Functorial Results for C^*-Algebras of Higher-Rank Graphs

by

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ABSTRACT

Higher-rank graphs, or k-graphs, are higher-dimensional analogues of directed graphs, and as with ordinary directed graphs, there are various C^*-algebraic objects that can be associated with them. This thesis adopts a functorial approach to study the relationship between k-graphs and their associated C^*-algebras. In particular, two functors are given between appropriate categories of higher-rank graphs and the category of C^*-algebras, one for Toeplitz algebras and one for Cuntz-Krieger algebras. Additionally, the Cayley graphs of finitely generated groups are used to define a class of k-graphs, and a functor is then given from a category of finitely generated groups to the category of C^*-algebras. Finally, functoriality is investigated for product systems of C^*-correspondences associated to k-graphs. Additional results concerning the structural consequences of functoriality, properties of the functors, and combinatorial aspects of k-graphs are also included throughout.

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Chapter 1

INTRODUCTION

Higher-rank graphs, or k-graphs, are higher-dimensional analogues of directed graphs, first introduced by Kumjian and Pask in [6] to provide a general combinatorial framework encompassing both ordinary graph algebras and the higher-rank Cuntz-Krieger algebras studied by Robertson and Steger in [15] and [16]. Much research has since been done on the various C^*-algebraic objects associated with k-graphs, including their Cuntz-Krieger and Toeplitz algebras, as well as certain product systems of C^*-correspondences. A persistent aim of such research is to relate properties of the C^*-algebraic object to properties of the underlying graph. For example, the ideal structure of the associated Cuntz-Krieger algebras can, for a large class of k-graphs, be determined by the structure of the graph (see [13]).

The current research is motivated from questioning how far the connections between the algebra and the graph actually extend. The combinatorial apparatus for determining such things as the simplicity and ideal structure of an algebra is somewhat involved, and it would be worth knowing whether, for example, the symmetries of a graph, which are usually apparent from simple inspection, reflect symmetries of the algebra. In this way, complexities of the C^*-algebraic world would be further reduced and deferred to the simpler graphical setting.

This thesis adopts a functorial approach to study the relationship between k-graphs and their associated C^*-algebras. There are three main sections after the preliminary material of chapter two. In the first (chapter 3), two functors are given between appropriate categories of higher-rank graphs and the category of C^*-algebras, one for Toeplitz algebras and one for Cuntz-Krieger algebras. Next, the Cayley graphs of finitely generated groups are used to define a class of k-graphs, and a functor is then given from a category of finitely generated groups to the category of C^*-algebras. Finally, functoriality is investigated for product systems of C^*-correspondences associated to k-graphs. Additional results concerning the structural consequences of functoriality, properties of the functors, and combinatorial aspects of k-graphs are also included throughout.

Chapter 2

PRELIMINARIES

2.1 Higher-Rank Graphs and their Algebras

A k-graph (Λ, d) is a countable small category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Hom}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$ that satisfies the *factorization property*: For all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m+n$, there are unique elements $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. For $n \in \mathbb{N}^k$, define $\Lambda^n := d^{-1}(n)$. With this notation, Λ^0 is the set of *vertices* of Λ, and the map $o \mapsto \mathrm{id}_o$ gives a bijection between $\mathrm{Obj}(\Lambda)$ and Λ^0; thus, we typically identify the two. Also, set $\Lambda^* := \bigcup_{n \in \mathbb{N}^k} \Lambda^n$. For $v \in \Lambda^0$, define $v\Lambda := \Lambda \cap r^{-1}(v)$ and $\Lambda v := \Lambda \cap s^{-1}(v)$. (Λ, d) is said to be *row finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and it has *no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Let $e_1, \ldots, e_k$ denote the standard basis for $\mathbb{N}^k$. Then, (Λ, d) is said to be *locally convex* if for all $1 \leq i < j \leq k$, $e \in \Lambda^{e_i}$, and $f \in \Lambda^{e_j}$ with $r(e) = r(f)$, there are paths ee' and ff' in $\Lambda^{e_i+e_j}$ that extend e and f, respectively.

There are two basic types of maps between higher-rank graphs. The difference is in whether the dimensions of the graphs are allowed to vary. A *morphism* is a degree preserving functor between two k-graphs: $\phi : (\Lambda_1, d_1) \to (\Lambda_2, d_2)$ with $d_1(\lambda) = d_2(\phi(\lambda))$. A *quasimorphism* from an l-graph (Λ_1, d_1) to a k-graph (Λ_2, d_2) is a functor $\phi : \Lambda_1 \to \Lambda_2$ together with a map $\psi : \mathbb{N}^l \to \mathbb{N}^k$ that intertwines the degree maps: $d_2(\phi(\lambda)) = \psi(d_1(\lambda))$. Also, a morphism $\phi : (\Lambda_1, d_1) \to (\Lambda_2, d_2)$ is said to be *saturated* if $\gamma \in \phi(\Lambda)$ whenever $r(\gamma) \in \phi(\Lambda^0)$, and a quasimorphism $\phi : \Gamma \to \Lambda$ from an l-graph to a k-graph with $l \leq k$ is said to be *weakly saturated* if for all $v \in \phi(\Gamma^0)$ and all $e_i \in \mathbb{N}^k$, either $v\phi(\Gamma) \cap \Lambda^{e_i} = v\Lambda^{e_i}$ or $v\phi(\Gamma) \cap \Lambda^{e_i} = \emptyset$.

Finally, given a k-graph Λ, a *subgraph* $\Gamma \subseteq \Lambda$ is taken to be a subcategory of Λ that is an l-graph for some $l \leq k$. We say that Γ is *weakly saturated* if the inclusion map is weakly saturated.

Definition 2.1.1. Let (Λ, d) be a row-finite k-graph with no sources. A *Cuntz-Krieger Λ-family* in a C^*-algebra B is a collection $T = \{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying the Cuntz-Krieger relations:

1. $\{t_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

2. $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

3. $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

4. $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$.

Call these relations (CK1) through (CK4).

Definition 2.1.2. Let $T = \{t_\lambda : \lambda \in \Lambda\}$ be a family of partial isometries satisfying (CK1) through (CK3). If instead of the (CK4) relation, we have that for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$, the operators in $\{t_\lambda t_\lambda^* : \lambda \in v\Lambda^n\}$ are mutually orthogonal and

$$t_v \geq \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*,$$

then T is said to be a *Toeplitz-Cuntz-Krieger* Λ-family. Call this condition (TCK4).

The Cuntz-Krieger algebra of (Λ, d) is defined as the C^*-algebra generated by a universal Cuntz-Krieger Λ-family. More specifically, we have the following proposition:

Proposition 2.1.3 (Prop. 10.9, [12])**.** Let Λ be a row-finite k-graph with no sources. Then there is a C^*-algebra $C^*(\Lambda)$ (called the C^*-algebra of Λ) generated by a Cuntz-Krieger E-family $\{s_\lambda\}$ such that for every Cuntz-Krieger Λ-family $t = \{t_\lambda\}$ in a

C^*-algebra B, there is a homomorphism $\pi_t : C^*(\Lambda) \to B$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda^*$.

Note also that the Cuntz-Krieger algebra is equal to the closed linear span of elements of the form $t_\lambda t_\mu^*$ where $\lambda, \mu \in \Lambda$.

Proposition 2.1.4 (Corollary 10.8, [12]). Let (Λ, d) be a row-finite k-graph with no sources, and let $\{t_\lambda)$ be a Cuntz-Krieger Λ-family. Then

$$C^*(\{t_\lambda\}) = \overline{\text{span}}\{t_\lambda t_\mu^* : s(\lambda) = s(\mu)\}.$$

Analogous results hold for the Toeplitz algebra $\mathcal{T}(\Lambda)$, which is generated by a universal Toeplitz-Cuntz-Krieger Λ-family.

When the algebras of higher-rank graphs were first introduced in [6], the k-graphs were required to be both row-finite and sourceless. Later, Raeburn et al. ([13]) defined Cuntz-Krieger relations that apply to any row-finite k-graph. The fourth relation, in particular, is the one at issue for more general classes of graphs, though its revision looks remarkably similar when using the following notation: For $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, define

$$\begin{aligned} \Lambda^{\leq n} &:= \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } s(\lambda)\Lambda^{e_i} = \emptyset \text{ if } d(\lambda) + e_i \leq n\}, \text{ and} \\ v\Lambda^{\leq n} &:= \Lambda^{\leq n} \cap r^{-1}(v). \end{aligned}$$

We can then replace (CK4) with

$$t_v = \sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^* \text{ for all } v \in \Lambda^0 \text{ and all } n \in \mathbb{N}^k.$$

However, Theorem 3.5 of [13] indicates that this generality is not needed to capture most of the interesting behavior afforded by Cuntz-Krieger algebras, since, by the theorem, given a row-finite k-graph (Λ, d), there exists a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ with each t_λ nonzero if and only if Λ is locally convex. Still, even though significant structural results can be obtained in this more general setting, the theory is most robust in the case of row-finite k-graphs with no sources, and much of the literature proceeds under these constraints. Thus, the standing assumption in what follows is that all k-graphs are row-finite with no sources, and we will use the Cuntz-Krieger relations as first presented in this section.

2.2 Gauge-Invariant Uniqueness Theorem

First proved by Kumjian and Pask ([6], Theorem 3.4), the gauge-invariant uniqueness theorem is one of the most significant general results about the Cuntz-Krieger algebras of higher-rank graphs. Given a k-graph (Λ, d), there is a canonical, strongly continuous action of the torus $\mathbb{T}^k$ on $C^*(\Lambda)$ called the *gauge action*:

$$\gamma : \mathbb{T}^k \to \text{Aut } C^*(\Lambda), \;\; \gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda,$$

where for $z = (z_1, \cdots, z_k) \in \mathbb{T}^k$ and $n = (n_1, \cdots, n_k) \in \mathbb{N}^k$, we set $z^n := z_1^{n_1} \cdots z_k^{n_k}$. The presence of a gauge action on a Λ-family $S = \{s_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B can reveal much about $C^*(\Lambda)$. Under modest assumptions, the $*$-homomorphism $\pi_S : C^*(\Lambda) \to B$, whose existence is a consequence of universality, is injective, and the abstractly generated algebra $C^*(\Lambda)$ can then be related to some more familiar C^*-algebra B. The result is therefore employed constantly when trying to find a more concrete realization of $C^*(\Lambda)$, as, for instance, in example 2.2.2.

Theorem 2.2.1. Gauge-Invariant Uniqueness Theorem Let (Λ, d) be a row-finite k-graph with no sources. Let $S = \{S_\lambda\}$ be a Cuntz-Krieger Λ-family in a C^*-algebra B with $S_v \neq 0$ for every $v \in \Lambda^0$. Suppose there is an action $\beta : \mathbb{T}^k \to \text{Aut } B$ such that $\beta_z(S_\lambda) = z^{d(\lambda)} S_\lambda$ for every λ. Then the homomorphism $\pi_S : C^*(\Lambda) \to B$ is injective.

Proof. See [6], Theorem 3.4. □

Example 2.2.2. Consider the 1-graph nC_k, which consists of a k-cycle with n (directed) edges between successive vertices. More specifically, let e_j^i denote an edge from v_j to v_{j+1} for $1 \leq j \leq k-1$, and let it denote an edge from v_k to v_1 for $j = k$. Then,

$$\begin{aligned} V(nC_k) &= \{v_1, \ldots, v_k\} \\ E(nC_k) &= \{e_j^1, \ldots, e_j^n : 1 \leq j \leq k\}. \end{aligned}$$

We'll show that

$$C^*(nC_k) \cong \mathcal{O}_n \otimes M_k(\mathbb{C}).$$

Let $S_1 \ldots S_n$ be partial isometries generating $\mathcal{O}_n$. Define vertex projections in $M_k(\mathcal{O}_n)$ by

$$(p_{v_j})_{pq} = \begin{cases} 1_{\mathcal{O}_n} & \text{if } p = j = q \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq j \leq k-1$, define partial isometries

$$(s_{e_j^i})_{pq} = \begin{cases} S_i & \text{if } p = j+1, q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, define

$$(s_{e_k^i})_{pq} = \begin{cases} S_i & \text{if } p = 1, q = k \\ 0 & \text{otherwise.} \end{cases}$$

As can be checked, this collection of operators is a Cuntz-Krieger family for nC_k, and $\gamma \otimes \mathrm{id}_{M_k}$ is a gauge action on $\mathcal{O}_n \otimes M_k$ where $\gamma : \mathbb{T} \to \mathcal{O}_n$ is the gauge action on $\mathcal{O}_n$. As such, the canonical homomorphism is an isomorphism onto the algebra generated by the above family of partial isometries. (See Theorem 2.2, [12] for the statement of the gauge-invariant uniqueness theorem in the case of ordinary directed graphs).

2.3 Covering Maps and the Fundamental Groupoid

In [9], Pask et al. defined the fundamental groupoid $\mathcal{G}(\Lambda)$ of a k-graph (Λ, d) to give an analogue of the fundamental groupoid of an ordinary graph. They later expanded on this work in [10], where they used the groupoid machinery to explore covering maps of k-graphs. Such work was of the first in a growing literature to treat k-graphs topologically. Unfortunately, the fundamental groupoid is not always well-behaved. In particular, the canonical functor $i : \Lambda \to \mathcal{G}(\Lambda)$ need not be injective. In Section 4.4, a necessary condition is given to detect whether a given k-graph does embed faithfully.

It will suffice to consider only the concrete presentation of the fundamental groupoid as given in Section 5 of [9]. Let (Λ, d) be a k-graph with 1-skeleton E.

Definition 2.3.1. For each $e \in E^1$, let e^{-1} be an edge with $r(e^{-1}) = s(e)$ and $s(e^{-1}) = r(e)$. Set $E^{-1} := \{e^{-1} : e \in E^1\}$. The *augmented graph* E^+ of E is defined as $E^+ = (E^0, E^1 \cup E^{-1}, r, s)$. The *cancellation relations* C for E^+ are all relations of the form $(e^{-1}e, s(e))$ where $e \in E^1$.

The fundamental groupoid is then given concretely as a quotient of the path category $\mathcal{P}(E^+)$ of the augmented graph E^+. More specifically, let $S = \{S_1, S_2, \dots\}$ be the collection of commuting squares of (Λ, d). Note that each S_j is a relation $ef = gh$ between bi-colored paths. Let $/S$ denote the quotient by the equivalence relation generated by S. Then, Pask et al. (Section 5, [9]) show that

$$\mathcal{G}(\Lambda) \cong \mathcal{P}(E^+)/(C \cup S) \cong (\mathcal{P}(E^+)/C)/S = \pi(E)/S$$

where

$$\pi(E) := \mathcal{P}(E^+)/C$$

is the fundamental groupoid of an ordinary directed graph.

2.4 C^*-Correspondences

Let B be a C^*-algebra and let X be a right B-module. A *B-valued inner-product* on X is a function $\langle \cdot, \cdot \rangle_B : X \times X \to B$ such that for all $\xi, \eta, \zeta \in X$, $b \in B$, and $\alpha, \beta \in \mathbb{C}$

1. $\langle \xi, \alpha\eta + \beta\zeta \rangle_B = \alpha \langle \xi, \eta \rangle_B + \beta \langle \xi, \zeta \rangle_B$,
2. $\langle \xi, \eta b \rangle_B = \langle \xi, \eta \rangle_B b$,
3. $\langle \xi, \eta \rangle_B = \langle \eta, \xi \rangle_B^*$,
4. $\langle \xi, \xi \rangle_B \geq 0$, and
5. $\langle \xi, \xi \rangle_B = 0$ if and only if $\xi = 0$.

X is said to be a (right) Hilbert B-module if it is complete with respect to the norm $\|\xi\|^2 = \|\langle \xi, \xi \rangle_B\|$. An operator $T : X \to X$ is said to be adjointable if there is $T^* : X \to X$ such that

$$\langle T\xi, \eta \rangle_B = \langle \xi, T^*\eta \rangle_B$$

for all $\xi, \eta \in X$, and the collection of all adjointable operators on X is denoted by $\mathcal{L}(X)$. When equipped with the standard operator norm, $\mathcal{L}(X)$ is a C^*-algebra.

Given $\xi, \eta \in X$, there is an operator $\Theta_{\xi,\eta}$ with adjoint $\Theta_{\eta,\xi}$ defined by

$$\Theta_{\xi,\eta}\zeta = \xi \langle \eta, \zeta \rangle_B.$$

$\mathcal{K}(X) = \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in X\}$ is the algebra of *compact operators* in $\mathcal{L}(X)$; it is an ideal, and for $T \in \mathcal{L}(X)$, $T\Theta_{\xi,\eta} = \Theta_{T\xi,\eta}$.

Let A be a C^*-algebra. An A-B C^*-correspondence is a right Hilbert B-module X together with a $*$-homomorphism $\phi_X : A \to \mathcal{L}(X)$, called the *left action*. An A-A C^*-correspondence is called either a C^*-correspondence over A or an A-correspondence.

Furthermore, the C^*-correspondence X is *nondegenerate* if $\overline{\operatorname{span}}\{\phi_X(a)\xi : a \in A, \xi \in X\} = X$, it is *full* if $\overline{\operatorname{span}}\{\langle \xi, \eta \rangle_A : \xi, \eta \in X\} = A$, and it is *faithful* if $\phi_X : A \to \mathcal{L}(X)$ is injective.

Example 2.4.1. Product systems of C^*-correspondences associated to higher-rank graphs will be the main object of study later, but the one-dimensional case is instructive. Given an ordinary directed graph $E = (E^0, E^1, r, s)$ with countable vertex set E^0, the graph correspondence of E is a nondegenerate C^*-correspondence $X(E)$ over $c_0(E^0)$ defined as follows:

$$X(E) = \{\xi : E^1 \to \mathbb{C} : v \mapsto \sum_{s(e)=v} |\xi(e)|^2 \text{ is in } c_0(E^0)\}.$$

Given $a, b \in c_0(E^0)$ and $\xi, \eta \in X(E)$, the module actions are given by

$$(a \cdot \xi \cdot b)(e) = a(r(e))\xi(e)b(s(e)),$$

and the $c_0(E^0)$-valued inner product is given by

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e).$$

Later, we will associate a product systems of C^*-correspondences to a higher-rank graph Λ. For more on ordinary graph correspondences, see chapter 8 of [12].

Example 2.4.2. Any C^*-algebra A can be taken as a C^*-correspondence over A. The left and right actions are given by left and right multiplication, and the A-valued inner product is given by

$$\langle a, b \rangle_A = a^*b.$$

Various C^*-algebras can be associated to C^*-correspondences. Pimsner, in [11], was the first to introduce such a construction; however, the left action in his work was required to be injective. In [5], Katsura showed that injectivity is not required.

In either case, the resulting C^*-algebras can either be built constructively or with representations. The latter approach is presented here.

Let (X, A) be a C^*-correspondence over A. Define an ideal J_X of A by

$$J_X = \{a \in A : \phi_X(a) \in \mathcal{K}(X), \text{ and } ab = 0 \text{ for all } b \in \ker(\phi_X)\}.$$

Note that if ϕ_X is injective on J_X, then $J_X = \phi_X^{-1}(\mathcal{K}(X))$.

Definition 2.4.3. Let X be a C^*-correspondence over A. A *Toeplitz representation* of X is a pair (ψ, π) where $\psi : X \to B$ is a linear map, $\pi : A \to B$ is a $*$-homomorphism, and where for $x, y \in X$ and $a, b \in A$,

1. $\psi(a \cdot x \cdot b) = \pi(a)\psi(x)\pi(b)$, and

2. $\psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A)$.

Given a Toeplitz representation (ψ, π), there is a $*$-homomorphism $\pi^{(1)} : \mathcal{K}(X) \to B$ satisfying

$$\pi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*.$$

The representation is said to be *Cuntz-Pimsner covariant* if for all $a \in J_X$,

$$\pi^{(1)}(\phi_X(a)) = \pi(a).$$

Definition 2.4.4. The *Toeplitz algebra* $\mathcal{T}_X$ of X is the C^*-algebra which is universal for Toeplitz representations of X, and the Cuntz-Pimsner algebra $\mathcal{O}_X$ is the C^*-algebra universal for Cuntz-Pimsner covariant Toeplitz representations of X.

Cuntz-Pimsner algebras can also be developed in a functorial framework, as was done by Robertson and Szymański in [14]. This perspective is presented below.

Let (X, A), (Y, B), and (Z, C) be C^*-correspondences over A, B, and C, respectively. As noted in [14], for any continuous linear map $\psi_X : X \to Y$, there is a

$*$-homomorphism $\psi_X^+ : \mathcal{K}(X) \to \mathcal{K}(Y)$ that satisfies

$$\psi_X^+(\theta_{\xi,\eta}) = \theta_{\psi_X(\xi),\psi_X(\eta)}.$$

Moreover, if $\psi_Y : Y \to Z$ is another such map, then

$$(\psi_Y \circ \psi_X)^+ = \psi_Y^+ \circ \psi_X^+.$$

Definition 2.4.5 (Def. 2.3 of [14])**.** A *morphism* from (X, A) to (Y, B) is a pair (ψ_X, ψ_A) consisting of a linear map $\psi_X : X \to Y$ and a $*$-homomorphism $\psi_A : A \to B$ such that

1. $\langle \psi_X(\xi), \psi_X(\eta) \rangle = \psi_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in X$,
2. $\psi_X(\phi_X(a)\xi) = \phi_Y(\psi_A(a))\psi_X(\xi)$ for all $\xi \in X$ and $a \in A$,
3. $\psi_A(J_X) \subseteq J_Y$, and
4. $\phi_Y(\psi_A(a)) = \psi_X^+(\phi_X(a))$ for all $a \in J_X$.

C^*-correspondences together with the above morphisms form a category, and the Cuntz-Pimsner algebra can be defined by using these morphisms as representations in the following manner.

Definition 2.4.6. Let (X, A) be a C^*-correspondence over A, and let B be a C^*-algebra. A *covariant representation* of (X, A) on B is a morphism $(\psi_X, \psi_A) : (X, A) \to (B, B)$, where (B, B) is the C^*-correspondence canonically associated to B, as in example 2.4.2.

Definition 2.4.7. Let (X, A) be a C^*-correspondence over A. Then, $\mathcal{O}_X$ is the C^*-algebra universal for covariant representations of (X, A). That is, there exists a universal covariant representation (π_X, π_A) of (X, A), and $\mathcal{O}_X$ is generated by the image of (X, A) under this representation.

Let $(\psi_X, \psi_A) : (X, A) \to (Y, B)$ be a morphism of C^*-correspondences. Using the universal property of the associated C^*-algebras, it follows (see [14], Proposition 2.9) that there is a map $F(\psi_X, \psi_A) : \mathcal{O}_X \to \mathcal{O}_Y$ such that

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\psi_X, \psi_A)} & (Y, A) \\ {\scriptstyle (\pi_X, \pi_A)}\downarrow & & \downarrow{\scriptstyle (\pi_Y, \pi_B)} \\ \mathcal{O}_X & \xrightarrow[F(\psi_X, \psi_A)]{} & \mathcal{O}_Y \end{array}$$

commutes.

By Proposition 2.11 of [14], the map F that takes (X, A) to $\mathcal{O}_X$ and (ψ_X, ψ_A) to $F(\psi_X, \psi_A)$ is a covariant functor from the category of C^*-correspondences to the category of C^*-algebras.

With these preliminaries in place, the main results of the thesis may begin. Functoriality for k-graphs and their associated Cuntz-Krieger and Toeplitz algebras is investigated first.

Chapter 3

FUNCTORIALITY FOR CUNTZ-KRIEGER AND TOEPLITZ ALGEBRAS

Functorial results concerning graphs and their associated analytic objects have been considered previously. In the case of ordinary directed graphs, Spielberg ([17]) defined a functor from a category of directed graphs with inclusions to a category of C^*-algebras with injective $*$-homomorphisms and proved a number of structural results from the construction. In the case of k-graphs, Kumjian et al. ([7]) defined k-morphs, which consist of data for combining a k- and an l-graph into a $k+l$-graph, and they then constructed a functor from a category of k-graphs with morphisms given by isomorphism classes of k-morphs to a category of C^*-algebras with morphisms given by isomorphism classes of C^*-correspondences.

The k-morph machinery is fairly involved, but functoriality also holds in certain cases for more standard categories. In [8], Maloney et al. showed that for an injective, saturated k-graph morphism $\phi : \Lambda \to \Gamma$, there is a $*$-homomorphism $\phi_* : C^*(\Lambda) \to C^*(\Gamma)$. Since injectivity and saturation are preserved under composition of k-graph morphisms, the assignments $\Lambda \mapsto C^*(\Lambda)$ and $\phi \mapsto \phi_*$, together with the composition $(\phi \circ \pi)_* = \phi_* \circ \pi_*$, define a functor from k-graphs with injective, saturated morphisms to the category $\mathbf{C^*alg}$ of C^*-algebras together with $*$-homomorphisms. As will be shown, the saturation condition here is stronger than is needed, though a similar constraint is still required for the Cuntz-Krieger relations to hold among the images of the generating operators.

Indeed, for Cuntz-Krieger algebras there seems to be no straightforward way to get functoriality with injective morphisms alone. To see this, suppose $\phi : \Lambda \to \Gamma$ is an injective morphism. Then, ϕ can be taken as a map $\phi_g : \Lambda_g \to \Gamma_g$ where Λ_g and

Γ_g are the universal Cuntz-Krieger families for the k-graphs. To satisfy (CK4), we might first try to send vertex projections to sums: For $v \in \Lambda^0$, let

$$\phi_g(p_v) = \sum_{\{\lambda \in \Lambda^n : r(\lambda) = v\}} s_{\phi(\lambda)} s^*_{\phi(\lambda)}, \text{ for all } n \in \mathbb{N}^k.$$

However, there is no way to guarantee that the sums will match for each choice of $n \in \mathbb{N}^k$. As such, saturation or a similar condition is needed so that projections can be sent to projections. The *weakly saturated* condition (recalled below) is exactly enough to ensure that projections satisfy (CK4), relative to its Γ-family definition.

Definition 3.0.8. Let (Γ, d) and (Λ, d) be l- and k-graphs, respectively, with $l \leq k$. A quasimorphism (ϕ, ψ) is said to be *weakly saturated* if for all $v \in \phi(\Gamma^0)$ and all $e_i \in \mathbb{N}^k$, either

$$v\phi(\Gamma) \cap \Lambda^{e_i} = v\Lambda^{e_i}, \text{ or}$$
$$v\phi(\Gamma) \cap \Lambda^{e_i} = \emptyset.$$

In the case of quasimorphisms, further constraints on the intertwining map are needed to preserve the Cuntz-Krieger relations. Namely, $\psi : \mathbb{N}^l \to \mathbb{N}^k$ must be injective, and it must send generators to generators so that factorizations still hold. This implies that we can only pass from graphs of lower rank to graphs of higher rank, which is to be expected, though, since if the dimension collapses information will be lost, whereas a graph of lower dimension can be embedded into one of higher dimension without corruption.

These considerations show that to get functoriality for Cuntz-Krieger algebras, the constraints are fairly rigid. Passing to the Toeplitz algebra is, as will be seen, the more natural choice. Still, there are various inductive constructions in the literature that implicitly employ functoriality to realize the associated Cuntz-Krieger algebra as an inductive limit of subalgebras, and it is thus worth knowing the precise cases in which functoriality holds.

Definition 3.0.9. Let $\phi : \Gamma \to \Lambda$ be a quasimorphism with intertwining map $\psi : \mathbb{N}^l \to \mathbb{N}^k$. Then, (ϕ, ψ) is said to be *Toeplitz-Cuntz-Krieger preserving* if ϕ and ψ are both injective and ψ sends generators to generators. The pair is called *Cuntz-Krieger preserving* if, in addition, ϕ is weakly saturated.

The result from [8] is extended somewhat in Corollary 3.0.12 by allowing the maps to be Cuntz-Krieger preserving quasimorphisms between higher-rank graphs. First, though, functoriality for Toeplitz algebras is characterized in Theorem 3.0.11. For Toeplitz algebras, it does suffice to send projections to a sum of (sums of) operators.

Proposition 3.0.10. The following are categories:

i. **HG**: The objects are row-finite higher-rank graphs with no sources, and the morphisms are Toeplitz-Cuntz-Krieger preserving quasimorphisms.

ii. **cHG** is a subcategory of **HG** obtained by restricting to Cuntz-Krieger quasimorphisms.

Proof. Let $\phi_1 : \Lambda_1 \to \Lambda_2$ and $\phi_2 : \Lambda_2 \to \Lambda_3$ be quasimorphisms with intertwining maps $\psi_1 : \mathbb{N}^{k_1} \to \mathbb{N}^{k_2}$ and $\psi_2 : \mathbb{N}^{k_2} \to \mathbb{N}^{k_3}$. If (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are Toeplitz-Cuntz-Krieger preserving then $(\phi_2 \circ \phi_1, \psi_2 \circ \psi_1)$ is as well since injectivity is preserved under composition and generators are still sent to generators. If (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are Cuntz-Krieger preserving, then, in particular, ϕ_1 and ϕ_2 are weakly saturated. Let

$$v = (\phi_2 \circ \phi_1)(w) \in (\phi_2 \circ \phi_1)(\Lambda_1^0),$$

and let $e_i \in \mathbb{N}^{k_3}$. Assume

$$v(\phi_2 \circ \phi_1)(\Lambda_1) \cap \Lambda_3^{e_i} = v(\phi_2 \circ \phi_1)(\Lambda_1)^{e_i} \neq \emptyset.$$

Then, there is $e_j \in \mathbb{N}^{k_2}$ with $\psi_2(e_j) = e_i$. Since ϕ_1 and ϕ_2 are weakly saturated and the intersection is nonempty, it follows that

$$\begin{aligned} v(\phi_2 \circ \phi_1)(\Lambda_1)^{e_i} &= (\phi_2 \circ \phi_1)(w)\phi_2(\phi_1(\Lambda_1)^{e_j}) \\ &= \phi_2(\phi_1(w)\phi_1(\Lambda_1)^{e_j}) \\ &= \phi_2(\phi_1(w)\Lambda_2^{e_j}) \\ &= (\phi_2 \circ \phi_1)(w)\phi_2(\Lambda_2^{e_j}) \\ &= v\phi_2(\Lambda_2)^{e_i} \\ &= v\Lambda_3^{e_i}. \end{aligned}$$

Thus, $\phi_2 \circ \phi_1$ is weakly saturated. Since the composition of quasimorphisms is associative, **HG** and **cHG** are categories. □

Let $\phi : \Gamma \to \Lambda$ be a quasimorphism in **HG**. Let $\gamma \in \Gamma \setminus \Gamma^0$, and let $v \in \Gamma^0$. Set $\phi_*(s_\gamma) = s_{\phi(\gamma)}$, and set

$$\phi_*(s_v) = \sum_{i=1}^{k} \sum_{\lambda \in \phi(v)\phi(\Gamma)^{e_i}} s_\lambda s_\lambda^*.$$

We'll show in Theorem 3.0.11 that ϕ_* extends to a $*$-homomorphism from $\mathcal{T}(\Gamma)$ to $\mathcal{T}(\Lambda)$ and that this assignment is functorial.

Theorem 3.0.11. The assignments

$$(\Lambda, d) \mapsto \mathcal{T}(\Lambda)$$

$$(\phi : \Gamma \to \Lambda) \mapsto (\phi_* : \mathcal{T}(\Gamma) \to \mathcal{T}(\Lambda))$$

give a functor from **HG** to $\mathbf{C^*alg}$.

Proof. Let (Λ, d) be a row-finite k-graph with no sources. For objects, (Λ, d) simply maps to $\mathcal{T}(\Lambda)$. For morphisms, let $\phi : \Gamma \to \Lambda$ be a quasimorphism in **HG** with intertwining map $\psi : \mathbb{N}^l \to \mathbb{N}^k$ where $l \le k$. Then, we can reduce to the case of a morphism by taking Γ as a k-graph in the following manner. First define $f : \mathbb{N}^{l+1} \to \mathbb{N}^l$ by projection onto the first l coordinates: $(n_1, \ldots, n_{l+1}) \mapsto (n_1, \ldots, n_l)$. Define

$$f^*\Gamma := \{(\gamma, (n_1, \ldots, n_l, 0)) \in \Gamma \times \mathbb{N}^{l+1} : d(\gamma) = f(n_1, \ldots, n_l, 0)\}.$$

The set $f^*\Gamma$ inherits edge factorizations from Γ, and it is an $l+1$-graph with

$$s(\gamma, (n, 0)) = s(\gamma),\ r(\gamma, (n, 0)) = r(\gamma),\ \text{and } d(\gamma, (n, 0)) = (n, 0)$$

for $n = d(\gamma) \in \mathbb{N}^l$. Note that the structure of the graph is unchanged, and, in particular, Γ_g is still a universal generating family for $f^*\Gamma$ under the correspondence between s_γ and $s_{\gamma,(n,0)}$ where $d(\gamma) = n$. Thus, $\mathcal{T}(f^*\Gamma)$ and $C^*(f^*\Gamma)$ are isomorphic to $\mathcal{T}(\Gamma)$ and $C^*(\Gamma)$, respectively, and by iterating this relabeling of the degrees, we may regard Γ as (isomorphic to) a k-graph.

Note that the graph $f^*\Gamma$ now technically has sources, but this object is only introduced for notational convenience. By supposing that Γ is isomorphic to a k-graph, we can bypass the work of renaming degrees with ψ and thereby exclude it from computations. Crucially, though, in what follows we still obtain a TCK Γ-family in $\mathcal{T}(\Lambda)$ through the correspondence between s_γ and $s_{\gamma,(n,0)}$.

Let $\gamma \in \Gamma \setminus \Gamma^0$, and let $v \in \Gamma^0$. Set $\phi_*(s_\gamma) = s_{\phi(\gamma)}$, and define vertex projections p_v in $\mathcal{T}(\Lambda)$ by

$$\begin{aligned} p_v := \phi_*(s_v) &= \sum_{i=1}^{k} \sum_{\gamma \in v\Gamma^{e_i}} s_{\phi(\gamma)} s_{\phi(\gamma)}^* \\ &= \sum_{i=1}^{k} \sum_{\lambda \in \phi(v)\phi(\Gamma)^{e_i}} s_\lambda s_\lambda^*. \end{aligned}$$

Then, for any $v \in \Gamma^0$ and $e_i \in \mathbb{N}^k$,

$$p_v \geq \sum_{\gamma \in v\Gamma^{e_i}} s_{\phi(\gamma)} s_{\phi(\gamma)}^*.$$

Hence, p_v is a projection in Λ satisfying the (TCK4) condition relative to the Γ-family definition of the relations. Furthermore, if $s(\gamma) = r(\mu)$, then

$$s_{\phi(\gamma)} s_{\phi(\mu)} = s_{\phi(\gamma)\phi(\mu)} = s_{\phi(\gamma\mu)},$$

and hence the second Toeplitz-Cuntz-Krieger relation is satisfied, while satisfaction of the third is inherited from Λ. Thus,

$$\{s_{\phi(\gamma)} : \gamma \in \Gamma \setminus \Gamma^0\} \cup \{p_v : v \in \Gamma^0\}$$

is a TCK Γ-family in Λ. As such, there is a $*$-homomorphism $\phi_* : \mathcal{T}(\Gamma) \to \mathcal{T}(\Lambda)$ where $s_\gamma \mapsto s_{\phi(\gamma)}$ for $\gamma \in \Gamma \setminus \Gamma^0$ and $s_v \mapsto p_v$ for $v \in \Gamma^0$. Moreover, since ϕ is injective and each $\phi_*(p_v)$ is nonzero for $v \in \Gamma^0$, it follows from the gauge-invariant uniqueness theorem that, in particular, ϕ_* is injective.

Finally, let $\phi : \Gamma \to \Lambda$ and $\psi : \Lambda \to \Sigma$ be morphisms in **HG**. Assume Γ, Λ, and Σ are all (isomorphic) to k-graphs for some k. Then, for $\gamma \in \Gamma \setminus \Gamma^0$,

$$\begin{aligned}
(\psi \circ \phi)_*(s_\gamma) &= s_{\psi\circ\phi(\gamma)} \\
&= \psi_*(s_{\phi(\gamma)}) \\
&= \psi_* \circ \phi_*(s_\gamma),
\end{aligned}$$

and for $v \in \Gamma^0$,

$$\begin{aligned}
(\psi \circ \phi)_*(s_v) &= \sum_{i=1}^{k} \sum_{\gamma \in v\Gamma^{e_i}} s_{\psi\circ\phi(\gamma)} s^*_{\psi\circ\phi(\gamma)} \\
&= \sum_{i=1}^{k} \sum_{\lambda \in \phi(v)\phi(\Gamma)^{e_i}} s_{\psi(\lambda)} s^*_{\psi(\lambda)} \\
&= \psi_* \left(\sum_{i=1}^{k} \sum_{\lambda \in \phi(v)\phi(\Gamma)^{e_i}} s_\lambda s^*_\lambda \right) \\
&= \psi_* \circ \phi_*(s_v).
\end{aligned}$$

Thus, there is a (covariant) functor from **HG** to **C*alg**. □

For Corollary 3.0.12, define $\phi_*(s_\gamma) = s_{\phi(\gamma)}$ for $\gamma \in \Gamma$. We'll show that this map extends to a $*$-homomorphism $\phi_* : C^*(\Gamma) \to C^*(\Lambda)$ and that this assignment is functorial.

Corollary 3.0.12. The assignments

$$(\Lambda, d) \mapsto C^*(\Lambda)$$
$$(\phi : \Gamma \to \Lambda) \mapsto (\phi_* : C^*(\Gamma) \to C^*(\Lambda))$$

give a functor from **cHG** to $\mathbf{C^*alg}$.

Proof. For objects, (Λ, d) maps to $C^*(\Lambda)$. For morphisms, let $\phi : \Gamma \to \Lambda$ be a quasimorphism in **cHG** with intertwining map $\psi : \mathbb{N}^l \to \mathbb{N}^k$ where $l \leq k$. Assume that Γ is isomorphic to a k-graph, as in Theorem 3.0.11. We'll show that $S := \{s_{\phi(\gamma)} : \gamma \in \Gamma\}$ is a Cuntz-Krieger Γ-family in $C^*(\Lambda)$. The first three Cuntz-Krieger relations are satisfied as in the case of Toeplitz algebras. It remains to check the fourth Cuntz-Krieger relation.

Since ϕ is weakly saturated and injective, $\phi(v)\phi(\Gamma) \cap \Lambda^{e_i} = \phi(v)\Lambda^{e_i}$ for $v \in \Gamma^0$ whenever $1 \leq i \leq l$. Thus, for $1 \leq i \leq l$,

$$s_{\phi(v)} = \sum_{\lambda \in \phi(v)(\Gamma) \cap \Lambda^{e_i}} s_\lambda s_\lambda^* = \sum_{\lambda \in \phi(v)\Lambda^{e_i}} s_\lambda s_\lambda^*,$$

and consequently, $s_{\phi(v)}$ is a projection in Λ satisfying (CK4) relative to the Γ-family definition of the Cuntz-Krieger relations. Thus, S is indeed a Cuntz-Krieger Γ-family in $C^*(\Lambda)$, and as such, there is a $*$-homomorphism $\phi_* : C^*(\Gamma) \to C^*(\Lambda)$ where $s_\gamma \mapsto s_{\phi(\gamma)}$. Moreover, since ϕ is injective and each $s_{\phi(v)}$ is nonzero for $v \in \Gamma^0$, it again follows from the gauge-invariant uniqueness theorem that ϕ_* is injective.

For compositions of morphisms $\phi : \Gamma \to \Lambda$ and $\psi : \Lambda \to \Sigma$ in **cHG**, we have that for $\gamma \in \Gamma$,

$$\begin{aligned}(\psi \circ \phi)_*(s_\gamma) &= s_{\psi\circ\phi(\gamma)} \\ &= \psi_*(s_{\phi(\gamma)}) \\ &= \psi_* \circ \phi_*(s_\gamma).\end{aligned}$$

Hence, there is a (covariant) functor from **cHG** to **C*alg**. $\square$

One feature of functoriality is that in certain cases we can characterize the C^*-algebra of a k-graph as an inductive limit of C^*-subalgebras associated to an increasing chain of (weakly saturated) subgraphs. (Recall that a subgraph of Γ is a subcategory of Γ that is itself a higher-rank graph, and it is called weakly saturated if the inclusion map is weakly saturated).

In what follows, the maps in the direct sequences are inclusion maps $i : \Lambda_n \to \Lambda_{n+1}$ and their induced $*$-homomorphisms.

Proposition 3.0.13. Let Λ be a k-graph. Suppose there is an increasing chain of subgraphs $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ such that $\bigcup_{n\geq 1} \Lambda_n = \Lambda$. Then, $\mathcal{T}(\Lambda) \cong \varinjlim \mathcal{T}(\Lambda_n)$.

Proof. By Theorem 3.0.11, there are injective $*$-homomorphisms $i_* : \mathcal{T}(\Lambda_n) \to \mathcal{T}(\Lambda_{n+1})$ for $n \geq 1$. As such, $(\mathcal{T}(\Lambda_n), i_*)_{n=1}^\infty$ is a direct sequence of increasing C^*-algebras, and since for large enough N, each generator of $\mathcal{T}(\Lambda)$ is in $\bigcup_{n=1}^N \mathcal{T}(\Lambda_n)$, it follows that $\bigcup_{n=1}^\infty \mathcal{T}(\Lambda_n)$ is dense in $\mathcal{T}(\Lambda)$. Thus, $\mathcal{T}(\Lambda) \cong \varinjlim \mathcal{T}(\Lambda_n)$. $\square$

Proposition 3.0.14. Let Λ be a k-graph. Suppose there is an increasing chain of weakly saturated subgraphs $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$ such that $\bigcup_{n\geq 1} \Lambda_n = \Lambda$. Then, $C^*(\Lambda) \cong \varinjlim C^*(\Lambda_n)$.

Proof. By Corollary 3.0.12, $(C^*(\Lambda_n), i_*)_{n=1}^\infty$ is a direct sequence of increasing C^*-algebras such that $\bigcup_{n=1}^\infty C^*(\Lambda_n)$ is dense in $C^*(\Lambda)$. Hence, $C^*(\Lambda) \cong \varinjlim C^*(\Lambda_n)$. $\square$

Example 3.0.15. The simplest example of this inductive structure is the realization of $\mathcal{K}(\ell^2(\mathbb{N}))$ as the direct limit of increasing matrix algebras:

$$C^*(E_\infty) \cong \mathcal{K}(\ell^2(\mathbb{N})) \cong \varinjlim M_n \cong \varinjlim C^*(E_n)$$

where E_n is a directed path on n vertices.

Chapter 4

COMBINATORIAL RESULTS

In this section, it is shown how to obtain k-graphs from the Cayley graphs of finitely generated groups. Moreover, it is shown that the process of going from a group to a graph algebra is functorial: there is a functor from a category of finitely generated groups to the category of C^*-algebras. Finally, a combinatorial result about the fundamental groupoid of a higher-rank graph is also presented.

4.1 Cayley k-Graphs

Definition 4.1.1. Let G be a group with generating set S. The *Cayley graph* $\Gamma(G, S)$ is a colored directed graph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{(g, sg) : g \in G, s \in S\}$. Each generator $s \in S$ is assigned a color c_s, and the edge (g, sg) is colored c_s. If $|S| = k$, then our convention is to assume that colors take their values in $\mathbb{N}^k$. More specifically, for a generating set $S = \{s_1, \ldots, s_k\}$, we set $c_{s_i} = e_i \in \mathbb{N}^k$ where, again, e_i is one of the standard basis elements of $\mathbb{N}^k$.

The Cayley graph of a finitely generated abelian group can be regarded as a 1-skeleton of a unique k-graph, and the C^*-algebras of such graphs are easily characterized by Proposition 4.2.3. When the group is not abelian, the Cayley graph no longer serves as a 1-skeleton, since there is, in general, no way to satisfy the factorization rules. If, however, a sufficiently large equivalence relation is placed on the vertices of the Cayley graph, the resulting quotient graph can again be used to define a k-graph.

One approach for defining the needed equivalence relation is to minimize the damage done to the structure of the original graph while still ensuring that factorizations are possible. Unfortunately, this approach leaves one with a number of arbitrary choices when defining factorizations. If instead the equivalence relation collapses everything to a single vertex, there is a straightforward way to define factorizations in a uniform manner using the original Cayley graph. Furthermore, in the single vertex case, the process of going from a finitely generated group to the Toeplitz algebra of the single vertex k-graph is functorial for appropriate categories.

The single vertex k-graphs are defined below, though examples of other graphs coming from Cayley graphs are also given in the next section.

Let $\Gamma := \Gamma(G, S)$ be a Cayley graph, and let $\sim := V(\Gamma) \times V(\Gamma)$ be the equivalence relation on Γ that reduces the vertex set to a single element v_0. The quotient graph $\Gamma/\sim$ has vertex set $V(\Gamma)/\sim$ and edge set

$$\{(x, y) \in V(\Gamma/\sim) \times V(\Gamma/\sim) : \exists\, (g, h) \in E(\Gamma) \text{ with } g \in x \text{ and } h \in y\}.$$

Note that there is a bijection $q : E(\Gamma) \to E(\Gamma/\sim)$ that sends (g, sg) to $([g], [sg])$. That is, $([g], [sg])$ is a copy of the edge (g, sg), but now $s([g], [sg]) = v_0 = r([g], [sg])$.

Let $\mathcal{P}$ be the set of all bicolored paths $P = f_1 f_2$ in $\Gamma/\sim$. Each edge f_j of any such path P is associated to a unique edge $q^{-1}(f_j) = (g_j, s_{i_j} g_j)$ of the Cayley graph, where $s_{i_j} \neq s_{i_l}$ for $j \neq l$. Define a map $F : \mathcal{P} \to \mathcal{P}$ by

$$F(P) = ([g_1], [s_{i_2} g_1])([g_2], [s_{i_1} g_2]),$$

where $P = ([g_1], [s_{i_1} g_1])([g_2], [s_{i_2} g_2])$. Call $F(P)$ the *Cayley factorization* of P, and call F the *Cayley factorization map.*

Since there is exactly one in-edge and one out-edge of each color at every vertex of the Cayley graph, and since there is a bijection between the edges of the original Cayley graph and the quotient graph, F is a well-defined bijection.

Proposition 4.1.2. Let (G, S) be a finitely generated group. Using Cayley factorizations, $\Gamma(G, S)/\sim$ is a 1-skeleton that determines a unique k-graph $\Lambda[G, S]$.

Proof. Let $\Gamma := \Gamma(G, S)$ be a k-colored Cayley graph where $S = \{s_1, \ldots, s_k\}$. Let $\sim$ be the equivalence relation from above.

The degree functor $d : \Lambda \to \mathbb{N}^k$ is defined on an edge $([g], [s_j g])$ of $\Gamma/\sim$ by

$$d(([g], [s_j g])) = c_{s_j} \in \mathbb{N}^k.$$

Then, d extends to all of Λ by factoring any $\lambda \in \Lambda$ (in the way described below) into a concatenation of edges from the 1-skeleton. That is, write

$$\lambda = ([g_1], [s_{j_1} g_1]) \cdots ([g_n], [s_{j_n} g_n]),$$

and set

$$d(\lambda) = c_{s_{j_1}} + \cdots + c_{s_{j_n}}.$$

As discussed in [13], to show that the factorization property holds, it suffices to check that multicolored paths in the 1-skeleton of length two or three factor appropriately. Furthermore, since the Cayley factorization map is a bijection on the collection of bicolored paths in $\Gamma/\sim$, we need only check that this bijection satisfies the commuting cube property for paths of length three.

More specifically, let $P = f_1 f_2 f_3$ be a tricolored path in $\Gamma/\sim$. Each edge f_j is uniquely associated to an edge $(g_j, s_{i_j} g_j)$ of the original Cayley graph where $s_{i_j} \neq s_{i_l}$ for $j \neq l$. Then, if we complete the commuting cube associated with P by first factoring $f_2 f_3$, we have

$$\begin{aligned}
&([g_2], [s_{i_2} g_2])([g_3], [s_{i_3} g_3]) = ([g_2], [s_{i_3} g_2])([g_3], [s_{i_2} g_3]),\\
&([g_1], [s_{i_1} g_1])([g_2], [s_{i_3} g_2]) = ([g_1], [s_{i_3} g_1])([g_2], [s_{i_1} g_2]), \text{ and}\\
&([g_2], [s_{i_1} g_2])([g_3], [s_{i_2} g_3]) = ([g_2], [s_{i_2} g_2])([g_3], [s_{i_1} g_3]),
\end{aligned}$$

which implies that

$$([g_1],[s_{i_1}g_1])([g_2],[s_{i_2}g_2])([g_3],[s_{i_3}g_3]) = ([g_1],[s_{i_3}g_1])([g_2],[s_{i_2}g_2])([g_3],[s_{i_1}g_3]).$$

Alternatively, if we complete the commuting cube by first factoring f_1f_2, we have

$$([g_1],[s_{i_1}g_1])([g_2],[s_{i_2}g_2]) = ([g_1],[s_{i_2}g_1])([g_2],[s_{i_1}g_2]),$$

$$([g_2],[s_{i_1}g_2])([g_3],[s_{i_3}g_3]) = ([g_2],[s_{i_3}g_2])([g_3],[s_{i_1}g_3]), \text{ and}$$

$$([g_1],[s_{i_2}g_1])([g_2],[s_{i_3}g_2]) = ([g_1],[s_{i_3}g_1])([g_2],[s_{i_2}g_2]),$$

which again implies that

$$([g_1],[s_{i_1}g_1])([g_2],[s_{i_2}g_2])([g_3],[s_{i_3}g_3]) = ([g_1],[s_{i_3}g_1])([g_2],[s_{i_2}g_2])([g_3],[s_{i_1}g_3]).$$

Since either way of completing the commuting cube associated with P gives the same factorization, it follows that Cayley factorizations determine a consistent collection of commuting squares. Hence, $\Gamma/\sim$ determines a unique k-graph $\Lambda[G,S]$. □

4.2 Examples

As noted, the Cayley graphs of finitely generated abelian groups can already be taken as 1-skeletons that determine unique k-graphs. The algebras yielded in this manner are easily classified by Proposition 4.2.3. Note first the following preliminary observation.

Definition 4.2.1. Let G and H be directed graphs. The *Cartesian product* of G and H is the graph $G \times H$ with underlying vertex set $V(G) \times V(H)$. An edge is directed from (v_1, w_1) to (w_1, w_2) if either $v_1 = v_2$ and $(w_1, w_2) \in E(H)$ or if $w_1 = w_2$ and $(v_1, v_2) \in E(G)$.

Remark 4.2.2. Let Λ and Γ be higher-rank graphs with 1-skeletons E_Λ and E_Γ. Then, the 1-skeleton $E_{\Lambda\times\Gamma}$ of the categorical product $\Lambda \times \Gamma$ coincides with the Cartesian product of the 1-skeletons. That is,

$$E_{\Lambda\times\Gamma} \cong E_\Lambda \times E_\Gamma.$$

This follows from the factorization property. More specifically, recall that a morphism from (V_1, W_1) to (V_2, W_2) in $\Lambda \times \Gamma$ is a pair (f, g) where $f : V_1 \to V_2$ and $g : W_1 \to W_2$ are morphisms in Λ and Γ, respectively. But any (f, g) can be factored as a composition of the form

$$(f, g) = (f, \mathrm{id}_{W_2}) \circ (\mathrm{id}_{V_1}, g) = (\mathrm{id}_{V_2}, g) \circ (f, \mathrm{id}_{W_1}).$$

Hence, the 1-skeleton of $\Lambda \times \Gamma$ includes only those edges corresponding to morphisms of the form (f, id_γ) or (id_λ, g). But this is just the Cartesian product $E_\Lambda \times E_\Gamma$ of the 1-skeletons.

For Proposition 4.2.3, we will use $\Lambda[G]$ to denote the k-graph obtained from treating the Cayley graph Γ of a finitely generated abelian group G (equipped with standard generators) as a 1-skeleton. Hence, in what follows $\Lambda[G]$ is not one of the single-vertex k-graphs defined previously, as here no quotient relation is imposed on the vertices of Γ.

Proposition 4.2.3. Let $G := \mathbb{Z}^l \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}$ be a finitely generated abelian group with standard generators. Then, the Cuntz-Krieger algebra of $\Lambda[G]$ is isomorphic to

$$\mathcal{K}(\ell^2(\mathbb{Z}^l)) \otimes C\left(\mathbb{T}^m, M_{n_1}(\mathbb{C}) \otimes \cdots \otimes M_{n_m}(\mathbb{C})\right).$$

Proof. Let G be a finitely generated group of the form above. There are $l + m$ generators of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$. That is, the first l positions generate $\mathbb{Z}^l$ and each position thereafter generates one of the $\mathbb{Z}_{n_j}$ factors. The Cayley graph associated to G can be written as the Cartesian product of the Cayley graphs associated to its factors:

$$\Gamma(G) \cong \Gamma(\mathbb{Z}^l) \times \Gamma(\mathbb{Z}_{n_1}) \times \cdots \times \Gamma(\mathbb{Z}_{n_m}).$$

From the above remark, and from the fact that the Cayley graphs of (finitely generated) abelian groups are already 1-skeletons of higher-rank graphs, we then have that

$$\begin{aligned} E_{\Lambda[G]} &\cong E_{\Lambda[\mathbb{Z}^l]} \times E_{\Lambda[\mathbb{Z}_{n_1}]} \times \cdots \times E_{\Lambda[\mathbb{Z}_{n_m}]} \\ &\cong E_{\Lambda[\mathbb{Z}^l]\times\Lambda[\mathbb{Z}_{n_1}]\times\cdots\times\Lambda[\mathbb{Z}_{n_m}]}. \end{aligned}$$

Since the factorizations are unique, isomorphic 1-skeletons determine isomorphic higher-rank graphs. Thus,

$$\Lambda[G] \cong \Lambda[\mathbb{Z}^l] \times \Lambda[\mathbb{Z}_{n_1}] \times \cdots \times \Lambda[\mathbb{Z}_{n_m}].$$

Then, by Corollary 3.5 in [6], we have

$$C^*(\Lambda[G]) \cong C^*(\Lambda[\mathbb{Z}^l]) \otimes C^*(\Lambda[\mathbb{Z}_{n_1}]) \otimes \cdots \otimes C^*(\Lambda[\mathbb{Z}_{n_m}]).$$

Each $\Lambda[\mathbb{Z}_{n_j}]$ is a cycle of length n_j, and $\Lambda[\mathbb{Z}^l]$ corresponds to the l-graph $\Delta_l := \{(m,n) \in \mathbb{Z}^l \times \mathbb{Z}^l : m \leq n\}$ where $r(m,n) = m$, $s(m,n) = n$, and $d(m,n) = n - m$. Hence,

$$\begin{aligned} C^*(\Lambda[G]) &\cong \mathcal{K}(\ell^2(\mathbb{Z}^l)) \otimes C(\mathbb{T}, M_{n_1}(\mathbb{C})) \otimes \cdots \otimes C(\mathbb{T}, M_{n_m}(\mathbb{C})) \\ &\cong \mathcal{K}(\ell^2(\mathbb{Z}^l)) \otimes C\left(\mathbb{T}^m, M_{n_1}(\mathbb{C}) \otimes \cdots \otimes M_{n_m}(\mathbb{C})\right). \end{aligned}$$

□

The following are examples of k-graphs obtained from Cayley graphs of finitely generated nonabelian groups where the quotient relation on the Cayley graph does not collapse the entire vertex set to a single point. Let $\Gamma := \Gamma(G,S)$ be the Cayley graph of a finitely generated group (G,S). Define $\sim_R$ to be the smallest equivalence relation on $V(\Gamma)$ containing the set

$$R = \{(s_i s_j g, s_j s_i g) \in V(\Gamma) \times V(\Gamma) : g \in V(\Gamma), s_i, s_j \in S, i \neq j\}.$$

Recall that nC_k is a k-cycle with n (directed) edges between successive vertices and that $C^*(nC_k) \cong \mathcal{O}_n \otimes M_k(\mathbb{C})$ (see example 2.2.2). In the following examples, each graph $\Lambda[G] := \Gamma/\sim_R$ can be taken as the categorical product of graphs of the form nC_k. Hence, $C^*(\Lambda[G])$ can be described as a tensor product of the C^*-algebras of the k-cycles.

Example 4.2.4. Let D_n be the dihedral group with presentation $D_n = \langle a, b \mid a^4 = b^2 = e, ab = ba^3 \rangle$. Then for n odd, $D_n/\sim_R = nC_1 \times nC_2$, whereas for n even, $D_n/\sim_R = \frac{n}{2}C_2 \times \frac{n}{2}C_2$. Thus, for n odd, $C^*(\Lambda[D_n]) \cong C^*(nC_1) \otimes C^*(nC_2)$, and $C^*(\Lambda[D_n]) \cong C^*(\frac{n}{2}C_2) \otimes C^*(\frac{n}{2}C_2)$ for n even. That is,

$$C^*(\Lambda[D_n]) \cong \begin{cases} \mathcal{O}_n \otimes \mathcal{O}_n \otimes M_2 & \text{for } n \text{ odd} \\ \mathcal{O}_{n/2} \otimes \mathcal{O}_{n/2} \otimes M_4 & \text{for } n \text{ even.} \end{cases}$$

Example 4.2.5. The symmetric group behaves similarly, except that the dimension of the matrix algebra now varies with n. Let S_n be generated by cyclic permutation and transposition. Then, for n even, $S_n/\sim_R = (n-1)C_n \times (n-1)C_2$, so $C^*(\Lambda[S_n])$ is isomorphic to

$$\mathcal{O}_{n-1} \otimes \mathcal{O}_{n-1} \otimes M_{2n}.$$

4.3 Functoriality

Here it is shown that the single-vertex k-graphs $\Lambda[G, S]$ of Proposition 4.1.2 can be used to define a functor from a category of finitely generated groups to the category of C^*-algebras. More specifically, it is shown that the sequence of assignments

$$(G, S) \mapsto \Lambda[G, S] \mapsto \mathcal{T}(\Lambda[G, S])$$

that takes the finitely generated group (G, S) to the Toeplitz C^*-algebra $\mathcal{T}(\Lambda[G, S])$ is functorial at each step under appropriate morphisms. The categories involved are listed below.

i. **FG**: The objects are pairs (G, S) where S is a finite generating set for G. A morphism from (G_1, S_1) to (G_2, S_2) is an injective group homomorphism $\phi : G_1 \to G_2$ such that $\phi(S_1) \subseteq S_2$.

ii. **HG**: The objects are row-finite higher-rank graphs with no sources, and the morphisms are Toeplitz-Cuntz-Krieger preserving quasimorphisms.

iii. **C*alg**: C^*-algebras together with $*$-homomorphisms.

Note that injective morphisms are not needed for **FG** to be a category, but injectivity is needed when passing to the C^*-algebraic side of things, and the assumption is therefore built in for convenience.

Proposition 4.3.1. **FG** is a category.

Proof. The morphisms of **FG** are injective group homomorphisms that send generators to generators. If $\phi : (F, R) \to (G, S)$ and $\sigma : (G, S) \to (H, T)$ are morphisms in **FG**, then the composition $\sigma \circ \phi : (F, R) \to (H, T)$ is an injective group homomorphism, and $\sigma \circ \phi(R) = \sigma(\phi(R)) \subseteq T$ since $\phi(R) \subseteq S$ and $\sigma(S) \subseteq T$. Since group homomorphisms are associative, **FG** is a category. $\square$

Let $\phi : (G, S) \to (H, T)$ be a morphism in **FG**. Define a map $\phi_\Lambda : \Lambda[G, S] \to \Lambda[H, T]$ on elements of the 1-skeleton by

$$\phi_\Lambda(([g], [sg])) = ([\phi(g)], [\phi(s)\phi(g)]).$$

In Lemma 4.3.2, we'll show that this map extends to a morphism in **HG** and that the assignment is functorial.

Lemma 4.3.2. The assignments

$$(G, S) \mapsto \Lambda[G, S]$$

$$(\phi : (G, S) \to (H, T)) \mapsto (\phi_\Lambda : \Lambda[G, S] \to \Lambda[H, T])$$

give a functor from **FG** to **HG**.

Proof. The functor is obtained by passing through the Cayley graphs of the groups. For maps, let $\phi : (G,S) \to (H,T)$ be a morphism in **FG**. Assume $S = \{s_1, \ldots, s_l\}$ and $T = \{t_1, \cdots, t_k\}$. The map ϕ determines a graph homomorphism $\phi_\Gamma : \Gamma(G,S) \to \Gamma(H,T)$ between Cayley graphs: ϕ_Γ acts on vertices by $\phi_\Gamma(g) = \phi(g)$, and the edge (g, sg) is sent to $(\phi(g), \phi(s)\phi(g))$, which is an edge of $\Gamma(H,T)$ by definition since $\phi(s) \in T$.

We can get a quasimorphism from $\Lambda[G,S]$ to $\Lambda[H,T]$ by reverting to the Cayley graph, applying the ϕ_Γ map, and then collapsing the vertices again under the equivalence relation. More specifically, let $\phi_\Lambda : \Lambda[G,S] \to \Lambda[H,T]$ be defined as above:

$$\phi_\Lambda(([g],[sg])) = ([\phi(g)],[\phi(s)\phi(g)])$$

where $([g],[sg])$ is an element of the 1-skeleton $\Lambda/\sim$. Since ϕ (and therefore ϕ_Γ) are injective, ϕ_Λ is also injective on the 1-skeleton. We can extend this map to the whole k-graph since repeated applications of the factorization property allow us to write an element λ as a concatenation of edges in the 1-skeleton. Moreover, the intertwining map ψ is defined on the standard basis of $\mathbb{N}^l$ by

$$d_2(\phi_\Lambda(([g],[sg]))) = c_{\phi(s)} = \psi(c_s) = \psi(d_1(([g],[sg]))).$$

Since ψ is an injection that sends generators to generators, (ϕ_Λ, ψ) is a morphism in **HG**.

Finally, if $\phi : (F,R) \to (G,S)$ and $\sigma : (G,S) \to (H,T)$ are morphisms in **FG**, then

$$\begin{aligned}(\sigma \circ \phi)_\Lambda([a],[b]) &= ([\sigma \circ \phi(a)],[\sigma \circ \phi(b)]) \\ &= \sigma_\Lambda([\phi(a)],[\phi(b)]) \\ &= \sigma_\Lambda \circ \phi_\Lambda([a],[b]).\end{aligned}$$

Thus, there is a (covariant) functor from **FG** to **HG**. □

Theorem 4.3.3. The assignments

$$(G, S) \mapsto \mathcal{T}(\Lambda[G, S])$$

$$(\phi : (G, S) \to (H, T)) \mapsto (\phi_* : \mathcal{T}(\Lambda[G, S]) \to \mathcal{T}(\Lambda[H, T]))$$

give a functor from **FG** to **C*alg**.

Proof. From Lemma 4.3.2 and Theorem 3.0.11, the following assignments are functorial at each step:

$$(G, S) \mapsto \Lambda[G, S] \mapsto \mathcal{T}(\Lambda[G, S])$$

$$\phi \mapsto \phi_\Lambda \mapsto \phi_*.$$

□

4.4 Faithful Groupoid Embeddings

In this section, a necessary condition is given to decide whether a k-graph embeds faithfully into its fundamental groupoid.

Let $\{n_i\}_{i=1}^k$ be the standard basis for $\mathbb{N}^k$, and let (Λ, d) be a k-graph with commuting squares $S_1, S_2, \dots$ and 1-skeleton E^1. Each S_j is a relation $ef = gh$ between bi-colored paths. For our purposes, we always assume that $i > j$ where $d(e) = n_i$ and $d(f) = n_j$.

Definition 4.4.1. The *factorization graph* of a k-graph (Λ, d) is a simple (i.e., no loops or multiple edges) undirected graph $G_\Lambda = (V(G_\Lambda), E(G_\Lambda))$ defined as follows:

$$V(G_\Lambda) = \{(e, g) \in E^1 \times E^1 : \exists\ S_j \text{ with } ef = gh \text{ or } fe = hg\}$$

$$E(G_\Lambda) = \{\{(e, g), (f, h)\} : \exists\ S_j \text{ with } ef = gh \text{ or } fe = hg\}$$

We will denote an edge between (e, g) and (f, h) by juxtaposition. Note also that the edges of each pair (e, g) have orthogonal degrees and either a shared source or a shared range. Now, let

$$P = (e_1, e_2)(e_3, e_4) \cdots (e_{2(m+1)-3}, e_{2(m+1)-2})(e_{2(m+1)-1}, e_{2(m+1)})$$

be a path of length m in G_Λ. Consider the odd and even sequences

$$e_1, e_3, \ldots, e_{2(m+1)-3}, e_{2(m+1)-1}, \text{ and}$$
$$e_2, e_4, \ldots, e_{2(m+1)-2}, e_{2(m+1)}.$$

Definition 4.4.2. Let P a path as above. P is called *alternating* if for any subpath $(e_{j-2}, e_{j-1})(e_j, e_{j+1})(e_{j+2}, e_{j+3})$ we have either

$$e_j e_{j+2} = e_{j+1} e_{j+3} \text{ if and only if } e_j e_{j-2} = e_{j+1} e_{j-1}, \text{ or}$$
$$e_{j-2} e_j = e_{j-1} e_{j+1} \text{ if and only if } e_{j+2} e_j = e_{j+3} e_{j+1}$$

Definition 4.4.3. We say that G_Λ *collapses* if there exists an alternating path P and $1 \leq i < j \leq 2(m+1) - 1$ such that $e_i = e_j$ and $e_{i+1} \neq e_{j+1}$ or such that $e_{i+1} = e_{j+1}$ and $e_i \neq e_j$.

Definition 4.4.4. We call a k-graph *faithful* if it embeds faithfully into its fundamental groupoid. That is, the canonical functor is injective.

Proposition 4.4.5. A k-graph (Λ, d) is faithful only if its factorization graph G_Λ does not collapse.

Proof. Let (Λ, d) be a k-graph. We'll show the contrapositive. Assume G_Λ collapses. Then, there is an alternating path

$$P = (e_1, e_2)(e_3, e_4) \cdots (e_{2(m+1)-3}, e_{2(m+1)-2})(e_{2(m+1)-1}, e_{2(m+1)})$$

in G_Λ with either $e_i = e_j$ and $e_{i+1} \neq e_{j+1}$ or $e_{i+1} = e_{j+1}$ and $e_i \neq e_j$ for some $i, j \in \{1, \ldots, 2(m+1) - 1\}$, $i < j$. Without loss of generality, assume m is even, and suppose $e_i \neq e_j$ and $e_{i+1} = e_{j+1}$. (The other possible combinations follow a parallel argument). In fact, it suffices to only consider the shortest path between the relevant vertices, and thus, we may assume that $e_i = e_1$ and that $e_j = e_{2(m+1)-1}$, or, setting $j := m$, we may assume that $e_1 \neq e_{j-1}$ while $e_2 = e_j$. We'll show that $e_1 = e_{j-1}$ in the fundamental groupoid.

Assume that $e_1 e_3 = e_2 e_4$. (If instead $e_3 e_1 = e_4 e_2$, the computation is similar). Since the path is alternating, the other commuting squares are

$$e_5 e_7 = e_6 e_8$$
$$e_9 e_7 = e_{10} e_8$$
$$\vdots$$
$$e_{j-1} e_{j-3} = e_j e_{j-2} = e_2 e_{j-2}.$$

Then, in the fundamental groupoid, we have

$$\begin{aligned} e_1 &= e_2 e_4 e_3^{-1} \\ &= e_2 e_6^{-1} e_5 \\ &= e_2 e_8 e_7^{-1} \\ &\vdots \\ &= e_2 e_{j-2} e_{j-3}^{-1} \\ &= e_2 e_j^{-1} e_{j-1} \\ &= e_2 e_2^{-1} e_{j-1} \\ &= e_{j-1}. \end{aligned}$$

Thus, Λ is not faithful. □

It is possible that the converse holds as well, but a proof is not yet known. The strategy for such a proof should presumably be as follows. Assume Λ is not faithful. Then, there are distinct elements e and f of Λ that are identified in $\mathcal{G}(\Lambda)$. In particular, we may assume that e and f are elements of the 1-skeleton. Since $\mathcal{G}(\Lambda)$ inherits degrees from Λ, $d(e) = n_i = d(f)$ for some $n_i \in \mathbb{N}^k$, and since the 1-skeleton of Λ, taken as a directed graph, does faithfully embed into $\mathcal{G}(\Lambda)$, e and f must be identified through substitutions involving the commuting squares. The problem then is to justify the existence of commuting squares making the computation

$$\begin{aligned} e &= h_2 h_3 h_1^{-1} \\ &\vdots \\ &= h_2 h_j h_{j-1}^{-1} \\ &= h_2 h_2^{-1} f \\ &= f \end{aligned}$$

go through. (The above assumes, at the least, that there is a square $eh_1 = h_2 h_3$. If there is a square $h_1 e = h_3 h_2$, the computation should again be similar). Such a computation would demonstrate that there is an alternating path in G_Λ between (e, h_2) and (f, h_2), and as such, the factorization graph would collapse, proving the converse.

Chapter 5

FUNCTORIALITY FOR C^*-CORRESPONDENCES

The C^*-correspondence associated to an ordinary directed graph was introduced in example 2.4.1. The analogous construction for higher-rank graphs requires product systems of C^*-correspondences, which are reviewed here. Functorial properties between k-graphs and product systems of C^*-correspondences are then explored.

5.1 Product Systems of C^*-Correspondences

Let (X, A) be a C^*-correspondence over A. Then, $X^{\otimes n}$, the n-fold internal tensor product of X, can also be taken as a C^*-correspondence over A, and the collection $\{X^{\otimes n} : n \in \mathbb{N}\}$ is the basic example of a product system. In [1], Fowler constructed a C^*-algebra associated to product systems fibred over more general semigroups.

Definition 5.1.1. Let X and Y be A-correspondences. The *balanced tensor product* $X \otimes_A Y$ of X and Y is the completion of the vector space spanned by $x \otimes_A y$ ($x \in X$, $y \in Y$) subject to the relation

$$x \cdot a \otimes_A y = x \otimes_A \phi(a)y,$$

where the norm is induced by the inner product

$$\langle x_1 \otimes_A y_1, x_2 \otimes_A y_2 \rangle_A = \langle y_1, \langle x_1, x_2 \rangle_A \cdot y_2 \rangle_A.$$

Define a right action of A on $X \otimes_A Y$ by

$$(x \otimes_A y) \cdot a = x \otimes_A (y \cdot a).$$

Given $T \in \mathcal{L}(X)$, the map defined by

$$(T \otimes \mathrm{id}_{\mathcal{L}(Y)})(x \otimes_A y) = Tx \otimes_A y$$

is an adjointable operator. As such, we can define a left action on $X \otimes_A Y$ by

$$a \mapsto \phi(a) \otimes \mathrm{id}_{\mathcal{L}(Y)}.$$

When equipped with these left and right actions, $X \otimes_A Y$ has the structure of a C^*-correspondence over A.

Let P be a countable semigroup with identity e, and let $p : X \to P$ be a collection of A-correspondences fibred over P. Denote $p^{-1}(s)$ by X_s, the C^*-correspondence over A with left action $\phi_s : A \to \mathcal{L}(X_s)$.

Definition 5.1.2 (Def. 2.1 of [1])**.** We say that $p : X \to P$ is a *(discrete) product system over* P if X is a semigroup, p is a semigroup homomorphism, X_s is an A-correspondence for each $s \in P \setminus \{e\}$, and the multiplication on X induces a collection of A-correspondence isomorphisms

$$\beta_{s,t} : X_s \otimes_A X_t \to X_{st}$$

such that $X_e = A$; $\beta_{e,s} : X_e \otimes_A X_s \to X_s$ and $\beta_{s,e} : X_s \otimes_A X_e \to X_s$ are induced by $a \otimes \xi \mapsto a \cdot \xi$ and $\xi \otimes a \mapsto \xi \cdot a$, respectively; and for all $r, s, t \in P$,

$$\beta_{rs,t} \circ (\beta_{r,s} \otimes \mathrm{id}_t) = \beta_{r,st} \circ (\mathrm{id}_r \otimes \beta_{s,t}).$$

Definition 5.1.3. Let $p : X \to P$ be a product system of A-correspondences, let B be a C^*-algebra, and let $\psi : X \to B$. Write ψ_s for the restriction of ψ to X_s. Then, ψ is a *Toeplitz representation* of X if

1. (ψ_s, ψ_e) is a Toeplitz representation of (X_s, A) on B for each $s \in P$,

2. $\psi_{st}(xy) = \psi_s(x)\psi_t(y)$ for $x \in X_s$ and $y \in X_t$.

If each (ψ_s, ψ_e) is also Cuntz-Pimsner covariant, then ψ is said to be a *Cuntz-Pimsner covariant Toeplitz representation.*

The Toeplitz algebra of $p : X \to P$ is the algebra $\mathcal{T}_X$ that is universal for Toeplitz representations of X. Similarly, the Cuntz-Pimsner algebra $\mathcal{O}_X$ is universal for Cuntz-Pimsner covariant Toeplitz representations. See Propositions 2.8 and 2.9 of [1] for more on these algebras.

5.2 The C^*-Correspondence of a Higher-Rank Graph

Let E and F be directed graphs over a vertex set V.

Definition 5.2.1. The fibred product of E and F is the directed graph $E * F$ with vertex set V, edge set

$$(E * F)^1 = \{(e, f) \in E^1 \times F^1 : r_F(f) = s_E(e)\},$$

and range and source maps given by

$$r(e, f) = r_E(e) \qquad \text{and} \qquad s(e, f) = s_F(f).$$

Definition 5.2.2. Let P be a countable semigroup with identity e. A *product system over P of graphs on V* is a collection $E = \{E_s : s \in P\}$ of directed graphs over V (where $E_e = (V, V, \mathrm{id}_V, \mathrm{id}_V)$) together with a collection $\alpha = \{\alpha_{s,t} : s, t \in S\}$ of vertex fixing graph isomorphisms $\alpha_{s,t} : E_s * E_t \to E_{st}$ satisfying *associativity* for all $r, s, t \in P$:

$$\alpha_{rs,t} \circ (\alpha_{r,s} * \mathrm{id}_t) = \alpha_{r,st} \circ (\mathrm{id}_r * \alpha_{s,t}).$$

See [2], Definition 1.1, for a more general formulation.

Let (Λ, d) be a k-graph. For $m \in \mathbb{N}^k$, $d^{-1}(m)$ is the edge set of an ordinary directed graph E_m over V where the range and source maps are inherited from Λ. Let X_m be the graph C^*-correspondence over $c_0(V)$ associated to E_m; (see example 2.4.1 for the definition of X_m). Let χ_e denote the characteristic function of $\{e\}$. Then, as noted in [4],

$$\{\chi_e \otimes \chi_f : (e, f) \in (E_m * E_n)^1\}$$

has dense span in $X_m \otimes_{c_0(V)} X_n$. Hence, there are isomorphisms $\beta_{m,n} : X_m \otimes_{c_0(V)} X_n \to X_{m+n}$ such that

$$\beta_{m,n}(\chi_e \otimes \chi_f) = \chi_{\alpha_{m,n}(e,f)} = \chi_{ef}$$

for $(e, f) \in (E_m * E_n)^1$.

Definition 5.2.3 (See Example 1.5 of [2])**.** The pair (X, β) where $X = \{X_m : m \in \mathbb{N}^k\}$ and $\beta = \{\beta_{m,n} : m, n \in \mathbb{N}^k\}$ is called the *k-graph correspondence* of (Λ, d).

Definition 5.2.4. Let (X, β) be a product system of A-correspondences fibred over $\mathbb{N}^k$. Set $Y_i := X_{e_i}$, and set

$$T_{i,j} := \beta^{-1}_{e_j,e_i} \circ \beta_{e_i,e_j} : Y_i \otimes_A Y_j \to Y_j \otimes_A Y_i.$$

Let $Y = \{Y_i : 1 \leq i \leq k\}$, and let $T = \{T_{i,j} : 1 \leq i < j \leq k\}$. Then, (Y, T) is called the *skeleton* of (X, β).

For $1 \leq i < j < l \leq k$, the *hexagonal equation* holds:

$$\begin{aligned}(T_{j,l} \otimes \mathrm{id}_i)(\mathrm{id}_j \otimes T_{i,l})(T_{i,j} \otimes \mathrm{id}_l)& \\ &= (\mathrm{id}_l \otimes T_{i,j})(T_{i,l} \otimes \mathrm{id}_j)(\mathrm{id}_i \otimes T_{j,l}).\end{aligned}$$

The product system (X, β) is uniquely determined up to isomorphism by the skeleton (Y, T); (see [2], Proposition 2.11, and [4], Section 2 for more on this).

5.3 k-Graph Correspondence Functor

In this section, we show that there is a functor from a category of k-graphs to a category of product systems. Since all C^*-correspondences are over a fixed C^*-algebra A, we can simplify the definitions of the needed morphisms as follows.

Definition 5.3.1. Let A be a C^*-algebra. Let X and Y be C^*-correspondences over A. An *A-correspondence morphism* is a linear map $\psi : X \to Y$ such that

i. $\psi(a \cdot \xi \cdot b) = a \cdot \psi(\xi) \cdot b$, and

ii. $\langle \psi(\xi), \psi(\eta) \rangle_A = \langle \xi, \eta \rangle_A$.

Definition 5.3.2. Let (X, α) and (Y, β) be A-correspondence product systems over a semigroup P. An *A-correspondence product morphism* is a collection $\psi = \{\psi_s : s \in P\}$ of A-correspondence morphisms fibred over P such that the diagrams

$$\begin{tikzcd}
X_m \otimes X_n \arrow[r, "\alpha_{m,n}"] \arrow[d, "\psi_m \otimes \psi_n"'] & X_{m+n} \arrow[d, "\psi_{m+n}"] \\
Y_m \otimes Y_n \arrow[r, "\beta_{m,n}"'] & Y_{m+n}
\end{tikzcd}$$

commute.

Let V be a countable set. We will work with the following categories:

i. $\mathbf{k}\mathcal{G}(\mathbf{V})$: The objects are k-graphs with vertex set V; the morphisms are injective, vertex-fixing k-graph morphisms.

ii. $\mathbf{p}\mathcal{C}_{\mathbf{0}}(\mathbf{V})$: The objects are $c_0(V)$-correspondence product systems over $\mathbb{N}^k$; the morphisms are $c_0(V)$-correspondence product morphisms.

Proposition 5.3.3. $\mathbf{k}\mathcal{G}(\mathbf{V})$ and $\mathbf{p}\mathcal{C}_{\mathbf{0}}(\mathbf{V})$ are categories.

Proof. It is known that k-graphs together with k-graph morphisms are a category, and since associativity, injectivity, and the vertex fixing property are all preserved for compositions of morphisms in $\mathbf{k}\mathcal{G}(\mathbf{V})$, it follows that $\mathbf{k}\mathcal{G}(\mathbf{V})$ is a category.

Let (X, α), (Y, β), and (Z, γ) be $c_0(V)$-correspondence product systems over $\mathbb{N}^k$, and let $\phi : X \to Y$ and $\psi : Y \to Z$ be $c_0(V)$-correspondence product morphisms. Then, for $m \in \mathbb{N}^k$,

$$(\psi \circ \phi)_m = \psi_m \circ \phi_m$$

is a $c_0(V)$-correspondence morphism from X_m to Z_m. Moreover, the diagrams

$$\begin{tikzcd} X_m \otimes X_n \arrow[r, "\alpha_{m,n}"] \arrow[d, "(\psi_m \otimes \psi_n)\circ(\phi_m \otimes \phi_n)"'] & X_{m+n} \arrow[d, "(\psi_{m+n})\circ(\phi_{m+n})"] \\ Z_m \otimes Z_n \arrow[r, "\gamma_{m,n}"'] & Z_{m+n} \end{tikzcd}$$

commute. Hence, the composition $\psi \circ \phi = \{\psi_m \circ \phi_m : m \in \mathbb{N}^k\}$ is a morphism in $\mathbf{p}\mathcal{C}_\mathbf{0}(\mathbf{V})$. Since $c_0(V)$-correspondence morphisms are associative, $\mathbf{p}\mathcal{C}_\mathbf{0}(\mathbf{V})$ is a category.

□

Theorem 5.3.4. The assignments

$$\Lambda \mapsto (X(\Lambda), \beta)$$

$$(\phi : \Lambda \to \Gamma) \mapsto (X(\phi) : X(\Lambda) \to X(\Gamma))$$

define a functor from $\mathbf{k}\mathcal{G}(\mathbf{V})$ to $\mathbf{p}\mathcal{C}_\mathbf{0}(\mathbf{V})$.

Proof. Let $\phi : \Lambda \to \Gamma$ be a morphism in $\mathbf{k}\mathcal{G}(\mathbf{V})$, and let $X(\Lambda)$ and $X(\Gamma)$ be the associated k-graph correspondences. For $m \in \mathbb{N}^k$, let $X(\phi)_m : X(\Lambda)_m \to X(\Gamma)_m$ be obtained by linearly extending the map defined on the indicator functions by

$$X(\phi)_m(\chi_e) = \chi_{\phi(e)}.$$

This gives us a family $X(\phi) = \{X(\phi)_m : m \in \mathbb{N}^k\}$ of C^*-correspondence morphisms. Moreover, these morphisms intertwine the transition maps since

$$\begin{aligned} X(\phi)_{m+n} \circ \alpha_{m,n}(\chi_e \otimes \chi_f) &= X(\phi)_{m+n}(\chi_{ef}) \\ &= \chi_{\phi(ef)} \\ &= \chi_{\phi(e)\phi(f)} \\ &= \beta_{m,n}(\chi_{\phi(e)} \otimes \chi_{\phi(f)}) \\ &= \beta_{m,n} \circ X(\phi)_m \otimes X(\phi)_n(\chi_e \otimes \chi_f). \end{aligned}$$

Thus, $X(\phi)$ is a morphism in $\mathbf{p}\mathcal{C}_\mathbf{0}(\mathbf{V})$.

Let $\phi : \Lambda \to \Gamma$ and $\sigma : \Gamma \to \Sigma$ be morphisms in $\mathbf{k}\mathcal{G}(\mathbf{V})$. Then, for any $m \in \mathbb{N}^k$ and any indicator function,

$$\begin{aligned} X(\sigma \circ \phi)_m(\chi_e) &= \chi_{(\sigma\circ\phi)(e)} \\ &= X(\sigma)_m(\chi_{\phi(e)}) \\ &= X(\sigma)_m \circ X(\phi)_m(\chi_e), \end{aligned}$$

and hence, by linearity,

$$X(\sigma \circ \phi)_m = X(\sigma)_m \circ X(\phi)_m = (X(\sigma) \circ X(\phi))_m.$$

Thus, $X(\sigma \circ \phi) = X(\sigma) \circ X(\phi)$. As such, there is a (covariant) functor from $\mathbf{k}\mathcal{G}(\mathbf{V})$ to $\mathbf{p}\mathcal{C}_\mathbf{0}(\mathbf{V})$. □

Theorem 5.3.6 proves that this functor is injective on objects and that it reflects isomorphisms. It uses the machinery of Hilbert systems, which were introduced in [4]. All definitions can be found therein.

Definition 5.3.5 (See section 4 of [4])**.**

1. A *Hilbert matrix over* V is a family $H = \{H_{uv}\}_{u,v\in V}$ of Hilbert spaces.

2. An *isomorphism* of Hilbert matrices H and K is a family $S = \{S_{uv}\}_{u,v\in V}$ of unitary operators $S_{uv} : H_{uv} \to K_{uv}$.

3. The *product* of Hilbert matrices H and K is the Hilbert matrix $H * K$ given by

$$(H * K)_{uv} = \bigoplus_{x\in V}(H_{ux} \otimes K_{xv}).$$

4. A *Hilbert system* is an $(k+1)$-tuple $(H^1, \ldots, H^n, S = \{S^{ij}\})$ where H^i is a Hilbert matrix (over V) and $S^{ij} : H^i * H^j \to H^j * H^i$ is an isomorphism.

5. An *isomorphism of Hilbert systems*,

$$\sigma : (H^1, \ldots, H^k, S) \to (F^1, \ldots, F^K, T),$$

is a collection of isomorphisms $\sigma = \{\sigma^{(i)}\}$, $\sigma^{(i)} : H^i \to F^i$, such that the diagrams

$$\begin{tikzcd} H^i * H^j \arrow[r, "S^{ij}"] \arrow[d, "\sigma^{(i)} * \sigma^{(j)}"'] & H^j * H^i \arrow[d, "\sigma^{(j)} * \sigma^{(i)}"] \\ F^i * F^j \arrow[r, "T^{ij}"'] & F^j * F^i \end{tikzcd}$$

commute, where

$$(\sigma^{(i)} * \sigma^{(j)})_{uv} = \bigoplus_{x \in V} (\sigma^{(i)}_{ux} * \sigma^{(j)}_{xv}).$$

The crucial fact needed for Theorem 5.3.6 is that there is a (categorical) equivalence between product systems and Hilbert systems. As such, a product system induces a Hilbert system (in the manner described below), and an isomorphism between product systems induces an isomorphism between the corresponding Hilbert systems.

Theorem 5.3.6. Let V be a countable set, and let Λ and Γ be objects in $\mathbf{k}\mathcal{G}(\mathbf{V})$.

i. If $X(\Lambda) = X(\Gamma)$ in $\mathbf{p}\mathcal{C}_{\mathbf{0}}(\mathbf{V})$, then $\Lambda = \Gamma$ in $\mathbf{k}\mathcal{G}(\mathbf{V})$.

ii. If $\phi : \Lambda \to \Gamma$ induces an isomorphism $X(\phi) : X(\Lambda) \to X(\Gamma)$ in $\mathbf{p}\mathcal{C}_{\mathbf{0}}(\mathbf{V})$, then ϕ is an isomorphism in $\mathbf{k}\mathcal{G}(\mathbf{V})$.

Proof. The proof of (i.) follows the case for ordinary directed graph correspondences, as proved in [3]. Suppose $\Lambda \neq \Gamma$ in $\mathbf{k}\mathcal{G}(\mathbf{V})$. If $\Lambda^* \neq \Gamma^*$, then there is χ_e such that either $\chi_e \in X(\Lambda) \setminus X(\Gamma)$ or $\chi_e \in X(\Gamma) \setminus X(\Lambda)$, and hence, $X(\Lambda) \neq X(\Gamma)$. Alternatively, if $\Lambda^* = \Gamma^*$, then since $\Lambda \neq \Gamma$, there is $e \in \Lambda^* = \Gamma^*$ such that either $r_\Lambda(e) \neq r_\Gamma(e)$ or $s_\Lambda(e) \neq s_\Gamma(e)$. Hence, setting $u = r_\Lambda(e)$ and $v = s_\Lambda(e)$, it follows that $p_u \cdot \chi_e \cdot p_v = \chi_e$ in $X(\Lambda)$, but $p_u \cdot \chi_e \cdot p_v = 0$ in $X(\Gamma)$, and thus $X(\Lambda) \neq X(\Gamma)$.

For (ii.), note that since ϕ is injective,

$$\phi(u\Lambda v) \subseteq u\Gamma v$$

for $u, v \in \Lambda^0$. Let $(H^1, \cdots, H^k, S = \{S^{ij}\})$ be the Hilbert system associated to $(X(\Lambda), \beta)$: $H^i = \{H^i_{uv}\}_{u,v \in V}$ is the Hilbert matrix given by

$$H^i_{uv} = p_u \cdot Y_i \cdot p_v$$

where $(Y_1, \cdots, Y_k)$ is the skeleton of $(X(\Lambda), \beta)$, and S is a family of Hilbert matrix isomorphisms $S^{ij} : H^i * H^j \to H^j * H^i$. That is, S^{ij}_{uv} is a unitary operator

$$\begin{aligned} S^{ij}_{uv} : (H^i * H^j)_{uv} = \bigoplus_{x \in V} (H^i_{ux} \otimes H^j_{xv}) \\ \longrightarrow \bigoplus_{x \in V} (H^j_{ux} \otimes H^i_{xv}) = (H^j * H^i)_{uv}. \end{aligned}$$

Let $(F^1, \ldots, F^k, T = \{T^{ij}\})$ be the Hilbert system associated to $(X(\Gamma), \alpha)$ where $(Z_1, \ldots, Z_k)$ is the skeleton of $(X(\Gamma), \alpha)$. Then, the sets

$$\Lambda^{e_j}_{uv} = \{\chi_e : e \in u\Lambda^{e_j} v\} \text{ and}$$
$$\Gamma^{e_j}_{uv} = \{\chi_f : f \in u\Gamma^{e_j} v\}$$

form orthonormal bases for the Hilbert spaces H^j_{uv} and F^j_{uv}, respectively. Note also that

$$X(\phi)(\Lambda^{e_j}_{uv}) = \{X(\phi)(\chi_e) : e \in u\Lambda^{e_j} v\} = \{\chi_{\phi(e)} : e \in u\Lambda^{e_j} v\} \subseteq \Gamma^{e_j}_{uv}.$$

Since $X(\phi)$ is an isomorphism, it induces a Hilbert system isomorphism

$$\sigma : (H^1, \ldots, H^k, S) \to (F^1, \ldots, F^K, T).$$

In particular, we get unitaries $\sigma^{(i)}_{uv}$ from H^i_{uv} to F^i_{uv}. Thus,

$$X(\phi)(\Lambda^{e_j}_{uv}) = \Gamma^{e_j}_{uv}.$$

As such,

$$X(\phi)(p_u \cdot X(\Lambda) \cdot p_v) = p_u \cdot X(\Gamma) \cdot p_v,$$

and this implies that $\phi(u\Lambda v) = u\Gamma v$. Since the morphisms are injective and vertex-fixing, it follows that ϕ is an isomorphism in $\mathbf{k}\mathcal{G}(\mathbf{V})$. $\square$

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