Essays in Market Design

by

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ABSTRACT

I study the design of two different institutions to evaluate the welfare implications of counterfactual policies. In particular, I analyze (i) the problem of assigning students to colleges (majors) in a centralized admission system; and (ii) an auction where the seller can use securities to determine winner's payment, and bidders suffer negative externalities. In the former, I provide a novel methodology to evaluate counterfactual policies when the admission mechanism is manipulable. In the latter, I determine which instrument yields the highest expected revenue from the class of instruments that combines cash and equity payments.

To my parents, who have always taught me by example how to be a good man, who have showed me that dreams are not impossible if one works hard, who have celebrated my achievements as much as have given me their unconditional support in bad moments, but more importantly, who have showed me that the real treasure is in heaven.

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Chapter 1

INTRODUCTION

Many economic applications involve institutions in which a planner is interested in maximizing a certain objective function, but cannot force the agents to (i) participate, and (ii) take the "right" actions that maximize its objective. Instead, it has to design the adequate "*rules of the game*" such that when the agents take the actions that satisfy their own interest, the final outcome is the one desired by the planner. In some situations the planner can establish side payments amongst the participant agents to achieve its objective (e.g. an auction, implementation of a public project, etc), whereas in others, it is legally and socially impossible to do so (e.g. allocation of students to colleges, assignment of kidney transplant donors to patients, etc). The former problems are studied under the literature of *mechanism design*, whereas the analysis of the latter have resulted in the emerging literature of *market design*.

The design of such institutions have a clear impact on the welfare of agents who participate, and therefore, it is very important to count with theoretical and empirical tools that permit to (i) rigorously analyze the current institutions, (ii) to design which would be the optimal frameworks, and (iii) provide policy recommendation to practitioners. This dissertation aims to this objective in two different environments: in the assignment of students to colleges (majors), and in an auction under securities and externalities.

In the first two chapters we analyze the problem of assigning students to colleges (majors) in a centralized system. This is relevant because in many countries around the world, university seats are allocated using a centralized protocol where students report preferences over colleges (majors) to a planner, and the planner uses such reports -a long with an admission score- as the inputs of a serial dictatorship mechanism to assign students to colleges (majors).

The design of such admission systems are under the scrutiny of policy makers, because the mismatch between students and colleges (majors) have an impact of the productivity of individuals and the economy itself. However, little is known of how different admission policies affect assignments. The first chapter addresses this important question. It provides a new methodology to evaluate counterfactual admission policies with respect to students' welfare.

As a sub-product, we introduce a novel methodology to recover students' preferences from their reports. The importance of this observation lies in the fact that normally, students are constrained to report fewer options than the total of options available, and hence they do not have a weakly dominant strategy. In fact, for many students their optimal strategy involves the manipulation of their reports (i.e report something different to their true preferences), which imposes a challenge for the inference problem. That is, how to obtain students' true preferences from the manipulated reports.

We implement our methodology using a unique data set from the University of Costa Rica. In particular we examine different counterfactuals such as the welfare effects of (i) increasing the number of options to report, (ii) reallocating seats across majors, and (iii) implementing different affirmative action programs. Finally, we go beyond the serial dictatorship algorithm and introduce a mechanism based on a simultaneous ascending auction.

The second chapter aims to answer a design question of the latter environment: Why do planners around the world restrict the students' choice? In other words, why do planners do not let students to report the full list of preferences? This question is relevant because when students can report the full list of preferences the serial dictatorship mechanism is well known to have desirable properties. For instance, it is a weakly dominant strategy for students to report the truth. Moreover, the final allocation is Pareto optimal and free of envy. However, as soon as students are restricted, these desirable properties disappear. The puzzle comes from the fact that with nowadays resources, the cost of letting students to report the full list is negligible.

The second chapter provides an answer to this puzzle. It establishes that when the planner itself has preferences over the assignment of students, and students are sufficiently risk averse, then it is optimal for the planner to restrict students' choice; provided that planner and students preferences are completely misaligned (i.e. the most preferred option from students' perspective is the worst from planner's perspective and viceversa.)

The last chapter analyzes an auction when the final payment from the winner to the seller is contingent to the realization of the project or asset being auctioned off by the seller, or the so-called, auctions under securities. In this context we introduce negative externalities to investigate how the interaction of the securities and the externalities impact seller's expected revenue. To simplify the analysis, we center our attention to second price auctions and four instruments (i) cash, (ii) equity (i.e where the bids are shares over future project's revenue), (iii) a fixed-equity hybrid, where the seller fixes the amount of cash he will request from the winner, but let the bidders to compete in cash in the auction; and (iv) a fixed-cash hybrid, where the seller fixes the equity that he will request from the winner, but let the bidders to compete in equity in the auction. We show that the fixed-equity hybrid is the instrument that yields the highest expected revenue, whereas equity yields the lowest. Absent of the externality, equity give the highest expected revenue, a result well known in the literature. That is, we show that when externalities are present, the conclusion that securities that are more sensitive to bidders' true types (i.e steeper securities) do not necessarily yield the highest expected revenue.

Chapter 2

CENTRALIZED ASSIGNMENT OF STUDENTS TO MAJORS: EVIDENCE FROM THE UNIVERSITY OF COSTA RICA

Many countries use a centralized admission system, by which students are admitted to a university. Examples include Chile, China, Costa Rica, Hungary, Iran, Turkey, The Netherlands, Spain, etc. Typically, students report preferences over colleges to a social planner, and the planner allocates students to colleges. Such admission systems are under constant scrutiny by policy makers, because inefficiencies in the assignments can have negative consequences on students' careers and, so, on economic productivity.

Countries exhibit variation in these centralized admissions policies. (For instance, there is variation in the number of options a student can report, as well as in the affirmative action programs considered.) However, little is known about how changes to such policies impact the allocation of students to colleges. This paper uses a novel dataset from the University of Costa Rica's (UCR's) 2008 admissions process, to evaluate the impact of different admissions policies on how students are assigned to majors. In doing so, we propose a new methodology to recover students' preferences, and conduct several counterfactual analyses.

In UCR, each student is assigned a score based on the results of a standardized test and high school grades. Students privately observe their score and then report an ordered list of majors to the Registrar Office (henceforth, RO). The RO uses the serial dictatorship mechanism to allocate students to majors. This mechanism assigns students to majors based on their scores and reported preferences. Roughly speaking, it orders students by their scores and assigns them to their best major, among those with available seats.

If students were allowed to report their complete preferences over all available majors, then they would have a weakly dominant strategy to report truthfully. Indeed, for any given reported list, the serial dictatorship mechanism starts revising the major in the highest position reported to try to allocate the student to that major. If it is not possible, it turns to the second highest major reported, and so on. Therefore, there is nothing better than report the true preferences when students can express preferences over all options available. However, UCR only allows students to report two options. As such, students may have an incentive to misreport their true preferences over majors.

This raises a challenge for the inference problem: To evaluate alternate policies, the researcher needs to obtain students' preferences over majors. Because the students are constrained to report only two options, there are two challenges. First, students report an ordered list over two majors, which can be manipulated. Second, even if the reports were truthful, the researcher only observes a truncated order over majors.

We address this challenge by introducing a multi-agent decision approach. The crucial observation is that the student's decision problem is conceptually simple. To understand why, we first note that the outcome of the admissions process induces a threshold score associated with each major. A threshold is a value such that students with lower scores cannot obtain a seat in the given major. UCR publishes information about past threshold scores. In the data, these scores are stable from year to year. Students can use this information along with their own scores to estimate the probability of admission to any major. Armed with this information, students report an ordered list that maximizes their expected utility.

This multi-agent decision approach is founded on the idea that students act in a large population of applicants. Therefore, they do not believe that their own reports affect their probability of admission to any given major.

We use the multi-agent decision approach to recover students' preferences. This is done in two steps. First, we recover the students' ordinal preferences over majors. Second, we use the ordinal preferences to recover the students' cardinal preferences. (The cardinal preferences are necessary, because students maximize their expected utility given the estimated probabilities.)

First, we provide a "revealed preference algorithm" to recover a minimal set of ordinal preferences that are compatible with the data. To do so, we assume that preferences are independent of scores. In addition, we impose two axioms of rationality on the student's choice: no cycles and no-dominated choice. This allows us to use the students' reports and past threshold scores to obtain their ordinal preferences. In particular, students with scores above the highest threshold are guaranteed admissions to any major. As a consequence, they have an incentive to report truthfully. Their report, thus, constitutes the beginning of a specific ordinal preference. We then look at students with scores just below the highest threshold, and use their reports to continue reconstructing ordinal preferences. This process is continued using the reports of students with lower and lower scores.

Second, we use the recovered ordinal preferences to obtain cardinal preferences. Specifically, we assign initial cardinal values based on the ordinal preferences obtained in the previous step. Then, we solve the student's utility maximization problem to obtain the set of optimal reports. With these reports in hand, we apply the serial dictatorship mechanism to simulate the assignments in each major. If, in each major, the simulated assignment matches the actual assignment, then the process is complete. If, in any given major, the simulated assignment is different from the actual assignment, then cardinal utilities are adjusted according to a simple rule. Importantly, the adjusted values have to respect the initial ordinal preference assignment.

An important caveat is in order: The procedure recovers a set of cardinal preferences that is consistent with the data. However, these preferences are not uniquely identified. There may be several preference profiles that are observationally equivalent. The non-uniqueness can arise in either of the two steps. Nonetheless, we show that our procedure is robust to alternate specifications of preferences.

Overview of the Results We apply our methodology to the UCR dataset. It contains information of approximately 10,000 students who applied for admission in any of the 82 majors offered on the main campus.

We can trivially match both the students' reports and assignments by allowing each student to have a different preference profile. (For instance, we can give each student a preference profile where their assigned major has a very large cardinal utility in comparison with the others.) However, that approach would be agnostic about the students ordinal preferences over the non-reported majors. As a consequence, it cannot be used to conduct counterfactuals and evaluate policies. Instead, we use the revealed preference algorithm. In doing so, we recover 205 different ordinal preferences. These are used to match the aggregate assignments per major. We show that the recovered cardinal preferences fit the aggregate assignments with 94% accuracy. This finding has merit *per se*, since the total number of possible preference profiles is of the order of 10^{122}.

We compare policies to a benchmark in which the RO uses a serial dictatorship mechanism and allows the students to report their complete ordinal preferences. This benchmark has three desirable properties. First, reporting truthfully is a weakly dominant strategy (Dubins and Freeman

For any given policy, we compare the assignment under that policy to the assignment under the benchmark. We do so by focusing on three measures of how the assignments differ. The first is the fraction of students that obtain a different assignment from they would obtain under the benchmark. The second is the euclidean norm between the cardinal utilities associated with the two assignments. The third is the cardinal aggregate welfare.

Within the context of the serial dictatorship mechanism, we evaluate three changes in policy: (i) an increase in the number of options to reports, (ii) affirmative action programs, and (iii) a reallocation of seats across majors. Later, we depart from the serial dictatorship and allow the RO to employ a broader class of mechanisms.

First, we look at the counterfactual in which the RO varies the number of option that can be reported. When the RO limits the list of options to two, 72% of the students receive a different allocation than they would obtain under the benchmark. Increasing the number of options to report decreases the differences in assignments. Likewise, it decreases the euclidean norm between the cardinal utilities of the two assignments and it increases the cardinal welfare. The fact that the cardinal welfare is increasing is non-trivial. When the RO increases the number of options that can be reported, it both increases the probability that middle-score students are assigned and decreases the probability that low-score students are assigned. So, there could be a loss in welfare if middle-score students are assigned to unpopular majors, for which low-score students have a higher cardinal utility. In fact, because of this potential loss in welfare, Miralles

Second, we look at the impact of two affirmative action programs. The first program sets quotas for the target population, and the second provides the target population with a bonus in their admissions scores. We find that the bonus is more effective in admitting students from the target population. However, it also produces larger distortions in the assignment of the overall population.

Third, we look at the effect of reallocating wasted seats (i.e. seats that were not filled). We consider allocating these seats in three different ways: (i) to the ten most demanded majors, (ii) to the five most demanded majors, or (iii) the most demanded major (medicine). The simulations show that the first scheme dominates the others. This likely indicates that there is high variability in the students' preferences.

Finally, we look at the effect of changing the mechanism, beyond the realm of the serial dictatorship mechanism. We study two of such mechanisms. The first is, what we call, the *Posting Scores Upfront* mechanism (PSU) and the second is an *Ascending Auction* mechanism (AA). In the PSU, the RO announces a minimum threshold required for admissions to any given major. Importantly, the RO commits to this threshold, even if this requires creating additional seats. In the AA, the RO announces a preliminary-threshold for admission to any given major, and the students submit their "demand" for a seat. The planner then computes the "excess demand" in each major, and adjusts the thresholds of some major with maximal excess demand. The process is repeated until there are no majors with excess demand. We show that the AA is highly desirable. In particular, under the AA, only 5% of the students receive a different assignment from the benchmark. Moreover, it delivers higher cardinal welfare than the benchmark.

Related Literature In a seminal paper, Balinski and Sonmez

There is also a growing literature that addresses the school choice problem empirically. See, e.g., Hwang

In the college assignment problem (or the problem of assigning students to majors) a student's priority is entirely determined by a score. Moreover, preferences are more idiosyncratic and can depend on both unobserved variables (e.g., taste) and observed variables (e.g., wages, attrition rates, etc). These features make it difficult to obtain unique identification, even if a parametric model were used. For this reason, we introduce a non-parametric methodology based on the multi-agent decision approach.

The multi-agent decision approach is based on the observation that students act in a large population. As such, they take the probabilities of admission as given. In other words, students do not believe that their own reports affect their admission probabilities. Sönmez and Unver (2010) take a similar approach to analyze the allocation of courses in business schools. The approach is also related to the literature that analyzes admission systems as large economies. See, e.g., Chade, Lewis and Smith

Organization of the Paper The paper is organized as follows. Section 2 presents the background about the higher education system in Costa Rica. Section 3 outlines the model and points to comparative statics. Section 4 introduces and implements the multi-agent decision approach. Section 5 describes the data and presents descriptive statistics. Section 6 shows the simulation results. Section 7 conducts counterfactual analyses in the realm of the serial dictatorship mechanism. Section 8 conducts counterfactuals beyond the serial dictatorship. Section 9 concludes.

2.1 Background

Higher Education in Costa Rica There are sixty three institutions of higher education in Costa Rica, of which five are public, fifty three are private and five are international. [1] All public universities absorbs roughly 60% of the total enrollment in the country, and the UCR about 25%. [2] Altogether, higher education institutions offer around 1,100 programs, but the academic offer is highly concentrated in the areas of social sciences, economic sciences and education. Moreover, all the majors in hard sciences, and the majority of majors in arts and natural resources are only offered in public universities. For instance, crucial majors for the technological development of the country such as: pure mathematics, physics and statistics are only offered by the UCR.

The admission process for the public universities is decentralized among universities but centralized within each university. That is, each institution uses their own admission system, but the allocation of majors is centralized. In particular the UCR uses a serial dictatorship mechanism to allocate students to majors. [3]

All public universities have well established systems of scholarships that give tuition waiver and financial assistance to all students who meet the requirements. Among them, the program of the UCR stands out. As an illustration, in 2013 the UCR transferred around 23 million dollars in scholarships, that benefited 50% of the enrolled population. The fraction of the beneficiaries increased to 60% if the stimulus scholarships (i.e. those granted to students who participate in athletic and cultural groups, or to students with excellent academic performance)

[1] The five public universities are: The University of Costa Rica (UCR) founded in 1940, the Costa Rica Institute of Technology (ITCR) founded in 1971, the National University (UNA) founded in 1973, the University of Distance Education (UNED) founded in 1977, and the Technical National University (UTN) founded in 2015. The UCR and UNA are comprehensive universities which offer a large menu of majors in all academic areas, while ITCR and UTN are specialized in engineering and technical majors. UNED is the main institution of distance education.

[2] For more details see `http://www.estadonacion.or.cr/educacion2015`

[3] The UNA shares the same admission standardized exam with UCR, but uses a complicated statistical-mathematical model that stratifies students by region and high-school of origin. The ITCR applies its own exam and a similar admission system to the UCR. Meanwhile, the other public universities, and all private universities do not require an admission exam.

are considered. The tuition for students who do not receive financial assistance is approximately 315 dollars per semester (for a maximum of twelve credits), irrespectively of the major enrolled. This amount is considerably lower than the mean cost of a semester in a private university, which is approximately 800 dollars in majors that belongs to education or economic sciences; but can ascend up to 4,000 dollars in the case of medicine. [4]

Finally, the UCR stands as the main actor in the production of innovation and scientific knowledge in the region. It concentrates more than 51% of the main researchers of the country -measured by the number and impact factor of their publications, counts with forty six research centers, and edits thirty two academic journals, covering all the academic areas. In contrast, private universities are mainly focused in instruction, and their participation in the research life of the country is negligible. [5]

In summary, the UCR offers the most comprehensive menu of majors, give scholarships to all students who meet the requirements, and is perceived as the best university in the country. As a consequence, the seats it offers in almost all majors are over demanded. Therefore, it has to use a mechanism to assign students into majors that respect sutdents' preferences and priorities.

Admission Process in the University of Costa Rica The admission process at UCR is a centralized nationwide process that takes approximately eleven months. For the academic year starting in March, the procedure starts in April of the previous year, when potential incoming students pay the fee and register to take the Academic Aptitude Test (APT); a standardized test that includes

[4]Recovered from `http://www.nacion.com/archivo/Matricula-cursos-privadas`. The amounts were converted to dollar using the average exchange rate of 2013.

[5]For more details see `http://www.estadonacion.or.cr/estado-ciencia-tecnologia`

logical-mathematical and verbal reasoning items. In June, students receive an appointment and practice material is distributed. The test is administered around the country during August and September. Admission scores –which determine students' priorities in the admission mechanism- are privately communicated to each student in November. [6]

Only students with a score greater than 442 points are considered eligible, and advance to the next phase in December, where they are asked to report an ordered list of two major choices. [7] These are the reports that are considered by the RO to run the admission mechanism and to generate the assignments, which are publicized in January. Students who are not assigned a major are out of the university

Once students are assigned a major they have to consolidate the courses they will take in the incoming semester, since being enrolled in a major is a *sine qua non* condition to take courses in the university. In fact, this requirement make many students to apply to less desired majors but with high probability of admission, with the sole purpose of being enrolled at the university, and thus able to take courses. [8]

[6] Admission scores are an equally weighted average of the GPA obtained in high school, and of the score in the APT. Its range is 200-800 points. There are some majors with special requirements like music, painting or architecture, where students also need to approve a specialized pretest in order to be eligible.

[7] Technically speaking, students have an additional option if they are willing to list his first option in two different campuses, but this option is valid for the reduced menu of majors that are also offered off the central campus. Moreover, students have also the possibility of deferring his score for at most the next two years.

[8] In particular, they can take the courses in humanities that are part of the program of study in all majors. Then, they could transfer to other majors retaking the APT, or taking part of a very competitive internal process based on the GPA obtained in the first year of study.

2.2 Theoretical Framework

We index students by s, $s = 1, 2, \cdots, S$, and by an abuse of notation let S represent the set of students; thus $s \in S$. Likewise, we index majors by m, $m \in \{1, 2, \cdots, M\}$, and by the same convention let M also represent the set of majors; thus $m \in M$. Each major m has a capacity of q_m, given by the number of seats available. We assume that $\sum_{m=1}^{M} q_m \leq S$, that is, at most there are as many slots as needed to exactly accommodate all students. We let $\emptyset$ denotes the no-major option, which vacuously satisfies $q_\emptyset = S$.

Student s has private information about his score x_s and his preferences over majors $\succeq_s$. The score x_s is a scalar, whereas preferences $\succeq_s$ are represented by a vector $\boldsymbol{u}_s = (u_s^1, \cdots, u_s^M)$, where u_s^ℓ represents the utility of student s if he gets major ℓ.

The score of each student is observed by the RO, who utilizes them to prioritize students in the admission process. Without loss of generality, we sort students in decreasing order with respect to their scores. Thus, $x_s > x_{s'}$ if and only if $s < s'$. Under this convention, the index of each student determines his priority, and so, a student s has a priority higher than or equal to k if $s \leq k$.

The RO has to choose a procedure to assign students into majors, such that priorities are not violated. The standard procedure used by many universities around the world works as follows. Students report an order of preferences to the RO, which normally allows to list fewer options than the total number of majors. Then, the RO runs a predefined algorithm to allocate majors, such that if a student is not assigned he is given the no-major option. This process involves a large degree of uncertainty for students, because they are given few options to report, and their assignment depends on preferences and priorities of all other students.

A feasible allocation in this environment corresponds to a *many-to-one matching*, namely a rule that assigns each student to at most one major, and such that the number of students assigned to a given major is less than or equal to its capacity. Here, each student's report corresponds to a k-tuple over the set of majors M, where $1 \leq k < M$. Thus, $\boldsymbol{m}_s \in M^k$. Therefore, given a profile of reports $\boldsymbol{m} = (\boldsymbol{m}_1, \cdots, \boldsymbol{m}_S)$, the matching mapping ϕ gives the allocation $\phi_s(\boldsymbol{m})$ to student s.

For each student s we write $\boldsymbol{m}_s(\ell)$ to refer the ℓth major reported in $\boldsymbol{m}_s$. When we want to make clear the dependence of the report to a specific parameter b, we write $\boldsymbol{m}_{[b]}(\ell)$, nonetheless we omit this notation when it can be inferred from the context.

Finally, we denote $\boldsymbol{o}_s$ as the vector that orders the majors according to student's true preferences $\boldsymbol{u}_s$, and $\boldsymbol{o}_s(\ell)$ as his ℓth preferred major. Therefore, letting $\boldsymbol{m}_s^*$ be the optimal report of student s, we say that an admission mechanism is strategy-proof if $\boldsymbol{m}_s^*(\ell) = \boldsymbol{o}_s(\ell)$ for all $\ell = 1, \cdots, k < M$, $s \in S$ and u_s.

Many of the countries listed in the introduction apply a serial dictatorship mechanism. The description of the algorithm is as follows:

In step 1. The student with the highest score is considered. He is assigned a seat at the major reported in the first position.

In step 2. The student with second highest score is considered. He is assigned a seat at the major reported in the first position if there are available seats; otherwise, he is assigned a seat at the major reported in the second position.

In step ℓ ($\ell > 2$). The student with the ℓth highest score is considered. He is assigned a seat at the major reported in the highest position that has available seats.

The algorithm terminates when the reports of all students have been considered or all the seats have been allocated.

When students are allowed to report a complete order over preferences, the serial dictatorship mechanism is strategy-proof. However, when students are constrained, they have an incentive to misreport. That is the reason for using the qualifier *reported*, instead of *preferred*, in the description of the serial dictatorship above. Furthermore, without constraints, the serial dictatorship is Pareto-efficient and free of justified envy. That is, no student with higher score prefers the assignment of a student with lower score over his own assignment.

Despite of these desirable properties, many universities do not let students to report a full list of preferences, and hence the game of incomplete information induced by the mechanism does not have an equilibrium in dominant strategies. In fact, the only players that have a dominant strategy are the students with a priority higher than or equal to the number of options to report. For the rest of students we have to model a sophisticated decision problem, as we show in the appendix 4.4.

2.2.1 Student's Decision Problem

One important characteristic of this process is that students participate under the same rules year after year, and so the series of equilibrium outcomes can reveal information about population preferences and scores. Although not all the information is available to students, normally the RO publicizes past threshold scores in each major, which arguably encompasses all the relevant information about preferences and scores contained in previous admission processes. If such data is available to students, it can be used along with their own current scores to compute their vector of admission probabilities to each possible major.

Fixing the history of past threshold scores in each major, we denote $\eta(m, x)$ the ex-ante probability of getting admission into major m given a score of x. Probabilities are non-decreasing in the score, that is, if $s < \tilde{s}$, then $\eta(m, x_s) \geq \eta(m, x_{\tilde{s}})$.

Given a vector of cardinal utilities $\boldsymbol{u}_s$ and a score x_s, student s chooses to report an ordered k-tuple of majors to maximize his expected utility.

Notice that for a given report of student s, $\boldsymbol{m}_s$, either he gets into the first major reported or he does not. If he is rejected, either he gets into the second major reported or does not; and so on. Hence, define

$$r^{\ell-1}(\boldsymbol{m}_s, x_s) = \begin{cases} \prod_{j=1}^{\ell-1} 1 - \eta(\boldsymbol{m}_s(j), x_s) & \text{for } \ell = 2, \cdots, k \\ 1 & \text{for } \ell = 1 \end{cases}$$

as the probability of not getting admission in any of the first $\ell - 1$ majors listed in the report $\boldsymbol{m}_s$. Notice that the order matters, since any major chosen imposes an externality on the following majors in the order.

Thus, the expected utility of each student s can be written recursively as:

$$V(\boldsymbol{u}_s, x_s) = \max_{\boldsymbol{m}_s \in M^k} \sum_{\ell=1}^{k} r^{\ell-1}(\boldsymbol{m}_s, x_s) \eta_s^\ell u_s^\ell \tag{2.1}$$

The structure of the decision problem is the same as in Chade and Smith (2006), hence, the solution can be obtained using a greedy algorithm.

We denote $\boldsymbol{m}_s^*$ a typical element of the arg max of $V(\boldsymbol{u}_s, x_s)$. Once all students solve their respective decision problem, it is possible to construct the profile of optimal reports $\boldsymbol{m}^* = (\boldsymbol{m}_1^*, \cdots, \boldsymbol{m}_S^*)$, which along with the selected mechanism, endogenously determines assignments $\boldsymbol{\phi}(\boldsymbol{m}^*) = (\phi_1(\boldsymbol{m}^*), \cdots, \phi_S(\boldsymbol{m}^*))$ and threshold scores $\boldsymbol{t}^* = (t_1^*, \cdots, t_M^*)$, where

$$t_\ell^* = \min\{x_s : \phi_s(\boldsymbol{m}^*) = \ell\}, \quad \forall \ell \in M \tag{2.2}$$

2.2.2 Comparative Statics

In this section we analyze what is the optimal behavior of a student if his vector of admission probabilities "improves" or if he is allowed to report a higher number of options.

Definition 1. *The report $\boldsymbol{m}_s$ is more aggressive than the report $\tilde{\boldsymbol{m}}_s$ if $\boldsymbol{m}_s \neq \tilde{\boldsymbol{m}}_s$ and $\boldsymbol{m}_s(\ell) \succeq \tilde{\boldsymbol{m}}_s(\ell)$ for all $\ell = 1, \cdots, k$.*

Theorem 1 (Chade and Smith **.** *Assume that $\boldsymbol{\eta}_s$ and $\tilde{\boldsymbol{\eta}}_s$ are two vector of admission probabilities such that (i) $\eta_s^{\boldsymbol{o}_s(\ell)} \geq \tilde{\eta}_s^{\boldsymbol{o}_s(\ell)}$ for all $\ell = 1, \cdots M$, and (ii) $\frac{\eta_s^{\boldsymbol{o}_s(\ell)}}{\tilde{\eta}_s^{\boldsymbol{o}_s(\ell)}} > \frac{\eta_s^{\boldsymbol{o}_s(\ell+1)}}{\tilde{\eta}_s^{\boldsymbol{o}_s(\ell+1)}}$ for all $\ell < M$. Then, $\boldsymbol{m}_s^*$ is more aggressive than $\tilde{\boldsymbol{m}}_s^*$.*

Suppose there are two vector of admission probabilities for student s, $\tilde{\boldsymbol{\eta}}_s$ and $\boldsymbol{\eta}_s$, such that the latter offers a higher probability of admission in each major, and relatively favors the more preferred majors by student s. Theorem two states that the optimal report of student s is more aggressive under $\boldsymbol{\eta}_s$ that under $\tilde{\boldsymbol{\eta}}_s$. That is, he will report a weakly preferred major in each of the k slots given, and will report a strictly preferred option in at least one of the positions. Notice that this result is relative just to student s, since even though $\tilde{\boldsymbol{\eta}}_s$ favors the more preferred majors by student s, it does not necessarily hold for all students $\tilde{s} \neq s$.

Proposition 1. *Fix a vector of admission probabilities $\boldsymbol{\eta}_s$ and let $k < \tilde{k}$ be two different number of options to report. Then, the optimal report $\boldsymbol{m}_{s,[\tilde{k}]}$ truncated to the first k majors is more aggressive than the report $\boldsymbol{m}_{s,[k]}$.*

Proof. Follows immediately from Theorem 1 in Chade and Smith

Proposition 1 says that given a vector of probabilities $\boldsymbol{\eta}_s$, if the number of options increases, student s will become more aggressive using the first k slots to report weakly preferred majors. The intuition for this result resides in the fact that if an additional option is given then the student either (i) will use it to report a safety school in the marginal slot given, which does not change the report of the first k positions, or (ii) will report a better option in the first k slots with respect to his previous report, and will use the marginal slot to report a safety option (i.e. a major with higher probability of admission).

2.2.3 Justified Envy and Strtegy Proofness in the Large

As we discussed in the introduction, when students are not allowed to report the complete list of preferences, the mechanism induced by the serial dictatorship algorithm is not strategy-proof. We now investigate if it satisfies a weaker notion of strategy-proofness called strategy-proofness in the large (SP-L).

The SP-L concept was introduced by Azevedo and Budish

As the authors point it out, this concept is stronger than a Bayesian Nash equilibrium, because truthful reporting is a best response against any probability distribution, and not only the distribution associated with the corresponding Bayesian Nash equilibrium. In fact, Budish and Azevedo show that several well known mechanisms that are not strategy-proof are SP-L.

In our realm, the admission mechanism would be SP-L if for any vector of admission probabilities, saying the truth were a dominant strategy. It is clearly not the case. Consider for example a student with cardinal utilities that strictly order all the majors, but such that the difference between the maximum and minimum cardinal utility is sufficiently small. Furthermore, suppose probabilities are strictly increasing in the reverse order of the students' preferences (i.e. the most preferred major has the lowest admission probability, and so on). Then, the student has an incentive to report the majors with higher probabilities of admission even though they are not the most preferred.

Another recurrent concept in the analysis of the admission mechanism is the no-justified envy property (cf. Balinski and Sonmez

Our admission mechanism also fails the no-justified envy test. Indeed, suppose that students are only allowed to report two options as in the UCR, and that being out of college is strictly worse than studying any major. Then, take two students s and $\tilde{s}$ with low adjacent scores (hence very similar probabilities of admission) but such that student s has very strong cardinal utilities for very popular (over demanded) majors, whereas student $\tilde{s}$ is indifferent among all majors. Then, it is very likely that, ex-post, the student s envies the allocation of student $\tilde{s}$. In fact, in section 4.4 we compute the proportion of students with ex-post justified envy in our simulation.

2.3 Multi-agent Decision Approach

In this section, we present how to implement the multi-agent decision problem when the researcher has access to a data set that contains admission scores, reports, assignments and threshold scores.

The main goal of this exercise is to recover the crucial non-observable variable of our model: students' preferences over majors. One uninteresting case is to assign a different profile of preferences to each student. In that way, for a given vector of probabilities it is always possible to properly select very high cardinal utilities for the majors reported, so that the solution of (2.1) coincides with the observed report. This option would violate the fact that some students tend to have similar preferences, and so that a few preference profiles can capture the decision process in a parsimonious way. Moreover, the preferences recovered would be useless to conduct counterfactual analysis in a robust way.

We propose the following algorithm to recover cardinal preferences.

2.3.1 Admission Probabilities

The distinctive feature of the model introduced in section 3.2 is that students take into account past threshold scores to compute their admission probabilities to each major: $\eta(m, x_s)$.

Let $\boldsymbol{T}_m$ be the series of past threshold scores. We assume that $\boldsymbol{T}_m$ follows a truncated normal distribution for each major m, and check such assumption by conducting a Jarque-Bera and a Wilk-Shapiro test on the series of values corresponding to the period $2000 - 2014$. In almost all the majors without missing values, the null hypothesis of normality is not rejected.

Now, given the normality assumption, students compute admission probabilities as follows

$$\eta(m, x_s) = \frac{\Phi((x_s - \mu_m))/\sigma_m) - \Phi((t_m^{\min} - \mu_m))/\sigma_m)}{\Phi((t_m^{\max} - \mu_m))/\sigma_m) - \Phi((t_m^{\min} - \mu_m))/\sigma_m)}$$

Here, Φ denotes the cumulative distribution function of standard normal variable, whereas μ_m, σ_m, $t_m^{\min}$ and $t_m^{\max}$ correspond –respectively– to the mean, standard deviation, minimum and maximum of $\boldsymbol{T}_m$ for each major m. [9]

2.3.2 Students' Optimal Portfolio

Once probabilities are obtained from the data, we solve student's decision problem stated in (2.1). In general, this problem is fairly complicated, since it entails

[9] The computed probabilities do not vary significantly if we assume that the series of past threshold scores follows a uniform distribution instead.

the maximization of a submodular function over a finite set of alternatives –an NP hard problem (Lovász, 1982). However, we can use the *Marginal Improvement Algorithm* (MIA) introduced by Chade and Smith (2006). In short, MIA is a greedy algorithm that looks for the option that yields the highest marginal increment in the expected utility at each stage. First, for given student s it sets the initial portfolio $\boldsymbol{m}_s^0 = \emptyset$ and searches for the option with the largest expected utility. That is, it chooses a major $m_s^1 \in \arg\max_{m \in M} \eta(m, x_s) u_s^m$ and then updates the optimal portfolio to $\boldsymbol{m}_s^1 = \boldsymbol{m}_s^0 \cup \{m_s^1\}$. In step ℓ, $\ell \geq 2$ it chooses the option $m_s^\ell \in \arg\max_{m \in M \setminus \{\boldsymbol{m}_s^{\ell-1}\}} \eta(m, x_s) u_s^m$ and updates the optimal portfolio to $\boldsymbol{m}_s^\ell = \boldsymbol{m}_s^{\ell-1} \cup \{m_s^\ell\}$. In other words, at each stage the algorithm picks the major that yields the largest marginal benefit over the portfolio of majors constructed so far, and updates the optimal portfolio recursively. It stops when it has chosen the k locally optimal options. The authors show that the optimal solution provided by MIA coincides with the global solution of the original maximization problem, and moreover, that it reaches its solution in a quadratic number of steps.

As pointed out by Fack et al.

To study the assignment problem of students to secondary schools in Ghana, Ajayi

2.3.3 Allocations and Threshold Scores

Now, when students' optimal reports $\boldsymbol{m}^*$ are computed as described above, they are considered the inputs of a serial dictatorship mechanism, whose solution becomes the final assignment. Given the final assignments, the threshold score for major m corresponds to the score obtained by the last student admitted to this major, as shown in equation (2.2).

2.3.4 *Adjustment of Preferences*

The initial cardinal utilities are adjusted according to the performance of the algorithm with respect to the actual assignment in the data. Specifically, for each major we compute the difference between the simulated aggregate assignment and the actual aggregate assignment. If for a given major such difference is positive, it means that the simulation is assigning more students than desired. Then, preferences of all students for such major are adjusted down by a factor of $\delta > 0$. That is, if for a given student s the preference for major m is given by u_s^m, the new preferences would be $\tilde{u}_s^m = u_s^m - \delta$ after the adjustment. If on the other hand the difference is negative, the new preferences would be $\tilde{u}_s^m = u_s^m + \delta$.

The adjustment procedure has to satisfy one additional requirement: it has to preserve the order of the majors in the profile (such order is obtained by the algorithm described in section 2.3.5). That is, it is not possible to adjust upward the cardinality of a major such that it surpasses the position of his neighbor above. Likewise, the cardinality cannot be adjusted downward so that is falls below its neighbor underneath.

In general, the process continues until the assignments observed in data are perfectly matched or ten thousand iterations are run, whatever happens first. [10] The final preferences of such numerical exercise are considered the students' preferences that rationalize choices, and thus they become the primitives of the model to conduct future counterfactual analysis. The algorithm 1 displays the pseudocode of the procedure described above.

[10] The number of iterations is not binding, since the algorithm converges in seven thousand iterations.

Algorithm 1 Recovering Students' Cardinal Preferences

Set probabilities of admission for each student: $\{\boldsymbol{\eta}_s\}_{s=1}^S$

Set initial preferences for each student: $\{\boldsymbol{u}_s^0\}_{s=1}^S$

Set aggregated assignments from actual data: $\{\boldsymbol{a}_m\}_{m=1}^M$

Set the maximum number of iterations N

Set the preference adjustment parameter $\delta > 0$

Set optimal preference $\{\boldsymbol{u}_s^*\}_{s=1}^S = \{\boldsymbol{u}_s^0\}_{s=1}^S$

Set the initial error: $\epsilon_0 = \infty$

while $j < N$ and $\epsilon_j \neq 0$ **do**

  Apply MIA to obtain each student optimal portfolio: $\{\boldsymbol{m}_{s,j}^*\}_{s=1}^S$

  Run a serial dictatorship algorithm using $\{\boldsymbol{m}_{s,j}^*\}_{s=1}^S$ as reports

  Determine simulated assignments: $\{\phi_{s,j}(\boldsymbol{m}^*)\}_{s=1}^S$

  Compute the assignment error per major $\epsilon_{m,j} = |\sum_{s=1}^S \mathbb{1}\{\phi_{s,j}(\boldsymbol{m}^*) = m\} - a_m|$

  Compute the total assignment error: $\epsilon_j = \sum_{m=1}^M \epsilon_{m,j}$

  if $\epsilon_j < \epsilon_{j-1}$ **then**

    Set Optimal preferences $\boldsymbol{u}_s^* \leftarrow \boldsymbol{u}_{s,j}$

  end if

  if $\epsilon_m < 0$ **then**

    $u_{s,j}^m = u_{s,j-1}^m + \delta$ if and only if the order in $\boldsymbol{u}_s^0$ is not violated

  else if $\epsilon_m > 0$ **then**

    $u_{s,j}^m = u_{s,j-1}^m - \delta$ if and only if the order in $\boldsymbol{u}_s^0$ is not violated

  end if

  $j \leftarrow j - 1$

end while

2.3.5 *Initial Preferences: "Revealed Preference" Algorithm*

We use the students' reported preferences and the history of past threshold scores to recover students preferences. I propose an algorithm to do so. One assumption underlying the algorithm is that students' preferences and scores are independent.

To understand why independence is a natural assumption, note that the score is determined partly by a grades in high school and partly by a standardized exam. There is evidence that students are still forming their preferences over majors, at the point where they are taking the admissions test. [11]

To present our argument we suppose that students are allowed to report only two majors, as it is the case in the UCR. We construct different tiers based on the information of past threshold scores to categorize students into classes. Because we have a series of past threshold scores instead of a single value, we take the $\tilde{t}_m$ as the maximum of the past threshold scores in major m. We say that if a student s has a score greater than the highest maximum threshold score, he belongs to the first tier. Students in the first tier can "afford" any major. Students with scores between the second highest and the highest maximum threshold score belong to the second tier, and they can afford all the majors but the major with the highest maximum threshold score. We classify all students into different tiers following the same procedure.

Students in tier 1 are the students for which there is certainty to get admission into any of the majors. Hence, for them it is a strictly dominant strategy to report their most preferred option in the first position, and a weakly dominant strategy

[11] According to university authorities, many students attend the Center of Vocational Orientation between September and November. The Center helps students discern their aptitudes towards each major. They take the exam in September and they have to report their preferences in November.

to report the truth in the second position. We use the reports of these students, as the beginning of all the potential ordinal preferences that students can have.

By the assumption of preferences and scores, for any student s in tier 1 there exist a student $\tilde{s}$ in tier 2 that shares the same preference profile. However, student $\tilde{s}$ does not necessarily report the same ordered list as student s, because he has lower priority, and so reporting less preferred colleges but with lower historical threshold scores could be optimal. Nonetheless, we can use his report to complete the report of student s, provided that two consistency conditions –namely, no dominated choice and no cycles in preferences– are satisfied. We can proceed recursively for the succeeding tiers in the same fashion.

Recall that for a any tier ℓ, a report of student s in tier ℓ, $\boldsymbol{m}_s^\ell$, is an ordered pair. Now, let $\boldsymbol{o}^\ell$ be a preference profile constructed by using the students' reports up to tier ℓ. That is, $\boldsymbol{o}^\ell$ is formed by concatenating the selected reports in each of the previous tiers, whenever it is "feasible". Hence, the dimension of $\boldsymbol{o}^\ell$ is weakly increasing in the index of the tier.

The concatenation procedure works as follows. We pick a report $\boldsymbol{m}^1$ in tier 1 (we will avoid to write the identity of the student for simplicity, and instead will use a superscript to keep track of the tier), and set it as the beginning of an ordinal preference profile. That is, $\boldsymbol{o}^1 = \boldsymbol{m}^1$. Then, if the report $\boldsymbol{m}^2$ in tier 2 is a "feasible" continuation of the profile $\boldsymbol{o}^1$, the new profile would be $\boldsymbol{o}^2 = \boldsymbol{o}^1 \oplus \boldsymbol{m}^2$, where $\oplus$ stands for concatenation. In general, $\boldsymbol{o}^\ell = \boldsymbol{o}^{\ell-1} \oplus \boldsymbol{m}^\ell$ for $\ell \geq 2$.

The report $\boldsymbol{m}^\ell$ is a feasible successor to the profile $\boldsymbol{o}^{\ell-1}$ if two conditions of minimum rationality are satisfied: (i) no-dominated choice and (ii) no cycles.

Definition 2 (No-dominated choice)**.** *We say that a report $\boldsymbol{m}_s^\ell$ in tier ℓ satisfies the no-dominated choice relative to the preference profile $\boldsymbol{o}^{\ell-1}$, if for all majors reported in $\boldsymbol{m}_s^\ell$ there is not a major in $\boldsymbol{o}^{\ell-1}$ that is strictly preferred, and whose* *threshold score is lower than the student score x_s.*

In other words, it says that a student in tier ℓ will not choose to report a less preferred major whenever a major in the profile $\boldsymbol{o}^{\ell-1}$ is "affordable."

Definition 3 (No cycles)**.** *We say that $\boldsymbol{m}^{\ell}$ satisfies the no-cycles property relative to the preference profile $\boldsymbol{o}^{\ell-1}$ if no major in $\boldsymbol{o}^{\ell-1} \oplus \boldsymbol{m}^{\ell}$ forms a cycle in the order of preferences.*

For a given report $\boldsymbol{m}^{\ell}$ in tier ℓ, we define the set of its antecessors, $\mathcal{A}(\boldsymbol{m}^{\ell})$, as the set of all profile preferences $\boldsymbol{o}^{\ell-1}$, for which it can be a feasible successor. The set of feasible successors for a given preference profile $\boldsymbol{o}^{\ell-1}$ can be defined in a similar manner.

The following example illustrates how the algorithm operates.

Example Suppose there are ten students and five majors: Medicine (M), Law (L), Engineering (E), History (H) and Arts (A). Each major has exactly one seat available. Students can be assigned at most one seat in each college. If a student is not assigned a seat in any college, we say he is out college, and denote this option by $\emptyset$.

Students are categorized in tiers, constructed from the historical maximum scores, as follows:

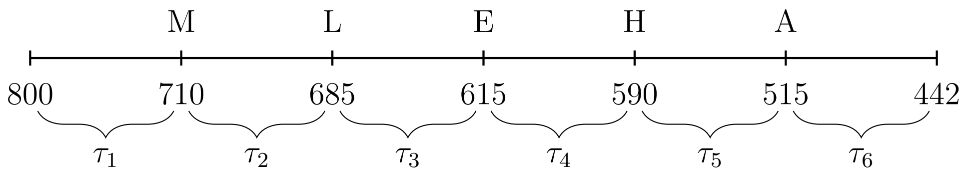


Figure 2.1: Definition of Tiers

Students' information is summarized in table 2.1.

Table 2.1: Students' Scores and Reports

Student	Score	Tier	Report
1	750	1	(M, L)
2	735	1	(M, E)
3	705	2	(L, M)
4	690	2	(E, H)
5	650	3	(E, A)
6	620	3	(H, A)
7	600	4	(L, A)
8	570	5	(A, H)
9	550	5	(A, E)
10	525	5	(L, A)

By the procedure described above, the two reports in tier 1, (M, L) and (M, E), are set as the commencement of the different preference profiles students can have.

The next step is to determine the feasible antecessors for each of the reports in tier 2, which are,

$$\mathcal{A}[(L, M)] = \emptyset \quad \text{and} \quad \mathcal{A}[(E, H)] = \{(M, E)\}$$

Notice that (M, L) cannot be a valid antecessor of (L, M) because it would form a cycle in preferences. Likewise, (M, E) would violate the no-dominated choice property, because the agent would be choosing to report L, as his first option, while E is still affordable and is preferred. For the same reason (M, L) can not be a valid antecessor of (E, H). Hence, the only possibility is to concatenate the profile (M, E) with the report (E, H); to discard the report (L, M); and to carry over the profile (M, L) to the next tier. Thus, the preference profiles up to tier 2 would be (M, L) and (M, E, H)

Using the reports in tier 3 we have that

$$\mathcal{A}[(E, A)] = \{M, L\} \quad \text{and} \quad \mathcal{A}[(H, A)] = \{(M, L), (M, E, H)\}$$

Although the profile (M, L) is a feasible antecessor for both reports, we give priority in the assignment of antecessors to the reports submitted by students with higher score. Then, (M, L) is assigned to (E, A), and thus the only possibility is to assign (M, E, H) to (H, A). Therefore, the preference profiles up to tier 3 would be (M, E, H, A) and (M, L, E, A).

Continuing in the following fashion, we can show that the final ordinal preference profiles would be (M, E, H, A) and (M, L, E, A, H). To utilize such profiles in the general algorithm it is necessary to assign cardinal utilities to both profiles. As an illustration, we assign a value of $2,000$ utils to the most preferred major in each profile, and then decrease the intensity in 400 utils until reaching the last. Notice that Law (L) is not ranked in the first profile, and so it can be assigned any value lower than 400 and greater than 0. We assign it a value of 100. Therefore, letting $\boldsymbol{u} = (u_M, u_L, u_E, u_H, u_A)$ be the vector of cardinal utilities, we can construct the following two vectors from the procedure above

$$\boldsymbol{u}' = (2,000; 200; 1,600; 1,200; 800) \quad \text{and} \quad \boldsymbol{u}'' = (2,000; 1,600; 1,200; 1,200; 800; 400)$$

Finally, each student is assigned any of these profiles with equal probability, and independently of their score.

In general, the algorithm to concatenate the reports across tiers is as follows. We identify all the reports in tier 1 and set them as the commencement of the different preference profiles students can have. Then, we sort the reports in tier

2 in decreasing order with respect to students' score, and determine the set of feasible antecessors for each report. To continue, we pick the first report and assign a preference profile randomly from the set of feasible antecessors. The new preference profile is constructed by concatenating the chosen antecessor with the current report.

We exclude the chosen antecessor from the set of antecessors of other reports, and continue recursively for the rest of the profiles in tier 2. Finally, we repeat the procedure for all tiers. If a profile cannot be continued by a report in tier ℓ, then it is carried over intact to the next iteration in tier $\ell + 1$. Furthermore, if in a particular tier ℓ, there exist more reports than the potential profiles constructed up to tier $\ell - 1$, we discard the reports that cannot be matched. This is fundamental to maintain the number of profiles equal to the ones encountered in tier 1.

Once we determine the different ordinal profiles, we need to assign a cardinal utility to them, so that they can be utilized as the initial preferences in the algorithm described below. In order to do so, we define a vector of cardinal utilities based on the position of each major in the profile. That is, we assign the maximal cardinal utility for the major in the first position, the second highest for the second position, and so on. Clearly, each major in each profile will be attached a different cardinal utility depending on the position it holds in the profile. Finally, by the assumption of independence between preferences and profiles, we assign the cardinal profile of preferences to each student uniformly random.

2.3.6 Evaluation of Equilibrium Assignments

To evaluate the different policies considered, we compare the resultant assignments with the ones obtained when students are allowed to report preferences over all the majors available. The reason to use the latter case as a benchmark lies in the fact that it has three desirable properties: (i) it is a strategy-proof mechanism, (ii) its final assignment is Pareto-optimal, and (iii) its final assignment is free of justified envy.

We use three measures to capture the dissimilarity between the assignments of the case under consideration and the benchmark. The first is the fraction of students with a different assignment from the benchmark. The second is the euclidean norm between the vectors of cardinal utilities associated to the assignments. The third is the aggregate welfare obtained by adding the cardinal utility associated to each student's assignment. Here, the relevance is the direction of the effect more on its magnitude, since it involves interpersonal comparisons of utilities that are hard to interpret.

2.4 Data Overview

We have data from the admission process at UCR in 2008, which includes all students who applied for admission. In particular. each student's record contains information such as: gender, age, high school of procedence, admission score, majors reported as first and second option, and the final assignment given by the admission algorithm.

Population There are three stages of the admission process at which students can be identified: (i) those who take the test, (ii) those who take the test and become eligible and (iii) those who become eligible and submit a report to the RO. Our data set corresponds to the population at the second stage in 2008, which corresponds to $19,621$ students. From these, $9,930$ -around 51%- did not apply to any major. From $9,691$ students who applied to some major (i.e. those who submit a report of preferences to the RO), $2,470$ of them -around 13%- did not get a seat in any major. It implies that $7,221$ students were assigned to some major in one of the campuses around the country, and specifically, $5,212$ were assigned to the main campus.

Academic offer The UCR counts with a central campus located in the capital, and five branches around the country. The main campus offered eighty two majors in 2008, which can be grouped in nine academic areas: (i) Arts, (ii) Literature, (iii) Hard Sciences, (iv) Education, (v) Economic Sciences (vi) Social Sciences, (vii) Engineering, (viii) Health, and (ix) Agro-alimentary Sciences. Some majors (e.g. piano, composition, physics and pure mathematics) are offered exclusively by UCR at the main campus. Although in recent years the UCR has made an effort to augment the menu in peripheral campuses, the academic offer continues to be restricted in comparison to the main campus, and historically has been designed to fulfill a requirement of professionals in region-specific economic activities.

We consider the main campus as the "main target" for students who apply to the UCR. In fact, from the $9,691$ students who applied for admission, $7,584$ listed a major in the main campus as their first option. This number increases to $8,219$ if considered those who put a major in the main campus as the second option. Moreover, given the different reasons introduced in section 2.1, for many students the UCR constitute the only option of higher education in Costa Rica. To support this assertion, table (4) shows the fraction of students who accepted to be enrolled in their second option.

Threshold Scores We have access to the series of threshold scores for all majors offered in the main campus for the period 2000-2014. Notice that if this information is stable over time, it provides a student with a notion of what are his chances to get admission in a particular major, given his current score. We plot in figure **??** the series of past threshold for selected majors.

2.5 Simulation Results

In this section we discuss the main results of applying the methodology introduced in section 2.3 to the data described in section 2.4.

First, we present the results of the auxiliary "revealed preference" algorithm implemented to determine the initial preferences of the general algorithm. We use the historical maximum score in each major to construct the different tiers, whose distribution is presented in figure 7 in the appendix 4.4. As it can be seen, there are 205 students in tier 1, whose reports are the first step to form the different ordinal preference profiles. After applying the procedure explained in section 2.3.5 to concatenate the reports, we obtain that the median length of the preference profiles (i.e. the number of majors the algorithm is capable to order) is 42, whereas the maximum is 50 and the minimum is 35. The standard deviation is 3.45. The reason for which it is not possible to order all majors within a profile is because the two rationality conditions are binding. All majors that were not ordered within a profile are considered the least preferred in such profile.

The next step is to assign the cardinal utilities to the ordinal profiles constructed. We assign 2075 utils to the most preferred major in each profile, and then decrease 25 utils to the next option below, until assigning a cardinal utility to the last major ranked in the profile. All majors that were not ranked by the profile are given 5 utils. The way of assigning cardinal utilities is chosen by convenience, however, the main insights that we obtain do not depend on it.

Given these preference profiles, and the admission probabilities obtained from the data, we run our general algorithm to recover the cardinal preference profiles

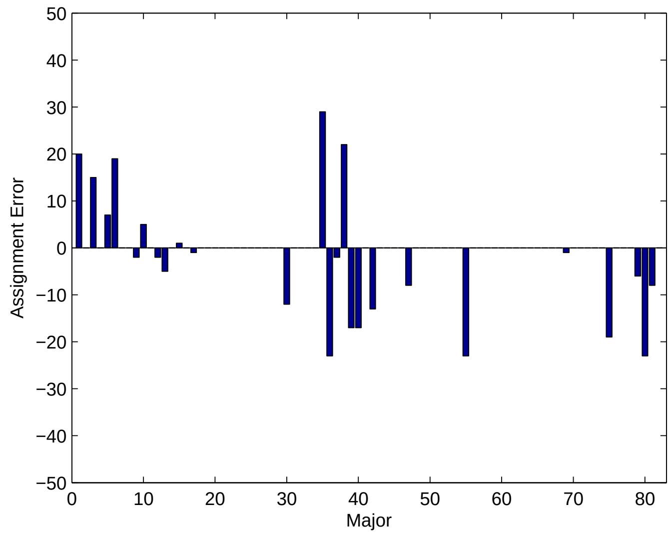


Figure 2.2: Assignment Error in the Baseline Model.

that rationalize the total assignments in each major. As such, we compute the absolute and relative error. The former is the absolute difference between the actual aggregate assignment and the simulated aggregate assignment in each major. Likewise, the latter is computed as the absolute error divided by the total of seats in each major. Following our two-step methodology, we find an absolute error of 300 seats, and a relative error of approximately 6%. Figure 2.2 depicts the difference between the simulated assignment and the actual assignment in the data, for each of the 82 majors. As it can be seen the maximum absolute error is bounded above by 32. Therefore, the algorithm not only shows a very good fit in the aggregate, but also when considering each major separately. Meanwhile, Figure 10 in appendix 4.4 shows the simulated threshold scores.

Robustness As a robustness check, we repeat the same exercise for different specifications of the initial preferences, as presented in table 2.2. For instance, in

Table 2.2: Results for Different Specification of Preferences

	Specification	Absolute Error	Relative Error
(1)	All Profiles (205)	300	5.8%
(2)	Truncated: 20 options	482	9.2%
(3)	Random Profiles	1263	24.2%
(4)	80 profiles: First	415	7.8%
(5)	80 profiles: Random	633	12.1%
(6)	40 profiles: First	687	13.2%
(7)	40 profiles: Random	661	12.7%

Note: (1) All profiles obtained in the algorithm described in section 2.3.5 are utilized. (2) The number of profiles is the same as in (1) but they are truncated to the first 20 majors ranked. (3) The number of profiles is the same as in (1), but majors within each profiles are randomly ordered. (4,5) Only 80 profiles from 205 are considered, chosen as the first or randomly. (6,7) The same procedure as in (4,5) but only for 40 profiles.

the second line we truncate all the 205 preference profiles to the first 20 majors ordered. That is, we assume that the individual is only able to rank the first 20 majors in each profile, and is indifferent among the remaining.

Meanwhile, in the third line we keep the original 205 profiles, but permute randomly the order within each profile. Notice that unlike the previous exercise we keep the number of major each profile ranks.

In lines four to seven, we reduce the number of profiles to investigate how sensible are the results when students are more homogeneous in their preferences. Specifically, instead of using the complete 205 profiles, we only use a selection of them. In lines four and five we select 80 profiles, whereas in lines six and seven we choose only 40 profiles. In both cases we use two criteria of selection: (i) using the first profiles ranked, and (ii) choosing the profiles uniformly random.

As we can see, using all profiles delivers the best adjustment among all specifications, as well as it maximizes the welfare and the caliber of the students admitted. On the other hand, using a random permutation in each of the original profiles produces the highest absolute error and the lowest level of welfare and innovation. We can observe that the lost in accuracy of passing from 205 profiles to 80 is lower than passing from 80 to 40, since the degree of heterogeneity in preferences is not sufficient to induce students to apply to all the majors. Moreover, given a particular number of profiles, using the first profiles is better than choosing them randomly. The explanation comes from the fact that the first reports are those from students with the highest scores, and so they are the most reliable to report truthfully.

Achieving a good fit of adjustment for the aggregated assignment with 205 has merit per se, because when the number of majors is 82, the total possible orderings –including the option of truncating the list at any point– is given by the expression

$$\sum_{j=1}^{81} \frac{82!}{(82-j)!} = 8.37 \times 10^{122}$$

These expression counts all possible preference profiles when the length of each profile varies. For example, when the index of the sum is equal to two, we are counting how many ordered pairs can be constructed. Such case corresponds to a situation when the student prefers to be out of college if he is not admitted to their two first options reported. Proceeding analogously for all possible lengths of the report, we get the number of all possible preference profiles students can have: a gigantic number.

2.6 Counterfactual Analysis

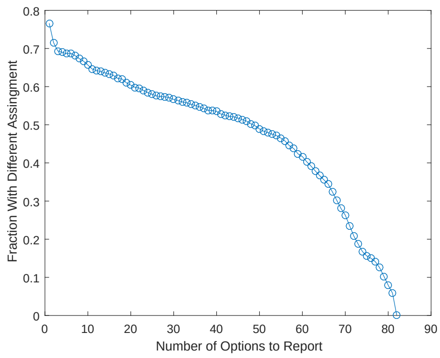
Once we recover the preferences that rationalize students' choices from the numerical simulation, the natural step is to conduct counterfactual exercises about different admission policies. The results of these analyses are vital to understand better how to organize the general admission process in order to produce more efficient allocations. In practice, authorities normally do not count with tools to conduct a rigorous study, and decision about policies are normally made, either based on conjectures or by inferences of related "markets".

With the theoretical and empirical tools developed in the previous sections we can be precise over how to evaluate the different tradeoffs present in the implemetation of different admission policies. In particular, we can evaluate what is the gain of letting students to report more options, how to reallocate seats across majors, or how to implement a second-best affirmative action program. Those are topics under high scrutiny by policy makers.

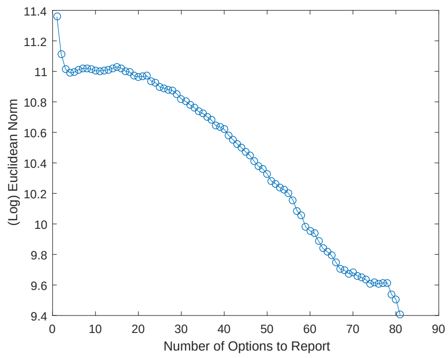
2.6.1 Augmenting the Options to Report

The RO controls how many options each student can report, and this parameter affects equilibrium allocations. Here, we evaluate to which extent the restriction makes students to adopt a different strategy from truth-telling, and how it is reflected in the final assignments.

Figure 2.3a shows that the fraction of students with different assignment from the benchmark is decreasing in k, as expected. That is, as the number of options to report increases, students report closer to their true order of preferences. Allowing students to report only two options implies that 72% of the students have a different assignment from the benchmark. Likewise, figure 2.3b shows that



(a) Students with Different Assignments



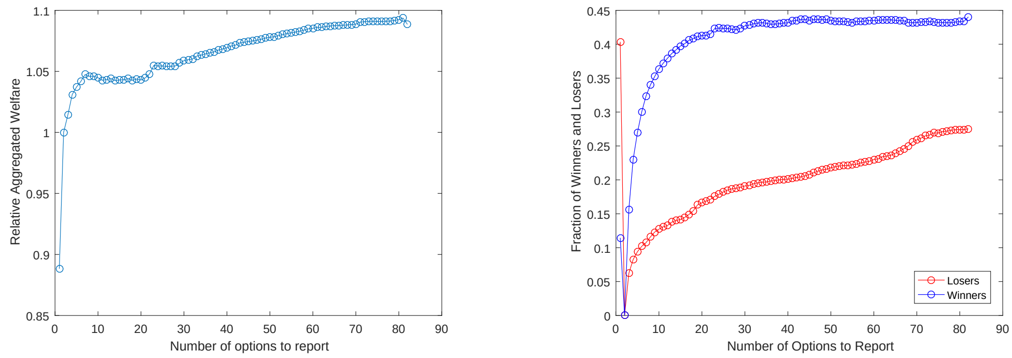
(b) Euclidean Norm

Figure 2.3: Dissimilarity with respect to the Benchmark

the euclidean norm is also decreasing in k, which implies that the dissimilarity in utilities is reduced as k approaches to the total of majors available.

We also investigate the effect to the aggregate welfare of increasing the number of options. Here, there are two effects to consider. First, a higher number of options k makes the students more aggressive in virtue of Proposition 1, and so if they get into college they will be admitted in a weakly preferred major, which increases their welfare. Second, augmenting k also has the effect of decreasing the probability of admission of students with lower priority, which can be detrimental for aggregated welfare. Such negative effect is magnified if the RO has some consideration of affirmative action programs.

Restricting the number k also introduces more uncertainty in students' decision problem, and forces students with high priority to play protective strategies by choosing options less desired but with higher probabilities of admission. At the same time, it may increase the probabilities of admission for students with low priority, which has the potential to increase welfare by the same argument as before.



(a) Relative Aggregate Welfare (b) Winners and Losers

Figure 2.4: Welfare and Innovation for Different Rearrangements Scenarios

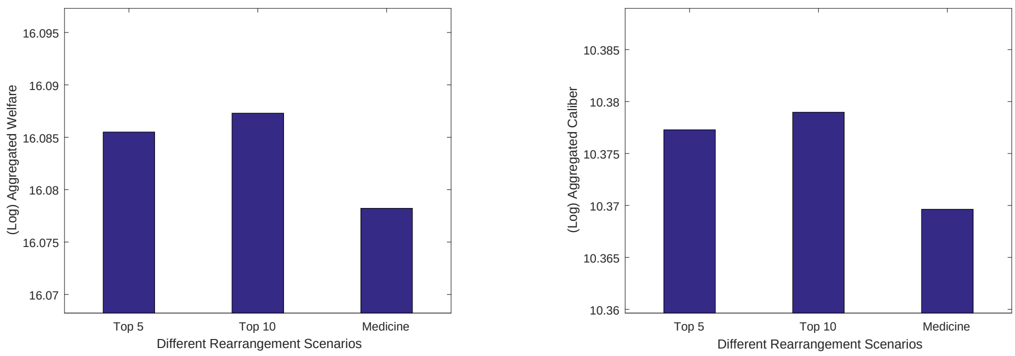
Figure 2.4b depicts the fraction of winners and losers as k increases. A winner [loser] is a student who is assigned a more [less] preferred major with respect to his assignment in the original mechanism with $k = 2$ options. The intersting finding here is that the difference between the fraction of winners and losers is obtained around a value of $k = 20$.

Another measure of welfare we can utilize is the proportion of students assigned to his most preferred option, which can be different from the first option reported. Increasing k decreases the uncertainty over the admission process, and so induces more students to report their first option truthfully, which a priori should increase the number of students assigned to their most preferred option. Nonetheless, as we can observe in Figure 9 the proportion of students assigned to their first major remains very stable when varying the number of options to report.

2.6.2 Reallocation of Seats across Majors

In the admission process of 2008, several majors did not fill all the seats that were available, whereas many others were highly over demanded. It happened mostly in the college of arts where all of the 13 majors wasted some their capacity.

This phenomenon has been persistent according to university authorities. Here, we analyze what would be the impact on students' welfare and caliber, if all wasted seats were proportionally reallocated to: (i) the top 10 most demanded majors, (ii) the top 5 most demanded majors, and (iii) the most demanded major, which is medicine. [12] In the case of (i) and (ii) we use the ratio of each major's demand –defined as how many students place the major as their first option– to the total demand within the correspondent group, as a weight to assign the extra seats.



(a) Aggregated Welfare (b) Aggregated Innovation

Figure 2.5: Welfare and Innovation for Different Rearrangements Scenarios

Figure 2.5 shows that the most effective policy in terms of welfare and innovation is to redistribute the wasted seats in the top ten most-demanded majors. Allocating all the seats in the most popular career, medicine, would not produce a cascade effect, like the one commented above, which reflects the fact that preferences exhibit a sufficient degree of variation.

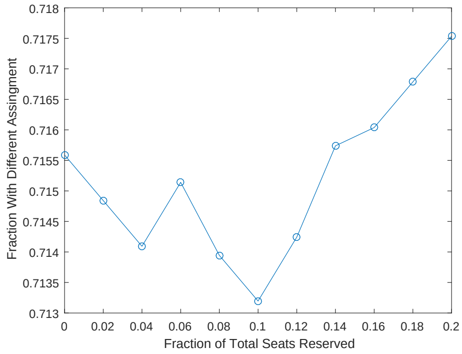
[12]Proportions are computed as the number of students who lists a particular major in the first position, with respect to the total of reports being considered. Then we renormalize the weights, so they can add up to one.

2.6.3 Affirmative Action Programs

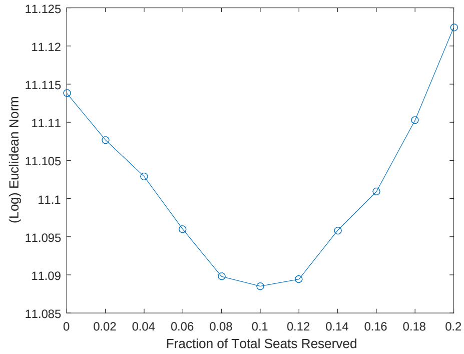
Affirmative action in higher education refers to the aim of reaffirming the rights of groups under-represented in the student body due to social, ethnic, gender or economic reasons. There are many ways to implement an affirmative action program, but in this section we will analyze two popular programs: (i) quotas (ii) and a bonus over the admission score (i.e. a "subsidy" over the score obtained). Students who come from high schools that have been historically under-represented in enrollment constitute our target population. We use the list of high schools provided by the RO to identify such students.

Quotas This program proposes to secure a predefined number of seats to the target population. In that sense, students within the target group will compete with their equals for a seat in a major. To implement this policy, the admission process is carried over via two sequential serial dictatorship algorithms. In the first, only students from the not favored population participate, and the number of seats in each major corresponds to the predefined quota. Those students who were not assigned in the first stage, will participate in a second serial dictatorship mechanism over the remaining seats, along with those students who do not meet the affirmative action condition. The assignments after this second stage are considered final assignments.

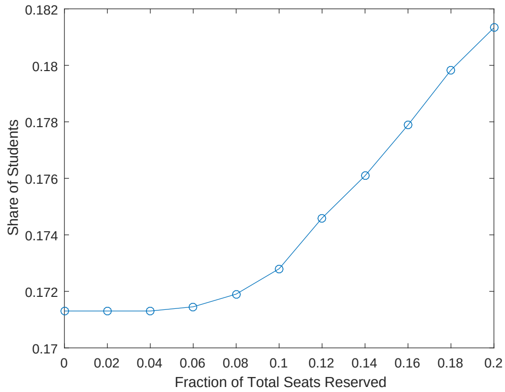
Notice form panel 2.6c that when the quota is lower than 10%, the proportion of targeted students admitted do not vary dramatically from the one obtained when such affirmative action is absent. As soon as it gets higher than 10% the fraction starts to increase sharply. However, notice that it is always under the quota reserved. For instance, notice that when the quota reserved is 20% of the spaces, the fraction admitted is lower than 20%. It happens because, although



(a) Students with Different Assignments



(b) Euclidean Norm



(c) Targeted Students Admitted

Figure 2.6: Counterfactual Analysis for a System of Quotas

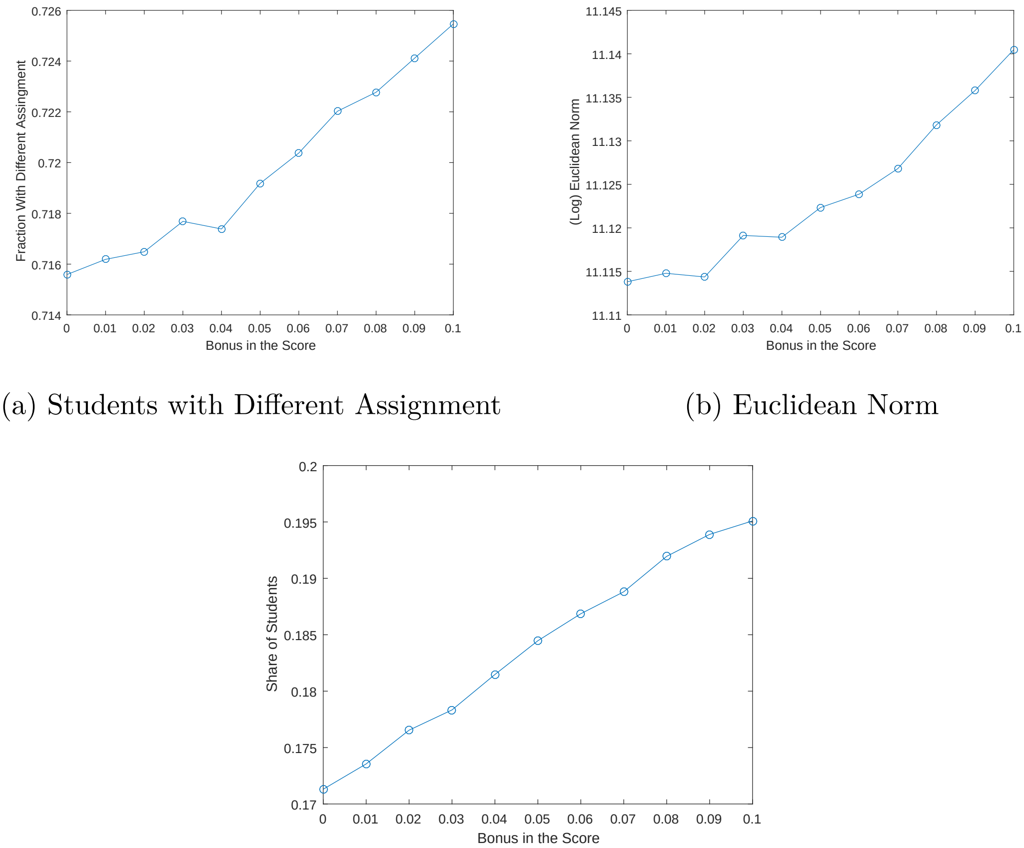
with a quota all target students only compete with their equals for a seat, there still some majors that are over-demanded, and hence some students do not get admission in any of the two majors reported to the RO. Students that are not assigned in the first stage are hardly assigned in the second stage, since now they have to compete with all regular students who have higher scores.

Figures 2.6a and 2.6b show the distortive effect of introducing the quotas. The first computes how the introduction of different quota levels affect the fraction of students that receive a different assignment from the benchmark. The second computes the respective euclidean norm. Both graphs are u-shape. It suggests that some students from the target population who have high score, are choosing very popular majors when given two options. When the RO starts using the quota, they are admitted at the major they would have been admitted if given the possibility to report the full list. When the quota start to increase beyond 10%, students who would not be admitted under the full list, are now admitted because the quota is sufficiently high. The similar intuition holds for the euclidean norm.

Bonus in the Admission Score The second program intends to give a bonus α in the admission score to all students from the target population, to later run a serial dictatorship mechanism over the adjusted scores. Specifically, if a student from the not favored population has a score of x_s points, his adjusted admission score would be $(1 + \alpha)x_s$. A similar exercise is analyzed in Chade et. al. (2014), when each college "discounts" the standard requested to admit a student from the target population.

Unlike the quota, the bonus scheme in panel 2.7c produces an immediate effect in increasing the admission of students from the target population. The explanation is that students who have the benefit of the bonus become more competitive with respect to all regular students. In particular, notice that when the planner uses a quota, the less competitive students from the target population are likely to lose their seat with their more competitive equals, and even more likely with the students from the regular population. Nonetheless, when the planner uses a bonus, those candidates become stronger and capable to compete with the less competitive students from the regular population for a seat. This effect makes that the proportion of students admitted be higher when using a bonus scheme relative to a quota.

With respect to the distortive effect of the bonus, we can see in figures 2.7a



(a) Students with Different Assignment

(b) Euclidean Norm

(c) Proportion of Target Students

Figure 2.7: Counterfactual Analysis for a Bonus System

and 2.7b that the bonus is more distortive than the quota, if the target of the RO is admitting a high fraction of the students from the target population.

2.7 Alternate Mechanisms

Alternatively to the use of an admission algorithm, where the student faces uncertainty over his possibilities to get admission in each major, the RO can use procedures where the threshold required to be admitted in each major are announced before students' make their decision. Here, we investigate two different mechanisms: the *Posting Scores Upfront* (PSU) and the *Ascending Auction* (AA) mechanisms.

2.7.1 Posting Scores Upfront (PSU)

The PSU is a mechanism akin to competitive equilibrium where the RO announces upfront the vector of threshold scores necessary to be admitted into a major, $\boldsymbol{t} = (t_1, \cdots, t_M)$; and commit to give a seat to every student who demands a seat in a particular major, and has a score greater than the posted score. Under this procedure, students observe posted scores, compute their set of feasible majors, and choose the best option within this set according to their true preferences. Unlike the matching mechanism, in the PSU mechanism a student knows ex-ante which majors are feasible, and hence, there is no uncertainty involved in his decision problem. All the uncertainty is transferred to the planner, whose decision problem becomes more cumbersome –since now it is the agent that has to form beliefs about students' preferences– whereas student's decision problem is simplified dramatically. The relevance of this trade-off should not be overlooked, because normally it is argued that the objective of the RO is to maximize students' welfare. Hence, using a mechanism where students face the highest level of uncertainty could be contradictory.

Students' Decision Problem Once a vector of scores $\boldsymbol{t}$ is announced by the RO, and admission scores are realized, the decision problem of student s consists of picking a maximal element with respect to his preferences, in the set of feasible majors. That is,

$$V(\boldsymbol{u}_s, x_s, \boldsymbol{t}) = \max_{m \in \{M(x_s, \boldsymbol{t}) \cup \{\emptyset\}\}} u_s^m \tag{2.3}$$

where $M(x, \boldsymbol{t}) = \{m \in M : x \geq t_m\}$ is precisely the set of feasible majors. We denote $\psi_s(\boldsymbol{u}_s, x_s, \boldsymbol{t})$ the $\arg\max$ of $V(\boldsymbol{u}_s, x_s, \boldsymbol{t})$, and let

$$\Psi_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}) = \left(\sum_{s=1}^{S} \mathbb{1}(\psi_s(\boldsymbol{u}_s, x_s, \boldsymbol{t}) = m) \right)$$

be the aggregate demand for major m.

Remark 1 (Law of demand)**.** *Let $\boldsymbol{t}$ and $\tilde{\boldsymbol{t}}$ be two posted admission scores vectors such that $\boldsymbol{t} \leq \tilde{\boldsymbol{t}}$. If $t_\ell = \tilde{t}_\ell$ for all $\ell \neq m$ and $t_m < \tilde{t}_m$, then $\Psi_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}) \geq \Psi_m(\boldsymbol{u}, \boldsymbol{x}, \tilde{\boldsymbol{t}})$. That is, ceteris paribus if the posted score of major m increases its demand cannot increase.*

Notice that since students have single-unit demands there are no *wealth effects* in their decision problem. Moreover, for a given student s all majors but his most preferred can be considered as *inferior goods.* That is, $\psi_s(\boldsymbol{u}_s, \tilde{x}_s, \boldsymbol{t}) \succeq_s \psi_s(\boldsymbol{u}_s, x_s, \boldsymbol{t})$ for all $\tilde{x}_s \geq x_s$.

The RO's problem in the PSU mechanism The RO can have different objectives at the time of designing an admission policy. One of this could be the maximization of students' aggregate welfare (i.e. the sum of the cardinal utilities associated to each student's assignment). In that sense the RO would be a benevolent planner that only cares about students well being. Alternatively, the RO could be interested in maximizing the caliber of the students admitted (i.e. maximizing the sum of the test score of the students admitted) under the assumption that better students produce higher innovation in any major. Such innovation produces spillovers to the rest of the economy, and then it as way to retribute tax payers for financing public education.

Therefore, the maximization problem of the RO resides in choosing the vector of admission scores $\boldsymbol{t}$ to maximize the predefined objective function, subject to a capacity constraint. Let $\boldsymbol{u} = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_S) \in U^S$ be a profile of preferences, which are drawn according to the density f, and let $\boldsymbol{x} = (x_1, \cdots, x_S) \in [\underline{x}, \overline{x}^M]$ be a profile of scores. Then, the problem of the RO is given by

$$\max_{\boldsymbol{t}\in[\underline{x},\overline{x}]^M} \int_{U^S} \left(\sum_{s=1}^{S} \gamma[u_s^{\psi_s(\boldsymbol{u}_s, x_s, \boldsymbol{t})}] + (1-\gamma)[x_s \cdot \mathbb{1}\{\psi_s^*(\boldsymbol{u}_s, x_s, \boldsymbol{t}) \neq \emptyset\}] \right) f(\boldsymbol{u}) d\boldsymbol{u} \quad (2.4)$$

subject to

$$\Psi_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}) \leq q_m \quad \forall m \ \ \forall \boldsymbol{u} \quad (2.5)$$

where $\gamma \in \{0, 1\}$. When $\gamma = 1$ the RO maximizes the aggregate welfare, whereas when $\gamma = 0$ it maximizes the aggregate caliber.

Condition 2.5 establishes that threshold scores have to be such that for any major, and any realization of the preferences, the aggregate demand is lower than or equal the number of seats in each major.

Now, suppose that the planner has the ability to create new seats in each major m at a cost c_m, and that he has a budget of B. Then, letting $z_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}) = \max\{\Psi_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}) - q_m, 0\}$ be the excess demand in major m, planner's optimization problem can be reconsidered as to maximize the objective in 2.4 subject to the budget constraint,

$$\sum_{m=1}^{M} z_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}) c_m \leq B \ \ \forall \boldsymbol{u} \quad (2.6)$$

Definition 4. *The allocation $\boldsymbol{\psi}^* = (\psi_1^*, \cdots, \psi_S^*)$ and vector of threshold scores $\boldsymbol{t}^* = (t_1^*, \cdots, t_M^*)$ constitutes an equilibrium of the posting admission mechanism if*

i) Given threshold scores $\boldsymbol{t}^*$, ψ_s^* solves student's s decision problem, for each s.

ii) Given students' demands $\boldsymbol{\psi}^*$, threshold scores $\boldsymbol{t}^*$ solves the RO's decision problem, subject to the capacity constraint (2.5), or the budget constraint (2.6).

The option of establishing threshold akin to a competitive market seems very appealing for three reasons (i) students' problem is reduced to choose the best option out of a finite set of feasible majors, (ii) demands are expressed simultaneously and (iii) infra-marginal points of the test are valuable, because the points earned in the test act like a budget constraint.

Simulating the PSU Mechanism Using the past threshold scores in each major, we simulate the equilibrium assignments under the PSU mechanism, for three different announcements of the threshold scores in each major: (i) the historical mean, (ii) the historical median, and (iii) the historical maximum. Figure 11 shows the excess demand per major, for the three scenarios considered before. As we can see, there are large excess demand in many majors, which suggests that none of these configuration constitute an equilibrium for the –degenerate case– when preferences are equal to the ones recovered in the previous section. We provide an algorithm to compute clearing market threshold scores in the next section.

2.7.2 Ascending Auction Mechanism

The main advantage of the PSU is that it removes completely the uncertainty from students' decision problem, but the planner needs to guarantee that either the capacity constraint (2.5), or the (2.6) has to be satisfied. This restriction may induce the planner to set thresholds scores conservatively high, which can reduce students' welfare.

An alternative to the PSU is to use an Ascending Auction mechanism (AA). Here, the planner sets an initial asking threshold for getting a seat in a given major, and students determine their optimal demand given their score and the asking thresholds. Once individual demands are submitted, the planner computes the aggregated demand in each major, and adjusts upward the asking price for a major with maximal demand. If there is more than one major with maximal demand, a major is selected randomly. Once thresholds are adjusted, students recompute their optimal demand. This process continues until no major has an excess of demand. This mechanism retains the advantage of removing students' uncertainty, but guarantees that the aggregated demand never exceeds the supply.

Our procedure resembles closely the algorithm introduced by Demange et al.

Proposition 2. *For a given vector of scores $\boldsymbol{x}$ and preferences $\boldsymbol{u}$, there exists an equilibrium of the AA mechanism $\boldsymbol{t}^* = (t_1^*, \cdots, t_M^*)$.*

Proof. Fix a profile of cardinal utilities $\boldsymbol{u} = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_S)$, and a profile of scores $\boldsymbol{x}$. Let the initial vector of scores be $\boldsymbol{t}^0 = (t_1^0 \cdots, t_M^0) = (\underline{x}, \cdots, \underline{x})$. That is, the RO the initial asking price is equal to the lowest score in all of the majors. Given this information, it is possible to compute the excess demand function $z_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^0)$.

Once the excess demand functions are computed for each major, a major with maximal excess demand is selected from the set,

$$Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^0) := \{m \in M : z_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^0) \geq z_{\bar{m}}(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^0) > 0 \ \ \forall \bar{m} \in M\}$$

and its score is adjusted by an $\epsilon > 0$ sufficiently small. Therefore, in the step $\ell+1$, $\boldsymbol{t}^{\ell+1} = (t_1^\ell, \cdots, t_{\tilde{m}}^\ell) + \epsilon, \cdots, t_M^\ell)$, where $\tilde{m} \in Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^\ell)$.

Formally, fix $0 < \epsilon < \min\{|x_s - x_{\tilde{s}}| : s \neq \tilde{s} \in S\}$ and define the operator $\xi : [\underline{x}, \overline{x}]^M \times U^S \mapsto [\underline{x}, \overline{x}]^M$ by

$$\xi(\boldsymbol{t}, \boldsymbol{u}) = \{\boldsymbol{y} \in [\underline{x}, \overline{x}]^M : y_m = t_m + \epsilon \ \text{ for } \ m \in Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}), \ \text{ and } \ y_\ell = t_\ell \ \text{ for all } \ \ell \neq m\}$$

Notice that $[\underline{x}, \overline{x}]^M$ is a complete lattice and $\xi(\cdot, \boldsymbol{u})$ is an isotone operator with respect to the natural order in $\mathbb{R}^n$. Then, by Tarki's fixed point theorem, there exists $\boldsymbol{t} \in \xi(\boldsymbol{t}, \boldsymbol{u})$. Moreover, $\xi(\boldsymbol{t}, \boldsymbol{u})$ is a complete lattice, and then we can see that $\wedge\xi(\cdot, \boldsymbol{u})$ and $\vee\xi(\cdot, \boldsymbol{u})$ are the lowest and largest equilibrium, respectively. ■

Let $z_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t})$ be the excess demand in major m, and let $Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t})$ be the set of majors with maximal demand. The algorithm 2 displays the pseudocode to find the equilibrium threshold and assignments.

Simulating the AA Equilibrium We use the 205 preference profiles recovered from the matching model to simulate the alternative AA mechanism. Once we obtain the new assignments we compare them with the full-list benchmark. The main finding is that only 5% of the students receive a different assignment under the auction than under the matching model when students can report the full list.

We can observe whether equilibrium scores produced by the PSU mechanism are higher than the endogenous scores obtained by the matching algorithm. Figure 12 depicts the equilibrium threshold scores in the AA and shows for all the unpopular majors the PSU determines a higher threshold score, whereas for the most popular it produces slightly lower threshold scores, which reflects the distortions introduced by increasing the uncertainty in students' decision process.

2.8 Conclusions

We study a major assignment problem where students report preferences over majors the Register Office (RO), and assignments are determined based on students' priorities and reports, via a serial dictatorship mechanism. Preferences and priorities are private information, and students are allowed to report fewer options than the total of majors available. Hence, the admission mechanism is *manipulable*, in the sense that reporting true preferences is not a weakly dominant strategy for every student. Under this environment we propose a tractable methodology

Algorithm 2 Finding Equilibrium Scores in the AA Mechanism

Set initial threshold scores: $t_m^0 = \underline{x}$ for all m

Set students' preferences: $\{\boldsymbol{u}_s\}_{s=1}^S$

Set students' scores in the test: $\{x_s\}_{s=1}^S$

Set the adjustment factor of the threshold $\epsilon > 0$

Set the maximum number of adjustment steps N_ϵ

while $j < N_\epsilon$ and $Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^j) \neq \emptyset$ **do**

    Obtain students' demand $\{\psi_s(\boldsymbol{u}_s, x_s, \boldsymbol{t}^j)\}_{s=1}^S$

    Compute aggregated demand in each major $\{\Psi_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^j)\}_{s=1}^S\}_{m=1}^M$

    Compute excess demand function in each major $\{z_m(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^j))\}_{m=1}^M$

    Construct the set of majors with maximal excess demand: $Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^j)$

    Select randomly a major m from $Z(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{t}^j)$

    Adjust the threshold score: $t_m^j = (1+\epsilon)t_m^{j-1}$

end while

$j \leftarrow j - 1$

for conducting several counterfactual analyses, without imposing truth-telling as an equilibrium.

We depart from a game theoretic approach, and propose a multi-agent decision approach, in which, given their cardinal utilities and admission probabilities, each student has to choose which majors to report in order to maximize their expected utility. Here, each student combines the history of past threshold scores with their own score, to obtain their conditional vector of admission probabilities. Past threshold scores are publicized by the RO, and let students to draw inferences about the preferences and priorities of other students in a robust way.

Following this approach, we calibrate students' preferences to match the aggregated assignments per major. That is, for a given vector of probabilities and cardinal utilities for each student, we solve their decision problem. Then we run the serial dictatorship mechanism and confront the simulated data with the actual assignment. If the simulated assignment is higher [lower] than the current assignment, we decrease [increase] the cardinal utility of this major in each profile of preferences. We proceed recursively until fitting exactly the data or convergence is reached.

Given that the computation is sensitive to the choice of the initial preferences, we use a "revealed preference" algorithm to recover the initial ordinal profiles. The key assumption in the procedure is the independence between scores and preferences. In addition, we impose two minimal consistency conditions to construct the profiles, namely, that students do not choose dominated options when a better option is available, and that profiles do not form cycles in preferences.

Using this procedure, we recover 205 preference profiles and fit the data with an error of approximately 6% (around 300 seats). This finding has merit, because the number of preference profiles that students can have is of the order of 10^{122}. We check the robustness of our finding for different configuration of the preferences. In all cases, the fitness to the data is significantly worse.

We consider that using a calibration in a major assignment problem is more reasonable than estimating a random utility model, which is standard in the recent empirical literature that tries to recover underlying preferences in school assignment problems. The reason is that students' cardinal preferences over majors are mostly idiosyncratic, and it is difficult to obtain data over the covariates that could affect students' decisions. Furthermore, priorities are derived from the score in a test, and are not related with the residence of the student, which does not make possible to use location as a shifter regressor to identify preferences.

Once we recover students' preferences we conduct several counterfactual analyses. First, we find that when students are allowed to report only two options, 72% percent of the students receive a different allocation from what they would have gotten in the absence of constraints. Second, we find that redistributing the wasted seats among the ten majors more demanded is better than a more concentrated policy of transferring all the seats to the most demanded major. Third, we find that a bonus scheme –where students from the target population are given a relative bonus in their admission score– is more effective. Nonetheless, it produces a higher distortion, with respect to the benchmark, in the overall population assignment.

Finally, we propose two alternative mechanisms that eliminate students' uncertainty. In the first, the RO posts upfront the threshold scores to be admitted into each major, and each student chooses the best option within their set of feasible majors. In the second, the RO uses an ascending auction. Using the preferences recovered in the matching mechanism, we show that the ascending auction produces an assignment where only 5% of the students receive a different major from the case without constraints.

Chapter 3

ON THE OPTIMALITY OF CONSTRAINED CHOICE IN COLLEGE ASSIGNMENT PROBLEMS

In many countries around the world the admission process to public universities is organized through a centralized mechanism that places students into colleges, taking into account students' preferences over colleges and their scores in an admission test. A recurrent feature in such mechanisms is to let students to report fewer options than the total of colleges available, or in other words, to constrain students' choice. Nonetheless, the intensity of the constraint varies from country to country. For instance, in Turkey and Chile students can report, respectively, up to thirty and ten options; whereas in other countries like in Costa Rica and the Netherlands they are allowed to report, respectively, as few as two and one options. Furthermore, in some cases like Hungary students can report three options but have to pay a fee starting the fourth reported option.

Despite of the burden a constrained choice imposes on students, as we will see later, it happens to be the rule rather than the exception in the design of admission mechanisms to public universities. One potential explanation for its pervasiveness could be the higher computational cost it demands. However, given the fact that universities count with modern computational resources to process all the relevant information, the marginal cost of increasing the quota bestowed to students would be negligible.

This series of observations open the question of why univerity authorities would be interested in constraining the number of options students can report. In this article we provide a rationale for this behavior, assuming that there is a planner (the Registrar Office) that has preferences over students' allocations, expressed through the innovation they can produce in their colleges assigned. Here, the innovation students produce can be conceived as the outcome of the research projects they undertake while enrolled at the university. Reasonably enough, we assume it is an increasing function of students' cognitive ability.

The reason of why a planner might care of this measure lies in the social role that public universities are meant to play in many countries. Actually, in the discussion between public universities and government authorities when bargaining about how to finance higher education institutions, it is mentioned frequently that one responsibility of public universities shall be the production of public goods to retribute to the society part of the cost incurred by taxpayers. [1] Precisely, one of such public goods is the production of the "innovation" realized through the projects that students undertake as part of their research programs. These projects produce spillovers to the rest of the economy but are normally "managed" by public universities under a no-seeking-rents policy. However, not all the innovation is equally valued by the planner. In fact, there could be strategic areas that the planner may want to incentivize because they have a high potential (e.g. nanotechnology, nuclear physics and computer science), whereas there are other (e.g. law and medicine) it may want to de-incentivize due to congestion.

The intuition of why constraining the choice may benefit the planner to maximize the expected innovation produced is as follows. Suppose students are risk averse and their preferences over colleges are private information. Moreover, assume that planner's preferences over colleges are completely misaligned with re-

[1] This *retribution* is more relevant in countries where public universities have bigger enrollment, higher quality and where students of higher income are overrepresented, and so the low-income taxpayers, subsidizes the upper tail of the income distribution.

spect to students' preferences, in the sense that the most preferred college by the students is the least preferred by the planner, the second most preferred college by students is the second least preferred by the planner, and so on. If the planner lets students to report the full list, they have an incentive to report their true preferences, which implies that in equilibrium the students with the highest scores will be seated in their most preferred colleges, but in the least preferred from the planner's perspective. If the planner constrains the choice, it introduces a higher risk, since the probability of being out of college increases. Under this scenario, students would report their less preferred, yet safer colleges, since they are risk averse. Hence, from planner's perspective, the probability to seat the best students in more preferred colleges increases with respect to the unconstrained case.

As an illustration, suppose all students prefer medicine to engineering, and in turn engineering to physics, but the planner values more the innovation produced in physics to engineering to medicine, perhaps because there is congestion in medicine and there are promising projects in physics that could boost the economic growth of the country. Thus, in the first best the planner would like to assign the best students to physics, the second set of students to engineering and the third set of students to medicine. However, it cannot be done if students are given the possibility to list all the possible colleges and priorities and preferences have to be respected in the assignment process. Nonetheless, given that students are risk averse and preferences are private, by constraining the choice the planner makes that, more often, better students choose physics over medicine to secure a seat in the university. In other words, it induces students to choose colleges that are more aligned with its preferences.

The latter mechanism seems reasonable, since public universities are crucial agents in the development of a country, and can use admission policies to induce allocations that seat the best students in strategic colleges. However, this approach differs from standard matching literature, which implicitly assumes that the agent that organizes the market is totally indifferent with respect to final assignments; and limits its role to set the “right” institutions to guarantee that the induced equilibrium allocations satisfy properties such as strategy-proofness and stability. However, this *dictum* is frequently violated, since most of the universities use a serial dictatorship mechanism, which is not strategy-proof when students are constrained in the number of options to report (c.f. Haeringer and Klijn). This feature demands from students to increase the sophistication of their strategic thinking in order to maximize the value of their final assignment.

The fact that constraining choice destroys strategy-proofness is not irrelevant. Indeed, strategy-proofness is normally considered as a desired property in any admission mechanism, to the point that sometimes designers recommend to give up other desirable properties like stability and Pareto-efficiency in order to guarantee its existence. [2] Therefore, it would be difficult to justify the use of a constrained mechanism on the light of standard matching literature.

Finally, it is important to remark that we do not pursue an optimal mechanism approach in this paper. Our endeavor is more humble, since we do not evaluate all possible admission mechanisms to determine which is the one that maximizes preferences over students’ innovation. Instead, we fix the serial dictatorship mechanism, which is ubiquitous in the centralized admission systems around the world, and try to find the conditions such that, relative to the objective of maximizing planner’s preference over innovation, giving the students the opportunity to report

[2] For instance, the admission system used in the school public system of Boston, called the Boston mechanism, was changed in favor of the Deferred Acceptance, under the idea that a strategy-proof algorithm tends to minimize the “damage” produced on decision-takers that do not strategize so well.

the full list is suboptimal.

Related Literature Our paper contributes to the literature of college-school assignment problems initiated by Balinski and Sönmez (1999), and Abdulkadiroğlu and Sönmez (2003). They introduced a mechanism design approach to formalize the admission process to public schools and public universities, in an environment where students' preferences and priorities are common knowledge. Their model has been known as the canonical school choice model, and later endeavors have been centered in validating its equilibrium properties under different set of assumptions. Nonetheless, and despite of its importance and recurrence, few papers have been interested in analyzing the implications of constraining students' choice. From those that occupy on this, Haeringer and Klijn (2009) analyze the implications for strategy-proofness and stability under the Deferred Acceptance and Boston Mechanism when preferences and priorities are public. Meanwhile, Casalmiglia, Haeringer and Klijn (2010) test the robustness of the constrained choice in the laboratory. They find that when the admission mechanism is the Deferred Acceptance, the proportion of individuals that behave rationally augments with the constraint, since the problem is more complex and students tend to think harder to understand the mechanism.

Our paper also contributes to the matching literature that considers private preferences and solve the induced Bayesian game, as in Abdulkadiroğlu, Che and Yasuda (2011) and Troyan, (2012). However, these papers assume that students coincide in the ranking of the schools, and only differ in the level of cardinal utility each college yields, whereas in our case, students differ in the way they rank colleges. We believe this is an important assumption, since in a college choice -rather than in a school choice- problem preferences are mostly idiosyncratic.

Furthermore, we assume students are risk averse expected utility maximizers.

A feature normally considered in insurance and auctions literature, but generally disregarded in the college choice literature. A related paper in this matter is Klijn *et al.* (2013) who study how constraining the choice affects equilibrium properties through the indirect effect caused by students' risk aversion. They show in the laboratory that more risk averse individuals are more likely to play a protective strategy (i.e. report a school less preferred but with a higher probability of admission) under the Deferred Acceptance than under Boston Mechanism. In our model students' incentive to play a safe strategy is precisely the key element that make useful the restriction of the number of options to the planner.

Organization of the Paper The rest of the paper is organized as follows. Section 2 illustrates the main insights of the problem at hand through a detailed example. Section 3 lays out the model. Section 4 presents the main result of the paper, under an assumption on preferences. Section 5 constructs a class of preferences that endogeneizes such assumption. Section 6 concludes.

3.1 A Motivating Example

In this section we develop a detailed example that illustrates how restricting the number of options can help the Registrar Office -RO hereafter- to maximize its preferences over students' assignments.

We assume there are four students indexed by $1, 2, 3, 4$, and four colleges named X, Y, W, Z. Each college has exactly one seat available. Students can be assigned at most one seat in each college. If the student is not assigned a seat in any college, we say he is out college, and denote this option by $\emptyset$.

Students are prioritized by the RO with respect to their score in an admission test. Without loss of generality, we assume their priority is decreasing in their index. Specifically, the vector $x = (x_1, x_2, x_3, x_4) = (0.9, 0.7, 0.5, 0.3)$ represents their score in the test.

Own preferences over colleges are known by each student, but the preferences of other students are private information. Nonetheless, it is common knowledge that they are drawn, identically and independently, from the set $\mathcal{V} = \{v' = (1, 0.9, 0.6, 0.3), v'' = (0.6, 0.9, 1, 0.3)\}$, according to the probabilities: $Pr(v = v') = 3/4$ and $Pr(v = v'') = 1/4$. Notice for example, that in v', X is preferred to Y, but in v'', Y is preferred to X. Students' utility over the college assigned c is given by a constant relative risk aversion, von Neumann-Morgenstern function $u(c, v, \rho) = v_c^{1-\rho}/(1-\rho) \triangleq u_c$. Here, ρ corresponds to students' risk aversion parameter, which is also common knowledge, and we assume is equal to 0.7.

We suppose the RO uses a serial dictatorship mechanism to determine final allocations based on the reports provided by the students. The description of the algorithm is as follows:

In step 1. The student with the highest score is considered. He is assigned a seat at the major reported in the first position.

In step 2. The student with second highest score is considered. He is assigned a seat at the major reported in the first position if there are available seats; otherwise, he is assigned a seat at the major reported in the second position.

In step ℓ ($\ell > 2$). The student with the ℓth highest score is considered. He is assigned a seat at the major reported in the highest position that has available seats.

A relevant observation is that, as the students are given fewer options to report, the uncertainty over final assignments increase.

Given the latter environment, we analyze how equilibrium assignments change when students are not allowed to report the full list, but instead are constrained in their choice. [3]

When students are entitled to report the full list, it is a well-known result in the literature that a serial dictatorship mechanism is strategy-proof (c.f. Dubins and Freeman, 1981; and Roth, 1982). Therefore, equilibrium assignments depend solely on the realization of preferences.

Denote μ a matching between students and colleges. Thus, the ex-ante distribution of equilibrium assignments when students are entitled to report four colleges, corresponds to

$$\begin{aligned} Pr(\mu^{(1)} = (X, Y, W, Z)) &= 9/16 \\ Pr(\mu^{(2)} = (X, W, Y, Z)) &= 3/16 \\ Pr(\mu^{(3)} = (W, X, Y, Z)) &= 3/16 \\ Pr(\mu^{(4)} = (W, Y, X, Z)) &= 1/16 \end{aligned} \tag{3.1}$$

First, notice that when students are allowed to report the full list, a student with priority ℓth will receive at least his ℓth preferred college in any equilibrium.

The first term in (3.1) corresponds to the case where students 1 and 2 draw the preference profile v'. By independence, this event occurs with probability 9/16. Now, because the assignment algorithm is a serial dictatorship, and all students report truthfully, student 1 gets his most preferred college, in this case college X, whereas the student 2 gets his second choice, college Y. Student 3 gets college W, independently of his preference profile; because W is strictly preferred than Z in both profiles. Likewise, the second term in (3.1) corresponds to the case where

[3] For the rest of the example we will focus in the case of weakly undominated strategies to analyze the respective equilibria of the admission game.

student 1 draws the preference profile v' and student 2 draws the profile v''. The probability of this event is 3/16. Here, since preferences of both students do not coincide, both get their best option, college X in the case of student 1, and college W in the case of student 2. It immediately secures a seat in college Y to student 3, independently of his preference profile, because Y is strictly preferred to Z. We can recover the other two terms following the same analysis.

Suppose now that the RO allows students to report only two options. Students 1 and 2 continue to report truthfully, because their priority is greater than or equal to the number of options to report, and so there will be a seat available for them in some of these colleges with certainty. Students affected by the restriction are students 3 and 4, who now do not have a trivial best response.

Denote by $\sigma_s^*(s, v_s)$ the optimal pure strategy of student s when his type is v_s, and $\eta_s(\sigma_{-s}^*) = (\eta_s(X, \sigma_{-s}^*), \cdots, \eta_s(Z, \sigma_{-s}^*))$ as his vector of admission probabilities; given that other students are following the profile of equilibrium strategies σ_{-s}^*.

Using the information over the distribution of preferences for students 1 and 2, we have that

$$\eta_3(\sigma_{-3}^*) = (1/16, 3/8, 9/16, 1)$$

Hence, we can convert the problem of finding the optimal response of student 3 into an individual decision problem in which, given his preferences and vector of admission probabilities, he finds the combination of colleges that maximizes his expected utility. In this case,

$$\sigma_3^*(3, v') = (Y, Z); \quad \sigma_3^*(3, v'') = (W, Z)$$

That is, student 3 protects himself incorporating college Z in their "optimal portfolio" when he has only two options to report. Recall that under the full list case, college Z was never reported among the first two options by student 3.

Taking into account the best response of student 3, it is possible to proceed in the similar fashion for student 4. His corresponding vector of admission probabilities would be

$$\eta_4(\sigma^*_{-4}) = (1/16, 6/64, 27/64, 27/64)$$

Notice that since student 3 is now moving college Z up in his report, the probability of student 4 to get admission there reduces to 27/64.

Now, his optimal response corresponds to

$$\sigma^*_4(4, v') = \sigma^*_4(4, v'') = (W, Z)$$

which implies he reports the two colleges with the highest probability of admission.

In summary, the ex-ante distribution over equilibrium outcomes when students can report only two options is given by

$$\begin{aligned}
Pr(\mu^{(1)} = (X, Y, W, Z)) &= 9/64 \qquad (3.2)\\
Pr(\mu^{(2)} = (X, Y, Z, W)) &= 27/64\\
Pr(\mu^{(3)} = (X, W, Y, Z)) &= 3/64\\
Pr(\mu^{(4)} = (X, W, Z, \emptyset)) &= 9/64\\
Pr(\mu^{(5)} = (W, X, Y, Z)) &= 9/64\\
Pr(\mu^{(6)} = (W, X, Z, \emptyset)) &= 3/64\\
Pr(\mu^{(7)} = (W, Y, Z, \emptyset)) &= 1/16
\end{aligned}$$

Observe that under the restriction, students 1 and 2 will be given the same ex-ante probability to receive colleges X, Y and W as in the full list case. However, student 3, who is never assigned to college Z under the full list case, has now a probability of 5/8 of being assigned there under the constrained choice. Likewise, student 4, who always –and only– gets a seat in college Z under the full list case, has now a probability of 27/64 of being assigned college W, and a probability 1/4 of being out of college.

Suppose further that the RO has preferences over the assignment of students. In particular, suppose it would like to seat the best student in college Z, the second best in college W, and so on, until seating the worst student in college X. Hence, *ceteris paribus*, the RO would be better off if it can switch assignments of two students, so that a better student get a seat in a more preferred college from its perspective.

In fact, by comparing the distribution of matchings under both scenarios, we can see that $\mu^{(1)}$ and $\mu^{(2)}$ in (3.2) corresponds to $\mu^{(1)}$ in (3.1). However, in the second matching of the constrained case, students 3 and 4 switch positions, placing the student with higher score in a more preferred college by the RO. Likewise, matchings $\mu^{(3)}$ and $\mu^{(4)}$ in (3.2) correspond to $\mu^{(2)}$ in (3.1). Here, in the second matching, student 3 is switched to college Z but at the expense of wasting a seat in college Y. The analysis is the same for the rest of the assignments.

Notice that, if the differences in the RO's cardinal preferences over the assignment of students are sufficiently big, the RO would be better off by restricting students' choice even at the expense of wasting a seat in one of the colleges.

Although a planner would rarely have preferences for students assignments *per se*, it is reasonable to assume that it has preferences over an indirect measure. Here, we consider the RO has the objective to maximize the aggregated innovation a student produce when he is seated in a given college c. Specifically, we assume that a student s assigned to college c will produce $\xi_c x_s^\alpha$ units of innovation for the planner, where $\xi = (\xi_X, \xi_Y, \xi_W, \xi_Z) = (0.2, 0.6, 1, 1.8)$ represents the RO's preferences. The parameter α is set equal to 0.5. As introduced before, the way the RO ranks the colleges is inversely related with the most probable preference type in the support of students' preferences.

Using the information provided by (3.1) and (3.2), and given the functional form of the innovation measure, it is possible to see that the RO would be better off by constraining the choice to two options. In this case, the expected aggregated innovation would be 2.47, which is higher than 2.28, the one obtained in the full list case. In fact, it can be shown that allowing students to report two options, is the level of the restriction that maximizes the RO's objective.

3.2 The Model

Students, Colleges. A student-college assignment problem consists of a finite set of colleges $\mathcal{C} = \{1, 2, \cdots, M\}$, each with one seat available, and a finite set of students, $\mathcal{S} = \{1, 2, \cdots, N\}$. We assume that $M \leq N$, and thus that, at most, there are many slots to accommodate exactly all students into a college. The outside option of students who do not attend college is denoted by $\emptyset$.

Scores. Students are prioritized by the score they obtained in a standardized test. The vector x denotes the list of scores obtained by students in such admission test, which belongs to the set $\mathcal{X} \triangleq \{x = (x_1, \cdots, x_N) \in \mathbb{R}^N_{++} : x_s \neq x_\ell \text{ for all } s \neq \ell\}$. That is, any profile of scores induces a unique priority order in the admission process. We assume scores are *public information* to all agents. Moreover, without loss of generality, we sort students in decreasing order with respect to their score. Thus, $x_s > x_\ell$ if and only if $s < \ell$. Under this convention, the index of each student determines his priority, and so a student s has a priority *higher than or* *equal* to K if $s \leq K$. [4]

Students' Preferences. Students' preferences over colleges in $\mathcal{C}$ are represented by a vector of cardinal utilities, v_s, that belongs to set $\mathcal{V} \triangleq \{v' = ({v_1}', \cdots {v_M}'), v'' = ({v_1}'', \cdots {v_M}'') \in [\underline{v}, \overline{v}]^M : v^c \neq v^\ell \text{ for all } c \neq \ell\}$. We assume that the utility of the outside option, $v^\emptyset$, is equal to zero for all students. We write $o^\ell(v)$ to denote the ℓth preferred college, according to v. Without loss of generality, we assume that $o_c(v') = c$ for all $c \in \mathcal{C}$. In that sense, under the profile v', c has a higher cardinal utility than $\tilde{c}$, if and only if $c < \tilde{c}$.

Different colleges are ranked according to the utility function $u(\cdot, \rho) : \mathcal{C} \times \mathcal{V} \to \mathbb{R}$, where ρ represents student's uniform level of risk aversion. A profile of preferences is denoted by $V = (v_1, \cdots, v_N) \in \mathcal{V}^N$. Likewise, for all $V \in \mathcal{V}^N$, $V_{-s} \equiv (v_1, \cdots, v_{s-1}, v_{s+1}, \cdots, v_N)$ denotes the projection of V over $\mathcal{V}^{\mathcal{S}\setminus\{s\}}$. We assume that each student's preference v_s is private information, and drawn identically and independently with probabilities $Pr_s(v') = p \in (0, 1)$ and $Pr_s(v'') = 1 - p$. Hence, $Pr(V) = \prod_{s=1}^N Pr_s(v_s)$ denotes the probability of drawing a profile of preferences V.

Planner's Preferences. Finally, the RO has cardinal preferences over students assignments given by $\xi = (\xi_1, \cdots, \xi_M)$, which satisfies two properties: (i) $\underline{\xi} < \xi_1 < \xi_2, \cdots, \xi_{M-1} < \xi_M < \overline{\xi}$ and (ii) $\xi_{c+1} - \xi_c$ increasing for $2 \leq c \leq M$. That is, RO's preferences are completely misaligned with students preference profile v', and display strict increasing differences.

Assignment Mechanism. Let μ denote a *matching*, namely a rule that assigns each student to at most one college, and such that the number of students assigned to

[4]Notice that as the index of the student becomes higher, his priority decreases.

a given college is less than or equal to the numbers of seats available. Denoting $\mathcal{M}$ as the space of all matchings; an assignment mechanism is then a function $\psi : (\mathcal{R})^N \to \mathcal{M}$, where $\mathcal{R}$ represents the space of all rank order lists over $\mathcal{C}$.

A mechanism ψ^K is called K-constrained mechanism if its domain is the set of all ranked ordered lists over $\mathcal{C}$ truncated to the first $K < M$ positions. In this case, we have $\psi^K : (\mathcal{R}^K)^N \to \mathcal{M}$. We consider constrained mechanisms to reflect the fact that normally students are not allowed to submit a full list, but instead are given limited options to report.

Every mechanism ψ^K induces a game of incomplete information in which the students report a rank order list of length K, and given this profile of reports, assignments are given according to ψ^K. Here, the mechanism ψ^k is a constrained serial dictatorship.

Timing. Figure 3.1 depicts the timeline of the game. First, nature draws preferences and scores for all students. Because scores are public information, each student can determine his priority in the assignment process. Preferences, on the other hand, are private information, and hence, students form beliefs over them according to the common prior. Second, the RO announces the number of options students are allowed to report, K. Once students observe this parameter, they sumbit the rank order list of preferences -of length K- to the RO. Finally, the RO runs a serial dictatorship algorithm and assignments and payoffs are realized.

3.2.1 Students Equilibrium Behavior

Given the environment introduced above, a pure strategy σ_s for each student is a mapping $\sigma_s : \mathcal{S} \times \mathcal{V} \to \mathcal{R}^K$. That is, given his index s and type preference v_s, a student chooses to report a rank order list to the RO. For simplicity, we denote

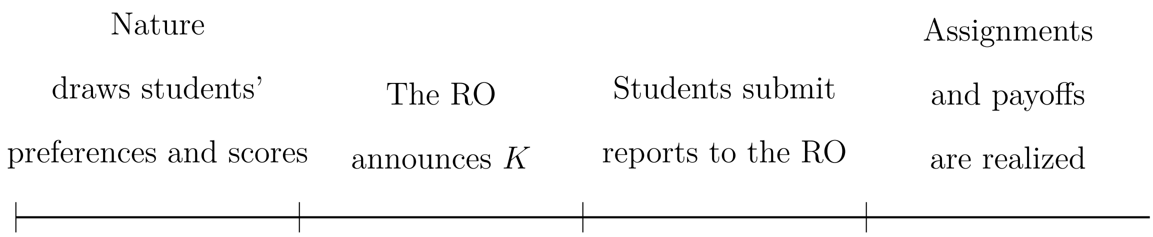


Figure 3.1: Timing of the admission game

$\sigma_s(s, u_s) = r_s$. Hence, a profile of reports is denoted by $R = (r_1, \cdots, r_N)$.

When the profile of pure strategies (σ_s, σ_{-s}) is played, the expected utility of student s under the mechanism ψ^k, corresponds to:

$$EU_s(\sigma_s, \sigma_{-s}) = \sum_{v \in \mathcal{V}^S} u(\psi_s^K(R), v_s, \rho) Pr(V) \tag{3.3}$$

A student with type v_s reports truthfully if $r_s^\ell = o^\ell(v_s)$ for any $1 \leq \ell \leq K$. [5] We say that a mechanism is strategy proof if truthfully reporting is a weakly dominant strategy for every student s and every type $v_s \in \mathcal{V}$.

When students are not allowed to report a complete order over $\mathcal{C}$ the serial dictatorship mechanism is not strategy proof (*c.f.* Haeringer and Klijn (2009)). Hence, it is not possible to use a dominant strategy equilibrium as a solution concept. Instead, we rely on the Bayesian Nash equilibrium in undominated strategies to solve the game.

Although there is no equilibrium in dominant strategies, the hierarchical nature of the admission decision process has the characteristic that, students with priority higher than K have always a weakly dominant strategy, as presented in the following proposition.

[5] Analogously with true preferences, for any pure strategy σ, we denote by r^ℓ the college reported in the ℓ-th position by a student with type v_s.

Proposition 3. *Fix $1 \leq K < M$. For any student $s \leq K$ truthfully reporting is a weakly dominant strategy.*

Proof. Fix a student $s \leq K$ and type v_s. First, notice that for any belief on the strategy profile of other students, reporting the first K preferred colleges constitutes a weakly dominant strategy. We have to show that for any non-truthful strategy $\tilde{\sigma}_s$ there exists a full support belief such that report according to $\tilde{\sigma}_s$ is dominated.

Fix an index $\ell \leq s \leq K$, and suppose that the preference type of all students with higher priority than s rank the first $\ell - 1$ colleges in the same way as student s, an event that has strict positive probability. Moreover, assume that such students report truthfully their preferences.

Now, suppose student s reports according to the pure strategy $\tilde{\sigma}_s(s, v_s) = \tilde{r}_s$, such that:

$$\tilde{r}_s^j = \begin{cases} o_j(v_s) & \forall\, j < \ell \\ o_h(v_s) & \text{for } j = \ell \text{ and some } h > \ell \end{cases}$$

Under this strategy, student s will be assigned to his hth preferred college, whereas if he reports truthfully he would be assigned to his ℓth preferred college. ■

Proposition 3 establishes that students with priority higher than or equal to K will always report truthfully. This results simplifies substantially the analysis of equilibrium, since for any value of the restriction K it is only necessary to inspect the behavior of the students with lower priority than K: those without a trivial best response.

We need to first find the best response of student $s = K + 1$. Toward this aim, let $\delta : \mathcal{S} \times (\mathcal{R}^K)^{N-1} \rightarrow \{0, 1\}^M$ be an operator such that $\delta(s, R_{-s})$ returns a vector with an entry of one in the ℓ-th position if college ℓ has an available seat for student s, when his fellow students report R_{-s}, and zero otherwise.

Thus, we can compute the vector of ex-ante admission probabilities for student s as follows

$$\eta_s(\sigma_{-s}) = \sum_{v_{-s}\in\mathcal{V}^{S-1}} \delta(s, R_{-s})Pr_{-s}(V_{-s}) \tag{3.4}$$

Hence, $\eta_s(c, \sigma_{-s})$ denotes the probability that student s gets admission into college c, when the profile of strategies σ_{-s} is played. Then, for any two students s and $\tilde{s}$ such that $s < \tilde{s}$, $\eta_s(c, \sigma_{-s}) \geq \eta_{\tilde{s}}(c, \sigma_{-\tilde{s}})$, for all $c \in \mathcal{C}$ and for any profile σ.

Even though for presentation purposes we consider the profile of strategies played by all fellows of student s, the reports of individuals with lower priority than him are irrelevant to calculate his chances to get admission in any college.

Let $q_s^{\ell-1}(r_s, \sigma_{-s})$ be the probability that the student s have been rejected by the first $\ell - 1$ colleges reported in r_s, given that the rest of the students are playing according to σ_{-s}. Such probability is computed as follows [6]

$$q_s^{\ell-1}(r_s, \sigma_{-s}) = \begin{cases} \prod_{j=1}^{\ell-1} 1 - \eta_s(r_s^j, \sigma_{-s}) & \text{for } \ell = 2, \cdots, K \\ 1 & \text{for } \ell = 1 \end{cases}$$

Once we derive the vector of admission probabilities, we can convert the problem of finding student s's optimal strategy into an individual decision problem where he solves an expected utility maximization problem. Specifically, the strategy σ_s is a best response to the profile σ_{-s}, if $\sigma_s(s, v_s)$ solves

$$\max_{r\in\mathcal{R}^K} \sum_{\ell=1}^{K} q_s^{\ell-1}(r_s, \sigma_{-s})\eta_s(r_s^\ell, \sigma_{-s})u(r_s^\ell, v_s, \rho) \tag{3.5}$$

[6] For instance, recall from the example in section 3.1 that when students are allowed to report only two options and they follow the equilibrium strategy prescribed there, student 3 has the following admission probabilities

$$\eta_3(\sigma_{-3}^*) = (1/16, 3/8, 9/16, 1)$$

Thus, if he reports the tuple (X, W) to the RO, his expected utility would be $(1/16)(9/16)u_3^X + (1-1/16)u_3^W$. In this case, since $K = 2$, the rejection function is only defined for the first option: $q^1((X, W), \sigma_{-3}^*) = (1 - 1/16)$.

We can proceed inductively, in the same fashion, to recover the best response of all succeeding students.

Once students find their optimal rank order list, they report it to the RO, who runs the algorithm and assign students to colleges. How the planner values such allocation is the matter of the next section.

3.2.2 Planer's Maximization Problem

As highlighted in the introduction and the example, one of the differences of the present paper with respect to standard matching literature, is that the planner (in this case the RO) evaluates students' assignments with respect to its preferences over the innovation produced by admitted students. Therefore, it is interested in designing an admission mechanism to maximize the value of the innovation produced. Formally, we assume that a student with score x_s who is assigned to college c produces $\xi_c \phi(x_s)$ units of innovation, where ξ_c is a parameter that denotes RO's preferences over colleges, or in other words, how the RO values the innovation produced in each of the colleges; and $\phi(x_s)$ represents the productivity of student s. [7] We assume the function $\phi(x_s)$ is strictly increasing.

Hence, fixing K, and assuming that students will report according to the equilibrium strategies $\sigma^* = (\sigma_1^*, \cdots \sigma_N^*)$, planner's expected innovation corresponds to

$$I(K) = \sum_{s=1}^{N} \sum_{v \in \mathcal{V}^N} \xi_{\psi_s^K(R^*)} \phi(x_s) Pr(V) \tag{3.6}$$

The main result of the paper, outlined in the next section, shows that the

[7] A planner might prefer to bring more capable students to certain colleges that it deems as crucial for the development of the economy, even though they are not the most preferred by the student body. In the definition of innovation we implicitly assume that a student with score x_s produces the same amount of innovation in all colleges. We could make it dependent on the specific college, but to maintain tractability we assume homogeneity.

optimal level of K is strictly lower than the total number of colleges M.

3.3 Optimal Constrained Choice

In this section we will show that under some conditions on preferences, distributions and students' risk aversion, letting students to report fewer options than the total of colleges available (i.e. constraining the choice) is optimal for the planner.

First, we will identify the set of values K that induce a different equilibrium assignment from the full list case. Then, restricting K to this set, we will show that if students are sufficiently risk averse, they will choose the K colleges with highest probability of admission, but will order them with respect to their true preference profile. Finally, we will show that if the differences in the intensity of planner's preferences are decreasing and sufficiently high, the higher probability to seat a better student in a more preferred college more than compensates the small probability of wasting a seat.

The first step consists in determining the number of options K that would induce a different allocation from the full list case. The relevant values to consider are those that effectively imposes a binding constraint for at least one student with priority lower than K.

Let $\Gamma_s = \{c \in \mathcal{C} : \eta_s(c, \sigma^*_{-s}) > 0\}$ be the set of colleges with positive admission probability for student s under the profile of equilibrium strategies σ^*.

Lemma 1. *The values of the restriction that potentially induce a different equilibrium allocation with respect to the full list case are those $K \leq \overline{K}$, where $\overline{K} = \max\{K :$ there exists $s > K$ with $|\Gamma_s| > K\}$.*

Proof. Suppose that the planner sets a value of K such that $|\Gamma_s| < K$ for all students with priority lower than K (i.e. all students $s > K$). Then, best strategy for those students with lower priority is to report all the colleges with positive probability according to his true preference profile, whereas the students with priority higher than K will report truthfully. Hence, there is no change in the equilibrium allocation with respect to the full-list case, which is a contradiction. ■

Lemma 1 says that setting the value of K close to the number of colleges available M, is not useful for the planner, because he will be "imposing a restriction" on students who do not have enough options with positive probability, and therefore, it will not have any effect on their decision problem.

Once the planner sets the restriction according to lemma 1, students have to choose the optimal portfolio of colleges given the collection of probabilities implied by the equilibrium played.

Proposition 4. *Fix $K \leq \bar{K}$. If students with priority lower than K are sufficiently risk averse, their best response corresponds to the following pure strategy: Choose the K colleges with the highest probability of admission, and report them truthfully using their true preference profile.*

Proof. Given the vector of admission probabilities and cardinal utilities, each student solves his correspondent optimization problem stated in equation (3.5), since for students with a priority lower than K reporting truthfully is not a weakly dominant strategy.

By theorem 1 in Chade and Smith (2006), we can proceed sequentially to find the solution of each student's combinatorial problem. Specifically, in the first stage we choose the option that yields the largest expected utility, and then in step ℓ, $\ell \geq 2$, we select the option that yields the largest marginal benefit over the portfolio of colleges constructed so far. [8]

The goal is to find a sufficiently high level of the risk aversion parameter, that leads student s to choose the portfolio consisting of the colleges with the K highest probabilities of admission.

Pick a student s with priority lower than K (i.e. $s > K$) and let Γ_s^K be the set of his K colleges with the highest probability of admission. Denote $r_s^1(v_s') = \arg\max\{u(c, v_s', \rho)\cdot\eta_s(c, \sigma_{-s}^*), c \in \mathcal{C}\}$ It can be shown that there exists a sufficiently high level of risk aversion $\rho_s^1(v_s')$, such that $r_s^1(v_s') \in \Gamma_s^K$. In the same fashion define recursively $r_s^j(v_s') = \arg\max\{u(c, v_s', \rho)\cdot\eta_s(c, \sigma_{-s}^*), c \in \mathcal{C}\backslash\{r_s^1(v_s'), \cdots r_s^{j-1}(v_s')\}\}$ for $j = 2, \cdots, K$. As before, we can find a sufficiently high level of risk aversion $\rho_s^j(v_s')$ such that $r_s^j(v_s') \in \Gamma_s^K$. Thus, there would be a collection of risk aversion parameters such that the optimal portfolio consists on the K colleges with higher probability of admission. Then, take $\rho_s(v_s') = \max\{\rho_s^1(v_s'), \cdots, \rho_s^K(v_s')\}$. We can proceed analogously for the profile v_s'' to find $\rho_s(v_s'')$. Finally, take $\rho_s = \max\{\rho_s', \rho_s''\}$. Hence, if $\rho > \max_{s>K}\{\rho_s\}$ the optimal strategy for those students where the constraint binds, is the prescribed in the proposition. [9]

■

For the moment we will assume that preferences v', v'' and the probability p of drawing the profile v', are such that, for the marginal student $s = K + 1$ the minimum probability of being admitted to any of the K least preferred colleges under v', is higher than the probability of being admitted into any of the other colleges. In the next section we will construct a class of preferences that satisfies this endogenous property. Recall that the profile v' is completely misaligned with

[8]Chade and Smith call this procedure the Marginal Improvement Algorithm (MIA) and show that it reach the same global solution as the original combinatorial problem in a quadratic number of steps.

[9]Later on we will assume that the preference profile v' is the most popular profile, and so, computing the risk aversion level $\rho_s(v_s')$ would suffice.

planner's preferences ξ, and so this condition says that it is safer for the marginal student to apply to the colleges that are more preferred by the RO.

Denote Λ^ℓ the set of the $1 \leq \ell \leq M$ least preferred colleges under the profile v'.

Assumption 1. *Let $s = k+1$ be the marginal student. We assume that* $\min\{\eta_s(c, \sigma^*_{-s}), c \in \Lambda^K\} \geq \max\{\eta_s(c, \sigma^*_{-s}), c \in \mathcal{C} \backslash \Lambda^K\}$.

Assumption 1 merely says that colleges preferred by the RO have higher probabilities of admission. In the same line, it is possible to define which report of majors is more aligned with RO's preferences. Recall that for any two colleges c and $\tilde{c}$ such that $c > \tilde{c}$, we have $\xi_c > \xi_{\tilde{c}}$. In other words, the RO prefers colleges with higher index.

Let $\underline{r}_s \triangleq \min\{r_s^\ell : 1 \leq \ell \leq K\}$ and $\overline{r}_s \triangleq \max\{r_s^\ell : 1 \leq \ell \leq K\}$. That is, $\underline{r}_s$ is the college with lowest index chosen by r_s, and likewise for $\overline{r}_s$.

Definition 5. *The report r_s is more aligned with planner's preferences than $\tilde{r}_s$ (viz. $r_s \succeq \tilde{r}_s$), if $\underline{\tilde{r}}_s \leq \underline{r}_s$ and $\overline{\tilde{r}}_s \leq \underline{r}_s$, with at least one strict inequality.*

Now, we have all the necessary pieces to set our main result: constraining the choice is optimal for the Registrar Office.

Theorem 2. *Suppose that assumption 1 is satisfied. Then, if students are sufficiently risk averse, there exist planner preferences ξ, such that $I(K) > I(M)$ for some $K \leq \bar{K}$.*

Proof. Denote $\Omega = \{s > K : |\Gamma_s| > K\}$. That is, Ω denotes the set of students with priority lower than K, who have more than K colleges with positive probability of admission, and thus affected by setting the number of options in K instead of M. This is the relevant set of students where the restriction induces a different assignment equilibrium.

Furthemore, there exists a subset $\hat{\Omega} \subseteq \Omega$, such that for all $s, \tilde{s} \in \hat{\Omega}$, with $s > \tilde{s}$, we have that $r_s \succeq r_{\tilde{s}}$. Indeed, by proposition 4 and assumption 1 student $s = K+1$ will report the K least preferred colleges under preference v', which represents the maximum level of alignment with planner's preferences. Now consider, student $s = K + 2$. Suppose by contradiction that $r_{K+2} \succ r_{K+1}$. Then, by assumption 1 and the fact that $\eta_s(c, \sigma^*) \geq \eta_{s+1}(c, \sigma^*)$ for all c, it would imply that student $s = K + 1$ is not choosing the K colleges with highest probability of admission, which is a contradiction. Therefore, the best students in $\hat{\Omega}$ report colleges more aligned with the planner.

For all students $s \in \Omega \backslash \hat{\Omega}$ it is possible to see that, in the worst case scenario for the RO, they will report according to the full list case.

Therefore, since planner's preferences satisfy increasing differences, and admission probabilities are decreasing in Ω, there exists a preference profile ξ such that the gains of seating better students in a more preferred college, more than compensates the potential loss of wasting a seat. ■

3.4 Justifying Monotnocity in Probabilities

In this section we will show a class of preferences for which assumption 1 is plausible. It is a class where the profile v' (the one misaligned with planner's preferences) is pivotal, and the profile v'' just rearranges the order of v' by segments. The intuition behind this construction resides in the fact that normally students agree on which groups of colleges are preferred to others, but may disagree in the way the rank colleges within each group. Such description is captured by the following assumption.

Assumption 2. *The profile v'' is obtained from v' as follows*

$$o^c(v'') = \begin{cases} \pi_1(c) & \textit{if} \quad c \in [1, z] \\ \pi_2(c) & \textit{if} \quad c \in [z+1, 2z] \\ & \vdots \\ \pi_n(c) & \textit{if} \quad c \in [(n-1)z+1, nz] \\ & \vdots \end{cases} \tag{3.7}$$

where z is the length of the each partition, and $\{\pi_1, \pi_2, \cdots, \pi_n, \cdots\}$ is a collection of permutations defined over the relevant domains. [10]

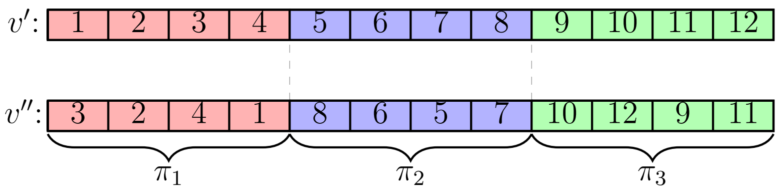


Figure 3.2: An example for the construction of preferences.

Here, there are 12 colleges divided in three partitions of three colleges each, and ordered increasingly under the pivotal profile v'. The alternative profile v'' is constructed from v' as a permutation in each partition. Specifically, both profiles agree in that colleges $1-4$ are better than colleges $5-8$, and in turn that these are better than colleges $9-12$. However, within the first partition the profile v'' ranks college 3 as the best and college 1 as the worst. Likewise, in the second parition, college 8 is the worst under v' but the best under v''.

That is, v'' is a piecewise permutation of v'. As mentioned before, the intervals of colleges can be interpreted as "academic areas" and the elements within that interval as colleges in such area. Hence, we assume that both preference profiles

[10] We assume that the number of partitions is greater than two, and that at least one permutation is different from the identity, such that the new order $o(v'')$ is different from $o(v')$.

rank the academic areas in the same way, but within each area the order of the colleges can vary. Furthermore, we assume that the cardinal utility associated with each position in the preference relation is fixed but the colleges that occupy each position can vary. For instance, in both profiles the cardinal utility of the third most preferred college is the same, but with respect to the profile v' such college is college 3, while with respect to the profile v'' it is the college given by $o^{\pi^{-1}(3)}(v')$.

This assumption is essential to guarantee that the least preferred colleges with respect to v' have always a positive probability of admission for the marginal student affected by the restriction.

Given that for students with priority higher than or equal to K, reporting truthfully is a weakly dominant strategy, the admission probabilities for all students with priority higher than K only depend on the distribution of preferences. Denoting $M(c, v')$ be the number of colleges preferred to college c with respect to v', and analogously for $M(c, v'')$; admission probabilities can be computed as shown in the following lemma.

Lemma 2. *For any student $s \leq k+1$, if $M(c, v') + M(c, v'') \geq s - 1$ the ex-ante probability that such student s get into college c, corresponds to*

$$\eta_s(c, \sigma^*{}_{-s}) = \begin{cases} \sum_{j=\max\{0, s-1-M(c,v'')\}}^{\min\{s-1, M(c,v')\}} \binom{s-1}{j} p^j (1-p)^{s-1-j} & \text{if} \quad M(c, v') \geq M(c, v'') \\ \sum_{j=\max\{0, s-1-M(c,v')\}}^{\min\{s-1, M(c,v'')\}} \binom{s-1}{j} p^{s-1-j} (1-p)^j & \text{if} \quad M(c, v') < M(c, v'') \end{cases} \tag{3.8}$$

otherwise if it is equal to zero.

Proof. By the result of proposition 3 any student with priority higher than or equal to K will report truthfully. Moreover, since the acceptance algorithm is a serial dictatorship, at least one student will be assigned to the college c if $M(c,v') + M(c,v'') < s-1$.

Now, if $M(c,v')+M(c,v'') \geq s-1$ it is possible to choose the preference profiles of all predecessors of student s in such a way that -taking into account they will report truthfully- leaves a seat open in college c. Clearly, we have to choose at most $M(c,v')$ students to assign them the profile v' and at most $M(c,v'')$ to assign them the profile v''. Moreover, if $M(c,v') \geq M(c,v'')$, we have to assign the profile v' to at least $s-1-M(c,v'')$ students whenever this number is positive. Since the profile v' occurs with probability p, the probability that student s gets a seat in college c is given by

$$\sum_{j=\max\{0,s-1-M(c,v'')\}}^{\min\{s-1,M(c,v'')\}} \binom{s-1}{j} p^j (1-p)^{s-1-j}$$

Likewise if $M(c,v') < M(c,v'')$. ■

Assuming this chracterization of preferences, notice that if $K \in [(n-1)z+1, nz]$ (i.e if K is in the nth partition), then for the marginal student the probability of getting admission into any college $c > nz$ is equal to one. It follows because no predecessor will be seated in any of those colleges, given their equilibrium strategies. If there are more than K of such colleges, or in other words if $M-nz \geq K$, the condition in 1 is satisfied.

Now, suppose that $M - nz < K$. In this case the result cannot be guaranteed from the sole configuration of preferences, however if $p > \bar{p}$ and $z \leq \bar{z}$ for suitably chosen $\bar{p}$ and $\bar{z}$, the result is guaranteed. That is, if the degree in the variability of the profile v'' with respect to the pivotal profile v' -capture by the length of each partition- is sufficiently low; and the probability of the pivotal profile is sufficiently high, the assumption 1 can be guaranteed, and so the argument in section 3.3.

3.5 Concluding Remarks

In centralized admission mechanisms, where a registrar office assigns seats in colleges to students -given preferences and priorities, a customary practice is to constrain the number of options students are able to report. That is, students are not authorized to report a full list over the set of available colleges. It is well known in the literature, however, that such practice distort many of the desirable properties of the mechanisms utilized, such as strategy-proofness, and yet the computational cost of letting students report the full list is negligible.

The present article reconciles this discrepancy by assuming that there is a planner that has preferences over the innovation students can produce in their assigned colleges. Innovation is a simple measure captured by an increasing function of students' cognitive ability, which is approximated in the model by students' score in the admission test. The crucial factor is that the planner does not value equally the innovation produced in the different colleges, and so is in their benefit to seat the best students in its preferred colleges.

Restricting the number of options to report serves the purpose of maximizing the expected aggregated innovation under two fundamental assumptions: (i) students are sufficiently risk averse, and (ii) planner preferences over colleges are misaligned with students' preferences. These two assumptions combined, assures that students will apply to the colleges with the higher probability of admission, which happen to be the least preferred from students' perspective, but the most preferred for the planner.

One important caveat is that a higher restriction also increases the probability of wasting a seat, however, if planner's preferences exhibit increasing differences, the gain of seating a better student in a more preferred college more than offset this potential loss.

Now, as suggested above, our main result is driven by the fact that admssion probabilities have to be higher for the less popular colleges among students, but the most preferred by the planner. This condition is sufficient to generate our result, however, it is not clear which assumptions over the model's primitives could generate it. To correct this weakness, we construct a class of preferences that generates this property. It is a class with two preference profiles, where one is pivotal, and the other is constructed from it by permutting the ranking of the colleges within intervals. The argument is that in both profiles students agree over which academic areas are preferred, but disagree in the way they rank colleges within each area.

Under the latter constuction we show that if the pivotal profile is sufficiently high, and the planner preferences are completely misaligned with respect to the pivotal (more popular) profile, letting students to report the full list is dominated with respect to the objective of maximizing the expected aggregated innovation.

Chapter 4

BIDDING WITH SECURITIES IN PROJECTS UNDER NEGATIVE EXTERNALITIES (WITH ANDRES FIORITI)

Over the last two decades, the sector of technological firms have witnessed a flourishment without precedence, boosted among others, by the presence of internet and a robust market of patents. The role of this market has been twofold. From one hand, it has allowed companies to monetize their inventions by auctioning them to a pool of interested firms, but at the same time has permitted the same companies to acquire patents to develop their own products. Such environment has made possible for start-up companies -unlike in any other market- to evolve into strong competitors, with a large market capitalization, in short time. Remarkable examples include Uber -which reached a capitalization of $41 billion in less than six years, the fastest in history- WhatsApp and Spnapchat. [1]

Therefore, if a competitor acquires “the right” portfolio of patents, an operating firm in a specific niche might promptly see its market share reduced, because it would enable the competitor to develop its own innovation. For this reason, many large firms acquire patents as a protective strategy: to preclude the development of nascent companies that may change the *status quo* of its market participation. Examples here include Facebook, Yahoo and Microsoft. [2] In addition, Hall and Ziedonis (2001) find that after 1982, the US semiconductor firms started *patent portfolio races*, not to appropriate R&D revenue, but to prevent other firms from

[1] For more details see http://www.wsj.com/articles/uber and http://www.wsj.com/articles/snapchat.

[2] Recently Facebook acquired a portfolio of 750 patents to defend itself from a lawsuit from Yahoo and other companies. See http://techcrunch.com/2012/03/23/facebook

getting these patents.

This scenario raises many interesting questions. First, if a start-up is selling its project -or innovation- through a standard second price auction and wants to maximize revenue, we could ask what the optimal method of payment is. Should the seller conduct the auction in cash, or should he use a security, contingent on project's return? This dichotomy has relevance, because if the innovation is allotted to a firm that intends not to implement the project, the seller would receive a payoff of zero if he uses a security. On the other hand, if the project only has value for the winner when he implements it, the seller might be better off using a contingent payment as it is more sensitive to bidder's true valuation (c.f. DeMarzo, Kremer and Skrzypacz, 2005).

A related question is how bidders' optimal strategies behave under the presence of a negative externality, given the method of payment. Here, the key observation is that the presence of a negative externality increases the eagerness of the bidders to win the auction, even when they attach a very low valuation to the project.

To answer these questions, we build a model where the seller sells the rights of a project through a standard second price auction, but where he can utilize two hybrids as methods of payment: (i) a fixed-equity hybrid where the seller fixes the fraction of equity requested, and let bidders compete in cash, and (ii) a fixed-cash hybrid, where the seller fixes the amount of cash the winner has to pay, and let bidders compete in equity. Notice that the former embeds pure cash whereas the latter embeds pure equity.

The reason for which including a fixed payment in the instruments may be beneficial for the seller, resides in the problem of adverse selection associated with the incentives of a buyer to participate in the auction. Specifically, a buyer may want to participate in the auction either to try to implement the project (because it is profitable to do so), or just to attempt to block the allocation of his rival. If the seller is paid upon the implementation of the project, allocating it to a buyer of the second class (i.e. "the bad type") would be detrimental for his revenue. In the absence of a fixed payment, the bad type have always an incentive to participate in the auction to try to destroy the equilibrium where the project is implemented. Thus, the fixed payment acts as a screening device among bidders. However, the seller faces a clear trade-off in his aim, because introducing a fixed payment decreases the profitability of the project for all buyers, which in turn leads to a lower probability of implementation. The goal of the present article is to determine the optimal fixed payment for both hybrids, and rank them with respect to seller's expected revenue.

Certainly, our article is not the first interested in exploring the relation between revenue and the method of payment used in an auction. In fact, De Marzo, Kremer and Skrzypacz (2005) has shown that if there are no externalities, the methods of payment can be ranked in revenue by their "steepness", or the sensitivity of bidder's true type to the instrument utilized. An insight first hinted by Hansen (1985) and Riley (1988). They also show that the auction format has only an impact on revenue by its ability of modifying the steepness of the particular instrument utilized. Nonetheless, to arrive to their conclusions it is crucial that bidders operate in an environment free of negative externalities. When we incorporate them into the model, their main result does not hold anymore, precisely because a winner of the auction may acquire the project not to implement it.

In order to isolate the effect produced by the interaction of the externalities with the method of payment, we focus on a simple model of two bidders, where the loser of the auction suffers a commonly known negative externality if the winner implements the project. [3] This framework arises naturally in industries where bidding firms are similar ex-ante, and the project gives a comparative advantage in the downstream market to the winner. Surprisingly, many of the insights can be captured with this simple version. First, we consider a simple model where both, externalities and valuations are public information. Even in this simple framework the characterization of equilibria is not trivial, because it depends on the interaction of the externality, the cost of the project and bidder's own valuation.

Our main result, stated in theorem 3, shows that under some mild technical conditions, the following is satisfied. First, the optimal fixed-equity requires a strictly positive equity payment, and the optimal fixed-cash hybrid involves a strictly positive payment in cash. Second, the optimal fixed-equity hybrid is the instrument that yields the highest expected revenue, followed by cash, which in turn is followed by the optimal fixed-cash hybrid. Equity is the worst instrument in the menu, despite of being the steepest.

The intuition of the latter results lies in the fact that with equity, a bidder will pay zero if he does not implement the project, but his bid will affect the profitability of implementing the project for his opponent. In that sense, a particular buyer can effectively use the threatening-power equity equips bidders with, to destroy the equilibria when the other buyer finds profitable to implement the project. When the seller uses cash as the instrument, this problem is mitigated by the fact that all payments are made upfront, rather than conditional on the implementation of the project. Therefore, the optimal instrument for the seller would be one that simultaneously features the screening benefits offered by cash,

[3] One important clarification is that we use the word implementation, because if an agent wins the object but does not implement it, no agent suffer any externality.

and the ability of equity to extract surplus. This design is precisely at the heart of the fixed equity hybrid. On the other hand, when the seller sets the fixed payment in terms of cash, and let buyers use equity to screen themselves, buyers conserve part of their power to block the implementation of the project, and so the adverse selection motive dominates. Surprisingly, this effect is so powerful that the optimal fixed-cash hybrid performs worse than cash for a sufficiently low implementation cost and a sufficiently high negative externality.

The ranking of the instruments is robust to the structure of information, as it is preserved for a large class of log concave distributions over private buyers' valuation. In particular, equity continues to deliver zero revenue despite of being the steepest instrument in the menu. Even though, we cannot deliver a general theorem as in the case of public information, we obtain very similar results via a simulation.

Finally, in a comparative statics exercise, theorem 4 finds that the fixed portion of both instruments is weakly increasing with respect to an improvement in the distributions, in the sense implied by the Monotone Likelihood Ratio (MLR) property. This result is clearly intuitive: as the probability of drawing higher valuation increases, the seller is less concerned of inducing participation, and can commit himself to extract a higher portion of revenue before the competition in the auction takes place.

Related Literature Our article is related to the literature of auctions with securities and to the literature of auctions with externalities. Nonetheless, as far as we know this is the first article connecting both strands of literature, to analyze how the interaction of negative externalities and securities impact bidding strategies and seller's expected revenue. Moreover, as we discussed before, due to the implementability incentives of buyers, our model can also be framed in the literature of auctions under adverse selection.

The literature of auctions with securities started with the seminal articles of Hansen (1985) and Riley (1988), who basically showed that a second price auction run in equity yields higher expected revenue to the seller than one run in cash. More recently, De Marzo, Kremer and Skrzypacz (2005) -hereafter DKS- generalize this framework by providing a methodology to rank securities with respect to revenue. Specifically, they characterize the "steepness" or sensitivity of several instruments via a single crossing property argument, and show that steeper instruments yield a higher revenue for the seller. Furthermore, they argue that the auction format is only relevant as long as it modifies the steepness of the instrument utilized. [4] Although DKS analyze a larger class of securities than what we do in this article, the main essence of their analysis is retained, because the distinction of the payment condition (i.e. contingent vs non-contingent) is the key ingredient to obtain our main result. As mentioned before, we focus in two hybrid instruments that are used in practice, and which include cash and equity as particular cases. [5]

Abishek, Hajek and Williams (2013) analyzes profit sharing contracts between a seller and the winning bidder, much in the spirit of our fixed-equity hybrid. Nonetheless, they conduct their analysis in a more general setting. In their model, signals over project's realized value can be interdependent, and bidders can be risk averse. They consider two types of securities, one where the seller shares only profits with the wining bidder, and one where the seller shares both profits and

[4] In particular, they prove that when the seller uses securities the Revenue Equivalence Theorem may not hold.

[5] Our fixed-equity hybrid resembles the way writers sell the rights of their books because there is a fixed royalty rate and publishers compete on cash. On the other side our fixed-cash hybrid captures the main feature of the oil rights auction in Mexico where buyers pay a fixed amount and compete on equity.

loses with the winner.

Abhishek et al. show that both securities yield a higher expected revenue to the seller than using a one-time payment in cash, which is also consistent with our findings. Furthermore, they show that using a security where the seller shares profits and loses, also yields a higher revenue than their counterpart with only profit sharing. This result does not necessarily depends on bidders' risk aversion. In fact, as long as signals are interdependent, it can also be generated with risk neutral bidders.

Extensions to this literature include the works of Gorbenko and Malenko (2011) and Che (2010). The former analyzes the predictions of a DKS model when the set of bidders is finite and many sellers compete for them. Their main result shows that sellers will not use the steepest instrument because they would not attract enough bidders. We also obtain the same result but for different reasons. In our case, using a pure security is detrimental for seller's revenue because it allows buyers, who do not intend to implement the project, to destroy the equilibria where good-type buyers would have implemented it otherwise.

Meanwhile, Che and Kim (2010) modify DKS framework by assuming that buyers with higher valuations also have a higher cost to implement the project. This simple modification leads to an adverse selection problem when the seller uses a security, because buyers with high valuation would bid a lower amount, and therefore, more often such buyers will win the auction. As the revenue of the seller is tied to bidders' true type when he uses a security, this adverse selection problem cause the revenue to decrease. We found that using securities when externalities are present can lead to the same result. Here, the low-valuation buyers would bid more aggressively because they want to avoid the negative externality, and can block implementation at no cost when the seller uses pure securities. Nonetheless, whereas Che and Kim (2010) makes assumptions on the cost structure of the model, we make assumptions on the after-market behavior of firms, which we consider more significant in many patent auctions where securities are normally utilized.

Our article also contributes -in minor extent- to the literature of auctions with externalities, initiated by Jehiel, Moldovanu and Stacchetti (1996, 1999). However, rather than proposing an optimal mechanism under an environment with externalities, we analyze a small but widely used class of instruments, which unlike Jehiel et al. also incorporates securities as a method of payment. We are able to show that under negative externalities, a second price auction in cash is no longer an optimal mechanism, because in our model we find that the best instrument is a fixed-equity hybrid.

Organization of the article The rest of the article is structured as follows. Section 4.1 states the environment of the model. In section 4.2 we introduce the case of complete information, derive the equilibrium bidding strategies, and rank the instruments with respect to revenue. Section 4.3 presents a robustness exercise for the case of private information. Section 4.4 concludes. Some of the proofs are relegated to the appendix.

4.1 The Environment

A seller is interested in allocating an indivisible asset -which can be thought as the rights of a project or innovation- among two different buyers. The winner is required to pay a cost of $c > 0$ in order to implement the project, which is considered as the initial investment to run the project, and is commonly known. We index buyers by $i = 1, 2$ whereas the seller is designated as player $i = 0$. Buyer

i's valuation v_i is drawn identically and independently from $[\underline{v}, \bar{v}]$, according to the distribution F which corresponding non-atomic density f.

If the project is implemented by a competitor, buyer i suffers a negative externality of $e \in [\bar{e}, 0]$, which we assume is symmetric and publicly known among buyers. One important aspect of our model is that externalities are contingent to the implementation of the allotted buyer. Second, private valuation refers to the gross return of the project, and so a rational winner i will only implement it if $v_i - c > 0$. Third, even if a buyer does not want to implement the project it would be beneficial for him to acquire it to preclude the implementation by other competitors, and thus avoiding the potential negative externality he might suffer.

The seller commits to use a second price auction to sell the project, but we assume he can utilize two different instruments: a fixed-equity hybrid and a fixed-cash hybrid. In the former the seller fixes the equity over project's return requested from the winner, and let bidders to compete in cash. The winner is the buyer who submits the highest bid in cash but pays the bid of his opponent. Clearly, a standard second price auction with cash corresponds to the case when the seller request zero equity. On the other hand, when the seller uses a fixed-cash hybrid, he fixes an amount in cash the winner of the auction has to pay, and let buyers to compete in equity. As before, the winner of the auction is the buyer who submits the highest equity bid, but pays the lowest bid. In this case, when the seller asks a fixed cash of zero, the auction is conducted in pure equity.

All players are risk neutral, and buyers' utility is additively separable. Let z_i be the return buyer i derives from the project after his implementation decision. That is, $z_i = v_i - c$ if he implements the project and zero otherwise. Thus, if buyer i, with type v_i, wins the auction his payoff is given by $z_i - t_i(v_i)$, where $t_i(v_i)$ represents the payment to the seller, which potentially depends on his valuation.

On the other hand, if the seller allocates the object to buyer j, then buyer i's payoff corresponds to e, provided his competitor implements the project; and zero otherwise. The value of the project for the seller is zero, and hence in any trade with buyer i his utility is $t_i(v_i)$. [6] If no trade occurs, the payoff is zero for all players.

Figure 4.1 depicts the timing of the game. First, seller chooses a payment instrument and commits to run a second price auction under this format. Then, buyers learn their valuations and submit their bids to the seller, who determines the winner of the auction. Next, the winner determines if he wants to implement or not the project. Finally, payoffs are realized contingent on the implementation decision.

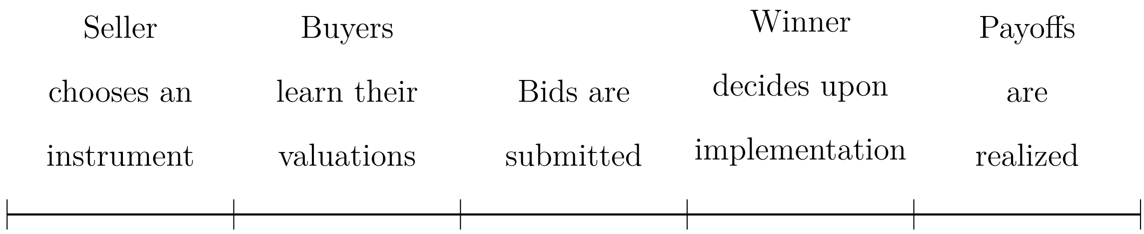


Figure 4.1: Timing of the auction

4.2 Public Buyer's Valuation

In this section we assume that before participating in the auction, each buyer learns his own valuation as well as the valuation of his opponent. Without loss of generality we will assume $v_1 > v_2$. The seller, on the other hand, only knows the distribution where buyers' valuations come from. Nonetheless, the negative externality is public information for all players. This setting plausibly corresponds

[6] Think for example in a seller who owns a patent over a specific productive process that by itself cannot be monetized, but can potentially enhance the productivity of the current technology held by the buyers.

to a situation where both buyers have been operating in a market for long time and have learned the technology of each opponent, but where a seller is an outsider of the industry who has developed an innovation that can enhance the technology of both buyers, but cannot evaluate to which extent.

The seller wants to maximize the ex-ante revenue and for that purpose has to choose which instrument to utilize. Once the seller chooses an instrument he commits to it. Thus, bidders are engaged in a game of public information, where they have to choose their bid b_i in the correspondent security space. In the case of the fixed-equity hybrid $b_i \in \mathbb{R}_+$, whereas in the case of the fixed-cash hybrid $b_i \in [0, 1]$.

A Motivating Example In this section we will go through an easy example that will highlight the main results of the article.

A. No Externalties Consider an auction in which two buyers, Alice and Bob, compete for a project. The project requires an initial fixed investment of $c > 0$ which can be interpreted as the minimum up-front cash payment required by the seller. Alice expects that if she undertakes the project, it would yield her a return of v_a, whereas Bob expects a cash flow of v_b. Without loss of generality, $c < v_b < v_a$. We assume that both valuations are common knowledge to both buyers. As the seller commits to use a second price auction, the weakly dominant strategy for both buyers is to bid their reservation value. As a result, Alice would bid $b_a(v_a) = v_a - c$ and Bob would bid $b_b(v_b) = v_b - c$. Hence, Alice wins the auction and pays Bob's bid, which implies seller's revenue would be $\Pi^{ca} = v_b - c$.

Now, suppose that rather than bidding with cash, the buyers compete by offering equity over the return of the project. As we discuss later, in this case it is also a weakly dominant strategy for both buyers to bid their reservation value. [7] Thus, Alice would make aa equity bid of $b_a(v_a) = \frac{v_a - c}{v_a}$, whereas Bob would make an equity bid of $b_b(v_b) = \frac{v_b - c}{v_b}$. As a result, Alice wins the auction and pays according to Bob's bid. Seller's revenue would be $\Pi^{eq} = \frac{v_b - c}{v_b} v_a$. By an easy algebraic manipulation, it is possible to see that sellers revenue under equity is higher than under cash, as

$$\Pi^{eq} = \frac{v_b - c}{v_b} v_a = (v_b - c)\frac{v_a}{v_b} > v_b - c = \Pi^{ca}$$

B. Externalities Consider the same auction as before but now with the modification that if buyer i wins the auction and implements the project, the payoff of buyer $-i$ will be $e < 0$.

Cash. When the payment instrument is cash, bidding the reservation value continue to be a weakly dominant strategy for both buyers. Nonetheless, it now should include the externality. Thus, $b_a(v_a) = v_a - c - e$ and $b_b(v_b) = v_b - c - e$. Seller's revenue becomes $\Pi^{ca} = v_b - c - e$.

Equity. If buyers compete by offering equity the analysis is more interesting. Here, Alice knows that if she bids $b_a(v_a) = \frac{v_a - c}{v_a}$ then Bob has no incentives to implement the project in case he wins, because $(1 - \frac{v_a - c}{v_a})v_b - c < 0$. [8] This implies that Alice will be willing to make the same offer as without externalities. For Bob, the incentives in the auction change. On one hand, he can bid his reservation value, lose the auction, let Alice implement the project, and obtain a payoff of $e < 0$. On the other hand, he can bid higher than Alice, win the auction, shut down the project, and obtain a payoff of 0. By comparing both scenarios, it

[7] The reservation value of buyer i is when his payoff equals 0: $(1 - b_i(v_i))v_i - c = 0$ thus $b_i(v_i) = \frac{v_i - c}{v_i}$.

[8] As $(1 - \frac{v_a - c}{v_a})v_a - c = 0$ and $v_b < v_a$.

is clear that Bob's optimal strategy is to bid anything on the interval $(b_a(v_a), 1]$ and secure for himself a payoff of 0. Seller's revenue becomes $\Pi^{eq} = 0$ in this case.

Fixed-Equity and Fixed-Cash. To conclude the example we will provide a rationale for introducing a fixed-equity and a fixed-cash hybrid as methods of payment. By definition, the revenue collected by both instruments depends on the selection of the fixed component. The challenge for the seller resides in choosing such fixed components when he only knows the distribution of valuations. For instance, if the seller sets a very high fixed equity $\bar{\alpha}$, buyers may lose the incentive to participate in the auction. Likewise, if he sets a very low fixed cash $\bar{b}$, he would not be extracting as much surplus as possible from the winner.

The following table shows the values of $\bar{\alpha}^*$, Π^{fe}, $\bar{b}^*$, Π^{fc}, Π^{ca} and Π^{eq} for different distributions of types when the cost of implementing the project is $c = 0.1$ and the externality is $e = -0.2$.

Table 4.1: Seller expected revenue under optimal securities: Public Info

			Expected Seller Revenue			
Distribution	$\bar{\alpha}^*$	$\bar{b}^*$	$\Pi^{fe}(\bar{\alpha}^*)$	Π^{ca}	$\Pi^{fc}(\bar{b}^*)$	Π^{eq}
$U[0,1]$	0.51	0.53	0.56	0.44	0.32	0
$B[2,2]$	0.49	0.43	0.57	0.47	0.3	0
$B[2,7]$	0.1	0.16	0.24	0.23	0.09	0
$IB[2,7]$[9]	0.73	0.63	0.90	0.80	0.56	0

By looking at the distributions and the revenues some facts can be highlighted:

[9]The Inverse-Beta distribution is computed from a former Beta distribution. If $f(x)$ represents the PDF of a Beta then the PDF of an Inverse-Beta would be $g(y) = f(-x+1)$. If the former Beta distribution had a right tail then the Inverse-Beta associated to it will have a left tail. When the former Beta is symmetric then the Inverse-Beta is exactly the same.

- When comparing symmetric distributions $\bar{\alpha}^*$, $\bar{b}^*$, $\Pi^{fe}(\bar{\alpha}^*)$, Π^{ca}, and $\Pi^{fc}(\bar{b}^*)$ are very similar.

- When the relative likelihood of high types to low types increases, $\bar{\alpha}^*$, $\bar{b}^*$, $\Pi^{fe}(\bar{\alpha}^*)$, Π^{ca}, and $\Pi^{fc}(\bar{b}^*)$ increase as well.

- Given $c = 0.1$ and $e = -0.2$ the rank of the instruments with respect to revenue is as follows $\Pi^{fe}(\bar{\alpha}^*) > \Pi^{ca} > \Pi^{fc}(\bar{b}^*) > \Pi^{eq}$.

In the succeeding section we will formally introduce the instruments, characterize the equilibrium bidding strategies, and obtain seller's expected revenue.

4.2.1 Fixed-Equity Hybrid

In the fixed-equity hybrid, seller fixes the equity $\bar{\alpha}$ the winner of the auction has to pay over the return of the project. Knowing this information buyers compete in cash for the allocation of the project. Thus, winner's payment to the seller consists of the lowest bid in cash, plus the fixed-equity fraction over projects' return.

Proposition 5. *The dominant-strategy equilibrium of the second price auction under a fixed-equity hybrid is characterized as follows:*

i) If $(1-\bar{\alpha})v_1 - c < 0$, *then* $b_1 = b_2 = 0$.

ii) If $(1-\bar{\alpha})v_1 - c > 0$ *and* $(1-\bar{\alpha})v_2 - c < 0$, *then* $b_1 = (1-\bar{\alpha})v_1 - c$ *and* $b_2 = -e$.

iii) If $(1-\bar{\alpha})v_2 - c \geq 0$, *then* $b_1 = (1-\bar{\alpha})v_1 - c - e$ *and* $b_2 = (1-\bar{\alpha})v_2 - c - e$.

Proof. In case *i)* the project is not profitable to implement for any of the buyers, and thus, their best strategy is to submit a bid of zero. On the other hand, in case *ii)* the project is profitable to implement for buyer 1 but not for buyer 2; hence, the best strategy for buyer 1 is to bid his reservation value, and implement the project if he is allocated. Given buyer 1's strategy, the best response of buyer 2 is to bid his reservation value, which in this case is the negative externality he knows will suffer if buyer 1 wins the auction. Finally, if the project is profitable for both bidders, both will bid their reservation value, which includes the avoidance of the externality. ■

There are several interesting observations that can be highlighted from proposition 5. First, the likelihood of allocations and payments are not necessarily weakly increasing in buyer's type. For instance, if buyer 2 -the one with the lowest valuation- bids the absolute value of the externality, wins the auction, and pays the reservation value of buyer 1. Moreover, if both buyers find profitable to implement the project, there cannot be an equilibrium in which buyer 2 implements the project, and therefore, his incentives to participate in the auction reside in avoiding the externality if he can win the auction at a price lower than the value of the externality e.

Figure 4.2 shows the bidding strategy of bidder 1 as a function of the valuation of bidder 2, given that $v_1 > \frac{c}{1-\tilde{\alpha}}$, and thus when only cases *ii)* and *iii)* are possible. [10]

It can be observed from figure 4.2 that as soon as the project becomes profitable for buyer 2 (i.e. when $v_2 \geq \frac{c}{1-\alpha}$) buyer 1 increases his bid by $-e$, to reflect the fact that he would suffer the externality in case he loses the auction.

The expected revenue generated by the fixed-equity hybrid under these equilibrium strategies correspond to

[10] If $v_1 < \frac{c}{1-\tilde{\alpha}}$ then he will bid $b_1(v_1) = 0$ when $v_2 < \frac{c}{1-\tilde{\alpha}}$, otherwise $b_1(v_1) = -e$.

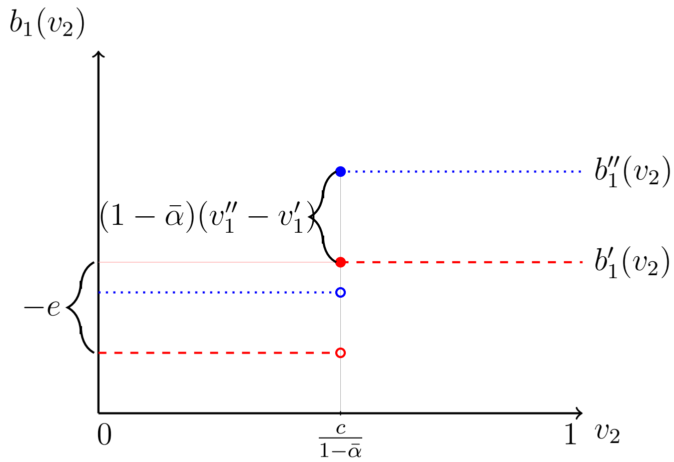


Figure 4.2: Bidding strategies with fixed-equity for buyer 1

$$\begin{aligned}\Pi^{fe}(\bar{\alpha}) &= 2F(\frac{c}{1-\bar{\alpha}})\int_{\frac{c}{1-\bar{\alpha}}}^{\frac{c-e}{1-\bar{\alpha}}}((1-\bar{\alpha})v_1-c)f(v_1)dv_1 \\ &+ 2F(\frac{c}{1-\bar{\alpha}})\int_{\frac{c-e}{1-\bar{\alpha}}}^{\bar{v}}(\bar{\alpha}v_1-e)f(v_1)dv_1 \\ &+ \int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}[(1-\bar{\alpha})\min\{v_1,v_2\}-c-e+\bar{\alpha}\max\{v_1,v_2\}]f(v_1)f(v_2)dv_1dv_2\end{aligned} \tag{4.1}$$

First, notice that if the project is not profitable for any buyer, the auction will generate zero revenue. In the case it is profitable for buyer 1 but not for buyer 2, we need to identify two sub-cases: one when $0 < (1-\bar{\alpha})v_1 - c < -e$, and the other one when $-e < (1-\bar{\alpha})v_1 - c$. In the former, buyer 2 wins the auction but does not implement the project, therefore the seller does not collect revenue from the equity portion of the hybrid, but will get a transfer of $(1-\bar{\alpha})v_1 - c$, which is the lowest bid in cash. This case corresponds to the first term in equation (1). Now, in the other case, buyer 1 will win and implement the project, which means the seller will collect a contingent revenue of $\bar{\alpha}v_1$ plus a transfer in cash of $-e$. This corresponds to the second term. Finally, when both buyers find profitable to implement the project, the seller collects the lowest reservation value in cash, plus the fraction of equity corresponding to the highest type. This is precisely the third term.

4.2.2 Fixed-Cash Hybrid

When the seller uses a fixed-cash hybrid he fixes the amount in cash the winner of the auction has to pay, $\bar{b}$. Knowing this information, bidders compete in equity for the allocation of the project, and it is allocated to the buyer with the highest bid in equity. Therefore, winner's final payment to the seller corresponds to the lowest bid in equity, times the return of the project when it is implemented by him, plus the fixed-amount in cash.

Proposition 6. *The Nash Equilibrium of the second price auction under a fixed-cash hybrid is characterized as follows:*

i) If $v_1 - c - \bar{b} < 0$*; then* $b_1 = b_2 = 0$*.*

ii,a) If $v_1 - c - \bar{b} > 0$ *and* $-\bar{b} \leq e$*; then* $b_1 = \frac{v_1 - c - \bar{b}}{v_1}$ *and* $b_2 = 0$*.*

ii,b) If $v_1 - c - \bar{b} > 0$ *and* $-\bar{b} > e$*; then* $b_1 = \frac{v_1 - c - \bar{b}}{v_1}$ *and* $b_2 = (\frac{v_1 - c - \bar{b}}{v_1}, 1]$*.*

Proof. In the first case the project is not profitable for any of the buyers and then no one will suffer the externality in case the project is allocated to his opponent. Moreover, bidding a positive equity will give the buyers a positive probability of winning the auction, which will force them to pay the amount $\bar{b}$ to the seller. Therefore, the best strategy for both buyers is to stay out of the auction. If the project is profitable for buyer 1 but not for buyer 2, and the fixed amount of cash $\bar{b}$ is higher or equal to the value of avoiding the externality $-e$, buyer 2 prefers to stay out of the auction and suffer the externality. On the other hand, if $-\bar{b} > e$, buyer 2 has an incentive to participate in the auction to bid high enough in order to destroy the incentives of buyer 1 to implement the project in case he wins the auction. In both cases, the best response of buyer 1 is to bid his reservation value, which does not take into account the avoidance of the externality, because he knows buyer 2 never will implement the project if he has the opportunity to do so. Finally if the project is profitable for both, there is no equilibrium in which buyer 2 wins the auction and implements the project. The reason is that as the reservation value of buyer 2 is lower than the one of buyer 1, if buyer 1 is not the winner then there is a profitable deviation in which he offers a slightly higher bid than buyer 2, wins the auction, and avoid the negative externality. Given that situation, the best response for buyer 2 is to bid 0 if $-\bar{b} \leq e$, or otherwise bid high enough to destroy the incentive of buyer 1 to implement the project in case he wins the project. Following the strategy of buyer 2, the best strategy for buyer 1 is to submit his reservation value. ■

Equity represents the particular case in which $\bar{b} = 0$. In this case implementation never takes place and blocking is always the best response of the weak buyer. Notice that this is true for any $e < 0$ and moreover this is one of the possible equilibrium for $e = 0$, being this equilibrium particularly robust. In figure 4.3 we present the bidding strategy of bidder 1 as a function of the valuation of bidder 2, given that it is profitable for him to implement (i.e. when $v_1 > c + \bar{b}$). In other words, we restrict attention to cases *ii,a)* and *ii,b.* [11]

Figure 4.3 shows that as v_1 increases $b_1(v_1, \cdot)$ increases as well ($v_1'' > v_1'$), which implies that the region of parameters under which buyer 1 just block the allocation decreases. [12]

[11] If $v_1 < c + \bar{b}$ then he will bid $b_1(v_1) = 0$ when $v_2 < c + \bar{b}$ or $\bar{b} > -e$, otherwise he will bid anything between 1 and the reservation value of buyer 2.

[12] Under v_1' he will block the allocation at the dashed plus dotted region whereas under v_1'' he will block only at the dotted region.

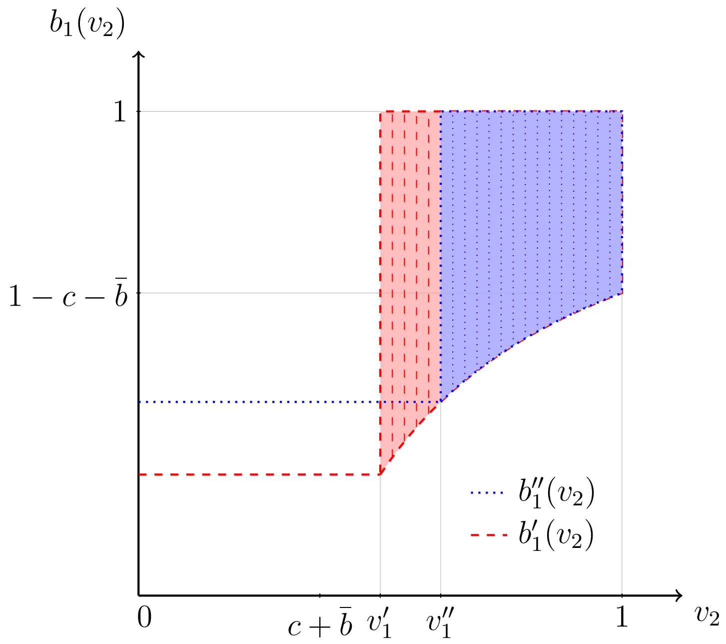


Figure 4.3: Bidding strategies with fixed-cash for buyer 1

The revenue generated by these equilibrium strategies corresponds to

$$\Pi^{fc}(\bar{b}) = (1 - F(c + \bar{b})^2)\bar{b} \tag{4.2}$$

It states that the seller will collect the fixed amount of cash $\bar{b}$ as long as at least one of the buyers find profitable to implement the project. The clear tradeoff for the seller is that increasing $\bar{b}$ diminishes the probability of implementation, but increases the surplus extracted conditional on implementation.

Once we considered the expected revenues of both instruments given by expressions (4.1) and (4.2), the natural following step is to determine how do they rank. This is precisely the matter of the following theorem.

Theorem 3. *For any log-concave density f, there exists a cutoff values $\bar{c}$ and $\underline{e}$, such that if $c \in (0, \bar{c})$ and $e < \underline{e}$ the instruments can be ranked in expected revenue* *as follows:*

$$\Pi^{fe}(\bar{\alpha}^*) > \Pi^{fe}(0) > \Pi^{fc}(\bar{b}^*) > \Pi^{fc}(0) \tag{4.3}$$

Theorem 3 states that if the cost of the project and the negative externality are sufficiently low, the seller is globally better off using a fixed-equity hybrid. The reason of this result is that, as we discussed before, when payments are made upfront in cash buyers have to face a sunk cost if they want to block the implementation of his opponent. Therefore, the willingness of a "bad type" to pay is bounded above by the absolute value of the negative externality. If the externality is so large that the bad type wins the auction, the seller secures for himself the reservation value of the highest type; otherwise, he receives the fixed equity from the good type, plus the value of the externality in cash. On the other hand, when buyers can bid in equity they can destroy more often the equilibrium in which the project is implemented. The effect is particularly dramatic when the seller uses a pure security, because blocking can be done at no cost. This problem can be mitigated by incorporating a fixed cash component $\bar{b}$; however as the theorem shows, its presence is not sufficient to offset the perverse incentives of the "bad type" buyers. The result holds for any log concave density, which suggests that the interaction between buyers is strong enough to hold under different distributional assumptions.

Figure 4.4 illustrates the result of theorem 3 when valuations are drawn independently and identically from a uniform distribution with support $[0, 1]$, with a negative externality of $e = -0.2$ and a cost of $c = 0.1$. As it can be seen, there is a large range for the parameter $\bar{\alpha}$ such that the fixed-equity hybrid renders a higher revenue than cash, which in turn yields a higher revenue than the fixed-cash hybrid. Noticeably, the revenue obtained by cash is 50% higher than the one

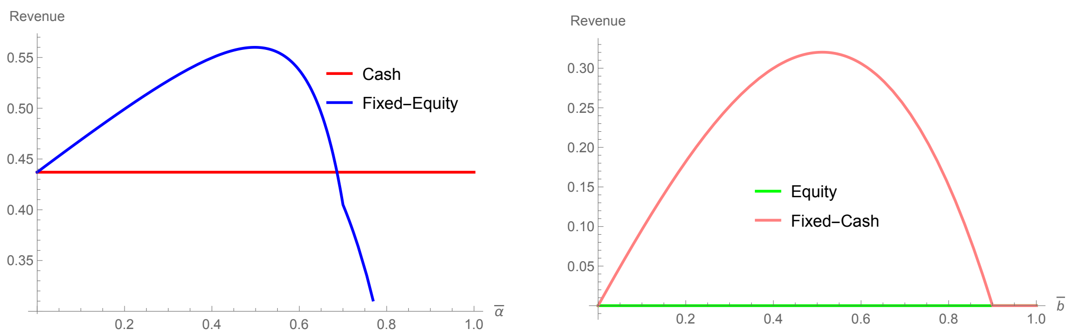


Figure 4.4: Revenue as a function of $\bar{\alpha}$ and $\bar{b}$ for $U[0, 1]$

yielded by the optimal fixed-cash hybrid. [13]

Monotone Comparative Statics Now we will inspect what happens to the optimal fixed parameters $\bar{b}^*$ and $\bar{\alpha}^*$ when the distribution improves in the sense of the Monotone Likelihood Ratio property. This analysis will provide an insight of how different distributions affect the design of both hybrids

Theorem 4. *Suppose f_1 dominates f_0 in the Monotone Likelihood Ratio (MLR), then $\bar{b}_1^* \geq \bar{b}_0^*$. If additionally, $\frac{F_1(y)}{F_0(y)} > \frac{f_1(y)}{f_0(y)}$ for all $y \in [c, \bar{v}]$, then $\bar{\alpha}_1^* \geq \bar{\alpha}_1^*$*

Proof. See the appendix. ■

Theorem 4 says that for a fixed cost and an externality, if the likelihood of getting higher values improve in the sense of MLR, the optimal fixed cash amount in its respective hybrid cannot decrease. If in addition the ratio of the densities is majorized by the ratio of the distributions for all values greater than the cost, the optimal fixed equity portion in its respective hybrid cannot decrease. It implies that when the seller is using the fixed-equity hybrid he will apply a higher equity portion over a higher expected return of the project. Likewise, when the seller uses a fixed-cash hybrid, it means that now the barrier a bad type has to surpass

[13]Similar figures for different distributions are presented in the appendix.

to enter the auction and block implementation is higher. Naturally, in both cases expected seller's revenue increases.

4.2.3 Other Variations

Deposit Insurance Notice that depending on the security design, the seller can collect a payment from buyers in two stages: after a buyer wins the auction, and after a winner implements the project. As mentioned before, the idea of introducing a payment in cash was a device to screen the low type buyers who otherwise would always have an incentive to enter into the auction to destroy the implementation incentives of the high type buyers. In particular, the fixed-cash hybrid forces the winner to make a payment in cash right after winning the auction. A variant of this instrument, is to introduce a cash deposit (or insurance). This device would work as follows. The seller fixes an amount each buyer has to deposit to participate in the auction. Then, a second price auction in equity is run. The loser gets the deposit back. If the winner implements the project, he has to pay the correspondent equity over project's return but the seller gives back the cash deposit. On the other hand, if the winner does not implement the project the seller retains the cash deposit. [14] Although the cash transfer is determined in a different stage, it can be shown that bidding strategies are the same as in the fixed-cash hybrid, and therefore the revenue for the seller does not change. In other words: If the cash deposit is below $-e$ then the "bad type" will block implementation, otherwise his bid will be zero thus revenue is the same as in the fixed-cash hybrid.

[14] Another way of doing the same is by fixing the size of the deposit the winner should pay upon winning the auction (only the winner pays) and he can claim it back upon implementation.

Unconstrained Bids The two hybrids presented before share the characteristic that the seller determines ex-ante the bid in one of the securities. For instance, in the fixed-equity hybrid, the seller fixes the fraction of equity asked but let buyers to compete in cash. Meanwhile, in the fixed-cash hybrid, the seller fixes the possible bids of cash but let buyers to compete in equity. Alternatively, one can think in a format where the seller decides to run a second price auction but without imposing any restriction on buyers' bids. Thus, each player bid consists on a tuple $b_i = (\alpha_i, \beta_i) \in \mathbb{R} \times [0,1]$, where α_i represents the equity promised on the return of the project and β_i corresponds to an upfront payment in cash. The critical difference of this approach with respect to the former is that now there is no trivial way to rank bids and determine the winner of the auction. Suppose the seller uses an order ψ such that $(\mathbb{R} \times [0,1], \psi)$ constitutes a linearly ordered set.

Proposition 7. *Fix an arbitrary ψ. The dominant strategy equilibrium of the second price auction under unconstrained bids corresponds to: $b_1 = (0, v_1 - c - e)$ if $v_1 - c > 0$ and $b_1 = (1, 0)$ otherwise; $b_2 = (1, 0)$.*

Proof. If the project is not profitable for buyer 2 his dominant strategy is to bid the whole equity and nothing in cash. Following this strategy, he makes sure the project is never implemented at no cost, and so, he never suffers the externality. When the project is profitable for both, bidder 1 offers his reservation value in the cheapest way, which involves only cash, as he is the highest type and any marginal fraction he bids in equity is only valued by the seller with respect to the expected type. Given this strategy, bidder 2 offers the whole equity and no cash, to block the allocation in which buyer 1 wins the auction and implements the project. ■

Notice that this equilibrium is obtained irrespectively of the order ψ the seller uses to rank the bids. The result follows because equity is the instrument that permits to avoid the externality without paying any cost. This is the worst case scenario for the seller, as the revenue under unconstrained bids is $\Pi^{ub} = 0$. The critical assumption is that the auction is a second price, because the buyer is forced to pay each component of his opponent bid.

4.3 Robustness: Private Buyer's Valuations

In this section we will analyze the set of securities under the assumption that buyers do not longer know the valuation of his opponent, which turns our model into a standard private values auction model. We will analyze how the information structure affects our main result.

4.3.1 Fixed-Equity Hybrid

Analogously to section 4.2.1 we characterize the equilibrium under private information. We use the Bayes-Nash equilibrium as the solution concept.

Proposition 8. *Bayes-Nash equilibrium bidding strategies of the second price auction when the seller uses fixed-equity, $\bar{\alpha}$, are characterized by*

i) $b_i(v_i) = 0$ *if* $(1-\bar{\alpha})v_i - c < 0$.

ii) $b_i(v_i) = (1-\bar{\alpha})v_i - c + (1 - F(\frac{c}{1-\bar{\alpha}}))(-e)$ *if* $(1-\bar{\alpha})v_i - c > 0$.

Proof. As the seller utilizes a second price auction, and buyers bid in cash, the best strategy for a buyer who finds profitable to implement the project is to bid their reservation value. Now, buyer i's reservation value depends on the implementation decision of his opponent. Thus, with probability $F(\frac{c}{1-\bar{\alpha}})$ it is not profitable for the other buyer to implement the project, and so buyer i's reservation value is equal to the net payoff of implementing the project: $(1-\bar{\alpha})v_i - c$. On the other hand, with probability $1-F(\frac{c}{1-\bar{\alpha}})$ it is profitable for the other buyer to implement the project, which implies that in case buyer i loses the auction he will suffer the externality e, and thus, such expected loss has to be added to his bid. For the buyer who does not want to implement the project his reservation value is given by $1 - F(\frac{c}{1-\bar{\alpha}})(-e)$. If he bids his reservation value, he will lose with probability one if he faces an opponent who wants to implement the project. In such case his payment will be e. On the other hand, when he faces an opponent who does not want to implement the project neither, his expected payoff will be $\frac{1}{2}(1-F(\frac{c}{1-\bar{\alpha}}))(e)$. Hence, there is clearly a profitable deviation to zero. By doing this the buyer will continue losing the auction when facing an opponent who wants to implement the project, and then will obtain the same payoff, but now will obtain a zero payoff if he faces an opponent who does not want to implement. ■

Following the same reasoning as with the public case, seller's ex-ante revenue is given by

$$\begin{aligned}\Pi^{fe}(\bar{\alpha}) &= 2F(\frac{c}{1-\bar{\alpha}})\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}(\bar{\alpha}v_1)f(v_1)dv_1 \\ &+ \int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}[(1-\bar{\alpha})\min\{v_1,v_2\} - c + (1-F(\frac{c}{1-\bar{\alpha}}))(-e) + \bar{\alpha}\max\{v_1,v_2\}]f(v_1)f(v_2)dv_1dv_2\end{aligned}$$

The first term in the integral corresponds to the case when one buyer finds profitable to implement the project and the other does not. In this scenario, the seller collects the fixed equity portion from the highest type and receives zero in cash. Meanwhile, the second term represents the expected revenue when both buyers want to implement the project. Here the seller receives the equity portion from the highest type, plus the cash embedded in the lowest bid.

4.3.2 Fixed-Cash Hybrid

In a similar fashion to section 4.2.2 we characterize the equilibrium under private information. We show that the existence of a Bayes-Nash equilibrium in pure strategies depends on the relationship between the fixed amount of cash $\bar{b}$

and the value of the externality e.

Proposition 9. *There are no equilibria in pure strategies in the fixed-cash hybrid if:*

i) $-\bar{b} > e$ *and* $(1 - F(c + \bar{b}))e < -\bar{b}$.

ii) $-\bar{b} > e$ *and* $(1 - F(c + \bar{b}))e > -\bar{b}$.

Proof. The problem to reach an equilibrium on case (i) resides in the optimal strategy of the buyer's type who does not want to implement the project: the "bad type." If both buyers of such type bid zero, any of them would find profitable to deviate and bid the smallest amount that guarantees him to be the winner of the auction. In such case the deviant buyer would get a payoff of $-\bar{b}$ which is greater than $(1 - F(c + \bar{b}))e$. For the same reason, the other buyer also deviates to the same bid, which yields a payoff of $F(c+\bar{b})(-\frac{\bar{b}}{2})(1 - F(c+\bar{b}))(-\bar{b})$ to both buyers. Notice that in such situation both buyers block the implementation with certainty and share the cost. Nonetheless, as soon as both buyers bid the same amount, any of them -say buyer i- has an incentive to bid an arbitrarily lower amount. It guarantees to suffer the externality with a very low probability in case he faces the good type of buyer $-i$, but saves his portion of the fixed amount of cash when faces his fellow bad type. The moment buyer i deviates, buyer $-i$ has two possible deviations, either to bid lower than buyer i, or returning to the initial bid. The former deviation is more profitable. Continuing with this analysis, some buyer will reach a level at which there is no downward deviation for his rival. That is, a point where if his opponent submits a lower bid, he will suffer a payoff lower than $-\bar{b}$. Or in other words, a bid $k \in (0, 1)$ such that $F(b^{-1}(k))(e) = \bar{b}$. Under this scenario, if buyer i bids k, the best deviation for buyer $-i$ is to return to the initial bid, which will start again the cycle of deviations. In order to prove case (ii) it is worth noting that "bad buyers" will make a bid of zero. Now the problem resides on the "good buyers". Consider the type $v_i = c + \bar{b}$. If he bids $b_i(v_i) = \frac{v_i - c - \bar{b}}{v_i}$ he wins against all the types that do now want to implement the project getting a payoff of zero but loses against all the other types that want to implement the project (it is clear that no bidder who wants to implement the project has incentives to bid below his reservation value without considering the externality). Whenever he loses, he gets e for sure (he only loses against types that are willing to implement at his reservation value) which is worse than paying $\bar{b}$ and not implementing. Hence, he is better off blocking every possible implementation: bidding the smallest amount that guarantees him to be the winner of the auction. Sufficiently many types will deviate to this bid as long as $-\bar{b} > e$, because they can block potential implementation. At this point the cyclical logic of case (i) comes into place, not for the "bad types" now but for the "good types", and no equilibrium is reached in pure strategies. ■

Proposition 10. *Bayes-Nash equilibrium bidding strategies of the second price auction under the fixed-cash hybrid, when $-\bar{b} \leq e$, are characterized by*

i) $b_i(v_i) = \frac{v_i - c - \bar{b}}{v_i}$, *if* $v_i - c - \bar{b} > 0$.

ii) $b_i(v_i) = 0$ *if* $v_i - c - \bar{b} < 0$.

Proof. In case (ii) the project is not profitable for the buyer, and moreover, the negative externality e is lower than the loss he would get by winning the auction and not implementing the project, $-\bar{b}$. As there is no way to prevent the implementation of the project by his competitor without winning the auction, the best strategy of the "bad type" is to bid zero in equity. In case (i), the buyer finds profitable to implement the project, and his best strategy is to bid his reservation value -which does not depends on the implementation decision of his opponent. If $b_i(v_i) > \frac{v_i - c - \bar{b}}{v_i}$ he will win whenever $b_i(v_i) > b_{-i}(v_{-i})$ but there are two different situations. When $b_{-i}(v_{-i}) < \frac{v_i - c - \bar{b}}{v_i}$ buyer i will win the auction and implement the project, guaranteeing for himself a payoff of at least zero. When $b_i(v_i) > b_{-i}(v_{-i}) > \frac{v_i - c - \bar{b}}{v_i}$ buyer i will win the auction but cannot implement the project, thus his payoff is $-\bar{b}$. By deviating to $b_i(v_i) = \frac{v_i - c - \bar{b}}{v_i}$ he keeps the positive payoffs (wins and implements in all the cases he wants to do so) and at most suffers a payoff of e upon losing which is better than $-\bar{b}$. ■

Once we have derived the equilibrium strategies we can state the expression for seller's expected revenue. Given we have equilibrium whenever $-\bar{b} < e$ we can state the revenue just for this particular case.

$$\begin{aligned}
\Pi^{fc}(\bar{b}) &= 2(1 - F(c + \bar{b}))F(c + \bar{b})\bar{b} \\
&+ \int_{c+\bar{b}}^{\bar{v}} \int_{c+\bar{b}}^{\bar{b}} (\min\{\frac{v_1 - c - \bar{b}}{v_1}, \frac{v_2 - c - \bar{b}}{v_2}\} \max\{v_1, v_2\} + \bar{b}) f(v_1) f(v_2) dv_1 dv_2
\end{aligned}$$

4.3.3 Equity

To analyze equity, we cannot simply take bidding strategies as particular cases of the fixed-cash hybrid, because now even for very low valuations the buyer can bid sufficiently high, and still avoid a positive payment to the seller.

Proposition 11. *Equilibrium bidding strategy when the seller uses equity is uniquely characterized by* $b_i(v_i) = 1$.

Proof. Clearly, if $v_i - c < 0$ buyer i will not implement the project if he wins, so winning the project only has value as long as it prevents the other agent to win and implement the project, because in this case buyer i avoids the negative externality it would entail. Now if $v_i - c > 0$, in principle buyer i optimal strategy is to bid his reservation value, as now he has the normal trade-off any buyer faces in an auction: increasing the bid increases the probability of winning but decreases the surplus. However, the presence of the externality biases buyer's incentives towards winning the auction. In concrete, if $v_i - c > 0$ but small, the buyer might be better off by bidding one in equity and avoiding the externality with certainty, than gambling on winning the auction and suffering the externality with positive probability.

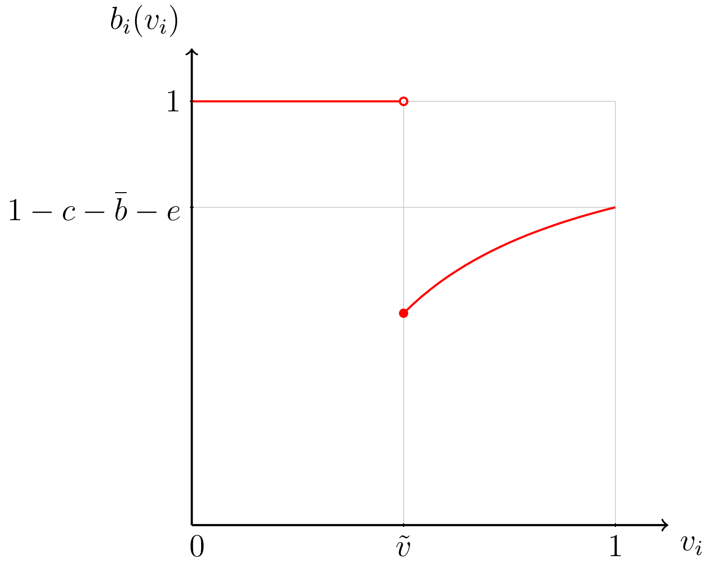


Figure 4.5: Threshold equilibrium under Equity

This behavior may give room for the possibility of having a cut-off strategy. If this were the case, there would exist a value $\tilde{v}$ such that if $v_i < \tilde{v}$ then $b_i(v_i) = 1$ and if $v_i \geq \tilde{v}$ then buyers bid their reservation value -which includes the externality he would suffer in case of his opponent implements the project. In such case the strategy of buyer i would have a discontinuity at $\tilde{v}$, as shown in figure 4.5. However, if it were the case, at $\tilde{v}$ the bid of the agent will be the lowest possible, which implies he loses the auction for sure and will suffer the externality with positive probability. Thus, bidding one is a profitable deviation. This observation holds for any value $\tilde{v} < 1$. Therefore, both agents will bid one in equilibrium and the project is never implemented. ■

4.3.4 Example Revisited

Following the example presented in section 4.2 we show the values of $\bar{\alpha}^*$, Π^{fe}, $\bar{b}^*$, Π^{fc}, Π^{ca} and Π^{eq} for different distributions of types when the cost of implementing the project is $c = 0.1$ and the externality is $e = -0.2$.

Table 4.2: Seller expected revenue under optimal securities: Private Info

			Expected Seller Revenue			
Distribution	$\bar{\alpha}^*$	$\bar{b}^*$	$\Pi^{fe}(\bar{\alpha}^*)$	Π^{ca}	$\Pi^{fc}(\bar{b}^*)$	Π^{eq}
$U[0,1]$	0.67	0.42	0.54	0.39	0.35	0
$B[2,2]$	0.58	0.32	0.55	0.45	0.34	0
$B[2,7]$	0.23	$**$	0.18	0.17	$**$	0
$IB[2,7]$	0.75	0.2	0.91	0.80	0.68	0

**: $\bar{b}$ is in the no equilibrium range

Even though the table computes Π^{fc} only for the case where we have an equilibrium in pure strategies, our results are robust: Most of the entries on table 4.2 are similar to the ones presented on table 4.1. The only difference is $\bar{b}$ for the $IB[2,7]$ because now the seller can force bidders to bid in equity without considering the externality and he seems willing to do so. [15] However the payoff he obtains is similar to the case of public information.

[15] 0.2 is the $\bar{\alpha}^*$ that maximizes revenue on $\bar{\alpha}^* \in [0.2, 1]$ but it is still possible (although unlikely) that $\bar{\alpha}^* < 0.2$.

4.4 Concluding Remarks

We analyzed a simple two-buyer second price auction, where the seller can use two different hybrids and the buyers suffer negative externalities upon the implementation of the project by their opponent. In particular, we consider a fixed-equity hybrid, where the seller fixes a portion of equity over project's return and buyers compete in cash; and a fixed-cash hybrid, where the buyers compete in equity and the winner has to pay an amount in cash predetermined by the seller.

Our main observation lies in the fact that pure securities equip low-valuation buyers (those who do not want to implement the project, or bad types) with a powerful tool to block the implementation from the good types, which impacts revenue negatively. Then, we find that in order to circumvent this problem the seller has to incorporate a fixed payment in the instruments to be used as a device to prevent "bad types" from blocking. However, mitigating this adverse selection problem poses a tradeoff on the seller: by increasing the fixed portion of the hybrid utilized, the project becomes less profitable for buyers, and thus, induces less participation.

The fixed-equity hybrid conducts the screening in cash, whereas the fixed-cash hybrid conducts the screening in equity. If the seller decides to use the latter, buyers retain the power of blocking the implementation, conditional on the fact that they decide to participate in the auction -which now depends on the fixed-amount of cash requested by seller to the winners. On the other hand, when the seller uses the former, the screening is realized in cash, which is the cheapest way "good types" can use to distinguish themselves. Therefore, the screening realized is more effective, and the seller ends up trading with the good types more often. This is the intuition that justifies the preeminence of the fixed-equity hybrid as the best instrument in the menu. At the same time, that is the reason why equity is the worst. More surprisingly is the result that the optimal fixed-cash hybrid performs worse than cash, if the value of the externality is sufficiently high (in absolute value). However, it reflects the fact that when buyers want to avoid a sufficiently high negative externality, their willingness to pay upfront more than offsets the potential extraction through equity.

An interesting feature of our result is that it seems to be robust to the structure of the information. That is, even when buyer's valuations are private information, the fixed-equity hybrid continues to be the best, and equity continues to be the worst. However, the fixed-equity instrument now does not always have an equilibrium in pure strategies, which increases the uncertainty over seller's revenue in the more general case.

Finally, we analyze what would happen to the optimal fixed-payment portion in both hybrids when the distribution improves in the Monotone Likelihood Ratio property. Intuitively, we obtain that the amount of cash in the fixed-cash hybrid is non-decreasing, and that under some condition of the distributions, the equity portion in the fixed-equity hybrid is also non-decreasing. These results state that when buyers draw better valuations, the seller is less concerned about inducing participation, and can extract a higher surplus from the winner of the auction.

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APPENDIX A - ASSIGNMENT PROBLEM AS A GAME OF INCOMPLETE INFORMATION

Assuming that the only piece of information students have at the time to make their decision is their preferences and scores, they are engaged in a game of incomplete information in which the ulterior objective is to maximize the expected utility derived of their major assigned in equilibrium. The space of actions of this game is M^k -the set of k-tuples defined over the set of majors- and so, a strategy σ for each student is a mapping $\sigma : [\underline{x}, \overline{x}]^S \times U \to \Delta(M^k)$, where $\Delta(M^k)$ denotes the set of probability distributions over M^k. Alternatively, we can express any mixed strategy as a randomization over the set of pure strategies. That is, letting B be the space of functions $\beta : [\underline{x}, \overline{x}] \times U \to M^k$, a mixed strategy σ corresponds to a probability distribution of $\Delta(B)$. Moreover, for a profile of mixed strategies $(\sigma_1, \cdots, \sigma_S) \in \Delta(B)^S$, the density assigned by the mixed strategy σ_s to the pure strategy β_s is given by $\lambda_{\sigma_s}(\beta_s)$. Then, for any profile of pure strategies $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_S)$, $\lambda_{\boldsymbol{\sigma}}(\boldsymbol{\beta}) = \prod_{s=1}^S \lambda_{\sigma_s}(\beta_s)$ denotes the density of playing the pure strategy $(\beta_1, \cdots, \beta_S)$ under the mixed strategy $\boldsymbol{\sigma}$.

Hence, when the profile of mixed strategies $(\sigma_s, \boldsymbol{\sigma}_{-s})$ is played, the expected utility of student s under the mechanism ϕ, corresponds to:

$$EU_s(\boldsymbol{\sigma}) = \int_{B^S} \left(\int_{U^S} \int_{[\underline{x}, \overline{x}]^M} u(\phi_s(\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{u}))h(\boldsymbol{u}, \boldsymbol{x})d(\boldsymbol{u}, \boldsymbol{x}) \right) \lambda_{\boldsymbol{\sigma}}(\boldsymbol{\beta})d\boldsymbol{\beta} \tag{4}$$

When students are not allowed to report a complete order over majors the mechanism is not strategy proof. Hence, it is not possible to use a dominant strategies equilibrium as a solution concept; instead, we rely on the Bayesian Nash equilibrium in un-dominated strategies to solve the game. However, not even the equilibrium can be guaranteed since the space of preferences and scores is a continuum, and therefore it could be the case that (4) is not well defined.

APPENDIX B - DATA ANALYSIS

Table 3: Assigned Seats in Each Area by High School Type and Sex

	High school type		Gender		
Academic Area	Private	Public	Male	Female	Total
Arts	44	57	39	62	101
Literature	202	236	167	271	438
Hard Sciences	195	284	282	197	479
Education	195	447	211	431	642
Economics Sciences	441	491	512	420	932
Social Sciences	378	424	353	449	802
Engineering	499	510	724	285	1,009
Health	285	269	200	354	554
Agroalimentary Sciences	115	140	141	114	255
Total	2,354	2,858	2,629	2,583	5,212
(%)	45.2%	54.8%	50.4%	49.6%	

Table 3 shows the distribution by gender and high school type among the different academic areas. There are not remarkable differences at the aggregate level between public and private high schools, since approximately 50% and 55% of the student body are male and from public schools, respectively. However, within areas it is possible to observe a pattern of this sort in education and engineering. In the former, 67% are women, while in the latter 71% are men. Education also presents a bias towards public high schools, with a representation over 69%.

Table 4: Students who Accepted Admission at their Second Option

Group	Total	Percentage
Arts	8	8%
Literature	191	42%
Hard Sciences	206	43%
Education	316	47%
Economics Sciences	392	37%
Social Sciences	325	38%
Engineering	250	23%
Health	194	35%
Agro alimentary Sciences	135	45%

Third column presents the percentage of students who where admitted, and consolidated their enrollment, at a major that was ranked second in their submitted report to the RO.

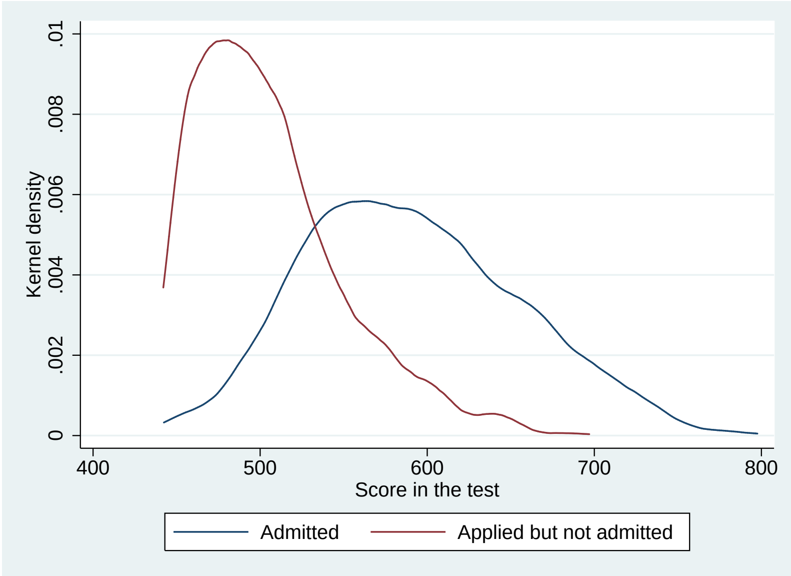


Figure 6: Distribution of Scores for Students who Applied for Admission.

Figure 6 shows the distribution of the admission scores for those students were admitted and for those who applied but were not admitted. As expected, the distribution of students who were not admitted is skewed to the left. One interesting aspect is that there is a non-negligible mass to the right of 600 under the non-admitted curve, and to the left of 500 under the admitted curve. The former group is the students who probably took "too much risk" and the latter is the group of played a protective strategy at the time to report his options to the RO. The figure presents clear evidence that the restriction in the number of options affects the caliber of the students admitted.

APPENDIX C - SIMULATION RESULTS

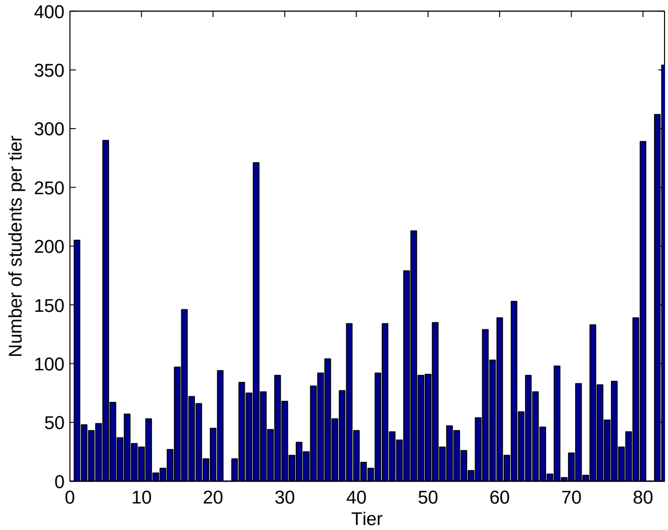


Figure 7: Distribution of Students in Tiers.

This figure shows the distribution of students in each of the tiers constructed from the maximum threshold scores. The reports of the students in the first tier are used as the commencement of the preference profiles. The frequency of students in the last tiers increase since we use the historical maximum scores to construct the tiers.

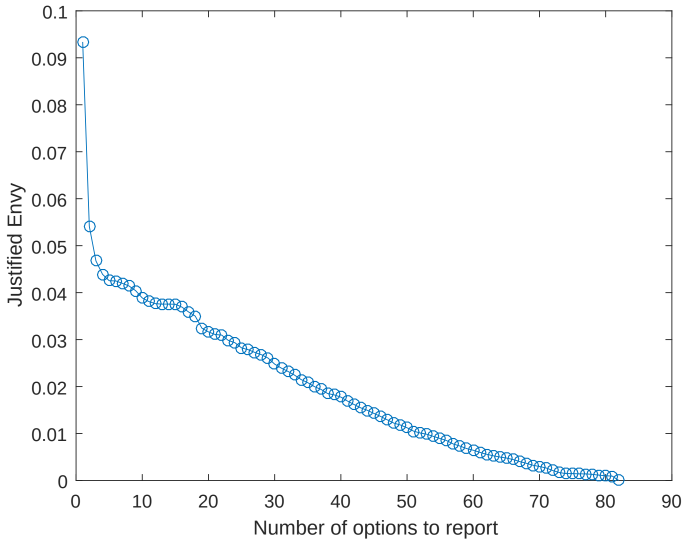


Figure 8: Ranking of the Assignments with respect to Students' Preferences.

The figure shows the fraction of cases of "justified envy" with respect to the total of possible pairwise comparisons. For example, taking the student with the highest score, we analyze if he envies the allocation of the student with second highest score, the third highest score, and so on, until to the student with the lowest score. Then, we repeat the process for all students. All possible comparisons conform the total of possible pairwise comparisons.

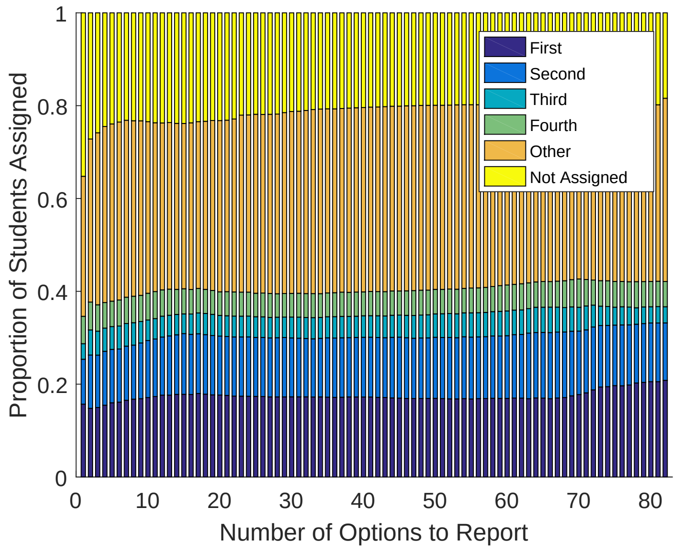


Figure 9: Ranking of the Assignments with respect to Students' Preferences.

This figure shows the distribution of the assignment with respect to students' true preferences, as a function of the number of options to report.

The figure shows the simulated threshold and different confidence intervals. Except for the majors in Arts and Literatures -the ones with lower index in the graph- which are less stable due to the extra ability requirements, for the rest of the majors the fit is satisfactory. This feature is a measure of the robustness of the algorithm, since this measure of the admission process was not targeted by the algorithm.

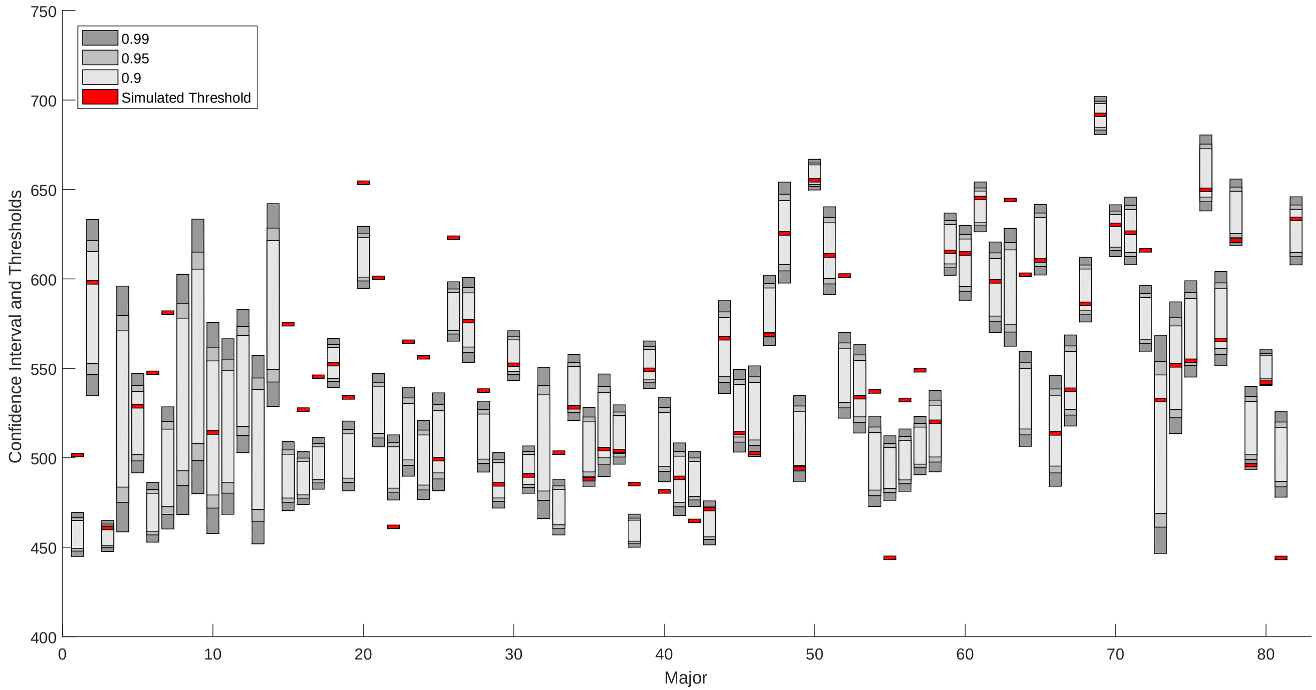
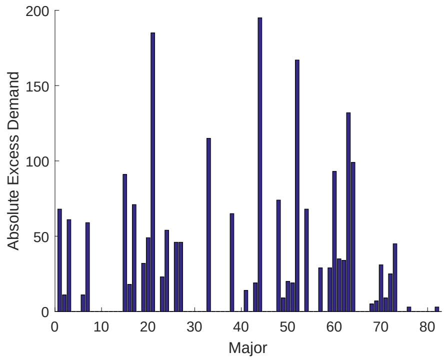
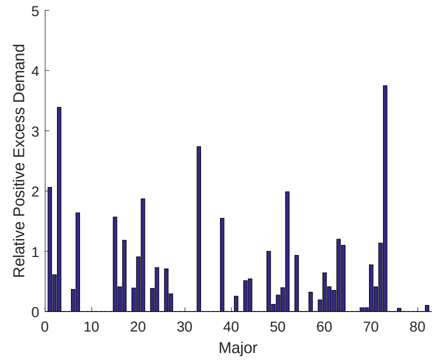


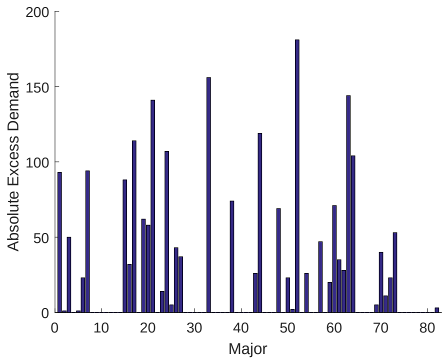
Figure 10: Confidence Interval of the Threshold Scores.



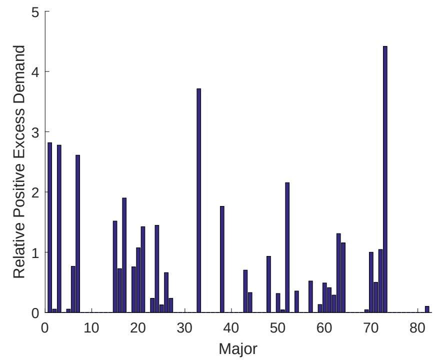
(a) Absolute Excess Demand: Mean



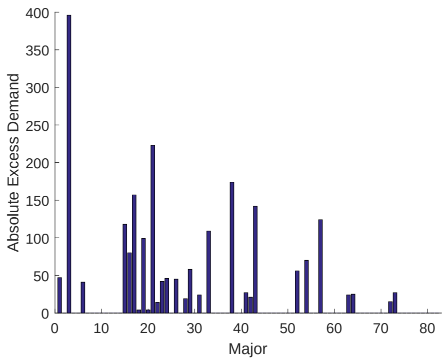
(b) Relative Excess Demand: Mean



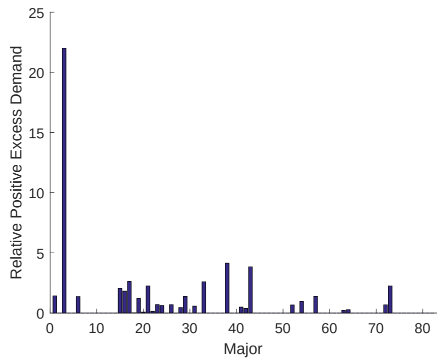
(c) Absolute Excess Demand: Median



(d) Relative Excess Demand: Median



(e) Absolute Excess Demand: Maximum



(f) Relative Excess Demand: Maximum

Figure 11: Excess Demand in the PSU Mechanism for Selected Threshold Scores

This panel shows the excess demand in the different majors when the planner announces three different configurations of the thresholds scores. In particular, figure 11a shows the absoulte excess demand when the planner announces the mean of the past threshold scores in each major. Likewise, figure 11b shows the excess demand relative to the number of seats offered in each major. Figures 11c to 11f do the same when the planner announces the median and the maximum of the past threshold scores.

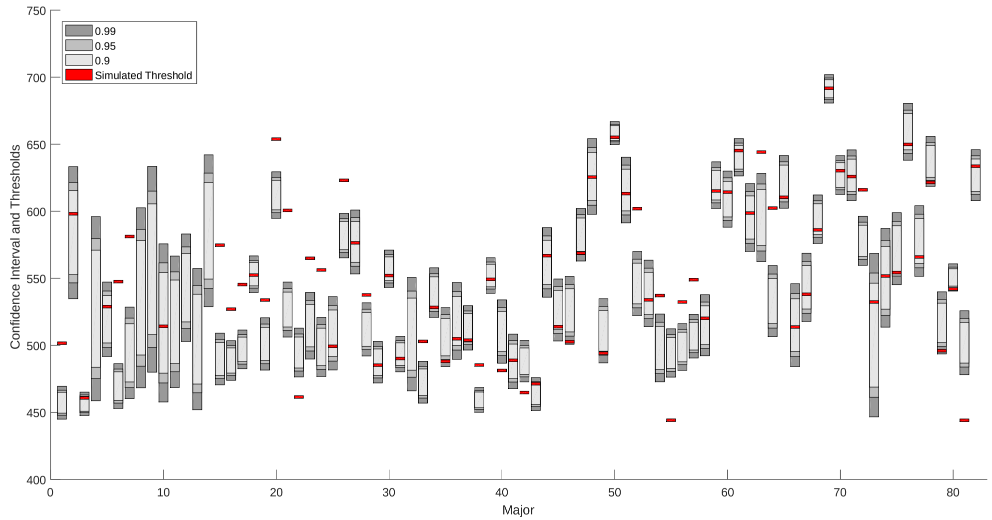


Figure 12: Assignment in the AA Mechanism.

The figure shows the equilibrium thresholds obtained by the Ascending Auction mechanism. We also depict the confidence intervals for the threshold scores, using the information of past threshold scores.

APPENDIX D - OMITTED PROOFS

Proof of Theorem 3. We will prove the theorem following three steps. First we will prove that the optimal fixed-equity hybrid involves a portion of equity $\bar{\alpha}^* \in (0,1)$, which immediately implies that the hybrid dominates pure cash in revenue. Analogously, in the second step we will show that the optimal fixed-cash hybrid involves a positive amount of cash (i.e. $\bar{b}^* > 0$), which in turn implies that it dominates pure equity in revenue. Finally, we will prove that the revenue under cash is higher than the revenue under the optimal fixed-cash hybrid.

Step 1 We take first order conditions by applying Leibniz' rule to the three different terms in (4.1). First derivative $D_1(\bar{\alpha}, c, e)$ corresponds to:

$$2F(\frac{c}{1-\bar{\alpha}})[\frac{(c-e)e}{(1-\bar{\alpha})^2} f(\frac{c-e}{(1-\bar{\alpha})}) - \int_{\frac{c}{1-\bar{\alpha}}}^{\frac{c-e}{1-\bar{\alpha}}} v_1 f(v_1) dv_1] - 2f(\frac{c}{1-\bar{\alpha}}) \frac{c}{(1-\bar{\alpha})^2} \int_{\frac{c}{1-\bar{\alpha}}}^{\frac{c-e}{1-\bar{\alpha}}} ((1-\bar{\alpha})v_1 - c) f(v_1) dv_1$$

Likewise, second derivative $D_2(\bar{\alpha}, c, e)$ is given by

$$2F(\frac{c}{1-\bar{\alpha}})[\frac{(\bar{\alpha}c - e)}{1-\bar{\alpha}} \frac{(c-e)}{(1-\bar{\alpha})^2} f(\frac{c-e}{(1-\bar{\alpha})}) + \int_{\frac{c-e}{1-\bar{\alpha}}}^{\bar{v}} v_1 f(v_1) dv_1] - 2f(\frac{c}{1-\bar{\alpha}}) \frac{c}{(1-\bar{\alpha})^2} \int_{\frac{c}{1-\bar{\alpha}}}^{\frac{c-e}{1-\bar{\alpha}}} (\bar{\alpha} v_1 - e) f(v_1) d$$

Finally, applying Leibniz rule twice in the third term and using the fact that valuations are independently and identically distributed, the third derivative $D_3(\bar{\alpha}, c, e)$ becomes:

$$2[\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} (\bar{\alpha} v_1 - e) f(v_1) dv_1] \frac{c}{(1-\bar{\alpha})^2} f(\frac{c}{1-\bar{\alpha}}) + \int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} \int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} (\max\{v_1, v_2\} - \min\{v_1, v_2\}) f(v_1) f(v_2) dv_1 dv_2$$

Letting $\tilde{D}(\alpha, c, e) = D_1(\alpha, c, e) + D_2(\alpha, c, e) + D_3(\alpha, c, e)$ we have that

$$\begin{aligned}
\tilde{D}(0, c, e) &= 2F(c)[\int_{c-e}^{\bar{v}} v_1 f(v_1) dv_1 - \int_{c}^{c-e} v_1 f(v_1) dv_1] \\
&- 2f(c)c[\int_{c}^{c-e} (v_1 - c) f(v_1) dv_1] - e(1 - 2F(c) + F(c-e))] \\
&+ \int_{c}^{\bar{v}} \int_{c}^{\bar{v}} (\max\{v_1, v_2\} - \min\{v_1, v_2\}) f(v_1) f(v_2) dv_1 dv_2
\end{aligned}$$

Now, we will explore the behavior of the first order condition when the cost tends to zero.

$$\lim_{c \downarrow 0} \tilde{D}(0, c, e) = \int_{0}^{\bar{v}} \int_{0}^{\bar{v}} (\max\{v_1, v_2\} - \min\{v_1, v_2\}) f(v_1) f(v_2) dv_1 dv_2 > 0 \;\; \forall e < 0$$

Notice that for a given e, as $\tilde{D}(0, c, e))$ is continuous, there exists a cut-off in the cost

$$\bar{c}_1 := \sup\{\tilde{c} > 0 : \tilde{D}(0,c,e) > 0 \text{ for all } c \in (0,\tilde{c})\}$$

Moreover,

$$\lim_{\bar{\alpha}\to 1} \Pi^{fe}(\bar{\alpha}) = 0$$

and,

$$\begin{aligned}\Pi^{fe}(0) &= 2F(c)[\int_c^{c-e}(v-c)f(v)dv + \int_{c-e}^{\bar{v}}(-e)f(v)dv] \\ &+ \int_c^{\bar{v}}\int_c^{\bar{v}}(\min\{v_1,v_2\}-c-e)f(v_1)f(v_2)dv_1dv_2 > 0\end{aligned}$$

Therefore, because revenue is strictly increasing at $\bar{\alpha} = 0$ and $\Pi^{fe}(0) > \Pi^{fe}(1)$ for all e, the optimal fraction of equity $\bar{\alpha}^* \in (0,1)$.

Step 2 Now we will prove that the optimal portion of cash in the fixed-cash hybrid is positive. That is, $\bar{b}^* > 0$.

Taking first order conditions of (4.2) with respect to $\bar{b}$ we have that

$$\bar{b}^* = \frac{1-F(c+\bar{b}^*)}{f(c+\bar{b}^*)}\frac{1+F(c+\bar{b}^*)}{2F(c+\bar{b}^*)} = \frac{1}{\lambda(c+\bar{b}^*)}\frac{1+F(c+\bar{b}^*)}{2F(c+\bar{b}^*)} \tag{5}$$

where $\lambda(\cdot)$ is the hazard ratio associated with f.

Now, as the density f is log-concave, by theorem 3 in Bagnoli and Bergstrom (2005) the hazard rate λ of F is an increasing function. Therefore, the second derivative of (4.2) is negative for all $\bar{b}$, and the expression in (5) corresponds to its unique global solution. Intuitively, if the seller raises marginally the fixed amount $\bar{b}$, his revenue increases by this amount only with probability $1-F(c+\bar{b})$, which is the likelihood that the project is profitable for a particular buyer. On the other hand, $f(\bar{b})$, measures the loss in implementation the seller will cause by rising the fixed amount of cash requested. That is, the seller will gain the marginal amount in the cash requested except in those cases where the winner was already indifferent between implementing or not the project. In those cases, if the seller raises $\bar{b}$ now the project is not profitable for the winner, and the seller will reduce participation. This expression is scaled by the factor at the right.

Step 3 In the last step we will show that the revenue under cash is higher than the revenue under the optimal fixed-cash hybrid.

Let $\bar{b}^*$ be the optimal fixed-cash amount when the cost is zero, and thus $\bar{b}^* = \frac{1-F(\bar{b}^*)}{f(\bar{b}^*)}\frac{1+F(\bar{b}^*)}{2F(\bar{b}^*)}$. Hence,

$$\begin{aligned}\lim_{c\downarrow 0}\Pi^{fe}(0,c,e) &= -e + \int_0^{\bar{v}}\int_0^{\bar{v}}\min\{v_1,v_2\}f(v_1)f(v_2)dv_1dv_2 \\ &> -e + \int_{\bar{b}^*}^{\bar{v}}\int_{\bar{b}^*}^{\bar{v}}\min\{v_1,v_2\}f(v_1)f(v_2)dv_1dv_2\end{aligned}$$

That is, when the cost approaches to zero from above, the expected revenue when the seller uses cash is higher than the expected revenue under the best fixed-cash hybrid.

Now, fix $c \in (c, \bar{c})$. Using the expressions of revenue for fixed-equity (4.1) and fixed-cash (4.2) hybrids, we need to show that

$$\begin{aligned}\Pi^{fe}(0, c, e) &= 2F(c)[\int_c^{c-e}(v_1 - c)f(v_1)dv_1 + \int_{c-e}^{\bar{v}}(-e)f(v_1)dv_1] \\ &+ \int_c^{\bar{v}}\int_c^{\bar{v}}[\min\{v_1, v_2\} - c - e]f(v_1)f(v_2)dv_1dv_2\end{aligned}$$

is greater than

$$\begin{aligned}\Pi^{fc}(\bar{b}^*, c, e) &= 2\int_0^{c+\bar{b}^*(c)}\int_{c+\bar{b}^*(c)}^{\bar{v}}\bar{b}^* f(v_1)f(v_2)dv_1dv_2 \\ &+ \int_{c+\bar{b}^*(c)}^{\bar{v}}\int_{c+\bar{b}^*(c)}^{\bar{v}}\bar{b}^* f(v_1)f(v_2)dv_1dv_2\end{aligned}$$

Or rearranging terms, we need that

$$\begin{aligned}2F(c)[\int_c^{c-e}(v_1 - c)f(v_1)dv_1 + \int_{c-e}^{\bar{v}}(-e)f(v_1)dv_1] + (1 - F(c))^2(-c - e) \\ + \int_{c+\bar{b}^*(c)}^{\bar{v}}\int_{c+\bar{b}^*(c)}^{\bar{v}}[\min\{v_1, v_2\} - \bar{b}^*(c)]f(v_1)f(v_2)dv_1dv_2 \\ \int_0^{c+\bar{b}^*(c)}\int_0^{c+\bar{b}^*(c)}\min\{v_1, v_2\}f(v_1)f(v_2)dv_1dv_2\end{aligned}$$

be greater than

$$\frac{(1 - F(c + \bar{b}^*(c))^2)}{f(c + \bar{b}^*(c))}(1 + F(c + \bar{b}^*(c)))$$

where the last expression is obtained by replacing the functional form of $\bar{b}^*(c)$. Hence, to show that $\Pi^{fe}(0) > \Pi^{fc}(\bar{b}^*(c))$ is sufficient that

$$-e > \frac{1 + F(c + \bar{b}^*(c))}{f(c + \bar{b}^*(c))} + c - \frac{2}{(1 - F(c))^2}\int_0^{c+\bar{b}^*(c)}(1 - F(v_1))f(v_1)v_1dv_1$$

Therefore we can define

$$-\underline{e} = \arg\max_{c\in(0,\bar{c})}\{\frac{1 + F(c + \bar{b}^*(c))}{f(c + \bar{b}^*(c))} + c - \frac{2}{(1 - F(c))^2}\int_0^{c+\bar{b}^*(c)}(1 - F(v_1))f(v_1)v_1dv_1\}$$

Figure 13 shows the behavior of $\bar{c}$ as a function of $|e|$. If $c < \bar{c}$ then theorem 3 holds thus $\Pi^{fc}(0) > \Pi^{fc}(\bar{b}^*)$, otherwise the reverse is true.

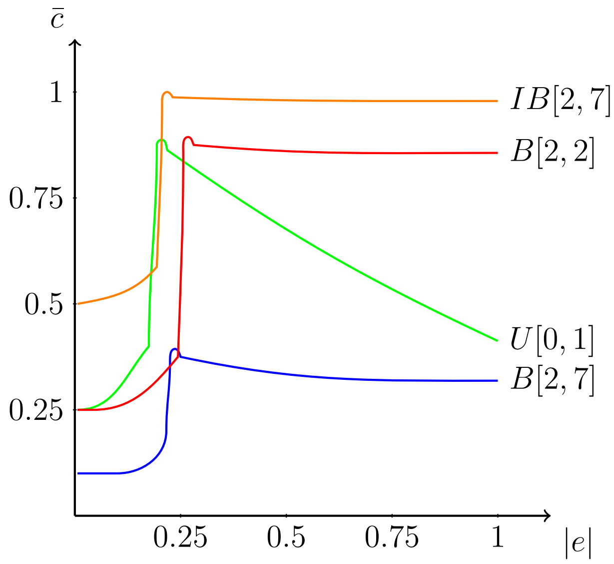


Figure 13: Upper bound of the cost for different distributions

Table 5: Revenue in Cash and Fixed-Cash as a function of e

		Expected Seller Revenue			
Distribution	e	$\Pi^{fe}(0)$	$\Pi^{fc}(\bar{b}^*)$	$\bar{b}^*$	$\underline{e}$
$U[0,1]$	-0.001	0.334333	0.3849	0.57735	-0.0512
	-0.01	0.343333			
	-0.1	0.433333			
$B[2,2]$	-0.001	0.372429	0.375	0.5	-0.0036
	-0.01	0.381429			
	-0.1	0.471429			
$B[2,7]$	-0.001	0.15002	0.152539	0.222329	-0.0035
	-0.01	0.15902			
	-0.1	0.24902			
$IB[2,7]$	-0.001	0.705575	0.656547	0.718398	$\nexists$
	-0.01	0.714575			
	-0.1	0.804575			

On table 5 we explore theorem 3 by showing the value of $\underline{e}$ for different distributions. Alongside, we present the values of $\Pi^{fe}(0)$ and $\Pi^{fc}(\bar{b}^*)$ for different values of e, to confirm why the bound is needed although it is rather low.

■

Proof of Theorem 4. We will prove the result using techniques of monotone comparative statics on lattice programming, for which we need to introduce some the definitions and results of this theory.

Definition 6 (Milgrom and Shannon (1994)). *Let X and T be non-empty subsets of $\mathbb{R}$ and let $g : X \times T \to \mathbb{R}$. We say g satisfies the* **strict single crossing property** *(SSCP) in (x,t) if for every x'', x' in X and t'', t' in T, with $x'' > x'$ and $t'' > t'$*

$$g(x'',t') \geq g(x',t') \text{ implies } g(x'',t'') > g(x',t'') \tag{6}$$

and we write $g(\cdot,t'') \succeq_{SSCP} g(\cdot,t')$.

Definition 7 (Quah and Strulovici (2009)). *Let X and T be non-empty subsets of $\mathbb{R}$, and let $\{g(\cdot,t)\}_{t\in T}$ be a family of real valued functions defined on X, we say that $g(\cdot,t')$ is interval order dominated by $g(\cdot,t'')$ -with the notation $g(\cdot,t'') \succeq_{IDO} g(\cdot,t'')$- if equation (6) holds for all $x' < x''$ whenever $g(x,t') < g(x'',t'')$ for all $x \in [x',x'']$.*

Proposition 12 (Quah and Strulovici (2009)). *Let X and T be respectively an interval and a non-empty subsets of $\mathbb{R}$, and suppose that $\{g(x,\cdot)\}_{t\in T}$ is a family of real valued functions, which are also absolutely continuous in intervals of X; and that there is a positive an increasing function $h : X \to \mathbb{R}$ such that $g'(x,t'') > h(x)g'(x,t')$ a.e. Then, $g(\cdot,t'') \succeq_{IDO} g(\cdot,t')$*

Theorem 5 (Quah and Strulovici (2009)). *Let X and T be non-empty subsets of $\mathbb{R}$ and let $g(\cdot,t''), g(\cdot,t')$ be two real valued functions defined on X, with $t'', t' \in T$ such that $t'' > t'$. If $g(\cdot,t'') \succeq_{IDO} g(\cdot,t')$ then*

$$argmax_{x\in J} g(\cdot,t'') > argmax_{x\in J} g(\cdot,t') \quad \text{for any interval } J \text{ of } X. \tag{7}$$

Furthermore, if (7) is satisfied then $g(\cdot,t'') \succeq_{IDO} g(\cdot,t')$

Suppose f_1 dominates f_0 in the monotone likelihood ratio (MLR) and rewrite (**??**) as

$$\begin{aligned}
\Pi^{fe}(\bar{\alpha},t) &= 2F_t\left(\frac{c}{1-\bar{\alpha}}\right)\int_{\frac{c}{1-\bar{\alpha}}}^{\frac{c-e}{1-\bar{\alpha}}} ((1-\bar{\alpha})v_1 - c)f_t(v_1)dv_1 \\
&+ 2F_t\left(\frac{c}{1-\bar{\alpha}}\right)\int_{\frac{c-e}{1-\bar{\alpha}}}^{\bar{v}} (\bar{\alpha}v_1 - e)f_t(v_1)dv_1 \\
&+ \int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} [(1-\bar{\alpha})\min\{v_1,v_2\} - c - e + \bar{\alpha}\max\{v_1,v_2\}]f_t(v_1)f_t(v_2)dv_1dv_2
\end{aligned} \tag{8}$$

with $t \in \{0,1\}$. It is sufficient to show that there exists a positive and increasing function $h(\alpha)$ such that $\Pi^{fe}_{\alpha}(\alpha,1) > h(\alpha)\Pi^{fe}_{\alpha}(\alpha,0)$ to show that $\alpha^*_1 \geq \alpha^*_0$, in virtue of proposition 12 and theorem 5.

Define $h(\alpha,c) = \frac{f_1(\frac{c-e}{1-\bar{\alpha}})}{f_0(\frac{c}{1-\bar{\alpha}})}$ and $g(\alpha,c) = \frac{F_1(\frac{c}{1-\bar{\alpha}})}{F_0(\frac{c}{1-\bar{\alpha}})}$. Notice that $h(\alpha,c)$ is increasing in α for all c, and hence, if we show that $\Pi^{fe}_{\alpha}(\alpha,1) - h(\alpha,c)\Pi^{fe}_{\alpha}(\alpha,0) > 0$ we can conclude that $\Pi^{fe}(\alpha,1) \succeq_{IDO} \Pi^{fe}(\alpha,0)$ In order to show that, we can proceed separately as we did with the derivative the proof of theorem 3.

Thus, for the first term we have

$$\frac{2(c-e)e}{(1-\bar{\alpha})^2} f(\frac{c}{(1-\bar{\alpha})}) \, [g(\alpha,c)-h(\alpha,c)] - 2\int_{\frac{c}{1-\bar{\alpha}}}^{\frac{c-e}{1-\bar{\alpha}}} v_1[g(\alpha,c)\frac{f_1(v_1)}{f_0(v_1)} - h(\alpha,c)]dv_1 \tag{9}$$

Likewise, the second term corresponds to

$$\frac{2(\bar{\alpha}c-e)}{1-\bar{\alpha}} \frac{c-e}{(1-\bar{\alpha})^2}[g(\alpha,c)-h(\alpha,c)] + 2\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} v_1[g(\alpha,c)\frac{f_1(v_1)}{f_0(v_1)} - h(\alpha,c)]dv_1 \tag{10}$$

The third term is equal to

$$\frac{2c}{(1-\bar{\alpha})^2}\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} (\bar{\alpha}v_1 - e)[\frac{f_1(v)}{f_0(v_1)} - h(\alpha,c)]dv_1 \tag{11}$$
$$+\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}}\int_{\frac{c}{1-\bar{\alpha}}}^{\bar{v}} (\max\{v_1,v_2\}-\min\{v_1,v_2\})[\frac{f_1(v_1)f_1(v_2)}{f_0(v_1)f_0(v_2)} - h(\alpha,c)]dv_1dv_2$$

Grouping the first terms in (9) and (10), respectively, we get

$$\frac{2\bar{\alpha}(c-e)^2}{(1-\bar{\alpha})^3}[g(\alpha,c)-h(\alpha,c)] \tag{12}$$

Likewise, adding the second terms in (9) and (10) we obtain

$$2\int_{\frac{c-e}{1-\bar{\alpha}}}^{\bar{v}} v_1[g(\alpha,c)\frac{f_1(v_1)}{f_0(v_1)} - h(\alpha,c)]dv_1 \tag{13}$$

Terms (11)-(13) imply the result because we assume that $g(\alpha,c) > h(\alpha,c)$, f_0 is dominated in MLR by f_1, and the inferior limit of all the integrals involved is greater than or equal to $\frac{c}{1-\bar{\alpha}}$.

Applying the same argument, we can see that $\bar{b}_1^* > \bar{b}_0^*$ if and only if

$$-2\bar{b} + \frac{1}{\lambda_0(c+\bar{b})}\frac{1+F_0(c+\bar{b})}{F_0(c+\bar{b})} > h(\bar{b})[-2\bar{b} + \frac{1}{\lambda_1(c+\bar{b})}\frac{1+F_1(c+\bar{b})}{F_1(c+\bar{b})}] \tag{14}$$

for $h(\cdot)$ increasing and positive.

Notice that as f_1 dominates f_0 in MLR then the hazard ratio is decreasing (i.e $\lambda_0 < \lambda_1$). Moreover, it implies that F_1 dominates F_0 in first stochastic dominance order (FOSD), which in turn implies that $\frac{1+F_0(c+\bar{b})}{F_0(c+\bar{b})} < \frac{1+F_1(c+\bar{b})}{F_1(c+\bar{b})}$. Therefore the condition in (14) is satisfied for $h(\cdot)$. $\blacksquare$

APPENDIX E - SIMULATION FOR DIFFERENT DISTRIBUTIONS

Following the results presented on figure 4.4, here we show the behavior of revenue as a function of $\bar{\alpha}$ and $\bar{b}$ for the main distributions considered in this article. Figure 14 has the functions for a $Beta[2,2]$, figure 15 has the functions for a $Beta[2,7]$ and figure 16 has the functions for an $InverseBeta[2,7]$

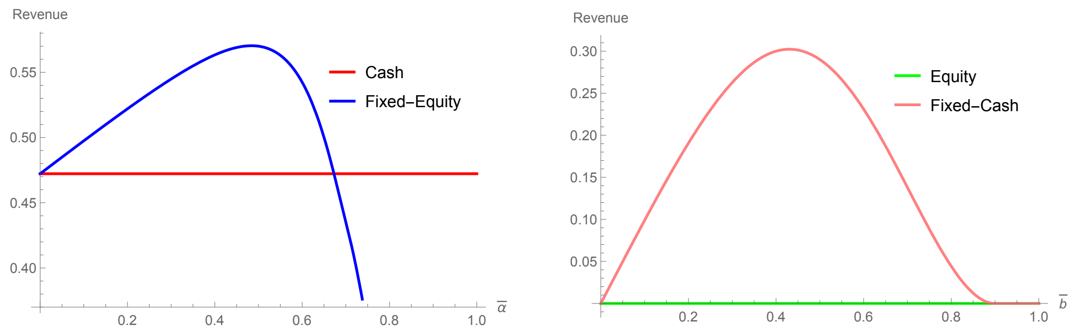


Figure 14: Revenue as a function of $\bar{\alpha}$ and $\bar{b}$ for $B[2,2]$

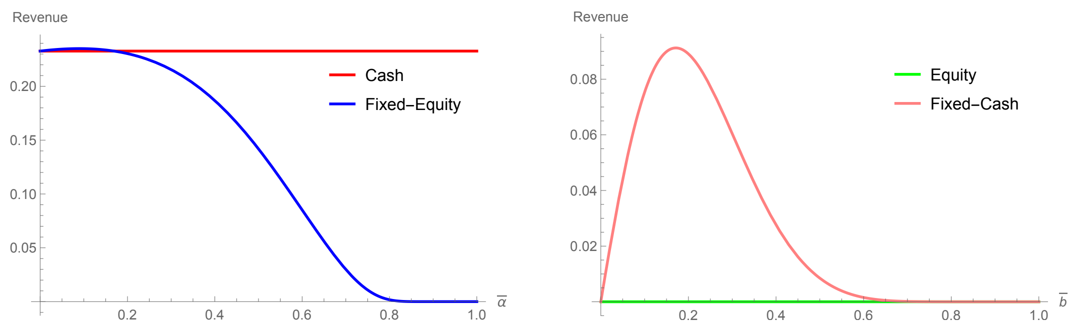


Figure 15: Revenue as a function of $\bar{\alpha}$ and $\bar{b}$ for $B[2,7]$

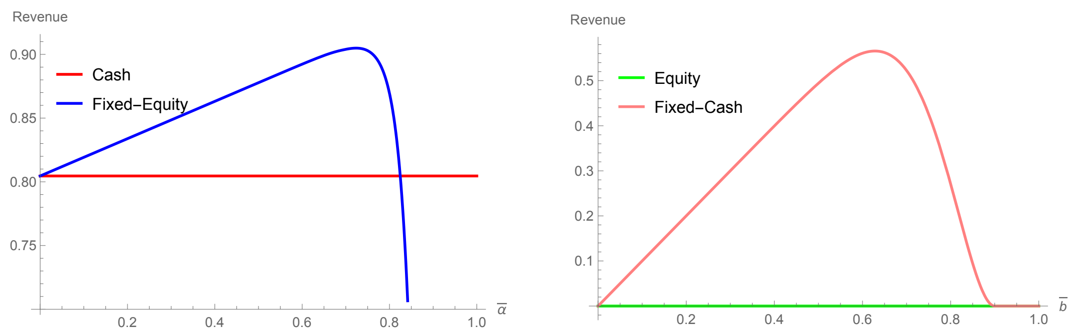


Figure 16: Revenue as a function of $\bar{\alpha}$ and $\bar{b}$ for $IB[2,7]$