose of this study was to examine how students come to engage in covariational reasoning and more specifically how they come to imagine smooth continuous variation, construct multiplicative objects, and conceptualize invariant relationships between varying quantities' measures. I engaged each participant in a teaching experiment to support him/her in making these constructions. As Steffe and Thompson (2000) explained, while the researcher must continuously revise his/her hypothesis of how students construct their mathematics, the researcher must begin a teaching experiment with a clear hypothesis to test (p. 275). To document my initial hypothesis I developed a hypothetical learning trajectory.

According to Simon (1995), a hypothetical learning trajectory (HLT) is a theoretical model that instructors, researchers, and curriculum developers create when designing mathematics instruction based in constructivist principles. An HLT consists of three components: a teacher's learning goal for students, a set of tasks to support students in achieving the learning goal, and a hypothesis about how students will come to achieve the learning goal by completing these tasks (Simon, 1995; Simon & Tzur, 2004). These components suggest that although each student's learning progression is unique, there are similarities between students' learning progressions and a single task can simultaneously benefit many students.

While each component of an HLT is independent, the set of tasks and the teacher's hypothesis about how students learn must co-emerge. As Simon (2014) explained, "The trajectory of students' learning is *not* independent of the instructional intervention used. Students' learning is significantly affected by the opportunities and constraints that are provided by the structure and content of the mathematics lessons" (p. 273). I interpret this to mean the teacher designs learning activities based on his/her hypothesized learning progression and a student's learning progression is dependent upon the activities in which the student engages.

Since the construct of an HLT is based in constructivist principles, the teacher's focus must remain on the student's understanding and activity. When identifying learning goals, the teacher needs to take into consideration the students' existing meanings and understandings as well as the students' constructions throughout the lesson. As Simon (1995) described, the only thing predictable about classroom instruction is that nothing will go as planned (p. 133). Thus, the teacher should use his/her understanding of classroom activities and his/her interactions with the students to constantly modify the HLT: including his/her learning goals, the hypothesized learning progression, and the task sequence to best support students' developing understandings.

The teacher/researcher must document any modifications she makes to the HLT and justifications for these modifications. This documentation can be used as a source of data at the end of the teaching interviews in order to see how the researcher's thinking changed over the course of the teaching sessions. In the rest of this section I will describe the hypothetical learning trajectory I designed at the beginning of this dissertation study in order to support students in developing robust ways of engaging in covariational reasoning.

Learning Goals

Before I could design tasks to support students in constructing new understandings, I had to first outline the understandings I wanted my students to develop. For this study I wanted my students to conceptualize graphs as representations of emergent relationships. As Moore and Thompson (2015) described, this involves "understanding a graph simultaneously as what is made (trace) and how it is made (covariation)" (p. 4). I conjectured that this would require the student conceptualize a point as a way to simultaneously represent the measures of two quantities. I thought one could do this by imagining two quantities' measures represented along the axes, extending these measures into the plane so that the intersection of these measures is a point that simultaneously represents the measures of the two quantities (Figure 16). Conceptualized this way, the intersection point in the Cartesian coordinate system would represent a static conception of a situation where a single measure of x is associated with a specific measure of y. If the student then introduced her conceptualization of variation to the situation, she could imagine the measure of x varying. Since the measures of x and y are united through the point, the multiplicative object, as the student imagined the measure of x varying she would remain aware that y also has a measure. By representing the relationship between a single x and its associated y through the intersection point, the student could imagine tracing the point to capture how x and y change together.

For a student to make these constructions, I anticipated the student would need to first conceptualize the smooth continuous variation of two quantities' measures, construct a multiplicative object to unite these measures, and then imagine this relationship as invariant so that the student could conceptualize the graph as a representation of this invariant relationship.

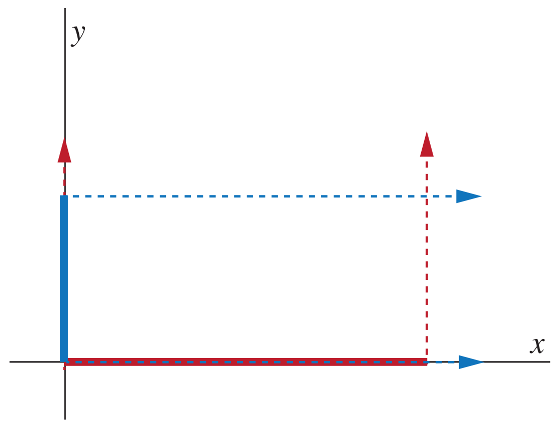


Figure 16: A point as the intersection of two quantities' values extended from the axes.

In order for the student to conceptualize two quantities' smooth continuous variation, he must first conceptualize two quantities in the situation. As Thompson (1994a, 2011b) described, to conceptualize a quantity one must construct an attribute of a situation that he understands has a measureable magnitude. While conceptualizing a quantity involves imaging a measurement process, one does not need to enact this measurement process. Thus, one can differentiate between the value of a quantity's measure – the result of the measurement process, and the magnitude of its measure – a general quantitative sense of the size of the measure[3]. In the context of covariational reasoning, this is a critical differentiation for students to make because researchers have documented that one can reason covariationally by coordinating magnitudes in flux with the anticipation that these magnitudes have specific measures (e.g., Moore et al., 2016; Saldanha & Thompson, 1998). As a result, for a student to engage in a robust form of covariational reasoning she must be able to differentiate between the value of a quantity's measure and the magnitude of its measure.

[3] See Thompson et al. (2014) for a description of different schemes individuals can hold for magnitude.

Once the student has conceptualized two quantities in the situation, the student can then imagine the quantities' measures varying continuously so that the student imagines change in progress and conceptualizes each quantities' measure sweeping over a continuum of values. To support students in engaging in smooth thinking I designed tasks where the relevant quantities' measures took on all real numbers. Additionally, I focused on continuous relationships so that the student could imagine continuous change in progress.

Given the prominence of smooth thinking in everyday life, one would expect students to be able to engage in smooth thinking in their mathematical activity. As Castillo-Garsow (2012) explained, students use smooth reasoning in their everyday life. For example,

> People learn in infancy that they cannot pass from point to another without passing through every point on the way, and similarly that one cannot simply leap into the future without passing through every moment of time between now and then. Continuous change in progress is part of our every day lives, and it is something that, on the sensorimotor level, students understand very well. Otherwise they would attempt to walk through walls much more often. (Castillo-Garsow, 2012, p. 68)

He goes on to explain that the problem is in school mathematics where students are encouraged to focus on discrete reasoning. For example, throughout elementary and secondary school we ask students to count, to solve equations, and to plot points. These activities do not require students to imagine change in progress but instead necessitate students imagine completed change. The types of problems that fill school mathematics are all based in chunky thinking. Thus, students experience cognitive dissonance between their daily experiences and the mathematics they experience in school. To support students in conceptualizing smooth continuous variation in the context of mathematics I built off of Castillo-Garsow's (2012) and Thompson and Carlson's (2017) recommendations and I designed tasks that (1) used dynamic animations throughout the teaching sessions, (2) I planned to ask students to reason about the magnitude of a quantity's measure, and (3) I planned to encourage students to use fictive motion to describe how quantities' measures vary.

To support students in reasoning about continuous variation I intended to ask students to engage with dynamic animations that represent continuously changing phenomena. As Castillo-Garsow (2012) recommended, modern technology allows educators to create representations that occur in parallel with a student's sense of experiential time which might help the student imagine continuous change of both measured time and other quantities that the student constructs from the situation. Additionally, computer animations can display numerical values that give the illusion of continuously changing numbers (p. 68).

Castillo-Garsow (2010, 2012) also explained that when students attend to specific numerical values they have stopped imagining change in progress. Thus, reasoning about smooth continuous variation does not involve reasoning about specific numerical values. Instead, smooth thinking involves reasoning about how a measure's magnitude varies with the anticipation that the measure takes on a continuously changing value. As a result, I designed tasks where the student attends to the variation of a quantity's magnitude as opposed to reasoning about the numerical values that measure takes on. I anticipated that students might be more likely to attend to *how* the quantity varies when they focused on the varying magnitude as opposed to focusing on the numerical values the quantity's measure can assume.

After conceptualizing two quantities' variation, I conjectured the student would need to coordinate how the two quantities vary together. According to Thompson and Carlson (2017) there are multiple ways for a student to coordinate two quantities' variation (see Table 3) where each level of coordination involves constructing a different multiplicative object. As Whitmire (2014) and Johnson (2013) documented, students often reason about how quantities vary by attending to change in one quantity as they witness it in their experiential time and then constructing a separate image of the other quantity's variation as they witness it in their experiential time. As a result, students asynchronously compare the changes in two quantities' measures. To support students in reasoning about how quantities' measures change together I anticipated the student needed to conceptualize an invariant quantitative relationship that constrains how the two quantities' measures vary together. If the student conceptualized this constraint, then as the student attended to changes in one quantity's measure he would be aware that the measure of the other quantity changed as well.

Additionally, to coordinate how two quantities change together, the student must also construct a multiplicative object that unites the two quantities' measures. As Thompson and Carlson (2017) explained, when engaging in covariational reasoning, students can construct a variety of multiplicative objects that relate quantities' (and changes in quantities') measures. Kevin Moore and I have documented that when reasoning about a contextual situation students are often able to engage in what Thompson and Carlson called a *gross coordination of values* (*ibid,* p. 23). For example, the student may be able to describe that as the measure of Quantity A increases the measure of Quantity B decreases then increases. These students often fail to represent their conceptualization graphically (Frank, 2016; Moore et al., 2016). Thus, to successfully support students in conceptualizing graphs as emergent representations I must support students in coordinating quantities' values beyond a *gross coordination of values*. The student must come to engage in what Thompson and Carlson (2017) called a *coordination of values*. This necessitates the student imagine the relationship between two quantities' measures at a given moment in his experiential time so that the student can construct a multiplicative object that unites these two quantities' measure. Finally, the student can coordinate his/her conception of the smooth variation of each quantity's measure in order to engage in what Thompson and Carlson called *smooth continuous covariation*.

The final phase of my learning progression focused explicitly on ways of thinking necessary to conceptualize graphs as emergent representations of continuously changing phenomena. I planned to support students in conceptualizing a graph as a way of representing the covariational reasoning they previously engaged in. In other words, I intended for the student to conceptualize a graph as more than a classroom activity but instead a response to the challenge of how one represents a dynamic relationship with a single and static representation. To do this, students would need to conceptualize representing the measures of two attributes along perpendicular axes, imagine extending these measures into the plane so that the intersection of these measures is a point in the plane, and then imagine keeping track of this intersection point as they witness it changing in their experiential time. In doing so, the student would have the opportunity to engage in emergent shape thinking.

I designed the instructional sequence below with the intent that these tasks would support students in making the constructions described above and thus would provide students with opportunities to engage in emergent shape thinking.

Instructional Sequence

My understanding of how students construct new understandings is rooted in Piaget's Theory of Reflective Abstraction. Thus, I designed this instructional sequence with the intent that the student's experience would engender reflecting abstractions. I implemented three design principles in order to support reflecting abstractions. First, I designed tasks that support students in differentiating and then coordinating their newly constructed understandings. I intended this to support the student in organizing his/her newly constructed understandings. Additionally, I included thought experiments throughout the tasks. Since imagination and imitation are acts of reflecting, tasks that encourage students to anticipate "What would happen if…?" likely support students in reflecting on their actions. von Glasersfeld (1995) called such tasks thought experiments and hypothesized "thought experiments constitute what is perhaps the most powerful learning procedure in the cognitive domain" (p. 69).

Part I: Conceptualizing and Coordinating Varying Quantities

In order to conceptualize the covariation of two quantities, the student must first construct quantities from a situation and attend to the measure of those varying quantities.

This was the purpose of the first task I designed (see Figure 17). I designed an animation depicting a plane flying from San Diego to Phoenix. I envisioned asking the student to use a modification of the finger tool to attend to the plane's distance above the ground. I anticipated that as the student focused on the distance between his pointer fingers the student would attend to the magnitude of the quantity, distance between plane and ground, as opposed to the location of the plane in the sky.

I planned to repeat this activity twice more. The second version introduced a helicopter into the animation where the helicopter was always directly below the plane; the distance between the plane and the helicopter varied. Again, I envisioned asking the student to use his pointer fingers to represent the distance between the plane and helicopter as they traveled from San Diego to Phoenix. Note that since the helicopter is always directly below the plane, the student would not need to construct this quantity as a difference. Instead, the student could attend to the perceptual quantity – the space between the planes. I anticipated that the student would want to move both hands as the aircraft travel but I hoped to encourage the student to keep one hand fixed so that he could attend to the varying magnitude of the quantity, distance between the plane and helicopter, as opposed to the location of both aircraft in the sky.

Finally, to determine how the student coordinates quantities, I designed a third version of this animation where the helicopter is no longer directly below the plane. Again, I envisioned asking the student to use the finger tool to attend to the vertical distance between the planes. This time the student would have to construct a new quantity that is the difference between two quantities' measures. The vertical distance between the planes could no longer be conceptualized from a perceptual quantity and thus to succeed in this task the student must begin to coordinate quantities' magnitudes.

A small plane got caught in a storm on its way from San Diego to Phoenix. To avoid the storm the pilot had to navigate the storm clouds and continuously change his elevation to avoid the storm.

a. What is changing as the plane flew from San Diego to Phoenix?
b. What is staying the same as the plane flew from San Diego to Phoenix?
c. I want you to focus on the plane's distance above the ground as it flew from San Diego to Phoenix.
    i. Using your pointer fingers (palms facing each other), I want you to move your hands so that the distance between your pointer fingers represents the plane's distance above the ground.
    ii. Why are you not moving your left hand? (ground/sea level does not change as the plane travels from San Diego to Phoenix)
d. (VERSION 2) A helicopter took off shortly after the small plane. Did the helicopter experience the same weather difficulties as the first? How do you know?
e. Use your hands so that the distance between your pointer fingers represents the distance between the plane and helicopter as they travel from San Diego to Phoenix.
f. (VERSION 3) Suppose the helicopter took off a few minutes after the plane. Use your hands so that the distance between your pointer fingers represents the distance between the plane and helicopter as they travel from San Diego to Phoenix.

Figure 17: Airplane problem: Task 1 – teaching experiment

After I have evidence that the student is able to construct quantities from his conceptualization of a situation, I planned to engage students in tasks I designed to support them in conceptualizing two quantities changing together such that their covariation is constrained by an invariant quantitative relationship. To do this, I included the box problem, a problem common in precalculus and calculus textbooks:

> Starting with an 11 inch x 13 inch sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

Students are often asked to write a formula that relates the volume of the box and the length of the side of the square cutout. As Moore and Carlson (2012) described, all nine of the precalculus students they interviewed had difficulty constructing an appropriate formula. The authors attributed this difficulty to the students' weak image of the situation and the relationship between the quantities in the situation. Thus, in my implementation of the box problem I envisioned focusing on students' attention to the quantitative relationship between the quantities' measures as opposed to the formula relating these measures.

I planned to present the student with a plan view of the unfolded box. I would ask the student to reason about why the cutout has to be square as well as conjecture what relationship exists between the length of the cutout and the length of the base of the box (see Figure 18). Ideally students would construct an invariant quantitative relationship where they conceptualized the original length of the paper being composed of two cutout lengths and the length of the base of the box. I intended to ask the student to anticipate what the largest cutout length can be and to anticipate how the length of the base of the box would change as the cutout length increases. I conjectured that when a student constructs a relationship between the length of the paper and the length of the base of the box, the student would either engage in pseudo-empirical abstractions by generalizing the pattern length of the base of the box is always the paper length minus two of the cutout lengths or the student would construct an invariant quantitative relationship where the length of the paper is necessarily made by combining 2 cutout lengths and the length of the base of the box. Thompson and Carlson's (2017) conception of covariational reasoning suggests that conceptualizing this invariant quantitative relationship is essential to engage in the most propitious forms of covariational reasoning.

Next I planned to ask the student to construct a slightly more complicated invariant relationship: the relationship between the surface area of the box and the length of the square cutout. Note that both the length of the box and the surface area of the box can be constructed as perceptual quantities. To see if the student's thinking is limited to reasoning about perceptual quantities, I would ask the student to construct a third relationship: the relationship between the volume of the box and the length of the side of the square cutout.

Finally, throughout this task I planned attempt to support students in conceptualizing variables as symbols representing the varying values a quantity assumes by encouraging students to name the quantities they construct and always reference the name of the quantity when describing that quantity's measure. As the student named attributes of the box, I would add these labels into the animation to support the student in speaking with meaning about the situation.

The Box Problem: Starting with an 11 inch x 13 inch sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

a. What do you see in this figure?
- i. What do you think the black solid lines represent? The brown solid lines? The dotted lines? The shaded part?

b. In the problem statement, it says that the box is formed by cutting equal-sized square cutouts from each corner.
- i. Why do the cutouts have to be square?
- ii. What would happen if the cutouts were rectangles?
- iii. Why do the cutouts have to be the same size in each corner?
- iv. What would happen if each corner had a different sized cutout?

c. Does the piece of paper also have to be square?
- i. Why do the cutouts have to be square but the piece of paper can be any dimension?

d. I am going to animate this image. If at any point you want to stop the animation, you can click the pause button in the bottom right corner.

i. What is changing as I animate this image?

ii. What stays the same as I animate this image?

e. How does the relationship between the cutout length and the length of the box change as the length of the cutout increases?

i. Is this relationship the same as the cutout length varies?

ii. Is this relationship the same as the dimensions of the piece of paper vary?

iii. Is this relationship the same if the paper were square instead of rectangular?

f. (VERSION 2) How does the relationship between the cutout length and the surface area of the box change as the length of the cutout increases?

i. When I did this task with another student, she said the surface area of the paper was like the area of the box but that it was too much because of the four squares you cutout. So she suggested that the surface area of the paper was the total of the surface area of the box and the 4 areas of the cutouts. Do you agree? Is this true as the cutout length varies? As the dimensions of the paper vary?

g. (VERSION 3) How does the relationship between the cutout length and the volume of the box change as the length of the cutout increases?

i.

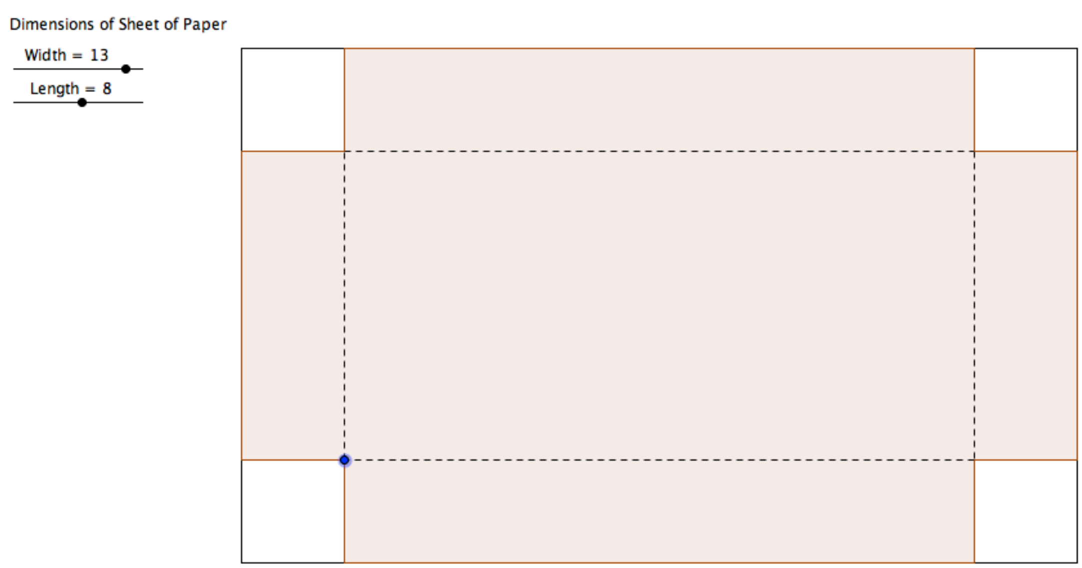


Figure 18: Box problem: Task 2 – teaching experiment.

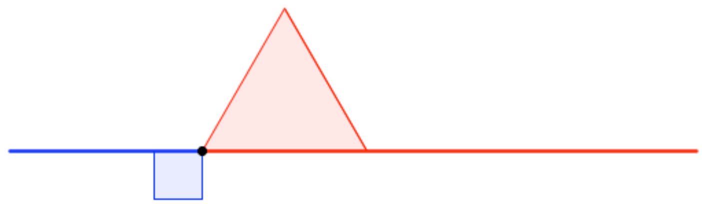
In designed the next task to support the student in constructing a more complicated invariant relationship. This task is based on the Two Polygons Applet designed by John Mason and Dan Meyer (2016). In this task I designed an animation where a black point moves along a horizontal line such that the point is a vertex for both a blue square and a red equilateral triangle, which sit along the horizontal line (see Figure 19). As the point travels along the line the side length of the square and triangle vary. I envisioned the student engaging with two versions of this task. From my perspective, in each version there is a constraint on how the perimeter of the square and the perimeter of the triangle vary together. In the first version the sum of the perimeter of the square and the perimeter of the triangle is constant. In the second version of the task the perimeter of the square and the perimeter of the triangle remain equal as the side lengths vary.

I would ask the student to describe what he noticed in the first version of the task. I anticipated that the student would attend to the varying side length and area of each polygon but might not attend to the constraint placed on the sum of their perimeters, perhaps because the student had not conceptualized the perimeter as a relevant quantity in the situation. For the student to conceptualize the invariant relationship between the sum of the two shapes' perimeters, the student must *believe* that the bold blue and red lines in the animation represent the perimeter of the square and triangle, respectively. Since perimeter is not a perceptually perceived quantity, I anticipated supporting the student in making this construction by displaying an animation where the bold blue and red lines wrap around the shapes so that the shape is constructed from its perimeter. Ideally, the shape must fall into the background of the students thinking so that she could reason about the relationship between the two shapes' perimeters. In other words, the shapes are actually a perceptual distractor from the quantities I ask the student to co-vary.

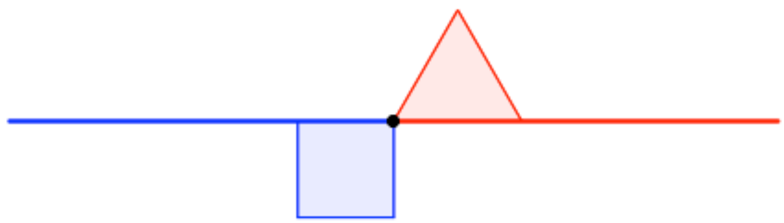
Finally, to support the student in conceptualizing varying measures satisfying an invariant relationship I would ask the student to reason about specific relationships between quantities' measures. For example, in the second version of the task, how does the sum of the perimeter compare to the perimeter of the square as the side length of the square varies? As I implement this task I knew I needed to be conscious to focus the student's attention to coordinating the varying measure of multiple quantities. In particular, I envisioned attending to how long the student focuses on one quantity, such as perimeter of triangle, before attending to the other quantity, the perimeter of the square. As Whitmire (2014) explained, students who persistently consider two quantities simultaneously are more likely to reason about how two quantities change together (p. 24).

Two Polygon Task (adapted from Mason and Meyer, 2016)

a. (Teacher/researcher displays Version 1 of animation). What do you see when you look at this animation?
    i. What do you think this bold blue/red line represents?
        i. Display animation of perimeter rotating to construct square and triangle to support student in believing the length of the bold lines represent perimeter of shape.
    ii. How are the perimeter of the square and the triangle related?
b. (Teacher/researcher displays Version 2 of the animation - with version 1 still playing). What do you see in this second version?
    i. How is the second version the same as the first?
    ii. Are they exactly the same? If not, how is the second version different than the first?
    iii. In this second animation, how does the sum of the perimeter compare to the perimeter of the square? Is this always true?
    iv. In this second animation, how does the sum of the perimeter compare to the side length of the square?
        i. Is this always true? In both animations?
        ii. Using two pointer fingers and table as reference point, represent side length of square with one finger's distance from table and perimeter of square with the other finger's distance from table. What are you attending to as you move your fingers?



ii. Screenshot from Version 1: As the black dot moves from left to right, the side length of the square and triangle vary so that the sum of the perimeter of the square and the perimeter of the triangle remains constant.



iii. Screenshot from Version 2: As the black dot moves from left to right, the side length of the square and triangle vary so that the perimeter of the square and the perimeter of the triangle remain equal.

Figure 19: Invariant relationship problem: Task 3 – teaching experiment (Task adapted from Mason and Meyer, 2016.

At this point in the teaching experiment I anticipated that the student would have experience constructing quantities and relating quantities' measures through an invariant quantitative relationship. So far in the teaching experiment, I would have asked the student to attend to quantities other than measured time in order to support the student in conceptualizing attributes of a situation and attending to how the measure of that attribute varies. However, as Castillo-Garsow (2012) suggested, students might need to conceptualize measured, or conceptual, time as a relevant quantity in order to conceptualize smooth variation. Thompson (2012) elaborated Castillo-Garsow's conjecture and explained,

> As Castillo-Garsow (2012) suggested, to operate with change happening in conceptual time, one must extract time from change, so that change happens in relation to time as opposed to happening because of the passing time. It could be that for someone to imagine change happening smoothly, they must have conceptualized time as passing smoothly and changes happening in relation to smooth-changing, measured, conceptual time (p. 11).

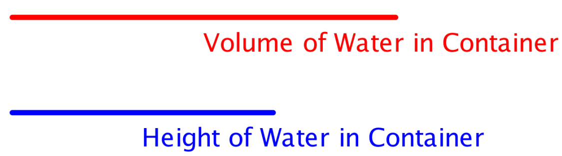
To gather empirical evidence to support or refute this conjecture, I designed the next task to understand how the student constructs measured time from his experiential time. In this fourth task (see Figure 20), I designed a dynagraph (Goldenberg, Lewis, & O'Keefe, 1992) representing the relationship between the varying height and volume of water in a container as the bottle fills with water. As Goldenberg et al. (1992) explained, a dynagrapah is:

> a class of function-visualizing tools that have as their common features that 1) the domain variable is dynamically mouse-manipulated by the user and 2) the domain variable and its image are represented each in its own space (p. 244)

Initially, I would present a static view of the dynagraph, which, from my perspective, represents a measure of the volume of water in the bottle and the associated height of water in the bottle. I would ask the student to describe how the container might be filling in order to get to these fixed lengths. Then I would play the animation so that the lengths of the two bars vary continuously with respect to experiential time. I would explain to the student that it took 5 hours for the bottle to fill with water and I planned to ask the student if it is possible that during those 5 hours the person took a break from pouring water into the container.

Suppose the length of the red horizontal bar represents the volume of water in a container and the length of the blue horizontal container represents the height of water in the bottle.

a. Can you describe how the container might have been filling in order to have this height and this volume of water?
b. [Animate bars]. Suppose that it took 5 hours for the container to completely fill with water. Can you describe how the container might have been filling?
c. Is it possible that during those 5 hours the person took a break from pouring water into the container?



Screenshot from Task 4: Animation depicts two horizontal bars whose lengths represent the varying values of two quantities (volume and height of water).

Figure 20: Experiential time problem: Task 4 – teaching experiment.

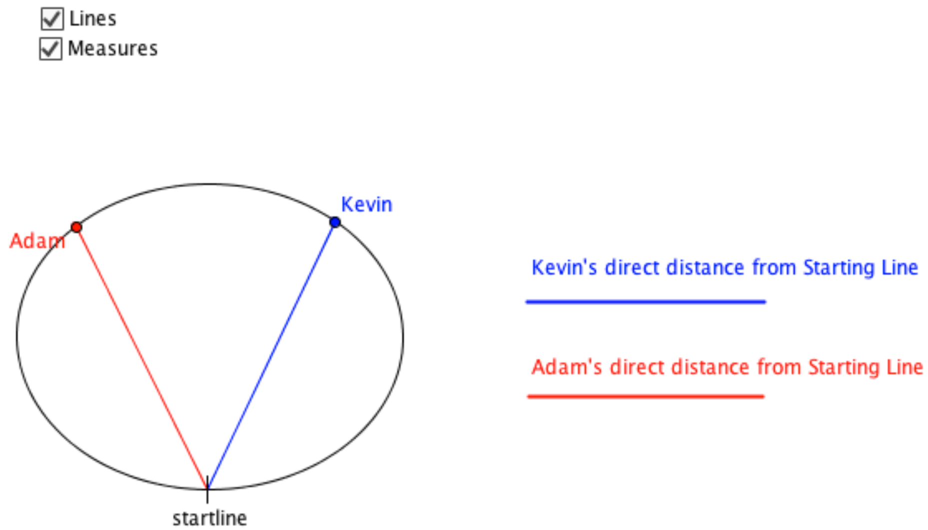
So far I designed tasks where the student could construct to at least one quantity that was either monotonically increasing or decreasing. When a student reasons about a monotonically increasing quantity, they have the opportunity to cognitively replace that quantity's variation with their sense of experiential time. To support students in coordinating two quantities' variation, I designed the next task to support students in conceptualizing more complicated relationships between quantities' measures where neither of the relevant quantities' measures are monotonically increasing or decreasing. I designed a dynamic representation of two people, Kevin and Adam, running around an ellipse shaped track (see Figure 21). I planned to ask the student to attend to Kevin's direct distance from the starting line and Adam's direct distance from the starting line. I thought that asking the student to attend to how one quantity changes as the other quantity reaches its maximum/minimum would support the student in coordinating their images of varying quantities.

In addition to supporting students in coordinating two quantities' measures, these types of questions might help the student develop what Silverman (2005) called two-dimensional landmark points. As Silverman described, a student conceptualizes a two-dimensional landmark point when he attends to when one quantity is maximum/minimum while also attending to what is happening to the other quantity's measure. Silverman conjectured that conceptualizing two-dimensional landmark points is essential when constructing a graph from one's conceptualization of a phenomenon. Thus, it is important to support students in making these constructions independent of their graphing activity early in the teaching sessions.

Kevin and Adam are both running around a quarter-mile ellipse shaped track. When Kevin starts running Adam is 100 meters ahead of Kevin.

- a. Drag the two people so that their starting positions match what is described.
- b. Okay, now I want you to imagine the boys running around the track. How is Kevin's direct distance from the starting line changing as he runs around the track?
  - i. Play animation – is this what you expected?
- c. How is the total number of meters Kevin has run changing as he runs around the track?
- d. As Kevin's distance from the starting line reaches its maximum value, what is happening to Adam's distance from the starting line? Is this always true as the boys continue to run multiple loops around the track?
- e. As Adam's distance from the starting line reaches its minimum value what is happening to Kevin's distance from the starting line?
  - i. Is this always true as the boys continue to run multiple loops around the track?
  - ii. Would this be true if the track were a mile loop instead of a 400m loop?
  - iii. What would have to change for this relationship to no longer hold?
- f. Determine whether the following statement is true or false. As Kevin's direct distance from the starting line increases; Adam's direct distance from the starting line also increases. Explain your reasoning.
  - i. What is happening to Adam's direct distance from the starting line as Kevin's direct distance from the starting line increases?

g. How can I record what is going on so that I know that whenever Adam's direct distance from the starting line was "this" long that Kevin's direct distance from the starting line was "this" long?



Screenshot from Task 5. The red and blue lines in the animation as well as the horizontal bars can be turned on/off depending on the amount of visual support the student needs when engaging in this task.

Figure 21: Kevin and Adam Problem: Task 5 – teaching experiment.

Part II: Constructing Representations of How Quantities Vary Together

At this point in the instructional sequence I anticipated students would have experience conceptualizing quantities, imaging their covariation constrained by an invariant relationship, and imagining their varying measures being represented by the varying length of horizontal bars. The next part of the teaching sessions focused on supporting the student in constructing a way to re-present how two quantities change together. In particular, I attempted to support the student in constructing a point in the plane as a multiplicative object that unites the measures of two quantities.

In Task 6 I planned to support the student in constructing the coordinate axes by suggesting that we orient the bars in the dynagraph from the Kevin & Adam task (Task 5) perpendicularly. I would initially orient these bars perpendicularly without the presence of axes and ask the student if anything about the relationship represented by the changing bars has changed. Ideally the students would focus on the invariant relationship between two quantities' measures and not the orientation of the bars. Finally, I envisioned displaying axes behind the bars (see Figure 22) and explaining that orienting the changing bars perpendicularly is actually the convention of the Cartesian coordinate axes. Ideally the student would understand that he could represent changing magnitudes along the axes and that quantities' measures are on the axes, not in the plane.

Kevin and Adam are both running around a quarter-mile ellipse shaped track. When Kevin starts running Adam is 100 meters ahead of Kevin.

a. Does it matter how I orient the blue and red bars? Does anything change if I orient them perpendicularly?
b. Introduce the convention of the coordinate axes as a way to organize thinking about two changing magnitudes.

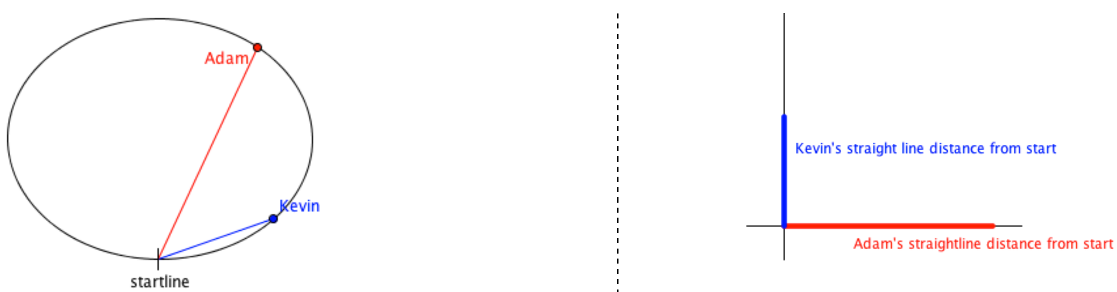
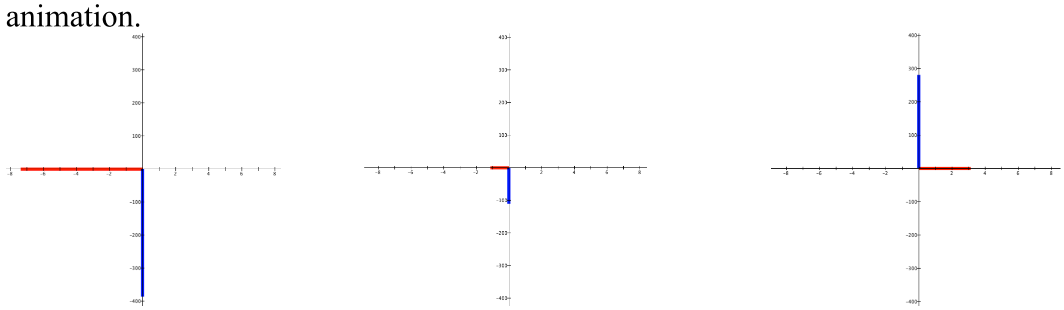


Figure 22: Construction of coordinate axes: Task 6 – teaching experiment.

Once the student has constructed what, from my perspective, is the coordinate axes, I planned to support the student in constructing the point in the plane as a multiplicative object. Task 7 is based on an item from a diagnostic instrument used to understand secondary mathematics teacher's meanings (Thompson, 2011a). In Task 7 I planned to present each student with an animation that depicts a red bar along the horizontal axis and a blue bar along the vertical axis. As the animation played, the lengths of the bars vary simultaneously with each bar having one end fixed at the origin (See

Figure 23 for selected screenshots from the animation). I would present each student with three versions of this task. In the first version the horizontal (red) bar's unfixed end varies at a steady pace from left to right while the vertical (blue) bar's unfixed end varies unsystematically. I would explain to each student that the length of the red bar along the horizontal axis represents the value of Quantity A and the length of the blue bar along the vertical axis represents the value of Quantity B. I would ask each student to construct a representation they could mail to a friend that represents how the two bars changed together. I would try not to use the word "graph" until I felt the student exhausted all of her ways of thinking about the task. In the second version of the task the horizontal bar would vary unsystematically while the vertical bar would decrease systematically from top to bottom. In the third version of the task both bars would vary unsystematically so that there are moments when the value of x is constant while the value of y varies aand there are moments when the value of y varies while the value of x is constant.

In the animation below, the length of the horizontal red bar represents the varying measure of x and the length of the vertical blue bar represents the varying measure of y. As the animation plays the lengths of the red and blue bars will vary together. Your job is to represent what is going on in this animation so that you could mail this representation to a friend and he would understand exactly what happened in the animation.



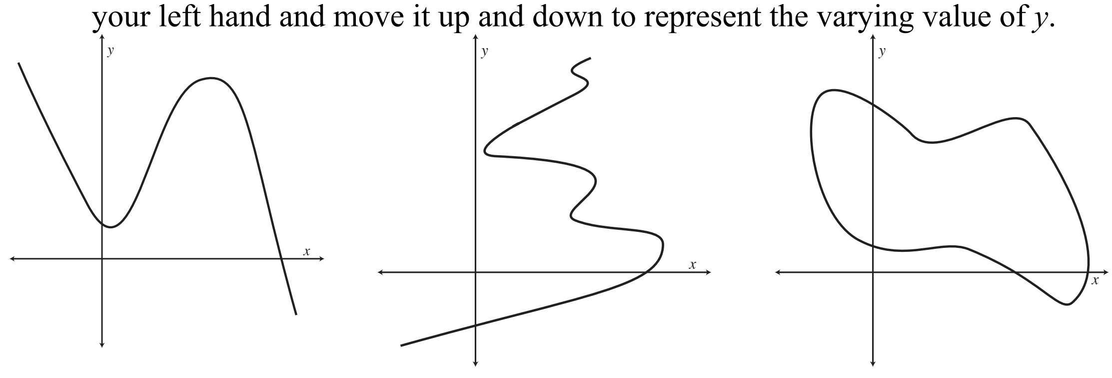
Selected screenshots from animation in Task 7.

Figure 23: Coordinating varying quantities on the axes: Task 7 – Teaching Experiment (Task adapted from Thompson (2011a).

After the student has constructed a point in the plane as a way to simultaneously represent the varying measure of two quantities, the student can begin to develop a reflexive relationship between his/her conceptualization of static graphs and coordinating varying measures represented along the axes. Thus, in Task 8 (see Figure 24) I envisioned presenting the student with three graphs and I would ask the student to imagine what the variation on the axes would look like. Finally, I would provide the student with animations of varying measures and I planned to ask the student to coordinate these with the given graphs. I would engage the student in three versions of the task, where each version represents a different relationship between the varying values of x and y.

I will present the student with one of the graphs below and then repeat this question sequence for each of the graphs displayed below.

a. How do the values of x and y change together?
b. Suppose I wanted to add the blue/red bars from the previous task to this graph. How would the blue/red bars vary along the axes so that their variation would represent the relationship depicted by this graph?
c. Use your palms to represent how the values are varying along the axes. Use your right hand and move it left to right to represent the varying value of x and use your left hand and move it up and down to represent the varying value of y.



Three graphs that I will use as the basis of the three versions of Task 8.

Figure 24: Conceptualizing varying quantities from a static representation: Task 8 – teaching experiment

The final task in the teaching sessions was designed to understand how students coordinate two distances where neither distance's measure increases/decreases monotonically. This task was developed independently by Swan (1982) and Saldanha and Thompson (1998) and has been used in numerous research studies (e.g., Bishop & John, 2008; Moore et al., 2016; Silverman, 2005; Whitmire, 2014). I envisioned presenting the student with a diagram of a straight road with two cities located near the road (see Figure 25). I would animate the diagram so that a car moves along the road. I would ask the student to sketch a graph of the car's distance from City B in terms of the car's distance from City A. If the student is unable to construct this graph, I would introduce levels of support, such as animating a line between the car and City A or City B, animating the measure of each quantity along a set of axes, and the final level of support would involve displaying a point that, from my perspective, simultaneously represents the measures of two quantities.

Researchers who use this task often describe the difficulties students encounter when completing this task. For example, Whitmire (2014) found calculus students who could not construct a graph because they wanted time to be a relevant quantity in the graph. Moore et al. (2016) found that pre-service secondary math teachers struggled to complete this task because they had developed graphing habits (e.g., graphs start on the vertical axis and pass the vertical line test) that contradicted the graphs they were constructing during the task. I hoped that the student's mathematical activity prior to this task would provide a foundation for the student to successfully engage in this task.

This animation depicts a car driving along a straight road. Notice that City B is on the east side of the road and City A is on the west side of the road.

a. As the car drives along the road, sketch a graph that represents the car's distance from City B in terms of the car's distance from City A.
b. What would happen if City B were located on the other side of the road?
c. (Version 2) Suppose Homer had to turn around for a little bit? How will the graph change
d. (Version 3) What if the road was not straight. How will the graph change?

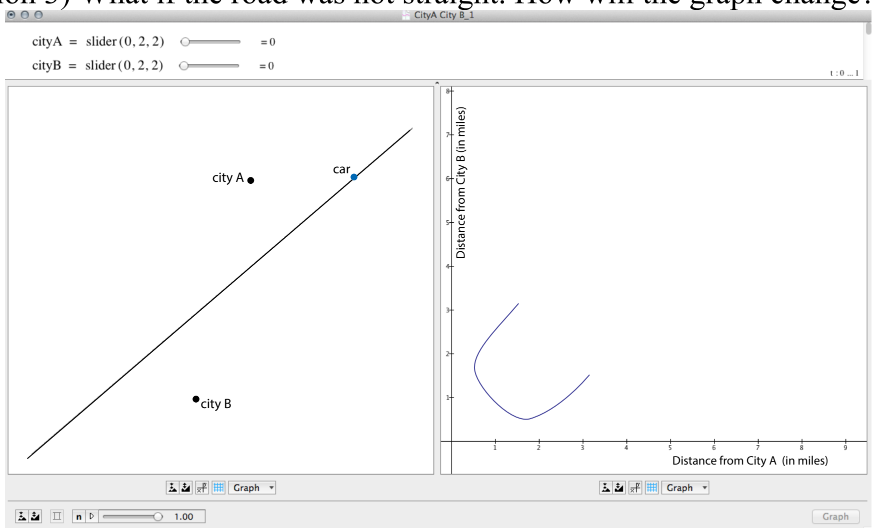


Figure 25: City A and City B Problem: Task 9 – teaching experiment (Task adapted from Saldanha and Thompson, 1998).

Concluding Remarks

This instructional sequence represents my pre-data collection thinking about how students construct understandings that enable them to engage in covariational reasoning. While I designed this instructional sequence based on my own and other researchers' hypotheses (e.g., Castillo-Garsow, 2012; Thompson & Carlson, 2017) about how students might conceptualize and coordinate images of smooth variation, it was almost guaranteed that students would engage in these tasks in unexpected ways. As a result, I anticipated that I would need to design new tasks both during the teaching sessions and between teaching sessions to enable me to better understand my students' thinking, and support them in both conceptualizing smooth variation and constructing invariant relationships and multiplicative objects. These ongoing revisions are an essential part of creating a hypothetical learning trajectory (Simon, 1995)

CHAPTER 6
A TEACHING EXPERIMENT WITH SUE

At the time of the study Sue had just completed her first year at the university and she was enrolled in summer semester precalculus after failing the course the previous semester. She was majoring in Space Exploration Systems Design, a STEM field, and she repeatedly expressed that she felt underprepared in mathematics. During her recruitment interview Sue demonstrated tendencies to think about graphs as iconic translations of an object's motion (Monk, 1992). Sue did not view a graph as a means of representing two quantities' measures. I selected Sue to participate in a teaching experiment to understand how an individual's scheme for quantitative reasoning informs his/her scheme for covariational reasoning.

Table 7

Sue's Schedule

Date	Event
May 16, 2016	Began Summer Session Precalculus (Traditional Curriculum)
June 1, 2016	Recruitment Interview
June 13, 2016	Pre-Teaching Experiment Clinical Interview
June 15, 2016	Teaching Experiment Session 1 (witness: P.W. Thompson)
June 17, 2016	Teaching Experiment Session 2 (witness: P.W. Thompson)
June 24, 2016	Ended Summer Session Precalculus with an A
June 28, 2016	Teaching Experiment Session 3 (witness: P.W. Thompson)
June 29, 2016	Began Fall Semester Calculus (Traditional Curriculum)
June 30, 2016	Teaching Experiment Session 4 (witness: P.W. Thompson)
June 30, 2016	Post-Teaching Experiment Clinical Interview

My teaching experiment with Sue was the first of three teaching experiments and her pre-teaching experiment clinical interview (pre-TECI) took place 12 days after her recruitment interview (Table 7). At the time of the pre-TECI, Sue was enrolled in a summer precalculus course and by the end of the teaching experiment she had completed the first two days of an introductory calculus course.

Sue's Initial Meanings for Graphs and Formulas

Sue participated in two clinical interviews—the recruitment interview and the pre-teaching experiment clinical interview—prior to her four-session teaching experiment. The purpose of these interviews was to establish a base-line characterization of Sue's meanings for graphs and formulas and to understand how Sue coordinated these meanings.

Sue's Initial Meanings for Graphs

During Sue's recruitment interview and pre-teaching experiment clinical interview (pre-TECI) she demonstrated two schemes of meanings for a graph. With one scheme she viewed the shape of a graph as a depiction of a situation. When Sue imagined the object in the situation moving through space her understanding of the graph was a picture of the object's motion, consistent with Monk's (1992) notion of *iconic translations*. If she did not imagine the object in the situation moving through space she understood the graph as a picture of the object. With the second scheme of meanings Sue imagined points on the curve and reasoned about a point in terms of the associated numerical values labeled on the axes. There is no evidence that Sue coordinated her meanings for points with her meaning for the shape of the graph. I will illustrate these non-coordinated meanings with two examples.

Example 1. The second task of the recruitment interview was a version of the Car A/Car B task (Monk, 1992). I presented Sue with a graph with two curves representing, from my perspective, Car A's speed relative to elapsed time and Car B's speed relative to elapsed time (see Figure 26). As Sue explained the graph she gestured along each curve and described two cars moving along two different roads that "come together from like another city". This suggests Sue imagined the graph as pictures of the roads each car drove along. Sue's activity is consistent with Carlson's (1998) findings that 88% of students who completed college algebra with an A interpreted the graph as a literal path of the car rather than interpreting the quantitative relationship displayed by the graph.

Task: Consider the graph below, which describes two cars' speeds in terms of the number of hours elapsed since they started traveling.

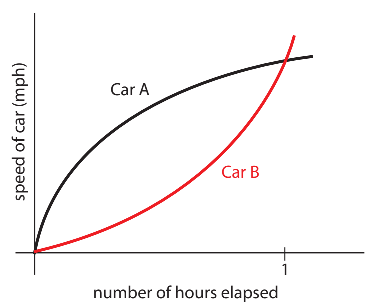


Figure 26: Car A and Car B problem: Task 2 – recruitment interview (Monk, 1992).

I included an additional prompt to see if Sue could reason about the graph in terms of two quantities' measures. I asked Sue if she could determine which car was traveling faster after half-an-hour of travel (Excerpt 1). Sue demonstrated two conflicting meanings for the graph. Initially Sue reasoned from the shapes of the curves and determined the symmetry of the two curves meant the cars travel the "exact same speed but in different directions" for the entire trip. Then she imagined points along the curve that had coordinates. She decided that since the curve for Car A was above the curve for

Car B at t=0.5, Car A was traveling faster than Car B. In Excerpt 1 Sue further explains the two ways she thought about the cars' speeds.

Excerpt 1: Sue recruitment interview, 00:15:49

KF: A half-hour after the cars started moving, which car is traveling faster?

Sue: So it would just be like somewhere here (*marks tick at 0.5 on horizontal axis*) I would just say they are traveling the same. Um. Like if I was just looking at it I would say they are traveling the same but according to the graph they are probably not because they are going to be on the same like if this is like a half (*labels tick mark on horizontal axis with 0.5*) then that's a point (*marks dot on car B's curve above 0.5 tick mark*) and then that's a point (*marks dot on Car A's curve above 0.5. tick mark*) but this (*traces from point on Car A's curve back to vertical axis*) is like obviously going to be like a greater number so they [Car A] were going like 60 miles per hour and they [Car B] were only going 20 miles per hour. Then Car A is going faster.

KF: A second ago you said they were going the same, what did you mean?

Sue: It looks like they are going the same amount of speed (*gestures path of curves with hands*). Like if this were like covered (*covers label "speed of car in mph" on vertical axis with left hand)* and I just saw there were two cars and this was the number of hours but they both met at the same point at the same time (*points to intersection point in the plane*) and they started at the same time (*points to origin*) then I would assume they are going the same speed but when I see this is the speed of the car (*moves pen up and down vertical axis*) then I would assume that these are lower numbers (*places pen on vertical axis near origin*) and these are higher numbers (*places pen further above origin on vertical axis)* so they are not going to go the same speed after all.

It seems that Sue's meaning for graphs changed as she focused on different aspects of the graph. When she focused on the lines in the plane she saw graphs as wire-like-shapes. She constructed properties of these shapes (e.g., symmetry) that she used to explain the phenomenon (e.g., the cars traveled the same speed the whole time). When she focused on the axis label, *speed of the car*, Sue attended to numerical values on both axes and imagined the graph containing points with coordinates.

Example 2. In the second task of the pre-TECI Sue again demonstrated thinking about a graph in two distinct ways: as a depiction of a situation and in terms of points with coordinates. The prompt in this task read:

> A company produces different size smart phones with rectangular screens. The screens dimensions are *w* and *h*, where the height of the screen (*h*) is half the width of the screen (*w*) for all sizes of smartphones.

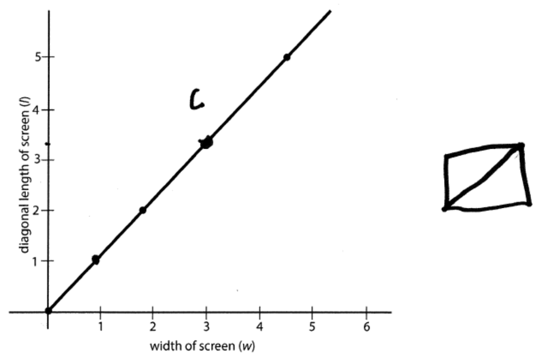


Figure 27: Sue relates a diagonal shaped graph to diagonal length (pre-TECI, Task 2)

I presented Sue with a graph of the company's cell phone screens' diagonal length in relation to their width under the constraint that any screen's height is half of its width (see Figure 27). When I asked Sue what the graph represented she drew a cell phone screen to the right of the graph. She said, "this [graph] is like a really zoomed in picture of that diagonal length" and explained that the graph is "an example for just one of the phones the company makes. A different phone gives you a different line". Sue apparently understood the shape of the graph as a picture of a cell-phone screen's diagonal length.

With prompting, Sue shifted her attention away from the overall shape of the graph and reasoned about the meaning of a point on the graph. When I highlighted a point on the graph Sue used the values on the axes to interpret the meaning of the point in terms of the dimensions of a single cell phone screen. Sue said, "the width of the screen is 3 and the diagonal length and is like 3.25". Sue was not perturbed by her conflicted interpretations as she shifted between thinking of the graph as a picture of a single screen's diagonal length and interpreting a point in terms of two numerical values.

Sue's Initial Meanings for Formulas

On three occasions in the pre-TECI I asked Sue to either construct or interpret a formula in relation to a described situation. In all of these tasks Sue reasoned about formulas as a way to calculate one number from another. For example, the last task was a homework problem from Connally et al. (2000). The problem read:

> The tuition cost (in dollars), T, for part-time students at Stonewall college is given by $T = 300 + 200C$ where C represents the number of credits taken.

Sue quickly determined that the value of T is 1900 when told that the value of C is 8, and determined the value of C is 7 when told the value of T was 1700, by "just plugging in what you know and solving for what you don't know." Sue interpreted the result of her calculations in terms of the situation, saying, "for 8 it costs 1900 dollars" and "for 1700 dollars you can take 7 credits", respectively. However, she was unable to explain what the coefficients 300 and 200 meant in relation to the situation. She said, "I don't think there is enough information to really know what 300 and 200 both mean because it is not telling you. It is just telling you the formula. … The formula is just a way to relate numbers." This suggests that Sue did not anticipate that the formula provided information about the situation or about how the quantities changed together.

Sue's Initial Coordination of Meanings for Graphs and Formulas

I designed the cell-phone screen task to understand how Sue coordinated her meanings for graphs and formulas. I asked Sue to construct a formula that gave the cell phone screen's diagonal length in terms of its width. Sue wrote $h=½w$ and $h^2+w^2=c^2$ and used these two formulas to imagine calculating a value of the diagonal length. For example, she imagined the screen having a width of 30 and she used her first formula, $h=½w$, to determine the screen's height was 15. Then she explained she would plug in 15 and 30 to her formula for h and w, respectively, to determine the value of c.

As described in Example 2, after Sue constructed these formulas, I presented her with a graph of the company's cell phone screens' diagonal length in relation to their width under the constraint that any screen's height is half of its width. I asked Sue to explain whether the formulas she wrote related to the graph. Sue explained that you could use the formulas, $h=½w$ and $h^2+w^2=c^2$, to calculate a value of c from a value of w and then use those numbers to plot a point on the graph.

To understand how Sue imagined these numbers relating to her graph, I asked Sue to suppose the width was 3. She found 3 on the horizontal axis, marked the point on the graph associated with $w = 3$, and then determined the diagonal length was about 3.25. Next I asked Sue to imagine c, the diagonal length, was 3. This time Sue used her formula to determine $c^2 = 9$. She drew a new set of axes, labeled 9 on the horizontal axis, plotted a point above her tick mark labeled 9, and then drew a line from the origin through that point (see Figure 28). She interpreted the resulting graph as a picture of the diagonal of the cell phone's screen.

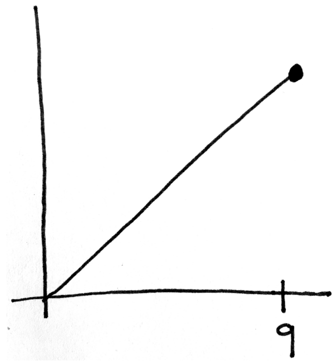


Figure 28: Sue's graph for cell phone screen task given $c = 3$ so $c^2 = 9$.

The above account suggests that Sue anticipated coordinating her meaning for plotting points with her meaning for formulas. However, she did not demonstrate a clear understanding of what that coordination entailed. When Sue reasoned from the graph she imagined the graph having points with coordinates, one of her two meanings for graphs. Then she interpreted these coordinates in terms of values in the formula. However, when she reasoned from the formula she imagined constructing a graph that was a picture of the cell phone screen; consistent with her meaning for graphs as a depiction of a situation/phenomena. Sue did not anticipate representing pairs of measures. I take this as evidence that Sue's coordination of formulas and plotting points was not reversible. She did not imagine constructing a point from a formula using the same system of actions that she used to interpret a point in terms of a formula.

Summary

Sue's actions during the initial clinical interviews suggested that she had two schemes of meanings for graphs. With the first scheme of meanings, Sue understood graphs as depictions of phenomenon. She imagined the shape as either a picture of an object (e.g., cell phone screen's diagonal length) or a picture of an object's motion over the course of an event (e.g., the path Car A and Car B traveled along). When Sue operationalized this scheme she confounded the shape of a graph with features of the situation. This suggests that she did not differentiate between motion as described in the text and the path of the graph. With the second scheme of meanings, Sue attended to the labels on axes and interpreted points in terms of two numerical values. In other words, Sue demonstrated a scheme for curves called graphs and a scheme for dots labeled on these curves. These uncoordinated schemes are consistent with Monk's (1992) finding that students often construct appropriate point-wise interpretations when they are otherwise unable to reason about the graph across time.

Sue's Teaching Experiment

I engaged Sue in four teaching experiment sessions. Each session lasted between 1.5 and 2 hours and was witnessed by Pat Thompson. The teaching experiments consisted of three series of tasks to help me understand (1) Sue's scheme for quantitative reasoning, (2) the ways of thinking that supported or inhibited Sue from engaging in emergent shape thinking, and (3) the generality of any constructions she made during the teaching experiment.

Teaching Experiment Phase I: Quantitative Reasoning

I began the teaching experiment by asking Sue to complete five tasks I designed to reveal the objects in Sue's images of quantitative situations that she acted upon when reasoning about those situations. My analysis revealed that the objects of Sue's reasoning were restricted to objects in her perceptual space. These objects included features she perceived in a dynamic animation and images of an object's motion that she constructed from a contextual description. By reasoning about elements in her perceptual space, Sue did not anticipate abstracting attributes of these perceptions. As a result, she did not anticipate constructing quantities and reasoning about their varying magnitudes.

As Piaget (1995) explained, one's perceptual space consists of what the student can see and conceive with the operations she has available. To reason from one's perceptual space one must systematically explore what she is looking at and choose what to look at and act upon (Piaget, 1985). This ability to focus on aspects of one's perceptions was evident in Sue's engagement in the initial clinical interviews. Sometimes Sue reasoned about a graph by attending to the shape of the curve while other times she attended to the axes' labels and numbers on the axes. From my perspective the visual stimulus did not change – Sue reasoned from the same graph – but what Sue saw in her perceptual space changed.

When Sue reasoned from her perceptual space she had a tendency to attend to the motion of an object instead of on the variation of an attribute's measure. For example, in the first task of the teaching experiment I presented Sue with a GeoGebra® animation depicting an airplane traveling from left to right on the screen (see Figure 29). When I asked Sue to describe how the distance between the airplane and the ground changed she described how the airplane moved on the screen. She said, "He goes up and then he stays the same for a little bit and then goes up again and then back all the way down." This suggests that Sue attended to a perceptual feature of the animation (the plane's location on the screen) that she tracked as the animation played. There was no evidence that Sue constructed the attribute *distance between the plane and the ground*.

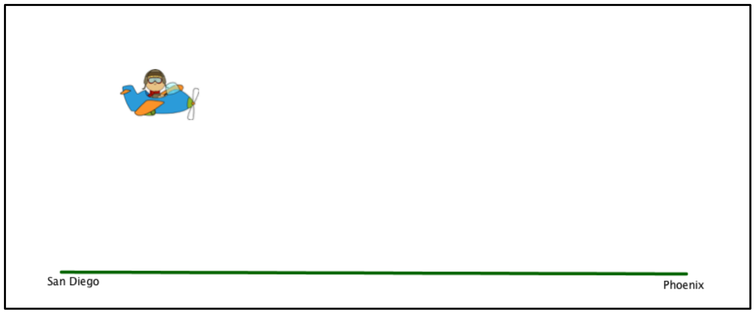


Figure 29: Screenshot of Airplane Task (Day 1, Task 1).

Sue also focused on an object's motion in the Kevin & Adam Task (Day 2, Task 5 adapted from Carlson, Oehrtman, & Moore, 2013). In this task, I presented Sue with a GeoGebra® animation depicting an ellipse shaped track with two dots, intended to represent Kevin's and Adam's locations on the track as they moved (see Figure 30). I asked Sue to describe how Kevin's straight-line distance from the starting line changed as he ran around the track. Instead of describing a distance that increased and decreased in measure, Sue focused on whether Kevin's location on the track was getting closer to or further from the starting point. In other words, instead of identifying and reasoning about an attribute's measure Sue reasoned about Kevin's physical proximity to the starting line saying,

> He is going to get further away from it for a very long time and then he'll start to get closer again. Like after I think 200 meter mark. Like he starts to come down. Like he starts to come closer to it after he has finished half of it.

This utterance suggests that Sue focused on Kevin's motion and she did not abstract the attribute of straight-line distance, a linear measure, to then coordinate with her image of Kevin's proximity to the starting line.

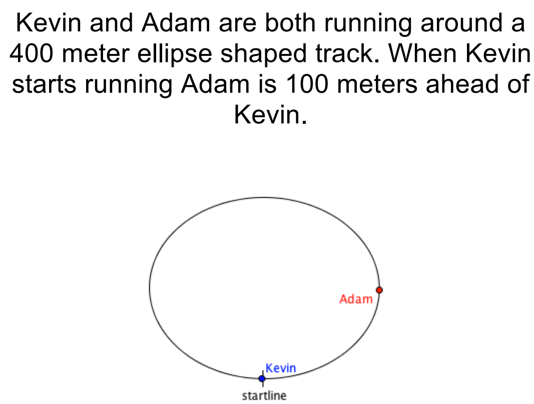


Figure 30: Screenshot of Kevin & Adam Task (Day 2, Task 5)

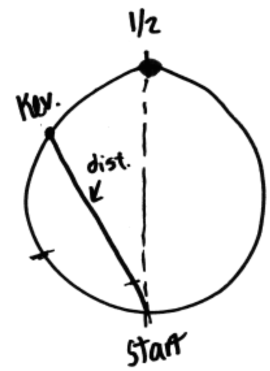


Figure 31: Sue's diagram of Kevin's straight-line distance from start.

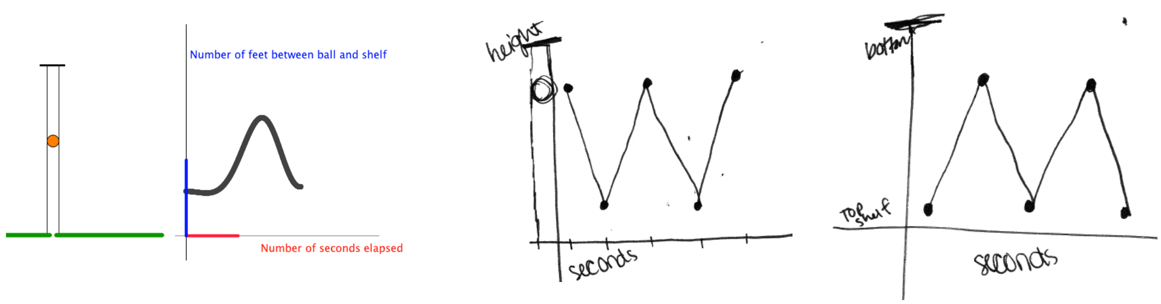
To better understand how Sue understood my utterance, "Kevin's straight-line distance from starting line", I asked Sue to draw the distance she was imagining (see Figure 31). From my perspective, Sue appropriately identified Kevin's straight-line distance from start. However, Sue explained this straight-line distance was not what she focused on. Sue explained that she focused on Kevin's motion around the track and she kept track of when he got closer to or further from the starting line. Sue explained,

> This is technically the distance from the starting line (*points to straight line distance*) but that is not how I like would measure it. I mean like it is but like I would look at this (*moves pen around track*) first for some reason. … So I am just looking at the point and imagining him running around or yeah running around the circle and I was keeping track of the point and as he starts to come closer to the starting line like he starts to come closer to the starting line. I am not really sure how to measure that, I just know that by looking at it you can see he is close to the starting line.

It is possible that Sue was imagining Kevin's distance from the starting line being measured along the track as opposed to his direct distance from the starting line measured across the infield. However, I am hesitant to claim that Sue was reasoning about an attribute's measure increasing and decreasing. Instead, it seems that Sue isolated two aspects of her perceptual space – Kevin and the starting line – and kept track of when the objects were moving away from each other and when they were getting closer to each other. This suggests that the attribute *distance between Kevin and the starting line* (either around track or across infield) was an implication of her reasoning about the space between the objects – not an explicit object of her reasoning. This interpretation is consistent with the thinking Sue exhibited in the context of the airplane task when she described the motion of the plane going up and down in the sky instead of attending to changes in the *distance* between the plane and the ground.

Testing my model of Sue's quantitative reasoning. To test my hypothesis that Sue reasoned from her image of an object's motion instead of images of attributes and their measures, I designed a task where one's perception of the object's motion does not match the quantity's variation. In this animated task I presented Sue with a depiction of a ball floating in a tube between a shelf and the ground (see Figure 32). I asked Sue to graph the distance of the ball from the top shelf relative to the number of seconds elapsed. I anticipated Sue's focus on her perceptions, the motion of the ball moving up and down, inhibited her from reasoning about the ball's distance from the top shelf. Sue's activity confirmed my hypothesis. Sue drew a graph by first drawing the tube along the vertical axis and then created a curve by "following the ball". Sue's placement of the tube on the axis suggests that Sue did not anticipate that a graph was the product of an abstraction; instead she saw a graph as a depiction of the activity in the phenomenon.

Since Sue labeled her vertical axis as *height* I decided it was possible she misunderstood my prompt. So, to understand how she would attend to distance from the shelf I asked Sue if her graph showed the ball's distance from the top shelf. Sue explained, "No it doesn't but it would be opposite. So like instead of starting high up it starts pretty low. So like the W should be an M." She drew a new graph by labeling the vertical axis by the origin with "top shelf" and then she labeled the top of the vertical axis with "bottom". Finally she drew an M shaped graph (Figure 32). Since Sue labeled the axes with features of the phenomenon, I hypothesize that Sue constructed her new graph by rotating her image of the phenomenon and then tracking the ball's movement as her image of the phenomenon changed in her experiential time. I take this as evidence that Sue's quantitative reasoning was constrained to her perception of an object's motion.



Screenshot from floating ball task & graph of actual covariation

Sue's initial graph of the ball's distance from top shelf in terms of number of seconds elapsed

Sue's second graph of the ball's distance from top shelf in terms of number of seconds elapsed

Figure 32: Screenshot of Floating Ball Task and Sue's solution (Day 4, Task 9)

While the development of quantitative reasoning was not the focus of this dissertation study, I hypothesized that Sue's tendency to reason from her perceptual space and not focus on attributes and their measures would influence her engagement in covariational reasoning. The most robust forms of covariational reasoning necessitate holding two attributes in thought simultaneously, an activity that requires constructing and coordinating two quantities (objects conceptualized as measurable and having a measure). Since Sue consistently reasoned from her perception in the moment of acting, I anticipated that she would experience difficulty constructing and coordinating quantities, reasoning instead about features of her perceptions.

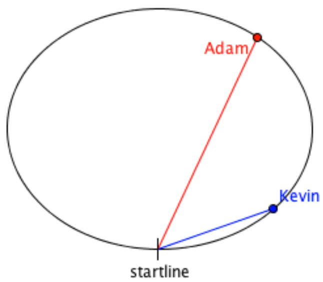
Teaching Experiment Phase II: Supporting Emergent Shape Thinking

The second phase of the teaching experiment lasted one session (Day 3, 1 hour 17 minutes). In this session I engaged Sue in tasks I designed to support her in engaging in emergent shape thinking (see learning trajectory in Chapter 5). Specifically, I designed tasks to support Sue in making three constructions:

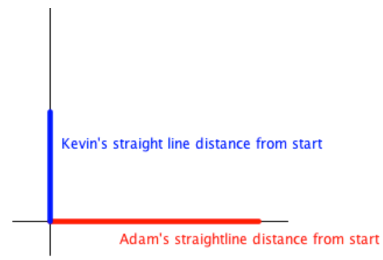
1. Imagine representing quantities' magnitudes along the axes;
2. Simultaneously represent these magnitudes with a point in the plane; and
3. Anticipate tracking the values of two quantities' attributes simultaneously.

As I describe in the following section, Sue did not make these constructions. Instead, Sue learned to use perceptual features of the tasks to complete the tasks. In this section I will document both my efforts to support Sue in making these constructions and the ways Sue's images of changing magnitudes inhibited her from engaging in emergent shape thinking.

I started the third teaching session with a teaching move aimed at supporting Sue in understanding numbers along the axes as quantities' measures. I situated the teaching move in the Kevin and Adam task from the previous teaching session and I introduced two new visualizations in the animation. First, in the depiction of the event, I displayed a blue line segment between Kevin and the starting line and a red line segment between Adam and the starting line (see Figure 33). I hoped this depiction would support Sue in thinking about the changing straight-line distance instead of the boy's location on the track. Next, I displayed a red bar and a blue bar oriented perpendicularly on the axes (see Figure 33). I explained that the length of the blue bar represented the measure of Kevin's straight-line distance from start and the length of the red bar represented the measure of Adam's straight-line distance from start.



Screenshot from Kevin and Adam Task where boys' straight-line distances from start are displayed in the depiction of the situation.



Screenshot from Kevin and Adam Task where the boys' straight line distances from start are represented as perpendicular magnitude bars.

Figure 33: Screenshot of Kevin & Adam Task (Day 3, Task 6)

To understand how Sue understood these new perceptual features I asked Sue to explain how the lengths of the bars on the axes would change as Kevin's straight-line distance from start increased. With the animation paused Sue used the computer pointer to explain, "The blue line is going to go to the highest point it can up here (*moves computer point up vertical axis*) and at the same time the red is going to go to its maximum and then decrease (*moves computer pointer right then left on horizontal axis*)." I took this as evidence that Sue coordinated her image of the length of each bar on the axes with her image of how each quantity's measure varied.

To understand if and how Sue anticipated representing two measures simultaneously I included an animated item adapted from Thompson (2016). I presented Sue with an animation that depicted a red bar along the horizontal axis and a blue bar along the vertical axis. As the animation played, the lengths of the bars varied simultaneously in such a way that each bar had one end fixed at the origin. (See Figure 34 for selected screenshots from the video). In the first version of this task the horizontal (red) bar's unfixed end varied at a steady pace from left to right while the vertical (blue) bar's unfixed end varied unsystematically. I explained to Sue that the length of the red bar represented the varying value of *u* and the length of the blue bar represented the varying value of *v*. Finally, I presented Sue with a printout that included a screen capture of the initial position of the bars in the animation and I asked Sue to graph the value of *v* relative to the value of *u.* The video played repeatedly until Sue completed the task.

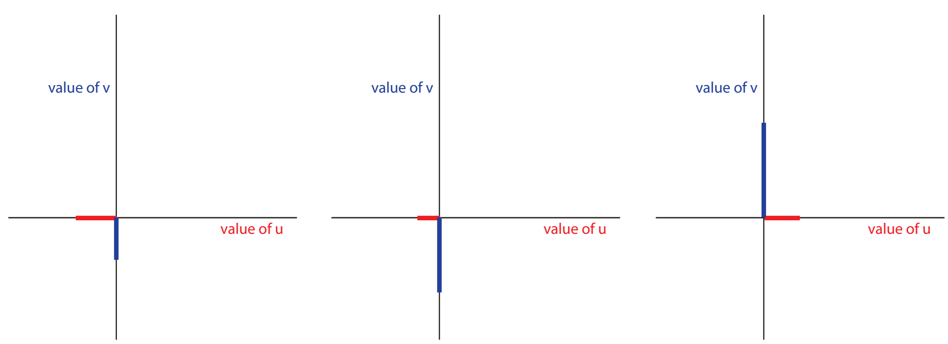
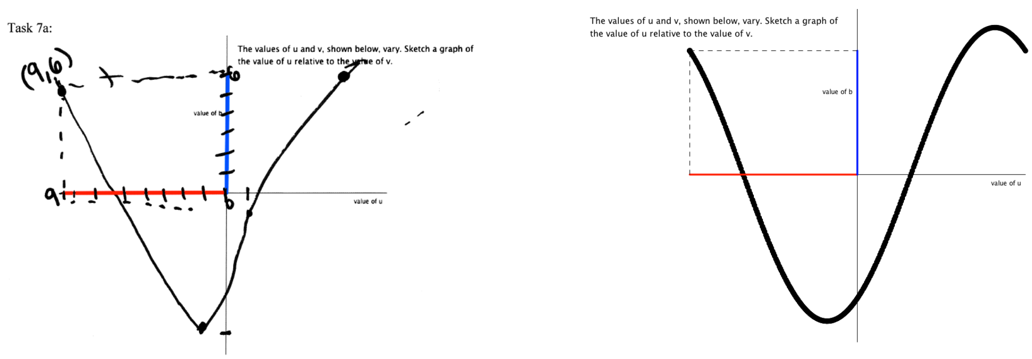


Figure 34: Three screenshots from U&V task (adapted from Thompson, 2016).

With the animation playing, Sue could not anticipate how to make a graph from the animation. She watched the animation play through three times and then said, "I'm not sure how to go about it and to make it into a graph. Like in my opinion you can't." Since Sue seemed to have no actions available to her in the moment, I paused the animation at the beginning and asked Sue if she could represent the paused moment. She assimilated the paused animation to her scheme for plotting points; she imagined the red bar and blue bar ending at tick marks labeled with numbers on the axes. Then she plotted a point that had those numbers as coordinates (see Figure 35). Note that Sue imagined both bars having a positive length. This suggests that Sue did not imagine the lengths of the bars as *directed* measures represented along the axes.



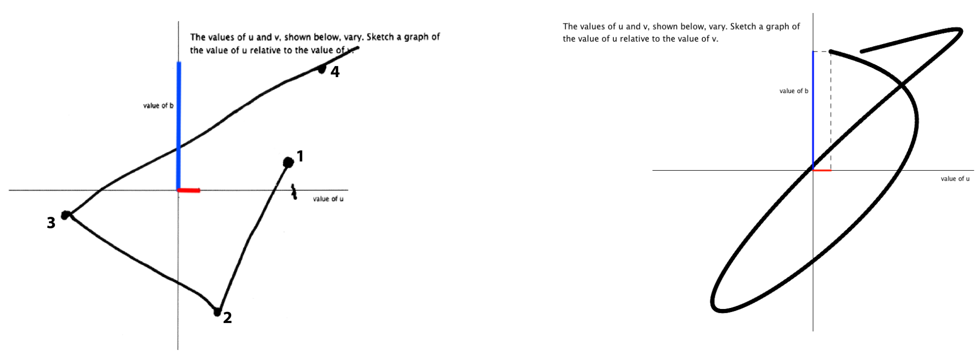
Sue's graph

Graph of actual covariation of bars' lengths (Sue never saw the computer make this trace).

Figure 35: U&V Task version 1: Sue's solution and graph of actual covariation (Day 3, Task 7.1)

Although Sue thought about the length of both bars when placing her initial point, this attention to both bars did not persist in her thinking. Instead of tracking both measures continuously as the animation played, Sue focused on the motion of the red bar and plotted three more points: one "right before the red gets to zero", another "right after the red crosses over so you can see it is increasing", and a final point "at like the maximums of *u* and *v* over here just to see that it is still increasing." Since the length of the blue bar decreased at the end of the animation, this last utterance suggests that Sue focused exclusively on the length of the red bar increasing. This is significant because it implies that Sue's conception of both her graph and a point in the plane favored her image of the red bar’s motion. This implies that her conception of a point was not multiplicative.

It seems that Sue drew a collection of points because her image of the animation focused on landmark states, maximums and minimums, the quantities attained over the course of the entire animation. As a result, she did not anticipate keeping track of how both lengths changed between the points she plotted. This thinking was exemplified in Sue’s engagement in the third version of the task when the ends of both the red bar and the blue bar varied unsystematically. With both bars moving unsystematically Sue no longer prioritized her image of the red bars motion. Instead, I claim that she constructed two images from the animation: a first image of landmark states the red bar attained and a second image of landmark states the blue bar attained. Then she coordinated these images by plotting four points and connecting these points with straight lines (Figure 36). In the following paragraphs I will provide evidence to support this claim.



Sue’s graph

Graph of actual covariation of bars’ lengths.
Sue never saw the computer make this trace.

Figure 36: U&V Task version 3: Sue’s solution and graph of actual covariation (Day 3, Task 7.3)

Sue explained the points she plotted by describing directional changes in the movement of each bar. For example, she explained her first point saying, “the red is increasing out here and the blue is decreasing, that is how I got this point". While this suggests Sue thought about both the red bar and the blue bar when plotting a point, I claim she thought about the changes in the lengths of the bars asynchronously. This was most evident in Sue's thinking about her third point.

Sue explained how she decided where to plot the third point saying, "eventually the blue bar will increase and the red bar will decrease". While the bars each eventually and individually behaved like this, Sue's characterization did not account for how the bars changed *together*. In the animation the length of the red bar never had a decreasing negative value while the length of the blue bar had an increasing negative value. This suggests that Sue's image of the animation was actually a loose coordination of two independent images: her image of landmark states the red bar attained and her image of landmark states the blue bar attained.

I suspect that as Sue watched the animation she looked for the next landmark state each quantity attained. At the beginning of the animation she noticed the red bar reached its maximum. She also noticed the blue bar flipped over the horizontal axis. It seems she plotted a point to show this combination of landmark states. The next set of landmark states she imagined was the red bar flipping over the vertical and the blue bar reaching its minimum. She represented this pair of landmark states with her second point. Then she saw the red bar reach its minimum. She saw the blue bar increase from its minimum. She represented this pair of images, red at its minimum and blue increasing, with her third point. Finally, she saw the red bar reach a maximum and she also saw the blue bar reach a maximum. Sue represented this last pair of images with her fourth point. This characterization suggests that it is the design of the covariation, not Sue's thinking, that made the location of Points 1, 2, and 4 seem like they were located at the intersection of the red and blue bars if they were extended into the plane.

This characterization suggests that Sue needed the to see the red and blue bars moving on the axes to conceptualize the next landmark state each quantity attained. Then she coordinated that pair of landmark states through a point. This implies that her activity coordinating landmark points happened in real time as she watched each bar move along the axes.

Two teaching moves. In the next part of the teaching experiment Pat, the witness to the teaching experiment, and I tried to support Sue in making two constructions. First, we tried to support Sue in constructing an image of the animation that focused simultaneously on the length of the red bar and the length of the blue bar so that she anticipated her graph represented how the lengths of the bars changed *together.* Second, we tried to support Sue in constructing an image of the changing bars that focused on more than landmark states so that she could anticipate a curve as tracking these in-between measures. In the following paragraphs I explain how Sue's focus on quantities' gross variations and landmark states inhibited her from conceptualizing her graph as a representation of the nuances in how two quantities' magnitudes changed together.

Coordinating two images of change. The first teaching move happened during the third version of the U&V task. From my perspective, Sue's graph (Figure 36) showed the value of v (blue bar) reaching it's minimum for a positive value of u (red bar) but the motion of the bars in the animation showed the value of v (blue bar) reaching its minimum for a negative value of u (red bar). Pat and I tried to get Sue to notice and then reconcile this difference by attending to how the lengths of the red bar and blue bar changed together.

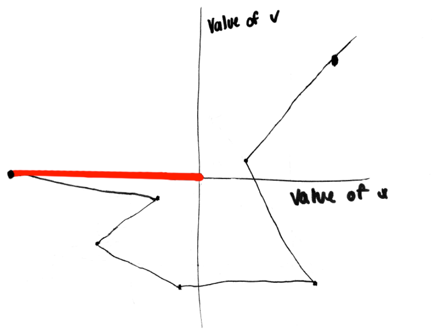
We asked Sue to study what she drew and then compare that to what she saw in the animation; Sue saw them as the same. Next, we asked Sue to explain how the red and blue bars would have to change in order make the graph she drew. She gave a description that appropriately matched her graph noting that after her second point the blue bar would be negative and increasing and the red bar would "flip over the vertical axis and is now negative and still decreasing." While Sue's explanation matched her graph, she noticed no difference when she compared this anticipation with the animation even though in the animation the blue bar was negative and decreasing, not increasing.

Finally, I manually controlled the animation and hovered the animation around $0 < u < 1$. From my perspective, over this interval the value of u was positive and decreasing and the value of v was negative and decreasing. While Sue watched the animation she gave an explanation that matched this motion of the bars: she explained that over that interval the red bar was decreasing and the blue bar was decreasing. When I asked her to identify that part of the animation on her graph she said, "So that [red bar positive and decreasing and blue bar negative and decreasing] isn't shown. I like didn't think it mattered". Sue justified her reasoning by describing how the bars' lengths would eventually change if the animation continued to play. As I explained above, Sue justified the location of her third point by saying that eventually the blue bar will be negative and increasing and the red bar will be negative and decreasing.

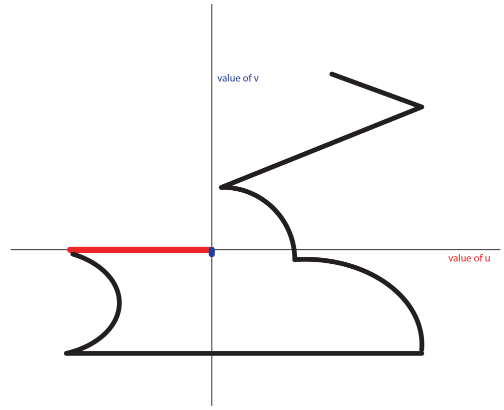
Pat and I spent over ten minutes trying to get Sue to experience a perturbation by seeing that her graph showed the value of v was negative and <u>increasing</u> while the value of u was positive and decreasing when the animation actually showed the value of v was negative and decreasing while the value of *u* was positive and decreasing. Sue kept saying, “I didn’t think it mattered … eventually the blue bar will increase [in value]”. I take this as evidence that from the animation Sue could construct an image of how the two bars changed together. However, she did not anticipate representing this image in her graphing actions. Instead, she represented her images of key landmark states the quantities attained.

Constructing and representing images of change in progress. In the fourth version of the task I tried to support Sue in attending to nuances in how the lengths of the bars changed together. I wanted to see if thinking of a line as a collection of points might help Sue conceptualize a graph as a representation of the nuances in how two quantities’ magnitudes changed together.

In previous versions of the U&V task the animation displayed a moving red bar and a moving blue bar. This time, in addition to both bars moving on the axes, I set GeoGebra® to trace the bars’ actual covariation. Then I asked Sue to compare how she made her graph with how the computer made its graph (Figure 37). Initially, Sue compared the shape she made and the shape the computer made noting that her third point (bottom left) was too high.



Sue's graph



Graph of actual covariation of bars' lengths
(Unlike previous versions of this task, Sue did see the computer make this trace).

Figure 37: U&V Task version 4: Sue's solution and graph of actual covariation (Day 3, Task 7.4)

Pat reminded Sue that she constructed her graph by first plotting points and then connecting them with a segment, then asked if that was what the computer did. Sue said, no the computer "made a line as it went along. It didn't make points it just made a line to where the maximums and minimums were. It was more like following the red and blue lines while drawing it." Sue went on to explain, "and like it has curves. I don't really like that. I like just straight lines, which is why I probably lean more towards points because you can just connect them easier."

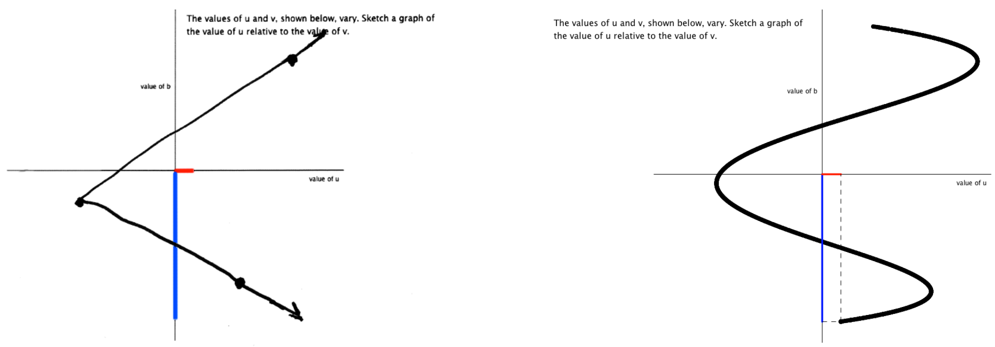
In explaining the computer's trace Sue attended to the computer's continuous tracking of the red and blue bars. However, she still prioritized the landmark states when she said "it just made a line to where the maximum and minimums were". This suggests that Sue did not imagine the trace as a collection of points such that each point represented a pair of magnitudes. Instead, Sue understood the computer's continuous trace as a way to get from one landmark state to another. This activity is consistent with my claim that Sue's image of the animation consisted of images of landmark states each quantity attained over the animation. Since Sue's image only contained landmark states there was nothing else in her thinking to represent. In other words Sue could not track something in-between the landmark states because there was nothing in her thinking to track.

Finally, Sue de-emphasized the curvature of the lines in the computer's trace saying, "I like just straight lines". This suggests that Sue conceptualized a line segment between points in the plane as no more than a visual connector between points. Since Sue saw the segment as a way to connect two points in space it did not matter to her whether it was curved or straight. As a result Sue did not anticipate the curvature was the product of capturing nuances in how the two quantities' magnitudes changed in relation to each other between critical points.

For Sue to construct a meaning for a curved graph grounded in representing how two magnitudes changed together she would need to construct an image of the lengths of the bars that captured more than landmark states; she would need to imagine change in progress. The U&V task did not support her in constructing this image.

Is it a graph? A final note about Sue's graphing activity. In the analysis above I focused on Sue's activity making a graph and the meanings she seemed to convey through her graphing activity. In this section I will discuss the meaning Sue had for the graph she made: the graph was a shape. On two occasions Sue interpreted the product of her graphing actions as a graph-as-wire shape and did not see in her completed graph the actions she engaged in to make the graph.

Example 1: In the second version of this task the vertical (blue) bar's unfixed end varied at a steady pace from bottom to top while the horizontal (red) bar's unfixed end varied unsystematically. Consistent with her engagement in the first version of the task, Sue created her graph by drawing three points "at the maximums" and connecting these points with straight lines (see Figure 38). Although Sue did not specify what maximum she was attending to, her choice of points is consistent with attending to the maximum lengths of the red bar.



Sue's graph for second version of U&V Task

Graph of actual covariation
(Sue never saw the computer make this trace).

Figure 38: U&V Task version 2: Sue's solution and graph of actual covariation (Day 3, Task 7.2)

After Sue connected the points she questioned whether what she made was even a graph. She was perturbed by the graph she made not looking like what she was accustomed to calling a graph. In order to assimilate the shape she created, Sue made an accommodation to her graphing scheme by thinking about all the shapes she had seen in math class. She determined what she drew was a graph because it was "a triangle thing, like an angle. … and angles are popular in math." I take this as evidence that while Sue constructed her graph by attending to the lengths of the red and blue bars at three moments of the animation, she attributed meaning only by trying to identify the <u>shape</u> she created. Sue did not anticipate reasoning about the shape she made in terms of the actions she used to make the shape. This suggests that Sue's meaning for her graph was a pseudo-empirical abstraction from her past activities of graphing and interpreting graphs.

Example 2: In the previous section I described Sue's engagement in the fourth version of the U&V task. After Sue constructed her graph I let GeoGebra® trace out the bars' actual covariation. Then I asked Sue to compare how she made her graph (Figure 37) with how the computer made its graph. Initially, Sue compared the shape she made and the shape the computer made noting that her third point (bottom left) was too high. This suggests that Sue saw the products of her graphing actions in the shape she created and did not anticipate comparing the actions she engaged in to make the graph with the computer's continuous trace.

These two examples suggest that Sue's meaning for her constructed graphs were empirical abstractions grounded in the final shape she produced.

Teaching Experiment Phase III: Operationalizing Emergent Shape Thinking

I designed the third phase of the teaching experiment to better understand Sue's thinking during the U&V task, in particular what aspects of her thinking were dependent upon two moving bars oriented perpendicularly on the axes? I engaged Sue in three context based graphing tasks (details in Appendix C) to study the ways Sue thought about representing changing magnitudes. I anticipated the ways Sue coordinated these tasks would provide insights into her thinking during the U&V task.

Throughout this third phase of the teaching experiment Sue needed to see the red and blue bars moving on the axes in order to operationalize the thinking she engaged in throughout the U&V task. She needed to reason within her perceptual space to construct graphs by representing and connecting landmark states. This suggests that when engaged in the U&V task she did not experience perturbation necessary to construct a way to represent and track two quantities' varying magnitudes. I will illustrate Sue's perception-based constructions by documenting her engagement in the Homer task.

In the Homer task I presented Sue with an animation depicting a straight road with City A located above the road and City B located below the road (see Figure 39). I asked Sue to graph Homer's distance from City B relative to his distance from City A and displayed labeled axes on the screen. Consistent with Sue's conception of graphs as a depiction of a situation, Sue initially understood this prompt as drawing Homer, City A, and City B on her axes. She felt this didn't make sense because she didn't know how to "compare two cities" and she didn't know where to put the two cities on her graph. There was no evidence that Sue constructed any quantities to reason about.

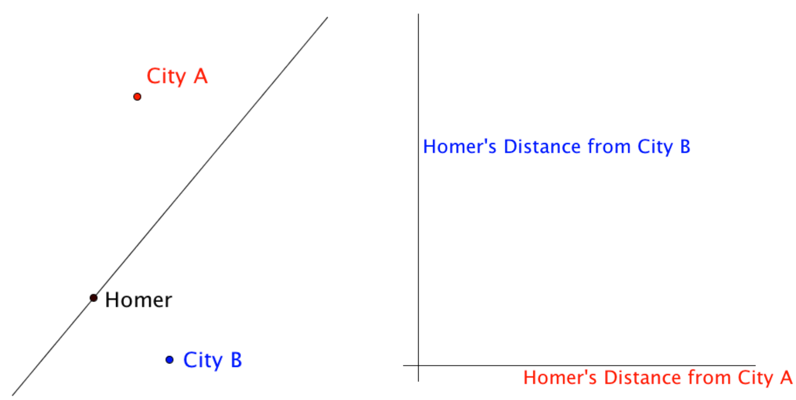


Figure 39: Screenshot 1 of Homer Task. At the beginning of the task the animation displayed (1) a depiction of the situation that showed the location of the cities (fixed) and Homer moving from the bottom of the road to the top of the road at a constant speed and (2) a set of axes labeled with Homer's distance from City B and Homer's distance from City A. (Day 4, Task 11.1)

Next I paused the animation and asked Sue to use the computer pointer to indicate Homer's distance from City A and Homer's distance from City B on the screen. Sue moved the pointer in a straight line between Homer and City A and then moved the pointer in a straight line between Homer and City B. After Sue imagined drawing these segments I let the animation play again. I asked Sue to describe how Homer's distance from City A changed. Sue described in real time how she saw the distance from City A changing saying, "the distance [from City A] is getting shorter and shorter and shorter and it's at its shortest then it starts growing again." Sue gave her explanation as the animation played so that her utterance "and it's [distance from City A] at its shortest" occurred when the animation showed Homer closest to City A. This suggests that Sue's activity tracking Homer's distance from City A was tied to her experience watching him move down the road and that she described the gross variation of the distance as she attended to it in her experiential time. This implies the context in which she was reasoning, her reasoning about how Homer's distance from City A changed, and the product of her reasoning – an image of the smooth variation of *Homer's distance from City A*, were all part of the same cognitive entity.

Sue constructed a similar image of Homer's changing distance from City B. As the animation showed Homer moving down the road Sue described in real time how she imagined the distance from City B changing saying, "it gets closer and keeps getting shorter then the distance increases and keeps increasing." Again, it seems that the context of her reasoning – seeing Homer moving down the road – was essential for Sue to construct an image of the smooth variation of *Homer's distance from City B.*

While Sue constructed an image of how each distance varied independently, she was still unsure how to put the two together. Sue explained: "I understand both by itself but like asking to put them together I don't know how to think about that." This was significant because it implied that Sue did not anticipate that a graph tracked how two quantities change together. I had hoped that a student's engagement in the U&V task would support him in coordinating his image of two changing quantities. I propose two possible explanations for why Sue's engagement in the U&V task did not help her think about "putting them together". First, if Sue's image of the U&V task focused on the motion of the red bar separately from the motion of the blue bar then she would not imagine that representational system as a way of "putting them [images of two quantities] together." A second possible explanation is that Sue did not imagine the lengths of the red and blue bars to represent quantities' changing magnitudes.

To better understand Sue's thinking in the U&V task and the meaning she constructed for the red and blue bars I asked Sue if she could relate the Homer task to the U&V task (see Excerpt 2). As I describe below, Sue's meaning for the U&V task was based in her perception of a moving red bar and a moving blue bar on the axes. With support from Pat and me, Sue came to imagine these bars representing quantities' changing measures. She could coordinate her image of Homer moving down the road with her anticipation of how the bars moved together. However, Sue imagined each bar moving independently of the other. Without the visual support of bars moving on the axes Sue had a hard time coordinating the landmark states each quantity attained. As a result, she had difficulty constructing a graph from her image of how the quantities' changed together.

Excerpt 2: Sue TE Day 4, 00:45:30

(*Animation playing: Homer moves along road at constant speed – see* Figure 39 *for screenshot)*
KF: Is this at all like what we were doing with the U&V task?
Sue: No. I think this is kind of opposite. Like the U and V the blue and the red lines were moving and all I had to do was make essentially one

relation. Just follow the two lines moving and make your graph out of that information. But this one the two points, the red and blue points, are stationary and you are following the one line (*points to road*) and you are asking to compare two things. So it is just different.
KF: In the U&V task were you comparing anything?
Sue: Like you were comparing two things but they were moving. But here the red dot and blue dot aren't moving.

In this excerpt Sue focused on what she saw in each task's animation. She explained that the U&V task was different than the Homer task because the former has two things moving (red and blue bars) and the latter has only one thing moving (Homer). She also explained that her job in each task was fundamentally different: make one curve from two changing things in the U&V task and relate two static things in the Homer task.

At this point I modified the animation to display a red segment between Homer and City A and a blue segment between Homer and City B (see Figure *40*). The lengths of these segments changed as Homer drove down the road. I wanted to see if the presence of a changing red line and changing blue line influenced Sue's coordination of the tasks.

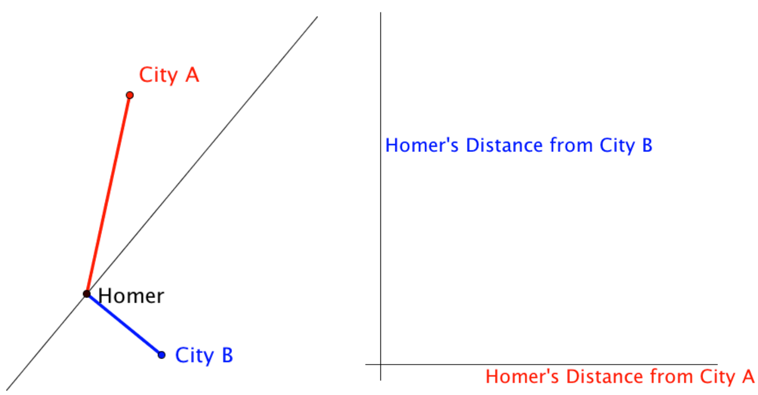


Figure 40*:* Screenshot 2 of Homer Task. I introduced a new feature of the animation - red and blue line segments between Homer and City A and City B, respectively (Day 4, Task 11.1).

Sue related the new visual display to her memory of the U&V task. She explained,

> I think it is similar to the thing we did on the other graphs where the two lines were moving. … The only thing that is similar is they are both asking to compare something that moves and changes together at the same time. So that is how it is similar. Um they are different. The graphs on the other ones had all four quadrants and you could see the lines move (*gestures up and down then left to right*) and this one is just a straight line (*gestures diagonal line parallel to the road*) and two lines moving through that line.

In this excerpt Sue explained the tasks are similar because both tasks have moving red lines, moving blue lines, and stationary black lines. That is, Sue focused on perceptual similarities when comparing the two tasks. This suggests that Sue's image of the U&V task focused on her perception of two bars' changing in length and not what those bars might represent.

While Sue now saw the tasks as similar, her coordination of the tasks did not support her in completing the Homer task. It seems that Sue did not construct a graph of Homer's distance from City B relative to his distance from City A because the orientation of the perceptual features were different: the red and blue bars in the Homer task were not oriented perpendicular to each other. Although there were axes depicted on the screen and on a piece of paper in front of her, Sue did not make any utterance or gesture to suggest that she imagined representing the red and blue line segments displayed between Homer and City A and City B on the axes. This suggests that Sue did not anticipate re-orienting the segments between Homer and each city so that they were perpendicular to each other. This implies that Sue needed the same perceptual stimulus in order to re-use the activity she learned during the U&V task in the Homer task.

At this point, it seemed that Sue was not able to make progress constructing a graph, so I suggested Sue imagine the lengths of the lines between Homer and the cities represented on the axes displayed in the animation. I paused the animation and displayed these bars in GeoGebra® (see Figure *41*). The animation now showed red and blue line segments between Homer and City A and B, respectively, as well as red and blue bars along the axes. From my perspective, the bars' lengths represented the magnitude of *Homer's distance from City A* and *Homer's distance from City B.*

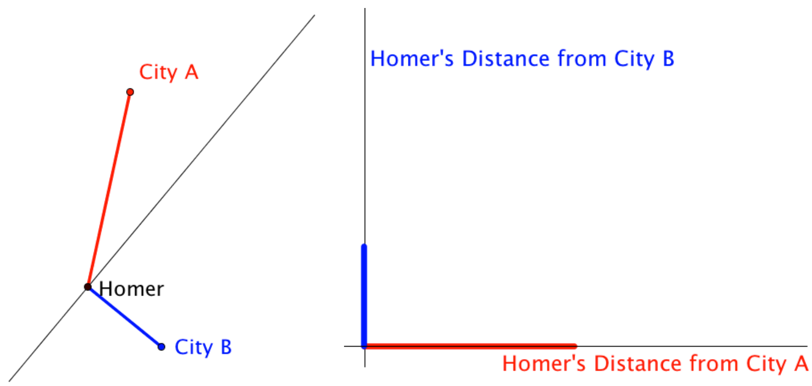


Figure 41: Screenshot 3 Homer Task. I introduced a new feature of the animation - red and blue bars on the axes to represent the measure of Homer's distance from City A and City B, respectively (Day 4, Task 11.1).

With the bars on the axes and the animation paused, Sue anticipated how the lengths of the bars would change together as Homer drove down the road. She explained,

> So the blue bar (*points to blue bar on axes*) is going to start [to] decrease because he comes closer to City B (*points to depiction of the road*) and the red line is also going to start to decrease. Like the blue line is going to further down than the red line is going to come close. Then eventually City B that's like Homer's shortest distance and then once he reaches that it is going to start to go up and start to increase because he is going further away from the City [B]. Then the red line is

going to continue to decrease until his shortest distance from City A and then it is going to start to increase again.

Sue at first seemed to attend to the bars' simultaneous variation ("the blue bar is going to decrease … and the red line is also going to decrease"), but quickly moved to describing each bar's variation in isolation of the other. This suggests that while Sue's language implies that she was thinking about how the bars change together, she likely thought about changes in the two quantities asynchronously. More specifically, Sue thought about how Homer's distance from City A changed from one landmark point to the next. Then, Sue thought about how Homer's distance from City B changed as she imagined Homer moving between the same locations on the road. As Sue thought about changes in Homer's distance from City B she could be confident that the red bar remained just where she left it. Next Sue went back to thinking about Homer's distance from City A while she imagined the blue bar staying exactly the same, etc. Since Sue thought about changes in each quantity separately, it was hard for her to maintain a focus on both quantities as Homer traveled the entire length of the road. Thus, by the end of her explanation Sue focused exclusively on how she imagined Homer's distance from City A to change.

Since the animation was paused Sue could control when and how she imagined Homer moving down the road. This is in contrast to Sue's activity earlier in the task when she constructed an image of how each distance changed in real-time with her experience watching animation. With the animation paused, Sue's thinking about how each quantity varied did not have to keep pace with a continuously changing animation. Instead, Homer did not move to the next section of the road until Sue imagined him moving there. As a result, Sue's thinking about asynchronous changes in each quantity's magnitude could keep pace with how she imagined Homer moving.

Sue's activity describing how the lengths of the bars would move along the axes did not support her in sketching a graph that showed how the bars' changed together. This suggests that Sue's images of each quantity's variation were not at a reflected level so she experienced difficulty coordinating her images of each quantity's variation with her graphing activity. Once the animation started playing Sue quickly drew a graph by identifying landmark states when either the red bar or blue bar was "at its shortest" and then connecting those points with straight lines (see Figure 42).

Sue's graph Graph of actual covariation

Figure 42: Sue's graph, constructed in the presence of Homer moving along the road, segments between Homer and City A and City B visible, and blue and red bars changing in length along the axes (see *Figure* 41).

With the animation playing the task was the same as the U&V task and no longer required additional reasoning. Sue no longer had to engage in the cognitively demanding work coordinating (1) her image of Homer moving down the road, (2) her image of asynchronous changes in the bars' lengths, and (3) her activity identifying and coordinating landmark points in the plane. She could construct the graph by coordinating her perceptions of the motion of two bars.

Sue’s engagement in the second version of the task exemplified her need to see the bars moving on the axes in order to identify landmark points, the same type of thinking she constructed in the U&V task. In this second version of the Homer task the shape of the road and the relative location of the cities remained the same. The only thing that changed from the first version was how Homer moved along the road. Instead of moving in one direction along the road now Homer drove forward, then back toward the start, drove forward again, then backward, and finally drove forward to the end of the road. From the very beginning of this second version Sue anticipated that the computer could make red and blue bars that moved on the axes. She anticipated using these moving bars to make a graph, but she had a hard time imagining these bars herself. With the animation playing Sue explained,

> So when I look at it like that (*points to animation of Homer driving along road*), I can’t really see a graph. When I look at it with the blue and red line [on axes] then I can see a graph. But even still I try and imagine the same setup with the two lines but I just can’t keep track of it. I am just trying to imagine the blue and red lines (*points to axes*) but I just can’t. … I know that after seeing the red and the blue lines it would make sense so then like I would be able to graph it. But when it is something like this (*points to Homer on the road*) I can’t relate them. I don’t know why.

In this utterance Sue explains that once she sees the red and blue bars moving on the axes she will know how to make the graph. She then describes that when she tries to imagine these bars from her image of Homer’s motion and changing distances between Homer and each city she “just can’t keep track of it.” This is significant because it suggests that with the animation playing, Sue’s activity can’t keep up with her perception of Homer moving along the road. More specifically, Sue is trying to maintain three constructions as the animation plays: (1) construct image of how each distance changes as Homer moves down the road, (2) orient these distances perpendicularly, and (3) identify landmark points in the motion of the bars. Sue’s activity constructing this imagery cannot keep up with her experience watching Homer move along the road.

Once the computer displays the bars moving, the Homer task has the same cognitive demand as the U&V task. Sue no longer has to think about her image of quantities’ varying measures and no longer has to imagine orienting the bars perpendicularly. With the bars moving on the axes Sue can use her perception of the bars’ movement to identify landmark points when either the red or blue bar changes direction.

Meanings that inhibit emergent shape thinking. Saldanha and Thompson (1998) provided one of the earliest conceptions of emergent shape thinking through their description of covariational reasoning. They explained,

> Our notion of covariation is of someone holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one’s understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity’s value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value (p. 1-2).

Sue's engagement in this teaching experiment highlights the importance of constructing an image of each quantity's variation so that while one thinks about variation of Quantity X she can anticipate how Quantity Y is changing. Since Sue did not anticipate keeping track of how Quantity X changed as she imagined Quantity Y changing, she had a hard time constructing pairs of landmark states without the seeing the bars moving on the axes.

Generalizations in Post-TECI

After the fourth session of the teaching experiment I engaged Sue in a 1-hour post teaching experiment clinical interview (post-TECI). In this interview I engaged Sue in the same tasks that I used in the recruitment interview and the pre-TECI. There was no noticeable difference in Sue's engagement in the graphing tasks. Sue still imagined the graph of Car A's speed and Car B's speed relative to elapsed time (Figure 43) as a picture of the roads the cars traveled along and used the lengths of these curves to reason that Car B traveled less distance than Car A over the first half an hour because the line to the left of the half mile marker was shorter for Car B.

Task: Consider the graph below, which describes two cars' speeds
in terms of the number of hours elapsed since they started traveling.

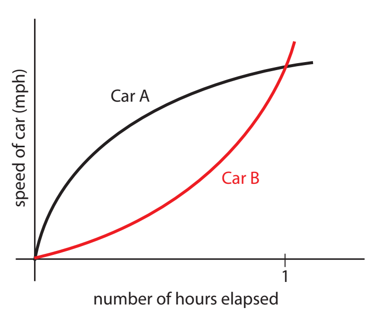


Figure 43: Car A and Car B problem: Task 2 – recruitment interview (Monk 1992).

I take this as evidence that Sue still imagined graphs as iconic translations and saw the curve as a depiction of an aspect of the situation. Sue also constructed graphs as iconic translations in the skateboard task. The prompt read:

> A skateboarder skates across a half-pipe and returns back to start. Graph his horizontal distance from start relative to his vertical distance above the ground.

Sue focused on the skateboarder's motion and tracked his movement up and down the ramp in order to graph the skateboarder's horizontal distance from start relative to his vertical distance above the ground. These examples highlight that Sue's graphing actions were still constrained to her image of an object's motion and she had not abstracted attributes to coordinate in her graphing activity.

This suggests that the constructions Sue made in the teaching experiment did not support her in constructing a graphing scheme where she understood graphs as a way to track how two quantities' magnitudes changed together. This is not surprising since Sue consistently thought about two quantities' variation asynchronously. Additionally, by imagining quantities' magnitudes changing from one landmark state to another, Sue did not construct images of change that supported her in thinking about the nuances in how two quantities changed together. This highlights the importance of constructing operative images of the nuances in how each quantity's magnitude changes in order to coordinate and then represent these images. Thompson (2013) claimed, "To construct stable understandings, one must repeatedly construct them anew" (p. 61). Sue's engagement in this teaching experiment suggests that students will need to repeatedly construct images of smooth variation in order to construct a stable image of change.

CHAPTER 7
A TEACHING EXPERIMENT WITH ALI

At the time of the study Ali had just completed spring semester precalculus with a B in the course. She had declared a double major in linguistics and global studies. As a liberal arts major, precalculus satisfied Ali's university math requirement. Ali did not plan to take another math course. During her recruitment interview Ali engaged in static shape thinking and described her graphing activity as "thinking of possible… shapes it could be". Thus, I selected Ai to participate in a teaching experiment to understand how students who conceptualize graphs as static shapes might come to reason about graphs as emergent representations.

My teaching experiment with Ali was the second of three teaching experiments. Her pre-teaching experiment clinical interview (pre-TECI) took place 51 days after her recruitment interview (see Table 8).

Table 8

Ali's Schedule

Date	***Event***
May 9, 2016	Completed Spring Semester Precalculus
May 23, 2016	Recruitment Interview
July 13, 2016	Pre-Teaching Experiment Clinical Interview
July 15, 2016	Teaching Experiment Session 1 (witness: P.W. Thompson)
July 20, 2016	Teaching Experiment Session 2 (witness: P.W. Thompson)
July 21, 2016	Teaching Experiment Session 3 (witness: P.W. Thompson)
July 22, 2016	Teaching Experiment Session 4 (witness: P.W. Thompson)
July 25, 20116	Post-Teaching Experiment Clinical Interview

Ali's Initial Meanings for Graphs and Formulas

Ali participated in two clinical interviews—the recruitment interview and the pre-teaching experiment clinical interview—prior to her four-session teaching experiment. The purpose of these interviews was to establish a base-line characterization of Ali's meanings for graphs and formulas and to understand how Ali coordinated these meanings.

Ali's Initial Meanings for Graphs

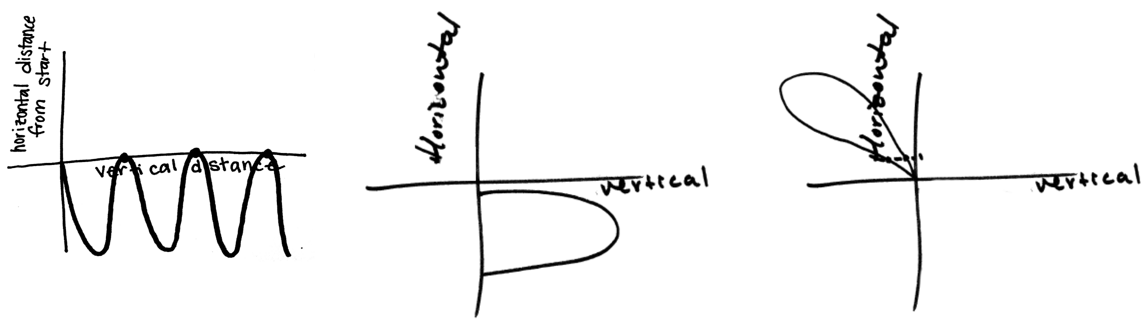
During Ali's recruitment interview and pre-TECI she demonstrated two schemes of meanings. The first scheme of meanings consisted of two distinct graphing activities: first generate a graph and then understand that sketched graph as a representation of the gross, and asynchronous, variation of Quantity X and Quantity Y. With the second scheme of meanings, Ali understood graphs as collections of points where each point represents a pair of values. There is no evidence that Ali coordinated these meanings. In this section I will illustrate these two schemes of meanings and I will describe the conditions under which Ali used each meaning.

Scheme 1: Two distinct graphing experiences. When Ali created a graph from a contextual description of a situation she engaged in two distinct activities. First, Ali generated a shape by tracking one quantity's variation as she imagined that variation in her experiential time. Then, Ali used the properties of the shape she created to reason asynchronously about the variation of the two quantities labeled on the graph's axes. If the shape she created did not match her anticipation of how each quantity varied, then she guessed shapes from her memory of past graphing activities until she picked a shape that matched her image of how each quantity varied. This suggests that Ali used distinct and uncoordinated systems of actions when generating graphs (drawing shapes) and understanding sketched graphs (reasoning about two quantities' asynchronous variation).

I will illustrate this scheme of meanings with Ali's engagement in the last task of the recruitment interview, the skateboard task. The task read:

> A skateboarder skates on a half-pipe like the one shown below. The skateboarder goes across the half-pipe and then returns to the starting position.

On the task sheet there was a picture of a skateboarding half-pipe ramp illustrating a starting point and a skateboarder at the bottom of the ramp (see Appendix A). I asked Ali to graph the skateboarder's horizontal distance to the right of the starting position relative to the skateboarder's vertical distance above the ground. Ali made three attempts to draw the graph (see Figure 44).



Ali's first attempt — Ali's second attempt — Ali's third attempt

Figure 44: Ali's three attempts to graph skateboarder's horizontal distance from start relative to his vertical distance above the ground.

On Ali's first attempt she drew an oscillating curve in the fourth quadrant. Since Ali imagined the half-pipe below ground, Ali made this graph by tracking how she imagined the skateboarder's vertical distance changing as she imagined the that variation in her experiential time. After drawing the curve, and without prompting, Ali determined her graph was incorrect because “the graph I drew is showing that the vertical distance is increasing the whole time.” She went on to draw two more shapes (Figure 44) and each time appropriately reasoned why her sketched graph was incorrect. For example, in her second attempt, Ali drew a side-ways U-shape in the third quadrant. After drawing the shape Ali indicated that her graph showed the vertical distance was positive when she wanted to show the vertical distance was negative. Ali ruled out her second graph and tried another shape.

After Ali rejected her third graph I asked her to explain her approach to graphing. She explained that she would “think of … shapes that can be drawn” (Excerpt 3).

Excerpt 3: Ali recruitment interview, 01:31:25

KF: What are you doing when you are trying to figure out what graph it could be?

Ali: Um. Well I think of like. I either focus. I go back and forth with like okay vertical distance and horizontal distance. So I think of potential like, I guess shapes, that can be drawn and then I'm like does this fit the characteristic of the horizontal distance. If it doesn't then it is out and I think of another one. And so. That's how I usually go about with graphing graphs until I eventually - I'm like this one fits both criteria

In this excerpt, Ali described her three-step approach to graphing: (1) draw a shape, (2) consider what the shape conveyed about the variation of each quantity separately, and (3) adjusting the shape to match her image of each quantity’s variation. I take this as evidence that Ali actually constructed two images of the quantities’ variation. Specifically, she compared her image of each quantity’s variation that she constructed from the sketched graph to her image of each quantity’s variation that she constructed from her understanding of the phenomenon. Ali decided that her graph was correct when her image of each quantity's variation matched her understanding of her sketched graph. In this task, Ali concluded that there was a correct graph out there but "she could not think of it [the shape]".

In summary, Ali engaged in two distinct graphing experiences – making a graph and reasoning from the sketched graph. As distinct experiences, Ali's reasoning was about the product of her actions, her sketched graph, and not the actions she engaged in to make the graph. This suggests that Ali's meaning for her graph was an empirical abstraction.

Scheme 2: Graphs as collections of points. When the problem statement included numerical values, either in a table or on the axes of a graph, Ali demonstrated a different scheme of meanings; Ali understood her mathematical activity as coordinating two quantities' values and interpreting graphs in terms of pairs of values.

Example 1. In the recruitment interview I presented Ali with a table of values situated in the context of a girl walking away from her house (Figure 45). Throughout Ali's graphing activity she attended to the values in both columns simultaneously.

Susie is walking away from her house. The table below represents her distance from home (in feet) in terms of the number of minutes elapsed since she left her house. Sketch a graph of this relationship.

t	***d***
1	2
2	5
3	10
4	17

Figure 45: Susie Walking Task (recruitment interview, Task 1)

Ali constructed her graph by plotting four points, one for each row in the table. She then connected these points with a curved line (Figure 46). When Ali explained what a point on her graph represented she attended to the values on both axes. For example, when Ali explained what the point (1, 2) represented she moved her hand upward from the first tick mark on the horizontal axis and moved her hand to the right from that tick mark that she had labeled as 2 on the vertical axis. By focusing on the values on each axis, instead of going over 1 and up 2, I take this as evidence that Ali imagined constructing the point by extending values from the axes into the plane to construct the point's coordinates. After plotting this point Ali explained, "from this one point (*points to dot in plane*) you can gather information about how much time has elapsed and how much distance she traveled." This revealed Ali understood that her point gave her information about two quantities' values simultaneously saying,

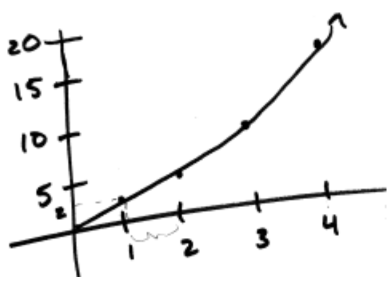


Figure 46: Graph Ali constructed from table of values in Susie Walking Task (recruitment interview, task 1)

At the time of the interview I took this as evidence that Ali constructed a point as a multiplicative object. However, as I will describe in the next section, Ali did not consistently think about a point as a representation of two measures *simultaneously*. This suggests Ali's construction of a point was a pseudo-multiplicative object. In the context of plotting points from a table of values, Ali understood the point's coordinates to simultaneously represent both values in the table. She was then able to interpret those values in terms of the contextual situation. I claim that Ali's point-plotting scheme and her meaning for a point's coordinates is what supported her in attending to two values simultaneously; she did not understand the point itself as simultaneously representing two quantities' values. I will provide evidence to support this claim in the remainder of this chapter.

In the next part of this task, Ali's remarks suggested that she understood the line connecting the points as an infinite collection of points. More specifically, as Ali imagined the event, Susie walking, happening continuously she imagined constructing a time-distance pair at every possible moment in time. Ali elaborated on this thinking in Excerpt 4.

Excerpt 4: Ali Recruitment Interview, 00:10:09

KF: Why did you connect the points with a somewhat curvy line
Ali: Because. Like there is time in-between. It is not just going to be exactly 1 minute and then 2 minutes. There is time in between like one minute and ten seconds. So you know that as she is always walking there is going to be a different amount of distance as well.
KF: How are those different distances represented?
Ali: Um since it is connected there is like the line is. I can think of it as being a whole bunch of little points being drawn.

In this excerpt Ali focused on pairs of values that were in-between those given in the table. Ali's utterance: "you know that as she is walking there is going to be a different amount of distance as well" (lines 4-5) suggests that Ali was imagining change in progress as she imagined Susie walking. Coupled with her persistent attention to both elapsed time and distance traveled, Ali anticipated creating time-distance pairs for every moment in time. Graphically she understood this as "a whole bunch of little points" (line 8) to show the Susie's distance at every moment in time. I take this as evidence that when

Ali understood a graph in terms of points with coordinates she understood lines in the plane as collections of such points.

Example 2. Ali's meaning for a point's coordinates conveying a pair of measures extended beyond the points that made up the curve. Ali also understood points in the plane not captured by the curve (i.e., $\{(x,y) \mid f(x) \neq y\}$) as having coordinates that represented pairs of measures that did not satisfy a given relationship. I will illustrate this thinking with Ali's engagement in the cell-phone task. This task read:

> A company produces different sized smart phones with rectangular screens. The screens' dimensions are w and h, where the height of the screen (h) is half the width of the screen (w) for all sizes of smartphones.

I presented Ali with a graph of the cell phone screen's diagonal length relative to its width (see Appendix B). I asked Ali if it was possible to use the graph to determine if there were more screens the company could make or more screens the company could not make. Without hesitation, Ali said there were more screens the company could not make. She elaborated,

> With this line (*moves pen along line in plane*) um it is giving you like a specific. There is only one amount of quantity specific for that quantity so that is the correct one (*uses right hand to gesture up from vertical axis to meet line and the left hand to gesture right from vertical axis to meet line at same place*). Whereas if I like created um just put random numbers together um like for example like if I pick any points from here (*draws squiggle above the line in graph*) or here (*draws squiggle below the line in the graph*) they are not going to be correct.

This utterance revealed that Ali imagined the plane full of points. She understood the points on the line had coordinates that showed the pairs of measures that were "correct". I interpreted Ali to understand that these points satisfy the specified relationship. Ali understood the other points in the plane, those not captured by the line, to have coordinates with random pairs of numbers that do not satisfy the constraint given by the company. Ali determined there were more screens the company could not make because she imagined more points off the line than on the line.

I take this as evidence that Ali constructed the coordinate plane as a pseudo-multiplicative object. When she imagined points to have coordinates she imagined creating every possible (x, y) pair. This supported her in understanding the graph as capturing the (x, y) pairs that satisfied the specified relationship. This implies that Ali's meaning for the graph extended beyond the curve; she situated her meaning for the curve in terms of all the possible coordinate pairs in the plane.

Real and imaginary points. My analysis of Ali's activity in these interviews revealed she held two meanings for graphs: (1) graphs are shapes that show the gross (and asynchronous) variation of Quantity X and Quantity Y and (2) graphs are a collection of points that have coordinates, which represent pairs of measures. The first meaning is non-multiplicative; Ali reasoned about the two gross variations asynchronously. On the other hand, her other meaning that involved her conceptualizing a graph as a collection of pairs of measures, is a multiplicative construction.

To avoid constructing two distinct meanings for a graph Ali differentiated between what she called *real points* and *imaginary points*. When Ali understood the graph as a depiction of the gross and asynchronous variation of two quantities she talked about the graph being made of imaginary points. Whereas when she understood the graph as a collection of points that represent pairs of measures she talked about the graph as being made of real points. In Excerpt 5 and Excerpt 6 Ali explains the difference between real and imaginary points.

Excerpt 5: Ali pre-TECI, 00:14:35 (Ali is reasoning about a graph she constructed from qualitative description: the bottle problem)

KF: So you said they are imaginary points. How is that different than a real point?

Ali: I would say it is because I am just picking a random one rather than having one being assigned to me. For me it is like an imaginary point I know it is there but I don't know the exact location of it, or you know I am just imagining it in my head.

KF: So what do you mean you don't know the exact location of it?

Ali: Like I don't know the exact quantity[4] (number) it would be, so it is just a random point.

Excerpt 6: Ali pre-TECI, 00:31:52 (Ali is reasoning about graph with numerical values on axes: the cell phone task)

KF: Before you talked about imaginary points and real points. So are these real points or imaginary points or some different type of point?

Ali: To me, these three points would be real points.

KF: Are there any imaginary points?

Ali: Uh, yeah. Um. Actually, no. Now that I have like quantities (*sweeps pen across horizontal axis and up vertical axis*) if I were to like draw it from a random point (*draws vertical line at w=1 up to the graphed line and then draws horizontal line back to vertical axis around l=1*) then I would still end up getting a precise quantity (*circles the '1' labeled on the vertical axis*) so for me these are all real points.

KF: So how many real points are there?

Ali: As many that make up the line (*sweeps pen across curve*).

In these excerpts Ali focused on whether or not she knew the numbers associated with the point's coordinates. If she could use the graph (table, or formula) to determine the values of the coordinates then it was a real point. In contextual situations, such as the

[4] I interpret Ali to use "quantity" synonymously with "number".

bottle problem, Ali had no way to imagine approximate measures of the quantities. As a result, could not imagine the point's coordinates and considered the point to be an imaginary point. It seems that Ali's distinction between real and imaginary points allowed her to separate her two meanings for graphs so that she only demonstrated one meaning for a graph on any given task.

I take Ali's distinction between real and imaginary points as evidence that her understanding of real points was a pseudo-multiplicative object. For Ali to think about a point in terms of two quantities' measures she needed to know the numerical values of the coordinates. This suggests that it is her meaning for a point's coordinates, not the point itself, which is multiplicative. As a result, when Ali imagined points that had no coordinates, what she called imaginary points, she did not understand the point in terms of two quantities' measures.

For Ali to coordinate her two meanings for graphs she would need to understand imaginary points as real points with specific, but unknown, coordinates. That is, Ali would need to conceptualize points as multiplicative objects that unite attributes' measures as opposed to uniting numbers.

Ali's Initial Meanings for Formulas

On three occasions in the pre-TECI I asked Ali to either construct or interpret a formula in relation to a described situation. On all of these tasks Ali reasoned about a formula as a way to "convert" one quantity to another.

For example, in the bathtub task the problem statement gave the total weight of the water and tub (875 pounds), the weight of one gallon of water (8.345 lb/gal) (adapted from Carlson et al., 2013). I asked Ali to define a formula that gave the total weight, in pounds, of the tub and water in terms of the number of gallons of water that had drained from the tub (see Appendix B). Ali constructed her formula by stringing together calculations that she understood as “converting” (her word) from one quantity to another. Ali explained her thinking:

> I take whatever number gallon of water that has drained and I times it by 8.345 to convert it into pounds. So then whatever that number is, um. I subtract it from 875 and that will give me the total weight in pounds of the tub and water.

Ali’s utterance suggests two aspects of her meaning for formulas. First, she seems to think about “converting” one measure at a time. She needed to imagine carrying out this sequence of actions for any number of gallons she thinks about. Thinking about one measure at a time inhibited Ali from understanding these conversions as quantitative relationships between quantities, not numbers. Additionally, the utterance suggests that Ali imagined one measure being transformed into another measure. This implies that both measures did not exist simultaneously in her thinking. Thinking about formulas as converting from one quantity to another inhibited Ali from constructing formulas as multiplicative objects.

Ali’s focus on converting from one quantity to another (or one unit of measure to another) supported her in understanding coefficients in terms of the situation. This was evident in Ali’s engagement in the last task of the pre-TECI: the tuition task. The problem read:

> The tuition cost (in dollars), *T*, for part-time students at Stonewall college is given by $T = 300 + 200C$ where *C* represents the number of credits taken (Connally et al., 2000).

Ali quickly determined that the value of *T* was 1900 when told that the value of *C* was 8, and determined the value of *C* was 7 when told the value of *T* was 1700. She also created a table of values for $C = 1, 2, 3, 4, 5, 6, 7$. At the end of the task I asked Ali what the 200 and 300 represented in the formula. Ali explained that the 300 represented the enrollment fee and the 200 was the cost per credit. Without prompting, she went on to explain that one can see the 200 in the table because when the number of credits increased by 1 the change in cost was $200. I interpret Ali's meaning for these coefficients as evidence that Ali understood formulas as a way to convert between quantities (or units of measure) as opposed to merely converting one number to another.

Ali's Initial Coordination of Meanings for Graphs and Formulas

I designed the cell-phone screen task to understand how Ali coordinated her meanings for graphs and formulas. I asked Ali to construct a formula that gave the cell phone screen's diagonal length in terms of its width. Ali used the Pythagorean Theorem to construct the formula $\left(w/2\right)^2 + (w)^2 = c^2$.[5] However, she could not construct a graph from her formula saying, "I can't remember how to draw a graph for these. I am trying to think back to my previous classes and trying to remember what we went through." I take this as evidence that when drawing graphs Ali tried to recall a collection of static shapes and experiences from math class.

[5] Ali included these parentheses in her formula

Since Ali could not construct a graph from her formula, I presented her with a graph of the company's cell phone screens' diagonal length in relation to their width under the constraint that any screen's height is half of its width. When illustrating Ali's point-wise meaning for graphs (p. 173), I described how Ali understood both the points and line on this graph. When I asked Ali if she could relate the graph with her formula she coordinated the formula with her meaning for plotting points. She explained that you could plug in a value of *w* to get a value of *c* and then use those numbers to determine the exact coordinates for each point on the line. There is no evidence that Ali coordinated the overall behavior of the graph (e.g., linear shape) with the structure of the formula.

Summary

Ali's actions during the preliminary clinical interviews revealed the complexity of her graphing scheme. She understood graphing tasks using two distinct schemes of meanings.

When Ali reasoned from a contextual description of a situation, her graphing actions were motivated by her anticipation to produce a shape to then reason about. With this scheme of meanings Ali's graphing activity consisted of two distinct experiences: drawing a graph and reasoning from that sketched graph. Although Ali constructed an image of how each quantity varied from the context, she did not use this image of the two quantities' variation while drawing the graph. Instead, she made the graph by either (1) imagining how one quantity's magnitude changed as she imagined the event unfolding or (2) remembering a shape from her past mathematical experiences. Then, Ali understood her sketched graph as a depiction of two quantities' gross (and asynchronous) variation. This is significant because it suggests that the meanings Ali conveyed in her drawing activity were not the meanings she attributed to the products of her graphing actions. Thus, Ali's meaning for the graph was an empirical abstraction.

When the problem statement included numerical relationships (e.g., cell phone screen's height is half the width), Ali's graphing actions were motivated by her anticipation to keep track of and represent two quantities' values simultaneously. With this scheme of meanings, Ali imagined a graph as a collection of points where each point had coordinates that represented the values of two quantities simultaneously.

Ali's Teaching Experiment

I engaged Ali in four teaching experiment sessions. Each session lasted approximately 1.5 hours and was witnessed by Pat Thompson. The teaching experiments consisted of three series of tasks to help me understand (1) Ali's scheme for quantitative reasoning, (2) the ways of thinking that supported or inhibited Ali from engaging in emergent shape thinking, and (3) the generality of any constructions she made during the teaching experiment.

Teaching Experiment Phase I: Quantitative Reasoning

The first part of the teaching experiment consisted of five tasks I designed to reveal the objects in Ali's images of quantitative situations that she acted upon when reasoning about those situations. My analysis revealed that Ali constructed and reasoned about measureable attributes – quantities. She differentiated these attributes from the objects themselves and then coordinated her image of the attribute with her image of the object's motion in order to construct an image of how the attribute's magnitude changed. I will illustrate this thinking with two examples.

The first task of the teaching session was the airplane task (Appendix C). In this task I presented Ali with a GeoGebra® animation depicting an airplane and a helicopter flying from left to right across the screen. I will focus on Ali's activity during the second version of the task when the airplane was directly above the helicopter (Figure 47). I asked Ali to use the distance between her left pointer finger and right pointer finger to represent the distance between the aircraft as they flew from San Diego to Phoenix (Figure 48).

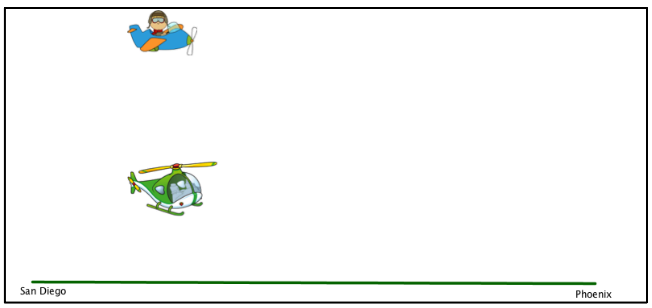


Figure 47: Screenshot of second version of airplane task (Day 1, Task 1).

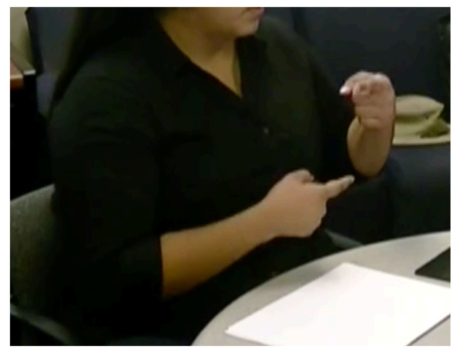


Figure 48: Ali uses the distance between her pointer fingers to represent distance between airplane and helicopter

Ali explained that she could keep track of the distance between the aircraft in two ways: (1) moving both of her fingers to show the motion of the helicopter (bottom finger) and the motion of the airplane (top finger), or (2) keeping her bottom finger fixed so that the "helicopter is a reference point" and you see "the plane's distance in relation to the helicopter's distance". Ali's first way of tracking the objects suggests that she constructed the distance between the aircraft not as a new cognitive object, but as an implication of tracking each object's motion. However, Ali's second way of tracking the objects suggests that she could construct the distance between the aircraft as a quantitative structure: *the distance between the aircraft* is how much larger *the distance between the airplane and the ground* is than *the distance between the helicopter and the ground*.

Ali's description of two ways to imagine the distance between the aircraft highlights the cognitive work necessary to consistently attend to quantities and relationships among quantities. While Ali could construct *the distance between the aircraft* as a quantitative difference, there were moments when she focused on tracking the two objects' motion and described how the helicopter and airplane moved up or down in the sky. It seems that as Ali's attention shifted between the motion of the objects and her image of the plane's distance above the ground in relation to the helicopter's distance above the ground she was constructing an image of how the quantitative difference changed over time.

Ali's image of a quantity's variation consistently focused on how the quantity's size changed, what Thompson and Carlson (2017) call gross variation. As Thompson et al. (2014) explained, when one has an *awareness of size*, she can make judgments about whether Quantity A is smaller than or larger than Quantity B. She can imagine these sizes getting smaller or larger but does not imagine the size having a measure as it changes.

Ali's engagement in the box problem (Day 1, Task 2) highlights her thinking about a quantity's varying size. In this task I asked Ali to imagine creating a box by cutting out equal sized square cutouts from the corners of an 11"x7" piece of paper (see Appendix C). To support Ali in imagining attributes of the box changing, I displayed an animation depicting an unfolded view of the box (see Figure 49 for selected screen shots).

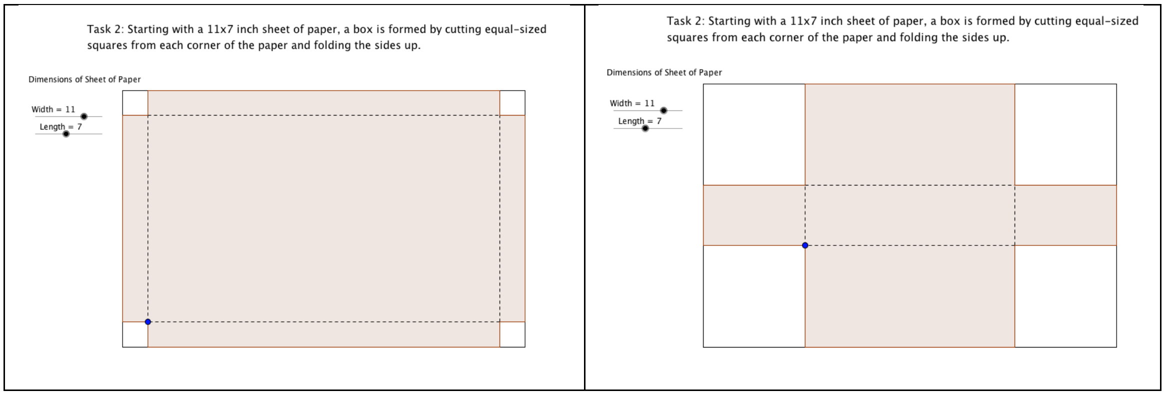


Figure 49: Selected screen shots from box problem (Day 1, Task 2)

As Ali watched the animation play she spoke of the width, length, and height increasing and decreasing. Despite being told the initial dimensions of the paper, Ali did not say anything to suggest she was thinking about these attributes taking on numerical values. This suggests that Ali's image of the quantity's variation focused on whether the quantity's size increased or decreased. As a result, when she went to relate two quantities (e.g., relate width of the box to cutout length) she was constrained to relating directional changes in each quantity's size. For example, she reasoned that as the cutout length increased the length of the box decreased. For Ali to relate these dimensions she needed to conceptualize the length of the paper as a quantity with a fixed value and then conceptualize the box's side length as the difference between the paper's length and twice the varying measures of the square cutout's side-length.

Teaching Experiment Phase II: Supporting Emergent Shape Thinking

The second phase of the teaching experiment lasted two sessions (Day 2 - 36 minutes, and Day 3 - 1 hour 17 minutes). In this session I engaged Ali in tasks I designed to support her in engaging in emergent shape thinking (see learning trajectory in Chapter 5). As I describe in the following section, Ali began to engage in emergent shape thinking and appeared to understand a graph as being made by coordinating how two quantities changed together. In this section I document both my efforts to support Ali in making these constructions and elaborate the constructions Ali made as she began to conceptualize graphs as emergent representations.

In the second day of the teaching experiment it became apparent that when Ali imagined change in progress she did not imagine constructing a point to simultaneously convey two measures. I will illustrate this thinking with Ali's engagement in the U&V task (adapted from Thompson, 2016).

I presented Ali with an animation that depicted a red bar along the horizontal axis and a blue bar along the vertical axis. As the animation played, the lengths of the bars varied simultaneously in such a way that each bar had one end fixed at the origin. (See Figure 34 for a sequence of screenshots from the video). In the first version of this task the horizontal (red) bar's unfixed end varied at a steady pace from left to right while the vertical (blue) bar's unfixed end varied unsystematically. I explained to Ali that the length of the red bar represented the varying value of u and the length of the blue bar represented the varying value of v. Finally, I presented Ali with a printout that included a screen capture of the initial position of the bars in the animation and then asked her to graph the value of v relative to the value of u. The video played repeatedly until Ali completed the task.

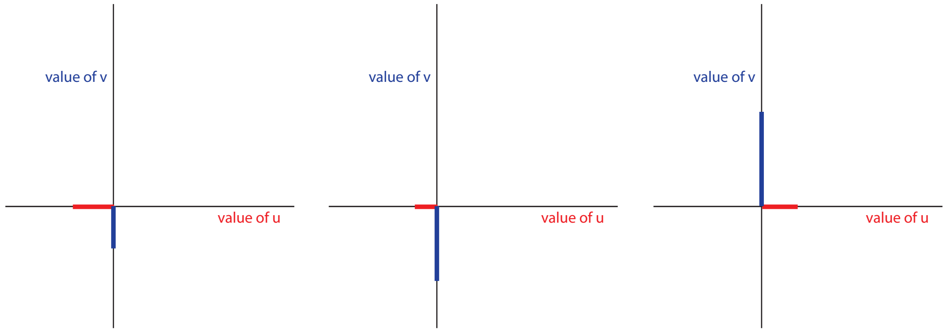
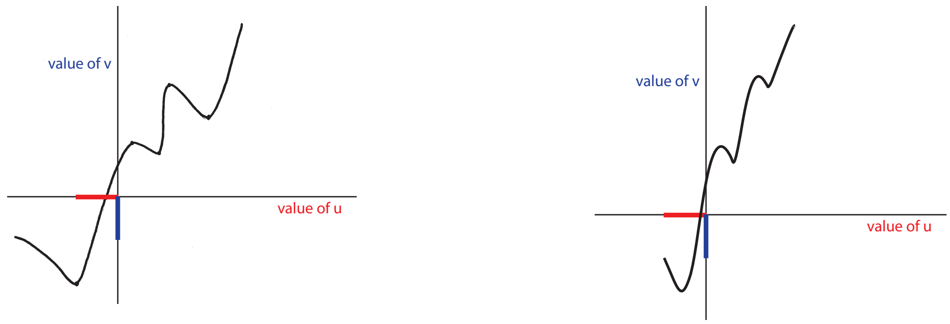


Figure 50: Three screenshots from U&V task (adapted from Thompson, 2016).

After watching the video play twice Ali made five dots in the plane and then connected these dots with a curved line (see Figure 51). She explained that she figured out the graph by “looking at the motion of how the blue line is increasing and decreasing” and made a dot each time “the blue line kinda stopped and the line kinda dipped down.” When I asked Ali about the red line she said, “since the whole time this red line is increasing it (the graph) is going to the right.” Ali’s focus on the motion of the blue bar supported her in making a shape similar to the graph of the actual covariation depicted by the varying lengths of the red and blue lines (Figure 51). However, she did not attend to the length of the red and blue bar simultaneously when constructing her graph. Instead Ali thought about the variation of each bar’s length asynchronously and understood a point on her graph as a representation of the length of the blue bar. In the following paragraphs I provide evidence to support this claim.



Ali's graph for U&V task with a generally correct shape but incorrect initial point.

Graph of actual covariation of bars' lengths (Ali never saw computer make this trace)

Figure 51: U&V Task Version 1: Ali's solution and graph of actual covariation (Day 2, Task 7.1)

First, consider Ali's initial point (in Q3, see Figure 51). Since Ali sketched her graph on axes that displayed the initial lengths of the red and blue bars, Ali's initial point should have been placed at the intersection of the extensions of these bars (see graph of actual covariation in Figure 51). However, Ali's initial point was only aligned with the extension of the blue bar. Ali explained how she thought about her first point saying, "I based it off of this blue line (*points to blue line on vertical axis*)." Notice that Ali did not mention the red bar when discussing how she placed her initial point. Ali's focus on only the blue bar when placing her initial point is evidence that she was not conceptualizing the mark in the plane as a way to unite two attributes simultaneously – as a multiplicative object.

This is significant because in the pre-TECI Ali demonstrated thinking that suggested she imagined a point representing two quantities' magnitudes simultaneously (I describe this thinking on p. 170). However, in the U&V task Ali focused on the length of the blue bar when marking a point. This suggests that Ali did not assimilate the U&V task to her point-plotting scheme where she understood a point in terms of measures represented on each axis. I claim that Ali did not imagine the lengths of the bars taking on values. As a result, she did not have numbers to imagine uniting through a point's coordinates. I provide evidence to support this claim in the following paragraph.

Notice that the axes I presented to Ali had no values or tick marks along the axes. For Ali to imagine the lengths of these bars taking on a value – a specific measure with respect to some unit – she would need to impose a measurement system on the axes. Without the measurement system one's image of the quantities' variation is restricted to a sense of the magnitude's size changing. It is possible that Ali imagined the lengths of these bars taking on a measure but she did not know what that measure was. Without knowing the specific measures each quantity took on, Ali was limited to thinking about her graph in terms of imaginary points.

Once she anticipated plotting imaginary points she understood her graph to show two quantities' gross variations asynchronously. With this thinking she explained her graph saying, "As the red is increasing (*traces hand left to right on horizontal axis*) it (the graph) is showing the motion or like the path of the blue line (*traces along curve from left to right*)." While she understood her graph to show each quantity's variation separately, she could not re-present two distinct images of variation in a single line. As a result, she focused on her image of the motion of the blue bar since it was the one that varied unsystematically. She made her graphing by tracking the end of the blue bar as she witnessed it in her experiential time.

This task was not designed to support Ali in making new constructions. Instead, Thompson designed it to assess the role of multiplicative thinking in one's covariational reasoning (see Thompson, Hatfield, Yoon, Joshua, & Byerley, under review). Since Ali's activity suggested that she had a non-multiplicative conception of her graph, Pat and I devised two didactic objects for the third teaching session to support Ali in simultaneously attending to two varying attributes when constructing and reasoning about a graph. As Thompson (2002) explained, a didactic object is "'a thing to talk about' that is designed with the intention of supporting reflective mathematical discourse" (p. 198). The two didactic objects we implemented had been previously conceptualized by Pat Thompson (see Thompson, 2002; Thompson et al., under review). They were intended to support students in (1) conceptualizing a correspondence point that simultaneously represented *two* attributes' measures and (2) conceptualizing a curve as a locus of such points.

Didactic object I: Conceptualizing correspondence points. The first didactic object was intended to support Ali in conceptualizing a point as a multiplicative object that unites attributes' measures as opposed to uniting numbers. At the beginning of the third teaching session I introduced the notion of a correspondence point as a way to simultaneously represent the value of u and the value of v. I modified the U&V animation so that at any moment I could pause the animation and display the correspondence point (see Figure 10). Following the recommendation of Thompson et al. (under review), I engaged Ali in an activity where I let the animation play, paused the animation, and asked Ali to use the pointer to show where the correspondence point would be. Each time I asked Ali to justify why the correspondence point would be in that specific location. Finally, I displayed the correspondence point to confirm Ali's conceptualization. I repeated this four times to support Ali in repeatedly constructing the location of the correspondence point given the lengths of the red and blue bars. I hoped that repeatedly placing the correspondence point would support Ali in coordinating her understanding of the correspondence point with both the length of the red bar *and* the length of the blue bar.

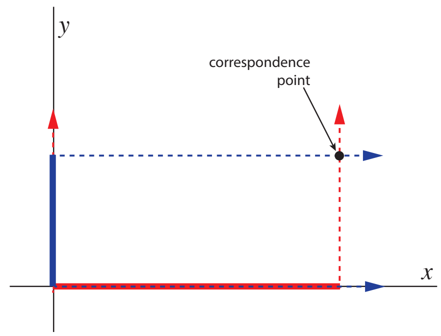


Figure 52: A point as the intersection of two quantities' values extended from the axes.

Next, I asked Ali to imagine tracking the correspondence point as the animation played and try to form a memory everywhere it had been. As the animation played Ali tracked the correspondence point with the computer pointer. Then she sketched a graph from her memory of where the correspondence point had been (Figure 53).

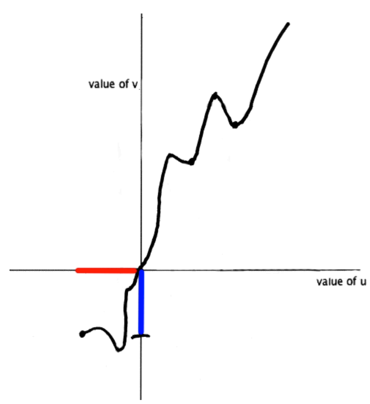


Figure 53: Ali's graph showing everywhere she remembered the correspondence point having been.

While Ali's graph now had the correct shape and correct initial point, Ali explained that her graph was made up of imaginary points that did not have coordinates.

She went on to describe her newly constructed graph by saying, "As *v* is increasing and decreasing, *u* is just going to the right." This suggests that Ali understood her sketched graph as a representation of two quantities' gross variation asynchronously. I take this as evidence that Ali had constructed a new meaning for constructing a graph – simultaneously track the lengths of both bars. However, this focus on the lengths of *both* bars did not persist in her memory of her graphing activity. As a result, she did not understand her sketched graph in terms of her graphing actions. Instead, she understood her sketched graphing with the same scheme of meanings that she demonstrated in the initial clinical interview; a graph represents two quantities' gross variation asynchronously.

Rethinking my meaning for emergent shape thinking. Moore and Thompson (2015) explained, "emergent shape thinking involves understanding a graph simultaneously as what is made (a trace) and how it is made (covariation)" (p. 4). At the outset of this study I thought that if one made a graph by simultaneously tracking two magnitudes then she engaged in emergent shape thinking. I had not considered that Ali's meaning for her sketched graph might not reflect the thinking she engaged in to make the graph. Ali's activity in this teaching session revealed that it is nontrivial for students to understand the product of their graphing actions in terms of their graphing actions. This required me to rethink what constituted emergent shape thinking so that I don't take a student's drawing activity as evidence of emergent shape thinking. Instead, I now understand emergent shape thinking to involve both constructing a graph by simultaneously tracking two magnitudes and also understanding a graph as having been made by tracking two magnitudes simultaneously. In the following section I detail constructions Ali made in order to construct an understanding in the moment that her graph as the result of tracking two magnitudes.

Didactic object II: Tinker Bell's pixie dust. The second didactic object involved having Ali imagine her sketched graph (Figure 53) having been made by Tinker Bell, the fairy from Peter Pan's Neverland. As Thompson (2002) explained, when students understand graphs to be made of pixie dust, they have an understanding of their sketched graph that supports them in imagining lines and curves as being composed of points – particles of pixie dust. Additionally, they can understand each particle represents the measures of two quantities simultaneously. Pat introduced this didactic object to support Ali in understanding her sketched graph as a collection of correspondence points as opposed to a representation two quantities' asynchronous gross variations.

Pat introduced this didactic object by first asking Ali if she knew of Tinker Bell (from Peter Pan) and what she left behind as she flew through the air (she did). Ali explained that Tinker Bell is special because she can fly and has pixie dust so that as she flies you "see where she has been in the pixie dust". Pat then asked Ali to imagine her graph having been made by Tinker Bell. He suggested that Ali imagine Tinker Bell flying along the path of the curve so that she left a trail of pixie dust marking everywhere she had been.

Next, Pat asked Ali to think about her pen as Tinker Bell and everything she drew (the curve) as pixie dust. When he asked if there was any pixie dust on her graph Ali explained that each particle of pixie dust looked like an imaginary point. It seems that thinking about an imaginary point as a particle of pixie dust supported Ali in constructing a "real" object in the plane. As a result she had a new cognitive object to operate upon –

her image of particles of pixie dust. As I describe below, imagining particles of pixie dust in the plane supported Ali in differentiating between the motion of the blue bar and the path of the graph. Additionally, Ali came to understand a point in terms of both the length of the blue bar *and* the length of the red bar.

After introducing the idea that Ali's graph was made of pixie dust there were two differences in her mathematical activity. First, when Ali described how the value of v changed she gestured along the vertical axis instead of gesturing along the curve. I take this as evidence that Ali differentiated between the value of v and a place on the curve. I claim that introducing the idea of pixie dust to her image of her sketched graph necessitated that Ali differentiate between the value of v (the length of the blue bar) and the point in the plane (the particle of pixie dust). This differentiation was evident in Ali's image of the blue bar's motion. Now when Ali described her image of the blue bar's motion she gestured along the vertical axis instead of along the path of the curve.

Having differentiated these objects, Ali needed a way to think about placing a particle of pixie dust that involved more than the blue bar. She responded to this intellectual need by coordinating her conception of the end of the red bar and the end of the blue bar in order to think about the location of the particle of pixie dust. I will illustrate this thinking in the following paragraphs.

I asked Ali to think about how a particle of pixie dust ended up in a certain place on the graph. She said that Tinker Bell put it there. She went on to reason that she could think about what Tinker Bell saw in order to "know where to fly". She determined that Tinker Bell needed to keep track of both the length of the red bar and the length of the blue bar in order to decide where to put a particle of pixie dust. This thinking led Ali to understand a point on the curve as showing "she [Tinker Bell] is a certain distance above the horizontal axis and a certain distance away from the vertical axis". I take this as evidence that thinking of a particle of pixie dust provided Ali with imagery that supported her in thinking of a point's location in the plane, as opposed to it's coordinates, as a representation of two attributes' measures. This suggests Ali had constructed an understanding in the moment that supported her in constructing a point as a multiplicative object. However, as I explain in the next section Ali's image of coordinating two measures at a given moment seems to have been dependent on seeing the red and blue bars oriented perpendicularly on the axes.

Ali's image of a particle of pixie dust (an imaginary point) now supported her in coordinating her image of the value of *v* (the length of the blue line) and the value of *u* (the length of the red line); Ali understood the overall behavior of the curve in terms of how these values changed together. More specifically, after introducing the notion of pixie dust Ali gave her first explanation of how *u* and *v* changed together that, from my perspective, coordinated two varying magnitudes. Ali explained,

> So as the value of *u* keeps on going towards the right the value of *v* um dips down. So *v* gets a bit closer to the value of *u* and then it dips down. Then as the value of *u* keeps going towards the right the value of *v* increases significantly (*moves pen up vertical axis*) then at a certain point where the value of *u* is about here (*points on horizontal axis*), the value of *v* decreases and then when the value of *u* is about here (*points on horizontal axis*), up until the value of *u* is around here the value of *v* increases and then dips down again (*moves finger up vertical axis*).

Then again when the value of *u* is around here (*points on horizontal axis*) then the value of *v* increases again.

In this explanation Ali coordinated her image of how the value of *u* changed with her image of how the value of *v* changed. One can coordinate two images of change by thinking of one then the other (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Ali, however, understood her graph to show how both values changed simultaneously. For example I take Ali's utterance, "then at a certain point where the value of *u* is about here (*points on horizontal axis*), the value of *v* decreases" as evidence that she identified landmark points in the value of *u* to coordinate with her image of directional changes in the value of *v*. This suggests that Ali was engaging in at least a gross coordination of values and coordinating directional changes in both quantities' variation (Thompson & Carlson, 2017).

At the end of this task Ali discussed her new understanding for her sketched graph. She explained that this was different than how she normally thought about graphs because, "I usually see both sides separately and then I compare one to the other if I need an answer, but I never think of them together". I take this as further evidence that up until now Ali's image of a graph was a pre-coordination of values. She separately compare her image of quantity X's variation with the behavior of the graph and then compare her image of quantity Y's variation with the behavior of the graph. The shift Ali describes to "think[ing] of them [two changing magnitudes] together" is an essential construction to engage in emergent shape thinking. Ali's consciousness of her new activity suggests that she was in the midst of constructing at reflected image of her activity coordinating two quantities' variation.

I engaged Ali in two more versions of this task that she completed without difficulty. Although she did not bring up Tinker Bell while completing these tasks she constructed, from my perspective, an appropriate curve. More importantly, she explained each curve in terms of how she imagined the two quantities vary together.

Implications of Tinker Bell's pixie dust. The imagery of a graph being made of particles of pixie dust seemed to support Ali in understanding her sketched graph in terms of how two quantities change together. While this imagery was essential for Ali to make this construction, as I explain in the next section, the construction Ali made was about coordinating two quantities' variation – not Tinker Bell and her pixie dust. I will elaborate on this construction in the next section when I discuss Ali's engagement in the Homer task. In this section I provide a possible explanation for why Tinker Bell and her pixie dust supported Ali in understanding her sketched graph in a new way.

As Ali imagined Tinker Bell flying around the plane to create a curve of pixie dust gave she had a new perspective on her graphing actions. Instead of being engrossed in her own graphing actions, Ali now imagined watching Tinker Bell create the curve with her pixie dust. This supported Ali in reflecting on the *actions* she was using to create the graph. For Ali to imagine Tinker Bell moving in the plane she needed to attend to both the path Tinker Bell made but also she needed to imagine *how* Tinker Bell made that path. This involved a crucial element for Ali – thinking about how Tinker Bell knew where to fly. As Ali explained, Tinker Bell knew where to fly by "noticing where the value of u and the value of v were". I take this as evidence that Ali was attending to the actions involved in constructing the graph, namely simultaneously attending to both the value of u and the value of v. Imagining Tinker Bell "knowing where to fly" seems to have been a way for Ali to externalize how *she* knew where to place points in the midst of coordinating two quantities' continuous variation.

Thinking about her graphing actions – or more specifically, the actions she imagined Tinker Bell engaged in to make the curve – supported Ali in seeing the product of her actions, the curve, in terms of the actions she used to create it. As a result, Ali explained her graph by attending simultaneously to both the value of u and the value of v as she imagined these values changing. This is a significant because it suggests Ali's understanding of her graph was reversible: she constructed her graph by tracking a correspondence point and she imagined her curve having been created by tracking a correspondence point.

According to Piaget there are two forms of reversibility: inversions and reciprocity. As an inversion, one imagines undoing the action +A with the action –A to return to the starting point. On the other hand reciprocity involves constructing the relation $A < B$ to be the same as $B > A$ (Piaget & Inhelder, 1966). Put another way, inversion involves images of acting and undoing actions whereas reciprocity involves relating the products of having acted.

In the case of emergent shape thinking one can determine how to undo each graphing action in the moment of acting. For example, in reversing one's graphing actions one might first think to highlight a point they drew and reason about how they made that point (extend two attributes' magnitudes into the plane) and then imagine making that construction for every point along the curve. This is an example of an inversion, which is constructed step by step in the moment of acting.

At a reflected level, one does not engage in the action of undoing, but instead sees an outline of how he/she reversed (or can reverse) these actions. In other words, one does not have to imagine isolating a point and thinking about how to undo the making of the point. Instead, one *anticipates* that the graph has been constructed by tracking a correspondence point as the quantities' magnitudes vary. This is an example of reciprocity, which is a reversing of a relation as opposed to an individual action. In other words, reciprocity is reversibility at a reflected level.

Imagining another actor (Tinker Bell) supported Ali in reflecting on her own graphing actions. As Ali imagined Tinker Bell moving around the plane she thought about what Tinker Bell needed to see in order to know where to fly. Ali reasoned that Tinker Bell needed to keep track of both the value of u and the value of v. Since Ali now had a way to reflect on her own graphing activity, she could imagine reversing the structure of her actions. She did not need to physically engage in undoing each action in order to anticipate that Tinker Bell made the curve by tracking how two quantities' changed together. I do not claim that this was a stable construction. Instead, it is likely that Ali would need to repeatedly reflect on her graphing actions, both in the context of Tinker Bell & her pixie dust as well as other tasks – both novel and familiar, in order to understand graphs as the result of having tracked two quantities' variation simultaneously.

As I explain in the following section, what persisted from Ali's engagement in this task was not her thinking about Tinker Bell and her pixie dust. Instead, Ali's engagement in the third phase of the teaching experiment suggests her memory of this activity focused on her new understanding of making a graph by re-presenting her images of how quantities changed together. Although Ali did not always anticipate coordinating continuous or even smooth images of change, she did consistently construct a graph by re-presenting her image how two quantities' changed together.

Teaching Experiment Phase III: Operationalizing Emergent Shape Thinking

I designed the third phase of the teaching experiment to better understand Ali's thinking during the U&V task, in particular what aspects of her thinking were dependent upon two moving bars oriented perpendicularly on the axes? I engaged Ali in three context based graphing tasks (details in Appendix C) to study the ways she thought about representing changing magnitudes. I anticipated the ways Ali coordinated these tasks with the U&V task would provide insights into her thinking during the U&V task.

I engaged Ali in three animated graphing tasks where I presented her with a GeoGebra® animation depicting a situation and asked her to sketch a graph relating two quantities from the situation. None of these task provided information about numerical relationships. My model of Ali's initial graphing scheme suggests that Ali would complete these tasks by either tracking one quantity's variation in experiential time or picking a shape from her memory of past graphing experiences. However, Ali engaged in these tasks by simultaneously attending to two quantity's varying magnitudes. I will illustrate this thinking with Ali's engagement in the last task of the teaching experiment – the Homer task.

In the Homer task I presented Ali with an animation depicting a straight road with City A located above the road and City B located below the road (Figure 39). I asked Ali to graph Homer's distance from City B relative to his distance from City A and displayed labeled axes on the screen.

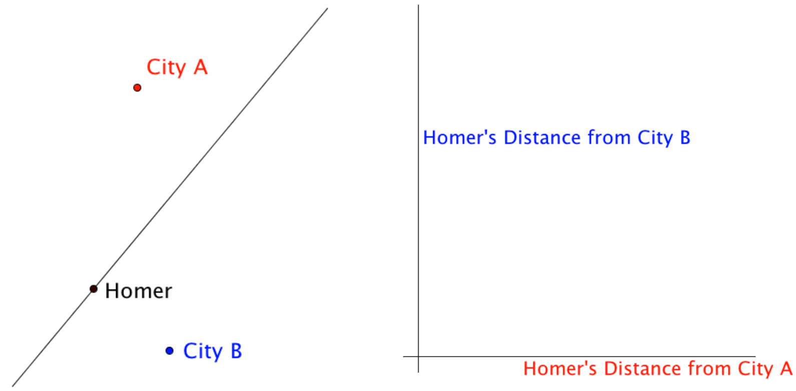
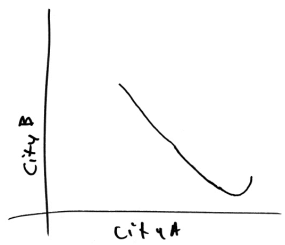
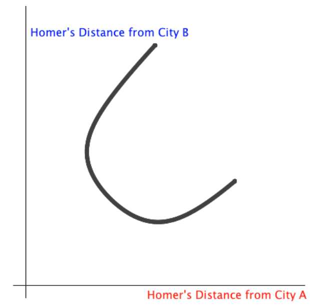


Figure 54: Screenshot 1 of Homer Task. At the beginning of the task the animation displayed (1) a depiction of the situation that showed the location of the cities (fixed) and Homer moving from the bottom of the road to the top of the road at a constant speed and (2) a set of axes labeled with Homer's distance from City B and Homer's distance from City A. (Day 4, Task 11)

As Ali watched the animation she drew a curve (Figure 42) that started on the right side of the first quadrant, decreased from right to left and then increased from right to left. Ali constructed her graph as the animation played by moving her pen a little bit at a time. This suggests that Ali decided what to draw by imaging how each distance changed over a small interval of time and then coordinating these two images of change with the orientation of a line in the plane. This is the first evidence that Ali coordinated two images of quantities changing in the moment of constructing a graph. However, as I explain below, Ali's focus on both quantities did not persist throughout the entirety of her graphing activity. As a result, the shape of Ali's graph did not match the actual covariation for the second half of the trip.



Ali's graph constructed in the presence of Homer moving along the road (see Figure 39)



Graph of actual covariation (Ali never saw computer make this graph)

Figure 55: Ali's graph of Homer's distance from City B relative to his distance from City A and graph of actual covariation (Day 4, Task 11).

Ali's activity drawing the graph little by little as she watched the animation is significant because it suggests she constructed her graph by re-presenting her image of how two quantities changed together. While it was not new for her to reason from situation about how each quantity varied, prior to the teaching experiment Ali did not have a way to re-present this thinking graphically. Now Ali had a way to construct her graph of Homer's distance from City B relative to his distance from City A by coordinating her images of how each quantity varied. This suggests that the imagery of the correspondence point and/or Tinker Bell and her pixie dust supported Ali in constructing a way to coordinate her images of quantities' gross variation. As a result, Ali was no longer limited to creating a graph by tracking her image of how one quantity's magnitude changed or trying to fit a known shape to her image of how each quantity changed.

In addition to making her graph by re-presenting her image of two changing quantities Ali also understood her sketched graph to show how the two distances changed together. She explained her first point by comparing Homer's initial distance from City A to his initial distance from City B. She reasoned "I know he is a lot closer [to City B] than

City A, like City A is farther away." She pointed to both axes simultaneously so that the vertical distance above the origin was less than the horizontal distance right of the origin. I take this as evidence that Ali imagined each distance having a size that she represented on the axes. Additionally, she understood a point to show both distances so that one can look at the point and compare which distance is longer than the other at a given moment in time.

As Ali imagined Homer moving down the road she understood her sketched graph to show how both quantities changed as Homer traveled along a stretch of the road. She explained,

> As Homer begins moving (Ali lets animation play until Homer is at his closest location to City B then pauses animation) the distance between Homer and City B gets closer (moves pen right to left along first section of curve). That's why it dips down. At the same time he is also getting closer to City A so that's why it also starts decreasing (moves pen right to left along first section of curve).

Ali's explanation, in particular her utterance "at the same time he is also getting closer", is evidence that Ali imagined two distances changing at the same time. She anticipated that as Homer's distance from City B was decreasing his distance from City A was also changing. Ali seemed to then focus on her image of Homer's distance from City A to reason that it was also decreasing. Thinking about how both distances changed supported her in understanding the directional change in her graph, down and right, in terms of how each quantity changed.

While Ali anticipated that both distances changed as Homer drove down the road, it took persistent attention for her to remember to keep track of how both distances changed. After Ali imagined Homer at the halfway point on the road she lost track of her image of Homer's changing distance from City A. As she explained her sketched graph she said, "As he keeps traveling (*plays animation*) the distance begins increasing from City B (*moves pen right to left along second section of curve*)." Ali correctly attended to how Homer's distance from City B changed, however, she did not attend to Homer's distance from City A over this part of the trip.

Since Ali maintained her focus on both quantities throughout all versions of the U&V task, this suggests that there was something different about Ali's understanding of the Homer task. Ali experienced difficulty because the red and blue bars were no longer displayed on the axes. This suggests that Ali's construction of the multiplicative object was dependent on seeing the animated bars oriented perpendicularly on the axes. In other words, for Ali to consistently imagine uniting two quantities' measures she needed to see these measures oriented on the axes and imagine extending those measures into the plane. Without that perceptual support, Ali's image of uniting attributes was something *she* had to maintain. With this added construction – the construction of coordination – Ali had a hard time constructing an image of each distance and then also coordinating those constructions in real time as the animation played. As a result, as she watched Homer move along the road she lost track of her image of how his distance from City A changed. This suggests it is nontrivial for one's construction of a multiplicative object to persist without the visual support of quantities' measures being displayed directly on axes.

I asked Ali to explain her graph again staying focused on both Homer's distance from City A *and* his distance from City B. Her explanation started off the same until Homer was halfway down the road. Then, she stopped talking and watched the animation play through from beginning to end. She explained the last part of the trip saying, "well I know for sure that his distance from Homer to City B is increasing but right now I am also realizing that Homer is also is getting farther away from City A." This suggests that while watching the animation she constructed both an image of Homer's changing distance from City B *and* his changing distance from City A.

She corrected her graph by adding a line segment that increased from left to right (see Figure 56). She appropriately explained her new sketched graph saying this segment showed Homer's distance from City B was increasing and his distance from City A was also increasing. When I asked her to explain how she saw both distances to be increasing she drew arrows on the axes to indicate that both values moved away from the origin (see arrows on axes in Figure 56).

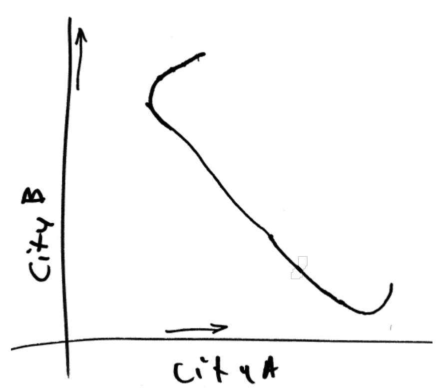


Figure 56: Ali's modified graph for first version of Homer task constructed in the presence of Homer moving along the road (see *Figure 39* for screenshot of animation).

At no point did Ali mention Tinker Bell, her pixie dust, or the red and blue bars. To understand if/how Ali coordinated her activity in the U&V task with her activity on the Homer task Pat asked Ali if there was anything that resembled how she was thinking about pixie dust. Ali explained that she imagined Homer carrying the pixie dust as he drove down the road. She understood this meant Homer's pixie dust ended up on the road and she could not imagine the pixie dust on the graph. Instead, she imagined the graph being created by strings she imagined between Homer and each of the cities.

This suggests that Ali's image of the pixie dust was tied to an imagined actor in a situation, initially Tinker Bell and in this task Homer. Since she did not see Homer in her graph she did not see the pixie dust in her graph. This reveals a possible limitation to the pixie dust didactic object. Ideally, one makes an abstraction to see the pixie dust emerging from the way she records her thinking. As a record of one's thinking, one can imagine their activity tracking two quantities' magnitudes as leaving pixie dust in the plane. More specifically, in the Homer task one can imagine every time she coordinates a distance from City A with a distance from City B she leaves a piece of pixie dust in the plane. As she coordinates these distances continuously she leaves a trace of pixie dust.

Ali's thinking about the pixie dust suggests one must differentiate the pixie dust from the actor in order to imagine *her* graphing actions leaving the pixie dust in the plane. Since Ali did not make this differentiation she saw the pixie dust in the phenomena, not the graph. This was not problematic in the U&V task because there was no phenomenon for Ali to coordinate with her graph. However in the Homer task, Ali could not reconcile her focus on Homer carrying pixie dust in order to imagine *her actions of graphing* as leaving pixie dust.

Although Ali did not imagine her graph being made of pixie dust she used the thinking she constructed when reasoning about Tinker Bell and her pixie dust. For example, when Pat marked a point on Ali's curve she explained that point saying, "like each point tells me he is like a certain distance away from City A and City B". I take this as evidence that she thought about a place on the curve as a representation of two quantities' magnitudes. Additionally, she explained her graph by coordinating her images of Homer's varying distance from City B with her image of his varying distance from City A. I take this as evidence that she understood the overall behavior of the sketched graph as a depiction of how two quantities changed *together*. This suggests that Ali's constructions in the U&V were not about Tinker Bell, but instead about relating two quantities' variation. This implies she abstracted her actions of coordinating two magnitudes from the context in which she first conceptualized these actions. This supported her in making a generalizing assimilation to see the Homer task as the similar to the U&V task even though the tasks were contextually and perceptually different.

Since Ali appropriately responded to the first version of the task I introduced two additional versions of the task, each with a new complexity, to investigate the generality of Ali's constructions. How Ali accommodated each complexity provided insights into her graphing scheme including the images she constructed from the situation, the images she anticipated representing in her graph, and the meanings she constructed from her sketched graph. Introducing these complexities supported me in identifying limitations to Ali's thinking about graphs. As a result I was able to refine my interpretation of Ali's graphing activity so as to not overstate her successes. I emphasize that the purpose of these complexities was to better understand Ali's constructions as opposed to studying what complexities would cause Ali to no longer be able to complete the task. In the following paragraphs I will use Ali's engagement in the third version of the Homer task, when the road was curved, to refine my characterization of Ali's graphing activity.

In the third version of the Homer task the road was curved and Homer moved at a constant speed along the road (Figure 57). From my perspective, the curved road introduced more nuances in how Homer's distance from each city changed; each distance increased and decreased numerous times as Homer drove along the road. I anticipated that the complexity in each quantity's variation would support me in understanding the images Ali constructed of each quantity's variation and how she anticipated coordinating these images.

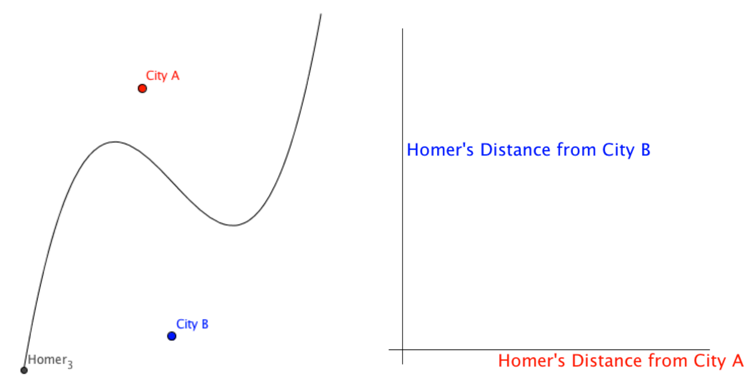
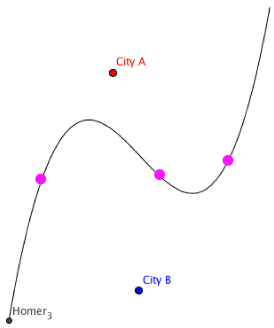
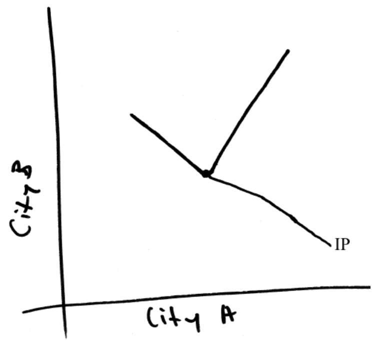


Figure 57: Screenshot of Homer Task version 3. At the beginning of the task the animation displayed (1) a depiction of the situation that showed the location of the cities (fixed) and Homer moving from the bottom of the road to the top of the road at a constant speed and (2) a set of axes labeled with Homer's distance from City B and Homer's distance from City A. (Day 4, Task 11.3)

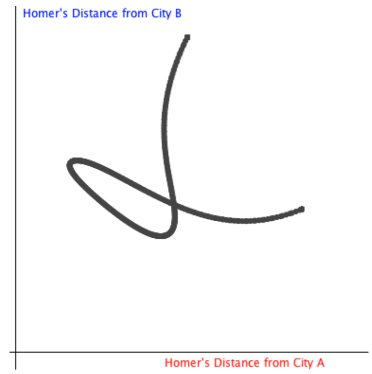
Ali engaged in this third version of the task by watching the animation play through five times before making any marks in the plane. Then she drew a point in the center of the plane and drew three line segments that met at that point (Figure 58). This was notably different than her engagement in the first version of the task where she drew her graph as the animation played.



Depiction of three landmark points Ali identified in situation (identified in pink for reader).



Ali's graph constructed in the presence of Homer moving along the road (see Figure 57). Initial point labeled "IP"



Graph of actual covariation

Figure 58: Ali's graph of Homer's distance from City B relative to his distance from City A and graph of actual covariation (Day 4, Task 11.3).

Ali explained that in this third version of the Homer task she noticed three places along the curved road where Homer's distance from City A was the same as his distance from City B (labeled for the reader in Figure 58). She marked a point in the plane to represent Homer's distance from City A was the same as his distance to City B. Then she focused on what happened to these distances as Homer traveled between these three points on the road. She had two images of what happened between these points on the road: (1) Homer's distance from each city varied as he traveled between landmark points and (2) as Homer moved away from one landmark point he got closer to another. As Ali made her graph her focus alternated between these two images.

For example, Ali explained her initial point (labeled "IP" in Figure 58) saying, "In the beginning Homer starts off a lot closer to City B than City A". This suggests she focused on coordinating two distances when making this point. But then, as she imagined

Homer moving down the road she reasoned, “he travels to a midpoint.” This suggests that Ali understood the line segment between her initial point and the center point showing Homer getting closer to the next landmark point as opposed to a decreasing distance from City A and an increasing distance from City B. Ali continued to alternate between these two images as she explained the rest of her graph. For example, after she imagined Homer at the first landmark point she said “then his distance to City A decreases as his distance to City B is still increasing” this suggests she understood her graph to show how two distances changed. Her next utterance, “but then he gets closer to the midpoint”, suggests she imagined her graph showing how Homer got closer to and away from the landmark points.

I take this as evidence that Ali had two different meanings for drawing a line in the plane: (1) a line coordinates the gross variation of two quantities’ magnitudes and (2) a line connects an object’s location at two moments in time. This suggests it is difficult to maintain a focus on coordinating quantities’ variation if one confounds the dynamic nature of coordinating two quantities’ changing with the dynamic motion of the phenomenon (e.g., coming to a point on the road). Ali’s engagement with this third version of the task suggests that students need repeated opportunities to construct the dynamism of their graphing activity as the *product of their reasoning* as opposed to the product of the object’s motion.

Additionally, Ali’s activity identifying landmark points in the situation prior to constructing a graph suggests she constructed a new understanding for her graphing activity. Instead of constructing her graph by representing her images of quantities’ smooth variation, as she did in the first version of the task, Ali now constructed three landmark points when Homer's distance to City A was the same as his distance to City B. Then she reasoned about how Homer moved between these points. As I explain in the following paragraph, Ali's activity constructing landmark points from a sketched graph supported her in attending to landmark points in her graphing activity.

It seems that the images Ali constructed from her sketched graphs began to inform the images she constructed from the situation. More specifically, since Ali reasoned about her sketched graphs in terms of gross covariation between landmark points the idea of a landmark point became more prominent in her thinking. Eventually, Ali anticipated making her graph by first identifying the landmark points.

From my perspective, when Ali constructed a landmark point from a sketched graph the landmark point was a moment when one quantity's variation switched from increasing to decreasing or positive to negative. Thus, each quantity strictly increased or decreased between landmark points. As a result, identifying these landmark points on the graph helped Ali reason about how the quantities' magnitudes changed together.

When Ali constructed a landmark point from the situation she identified moments she deemed significant relative to the object's activity. In the third version of the Homer task she constructed landmark points Homer's distance from City A was the same as the distance from City B. These landmark points did not align with where a there was a directional change in a quantity's varying magnitude. As a result, when Ali went to imagine the directional change of each quantity in between her landmark points she had a difficult time coordinating how the quantities' changed together because both quantities increased and decreased over a given interval. This suggests that in order for Ali activity tracking two quantities' magnitudes to keep up with her image of Homer's motion between landmark points she imagined graphing Homer's motion and not a covariational relationship.

I want to emphasize that it is the way Ali constructed landmark points from the situation that inhibited her from constructing and representing smooth images of change. (Silverman, 2005) explained that when one constructs a *two-dimensional landmark point* by identifying locations where there is a noteworthy change in how either quantity varies, then she is positioned to reason about the smooth variation of each quantity between these points (p. 98). However, Ali did not identify landmark points based off of how the quantities' changed. Instead, she focused on specific measures the quantities' took on in order to construct her landmark points. As a result, her landmark points did help Ali organize her thinking about how the quantities' changed together.

Revisiting my Meaning for Emergent Shape Thinking

At the outset of this study I proposed three constructions students would need to make in order to engage in emergent shape thinking:

1. Imagine representing quantities' magnitudes along the axes
2. Simultaneously represent these magnitudes with a point in the plane, and
3. Anticipate tracking the values of two quantities' attributes simultaneously

Ali's engagement in the teaching experiment provided insights into these constructions and also highlighted a fourth construction.

Prior to engaging in the teaching experiment Ali understood a point's coordinates to represent two quantities' magnitudes simultaneously. However, this thinking required that Ali unite (or imagine uniting) numbers. As a result, in the U&V task she did not construct a point to represent the lengths of both bars simultaneously. This highlights the importance of conceptualizing the location of a point in the plane as a way to represent attributes' measures, not numbers. Ali's engagement in the teaching experiment suggests that it takes explicit instruction for students to understand the location of a point in the plane, not its numerical coordinates, as a representation of two attributes' measures simultaneously.

Second, to anticipate tracking the values of two quantities' simultaneously one must construct and anticipate re-presenting smooth images of change. At the outset of this study I thought that reasoning about continuously changing phenomena presented in a dynamic animation would support students in conceptualizing and representing smooth images of change. However, Ali's engagement in the Homer task suggests that one is not likely to reason about smooth images of change when constructing a graph if she does not *anticipate* reasoning about her graph in terms of smooth images of change. This suggests that it is essential that educators and researchers aim for students to feel an intellectual need to construct and represent smooth images of change when constructing their graph and reasoning from their sketched graphs. Having students watch dynamic animations alone does not provide this intellectual need.

Additionally, students need repeated opportunities to coordinate their images of each quantity's variation so that their image of the coordination can persist under variation. In the U&V task, when there were red and blue bars moving on the axes, Ali maintained her focus on both quantities by focusing on the ends of each bar. However, when she engaged in the Homer task she had a hard time keeping track of her image of Homer's distance from City A and her image of Homer's distance from City B simultaneously. Ali experienced difficulty because the red and blue bars were no longer displayed on the axes. Without that perceptual support, Ali's image of uniting attributes was something *she* had to maintain. With this added construction, Ali had a hard time making these coordinations in real time as the animation played. As a result, as she watched Homer move along the road she lost track of her image of how his distance from City A changed. This suggests it is nontrivial for one's construction of a multiplicative object to persist without the visual support of quantities' measures being displayed directly on axes. As Thompson (2013) explained, "To construct stable understandings, one must repeatedly construct them anew" (p. 61). In the case of emergent shape thinking it is likely that students must repeatedly coordinate their images of each quantities' variation in order to construct a stable understanding that can keep up with their image of an object's motion in a phenomenon.

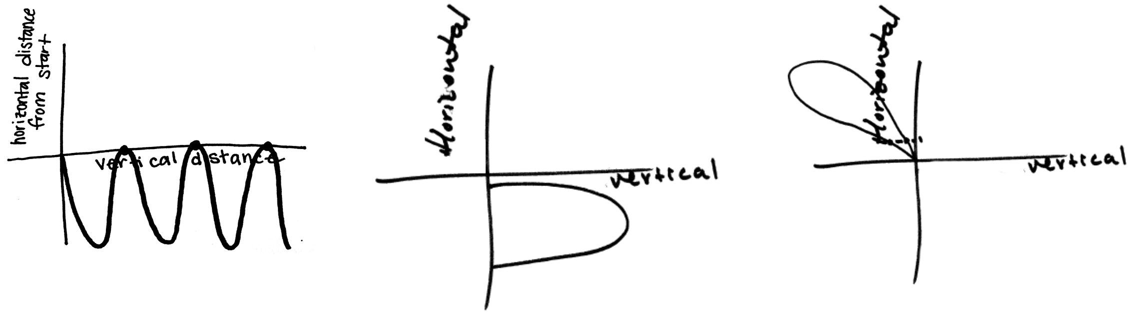
Finally, Ali's engagement with the Tinker Bell scenario highlighted the role of reflecting abstraction in the construction of emergent shape thinking. After introducing the notion of a correspondence point Ali could construct a graph by tracking two quantities' magnitudes simultaneously. However, she did not understand a sketched graph in terms of the actions she used to make it. Instead, she understood her graph as a representation of two quantities' gross and asynchronous variation. This highlights that emergent shape thinking involves more than ways to think about making a graph; a student engaged in emergent shape thinking would also understand a graph as having been made by tracking two quantities' changing magnitudes simultaneously. For Ali to engage in emergent shape thinking she needed a way to reason about her own graphing actions in order to see her completed graph in terms of the actions she used to make it.

Thinking about the actions Tinker Bell engaged in to make the graph supported Ali in taking her actions as the object of her constructions.

Generalizations in Post-TECI

After the fourth teaching session I engaged Ali in a one-hour post teaching experiment clinical interview (post-TECI). In this interview I engaged Ali in the same tasks that I used in the recruitment interview and the pre-TECI. Ali's engagement in these tasks revealed that she anticipated making a graph by representing how two quantities changed. This is in contrast to her activity in the pre-TECI where she drew graphs by tracking one quantity's variation or guessing shapes from past mathematical experiences. While Ali demonstrated a new understanding of her graphing activity, she demonstrated the same meaning for formulas in the pre- and post- TECI; formulas convert one quantity to another. This suggests that Ali's new understanding of her graphing activity did not support her in constructing a new understanding of formulas. I will provide two examples to highlight the implication of the teaching experiment on Ali's meaning for graphs.

Example 1: In the skateboard task (Task 2, post-TECI), I asked Ali to graph the skateboarder's horizontal distance from start relative to his distance above the ground. Prior to the teaching experiment Ali engaged in this task by guessing and checking shapes from her past mathematical experiences (see Figure 59 for three graphs shapes Ali made trying to complete this task in recruitment interview).

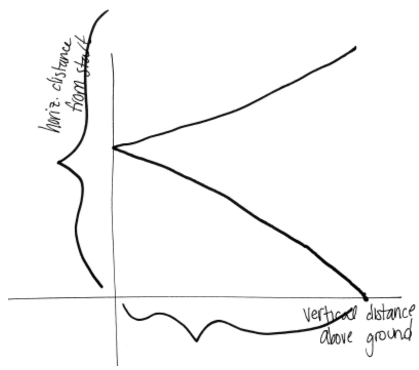


Ali's first attempt Ali's second attempt Ali's third attempt

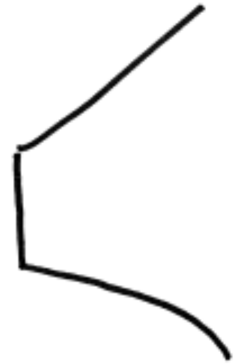
Figure 59: Ali's three attempts graphing skateboarder's horizontal distance from start relative to his vertical distance above the ground.

In the post-TECI Ali used her way of thinking about landmark points to identify three key moments in the phenomenon: when the skateboarder was at the top of the ramp, when he was at the bottom center of the ramp, and when he was at the top right of the ramp. While these were significant moments in the phenomenon, from my perspective, they did not correspond with directional changes in the quantities' variation.

As Ali explained her graph (Figure 60) it was evident that she thought about both quantities' size at each of these points (e.g., initially the vertical distance is maximum and horizontal distance is 0) and connected these points with straight lines to show how each quantity's magnitude changed (e.g., "he begins to get closer and closer to the ground and so he also starts having a horizontal distance from start"). This suggests she anticipated each quantity either increased or decreased between these landmark points.



Ali's initial graph of skateboarder's horizontal distance from start relative to vertical distance above ground



Ali's revised graph of skateboarder's horizontal distance from start relative to vertical distance above ground

Figure 60: Ali's initial and revised graph in skateboard task (post-TECI, Task 2)

Consistent with her engagement in the initial clinical interviews, after Ali drew her curve she compared her image of the each quantity's variation to her objective reading of her sketched graph. She reasoned that she needed to "add a vertical line" (see revised graph in Figure 60) to show that "his vertical distance above the ground isn't changing here (*points along the bottom of the ramp*) but his horizontal distance keeps changing". This suggests that Ali understood her graph to represent the smooth variation of each quantity. The images she constructed from her sketched graph did not match the smooth images of change she constructed from her image of the situation. In order for these images to match she modified her graph by tracking the smooth variation of each quantity simultaneously.

Although Ali did not initially track both quantities' magnitudes continuously as she made her graph, Ali's activity revealed that she was no longer limited to guessing and checking shapes from her past mathematical experiences. This suggests that Ali's engagement in the teaching experiment supported her in constructing a way to construct a graph based on her image of two varying quantities and how they change together.

Example 2: While Ali attended to both distances simultaneously in the skateboard task, she did not attend to both quantities simultaneously on all of the tasks. For example, in the bottle problem I presented Ali with a picture of a spherical bottle and asked her to graph the height of the water in the bottle relative to the volume of water in the bottle. In the pre-TECI Ali engaged in this task by tracking how the height of the water in the bottle changed as she imagined it filling with water. She did not mention the volume of water at all in her graphing activity.

In the post-TECI Ali attended to her image of each quantity's variation asynchronously. Before drawing anything in the plane Ali explained her approach saying,

> The volume is always going to be increasing because as more water is going in there is more volume of water so I established that I am not thinking about the volume of water because I know it is always going to the right so now I am focusing on the height of the water.

This suggests Ali did not anticipate her drawing activity as a way to track two magnitudes' measures simultaneously. Instead, she anticipated showing the directional change in each quantity's magnitude. Since she knew volume kept going to the right, she reduced her cognitive demand by only focusing on the directional change in the height – it increased as the bottle filled with water. This suggests that Ali's understanding of a correspondence point as a way to represent both a value of x and a value of y did not persist in her understanding of graphs.

Although Ali did not engage in emergent shape thinking in the post-TECI, she did anticipate making a graph by representing how quantities changed together. The images she anticipated representing were either pairs of values at landmark points (e.g., skateboard task) or a pre-coordination of values (bottle problem).

Ali's thinking about representing two changing quantities still did not generalize to her meaning for formulas. She still discussed formulas as "converting from one to the other." To see if Ali could coordinate her new understanding of graphs with her thinking about formulas I asked Ali if she could imagine the red and blue bars in the context of a formula. She said, "I honestly cannot imagine the red and blue bars with the formulas. But the other ones, like the graphing ones, I did think about them."

There are two possible explanations for Ali's difficulty coordinating her meaning for formulas with her image of the red and blue bars. First, it is possible Ali did not imagine the red and blue bars with taking on values. Instead, she imagined these red and blue bars having sizes that got smaller and larger but did not imagine them always having a measure. Without imaging values, Ali could not imagine the red and blue bars in her number-centered meaning for formulas. A second possible explanation is that Ali's meaning for variables inhibited her from thinking about a variable's value varying continuously. This would suggest that Ali could not coordinate her image of a changing value of x and the symbol x in the formula. Either way, Ali did not coordinate her new understanding of graphs with her thinking about formulas. This supports Thompson's (1994c) claim that students do not see graphs and formulas as representations of the same thing.

From my perspective, Ali had developed a way to represent her images of two varying quantities (whether those smooth or chunky images of change). To understand if

Ali was aware of these constructions, I asked Ali if she had any thoughts about her experience working with Pat and me.

Excerpt 7: Ali post-TECI, 1:19:45

KF: So what did you think of this whole process working with me and Pat?

Ali: It really opened my eyes a lot to math in general. I feel like a lot of. How do I say this? I would say that if I had this type of experience before precalc or any type of math class I probably would have been thinking about math differently while in class. I feel that since we have English, math, you know our core classes as we get older we just program ourselves in math to do this similar to the example in class. We don't really think about it. I think it complicates. Or we complicate it for ourselves because we aren't thinking of math in an abstract ways it is just a repetitive process just keep going.

KF: Are there any tasks that stood out to you?

Ali: I think the red and blue bars. Like learning– it is such a simple concept – it really is. But it blew my mind. Honestly. Like I even went home and spoke about this with my sisters. You wouldn't think about math this way. But it is such a simple concept that can really put the glue to everything together. Now when I see a graph this will probably help me when I am looking at the graph because I have a clearer perception of what is actually happening rather than just okay, plot the points and the robotic process of connecting them.

In this excerpt Ali discussed the power the red and blue bars had on her thinking about graphs saying "I have a clearer perception of what is actually happening rather than just okay, plot the points and the robotic process of connecting them." This is significant because it revealed that she has constructed a new meaning for graphs where the shape is governed by the behavior of the bars – the quantities – and is not predetermined. Additionally, this excerpt revealed Ali had a new appreciation for math that is about one's thinking instead of "robotic" and "repetitive process[es]". This suggests Ali was aware that she was now attending to her acts of construction and acts of reasoning instead of the products of her actions. Thus, Ali constructed a meaning for graphs as representations of how two quantities changed together at a reflected level.

CHAPTER 8
A TEACHING EXPERIMENT WITH BRYAN

At the time of the study Bryan was enrolled in summer precalculus and about to begin his first year at the university. He had declared a major in civil engineering and was confident in his mathematics ability; he expected to earn A's in all his mathematics coursework. During his recruitment interview Bryan constructed both smooth and chunky images of a quantity's varying value and engaged in multiple forms of covariational reasoning. Bryan's engagement in these varied ways of thinking revealed he had multiple non-coordinated graphing schemes. I selected Bryan to participate in a teaching experiment to understand how one might coordinate graphing schemes grounded in different images of change.

My teaching experiment with Bryan was the last of the three teaching experiments and his pre-teaching experiment clinical interview (pre-TECI) took place 59 days after his recruitment interview (Table 9).

Table 9

Bryan's Schedule

Date	***Event***
May 16, 2016	Began Summer Session Precalculus (Traditional Curriculum)
June 6, 2016	Recruitment Interview
June 24, 2016	Ended Sumer Session Precalculus with A
August 4, 2016	Pre-Teaching Experiment Clinical Interview
August 9, 2016	Teaching Experiment Session 1 (witness: P.W. Thompson)
August 15, 2016	Teaching Experiment Session 2 (witness: P.W. Thompson)
August 17, 2016	Teaching Experiment Session 3 (witness: P.W. Thompson)
August 22, 20116	Post-Teaching Experiment Clinical Interview

Initial Model of Bryan's Meanings for Graphs and Formulas

Bryan participated in two clinical interviews – the recruitment interview and the pre-teaching experiment clinical interview – prior to his three-session teaching experiment. The purpose of these interviews was to establish a base-line characterization of Bryan's meanings for graphs and formulas and to understand how Bryan coordinated these meanings.

Bryan's recruitment interview occurred midway through his enrollment in a summer-session precalculus course and he completed the summer-session precalculus course between the recruitment interview and his pre-TECI. To determine if his meanings for graphs and formulas changed as a result of completing this course I constructed independent characterizations of his meanings for graphs and formulas in his recruitment interview and his pre-teaching experiment interview. There was no difference in my characterizations. Thus, I will use Bryan's engagement in both of these interviews to illustrate my characterization of his meanings for graphs and formulas.

Bryan's Initial Meaning for Graphs

Bryan demonstrated two distinct meanings for graphs: (1) graphs are a collection of (x,y) pairs where x and y take on specific values, and (2) graphs are a continuous trace of a quantity's increasing and decreasing magnitude across time. As I explain below, neither of these meanings for graphs supported Bryan in thinking about drawing a graph as a way to represent an infinite collection of points, nor did Bryan's meanings support him in thinking about a point as a multiplicative object, in the sense that it united two quantities' measures as they varied simultaneously.

Scheme 1: Graphs as collections of (x, y) pairs. Anytime Bryan sketched a graph from a contextual description he plotted points to represent pairs of measures. Then he connected these points with a line or curve. I will illustrate this thinking with Bryan's engagement in the last task of the recruitment interview, the skateboard task. The task read:

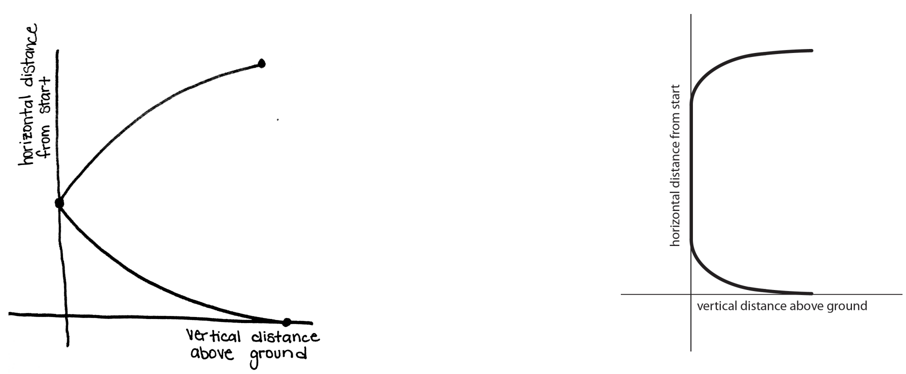
> A skateboarder skates on a half-pipe like the one shown below. The skateboarder goes across the half-pipe and then returns to the starting position.

On the task sheet there was a picture of a skateboarding half-pipe ramp illustrating a starting point and a skateboarder at the bottom of the ramp (see Appendix A). I asked Bryan to graph the skateboarder's horizontal distance to the right of the starting position relative to the skateboarder's vertical distance above the ground.

Before constructing a graph Bryan reasoned about the skateboarder's vertical distance above the ground at three locations on the ramp: the starting position, the bottom center of the ramp, and the top right of the ramp. He explained that he picked these locations because it was where the skateboarder's vertical distance was maximum, zero, and maximum again; Bryan did not attend to the vertical distance being zero as the skateboarder traveled across the bottom of the ramp.

Next Bryan went to plot three points, one for each of these locations. He explained he made a point by, "take[ing] the horizontal distance from start in respect to the vertical distance". I take this as evidence that Bryan's point-plotting scheme supported him in understanding a point to relate two quantities' measures. I claim that it was Bryan's anticipation of plotting a point that supported him in attending to two quantities' measures; he did not imagine two related measures prior to thinking about plotting a point. This suggests his meaning for a point was likely a pseudo-multiplicative object; Bryan needed to think about plotting a point in order to attend to two quantities measures.

After plotting these three points Bryan attended to the skateboarder's return trip across the ramp. He reasoned that since the skateboarder passed through the same three locations on the ramp as he returned to the starting position he did not need to plot any more points. Finally, Bryan connected these three points with a curved line (see Figure 61) to show that "when I first started thinking my first plot was this (*moves hand to first point on horizontal axis*), second was this (*moves hand to point on vertical axis*), and third was this (*moves hand to point in first quadrant*)." This suggests Bryan connected the points in order to show the order in which he plotted them. I interpret this as evidence that Bryan understood the curve as a way to connect moments in time. There is no evidence that while drawing the line Bryan imagined he was showing pairs of related measures.



Bryan's graph Graph of actual covariation

Figure 61: Bryan's graph and graph of actual covariation of skateboarder's horizontal distance from start relative to vertical distance above ground (recruitment interview, task 4).

Bryan also made a graph by plotting and connecting pairs of measures when the problem statement included numerical relationships. For example, in the pre-TECI, I asked Bryan to construct a formula relating a cell phone screen's diagonal length and width under the constraint that the height of the screen was always half the width (see Appendix B). Bryan used the Pythagorean Theorem to construct the formula $c^2 = w^2 + \left(\frac{1}{2}w\right)^2$ which he appropriately simplified to $c = 1.118w$. After Bryan wrote this formula I asked him to sketch a graph of the diagonal length relative to the width of the cell phone screen. Bryan constructed a graph by attending to specific measures. He substituted 0 and 3 into his formula for w to get the points (0, 0) and (3, 3.354). Then, he plotted these points and drew a straight line between the points (Figure 62). Bryan explained his decision to draw a straight line saying, "it is a straight line equation because it is w not w squared or cubed." This suggests Bryan engaged in a form of static shape thinking; he associated the shape of a line with a formula defined in terms of w. There is no evidence that Bryan imagined either quantity's value varying when he connected the two points.

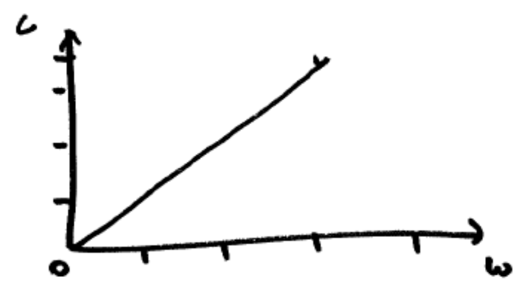


Figure 62: Bryan's graph of cell phone's diagonal length in terms of width of cell phone screen (pre-TECI, task 2)

In the two examples above, Bryan constructed a graph by plotting at least two points and then connecting these points. In the moment of drawing the line Bryan focused on connecting places in space in order to create a shape or connect moments in time. This is in contrast to his engagement in the bottle evaporation task, described below, where Bryan drew a line in to track one quantity's varying magnitude across time. While Bryan constructed his graph by representing a smooth image of a quantity's varying magnitude, he did not reason about his graph in terms of a quantity's smooth variation – the image of change he represented in his graphing activity.

In the pre-TECI I asked Bryan to imagine a spherical bottle filled with water that was left outside to evaporate (see Appendix B). I asked him to graph the height of water in the bottle relative to the volume of water in the bottle. Before Bryan constructed a graph he reasoned, "When volume is maximum the height should be maximum and when volume is zero height should be zero." This suggests Bryan coordinated two magnitudes' sizes at two moments in time. He proceeded to draw a straight line from the top middle of the plane that fell from left to right (see Figure 63, red line).

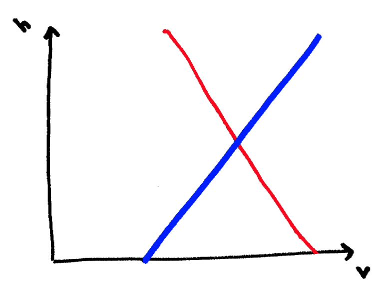


Figure 63: Bryan's initial (red) and revised (blue) graph for the evaporating water problem (pre-TECI, task 1b)

From my perspective, the line Bryan drew was not a representation of his image of two pairs of magnitudes. Instead, after Bryan made his initial point with the anticipation of showing the simultaneous state of maximum height and maximum value, he drew a line by representing his image of the height of the water decreasing as he imagined the water in the bottle evaporating. This suggests that Bryan constructed his line by imagining the gross variation of the height of the water as he imagined the variation within experiential time.

After Bryan drew the line he reconstructed his initial image of pairs of magnitudes to reason that his graph should show maximum height and maximum volume. He determined that his graph did not represent this image saying, “It doesn’t make sense. Because over here (*points to start of line in top middle of plane*) it says height is maximum but volume is not maximum (*points to intersection of line with horizontal axis*).” Bryan drew a new graph that was a vertical reflection of his original graph about its midpoint; his graph now decreased from right to left (see Figure 63, blue line). Bryan explained that now he understood his graph to show the height is maximum when the volume is maximum and also show the height is minimum when the volume is minimum.

In summary, Bryan’s engaged in three distinct graphing activities when completing the bottle evaporation task. First he constructed an image of each quantity’s (discrete) variation and coordinated these images to construct pairs of measures. Then he constructed a line by representing his image of one quantity’s gross variation as he attended to that variation in his experiential time. Finally, he reconstructed his initial image of pairs of measures to determine if the behavior of the sketched graph matched his anticipation of the relationship between the quantities’ measures.

These phases are significant because Bryan demonstrated two different images of varying quantities in his graphing activity. When reasoning about the situation and his sketched graph Bryan attended to pairs of measures. However, Bryan did not represent this image of pairs of measures when drawing his graph. Instead, he represented his image of one quantity's gross variation in his experiential time. Since the images of change Bryan represented when drawing the graph (gross variation of one quantity) were different than the images of change he constructed from his sketched graph (coordination of values), I take this as evidence that Bryan's understanding of his sketched graph was an empirical abstraction. In other words, the understandings Bryan constructed from his sketched graph were understandings about the shape he produced and not the actions he engaged in to make the shape.

Scheme 2: Graphs track one quantity's variation across time. When the problem statement included both a graph and a contextual description Bryan demonstrated a different meaning for graphs. Instead of reasoning about the graph in terms of pairs of measures, Bryan understood the graph to show one quantity's gross variation as he attended to that variation in his experiential time.

For example, in the pre-TECI I included the racetrack problem (Task 4, pre-TECI, Figure 64) from Bell and Janvier (1981). In this task I presented Bryan with a graph of a car's speed in relation to the number of kilometers the car traveled along the track. The problem statement asked Bryan to determine which race-track the car traveled along to produce the given graph.

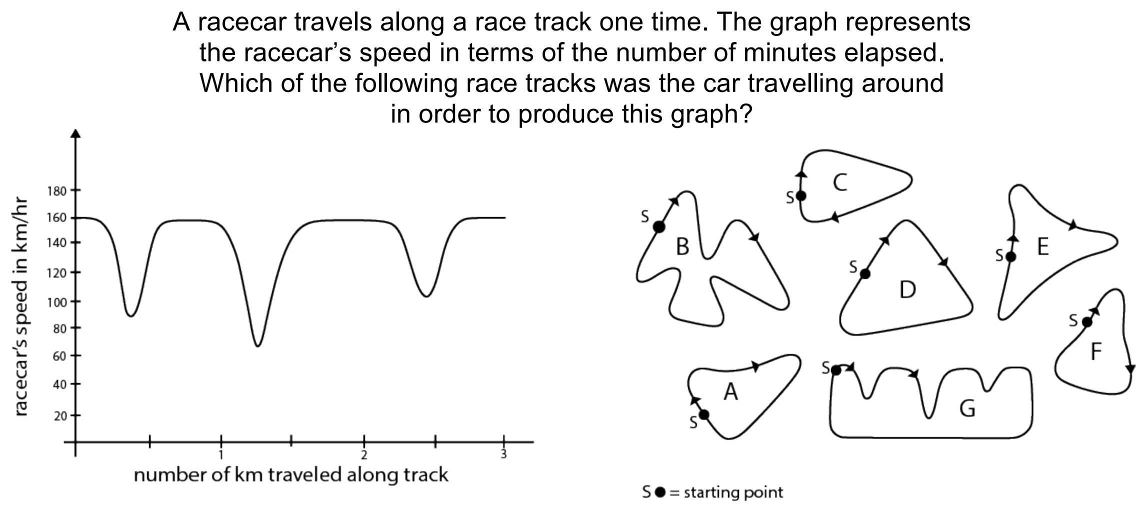


Figure 64: Racetrack problem from Bell and Janvier (1981)

Bryan explained the graph by describing how he imagined the car to speed up and slow down as it went around the track. He said,

> So the car is at 160 [kilometers per hour] at the starting point so that means it is coming fast. Then it brakes and slows down (*moves pen along curve as he gives his explanation*). Then after slowing down it speeds up again then travels at a constant speed. Then slows down the most at the second turn. Then speeds up, then slows down, and is constant. So according to this there are three turns (*points to three dips in curve*) on the circuit.

Bryan's focus on how he imagined the car to move suggests that Bryan used his understanding of the graph to imagine an event occurring in his experiential time. He then described the graph by attending to what he witnessed in his experiential time. This is distinct from Monk's (1992) notion of an *iconic translation* because Bryan did not imagine the shape of the graph as the road the car traveled along. Instead, Bryan constructed an image of how the car's speed varied and then used this image to construct an image of the situation.

In this explanation Bryan focused on the speed of the car and he only attended to the distance the car traveled (the quantity represented on the horizontal axis) at the very beginning of his explanation. This suggests that Bryan understood the graph as a representation of a single quantity's variation as he imagined that quantity changing in his experiential time. Since distance traveled increased monotonically with Bryan's experiential time, this did not cause any problems in Bryan's reasoning.

Since Bryan reasoned about the car's changing speed by imagining it changing in his experiential time he could not maintain a persistent focus on two changing quantities. In other words Bryan's image of the situation out-paced his ability to coordinate his image of a changing distance with a changing speed. This is significant because it suggests that Bryan understood the graph in terms of a single quantity's variation and thus did not engage in covariational reasoning when reasoning about the graph.

Bryan's Initial Meanings for Formulas

On three occasions in the pre-TECI I asked Bryan to either construct or interpret a formula in relation to a described situation. Bryan's engagement in these tasks revealed that he understood variables as a letter that stands for a number, mathematical operations as numerical calculations, and formulas as a way to get one number from another.

For example, in the bathtub task (Task 6, pre-TECI adapted from Carlson et al. (2013)) the problem statement gave the total weight of the water and tub (875 pounds) and the weight of one gallon of water (8.345 lb/gal). I asked Bryan to define a formula that gave the total weight, in pounds, of the tub and water, *w*, in terms of the number of gallons of water that had drained from the tub, *g* (see Appendix B).

Bryan wrote $w = g/8.345$ to "just write something down to relate to the question". Then he imagined substituting 10 for *g* and decided his formula didn't make sense. He decided the operation should be multiplication, not division, saying, "for any gallon of water I have to multiply, not divide, to get the weight". He then wrote a new formula, $w = g*8.345$. This suggests that Bryan used trial and error to determine whether to multiply or divide the number of gallons by 8.345. I take this as evidence that Bryan understood the operations of multiplication and division as numerical calculations and he did not construct them as quantitative operations. Bryan's focus on numerical calculations suggests that he understood his formula as a way to convert one number to another number. He interpreted the product of this calculation in terms of the situation, saying that he "got the weight". However, there is no evidence that he viewed his formula as a representation of the quantitative relationship between the number of gallons of water and the weight of that water.

Additionally, Bryan's utterance, "for any gallon of water I have to multiply, not divide, to get the weight" suggests that Bryan imagined substituting a discrete (possibly infinite) collection of values in for *g* to get an associated weight. This implies Bryan understood variables as a placeholder for unknown values and suggests Bryan's meaning for formulas was based in discrete images of change.

Bryan's Initial Coordination of Meanings for Graphs and Formulas

I designed the cell-phone screen task (Task 2, pre-TECI) to understand how Bryan coordinated his meanings for graphs and formulas. As I described above, Bryan constructed the formula $c = 1.118w$ to relate a cell phone screen's diagonal length to its width. After Bryan wrote this formula he graphed this relationship by determining the coordinates of two points on the graph. Then he reasoned that it was a straight-line equation so he should connect the two points with a straight line.

This suggests two ways that Bryan coordinated his meanings for graphs and formulas. First, Bryan coordinated the action of substituting a value into his formula with his meaning for a point's coordinates. Additionally, Bryan coordinated his meaning for the overall behavior of a graph (a line) with his meaning for formulas (defined in terms of w) by recalling pre-determined associations that he memorized in math class.

Summary

Bryan's actions during the initial clinical interviews revealed that he had two schemes of meanings for displayed graphs. With the first scheme of meanings, Bryan understood his sketched graph in terms of a few (x, y) pairs. He imagined connecting these points to connect moments in time; there is no evidence that he thought about the line between the points representing other (x, y) pairs. With the second scheme of meanings, Bryan understood a graph to show one quantity's gross variation across time. There is no evidence that he imagined points on the graph that conveyed pairs of measures.

While these were the meanings Bryan demonstrated when reasoning from a graph, Bryan's engagement in the bottle evaporation task revealed that he had two images of quantities' co-variation to represent in his graphing actions: a coordination of quantities' sizes at select moments in time (a coordination of values), and an image of each quantity's gross variation in his experiential time (a pre-coordination of gross variations). Bryan's attention shifted between these two images of changing quantities so that the image of change Bryan represented when drawing the graph (a single quantity's gross variation) was different than the image of change he constructed from his sketched graph (a coordination of values). I take this as evidence that Bryan's understanding of his sketched graph was an empirical abstraction. In other words, the understandings Bryan constructed from his sketched graph were understandings about the shape he produced and not the images he re-presented when making the graph.

Bryan's Teaching Experiment

I engaged Bryan in three teaching experiment sessions. Each session lasted between 1.5 and 2 hours and was witnessed by Pat Thompson. The teaching experiments consisted of three series of tasks to help me understand (1) Bryan's scheme for quantitative reasoning, (2) the ways of thinking that supported or inhibited Bryan from engaging in emergent shape thinking, and (3) the generality of any constructions he made during the teaching experiment.

Teaching Experiment Phase I: Quantitative Reasoning

I began the teaching experiment by asking Bryan to complete five tasks I designed to reveal the objects in Bryan's images of quantitative situations that he acted upon when reasoning about those situations. While Bryan constructed attributes of a situation to measure, as I describe below, he experienced difficulty maintaining his focus on attributes and shifted to speaking about objects themselves.

Bryan's engagement in the first task of the teaching experiment, the airplane task, revealed that he experienced difficulty maintaining his focus on attributes. In this task I presented Bryan with a GeoGebra® animation depicting an airplane and a helicopter flying from left to right across the screen (see Appendix C). I will focus on Bryan's activity during the second version of the task when the airplane was directly above the helicopter (Figure 65). I asked Bryan to use the distance between his left pointer finger and right pointer finger to represent the distance between the aircraft as they flew from San Diego to Phoenix (Figure 66).

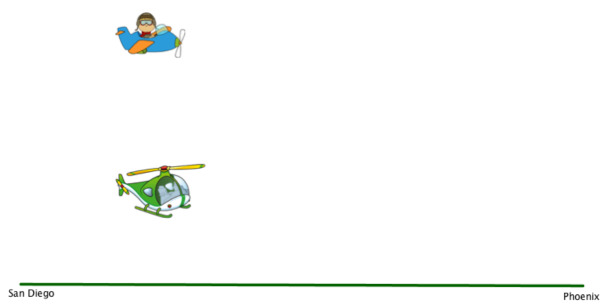


Figure 65: Screenshot of second version of airplane task (Day 1, Task 1).

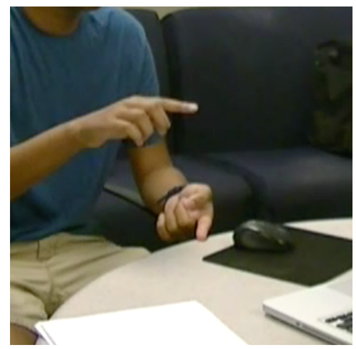


Figure 66: Bryan uses the distance between his pointer fingers to represent distance between airplane and helicopter

As Bryan used his fingers to represent the distance between the aircraft his attention shifted from his image of the distance between the aircraft to representing the motion of the aircraft. For example in the second version of this task Bryan explained, "from the start the distance is increasing and then it is decreasing again, it is decreasing, decreasing, then it is increasing, and then they come to the same spot." In this explanation Bryan shifted his attention from his image of how an attribute, the distance, changes (increases/decreases) to attending to the motion of the aircraft (they come together). This suggests that Bryan had not completely differentiated his image of the attribute's variation from his image of the two object's motion.

Although Bryan experienced difficulty maintaining his focus on attribute's varying measures, when he did focus on an attribute's varying measure he repeatedly constructed images of gross variation. For example, in the box problem (Task 2, Day 1) I asked Bryan to imagine creating a box by cutting out equal sized square cutouts from the corners of an 11"x7" piece of paper (see Appendix C). To support Bryan in imagining attributes of the box changing, I displayed an animation depicting an unfolded view of the box (see Figure 49 for selected screen shots).

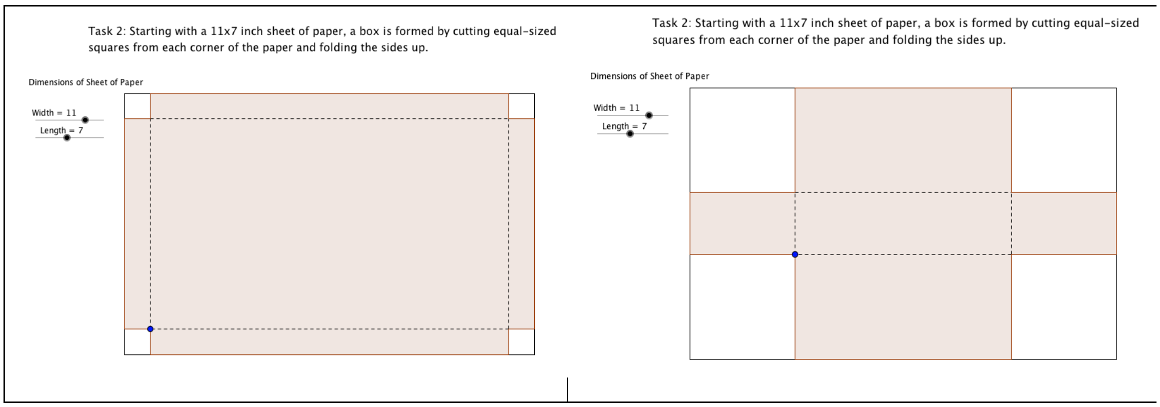


Figure 67: Selected screen shots from box problem (Day 1, Task 2)

As Bryan watched the animation play he spoke of the width of the box, length of the box, and height of the box increasing and decreasing. Despite being told the initial dimensions of the paper, Bryan did not say anything to suggest he was thinking about these attributes taking on numerical values. This suggests that Bryan's image of each quantity's variation focused on how the quantity's size increased or decreased. Bryan's images of quantity's gross variations supported him in relating two quantities (e.g., relate width of the box to cutout length) by relating directional changes in each quantity's size. For example, he reasoned that as the cutout length increased the length of the box decreased. Bryan would need to construct a more nuanced image of each quantity's variation in order to engage in a more robust form of covariational reasoning.

Teaching Experiment Phase II: Supporting Emergent Shape Thinking

The second phase of the teaching experiment began near the end of the first session (lasting 25 minutes) and continued into the second session (lasting 60 minutes). The purpose of Phase II was to support Bryan in engaging in emergent shape thinking (see learning trajectory in Chapter 5). As I describe in the following section, Bryan did not engage in emergent shape thinking. Instead, he constructed graphs by plotting and connecting landmark points. In this section I will document both my efforts to support Bryan in engaging in emergent shape thinking and the ways Bryan's meanings for graphs inhibited him from viewing a graph as having emerged from covarying the values of two quantities' values.

A graph's shape: It is just the way it is. The preliminary clinical interviews with Bryan revealed that he attended to two quantities' magnitudes when plotting points. Then, Bryan connected these points. It seems Bryan connected these points in order to connect two moments in time or draw a shape he was familiar with. He did not anticipate making a collection of points as he drew the line. In other words, he did not attend to two quantities' varying magnitudes simultaneously as he connected the points. Bryan engaged in this same activity during the first session of the teaching experiment. This provided Pat and me with opportunities to try to support Bryan in attending to variation in two quantities' magnitudes simultaneously as he sketched a graph.

I presented Bryan with a GeoGebra® animation at the end of Day 1. The animation depicted Adam and Kevin running at a constant speed around an ellipse shaped track so that Adam was always 100 meters (1/4 of the track) ahead of Kevin. I asked Bryan to graph Kevin's straight-line distance from start relative to Adam's straight-line distance from start. To understand how Bryan coordinated his image of each quantity's variation I asked Bryan to anticipate what would happen to Adam's straight line distance from start as Kevin's straight line distance from start increased. Bryan explained,

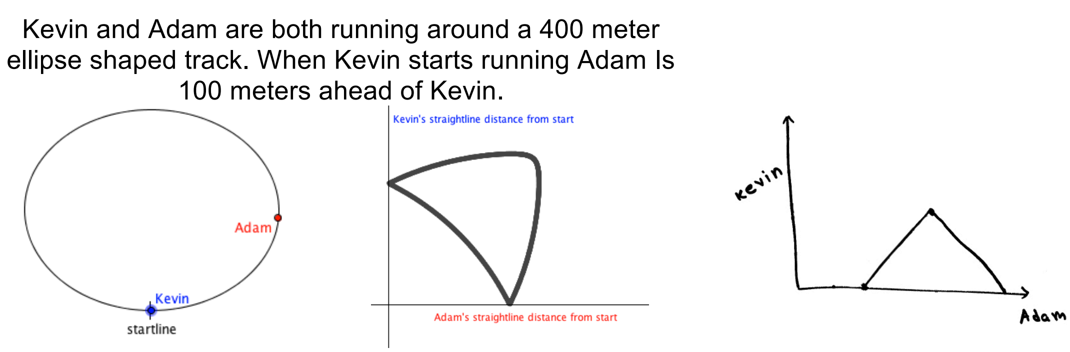
> (Bryan clicks play so animation shows Kevin moving to 100-meter mark and Adam at moving to maximum) As Kevin's straight-line distance from start increases Adam's is also increases [sic] (Bryan pauses animation when Adam is at top of track). Then when Kevin is at 100-meter mark, Adam's distance is at the maximum. (Bryan clicks play and pauses again when Kevin is at top of track) Then Kevin's keeps increasing and Adam's is decreasing.

Bryan's description suggests that he engaged in gross covariation of the two distances. He used the play/pause feature of the animation so that he could coordinate his activity relating two changing distances with his experience of imagining these distances change in the animation. In other words, he described how the quantities changed together in real time with his experience imagining them change. This supported him in identifying landmark locations (e.g., 100 meter mark, top of track) when he saw either quantity's size start decreasing.

Bryan attended to both his image of Kevin's changing straight-line distance *and* Bryan's changing straight-line distance as he described what he saw in the animation.

This suggests that Bryan anticipated that as he attended to one quantity's variation the other quantity changed too. This implies he kept his images of both quantities' gross variation in mind, which supported him in maintaining his focus on two changing quantities throughout his explanation.

The animation was paused from the previous activity so that Kevin was at the starting line and Adam at the 100-meter mark. I asked Bryan to graph Kevin's straight-line distance from start relative to Adam's straight-line distance from start. Instead of playing and pausing the animation, Bryan constructed a graph from his recollection of the animation. He placed a point on the middle of the horizontal axis saying, "when Kevin's distance is 0 Adam's has some value". Then he drew a graph that started at his point on the middle of the horizontal axis and then increased and decreased (Figure 68). Consistent with his reasoning in the pre-TECI, Bryan attended to both quantities' magnitudes when placing his initial point. However, his thinking about both quantities did not persist. Instead, it seems Bryan drew his graph by representing his image of how Kevin's straight-line distance from start changed as he attended to that distance in his experiential time – it increased then decreased.



Screen capture of depiction of phenomenon including, for the reader, a graph of actual covariation

Bryan's initial graph

Figure 68: Screenshot of Kevin & Adam Task, graph of actual covariation, and Bryan's solution

In the recruitment interviews Bryan demonstrated a meaning for graphs based in plotting points that represented two quantities' measures. It seems that this image of a point as way to pair two quantities' measures could not keep up with his image of the variation of both distances as he imagined them in his experiential time. For Bryan to think about making a point he would need to repeatedly make three constructions. First he would need to imagine the relative location of both boys on the track, then he would need to imagine each boy's straight line distance from start, and finally he would need to use his image of a point to coordinate these two distances. Since each of these three constructions takes experiential time Bryan could not consistently make these three constructions as his image of the phenomenon was continuously changing in his experiential time. As a result, Bryan could only consistently make one of these constructions – imagine Kevin's straight-line distance from start – as he imagined the event unfolding in his experiential time. Thus, Bryan's graph is a representation of his image of Kevin's changing straight-line distance as he imagined this distance changing in his experiential time.

With the animation still paused, Bryan explained his sketched graph saying,

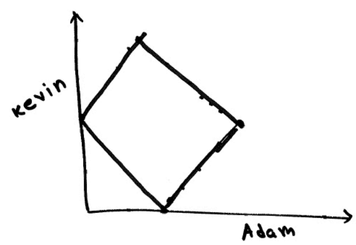
> This is the starting point (*points to point on middle of horizontal axis – his initial point*) so when Kevin is at the starting point Adam is already at the 100-meter mark (*points toward computer screen*) so his distance would have some value. Then when Kevin gets to say 100 meters (*points to middle of vertical axis*) Adam becomes. No I don't think this is correct because when Kevin goes 100 meters (*points to middle of vertical distance keeps left finger on vertical axis*) Adam is at

> the maximum (*uses other hand to simultaneously point to right side of horizontal axis*). I don't know how to draw this.

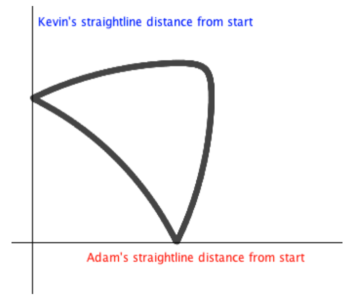
In this explanation Bryan no longer imagined the boys moving continuously around the track. Instead, he imagined the boys' at fixed locations on the track and then related their straight-line distances. Since his imagery was no longer changing continuously Bryan could construct and coordinate the two quantities' measures. Bryan imagined the boys at their starting position (depicted in paused animation) to reason that initially Kevin's straight-line distance was 0 while Adam's had some value. He understood his initial point to represent this pair of measures. Then he imagined Kevin at the 100-meter mark and Adam at the top of the track – another static image. With his image of the situation paused, Bryan reasoned that Adam's straight-line distance was maximum Kevin's straight-line distance was nonzero. He compared this image of a specific pair of measures to his understanding of his sketched graph and concluded that his sketched graph did not show that Kevin had some distance from start when Adam was at his maximum distance from start. In other words, while Bryan attended to just Kevin's distance from start when sketching his graph, he attended to the pair of distances when reasoning about points on the graph he sketched.

Bryan attended to both quantities' magnitudes when placing his initial point and when determining the validity of his graph. This suggests his meaning for points supported him in understanding his graph as representing pairs of measures. However, while listening to Bryan's explanations, I hypothesized that he had a hard time imagining plotting points as his image of the phenomenon was continuously changing in his experiential time.

To test this hypothesis Pat asked Bryan to imagine the boys walking around the track. If Bryan imagined the boys walking then it would slow down his experience of the event so he didn't imagine the boys moving until he had imagined their straight-line distances taking on a measure and coordinating these measures by plotting a point.



Bryan's new graph



Graph of actual covariation

Figure 69: Bryan's second attempt graphing Kevin's straight-line distance from start relative to Adam's straight-line distance from start as he imagined the boys moving a couple steps at a time.

When Pat first suggested Bryan imagine the boys walking, it was to support Bryan in coordinating his images of both quantities' variation. Bryan did not anticipate attending to how the two distances changed together. For example, Bryan said, "Kevin's will increase a little" when Pat first asked Bryan to imagine the boys walking. Pat had to explicitly ask how Adam's distance changed before Bryan said, "Adam's would go up a little bit". This suggests that Bryan imagined the two distances changing asynchronously. He coordinated these two images of change when he thought about plotting a point to show "Adam is at something and Kevin is also at something". It seems that Bryan represented his image that both Adam and Kevin's distances increased by imagining both measures increasing a little bit along the axes to plot a new point.

Bryan thought about a little bit of change, plotted a point, thought about a little more change, and plotted a point. I claim that when Bryan's image included an anticipation of little bits of change the points he plotted (or imagined plotting) were connected in his thinking as he plotted them. This makes his activity distinct from plotting a collection of points and then retrospectively connecting them. After plotting four points Bryan no longer imagined the boys moving a little at a time and instead imagined Adam at his maximum straight-line distance from start. This suggests that Bryan noticed a pattern in his reasoning and anticipated that so long as both boys' distances were increasing he would continue to plot points going up and to the right in the plane.

After Bryan's imagery shifted from attending to little bits of change in each boys' distance to imagining when the variation in one quantity will change Bryan expressed, "I'm pretty lost". I interpret this as evidence that Bryan's image of two distances could not keep up when he jumped to thinking about Adam's distance reaching its maximum. To support Bryan in reconstructing his image of both distances I let the animation play until Adam was at his maximum (and Kevin was at the 100-meter mark). Then Bryan reasoned, "the point after that Kevin's is increasing and Adam's is decreasing". This suggests Bryan went back to imagining small increments of change so he could maintain his focus on his image of both distances. He repeated this activity plotting three points showing, from my perspective (and I suspect Bryan's perspective), that Kevin's distance is increasing and Adam's is decreasing. Then Bryan's image again switched to when Kevin reached his maximum distance. Unlike his first attempt, Bryan gave no indication that he became "pretty lost". Instead, he imagined the boys taking a few steps in order to plot the next point after he plotted a prior point that showed Kevin at his maximum distance. Bryan continued to alternate between his image of the boys taking a few steps and his image of the boys at the landmark locations until he completed his graph.

I claim that in the episode described above, Bryan was in the process of making two constructions. First, Bryan was constructing a chunky continuous image of covariation by creating chunks (a couple steps) in his continuous image of the phenomenon. Within a small chunk he could rapidly switch his attention between his images of each quantity's variation in order to maintain his focus on both boys' distances as they changed. Bryan's new focus on both distances as they changed over a small chunk suggests that Bryan's image of the distances was projected into the foreground of his thinking so that his image of the situation and the boys' motion was secondary.

Bryan also seemed to construct an initial image of coordinated magnitudes by coordinating two magnitudes at the end of each chunk. When constructing his graph, Bryan switched from imagining little bits of change to focusing on when one of the boys' reached a landmark location. I interpret this as evidence that Bryan used his anticipation of little bits of change in both quantities to locate a new point relative to a prior point—without having to first imagine amounts of change in each quantity individually and then plot a point using his "over and up" scheme. I take Bryan's activity of jumping ahead to a landmark location as evidence that his activity of constructing and coordinating magnitudes over little chunks became an activity that he could envision carrying out. In the next paragraph I will provide additional evidence to support this claim.

After Bryan completed his graph he explained that he could have been more diligent by plotting all the points, but he didn't need to do that to understand what was going on. The reason Bryan no longer needed to invoke his point-plotting scheme seems to be that he now understood a line in his graph to contain intermediate simultaneous measures, and that he could reason about smaller changes in the boys' positions if necessary. Therefore, he no longer needed to carry out the concrete activity of pausing his image of the situation, sketching or imagining two segments and coordinating their magnitudes, and plotting a point in order to attend to the two distances' magnitudes or changes in magnitudes. It seems that Bryan could imagine carrying out his point-plotting scheme at any moment in time because he could anticipate his graph being made by attending to smaller chunks of change in both boys' positions.

Consistent with Piaget's claim that, "Assimilation... is the source of schemes.... Assimilation is the operation of integration of which the scheme is the result" (Piaget, 1977, p. 70), Brian's activity of repeatedly constructing and coordinating two magnitudes at the end of a chunk by plotting a point supported him in developing a reflected image of his actions that embodied the structure of his actions. His image of plotting points now existed at a reflected level, at least momentarily. This implies that Bryan's image of plotting a point was no longer dependent on an imagined concrete location of the boys' on the track. Instead, he could draw a line while thinking, 'I could also make points here, and here, and here' by tracking how the distances changed individually and together. Since Bryan explained that he could have been more diligent to plot more points, I interpret him as thinking that he was representing pairs of measures, but not of measures that varied continuously. Bryan seemed to have constructed, at least, chunky continuous covariation of the boys' distances, but I cannot claim that his thinking entailed smooth continuous covariation.

After completing his point-plotting and line drawing activity in the context of the Kevin and Adam task, Bryan questioned whether what he drew was even a graph. Bryan explained that his graph didn't look like any of the shapes he had seen in math class. He tried to make sense of his graph by focusing on the points that he drew. He reasoned that points usually represent x and y values and his points represented values of Kevin's straight-line distance from start and Adam's straight-line distance from start. He concluded that what he drew was probably a graph saying, "it [the shape of the curve] is just the way it is … according to my thinking." This suggests that, at this moment, Bryan understood his graph not as a shape but instead as a product of *his thinking*. This is the first evidence that *Bryan* understood his sketched graph in terms of his graphing actions. However, as we will see in my analysis of the next task, Bryan's projection of his graphing activity to a reflected level was still tied to the context of Kevin and Adam walking a track.

U&V Task. The next task in the teaching session was the U&V task. I presented Bryan with an animation that depicted a red bar along the horizontal axis and a blue bar along the vertical axis. As the animation played, the lengths of the bars changed simultaneously in such a way that each bar had one end fixed at the origin. (See Figure 34 for selected screenshots from the animation). In the first version of this task the horizontal (red) bar's unfixed end varied at a steady pace from left to right while the vertical (blue) bar's unfixed end varied unsystematically. I explained to Bryan that the length of the red bar represented the varying value of u and the length of the blue bar represented the varying value of v. Finally, I presented Bryan with a printout that included a screen capture of the initial position of the bars in the animation and then asked him to graph of the value of *v* relative to the value of *u.* The video played repeatedly until Bryan completed the task. Bryan completed three versions of this task that increased in complexity in how the bars varied together.

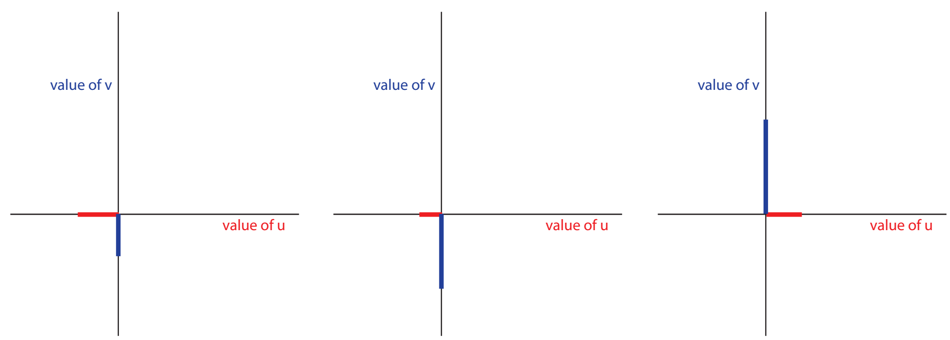
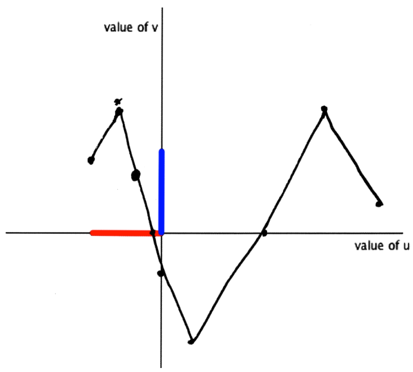


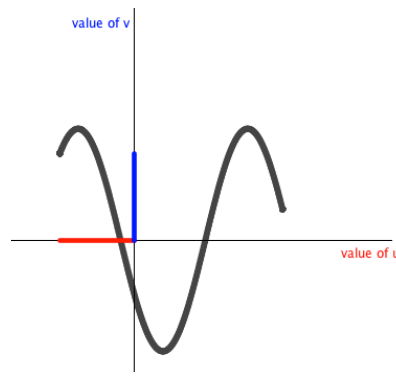
Figure 70: Three screenshots from the U&V task (adapted from Thompson, 2016).

In the first two versions of the task the length of one of the bars increased monotonically. In the first version the end of the red bar varied at a steady pace from left to right and in the second version the end of blue bar varied at a steady pace from bottom to top. Bryan used the play and pause feature of the animation to pause the animation anytime one of the bars reached a local maximum, a local minimum, or 0. Then with the animation paused he plotted a point where the extension of the red and blue bar would intersect in the plane. Bryan's need to pause the animation in order to plot a point suggests that the abstractions he made in the Kevin and Adam task arose from a functional accommodation in his thinking in regard to that task; Bryan did not anticipate making his graph by tracking how the bars change over small intervals, as he did in the Kevin & Adam task. In this new task, Bryan's point-plotting scheme still necessitated a concrete image.

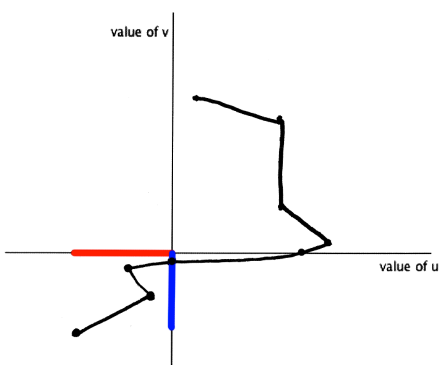
After plotting all of his points he asked, “Should I plot the thing with straight lines or curved lines?” Bryan looked at me for an answer and said, “You aren’t going to say that are you?” Before I could say anything he laughed and drew straight lines (see Figure 71). In Excerpt 8 Bryan discussed how he thought about what makes something curved/straight.



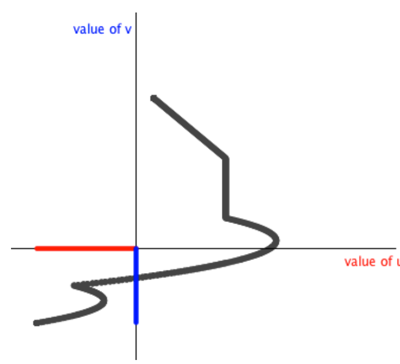
Bryan’s graph for first version of U&V



Graph of actual covariation for first version of U&V



Bryan’s graph for second version of U&V



Graph of actual covariation for second version of U&V

Figure 71: Bryan’s solutions for first two versions of U&V Task and graphs of actual covariation (Day 2, Task 7)

Excerpt 8: Bryan TE3-Day 2, 00:33:18

Pat: What makes something curved as opposed to straight?
Bryan: Um. I don’t know. I know like when it comes to like terms like if you plot an equation and stuff. Like I know if there is an x^2 you gotta make that curved and stuff like that.
Pat: But you don’t know why it has to be curved or what about x^2 makes it

curved?
Bryan: No.

In this excerpt Bryan explained that he used formulas to determine whether a graph should be straight or curved. This is consistent with his engagement in the pre-TECI when he reasoned that the graph of c=1.118w is a straight line because the formula is just w, as opposed to w^2. This suggests Bryan did not have a meaning for straight and curved graphs outside of his association between a graph and its formula.

Bryan's activity coordinating a graph's shape with its associated formula suggests that his understanding of a graph's shape being "just the way it is", a construction he made in the Kevin and Adam task, was a functional accommodation. His understanding of a sketched graph as the product of tracking two quantities' magnitudes did not persist in his thinking. Instead, Bryan engaged in static shape thinking and associated the shape of a graph with its formula. This suggests Bryan's meaning for the graph was about the object he created (a shape) and not the thinking he used to make the shape.

Bryan engaged in the first two versions of the U&V task by identifying landmark points, plotting these landmark points, and then connecting them with straight lines. This suggests Bryan did not attend to how each quantity changed between these landmark points. To better understand Bryan's meaning for the line between the points I asked Bryan to explain what was happening in-between the points he drew. Bryan explained each line segment by describing how each bar would have to change "to go from this point (*indicates left end of line segment*) to this point (*indicates right end of line segment*)". For example, he determined that to go from the first point to the second point (the endpoints of an increasing line segment in the 4th quadrant – see Figure 71) "the value of both u and v are decreasing [in length]". This suggests that Bryan understood his sketched graph to show each quantity's gross variation between landmark points. While Bryan understood his graph to show how each quantity varied, the only points Bryan attended to on his graph were the landmark points – the endpoints of each line segment.

Since Bryan had yet to attend to any points between the landmark points I explicitly asked Bryan if there were any points between the ones he drew. He said, "Of course, it is a straight line so there can be any point on the straight line." This suggests seeing the straight line, a shape seen often in school mathematics, supported Bryan in constructing a new understanding of his sketched graph – his graph is made up of points because lines are made of points. When I asked what these points represented Bryan said, "they show the movement of the bars". This suggests Bryan understood the points to help show the gross variation of each quantity; he did not understand the points conveyed a pair of magnitudes. This means that while Bryan understood his graph contained an infinite number of points, he did not anticipate these points emerged from tracking two quantities' magnitudes simultaneously.

To support Bryan in understanding that each point conveyed a pair of magnitudes I displayed a correspondence point (Figure 10). I explained that the point represented the value of u and the value of v simultaneously. Following the recommendation of Thompson et al. (under review), I engaged Bryan in an activity where I let the animation play, I paused the animation, and then asked Bryan to use the pointer to show where the correspondence point would be. Each time I asked Bryan to justify why the correspondence point would be in that specific location. Bryan explained, "this is what I was doing before." I interpret Bryan to mean that he used the length of the red and blue bar to plot his landmark points.

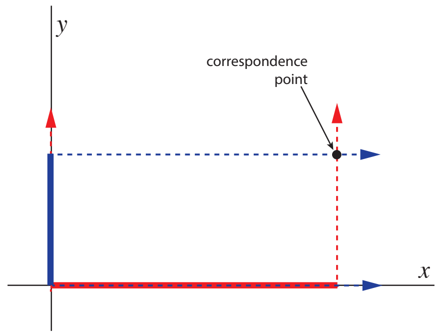


Figure 72: A point as the intersection of two quantities' values extended from the axes.

I asked Bryan to anticipate would happen if the computer kept track of the correspondence point as the animation played. Bryan said, "You would get a more accurate graph". This suggests that Bryan anticipated that the computer could make a graph by plotting all possible pairs of measures. As I describe below, I claim that Bryan did not imagine the computer continuously tracking two varying magnitudes to plot these points. With the correspondence point displayed I had GeoGebra® trace out a graph (see for sequence of screen shots).

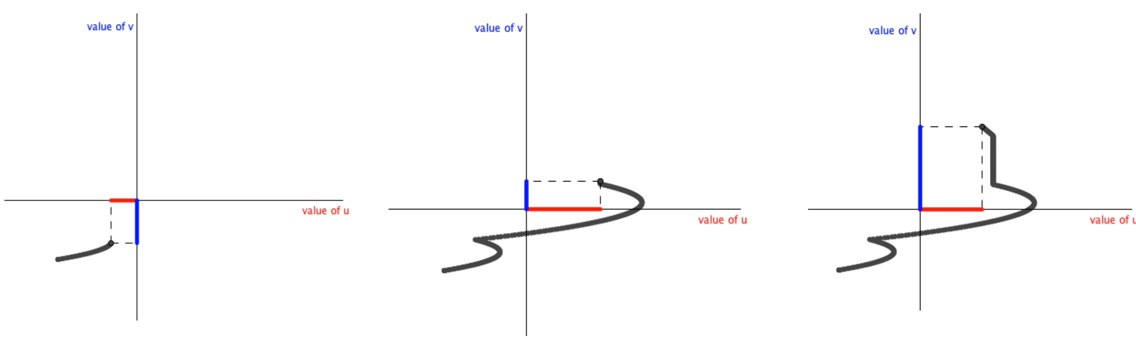


Figure 73: Selected screenshots from U&V task showing correspondence point.

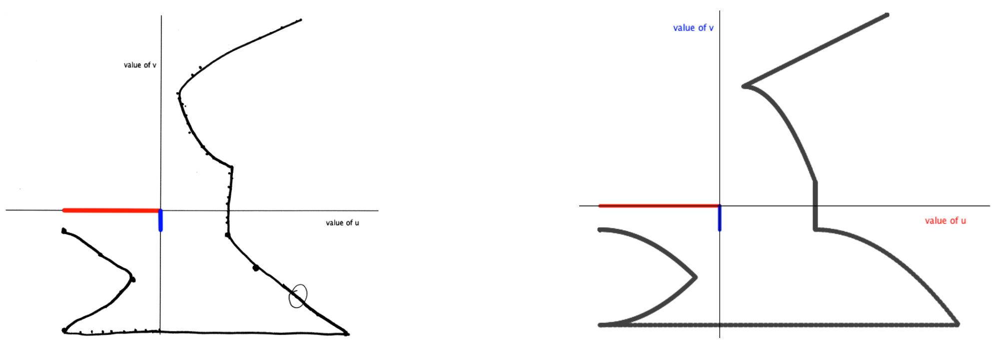
After watching the computer trace out the graph Bryan said, "it feels good to know I drew the right thing." This suggested Bryan focused on the shape the computer produced and not the way the computer made the graph. To see how Bryan understood the computer's continuous trace I asked Bryan if the computer made its graph the same way he did (see Excerpt 9).

Excerpt 9: Bryan TE3-Day 2, 00:34:22

KF: Did the computer make this graph the same way you made yours?
Bryan: Yeah.
Pat: So how did the computer make it's graph.
Bryan: The correspondence points.
KF: How many correspondence points did the computer keep track of?
Bryan: Every one of them. Because for example it was plotting everything as it goes. But for me it is not possible to do that. So I just make a few points.
KF: Okay, so you couldn't keep track of all the points.
Bryan: I mean I could but it would take a long time to keep track every movement, play it, pause it, turn it back, pause it, play it. It would take all day (*laughs*).
Pat: Are you envisioning you would pause it so you could plot a point?
Bryan: Yeah.
Pat: Would there be another way to track what is going on without having to pause it?
Bryan: I don't think so.

In Excerpt 9 Bryan explained that the computer made its graph the same way he did. This suggests that Bryan did not imagine the computer making its graph by continuously tracking two changing magnitudes. Instead, it seems Bryan imagined the computer making its graph by plotting every possible point where he imagined plotting a point from a paused image of the animation. Bryan did not imagine engaging in this activity himself. This suggests his activity attending to two specific magnitudes could not keep up with his experience watching the animation. Instead, Bryan needed to pause (or imagine pausing) the animation in order to plot a point. This imagery inhibited him from imagining continuously tracking a correspondence.

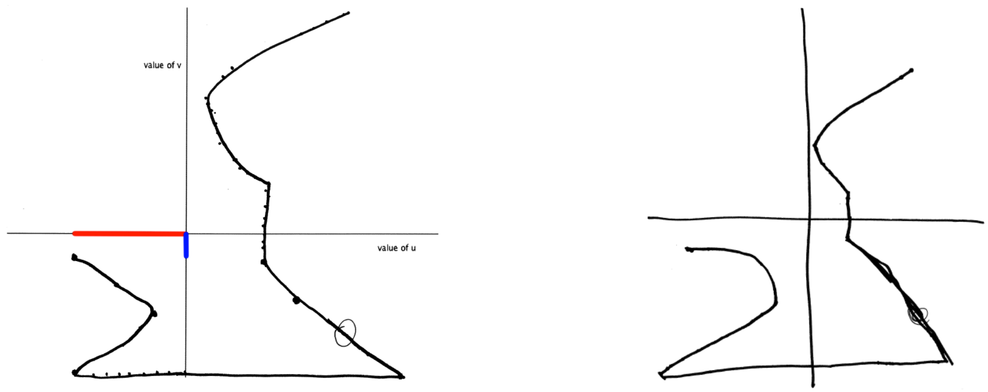
I engaged Bryan in one more version of the U&V task to see if he would use the idea of tracking a correspondence point to create his graph; he did not. The animation displayed red and blue bars moving on the axes so that the end of each bar varied unsystematically; the correspondence point was not displayed in the animation. Consistent with his engagement in the first two versions of the task Bryan played and paused the animation to support him in plotting points. Although Bryan plotted more than landmark points, likely to capture his image of the length of one of the bar's staying the same while the length of the other bar changed, Bryan still made his graph by plotting points and then connecting these points (Figure 74). From my perspective, he did not anticipate representing all possible correspondence points when making his graph.



Bryan's graph for third version of U&V   Normative solution for third version of U&V

Figure 74: U&V Task version 3: Bryan's solution and graph of actual covariation (Day 2, Task 7.3)

I asked Bryan to make another graph by keeping track of the correspondence point as the animation played continuously. I did not display the correspondence point so Bryan needed to imagine the correspondence point and then track his image of the correspondence point as the animation played continuously. While Bryan's new graph (Figure 74) had a similar shape to his original graph, from my perspective, the images of change Bryan represented in each graph were very different. In this new graph Bryan attended to two continuously changing magnitudes while imagining and tracking his image of a correspondence point. This is in contrast to Bryan's initial graph where he represented chunky images of change. However, as I explain below, Bryan understood his activity representing change in progress to convey the same information as his activity representing completed amounts of change.



First attempt made by playing and pausing animation to plot points and then connecting these points

Second attempt made by imagining tracking the correspondence point. Animation displayed red and blue bars moving on the axes – correspondence point was not displayed.

Figure 75: Bryan's graphs for third version of U&V task (Day 2, Task 7.3)

To better understand the meaning Bryan had for his graphing actions, Pat asked Bryan to attend to the meaning of a point on each of his graph. Pat circled a point in a similar location on each of Bryan's graphs (see Figure 75) and asked Bryan if these two points had the same meaning. Bryan said both points showed, "where the segments are". He saw no difference in the meanings of a point on either graph. This is significant because from my perspective, Bryan did not attend to where the segments were as he connected the points in his first graph.

Pat reminded Bryan that he made the first graph by connecting two points. Then he asked if Bryan thought about the bars when making the point circled on the first graph (the graph where Bryan connected two points with a straight line). Bryan explained,

> That exact point on that graph (*points to graph made by connecting landmark points*) had no meaning because I didn't take that point into consideration. I just took some other point and like plotted it. But over here (*points to graph made by tracking correspondence point*) I was looking at that thing [computer animation] and like going as the picture goes.

Although Bryan recognized that the meaning he employed when making a point on each graph was different he did not see this as significant. He explained, "I mean the thinking process is different. But I think the end result is the same." This suggests that Bryan prioritized the products of his actions (the shape of the graph), and the meanings he could impose on his graph (a collection of points), over the images he represented when making each graph.

This conversation went on for another ten minutes as Pat and I tried to get Bryan to experience a conflict between his meaning for a point between landmarks on his first graph and a point made by tracking the correspondence point on his second graph. Bryan was not perturbed. He ended the conversation saying,

> There are millions of points you can plot. But I don't have to plot everything to get as close to it as possible. Like in 10 minutes I could do this (point to graph made by connecting landmark points), I could take two hours and plot you a more accurate graph if you wanted but I don't think that is needed to understand the concept of things.

Bryan's utterance of "millions of points you can plot" implies that he anticipated that he could make his graph by more diligently attending to all pairs of measures, likely by playing by pausing the animation and plotting a point. This suggests that he was still thinking of capturing static states of the distances' covariation. However, he did not necessitate plotting all possible points to "get as close to ***it*** as possible". This suggests Bryan imagined graphs to be shapes out there to match. So long as Bryan made a shape close to the correct one it did not matter what he attended to while making it. As a result, Bryan experienced no intellectual need to attend to the meanings he employed while representing a graph.

Summary. Bryan's engagement in the teaching experiment revealed that his image of plotting points entailed capturing static states in each quantity's covariation. As a result, on two occasions his imagery of coordinating static pairs of measures was "outpaced" (to use Piaget's word) by his perception (either witnessed or imagined) of variation in both quantities. For example, in the Kevin & Adam Task Bryan started his graphing activity by plotting a point to coordinate Kevin and Adam's initial distances from start. He ended up constructing his graph by tracking one quantity's magnitude as he imagined it in his experiential time. I interpret this as evidence that his activity of imagining a segment between the boy and the starting line and coordinating these distances with a point could not keep up with his image of the boys running around the track.

Bryan's comment in the U&V task, that one constructs a graph by plotting "millions of points", suggests that he could not anticipate tracking a correspondence point as the animation played to represent all possible pairs of measures. Although Bryan did track a correspondence point per our request (Figure 75), he did not engage in this activity without prompting. This suggests that a correspondence point was not part of

Bryan’s image of the animation. As I describe in the following section, Bryan understood the tracking of the correspondence point to be the concrete activity of tracing the corner of a rectangle. He did not understand the position of the rectangle’s corner as representing two measures simultaneously. The rectangle’s corner was the focus of his attention, and thinking of the rectangle allowed him to think of its corner. However, the rectangle’s sides were sides of a rectangle. They were not representations of two quantities’ magnitudes. Thus, I claim that Bryan did not see his activity of tracking a correspondence point as tracking two magnitudes simultaneously. Put another way, Bryan’s image of tracking a correspondence point was distinct from his image of plotting “a million points”. However, since both activities produced a curve, Bryan could anticipate making a point at any particular location. As a result, he could assimilate two completed graphs as having the same meaning (in regard to points on them) regardless of the imagery he employed when making the graphs.

Teaching Experiment Phase III: Operationalizing Emergent Shape Thinking

I designed the third phase of the teaching experiment to better understand Bryan’s thinking during the U&V task, in particular what aspects of his thinking were dependent upon his perception of two moving bars oriented perpendicularly on the axes? I engaged Bryan in three context based graphing tasks (details in Appendix C) to study the ways he thought about representing changing magnitudes. I anticipated that the ways Bryan related these tasks with the U&V task would provide insights into his thinking during the U&V task.

I engaged Bryan in three animated graphing tasks in which I presented him with a GeoGebra® animation depicting a situation and asked him to sketch a graph relating two quantities from the situation. None of these task provided information about numerical relationships. As I describe below, Bryan's engagement in these tasks revealed that his activity attending to the correspondence point was dependent on seeing the bars on the axes – it was a concrete activity. Without that stimulus Bryan focused on plotting and connecting landmark points – just as he did in the pre-TECI. I will illustrate Bryan's thinking with his engagement in the last task of the teaching experiment, the Homer task.

In the Homer task I presented Bryan with a GeoGebra® animation depicting a straight road with City A located above the road and City B located below the road (Figure 39). As the animation played Homer moved from the bottom of the road to the top of the road. As the animation played I asked Bryan to graph Homer's distance from City B relative to his distance from City A. I displayed labeled axes on the screen.

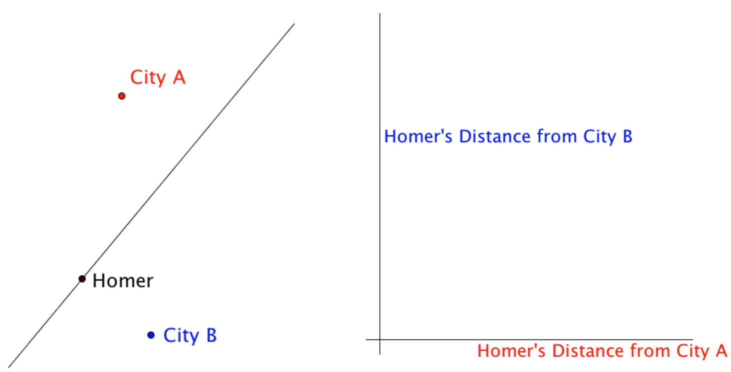
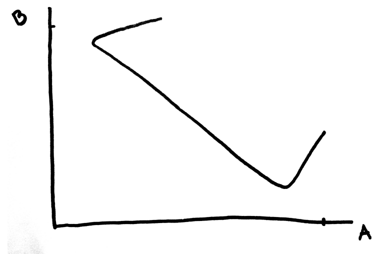
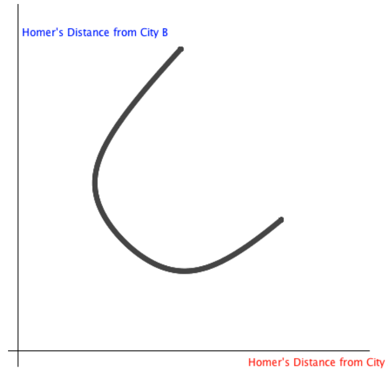


Figure 76: Screenshot 1 of Homer Task. At the beginning of the task the animation displayed (1) a depiction of the situation that showed the location of the cities (fixed) and Homer moving from the bottom of the road to the top of the road at a constant speed and (2) a set of axes labeled with Homer's distance from City B and Homer's distance from City A. (Day 3, Task 11)



Bryan's graph of Homer's distance from City B relative to his distance from City A



Graph of actual covariation (Bryan never saw the computer make this trace)

Figure 77: Bryan's graph of Homer's distance from City B relative to his distance from City A and graph of actual covariation (Day 3, Task 11).

Instead of plotting points and then connecting these points Bryan made his graph by drawing four connected curves (Figure 77). Although Bryan's behavior seemed to focus on smooth images of change, he still constructed his graph by focusing on pairs of measures at landmark points. He explained,

> I was just looking at Homer's black dot over there just trying to like get the best distance measurements I could get out of my head. … I just looked at like when he was over here (*points to start of road*), I was taking the distance and thinking about from City A and then City B. And then I just moved him to this point (*points to road closest to City B*) and I thought about the distance from City A and City B thing. And then I took this point (*points to road closest to City A*). I broke it down in segments and like took the measurements.

In this utterance Bryan explained that he made his graph by creating pairs of measures at landmark points on the road. Instead of plotting these points and then connecting them, he connected them as he made them. It does not seem like Bryan understood himself to be conveying a collection of pairs of measures with his line. Instead, it seems drew a line segment in order to get to the next pair of measures he imagined.

After Bryan completed his graphing activity I asked him to explain what his graph meant. In his explanation Bryan focused on more than the landmark points; he attended to the gross variation of each quantity between the landmark points. He explained,

> As it [Homer] moves closer to City B you can see that the distance from City B is decreasing (*moves pen from right to left down along right-most line segment*). Then as it [Homer] goes up and crosses this part [of the road] (*points to road closest to City B*) so its distance is increasing from B so the thing – the graph – is going up (*moves pen from right to left along middle line segment*). Then when it [Homer] comes to this point [on the road] (*points to road closest to City A*) it is the closest to City A. Then it is again increasing from City A and the distance is also increasing from City B (*moves pen from left to right along top left line segment*).

In this explanation Bryan coordinated his understanding of his sketched graph with his image of Homer's motion along the road. For most of his explanation, Bryan only attended to Homer's changing distance from City B. It was not until the very last part of this explanation that Bryan explained a part of his graph in terms of *two* changing distances. This is significant for two reasons. First, it suggests that reasoning from the sketched graph supported Bryan in constructing a new image of each quantity's variation

– a gross image of how each quantity changed in relation to the other. This is in contrast to the discrete pairs of measures he conveyed when making the graph. This suggests that Bryan understood a graph to show how a quantity's measure increases or decreases only *after* the line had been drawn. Bryan did not anticipate conveying this image of quantities' gross variation when making the graph.

Additionally, Bryan's focus on one quantity's gross variation at a time suggests that Bryan's image of situation did not focus on both changing quantities' simultaneously. It seems Bryan's image of the situation influenced the quantities he attended to when reasoning about the graph. Initially, Bryan's image of the phenomenon focused on Homer getting closer to and further from City B. Thus, he explained his graph in terms of how Homer's distance from City B changed. It was not until Bryan imagined Homer close to City A that Homer's distance from City A became part of his understanding of his sketched graph. Now Bryan's image of the situation attended to both City A and City B, thus he reasoned about the last part of his graph in terms of both changing quantities. Bryan needed to construct an image of the phenomena that entailed both City A and City B in order to coordinate his images of how each quantity changed.

Bryan completed the Homer task without referring to any of the tasks or visualizations from the teaching experiment. I asked Bryan to compare how he made his graph on the Homer task to his activity tracking the correspondence point in the last version of the U&V task to understand if/how Bryan related his activity in the U&V task to his activity on the Homer task. Bryan explained:

> I think before there were rectangles or squares or stuff like that. It was rectangles, right? Like the blue and the red bar I could take that as squares or rectangles and take the other point. Over here I wasn't thinking like that.

In this utterance Bryan describes tracking the corner of a rectangle to make a graph. This suggests that Bryan did not understand the correspondence point as a way to represent two quantities' magnitudes, as I intended. Instead, Bryan understood the correspondence point as the product of a concrete experience—the point was the corner of the rectangle created from extending the ends of the red and blue bars into the plane. This suggests that Bryan's memory of tracking the correspondence point was non-quantitative. He remembered keeping track of the corner a rectangle he could imagine on the screen but he did not understand his tracking actions as a way to simultaneously track two quantities' magnitudes. Although Bryan did not make the anticipated constructions in the U&V task, he was able to complete all versions of the Homer task using his pre-existing meanings for graphs.

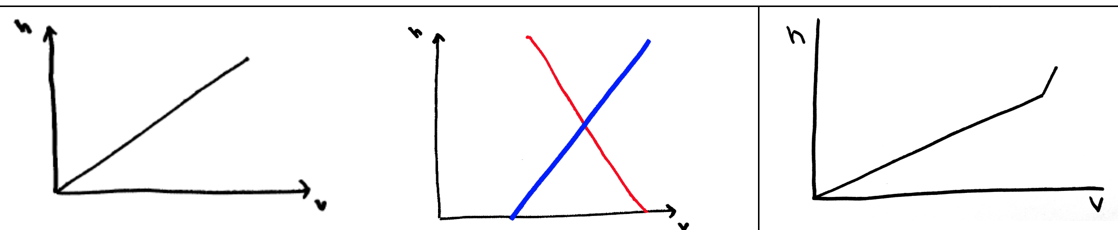
Generalizations in Post-TECI

Four days after the third, and final, teaching session I engaged Bryan in a one-hour post teaching experiment clinical interview (post-TECI). In this interview I engaged Bryan in the same tasks that I used in the recruitment interview and the pre-TECI. Bryan demonstrated two different images of how to coordinate quantities' measures in one's graphing activity.

Graphs and Pairs of Measures

Throughout the initial clinical interviews and teaching experiment Bryan would construct a graph by tracking his image of one quantity's variation as he constructed that image in his experiential time. Bryan's focus on a changing quantity in his experiential time inhibited Bryan from consistently attending to two quantities when constructing and reasoning about a graph. In the post-TECI Bryan's image of could anticipate a graph showing pairs of measures even when he imagined the phenomenon changing in his experiential time.

In both the pre-TECI and post-TECI I included two versions of the bottle problem. In the first version I asked Bryan to graph the height of water in the bottle relative to the volume of water in the bottle as the bottle of water filled (Stevens & Moore, 2017). In the second version I asked Bryan to imagine the water in the bottle evaporating and again graph the height relative to volume (Carlson et al., 2002). In the pre-TECI Bryan drew two separate graphs, one for the bottle filling task and one for the bottle evaporation task (see Figure 78); he made each graph by tracking the height of the water in his experiential time as he imagined the water filling/evaporating. As a result, he drew a decreasing graph as he imagined the water evaporating (red line, Figure 78).



Bryan's pre-TECI graph for bottle filling task

Bryan's pre-TECI graph for water evaporation task (initial graph in red, revised in blue)

Bryan's post-TECI graph for water filling task & water evaporation task (he drew one graph for both phenomenon

Figure 78: Comparison of Bryan's graphs for bottle task in pre-TECI and post-TECI

In the post-TECI Bryan quickly determined that the two graphs would be the same and did not draw a second graph for the bottle-evaporating situation. Instead, he reasoned that when height is maximum the volume is maximum regardless of whether the water is filling or evaporating. He concluded that the same point on both graphs represented the same pair of measures. It seems that Bryan imagined his graph to be made up of a collection of points that each represented a pair of measures that fit both the filling and evaporating situation. This thinking involved him imagining static states in each quantity's variation when thinking about a point. Bryan's focus on the measures represented by each point suggests that, in the post-TECI, Bryan's image of pairs of measures dominated his image of his graph as opposed to his pre-TECI image of a varying quantity in his experiential time.

Images of Asynchronous Coordination

Bryan's focus on constructing pairs of measures did not persist through his graphing activity. For example, in the post-TECI I asked Bryan to graph a skateboarder's speed relative to total distance traveled as he skated across a half-pipe ramp and returns to the starting position. Bryan sketched an oscillating curve in the first quadrant, from my perspective an appropriately shaped graph. However, Bryan did not explicitly attend to both quantities throughout his graphing activity. He explained,

> The total distance traveled is always increasing so I was thinking the graph is always moving to the right (*gestures left to right on horizontal axis*). For the speed, the speed is zero here (*points to start of ramp*), and then maximum over here (*points to bottom of ramp*) so that is that (*points to first maximum in graph*).

> Then it is minimum over here (*points to top right of ramp*) so that is that (*points to minimum in middle of curve*). Then it becomes maximum again (*points to second maximum in graph*) and then stops there (*points to starting point of ramp and then far right point of graph*).

In this explanation Bryan explained that he imagined the distance always increasing and understood this meant the shape of his graph moved to the right. Then he reasoned about the skateboarder's speed at five moments in time. He imagined plotting these five speeds, a discrete image of variation, as he moved his pen to the right. This suggests that when Bryan placed a point in the plane he was focused on his image of the varying speed as he imagined it changing in his experiential time. By imaging distance to be always increasing Bryan did not have to attend to its measure and instead could cognitively replace an increasing distance with his increasing experiential time. This implies that in the moment of marking a point Bryan explicitly attended to the speed of the skateboarder and implicitly attended to the amount of experiential time that had passed in his image of the event unfolding.

Rethinking my Meaning for Emergent Shape Thinking

At the outset of this study I proposed three constructions students would need to make in order to engage in emergent shape thinking:

1. Imagine representing quantities' magnitudes along the axes
2. Simultaneously represent these magnitudes with a point in the plane, and
3. Anticipate tracking the values of two quantities' simultaneously

Bryan's engagement in the teaching experiment provided insights into the constructions one must make in order to track two quantities' magnitudes simultaneously.

Throughout this study Bryan needed to pause (or imagine pausing) his image of the situation in order to create a point. When Bryan engaged in this point-plotting activity he created graphs by constructing and representing landmark points. Then he connected these points with straight lines. Since Bryan needed to imagine static states in each quantity's variation in order to plot a point, he could not imagine making a graph by tracking two magnitudes simultaneously. In fact, in the second version of the U&V task Bryan expressed that he didn't know how to create points without pausing his image of the situation (Excerpt 9, p. 250).

Bryan's engagement in the Kevin & Adam task provides some insights into how one might come to imagine keeping track of change in progress. In this task Pat asked Bryan to imagine two boys *walking* around a track in order to graph Kevin's straight-line distance from start relative to Adam's straight-line distance from start. Imagining the boys walking provided essential imagery for Bryan. First, it supported him in chunking his image of the continuously changing phenomenon so that he could imagine pairs of measures at the end of each chunk. He understood that he could represent those pairs of measures with a point. Since these chunks were created from his image of smooth phenomenon, I claim that this point-plotting activity is distinct from imagining a discrete collection of points at select moments throughout the event and instead supported Bryan in constructing a chunky continuous image of quantities' covariation.

I claim that Bryan's activity coordinating two distances at the ends of these chunks supported him in constructing momentary states of simultaneity. In a moment he imagined that Kevin had a distance from start *and* Adam had a distance from start. It is likely that repeatedly constructing and coordinating two magnitudes by plotting a point supported Bryan in developing a reflected image of his actions that embodied the structure of his actions. His image of plotting points now existed at a reflected level, at least momentarily, so that he understood his activity to be about coordinating two measures as opposed to plotting a point. This suggests Bryan's image of plotting a point was no longer dependent on the imagery of concrete location of the boys' on the track. Instead, he could draw a line while thinking 'I could also make points here, and here, and here' by keeping track of how each boy's distance changed.

I hypothesize that if a student repeatedly coordinates quantities' measures at the ends of chunks (created from his awareness of smooth change) then he has the opportunity to repeatedly construct the relation *a* and *b* in the moment until the relation is no longer dependent on the imagery in which it was created. Then one can construct an anticipation of relating *a* and *b*. This anticipation, in turn, will support student in imagining making this construction as *a* (or *b* varies). In other words, when the operation of "and" is no longer dependent on the imagery of plotting a point the student can keep both quantities in mind, in the sense that he can coordinate the changes in one quantity with changes in the other quantity by way of moving his attention rapidly between the two. Note that when the operation of "and" is no longer dependent on the imagery in which it was created, the images of change the student coordinates are not perceptions of change. Instead, he is coordinating reflected images of change.

I do not claim that Bryan achieved this reflected image of coordination. In fact, since Bryan never spontaneously engaged in imagining small bits of change, it seems he did not come to imagine that constructing small bits of change so that one can rapidly move his attention between images of quantities are part of keeping track of and coordinating two changing distances. It is likely that students will need repeated opportunities to imagine small intervals of change (and in many contexts) to construct a reflected image of coordinating two quantities' magnitudes as they change together.

CHAPTER 9
DISCUSSION & CONCLUDING REMARKS

In this chapter I present my retrospective analysis, the third phase of my analytical methodology. In this phase of the analysis, I compared the models I created for each student by looking for patterns in how I characterized each student's mathematical activity. I repeatedly leveraged Piaget's notion of developmental images in making sense of student's momentary successes and difficulties. I conclude the chapter by describing the development of my three students' images of variation and elaborate on how these students developed new ways to coordinate those images of variation.

Role of Imagery in Covariational Reasoning

This section elaborates my approach to examining students' imagery. I began the analysis by examining how students imagined quantities to co-vary. This resulting data was useful for investigating and characterizing how my subjects' images of covariation influenced their graphing activity. In this characterization I discuss the accommodations each student made in order to construct a graph from his/her image of how quantities changed together.

Next, I summarize how imagery (e.g., a correspondence point, Tinker Bell's pixie dust) supported students in developing images of variation and new ways to coordinate these new images of two quantities' variation. I address why different imagery might support different students in different ways. For example, students might need to understand a point as a way to coordinate simultaneous states in quantities variation in order to engage in continuous covariational reasoning. By documenting how students develop new ways to coordinate quantities' variation I hope to contribute insights into how one moves between levels of covariational reasoning proposed by Carlson et al. (2002) and Thompson and Carlson (2017).

The Story of Sue

At the beginning of the teaching session Sue attended to the path of an object's motion when making her graph. Then she reasoned about her graph as if it were a picture of the event. This suggests that images Sue constructed from a situation were pre-quantitative images of an object's motion. Since this was the only image Sue had of the situation, she used it when both making her graph and reasoning about that graph. In other words, the meanings Sue constructed from the products of her actions were consistent with the images she intended to show in her graph. As I explain, it is non-trivial for students to re-present images of two varying quantities in their graphing actions.

Developing acts of covariation. The imagery of moving red and blue bars in the U&V task supported Sue in constructing a new image of quantities' variation. Instead of focusing on the motion of an object as she had done in all previous tasks, Sue attended to when the motion of the bar changed direction. For example, she imagined a landmark state when motion of the red bar switched from moving right to moving left. Sue constructed two images of landmark states: one of landmark states the red bar achieved and one of landmark states the blue bar achieved. Sue's images of landmark states gave her new objects to coordinate. She coordinated her images of landmark states by watching the animation to see the next landmark state each quantity attained. Then she conveyed this pair of landmark states with the location of a point in the plane. Since the landmark states Sue coordinated in a point happened asynchronously, her meaning for a point was not multiplicative; Sue did not understand a point to convey simultaneous states in two quantities' variation.

Limitations to Sue's act of covariation. While Sue's thinking about landmark states supported her in constructing graphs for all of the U&V tasks, it did not support her in constructing graphs from the images of quantities' variation that she constructed from a situation. Sue's engagement in the Homer task suggested that she needed to consistently make three constructions to construct a graph by coordinating landmark states. First, she needed to construct an image of each quantity's variation. For Sue, this involved constructing an image of how each quantity's gross variation as she witnessed it changing in her experiential time. Next, Sue had to imagine orienting the magnitudes of these quantities' perpendicularly. Finally, Sue needed to identify landmark points in the motion of the bars. Each of these constructions happens in Sue's experiential time. This means that at soon as Sue tried to imagine orienting the bars on the axes her image of the quantity's variation had already changed. As a result, Sue could not repeatedly make these constructions. As Sue explained, she "just can't keep track of it".

This remark suggests that Sue anticipated coordinating images of variation in her graphing activity. However, Sue's activity constructing images of landmark states – the images she knew how to coordinate – could not keep up with her experience watching Homer move along the road. It seems that Sue would need to construct a reflected image of each quantity's variation in order to construct and coordinate landmark states without seeing the quantities' variation displayed on the axes.

Without seeing the red and blue bars on the axes Sue could not coordinate landmark states to make her graph. Instead, Sue constructed graphs by tracking an object's motion. She completed the Homer task by tracking his motion along the road keeping track of Homer's distance from the start of the road as he moved closer to and further from the starting point. With this meaning for graphs, Sue did not attend to landmark states or quantities' variation when making her graph. This suggests that the way one imagines coordinating two quantities' variation cannot keep up with her experience witnessing the quantities' variation she is limited to engaging in static shape thinking. For Sue this entailed re-presenting pre-quantitative images of the situation.

Final thoughts about Sue. Sue's engagement in the teaching experiment suggests that it is nontrivial for students to coordinate their images of variation. While the perceptual stimulus of animated red and blue bars supported Sue in constructing an image of landmark states and an anticipation of coordinating those landmark states in real time, she could only construct this imagery in the presence of the animated red and blue bars. This suggests that an image of coordinating landmark states in each quantity's variation is not, in itself, sufficient to conceive those landmark states, because the quantities' landmark states do not necessarily happen at the same time. Instead, it seems that students need to imagine how quantities' values change between landmark states in order to construct an image of variation that they can coordinate in real time with their construction of that image.

The Story of Ali

This section summarizes how Ali imagined quantities to change together at the start of the teaching experiment – what I call her preliminary acts of covariation. Then I focus on the role of imagery, in particular the imagery of the correspondence point and Tinker Bell's pixie dust to characterize the progress Ali made in developing new ways to coordinate her images of quantities' variation.

Ali's preliminary acts of covariation. Throughout the initial clinical interviews Ali imagined the variation of two quantities happening asynchronously. For example, she would imagine how a skateboarder's horizontal distance changed: it increased then decreased. Then, she would imagine how the skateboarder's vertical distance changed: it decreased, then increased, etc. It seems that Ali did not have a way to coordinate her two images of quantities' variation.

Ali anticipated that a graph would convey both of her images of change. In other words, Ali anticipated she could use the shape of her graph to see how the skateboarder's horizontal distance changed and also see how the skateboarder's vertical distance changed. However, Ali did not have a way to think about making one shape that would convey both of her images of change. It seems that Ali could not imagine two quantities changing together. She did not have a single image from having coordinated two quantities' variation that she could attend to when making her graph. This suggests that Ali could not form of a multiplicative object that united the two quantities' values as they varied together.

Since Ali did not have a way to attend to both of her images of change when making her graph (she had not constructed a multiplicative object), she made her graph by focusing on only one of her images of variation as she imagined it in her experiential time. When Ali's activity tracking one quantity's variation produced a graph she determined was incorrect she made her graph by guessing and checking shapes. I claim Ali engaged in static shape thinking to make her graph because she did not have a way to coordinate and re-present her image of two quantities asynchronous variation. As a result, the meanings Ali constructed from her graph were about the shape she made – they were empirical abstractions.

In summary, since Ali had two images of changing quantities that she could not coordinate and re-present in her graphing actions her graphing scheme consisted of two distinct experiences. First she made a graph (a shape). Then she reasoned about that graph by constructing images of quantities' gross and asynchronous images of change – images consistent with those she anticipated representing. While these are distinct activities I claim that the same act of covariation – imagining two quantities' changing asynchronously – can account for both Ali's activity making the graph and reasoning about that graph.

Developing acts of covariation. Pat and I used two didactic objects in Ali's teaching experiment. First we introduced the notion of a correspondence point with the intent that this new imagery would support Ali in coordinating two quantities simultaneously. Then we asked Ali to imagine her graph being made of Tinker Bell's pixie dust with the intent that Ali would see her graph as being made of correspondence points. In the following paragraphs I will describe the images that Ali constructed in this part of the teaching experiment.

The imagery of the correspondence point. As I described above, at the beginning of the teaching experiment Ali constructed images of how two quantities changed and she attended to these images asynchronously. I introduced the imagery of a correspondence point with the intent that it would support Ali in attending to two quantities' measures simultaneously. After repeatedly constructing the correspondence point at paused moments in the animation, Ali was able to successfully imagine and track the correspondence point in order to make her graph. It seems that the imagery of the correspondence point focused Ali's attention on the location of the end of each bar as opposed to the gross motion of the bars along the axes. Ali's focus on the ends of the bars gave her new images to coordinate; she understood a correspondence point as a way to coordinate the ends of these bars. As a result, as she witnessed the ends of the bars moving in the animation she coordinated those ends with the location of a correspondence point. As I explain in the next section, Ali needed to see the bars on the axes in order to continuously imagine uniting two quantities' measures. Without the perceptual support of seeing the bars on the axes, Ali's ability to unite her images of variation did not persist since it was something *she* had to maintain.

While Ali coordinated the ends of each bar as they changed together to make her graph, she did not see the graph she created in terms of her acts of coordinating. This suggests she did not have an image of having coordinated each bar's changing length. As a result, she explained her graph as a depiction of two quantities' gross and asynchronous variation. In other words, she still engaged in her initial acts of covariation – imagining two quantities' changing asynchronous – when reasoning about her sketched graph.

The imagery of Tinker Bell & her Pixie Dust. Ali constructed an image of having coordinated the ends of the bars by imagining Tinker Bell making the graph with particles of pixie dust. In other words, imagining Tinker Bell creating a curve with pixie dust supported Ali in attending to the actions she engaged in to make the graph.

Instead of being engrossed in her own graphing actions, Ali now imagined watching Tinker Bell create the curve with her pixie dust. For Ali to control how she imagined Tinker Bell to move in the plane she needed to attend to both the path Tinker Bell made but also she needed to imagine *how* Tinker Bell made that path. This involved a crucial element for Ali – thinking about how Tinker Bell knew where to fly. As Ali explained, Tinker Bell knew where to fly by "noticing where the value of u and the value of v were". This suggests Ali attended to the actions involved in constructing the graph. More specifically Ali was constructing a reflected image of coordinating the value of u and the value of v through the location of a particle of pixie dust.

Ali's imagining Tinker Bell "knowing where to fly" appears to have provided her a way to externalize how *she* knew where to place points in the midst of tracking two quantities' simultaneous variation. In other words, it supported Ali in understanding that she made her graph by coordinating changes in two quantities' magnitudes. This suggests that students need opportunities to reflect on their graphing actions in order to construct a reflected image of having coordinated two quantities' values. It is possible that imagery like Tinker Bell and her pixie dust gives students a new perspective on their graphing actions that supports them in constructing this reflected image, at least momentarily.

Limitations to Ali's acts of covariation. In the U&V task Ali coordinated the changing values of each quantity (the location of the ends of the bar) by tracking a correspondence point. This supported her in completing all versions of the U&V task. However, without seeing the moving red and blue bars, Ali had a hard time maintaining her image of a correspondence point. This was evident in Ali's engagement in the Homer task.

In the Homer task Ali anticipated coordinating her images of variation. However, without the presence of the continuously changing red and blue bars on the axes the images she had to coordinate were images of gross variation – not the ends of bars. As a result, when Ali graphed Homer's distance from City B relative to his changing distance from City A she coordinated her two images of gross variation with the direction of a line. She understood a line going to the left and down shows both a decreasing distance from City A and City B. However, Ali's could not maintain her focus on both images of gross variation; when Ali imagined Homer at the halfway point on the road she lost track of her image of Homer's changing distance from City A. As a result Ali attended to her image of Homer's changing distance from City B as she made her graph and as she reasoned about that sketched graph.

This suggests that Ali's thinking that a point is made from coordinating the value of u and the value of v in the U&V task was dependent on seeing the animated bars oriented perpendicularly on the axes. In other words, for Ali to consistently imagine uniting two quantities' measures she needed to see these measures oriented on the axes and imagine extending those measures into the plane. Without that perceptual support, Ali's image of uniting attributes was something *she* had to maintain. With this added construction – the construction of coordination – Ali continued to have difficulty constructing an image of each distance and then also coordinating those constructions in real time as the animation played. As a result, as she watched Homer move along the road she lost track of his distance from City A changed. This suggests it is nontrivial for one's image of a multiplicative object to persist without the visual support of quantities' measures being displayed directly on coordinate axes. In summary, while Ali anticipated coordinating how two quantities changed together to make her graph, her ability to coordinate her images of two quantities' gross variation was "outpaced" (to use Piaget's term) by the variation she imagined in each quantity.

Final thoughts about Ali. Ali's engagement in the teaching experiment suggests that it is essential that students have an opportunity to construct a reflected image of how they coordinate images of variation. At the outset of this study I thought that if one made a graph by simultaneously tracking two changing quantities (i.e., tracking a correspondence point), then she would understand her sketched graph having been made by tracking two magnitudes simultaneously. I had not considered that Ali could construct a graph by tracking a correspondence point and then not reason about her graph as a re-presentation of coordinating images of changing quantities. This suggests that the meaning Ali had for her sketched graph did not reflect the thinking she engaged in to make the graph. This implies that it is nontrivial for students to construct a reflected image of coordination where they understand the products of having coordinated (a graph) in terms of the way they coordinated images of variation to make that graph.

More generally speaking, this suggests that a researcher must not take a student's thinking when making a graph, by itself, as evidence of emergent shape thinking. Instead, emergent shape thinking involves both constructing a graph by attending simultaneously to two changing magnitudes and also understanding a graph as having been made by tracking two magnitudes simultaneously. This means that researchers must be attentive to the images that govern both students graphing actions as well as the images that govern the meanings the student has for the products of those graphing actions. Ali's thinking about Tinker Bell and her pixie dust suggests that students need opportunities to attend to *how* they made their graph so that they can take their graphing actions as objects of thought resulting in order to understand their sketched graph as showing/representing how two quantities change together.

Comparing Ali & Sue's acts of covariation. I expect that Ali was able to develop more sophisticated acts of covariation than Sue because she had different images of variation to coordinate: Sue coordinated landmark states and Ali coordinated smooth images of variation. Since Sue coordinated asynchronous landmark states she could only engage in this thinking in the presence of perceptual stimuli like moving red and blue bars on the axes. Ali, on the other hand, coordinated images of change in progress. As a result, she could coordinate her images of quantities' variation as she made them. In other words, Ali did not need to construct a reflected image of change in which she could identify landmark states. While it was cognitively demanding for Ali to maintain her focus on both images of variation, she constructed a way of thinking that supported her in making graphs by re-presenting how she imagined quantities to change together. Simply stated, this suggests students must have images of quantities' smooth and gross variation in order to coordinate two quantities' values changing simultaneously.

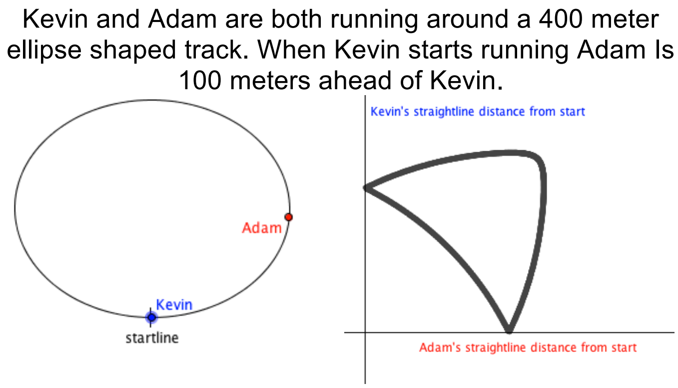
The Story of Bryan

In this section I summarize Bryan's acts of covariation at the start of the teaching experiment – what I call his preliminary act of covariation. Then I describe progress Bryan made engaging in continuous covariational reasoning and highlight imagery that supported Bryan in coordinating his images of covarying quantities – imaging little bits of change. I conclude by hypothesizing elements of Bryan's images of coordination that supported him in developing a more sophisticated way to reason about covarying quantities than either Ali or Sue.

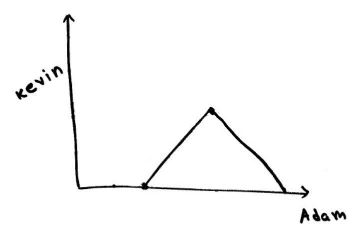
Bryan's preliminary engagement in covariational reasoning. At the beginning of this study, Ali and Sue did not seem to coordinate two changing quantities. Bryan, however, coordinated two quantities' measures by imagining a static state in each quantity's variation and then plotting a point to represent both quantities' magnitudes.

Since Bryan needed to imagine a static state in order to coordinate quantities' measures, he could not imagine plotting points when he imagined the quantities' to change continuously. As a result, Bryan could only coordinate measures at specific moments of the phenomenon. This suggests Bryan's image of coordinating static states in quantities' variations could not keep up with his experience witnessing (or imagining) quantities' variation. Thus, when Bryan attempted to re-present his image of two changing quantities he could not maintain his focus on both quantities as they varied.

Since Bryan was unable to coordinate his images of two continuously changing quantities he made his graph by imagining one of the quantities varying in his experiential time. For example in the Kevin & Adam task I asked Bryan to graph Kevin's straight-line distance from start relative to Adam's straight-line distance from start as the boys ran around an ellipse shaped track. Initially, Bryan responded to this task by attending to a pair of measures. He reasoned, “when Kevin’s distance is 0 Adam’s has some value” and represented this with a point in the middle of the horizontal axis (see Figure 68). However, Bryan’s activity coordinating his two images of variation could not keep up with his image of two boys moving around the track. As a result, he drew the rest of his graph by attending to only one quantity’s (Kevin’s distance) variation as he imagined it changing in his experiential time.



Screen capture of depiction of phenomenon including, for the reader, a graph of actual covariation



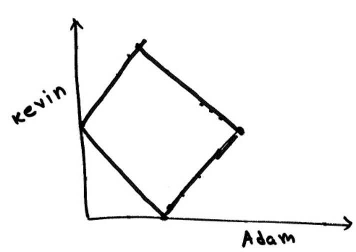
Bryan’s initial graph

Figure 79: Screenshot of Kevin & Adam Task, graph of actual covariation, and Bryan’s solution

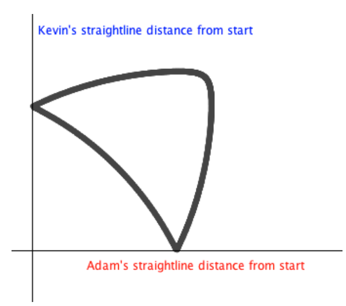
While Bryan coordinated static states in each quantity’s variation, he did not coordinate his images of changing quantities. This was evident when Pat asked Bryan to explain what happened when the boys in the Kevin & Adam Task took just a few steps. Bryan said, “Kevin’s will increase a little”. Pat had to explicitly ask how Adam’s distance changed before Bryan said, “Adam’s would go up a little bit”. This suggests that Bryan imagined the two distances changing asynchronously. He coordinated these two images of change when he thought about plotting a point to show “Adam is at something and Kevin is also at something”.

Developing acts of covariation. It seems that imagining little bits of change in each quantity's value supported Bryan in transition from coordinating static states to coordinating two quantities' measures as they were changing. In the following paragraphs I will provide evidence to support this claim.

As Bryan imagined little bits of change he made a graph (Figure 80) by imagining a little bit of change in each quantity, plotting a point, imagining another little change in both quantities, and plotting a point. This activity was significant because it suggests that Bryan coordinated the ends of the chunk not by imagining specific measures but instead by imagining the little bits of change in each quantity over the chunk. This implies his image of points was no longer dependent on imagining static states to convey simultaneously. Instead, Bryan understood a point to show how quantities' changed together. More specifically, he understood that he could determine the location of a new point relative to a prior point by imaging how the quantities changed between those points. As a result, Bryan could imagine making points without needing to imagine the boys at a concrete location on the track.



Bryan's new graph



Graph of actual covariation

Figure 80: Bryan's second attempt graphing Kevin's straight-line distance from start relative to Adam's straight-line distance from start as he imagined the boys moving a couple steps at a time.

It seems that imagining little bits of change supported Bryan in imagining a quantities' gross variation to be made up of accumulating changes in measures (or magnitudes). This supported him in coordinating his image of a quantity's gross variation with his image of static states in a quantity's variation so that he imagined quantities' values emerging from little bits of change in each quantity. This suggests imagining little bits of change is essential to construct an image of a quantities' chunky continuous variation

Constructing an image of having coordinated. Bryan could anticipate imagining little bits of change in both quantities to locate a new point relative to a prior point as he imagined the quantities' values to continue changing. This was evident when Bryan switched from imagining little bits of change to focusing on when one of the boys' reached a landmark location. I take this as evidence that Bryan's had an image of plotting points that existed at a reflected level, at least momentarily, so that he understood his activity to be about coordinating how two measures changed as opposed to plotting a point. This implies Bryan's image of plotting a point was no longer dependent on the imagery of concrete location of the boys' on the track. Instead, he could draw a line while thinking 'I could also make points here, and here, and here' by imagining an even smaller chunk size in order to plot more points along the line. This suggests Bryan constructed, at least, chunky continuous covariation of the boys' distances, but I cannot claim that his thinking entailed smooth continuous covariation where Bryan anticipated representing pairs of measures as they varied continuously and simultaneously.

It is likely that repeatedly constructing chunks and imaging a point by coordinating little bits of change supported Bryan in developing a reflected image of his actions that embodied the structure of his actions. His image of plotting points now existed at a reflected level, at least momentarily, so that he understood his activity to be about coordinating how two measures changed together and not plotting a static point.

Limitation to Bryan's act of covariation. Bryan's image of plotting one point relative to another by imagining a little bit of change in each quantity was a functional accommodation; it did not persist in Bryan's thinking about graphs. This was most evident in Bryan's engagement in the U&V task; Bryan explained that the only way he could envision making a point was by pausing the animation to plot a point. I take this as evidence that the image Bryan constructed of coordinating little bits of change in order to coordinate two changing quantities was not a reflected image of coordination Bryan did not come to imagine that constructing small bits of change so that one can rapidly move his attention between images of quantities is essential to keeping track of and coordinating two changing quantities.

Bryan's developmental acts of covariation. In this section I hypothesize why Bryan's activity coordinating quantities' measures with a point might have be essential for him to construct an image of chunky continuous covariation

When Bryan imagined the boys taking a couple steps he created an image of a little bit of change in quantity X and a little bit of change in quantity Y. It seems his focus on quantities' measures supported him in anticipating new measures: x and y. With two measures in mind Bryan coordinated these two measures with the location of a point. By repeatedly making this construction Bryan came to understand that he could determine the location of a new point relative to a prior point by imaging how the quantities changed between those points. This was significant because Bryan no longer needed to imagine a static state in order to plot a point and coordinate two measures. I claim that Bryan's focus on the relationship between the points supported Bryan in constructing an image of *related* points, as opposed to a collection of isolated points. By thinking about how points are related (by imagining little bits of change) Bryan could coordinate magnitudes as they varied together.

Constructing an image of the relationship between the points was essential for Bryan to anticipate plotting points as he drew a line. Since Bryan could imagine the little bits of change to be any size he could imagine those points being as close as possible. As a result, he could imagine capturing all possible points. Thus, as he drew a line he thought, 'I could have made a point here, and here, and here' by attending smaller amounts of change.

I want to emphasize that Bryan's act of coordination was always about plotting points and coordinating pairs of measures. Thus, I hypothesize that one needs to have an image of coordinating measures in a point – an image of a multiplicative object – in order to construct an image of the relationship between these points. In other words, it seems to have been essential that Bryan understood a point as a coordination of static states in quantities' variation for him to come to coordinate little bits of change with the relative location of two points.

Since Ali needed to see the red and blue bars moving on the axes in order to maintain her construction of a multiplicative object, she did not have an image of coordination that persisted under variation. In other words, she did not coordinate quantities' measures independent of imagining and coordinating the red and blue bars. As a result, my hypothesis would suggest that Ali would not have been able to construct an image of chunky continuous covariation. Since my teaching experiment with Bryan was the last of the three I did not have an opportunity to test this hypothesis.

The Role of Coordination in Covariational Reasoning

The findings presented in this section highlight the cognitive work involved in constructing even the earliest image of covariation where one anticipates coordinating two quantities' variation. At the outset of this study neither Ali nor Sue coordinated their images of two varying quantities. It took explicit instruction for these university precalculus students to coordinate their images of quantities' variation. This suggests that researchers should be mindful of a Level 0 image of covariation – no coordination.

It is possible that students might need to coordinate amounts of change in each quantity (what Carlson et al. (2002) call MA3) in order to construct an initial image of covariation. Bryan's activity imagining little bits of change suggests that one might need to coordinate small changes in each quantity's magnitude in order to construct an initial image of covariation that persists under variation. This suggests students might need to imagine little bits of change in order to construct an image of covariation where they imagine changes in both quantities happening together.

Finally Ali's and Bryan's image of the correspondence point was dependent on seeing the red and blue bars on the axes in order to imagine a rectangle and its corner to track. This had different implications for each student. For Ali, the imagery of tracking a correspondence point supported her in understanding a graph as a re-presentation of how quantities change *together*. However, her image of the multiplicative object – how to unite quantities' variation – was dependent on seeing the red and blue bars on the axes.

As a result, without this perceptual support she had a hard time maintaining her focus both her images of quantities' variation.

Bryan, on the other hand, had a non-quantitative image of the correspondence point. He understood the tracking of the correspondence point to be the concrete activity of tracing the corner of a rectangle. He did not understand the position of the rectangle's corner as representing two measures simultaneously. The rectangle's corner was the focus of his attention, and thinking of the rectangle allowed him to think of its corner. However, the rectangle's sides were sides of a rectangle. They were not representations of two quantities' magnitudes. Thus, I claim that Bryan did not see his activity of tracking a correspondence point as tracking two magnitudes simultaneously.

Ali's and Bryan's image of a correspondence point have consequences for task design and instruction. When I designed the tasks for this teaching experiment I anticipated that students would abstract their construction of extending two magnitudes' from the axes to imagine uniting quantities' measures. I did not anticipate the extent to which students would need to rely on a figurative construction from the red and blue bars on the axes. This finding suggests that researchers and educators should make conscious efforts to avoid attributing conceptual operations to students' tracking of a correspondence point from bars labeled on the axes.

Summary of Main Findings

In this section I summarize the main findings of this dissertation study. Specifically, I discuss the importance for researchers to attend to the images of covarying quantities students intend to convey in their graphing activity and the images students re-present in their graphing activity. In doing so, I document the difficulty students' had maintaining acts of covariation as they imagined change in progress. Finally, I discuss how acts of covariation constructed in the context of graphing do not naturally generalize to understanding formulas in terms of images of covarying quantities.

Differentiating Between Images of Constructed and Images Re-Presented

Sue, Ali, and Bryan engaged in multiple forms of variational and covariational reasoning when engaged in a single task. More specifically, the acts of covariation they engaged in making a graph were often different than the acts of covariation they engaged in when reasoning about their sketched graph. While this highlights the meanings students learn to impose on the products of their graphing actions, the findings from this study suggest that the meanings students construct from their sketched graph might be consistent with the images of covarying quantities they intended to re-present in their graph. This is significant because it implies that students engage in different forms of covariational reasoning because they are unable to re-present how they imagine quantities changing together in their graphing actions.

The findings from this study provide insights into two reasons a student might be unable to re-present his actions of variation or covariation. First, the student might attend to two quantities' variation asynchronously. As a result, the student does not have a single coordinated image to attend to when making her graph. For example, Ali consistently imagined the variation of each quantity happening separately from the other in her experiential time: first she imagined the variation of Quantity X and then she imagined the variation of Quantity Y. When she attempted to construct her graph she anticipated that she could use whatever shape she made to see the variation of each quantity, but she did not have a way to think about how to make that shape. Instead, she made her graph by guessing shapes until she picked one that appropriately matched how she imagined each quantity's gross variation.

Another reason a student might be unable to re-present his actions is that the student's ability to coordinate two varying quantities' cannot keep up with his experience imagining the quantities changing in his experiential time. For example Bryan coordinated static states in quantities' variation with coordinates of a point. However, as soon as he imagined one of the quantities varying he no longer had an image of a static state in which he could coordinate two measures. As a result, when he attempted to construct his graph he did not continuously coordinate quantities' measures. Instead, Bryan made his graph by imagining one quantity changing in his experiential time. After making his graph, however, Bryan imagined coordinating measures to reason about what his sketched graph represented; he appeared to reason about an infinite collection of points on his graph. For example, after drawing a line he would describe the "millions of points" on he imagined on that line. In summary, since Bryan's image of plotting points did not persist under variation, Bryan could not re-present this image of covariation when he imagined a continuously changing phenomenon.

Initially I attributed students' engagement in multiple forms of variational and covariational reasoning to uncoordinated graphing schemes. However, it is more coherent to think about the acts of covariation students engage in and how they might re-present those acts as they imagine change in progress. The examples described above suggest that while a student might have distinct experiences making a graph and reasoning about that graph these experiences are actually governed by the same scheme; the student's activity making a graph is the result of an accommodation to their scheme for covariational reasoning in order to have actions available to them that persist under variation.

Coordinating Images of Change in Progress

As I described above, it is nontrivial for one to coordinate her image of two quantities' values as the values of the two quantities vary in tandem. In fact, Bryan's construction of a coordination of values inhibited him from coordinating images of variation. This was evident when Bryan was responding to the Kevin and Adam task. Recall that Pat eventually directed Bryan to focus on both Kevin and Adam's straight-line distance from start as they moved a few steps from the starting line. When Bryan could not coordinate two quantities' measures as they changed continuously, he focused on one quantity's variation as he imagined it in his experiential time.

Ali also had a hard time imagining a gross coordination of values as she imagined change in progress. In the Homer task she started to make her graph by attending to gross changes in both Homer's distance from City A and his distance from City B. Half way through her graphing activity she stopped imagining the variation in both quantities' values, and only maintained a focus on Homer's changing distance from City B. Thus, she created the rest of her graph by tracking one quantity's variation as she imagined it in her experiential time.

These two examples suggest that when a student's image of covarying quantities does not persist as they imagine change in progress they will end up re-presenting an image of one quantity's magnitude (or one object) changing in their experiential time. Bryan's engagement in the Kevin and Adam task provides some insights into how one might construct an image of covariation that can persist under variation. In this task it seemed essential for Bryan to imagine little bits of change so that he could create chunks in which he rapidly switched his focus between quantities' measures. This is similar to the reasoning that Carlson et al. (2002) reported students using when responding to the bottle task and suggests that constructing a systematic approach to coordinate change in progress might support students in staying focused on both changing quantities as they imagine them both changing in their experiential time.

Reasoning Covariationally about Formulas

In theory, the images of covariation one constructs in the context of graphing are not constrained to his graphing activity; if one constructs the operation of "and" at an operative level it is not constrained to the context in which it was constructed. In this study I sought to understand the ways in which students generalized the constructions they made in the context of graphing to their reasoning about formulas. The students in this study did not come to reason covariationally when reasoning about formulas.

In fact, only Ali reasoned covariationally without the support of the research team. (Sue's constructions were constrained to the perceptual stimulus of moving bars on the axes, and Bryan's constructions were functional accommodations.) This suggests that students need repeated opportunities to coordinate their images of variation in order to construct a stable image of coordination that they can operationalize in other graphing tasks.

While Ali came to consistently construct graphs by coordinating her images of change in progress, Ali did not coordinate her new understanding of graphs with her meaning for formulas. When I asked Ali if she could relate the red and blue bars to formulas she said, "I honestly cannot imagine the red and blue bars with the formulas."

This is significant because Ali had previously volunteered how productive it had been for her to think about graphs in terms of the red and blue bars on the axes. She explained the imagery of the red and blue bars on the axes gave her “a clearer perception of what is actually happening rather than just okay, plot the points and the robotic process of connecting them.”

Ali’s inability to imagine the red and the blue bars in the context of understanding formulas suggests that Ali did not construct graphs and formulas as representations of the same covariational relationship. I will address two possible explanations for why Ali did not understand a formula to coordinate two varying quantities.

In the pre-TECI, Ali demonstrated thinking that suggests she understood formulas to relate quantities’ measures; she explained numerical operations as “converting” from one quantity to another. This suggests she had constructed at least a preliminary understanding of a quantitative relationship. As a result, Ali’s understanding of the red and blue bars as representing quantities’ measures should align with her meaning for formulas relating quantities’ measures.

It seems that her problem might instead be in how she understood variables. In the pre-TECI Ali thought about a variable as a placeholder for a single number. This suggests that she did not anticipate variables representing varying magnitudes. This static meaning for variables inhibited Ali from coordinating the dynamic imagery of the red and blue bars with her static image of formulas.

Directions for Future Work

In this section I provide suggestions for how to extend the work presented in this dissertation. Earlier in this chapter I hypothesized that one must construct an image of a point as a way to unite static states in quantities' measures – an image of a multiplicative object – in order to construct an image of continuous covariation. Since I developed this hypothesis in the context of my final teaching experiment with Bryan I did not have an opportunity to test this hypothesis. I suspect that testing this hypothesis would provide insights into developmental constructions necessary to engage in continuous covariational reasoning.

A second line of study could examine the role of dynamic animations in the development of students' images of covariation. Throughout this chapter I described how students had a hard time maintaining images of how to coordinate two quantities as they imagined change in progress. Much of the participants' difficulty maintaining an image of covariation happened when they tried to reason about changing quantities in the presence of a dynamic animation. This suggests that while dynamic animations might support students in imagining change in progress, these animations alone might actually inhibit students from constructing images of how to coordinate two changing quantities. What conversations can educators design that turn dynamic animations into didactic objects – "things to talk about" that support students in constructing images of change in progress but do not inhibit them from maintaining their images of covariation as they imagine change in progress?

One could also study how to support students in engaging in covariational reasoning when reasoning about formulas. I suspect that there are two aspects to this line of inquiry. The first is in supporting students in seeing formulas as expressions of quantitative relationships – relationships among quantities' measures as opposed to means to get one number from another. One could also investigate if acts of covariation constructed in the context of formulas generalize more naturally to thinking about graphs.

Another area of focus could address the students' internal motivation to *try* and coordinate quantities measures. How do educators support students in experiencing the intellectual need to understand the task in a way that is about coordinating quantities measures? I hypothesize that this is related to constructing an invariant covariational relationship.

Finally, this study highlights the complexities in students' graphing schemes. Throughout my analysis it was essential for me to differentiate the images students attended to when making a graph and the images they operationalized when reasoning about their sketched graph. This provides a challenge for educators. If one wants to understand students' graphing schemes what tasks might produce insights into both the images students intend to convey and the images they use when reasoning about their sketched graph?

Addressing "The Problem"

At the outset of this study I thought that students' focus on static relationships inhibited them from engaging in covariational reasoning, and thus emergent shape thinking. The findings from this study provide a more nuanced understanding of this problem – students focus on static relationships between quantities' measures because they do not have ways to coordinate their images of changing quantities. Before a student can reason about a graph (or formula) as representations of how quantities change together, they need to develop ways to coordinate their images of varying quantities. Sue, Ali, and Bryan's engagement in the teaching experiment suggests that students have imagine quantities changing in their image of the situation, however they need new imagery, such as imagining the quantities changing in little chunks, to help them coordinate two quantities measures as they imagine those measures to change continuously.

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APPENDIX A

PROTOCOL FOR RECRUITMENT INTERVIEW

I recruited 10 students from a large southwestern university to participate in a Recruitment Interview. All recruited students had either completed university Precalculus or were currently enrolled in university Precalculus. The recruitment interviews lasted between one and two hours. In addition to completing four mathematical tasks students also completed a survey about their mathematical background.

I designed the tasks used in this interview to gain insights into:

- The ways in which the student engages in static/emergent shape thinking
- How the student conceptualizes and attends to quantities' measures varying in a situation
- How the student coordinates two varying quantities' measures in their graphing activity

Task 1: *I designed this task for my pilot study to help me better understand the meanings a student has for points and lines on the graph and how the student coordinates these meanings.*

Susie is walking away from her house. The table below represents her distance from home (in feet) in terms of the number of minutes elapsed since she left her house. Sketch a graph of this relationship

t	*d*
1	2
2	5
3	10
4	17

a. Before I ask you to graph the relationship, can you explain to me what you see here?

   i. What does this column represent (point to column on right)? What about this column?

   ii. What about this row?

   iii. What does *t* mean? *d*?

b. Okay, now can you graph the relationship?

   i. Can you explain what you did?

   ii. How did you decide how to graph this row that says 1, 2?

   Focus on the student's conceptualization of a point. For example, is the point a place in space, the product of over and up, or a multiplicative object?

c. Why did you connect the points you graphed with lines?

   Focus on how the student attends to x values between 1 and 2.

d. What is the difference between points and lines?

   Focus on the extent to which the student conceptualizes a graph as an emergent trace and whether the student imagines lines as objects or as collections of points.

e. How many points are on your graph? Could you plot more points?

   Focus on the generalizations the student makes for other values of x. Also attend to other representations the student mentions (i.e. formulas), what the student imagines these to be representations of, and how the student sees these representations as similar and different.

f. What does graphing get me?

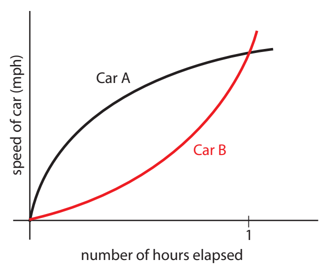
   The purpose of this question is to attend to the extent to which a student conceptualizes a graph as a representation of something. For example, is the graph the product of a command to act or a desire to represent?

g. Did you learn anything by graphing the values represented in this table?

h. Why do you think your teachers ask you to draw graphs?

Task 2. *The second task comes from Monk (1987) and Carlson (1998). These authors have documented that students often engage in static shape thinking while engaging in this task.*

Consider the graph below, which describes two cars' speeds in terms of the number of hours elapsed since they started traveling.



a. Can you describe what you see in this graph?

   Attend to the quantities the student does/does not conceptualize in the situation.

b. How many points are plotted on this graph?

   Does the student place more emphasis on intersection points?

c. Why do you think both curves start at the same place? What does this say about the situation?

   To what extent does the student coordinate aspects of the situation & his experience driving with his interpretation of the graph?

d. 0.5 hours after the cars started moving, which car is traveling faster?

   How does the student interpret point-wise features of the graph? Does the student continue to attend to the quantities labeled on the axes, or does the student begin reasoning about distance with respect to time?

e. 0.5 hours after the cars started moving, which car has traveled the furthest?

   How does the student engage in a point-wise comparison between two relationships?

f. How is the speed that Car A is traveling changing as the car travels along the road?

   Attend to the way in which the student engages in covariational reasoning and if the student coordinates Car A's speed with measured or experiential time.

g. How is the distance that Car A has traveled changing as the car travels along the road?

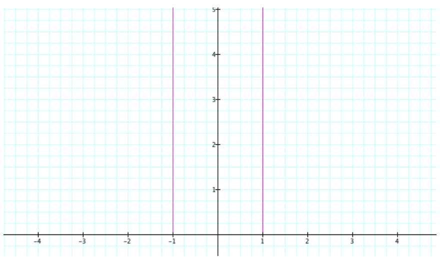
   Attend to the extent that the student is able to construct quantitative relationships from the relationship represented graphically. (Or even more general, Does the student conceptualize the graph representing a relationship?)

h. What is the relative location of the cars after 1 hour?

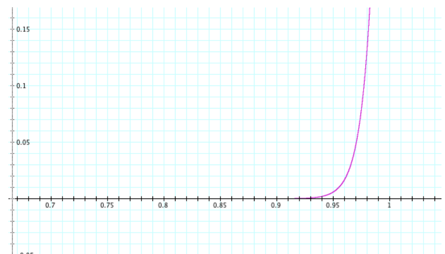
   To what extent does the student engage in iconic translation when making across-time comparisons between relationships between two pairs of varying quantities?

Task 3: *Patrick Thompson designed this task. My pilot study data suggests that static shape thinkers are not able to assimilate the zoomed in behavior of the graph without conceptualizing an entirely new shape – a new graph.*

Using Graphing Calculator, researcher depicts following graph:



Initial display in graphing calculator



Zoomed in view around $x = 1$ and the horizontal axis.

a. Can you describe what you see?

b. What do you think is happening right here? (Point to x = 1 on horizontal axis)

   Attend to whether the student describes shapes or relationships between values.

c. Is there a y value associated with x = 0.5?

   Attend to whether the student seems to already be thinking about x and y values or if this question causes the student to think about the graph in a new way. Does the student think about graphs as shapes or representations of relationships between x and y values.

d. Is there a y value associated with x = 1?

   Does the student experience any perturbation thinking about infinitely many y values associated with a single x value? If so, how does the student resolve this perturbation?

e. What do you think we will we see if we zoom in around x = 1 on horizontal axis? Go ahead and zoom in (student uses computer based graphing calculator to zoom in.) *Be sure to have student do the zooming in so that she does not think the researcher somehow changed the graph when zooming in.*

f. Is this what you expected to happen? How is this possible? *This question will help me understand the ways in which the student thinks variationally and imagines values of x varying.*

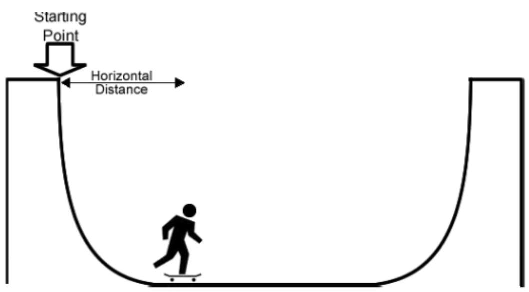
   i. Is there a y value associated with x = 0.5?

   ii. Is there a y value associated with x = 1?

g. Suppose your friend is going to come and complete this task tomorrow. How would you want her to think about the initial graph? *This question will help me understand how the student's initial image of the situation has changed as a result of their zooming activity. Specifically, I will attend to the measures of the quantities the student will attend to.*

Task 4: *The skateboard context comes from the Pathways Precalculus materials. However, I have adapted the task in order to understand the extent to which the student engages in variational reasoning with respect to experiential time versus coordinating two quantities' measures.*

A skateboarder skates on a half-pipe like the one shown below. The skateboarder goes across the half-pipe and then returns to the starting position (Carlson et al., 2013)



a. Can you describe what is going on in the situation?

b. Sketch a graph that represents the skateboarder's speed in terms of the total distance traveled by the skateboarder since leaving the starting point as the skateboarder goes to the far end of the half-pipe and back to the starting point.

    i. Can you explain what you decided to draw?

c. Sketch a graph that represents the skateboarder's speed in terms of the number of seconds elapsed since leaving the starting point as the skateboarder goes to the far end of the half-pipe and back to the starting point.

    Attend to whether the student seems perturbed by this question. I anticipate that if the student sketched the graph in part (b) by attending to experiential time, then the student will not understand how the task in part (c) is different than part (b).

d. How is this graph similar to/different from the graph you first drew?

Attend to whether the student describes shape similarities or similarities between how quantities vary together.

e. Sketch a graph that represents the skateboarder's horizontal distance from the starting point in terms of the skateboarder's vertical distance above the ground as the skateboarder goes to the far end of the half-pipe and back to the starting point.
   Attend to the student's graphing habits and whether the student experiences perturbation "starting" the graph not on the vertical axis. Also attend to how the student reasons about the graph as the skateboarder comes back across the half-pipe.

f. Is it weird to draw three graphs from the same situation?
   This is a general question to try and understand the student's motivation for graphing.

   i. Have you ever done something like this before?

   ii. What do we gain by drawing all three of these graphs?

Part II: Mathematical Background Questionnaire

Name: ___________________________________

Freshman Sophomore Junior Senior Graduate Student

Major: __________________________________

Expected Graduation Date: _________________

1. When did you take MAT 170 (Precalculus)? ________________________

2. Who was your MAT 170 Instructor? ______________________________

3. **Optional:** What grade did you earn in MAT 170? ____________________

4. Have you previously taken Precalculus? If so, when and where?

____

5. Have you ever taken Calculus? If so, when and where?

____

6. Have you taken any physics courses? If so, when and where?

____

7. Do you plan on taking any more math, statistics, or physics courses?

____

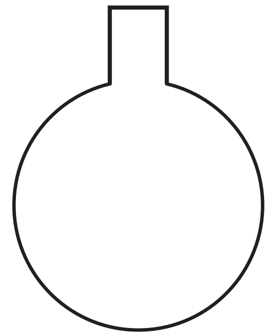
8. Where did you go to high school?

____

APPENDIX B

PROTOCOL FOR PRE-TEACHING EXPERIMENT CLINICAL INTERVIEW

Task 1: Water Filling - Bottle Problem (in Carlson, 1998; Carlson et al., 2002; Johnson, 2015; Paoletti & Moore, 2016)

Imagine this bottle is being filled with water. Sketch graph of height of water in the bottle in terms of volume of water in the bottle.	

Attend to how student labels axes, student's description of quantities, and student's conception of time while imagining the bottle filling with water. Is the student perturbed since time elapsed is not one of the quantities being compared?

a. Can you explain how you decided what to draw?

b. You leave the bottle of water outside on a hot Arizona summer day and the water evaporates. Sketch a graph of the height of the water in the bottle in terms of the volume of water as the water evaporates. (From Paoletti & Moore, 2016)

   Does the student see this task as asking for the same relationship as part (a)? Is the student able to reason about the behavior of the graph as you imagine moving from right to left on the horizontal axes?

   i. Can you explain how you determined your graph?

   ii. How are the situations in part (a) and (b) similar/different?

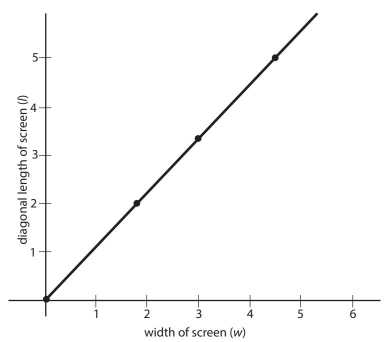
   Does the student differentiate between experientially different situations and quantitatively different situations?

Task 2: A company produces different sized smart phones with rectangular screens. The screens dimensions are *w* and *h*, where the height of the screen (*h*) is half the width of the screen (*w*) for all sizes of smartphones.

a. Define a formula that gives the diagonal length of the smartphone in terms of the width of the smartphone.

   Attend to what student imagines "diagonal length" means. Perhaps have student draw a picture to show what they are thinking about and ask the student to describe what he/she means by each variable/symbol he/she writes.

b. Cassie sketched the graph below.



   i. How many points did Cassie plot? What would Cassie need to do to plot more points?

   Attend to whether student talks about extending line and putting point on line (shapes) or if the student attends to the quantities and restriction placed on how quantities vary.

   ii. What does this point represent? (*Researcher points to point where w = 3*).

   iii. How are the formula and graph related?

How does the student accommodate a linear graph from his/her formula (which I anticipate is not simplified)? Does student attend to procedures to simplify shapes of linear relationships or does student attend to quantities' varying?

c. Assuming Cassie's graph appropriately represents the diagonal length of the screen in terms of the width of the screen, can you represent a screen that you cannot create?

   i. What does it mean to have a screen I cannot create?

   ii. How does what you did represent a screen I cannot create?

I anticipate students will not conceptualize a graph as the set of points that satisfy a relationship and thus any point in the plane that does not fall along the graph represents a screen I cannot construct.

d. Are there more screens you can create or more screens you cannot create? How do you know?

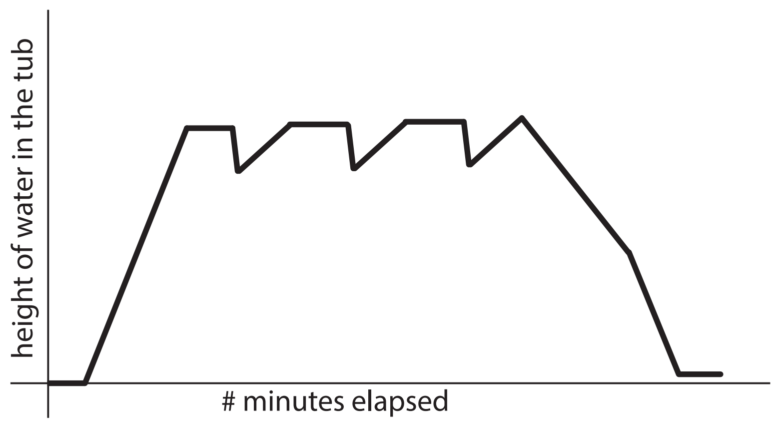
Although the answer is technically infinite for both situations, I want to see if students imagine all the points not represented by the graph as "impossible" and then the selection of points that create the graph as "possible".

Task 3: Task adapted from Swan (1982).

A man is taking a bath. The graph below represents the height of the water in the tub in terms of the number of minutes since the man turned on the faucet.

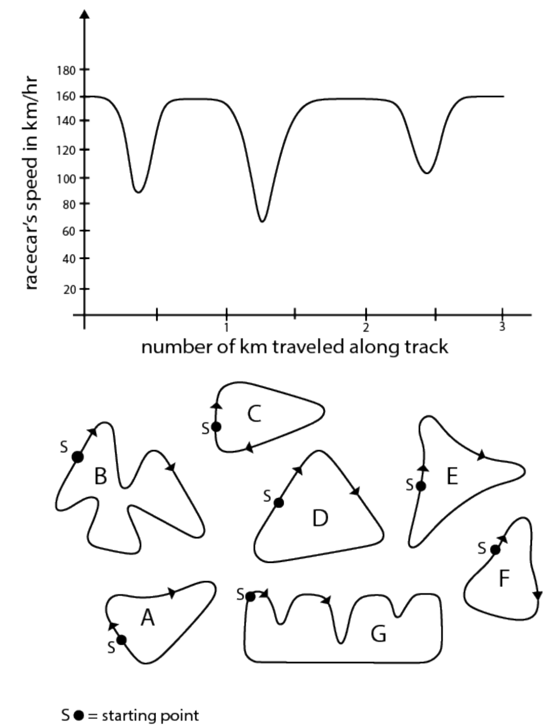
a. What events could have resulted in this graph?

b. Did the man ever get out of the tub?

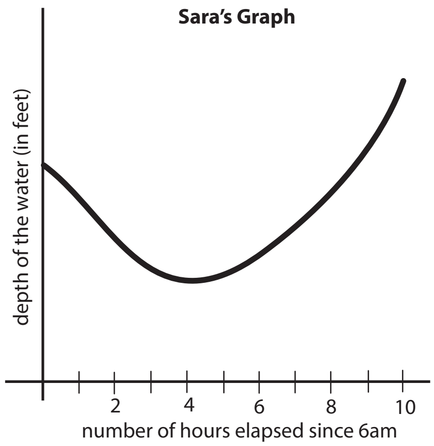


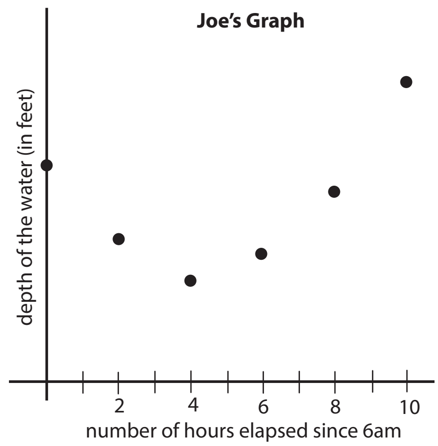
Task 4: Task from Bell and Janvier (1981).

A racecar travels along a race track one time. The graph represents the racecar's speed in terms of the number of minutes elapsed. Which of the following race tracks was the car travelling around in order to produce this graph?



Task 5: Two students were asked to go to Lake Powell one day in November and keep track of the depth of the water at Wahweap Point. The students produced the following graphs.





a. How are these graphs similar/different?

Attend to how dependent the student's graphing scheme is on specific measures as opposed to magnitudes.

b. How do you think Sara produced her graph? What about Joe?

Does the student attend to continuously monitoring change in progress versus attending to values at specific moments in time?

Task 6: A bathtub made of cast iron and porcelain contains 60 gallons of water and the total weight of the tub and water is 870 pounds. You pull the plug and water begins to drain. (Note that water weighs 8.345 pounds per gallon). (From Carlson et al., 2013)

a. Define a formula that determines the weight of the water that has drained from the tub, h, in terms of the number of gallons of water that have drained from the tub, g.

b. Define a formula that determines the total weight in pounds of the tub and water, w, in terms of the number of gallons of water that have drained from the tub, g.

c. How much does the tub weigh when there is no water in the tub?

d. If the weight of the tub and water is 566.935 pounds, how many gallons of water are in the tub?

Task 7: Tuition cost (in dollars), T, for part-time students at Stonewall College is given by $T = 300 + 200C$ where C represents the number of credits taken (From Connally et al., 2000, p. 44)

a. Find the tuition cost for eight credits.

b. How many credits were taken if the tuition was $1700?

c. Make a table showing costs for taking from one to twelve credits. For each value of C, give both the tuition cost, T. How are the table and the formula related?

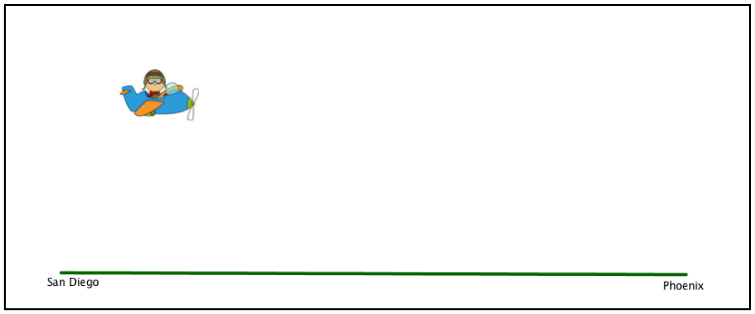
d. What does the 300 represent in the formula for T?

e. What does the 200 represent in the formula for T?

APPENDIX C

PROTOCOL FOR TEACHING EXPERIMENTS

Task 1: *Purpose: How does student conceptualize quantities from a situation? Is the student able to attend to varying magnitudes of quantities or is the student limited to thinking about locations of objects as they move? Additionally, to what extent is the student able to coordinate the varying measures of two quantities?*



Screenshot Task 1 v1

A small plane got caught in a storm on its way from San Diego to Phoenix. To avoid the storm the pilot had to navigate the storm clouds and continuously change his elevation to avoid the storm.

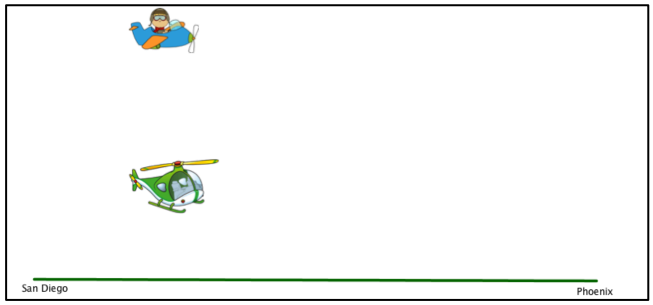
a. What is changing as the plane flew from San Diego to Phoenix?

b. What is staying the same as the plane flew from San Diego to Phoenix?

c. I want you to focus on the plane's distance above the ground as it flew from San Diego to Phoenix.

   i. Using your pointer fingers (palms facing each other), I want you to move your hands so that the distance between your pointer fingers represents the plane's distance above the ground.

   ii. Why are you not moving your left hand? (ground/sea level does not change as the plane travels from San Diego to Phoenix)

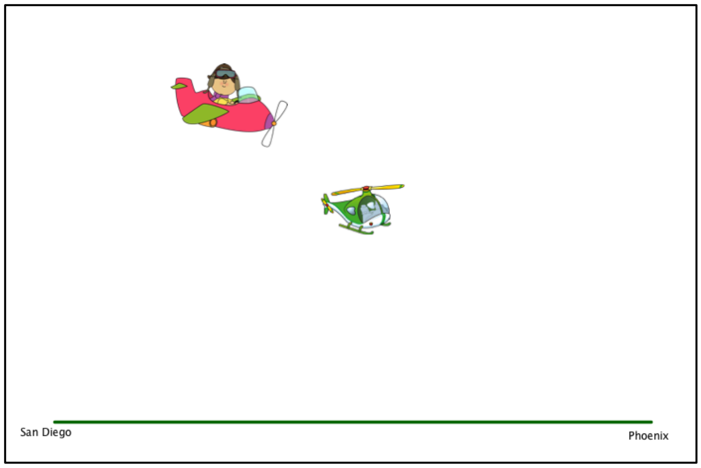


Screenshot Task 1 V2

(**VERSION 2**) A helicopter took off shortly after the small plane. Did the helicopter experience the same weather difficulties as the first? How do you know?

i. Use your hands so that the distance between your pointer fingers represents the distance between the plane and helicopter as they travel from San Diego to Phoenix.

   Want the student to keep bottom hand fixed so that the student is attending to the varying magnitude of this quantity instead of the location of the two planes in the sky.



Screenshot Task 1 V3

(**VERSION 3**) Suppose the helicopter took off a few minutes before the plane. Use your hands so that the distance between your pointer fingers represents the vertical distance between the plane and helicopter as they travel from San Diego to Phoenix.

i. What makes these activities easy/hard to think about?

Task 2: The Box Problem: Starting with an 11 inch x 13 inch sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

a. What do you see in this figure?

    i. What do you think the black solid lines represent?

    ii. The brown solid lines?

    iii. The dotted lines?

    iv. The shaded part?

b. In the problem statement, it says that the box is formed by cutting equal-sized squares from each corner.

    i. Why do the cutouts have to be square?

    ii. What would happen if the cutouts were rectangles?

    iii. Why do the cutouts have to be the same size in each corner?

    iv. What would happen if each corner had a different sized cutout?

c. Does the piece of paper also have to be square?

    i. Why do the cutouts have to be square but the piece of paper can be any dimension?

d. I am going to animate this image. If at any point you want to stop the animation, you can click the pause button in the bottom right corner.

    i. As I move this point, what is changing?

    ii. As I move this point, what is staying the same?

e. (**Version 2 of animation)** How does the relationship between the cutout length and the length of the box change as the length of the cutout increases?

i. Is this relationship the same as the cutout length varies?

ii. Is this relationship the same as the dimensions of the piece of paper vary?

iii.Is this relationship the same if the paper were square instead of rectangular?

f. **(Part 2 of task)** How does the relationship between the cutout length and the surface area of the box change as the length of the cutout increases?

i. When I did this task with another student, she said the surface area of the paper was like the area of the box but that it was too much because of the four squares you cutout. So she suggested that the surface area of the paper was the total of the surface area of the box and the 4 areas of the cutouts. Do you agree? Is this true as the cutout length varies? As the dimensions of the paper vary?

g. **(Part 3 of task)** How does the relationship between the cutout length and the volume of the box change as the length of the cutout increases?

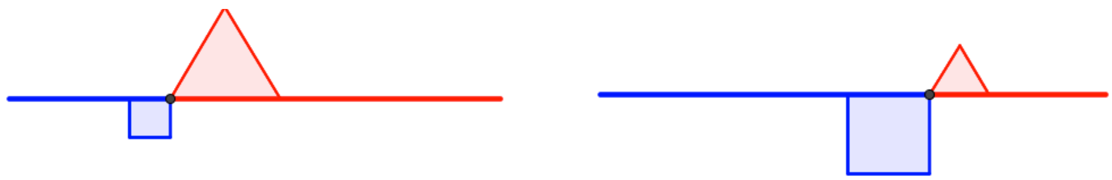
__Task 3a:__ *Designed during teaching experiment with Sue to determine how she reasoned about quantities' sizes that were not directly perceived (e.g., area).*

Selected Screenshots from Task 3a

Begin by explaining that I constructed animation so that the length of the black segment to the left of the point was the height of the rectangle and this would be true as the black dot moved from left to right across the screen.

a. Can you describe what you see here?
b. I am going to push play. What do you think will happen to the figure as I click play?
   Check student's conception of how I constructed figure.
c. What happens to the perimeter of the rectangle as the animation plays? Is there anything in the picture that helps me see perimeter?
d. What happens to the area of the rectangle?
e. Is the rectangle ever a square? How do you know?
   Goal: focus on a specific measure or at least the idea of a measure.

<u>Task 3b</u>: *Two Polygon Task (adapted from Mason and Meyer, 2016)*



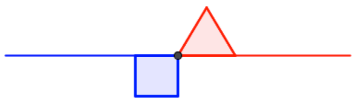
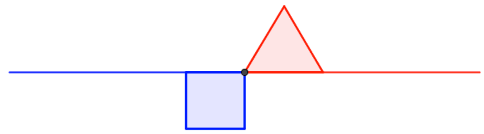
Selected Screenshots from Task 3b v1

a. (Teacher/researcher displays Version 1 of animation paused.) What do you see when you look at this figure?

b. Teacher/researcher animates the figure?

   i. What do you think this bold blue/red line represents?

   ii. (Display animation of perimeter rotating to construct square and triangle to support student in believing the length of the bold lines represent perimeter of shape.) How are the perimeter of the square and the perimeter of the triangle related?

Selected Screenshots from Task 3b v2

a. (Teacher/researcher displays Version 2 of the animation) What do you see in this second version?

b. How are the second version and the first version similar?

   i. Are they exactly the same? If not, how is the second version different than the first?

c. In this second animation, how does the sum of the two shapes' perimeters compare to the perimeter of the square? Is this always true as the animation plays?

d. In this second animation, how does the sum of the perimeter compare to the side length of the square?

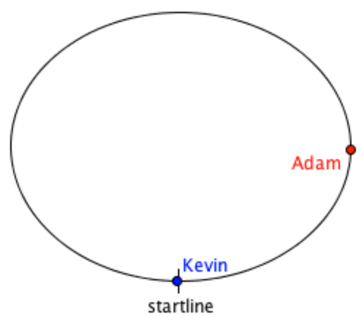
e. Is this always true?

f. In both animations?

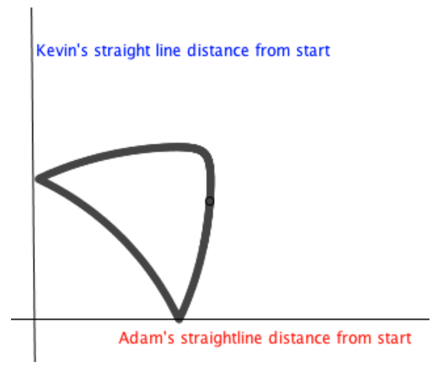
Task 4: Suppose the length of the red horizontal bar represents the volume of water in a bottle and the length of the blue horizontal bar represents the height of water in the bottle.

a. Can you describe how the container might have been filling in order to have this height and this volume of water?
b. (Animate bars.) Suppose that it took 5 hours for the bottle to completely fill with water. Can you describe how the container might have been filling?
c. Is it possible that during those 5 hours the person took a break from pouring water into the container?

Task 5: Kevin and Adam are both running around a quarter-mile ellipse shaped track. When Kevin starts running Adam is 100 meters ahead of Kevin.



Screenshot



Graph of actual covariation

a. Drag the people so that their starting positions match what is described.

b. Okay, now I want you to imagine the boys running around the track. How is Kevin's direct distance from the starting line changing as he runs around the track?

   i. Play animation – is this what you expected?

c. How is the total number of meters Kevin has run changing as he runs around the track?

d. As Kevin's distance from the starting line reaches its maximum value, what is happening to Adam's distance from the starting line? Is this always true as the boys continue to run multiple loops around the track?

e. As Adam's distance from the starting line reaches its minimum value what is happening to Kevin's distance from the starting line?

   i. Is this always true as the boys continue to run multiple loops around the track?

   ii. Would this be true if the track were a mile loop instead of a 400m loop?

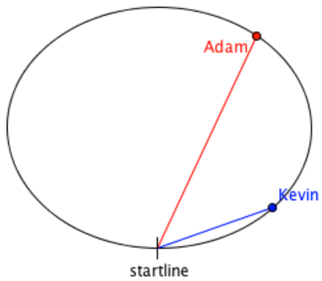
   iii. What would have to change for this relationship to no longer hold?

f. Determine whether the following statement is true or false, As Kevin's direct distance from the starting line increases; Adam's direct distance from the starting line also increases. Explain your reasoning.

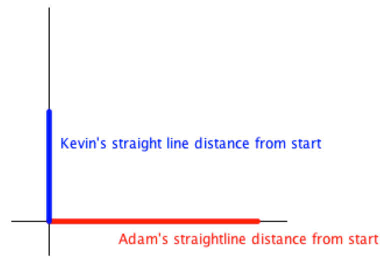
i. What is happening to Adam's direct distance from the starting line as Kevin's direct distance from the starting line increases?

g. How can I record what is going on so I know that whenever Adam's direct distance from the starting line was "this" long that Kevin's direct distance from the starting line was "this" long?

Task 6: Introduce Conventions of Cartesian Coordinate System in context of Adam and Kevin running around the track. *Purpose of the task is to introduce a new way of thinking about representing relationships.*



Screenshot from Kevin and Adam Task where boys' straight-line distances from start are displayed in the depiction of the situation.



Screenshot from Kevin and Adam Task where the boys' straight line distances from start are represented as perpendicular magnitude bars.

Researcher introduces lines in the situation to support student in attending to attribute and then orients these bars perpendicularly to introduce convention of representing changing measures on the axes.

a. Ask student to anticipate how lengths of bars will change as Kevin's straight-line distance increases to its maximum.
b. (Play Animation.) Is this what you expected?
c. What are you attending to – situation or diagram?

   Try to understand how student conceptualizes the lengths of the bars on the axes - perceptual or quantitative?

<u>Task 7:</u> U&V Task (adapted from Project Aspire Thompson, 2011a)

In the animation, the length of the horizontal red bar represents the varying measure of *x* and the length of the vertical blue bar represents the varying measure of *y*. As the animation plays the lengths of the red and blue bars will vary together. Your job is to represent what is going on in this animation so that you could mail this representation to a friend and he would understand exactly what happened in the animation. (There will be three versions of this task.)

<u>Task 7b</u>: Introduce Didactic Object of Correspondence Point if student does not imagine placing point at projection of red and blue bar.

a. **(**Let the animation play through and remind student what we are looking at.)

b. (Pause the animation towards the beginning and propose the correspondence point.)

   **teaching part of teaching experiment

   "This is a way that some people have come up with to try and keep track of the ends at the same time and we are going to call it a correspondence point because it shows how the end of one bar corresponds to the end of the other. This correspondence point gives us a way to represent the value of u (the length of the red bar) and the value of v (the length of the blue bar) simultaneously."

c. (Have student finger point at the correspondence point.) Why is the correspondence point where it is?

d. (Hide correspondence point, move the bars. Show correspondence point)

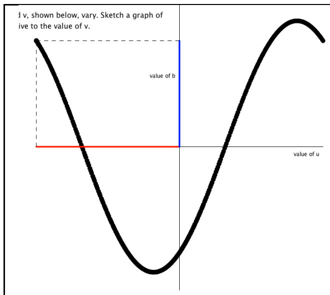
e. (Drag again.) Now before I show you the correspondence point, can you point to where you think it might be? Why there? (Show correspondence point. Hide correspondence point.)

f. (Repeat E until she gets comfortable.)

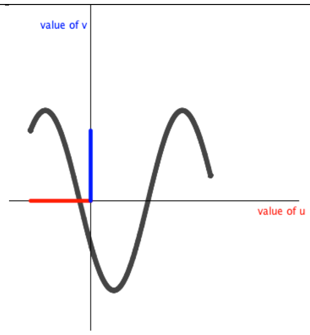
g. Okay now I am going to play this from the beginning and I want you to imagine the correspondence point and keep track of it. It's not going to be there, but I want you to imagine it and keep track of it with your finger.

h. Now can you sketch the graph you just traced?

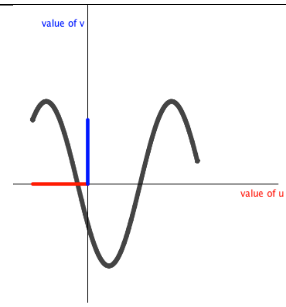
i. Does the graph you just drew have any points on it?



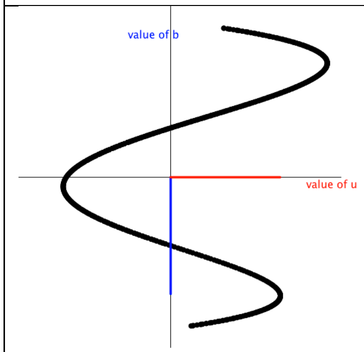
Sue V1



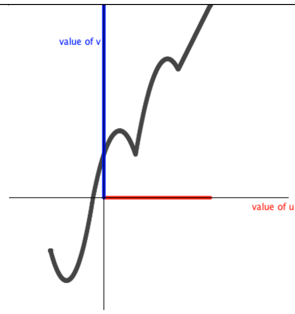
Ali V1
I introduced more



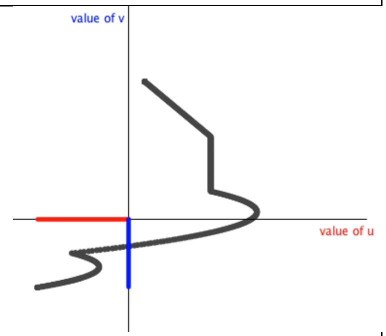
Bryan V1



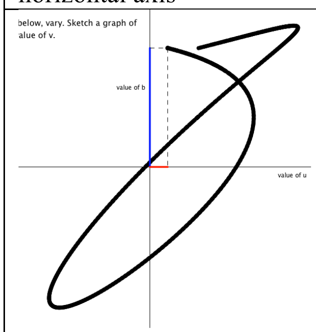
Sue V2
*value of u no longer increases monotonically on horizontal axis



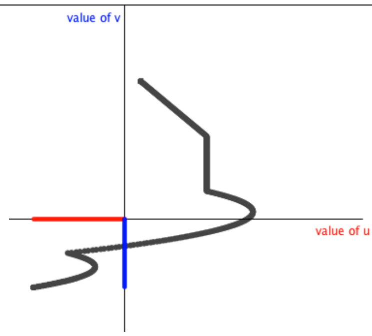
Ali V2



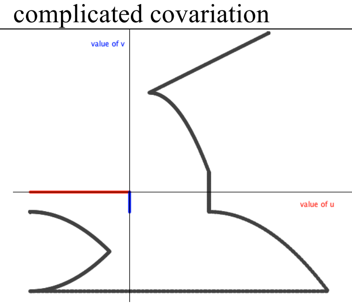
Bryan V2
*since Bryan completed V1 quickly I skipped to a more complicated covariation



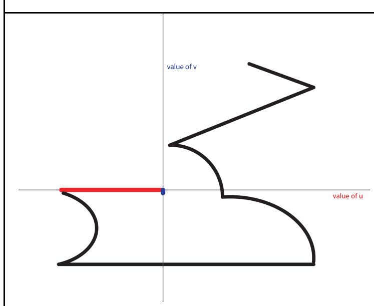
Sue V3



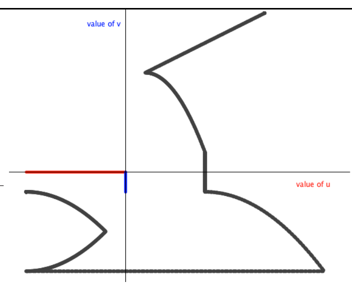
Ali V3



Bryan V3



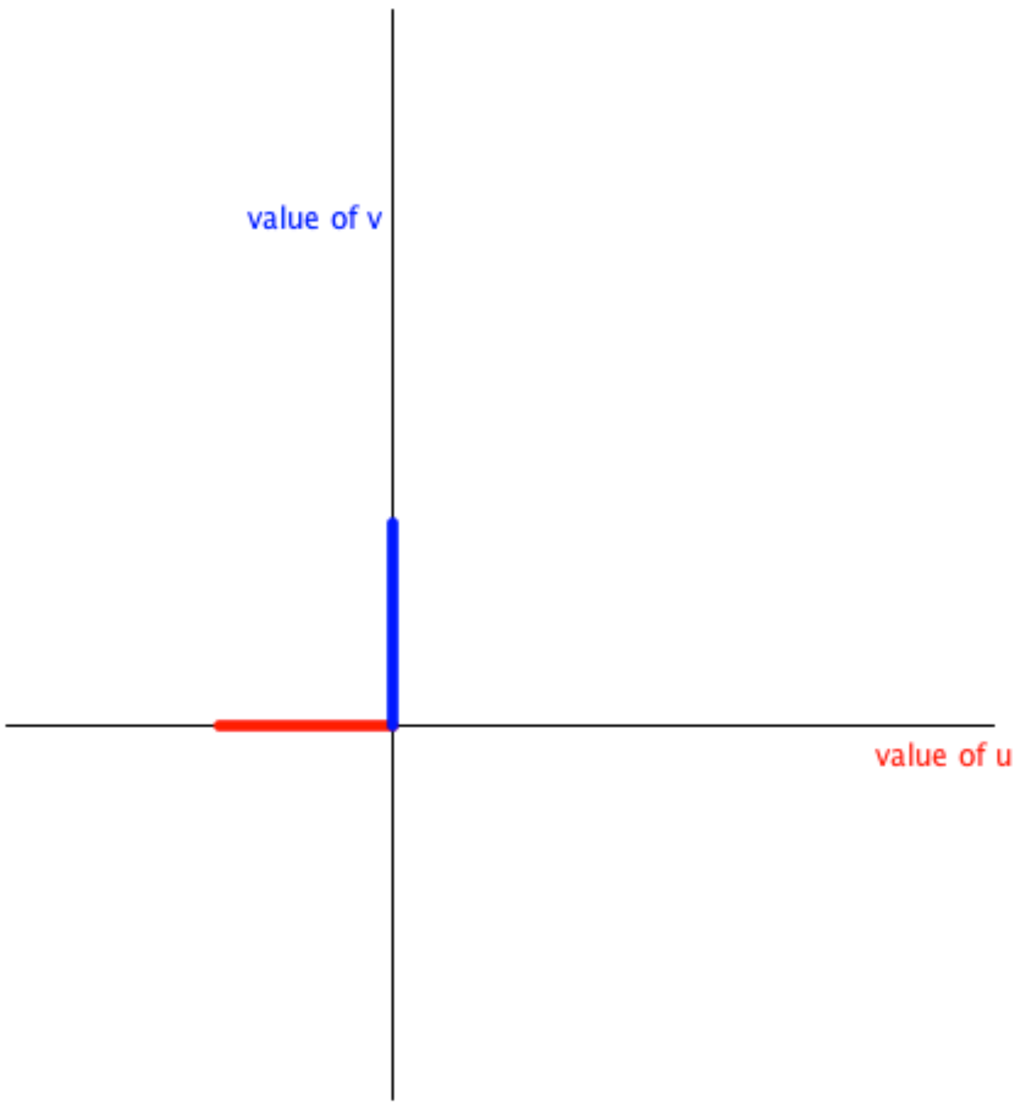
Sue V4
*tried to design a covariational relationship that could not be graphed by focusing on max/mins



Ali V4
*Different than Sue V3 because I learned to parametrize in GeoGebra so I could control the slider

Example of Task sheet for Task 7 in Teaching Experiment:

The values of *u* and *v* vary. Sketch a graph of the value of *v* relative to the value of *u* in the diagram below. The diagram presents the initial values of *u* and *v*.

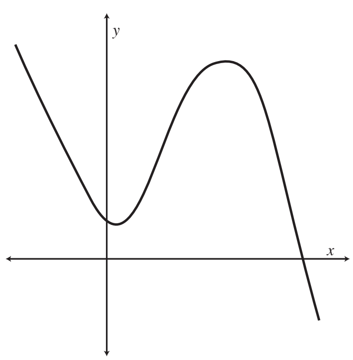


<u>Task 8:</u> I will present student with graphs below and then repeat this question sequence for each of the graphs displayed below.

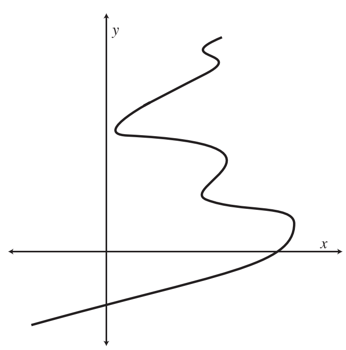
a. How do the values of *x* and *y* change together?

b. Suppose I wanted to add the blue/red bars from the previous task to this graph. How would the blue/red bars vary along the axes so that their variation would represent the relationship depicted by this graph?

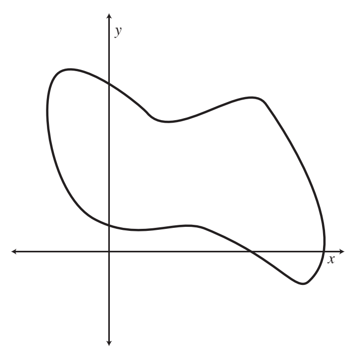
c. Use your pointer fingers to represent how the values are varying along the axes. Use your right pointer finger and move it left to right to represent the varying value of *x* and use your left pointer finger and move it up and down to represent the varying value of *y*.



Version 1



Version 2: student needs to construct horizontal axis as representation of measure, not experiential time

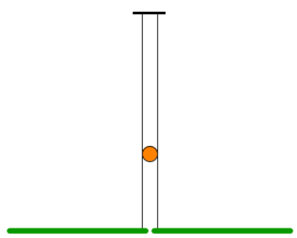


Version 3: student needs to construct starting point as arbitrary choice

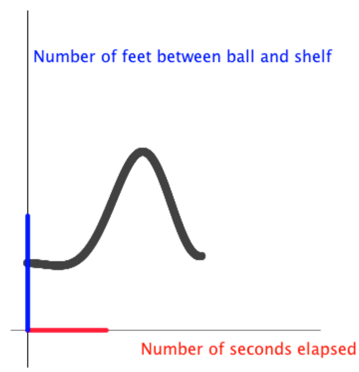
<u>Task 9:</u> Floating Ball Task

Animation depicts a ball that is floating on a stream of air being pumped from below. There is a shelf above the ball.

Prompt: Sketch distance of the ball from shelf in terms of time elapsed.



Screenshot from floating ball task

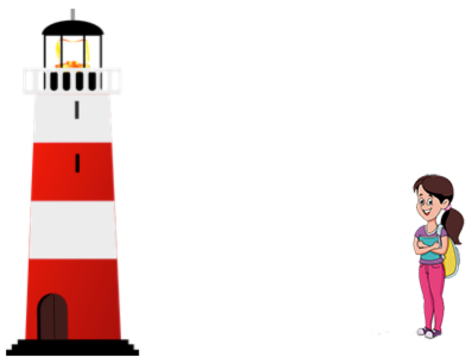


Graph of actual covariation

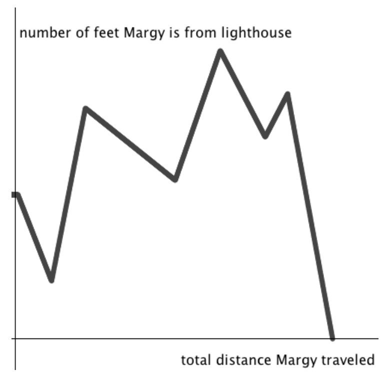
I introduced this task to study Sue's reliance on her perceptions when constructing graphs. Student's perception of the ball motion (going up and down) does not match the variation I am asking her to represent. Thus, student must first differentiate the distance of the ball from the shelf from the motion of the ball so that she has something to reason about that is increasing/decreasing.

<u>Task 10:</u> Margie Walking Task

Prompt: Margie is walking up and down the beach trying to decide whether to go see the lighthouse. Sketch a graph that gives Margie's distance from the lighthouse in terms of the total distance Margie has traveled.



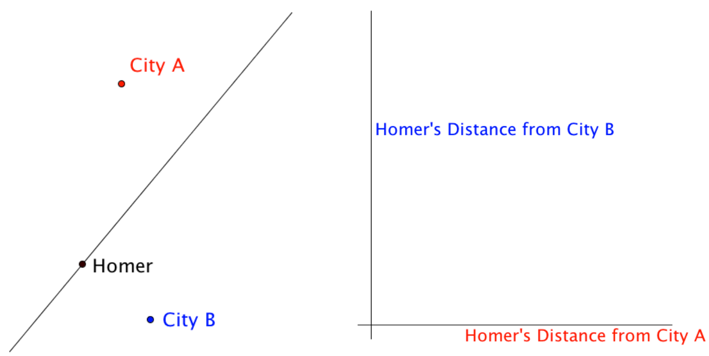
Screenshot from Margie Task



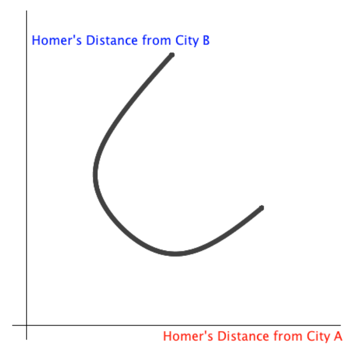
Graph of actual covariation

<u>Task 11:</u> Homer Task

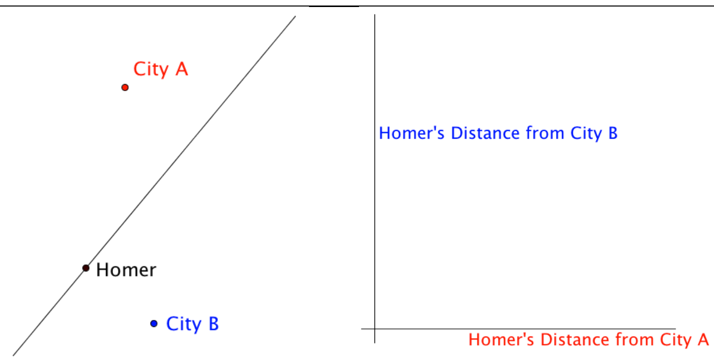
This animation depicts Homer driving along a road. Notice that City B is on the east side of the road and City A is on the west side of the road. As the car drives along the road, sketch a graph that represents the car's distance from City B in terms of the car's distance from City A.



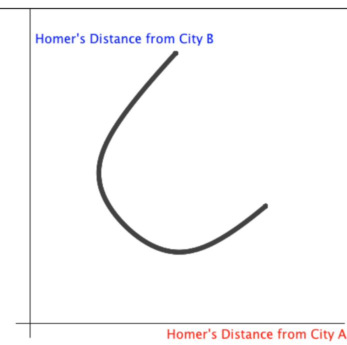
Homer Version 1 – screenshot (Homer moved in one direction along the road at a constant speed)



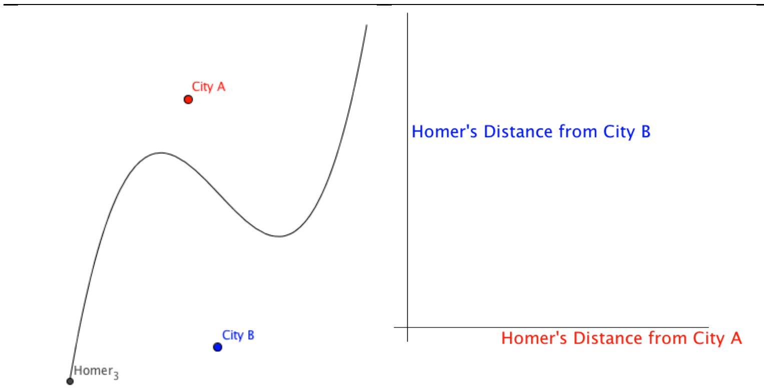
Homer Version 1 – Graph of Actual Covariation



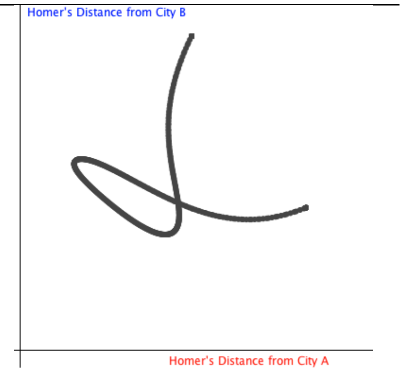
Homer Version 2 – screenshot (Homer moved up and down the road) *study the role of one's experiential time in their graphing activity*



Homer Version 2 – Graph of Actual Covariation



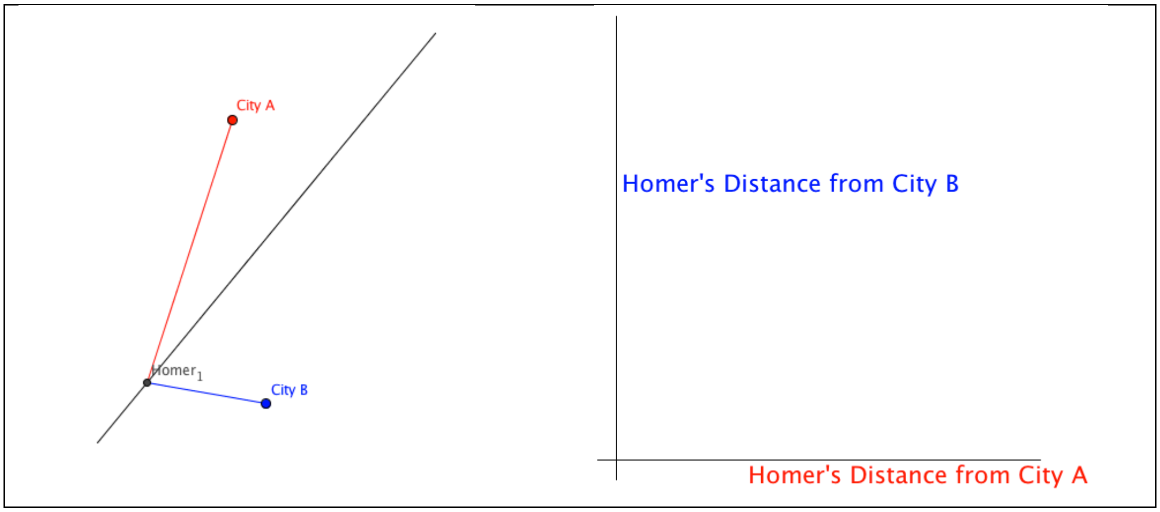
Homer Version 3 – screenshot *introduce more complicated images of change to understand how student attended to change in progress*



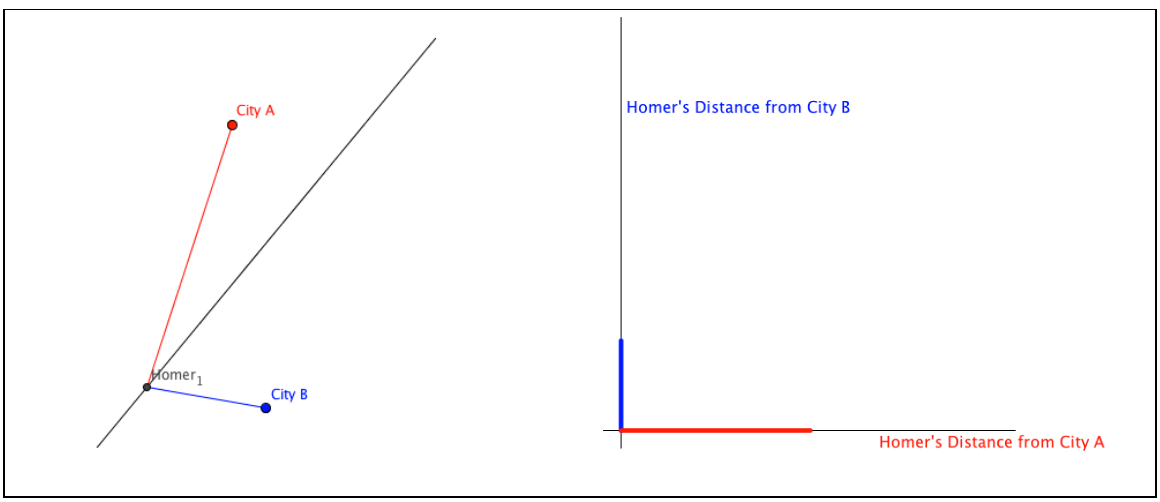
Homer Version 3 – Graph of Actual Covariation

GeoGebra File includes four levels of support for Homer Task

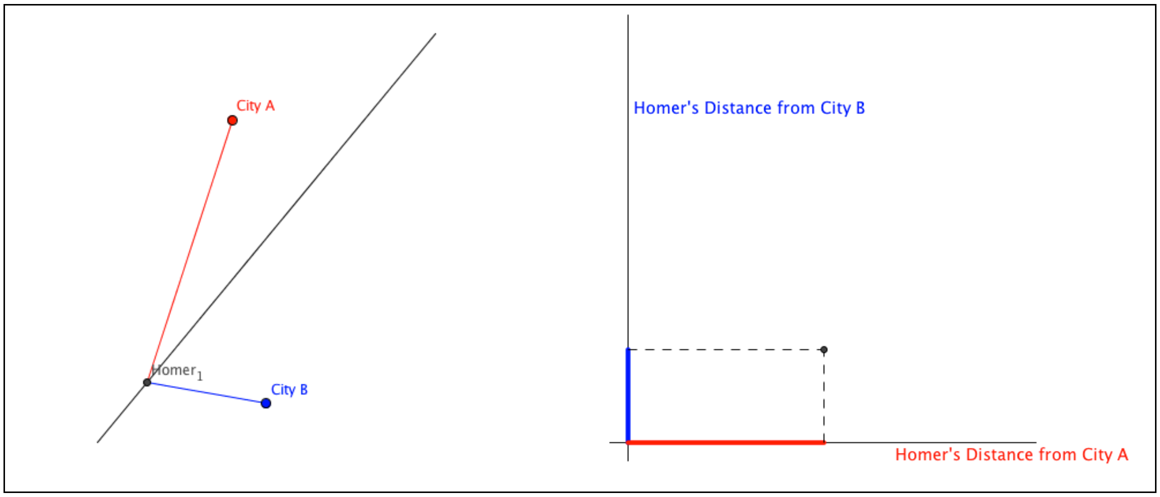
Support 1: Red and blue lines depicted between Homer and each city to support the student in conceptualizing an attribute of the situation to attend to.



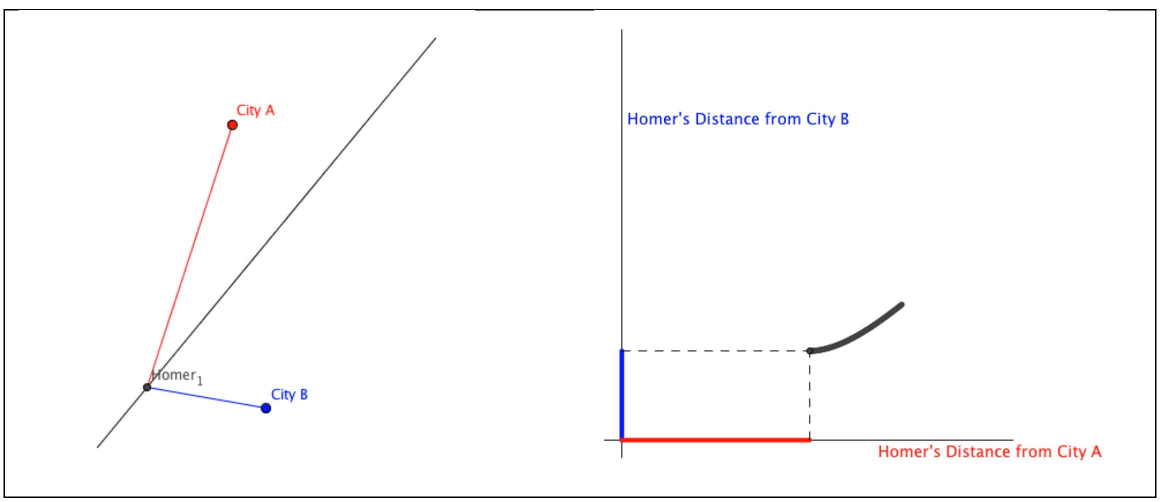
Support 2: Red and blue bars displayed on the axes to support student in imagining varying magnitudes represented along the axes.



Support 3: Display correspondence point in the plane to support student in coordinating two magnitudes with a point's location in the plane.



Support 4: Display trace so that GeoGebra tracks the correspondence point as the animation plays.



APPENDIX D

HUMAN SUBJECTS APPROVAL LETTER

EXEMPTION GRANTED

Patrick Thompson
Mathematics and Statistical Sciences, School of
480/965-2891
Pat.Thompson@asu.edu

Dear Patrick Thompson:

On 4/4/2016 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Students Engagement in Covariational Reasoning: Examining Ways of Thinking that Transfer Across Problem Types
Investigator:	Patrick Thompson
IRB ID:	STUDY00004211
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Interview Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Kristin Oral Recruitment Spring 2016.pdf, Category: Recruitment Materials; • Kristin IRB Proposal.v2.20160401.docx, Category: IRB Protocol; • Kristin Paricipant Consent Form.v2.20160401.pdf, Category: Consent Form;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings, (2) Tests, surveys, interviews, or observation on 4/4/2016.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Kristin Frank
Patrick Thompson
Kristin Frank
Marilyn Carlson