Potential Games and Competition in the Supply of Natural Resources

by

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ABSTRACT

This dissertation discusses the Cournot competition and competitions in the exploitation of common pool resources and its extension to the tragedy of the commons. I address these models by using potential games and inquire how these models reflect the real competitions for provisions of environmental resources. The Cournot models are dependent upon how many firms there are so that the resultant Cournot-Nash equilibrium is dependent upon the number of firms in oligopoly. But many studies do not take into account how the resultant Cournot-Nash equilibrium is sensitive to the change of the number of firms. Potential games can find out the outcome when the number of firms changes in addition to providing the "traditional" Cournot-Nash equilibrium when the number of firms is fixed. Hence, I use potential games to fill the gaps that exist in the studies of competitions in oligopoly and common pool resources and extend our knowledge in these topics. In specific, one of the rational conclusions from the Cournot model is that a firm's best policy is to split into separate firms. In real life, we usually witness the other way around; i.e., several firms attempt to merge and enjoy the monopoly profit by restricting the amount of output and raising the price. I aim to solve this conundrum by using potential games. I also clarify, within the Cournot competition model, how regulatory intervention in the management of environmental pollution externalities affects the equilibrium number of polluters. In addition, the tragedy of the commons is the term widely used to describe the overexploitation of open-access common-pool resources. Open-access encourages potential resource users to continue to enter the resource up to the point where rents are exhausted. The resulting level of resource use is higher than is socially optimal, and in extreme cases can lead to the collapse of the resource and the communities that may depend on it. In this paper I use the concept of potential games to evaluate the relation between the cost of resource use and the equilibrium number of resource users in open access regimes. I find that costs of access and costs of production are sufficient to determine the equilibrium number of resource users, and that there is in fact a continuum between Cournot competition and the tragedy of the commons. I note that the various common pool resource management regimes identified in the empirical literature are associated with particular cost structures, and hence that this may be the mechanism that determines the number of resource users accessing the resource.

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INTRODUCTION

1.1 Motivation

This dissertation discusses the Cournot competition and competitions in the exploitation of common pool resources and its extention to the tragedy of the commons. I address these models by using potential games and inquire how these models reflect the real competitions for provisions of environmental resources.

There are three important empirical points that need to be addressed in the theory of oligopoly. First, it is of our general knowledge that the number of suppliers of environmental public goods has a significant impact on well-being via the effects it has on prices, quantities, cooperation, and coordination. However, many studies assume that the number of suppliers is fixed, and do not consider how the Cournot-Nash equilibrium is sensitive to the change of the number of suppliers. Second, the Cournot model makes it possible to identify the conditions under which monopoly or oligopoly might be observed, but there is no general theory of the equilibrium number of parties to an agreement or the equilibrium number of suppliers in resource access game. Third, by extension of the Cournot model, it is possible to identify the equilibrium number of firms as a function of the costs of engagement. However, the theory of potential games can answer these points; i.e., potential games make it possible to determine the number of firms involved in providing a public good.

There are several studies that analyze the oligopolistic industires. For instance, the world oil market is a typical example of oligopoly, and it is often analyzed by the Cournot model (Loury, 1986; McMillan and Sinn, 1984; Salant, 1976, 1982).

Because these models are dependent upon how many firms there are, the resultant Cournot-Nash equilibrium is dependent upon the number of firms in oligopoly. But these studies do not take into account how the resultant Cournot-Nash equilibrium is sensitive to the change of the number of firms. In fact, as mentioned above, potential games can find out the outcome when the number of firms changes in addition to providing the "traditional" Cournot-Nash equilibrium found in these studies.

Hence, I use potential games to fill the gaps that exist in the studies of competitions in oligopoly and common pool resources and extend our knowledge in these topics. In addition, the tragedy of the commons is the term widely used to describe the overexploitation of open-access common-pool resources. Open access is argued to encourage entry up to the point where rents are exhausted. In this dissertation, I reexamine the number of users of common pool resources through the concept of potential games. I show that the equilibrium number of users is sensitive to the structure of costs of entry. I find that costs of access and costs of production are sufficient to determine the equilibrium number of resource users, and that there is in fact a continuum between Cournot competition and the tragedy of the commons. I note that the various common pool resource management regimes identified in the empirical literature are associated with particular cost structures, and hence that this may be the mechanisms that determine the number of resource users accessing the resource. Finally, I briefly consider the implication of my results to the recently discussed issue of the common pool resource based collapse of societies (Butzer, 2012; Dasgupta and Heal, 1979; Diamond, 2005).

1.2 Cournot Competition

1.2.1 Description of Cournot Competition

In the mid-19th century, Antoine Augustine Cournot modeled competitions in a market for environmental resources (spring water) that was dominated by two suppliers (Cournot and Fisher, 1897). Today, economists use the Cournot model as a model of oligopoly (i.e., a few firms competing in the same market) as well as a model of competition in the provisions of environmental resources. In this model, each firm adjusts output (supply) to maximize its profit. Unlike perfect competition, the demand curve does not stay constant. Rather, as the quantity demanded increases, the price decreases.

In the Cournot model, economists usually model the demand curve as a decreasing linear function of the quantity demanded although other forms of demand curves are possible. There is at least one attempt to model a Cournot model with a nonlinear demand function in potential games (Dragone *et al.*, 2012). However, to make the problems tractable, the authors assume that the cost function is nil, which is unrealistic. I am unaware of Cournot models that assume both non-linear demand and positive costs. In this dissertation, I use linear demand functions in accordance with studies that discuss Cournot competition in the context of potential games (Monderer and Shapley, 1996; Slade, 1994; Ui, 2000).

Here is the basic mathematical structure of the Cournot model. Suppose π_i is the profit of firm i, q_i is the output of firm i, p is the price, and c is a positive constant. This model assumes that each firm chooses its output supposing that the other firms will not vary their output. Then

$$\begin{aligned}\pi_i &= pq_i - cq_i \\ &= \left(l - \sum_j q_j\right) q_i - cq_i\end{aligned}$$

where l is a positive constant and sufficiently large so that the equilibrium makes sense (i.e., the output and the profit do not become negative).

In case of two firms, we have

$$\begin{aligned}\pi_1 &= (l - q_1 - q_2)q_1 - cq_1 \\ \pi_2 &= (l - q_1 - q_2)q_2 - cq_2\end{aligned}$$

The solution, (q_1^*, q_2^*), that maximizes a firm's profit given the amount of output by its opponent is called the Cournot-Nash equilibrium.

1.2.2 Unresolved Issues of the Cournot Model

Cournot competition is extremely popular among economists, partly because its mathematical expression is relatively simple. As a result, it has appeared in many economic studies of oligopoly, including some studies of oligopoly in the supply of environmental goods and services (especially in the study of the world oil market) (Loury, 1986; McMillan and Sinn, 1984; Salant, 1976, 1982). Most such studies assume that the marginal cost is constant although we frequently observe that marginal costs are an increasing function of output. There has been some limited effort to analyze the Cournot model with increasing marginal costs (Canton *et al.*, 2008; Lee, 1999; McKitrick, 1999; Okuguchi, 2004; Schernikau, 2010). Most studies also assume that the cost function insensitive to the number of firms in the game; i.e., suppose the cost function in case of monopoly is cq where q is output and c is a constant. Most treatments apply the same cost function for a firm when it is in duopoly or triopoly;

i.e., cq_i, where q_i is the output of firm i. As the number of firms changes, the cost function of a firm is unlikely to stay the same because of the change of the scale of its operation, the change of its transaction cost, etc.

The rational conclusion from the Cournot model is that a firm's best policy is frequently to split into separate firms (Rasmusen, 2007). For instance, suppose there are two firms, *Apex* and *Brydox*. Obviously, *Apex* gets half the industry profits in a duopoly game. If *Apex* splits into firms *Apex 1* and *Apex 2*, the Cournot model predicts that it would get two thirds of the profit in the Cournot triopoly game, even though industry profit falls implying that *Apex* would indeed split into separate firms.

Furthermore, Gibbons (1992) argues that when there are two firms, both firms have an incentive to deviate from the strategy of producing the monopoly output. Both firms try to produce more output so that they can earn more profit. On the other hand, when they are producing at the duopoly level (Nash equilibrium), neither firm has an incentive to deviate because deviation does not change their profit.

Since the evidence indicates that environmental public goods are supplied in a number of ways (i.e., extending from monopoly all the way to unlimited entry), we need a more systematic understanding of what it is that determines the numbers of firms.

1.2.3 Research Questions for Cournot Competitions

Based on the argument above, I address two research questions.

Research Question 1. In what conditions does Cournot competition in the supply of environmental resources result in monopoly, and in what conditions does it result in duopoly?

In reality, monopoly emerges far more frequently than a firm splitting into smaller firms. Hence, in Chapter 3, I address how the Nash equilibrium changes as the number of firms changes if I assume that the cost function of a firm becomes larger as the number of firms increases. I also address how the equilibrium changes as the number of firms changes if I assume that the marginal cost is an increasing function of the output. When I address these issues, I aim to determine, under what forms of cost functions, monopoly becomes the only Nash equilibrium.

Research Question 2. Within the Cournot competition model, how does regulatory intervention in the management of environmental pollution externalities affect the equilibrium number of polluters?

In Chapter 4, first I aim to determine how the equilibrium changes under the presence of pollution and also determine the optimal rate of tax on effluent. In fact, there have been many studies conducted on this issue; however, most of them do not consider how the equilibrium and the optimal rate of taxation change as the number of firms changes (Canton *et al.*, 2008; Kennedy, 1994; Lee, 1999; Levin, 1985; McKitrick, 1999; Okuguchi, 2004; Requate, 1993; Simpson, 1995). These studies inquire the output of each firm and the optimal rate of taxation at the Nash equilibrium given the number of firms fixed. In Chapter 4, I identify the Nash equilibrium and the optimal rate of taxation, too. However, my main focus is on the research question mentioned above, not the appropriate amount of tax per se. In Chapter 3, I address the research question without taking into account the abatement cost of pollution and the tax on the amount of effluent. Hence, in Chapter 4, I inquire how regulatory intervention in the management of environmental pollution externalities affects the equilibrium number of polluters. As a result, I hope my result will provide a new insight into the issue of pollution and tax on effluent in oligopoly as the number of firms changes.

1.3 The Tragedy of the Commons

1.3.1 Description of the Tragedy of the Commons

Ever since publication of Garrett Hardin's influential paper (Hardin, 1968), the causes and consequences of the tragedy of the commons have been a topic of debate among social and life scientists. If there is a consensus, it is that the tragedy of the commons involves the overexploitation of environmental resources that are open access but also scarce (Dasgupta, 2001; Feeny *et al.*, 1990; Libecap, 2009; McWhinnie, 2009; Ostrom, 2015; Ostrom *et al.*, 2002; Perrings, 2014). Open access is critical. Anderies and Janssen (2013) argue that it should properly be called the "tragedy of open access" because the "tragedy" happens only if access to the commons is open to anyone able to meet the cost of access. The same literature, however, draws attention to the fact that in very many cases the number of resource users who actually access the commons is considerably less than the number entitled to do so. Indeed, there is now a large empirical literature on the conditions under which common pool resources have been exploited around the world, focusing on the mechanisms that limit the number of users (Berkes *et al.*, 1989; Dolsak and Ostrom, 2003; Ostrom, 2015; Ostrom *et al.*, 2002; Seabright, 1993). From a theoretical and experimental perspective, there is an equally large literature exploring the effect of variations in incentives on entry in commons games (see for example Dragone *et al.* (2013); Mason and Polasky (1997)). In this paper I revisit the relationship between costs of access and production in common pool resources, and the equilibrium number of resource users.

The archetype of the tragedy of the commons is as follows. Suppose, in a rural village, there are cattle herders sharing a common parcel of land on which each is entitled to let his or her cattle graze. If there is open access to the resource, those entitled to access the resource have an incentive to add cattle as long as it is profitable to do so (Hardin, 1968). It follows that the number of cattle on the commons will depend on the factors that determine profitability: the marginal costs and benefits of grazing. In what follows I focus on the structure of costs in the case where the commons game is symmetric. That is, I consider common pool resources in which resource users are homogeneous, so all users produce at the same level and face the same costs, but costs vary with the number of users. I show that the equilibrium number of entrants to the commons depends solely on costs. For particular cost structures I find that the equilibrium number of users may be infinite, but for most cost structures the number will be finite and decreasing in the cost of access or production. In the limit, as costs of access or production rise, the equilibrium number of users is one. Cournot competition and the tragedy of the commons belong to a continuum in which the structure of costs uniquely determines the number of resource users. If the institutional mechanisms explored by Ostrom and colleagues (Ostrom, 2015) have implications for cost structures, I can analyze the consequences for the equilibrium number of resource users. But I do not concern myself with drivers of changes in cost structures beyond the number of resource users.

The general problem of the tragedy of the commons is important precisely because many environmental resources are open access—fisheries, forests, rangelands, water resources, and the atmosphere all contain frequently cited examples (Anderies and Janssen, 2013). A number of the regulating services, such as storm buffering, erosion control, or pest predation, are also exploited in similar ways (Perrings, 2014). In all cases, overexploitation of resources is argued to be a common consequence of open access. Indeed, in the wake of Diamond (2005), it has been argued that the depletion of common pool resources due to open access has been the trigger for more fundamental societal collapse (Dasgupta *et al.*, 2016). Since we know that open access has not always had the dire consequences for common pool resources described by Dasgupta *et al.* (2016), it is worth asking when open access leads to the degradation of resources, and when it does not.

One implication of the approach stimulated by Ostrom (2015) is that a necessary condition for common pool resources to be conserved is the emergence of regulatory institutions. However, there are cases where open access does not lead to large numbers of resource users or to the exhaustion of rents even without such institutions. Indeed, it is arguable that the regulatory mechanisms described by Ostrom and colleagues work precisely because they change the structure of costs, and that other unrelated factors might have the same effect. Since the cost functions faced by resource users are accessible to policy makers, this significantly widens the set of instruments available to manage environmental resources in the public domain.

My approach treats the decision to enter the commons strategically. In particular, I treat use of common pool resources as a congestion game, and exploit the potential function corresponding to the game. The management of common pool resources has often been modeled as a cooperative game (Funaki and Yamato, 1999; Uzawa, 2005). While the approach works well for some regulated access resources, however, it is less helpful when access is open and there are no restrictions on entry to the resource. The number of resource users matters. Establishing and enforcing binding agreements among a small number of resource users is usually not difficult (Ostrom, 2015). However, as the number of resource users increases, cooperation is increasingly less likely. Thus, it is often appropriate to model the tragedy of the commons as a noncooperative game.

Among studies that have taken this approach, the study by Sandler and Arce M. (2003) considers open access fisheries and concludes that the Nash equilibrium involves overfishing as long as the average product of fishery assets exceeds the marginal product. They show that if each fishing firm receives a share of output equal to its share of effort, then it will equate the price of fish to the weighted sum of its marginal and average product. As the number of fishing firms increases, the weight assigned to average product increases, and as the number of firms approaches infinity, profits approach zero. As in most other papers treating the tragedy of the commons as a noncooperative game (e.g., Dasgupta and Heal (1979); McCarthy *et al.* (2001)), they treat both relative prices and the number of fishing firms as exogenous. In this paper I follow these authors in treating strategic behavior as non-cooperative, but I also treat the number of resource users in open access resources as endogenous. More particularly, I identify the number of resource users at the Nash equilibrium corresponding to the particular cost structure.

1.3.2 Research Question for the Tragedy of the Commons

In Chapter 5, based on the argument above, I address the following research question.

Research Question 3. Under the condition of open access, what is the equilibrium number of resource users in a common and how does it change with the costs?

1.4 Outline of the Dissertation

The outline of the dissertations is as follows. In Chapter 2, I briefly explain what the potential game is and provide examples of stochastically stable equilibrium. In Chapter 3, I address, in Cournot competition, how the Nash equilibrium changes as the number of players changes under various assumptions on cost functions so that I determine the conditions of the emergence of monopoly. The result obtained in this chapter is compared with the empirical case of the merger of non-profit organizations in the field of environmental protection. In Chapter 4, I address Cournot competition with pollution and taxation on effluent and determine how regulatory intervention in the management of environmental pollution externalities affects the equilibrium number of polluters. The analysis in this chapter indicates that the tax rate imposed by the government is not influential on firms' decisions as to whether they should merge or stay split; i.e., as long as the profit per firm is highest for a monopoly, then the firm "endures" the high tax rate so that the government's "punishment" against monopoly will fail. This result is in accordance with the empirical cases of mergers and acquisitions in the oil and natural gas industry that are mentioned in Chapter 4. However, the important thing to note is that the analysis in Chapter 4 is valid only when the price is stable. For a complete analysis, I need to take into account environmental stochasticity of the price in the industry, and this is the topic I will tackle in the future. In Chapter 5, I address competitions in the exploitation of common pool resources. The base model is provided by Gibbons (1992), and by using potential games, I analyze the case that the number of potential entrants to the common is unlimited, and reconsider the relationship between the number of resource users and the cost functions; i.e., the number of resource users is endogenous, and I inquire how that number varies with the structure of costs. The result obtained in this chapter is compared with the empirical example of Japanese fishery and the lobster industry in Maine. Also I seek the implication of the results obtained in this chapter to more general societal collapse mentioned in other studies (Butzer, 2012; Dasgupta *et al.*, 2016; Diamond, 2005). Lastly, in Chapter 6, I provide a general conclusion.

Chapter 2

POTENTIAL GAMES: DEFINITION AND THE EXISTENCE AND STABILITY OF NASH EQUILIBRIA

2.1 Definition

This dissertation uses potential games to answer the research questions. Although there are precursors to the idea of potential games in the literature on strategic behavior, it was the paper by Monderer and Shapley (1996) that formally organized ideas about potentials that had been scattered across various disciplines and structured those ideas as a unified theory. Suppose $\Gamma = (N, A, u)$ denotes a strategic form game, and that there is a finite number of players, $N = 1, ..., n$. A_i is a set of strategies for player $i \in N$ where $A = (A_i)_{i \in N}$, and $u_i \in \mathbf{R}$ is a payoff function for player $i \in N$. I use customary game-theoretic notations of $a = (a_1, a_2, ..., a_n)$ and $a_{-i} = (a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_n)$ where $a_i \in A_i$.

Definition. (Monderer and Shapley, 1996) $\Gamma = (N, A, u)$ is called an exact potential game if there exists a function $\Pi : A \rightarrow \mathbf{R}$ such that

$$u_i(a_i^{'}, a_{-i}) - u_i(a_i, a_{-i}) = \Pi(a_i^{'}, a_{-i}) - \Pi(a_i, a_{-i})$$

for any $i \in N$, $a_i, a_i^{'} \in A_i$ and $a \in A$.

If set A is continuous and u_i and Π are differentialble on A_i, then the differences in the payoff function and the potential function are replaced by derivatives; i.e., $\Gamma = (N, A, u)$ is an exact potential game if there exists a function $\Pi : A \rightarrow \mathbf{R}$ such that

$$\frac{\partial u_i}{\partial a_i} = \frac{\partial \Pi}{\partial a_i}$$

for any $i \in N$ and $a_i \in A_i$. Moreover, Monderer and Shapley (1996) clarify that the necessary and sufficient condition for the continuous game to have a potential function is

$$\frac{\partial^2 u_i}{\partial a_i \partial a_j} = \frac{\partial^2 u_j}{\partial a_i \partial a_j}$$

for all i and j. This condition provides a convenient criterion for testing whether any continuous game is a potential game. Together with the relationship of the first-order derivatives above, it can be used as a tool to find the potential function by taking anti-derivatives. I note that this condition is related to the fact that costs and revenues are shared by all resource users equally in symmetric games.

The usefulness of the potential game rests upon the following theorem.

Theorem. *(Monderer and Shapley, 1996) Let* $\Gamma = (N, A, u)$ *be an exact potential game with a potential function* Π. *Let* $\overline{\Gamma}$ *be the game with* (N, A, Π) *in which every player's payoff function is* Π. *Then, the set of Nash equilibria of* Γ *coincides with that of* $\overline{\Gamma}$.

This theorem ensures that a single potential function can be used to find all the Nash equilibria of a game, so simplifying analysis of the game. Moreover, for continuous exact potential game, if A_i is compact for all $i \in N$, then the game has at least one pure strategy Nash equilibrium; i.e., $a^* \in \underset{a \in A}{arg\,max}\, \Pi(a)$ is a Nash equilibrium (Foster and Young, 1990; Monderer and Shapley, 1996; Ui, 2000). The implication is that potential games can be studied from two different perspectives. First, they can be studied within the classical framework of game theory. Second, they can be studied through optimization of the potential function (Goyal, 2012; Monderer and Shapley, 1996; Slade, 1994). In this paper, I use the optimization framework; i.e., the Nash equilibrium is found by identifying the *argmax* of the potential function.

2.2 Stochastically Stable Nash Equilibria and Potential Functions

2.2.1 Definition

I exploit the fact that the Nash equilibria found from potential functions have a "stronger" property; i.e., the equilibria found from potential functions are "stochastically stable equilibria," a refinement of the evolutionarily stable strategies in conventional game theory (Foster and Young, 1990). More particularly, a state P is a stochastically stable equilibrium (SSE) if, in the long run, it is nearly certain that the system lies within every small neighborhood of P as noise tends to zero. That is, the stochastically stable set (SSS) is the set of states S such that, in the long run, it is nearly certain that the system lies within every open set containing S as noise tends to zero. This concept of equilibrium relates to the process of adaptive learning in games. A player observes the history of how other players have played against him or her in the past, and chooses a strategy for the future that is a best response to the past play of others (Gintis, 2009). Adaptive learning in games allows players to make errors when they perceive how others have behaved given past realizations of the system. But as the game continues, players learn from these mistakes and the frequency of errors becomes lower. In this way, the stochastically stable equilibrium is attained.

What is significant is that the potential function of the game attains the global maximum for maximization problems at the stochastically stable equilibrium (Alós-Ferrer and Netzer, 2010; Foster and Young, 1990; Goyal, 2012). There is, however, a restriction to this result. Alós-Ferrer and Netzer (2010) find that, for some exact potential games, the stochastically stable equilibrium coincides with the argmax of the potential function only if players revise their strategies based on "asynchronous learning,"; i.e., exactly one player is randomly selected every period to revise his or her strategy. If revisions of the strategies are not asynchronous (e.g., every player revises his or her strategy at the same time), the realized Nash equilibrium may not maximize or minimize the potential.

Despite this limitation, potential functions can be used as equilibrium selection tools when there are multiple Nash equilibria. The stochastically stable Nash equilibrium is the one that maximizes the potential function when players revise their strategies asynchronously. As a result, among all possible Nash equilibria, the stochastically stable equilibrium that makes the potential function maximum is more likely to emerge than other Nash equilibria when the game is played. Thus, throughout this dissertation, I assume that players revise their strategies asynchronously.

2.2.2 Examples

Adaptive Learning

The example in this section is from Ui (2000) and called the bilateral symmetric interaction game. Consider the strategic form game $\Gamma = (N, A, u)$ with $i, j \in N$ and $i \neq j$. The functions w_{ij} and h_i are defined such that $w_{ij} : A_i \times A_j \to \mathbf{R}$ and $h_i : A_i \to \mathbf{R}$. In addition, w_{ij} is assumed to be symmetric; i.e., $w_{ij}(a_i, a_j) = w_{ji}(a_j, a_i)$. Then, the payoff function for player i is defined as

$$u_i(a) = \sum_{j \in N \setminus \{i\}} w_{ij}(a_i, a_j) - h_i(a_i)$$

With this payoff function, the potential function is defined as

$$V(a) = \sum_{i<j} w_{ij}(a_i, a_j) - \sum_{i} h_i(a_i)$$

For the case of 2×2 games with two players, suppoese $A_1 = A_2 = \{0, 1\}$. Let's assume that w_{ij} and h_i take the following values.

$$w_{12}(0,0) = w_{21}(0,0) = x$$

$$w_{12}(0,1) = w_{21}(1,0) = z$$

$$w_{12}(1,0) = w_{21}(0,1) = z$$

$$w_{12}(1,1) = w_{21}(1,1) = y + z - w$$

$$h_1(0) = h_2(0) = 0$$

$$h_1(1) = h_2(1) = z - w$$

Then, Table 2.1 is the game matrix.

Table 2.1: Bilateral Symmetric Interaction Game (General Form)

player 1 \player 2	0	1
0	x, x	z, w
1	w, z	y, y

(player 1's payoff, player 2's payoff)

With this construction of a game matrix, many games that are familiar to us (e.g., coordination games, prisoner's dilemma, etc.) can be constructed. For instance, suppose $x = 5$, $y = 5$, and $w = z = 0$. Then, I obtain Table 2.2. This is obviously a coordination game. Now, if I change the value of y from 5 to 3, I obtain Table 2.3.

Table 2.2: Bilateral Symmetric Interaction Game (Two Nash Equilibria with $x = y = 5$, and $w = z = 0$.)

player 1 \player 2	0	1
0	$5, 5$	$0, 0$
1	$0, 0$	$5, 5$

(player 1's payoff, player 2's payoff)

Table 2.3: Bilateral Symmetric Interaction Game (Two Nash Equilibria with $x = 5$, $y = 3$, and $w = z = 0$.)

player 1 \player 2	0	1
0	$5, 5$	$0, 0$
1	$0, 0$	$3, 3$

(player 1's payoff, player 2's payoff)

It is intuitive that, in case of Table 2.2, $(0,0)$ and $(1,1)$ will emerge with the same frequency when players keep playing the game repeatedly, and in case of Table 2.3, $(0,0)$ will emerge far more frequently than $(1,1)$. This intuition is correct, and there are two ways to verify it. One is by using Markov chains, and the other is by using potential functions.

First, I address the method by Markov chains. Before I begin the argument, I define the states; i.e., state 1 is both players choose 0, state 2 is when one player chooses 0 and the other player chooses 1, and state 3 is when both players choose 1. Suppose each player remembers the opponent's last action. In addition, I assume that each player knows the payoff matrix, and each player knows that the other player knows the payoff matrix. Then, the transition matrix is defined as

$$M = (P_{ij})$$

where $i, j = 1, 2, 3$. Hence P_{ij} is the probability that the state changes from i to j. In case of Table 2.2, when the opponent played 0 and the player played 0 last time, then the player should play 0 now. The same argument holds when the opponent played 1 and the player played 1 last time: then the player should play 1 now. When the opponent played 0 and the player played 1 last time or vice versa, then the player should play 0 with probability 0.5 and 1 with probability 0.5 now. As a result, there is probability 0.5 that the state is in 2 again, probability 0.25 that the state changes to 1, and probability 0.25 that the state changes to 3. Consequently, the transition matrix becomes as below.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0 & 1 \end{pmatrix}$$

If players keep playing the game for sufficiently many times, the transition matrix converges. For instance, when the game is repeated 20 times, I obtain

$$M^{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, our intuition is correct; i.e., when players started the game in states 1 or 3, they would stay in 1 or 3. However, when they started the game in 2, there is a 50% of chance that players will end up in either state 1 or 2.

For the case of Table 2.3, the situation is a little different. When the opponent played 0 and the player played 0 last time, then the player should play 0 now. The same argument holds when the opponent played 1 and the player played 1 last time: then, the player should play 1 now. But when the opponent played 0 and the player played 1 last time, then, the player should play 0. When the opponent played 1 and the player played 0 last time, the player should still play 0 because they can increase their payoff when the opponent changes his or her strategy. Hence, there is probability 0 that the state changes from 2 to 3. Thus, the transition matrix becomes as follows:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If players keep playing the game for sufficiently many times, the transition matrix converges. For instance, when the game is repeated 20 times, I obtain

$$M^{20} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, when the game starts at states 1 or 3, then the game will stay there. However, if the game starts at state 2, then the game will converge to state 1. Consequently, our intuition is once again correct; i.e., state 1 emerges unless the game starts at state 3.

Now, suppose that each player finds a best response to the other player's previous move with probability $1-\varepsilon$ but chooses incorrectly with probability $\varepsilon > 0$ (Gintis, 2009). For the case of Table 2.2, when the other player played 0 last time, the players responds "correctly" by choosing 0 with probability $1-\varepsilon$, and this is the same for the other player. Hence, the probability that players stay at state 1 is $(1-\varepsilon)^2$. In addition, when both players played 0, there is probability ε^2 so that they both incorrectly respond by choosing 1. The values of a row must sum up to 1, so M_{12} has to be $2\varepsilon(1-\varepsilon)$. The same argument applies for the determination of the values of row 3. As for row 2, suppose player 1 played 0 and player 2 played 1 last time. Then, the probability that the state changes from 2 to 1 is $\varepsilon(1-\varepsilon)$ because player 1 needs to make an incorrect decision that player 2 chose 0, and player 2 needs to make a correct decision that player 1 chose 0. So, the probability from state 2 to 1 is $M_{21} = 2\varepsilon(1-\varepsilon)$. The same conclusion holds for the state changing from 2 to 3. Since a row must add up to 1, $M_{22} = 1 + 4\varepsilon^2 - 4\varepsilon$. As a result, the transition matrix becomes as follows:

$$M = \begin{pmatrix} (1-\varepsilon)^2 & 2\varepsilon(1-\varepsilon) & \varepsilon^2 \\ 2\varepsilon(1-\varepsilon) & 1+4\varepsilon^2-4\varepsilon & 2\varepsilon(1-\varepsilon) \\ \varepsilon^2 & 2\varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

Suppose $\varepsilon = 0.01$. Then, M becomes

$$M = \begin{pmatrix} 0.9801 & 0.0198 & 0.0001 \\ 0.0198 & 0.9604 & 0.0198 \\ 0.0001 & 0.0198 & 0.9801 \end{pmatrix}$$

If players keep playing the game for sufficiently many times, the transition matrix converges. For instance, when the game is repeated 1,000 times, the game matrix becomes

$$M^{1000} = \begin{pmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \end{pmatrix}$$

Hence, in this case the probability that if the players start at state 1 or 3, they will stay in state 1 or 3 with probability 0.6666. This happens because the payoff for $(0, 0)$ and $(1, 1)$ are the same.

As for Table 2.3, there are notable changes from Table 2.2; i.e., $M_{13} = M_{23} = 0$ because $(1, 1)$ has a lower payoff for both players than $(0, 0)$. Hence, changes from state 1 to 3 and from 2 to 3 do not occur at all. Instead, these probabilities that exist for Table 2.2 are now added to the probabilities from 1 to 1 and 2 to 1. Hence, the transition matrix becomes as follows:

$$M = \begin{pmatrix} \varepsilon^2 + (1-\varepsilon)^2 & 2\varepsilon(1-\varepsilon) & 0 \\ 4\varepsilon(1-\varepsilon) & 1 + 4\varepsilon^2 - 4\varepsilon & 0 \\ \varepsilon^2 & 2\varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

Suppose $\varepsilon = 0.01$. Then, M becomes

$$M = \begin{pmatrix} 0.9802 & 0.0198 & 0 \\ 0.0396 & 0.9604 & 0 \\ 0.0001 & 0.0198 & 0.9801 \end{pmatrix}$$

If players keep playing the game for sufficiently many times, the transition matrix converges. For instance, when the game is repeated 1,000 times, the transition matrix becomes

$$M^{1000} = \begin{pmatrix} 0.6667 & 0.3333 & 0 \\ 0.6667 & 0.3333 & 0 \\ 0.6667 & 0.3333 & 0 \end{pmatrix}$$

Hence, the systems spends 66.67% of the time in $(0, 0)$, and all of the remaining time (33.33%) in the nearby state (i.e., not in $(1, 1)$). Consequently, the only stochastically stable equilibrium is $(0, 0)$.

Potentials of the Games

Now, I find the potentials of Tables 2.2 and 2.3. The potential function is as given earlier:

$$V(a) = \sum_{i<j} w_{ij}(a_i, a_j) - \sum_i h_i(a_i)$$

For Table 2.2, $x = 5$, $y = 5$, and $w = z = 0$. For Table 2.3, $x = 5$, $y = 3$, and $w = z = 0$. Hence, the potentials become as shown in Tables 2.4 and 2.5.

These results are consistent with the results obtained from Markov chains. For Table 2.2, strategies $(0, 0)$ and $(1, 1)$ yield the same payoffs for both players. Hence, as expected, the potentials are the same for both. In addition, the result from adaptive learning confirms that $(0, 0)$ and $(1, 1)$ occur equally; i.e., if players start at state 1 or state 3, then they stay at these states. But when they start at state 2, the state will change to state 1 with 50% of probability and to state 3 with 50% of probability. Furthermore, when the possibility of error/noise is included, all entries of transition matrix become identical, which is consistent with the fact that the potentials are

Table 2.4: Potentials of Table 2.2

player 1 \player 2	0	1
0	5	0
1	0	5

Table 2.5: Potentials of Table 2.3

player 1 \player 2	0	1
0	5	0
1	0	3

equal for pairs of strategies $(0, 0)$ and $(1, 1)$. Since there is no difference in potentials, each state (1, 2, or 3) occurs equally, and this is represented in the identical entries of the transition matrix.

For Table 2.3, each player earns more when both of them choose $(0, 0)$ than when they choose $(1, 1)$ although $(1, 1)$ is also a Nash equilibrium. If players play the game repeatedly, then we expect that $(0, 0)$ emerges more frequently than $(1, 1)$. This is verified from the transition matrix. If players start at state 1 or 3, then they will remain in that state. However, when players start at state 2, they will eventually converge on state 1. Moreover, when the possibility of error/noise is considered, the transition matrix indicates that, for the 66.67% of the time, players are in state 1, and for the 33.33% of the time, players are in nearby states (as argued by Foster and Young (1990)). Note that it is shown above that $(0, 0)$ has higher potential than state $(1, 1)$.

Consequently, as time passes, players settle at the Nash equilibrium or the neighborhood of that Nash equilibrium that has the highest potential. Hence, in potential games, rather than modeling the adaptive learning process, what we need to do is to find the pair of strategies that has the highest potential, and that is the stochastically stable equilibrium. Thus, in this research, I seek the stochastically stable equilibria by finding the arguments of maximum of the potential functions.

Chapter 3

COURNOT COMPETITION

3.1 Setting of the Game

In Chapters 3 and 4, sequential games are played. There are two firms, *Apex* and *Brydox*, as in Rasmusen (2007). They play the sequential game as depicted in Figure 3.1. *Apex* moves first and decides whether to merge with *Brydox*, to split into *Apex 1* and *Apex 2*, or neither to merge nor to split. Then, in response to the move by *Apex*, *Brydox* moves. At node 2, *Brydox* decides whether to break away from the merger with *Apex* or to not break away (i.e., *Brydox* keeps the merger with *Apex*). At node 3, *Brydox* decides whether to merge with *Apex*, to split into *Brydox 1* and *Brydox 2*, or neither to merge nor to split. At node 4, *Brydox* decides whether to split into *Brydox 1* and *Brydox 2* or not to split. π_A is the profit of *Apex*, and π_B is the profit of *Brydox*. Next to the profits of each firm is the outcome of the game. Here, I need to remind you that if *Apex* or *Brydox* splits into two firms, they do not become completely unrelated firms; i.e., *Apex* (*Brydox*) splits into *Apex 1* (*Brydox 1*) and *Apex 2* (*Brydox 2*) to increase the profit as *Apex* (*Brydox*); therefore, at the end of the game, the profits of *Apex 1* (*Brydox 1*) and *Apex 2* (*Brydox 2*) are combined and represented as the profit of *Apex* (*Brydox*).

After *Apex* and *Brydox* make their moves, the profit of each firm is determined by the Cournot model of the fixed number of firms. The calculations are shown in the following sections.

In Chapter 3, I address Research Question 1 as the cost function takes various forms. First, in Section 3.2, I analyze the "traditional" Cournot competition; i.e.,

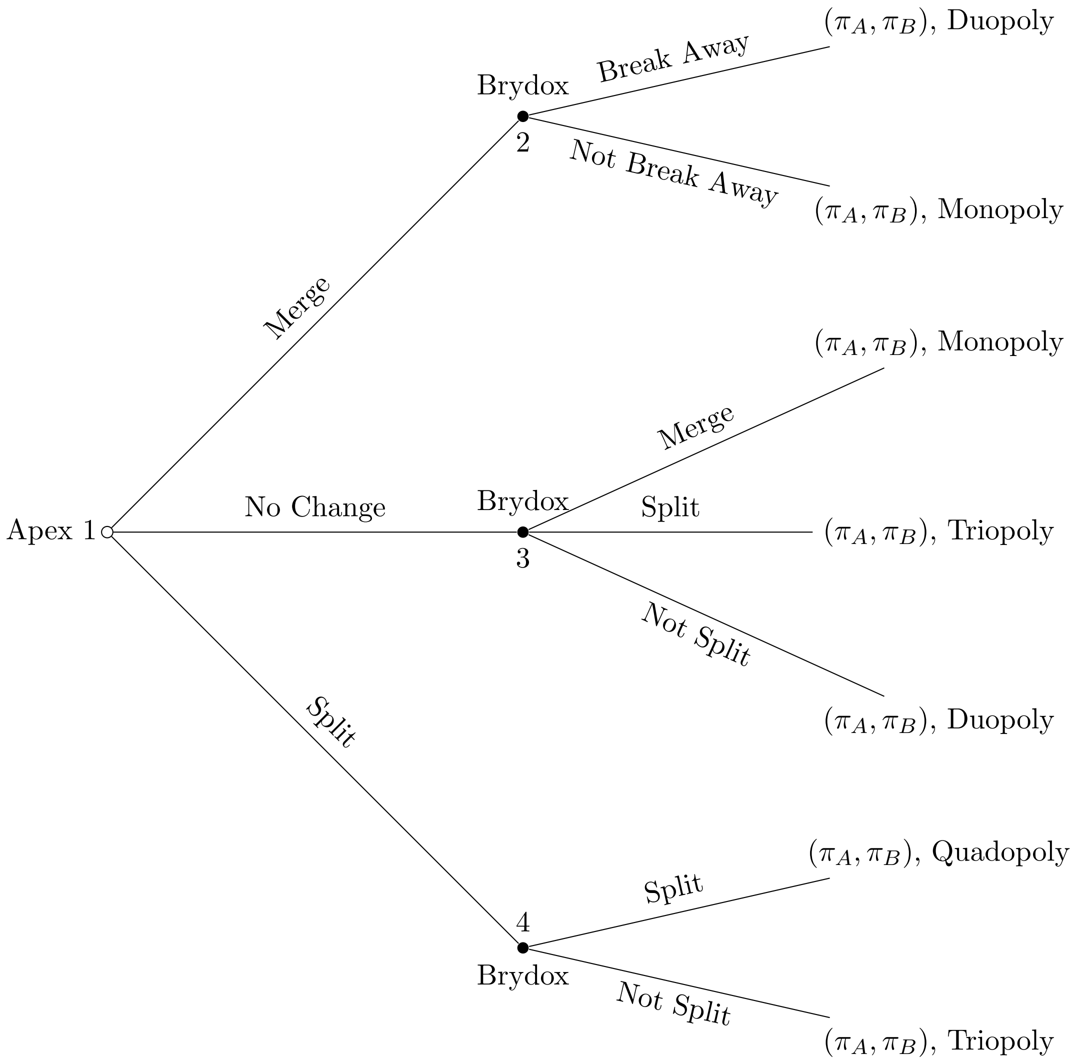

Figure 3.1: The Sequential Game of Merger/Split

the marginal cost is assumed to be constant and identical for the cases of monopoly, duopoly, triopoly and quadopoly. In other words, regardless of the number of firms in the market, a firm incurs a cost function cq_i where q_i is the output of firm i and c is a constant. Hence, the result of this section will verify what we already know about Cournot competitions from various literature; i.e., the result of this section should reveal the issue Rasmusen (2007) has raised about the splitting of a firm into separate firms as its best response.

In Section 3.3, I analyze the Cournot model when the marginal cost is an increasing function of the output, as we always assume in microeconomic analyses. Hence, the cost function is of the form $C(q) = cq^2$. In this section, the cost function is of the same form regardless of the number of firms in the market. Hence, for monopoly, duopoly, triopoly and quadopoly, the cost function is cq^2 for a firm.

In Section 3.4, as in Section 3.3, the marginal cost is increasing as the output increases. In addition, I assume that the cost function becomes larger as the number of firms increases; i.e., for monopoly, the cost function is cq^2, but for duopoly, the cost function is αcq_i^2 where $\alpha > 1$, and for triopoly, the cost function is βcq_i^2 where $\beta > \alpha$. The assumption of a larger cost function as the number of firms increases is consistent both with existence of scale economies at the level of the firm, and with the existence of transaction costs at the level of the industry.

Sections 3.3 and 3.4 use a specific quadratic expression of the cost function, rather than a general function. For instance, instead of cq^2, I can use a smooth function, $C(q)$, such that $C^{'}(q) > 0$ and $C^{''}(q) > 0$ for $q > 0$. However, in this case, I cannot obtain the specific form of q^*, and this may hinder the clarity of my arguments. Moreover, under the conditions of $C^{'}(q) > 0$ and $C^{''}(q) > 0$ for $q > 0$, the marginal cost function (the first derivative of the cost function) can be linearized around the Nash equilibrium. For the analyses of Chapters 3 and 4, I do not concern myself with the global behavior of the marginal cost function. Rather, I am interested in the intersection of the marginal revenue and the marginal cost curves, and in such a case, the linearization of the marginal cost function can provide me with fair approximations due to its smoothness at the intersection. Hence, I choose $C(q)$ to be a quadratic function so that the marginal cost function becomes a strictly increasing linear function.

Now, based on the arguments in Chapters 1 and 2, I make the three claims below.

Claim (1)**.** Suppose that marginal cost of production is constant and the cost function stays the same as the number of firms increases (i.e., the cost function is cq_i for monopoly, duopoly, triopoly and quadopoly for firm i). Then the solution of Cournot competition model is a quadopoly (i.e., firms keep splitting).

Claim (2)**.** If the cost function is identical regardless of the number of firms but marginal cost increases as an output increases, then the solution of Cournot Competition model is a monopoly (Schernikau, 2010).

Claim (3)**.** Suppose the marginal cost becomes larger as the number of firms increases and as output increases (i.e., the cost function is cq_i^2 for monopoly, αcq_i^2 for duopoly and βcq_i^2 for triopoly for firm i). Then there is a threshold value of α such that the solution of Cournot Competition model is a monopoly if α is greater than the threshold value, and there is a threshold value of β such that the solution of Cournot competition model is a duopoly, not triopoly, if β is greater than the threshold value.

3.2 Identical and Constant Marginal Costs for Monopoly, Duopoly and Triopoly

3.2.1 Monopoly/Collusion

Consider the following profit function

$$\pi = pq - cq$$

and let $p = l - q$ and $l > c > 0$ so that

$$\pi = (l - q)q - cq$$

The first-order condition is

$$\begin{aligned} \frac{d\pi}{dq} &= l - 2q - c \\ &= 0 \end{aligned}$$

Hence,

$$q_m^* = \frac{l - c}{2}$$

where the subscript denotes that it is an output for monopoly.

I need to check the second-order condition to determine whether local maximum or minimum occurs at q_m^*.

The second-order condition is

$$\frac{d^2\pi}{dq^2} = -2$$

Hence, π attains its local maximum at q_m^*.

Now, I find the optimal profit.

$$\begin{aligned} \pi_m^* &= l\left(\frac{l - c}{2}\right) - \left(\frac{l - c}{2}\right)^2 - c\left(\frac{l - c}{2}\right) \\ &= \frac{(l - c)^2}{4} \end{aligned}$$

where the subscript denotes that it is a profit function for monopoly.

Note that, in case of monopoly, there is only one player. Thus, the potential function for this case is the profit function.

3.2.2 *Duopoly*

Below is a two-player Cournot model.

$$\pi_i = pq_i - cq_i$$

where $p = l - q_1 - q_2$ and $i = 1, 2$. Then,

$$\pi_i = (l - q_1 - q_2)q_i - cq_i$$

for $i = 1, 2$. The first-order derivatives are

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_i - c$$

for $i = 1, 2$. The possible potential function, Π_d, is

$$\Pi_d = (l - q_1 - q_2)q_1 + (l - q_1 - q_2)q_2 - cq_1 - cq_2 + q_1 q_2$$

where the subscript denotes that it is a possible potential function for duopoly.

The first-order derivative of Π_d with respect to q_1 is

$$\begin{aligned} \frac{\partial \Pi_d}{\partial q_1} &= l - q_1 - q_2 - q_1 - q_2 - c + q_2 \\ &= \frac{\partial \pi_1}{\partial q_1} \end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_d}{\partial q_2} = \frac{\partial \pi_2}{\partial q_2}$. Hence, Π_d is indeed a potential function for this game.

The first-order conditions are

$$\begin{aligned} l - 2q_1 - q_2 - c &= 0 \\ l - q_1 - 2q_2 - c &= 0 \end{aligned}$$

Since the game is symmetric, $q_1^* = q_2^* = q_d^*$ where the subscript denotes that it is an output for duopoly. Hence,

$$\begin{aligned} & l - 3q_d^* - c = 0 \\ \Leftrightarrow \quad & q_d^* = \frac{l - c}{3} \end{aligned}$$

The second-order conditions for Π are

$$\frac{\partial^2 \Pi_d}{\partial q_i^2} = -2$$

for $i = 1, 2$ and

$$\frac{\partial \Pi_d^2}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

The eigenvalues are -3 and -1. Hence, H is negative definite so that Π_d and π_i ($i = 1, 2$) attain their local maxima at q_d^*.

Since the game is symmetric, $\pi_1^* = \pi_2^* = \pi_d^*$ where the subscript denotes that it is a profit for duopoly. Hence,

$$\begin{aligned}
\pi_d^* &= (l - 2q_d^*)q_d^* - cq_d^* \\
&= \left(l - \frac{2(l-c)}{3}\right)\left(\frac{l-c}{3}\right) - \frac{c(l-c)}{3} \\
&= \frac{(l-c)^2}{9}
\end{aligned}$$

Next, the maximum value of the potential function is

$$\begin{aligned}
\Pi_d^* &= 2q_d^*(l - 2q_d^*) - 2cq_d^* + q_d^{*2} \\
&= 2\left(l - \frac{2(l-c)}{3}\right)\left(\frac{l-c}{3}\right) - 2c\left(\frac{l-c}{3}\right) + \left(\frac{l-c}{3}\right)^2 \\
&= \frac{(l-c)^2}{3}
\end{aligned}$$

3.2.3 Triopoly

Below is a three-player Cournot model.

$$\pi_i = pq_i - cq_i$$

where $p = l - q_1 - q_2 - q_3$ and $i = 1, 2, 3$. Then,

$$\pi_i = (l - q_1 - q_2 - q_3)q_i - cq_i$$

for $i = 1, 2, 3$. The first-order derivatives are

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_3 - q_i - c$$

for $i = 1, 2, 3$. The possible potential function, Π_t, is

$$\begin{aligned} \Pi_t &= (l - q_1 - q_2 - q_3)q_1 + (l - q_1 - q_2 - q_3)q_2 + (l - q_1 - q_2 - q_3)q_3 \\ &\quad -cq_1 - cq_2 - cq_3 + q_1q_2 + q_1q_3 + q_2q_3 \end{aligned}$$

where the subscript denotes that it is a possible potential function for triopoly.

The first-order derivative of Π_t with respect to q_1 is

$$\begin{aligned} \frac{\partial \Pi_t}{\partial q_1} &= l - q_1 - q_2 - q_3 - q_1 - q_2 - q_3 - c + q_2 + q_3 \\ &= l - q_1 - q_2 - q_3 - q_1 - c \\ &= \frac{\partial \pi_1}{\partial q_1} \end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_t}{\partial q_i} = \frac{\partial \pi_i}{\partial q_i}$ for $i = 2, 3$. Hence, Π_t is indeed a potential function for this game.

The first-order conditions are

$$\begin{aligned} l - q_1 - q_2 - q_3 - q_1 - c &= 0 \\ l - q_1 - q_2 - q_3 - q_2 - c &= 0 \\ l - q_1 - q_2 - q_3 - q_3 - c &= 0 \end{aligned}$$

Since the game is symmetric, $q_1^* = q_2^* = q_3^* = q_t^*$ where the subscript denotes that it is an output for triopoly. Hence, the first-order conditions become

$$\begin{aligned} & l - 4q_t^* - c = 0 \\ \Leftrightarrow \quad & q_t^* = \frac{l - c}{4} \end{aligned}$$

The second-order conditions for Π_t are

$$\frac{\partial^2 \Pi_t}{\partial q_i^2} = -2$$

for $i = 1, 2, 3$, and

$$\frac{\partial^2 \Pi_t}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

The eigenvalues are -4 and two multiplicities of -1. Hence, H is negative definite so that Π_t and π_i $(i = 1, 2, 3)$ attain their local maxima at q^*.

Since the game is symmetric, $\pi_i^* = \pi_t^*$ where $i = 1, 2, 3$ and the subscript denotes that it is a profit for triopoly. Hence,

$$\begin{aligned} \pi_t^* &= \left\{l - 3\left(\frac{l-c}{4}\right)\right\}\left(\frac{l-c}{4}\right) - c\left(\frac{l-c}{4}\right) \\ &= \left(\frac{l+3c}{4}\right)\left(\frac{l-c}{4}\right) - c\left(\frac{l-c}{4}\right) \\ &= \frac{(l-c)^2}{16} \end{aligned}$$

Next, the maximum value of the potential function is

$$\begin{aligned} \Pi_t^* &= 3(l - 3q_t^*)q_t^* - 3cq_t^* + 3q_t^{*2} \\ &= 3lq_t^* - 6q_t^{*2} - 3cq_t^* \\ &= 3l\left(\frac{l-c}{4}\right) - 6\left(\frac{l-c}{4}\right)^2 - 3c\left(\frac{l-c}{4}\right) \\ &= \frac{3(l-c)^2}{8} \end{aligned}$$

3.2.4 Quadopoly

Below is a four-player Cournot model.

$$\pi_i = pq_i - cq_i$$

where $p = l - q_1 - q_2 - q_3 - q_4$ and $i = 1, 2, 3, 4$. Then,

$$\pi_i = (l - q_1 - q_2 - q_3 - q_4)q_i - cq_i$$

The first-order derivative is

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_3 - q_4 - q_i - c$$

for $i = 1, 2, 3, 4$. The possible potential function, Π_q, is

$$\begin{aligned}\Pi_q &= (l - q_1 - q_2 - q_3 - q_4)q_1 + (l - q_1 - q_2 - q_3 - q_4)q_2 + (l - q_1 - q_2 - q_3 - q_4)q_3 \\ &\quad + (l - q_1 - q_2 - q_3 - q_4)q_4 - cq_1 - cq_2 - cq_3 - cq_4 \\ &\quad + q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4\end{aligned}$$

where the subscript denotes that it is a possible potential function for quadopoly.

The first-order derivative of Π_q with respect to q_1 is

$$\begin{aligned}\frac{\partial \Pi_q}{\partial q_1} &= l - q_1 - q_2 - q_3 - q_4 - q_1 - q_2 - q_3 - q_4 - c + q_2 + q_3 + q_4 \\ &= l - q_1 - q_2 - q_3 - q_4 - q_1 - c \\ &= \frac{\partial \pi_1}{\partial q_1}\end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_q}{\partial q_i} = \frac{\partial \pi_i}{\partial q_i}$ for $i = 2, 3, 4$. Hence, Π_q is indeed a potential function for this game.

The first-order conditions are

$$\begin{aligned} l - q_1 - q_2 - q_3 - q_4 - q_1 - c &= 0 \\ l - q_1 - q_2 - q_3 - q_4 - q_2 - c &= 0 \\ l - q_1 - q_2 - q_3 - q_4 - q_3 - c &= 0 \\ l - q_1 - q_2 - q_3 - q_4 - q_4 - c &= 0 \end{aligned}$$

Since the game is symmetric, $q_1^* = q_2^* = q_3^* = q_4^* = q_q^*$ where the subscript denotes that it is an output for quadopoly. Hence, the first-order conditions become

$$\begin{aligned} & l - 5q_q^* - c = 0 \\ \Leftrightarrow \quad & q_q^* = \frac{l-c}{5} \end{aligned}$$

The second-order conditions for Π_q are

$$\frac{\partial^2 \Pi_q}{\partial q_i^2} = -2$$

for $i = 1, 2, 3, 4$ and

$$\frac{\partial^2 \Pi_q}{\partial q_i \partial q_j} = -1$$

when $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2 & -1 & -1 & -1 \\ -1 & -2 & -1 & -1 \\ -1 & -1 & -2 & -1 \\ -1 & -1 & -1 & -2 \end{pmatrix}$$

The eigenvalues are -5 and three multiplicities of -1. Hence, H is negative definite so that Π_q and π_i $(i = 1, 2, 3, 4)$ attain their local maxima at q_q^*. Since the game is symmetric, $\pi_i^* = \pi_q^*$ for $i = 1, 2, 3, 4$ and the subscript denotes that it is a profit for quadopoly.

$$
\begin{aligned}
\pi_q^* &= \left\{l - 4\left(\frac{l-c}{5}\right)\right\}\left(\frac{l-c}{5}\right) - c\left(\frac{l-c}{5}\right) \\
&= \left(\frac{l+4c}{5}\right)\left(\frac{l-c}{5}\right) - c\left(\frac{l-c}{5}\right) \\
&= \frac{(l-c)^2}{25}
\end{aligned}
$$

Next, the maximum value of the potential function is

$$
\begin{aligned}
\Pi_q^* &= 4(l - 4q^*)q^* - 4cq^* + 6q^{*2} \\
&= 4lq^* - 10q^{*2} - 4cq^* \\
&= 4l\left(\frac{l-c}{5}\right) - 10\left(\frac{l-c}{5}\right)^2 - 4c\left(\frac{l-c}{5}\right) \\
&= \frac{2(l-c)^2}{5}
\end{aligned}
$$

3.2.5 Solution of the Game

The results of the calculations are summarized in Table 3.1, and using the values of the profits from the table with $l = 2$ and $c = 1$, the game tree becomes as in Figure 3.2.

Note that when there is a monopoly, according to Table 3.1, the profit per firm is $(l-c)^2/4$. However, this monopoly is a result of the merger of *Apex* and *Brydox* so that after they earn the monopoly profit as one firm, they will divide the profit equally. This is the reason that the profit for monopoly in the game tree is $(l-c)^2/8 = 1/8$ for *Apex* and *Brydox*.

By using the well-known method of the backward induction, I can identify the subgame perfect equilibrium: *Apex* chooses Merge, and *Brydox* chooses Not Break Away at node 2, Merge or Split at node 3 and split at node 4. Thus, the subgame perfection predicts that the likely outcome is a monopoly. However, Table 3.1 clearly

Table 3.1: Summary of Section 3.2

	Monopoly	Duopoly	Triopoly	Quadopoly
Equilibrium Output per Firm	$\frac{l-c}{2}$	$\frac{l-c}{3}$	$\frac{l-c}{4}$	$\frac{l-c}{5}$
Equilibrium Price	$\frac{l+c}{2}$	$\frac{l+2c}{3}$	$\frac{l+3c}{4}$	$\frac{l+4c}{5}$
Combined Equilibrium Output	$\frac{l-c}{2}$	$\frac{2(l-c)}{3}$	$\frac{3(l-c)}{4}$	$\frac{4(l-c)}{5}$
Profit per Firm	$\frac{(l-c)^2}{4}$	$\frac{(l-c)^2}{9}$	$\frac{(l-c)^2}{16}$	$\frac{(l-c)^2}{25}$
Combined Profit	$\frac{(l-c)^2}{4}$	$\frac{2(l-c)^2}{9}$	$\frac{3(l-c)^2}{16}$	$\frac{4(l-c)^2}{25}$
Potential	$\frac{(l-c)^2}{4}$	$\frac{(l-c)^2}{3}$	$\frac{3(l-c)^2}{8}$	$\frac{2(l-c)^2}{5}$

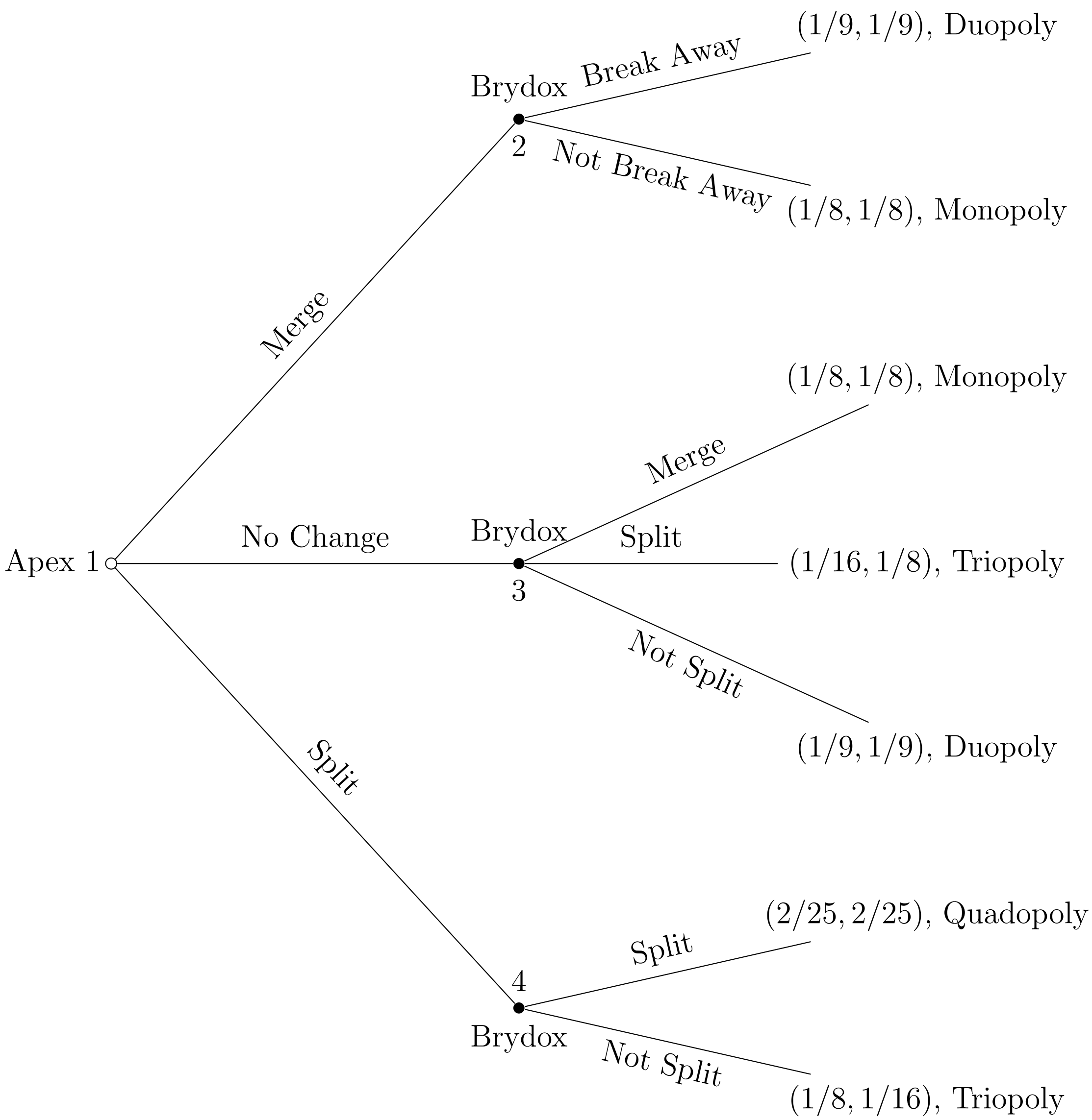

Figure 3.2: The Sequential Game with $l = 2$ and $c = 1$

shows that the potential attains the maximum value at the quadopoly and the minimum value at the monopoly. This is not surprising because some subgame perfect equilibria are not evolutionarily stable (Samuelson, 1998); i.e., the equilibrium found from the potentail functions are stochastically stable equilibria (refinement of the evolutionarily stable strategies) so that they are not necessarily subgame perfect. For instance, consider the game matrix of the same game below (Table 3.2).

Table 3.2: Game Matrix of Section 3.2

Apex \ Brydox	B-M-S	B-M-N	B-S-S	B-S-N	B-N-S	B-N-N	N-M-S	N-M-N	N-S-S	N-S-N	N-N-S	N-N-N
Merge	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$
No Change	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{16},\frac{1}{8}\right)$	$\left(\frac{1}{16},\frac{1}{8}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{1}{16},\frac{1}{8}\right)$	$\left(\frac{1}{16},\frac{1}{8}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$	$\left(\frac{1}{9},\frac{1}{9}\right)$
Split	$\left(\frac{2}{25},\frac{2}{25}\right)$	$\left(\frac{1}{8},\frac{1}{16}\right)$	$\left(\frac{2}{25},\frac{2}{25}\right)$	$\left(\frac{1}{8},\frac{1}{16}\right)$	$\left(\frac{2}{25},\frac{2}{25}\right)$	$\left(\frac{1}{8},\frac{1}{16}\right)$	$\left(\frac{2}{25},\frac{2}{25}\right)$	$\left(\frac{1}{8},\frac{1}{16}\right)$	$\left(\frac{2}{25},\frac{2}{25}\right)$	$\left(\frac{1}{8},\frac{1}{16}\right)$	$\left(\frac{2}{25},\frac{2}{25}\right)$	$\left(\frac{1}{8},\frac{1}{16}\right)$

Note that *Brydox's* strategies are represented in a format of the combination of three letters with a dash between the letters; e.g., B-N-S. The first letter represents *Brydox's* choice at node 2, and it can be either "Break Away" (B) or "Not Break Away" (N). The second letter represents Brydox's choice at node 3. It can be "Merge" (M), "Split" (S) or "Not Split" (N). The third letter represents Brydox's choice at node 4. It can be "Split" (S) or "Not Split" (N).

From Table 3.2, it is evident that the Nash equilibria are: (*Apex*, *Brydox*) = (Merge, N-M-S), (Merge, N-M-N), (Merge, N-S-S), (Merge, N-S-N), (Merge, N-N-S), (Merge, N-N-N), (No Change, B-M-S), (No Change, B-M-N), (No Change, N-M-S), (No Change, N-M-N). Clearly, a quadopoly is included in some of the Nash equilibria. Consequently, this finding indicates that the subgame perfect equilibrium is not stochastically stable. Indeed, Slade (1994) argues, "Nash equilibria of the game that are not maxima of the (potential) function are shown to be generically unstable" (45). In Chapter 2, it is shown that if I consider the adaptive learning processes, I will find that the likely outcome corresponds with the strategies with the highest potential. Hence, in this case, although a monopoly is the outcome from subgame perfection, it does not necessarily mean that we will witness a monopoly when the game is played because the potential is minimal for the monopoly.

3.2.6 Conclusion of Section 3.2

For the case I consider in this section, we may observe the outcome other than a monopoly when the game is played; however, without any noise and/or misperception, the subgame perfection predicts that the monopoly is the most rational outcome. Hence, this model does not clearly answer the "mystery" raised by Rasmusen (2007). In fact, this model is based on one unrealistic assumption; i.e., constant marginal cost. Consequently, I need to inquire the case that the marginal cost is an increasing function of output and see whether the "mystery" raised by Rasmusen (2007) occurs.

3.3 Identical Cost Function regardless of the Number of Firms, but Marginal Cost Increasing in Output

3.3.1 Introduction

In this section, I reconsider $\pi = pq - C(q)$, but this time, the marginal cost, $C'(q)$, is not constant. Rather, the marginal cost is a function of q. Specifically, I consider $C(q) = cq^2$ where c is a positive constant so that the marginal cost is $2cq$; i.e., the marginal cost increases as q increases.

3.3.2 Monopoly/Collusion

First, I consider $\pi = pq - cq^2$ for one player where $p = l - q$ and $l > q$.

$$\begin{aligned} \pi &= pq - cq^2 \\ &= (l-q)q - cq^2 \end{aligned}$$

The first-order condition is

$$\begin{aligned} \frac{d\pi}{dq} &= l - 2(c+1)q = 0 \\ \Leftrightarrow \quad q_m^* &= \frac{l}{2(c+1)} \end{aligned}$$

where the subscript denotes that it is an output for monopoly.

The second-order condition is

$$\frac{d^2\pi}{dq^2} = -2(c+1) < 0$$

Hence, π attains its local maximum at q_m^*. Consequently,

$$\begin{aligned} \pi_m^* &= (l - q_m^*)q_m^* - cq_m^{*2} \\ &= \left(l - \left(\frac{l}{2(c+1)}\right)\right)\left(\frac{l}{2(c+1)}\right) - c\left(\frac{l^2}{(2(c+1))^2}\right) \\ &= \frac{l^2}{4(c+1)} \end{aligned}$$

where the subscript denotes that it is a profit for monopoly

3.3.3 Duopoly

Below is a two-player Cournot model.

$$\pi_i = pq_i - cq_i^2$$

where $p = l - q_1 - q_2$ and $i = 1, 2$. Then

$$\pi_i = (l - q_1 - q_2)q_i - cq_i^2$$

for $i = 1, 2$. The first-order derivative is

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_i - 2cq_i$$

for $i = 1, 2$. The possible potential function, Π_d, is

$$\Pi_d = (l - q_1 - q_2)q_1 + (l - q_1 - q_2)q_2 - cq_1^2 - cq_2^2 + q_1q_2$$

where the subscript denotes that it is a potential function for duopoly.

The first-order derivative of Π_d with respect to q_1 is

$$\begin{aligned} \frac{\partial \Pi_d}{\partial q_1} &= l - q_1 - q_2 - q_1 - q_2 - 2cq_1 + q_2 \\ &= \frac{\partial \pi_1}{\partial q_1} \end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_d}{\partial q_2} = \frac{\partial \pi_2}{\partial q_2}$. Hence, Π_d is a potential function of the game.

The first-order conditions are

$$l - 2(c+1)q_1 - q_2 = 0$$

$$l - 2(c+1)q_2 - q_1 = 0$$

Since the game is symmetric, $q_1^* = q_2^* = q_d^*$ where the subscript denotes that it is an output for duopoly. Hence,

$$\begin{aligned} & l - 2(c+1)q_d^* - q_d^* = 0 \\ \Leftrightarrow \quad & q_d^* = \frac{l}{2c+3} \end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 \Pi_d}{\partial q_i^2} = -2(c+1)$$

for $i = 1, 2$ and

$$\frac{\partial^2 \Pi_d}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Hence, the Hessian matrix is

$$A = \begin{pmatrix} -2(c+1) & -1 \\ -1 & -2(c+1) \end{pmatrix}$$

The eigenvalues are $-2c-1$ and $-2c-3$: both of them are negative. Hence, Π_d and π_i ($i = 1, 2$) attain their local maxima at q_d^*. Since the game is symmetric, $\pi_i^* = \pi_d^*$ for $i = 1, 2$ and the subscript denotes that it is a profit for duopoly.

$$\begin{aligned} \pi_d^* &= (l - 2q_d^*)q_d^* - cq_d^{*2} \\ &= \left(l - 2\left(\frac{l}{2c+3}\right)\right)\left(\frac{l}{2c+3}\right) - c\left(\frac{l}{2c+3}\right)^2 \\ &= \frac{(c+1)l^2}{(2c+3)^2} \end{aligned}$$

Next, the maximum value of the potential function is

$$\begin{aligned} \Pi_d^* &= 2(l - 2q_d^*)q_d^* - 2cq_d^{*2} + q_d^{*2} \\ &= 2\left(l - 2\left(\frac{l}{2c+3}\right)\right)\left(\frac{l}{2c+3}\right) - 2c\left(\frac{l}{2c+3}\right)^2 + \left(\frac{l}{2c+3}\right)^2 \\ &= \frac{l^2}{2c+3} \end{aligned}$$

3.3.4 Triopoly

Below is a three-player Cournot model.

$$\pi_i = pq_i - cq_i^2$$

where $p = l - q_1 - q_2 - q_3$ and $i = 1, 2, 3$. Then

$$\pi_i = (l - q_1 - q_2 - q_3)q_i - cq_i^2$$

for $i = 1, 2, 3$. The first-order derivative is

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_3 - q_i - 2cq_i$$

for $i = 1, 2, 3$. The possible potential function, Π_t, is

$$\begin{aligned} \Pi_t &= (l - q_1 - q_2 - q_3)q_1 + (l - q_1 - q_2 - q_3)q_2 + (l - q_1 - q_2 - q_3)q_3 \\ &\quad -cq_1^2 - cq_2^2 - cq_3^2 + q_1q_2 + q_1q_3 + q_2q_3 \end{aligned}$$

where the subscript denotes that it is a possible potential function for triopoly.

Then, the first-order derivative of Π_t with respect to q_1 is

$$\begin{aligned} \frac{\partial \Pi_t}{\partial q_1} &= l - q_1 - q_2 - q_3 - q_1 - q_2 - q_3 - 2cq_1 + q_2 + q_3 \\ &= \frac{\partial \pi_1}{\partial q_1} \end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_t}{\partial q_i} = \frac{\partial \pi_i}{\partial q_i}$ for $i = 2, 3$. Hence, Π_t is the potential function of the game.

Since the game is symmetric, $q_1^* = q_2^* = q_3^* = q_t^*$ where the subscript denotes that it is an output for triopoly. Hence, the first-order conditions become

$$\begin{aligned} &l - 2q_t^* - 2(c+1)q_t^* = 0 \\ \Leftrightarrow \quad & q_t^* = \frac{l}{2(c+2)} \end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 \Pi_t}{\partial q_i^2} = -2(c+1)$$

for $i = 1, 2, 3$, and

$$\frac{\partial^2 \Pi_t}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2(c+1) & -1 & -1 \\ -1 & -2(c+1) & -1 \\ -1 & -1 & -2(c+1) \end{pmatrix}$$

The eigenvalues are $-2c-4$ and two multiplicities of $-2c-1$: all of them are negative. Hence, Π_t and π_i for $i = 1, 2, 3$ attain their local maxima at q_t^*. Since the game is symmetric, $\pi_i^* = \pi_t^*$ for $i = 1, 2, 3$ and the subscript denotes that it is a profit for triopoly.

$$\begin{aligned} \pi_t^* &= (l - 3q_t^*)q_t^* - cq_t^{*2} \\ &= lq_t^* - 3q_t^{*2} - cq_t^{*2} \\ &= \frac{l^2}{2(c+2)} - \frac{3l^2}{4(c+2)^2} - \frac{cl^2}{4(c+2)^2} \\ &= \frac{(c+1)l^2}{4(c+2)^2} \end{aligned}$$

Next, the maximum value of the potential function is

$$\begin{aligned} \Pi_t^* &= 3(l - 3q_t^*)q_t^* - 3cq_t^{*2} + 3q_t^{*2} \\ &= 3lq_t^* - 6q_t^{*2} - 3cq_t^{*2} \\ &= 3l\left(\frac{l}{2c+4}\right) - 6\left(\frac{l}{2c+4}\right)^2 - 3c\left(\frac{l}{2c+4}\right)^2 \\ &= \frac{3l^2}{4(c+2)} \end{aligned}$$

3.3.5 *Quadopoly*

Below is a four-player Cournot model.

$$\pi_i = pq_i - cq_i^2$$

where $p = l - q_1 - q_2 - q_3 - q_4$ and $i = 1, 2, 3, 4$. Then

$$\pi_i = (l - q_1 - q_2 - q_3 - q_4)q_i - cq_i^2$$

for $i = 1, 2, 3, 4$. The first-order derivative is

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_3 - q_4 - q_i - 2cq_i$$

for $i = 1, 2, 3, 4$. The possible potential function, Π_q, is

$$\begin{aligned}\Pi_q &= (l - q_1 - q_2 - q_3 - q_4)q_1 + (l - q_1 - q_2 - q_3 - q_4)q_2 + (l - q_1 - q_2 - q_3 - q_4)q_3 \\ &\quad +(l - q_1 - q_2 - q_3 - q_4)q_4 - cq_1^2 - cq_2^2 - cq_3^2 - cq_4^2 \\ &\quad +q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4\end{aligned}$$

where the subscript denotes that it is a possible potential function for quadopoly.

The first-order derivatives of Π_q with respect to q_1 is

$$\begin{aligned}\frac{\partial \Pi_q}{\partial q_1} &= l - q_1 - q_2 - q_3 - q_4 - q_1 - q_2 - q_3 - q_4 - 2cq_1 + q_2 + q_3 + q_4 \\ &= l - q_1 - q_2 - q_3 - q_4 - q_1 - 2cq_1 \\ &= \frac{\partial \pi_1}{\partial q_1}\end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_q}{\partial q_i} = \frac{\partial \pi_i}{\partial q_i}$ for $i = 2, 3, 4$. Hence, Π_q is indeed a potential function for this game.

The first-order condition is

$$l - q_1 - q_2 - q_3 - q_4 - q_i - 2cq_i = 0$$

where $i = 1, 2, 3, 4$. Since the game is symmetric, $q_1^* = q_2^* = q_3^* = q_4 = q_q^*$ where the subscript denotes that it is an output for quadopoly. Hence, the first-order condition becomes

$$\begin{aligned} & l - 5q_q^* - 2cq_q^* = 0 \\ \Leftrightarrow \quad & q_q^* = \frac{l}{5 + 2c} \end{aligned}$$

The second-order conditions for Π_q are

$$\frac{\partial^2 \Pi_q}{\partial q_i^2} = -2 - 2c$$

for $i = 1, 2, 3, 4$ and

$$\frac{\partial^2 \Pi_q}{\partial q_i \partial q_j} = -1$$

when $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2c-2 & -1 & -1 & -1 \\ -1 & -2c-2 & -1 & -1 \\ -1 & -1 & -2c-2 & -1 \\ -1 & -1 & -1 & -2c-2 \end{pmatrix}$$

The eigenvalues are $-2c - 5$ and three multiplicities of $-2c - 1$: all of them are negative. Hence, Π_q and π_i $(i = 1, 2, 3, 4)$ attain their local maxima at q_q^*. Since the game is symmetric, $\pi_i^* = \pi_q^*$ for $i = 1, 2, 3, 4$ and the subscript denotes that it is a profit for quadopoly.

$$\begin{aligned} \pi_q^* &= (l - 4q_q^*)q_q^* - cq_q^{*2} \\ &= lq_q^* - 4q_q^{*2} - cq_q^{*2} \\ &= \frac{l^2}{2c+5} - \frac{4l^2}{(2c+5)^2} - \frac{cl^2}{(2c+5)^2} \\ &= \frac{(c+1)l^2}{(2c+5)^2} \end{aligned}$$

Next, the maximum value of the potential function is

$$\begin{aligned}
\Pi_q^* &= 4(l - 4q_q^*)q^* - 4cq_q^{*2} + 6q_q^{*2} \\
&= 4lq_q^* - 10q_q^{*2} - 4cq_q^{*2} \\
&= 4l\left(\frac{l}{2c+5}\right) - 10\left(\frac{l}{2c+5}\right)^2 - 4c\left(\frac{l}{2c+5}\right)^2 \\
&= \frac{2l^2}{(2c+5)^2}
\end{aligned}$$

3.3.6 Solution of the Game

The results of the calculations are summarized in Table 3.3, and using the values of the profits from the table with $l = 2$ and $c = 1$, the game tree becomes as in Figure 3.3.

Note that, as in Section 3.2, when there is a monopoly, according to Table 3.3, the profit per firm is $l^2/4(c+1)$. However, this monopoly is a result of the merger of *Apex* and *Brydox* so that after they earn the monopoly profit as one firm, they will divide the profit equally. This is the reason that the profit for monopoly in the game tree is $l^2/8(c+1) = 1/4$ for *Apex* and *Brydox*.

By using the well-known method of the backward induction, I can identify the subgame perfect equilibrium: *Apex* chooses Split, and *Brydox* chooses Break Away at node 2, Split at node 3 and split at node 4. Thus, the subgame perfection predicts that the likely outcome is a quadopoly, and this is where the potential attains the maximum value (Note: From Table 3.3, it is straightforward to show that $\pi_m^* < \Pi_d^* < \Pi_t^* < \Pi_q^*$).

3.3.7 Conclusion of Section 3.3

As evident from Table 3.3, the combined equilibrium output increases as the number of players increases; consequently, the equilibrium price falls as the number of players increases. Moreover, the profit per player and the combined profit both de-

Table 3.3: Summary of Section 3.3

	Monopoly	Duopoly	Triopoly	Quadopoly
Equilibrium Output per Firm	$\frac{l}{2c+2}$	$\frac{l}{2c+3}$	$\frac{l}{2c+4}$	$\frac{l}{2c+5}$
Equilibrium Price	$\frac{(2c+1)l}{2c+2}$	$\frac{(2c+1)l}{2c+3}$	$\frac{(2c+1)l}{2c+4}$	$\frac{(2c+1)l}{2c+5}$
Combined Equilibrium Output	$\frac{l}{2c+2}$	$\frac{2l}{2c+3}$	$\frac{3l}{2c+4}$	$\frac{4l}{2c+5}$
Profit per Firm	$\frac{l^2}{4(c+1)}$	$\frac{(c+1)l^2}{(2c+3)^2}$	$\frac{(c+1)l^2}{4(c+2)^2}$	$\frac{(c+1)l^2}{(2c+5)^2}$
Combined Profit	$\frac{l^2}{4(c+1)}$	$\frac{2(c+1)l^2}{(2c+3)^2}$	$\frac{3(c+1)l^2}{4(c+2)^2}$	$\frac{4(c+1)l^2}{(2c+5)^2}$
Potential	$\frac{l^2}{4(c+1)}$	$\frac{l^2}{2c+3}$	$\frac{3l^2}{4(c+2)}$	$\frac{2l^2}{2c+5}$

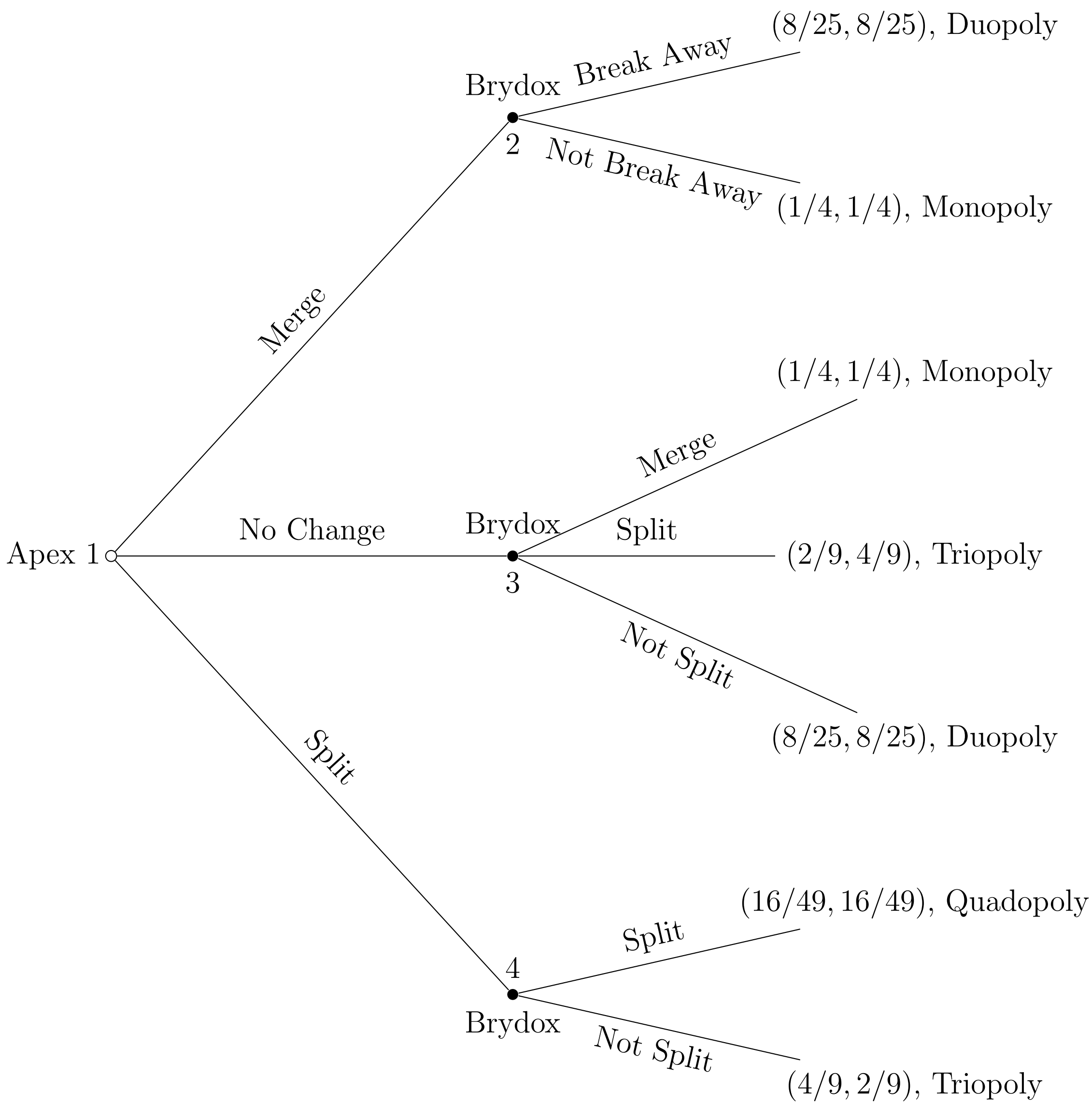

Figure 3.3: The Sequential Game with $l = 2$ and $c = 1$

crease as the number of players increases. Nevertheless, the only stochastically stable equilibrium is quadopoly, and the subgame perfection coincides with the stochastic stability. Hence, the model of this section, not the one from the previous section with the consitant marginal cost, is consistent with what Rasmusen (2007) observes; i.e., a firm's best policy is often to split into separate firms (Rasmusen, 2007). Schernikau (2010) suggests that the marginal cost that increases as the output increases may rectify the "mystery" pointed out by Rasmusen (2007), but the results of this section indicate that the model of this section is actually a stereotypical model that exhibits the "mystery" of a firm splitting in oligopoly.

3.4 The Marginal Cost of Production Increasing in Both Output and the Number of Firms

3.4.1 Introduction

In this section, the marginal cost of production is assumed to be increasing as output increases, and each firm faces the largest cost when there are three firms (triopoly) and the smallest cost function when there is only one firm (monopoly). Specifically, for the case of monopoly (one firm), a firm has a cost function cq^2 where c is a positive constant. For the case of duopoly (two firms), each firm has a cost function αcq^2 where $\alpha > 1$. For the case of triopoly (three firms), each firm has a cost function βcq^2 where $\beta > \alpha$. Thus, the profit function for the case of monopoly becomes

$$\pi = pq - cq^2$$

For the case of duopoly, firm i has a profit function

$$\pi_i = pq_i - \alpha cq_i^2$$

where $\alpha > 1$, and for the case of triopoly, firm i has a profit function

$$\pi_i = pq_i - \beta c q_i^2$$

where $\beta > \alpha$.

Note that if both *Apex* and *Brydox* split, then there are four firms in the market: *Apex 1*, *Apex 2*, *Brydox 1* and *Brydox 2*. But what we observe in this situation is a "quasi" duopoly because *Apex 1* (*Brydox 1*) and *Apex 2* (*Brydox 2*) together occupy half of the market and try to maximize their profit jointly as *Apex* (*Brydox*). Moreover, as discussed above, it is more costly for firms if they split. As a result, in this "quasi" duopoly of the four firms, *Apex 1* (*Brydox 1*) and *Apex 2* (*Brydox 2*) have incentives to merge to reduce the cost function so that we will witness the duopoly of *Apex* and *Brydox*.

3.4.2 Monopoly/Collusion

The results for the monopoly/collusion have been provided in Section 3.3.2. $q_m^* = \dfrac{l}{2(c+1)}$ and $\pi_m^* = \dfrac{l^2}{4(c+1)}$.

3.4.3 Duopoly

Below is a two-player Cournot model.

$$\pi_i = pq_i - \alpha c q_i^2$$

where $p = l - q_1 - q_2$, $\alpha > 1$ and $i = 1, 2$. Then

$$\pi_i = (l - q_1 - q_2)q_i - \alpha c q_i^2$$

for $i = 1, 2$.

The first-order derivative is

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_i - 2\alpha c q_i$$

for $i = 1, 2$. The possible potential function, Π_d, is

$$\Pi_d = (l - q_1 - q_2)q_1 + (l - q_1 - q_2)q_2 - \alpha c q_1^2 - \alpha c q_2^2 + q_1 q_2$$

where the subscript denotes that it is a possible potential for duopoly.

The first-order derivative of Π_d with respect to q_1 is

$$\begin{aligned} \frac{\partial \Pi_d}{\partial q_1} &= l - q_1 - q_2 - q_1 - q_2 - 2\alpha c q_1 + q_2 \\ &= l - 2q_1 - q_2 - 2\alpha c q_1 \\ &= \frac{\partial \pi_1}{\partial q_1} \end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi_d}{\partial q_2} = \frac{\partial \pi_2}{\partial q_2}$. Hence, Π_d is a potential function of the game.

The first-order conditions are

$$l - 2q_1 - q_2 - 2\alpha c q_1 = 0$$
$$l - q_1 - 2q_2 - 2\alpha c q_2 = 0$$

Since the game is symmetric, $q_1^* = q_2^* = q_d^*$ where the subscript denotes that it is an output for duopoly. Hence,

$$\begin{aligned} & l - 3q_d^* - 2\alpha c q_d^* = 0 \\ \Leftrightarrow \quad & q_d^* = \frac{l}{3 + 2\alpha c} \end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 \Pi_d}{\partial q_i^2} = -2 - 2\alpha c$$

for $i = 1, 2$ and

$$\frac{\partial^2 \Pi_d}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Consequently, the Hessian matrix is

$$H = \begin{pmatrix} -2-2\alpha c & -1 \\ -1 & -2-2\alpha c \end{pmatrix}$$

The eigenvalues are $-2\alpha c - 1$ and $-2\alpha c - 3$, and both of them are negative because $\alpha > 1$ and $c > 0$. So, H is negative definite. Hence, Π_d and π_i $(i = 1, 2)$ attain their local maxima at q_d^*.

Since the game is symmetric, $\pi_i^* = \pi_d^*$ for $i = 1, 2$ and the subscript denotes that it is a profit for duopoly.

$$\begin{aligned}
\pi_d^* &= (l - 2q_d^*)q_d^* - \alpha c q_d^{*2} \\
&= \left(l - \frac{2l}{3+2\alpha c}\right)\left(\frac{l}{3+2\alpha c}\right) - \alpha c\left(\frac{l}{3+2\alpha c}\right)^2 \\
&= \left(\frac{l+2\alpha c l}{3+2\alpha c}\right)\left(\frac{l}{3+2\alpha c}\right) - \alpha c\left(\frac{l}{3+2\alpha c}\right)^2 \\
&= \frac{l^2(1+\alpha c)}{(3+2\alpha c)^2}
\end{aligned}$$

Next, the maximum value of the potential function is

$$\begin{aligned}
\Pi_d^* &= 2(l - 2q_d^*)q_d^* - 2\alpha c q_d^{*2} + q_d^{*2} \\
&= 2l q_d^* - 2\alpha c q_d^{*2} - 3q_d^{*2} \\
&= 2l\left(\frac{l}{3+2\alpha c}\right) - 2\alpha c\left(\frac{l}{3+2\alpha c}\right)^2 - 3\left(\frac{l}{3+2\alpha c}\right)^2 \\
&= \frac{2l^2(3+2\alpha c) - 2\alpha c l^2 - 3l^2}{(3+2\alpha c)^2} \\
&= \frac{l^2}{2\alpha c + 3}
\end{aligned}$$

3.4.4 Triopoly

Below is a three-player Cournot model.

$$\pi_i = p q_i - \beta c q_i^2$$

where $p = q_1 - q_2 - q_3$. $\beta > \alpha > 1$ and $i = 1, 2, 3$. Then

$$\pi_i = (l - q_1 - q_2 - q_3)q_i - \beta c q_i^2$$

for $i = 1, 2, 3$. The first-order derivative is

$$\frac{\partial \pi_i}{\partial q_i} = l - q_1 - q_2 - q_3 - q_i - 2\beta c q_i$$

for $i = 1, 2, 3$. The possible potential function, Π_t, is

$$\begin{aligned} \Pi_t &= (l - q_1 - q_2 - q_3)q_1 + (l - q_1 - q_2 - q_3)q_2 + (l - q_1 - q_2 - q_3)q_3 \\ &\quad -\beta c q_1^2 - \beta c q_2^2 - \beta c q_3^2 + q_1 q_2 + q_1 q_3 + q_2 q_3 \end{aligned}$$

where the subscript denotes that it is a possible potential for triopoly.

The first-order derivatives of Π_t with respect to q_1 is

$$\begin{aligned} \frac{\partial \Pi_t}{\partial q_1} &= l - q_1 - q_2 - q_3 - q_1 - q_2 - q_3 - 2\beta c q_1 + q_2 + q_3 \\ &= l - 2q_1 - q_2 - q_3 - 2\beta c q_1 \\ &= \frac{\partial \pi_1}{\partial q_1} \end{aligned}$$

Similarly, it is straightforward to show that $\frac{\partial \Pi}{\partial q_i} = \frac{\partial \pi_i}{\partial q_i}$ for $i = 2, 3$. Hence, Π_t is a potential function of the game.

The first-order conditions are

$$l - 2q_1 - q_2 - q_3 - 2\beta c q_1 = 0$$

$$l - q_1 - 2q_2 - q_3 - 2\beta c q_2 = 0$$

$$l - q_1 - q_2 - 2q_3 - 2\beta c q_3 = 0$$

Since the game is symmetric, $q_1^* = q_2^* = q_3^* = q_t^*$ where the subscript denotes that it is an output for triopoly. Hence,

$$\begin{aligned} & l - 4q^* - 2\beta c q_t^* = 0 \\ \Leftrightarrow \quad & q_t^* = \frac{l}{2\beta c + 4} \end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 \Pi_t}{\partial q_i^2} = -2 - 2\beta c$$

for $i = 1, 2, 3$ and

$$\frac{\partial^2 \Pi_t}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Consequently, the Hessian matrix is

$$H = \begin{pmatrix} -2-2\beta c & -1 & -1 \\ -1 & -2-2\beta c & -1 \\ -1 & -1 & -2-2\beta c \end{pmatrix}$$

The eigenvalues are two multiplicities of $-2\beta c - 1$ and $-2\beta c - 4$. Since $\beta > \alpha > 1$, they are all negative. So, H is negative definite. Hence, Π_t and π_i $(i = 1, 2, 3)$ attain their local maxima at q_t^*.

Since the game is symmetric, $\pi_i^* = \pi_t^*$ for $i = 1, 2, 3$ and the subscript denotes that it is a profit for triopoly.

$$\begin{aligned}
\pi_t^* &= (l - 3q_t^*)q_t^* - \beta c q_t^{*2} \\
&= l q_t^* - 3q_t^{*2} - \beta c q_t^{*2} \\
&= l\left(\frac{l}{2\beta c + 4}\right) - 3\left(\frac{l}{2\beta c + 4}\right)^2 - \beta c\left(\frac{l}{2\beta c + 4}\right)^2 \\
&= \frac{2\beta c l^2 + 4l^2 - 3l^2 - \beta c l^2}{(2\beta c + 4)^2} \\
&= \frac{l^2(1 + \beta c)}{(2\beta c + 4)^2}
\end{aligned}$$

Next, the maximum value of the potential function is

$$
\begin{aligned}
\Pi_t^* &= 3(l-3q_t^*)q_t^* - 3\beta c q_t^{*2} + 3q_t^{*2} \\
&= 3\left\{l - 3\left(\frac{l}{2\beta c+4}\right)\right\}\left(\frac{l}{2\beta c+4}\right) - 3\beta c\left(\frac{l}{2\beta c+4}\right)^2 + 3\left(\frac{l}{2\beta c+4}\right)^2 \\
&= 3\left(\frac{l+2\beta cl}{2\beta c+4}\right)\left(\frac{l}{2\beta c+4}\right) - \frac{3\beta cl^2}{(2\beta c+4)^2} + \frac{3l^2}{(2\beta c+4)^2} \\
&= \frac{6l^2+3\beta cl^2}{(2\beta c+4)^2} \\
&= \frac{3l^2}{4(\beta c+2)}
\end{aligned}
$$

3.4.5 Solution of the Game

The results of the calculations are summarized in Table 3.4. To find the solution of the game, first I need to compare π_m^* (monopoly) and Π_d^* (potential of duopoly).

$$
\begin{aligned}
\pi_m^* - \Pi_d^* &= \frac{l^2}{4(c+1)} - \frac{l^2}{2\alpha c+3} \\
&= \frac{l^2(2\alpha c+3) - 4(c+1)l^2}{4(c+1)(2\alpha c+3)} \\
&= \frac{l^2(2\alpha c-4c-1)}{4(c+1)(2\alpha c+3)} \\
&> 0
\end{aligned}
$$

if and only if $2\alpha c - 4c - 1 > 0$; i.e., if and only if $\alpha > \dfrac{4c+1}{2c}$ (Note: α is greater than 1). Hence, if $\alpha > \dfrac{4c+1}{2c}$, then π_m^* is greater than Π_d^*.

Next, I need to compare Π_d^* (duopoly) and Π_t^* (triopoly).

$$
\begin{aligned}
\Pi_d^* - \Pi_t^* &= \frac{l^2}{2\alpha c+3} - \frac{3l^2}{4(\beta c+2)} \\
&= \frac{4l^2(\beta c+2) - 3l^2(2\alpha c+3)}{4(2\alpha c+3)(\beta c+2)} \\
&= \frac{l^2(4\beta c+8-6\alpha c-9)}{4(2\alpha c+3)(\beta c+2)} \\
&> 0
\end{aligned}
$$

Table 3.4: Summary of Section 3.4

	Monopoly	Duopoly	Triopoly
Equilibrium Output per Firm	$\frac{l}{2(c+1)}$	$\frac{l}{2\alpha c+3}$	$\frac{l}{2\beta c+4}$
Equilibrium Price	$\frac{(2c+1)l}{2(c+1)}$	$\frac{(2\alpha c+1)l}{2\alpha c+3}$	$\frac{(2\beta c+1)l}{2\beta c+4}$
Combined Equilibrium Output	$\frac{l}{2(c+1)}$	$\frac{2l}{2\alpha c+3}$	$\frac{3l}{2\beta c+4}$
Profit per Firm	$\frac{l^2}{4(c+1)}$	$\frac{l^2(\alpha c+1)}{(2\alpha c+3)^2}$	$\frac{l^2(\beta c+1)}{(2\beta c+4)^2}$
Combined Profit	$\frac{l^2}{4(c+1)}$	$\frac{2l^2(\alpha c+1)}{(2\alpha c+3)^2}$	$\frac{3l^2(\beta c+1)}{(2\beta c+4)^2}$
Potential	$\frac{l^2}{4(c+1)}$	$\frac{l^2}{2\alpha c+3}$	$\frac{3l^2}{4(\beta c+2)}$

if and only if $4\beta c + 8 - 6\alpha c - 9 > 0$; i.e., if and only if $\beta > \dfrac{6\alpha c + 1}{4c}$. Hence, if $\beta > \dfrac{6\alpha c + 1}{4c}$, then Π_d^* is greater than Π_t^*.

For ease of understanding, I specify some values of c, l, α and β that satisfy the conditions found above, and the results are shown in Table 3.5. Now, I use the specific values of the profits provided in Table 3.5 to find the solution of the game. The game tree is provided in Figure 3.4 with the values of profits available from Table 3.5.

Note that when there is a monopoly, according to Table 3.5, the profit per firm is 12.5. However, this monopoly is a result of the merger of *Apex* and *Brydox* so that after they earn the monopoly profit as one firm, they will divide the profit equally. This is the reason that the profit for monopoly is 6.25 for *Apex* and *Brydox*.

By using the well-known method of the backward induction, I can identify the subgame perfect equilibrium: *Apex* chooses either Merge or No Change, and *Brydox* chooses Not Break Away at node 2, Merge at node 3 and split at node 4. Thus, the outcome of the game is a monopoly, and as seen above, the potential function attains the maximum value when there is a monopoly and the α and β satisfy the conditions found above.

The argument above seems to suggest that the outcomes of subgame perfect equilibria are stochastically stable equilibria, but as explained in Section 3.2, this is not true because some subgame perfect equilibria are not evolutionarily stable (Samuelson, 1998). Below, I show the example.

Let's consider the case that $c = 1$, $l = 10$, $\beta = 5$, but α is changed to 2. In this case, the per-firm-profit of monopoly is still larger than that of duopoly (the profit is larger as long as $\alpha > 1.91$); however, the potential of duopoly is now 14.29 while the one of monopoly stays at 12.5. Figure 3.5 is the game tree for this case.

A simple inspection yields that the subgame perfect equilibrium is the same with the case of Figure 3.4. However, this time, the potential is maximized at duopoly,

Table 3.5: Summary of Section 3.4: $c = 1$, $l = 10$, $\alpha = 3$, $\beta = 5$

	Monopoly	Duopoly	Triopoly
Equilibrium Output per Firm	2.5	1.11	0.71
Equilibrium Price	7.5	7.78	7.86
Combined Equilibrium Output	2.5	2.22	2.14
Profit per Firm	12.5	4.94	3.06
Combined Profit	12.5	9.88	9.18
Potential	12.5	11.11	10.71

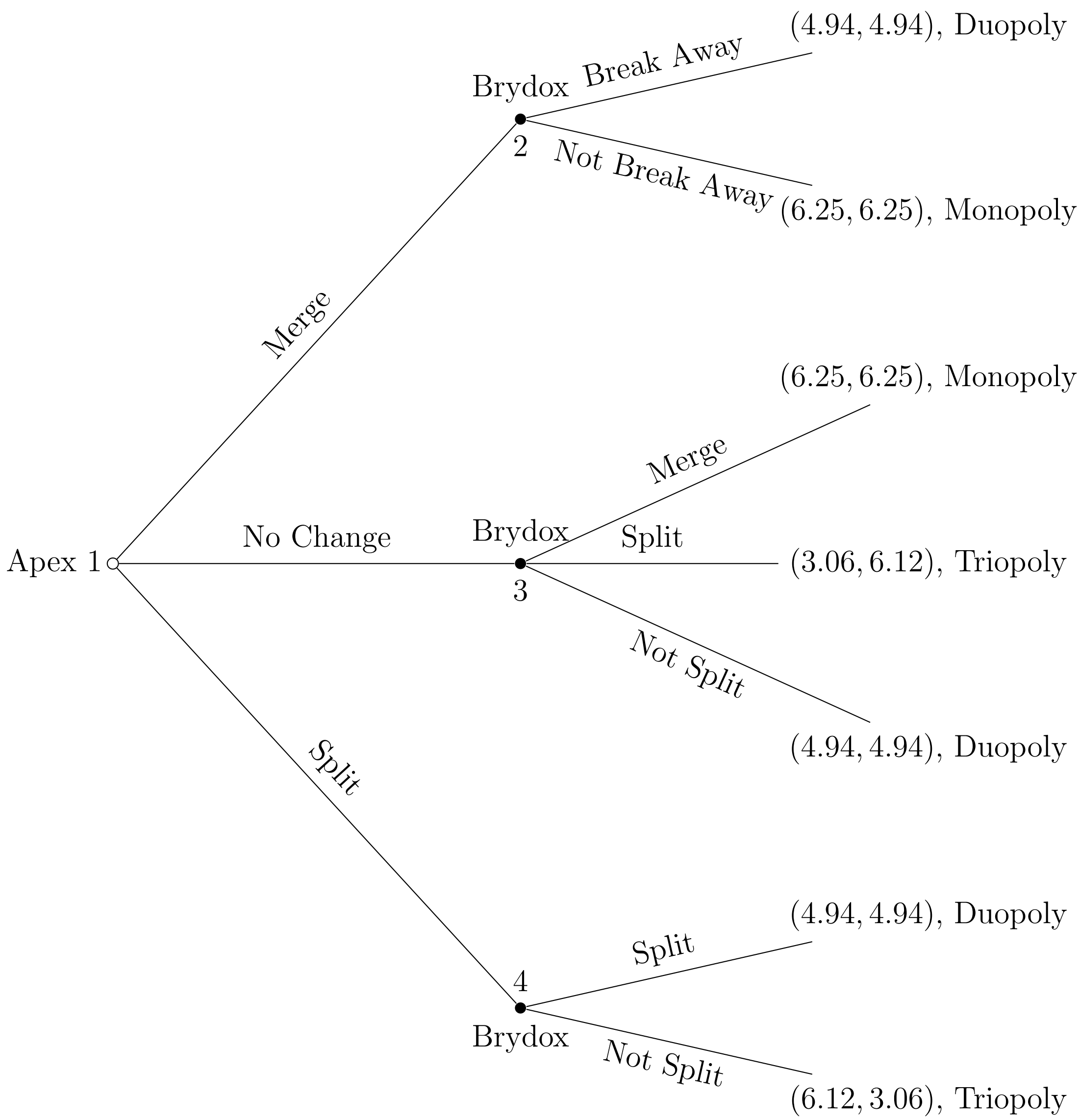

Figure 3.4: The Sequential Game with $c = 1, l = 10, \alpha = 3, \beta = 5$

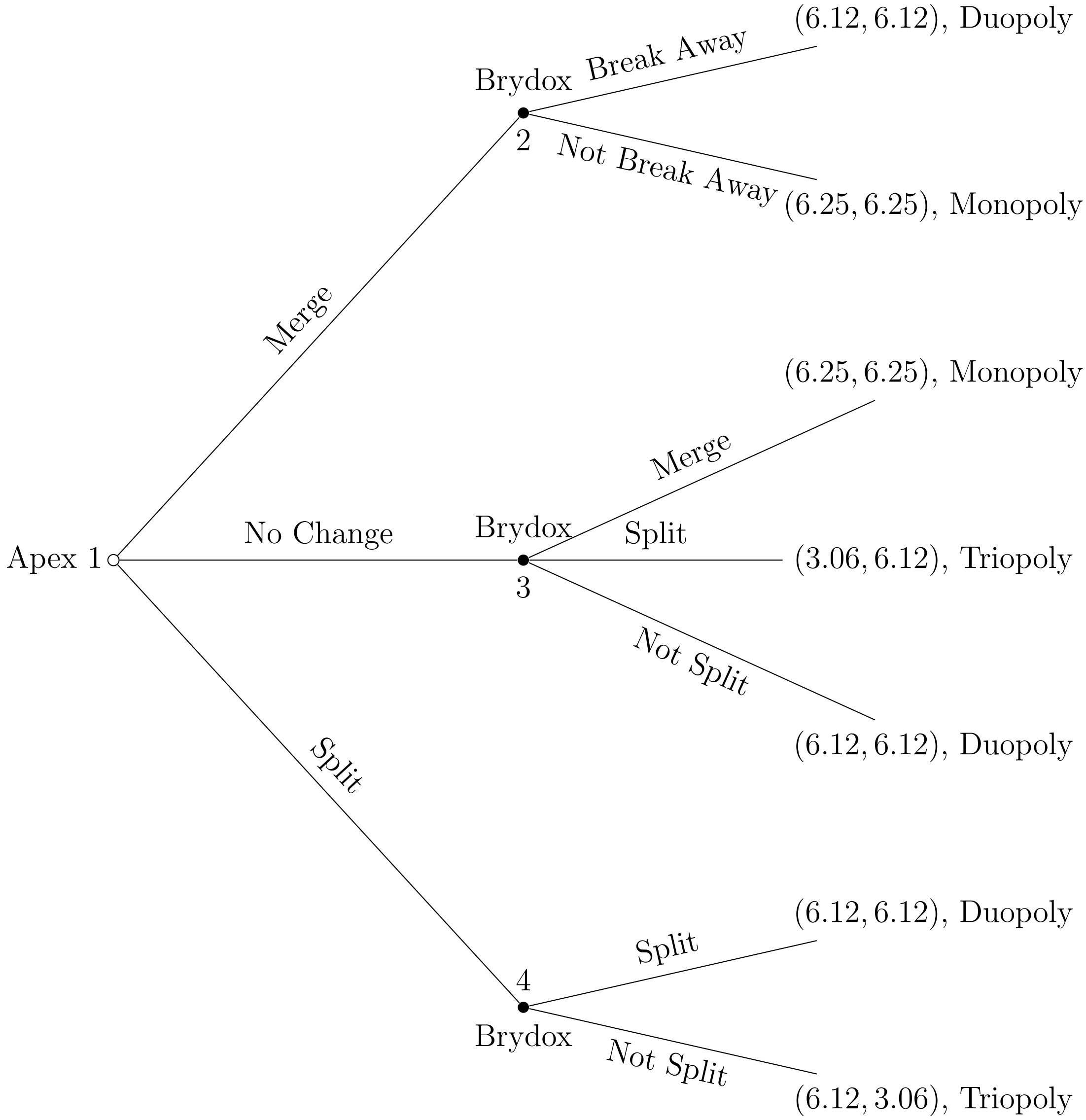

Figure 3.5: Game Tree When $\alpha = 2$

not at monopoly. This means, under the presence of noise, *Brydox* may make an error of choosing "Break Away" at node 2. But if $\alpha > 2.5$, then *Brydox* chooses "Not Break Away" at node 2 even if there is noise. Hence, if α and β are greater than the threshold values found above, then the outcome of the game is the subgame perfect equilibrium and stochastically stable equilibrium.

3.4.6 Extension to the Case of n Firms

So far, I have found the potentials for duopoly and triopoly and determined the coefficient of the cost function for each of them so that a monopoly becomes the outcome of the game. Of course, there is no reason that I should stop at a triopoly; rather, I can extend this case to the general case of n firms. This is the goal of this subsection; i.e., I determine whether there is any general pattern that the coefficient of the cost function satisfies for any number of firms in the market.

For the sake of argument, I need to make one assumption; i.e., in the previous section, when there are four firms (i.e., *Apex 1*, *Apex 2*, *Brydox 1* and *Brydox 2*), I assume that this state should rather be considered as a duopoly rather than a quadopoly because of their incentives to reduce costs. However, for the extension to a general case, I assume that firms do not have incentives to re-merge (e.g., for the case of the quadopoly mentioned above, *Apex 1* and *Apex 2* do not have incentives to internalize the cost by re-merging). Consequently, the case of *Apex 1* & *2* and *Brydox 1* & *2* is a quadopoly, not a duopoly. And the same reasoning applies for other cases such as *Apex 1, 2,* & *3* and *Brydox 1, 2,* & *3*, etc.

In the previous subsection, I use α for duopoly and β for triopoly. For the sake of ease of argument, I change these notations. The coefficient for cq^2 in case of duopoly (triopoly) is α_2 (α_3). In the previous subsection, I show that

$$\begin{aligned}\alpha_2 &> \frac{4c+1}{2c}\\ &= 2+\frac{1}{2c}\end{aligned}$$

and

$$\alpha_3 > \frac{6\alpha_2 c+1}{4c}$$

Now, I can work on α_3 as below:

$$\begin{aligned}\alpha_3 &> \frac{6\alpha_2 c+1}{4c}\\ &= \frac{3\alpha_2}{2}+\frac{1}{4c}\\ &> \frac{3}{2}\left(\frac{4c+1}{2c}\right)+\frac{1}{4c}\\ &= 3+\frac{1}{c}\end{aligned}$$

The Nash equilibrium, q_q^*, and the potential, Π_q^*, for a quadopoly can be easily found from the method used in the previous subsection. Hence,

$$q_q^* = \frac{l}{2\alpha_4 c+5}$$

and

$$\Pi_q^* = \frac{2l^2}{2\alpha_4 c+5}$$

where α_4 is the coefficient of cq^2 in case of a quadopoly.

As in the previous section,

$$\begin{aligned}\Pi_t^* - \Pi_q^* &= \frac{3l^2}{4(\alpha_3 c+2)} - \frac{2l^2}{2\alpha_4 c+5}\\ &= \frac{3l^2(2\alpha_4 c+5) - 8l^2(\alpha_3 c+2)}{4(2\alpha_4 c+5)(\alpha_3 c+2)}\\ &> 0\end{aligned}$$

if and only if $3l^2(2\alpha_4 c+5) - 8l^2(\alpha_3 c+2) > 0$; i.e., if and only if $\alpha_4 > \dfrac{8\alpha_3 c+1}{6c}$. Hence, a triopoly emerges over a quadopoly if and only if $\alpha_4 > \dfrac{8\alpha_3 c+1}{6c}$ holds. But it is already shown that $\alpha_3 > 3 + \dfrac{1}{c}$. Hence,

$$\begin{aligned}\alpha_4 &> \frac{4}{3}(3+\frac{1}{c}) + \frac{1}{6c}\\ &= 4 + \frac{3}{2c}\end{aligned}$$

Consequently, a monopoly emerges if and only if

$$\begin{aligned}\alpha_2 &> 2 + \frac{1}{2c}\\ \alpha_3 &> 3 + \frac{1}{c}\\ \alpha_4 &> 4 + \frac{3}{2c}\end{aligned}$$

and the potentials are

$$\begin{aligned}\pi_m^* &= \frac{l^2}{4c+4}\\ \Pi_d^* &= \frac{l^2}{2\alpha_2 c+3}\\ \Pi_t^* &= \frac{3l^2}{4\alpha_3 c+8}\\ \Pi_q^* &= \frac{2l^2}{2\alpha_4 c+5}\end{aligned}$$

Hence, it is not difficult to show that, for any n,

$$\Pi_n^* = \frac{nl^2}{4\alpha_n c + 2(n+1)}$$

The results above are summarized in the proposition below.

Proposition 1. Assume there are two firms. They in turn decide to split or merge so that the possible number of firms is $1, 2, 3, \ldots$. Then, a monopoly emerges if and only if $\alpha_n > n + \dfrac{n-1}{2c}$ for all $n = 2, 3, \ldots$ where α_n is a coefficient of cost function, cq^2, of each firm when there are n firms in the market.

Proof. I prove by induction. When $n = 2$, it is already shown that the proposition is true. So, I assume that the proposition is true when $n = k$; i.e., I assume a monopoly emerges if and only if $\alpha_k > k + \frac{k-1}{2c}$.

Now,

$$\begin{aligned} \alpha_k + 1 + \frac{1}{2c} &> k + \frac{k-1}{2c} + 1 + \frac{1}{2c} \\ &= k + 1 + \frac{k}{2c} \end{aligned}$$

But I know, from the argument preceding this proposition, that $\alpha_{k+1} - \alpha_k > 1 + \frac{1}{2c}$. Hence, a monopoly emerges if and only if $\alpha_{k+1} > k + 1 + \frac{k}{2c}$. Thus, I have shown that the proposition is true when $n = k + 1$. This proves the proposition. □

The corollary of Proposition 1 is as below.

Corollary. Assume there are two firms. They in turn decide to split or merge so that the possible number of firms is $1, 2, 3, ...$ Then, firms keep splitting if and only if $\alpha_n < n + \frac{n-1}{2c}$ for all $n = 2, 3, ...$ where α_n is a coefficient of cost function, cq^2, of each firm when there are n firms in the market.

Proof. Appropriate changes of the order of subtractions and the direction of inequalities in the argument that precedes the proposition and in the proof of Proposition 1 prove the corollary. □

Proposition 1 and the corollary can be combined to show the condition of the emergence of an n-poly. Note that the emergence of an *n-poly* is equivalent to the *argmax* of the potential function being an *n-poly*.

Proposition 2. Assume there are two firms. They in turn decide to split or merge so that the possible number of firms is $1, 2, 3, ...$ Then, an *n-poly* emerges if and only if $\alpha_k < k + \frac{k-1}{2c}$ for $k = 2, 3, 4, ..., n$ and $\alpha_k > k + \frac{k-1}{2c}$ for $k = n + 1, n + 2, ...$

where α_k is a coefficient of cost function, cq^2, of each firm when there are k firms in the market.

Proof. That Π_k^* for $k = 2, 3, 4, ..., n$ attains the maximum value at n if and only if $\alpha_k < k + \dfrac{k-1}{2c}$ for $k = 2, 3, 4, ..., n$ directly follows from the corollary.

Now, I need to show that, for $k = n+1, n+2, ...$, the argmax of Π_k^* is n if and only if $\alpha_k > k + \dfrac{k-1}{2c}$. But this directly follows from Proposition 1. Hence, Proposition 2 is proved. $\square$

Consequently, identifying the outcome of the merger/split game by potential games can be generalized to any number of firms; i.e., rather than constructing a game tree for n-poly, we can identify the outcome by calculating the potential of the game.

3.4.7 Conclusion of Section 3.4

The significant result obtained in this section is whether monopoly, duopoly or triopoly occurs depends only on the form of cost functions. In addition, this result can be generalized to n-poly. Regardless of the resulting number of firms after merger/split, we can identify the outcome of the game by finding the *argmax* of the potential function rather than constructing a game tree. The number of firms involved depends on how rapidly costs increase with output or the number of players; i.e., marginal cost needs to be an increasing function of output, and the cost function needs to be larger as the number of firms increases and the scale of operation of each firm becomes smaller. Hence, the results of this section are likely an answer to the "mystery" raised by Rasmusen (2007); i.e., where costs increase in a particular way, I have an explanation for this finding.

3.5 Conclusion of Chapter 3

At the beginning of Chapter 3, I raised these three claims:

Claim (1)**.** Suppose the marginal cost is constant and the cost function stays the same as the number of firms increases (i.e., the cost function is cq_i for monopoly, duopoly, triopoly and quadopoly for firm i). Then the solution of Cournot competition model is a quadopoly.

Claim (2)**.** If the cost function is identical regardless of the number of players but the marginal cost increases as an output increases, then the solution of Cournot Competition model is a monopoly (Schernikau, 2010).

Claim (3)**.** Suppose the marginal cost becomes larger as the number of players increases and as output increases (i.e., the cost function is cq_i^2 for monopoly, αcq_i^2 for duopoly and βcq_i^2 for triopoly for player i). Then there is a threshold value of α that the solution of Cournot Competition model is a monopoly if α is greater than the threshold value, and there is a threshold value of β that the solution of Cournot competition model is a duopoly, not triopoly, if β is greater than the threshold value.

As for Claim 1, the outcome of the subgame perfect equilibrium is a monopoly; however, the potential is highest at a quadopoly. Hence, as Slade (1994) argues, this subgame perfect equilibrium is "unstable"; i.e., under the presence of errors and noise, it is unlikely that a monopoly emerges although the subgame perfect equilibrium predicts that it is the most rational outcome. Hence, when this game is played, we will witness that firms keep splitting because the subgame perfect equilibrium and the argmax of the potential function do not coincide. This "anomaly" occurs because the assumption of the constant marginal cost is unrealistic. As a result, I need to consider the cases that the marginal cost is increasing in the number of firms and/or output of a firm.

As for Claim 2, if the marginal cost is increasing in output but not in the number of firms (i.e., the marginal cost is cq^2 regardless of the number of firms in the market), then the outcome of the subgame perfect equilibrium and the argmax of the potential function coincide at the quadopoly. Thus, Claim 2 turns out to be false; however, this case depicts what many economists (e.g., Rasmusen (2007)) observe as the theoretical consequence of Cournot competitions; i.e., firms keep splitting into smaller firms. Thus, this case should be considered as the "typical" Cournot competitions rather than the case of the constant marginal cost.

As for Claim 3, the result indicates that this claim is compatible with the reality of the cost functions (i.e., increasing as an output increases and becoming costly for a smaller scale of operation). Hence, this claim answers Research Question 1. The reason behind the "success" of Claim 3 is that it seems to incorporate two aspects of cost function that I mentioned earlier: economies of scale and transaction costs. Especially I believe transaction costs are the keys to explaining the validity of Claim 3. When a firm splits into separate firms, coordination between them will be difficult and costly. On the other hand, when separate firms merge and become a single firm, they can internalize the transaction cost that they used to incur for coordinating their acts so that their cost function will be cheaper after merger. In reality, many firms merge so that they attempt to enjoy the monopoly profit; however, our casual observation suggests that these firms also try to save a significant amount of transaction cost, and their operation becomes much more efficient after merger because they succeed to internalize these transaction costs they used to incur before merger.

Now it is unlikely that for-profit firms release the information about their business operations; i.e., even if these firms do merge, they probably do not release the information about the real reasons of their merger. However, for-profits organizations are not the only organizations operating in the field of environment and natural resources.

Indeed, there are numerous non-profit organizations operating in this field, and some of them do merge frequently. They have no incentive to hide their true motives for merging because they are not operating for profits. For instance, in January, 2015, three environmental groups in North Carolina merged and became a single environmental non-profit organization called the MountainTrue. According to their web site (MountainTrue, 2016), the reasons they merge are

- to have a stronger influence on policy at all levels of government through increased local presence
- to build a stronger organization and increase their geographic reach
- to strengthen their grassroots engagement and involve a broader spectrum of the population

These statements are abstract; however, a common theme among these goals can be identified; i.e., their merger is related to the issue of efficiency of their operation.

Wilder Research (2011) conducted a research on the operations of non-profit organizations, and they identified several reasons that non-profit organizations would consider a merger. One of the main reasons they identified is that non-profit organizations choose to merge when they wish to develop greater organizational efficiency. They argue that "[e]fficiencies can be related to programming, administrative capacity, or fundraising" (13), and they also emphasize that "cost savings is not a reason to merge" (14); i.e., "saving money should not be a key motivator for merging" (14).

In this chapter, I draw the main result (Claim 3) according to the form of the cost function, and as argued above, this cost function is not wholly a representation of the monetary payment. Rather, the cost function of this chapter is an expression of the transaction cost; i.e., if firms are separated, the transaction cost is high for cooperating and/or coordinating their operations so that, by merging, they internalize such transaction costs and their operations become more efficient. Thus, if the transaction cost is included in the argument of this chapter, then the empirical case of the merger of non-profit organizations validates the result obtained in this chapter.

Chapter 4

THE EFFECT OF GOVERNMENT INTERVENTION VIA A POLLUTION TAX ON THE STOCHASTICALLY STABLE NASH EQUILIBRIUM

4.1 Introduction

There are several studies that discuss the Cournot model with pollution and the taxation on effluent (Canton *et al.*, 2008; Katsoulacos and Xepapadeas, 1995; Kennedy, 1994; Lee, 1999; Levin, 1985; McKitrick, 1999; Okuguchi, 2004; Requate, 1993; Simpson, 1995). The goal of all these is to find the optimal rate of taxation when there is pollution. In Chapter 4, I find the optimal taxation rate; however, my main objective is to tackle a different research question from Chapter 3; i.e., what is the effect of Pigovian taxes on the stochastically stable Nash equilibrium? Therefore, as in Chapter 3, I observe how the Nash equilibrium changes as the number of players changes. The studies mentioned above analyze the optimal taxation rate given a fixed number of players and do not observe how the optimal rate of taxation changes as the number of players changes. Thus, to my knowledge, an analysis of the Cournot model with pollution and taxation when the number of players changes is unprecedented.

In addition, I contribute to our understanding of the relationship between the marginal damage to the environment and the rate of taxation. Some studies argue that the optimal tax rate could exceed the marginal damage (Katsoulacos and Xepapadeas, 1995; Simpson, 1995); some studies argue that the optimal tax rate is equal to the marginal damage (Lee, 1999); others argue that the marginal damage exceed the optimal tax rate (Kennedy, 1994; Okuguchi, 2004). In the course of answering the research question mentioned above, I also ascertain the relationship between the marginal damage to the environment and the optimal tax rate.

As in Section 3.4, I assume a production cost function of the form $C(q_i) = cq_i^2$ where c is a positive constant. As in a number of other studies of pollution taxes, I assume that the marginal abatement cost of firm i, $\frac{dC_i^a}{du_i}$, is of the form $\frac{dC_i^a}{du_i} = b - du_i \geq 0$ where b and d are positive constants, u_i is the amount of effluent, and $0 \leq u_i \leq \frac{b}{d}$. Furthermore, $u_i = \varepsilon q_i$ where ε is emission per output. Consequently, I consider the following model:

$$\begin{aligned}
\pi_i &= \big(l - \sum_i q_i\big)q_i - cq_i^2 - C_a - \tau u \\
&= \big(l - \sum_i q_i\big)q_i - cq_i^2 - C_a - \tau\varepsilon q_i
\end{aligned}$$

where π_i is the profit for firm i, l is a positive constant, and τ is the tax per unit effluent.

The setting of the game is identical with that of Chapter 3; i.e., as in Figure 4.1, first *Apex* decides its move, and then *Brydox* moves. The only difference with the cases in Chapter 3 is that each firm in this chapter causes emissions. Note that, as in Section 3.4, if both *Apex* and *Brydox* split, there will be four firms (*Apex 1*, *Apex 2*, *Brydox 1* and *Brydox 2*) so that the outcome is quadopoly. However, in this case, *Apex* occupies 50% of the market, and so does *Brydox*; hence, in reality, this case is a duopoly. As a result, they have incentive to merge again and internalize the cost because, in Chapter 4, the assumption on cost functions is that they become larger as the scale of operation becomes smaller (i.e., as a firm splits).

As in Chapter 3, I find profits of the firms and values of the potential functions for monopoly, duopoly and triopoly and determine under what conditions of the tax adjusted cost function, the solution of the game is a monopoly. Specifically, the cost function is $cq^2 + C^a$ for monopoly, $\alpha_1 cq_i^2 + \alpha_2 C_i^a$ for duopoly where $\alpha_1 > 1$ and $\alpha_2 > 1$, and $\beta_1 cq_i^2 + \beta_2 C_i^a$ for triopoly where $\beta_1 > \alpha_1$ and $\beta_2 > \alpha_2$. Hence, I detemine how

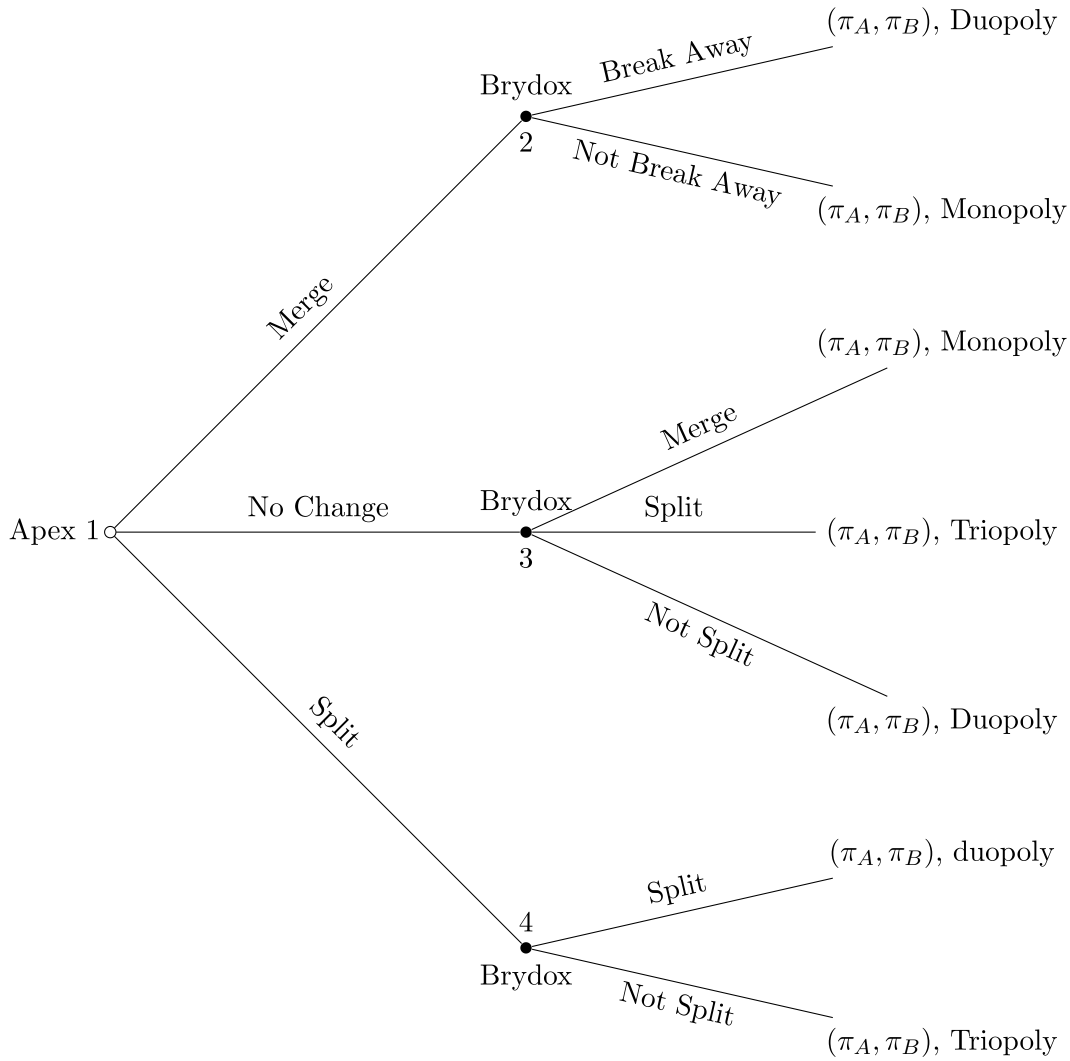

Figure 4.1: The Sequential Game of Merger/Split

large the α_i's and β_i's must be so that the solution of the game becomes a monopoly.

4.2 Model and Analyses

4.2.1 Monopoly

The profit function for monopoly is

$$\pi = (l - q)q - cq^2 - C^a - \tau u$$

As mentioned earlier, $\dfrac{dC_i^a}{du_i}$ is a marginal abatement cost and

$$\frac{dC_i^a}{du} = \begin{cases} b - du & (0 \leq u \leq \dfrac{b}{d}) \\ 0 & (otherwise) \end{cases}$$

where $u = \varepsilon q$. Hence, $0 \leq \varepsilon q \leq \dfrac{b}{d}$ must hold for any positive marginal abatement cost; i.e., $0 \leq q \leq \dfrac{b}{d\varepsilon}$ must hold any positive marginal cost. In this chapter, I restrict my analysis to the case where $0 \leq q \leq \dfrac{b}{d\varepsilon}$.

The first-order condition is

$$\begin{aligned} \frac{d\pi}{dq} &= l - 2q - 2cq - \frac{dC^a}{du}\frac{du}{dq} - \tau\frac{du}{dq} \\ &= l - 2q - 2cq - (b - du)\varepsilon - \tau\varepsilon \\ &= l - 2q - 2cq - (b - \varepsilon dq)\varepsilon - \tau\varepsilon \\ &= l - 2q - 2cq + d\varepsilon^2 q - b\varepsilon - \tau\varepsilon \\ &= 0 \end{aligned}$$

Hence,

$$q_m^* = \frac{l - b\varepsilon - \tau\varepsilon}{2 + 2c - d\varepsilon^2}$$

where the subscript denotes that it is an output for monopoly.

The second-order condition is

$$\begin{aligned}\frac{d^2\pi}{dq^2} &= -2-2c+d\varepsilon^2 \\ &< 0\end{aligned}$$

if $d\varepsilon^2 < 2+2c$. Hence, $l-b\varepsilon-\tau\varepsilon>0$ must hold; i.e., l must satisfy $l > b\varepsilon+\tau\varepsilon$.

To calculate the profit at the optimal level of output, first I need to know the abatement cost; i.e.,

$$C^a = bu - \frac{d}{2}u^2 + e$$

where e is a constant of integration. But since pollution is zero when output is zero, abatement cost is also zero when output is zero. Hence $e = 0$. As a result,

$$C^a = b\varepsilon q - \frac{d}{2}\varepsilon^2 q^2$$

Consequently, the optimal profit is

$$\begin{aligned}\pi_m^* &= lq^* - q^{*2} - cq^{*2} - b\varepsilon q^* + \frac{d}{2}\varepsilon^2 q^{*2} - \tau\varepsilon q^* \\ &= (l-b\varepsilon-\tau\varepsilon)q^* + (\frac{d\varepsilon^2}{2} - 1 - c)q^{*2} \\ &= \frac{(l-b\varepsilon-\tau\varepsilon)^2}{2(2+2c-d\varepsilon^2)}\end{aligned}$$

where the subscript denotes that it is a profit for monopoly.

Next, I need to determine the optimal tax rate. Hence, I define the social welfare function, W.

$$W = \pi_m^* - D + R$$

where D is the damage caused by pollution to the society and R is the revenue from taxation. The optimal rate of taxation is the rate that maximizes the social welfare, W. Damage caused by pollution is a “bad” (opposite of a good) to the society;

therefore, the opposite of the "Law of Diminishing Marginal Utility" holds; i.e., the "Law of Increasing Marginal Disutility" holds. Thus, many economic studies assume that a damage function is an increasing function of pollution, and it is also convex (Kennedy, 1994; Lee, 1999; McKitrick, 1999; Requate, 1993). Hence, in this chapter, I assume $D = mu^2 = m\varepsilon^2 q_m^{*2}$ where m is a positive constant and $R = \tau\varepsilon q_m^*$.

$$\begin{aligned} W &= \pi_m^* - m\varepsilon^2 q_m^{*2} + \tau\varepsilon q_m^* \\ &= \frac{(l - b\varepsilon - \tau\varepsilon)^2}{2(2 + 2c - d\varepsilon^2)} - m\varepsilon^2 \frac{(l - b\varepsilon - \tau\varepsilon)^2}{(2 + 2c - d\varepsilon^2)^2} \\ &\quad + \tau\varepsilon \frac{l - b\varepsilon - \tau\varepsilon}{2 + 2c - d\varepsilon^2} \end{aligned}$$

The first-order condition is

$$\begin{aligned} \frac{dW}{d\tau} &= \frac{2(l - b\varepsilon - \tau\varepsilon)(-\varepsilon)}{2(2 + 2c - d\varepsilon^2)} - \frac{2m\varepsilon^2(l - b\varepsilon - \tau\varepsilon)(-\varepsilon)}{(2 + 2c - d\varepsilon^2)^2} + \tau\varepsilon \frac{-\varepsilon}{2 + 2c - d\varepsilon^2} \\ &\quad + \frac{\varepsilon(l - b\varepsilon - \tau\varepsilon)}{2 + 2c - d\varepsilon^2} \\ &= \frac{2m\varepsilon^3(l - b\varepsilon - \tau\varepsilon) - \tau\varepsilon^2(2 + 2c - d\varepsilon^2)}{(2 + 2c - d\varepsilon^2)^2} \\ &= 0 \end{aligned}$$

Consequently,

$$2m\varepsilon(l - b\varepsilon - \tau\varepsilon) - \tau(2 + 2c - d\varepsilon^2) = 0$$

Hence,

$$\tau_m^* = \frac{-2lm\varepsilon + 2bm\varepsilon^2}{d\varepsilon^2 - 2 - 2c - 2m\varepsilon^2}$$

where the subscript denotes that it is a tax rate for monopoly.

Previously, I establish that $d\varepsilon^2 < 2 + 2c$. Hence, $d\varepsilon^2 - 2 - 2c - 2m\varepsilon^2 < 0$. Consequently, for $\tau_m > 0$ to hold, I need to have $-2lm\varepsilon + 2bm\varepsilon^2 < 0$; i.e., $l > b\varepsilon$ must hold. But previously, I establish that $l > \varepsilon(b + \tau_m)$ holds. Thus, $l > b\varepsilon$ always holds.

The second-order condition is

$$\begin{aligned}
\frac{d^2W}{d\tau^2} &= \frac{2m\varepsilon^3}{(2+2c-d\varepsilon^2)^2}(-\varepsilon) - \frac{\varepsilon^2}{2+2c-d\varepsilon} \\
&= \frac{-2m\varepsilon^4 - \varepsilon^2(2+2c-d\varepsilon^2)}{(2+2c-d\varepsilon^2)^2} \\
&< 0
\end{aligned}$$

because $d\varepsilon^2 < 2+2c$. Hence, τ_m^* is indeed an optimal tax rate.

Now, given the optimal rate of taxation, the optimal profit becomes

$$\begin{aligned}
\pi_m^* &= \frac{(l-b\varepsilon-\tau_m\varepsilon)^2}{2(2+2c-d\varepsilon^2)} \\
&= \frac{(ld\varepsilon^2-2l-2cl-bd\varepsilon^3+2b\varepsilon+2bc\varepsilon)^2}{2(d\varepsilon^2-2-2c-2m\varepsilon^2)^2(2+2c-d\varepsilon^2)} \\
&= \frac{(2+2c-d\varepsilon^2)(b\varepsilon-l)^2}{2(d\varepsilon^2-2-2c-2m\varepsilon^2)^2}
\end{aligned}$$

Note that since $d\varepsilon^2 < 2+2c$, $\pi_m^* > 0$ holds. Also, q_m^* becomes

$$\begin{aligned}
q_m^* &= \frac{l-b\varepsilon-\tau_m\varepsilon}{2+2c-d\varepsilon^2} \\
&= \frac{ld\varepsilon^2-2l-2cl-bd\varepsilon^3+2b\varepsilon+2bc\varepsilon}{(2+2c-d\varepsilon^2)(d\varepsilon^2-2-2c-2m\varepsilon^2)} \\
&= \frac{l-b\varepsilon}{2m\varepsilon^2+2+2c-d\varepsilon^2}
\end{aligned}$$

Since $l > b\varepsilon$ and $d\varepsilon^2 < 2+2c$, $q^* > 0$ holds. Also, at the beginning, I restricted q_m^* to be in $0 \leq q_m^* \leq \frac{b}{d\varepsilon}$. Hence, I need to check whether the value of q_m^* is indeed less than or equal to $\frac{b}{d\varepsilon}$.

$$\begin{aligned}
&\frac{b}{d\varepsilon} > \frac{l-b\varepsilon}{2m\varepsilon^2+2+2c-d\varepsilon^2} \\
\Leftrightarrow\ & l < \frac{b}{d\varepsilon}(2m\varepsilon^2+2+2c-d\varepsilon^2)+b\varepsilon
\end{aligned}$$

Hence, unlike the cases in Chapter 3, the Cournot model with pollution and optimal taxation requires that l be bounded above.

To clarify the intuition behind this result, consider the following parameter values. Suppose $b = c = d = m = \varepsilon = 1$. Then, l must satisfy $l < 6$. So, let's choose $l = 5$. Then, $\tau_m^* = 1.6$, $\pi_m^* = 0.96$ and $q_m^* = 0.8$. Note that $q_m^* < \frac{b}{d\varepsilon}$ and $\varepsilon(b + \tau_m) = 2.6 < 5 = l$. This shows that the results I have obtained do not have contradictions.

Lastly, I analyze the relationship between the marginal damage to the environment $\left.\frac{dD}{dq}\right|_{q_m^*}$ and the optimal tax rate, τ_m^*.

$$\begin{aligned}\left.\frac{dD}{dq}\right|_{q_m^*} &= \left.\frac{d}{dq} m\varepsilon^2 q^2\right|_{q_m*} \\ &= 2m\varepsilon^2 q_m^* \\ &= 2m\varepsilon^2 \left(\frac{l - b\varepsilon}{2m\varepsilon^2 + 2 + 2c - d\varepsilon^2}\right)\end{aligned}$$

and it is already shown that

$$\tau_m^* = \frac{2lm\varepsilon - 2bm\varepsilon^2}{2m\varepsilon^2 + 2 + 2c - d\varepsilon^2}$$

Hence,

$$\begin{aligned}\left.\frac{dD}{dq}\right|_{q_m^*} - \tau_m^* &= \frac{2m\varepsilon^2(l - b\varepsilon) - 2lm\varepsilon + 2bm\varepsilon^2}{2m\varepsilon^2 + 2 + 2c - d\varepsilon^2} \\ &= \frac{2m\varepsilon(l - b\varepsilon)(\varepsilon - 1)}{2m\varepsilon^2 + 2 + 2c - d\varepsilon^2}\end{aligned}$$

I have already shown that $l - b\varepsilon > 0$. Also, the second-order condition is $-2 - 2c + d\varepsilon^2 < 0 \Leftrightarrow 2 + 2c - d\varepsilon^2 > 0$. Hence, $2m\varepsilon^2 + 2 + 2c - d\varepsilon^2 > 0$. As a result, the marginal damage to the environment is greater (less) than the optimal tax rate if $\varepsilon > 1$ ($\varepsilon < 1$) and equal if $\varepsilon = 1$.

4.2.2 Duopoly

The duopoly model of Cournot competition with pollution and taxation on effluent is as follows

$$
\begin{aligned}
\pi_i &= (l - q_1 - q_2)q_i - \alpha_1 c q_i^2 - \alpha_2 C_i^a - \tau u_i \\
&= (l - q_1 - q_2)q_i - \alpha_1 c q_i^2 - \alpha_2 C_i^a - \tau \varepsilon q_i
\end{aligned}
$$

for $i = 1, 2$ and

$$
\frac{dC_i^a}{du_i} = \begin{cases} b - du_i & (0 \leq u_i \leq \dfrac{b}{d}) \\ 0 & (otherwise) \end{cases}
$$

where $u_i = \varepsilon q_i$ and $\alpha_1, \alpha_2 > 1$. Hence, $0 \leq \varepsilon q_i \leq \dfrac{b}{d}$ must hold for a positive marginal abatement cost; i.e., $0 \leq q_i \leq \dfrac{b}{d\varepsilon}$ must hold a positive marginal cost. In this subsection, as in the previous subsection, I restrict my analysis to the case where $0 \leq q_i \leq \dfrac{b}{d\varepsilon}$ so that marginal abatement cost is always positive. Note that for the same reason from the previous subsection, $C^a = b\varepsilon q - \dfrac{d}{2}\varepsilon^2 q^2$.

The possible potential function, Π_d, is

$$
\begin{aligned}
\Pi_d &= (l - q_1 - q_2)q_1 - \alpha_1 c q_1^2 - \alpha_2 C_1^a - \tau u_1 \\
&\quad + (l - q_1 - q_2)q_2 - \alpha_1 c q_2^2 - \alpha_2 C_2^a - \tau u_2 \\
&\quad + q_1 q_2
\end{aligned}
$$

where the subscript denotes that it is a possible potential for duopoly.

The first-order derivative with respect to q_1 is

$$
\begin{aligned}
\frac{\partial \pi_1}{\partial q_1} &= l - q_1 - q_2 - q_1 - 2\alpha_1 c q_1 - \alpha_2 \frac{dC_1^a}{du_1}\frac{du_1}{dq_1} - \tau\varepsilon \\
&= l - 2q_1 - q_2 - 2\alpha_1 c q_1 - \alpha_2(b - d\varepsilon q_1)\varepsilon - \tau\varepsilon \\
&= \frac{\partial \Pi}{\partial q_1}
\end{aligned}
$$

and we can similarly show that $\dfrac{\partial \pi_2}{\partial q_2} = \dfrac{\partial \Pi}{\partial q_2}$. Hence, Π_d is a potential function.

The first-order necessary conditions are

$$l - 2q_1 - q_2 - 2\alpha_1 c q_1 - \alpha_2 \varepsilon (b - d\varepsilon q_1) - \tau\varepsilon = 0$$

$$l - q_1 - 2q_2 - 2\alpha_1 c q_2 - \alpha_2 \varepsilon (b - d\varepsilon q_2) - \tau\varepsilon = 0$$

Since the game is symmetric, $q_1^* = q_2^* = q_d^*$ where the subscript denotes that it is an optimal output for duopoly. Hence,

$$\begin{aligned} & l - 3q_d^* - 2\alpha_1 c q_d^* - \alpha_2 b\varepsilon + \alpha_2 d\varepsilon^2 q_d^* - \tau\varepsilon = 0 \\ \Leftrightarrow \quad & q_d^* = \frac{l - \alpha_2 b\varepsilon - \tau\varepsilon}{3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2} \end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 \Pi}{\partial q_i^2} \;=\; -2 - 2\alpha_1 c + \alpha_2 d\varepsilon^2$$

for $i = 1, 2$ and

$$\frac{\partial^2 \Pi}{\partial q_i \partial q_j} = -1$$

for $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2 - 2\alpha_1 c + \alpha_2 d\varepsilon^2 & -1 \\ -1 & -2 - 2\alpha_1 c + \alpha_2 d\varepsilon^2 \end{pmatrix}$$

The eigenvalues are $\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 1$ and $\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 3$. For Π_d to have a local maximum at q_d^*, both eigenvalues have to be negative. In particular, $\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 3 < 0$ must hold. But this implies that for q_d^* to be positive, $l - \alpha_2 b\varepsilon - \tau\varepsilon > 0$ must hold; i.e., l is bounded below by $\alpha_2 b\varepsilon + \tau\varepsilon$.

Since, the game is symmetric, $\pi_1^* = \pi_2^* = \pi_d^*$ where the subscript denotes that it is a profit for duopoly. Hence,

$$
\begin{aligned}
\pi_d^* &= (l-2q^*)q_d^* - \alpha_1 c q_d^{*2} - \alpha_2\left(bu - \frac{d}{2}u^2\right) - \tau\varepsilon q_d^* \\
&= lq_d^* - 2q_d^{*2} - \alpha_1 c q_d^{*2} - \alpha_2 b\varepsilon q_d^* + \frac{1}{2}\alpha_2 d\varepsilon^2 q_d^{*2} - \tau\varepsilon q_d^* \\
&= (l - \alpha_2 b\varepsilon - \tau\varepsilon)q_d^* + \left(\frac{\alpha_2 d\varepsilon^2}{2} - \alpha_1 c - 2\right)q_d^{*2} \\
&= \frac{(l-\alpha_2 b\varepsilon - \tau\varepsilon)^2}{3+2\alpha_1 c - \alpha_2 d\varepsilon^2} + \left(\frac{\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 4}{2}\right)\frac{(l-\alpha_2 b\varepsilon - \tau\varepsilon)^2}{(3+2\alpha_1 c - \alpha_2 d\varepsilon^2)^2} \\
&= (l-\alpha_2 b\varepsilon - \tau\varepsilon)^2\left(\frac{2+2\alpha_1 c - \alpha_2 d\varepsilon^2}{2(3+2\alpha_1 c - \alpha_2 d\varepsilon^2)^2}\right)
\end{aligned}
$$

Since the eigenvalues are negative,

$$
\begin{aligned}
\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 1 &< 0 \\
\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 3 &< 0
\end{aligned}
$$

must be true. Hence,

$$
\begin{aligned}
1 + 2\alpha_1 c - \alpha_2 d\varepsilon^2 &> 0 \\
3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2 &> 0
\end{aligned}
$$

As a result,

$$2 + 2\alpha_1 c - \alpha_2 d\varepsilon^2 > 0$$

so that $\pi_d^* > 0$.

Next, the optimal value of potential function, Π_d^*, is

$$\begin{aligned}
\Pi_d^* &= (l - q_1 - q_2)q_1 - \alpha_1 c q_1^2 - \alpha_2 C_1^a - \tau u_1 \\
&\quad +(l - q_1 - q_2)q_2 - \alpha_1 c q_2^2 - \alpha_2 C_2^a - \tau u_2 \\
&\quad +q_1 q_2 \\
&= \pi_1^* + \pi_2^* + q_d^{*2} \\
&= (l - \alpha_2 b\varepsilon - \tau\varepsilon)^2 \left(\frac{2 + 2\alpha_1 c - \alpha_2 d\varepsilon^2}{(3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)^2} \right) + \frac{(l - \alpha_2 b\varepsilon - \tau\varepsilon)^2}{(3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)^2} \\
&= \frac{(l - \alpha_2 b\varepsilon - \tau\varepsilon)^2}{3 + 2\alpha_1 - \alpha_2 d\varepsilon}
\end{aligned}$$

Now, I need to determine the optimal tax rate. Hence, as in the previous subsection, I define the social welfare function, W.

$$\begin{aligned}
W &= \pi_1^* + \pi_2^* - m\varepsilon^2 q_1^{*2} - m\varepsilon^2 q_2^{*2} + \tau\varepsilon q_1^* + \tau\varepsilon q_2^* \\
&= \frac{(l - \alpha_2 b\varepsilon - \tau\varepsilon)^2 (2 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)}{(3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)^2} - 2m\varepsilon^2 \left(\frac{l - \alpha_2 b\varepsilon - \tau\varepsilon}{3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2} \right)^2 \\
&\quad +2\tau\varepsilon \left(\frac{l - \alpha_2 b\varepsilon - \tau\varepsilon}{3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2} \right)
\end{aligned}$$

The first-order necessary condition on τ is

$$\begin{aligned}
\frac{\partial W}{\partial \tau} &= \frac{-2\varepsilon(2 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)(l - \alpha_2 b\varepsilon - \tau\varepsilon)}{(3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)^2} + 4m\varepsilon^3 \left(\frac{l - \alpha_2 b\varepsilon - \tau\varepsilon}{(3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)^2} \right) \\
&\quad +2\varepsilon \left(\frac{l - \alpha_2 b\varepsilon - \tau\varepsilon}{3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2} \right) - 2\tau\varepsilon \left(\frac{\varepsilon}{3 + 2\alpha_1 c - \alpha_2 d\varepsilon^2} \right) \\
&= 0
\end{aligned}$$

Consequently, by using Matlab,

$$\tau_d^* = \frac{(l - \alpha_2 b\varepsilon)(2m\varepsilon^2 + 1)}{\varepsilon(2\alpha_1 c + 2m\varepsilon^2 - \alpha_2 d\varepsilon^2 + 4)}$$

where the subscript denotes that it is an optimal tax rate for duopoly.

Since one of the eigenvalues, $\alpha_2 d\varepsilon^2 - 3 - 2\alpha_1 c$, is negative, $2\alpha_1 c + 2m\varepsilon^2 - \alpha_2 d\varepsilon^2 + 4 > 0$ holds. Hence, $\tau_d^* > 0$.

The second-order condition on τ is

$$\begin{aligned}
\frac{\partial^2 W}{\partial \tau^2} &= \frac{2\varepsilon^2(2+2\alpha_1 c-\alpha_2 d\varepsilon^2)}{(3+2\alpha_1 c-\alpha_2 d\varepsilon^2)^2}-\frac{4m\varepsilon^4}{(3+2\alpha_1 c-\alpha_2 d\varepsilon^2)^2}-\frac{2\varepsilon^2}{3+2\alpha_1 c-\alpha_2 d\varepsilon^2} \\
&\quad -\frac{2\varepsilon^2}{3+2\alpha_1 c-\alpha_2 d\varepsilon^2} \\
&= \frac{2\varepsilon^2(2+2\alpha_1 c-\alpha_2 d\varepsilon^2)-4m\varepsilon^4}{(3+2\alpha_1 c-\alpha_2 d\varepsilon^2)^2}-\frac{4\varepsilon^2}{3+2\alpha_1 c-\alpha_2 d\varepsilon^2} \\
&= \frac{2\varepsilon^2(-4-2\alpha_1 c+\alpha_2 d\varepsilon^2-2m\varepsilon^2)}{(3+2\alpha_1 c-\alpha_2 d\varepsilon^2)^2}
\end{aligned}$$

It has been shown that one of the eigenvalues satisfies $\alpha_2 d\varepsilon^2 - 2\alpha_1 c < 3$. Hence, $-4-2\alpha_1+\alpha_2 d\varepsilon^2-2m\varepsilon^2 < 0$ so that $\dfrac{\partial^2 W}{\partial \tau^2} < 0$. As a result, W attains a local maximum at τ_d^*.

It has been shown that $\pi_d^* = (l-\alpha_2 b\varepsilon-\tau\varepsilon)^2\left(\dfrac{2+2\alpha_1 c-\alpha_2 d\varepsilon^2}{2(3+2\alpha_1 c-\alpha_2 d\varepsilon^2)^2}\right)$. Hence, if I plug in τ_d^* found above, I obtain

$$\pi_d^* = \frac{(l-\alpha_2 b\varepsilon)^2(-\alpha_2 d\varepsilon^2+2\alpha_1 c+2)}{2(2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4)^2}$$

One of the eigenvalues satisfies $\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 1 < 0 \Leftrightarrow 1+2\alpha_1 c-\alpha_2 d\varepsilon^2 > 0$. Hence, $2+2\alpha_1 c-\alpha_2 d\varepsilon^2 > 0$. As a result, $\pi_d^* > 0$.

It has been shown that $\Pi_d^* = \dfrac{(l-\alpha_2 b\varepsilon-\tau\varepsilon)^2}{3+2\alpha_1 c-\alpha_2 d\varepsilon}$. Hence, if I plug in τ_d^* found above, I obtain

$$\Pi_d^* = \frac{(l-\alpha_2 b\varepsilon)^2(-\alpha_2 d\varepsilon^2+2\alpha_1 c+3)}{(2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4)^2}$$

One of the eigenvalues satisfies $\alpha_2 d\varepsilon^2 - 2\alpha_1 c - 3 < 0 \Leftrightarrow 3+2\alpha_1 c-\alpha_2 d\varepsilon^2 > 0$. Hence, $\Pi_d^* > 0$.

It has been shown that $q_d^* = \dfrac{l-\alpha_2 b\varepsilon-\tau\varepsilon}{3+2\alpha_1 c-\alpha_2 d\varepsilon^2}$. Hence, if I substitute τ_d^* found above, I obtain

$$q_d^* = \frac{l-\alpha_2 b\varepsilon}{2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4}$$

It has been verified that $2\alpha_1 c + 2m\varepsilon^2 - \alpha_2 d\varepsilon^2 + 4 > 0$. Hence, $q_d^* > 0$. Note that the values of π_d^*, Π_d^* and q_d^* are computed by Matlab.

Next, by definition of the potential function, the potential for the case of monopoly is π_m. Now, I compute $\pi_m^* - \Pi_d^*$ to determine whether the potential is larger at a monopoly or at a duopoly.

$$\pi_m^* - \Pi_d^* = \frac{(2 + 2c - d\varepsilon^2)(l - b\varepsilon)^2}{2(2m\varepsilon^2 - d\varepsilon^2 + 2 + 2c)^2} - \frac{(l - \alpha_2 b\varepsilon)^2(2\alpha_1 c - \alpha_2 d\varepsilon^2 + 3)}{(2\alpha_1 c + 2m\varepsilon^2 - \alpha_2 d\varepsilon^2 + 4)^2}$$

Computation of the above expression by Matlab does not provide a tractable expression. Hence, I check the above subtraction numerically. In the previous subsection, I have obtained that when $b = c = d = m = \varepsilon = 1$ and $l = 5$, $\pi_m^* = 0.96$. Under the same condition, for the case of duopoly, α_1 and α_2 must satisfy $\alpha_2 < 2\alpha_1 + 1$ from the condition of eigenvalues. Hence, if $\alpha_1 = 2$, then $\alpha_2 < 5$ must be satisfied. So, let's choose $\alpha_2 = 2$. In this case, I obtain $\tau_d^* = 1.125$, $\pi_d^* = 0.28125$, $\Pi_d^* = 0.703125$ and $q_d^* = 0.375$. Hence, $\pi_m^* - \Pi_d^* > 0$. As a result, the value of the potential function at monopoly is larger than its value at duopoly.

Lastly, I analyze the relationship between the marginal damage to the environment $\left.\frac{dD}{dq_i}\right|_{q_d^*}$ and the optimal tax rate, τ_d^*.

$$\begin{aligned} \left.\frac{dD_i}{dq_i}\right|_{q_d^*} &= \left.\frac{d}{dq_i} m\varepsilon^2 q_i^2\right|_{q_d^*} \\ &= 2m\varepsilon^2 q_d^* \\ &= 2m\varepsilon^2 \left(\frac{l - \alpha_2 b\varepsilon}{2m\varepsilon^2 + 4 + 2\alpha_1 c - \alpha_2 d\varepsilon^2}\right) \end{aligned}$$

It is already shown that

$$\tau_d^* = \frac{(l - \alpha_2 b\varepsilon)(2m\varepsilon^2 + 1)}{\varepsilon(2m\varepsilon^2 + 4 + 2\alpha_1 c - \alpha_2 d\varepsilon^2)}$$

Hence,

$$\left.\frac{dD_i}{dq_i}\right|_{q_d^*} - \tau_d^* = \frac{2m\varepsilon^3(l-\alpha_2 b\varepsilon) - (l-\alpha_2 b\varepsilon)(2m\varepsilon^2+1)}{\varepsilon(2m\varepsilon^2+4+2\alpha_1 c-\alpha_2 d\varepsilon^2)}$$
$$= \frac{(l-\alpha_2 b\varepsilon)(2m\varepsilon^3-2m\varepsilon^2-1)}{\varepsilon(2m\varepsilon^2+4+2\alpha_1 c-\alpha_2 d\varepsilon^2)}$$

I have already shown that $l-\alpha_2 b\varepsilon > 0$. Also, the eigenvalues are negative; hence, $-3-2\alpha_1 c+\alpha_2 d\varepsilon^2 < 0 \Leftrightarrow 3+2\alpha_1 c-\alpha_2 d\varepsilon^2 > 0$. Hence, $2m\varepsilon^2+4+2\alpha_1 c-\alpha_2 d\varepsilon^2 > 0$. As a result, the marginal damage to the environment is greater than the optimal tax rate if $2m\varepsilon^3-2m\varepsilon^2-1 > 0$; i.e., there is a threshold value of ε such that the marginal damage is greater than the optimal tax rate if $\varepsilon^2(\varepsilon-1) > \frac{1}{2m}$. But it is known that m is a positive constant; therefore, the threshold value of ε is greater than 1.

4.2.3 Triopoly

The triopoly model of Cournot competition with pollution and taxation on effluent is as follows:

$$\pi_i = (l-q_1-q_2-q_3)q_i - \beta_1 c q_i^2 - \beta_2 C_i^a - \tau u_i$$
$$= (l-q_1-q_2-q_3)q_i - \beta_1 c q_i^2 - \beta_2 C_i^a - \tau\varepsilon q_i$$

for $i = 1, 2, 3$ and

$$\frac{dC_i^a}{du_i} = \begin{cases} b - du_i & (0 \le u_i \le \frac{b}{d}) \\ 0 & (otherwise) \end{cases}$$

where $u_i = \varepsilon q_i$ and $\beta_1 > \alpha_1 > 1$ and $\beta_2 > \alpha_2 > 1$. Hence, $0 \le \varepsilon q_i \le \frac{b}{d}$ must hold for a positive marginal abatement cost; i.e., $0 \le q_i \le \frac{b}{d\varepsilon}$ must hold a positive marginal cost. In this subsection, as in the previous subsection, I restrict my analysis for $0 \le q_i \le \frac{b}{d\varepsilon}$ so that the the marginal abatement cost always exists. Note that for the same reason from the previous subsection, $C^a = b\varepsilon q - \frac{d}{2}\varepsilon^2 q^2$.

The possible potential function, Π_t, is

$$\begin{aligned}\Pi_t &= (l - q_1 - q_2 - q_3)q_1 - \beta_1 c q_1^2 - \beta_2 C_1^a - \tau u_1 \\ &\quad + (l - q_1 - q_2 - q_3)q_2 - \beta_1 c q_2^2 - \beta_2 C_2^a - \tau u_2 \\ &\quad + (l - q_1 - q_2 - q_3)q_3 - \beta_1 c q_3^2 - \beta_2 C_3^a - \tau u_3 \\ &\quad + q_1 q_2 + q_2 q_3 + q_1 q_3\end{aligned}$$

where the subscript denotes that it is a possible potential for triopoly.

The first-order derivative with respect to q_1 is

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= l - q_1 - q_2 - q_3 - q_1 - 2\beta_1 c q_1 - \beta_2 \frac{dC_1^a}{du_1}\frac{du_1}{dq_1} - \tau\varepsilon \\ &= l - 2q_1 - q_2 - q_3 - 2\beta_1 c q_1 - \beta_2(b - d\varepsilon q_1)\varepsilon - \tau\varepsilon \\ &= \frac{\partial \Pi_t}{\partial q_1}\end{aligned}$$

Similarly, we can show that $\frac{\partial \pi_i}{\partial q_i} = \frac{\partial \Pi_t}{\partial q_i}$ for $i = 2, 3$. Hence, Π_t is indeed a potential function.

The first-order conditions are

$$l - 2q_1 - q_2 - q_3 - 2\beta_1 c q_1 - \beta_2 \varepsilon(b - d\varepsilon q_1) - \tau\varepsilon = 0$$

$$l - q_1 - 2q_2 - q_3 - 2\beta_1 c q_2 - \beta_2 \varepsilon(b - d\varepsilon q_2) - \tau\varepsilon = 0$$

$$l - q_1 - q_2 - 2q_3 - 2\beta_1 c q_3 - \beta_2 \varepsilon(b - d\varepsilon q_3) - \tau\varepsilon = 0$$

Since the game is symmetric, $q_1^* = q_2^* = q_3^* = q_t^*$ where the subscript denotes that it is an output for triopoly. Hence,

$$\begin{aligned} & l - 4q_t^* - 2\beta_1 c q_t^* - \beta_2 b\varepsilon + \beta_2 d\varepsilon^2 q_t^* - \tau\varepsilon = 0 \\ \Leftrightarrow \quad & q_t^* = \frac{l - \beta_2 b\varepsilon - \tau\varepsilon}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2}\end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 \Pi_t}{\partial q_i^2} = -2 - 2\beta_1 c + \beta_2 d\varepsilon^2$$

for $i = 1, 2, 3$ and

$$\frac{\partial^2 \Pi_t}{\partial q_i \partial q_j} = \frac{\partial^2 \Pi_t}{\partial q_j \partial q_i} = -1$$

for $i \neq j$. Hence, the Hessian matrix is

$$H = \begin{pmatrix} -2 - 2\beta_1 c + \beta_2 d\varepsilon^2 & -1 & -1 \\ -1 & -2 - 2\beta_1 c + \beta_2 d\varepsilon^2 & -1 \\ -1 & -1 & -2 - 2\beta_1 c + \beta_2 d\varepsilon^2 \end{pmatrix}$$

The eigenvalues are two multiplicities of $\beta_2 d\varepsilon^2 - 2\beta_1 c - 1$ and $\beta_2 d\varepsilon^2 - 2\beta_1 c - 4$. For Π_t to have a local maximum at q_t^*, both eigenvalues have to be negative. Especially, $\beta_2 d\varepsilon^2 - 2\beta_1 c - 4 < 0$ must hold. But this implies that for q_t^* to be positive, $l - \beta_2 b\varepsilon - \tau\varepsilon > 0$ must hold; i.e., l is bounded below by $\beta_2 b\varepsilon + \tau\varepsilon$.

Since the game is symmetric, $\pi_1^* = \pi_2^* = \pi_3^* = \pi_t^*$ where the subscript denotes that it is a profit for triopoly. Hence,

$$\begin{aligned}
\pi_t^* &= (l - 3q_t^*)q^* - \beta_1 c q_t^{*2} - \beta_2\left(bu - \frac{d}{2}u^2\right) - \tau\varepsilon q_t^* \\
&= l q_t^* - 3q_t^{*2} - \beta_1 c q_t^{*2} - \beta_2 b\varepsilon q_t^* + \frac{1}{2}\beta_2 d\varepsilon^2 q_t^{*2} - \tau\varepsilon q_t^* \\
&= (l - \beta_2 b\varepsilon - \tau\varepsilon)q_t^* + \left(\frac{\beta_2 d\varepsilon^2}{2} - \beta_1 c - 3\right)q_t^{*2} \\
&= \frac{(l - \beta_2 b\varepsilon - \tau\varepsilon)^2}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} + \left(\frac{\beta_2 d\varepsilon^2 - 2\beta_1 c - 6}{2}\right)\frac{(l - \beta_2 b\varepsilon - \tau\varepsilon)^2}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} \\
&= (l - \beta_2 b\varepsilon - \tau\varepsilon)^2\left(\frac{2 + 2\beta_1 c - \beta_2 d\varepsilon^2}{2(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2}\right)
\end{aligned}$$

Since the eigenvalues are negative,

$$\begin{aligned}
\beta_2 d\varepsilon^2 - 2\beta_1 c - 1 &< 0 \\
\beta_2 d\varepsilon^2 - 2\beta_1 c - 4 &< 0
\end{aligned}$$

Hence,

$$\begin{aligned}
1 + 2\beta_1 c - \beta_2 d\varepsilon^2 &> 0 \\
4 + 2\beta_1 c - \beta_2 d\varepsilon^2 &> 0
\end{aligned}$$

As a result,

$$2 + 2\beta_1 c - \beta_2 d\varepsilon^2 > 0$$

so that $\pi_t^* > 0$.

Next, the optimal value of the potential function, Π_t^*, is

$$\begin{aligned}
\Pi_t^* &= (l - q_1 - q_2 - q_3)q_1 - \beta_1 c q_1^2 - \beta_2 C_1^a - \tau u_1 \\
&\quad +(l - q_1 - q_2 - q_3)q_2 - \beta_1 c q_2^2 - \beta_2 C_2^a - \tau u_2 \\
&\quad +(l - q_1 - q_2 - q_3)q_3 - \beta_1 c q_3^2 - \beta_2 C_3^a - \tau u_3 \\
&\quad +q_1 q_2 + q_2 q_3 + q_1 q_3 \\
&= \pi_1^* + \pi_2^* + \pi_3^* + 3q_t^{*2} \\
&= 3(l - \beta_2 b\varepsilon - \tau\varepsilon)^2 \left(\frac{2 + 2\beta_1 c - \beta_2 d\varepsilon^2}{2(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} \right) + \frac{3(l - \beta_2 b\varepsilon - \tau\varepsilon)^2}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} \\
&= \frac{3(l - \beta_2 b\varepsilon - \tau\varepsilon)^2}{2(4 + 2\beta_1 c - \beta_2 d\varepsilon)}
\end{aligned}$$

Now, I need to determine the optimal tax rate. Hence, as in the previous subsection, I define the social welfare function, W.

$$\begin{aligned}
W &= \pi_1^* + \pi_2^* + \pi_3^* - m\varepsilon^2 q_1^{*2} - m\varepsilon^2 q_2^{*2} - m\varepsilon^2 q_3^{*2} + \tau\varepsilon q_1^* + \tau\varepsilon q_2^* + \tau\varepsilon q_3 * \\
&= \frac{3(l - \beta_2 b\varepsilon - \tau\varepsilon)^2(2 + 2\beta_1 c - \beta_2 d\varepsilon^2)}{2(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} - 3m\varepsilon^2 \left(\frac{l - \beta_2 b\varepsilon - \tau\varepsilon}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \right)^2 \\
&\quad +3\tau\varepsilon \left(\frac{l - \beta_2 b\varepsilon - \tau\varepsilon}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \right)
\end{aligned}$$

The first-order necessary condition on τ is

$$\begin{aligned}
\frac{\partial W}{\partial \tau} &= \frac{-3\varepsilon(2 + 2\beta_1 c - \beta_2 d\varepsilon^2)(l - \beta_2 b\varepsilon - \tau\varepsilon)}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} + 6m\varepsilon^3 \left(\frac{l - \beta_2 b\varepsilon - \tau\varepsilon}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} \right) \\
&\quad +3\varepsilon \left(\frac{l - \beta_2 b\varepsilon - \tau\varepsilon}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \right) - 3\tau\varepsilon \left(\frac{\varepsilon}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \right) \\
&= 0
\end{aligned}$$

The usage of a computational software (Matlab) yields

$$\tau_t^* = \frac{(l - \beta_2 b\varepsilon)(2m\varepsilon^2 + 2)}{\varepsilon(2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6)}$$

where the subscript denotes that it is a tax rate for triopoly.

Since one of the eigenvalues, $\beta_2 d\varepsilon^2 - 4 - 2\beta_1 c$, is negative, $2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6 > 0$ holds. Hence, $\tau_t^* > 0$.

The second-order condition on τ is

$$\begin{aligned}
\frac{\partial^2 W}{\partial \tau^2} &= \frac{3\varepsilon^2(2 + 2\beta_1 c - \beta_2 d\varepsilon^2)}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} - \frac{6m\varepsilon^4}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} - \frac{3\varepsilon^2}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \\
&\quad - \frac{3\varepsilon^2}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \\
&= \frac{3\varepsilon^2(2 + 2\beta_1 c - \beta_2 d\varepsilon^2) - 6m\varepsilon^4}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} - \frac{6\varepsilon^2}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2} \\
&= \frac{3\varepsilon^2(-6 - 2\beta_1 c + \beta_2 d\varepsilon^2 - 2m\varepsilon^2)}{(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2}
\end{aligned}$$

It has been shown that one of the eigenvalues satisfies $\beta_2 d\varepsilon^2 - 2\beta_1 c < 4$. Hence, $-6 - 2\beta_1 c + \beta_2 d\varepsilon^2 - 2m\varepsilon^2 < 0$ so that $\frac{\partial^2 W}{\partial \tau^2} < 0$. As a result, W attains a local maximum at τ_t^*.

It has been shown that $\pi_t^* = (l - \beta_2 b\varepsilon - \tau\varepsilon)^2 \left(\frac{2 + 2\beta_1 c - \beta_2 d\varepsilon^2}{2(4 + 2\beta_1 c - \beta_2 d\varepsilon^2)^2} \right)$. Hence, if I plug in τ_t^* found above, I obtain

$$\pi_t^* = \frac{(l - \beta_2 b\varepsilon)^2(-\beta_2 d\varepsilon^2 + 2\beta_1 c + 2)}{2(2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6)^2}$$

One of the eigenvalues satisfies $\beta_2 d\varepsilon^2 - 2\beta_1 c - 1 < 0 \Leftrightarrow 1 + 2\beta_1 c - \beta_2 d\varepsilon^2 > 0$. Hence, $2 + 2\beta_1 c - \beta_2 d\varepsilon^2 > 0$. As a result, $\pi_t^* > 0$.

It has been shown that $\Pi_t^* = \frac{3(l - \beta_2 b\varepsilon - \tau\varepsilon)^2}{2(4 + 2\beta_1 c - \beta_2 d\varepsilon)}$. Hence, if I plug in τ_t^* found above, I obtain

$$\Pi_t^* = \frac{3(l - \beta_2 b\varepsilon)^2(-\beta_2 d\varepsilon^2 + 2\beta_1 c + 4)}{2(2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6)^2}$$

One of the eigenvalues satisfies $\beta_2 d\varepsilon^2 - 2\beta_1 c - 4 < 0 \Leftrightarrow 4 + 2\beta_1 c - \beta_2 d\varepsilon^2 > 0$. Hence, $\Pi_t^* > 0$.

It has been shown that $q_t^* = \dfrac{l - \beta_2 b\varepsilon - \tau\varepsilon}{4 + 2\beta_1 c - \beta_2 d\varepsilon^2}$. Hence, if I plug in τ_t^* found above, I obtain

$$q_t^* = \frac{l - \beta_2 b\varepsilon}{2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6}$$

It has been verified that $2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6 > 0$. Hence, $q_t^* > 0$. Note that the values of π_t^*, Π_t^* and q_t^* are computed by Matlab.

Next, the potential for the case of duopoly is Π_d, that is found in the previous subsection. I compute $\Pi_d^* - \Pi_t^*$ to determine whether the potential is larger at a duopoly or at a triopoly.

$$\Pi_d^* - \Pi_t^* = \frac{(l - \alpha_2 b\varepsilon)^2(2\alpha_1 c - \alpha_2 d\varepsilon^2 + 3)}{(2\alpha_1 c + 2m\varepsilon^2 - \alpha_2 d\varepsilon^2 + 4)^2} - \frac{3(l - \beta_2 b\varepsilon)^2(2\beta_1 c - \beta_2 d\varepsilon^2 + 4)}{2(2\beta_1 c + 2m\varepsilon^2 - \beta_2 d\varepsilon^2 + 6)^2}$$

Computation of the above expression by Matlab does not provide a tractable expression. Hence, I check the above subtraction by plugging in some appropriate values. In the previous subsection, I have obtained that when $b = c = d = m = \varepsilon = 1$ and $l = 5$, $\Pi_d^* = 0.703125$. Under the same condition, for the case of triopoly, β_1 and β_2 must satisfy $\beta_2 < 2\beta_1 + 1$ from the condition of eigenvalues. Hence, if $\beta_1 = 3$, then $\beta_2 < 7$ must be satisfied. So, let's choose $\beta_2 = 3$. In this case, I obtain $\tau_t^* = 0.727$, $\pi_t^* = 0.0826$, $\Pi_t^* = 0.347$ and $q_t^* = 0.182$. Hence, $\Pi_d^* - \Pi_t^* > 0$. As a result, the value of the potential function at duopoly is larger than its value at triopoly.

The results obtained so far are summarized in Table 4.1, and Table 4.2 for specific parameter values.

Lastly, I analyze the relationship between the marginal damage to the environment

Table 4.1: Summary of Chapter 4

	Monopoly	Duopoly	Triopoly
Equilibrium Output per Firm	$\frac{l-b\varepsilon}{2m\varepsilon^2+2+2c-d\varepsilon^2}$	$\frac{l-\alpha_2 b\varepsilon}{2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4}$	$\frac{l-\beta_2 b\varepsilon}{2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+6}$
Equilibrium Price	$\frac{l(2m\varepsilon^2+1+2c-d\varepsilon^2)+b\varepsilon}{2m\varepsilon^2+2+2c-d\varepsilon^2}$	$\frac{l(2m\varepsilon^2+2+2\alpha_1 c-\alpha_2 d\varepsilon^2)+2\alpha_2 b\varepsilon}{2m\varepsilon^2+4+2\alpha_1 c-\alpha_2 d\varepsilon^2}$	$\frac{l(2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+3)+3\beta_2 b\varepsilon}{2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+6}$
Combined Equilibrium Output	$\frac{l-b\varepsilon}{2m\varepsilon^2+2+2c-d\varepsilon^2}$	$\frac{2(l-\alpha_2 b\varepsilon)}{2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4}$	$\frac{3(l-\beta_2 b\varepsilon)}{2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+6}$
Profit per Firm	$\frac{(2+2c-d\varepsilon^2)(b\varepsilon-l)^2}{2(d\varepsilon^2-2-2c-2m\varepsilon^2)^2}$	$\frac{(l-\alpha_2 b\varepsilon)^2(-\alpha_2 d\varepsilon^2+2\alpha_1 c+2)}{2(2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4)^2}$	$\frac{(l-\beta_2 b\varepsilon)^2(-\beta_2 d\varepsilon^2+2\beta_1 c+2)}{2(2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+6)^2}$
Combined Profit	$\frac{(2+2c-d\varepsilon^2)(b\varepsilon-l)^2}{2(d\varepsilon^2-2-2c-2m\varepsilon^2)^2}$	$\frac{(l-\alpha_2 b\varepsilon)^2(-\alpha_2 d\varepsilon^2+2\alpha_1 c+2)}{(2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4)^2}$	$\frac{3(l-\beta_2 b\varepsilon)^2(-\beta_2 d\varepsilon^2+2\beta_1 c+2)}{2(2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+6)^2}$
Potential	$\frac{(2+2c-d\varepsilon^2)(b\varepsilon-l)^2}{2(d\varepsilon^2-2-2c-2m\varepsilon^2)^2}$	$\frac{(l-\alpha_2 b\varepsilon)^2(-\alpha_2 d\varepsilon^2+2\alpha_1 c+3)}{(2\alpha_1 c+2m\varepsilon^2-\alpha_2 d\varepsilon^2+4)^2}$	$\frac{3(l-\beta_2 b\varepsilon)^2(-\beta_2 d\varepsilon^2+2\beta_1 c+4)}{2(2\beta_1 c+2m\varepsilon^2-\beta_2 d\varepsilon^2+6)^2}$
Tax Rate per Firm	$\frac{2bm\varepsilon^2-2lm\varepsilon}{d\varepsilon^2-2-2c-2m\varepsilon^2}$	$\frac{(l-\alpha_2 b\varepsilon)(2m\varepsilon^2+1)}{\varepsilon(2m\varepsilon^2+4+2\alpha_1 c-\alpha_2 d\varepsilon^2)}$	$\frac{(l-\beta_2 b\varepsilon)(2m\varepsilon^2+2)}{\varepsilon(2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2)}$
Marginal Damage per Firm	$\frac{2m\varepsilon^2(l-b\varepsilon)}{2m\varepsilon^2+2+2c-d\varepsilon^2}$	$\frac{2m\varepsilon^2(l-\alpha_2 b\varepsilon)}{2m\varepsilon^2+4+2\alpha_1 c-\alpha_2 d\varepsilon^2}$	$\frac{2m\varepsilon^2(l-\beta_2 b\varepsilon)}{2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2}$

Table 4.2: Summary of Chapter 4 ($b = c = d = m = \varepsilon = 1, l = 5, , \alpha_1 = \alpha_2 = 2, \beta_1 = \beta_2 = 3$)

	Monopoly	Duopoly	Triopoly
Equilibrium Output per Firm	0.8	0.375	0.182
Equilibrium Price	4.2	4.25	4.454
Combined Equilibrium Output	0.8	0.75	0.546
Profit per Firm	0.96	0.28125	0.0826
Combined Profit	0.96	0.5625	0.2478
Potential	0.96	0.703125	0.347
Tax Rate per Firm	1.6	1.125	0.727
Total Tax Revenue	1.6	2.25	2.181
Total Marginal Damage	1.6	1.5	1.092

$\left.\frac{dD_i}{dq_i}\right|_{q_t^*}$ and the optimal tax rate, τ_t^*.

$$\begin{aligned}\left.\frac{dD_i}{dq_i}\right|_{q_t^*} &= \left.\frac{d}{dq_i}m\varepsilon^2 q_i^2\right|_{q_t^*}\\ &= 2m\varepsilon^2 q_t^*\\ &= 2m\varepsilon^2\left(\frac{l-\beta_2 b\varepsilon}{2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2}\right)\end{aligned}$$

It is already shown that

$$\tau_t^* = \frac{(l-\beta_2 b\varepsilon)(2m\varepsilon^2+2)}{\varepsilon(2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2)}$$

Hence,

$$\begin{aligned}\left.\frac{dD_i}{dq_i}\right|_{q_t^*} - \tau_t^* &= \frac{2m\varepsilon^3(l-\beta_2 b\varepsilon)-(l-\beta_2 b\varepsilon)(2m\varepsilon^2+2)}{\varepsilon(2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2)}\\ &= \frac{(l-\beta_2 b\varepsilon)(2m\varepsilon^3-2m\varepsilon^2-2)}{\varepsilon(2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2)}\end{aligned}$$

I have already shown that $l-\beta_2 b\varepsilon > 0$. Also, the eigenvalues are negative; hence, $-4-2\beta_1 c+\beta_2 d\varepsilon^2 < 0 \Leftrightarrow 4+2\beta_1 c-\beta_2 d\varepsilon^2 > 0$. Hence, $2m\varepsilon^2+6+2\beta_1 c-\beta_2 d\varepsilon^2 > 0$. As a result, the marginal damage to the environment is greater than the optimal tax rate if $2m\varepsilon^3-2m\varepsilon^2-2 > 0$; i.e., there is a threshold value of ε such that the marginal damage is greater than the optimal tax rate if $\varepsilon^2(\varepsilon-1) > \frac{1}{m}$. But it is known that m is a positive constant; therefore, the threshold value of ε is greater than 1.

4.3 Solution of the Game

Now, I use the specific values of the profits provided in Table 4.2 to find the solution of the game. Note that when there is a monopoly, according to Table 4.2, the profit per firm is 0.96. However, this monopoly is a result of the merger of *Apex* and *Brydox* so that after they earn the monopoly profit as one firm, they will divide the profit equally. This is the reason that the profit for monopoly is 0.48 for *Apex* and *Brydox*.

By using the well-known method of the backward induction, I can identify the subgame perfect equilibrium: *Apex* chooses either Merge or No Change, and *Brydox* chooses Not Break Away at node 2, Merge at node 3 and split at node 4. Thus, the outcome of the game is a monopoly, and as seen above, the potential function attains the maximum value when there is a monopoly and that the α and β satisfy the conditions found above.

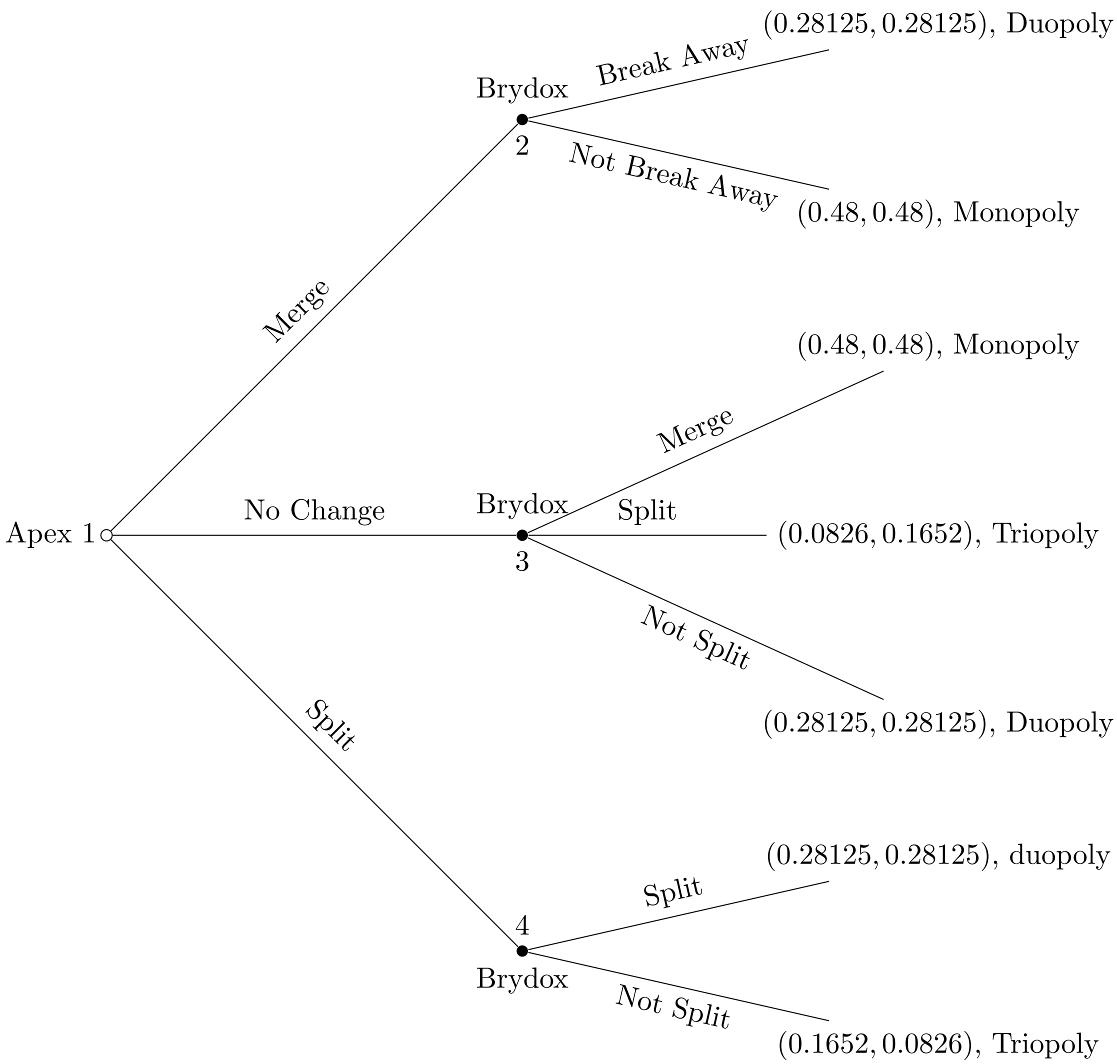

Figure 4.2: The Sequential Game of Merger/Split

Note that if α_i and/or β_i $(i = 1, 2)$ were less than the threshold values, the outcome of the subgame perfection could still be a monopoly as in Section 3.4; however, the potential would attain its maximum value either at a duopoly or at a triopoly. In such a case, the subgame perfection is not stochastically stable so that a duopoly or a triopoly will emerge if the game is actually played out although the most rational/efficient solution is a monopoly.

4.4 Conclusion of Chapter 4

As shown in Chapter 3, when the marginal cost is an increasing function of the output and the cost function becomes larger as the scale of operation becomes smaller, then there are threshold values for α_i and β_i $(i = 1, 2)$ such that monopoly is a solution. When monopoly is a solution, the total marginal damage is larger than that under duopoly or triopoly, as evident from Table 4.2. In addition, what is significant in Table 4.2 is that the tax rate per firm is highest for a monopoly and lowest for a triopoly. The implication of this result is that even if the tax rate is highest for a monopoly, a firm does not consider splitting into separate firms to avoid the high tax rate; i.e., as long as the profit per firm is highest for a monopoly, then the firm "endures" the high tax rate. There is a presumption that "tax considerations can be a key enabler or inhibitor of M&A activity in specific cases, as companies look to incorporate optimization of their tax exposure as an integral part of their assessments of deal valuation and strategic fit" (Deloitte Center for Energy Solutions, 2015, 4); however this presumption turns out to be false. In fact, tax considerations are probably not prime drivers of M&A activities (Deloitte Center for Energy Solutions, 2015), and the reason is as discussed above: A firm endures the high tax rate as long as they earn enough profit. As a result, this result suggests that the tax rate may not enable the government to control the number of firms in industry.

As for the relationship between the marginal damage to the environment and the optimal tax rate, there is a threshold value of ε (emissions per unit of output). For a monopoly, the marginal damage is greater if $\varepsilon > 1$. For a duopoly, marginal damage is greater than the marginal tax rate if $\varepsilon(\varepsilon-1) > \dfrac{1}{2m}$. For a triopoly, $\varepsilon(\varepsilon-1) > \dfrac{1}{m}$. Since m determines the rate of growth of the marginal damage ($2m\varepsilon^2 q$), the relationship between the marginal damage to the environment and the optimal tax rate depends on the rate of growth of the marginal damage for duopoly and triopoly; i.e., if m becomes larger, then the marginal damage is likely to be more than the optimal tax rate. On the other hand, if m becomes smaller, then the optimal tax rate is likely to be more than the marginal damage.

Lastly, as for the empirical examples, first I observe the mining industry. According to PricewaterhouseCoopers (2008), year 2007 was the unprecedented "eat or be eaten" year in the mining industry; i.e., the number and value of M&A in the industry were record high. PricewaterhouseCoopers (2008) argues the reasons of extremely active M&A as below:

> With less recent exploration, resource pipelines need filling. At the same time, exploration costs are at all-time highs, permitting is taking longer and companies also face skills' shortages. These are significant barriers to meeting what is a major upturn in world demand. With companies sitting on big cash positions, M&A is an important way of overcoming these challenges. In addition, it is key to enabling companies to diversify portofolios, both across geographies and commodities (5).

The above argument suggests that one of the major reasons of M&A is to reduce and internalize the transaction costs (costs of obtaining permits, skilled laborers, etc.). Hence, the analyses in this chapter is in accordance with this empirical result.

In addition, in 2015, Williams (2015) predicts that the coal industry will be witnessing a huge number of M&A in 2016 and beyond. He argues the reasons as below:

> In terms of thousand megawatt hours, coal comprised 38.7% of net energy generation in 2014. In 2005, coal represented nearly half of net energy generation! In short, coal is being replaced by cheap and cleaner natural gas, which is hurting demand and both thermal and metallurgical coal pricing. Coal companies can choose to cut costs and production a bit, but the smarter move may be to combine their forces in order to reduce competition, and enhance savings until coal prices eventually find a bottom.

The above statement is exactly in accordance with the basic theory of oligopoly that is used throughout this dissertation; i.e., if the price is low, firms should merge so that the number of suppliers decreases; consequently, the price will be higher. Thus, although the effects of emission and pollution tax are not explicitly mentioned in these reports, these reports are deemed to warrant the validity of the analysis in this chapter.

However, other reports suggest the validity of my analysis in this chapter may be limited. For instance, according to EY (2015), the number and value of M&A were low in 2014 because "[w]eak commodity prices and the uncertain outlook have created nervousness around valuations" (27) so that the M&A activities were constricted "until price stability, and in turn, confidence, returns" (7). Moreover, reports argue that, in the oil and natural gas industry, the oil price was low, but the number of M&A (mergers and acqusitions) was low in 2015 (A.T.Kearney, 2016; Deloitte Center for Energy Solutions, 2015). However, this is a little puzzling because, when the oil price is low, the number of M&A should increase (Agnihotri, 2016) so that, as evident from the oligopoly theory, the oil price will increase. However, in 2015, we did not observe the increase of the number of M&A (A.T.Kearney, 2016; Deloitte Center for Energy Solutions, 2015). Reports argue that the reason of the low number of M&A was due to the volatility of oil price and the deep uncertainty it engendered (A.T.Kearney, 2016); i.e., according to Deloitte Center for Energy Solutions (2015), “the psychology of the market saw many participants anticipating that the price downturn would be relatively short-lived, such that short-term activity and cost adjustments would be sufficient to ride out a limited period of reduced cash flow” (4). As a result, “oil price volatility and differences in valuation expectations between buyers and sellers have hindered deal-making (of M&A)” (A.T.Kearney, 2016, 7).

The model discussed in this chapter does not take into account the price volatility and people’s belief on how long the volatility will last. Hence, this is the limitation of my analysis. Indeed, the Nash equilibria discussed in this chapter are stochastically stable equilibria so that stochasticity is inherent in the model. However, this stochasticity is associated with the learning processes of firms. The stochasticity the participants of the mining and the oil and natural gas industry are observing is the environmental stochasticity; i.e., the stochasticity is inherent in environment, not in firm’s learning processes. To model the environmental stochasticity, I probably need to change the Cournot model as below. For firm i,

$$\pi_i = pq_i - cq_i^2$$

where $p = l - \sum_i q_i$ is deterministic if the economy is stable. But if the outlook of economy is uncertain, we will end up with a stochastic differential equation because the price becomes $p = l - \sum_i q_i + \varepsilon$ where ε describes environmental stochasticity. Unfortunately, to my knowledge, using stochastic differential equations is unprecedented in game theory. Moreover, there is another difficulty. Gintis (2009) argues that “stochastic differential equations with more than one independent variable virtually never have a close-form solution” (311) so that the analysis of a model will be extremely difficult. Furthermore, the relationships between potential functions and stochastic differential equations are unknown.

However, it is worthwhile to consider the effect of ε in the price curve no matter how difficult it is to include ε in the analysis: after all, empirical cases suggest that the effect of ε is not negligible. Thus, economists probably need to use Itô's Lemma, which is frequently used in financial engineering, to extend our understanding on firm's behavior in oligopoly under environmental stochasticity.

Chapter 5

THE TRAGEDY OF THE COMMONS

5.1 Potential Commons Games

5.1.1 Definition and Theory

My approach differs from the recent study by Dasgupta *et al.* (2016), which discusses commons games using the concept of Markov Perfect Equilibrium, a refinement of subgame perfect equilibrium (Fudenberg and Tirole, 1991). In general, subgame perfect equilibria are not evolutionarily stable (Samuelson, 1998). As a result, "the set of Markov perfect equilibria can change discontinuously when the payoffs are perturbed" (Fudenberg and Tirole, 1991, 502). Exploiting this property, Dasgupta *et al.* (2016) argue that, in case of the depletion of common pool resources, "[a] sudden crash in productivity, population overshoot, or decline in harvesting costs can tip an unmanaged common into ruin" (1). The problem I consider is slightly different. I am not concerned with the likelihood that some perturbation may induce the collapse of an open access resource, but the conditions in which open access makes collapse inevitable. This is partly motivated by the evidence that collapse has rarely been abrupt (Butzer, 2012). I look instead for properties of the system that lead it to collapse, potentially over much longer periods of time.

Specifically, I consider a common pool resource in which potential users decide sequentially whether to enter the resource. One potential reason why this might occur is that individuals located nearer or further from the resource may face differential costs of access. I make the following claim.

Claim. If the marginal cost of access or effort increases as the number of resource users increases, then we can identify the equilibrium number of resource users in the commons.

In what follows I make use of an additional important property of potential games. It is that all potential games with a finite number of players are congestion games with the same potential function (Monderer and Shapley, 1996). The commons game is a finite potential (congestion) game; i.e., the number of players, n, is finite, albeit large. If the number of players is finite, potential games are always isomorphic to congestion games. However, if the number of players is unlimited, I should ascertain whether the result above is valid. Sandholm (2001) shows that if players are anonymous and identical, then continuous congestion games can be defined as the limit of atomistic congestion games (i.e., in which the number of players is finite). It follows that my potential commons games are indeed isomorphic to congestion games.

5.1.2 Modeling the Commons Game

Our starting point is a model of the commons game by Gibbons (1992) in which the number of players, n, is assumed to be fixed and finite. The net benefit to the i^{th} resource user is given by:

$$\pi_i \quad = \quad v(G) \cdot q_i - cq_i$$

where $v^{'}(G) < 0$, $v^{''}(G) < 0$ and $G = \sum q_i$. In the open access, however, this model is too restrictive. First, since the number of resource users is fixed, the Gibbons model cannot address the case where the proportion of potential resource users who choose to access the resource is endogenous. Second, since costs are also assumed to be fixed, it cannot address the case where costs are sensitive to the number of resource users. I therefore modify the Gibbons model to allow both variable cost and variable levels of resource use. Specifically, I work with two models, in which the net benefit functions take the form:

$$\begin{aligned}\pi_i &= v(G)\cdot q_i - cn^\gamma q_i - \delta \\ &= (l - f(G))\cdot q_i - cn^\gamma q_i - \delta \end{aligned} \tag{5.1.1}$$

and

$$\begin{aligned}\pi_i &= v(G)\cdot q_i - cn^\gamma q_i^2 - \delta \\ &= (l - f(G))\cdot q_i - cn^\gamma q_i^2 - \delta \end{aligned} \tag{5.1.2}$$

where $f(G) = (\sum_{k=1}^{n} q_k)^2$, $0 \le \delta \le l$, $\gamma \ge 0$, and $n = 1, 2, ..., n(0)$ where $n(0)$ is the number of potential users. In the limit, $n(0) \to \infty$. In both models, costs are increasing in n. In model (5.1.1), however, marginal cost is constant for a given n, while in model (5.1.2) costs are also increasing in q. In both models, I include a non-negative cost of access, δ, which I take to be constant (although it could also be increasing in n). Note that $v(G) = l - f(G)$ satisfies $v^{'}(G) < 0$ and $v^{''}(G) < 0$.

In what follows I refer to δ as a cost of access, and $cn^\gamma q_i$ and $cn^\gamma q_i^2$ as costs of production. The costs of production are congestion costs since they are increasing in the number of users.

Now, for our models to be a potential game, I linearize $f(G)$. Since the game is symmetric, for a fixed n, the Nash equilibrium is $q_1^* = q_2^* = \cdots = q_n^* = q^*$.

$$\begin{aligned} f(G) &\approx f(G)\Big|_{q^*} + \frac{\partial f}{\partial q_1}\Big|_{q^*}(q_1 - q^*) + \frac{\partial f}{\partial q_2}\Big|_{q^*}(q_2 - q^*) + \cdots + \frac{\partial f}{\partial q_n}\Big|_{q^*}(q_n - q^*) \\ &= n^2 q^{*2} + 2nq^*(q_1 - q^*) + 2nq^*(q_2 - q^*) + \cdots + 2nq^*(q_n - q^*) \\ &= n^2 q^{*2} + 2nq^*\left(\sum_{k=1}^{n} q_k - nq^*\right) \end{aligned}$$

Hence,

$$\pi_i = \left\{l - n^2 q^{*2} - 2nq^*\left(\sum_{k=1}^{n} q_k - nq^*\right)\right\} q_i - cn^\gamma q_i - \delta \tag{5.1.3}$$

for (5.1.1), and

$$\pi_i = \left\{ l - n^2 q^{*2} - 2nq^* \left(\sum_{k=1}^{n} q_k - nq^* \right) \right\} q_i - cn^\gamma q_i^2 - \delta \tag{5.1.4}$$

for (5.1.2).

Next, I examine whether profit function (5.1.4) has a potential. The verification for (5.1.3) is similar. For (5.1.4),

$$\frac{\partial \pi_i}{\partial q_j} = -2nq^* q_i \Rightarrow \frac{\partial^2 \pi_i}{\partial q_i \partial q_j} = -2nq^*$$

and

$$\begin{aligned} \frac{\partial \pi_j}{\partial q_j} &= l - n^2 q^{*2} - 2nq^* \left(\sum_{k=1}^{n} q_k - nq^* \right) - 2nq^* q_j - 2cn^\gamma q_j \\ \Rightarrow \quad \frac{\partial^2 \pi_j}{\partial q_i \partial q_j} &= -2nq^* \end{aligned}$$

Consequently,

$$\frac{\partial^2 \pi_i}{\partial q_i \partial q_j} = \frac{\partial^2 \pi_j}{\partial q_i \partial q_j}$$

so that an exact potential exists for (5.1.4) for a given n. As a result, the models we consider are potential games.

For profit function (5.1.3), I propose the following potential function

$$\begin{aligned} \Pi \quad &= \quad \sum_{k=1}^{n} \left\{ l - n^2 q^{*2} - 2nq^* \left(\sum_{m=1}^{n} q_m - nq^* \right) \right\} q_k - cn^\gamma \sum_{k=1}^{n} q_k \\ &\quad + nq^* \left(\sum_{k=1}^{n} q_k \right)^2 - nq^* \sum_{k=1}^{n} q_k^2 \end{aligned} \tag{5.1.5}$$

and for profit function (5.1.4),

$$\begin{aligned} \Pi \quad &= \quad \sum_{k=1}^{n} \left\{ l - n^2 q^{*2} - 2nq^* \left(\sum_{m=1}^{n} q_m - nq^* \right) \right\} q_k - cn^\gamma \sum_{k=1}^{n} q_k^2 \\ &\quad + nq^* \left(\sum_{k=1}^{n} q_k \right)^2 - nq^* \sum_{k=1}^{n} q_k^2 \end{aligned} \tag{5.1.6}$$

For both models, Π is indeed a potential function, and the Nash equilibrium for a given n is

$$q^* = \begin{cases} \sqrt{\dfrac{l - cn^\gamma}{n^2 + 2n}} & (n \leq (l/c)^{1/\gamma}) \\ 0 & (otherwise) \end{cases}$$

for (5.1.5) and

$$q^* = \frac{\sqrt{c^2 n^{2\gamma} + l(n^2 + 2n)} - cn^\gamma}{n^2 + 2n}$$

for (5.1.6). The calculations for finding q^* for (5.1.6) are as follows. The calculations for finding q^* for (5.1.5) are similar.

First, we verify the first-order condition for Π to be a potential.

$$\begin{aligned} \frac{\partial \Pi}{\partial q_i} &= l - n^2 q^{*2} - 2nq^* \left(\sum_{m=1}^{n} q_m - nq^* \right) - 2nq^* q_i - 2cn^\gamma q_i \\ &\quad -2nq^*(q_1 + q_2 + \cdots + q_{i-1} + q_{i+1} \cdots + q_n) \\ &\quad +2nq^*(q_1 + \cdots \cdot + q_n) - 2nq^* q_i \\ &= l - n^2 q^{*2} - 2nq^* \left(\sum_{m=1}^{n} q_m - nq^* \right) - 2nq^* q_i - 2cn^\gamma q_i \\ &= \frac{\partial \pi_i}{\partial q_i} \end{aligned}$$

Hence, Π is a potential function.

To find the Nash equilibrium for a fixed n, first we examine the first-order condition.

$$\begin{aligned} \frac{\partial \Pi}{\partial q_i} &= l - n^2 q^{*2} - 2nq^* \left(\sum_{m=1}^{n} q_m - nq^* \right) - 2nq^* q_i - 2cn^\gamma q_i \\ &= 0 \end{aligned}$$

Since the game is symmetric, $q_1^* = \cdots = q_n^* = q^*$. Hence, the first-order condition becomes

$$\begin{aligned} & l - n^2 q^{*2} - 2nq^*(nq^* - nq^*) - 2nq^{*2} - 2cn^\gamma q^* = 0 \\ \Leftrightarrow \quad & (n^2 + 2n) q^{*2} + 2cn^\gamma q^* - l = 0 \end{aligned}$$

Consequently, the critical point is

$$q^* = \frac{\sqrt{c^2n^{2\gamma} + l(n^2+2n)} - cn^\gamma}{n^2+2n}$$

Note that $\sqrt{c^2n^{2\gamma} + l(n^2+2n)} - cn^\gamma > 0$ always holds.

Next, we verify the second-order condition.

$$\frac{\partial^2\Pi}{\partial q_i^2} = -4nq^* - 2cn^\gamma$$
$$\frac{\partial\Pi}{\partial q_j \partial q_i} = -2nq^*$$

Hence, the Hessian matrix is

$$H = \begin{pmatrix} -4nq^* - 2cn^\gamma & -2nq^* & \dots & -2nq^* \\ -2nq^* & -4nq^* - 2cn^\gamma & \dots & -2nq^* \\ \vdots & \vdots & \ddots & \vdots \\ -2nq^* & -2nq^* & \dots & -4nq^* - 2cn^\gamma \end{pmatrix}$$

The eigenvalues are $n-1$ multiplicities of $-2cn^\gamma - 2nq^*$ and $-2cn^\gamma - 2n(n+1)q^*$, and clearly, both of them are negative. Hence, H is negative definite so that Π attains a local maximum at q^*.

5.2 The Equilibrium Number of Resource Users under Open Access

5.2.1 Model 1: Marginal Cost Constant in Output

Zero Access Cost

First, I consider the case the access cost being nil; i.e., $\delta = 0$. For equation (5.1.5), $q^* = \sqrt{(l - cn^\gamma)/(n^2+2n)}$. Hence, $l - cn^\gamma \geq 0$ must hold; i.e., $n \leq (l/c)^{1/\gamma}$ must hold for the real value of $q^* \geq 0$ to exist. Figure 5.1 shows the graph of equation (5.1.3), a linear approximation of equation (5.1.1) as a function of n, for different levels of congestion (γ) given $\delta = 0$. If the number of people potentially having access

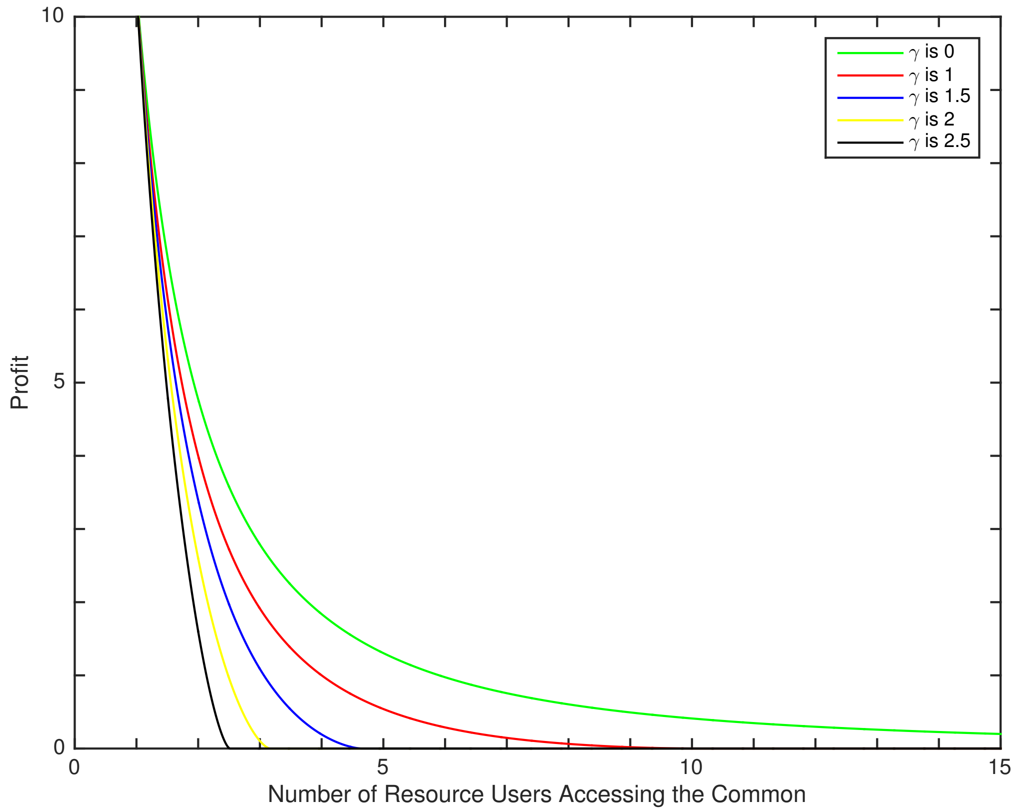

Figure 5.1: Model 1

Profit of Access to the Common: $l = 10$, $c = 1$ and $\delta = 0$

the resource is denoted $n(0)$, and the equilibrium number of resource users is denoted $n^* = n^*(\gamma, \delta)$, it follows that $n^* \leq n(0)$. Figure 5.1 shows that for $\gamma = 0$, $n^* = n(0)$, whereas for $\gamma > 0$, $n^* < n(0)$ given $\delta = 0$.

Two results follow directly from the restriction on n:

i If $n = (l/c)^{1/\gamma}$ then $q^* = 0$, implying that $\pi_i(q^*) = 0$.

ii n^* increases as γ decreases, and $n^* \to n(0)$ as $\gamma \to 0$, and $n^* \to 0$ as $\gamma \to \infty$

Positive Access Cost

Next, I consider the case the access cost being positive; i.e., $\delta > 0$. Figure 5.2 shows the graph of equation (5.1.3), a linear approximation of equation (5.1.1) as a function of n, for different levels of congestion (γ) given $\delta = 5$. In this case, for all values of

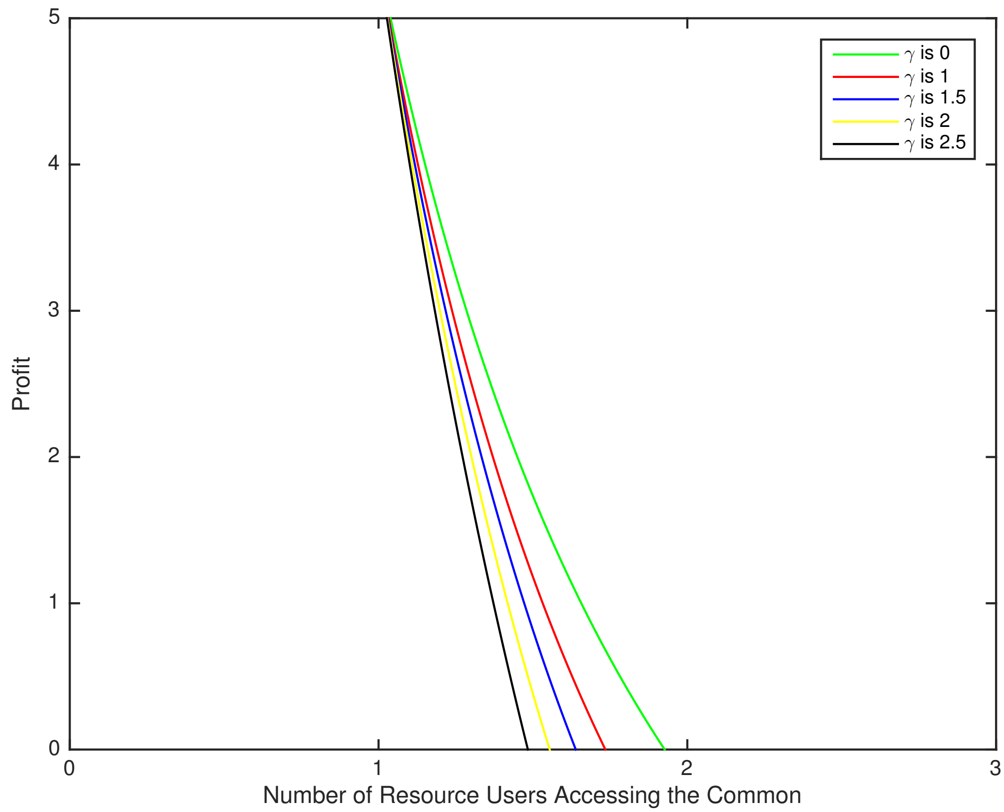

Figure 5.2: Model 1

Profit of Access to the Common: $l = 10$, $c = 1$ and $\delta = 5$

$\gamma \geq 0$, $n^* < n(0)$ so that unlimited entry never occurs. Otherwise, the results noted for the zero access cost still hold.

5.2.2 *Model 2: Marginal Cost Increasing in Output*

Zero Access Cost

First, I consider the case the access cost being nil; i.e., $\delta = 0$. For equation (5.1.6), $q^* = (\sqrt{c^2n^{2\gamma} + l(n^2 + 2n)} - cn^\gamma)/(n^2 + 2n)$. Note that $q^* \geq 0$ if and only if $\sqrt{c^2n^{2\gamma} + l(n^2 + 2n)} - cn^\gamma \geq 0$; i.e., if and only if $c^2n^{2\gamma} + l(n^2 + 2n) \geq c^2n^{2\gamma}$, and this is always true regardless of the values of c, l, n and γ. Thus, n is not limited by the restriction $n = (l/c)^{1/\gamma}$. Consequently, for all values of $\gamma \geq 0$, each resource user's profit becomes positive for all values of $n \geq 0$. Figure 5.3 shows the graph of equation (5.1.4), a linear approximation of equation (5.1.2), for different levels of congestion (γ) given $\delta = 0$. Once again, let the number of people entitled to enter the resource be denoted $n(0)$, and the equilibrium number of resource users is denoted n^*. It then follows that, for $\delta = 0$, the equilibrium number of resource users is always the maximum potential number of users regardless of the value of γ because each resource user's profit remains positive for all positive values of n. Moreover, for all values of $\gamma \geq 0$, q tends to 0 as $n(0)$ approaches infinity. In addition, note that when there is unlimited entry, each resource user is still producing optimally in the sense that they are equating marginal revenue and marginal cost, and that this is independent of the number of resource users in the commons.

Positive Access Cost

Next, I consider the case the access cost being positive; i.e., $\delta > 0$. Figure 5.4 shows the graph of equation (5.1.4), a linear approximation of equation (5.1.2), for different levels of congestion (γ) given $\delta = 5$. This time, δ shifts down all the curves so that $n^* < n(0)$ always holds; i.e., unlimited entry does not occur. As before, each resource user is producing optimally at n^*.

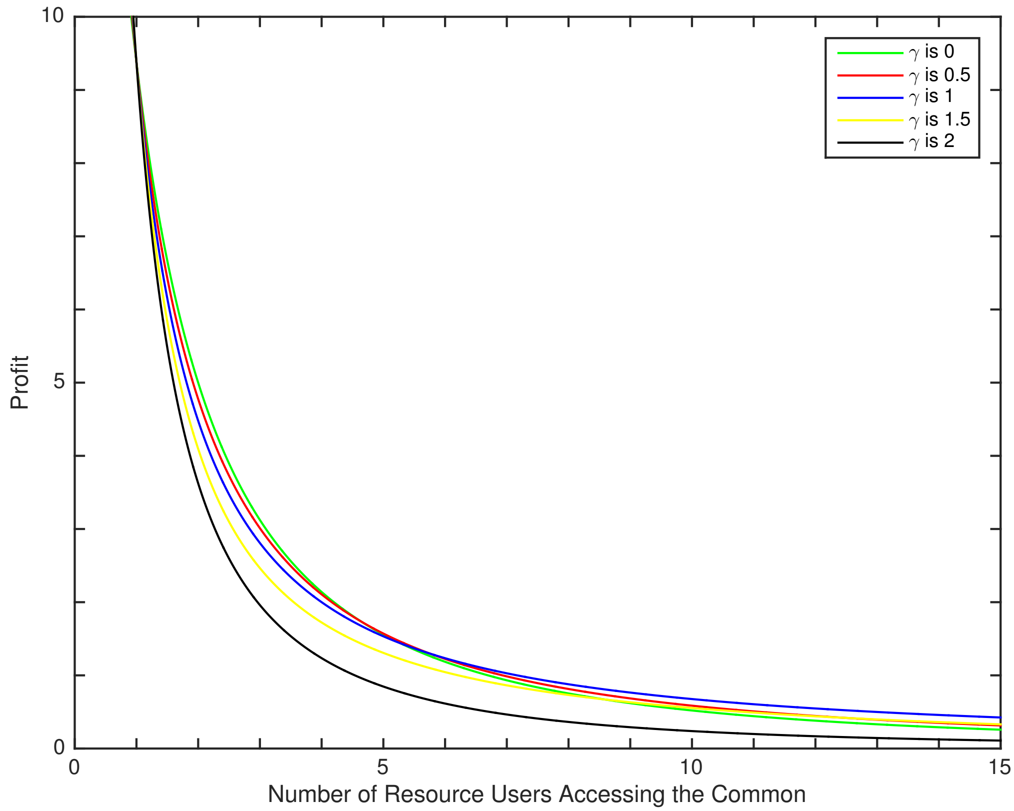

Figure 5.3: Model 2

Profit of Access to the Common: $l = 10$, $c = 1$ and $\delta = 0$

One intriguing example we should pay attention to is the following case. When $l = 10, c = 1, \gamma = 2$ and $\delta = 0$, we obtain

$$\begin{aligned}
\pi_i &= \left\{ l - n^2 q^{*2} - 2nq^* \left(\sum_{k=1}^{n} q_k - nq^* \right) \right\} q_i - cn^\gamma q_i^2 - \delta \\
&= (10 - n^2 q^{*2}) q^* - n^2 q^{*2} - \delta
\end{aligned}$$

because resource user i produces at q^*. At these values of l, c and γ, $q^* = 1.52$ when $n = 1$ and $\pi_i = 9.378$ when δ does not exist. Hence, if $\delta = 9.378$, $\pi_i = 0$ at $n = 1$ so that monopoly emerges.

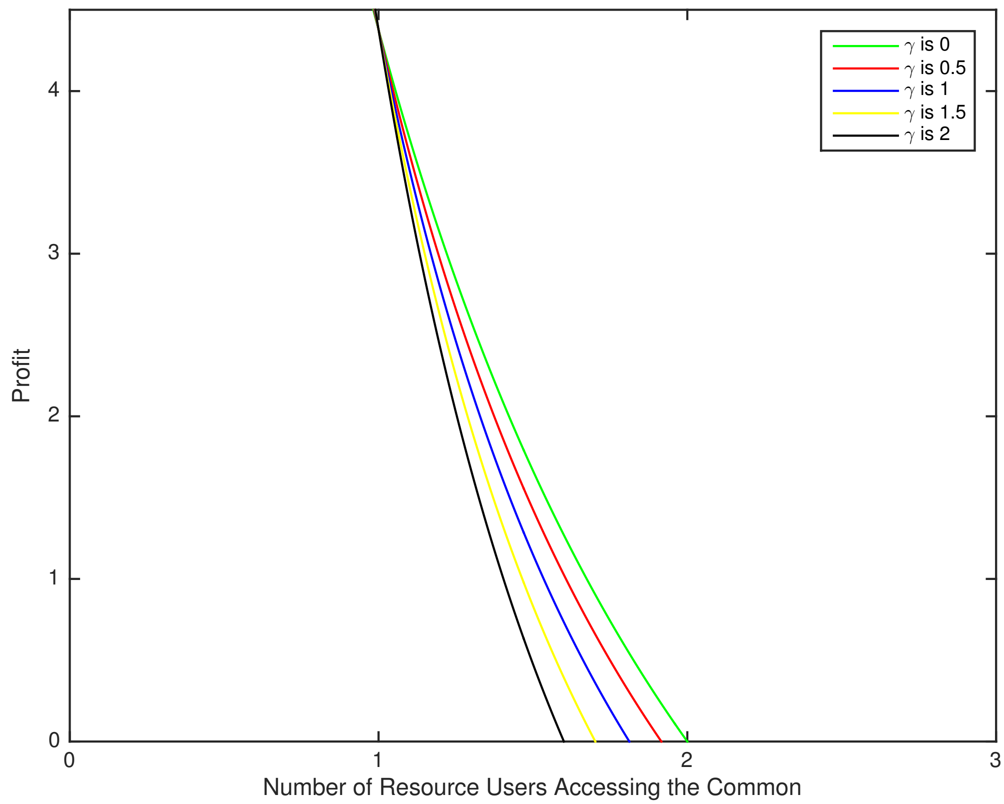

Figure 5.4: Model 2

Profit of Access to the Common: $l = 10$, $c = 1$ and $\delta = 5$

5.3 Conclusion of Chapter 5

In this chapter, I have shown how the potential function in a commons game allows me to identify the equilibrium number of resource users in an open access common pool resource. Specifically, I have verified my claim that when the cost function takes the form, $cn^\gamma q_i$ (i.e., the marginal cost is constant for a given n), I can identify the finite equilibrium number of resource users in the commons. However, I have also shown that if the cost function takes the form $cn^\gamma q_i^2$, the equilibrium number of resource users is the number that is entitled to access the commons. This is because each resource user's profit is positive for all $n \geq 0$.

My results provide a different perspective on the management of the commons than that suggested by the work of Ostrom and colleagues, who tend to focus on the institutional arrangements that structure access to common pool resources $n(0)$. I focus instead on the structure of costs, and the way that costs vary with the level of output of resource users and the number of resource users. I find that the tragedy of the commons is the product of a very particular set of cost structures in which either:

a) the cost of production is not increasing in the number of resource users or the level of output, or

b) the cost of production is increasing in the number of resource users or the level of output but at a lower rate than the increase in revenue.

I find that a cost function of the form $cn^{\gamma}q_i$, $\gamma > 0$, always generates a finite equilibrium number of resource users. There is, however, almost certainly a relationship between the cost structures that determine the equilibrium number of resources users under open access and institutional arrangements for managing common pool resources observed in real systems. Since the regulatory regimes established by resource users have implications for the cost of access, Ostrom (2015) might well be interpreted in terms of the cost structures they involve.

Aside from the many cases identified by Ostrom (2015), there are numerous examples of efforts to manage common pool resources through restrictions on $n(0)$, the maximum number of users potentially having access to the commons. An example of a commons to which access is independent of costs is licensed common pool fisheries in Japan (Government of Japan, 2016). Fishery licenses are issued by the governor of a prefecture. Applicants are divided into two groups. The first group comprises fishers who reside in the local area and have historically practiced aquaculture, while the second group comprises of fishers from elsewhere who plan to start aquaculture. The first group is automatically given entry, but members of the second group must wait until there is a vacancy. Other examples identified in the literature cover common pool forests, grasslands, wetlands, water resources, and hunting areas (Berkes *et al.*, 1989; Feeny *et al.*, 1990; McWhinnie, 2009).

There are also many examples of common pool resources in which the number of potential users has not been restricted, but in which the number of those who actually access the resource has depended on costs —whether costs of access or costs of production. One well known example of a common pool resource where the number of resource users is effectively determined by the cost of access is the lobster industry in Maine. Acheson (2003) described the system at that time as follows:

> To go lobstering, one needs a state license, which ostensibly allows a person to fish anywhere in state waters. In reality, more is required. One also needs to gain admission to a "harbor gang" that maintains a fishing territory for the use of its members (24).

Each harbor gang comprised a small group of fishers, perhaps as few as six or eight boats, controlling territories 100 square miles or less in area. There were two types of territories: nucleated and perimeter-defended. Acheson (2003) noted that entry into the harbor gangs that controlled nucleated territories was easier than entry into the perimeter-defended areas. Nevertheless, there were a range of informal "costs" associated with entering both nucleated and perimeter defended territories, and these were increasing in the number of fishers.

The reduction in the numbers of fishers allowed by the system has had positive effects the productivity in the fishery. Catches that were reported to be at record-high levels at the beginning of the Century (Acheson, 2003) have continued to rise. In

2016, fishers landed more than 130 million pounds of lobster (valued at $533 million), nearly three times the catch level in 2000 (centralmaine.com, 2017).

In addition, throughout history, we frequently observe several examples of successfully limiting access to common pool resources by enforcing informal "costs." Uzawa (2015) argues that, historically, open access to common pool resources have been rare, and Ostrom (2015) provides several examples of such non-open access commons including those of Japanese agrarian villages of Hirano, Nagaike and Yamanoka. The common understanding about the common pool resources in Japanese agrarian villages is that resources were strictly for use by the villagers, and each villager must perform a significant amount of cooperative work for the village. In addition, the common understanding is that there was virtually no migration between their villages and other regions so that every household in a village was believed to have been in the same village for hundreds of years. However, Miyamoto (2005) argues that this understanding is inaccurate.

Miyamoto (2005) argues that, during the Tokugawa period (from the early 17th century to the mid-19th century), villagers kept living in their villages when the climate was favorable for their agrarian activities. However, once the climate became unfavorable for raising crops, villagers abandoned their villages and migrated to other regions although the Tokugawa shōgunate prohibited such migrations with severe penalties. However, despite these penalties, villagers migrated, and it was not uncommon that when the climate was unfavorable, villages were completely deserted. However, once the climate recovered, people (not the former villagers) came to once abandoned villages from somewhere and started living there. This was possible because, when the former villagers deserted their villages, their houses, fields and commons were left intact. As a result, new comers could come to these once deserted villages and start living in those deserted houses. In other words, the "costs" of starting a new life in an once deserted village was very low, and the costs remained low until all houses were occupied. Once all houses were occupied, the "costs" of starting a life increased sharply because a new comer would have to build a new house and cultivate a new field, and this refrained a further entry into a village. Thus, these cases from Tokugawa period are the examples that fairly describe the effect of access and/or production cost on limiting entry under the condition of open access as discussed in the previous section.

Symmetrically, there are several common pool resources that have been depleted because access and/or production costs fell as a result of either technological developments or government subsidies on effort or capital equipment. Several notable examples stem from the exploitation of sea areas beyond national jurisdiction. Take the case of whales. All countries have open access to the High Seas, and many countries have actively hunted whales in the past. Several whale species were severely depleted in the 19th century. Baleen Whales targeted for their blubber included the Bowhead, Grey, Humpback, and Right Whales. Amongst the toothed whales, the Sperm Whale was hunted for spermaceti until the discovery of kerosene in the 1840s. In the 20th century the range of whales exploited widened, the number of firms accessing whale fisheries increased, and the rate at which whale populations were harvested rose dramatically. Several stocks were driven down to commercial extinction. Aside from the Baleen Whales targeted in the 19th century these include the Blue, Fin, Sei and Beluga Whales. It is estimated that just under 3 million whales were harvested between 1900 and 1986 when the International Whaling Commission approved a moratorium (Rocha *et al.*, 2014).

The driver of changes in number of whaling firms, the species targeted and the level of harvest was, in every case, a change in profitability caused by changes in the cost of access or production, or by changes in demand (Davis *et al.*, 2007). The rapid decrease of Bowhead Whale in Eastern Arctic between the late 18th and early 19th centuries, for example, was due both to the payment of 'revenue bounties' aimed at increasing the size of whaling vessels, and productivity improvements due to changes in hull design that reduced the cost of whale hunting (Allen and Keay, 2001). In the 20th century, the introduction of diesel engines, factory ships, and explosive harpoons were amongst the supply side drivers of the growth in the numbers of whalers, but profitability was also affected by demand side factors. At the time when the moratorium was declared, whaling was a rapidly declining industry due to the combined effects of declining stocks (which increased production costs), the emergence of substitute products, rising incomes, and internationally increasing environmentalism. Regulation was argued to follow, rather than lead, catch changes (Schneider and Pearce, 2004).

Other examples of cost-led declines in common pool marine resources include the Atlantic Cod fishery in which overexploitation was due in part to the fact that the predicted rate of growth of the stock was greatly overestimated (Hutchings and Myers, 1994) which induced a significant amount of industrial investment, and in part to the effect of government subsidies for new vessels and for upgrades to fishing capacity particularly after 1985/1986 (Finlayson and McCay, 1998). Similarly, the overexploitation of the Atlantic and Mediterranean Bluefin Tuna was due to the effect on costs of an increase in the size and power of French seiners, the introduction of new and powerful positioning and prospecting equipment, and the introduction of new storage equipment from the late 1980s to mid-1990s (Fromentin and Ravier, 2005).

In terrestrial systems, there are many parallel examples of changes in the rate at which common pool resources have been extracted that are driven by cost induced changes in effort, rather than by changes in the number of those entitled to access the resource. To take just one example, groundwater reserves are frequently available to anyone with the capacity to drill to the water table (the cost of access). The cost of production in such cases is simply the cost of pumping plus the cost of surface storage and distribution systems. A study of groundwater use by farmers in the Hamadan-Bahar plain in Iran, for example, argued that groundwater depletion in the area is due both to the fact that farmers are not required to pay for water, and to the existence of a range of subsidies for agricultural production. The net effect is a reduction in the cost of production that has led both to the sinking of new wells and an increase in the rate at which water is pumped from existing wells (Balali *et al.*, 2011).

It is important to underline the fact that resource users may be behaving efficiently even in the case where the structure of costs induce the tragedy of the commons. The Nash equilibrium identified by the argmax of the potential function of the commons game is one at which resource users equate marginal revenue and marginal cost. If common pool resources or the societies dependent on them have collapsed, it is because either access or congestion costs were simply inconsistent with the sustainable use of the resources. I have made the point that congestion aside, the institutional arrangements for the management of common pool resources recorded in the literature are likely to have implications for the structure of both access and production costs. The Maine lobster fishery is a case in point. There are, however, few attempts to identify the consequences of different common pool resource management regimes for access or production costs.

Uzawa (2015) argues that today's discussions on commons overlook four issues. First, historically, few commons have been genuinely "open access." Second, resource users are not always profit maximizing, but rather obey the rules of their commons. Third, not all commons are destined to be ruined. Whether or not they are depleted depends on the conditions ruling in each case. Fourth, the concept of commons includes the historical institutions and communities that govern the commons as well as the commons themselves; i.e., commons do not exist in a vacuum; rather, they are embedded in society so that when we analyze the fate of commons, we need to take into account the micro-macro interactions between commons and society. I suggest that the costs of access and production are, at least in part, functions of such institutional arrangements. There may be cases where marginal costs do not increase with the number of resource users in the common (i.e., $\gamma = 0$) or with output, and these may reflect weaknesses of the institutional arrangements governing access and use as much as the characteristics of common pool resources themselves. I nevertheless leave the relation between institutions and cost structures to future research.

Chapter 6

CONCLUSION

In this dissertation, the common theme I have discussed is the important role cost functions play in Cournot competitions and the tragedy of the commons. For Cournot competitions, my results indicate that the structure of cost functions determines whether the market witnesses the emergence of a monopoly or an oligopoly. Moreover, when I take into account taxation on effluent, the outcome is that the government cannot control the emergence of a monopoly or an oligopoly by varying the taxation rate on firms; i.e., even if the government wishes that a monopoly needs to be dissolved so that it enforces the highest taxation rate on a monopoly, my finding indicates that the firm survives such heavy taxation. Thus, if a policy maker wishes to influence on the number of firms in a market, he or she should set up institutions that directly influence on each firm's cost function rather than establishing laws and/or measures that blatantly aim to determine the number of firms in the market (e.g., anti-trust law).

Speaking of the tragedy of the commons, the institutional approach by Ostrom (2015) is popular among social and environmental scientists. The crux of her institutional approach is what we call the design principles, which Ostrom argues that many successful commons have fulfilled with. These are her eight design principles (Anderies and Janssen, 2013; Ostrom, 2015).

- The boundaries of the resource system (e.g., irrigation system or fishery) and the individuals or households with rights to harvest resource units are clearly defined.

- Rules specifying the amount of resource products that a user is allocated are related to local conditions and to rules requiring labor, materials, and/or money inputs.

- Many of the individuals affected by harvesting and protection rules are included in the group that can modify these rules.

- Monitors, who actively audit biophysical conditions and user behavior, are at least partially accountable to the users and/or are the users themselves.

- Users who violate rules-in-use are likely to receive graduated sanctions (depending on the seriousness and context of the offense) from other users, from officials accountable to these users, or from both.

- Users and their officials have rapid access to low-cost, local action situations to resolve conflict among users or between users and officials.

- The rights of users to devise their own institutions are not challenged by external governmental authorities, and users have long-term tenure rights to the resource.

- Appropriation, provision, monitoring, enforcement, conflict resolution, and governance activities are organized in multiple layers of nested enterprises.

Anderies and Janssen (2013) argue that these "design principles hold up when challenged with data" (72) so that these principles meet scientific scrutiny.

But the problem about these principles is that it is still not clear, under what mechanism, these principles enable resource users around the commons to maintain the sustainable equilibrium number of users of natural resources for generations. We can intuitively understand that these principles contribute to the persistence of sustainable equilibrium number of users for generations, but Ostrom (2015) is not clear how these principles have contributed to its persistence. However, as mentioned earlier, since the regulatory regimes established by resource users have implications for the cost of access, the design principles might well be interpreted in terms of the cost structures they involve.

My results on the tragedy of the commons indicate that if a policy maker is able to influence on each resource user's cost function, then the sustainable equilibrium number of resource users can be identified when the cost function becomes costly enough as more resource users enter the common. Hence, it is likely that these principles could affect on each resource user's cost function so that the sustainable equilibrium number of resource users can be identified and maintained for generations.

But both my results and that of Ostrom (2015) suggest that identifying the sustainable equilibrium number of resource users is not enough; i.e., this number needs to be enforced and if there are violators, we need to have a system of punitive actions. To consider the issue of enforcement, Uzawa (2005) provides an insightful suggestion.

Uzawa (2005) provides us an insight into the issue of the tragedy of the commons by proposing the concept of "social common capital." According to Uzawa, social common capitals can be classified into three categories: natural capital, social infrastructure and institutional capital. In particular, "[n]atural capital consists of the natural environment and natural resources such as forests, rivers, lakes, wetlands, coastal seas, oceans, water, soil, and above all, the earth's atmosphere" (Uzawa, 2005, vii). Hence, commons that have been discussed so far fall into the category of natural capital of social common capitals. Uzawa (2005) argues that "[t]he management of social common capital thus is entrusted on a fiduciary basis to autonomous social institutions, to provide the environmental framework within which all human activities are conducted and the allocative mechanism through which market institutions work" (8). However, Uzawa is not explicit about what these autonomous social institutions do on a fiduciary basis to protect the commons. Also he argues that social common capitals are to be managed by experts who are professionally trained in such capitals (Uzawa, 2000); however, Uzawa is not clear on the roles of these experts. But, according to my results in Chapter 5, the roles played by Uzawa's "entrusted autonomous social institutions" and "experts of the commons" are clear; i.e., they have to influence on each resource user's cost function and enforce the sustainable equilibrium number of resource users.

But economists who believe in laissez-faire may feel uncomfortable about the arguments of the identification and enforcement of the sustainable equilibrium number of resource users. Especially, they may feel uncomfortable with the need of "entrusted autonomous social institutions" and "experts of the commons" to manage the commons because they believe that pareto-optimality can be reached only by laissez-faire so that the only solution to the tragedy of the commons is privatization of the commons. However, Roemer (1989) shows that laissez-faire is not the only solution. He argues, "[P]ublic ownership can bring about Pareto-efficient solutions to the tragedy of the commons, which are superior to private ownership solutions from a distributional point of view" (75). Specifically, he considers the case of a lake and fishermen and argues that "if each fisherman knows the preferences and labor capacities of the others, then there is an allocation rule that is decentralizable..., and that implements a conception of public ownership"(92) where "decentralizable" means an "economy" that "the planner need only specify the rules of some non-cooperative game, but need not elicit information from the players about their preferences, skills, and endowment" (85). This is exactly what I have shown in this dissertation. If the planner specifies cost functions, then the sustainable equilibrium number of resource users can be identified and the tragedy of the commons can be avoided if there is an institution of enforcement. As Uzawa (2000) claims, the planner needs to be an expert who can figure out the collect form of cost functions, and the contribution of my dissertation is that I have shown such planning can realize the Nash equilibria of the finite number of resource users in a common.

Lastly, as Diamond (2005) argues, the collapse of societies and civilizations are (at least partly) attributed to their failure of management of common pool resources. Dasgupta *et al.* (2016) attempt to relate the tragedy of the commons to such collapse by using Markov Perfect Equilibrium (MPE). However, as argued earlier, MPE implies that many changes are discontinuous although many cases of societal collapse have gone through rather gradual processes (Butzer, 2012; Diamond, 2005). Hence, my approach of using potential games seems to be more appropriate for the explanation of gradual societal collapse. To my knowledge, there are not enough scientific evidences to fully determine the causes of of such collapse at this moment. Hence, it is one of my future agenda to work on the issues of societal collapse and provide its convincing model.

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