On Chains in the Tamari Lattice

by

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ABSTRACT

The Tamari lattice T_n was originally defined on bracketings of a set of $n+1$ objects, with a cover relation based on the associativity rule in one direction. Since then it has been studied in various areas of mathematics including cluster algebras, discrete geometry, algebraic combinatorics, and Catalan theory. Although in several related lattices the number of maximal chains is known, the enumeration of these chains in Tamari lattices is still an open problem.

This dissertation defines a partially-ordered set on equivalence classes of certain saturated chains of T_n called the Tamari Block poset, $\mathcal{TB}_\lambda$. It further proves $\mathcal{TB}_\lambda$ is a graded lattice. It then shows for $\lambda = (n-1, ..., 2, 1)$ $\mathcal{TB}_\lambda$ is anti-isomorphic to the Higher Stasheff-Tamari Orders in dimension 3 on $n+2$ elements. It also investigates enumeration questions involving $\mathcal{TB}_\lambda$, and proves other structural results along the way.

To Erica, Eben, and Abel

The Road goes ever on and on

Down from the door where it began.

Now far ahead the Road has gone,

And I must follow, if I can,

Pursuing it with eager feet,

Until it joins some larger way

Where many paths and errands meet.

And whither then? I cannot say.

J.R.R. Tolkien

ACKNOWLEDGMENTS

My love for math began in the seventh grade when my algebra teacher, Mr. Larry Littlefield, assigned challenging extra credit problems. We were allowed to work with others, so I brought them home to my dad and the two of us struggled through them together. It was not long before my dad was hurrying home from work to find out if more extra credit problems had been given. For me, this is the kernel of math's appeal. Over and over again it has brought me to the end of myself, led me into communion with others, and together we have wrestled the adversary.

I remember sitting in Mr. Mark Young's high school Calculus class and deciding that I wanted to be Mr. Young when I grew up. That was a testament to him as a person, as well as a recognition of my love for math and for teaching. That decision was reinforced as an undergrad when Dr. Ed Thurber and Dr. Walt Stangl took the time to mentor me in mathematics and teach me some of life's important lessons.

My experience at Arizona State University would not have been possible without my sponsorship from the United States Air Force Academy. I am so grateful to Dr. Brad Warner for mentoring me as a young instructor, and for challenging me to always focus my classroom on my students. I am most grateful to Colonel John Andrew for believing in me by selecting me for this opportunity. The views expressed in this dissertation are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

I have been particularly blessed at ASU to have professors on my dissertation committee who have not only pushed me beyond my capabilities in class, but taken the time to wrestle the course material with me outside of class. Chief among them is my advisor, Dr. Susanna Fishel, who took me under her wing despite the unique challenges associated with my plan of study, and has traveled with me along this often dark and sometimes barren, but ultimately fruitful, road.

In addition to Dr. Fishel and the other members of my committee, so much of my success at ASU is due to my fellow combatants, Roy Oursler and Luke Nelson. Roy and I made it through our qualifiers together, studying, encouraging each other, and teaching each other combinatorics. Luke and I collaborated together. I owe my dissertation topic to him. I am so grateful for both of their contributions to this accomplishment, as well as their friendship on the road.

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INTRODUCTION

Since Dov Tamari defined the Tamari lattices, T_n, over 50 years ago (Tamari 1962) they have proven to be fascinating in their own right, but they have also shown an uncanny ability to appear in a wide range of mathematics, often times where they are least expected. One example of the former property that will be evident in this dissertation is the diversity of ways the elements of T_n can be defined. Dyck paths, Young diagrams, scope sequences, forests, binary trees, triangulations of polygons, 312-avoiding permutations, and cluster algebras of a particular type are just a few of the objects that the Tamari order can be placed on. This dissertation will focus on many of these realizations. The difficulty of many enumeration problems surrounding T_n is another fascinating aspect of these lattices. In fact the genesis of much of this work was the open problem of enumerating the maximal chains of T_n.

The Tamari lattices also appear alongside some other important orders many of which I will discuss in this dissertation. It is a quotient lattice and sublattice of the *weak Bruhat order on* $\mathfrak{S}_n$ (see Bjorner and Wachs 1997; Reading 2006), a property I will touch on in Chapter 5. Related to this, T_n is also a *Cambrian lattice* (Reading 2006). The Tamari lattices are one of the initial cases of the *higher Stasheff-Tamari orders* (see Kapranov and Voevodsky 1991; Edelman and Reiner 1996 for more details), two important orders in Chapter 2 of this dissertation. Although there are other areas where T_n appears, the last one I will mention is the Hasse diagram of T_n is the 1-skeleton of the associahedron (Devadoss and Read 2001) which is an important geometric connection for much of this work.

In this dissertation I will first lay the foundation by providing the background and notation in the remainder of this chapter. Chapter 2 is a paper I coauthored with Mahir Bilen Can, Susanna Fishel, and Luke Nelson titled *On Faces of Associahedra and Maximal Chains in Tamari Lattices* in which we first consider maximal chains in T_n and from there we define a new order on these maximal chains called the *Tamari Block poset*, $\mathcal{TB}_\lambda$. The main results of the paper are Theorem 2.1 and Theorem 2.2 that show that $\mathcal{TB}_\lambda$ is a graded lattice and there is an order-reversing bijection between $\mathcal{TB}_\lambda$, for certain λ, and certain higher Stasheff-Tamari orders. In Chapter 3 I generalize our proof that $\mathcal{TB}_\lambda$ is a lattice in Section 2.5 into lattice tests for general posets. Next I focus on enumeration questions surrounding $\mathcal{TB}_\lambda$ in Chapter 4, and finally I highlight some future areas I want to explore in Chapter 5, including how $\mathcal{TB}_\lambda$ is connected to the higher Bruhat orders.

1.1 Integer Partitions and Young Tableaux

The following information can be found in Richard P. Stanley 2011 and Adin and Roichman 2015.

Let $[n] = \{1, 2, ..., n\}$ for an integer n where $[n] = \emptyset$ if $n \leq 0$, and let $[m, n] = \{m, m+1, ..., n\}$ for integers m and n with the understanding that $[m, n] = \emptyset$ if $m > n$.

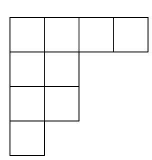
Define an *integer partition* of $n \in \mathbb{N}$ to be a sequence $\lambda = (\lambda_1, \lambda_2 ...)$ of integers λ_i satisfying $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq .... \geq 0$. I will also write $\lambda = (\lambda_1, ..., \lambda_r)$ if $\lambda_{r+1} = \lambda_{r+2} = ... = 0$. Because the integers of the sequence are weakly decreasing an integer partition is completely determined by the number of times each integer appears in the sequence, this means I can also represent a partition by $\lambda = 1^{k_1} 2^{k_2} ...$ where each k_i is the number of times i occurs in λ. So for example

$$(5, 3, 2, 2, 1, 1, 1, 0, ...) = (5, 3, 2, 2, 1, 1, 1, 0) = (5, 3, 2, 2, 1, 1, 1) = 1^3 2^2 3^1 5^1$$

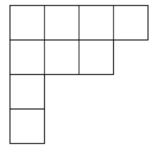
as partitions of 15. By the *null partition* I mean the integer partition $(0, 0, ...)$. If λ is a partition of n then I will write $|\lambda| = n$. The nonzero terms λ_i are called the *parts* of λ, and I will say λ has k parts where $k = \#\{i \mid \lambda_i > 0\}$. The number of parts of λ is also called the *length* of λ and denoted $\ell(\lambda)$. If λ is an integer partition with *distinct parts*, then $\lambda_1 > \lambda_2 > .... > 0$.

For two integer partitions $\lambda = (\lambda_1, \lambda_2 ...)$ and $\mu = (\mu_1, \mu_2, ...)$ I will say $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for every i. This puts a partial order on the integer partitions by containment (see Section 1.2 for more on partial orders).

For an integer partition λ its *Young diagram* is an array of left-justified boxes, where there are λ_i boxes in row i (where subsequent rows are added below). Throughout this dissertation I do not explicitly differentiate between an integer partition and its Young diagram, thus relying on the context to determine to which I am referring. The *conjugate partition*, denoted λ', is the integer partition whose Young diagram is obtained from that of λ by interchanging rows and columns. For example, for $\lambda = (4, 2, 2, 1)$, λ's Young diagram is

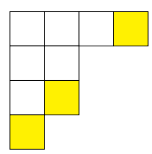


and $\lambda' = (4, 3, 1, 1)$ where



is λ''s associated Young diagram.

Although every integer partition has an associated Young diagram, not every diagram has an associated integer partition. I will call the diagrams that do have an associated partition Young diagrams (they are also called *ordinary* diagrams in the literature). They are the best known and most common diagrams. I will sometimes need to refer to a particular box of a diagram D, so let $b(i, j)$ of D be the box of D in the ith row from the top and in the jth column from the left. Of particular importance are the *corner boxes* of a diagram, by which I mean those boxes of a diagram such that no box is below them or to their right. So, to continue the previous example, the corner boxes of $\lambda = (4, 2, 2, 1)$ are $b(1, 4)$, $b(3, 2)$, and $b(4, 1)$, highlighted here:



A *tableau* of shape D is a filling of the diagram D by positive integers. When the labels of a tableau are strictly increasing along rows (respectively, columns) and weakly along columns (respectively, rows) then the tableau is called a *row-strict tableau* (respectively, *column-strict tableau*) (sometimes column-strict Young tableaux are called *semi-standard Young tableaux* in the literature). Furthermore, when the labels are the integers $[|\lambda|]$ and are both strictly increasing along rows and along columns then the tableau is called a *standard tableau.* In this dissertation I will deal most with row-strict Young tableaux, so when I simply say 'tableaux' I mean row-strict Young tableaux unless otherwise specified. The shape of a tableau T is the associated diagram, denoted $sh(T)$. Let $\mathrm{SYT}(\lambda) = \{T \text{ a standard Young tableau} \mid sh(T) = \lambda\}$. So

$$\begin{array}{|c|c|c|c|}\hline 1 & 5 & 3 & 2 \\ \hline 9 & 1 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 \\ \cline{1-2} 2 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

are all Young tableaux of shape $(4, 2, 2, 1)$. The left Young tableau is neither row-strict or standard, the middle Young tableau is row-strict, but not standard, and the right is a standard Young tableau.

There is a natural component-wise partial order that can be placed on the boxes of D where $b(i,j) \leq b(i',j')$ precisely when $i \leq i'$ and $j \leq j'$. With this in mind, a standard tableau of shape D can be considered a map $\phi: D \to [|D|]$ which is an order-preserving bijection, meaning ϕ satisfies

$$b(i,j) \neq b(i',j') \text{ implies } \phi(b(i,j)) \neq \phi(b(i',j')),$$

as well as

$$b(i,j) \leq b(i',j') \text{ implies } \phi(b(i,j)) \leq \phi(b(i',j')).$$

(for more on partial orders see Section 1.2). Figure 1 shows an example of this order-preserving bijection. This perspective of standard tableaux will be important in the last two sections of Chapter 4.

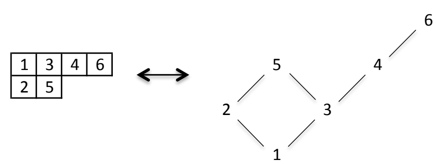


Figure 1: *An example of the map between standard Young tableaux and posets.*

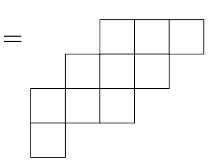
Enumerating the standard Young tableaux of shape λ is a classical problem in combinatorics. Define f^λ to be this number. In other words,

$$f^\lambda = |\mathrm{SYT}(\lambda)|.$$

In addition to Young diagrams there are other less common diagrams I will need to use. Suppose $\lambda = (\lambda_1, ..., \lambda_r)$ and $\mu = (\mu_1, ..., \mu_s)$ are integer partitions. If $\mu \subseteq \lambda$, then the *skew Young diagram*, denoted λ/μ, is the Young diagram λ with the boxes of μ removed starting in the top left corner, so

$$\lambda/\mu = \{b(i,j) \in \lambda \mid i \in [r], j \in [\mu_i + 1, \lambda_i]\}.$$

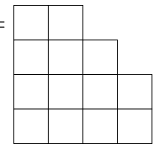
For example,

$(5,4,3,1)/(2,1) =$ 

Also, the *truncated Young diagram* $\lambda \setminus \mu$ is the diagram λ with the boxes of μ removed starting in the top right corner, so

$$\lambda \setminus \mu = \{b(i,j) \in \lambda \mid i \in [r], j \in [\lambda_i - \mu_i]\}.$$

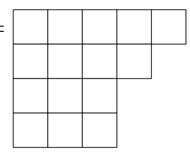
For example,

$$3^4 \setminus (2,1) =$$ 

If I no longer restrict μ to be contained in λ, but instead require $s \leq r$ the Young diagram $\lambda \cup \mu$ is the diagram associated to the integer partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, ..., \lambda_r + \mu_r)$. Finally, if λ has distinct parts, then the *shifted Young diagram of shape* λ is the array of boxes with λ_i boxes in row i such that row i begins in column i, in other words

$$\{b(i,j) \mid i \in [r], j \in [i, \lambda_i + i - 1]\}.$$

For example,

$$3^4 \cup (2,1) =$$ 

1.2 Partially-ordered Sets

The following information can be found in Richard P. Stanley 2011.

A *partially ordered set* P, or *poset*, is a set P together with a binary relation denoted $\leq$ (or $\leq_P$ when there is a possibility of confusion), satisfying the following three axioms:

1. For all $t \in P$, $t \leq t$ *(reflexivity)*.
2. If $s \leq t$ and $t \leq s$, then $s = t$ *(antisymmetry)*.
3. If $s \leq t$ and $t \leq u$, then $s \leq u$ *(transitivity)*.

I will use the obvious notation $t \geq s$ to mean $s \leq t$, $s < t$ to mean $s \leq t$ and $s \neq t$, and $t > s$ to mean $s < t$. Two elements $s, t \in P$ are *comparable* if $s \leq t$ or $t \leq s$; otherwise s and t are *incomparable*. If $s < t$ and $s \leq u \leq t$ only if $u = s$ or $u = t$, then s is covered by t which I will write $s \lessdot t$. If P has an element greater than (respectively, less than) or equal to every element of P, that element is denoted $\hat{1}$ (respectively, $\hat{0}$).

The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $s < t$ then t is drawn above s (with a higher vertical coordinate).

Example 1.1 Let $n \in \mathbb{N}$ and define the poset $\mathcal{B}_n$, called the *Boolean lattice on n elements*, to be the set of all subsets of $[n]$ such that $S \leq T$ in $\mathcal{B}_n$ if $S \subseteq T$ as sets. One says that $\mathcal{B}_n$ consists of the subsets of $[n]$ *ordered by inclusion.* The Hasse diagram of $\mathcal{B}_3$ is shown in Figure 2. Clearly $\hat{0} = \emptyset$ and $\hat{1} = \{1, 2, 3\}$.

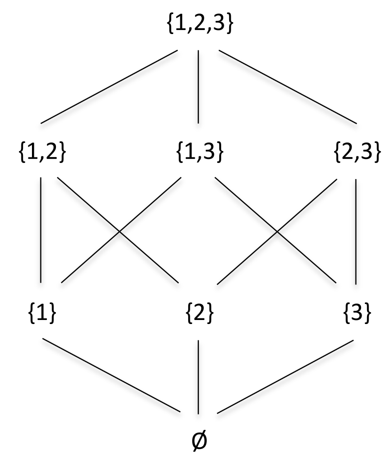


Figure 2: *The Hasse diagram of the Boolean lattice* $\mathcal{B}_3$.

A *chain* (or *totally ordered set* or *linearly ordered set*) is a poset in which any two elements are comparable. A chain $t_1 < t_2 < ... < t_k$ in P is *saturated* if and only if $t_i \lessdot t_{i+1}$ for every $i \in [k-1]$. A chain is *maximal* if it is not contained in a larger chain of P. Thus maximal chains are saturated, but not conversely. In this dissertation, a *top chain* is a saturated chain such that $t_k = \hat{1}$. Thus in Example 1.1 $\{1\} < \{1, 2, 3\}$ is a chain, $\{1\} \lessdot \{1, 3\}$ is a saturated chain, $\{1\} \lessdot \{1, 3\} \lessdot \{1, 2, 3\}$ is a top chain, and $\emptyset \lessdot \{1\} \lessdot \{1, 3\} \lessdot \{1, 2, 3\}$ is a maximal chain.

Two posets P and Q are *isomorphic*, denoted $P \cong Q$, if there exists an *order-preserving bijection* $\phi : P \to Q$ whose inverse is also order-preserving; that is,

$$s \leq t \text{ in } P \text{ if and only if } \phi(s) \leq \phi(t) \text{ in } Q.$$

By *subposet of* P, I mean a subset Q of the elements of P and a partial ordering of Q such that if $s \leq t$ in Q, then $s \leq t$ in P. By an *induced subposet of* P, I mean a subset Q of P and a partial ordering of Q such that for $s, t, \in Q$, $s \leq t$ in Q if and only if $s \leq t$ in P. A special type of subposet of P is the *interval* $[s, t] = \{u \in P \mid s \leq u \leq t\}$, defined whenever $s \leq t$. The interval $[s, s]$ consists of the single point s.

The *length* $\ell(C)$ of a finite chain is defined by $\ell(C) = \#C - 1$. The length of a finite poset P is

$$\ell(P) = \max\{\ell(C) \mid C \text{ is a chain of } P\}.$$

The length of an interval $[s, t]$ is denoted $\ell(s, t)$. If every maximal chain of P has the same length n, then we say that P is *graded of rank* n. In this case there is a unique rank function $\rho : P \to [0, n]$ such that $\rho(s) = 0$ if s is a minimal element of P, and $\rho(t) = \rho(s) + 1$ if $s \lessdot t$ in P. If $s \leq t$ then I will also write $\rho(s, t) = \rho(t) - \rho(s) = \ell(s, t)$. If $\rho(s) = i$, then s has rank i. If P is graded of rank n and has p_i elements of rank i, then the polynomial

$$F(P, x) = \sum_{i \in [0,n]} p_i x^i$$

is called the *rank-generating function of* P.

A *multichain* of the poset P is a chain with repeated elements; that is, a multiset whose underlying set is a chain of P. A multichain of length n may be regarded as a sequence $t_0 \leq t_1 \leq ... \leq t_n$ of elements of P.

If s and t belong to a poset P, then an *upper bound* of s and t is an element $u \in P$ satisfying $u \geq s$ and $u \geq t$. A *least upper bound* (or *join*) of s and t is an upper bound u of s and t such that every upper bound v of s and t satisfies $v \geq u$. If a least upper bound of s and t exists, then it is clearly unique and is denoted $s \vee t$. Dually the *greatest lower bound* (or *meet*) $s \wedge t$, when it exists. A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound.

1.3 The Symmetric Group

The following information can be found in Richard P. Stanley 2011 and Bjorner and Brenti 2006.

There are whole books written about specific aspects of the symmetric group so I have no hope of providing an exhaustive explanation of this hugely important mathematical object. Instead I want to focus on two combinatorial aspects of the symmetric group that are important for this dissertation. Even with this limited goal this introduction will not be exhaustive. The first part of this overview is an explanation of four ways to encode a permutation that will each be helpful in this dissertation. The second main idea is realizing the symmetric group as a Coxeter group and highlighting the weak Bruhat order of the symmetric group.

The *symmetric group* $\mathfrak{S}_n$ is the group of permutations on n elements. Before getting into the group itself I want to focus on its elements, the permutations. If S is an n-set, then a permutation σ of S can be defined as a linear ordering $\sigma_1, \sigma_2, ..., \sigma_n$ of the elements of S. Think of σ as a word $\sigma_1\sigma_2...\sigma_n$ in the alphabet S. If $S = \{x_1, ..., x_n\}$, then such a word corresponds to the bijection $\sigma : S \to S$ given by $\sigma(x_i) = \sigma_i$, so that a permutation of S may also be regarded as a bijection $S \to S$.

There are various ways to represent a permutation. In the previous paragraph I mentioned the most basic way to represent a permutation, as a word in the alphabet S. This is called the *single-line notation* of σ. When σ is written in single line notation it is clear that every element of S appears precisely one time in σ.

Now regard a permutation σ as a bijection $\sigma : S \to S$. It is natural to consider for each $x \in S$ the sequence $x, \sigma(x), \sigma^2(x), ...$ which must eventually return to x (since σ is a bijection and S is assumed to be finite). Thus for some unique $\ell \geq 1$ I have $\sigma^{\ell}(x) = x$ and the elements $x, \sigma(x), ..., w^{\ell-1}(x)$ are distinct. The sequence $(x, \sigma(x), ..., \sigma^{\ell-1}(x))$ is called a *cycle* of σ of length ℓ. The cycles $(x, \sigma(x), ..., \sigma^{\ell-1}(x))$ and $(\sigma^i(x), \sigma^{i+1}(x), ..., \sigma^{\ell-1}(x), x, ..., \sigma^{i-1}(x))$ are considered the same. Every element of S then appears in a unique cycle of σ, and so σ can be represented as a disjoint union or product of its distinct cycles $C_1, ..., C_k$, written $\sigma = C_1...C_k$. This is called the *cycle notation* of σ.

It is common to leave out the cycles of length one from a permutation's cycle notation. Cycles of length two are called *transpositions*, and the *simple transpositions* of $\mathfrak{S}_n$ are the permutations of the form $s_i = (i\ i+1)$ for $i \in [n-1]$ which I will discuss in greater detail later. It is a classical result that $\mathfrak{S}_n$ (as a group) is generated by its set of simple transpositions. In other words, every permutation can be represented as a product of simple transpositions. These representations are not unique.

A fourth representation of a permutation is by its set of inversions. A pair (σ_i, σ_j) is called an *inversion* of the permutation $\sigma = \sigma_1\sigma_2...\sigma_n$ if $i < j$ and $\sigma_i > \sigma_j$. Let $\mathrm{Inv}(\sigma) = \{(\sigma_i, \sigma_j) \text{ an inversion of } \sigma\}$. It is another classical result that the map from $\mathfrak{S}_n$ to $\{\mathrm{Inv}(\sigma) \mid \sigma \in \mathfrak{S}_n\}$ by $\sigma \mapsto \mathrm{Inv}(\sigma)$ is a bijection (Richard P. Stanley 2011, Proposition 1.3.12). Inversions of a permutation are also called *2-packets* for reasons that I make clear in Section 5.2.

Example 1.2 Let $S = [9]$ and consider the permutation $\sigma = 432761859$ in single-line notation. This means as a function $\sigma : [9] \to [9]$ by $\sigma(1) = 4$, $\sigma(2) = 3$, and so on. To view σ as a disjoint product of cycles just build the cycles. First, $\sigma(1) = 4$, $\sigma(4) = 7$, $\sigma(7) = 8$, $\sigma(8) = 5$, $\sigma(5) = 6$, and $\sigma(6) = 1$, so one cycle of σ is (147856). Next, $\sigma(2) = 3$ and $\sigma(3) = 2$, so the next cycle of σ is (23), the simple transposition s_2. Finally, $\sigma(9) = 9$, so (9) is the last cycle, and

$$\sigma = (147856)(23)(9) = (147856)(23)$$

in cycle notation. Notice that the cycle notation of a permutation is not unique. In this example $\sigma = (478561)(9)(32)$ as well. I leave it to the reader to confirm that

$$\sigma = s_1s_2s_3s_1s_2s_1s_5s_6s_7s_4s_5s_4 = s_3s_6s_5s_6s_1s_2s_7s_3s_4s_1s_5s_2$$

are two ways to represent σ as a product of simple transpositions.

To build $\mathrm{Inv}(\sigma)$ I will start with the inversions with 1. Every element to the left of 1 in single-line notation forms an inversion with 1, so $(1,4)$, $(1,3)$, $(1,2)$, $(1,7)$, and $(1,6)$ are the inversions with 1. Next, every element to the left of 2 except for 1 forms an inversion with 2, so $(2,4)$ and $(2,3)$ are also inversions of σ. I can continue in this way and get

$$\begin{aligned}\mathrm{Inv}(\sigma) = \{&(1,4),(1,3),(1,2),(1,7),(1,6),\\ &(2,4),(2,3),(3,4),(5,7),(5,6),(5,8),(6,7)\}.\end{aligned}$$

I now want to shift gears and consider $\mathfrak{S}_n$ as a Coxeter group. Let e be the identity element of a group. A *Coxeter group* W is defined by its set of generators $S = \{s_1, ..., s_k\}$ and the relations on the generators $(s_is_j)^{m(s_i,s_j)} = e$ such that for every $i, j \in [k]$ the following hold:

1. $m(s_i, s_j) \in \{1, 2, ..., \infty\}$,
2. $m(s_i, s_j) = m(s_j, s_i)$, and
3. $m(s_i, s_j) = 1$ if and only if $i = j$.

It is clear from this definition that the order of every generator is two, such elements are called the *reflections* of the group which explains why the elements of S are called the *simple reflections*. It is also clear that $m(s_i, s_j) = 2$ if and only if s_i and s_j commute in W meaning $s_is_j = s_js_i$.

That $\mathfrak{S}_n$ is the Coxeter group with generators $s_i = (i\ i+1)$ for $i \in [n-1]$ and relations

$$m(s_i, s_i) = \begin{cases} 1 & \text{if } j = i \\ 3 & \text{if } j = i+1 \\ 2 & \text{otherwise} \end{cases}$$

is proven in Bjorner and Brenti 2006, Proposition 1.5.4. These relations can be restated as the following:

1. $s^2 = e$ for every $s \in S$,
2. $s_i s_j = s_j s_i$ if $|j-i| \geq 2$, and
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The second relation is called the *commuting relation* and the third relation is called the *braid relation* of the symmetric group.

Coxeter groups are classified into different types and the symmetric groups are equivalent to the type A Coxeter groups, so in the context of Coxeter groups $\mathfrak{S}_n$ is typically referred to as A_{n-1} because there are $n-1$ generators. I will try to use this difference in titles to distinguish the context I am considering the symmetric group under at a particular time.

Every element $\sigma \in A_{n-1}$ can be written as a product of generators, and thus as a word in the alphabet of simple transpositions. There is not a unique word for σ. If $\sigma = s_{i_1}...s_{i_k}$ and k is minimal, then k is called the *length of* σ (written $\ell(\sigma) = k$) and the word $s_{i_1}...s_{i_k}$ is called a *reduced word* (or *reduced decomposition*) for σ. Bjorner and Brenti 2006, Proposition 1.5.2 shows that $\ell(\sigma) = |\text{Inv}(\sigma)|$ which brings together two important presentations of σ (compare this statement to the elements in the middle and right Hasse diagrams in Figure 21 as well as to Example 1.2 above). Because A_{n-1} is a finite Coxeter group there exists a *longest word* denoted w_0 of A_{n-1}. This is a unique element of A_{n-1} that has the maximum length of all elements of A_{n-1} ($\ell(w_0) = \binom{n}{2}$).

Two fundamental orders on a Coxeter group are the *strong Bruhat order* and the *weak Bruhat order* (sometimes just called the *Bruhat order* and the *weak order*). The strong Bruhat order is a *refinement* of the weak Bruhat order, meaning that $\sigma \leq \tau$ in the weak Bruhat order implies $\sigma \leq \tau$ in the strong Bruhat order. However I will only focus on the weak Bruhat order for my purposes here. Use $\leq_R$ as the relation symbol on the weak Bruhat order and define the weak Bruhat order to be the transitive closure of the cover relation $\sigma \leq_R \tau$ if and only if $\tau = \sigma s_i$ for some simple transposition s_i and $\ell(\tau) = \ell(\sigma) + 1$. This is technically the right weak Bruhat order and the left weak Bruhat order can be similarly defined, however the two orders are isomorphic to each other and I will only use the right, so I will simply call it the weak Bruhat order. Figure 21 and Figure 3 show the weak Bruhat order of A_2 and A_3 respectively.

As can be seen in Figure 21 and Figure 3, $\hat{0} = e$ and $\hat{1} = w_0$ in the weak Bruhat order. It is also easy to see the weak Bruhat order is graded with $\rho(\sigma) = \ell(\sigma)$. Another important property of the weak Bruhat order is that every cover relation can be uniquely associated to a simple transposition

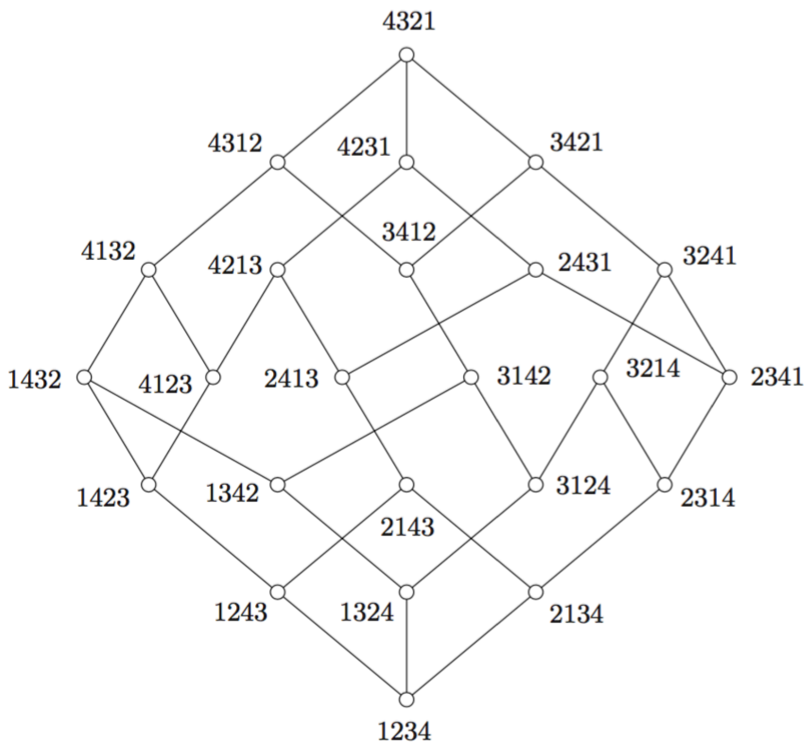


Figure 3: *The Hasse diagram of the weak Bruhat order of A_3 in single-line notation. This is Bjorner and Brenti 2006, Figure 3.2.*

by the map $\sigma \lessdot_L \tau \mapsto s_i$ if and only if $\tau = \sigma s_i$. This implies that maximal length chains in the weak Bruhat order can be uniquely viewed as reduced decompositions of w_0. This is important in Section 5.2.

1.4 The Tamari Lattice

In 1962 Tamari defined an order on bracketings of a set of $n+1$ objects with a cover relation based on the associativity rule in one direction (see Tamari 1962). This order eventually became known as the Tamari lattice T_n. A few years later T_n was shown to be a lattice in Friedman and Tamari 1967; Huang and Tamari 1972.

The number of elements of T_n is the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers also have a rich and fascinating history in combinatorics. Richard P Stanley 2015 describes this history and shows well over 200 mathematical objects that can be enumerated with C_n. In addition to such a nice formula, the Catalan numbers also satisfy the recurrence relation

$$C_{n+1} = \sum_{i\in[0,n]} C_i C_{n-i}$$

with $C_0 = 1$, and have the generating function

$$\frac{1-\sqrt{1-4x}}{2x}.$$

The ubiquitous nature of the Catalan numbers explains why T_n also appears in so many different places and can be put on so many different objects, but the Tamari lattices are not the only orders enumerated by the Catalan numbers. The *Stanley lattices* and *Kreweras lattices* are also enumerated by C_n (hence all three families of lattices are called *Catalan lattices*). The Kreweras lattices are a refinement of the Tamari lattices which are a refinement of the Stanley lattices. In order to better see how these lattices relate to each other I need to pick a set of objects to order. Just as is done in Bernardi and Bonichon 2009 I will use the set of *Dyck paths* of length $2n$ as my objects (what follows concerning these Catalan lattices can be found in Bernardi and Bonichon 2009).

A Dyck path of length $2n$ is a lattice path in the x, y-plane made of $(1,+1)$-steps and $(1,-1)$-steps that starts at $(0,0)$, never goes below the x-axis, and ends at $(2n, 0)$ (see Figure 4). To realize the Stanley lattice, order the set of Dyck paths of length $2n$ by the relation $P \leq_S Q$ if P never goes above Q. The Hasse diagram of the Stanley lattice for $n = 3$ is shown in Figure 5(a).

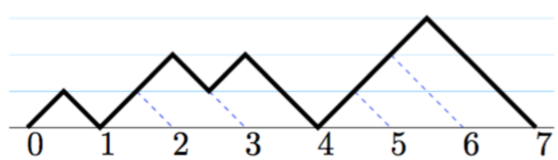


Figure 4: *A Dyck path for $n = 7$. This is Bernardi and Bonichon 2009, Figure 1(a).*

A set of consecutive steps of a Dyck path are a *subpath* if they form a Dyck path when they are shifted down and to the left to begin at $(0,0)$. Furthermore, a Dyck path that stays strictly positive (excluding its endpoints) is called a *prime Dyck path*. Now consider a prime subpath Q of a Dyck path P that begins at the point (x_0, y_0). To be a subpath Q must begin with a $(1,1)$-step, and since it is prime Q must end the first time its y-value equals y_0 after its initial step. This means every prime subpath can be uniquely determined by its first $(1,1)$-step, and since a Dyck path of length $2n$ has n such steps, there are n prime subpaths of each Dyck path of length $2n$. This also means if there is a $(1,-1)$-step followed by a (1,1)-step I can interchange the $(1,-1)$-step and the prime subpath associated with the $(1,1)$-step by following the prime subpath first and then making the $(1,-1)$-step afterwards.

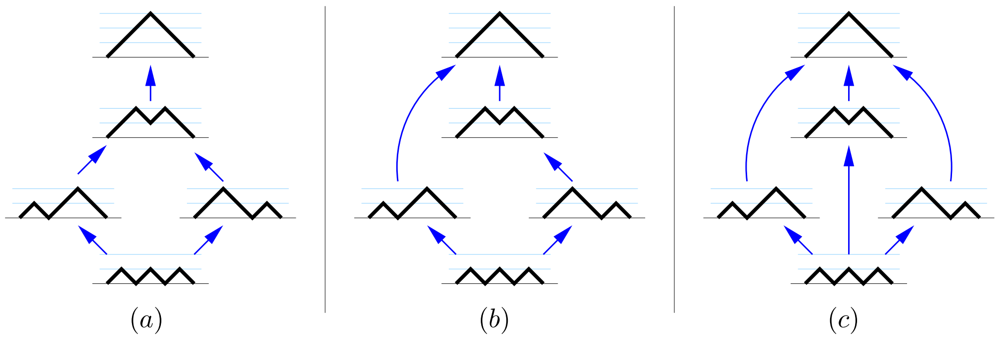


Figure 5: *The Hasse diagrams of* (a) *the Stanley lattice,* (b) *the Tamari lattice, and* (c) *the Kreweras lattice for* $n = 3$. *This is Bernardi and Bonichon 2009, Figure 2.*

I can now realize the Tamari lattice by defining its cover relations in these terms. Let P and P' be two Dyck paths. Then $P \leq_T P'$ if and only if there exists a $(1,-1)$-step denoted e in P, followed by a $(1,1)$-step, such that P' is obtained from P by interchanging e and the prime subpath following e. The Hasse diagram of the Tamari lattice for $n = 3$ is shown in Figure 5(b).

By using N to represent a $(1,1)$-step in a Dyck path and S as a $(1,-1)$-step, I can represent a Dyck path uniquely as a word on $\{N, S\}$. Furthermore choosing the location of the N's is my only real choice, so let $P = NS^{\alpha_1}NS^{\alpha_2}N...S^{\alpha_{n-1}}NS^{\alpha_n}$ be a Dyck path where each S^{α_i}, called the ith *descent*, is a possibly empty sequence of S's. I can realize the Kreweras lattice by defining its cover relations in these terms. Let P and P' be two Dyck paths. Then $P \leq_K P'$ if and only if P' is obtained from P by interchanging a non-empty descent with a subpath following it. The Hasse diagram of the Kreweras lattice for $n = 3$ is shown in Figure 5(c).

There is a bijection from the set of Dyck paths of length $2n$ to Young diagrams that are contained in $\delta_n = (n-1, ..., 2, 1)$. This bijection is easily seen by rotating a Dyck path 45° counterclockwise so that the first step from $(0,0)$ is $(0,1)$ instead of $(1,1)$, and then examining the shape between the path and the y-axis. In light of this bijection, the vertices of any of the Catalan lattices, and in particular T_n, can be viewed as Young diagrams that are contained in δ_n (compare Figure 5(b) to Figure 6). Under this encoding $\hat{0} = \delta_n$ and $\hat{1} = \emptyset$ (the empty Young diagram).

I want to convert the cover relations for T_n into this context. With Dyck paths as elements a cover relation was determined by a $(1,-1)$-step followed by a $(1,1)$-step. These two steps become a corner box of the Young diagram when the Dyck path is rotated 45°. Furthermore, to go from a covered Dyck path to the covering Dyck path I interchanged the $(1,-1)$-step with the prime subpath that followed. If I designate every box in the Young diagram that has a right edge that is a part of the prime subpath, then this interchange corresponds to removing these boxes (see Figure 7 for examples of this cover relation).

Chapter 2

ON FACES OF ASSOCIAHEDRA AND MAXIMAL CHAINS IN TAMARI LATTICES

This chapter is a paper I coauthored with Mahir Bilen Can, Susanna Fishel, and Luke Nelson. I have made minor formatting changes so that it conforms to the overall dissertation, but I have left it in the first person plural to emphasize the fact that it is a collaboration. I have also removed some of the basic definitions and introduction to avoid unnecessary repetition within the dissertation. All of my collaborators have agreed to have this paper included in my dissertation.

2.1 Introduction

In this paper, we generalize maximal chains (viewed as tableaux) to saturated chains that end at the top element of T_n, called *top chains*, and then partition the set of top chains of a certain shape, say λ. We then put a partial order on the equivalence classes of these top chains, which we name the Tamari Block poset for λ, $\mathcal{TB}_\lambda$. Stated differently in geometric terms, here we are investigating an order on the equivalence classes of walks on the associahedron (Stasheff-Tamari polytope); the equivalence relation is generated by the relations $P \sim Q$, where P and Q are two walks on the associahedron which agree at all vertices except two.

Our first main result, whose proof is built step-by-step in Sections 2.3 and 2.5, shows some important structure on $\mathcal{TB}_\lambda$.

Theorem 2.1 Let λ be a Young diagram,

1. $\mathcal{TB}_\lambda$ has a unique top and bottom element.
2. $\mathcal{TB}_\lambda$ is graded with rank function $\rho(\overline{S_i}) = \ell(\overline{S_i}) - \ell(\hat{0})$ (where $\overline{S_i}$ is an equivalence class of top chains, $\ell(\overline{S_i})$ is the length of any chain in the class, and $\ell(\hat{0})$ is the length of any chain in the bottom class).
3. $\mathcal{TB}_\lambda$ is a lattice.

Although working with general top chains in $\mathcal{TB}_\lambda$ simplifies many of our arguments, we pay special attention to the maximal case, $\lambda = \delta_n$. To ease our notation, we refer to $\mathcal{TB}_{\delta_n}$ as $\mathcal{TB}_n$. We are going to use similar notation for other objects in which we wish to set apart the maximal chain case. Recall that the Tamari lattices are the $d = 2$ case of the higher Stasheff-Tamari order $\mathrm{HST}_1(n, d)$. Rambau 1997 partitions maximal chains in $\mathrm{HST}_1(n, d)$ and shows the equivalence classes are in bijection with elements of $\mathrm{HST}_1(n, d+1)$. Furthermore, $\mathrm{HST}_1(n, 3)$ is known to be a graded lattice (see Cor 1.2 and Edelman and Reiner 1996, Thm 4.9 and Thm 4.10). In the light of our second main result, which we prove in Section 2.6, our Theorem 2.1 becomes a generalization of the results of Edelman-Reiner and Rambau from maximal chains to top chains:

Theorem 2.2 There exists an order-reversing bijection between $\mathcal{TB}_n$ and $\mathrm{HST}_1(n + 2, 3)$.

We should mention that, even though much of what we do here with respect to maximal chains is already known under the auspices of triangulations of the 3-dimensional cyclic polytope, our perspective and methodology are very different. First we are able to make the order on the equivalence classes of maximal chains more explicit than Rambau 1997 does. For another example, the proof of Theorem 2.1, relies on a fruitful new concept that we call the "r-stat" of a top chain. This new combinatorial encoding of a top chain seems to hold keys to understanding other structural properties related to our generalized lattices. There are still many open problems concerning the higher Stasheff-Tamari orders, and so, it is our hope that this new perspective will not only shed light on maximal chains of the Tamari lattice, but that it will also do so for the higher Stasheff-Tamari orders.

We conclude our introduction by posing an open problem that we plan to tackle in a future work: *is there a generalization of our Tamari Block poset for multi-associahedra?* (If there is one, the generalized Tamari Block posets can be used for further investigation of the subword complexes in the cases that are considered in Bergeron, Ceballos, and Labbé 2015.)

We organized our paper as follows: In Section 2.2 we give background information on T_n and explicitly define $\mathcal{TB}_\lambda$. In Section 2.3 we develop a statistic on top chains, called the diagonal sentence, and use it to prove some important properties of $\mathcal{TB}_\lambda$ including properties (1) and (2) of Theorem 2.1. The diagonal sentence is not constant on elements of $\mathcal{TB}_\lambda$ (which we often call blocks), so in Section 2.4 we define the r-stat of a top chain that characterizes the blocks and leads to our proof of Theorem 2.1(3) in Section 2.5. Finally, in Section 2.6 we provide background information on higher Stasheff-Tamari orders and prove Theorem 2.2 by showing an explicit order-reversing bijection between $\mathcal{TB}_n$ and $\mathrm{HST}_1(n + 2, 3)$.

As we have already mentioned, the number of vertices of the Tamari lattice, T_n, is the Catalan number C_n, which counts many interesting mathematical objects called Catalan objects (Richard P Stanley 2015 lists 214 Catalan objects). This has led mathematicians to encode T_n in many different ways. Huang and Tamari 1972 describes T_n as a poset of n-tuples $a_1, ..., a_n$, that have special properties where the ordering is by coordinate. Knuth describes T_n as the set of forests on n nodes (Knuth 2013). As we mentioned in the introduction, T_n can also be represented as triangulations of an $(n+2)$-gon with covering relations involving edge flips, as is done in Rambau and Reiner 2012. A very natural encoding of T_n views the vertices of T_n as binary trees, and the covering relations correspond to right rotations (see Bernardi and Bonichon 2009 for example).

Bernardi and Bonichon 2009 represents T_n as the set of Dyck paths of length $2n$, and expresses the covering relations in the Tamari lattice in these terms. Since this is the encoding of T_n we use the most, next, we are going to define T_n in these terms. The following proposition gives the covering relation of T_n in terms of Dyck paths.

Proposition 2.3 (Bernardi and Bonichon 2009, Proposition 2.1) Let P and P' be two Dyck paths. Then $P \lessdot P'$ if and only if there exists a $(1,-1)$-step e in P, followed by a $(1,1)$-step, such that P' is obtained from P by swapping e and the prime Dyck subpath following it.

There is a bijection from the set of Dyck paths of length $2n$ to Young diagrams that fit into δ_n. This bijection is easily seen by rotating a Dyck path 45° counterclockwise so that the first step from $(0,0)$ is $(0,1)$ and examining the shape between the path and the y-axis. In light of this bijection, the vertices of T_n can be viewed as Young diagrams that fit into δ_n. It is clear that $\hat{0} = \delta_n$ and $\hat{1} = \emptyset$ (the empty Young diagram). It is also clear that each covering relation can be associated with a corner box in the lesser diagram that is removed (along with all the boxes associated to its prime subpath) to get the greater diagram. If B is such a corner box then we call the boxes associated to its prime subpath the *B-strip*. For more details on this particular encoding of T_n, see Nelson 2015, Section 2. Figure 6 shows T_3 and Figure 7 has examples of the cover relations using this encoding.

Fishel and Nelson 2014 defines an injective map from the set of maximal chains of T_n to the set

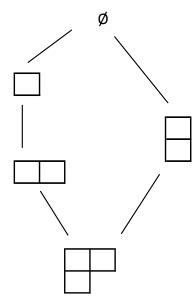


Figure 6: T_3 *with Young diagrams as vertices.*

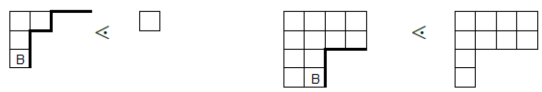


Figure 7: *Examples of Tamari cover relations with Young diagrams as vertices.*

of tableaux of shape δ_n. Nelson 2015 generalizes this map to send top chains to tableaux of general shape.

Definition 2.4 (Nelson 2015, Definition 3.1) Let $C = \{\hat{1} = \lambda_0 \gtrdot \lambda_1 \gtrdot ... \gtrdot \lambda_\ell\}$ be a top chain. As the chain is traversed from λ_ℓ to λ_0, boxes are removed. Fill the boxes removed in moving from λ_i to λ_{i-1} with i. The resulting tableau is the associated tableau of C.

If C is maximal then this definition is equivalent to the map in Fishel and Nelson 2014. However, if C is not maximal, then the resulting tableau does not have the shape δ_n, although it fits into δ_n. See Figure 8 for examples of top chains encoded as tableaux. Although this map is injective, it is clearly not surjective, so we refer to tableaux in the range of this map as *Tamari tableaux* (these tableaux are called *ψ-tableaux* in Nelson 2015). Also, just as we do in this definition, throughout the paper we will refer to moving up in the Tamari lattice as removing boxes in the associated Tamari tableau.

Before defining the Tamari Block lattices we need to discuss the associahedron, K_{n+1}, whose 1-skeleton is the Hasse diagram of T_n. The following summary of K_{n+1} is derived from Devadoss and Read 2001. K_{n+1} is a convex polytope of dimension $n-1$. Furthermore, every k-codimensional face corresponds to a set of non-intersecting diagonals on an $(n+2)$-gon of size k. As such, K_{n+1} has C_n vertices, and, when viewed as a graph, its 1-skeleton is regular with each vertex having degree $n-1$. Furthermore, a straightforward application of Theorem 2.3.2 proves all 2-dimensional faces of K_{n+1}

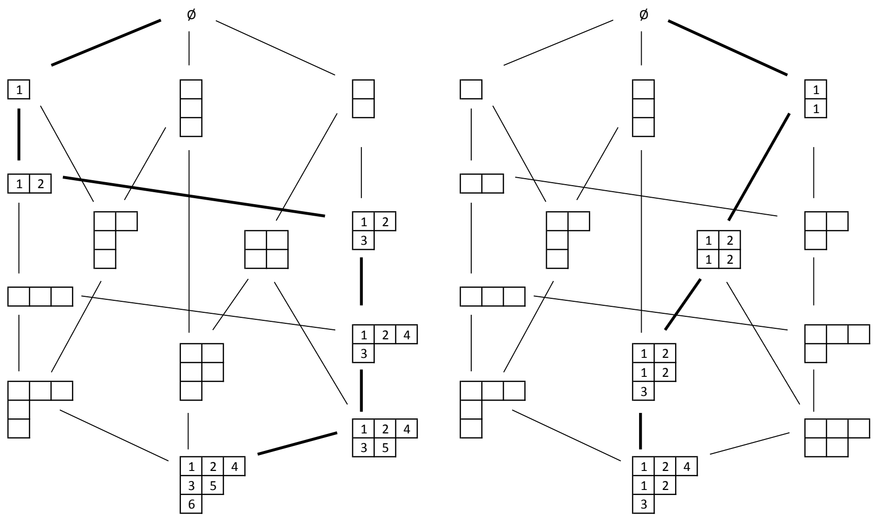


Figure 8: *Examples of maximal chains encoded as tableaux in T_4. For an example of a top chain that is not maximal, simply begin at a Young diagram that is not δ_4.*

are either squares or pentagons, and a similar application of Devadoss and Read 2001, Theorem 2.3.3 proves the number of pentagons is $\binom{2n-2}{n-3}$ and the number of squares is $\frac{n-3}{2}\binom{2n-2}{n-3}$.

Since the Hasse diagram of T_n is the 1-skeleton of K_{n+1} these results on the associahedron have the obvious implications for T_n. For our purposes, the most important implication is the fact that the 2-dimensional faces are either squares or pentagons. In the following lemma we state this explicitly and take it a step further by determining when the face is a square or pentagon.

Lemma 2.5

1. Suppose μ and ν are two distinct vertices that cover a vertex λ in the Tamari lattice. The interval $[\lambda, \mu \vee \nu]$ is a square or pentagon.
2. Specifically, suppose B_1 and B_2 are two distinct corner boxes in a Young diagram λ, where B_1 is of greater row index than B_2. Let μ and ν be the two distinct vertices corresponding to B_1 and B_2 that cover λ. If the prime path of B_1 ends at the point where the prime path of B_2 begins, then $[\lambda, \mu \vee \nu]$ is a pentagon; otherwise, the prime paths of B_1 and B_2 are disjoint or one is a subpath of the other, and $[\lambda, \mu \vee \nu]$ is a square.

Proof. According to Nelson 2015, Proposition 2.3 there are three cases to consider for part (2).

Case 1. Suppose the prime path of B_1 ends at the point where the prime path of B_2 begins. This means if we remove the B_2-strip first, all the boxes immediately to the left of these boxes will be a part of the B_1-strip and so will be removed when we remove the B_1-strip (see the right-hand path in Figure 9), the resulting diagram is $\mu \vee \nu$. However, if we remove the B_1-strip first, we have not affected the B_2-strip, so we need to remove the B_2-strip next and then remove the boxes associated with the box immediately to the left of B_2 to get to $\mu \vee \nu$.

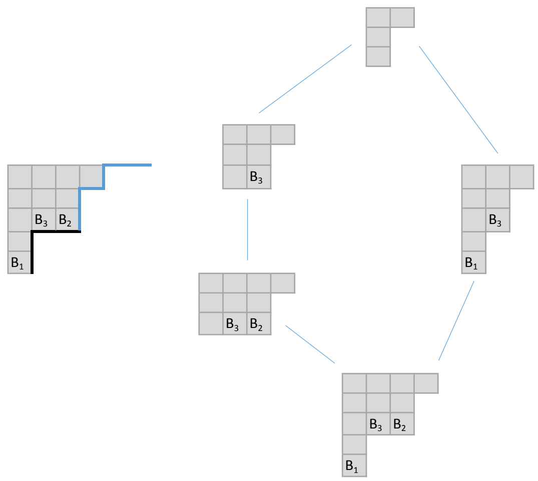


Figure 9: *Case 1 results in a pentagon.*

Case 2. Now suppose the prime paths of B_1 and B_2 are disjoint. This means removing the B_1-strip has no affect on the B_2-strip and vice versa. So remove one of the B_i-strips and then remove the other to get to $\mu \vee \nu$ (see Figure 10).

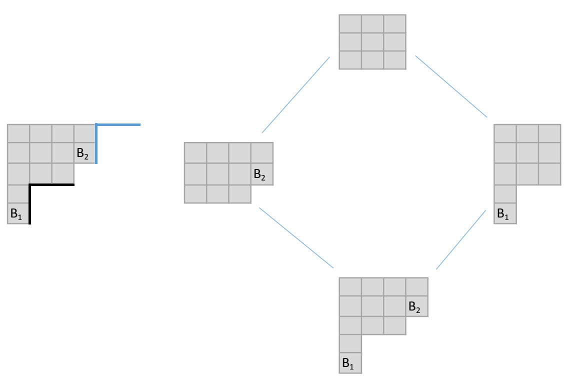


Figure 10: *Case 2 results in a square.*

Case 3. The final case is the prime path of B_1 contains the prime path of B_2. In this case, removing the B_2-strip first exposes the boxes immediately to the left of these boxes for the B_1-strip, so $\mu \vee \nu$ is the diagram with the B_2-strip removed and the B_1-strip removed which includes the boxes immediately to the left of the B_2-strip. However, unlike Case 1, if we remove the B_1-strip first, we remove the B_2-strip as well, so the only move left to make is to remove the boxes immediately to the left of the B_2-strip which are all associated with the box immediately to the left of B_2 (see Figure 11).

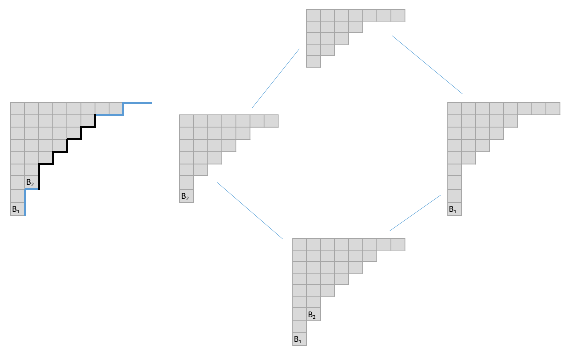


Figure 11: *Case 3 results in a square.*

$\square$

Since the squares and pentagons are intervals in T_n, we identify them by the interval end points. Furthermore, for each top chain in T_n we can associate the 2-dimensional faces that have both of their end points contained in that top chain. This leads us to the following definitions:

Definition 2.6 Let λ be a Young diagram.

1. Define the set $\mathcal{M}_\lambda = \{C \mid C \text{ a Tamari tableau of shape } \lambda\}$.
2. Let S (respectively, P) be a square (respectively, pentagon) contained in top chains C_1 and C_2 and suppose that $C_1 = C_2$ except on S (respectively, P) where they differ. A square move (respectively, pentagon move) on C_1 is the map that sends C_1 to C_2.
3. The Tamari Top Chain Graph, $TG_\lambda = (V, E)$, is a simple graph with $V = \mathcal{M}_\lambda$ and $C_1C_2 \in E$ if and only if C_2 can be obtained from C_1 by making a square or pentagon move.

Again, because we are particularly interested in the maximal chain case, and to ease notation, we refer to $\mathcal{M}_{\delta_n}$ and TG_{δ_n} as $\mathcal{M}_n$ and TG_n. See Figure 12 for an example of the top chain graph for $n = 4$.

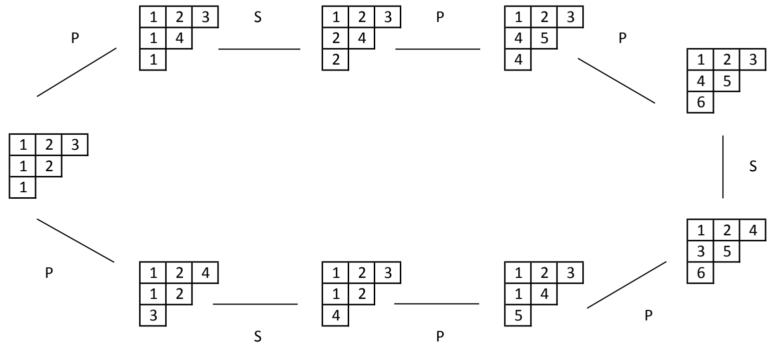


Figure 12: TG_4.

It is clear from these definitions that square moves do not change the length of a top chain, and similarly pentagon moves increase or decrease the length of a top chain by one. In light of these definitions, there is an equivalence relation on $\mathcal{M}_\lambda$ by $C_1 \sim C_2$ if and only if C_1 can be obtained from C_2 by making a set of square moves. We call the equivalence classes *square blocks.* Since every element of a square block has the same length, let $\ell(\overline{S})$ be the length of any of the chains in square block $\overline{S}$. This leads to a well-defined new poset.

Definition 2.7 The square blocks of $\mathcal{M}_\lambda$ form a poset which we call the Tamari Block Poset, $\mathcal{TB}_\lambda$, with $\overline{S_1} \lessdot \overline{S_2}$ if and only if there is a $C_1 \in \overline{S_1}$ and a $C_2 \in \overline{S_2}$ such that C_2 can be obtained from C_1 by making a pentagon move and $\ell(C_1) = \ell(C_2) - 1$. As before, use $\mathcal{TB}_n$ for $\mathcal{TB}_{\delta_n}$.

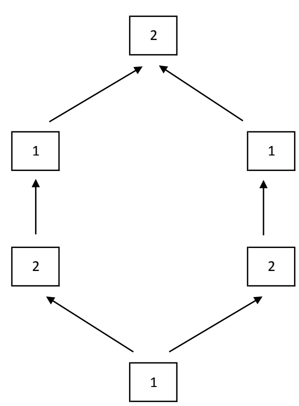


Figure 13: $\mathcal{TB}_4$ *with the sizes of each square block.*

2.3 The Diagonal Sentence of a Top Chain

Our goal in this section is to prove (1) and (2) of Theorem 2.1. To do this we develop a statistic on top chains called the diagonal sentence. But first, when we refer to a diagonal of a Young diagram we mean a diagonal that goes from the southwest to the northeast of the diagram. Furthermore, if $\lambda \neq \emptyset$ is a Young diagram and m is the maximum element of $\{i + j \mid (i, j) \in \lambda\}$, then we call the set of boxes $\{(i, j) \in \lambda \mid i + j = m\}$ the *outer diagonal* of λ. Finally, Nelson 2015 defines an *r-set* of a top chain to be the set of boxes of the tableau of the chain labeled r. We say an r-set *begins* (respectively *ends*) in row i when i is the lowest (respectively highest) row index such that r is a label of a box in row i, and we call the box where an r-set ends the *end box* of the r-set.

Definition 2.8 Consider the tableau of a top chain C of a Tamari lattice. We define an ordered sequence of sequences, called the diagonal sentence of C and denoted S_C, such that each sequence is the contents of a diagonal of C starting with the outer diagonal and such that each element of $[\ell(M)]$ only appears once (it appears in the sequence of the rightmost diagonal that the element is in). Each sequence of S_C is called a word of S_C.

Example 2.9 If

$$C_1 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 & 7 & 9 \\ \cline{1-5} 1 & 2 & 4 & 7 \\ \cline{1-4} 8 & 10 & 11 \\ \cline{1-3} 8 & 10 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \text{ and } C_2 = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 2 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

then $S_{C_1} = (8, 10, 11, 7, 9, 6)(4, 5)(2)(1, 3)()()$ and $S_{C_2} = (5)(2, 3, 4)()(1)()$.

Remark 2.10 It is not hard to show that the map from top chains to diagonal sentences is injective. It is also worth noting that diagonal sentences put a total order on the chains of each square block by using the lexicographic order of the diagonal sentences.

One of the key aspects of our methodology is to view square moves and pentagon moves as an "interchange" of consecutive labels in a Tamari tableau. What this means can be seen from the different cases in Lemma 2.5. For example, suppose a Tamari tableau has two corner boxes with consecutive labels and the prime paths of the boxes are disjoint (meaning they have no overlapping vertices or edges). This means we can interchange the labels of all the boxes with these labels to get a different Tamari tableau and this interchanging amounts to a square move (remove boxes labeled with 10 and 11 in C_1 in Example 2.9 and then interchange 8 and 9 for an example of this type of interchange). Interchanging labels does not always mean interchanging all the labels, though. If the corner boxes of consecutive labels have overlapping prime paths, then the interchange is a square move again, but the boxes within the overlap keep their labels and only the boxes outside the overlap are interchanged (interchanging 2 and 3 in C_1 in Example 2.9 is an example of this type).

It is important to note that square and pentagon moves can only happen through this interchanging of consecutive labels, which we denote by $b \leftrightarrow b+1$. However, in general, when we consider interchanging two consecutive labels of a top chain there are four possible outcomes: a square move, a pentagon move that interchanges a longer chain for a shorter chain (called a *decreasing pentagon move*), a pentagon move that interchanges a shorter chain for a longer chain (called an *increasing pentagon move*), or a non-move. There are two types of non-moves. First, if b and $b+1$ are two labels in a Tamari tableau such that b and $b+1$ appear in precisely the same rows, then $b \leftrightarrow b+1$ would not be a Tamari tableau, since interchanging any of the labels would cause the associated row to not be strictly increasing. In this case we say b is *trapped* by $b+1$. For example, 1 is trapped by 2 in C_1 in Example 2.9, and 4 is trapped by 5 in C_2. The other non-move occurs when b and $b+1$ are in the same word and $b+1$'s prime path ends at the same point that b's prime path begins. In this case the end box of the b-set cannot be removed before the boxes labeled $b+1$, so $b \leftrightarrow b+1$ cannot happen (consider $7 \leftrightarrow 8$ in C_1 in Example 2.9 to see this type). The next lemma characterizes the four possibilities under one restriction.

Lemma 2.11 Let C be a top chain of a Tamari lattice, let $b \in [\ell(C)]$, and suppose b is in word W_b and $b+1$ is in word W_{b+1} of S_C. Let W_* be W_b if $W_b = W_{b+1}$ or W_b is to the left of W_{b+1}, and let

W_* be W_{b+1} otherwise. If every element of the words to the left of W_* are greater than $b+1$, then

$$b \leftrightarrow b+1 \begin{cases} \text{increasing pentagon move} & \text{if } ..., b, Y, (b+1), ... \subseteq W_b, \text{and } y > b \text{ for} \\ & \quad \text{every } y \in Y. \\ \text{decreasing pentagon move} & \text{if } ..., (b+1), Y, b, ... \subseteq W_b,\ y > b \text{ for every} \\ & \quad y \in Y, \text{ and } b-1 \text{ trapped by } b. \\ \text{square move} & \text{if } W_{b+1} \neq W_b \text{ and b not trapped by } b+1, \text{or} \\ & \quad W_{b+1} = W_b \text{ and there is a } y < b \text{ between} \\ & \quad b \text{ and } b+1. \\ \text{non-move} & \text{otherwise.} \end{cases}$$

Proof. Each of the cases can be shown using Lemma 2.5 and the definition of a Tamari tableau. We do this for the first case only since the other cases are similar. Suppose $..., b, Y, (b+1), ... \subseteq W_b$, $y > b$ for every $y \in Y$, and every element of the words to the left of W_b are greater than $b+1$. This implies the end boxes of the b-set and $b+1$-set are on the same diagonal, say d_b, of C. Let $C^{(b+1)}$ be the tableau obtained from C by removing the boxes labeled with elements of the interval $[b+2, \ell(M)]$. Since $y > b$ for every $y \in Y$ if there were a box between b and $b+1$ on d_b in $C^{(b+1)}$ then it would have to be labeled with an element from a word to the left of W_b. This is not possible since all of these elements are removed from $C^{(b+1)}$ by assumption. This implies there are not any boxes between b and $b+1$ on d_b in $C^{(b+1)}$ so the prime path of b meets the prime path of $b+1$ at a point. By Lemma 2.5 $b \leftrightarrow b+1$ is an increasing pentagon move. $\square$

Example 2.12 Consider C_1 from Example 2.9. According to Lemma 2.11, $4 \leftrightarrow 5$ should be an increasing pentagon move. To see this, remove the six labels $[6, 11]$, which leaves the top chain

1	2	3	4	5
1	2	4		
1	2	4		

.

Now C_1 continues by removing the box labeled 5, and then removing the boxes labeled 4. If we instead interchange the order of the removal of those two corner boxes we get the top chain

1	2	3	4	5
1	2	6		
1	2	6		

,

which has length one more than the length of the previous top chain. It is easy to add back the boxes removed in the first step to see that when we interchange 4 and 5 in C_1 we get

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 7 \\
\hline
1 & 2 & 6 & 8 & 10 \\
\cline{1-5}
1 & 2 & 6 & 8 \\
\cline{1-4}
9 & 11 & 12 \\
\cline{1-3}
9 & 11 \\
\cline{1-2}
9 \\
\cline{1-1}
\end{array}.$$

Example 2.13 To make the importance of the restriction in Lemma 2.11 clear, consider the top chain

$$C = \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 6 \\
\hline
1 & 2 & 3 & 4 \\
\cline{1-4}
1 & 4 & 5 \\
\cline{1-3}
4 \\
\cline{1-1}
4 \\
\cline{1-1}
4 \\
\cline{1-1}
\end{array}$$

with $S_C = (4)(5,6)(3)(1,2)()()$. It is clear that the prime paths of the boxes labeled with 5 and 6 are disjoint and so by Lemma 2.5 $5 \leftrightarrow 6$ is a square move. However the characterization in Lemma 2.11 says $5 \leftrightarrow 6$ should be an increasing pentagon move. This conflict is resolved by insisting that Lemma 2.11 only be used if no word to the left of the word with 5 has any element less than 5 (a restriction that C violates since the first word has a 4). Fortunately this restriction is met in many important cases, in particular when the interchange we are interested in occurs on the outer diagonal making the restriction vacuous.

Our next goal is to prove that $\mathcal{TB}_\lambda$ has a unique top element and a unique bottom element (Theorem 2.1(1)). The next proposition is helpful for this purpose and it provides a key insight into the structure of square blocks which will be important for the remainder of the paper. The proposition says that every top chain is square equivalent to a top chain whose diagonal sentence has interval words (when viewed as sets rather than sequences) and the intervals are ordered starting with w_1 being the interval containing the largest letter and so on.

Proposition 2.14 For a Young diagram λ, every square block of $\mathcal{TB}_\lambda$ has a top chain C with diagonal sentence $S_C = W_1 W_2 ... W_m$ such that

$$\{b \mid b \in W_i\} = \begin{cases} [\ell(C) - s_i + 1, \ell(C) - s_{i-1}] & \text{if } s_i \neq s_{i-1}, \\ \emptyset & \text{otherwise} \end{cases}$$

for every $i \in [m]$, where $s_i = \sum_{j \in [i]} |W_j|$ and $s_0 = 0$.

Proof. Let C' be a top chain of shape λ, and suppose not every word of $S_{C'}$ satisfies the criterium. Let W be the first word of $S_{C'}$ that does not satisfy it, and let i be the largest letter missing from W (and hence i is the largest letter in $S_{C'}$ that is not in the right word). This means all the letters in the words before W are larger than i. Let k be the largest letter in W that is smaller than i. This implies $k+1$ is in a word that comes after W, and since the letters in the words before W are larger than i they must also be larger than k, so by Lemma 2.11 $k \leftrightarrow k+1$ is a square move.

If $k+1 = i$ then we have successfully inserted i into W while leaving the letters greater than i in the words they were originally in. By an inductive argument on i we are done. If $k+1 < i$ then $k+1$ is now the largest letter in W that is smaller than i, we can then repeat the process above until we insert i into W. This process must terminate in $i-k$ steps. $\square$

In general, there are more than one top chain in a square block that satisfies this proposition. We will often want to refer to such a top chain specifically, so we denote C_{int} as the top chain in a square block that the algorithm in the proof of Proposition 2.14 terminates at. The *int* in C_{int} is for *intervals* since the elements of each word of $S_{C_{\text{int}}}$ form an interval. C_{int} is useful because the outer diagonal of C_{int} can be *peeled* away by removing the boxes with labels from the first word of $S_{C_{\text{int}}}$, and the result is a top chain that is a subchain of C_{int}. Furthermore, this can be done repeatedly until the peeled top chain is just the null diagram in T_n. We note in passing that if n is minimal such that the shape of C_{int} is contained in δ_n, then the ith peeling away of a diagonal sentence word results in a top chain with shape that is contained in δ_{n-i}.

Proposition 2.15 For a Young diagram λ, there is one square block of minimum length top chains and one square block of maximum length top chains in $\mathcal{TB}_\lambda$.

Proof. We start by showing that every minimum length top chain is in the same block of $\mathcal{TB}_\lambda$. If $\lambda = \delta_n$, then there is only one maximal chain of minimum length. For general top chains this may not be the case. However, any top chain that removes an entire column at each step is of minimum length. So, fix a λ and let C be a minimum length top chain of shape λ. Consider C_{int}. In order for C_{int} to remove an entire column at each step, the prime path for every corner box on the outer diagonal must go all the way to the top of the diagram, which means the corner boxes of the outer diagonal must be removed in order starting from the corner box with the lowest row index. This implies the first word, W_1, of $S_{C_{\text{int}}}$ is strictly increasing. Now we can peel away the outer diagonal of C_{int} and repeat the argument to show that every word of $S_{C_{\text{int}}}$ is strictly increasing. There is only one such top chain for λ, so the square block of minimum length is unique.

Now we make a similar argument for maximum length top chains. An induction argument on m shows that any maximum length top chain of a shape with m diagonals must remove as few boxes as possible at each step. So, fix a λ and let C be a maximum length top chain of shape λ. Again consider C_{int}. In order for C_{int} to remove as few boxes as possible at each step, the prime path for every corner box on the outer diagonal must stop at the first corner box above it on the diagonal, which means the corner boxes of the outer diagonal must be removed in order starting from the corner box with the highest row index. This implies the first word, W_1, of $S_{C_{\text{int}}}$ is strictly decreasing. Just as we did in the minimum length case, we can now peel the outer diagonal of C_{int} and repeat our argument. This shows that C is square equivalent to the top chain with diagonal sentence that has interval words that are strictly decreasing. There is only one such top chain, so the square block of maximum length is unique. $\square$

Because every edge in $\mathcal{TB}_\lambda$ is a pentagon move and pentagon moves change the length of a chain by exactly one, chain length is a natural rank for $\mathcal{TB}_\lambda$. This means once we show every square block of $\mathcal{TB}_\lambda$ that is not the maximum length square block has an increasing pentagon move (Proposition 2.16) and every square block of $\mathcal{TB}_\lambda$ that is not the minimum length square block has a decreasing pentagon move (Proposition 2.18), it is clear that the maximum length square block is the $\hat{1}$ of $\mathcal{TB}_\lambda$ and the minimum length square block is the $\hat{0}$ of $\mathcal{TB}_\lambda$ (Theorem 2.1(1)). It is also clear that given these two propositions, $\mathcal{TB}_\lambda$ is graded, and that the rank function is the length of the chains of a square block minus the length of the bottom element chains (Theorem 2.1(2)). For $\mathcal{TB}_n$, this leads to the rank function

$$\rho(\overline{S_i}) = \ell(\overline{S_i}) - n + 1,$$

(since the unique minimum length chain in T_n has length $n-1$).

Recall that a descent of a sequence $(a_1, a_2, ...)$ is any element a_i such that $a_i > a_{i+1}$.

Proposition 2.16 Let λ be a Young diagram. For every square block of $\mathcal{TB}_\lambda$ that is not the maximum length square block, there is a set of square moves (possibly empty) that transform C_{int} into a top chain C' such that C' has an increasing pentagon move.

Proof. From the proof of Proposition 2.15 we know C_{int} of the maximum length square block has the diagonal sentence with every word strictly decreasing. So the diagonal sentence of C_{int} of a square block that is not the maximum length square block must have a word that is not strictly decreasing. Pick $W = (w_1, w_2, ...,)$ of $S_{C_{\text{int}}}$ to be the first (left-most) word with this property. We can peel away all the preceding words of $S_{C_{\text{int}}}$ to get a top chain with fewer words in its diagonal sentence and such that W is the outer diagonal of this new top chain. So, by an inductive argument on the number of words of $S_{C_{\text{int}}}$, it is enough to show the proposition in the case where W is the first word of $S_{C_{\text{int}}}$.

Let w_i be the first element of W that is not a descent, so $w_1 > w_2 > ... > w_i < w_{i+1}$. Also if w_i is not the minimum element of W, then let $j > i$ be the smallest integer such that $w_j < w_i$. Now we define

$$X = (w_1, ..., w_{i-1}),\ Y = (w_{i+1}, ..., w_{j-1}),\ \text{and } Z = (w_{j+1}, ...),$$

with the understanding that if w_i is the minimum element of W then Y is the subsequence of W from w_{i+1} to the end of W and Z is the empty sequence. So

$$W = (X, w_i, Y, w_j, Z),$$

such that Y is clearly nonempty, and every element of X and Y is greater than w_i. Because the elements of W form an interval we know $w_i + 1 \in W$. We want to make $w_i \leftrightarrow w_i + 1$ an increasing pentagon move. First, if $w_i + 1 \in Y$ then it is the minimum element of Y and so by Lemma 2.11 $w_i \leftrightarrow w_i + 1$ is an increasing pentagon move. Next suppose $w_i + 1 \notin Y$ and let y be the minimum element of Y. This means $y - 1$ must be in X or Z (since it can't be w_i or w_j by definition), so either w_i or w_j is between y and $y - 1$ which means $y - 1 \leftrightarrow y$ is a square move by Lemma 2.11. Making that square move puts $y - 1$ in Y. If $w_i + 1 = y - 1$ then $w_i + 1 \in Y$ and $w_i \leftrightarrow w_i + 1$ is an increasing pentagon move. Otherwise we use the same argument to show $y - 2 \leftrightarrow y - 1$ is a square move that puts $y - 2$ into Y, and by an inductive argument we can continue to make these square moves until $w_i + 1 \in Y$. Once $w_i + 1 \in Y$ we know $w_i \leftrightarrow w_i + 1$ is an increasing pentagon move. $\square$

Example 2.17 Let

$$C = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 & 7 & 8 & \\ \cline{1-5} 1 & 2 & 4 & 7 & & \\ \cline{1-4} 1 & 4 & 9 & & & \\ \cline{1-3} 1 & 4 & & & & \\ \cline{1-2} 1 & & & & & \\ \cline{1-1} \end{array}.$$

We write down the diagonal sentence after each move of the algorithm of Proposition 2.14 to show

how we find C_{int}.

$$\begin{aligned} S_C &= (1,4,9,7,8,6)(5)(2)(3)()() \\ 4 \leftrightarrow 5 &: (1,5,9,7,8,6)()(2,4)(3)()() \\ 1 \leftrightarrow 2 &: (2,5,9,7,8,6)()(4)(1,3)()() \\ 2 \leftrightarrow 3 &: (3,5,9,7,8,6)()(4)(1)(2)() \\ 3 \leftrightarrow 4 &: (4,5,9,7,8,6)()()(1,3)(2)() \\ 1 \leftrightarrow 2 &: (4,5,9,7,8,6)()()(2,3)()(1) \end{aligned}$$

which means

$$C_{\text{int}} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 5 & 7 & 8 & \\ \cline{1-5} 2 & 4 & 5 & 7 & & \\ \cline{1-4} 4 & 5 & 9 & & & \\ \cline{1-3} 4 & 5 & & & & \\ \cline{1-2} 4 & & & & & \\ \cline{1-1} \end{array},$$

and $4 \leftrightarrow 5$ is an increasing pentagon move by Lemma 2.11, so the maximal chain with diagonal sentence $(6,5,10,8,9,7)(4)()(2,3)()(1)$ is in a square block that covers the square block of C.

Proposition 2.16 says we can always move up in our poset, we now need to show we can always move down.

Proposition 2.18 For a Young diagram λ, every square block of $\mathcal{TB}_\lambda$ that is not the minimum length square block has a decreasing pentagon move.

Proof. Consider a square block of $\mathcal{TB}_\lambda$ that is not the minimum length square block, say $\overline{S}$. For ease of notation let $\ell = \ell(\overline{S})$. We will induct on ℓ. If $\ell = 1$ then there is only one block of $\mathcal{TB}_\lambda$ and it is the minimum length block, which is a contradiction. Similarly, if $\ell = 2$, then λ must have two columns and $\overline{S}$ must be the minimum length block, which is again a contradiction. So, let $\ell = 3$. If λ has three columns then $\overline{S}$ is the minimum length block, so λ must have two columns. This means

$$\overline{S} = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} \\ \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\},$$

which means $2 \leftrightarrow 3$ is a decreasing pentagon move.

Now suppose the proposition is true for all non-minimum length square blocks of length less than ℓ in $\mathcal{TB}_\lambda$ for any λ. Let $\overline{S}$ be a non-minimum length square block of length ℓ in $\mathcal{TB}_\lambda$ for some λ. For any chain $C \in \overline{S}$ we can remove the boxes labeled ℓ to get a new top chain of length $\ell - 1$, call this new chain C' in square block $\overline{S'}$. There are two cases:

Case 1. Suppose $\overline{S'}$ is not the minimum length square block. By the induction hypothesis $\overline{S'}$ has a chain with a decreasing pentagon move that is square equivalent to C'. Clearly none of these square moves involve ℓ, so these same moves take C to a top chain in $\overline{S}$ with a decreasing pentagon move.

Case 2. Suppose $\overline{S'}$ is the minimum length square block. This implies C' is square equivalent to C'_{bottom}, the chain that removes the boxes in the ith column at step i:

$$C'_{\text{bottom}} = \begin{array}{cccccc} 1 & 2 & & \ell-3 & \ell-2 & \ell-1 \\ 1 & 2 & \cdots & \ell-3 & \ell-2 & \\ 1 & 2 & & \ell-3 & & \\ 1 & 2 & & \ell-3 & & \\ 1 & 2 & & & & \\ 1 & 2 & & & & \\ 1 & & & & & \\ 1 & & & & & \end{array}$$

This means C is square equivalent to C'_{bottom} with the boxes labeled ℓ added back. Consider the diagonal sentence, $W_1 ... W_m$, of this top chain in $\overline{S}$. Let $\ell \in W_k$. If ℓ is at the end of W_k, then each word of the diagonal sentence would be strictly increasing, which would mean $\overline{S}$ is the minimum length square block, which is a contradiction. So,

$$W_k = \{..., \ell, b, ...\},$$

where the elements of W_k, except ℓ, are in increasing order. From the tableau perspective this says the ℓ-set does not extend to the top row of this top chain, and in fact, the prime path of ℓ ends at the end box of the b-set.

We claim $b - 1$ is trapped by b. It is clear by construction that every box labeled b has a box labeled $b - 1$ to its left. Assume the box below and to the left of the end box of the b-set is labeled $b - 1$. This implies $b - 1$ is between ℓ and b in W_k (a contradiction) or ℓ is in a word to the left of W_k (also a contradiction), so $b - 1$ is trapped by b.

We further claim we can make the square moves

$$\ell \leftrightarrow \ell - 1, \ell - 1 \leftrightarrow \ell - 2, ..., b + 1 \leftrightarrow b + 2$$

to put $b+1$ in the boxes ℓ originally occupied. To see this notice that for each of these square moves the end box of the b-set forces the prime paths of the two labels of each of the moves to be disjoint, and so by Lemma 2.5(2) each move is a square move. It is now easy to use Lemma 2.5(2) to show $b \leftrightarrow b+1$ is a decreasing pentagon move. $\square$

2.4 The r-stat of a Top Chain

To prove the lattice property we need to better characterize each square block. Another statistic, the r-stat of a top chain, does this.

Definition 2.19 The r-stat of a top chain C in a Tamari lattice is the array $\pi(C) = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that the ith row of $\pi(C)$ is the arrangement of the lengths of the r-sets of C that end in the ith row as read from left to right. Each nonzero entry of $\pi(C)$ is called a part of $\pi(C)$. Also, the shape of an r-stat π, denoted $sh(\pi)$, is the shape of the tableau of a top chain whose r-stat is π.

It is easy to show each row on an r-stat must be weakly decreasing (see Nelson 2015 for a proof). This means the rows of the r-stat can be viewed as integer partitions, however, the r-stat is not a plane partition because the columns may not be weakly decreasing. Clearly, the length of a top chain C is equal to the number of parts of $\pi(C)$. It is not hard to show $sh(\pi)$ is well-defined, meaning no two top chains of different shapes have the same r-stat.

Example 2.20 If

$$C = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 & 7 & 8 \\ \cline{1-5} 1 & 2 & 4 & 7 \\ \cline{1-4} 1 & 4 & 9 \\ \cline{1-3} 1 & 4 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

then

$$\pi(C) = \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \\ 3 & 2 & \\ 1 & & \\ 5 & & \\ 6 & & \end{array},$$

and $sh(\pi) = \delta_7$.

Next, we are going to show that the elements of $\mathcal{TB}_\lambda$ are in one-to-one correspondence with the r-stats. See Figure 15 for $\mathcal{TB}_5$.

Proposition 2.21 The r-stat of a top chain of a Tamari lattice is invariant under square moves. Therefore, in a square block, all elements have the same r-stat.

Proof. Let C be a top chain with r-stat $\pi(C)$ and C' be the top chain obtained from C by making the square move $b \leftrightarrow b+1$. Let B_1 be the end box of the b-set in C and B_2 be the end box of the $b+1$-set in C. By Lemma 2.5 there are two possibilities:

Case 1. Suppose the prime paths of B_1 and B_2 are disjoint. By the discussion before Lemma 2.11 this square move relabels every b box with a $b+1$ and every $b+1$ box with b. No other labels are changed since this is a square move, so it is clear that $\pi(C) = \pi(C')$.

$$\begin{array}{|c|c|c|c|c|} \hline & & & & b \\ \hline & & & & b \\ \hline & b+1 \\ \cline{1-2} & b+1 \\ \cline{1-2} \end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|} \hline & & & & b+1 \\ \hline & & & & b+1 \\ \hline & b \\ \cline{1-2} & b \\ \cline{1-2} \end{array}$$

Case 2. Now suppose one of the prime paths is contained in the other. Because the argument is the same for either possibility, we assume that the prime path of B_1 contains the prime path of B_2. In this case, every box of C with a $b+1$ has a b in the box to the immediate left. This means the square move from C to C' fixes the b's in C that have a $b+1$ to their right and interchanges all the b's that do not for $b+1$'s. The length of the b-set in C is equal to the length of the $(b+1)$-set in C' and the last row with b in C is the last row with $b+1$ in C'. Similarly, the length of the $(b+1)$-set in C is equal to the length of the b-set in C' and the last row with $b+1$ in C is the last row with b

in C'. Therefore, $\pi(C) = \pi(C')$.

$$\begin{ytableau} \; & \; & \; & \; & \; & b \\ \; & \; & b & b+1 \\ \; & b & b+1 \\ \; & b & b+1 \\ b \\ b \\ b \end{ytableau} \leftrightarrow \begin{ytableau} \; & \; & \; & \; & \; & b+1 \\ \; & \; & b & b+1 \\ \; & b & b+1 \\ \; & b & b+1 \\ b+1 \\ b+1 \\ b+1 \end{ytableau}$$

□

Proposition 2.22 If $\pi(C) = \pi(D)$ for top chains C and D of a Tamari lattice, then there exists a set of square moves from C to D. In other words, top chains with the same r-stat form a square block in $\mathcal{TB}_\lambda$.

Before proving Proposition 2.22 we need a lemma that relates the diagonal sentence and the r-stat.

Lemma 2.23 For a top chain C of a Tamari lattice with first word $W_1 = (w_1, ..., w_k)$ and r-stat $\pi(C)$,

1. $\pi(C)$ determines the descents of W_1, and
2. if w_j is the leftmost descent of W_1, then there exists a set of square moves that transform C into a top chain C' such that $w_j(C') = \ell(C') = \ell(C)$.

Proof.

1. Let B_{r_i} be the end box of the w_i-set where r_i is the row of $sh(C)$ containing B_{r_i}. If w_i is a descent then $w_i > w_{i+1}$, and so B_{r_i} is removed from C before $B_{r_{i+1}}$. Thus the length of the w_i-set is $r_i - r_{i+1}$ (since the prime path of B_{r_i} must end at $B_{r_{i+1}}$). This determines the last part of the r_ith row of $\pi(C)$.
   Furthermore, if the last part of the r_ith row of $\pi(C)$ is $r_i - r_{i+1}$, then w_i is a descent of W_1. (see Figure 14)

2. Since w_j is the leftmost descent of W_1, we have
   $$w_1 < w_2 < ... < w_j > w_{j+1}.$$
   Furthermore, $w_{j+1} < w_j + 1 \leq \ell(C)$, so by Lemma 2.11 $w_j \leftrightarrow w_j + 1$ is a square move. If $w_j + 1 = \ell(C)$ then we are done; otherwise, the proof follows by an inductive argument.

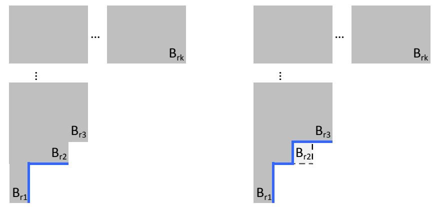


Figure 14: *B_{r+1}'s prime path if $w_1 > w_2$ on the left. The right figure shows the alternative, $w_1 < w_2$.*

□

Proof.[Proof of Proposition 2.22] We are going to use induction on $\ell(C)$. First, observe that

$$\ell(C) = \#\{(i,j) \mid \pi_{ij}(C) \neq 0\} = \#\{(i,j) \mid \pi_{ij}(D) \neq 0\} = \ell(D)$$

Now suppose $\ell(C) = 1$. This means C is a single column tableau (say with k rows) with 1 in every box, and that $\pi(C)$ has one part with entry k in the kth row. There is only one top chain with this r-stat, and so $D = C$.

Now suppose $\ell(C) = L$ and that the statement is true for all top chains of length less than L. As we have mentioned before we know C and D have the same shape. There are two possibilities:

Case 1. Suppose the L-sets of C and D are equal. Let C' and D' be the top chains obtained from C and D respectively by removing the L-sets of each. This means $\ell(C') = L - 1 = \ell(D')$. Furthermore, the r-sets of both C' and D' are the same as the original r-sets except that they are both missing the L-set entry, which implies $\pi(C') = \pi(D')$. So, by the inductive hypothesis there exists a set of square moves from C' to D'. Clearly none of these square moves involve L, and C' and D' are the subchains of C and D starting from each's second element by construction. This means these square moves are valid for C and D as well, so the statement is proved in this case.

Case 2. Now suppose the L-sets of C and D are not the same. Let $W_1(C)$ and $W_1(D)$ be the first words of S_C and S_D. By Lemma 2.23(2) we can transform C and D into top chains, C' and D', with L in the rightmost descent of each. Furthermore, Lemma 2.23(1) implies the descents of $W_1(C')$ and $W_1(D')$ are the same, so the L-sets of C' and D' end in the same box, which means the L-sets are the same, and so we are again in Case 1 in which the statement has already been proved. □

Since we now view the elements of $\mathcal{TB}_\lambda$ as r-stats, it is desirable to understand the cover relations in this context. As the next proposition makes precise, if $\pi \lessdot \rho$ for two r-stats, then we can view the

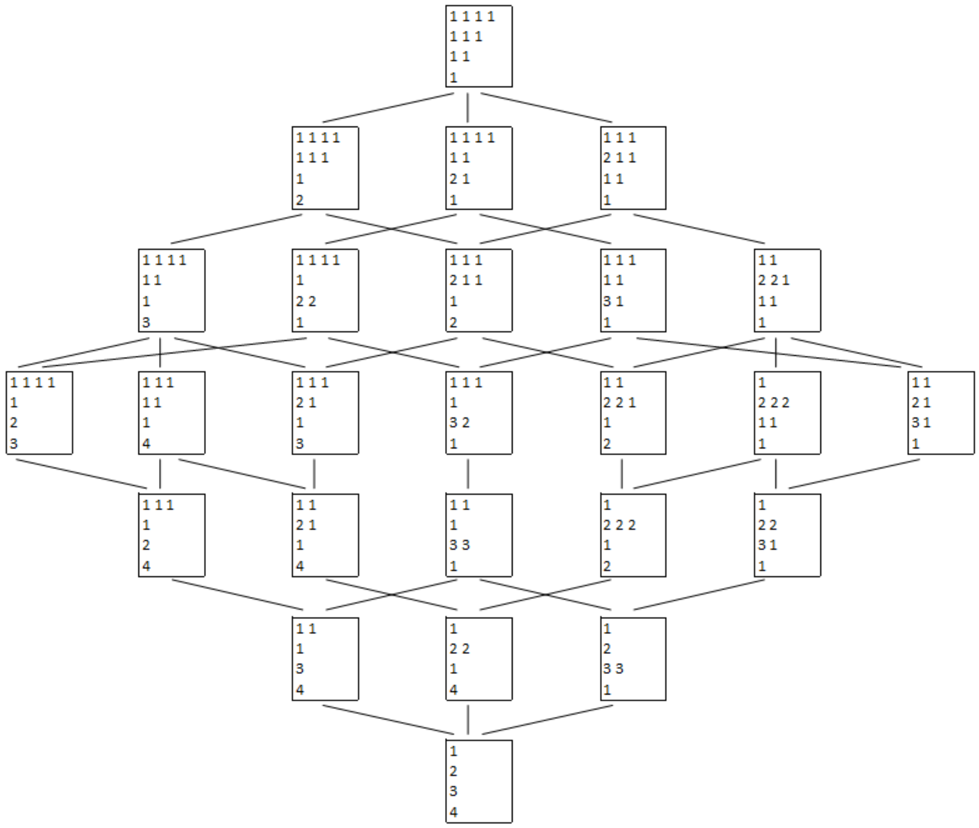


Figure 15: $\mathcal{TB}_5$ *with the* r*-stats of each block.*

pentagon move from π to ρ as π giving a portion of one of its parts to create a new part in ρ. Other than this giving of π and the new part created in ρ, the parts of π and ρ are the same.

Proposition 2.24 For a Young diagram λ, let $\pi, \rho \in \mathcal{TB}_\lambda$, $\pi \lessdot \rho$ implies that

1. ρ has one more nonzero part than π, say in row i,
2. There exist j, k with $i < j$ such that $\pi_{jk} > \rho_{jk} = j - i$,
3. Other than these two exceptions π and ρ are equal,
4. ρ's additional part is not unique in row i and is equal to $\pi_{jk} - \rho_{jk}$, and
5. The increasing pentagon move from π to ρ involves corner boxes in rows i and j.

Proof. This is easily seen by examining pentagon moves. If $b \leftrightarrow b+1$ is an increasing pentagon move then making this switch splits the old b-set into two sets (the set of boxes to the left of the $b+1$-set and the set of boxes below the $b+1$-set). After the move the three resulting sets are labeled with $b+2, b+1, b$. In fact, this explains why the length is increased by 1. (See Lemma 2.5 and the figure in Case 1 of the proof.) Thus (1) is clear. (2) follows from the splitting of the b-set. Since the b-set splitting is the only change that affects set size (3) is true. In order for $b \leftrightarrow b+1$ to be an increasing pentagon move the b-set and $b+1$-set must begin in the same row (the lowest row index with a b must also be the lowest row index with a $b+1$). This leads to (4). (5) is straightforward. $\square$

Example 2.25 We continue Example 2.17. It is stated that $4 \leftrightarrow 5$ is an increasing pentagon move for

$$C_{\text{int}} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 5 & 7 & 8 \\ \cline{1-5} 2 & 4 & 5 & 7 \\ \cline{1-4} 4 & 5 & 9 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array},$$

When we make this move we remove the corner box in row 6 before the corner box in row 5. This leads to the maximal chain

$$D = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 7 \\ \hline 2 & 4 & 5 & 8 & 9 \\ \cline{1-5} 2 & 4 & 5 & 8 \\ \cline{1-4} 4 & 5 & 10 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array},$$

which covers C_{int}. When we pass to r-stats, we obtain

$$\pi(C_{\text{int}}) = \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \\ 3 & 2 & \\ 1 & & \\ 5 & & \\ 6 & & \end{array} \quad \text{and } \pi(D) = \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \\ 3 & 2 & \\ 1 & & \\ 5 & 5 & \\ 1 & & \end{array},$$

Now we paraphrase the proposition for our example: $\pi(D)$ and $\pi(C_{\text{int}})$ agree in every part except in the last two rows. They both have a single part in the sixth row, but the 6 in $\pi(C_{\text{int}})$ is 5 more than the 1 in $\pi(D)$. That difference of 5 shows up as a part in the fifth row of $\pi(D)$ that $\pi(C_{\text{int}})$ does not have. And finally the 5 is not unique in the fifth row of $\pi(D)$, in fact the other 5 (which is in both r-stats) forced the new part to be a 5 as well.

Corollary 2.26 For a Young diagram λ, let $\pi, \rho \in \mathcal{TB}_\lambda$ be such that $\pi \lessdot \rho$. Suppose there is an increasing pentagon move from $C \in \pi$ to $D \in \rho$ involving corner boxes in rows $i < j$. Then every increasing pentagon move from π to ρ involves corner boxes in rows i and j. Furthermore, if $C' \in \pi$ has an increasing pentagon move involving corner boxes in rows i and j, then that move results in a top chain in ρ.

Proof. The first statement immediately follows from Proposition 2.24(5). Now, suppose $C' \in \pi$ has an increasing pentagon move involving corner boxes in rows i and j that results in $D' \in \rho'$. By Proposition 2.24, ρ' equals π everywhere except there exists a k' with $\pi_{jk'} > \rho'_{jk'} = j - i$ and an additional part in row i of ρ', $\pi_{jk'} - \rho_{jk'}$. This means $\rho'_{jk'} = j - i = \rho_{jk}$, hence, ρ and ρ' disagree at most at their additional parts.

Without loss of generality suppose $k' < k$, which implies $\pi_{jk} \leq \pi_{jk'}$ since rows of r-stats are weakly decreasing. If $\pi_{jk} < \pi_{jk'}$ then

$$\rho'_{jk} = \pi_{jk} > \rho_{jk} = j - i = \rho'_{jk'}$$

which is a contradiction since rows of r-stats are weakly decreasing. So $\pi_{jk} = \pi_{jk'}$, which means ρ''s additional part equals ρ's additional part which means $\rho = \rho'$. $\square$

Recall from Proposition 2.16 that if we are not in the maximum length square block then we can make square moves from C_{int} to a top chain with an increasing pentagon move. We can actually do better than that. As we have already mentioned we can put a total order on each square block using the lexicographic order of the diagonal sentence of each top chain. We can now associate a fixed square block with an r-stat, say π. Let $<_{\text{lex}(\pi)}$ denote the lexicographic order on the top chains in π, and let $C_{\max}(\pi)$ be the top chain with the diagonal sentence of maximum lexicographic value in π, so $C_{\max}(\pi)$ is the maximum element of the square block π under $<_{\text{lex}(\pi)}$. It is clear by this definition that $S_{C_{\max}(\pi)}$ is one of the diagonal sentences that satisfies the condition of Proposition 2.14, which means the words of $S_{C_{\max}(\pi)}$ viewed as sets are ordered intervals with the largest letters in the first word etc.

Proposition 2.27 For a Young diagram λ, let $\pi \in \mathcal{TB}_\lambda$. For every cover, ρ, of π, there exists an increasing pentagon move from $C_{\max}(\pi)$ to a top chain in ρ. In other words, $\pi \lessdot \rho$ if and only if $C_{\max}(\pi)$ has an increasing pentagon move to a top chain in ρ.

Proof. Let $\pi \lessdot \rho$ and suppose $C \in \pi$ and $D \in \rho$ such that $b \leftrightarrow b + 1$ is an increasing pentagon move from C to D involving corner boxes in rows $i < j$. By Corollary 2.26 it is enough to show $C_{\max}(\pi)$ has an increasing pentagon move involving corner boxes in rows i and j. There are two cases:

Case 1. The words of S_C when viewed as sets are ordered intervals in the sense of Proposition 2.14. Notice that this allows us to use Lemma 2.11. Because $b \leftrightarrow b+1$ is an increasing pentagon move we know b and $b+1$ are in the same word of S_C, and so they are also in the same word of $S_{C_{\max}(\pi)}$. Furthermore, by Lemma 2.11 moving from C to $C_{\max}(\pi)$ involves square moves that all occur within words, which means $c \leftrightarrow c+1$ is a move in this list if and only if

1. c and $c+1$ are in the same word,
2. c is before $c+1$, and
3. there exists an x between c and $c+1$ such that $x < c$.

Since $b \leftrightarrow b+1$ is an increasing pentagon move, by Lemma 2.11 every y between them is greater than $b+1$. If $y \leftrightarrow y+1$ is a square move toward $C_{\max}(\pi)$, then $b \leftrightarrow b+1$ is unaffected. If $y-1 \leftrightarrow y$ is a square move toward $C_{\max}(\pi)$, then $b \leftrightarrow b+1$ could only be affected if $y = b+2$, but this is not possible since no letter between b and $b+1$, and therefore between y and $b+1$, is less than b. Hence, square moves involving letters between b and $b+1$ preserve the pentagon move.

The only other square moves to consider are $b+1 \leftrightarrow b+2$ and $b-1 \leftrightarrow b$. Both cases use the same argument so we are going to argue for the former case only and skip the other. If $b+1 \leftrightarrow b+2$ is a square move then there exists an $x < b$ between $b+1$ and $b+2$. So make the $b+1 \leftrightarrow b+2$ move. Now the $b+2$-set ends in row i, but x is now between b and $b+1$, so $b \leftrightarrow b+1$ is a square move that increases the lexicographic value. Making this move puts the last box of the $b+1$-set in row j. Clearly $b+1 \leftrightarrow b+2$ is now an increasing pentagon move involving corner boxes in rows i and j.

Case 2. The words of S_C are not ordered intervals in the sense of Proposition 2.14. In this case we are going to show C_{int} has an increasing pentagon move involving corner boxes in rows i and j. By Case 1 this implies $C_{\max}(\pi)$ also has an increasing pentagon move involving corner boxes in rows i and j.

We are going to follow the algorithm in the proof of Proposition 2.14. So, let W be the first word of S_C that does not satisfy the condition, and let i be the largest letter missing from W. Let k be the largest letter in W that is smaller than i. The algorithm iterates on k and makes $k \leftrightarrow k+1$ square moves to bring $k+1$ into W until $k+1 = i$. Then it starts over with a new i or goes to the next word. Suppose first that b and $b+1$ are not in W. If they are in a word before W or neither ever becomes $k+1$ then the increasing pentagon move is unaffected by the ordering of W. Suppose they are in a word after W and b becomes $k+1$. This means the algorithm's next move is $b-1 \leftrightarrow b$ which puts b in W and leaves $b+1$ in it's original word (not W). This ruins our pentagon move temporarily, but the algorithm cannot be done yet. The algorithm finishes an iteration once i has been inserted into W, but $b \neq i$ since $b+1 \notin W$ and i is the largest letter in a word after W. So the next move is $b \leftrightarrow b+1$. Because $b \leftrightarrow b+1$ was a pentagon move before moving b into W we know the prime path of b contained the prime path of $b+1$ after the $b-1 \leftrightarrow b$ move. Since $b-1 \leftrightarrow b$ was a square move either the prime path of $b-1$ contained the prime path of b or their prime paths were disjoint. In either case, making both square moves leaves the end boxes of the $b-1$-set and the b-set on the same diagonal in rows i and j and so the increasing pentagon move is preserved, although now the move is $b-1 \leftrightarrow b$.

Finally, suppose $b, b+1 \in W$. Again if neither b nor $b+1$ ever become k then the increasing pentagon move is unaffected by this iteration of i. So suppose at some point of the algorithm $b+1 = k$, which means the next square move is $b+1 \leftrightarrow b+2$ which takes $b+1$ out of W and puts $b+2$ in its place (the prime paths must be disjoint in this case). However the algorithm cannot finish with W while $b \in W$ and $b+1 \notin W$. At some point $b = k$ and so $b \leftrightarrow b+1$ will be the next square move that puts $b+1$ in b's place in W. This means the end boxes of the $b+1$-set and the $b+2$-set are corner boxes in rows i and j, furthermore any boxes between them have been left unchanged or increased in value, so $b+1 \leftrightarrow b+2$ is an increasing pentagon move involving corner boxes in rows i and j. $\square$

Example 2.28 Consider first an example from Case 1 of Proposition 2.27. Let

$$C = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 12 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \cline{1-7} 1 & 3 & 5 & 8 & 10 & 11 \\ \cline{1-6} 3 & 5 & 8 & 10 & 13 \\ \cline{1-5} 3 & 5 & 8 & 10 \\ \cline{1-4} 5 & 8 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array}$$

with $S_C = (9, 8, 10, 13, 11, 7, 12)(5, 6)()(3, 4)()(1, 2)()()$. The words of S_C are intervals. Notice the increasing pentagon move $10 \leftrightarrow 11$ involving corner boxes in rows 3 and 5. There is a square move involving 10 and one involving 11, but the pentagon move is preserved in $C_{\max}(\pi(C))$,

$$C_{\max}(\pi(C)) = \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 9 & 10 & 11 \\ 3 & 5 & 9 & 10 & 12 \\ 3 & 5 & 9 & 10 \\ 5 & 9 \\ 5 & 9 \\ 13 \end{ytableau}$$

Now consider two examples from Case 2 of Proposition 2.27. Let C and D denote

$$C = \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 7 & 8 \\ 1 & 3 & 7 \\ 1 & 9 \\ 1 \end{ytableau}, \quad D = \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 8 \\ 1 & 2 & 3 & 5 \\ 9 & 10 & 11 \end{ytableau}$$

Then we have

$$S_C = (1,9,7,8,5,6)(3,4)()(2)()(), \; S_D = (11,5,8,6,7)(10,3)(9,2,4)(1)()()().$$

The first words of S_C and S_D are not intervals. C has an increasing pentagon move $3 \leftrightarrow 4$ involving corner boxes in rows 2 and 4. Note that 3 and 4 are not in the first word. By examining C_{int} we see that $2 \leftrightarrow 3$ is an increasing pentagon move in rows 2 and 4, so the pentagon move is preserved. D has an increasing pentagon move $5 \leftrightarrow 6$ involving corner boxes in rows 2 and 4 on its outer diagonal. Examining D_{int}, we see that $7 \leftrightarrow 8$ is an increasing pentagon move in rows 2 and 4, so once again the pentagon move is preserved.

$$C_{\text{int}} = \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 7 & 8 \\ 2 & 4 & 7 \\ 4 & 9 \\ 4 \end{ytableau}, \quad D_{\text{int}} = \begin{ytableau} 1 & 2 & 3 & 5 & 7 & 8 & 9 \\ 1 & 2 & 3 & 5 & 7 & 8 \\ 2 & 3 & 5 & 7 & 10 \\ 2 & 3 & 5 & 7 \\ 4 & 6 & 11 \end{ytableau}$$

2.5 $\mathcal{TB}_n$ Is a Lattice

Our goal in this section is to prove the last assertion of Theorem 2.1, that $\mathcal{TB}_\lambda$ is a lattice. The first step of this proof is to show $\mathcal{TB}_\lambda$ can be partitioned into a poset, which we will call the *peel* *poset of* λ that is isomorphic to one of the Tamari lattices. This implies that the peel poset is a lattice itself. We will then use this to find the join of any two elements of $\mathcal{TB}_\lambda$. Since $\mathcal{TB}_\lambda$ is finite and has a $\hat{0}$, it is easily seen that every two elements have a meet also, and this will complete the proof. To this end, we start with some definitions.

Definition 2.29 Let C be a top chain of a Tamari lattice with r-stat $\pi(C)$, and suppose $B_1, ..., B_m$ are the set of corner boxes of the outer diagonal of C with row indices $r_1 < ... < r_m$. Define the *outer peel* of π to be the sequence $p(\pi) = (p_1, ..., p_m)$ such that p_i is the last nonzero part of π in row r_i.

By Proposition 2.14, C_{int} in C's square block is a top chain with r-stat $\pi(C)$ that removes the corner boxes of the outer diagonal before any other corner boxes. As a result, we can "peel" away this outer diagonal from π to obtain a new r-stat.

Definition 2.30 For a Young diagram λ, let π be an r-stat in $\mathcal{TB}_\lambda$ with outer peel $p(\pi) = (p_1, ..., p_m)$ and suppose $B_1, ..., B_m$ are the set of corner boxes of the outer diagonal of λ with row indices $r_1 < ... < r_m$. Define *peel*(π) to be the r-stat obtained by removing p_i from row r_i in π for every $i \in [m]$.

Example 2.31 Recall C_{int} from Example 2.25 with r-stat:

$$C_{\text{int}} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\\hline 2&4&5&7&8\\\cline{1-5} 2&4&5&7\\\cline{1-4} 4&5&9\\\cline{1-3} 4&5\\\cline{1-2} 4\\\cline{1-1}\end{array}, \qquad \pi(C_{\text{int}}) = \begin{array}{ccc} 1&1&1\\ 1&&\\ 3&2&\\ 1&&\\ 5&&\\ 6&&\end{array}$$

Since the shape of C_{int} is δ_7, every row has a corner box on the outer diagonal, so

$$p(\pi(C_{\text{int}})) = (1,1,2,1,5,6), \text{ and } \operatorname{peel}(\pi(C_{\text{int}})) = \begin{array}{cc} 1&1\\ &\\ 3&\\ &\\ &\\ &\end{array}$$

Notice that removing the boxes with labels $[4, 9]$ from C_{int} results in the top chain

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1} 2\\\cline{1-1}\end{array},$$

which has r-stat $\operatorname{peel}(\pi(C_{\text{int}}))$.

At the beginning of this section we wanted to partition $\mathcal{TB}_\lambda$. To do this we set $\pi \sim \rho$ if and only if $p(\pi) = p(\rho)$. It is not hard to see this is an equivalence relation. Now we define a poset on these equivalence classes.

Definition 2.32 For a Young diagram λ, the associated peel poset, denoted by $\mathcal{P}_\lambda$, is defined as the poset obtained by:

1. Partitioning the elements of $\mathcal{TB}_\lambda$ into equivalence classes of outer peels.
2. For $P, R \in P_\lambda$, $P \lessdot_{\mathcal{P}_\lambda} R$ if and only if there are r-stats $\pi \in P$ and $\rho \in R$ such that $\pi \lessdot_{\mathcal{TB}_\lambda} \rho$.

The peel poset for $\lambda = \delta_4$ is depicted in Figure 16.

Proposition 2.33 $\mathcal{P}_\lambda$ is well-defined.

Proof. We argue by contradiction to show the cover relations are well-defined. Assume there exist $\pi_1, \pi_2 \in P$ and $\rho_1, \rho_2 \in R$ such that $\pi_1 \lessdot_{\mathcal{TB}_\lambda} \rho_1$, but $\rho_2 \lessdot_{\mathcal{TB}_\lambda} \pi_2$. By the definition of $\mathcal{P}_\lambda$ the outer peels of π_1 and π_2 are the same, as are the outer peels of ρ_1 and ρ_2. However, the outer peels of the r-stats in P are different than the outer peels of the r-stats in R, so by Proposition 2.24(3) the pentagon moves from π_1 to ρ_1 and from π_2 to ρ_2 must involve the outer peels. So by Proposition 2.24(2) there exist j, k with

$$(\pi_1)_{jk} < (\rho_1)_{jk} = (\rho_2)_{jk} < (\pi_2)_{jk} = (\pi_1)_{jk}$$

which is a contradiction. Since the cover relations are well-defined, so is $\mathcal{P}_\lambda$. $\square$

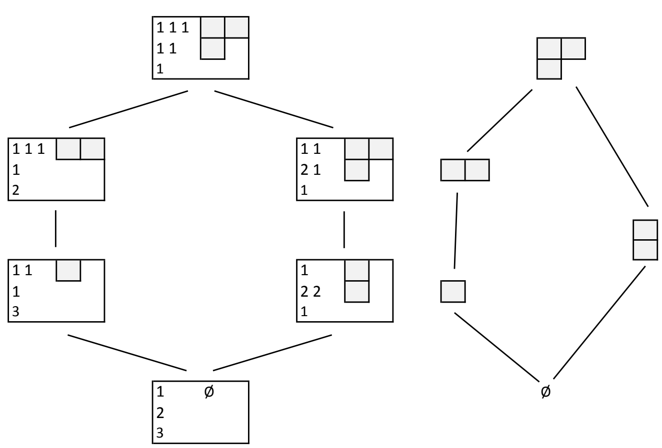


Figure 16: $\mathcal{TB}_4$ *with* r*-stats and peel shapes;* $\mathcal{P}_4$ *with shapes that fit in* δ_3.

Remark 2.34 A natural question is whether $\mathcal{P}_\lambda$ is a quotient of $\mathcal{TB}_\lambda$ with respect to the partitioning by outer peels (see Reading 2006; Chajda and Snášel 1998; Reading 2002 for a discussion of poset quotients). $\mathcal{P}_\lambda$ is not a quotient since the partitioning by outer peels is not an order congruence. For example, the projection map from r-stats in $\mathcal{TB}_\lambda$ to the minimal element of each equivalence class with respect to the outer peel partitioning is not order-preserving (again see Reading 2006, Chapter 3 for details).

Our first goal is to count the number of equivalence classes for a fixed shape λ. Although the elements of $\mathcal{P}_\lambda$ are defined in terms of equivalence classes of r-stats, we view them as the set of outer peels of the r-stats for a given λ:

$$\mathcal{P}_\lambda = \{p(\pi) \mid \pi \text{ is an } r\text{-stat of shape } \lambda\}.$$

With this perspective in mind we enumerate $\mathcal{P}_\lambda$.

Proposition 2.35 Let λ be a Young diagram with m corner boxes on its outer diagonal, and let $r_1 < r_2 < ... < r_m$ be the row indexes of those boxes (define $r_0 = 0$). If $\mathcal{P}_\lambda$ is the set of outer peels of r-stats with shape λ, then $\#\mathcal{P}_\lambda$ is the m-th Catalan number $C_m = \frac{1}{m+1}\binom{2m}{m}$, and $\mathcal{P}_\lambda$ is the set of m-tuples $(p_1, p_2, ..., p_m)$ such that for each $i \in [m]$,

1. $p_i \in \{r_i - r_j \mid 0 \leq j < i\}$, and
2. If $p_i = r_i - r_j$, then $p_k \leq r_k - r_j$ for $j < k < i$.

Proof. As we have done before, let $B_1, ..., B_m$ be the corner boxes on the outer diagonal of λ. Now condition (1) just says that when we consider the prime path of B_i the only possible impediments to this prime path are other corner boxes on the outer diagonal that are in rows above row r_i. Thus, condition (1) is clear.

Similarly, suppose $p_i = r_i - r_j$. This means B_i is removed before B_j, but after $B_{j+1}, ..., B_{i-1}$. We know this since B_i's prime path is stopped by B_j, and it is not stopped by any of these other corner boxes. But this means $B_{j+1}, ..., B_{i-1}$ were all removed before B_j, which means each of the associated prime paths are stopped by B_j or by a corner box below B_j. This implies condition (2).

To prove $\#\mathcal{P}_\lambda = C_m$ we are going to use induction on m. In other words, we are going to show that $\#\mathcal{P}_\lambda$ is counted by the Catalan recurrence:

$$C_m = \sum_{i=0}^{m-1} C_i C_{m-i-1}.$$

If $m = 1$ then the only outer peel is the sequence (r_1), which implies $\#\mathcal{P}_\lambda = C_1$. Next suppose $m > 1$. The prime path of B_m can be stopped by any of the other corner boxes or by the top of the diagram, so there are m choices for p_m. Now fix a choice for p_m, say the prime path of B_m is stopped by B_i, so $p_m = r_m - r_i$ (notice $i = 0$ is the case when the prime path is not stopped). This implies $B_{i+1}, ..., B_{m-1}$ come off before B_m which comes off before B_i. So, by the induction hypothesis, there are C_{m-i-1} ways to order the removal of $B_{i+1}, ..., B_{m-1}$, and C_i ways to order the removal of $B_1, ..., B_i$. By condition (2) these two orderings are independent of each other. This gives the promised recursion. $\square$

If $\lambda = \delta_n$ let $\mathcal{P}_n = \mathcal{P}_\lambda$ (as we have done in similar cases). This leads to the next corollary, which is a Catalan set found in Richard P Stanley 2015, Chapter 2 #85

Corollary 2.36 $\#\mathcal{P}_n = C_{n-1}$, and $\mathcal{P}_n$ is the set of $(n-1)$-tuples $(p_1, p_2, ..., p_{n-1})$ such that for each $i \in [n-1]$,

1. $1 \leq p_i \leq i$, and
2. if $p_i = j$, then $p_{i-r} \leq j - r$ for $1 \leq r \leq j - 1$.

So far we have considered the elements of $\mathcal{P}_\lambda$ as equivalence classes of r-stats and as outer peels. We will use $P \in \mathcal{P}_\lambda$ for the equivalence class, and $p(P)$ for the outer peel associated with P. There is a third way to view the elements of $\mathcal{P}_\lambda$. For a Young diagram $\lambda = (\lambda_1, ..., \lambda_t)$ with corner boxes $B_1, ..., B_m$ on the outer diagonal in rows $r_1 < ... < r_m$ and a given r-stat π, instead of making an m-tuple of the last nonzero elements of π in rows $r_1, ..., r_m$, we can make a t-tuple of the number of boxes removed in each row of λ by peeling $p(\pi)$; call this t-tuple $q(\pi)$ (see Example 2.39). Call the set of such t-tuples for all r-stats of shape λ $\mathcal{Q}_\lambda$. By utilizing the top chain C_{int} and examining the prime paths of $B_1, ..., B_m$ it is not hard to show $(q_1, ..., q_t) \in \mathcal{Q}_\lambda$ if and only if:

Q1. $q_{r_m} = 1$.

Q2. $1 \leq q_{r_i} \leq q_{r_{i+1}} + 1$ for every $i \in [m-1]$.

Q3. $q_k = q_{r_i}$ for every $r_{i-1} < k < r_i$ and $i \in [m]$.

Q4. $q_k = 0$ for every $r_m < k \leq t$.

The next proposition proves this is another characterization of $\mathcal{P}_\lambda$.

Proposition 2.37 Let $\lambda = (\lambda_1, ..., \lambda_t)$ be a Young diagram with m corner boxes on its outer diagonal, and let $r_1 < r_2 < ... < r_m$ be the row indexes of those boxes. The map

$$\begin{aligned} \phi: \quad & \mathcal{P}_\lambda \to \mathcal{Q}_\lambda \\ & (p_1, ..., p_m) \mapsto (q_1, ..., q_t) \end{aligned}$$

defined by

$$q_k = \#\{j \in [m] \mid r_j - p_j + 1 \le k \le r_j\} \tag{2.1}$$

is a bijection.

Proof. Let $C_{\text{int}}(\pi)$ be the $C_{\text{int}} \in \pi$. The number of boxes removed from row k of $C_{\text{int}}(\pi)$ by peeling $p(\pi)$ is clearly the right hand side of Equation (2.1), so ϕ is well-defined. This also implies ϕ is surjective, since $q \in \mathcal{Q}_\lambda$ by definition has at least one associated r-stat and the outer peel of any such r-stat satisfies Equation (2.1).

To prove injectivity, we are going to use another Catalan identity to complete the proof. In light of Q3 and Q4, every $q \in \mathcal{Q}_\lambda$ is completely determined by $q_{r_1}, q_{r_2}, ..., q_{r_m}$ which satisfy Q1 and Q2. Define a new m-tuple $q' = (q'_1, q'_2, ..., q'_m)$ where $q'_j = q_{r_{m-j+1}} - 1$. This implies the set of all of the q' is the set of m-tuples that satisfy

$$\begin{aligned} & q'_1 = 0, \text{ and} \\ & 0 \le q'_{j+1} \le q'_j + 1, \end{aligned}$$

which is a set of objects enumerated by the mth Catalan number (Richard P Stanley 2015, Chapter 2 #80). Therefore $|\mathcal{Q}_\lambda| = C_m = |\mathcal{P}_\lambda|$. $\square$

The following corollary is a direct consequence of Proposition 2.37, and more specifically, of Equation (2.1).

Corollary 2.38 Let λ be a Young diagram, if $\pi_1, \pi_2 \in \mathcal{TB}_\lambda$, then $p(\pi_1) = p(\pi_2)$ if and only if $sh(\text{peel}(\pi_1)) = sh(\text{peel}(\pi_2))$. In particular, as a set,

$$\mathcal{P}_n = \{\lambda \text{ a Young diagram} \mid \lambda \subseteq \delta_{n-1}\}.$$

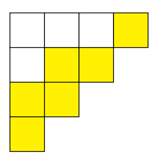
This corollary means there is a natural way to view the elements of $\mathcal{P}_\lambda$ as Young diagrams. Specifically, if $P \in \mathcal{P}_\lambda$ with an r-stat $\pi \in P$, then we can associate P with $sh(\text{peel}(\pi))$, which we denote by $sh(P)$. Since a pentagon move between two r-stats in a fixed $P \in \mathcal{P}_\lambda$ necessarily is a pentagon move not on the outer diagonal, it is easy to show that P when viewed as a subposet of $\mathcal{TB}_\lambda$ is isomorphic to $\mathcal{TB}_{sh(P)}$. We can take it one step further, if there is an r-stat $\rho \notin P$ such that $\pi \lessdot_{\mathcal{TB}_\lambda} \rho$, then the pentagon move between π and ρ necessarily happens on the outer diagonal, which means for every $\pi' \in P$ there exists a distinct $\rho' \in R$, where R is ρ's equivalence class, such that $\pi' \lessdot_{\mathcal{TB}_\lambda} \rho'$. In other words, if $P \leq_{\mathcal{P}_\lambda} R$, then there is an embedded copy of $\mathcal{TB}_{sh(P)}$ in $\mathcal{TB}_{sh(R)}$. See Figure 17 for examples of this.

This brings the number of ways to view the elements of $\mathcal{P}_\lambda$ to four. We describe each in Example 2.39.

Example 2.39 Let $\lambda = \delta_5$ and consider the quadruple $p = (1, 1, 1, 3)$. Clearly p satisfies the conditions of Proposition 2.35 so there exists a $P \in \mathcal{P}_5$ with $p = p(P)$. By examining Figure 15 we also see that p is the outer peel of r-stats

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & & & \\ 3 & & & \end{array}, \text{ and } \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & \\ 1 & & \\ 3 & & \end{array},$$

so P is the equivalence class of these two r-stats, and this equivalence class is isomorphic to $\mathcal{TB}_{(3,1)}$. For the third view of this element, we can peel p away from δ_5 to get the Young diagram $(3, 1)$:



which is $sh(P)$. And finally, we can apply Equation (2.1) to p which just records the difference of each row between δ_5 and $(3, 1)$ to get the quadruple $q(P) = (1, 2, 2, 1)$, which is the final view of P.

Lemma 2.40 Let λ be a Young diagram, let $\pi \in \mathcal{TB}_\lambda$, and suppose $\pi \in P \in \mathcal{P}_\lambda$ with $p(\pi) = (p_1, ..., p_m)$. For every p_j such that there is a $i < j$ with

$$p_i = p_j - j + i, \tag{2.2}$$

the following are true:

1. For the largest i that satisfies Equation 2.2, there exists an increasing pentagon move from π on the outer diagonal involving rows i and j.
2. This move adds p_i boxes to $sh(P)$ in rows $i - p_i + 1$ through i.
3. This move does not change any element of $p(\pi)$ except the jth element becomes $j - i$.

4. These are the only increasing pentagon moves on π's outer peel.

Proof. Suppose there exist p_j and p_i that satisfy Equation 2.2 and pick i to be the largest for a fixed j. Let B_i and B_j be the corner boxes on the outer diagonal of λ in rows i and j. Equation 2.2 implies the prime paths of B_i and B_j end at the same row which implies B_i is removed before B_j. Furthermore, by Proposition 2.35(2) every corner box between B_i and B_j on the outer diagonal is removed before B_j. Altogether these mean that there is a chain in π with label b in B_i and $b+1$ in B_j and so by Lemma 2.11 there exists an increasing pentagon move from π on the outer diagonal involving rows i and j.

(2) and (3) are consequences of Proposition 2.24. For (4) assume there is an increasing pentagon move from π involving the outer peel. It is not hard to show Equation 2.2 follows from Proposition 2.24. $\square$

Remark 2.41 Knuth 2013 shows there are C_n number of forests on n nodes. This is done by creating a bijection between a forest and a *scope sequence* of length n and showing that there are C_n scope sequences. These scope sequences appear in Bjorner and Wachs 1997 as well (see Definition 9.1). It turns out that outer peels are almost exactly the same object as scope sequences. If $P = (p_1, ..., p_{n-1})$ is an outer peel in $\mathcal{TB}_n$ then $(p_1 - 1, ..., p_{n-1} - 1)$ is a scope sequence and this can easily be generalized to outer peels in $\mathcal{TB}_\lambda$ (compare Definition 9.1 to Proposition 2.35 and Corollary 2.36). Furthermore, for outer peels $P = (p_1, ..., p_r), Q = (q_1, ..., q_r) \in \mathcal{P}_\lambda$ it is not hard to show $P \leq_{\mathcal{P}_\lambda} Q$ if and only if $p_i \geq q_i$ for every $i \in [r]$ using an inductive argument on the number of elements of P and Q that are not equal and Lemma 2.40. This statement is the Tamari lattice anti-relation on scope sequences (again refer to Definition 9.1). The proof of Lemma 2.40 does not rely on Proposition 2.37 or the q-vectors of $\mathcal{Q}_\lambda$ which means Proposition 2.42 can be proved using Bjorner and Wachs 1997 without reference to q-vectors. On the other hand, the q-vectors are the means to relate outer peels to Young diagrams, so are an important perspective aside from their usefulness in proving Proposition 2.42.

Proposition 2.42 Let λ be a Young diagram with m corner boxes on its outer diagonal. There is an order-reversing bijection from $\mathcal{P}_\lambda$ to T_m.

Proof. To show this bijection exists we will view the elements of T_m as Young diagrams that fit in δ_m as we have done throughout this paper. Corollary 2.38 allows us to view elements P of $\mathcal{P}_\lambda$ as

Young diagrams too, however, for general λ, $sh(P) \not\subseteq \delta_m$. To deal with this we will first find an order-preserving bijection from $\mathcal{P}_\lambda$ to $\mathcal{P}_{m+1}$, and then find an order-reversing bijection from $\mathcal{P}_{m+1}$ to T_m.

Consider the map

$$\psi : \quad \mathcal{P}_\lambda \to \mathcal{P}_{m+1}$$

such that if $q(P) = (q_1, ..., q_t)$ then $q(\psi(P)) = (q_{r_1}, ..., q_{r_m})$. By Q1-Q4, ψ is clearly injective and since $|\mathcal{P}_\lambda| = |\mathcal{P}_{m+1}|$ by Proposition 2.35, ψ is a bijection. Now suppose $P \lessdot_{\mathcal{P}_\lambda} P'$ and $q(P) = (q_1, ..., q_t)$. This implies there is an increasing pentagon move on the outer diagonal of λ, say in rows r_j and r_k with $j < k$, that sends r-stats in P to r-stats in P'. In light of Proposition 2.35(1) set the jth element of $p(P)$ to be $r_j - r_i$ for $i < j$. This implies the jth element of $p(\psi(P))$ is $j - i$.

By Lemma 2.40(2)

$$q(P') = (q_1, ..., q_{r_i}, q_{r_i+1} - 1, ..., q_{r_j} - 1, q_{r_j+1}, ..., q_t),$$

so

$$q(\psi(P')) = (q_{r_1}, ..., q_{r_i}, q_{r_{i+1}} - 1, ..., q_{r_j} - 1, q_{r_{j+1}}, ..., q_{r_m}),$$

but $q(\psi(P)) = (q_{r_1}, ..., q_{r_m})$ which means $\psi(P)$ has an increasing pentagon move on the outer diagonal of δ_{m+1} involving rows j and k. Furthermore, this move clearly leads to $\psi(P')$, so $\psi(P) \lessdot_{\mathcal{P}_{m+1}} \psi(P')$. A similar argument can be used to show $\psi(P) \lessdot_{\mathcal{P}_{m+1}} \psi(P')$ implies $P \lessdot_{\mathcal{P}_\lambda} P'$.

Next, we consider the map

$$\begin{aligned} \Psi : \quad & \mathcal{P}_{m+1} \to T_m \\ & P \mapsto sh(P) = (m - q_1, m - q_2 - 1, ..., m - q_i - i + 1, ...), \end{aligned}$$

where $q(P) = (q_1, ..., q_m)$. By Corollary 2.38 Ψ is a bijection. Suppose $P \lessdot_{\mathcal{P}_\lambda} P'$ and $q(P) = (q_1, ..., q_t)$. This implies there is an increasing pentagon move on the outer diagonal of δ_{m+1}, say in rows j and k with $j < k$, that sends r-stats in P to r-stats in P', set the jth element of $p(P)$ to be $j - i$ for $i < j$. So by Lemma 2.40(2)

$$sh(P') = (m - q_1, ..., m - q_i - i + 1, m - q_{i+1} - i + 1, ..., m - q_j - j + 2, m - q_{j+1} - j, ...).$$

We want to show $sh(P') \lessdot_{T_m} sh(P)$. The box added in row j of $sh(P')$ is clearly a corner box, so we need to show the prime path of this box is stopped by row i. By Proposition 2.3 this is equivalent to showing

1. $q_{j-x} \geq q_j + 1$ for every $x \in [i+1]$, and
2. $q_i < q_j$.

Let $B_1, ..., B_m$ be the corner boxes on the outer diagonal of δ_{m+1}. Assume there exists an $x \in [i+1]$ such that $q_{j-x} < q_j + 1$. This implies that for an r-stat in P, the number of prime paths from outer diagonal corner boxes that go through row $j - x$ is at most the number of prime paths from outer diagonal corner boxes that go through row j. But by Lemma 2.5(2) every prime path that goes through row j goes through row $j - x$ and the prime path of B_{j-x} goes through row $j - x$ and not row j. This is a contradiction, so (1) is true.

Similarly, assume $q_i \geq q_j$. Since we are considering r-stats in P every prime path of corner boxes from B_{j+1} to B_k stop before B_j and every prime path of corner boxes from B_{i+1} to B_j stop before B_i. This means only prime paths that go through row j go through row i and the prime paths of B_j and B_k go through row j and not row i, which is also a contradiction, so (2) is true.

To show $sh(P') \lessdot_{T_m} sh(P)$ implies $P \lessdot_{\mathcal{P}_\lambda} P'$ just reverse the argument and use Lemma 2.40. □

Figure 16 illustrates Proposition 2.42 in the case $\lambda = \delta_4$.

We are almost ready to prove $\mathcal{TB}_\lambda$ is a lattice. In order to find the join of any two elements we prove two technical lemmas about certain chains in $\mathcal{TB}_\lambda$.

Lemma 2.43 Let λ be a Young diagram with m corner boxes on its outer diagonal, and suppose $\sigma_0 \lessdot \sigma_1 \lessdot ... \lessdot \sigma_k$ is a minimum length chain from σ_0 to σ_k in $\mathcal{TB}_\lambda$ such that every edge of the chain is an edge between equivalence classes of $\mathcal{P}_\lambda$. Then every minimum length chain from σ_0 to an element in σ_k's equivalence class in $\mathcal{TB}_\lambda$ with this property, ends at σ_k.

Proof. Let σ_i be in the equivalence class P_i for $i \in [0, k]$. It will be useful to view the equivalence classes as Young diagrams, $sh(P_i)$. By the bijection between $\mathcal{P}_\lambda$ and T_m we can view a minimum length chain from σ_0 to $\sigma'_k \in P_k$ as a chain in T_m. From this perspective such a chain begins at $sh(P_0)$ in T_m and each step of the chain means a step in T_m that adds boxes to the Young diagram. For example, the move from σ_3 to σ_4 is just an edge in T_m from $sh(P_3)$ to $sh(P_4)$. So we can view such a chain as a skew tableau of shape $sh(P_k)/sh(P_0)$ such that the boxes added in the move $\sigma_{i-1} \lessdot \sigma_i$ are labeled with i for $i \in [k]$.

We are going to use induction on k to show $\sigma'_k = \sigma_k$. If $k = 1$ then σ_0 is covered by σ_k and σ'_k. However, σ_0 can only be covered by one element of each equivalence class since by Proposition 2.24

any cover differs from σ_0 on the outer peel and the additional element which is fixed by the move. So in the base case $\sigma'_k = \sigma_k$.

Now suppose the statement is true for minimum length chains from σ_0 of length less than k (where all the edges are between equivalence classes), and consider a minimum length chain from σ_0 to σ'_k as before. Since both this chain and the chain from σ_0 to σ_k have minimum length they both have length k. In terms of Young diagrams and skew tableaux this means the statement is true for every skew tableau of length less than k and we want to consider a skew tableau that represents a chain that terminates at σ'_k and one that represents a chain that terminates at σ_k, both of length k. The proof follows the same format as the proof of Proposition 2.22. If the k-set of the original chain (viewed as a skew tableau) ends in the same row as the k-set of the chain that ends at σ'_k then we can remove the k-sets from each and use the induction hypothesis. The induction hypothesis says both of these chains end at the same r-stat, so $\sigma_{k-1} = \sigma'_{k-1}$. Now there is only one edge from σ_{k-1} to an r-stat in P_k and this edge must go to σ_k which implies $\sigma_k = \sigma'_k$.

Finally, if the k-set of the original chain does not end in the same row as the k-set of the chain that ends at σ'_k then we would like to use Lemma 2.23(2) just as we did in the proof of Proposition 2.22. Although the proof of Lemma 2.23 did not consider skew tableaux, it is not hard to see that the lemma still applies to skew tableaux. We give a sketch here. If none of the boxes of $sh(P_0)$ are on the outer diagonal of $sh(P_k)$ then the proof is exactly the same. If, however, some of the boxes of $sh(P_0)$ reach the outer diagonal of $sh(P_k)$, we can pick any r-stat in P_k to show this implies the corner box immediately below the lowest removed box is a descent of the first word of the diagonal sentence and the proof follows.

So, by Lemma 2.23(2) there are square moves from the chain that ends at σ_k (viewed as a skew tableau) and from the chain that ends at σ'_k to chains that have k-sets that end in the same row. These square moves are actually square moves on the $sh(P_i)$ in T_m, so the chains are being changed with each square move and we need to show making such a square move on a chain does not change the terminal vertex (either σ_k or σ'_k) of the chain. This is easy to see since every square move changes a vertex between two other vertices in the chain, so the terminal vertex will always be fixed. So we make these square moves, which put the k-sets ending in the same row, and use what we showed above to conclude that $\sigma'_k = \sigma_k$. $\square$

Example 2.44 Consider

$$\sigma_0 = \begin{array}{ll} 1 & 1 \\ 1 & \\ 3 & \\ 4 & \end{array} \in \mathcal{TB}_5$$

and suppose we want to show all minimum length chains to the equivalence class with outer peel $(1, 1, 1, 2)$ end at the same r-stat. By examining Figure 15 we can see every minimum length chain will have length four, so we will use the same notation from the proof above and say $P_4 = (1, 1, 1, 2)$ is the terminal equivalence class. We can also see

$$sh(P_4) = \begin{array}{|c|c|c|} \hline & & \\ \hline & \\ \cline{1-2} \end{array}$$

There are 2 minimum length chains from σ_0 to an r-stat in P_4 (they both go to the same r-stat as the lemma says they will). Because peel$(\sigma_0) = 1$ we get $P_0 = \boxed{}$. So the skew tableaux of the 2 chains are:

$$\begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \\ \cline{1-2} \end{array} \quad \text{and} \quad \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \\ \cline{1-2} \end{array}$$

Here the 4-sets end in the same row (which means we are in the 1st case of the proof), so we can remove those sets to invoke the induction hypothesis. To build the terminal r-stat (σ_4) we start with peel(σ_0) and then add the r-stat part associated with the boxes labeled with 1,2, and 3, to get:

$$\begin{array}{lll} 1 & 1 & 1 \\ 1 & & \end{array}$$

Next we add the r-stat part associated with the 4-set (which is a 1 in row 2) to get:

$$\begin{array}{lll} 1 & 1 & 1 \\ 1 & 1 & \end{array}$$

And then add the outer peel associated with P_4 to get:

$$\begin{array}{llll} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & & & \\ 2 & & & \end{array}$$

This is the terminal r-stat for both chains.

For an example of the 2nd case we let σ_0 stay the same, but let the new terminal equivalence class have outer peel $(1,1,1,3)$. The minimum length chains in this case will have length 3 so label this equivalence class P_3. We calculate

$$sh(P_3) = \ydiagram{3,1}$$

Again there are two chains from σ_0 to P_3 with associated skew tableaux:

$$\begin{ytableau} \none & 1 & 3 \\ 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} \none & 1 & 2 \\ 3 \end{ytableau}$$

To build the terminal r-stat of both chains we begin, as before, with $\mathrm{peel}(\sigma_0) = 1$. We see the two chains (as skew tableaux) are a square move apart which means the r-stats associated to the skew tableaux are the same. So we can attach the r-stat of both chains to 1 to get:

$$\begin{array}{lll} 1 & 1 & 1 \\ 1 & & \end{array}$$

Finally, we attach the outer peel $(1,1,1,3)$ to get:

$$\begin{array}{llll} 1 & 1 & 1 & 1 \\ 1 & 1 & & \\ 1 & & & \\ 3 & & & \end{array}$$

This is the terminal r-stat for both chains.

Lemma 2.45 Let λ be a Young diagram, and suppose every element of $\mathcal{P}_\lambda$, viewed as an induced subposet of $\mathcal{TB}_\lambda$, is a lattice. Let $P \in \mathcal{P}_\lambda$. If $\pi, \rho, \sigma \in P$ are such that $\sigma = \pi \vee_{\mathcal{TB}_{sh(P)}} \rho$, then $\sigma = \pi \vee_{\mathcal{TB}_\lambda} \rho$.

Proof. It is clear by construction that $\sigma \geq_{\mathcal{TB}_\lambda} \pi$ and $\sigma \geq_{\mathcal{TB}_\lambda} \rho$. Suppose there is a σ' such that $\sigma' \geq_{\mathcal{TB}_\lambda} \pi$ and $\sigma' \geq_{\mathcal{TB}_\lambda} \rho$ and let $S' \in \mathcal{P}_\lambda$ with $\sigma' \in S'$. If $S' = P$ then, by the definition of a join, $\sigma' \geq_{\mathcal{TB}_\lambda} \sigma$. Otherwise, $S' \geq_{\mathcal{P}_\lambda} P$. This means there is an embedded copy of $\mathcal{TB}_{sh(P)}$ in S', call it $\tilde{P} \subseteq S'$ and say $\tilde{\pi}, \tilde{\rho}, \tilde{\sigma} \in \tilde{P}$ are the copies of the original elements. By Lemma 2.43 there are minimal length chains from π, ρ, and σ to S' and they each terminate at $\tilde{\pi}$, $\tilde{\rho}$, or $\tilde{\sigma}$ respectively. This implies $\sigma' \not< \tilde{\sigma}$. Furthermore, S' has all the embedded copies of the Tamari block posets on shapes between P and S', so $\tilde{\pi} < \sigma'$ and $\tilde{\rho} < \sigma'$. Since S' is a lattice by assumption this implies σ' and $\tilde{\sigma}$ must be comparable, so $\sigma \leq_{\mathcal{TB}_\lambda} \tilde{\sigma} \leq_{\mathcal{TB}_\lambda} \sigma'$. $\square$

Proof.[Proof of Theorem 2.1(3)] We are going to induct on n. It is easily seen by inspection that $\mathcal{TB}_\lambda$ is a lattice for every $\lambda \subseteq \delta_n$ for $n \in [4]$. Now suppose the statement is true for every $\lambda \subseteq \delta_n$ for some $n > 4$ and consider $\mathcal{TB}_\lambda$ where $\lambda \subseteq \delta_{n+1}$, but $\lambda \not\subseteq \delta_n$.

Let $\pi, \rho \in \mathcal{TB}_\lambda$. We are going to show $\pi \vee \rho$ exists. Suppose $P, R \in \mathcal{P}_\lambda$ are such that $\pi \in P$ and $\rho \in R$. If $P = R$ then the outer peel of π is equal to the outer peel of ρ. $sh(P) \subseteq \delta_n$ so by the inductive hypothesis peel$(\pi) \vee_{\mathcal{TB}_{sh(P)}}$ peel(ρ) exists (recall P as a subposet of $\mathcal{TB}_\lambda$ is isomorphic to $\mathcal{TB}_{sh(P)}$). Because the outer peels are in bijection with the vertices of $\mathcal{P}_\lambda$ we can add the outer peel of π to peel$(\pi) \vee_{\mathcal{TB}_{sh(P)}}$ peel(ρ) to get an element of $\mathcal{TB}_\lambda$, say σ. By Lemma 2.45 $\sigma = \pi \vee_{\mathcal{TB}_\lambda} \rho$.

The other possibility is that $P \neq R$. By Proposition 2.42 $\mathcal{P}_\lambda$ is a lattice, so $P \vee_{\mathcal{P}_\lambda} R$ is an element of $\mathcal{P}_\lambda$, say S. Furthermore, if $P = P_0 \lessdot P_1 \lessdot ... \lessdot P_{k_1} = S$ and $R = R_0 \lessdot R_1 \lessdot ... \lessdot R_{k_2} = S$ are chains in $\mathcal{P}_\lambda$ then each edge of the chain is an edge in $\mathcal{TB}_\lambda$ between equivalence classes. So there exists an isomorphic chain in $\mathcal{TB}_\lambda$ from π or ρ to an element in S. Let $\sigma_\pi, \sigma_\rho \in S$ such that σ_π and σ_ρ are the last elements of minimum length chains from π and ρ (respectively) that change equivalence classes at every step. By Lemma 2.43 σ_π and σ_ρ are unique. Furthermore, $S \subseteq \delta_n$ so by the inductive hypothesis peel$(\sigma_\pi) \vee_{\mathcal{TB}_{sh(S)}}$ peel(σ_ρ) exists. Because the outer peels are in bijection with the vertices of $\mathcal{P}_\lambda$ we can add the outer peel associated to S to peel$(\sigma_\pi) \vee_{\mathcal{TB}_{sh(S)}}$ peel(σ_ρ) to get an element of $\mathcal{TB}_\lambda$, say σ.

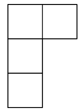
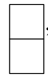
It remains to show that $\sigma = \pi \vee \rho$. It is clear by construction that $\sigma \geq \pi$ and $\sigma \geq \rho$. Suppose $\sigma' \geq \pi$ and $\sigma' \geq \rho$, and $\sigma' \in S' \in \mathcal{P}_\lambda$. If $S' = S$ then by Lemma 2.45 $\sigma' \geq \sigma$. If $S' > S$ it is not hard to show $\sigma' \geq \sigma$ by using a similar argument to the one used in Lemma 2.45. On the other hand, if $S' \not\geq S$, then since $\mathcal{P}_\lambda$ is a lattice, there cannot be chains from P and R to S', which is a contradiction. So $\sigma = \pi \vee \rho$ in $\mathcal{TB}_\lambda$.

Finally, since any two elements of $\mathcal{TB}_\lambda$ have a join and since $\mathcal{TB}_\lambda$ is a finite poset with $\hat{0}$, it is clear any two elements of $\mathcal{TB}_\lambda$ also have a meet (find the join of the nonempty set of lower elements of the two elements). This implies $\mathcal{TB}_\lambda$ is a lattice. $\square$

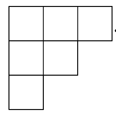
Example 2.46 Consider r-stats

$$\pi = \begin{array}{cc} 1 & 1 \\ 1 & \\ 3 & 3 \\ 1 & \end{array}, \rho = \begin{array}{cc} 1 & \\ 2 & 2 \\ 1 & \\ 4 & \end{array} \in \mathcal{TB}_5.$$

They have different outer peels which we represent as Young diagrams

 and ,

respectively. The join of these elements in $\mathcal{P}_4$ is

.

The terminal vertex of the respective chains from π and ρ to this equivalence class is highlighted in Figure 17. Once we are in the equivalence class we inductively make our way to $\pi \vee \rho$ which has also been highlighted in Figure 17.

$$\pi \vee \rho = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & 1 & & \\ 1 & & & \end{array}$$

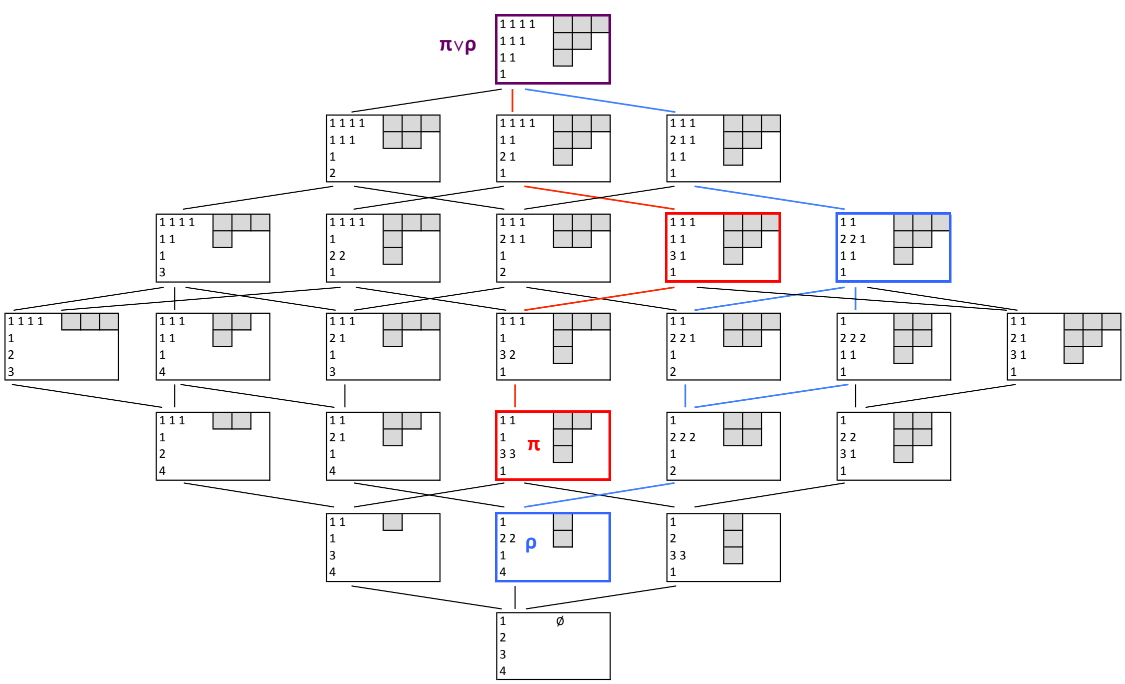


Figure 17: $\mathcal{TB}_5$ *with* $\pi \vee \rho$ *from Example 2.5.16 highlighted.*

Ziegler 1995, Example 0.6 defines the *cyclic polytope*, $\mathbf{C}_d(t_1, ..., t_n)$, by first defining the d-dimensional *moment curve* in $\mathbb{R}^d$ as the image of the parametrization

$$\begin{aligned} \mathbf{x}: \quad & \mathbb{R} \to \mathbb{R}^d \\ & t \mapsto \mathbf{x}(t) = (t, t^2, ..., t^d). \end{aligned}$$

$\mathbf{C}_d(t_1, ..., t_n)$ is the convex hull

$$\mathbf{C}_d(t_1, ..., t_n) = \operatorname{conv}\{\mathbf{x}(t_1), \mathbf{x}(t_2), ..., \mathbf{x}(t_n)\}$$

of $n > d$ distinct points $\mathbf{x}(t_i)$, with $t_1 < t_2 < ... < t_n$, on the moment curve. He goes on to show the combinatorial properties of this polytope do not depend on a specific choice of the parameters t_i, which explains our notation, $\mathbf{C}(n, d)$, for the d-dimensional cyclic polytope on n vertices. In light of this, we refer to faces of $\mathbf{C}(n, d)$ by the sets of subscripts $\{i_0, i_1, ..., i_k\}$ without reference to the choice of coordinates.

A *triangulation* of a polytope P is a subdivision of P such that every polytope of the subdivision is a simplex. We obtain a canonical *upper* (respectively *lower*) triangulation of $\mathbf{C}(n, d)$ by projecting via $\mathbf{C}(n, d+1) \to \mathbf{C}(n, d)$ the boundary facets of $\mathbf{C}(n, d+1)$ visible from points with large (respectively small) x_{d+1} coordinate. See Rambau and Reiner 2012 for a more nuanced discussion of this projection. The upper triangulation of $\mathbf{C}(5, 2)$ is the bottom element of the left poset in Figure 18, and the lower triangulation is the top element of the same poset (the poset is reversed from the more natural orientation for reasons we give later).

As we mentioned in Section 2.1, Kapranov and Voevodsky 1991 and later Edelman and Reiner 1996 introduced the Higher Stasheff-Tamari Orders, $\mathrm{HST}_1(n, d)$ and $\mathrm{HST}_2(n, d)$. (We are following the notation from the survey article Rambau and Reiner 2012). The elements of these orders are the triangulations of $\mathbf{C}(n, d)$. It was shown in Edelman and Reiner 1996 that the two orders are the same for $d = 1, 2, 3$, however whether this is true in general is still an open problem. Because we are most interested in $\mathrm{HST}_1(n, d)$, and specifically in the case $d = 3$, we will omit the definition of $\mathrm{HST}_2(n, d)$ and refer the interested reader to Rambau and Reiner 2012, Definition 3.1.

Definition 2.47 (Rambau and Reiner 2012, Definition 3.2) To define the *first higher Stasheff-Tamari order* $\mathrm{HST}_1(n, d)$ on triangulations of $\mathbf{C}(n, d)$, first define when a triangulation $\mathcal{T}'$ is obtained from a triangulation $\mathcal{T}$ by an *upward flip*: this means that there exists a

$(d+2)$-subset $t_{i_1} < t_{i_2} < ... < t_{i_{d+2}}$ of the vertices of $\mathbf{C}(n,d)$ whose convex hull gives a subpolytope $\mathbf{C}(d+2,d)$ of $\mathbf{C}(n,d)$ with the property that $\mathcal{T}, \mathcal{T}'$ restrict to the lower, upper triangulations of this $\mathbf{C}(d+2,d)$, and otherwise $\mathcal{T}$, $\mathcal{T}'$ agree on all of their other simplices not lying in this $\mathbf{C}(d+2,d)$.

Then define $\mathcal{T} \leq_{\mathrm{HST}_1(n,d)} \mathcal{T}'$ if there is a sequence of upward flips starting with $\mathcal{T}$ and ending with $\mathcal{T}'$. That is, $\mathrm{HST}_1(n,d)$ is the transitive closure of the upward flip relation.

Examples of these upward flips between triangulations of $\mathbf{C}(5,2)$ can be seen by comparing adjacent elements in the left poset of Figure 18. The higher Stasheff-Tamari orders are familiar for smaller dimensions. $\mathrm{HST}_1(n,0)$ is the linear order $1 < 2 < ... < n$ (Rambau and Reiner 2012, Example 3.4). $\mathrm{HST}_1(n,1)$ is isomorphic to the Boolean lattice on $n-2$ objects (Example 3.5). And as we have already mentioned $\mathrm{HST}_1(n,2)$ is isomorphic to the Tamari lattice, T_{n-2} (Example 3.6).

Just as there are many ways to encode the Tamari lattices, the higher Stasheff-Tamari orders have been encoded in many ways. In Edelman and Reiner 1996 and Rambau 1997 triangulations of $\mathbf{C}(n,d)$ are characterized by certain *submersion sets* which are collections of subsets $\{i_1, ..., i_{k+1}\}$ indexing k-simplices that have special significance to a particular triangulation. $\mathrm{HST}_1(n,3)$ was first proved to be a graded lattice using this encoding. Thomas 2002b shows a bijection between the elements of $\mathrm{HST}_2(n,d)$ and certain d-fold Cartesian products called *snug rectangles.* This bijection is then used to show a poset embedding

$$\mathrm{HST}_2(n,d) \hookrightarrow \prod_{j=d}^{n-1} \mathrm{HST}_2(j,d-2).$$

And finally, in Oppermann and Thomas 2010 triangulations of $\mathbf{C}(n,d)$ are encoded by certain $\frac{d}{2}$-faces called *non-interlacing separated $\frac{d}{2}$-faces.* Through this encoding the authors study triangulations of $\mathbf{C}(n,d)$ in terms of cluster algebras.

Our goal in this section is to show an order-reversing bijection between $\mathcal{TB}_n$ and $\mathrm{HST}_1(n+2,3)$. We begin in the 2-dimensional case with a bijection between Young diagrams that fit in δ_n and triangulations of an $(n+2)$-gon. This is a restatement of Fishel, Kallipoliti, and Tzanaki 2013, Theorem 4.2 in the case where $m=1$, and can also be derived from Hilton and Pedersen 1991 (a similar map is found in Gorsky 2011, Proposition 2.1). A triangulation of an $(n+2)$-gon has $n-1$ diagonals, we call the *initial points* (respectively *terminal points*) the lesser (respectively, greater) endpoints of the diagonals.

Proposition 2.48 Let P be an $(n+2)$-gon with vertices labeled by the integers in $[0, n+1]$ in counterclockwise order. Let T be a triangulation of P with initial points $\{i_1, i_2, ..., i_{n-1}\}$, where we assume that $i_1 \geq i_2 \geq ... \geq i_{n-1}$. The map θ_2 from the set of all triangulations of P to Young diagrams defined by

$$\theta_2(T) = (i_1, ..., i_{n-1})'$$

is a bijection.

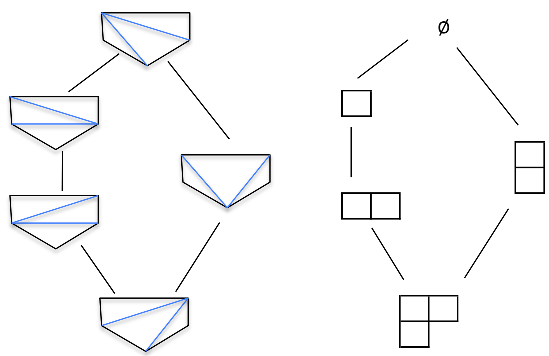


Figure 18: *An order-reversing bijection between triangulations of a pentagon and Young diagrams that fit in* δ_3.

Figure 18 shows the map for triangulations of a pentagon. The map in Proposition 2.48 is an order-reversing map, but we could not find this in the literature, so we prove it in Proposition 2.49. T_n is self dual so the order-reversing property is somewhat of a formality, but it is significant for us in at least two ways. First, θ_2 reverses the purposeful choice of the direction of the order for one of the encodings of T_n (we reverse the triangulation encoding and leave the Young diagram encoding alone in Figure 18), and second it foreshadows the 3-dimensional case that is not self dual.

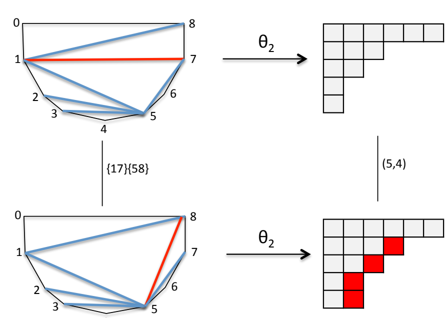


Figure 19: *A cover relation in* $\mathrm{HST}_1(9,2)$ *and the corresponding cover relation in* T_7. *Notice the greater triangulation is below the lesser one to match the Young diagrams.*

Before stating the next proposition, we note that edges in $\mathrm{HST}_1(n, d)$ can be labeled with the indices of the simplex removed from the covered triangulation combined with the indices of the simplex added to the covering triangulation (Rambau 1997; Edelman and Reiner 1996; Rambau and Reiner 2012). For $d = 2$ this means labeling edges with the initial point and end point of both the removed and added line segment. For example, in Figure 19 we label the edge with $\{17\}\{58\}$ because we remove the diagonal 17 and replace it with the diagonal 58. Let $\widetilde{T}_n$ be the Young diagram encoding of T_n to avoid confusion. We label the edges of $\widetilde{T}_n$ with an ordered pair, (r, x), where r is the row of the corner box removed from the covered Young diagram and x is the number of boxes removed with the corner box, again see Figure 19 for an example.

Proposition 2.49 θ_2 in Proposition 2.48 is an order-reversing map.

Proof. Since we already know both orders are encodings for T_n it is enough to show $\mathcal{T} \lessdot_{\mathrm{HST}_1(n+2,2)} \mathcal{T}'$ implies $\theta_2(\mathcal{T}') \lessdot_{\widetilde{T}_n} \theta_2(\mathcal{T})$ for every $\mathcal{T}, \mathcal{T}' \in \mathrm{HST}_1(n + 2, 2)$.

To show this implication we consider the edge labeling of both encodings. Suppose the move from triangulation $\mathcal{T}'$ to $\mathcal{T}$ interchanges initial point j for initial point i. It is not hard to show $\theta_2(\mathcal{T}')$ has a corner box in row j. It is also clear that $\theta_2(\mathcal{T})$ has $j - i$ fewer boxes that have been removed from rows $i + 1, ..., j$. However, we still need to show the prime path of the corner box in row j of $\theta_2(\mathcal{T}')$ ends at row i.

We will show an equivalent statement. If $\{i_1, ..., i_{n-1}\}$ is the set of initial points of $\mathcal{T}'$ where we assume that $i_1 \geq i_2 \geq ... \geq i_{n-1}$, then let $j = i_\alpha$. Clearly, $i < j$, say $i = i_{\alpha+\beta}$. We claim showing the prime path of the corner box in row j of $\theta_2(\mathcal{T}')$ ends at row i is equivalent to showing β is the minimum integer such that

$$i_{\alpha+\beta} = i_\alpha - \beta. \tag{2.3}$$

The statements can be shown to be equivalent by examining the relationship between rows in the Young diagram and the values of the initial points under θ_2 and using the definition of a prime path of a corner box.

Now suppose the edge between $\mathcal{T}$ and $\mathcal{T}'$ is labeled $\{i_{\alpha+\beta}, t_{\alpha+\beta}\}\{i_\alpha, t_\alpha\}$. This implies $i_{\alpha+\beta} < i_\alpha < t_{\alpha+\beta} < t_\alpha$ and these four points form a quadrilateral in $\mathcal{T}'$ that is only crossed by $\{i_\alpha, t_\alpha\}$. This implies $\{i_{\alpha+\beta}, i_\alpha\}$ is a face or a diagonal of $\mathcal{T}'$. If it is a face then $\beta = 1$, $i_{\alpha+1} = i_\alpha - 1$, and Equation 2.3 is true. If it is a diagonal, then there are $i_\alpha - i_{\alpha+\beta} - 2$ diagonals with initial points in $[i_{\alpha+\beta}, i_\alpha]$ (not including $\{i_{\alpha+\beta}, i_\alpha\}$ and $\{i_{\alpha+\beta}, t_{\alpha+\beta}\}$). Furthermore, $i_\alpha - k$ can be an initial vertex for at most $k-1$ diagonals for $k \in [0, i_\alpha - i_{\alpha+\beta}]$, so the initial points are

$$..., i_\alpha, i_{\alpha+1}, ..., i_{\alpha+i_\alpha-i_{\alpha+\beta}-2}, i_{\alpha+\beta}, i_{\alpha+\beta}, ...$$

and Equation 2.3 is true. $\square$

Example 2.50 Suppose the red diagonal, $\{6, 10\}$, in Figure 20 is the diagonal to be flipped to move down in $\mathrm{HST}_1(12, 2)$. So we can view Figure 20 as $\mathcal{T}'$ with only the diagonals forming the quadrilateral of the flip shown. This means $\mathcal{T}'$ has initial points $\{..., 6, 6, ..., 1, 1, ...\}$. According to the proof of Proposition 2.49 there are $6 - 1 - 2 = 3$ diagonals inside the polygon $\{1, 2, ..., 6\}$ which is clear from the figure. Furthermore, 5 can not be an initial point, 4 can be an initial point at most once, 3 at most twice, and 2 can be an initial point at most three times. Together this shows that the second 1 in the set of initial points is the largest initial point that satisfies Equation 2.3. Finally, it is clear from the figure that flipping $\{6, 10\}$ results in the diagonal $\{1, 9\}$, so the initial point 6 is replaced with initial point 1, so $\theta_2(\mathcal{T}')$ will have five more boxes than $\theta_2(\mathcal{T})$, one box extra in each of rows $2, 3, ..., 6$.

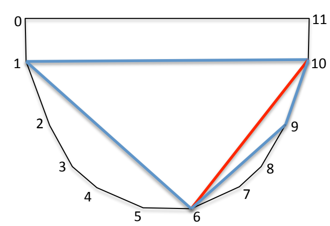


Figure 20: *Diagonal flip example for Proposition 2.6.3*

Rambau 1997 defines an equivalence relation on maximal chains of $\mathrm{HST}_1(n, d)$. For a maximal chain M of $\mathrm{HST}_1(n, d)$ define the set $E(M)$ to be the set of edges of M. This leads to the equivalence relation $M_1 \sim_E M_2$ if and only if $E(M_1) = E(M_2)$. Rambau then proves that the elements of $\mathrm{HST}_1(n, d+1)$ are in one-to-one correspondence with the equivalence classes of maximal chains in $\mathrm{HST}_1(n, d)$ (Theorem 1.1(ii)). Specifically, every element of $\mathrm{HST}_1(n, 3)$ corresponds to a set of edges in $\mathrm{HST}_1(n, 2)$ that form at least one maximal chain. By Proposition 2.49 θ_2 gives an edge-labeling of $\widetilde{T}_n$ from the edge-labeling of $\mathrm{HST}_1(n+2, 2)$. Explicitly, if $\{i_1, t_1\}\{i_2, t_2\}$ is an edge in $\mathrm{HST}_1(n+2, 2)$ with $i_1 < i_2$, then the associated edge of $\widetilde{T}_n$ is labeled $(i_2, i_2 - i_1)$. See Figure 19 for an example. We will use $\theta_2(M)$ to refer to the maximal chain in $\widetilde{T}_n$ whose elements are associated to the elements of

M. Let $\sim_{E'}$ be the equivalence relation on the maximal chains of $\widetilde{T}_n$ such that $M_1 \sim_{E'} M_2$ if and only if $E'(M_1) = E'(M_2)$ where $E'(M_i)$ is the multiset of edges of M_i in $\widetilde{T}_n$. The next proposition and the following corollary show that $\sim_{E'}$ is the same equivalence relation that we used to define square blocks in Section 2.2, and therefore the elements of $\mathrm{HST}_1(n+2,3)$ are in bijection with the elements of $\mathcal{TB}_n$.

Proposition 2.51 Let M_1 and M_2 be maximal chains in $\widetilde{T}_n$. $M_1 \sim_{E'} M_2$ if and only if M_1 and M_2 are in the same square block of $\mathcal{TB}_n$.

Proof. To prove this we will show there is a bijective map from the edge sets of maximal chains in $\widetilde{T}_n$ to r-stats in $\mathcal{TB}_n$. Define the map by sending the edge set

$$E'(M) = \{(r_1, x_1), ..., (r_k, x_k)\}$$

to the r-stat in $\mathcal{TB}_n$ with x_i in row j if and only if $r_i = j$. Comparing this operation to the definition of an r-stat shows that it is well-defined and surjective. The usual test shows this map is also injective. $\square$

In light of Proposition 2.48 and Proposition 2.51 the next corollary is clear.

Corollary 2.52 The map

$$\theta_3 : \quad \mathrm{HST}_1(n+2,3) \to \mathcal{TB}_n$$

that sends $E(M)$ to the r-stat associated with $E'(\theta_2(M))$ is a bijection.

One of the beautiful properties of Rambau 1997, Theorem 1.1(ii) is that the subpolytope where the flip associated to an edge in $\mathrm{HST}_1(n,d)$ occurs ends up being one of the simplices in any triangulation in $\mathrm{HST}_1(n,d+1)$ associated to a maximal chain in $\mathrm{HST}_1(n,d)$ that contains that edge. In other words, for a maximal chain $M \in \mathrm{HST}_1(n,d)$ such that $E(M)$ is associated with triangulation $\mathcal{T} \in \mathrm{HST}_1(n,d+1)$, the simplices that form $\mathcal{T}$ are the subpolytopes where each flip of $E(M)$ takes place. Specifically, if $\{i_1,t_1\}\{i_2,t_2\}$ is an edge label of a maximal chain $M \in \mathrm{HST}_1(n,2)$ and $\mathcal{T}$ is the triangulation in $\mathrm{HST}_1(n,3)$ associated with M's equivalence class, then $\mathcal{T}$ contains the simplex $\{i_1,i_2,t_1,t_2\}$.

Example 2.53 Let $n = 5$ and consider the set of edge labels

$$E = \{\{02\}\{13\}, \{03\}\{14\}, \{04\}\{16\}, \{05\}\{46\}, \{13\}\{24\}, \{14\}\{26\}, \{24\}\{36\}\}.$$

There are seven maximal chains in $\mathrm{HST}_1(7,2)$ with these edge labels. The triangulation in $\mathrm{HST}_1(7,3)$ associated with E is

$$\mathcal{T} = \{\{0123\}, \{0134\}, \{0146\}, \{0456\}, \{1234\}, \{1246\}, \{2346\}\}.$$

Furthermore, if we convert the edge labels of E using θ_2 we get edge labels

$$E' = \{(1,1), (1,1), (1,1), (4,4), (2,1), (2,1), (3,1)\}.$$

By Proposition 2.51, E' is associated with r-stat

$$\pi = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \\ 4 & & \end{array} \in \mathcal{TB}_5.$$

And so $\theta_3(\mathcal{T}) = \pi$.

This relationship between subpolytopes associated to flips in $\mathrm{HST}_1(n,2)$ and simplices of triangulations of $\mathrm{HST}_1(n,3)$ makes the relationship between $\mathrm{HST}_1(n+2,3)$ and $\mathcal{TB}_n$ clear. By Definition 2.47, $\mathcal{T} \lessdot_{\mathrm{HST}_1(n+2,3)} \mathcal{T}'$ if and only if $\mathcal{T}$ and $\mathcal{T}'$ agree on all of their simplices except those involving a subpolytope $\mathbf{C}(5,3)$ with indices $i_1 < i_2 < ... < i_5$ such that

1. $\mathcal{T}$ (respectively $\mathcal{T}'$) restricted to $\{i_1, i_2, ..., i_5\}$ is the lower (respectively higher) triangulation, and as a result,
2. $\mathcal{T}$ contains the simplices $\{i_1, i_2, i_3, i_4\}$, $\{i_1, i_2, i_4, i_5\}$, and $\{i_2, i_3, i_4, i_5\}$, which are replaced with the simplices $\{i_1, i_2, i_3, i_5\}$ and $\{i_1, i_3, i_4, i_5\}$ in $\mathcal{T}'$.

See Rambau and Reiner 2012, Figure 10 for a depiction of these simplices. So $\mathcal{T}$ and $\mathcal{T}'$ represent equivalence classes of maximal chains in $\mathrm{HST}_1(n+2,2)$ that are the same everywhere except $\mathcal{T}$ has three edges that are replaced by two different edges in $\mathcal{T}'$. This sounds eerily like a pentagon move, and we have already shown the equivalence classes of $\mathrm{HST}_1(n+2,3)$ are exactly the equivalence classes of $\mathcal{TB}_n$. We are now ready to prove $\mathcal{TB}_n$ is anti-isomorphic to $\mathrm{HST}_1(n+2,3)$, where *anti-isomorphic* means isomorphic to the dual of the poset. We will do this by proving something stronger, that θ_3 is an order-reversing map, which also proves the claim of Theorem 2.2.

Proof.[Proof of Theorem 2.2] First suppose $\mathcal{T} \lessdot_{\mathrm{HST}_1(n+2,3)} \mathcal{T}'$, say the subpolytope containing the flip has indices $i_1 < i_2 < ... < i_5$. We will show this implies there is an increasing pentagon move from $\mathcal{T}'$ to $\mathcal{T}$ (viewed as square blocks by Proposition 2.51). View $\mathcal{T}'$ as an equivalence class of maximal chains in $\mathrm{HST}_1(n+2,2)$ and let $M=(f_1,...,f_\ell)\in\mathcal{T}'$ where each $f_k=\{j_1,j_3\}\{j_2,j_4\}$ is an upward flip in $\mathrm{HST}_1(n+2,2)$. This means $\{i_1,i_4\}\{i_3,i_5\}$ and $\{i_1,i_3\}\{i_2,i_5\}$ are in M, and they must appear in that order since $\{i_3,i_5\}$ is a face of the subpolytope where the second flip takes place. This implies $\{i_1,i_3,i_5\}$ is a triangle in every triangulation of M between the $\{i_1,i_4\}\{i_3,i_5\}$ flip and the $\{i_1,i_3\}\{i_2,i_5\}$ flip. Let M_1 be the subchain of M before $\{i_1,i_4\}\{i_3,i_5\}$, M_2 be the subchain of M between $\{i_1,i_4\}\{i_3,i_5\}$ and $\{i_1,i_3\}\{i_2,i_5\}$, and M_3 be the subchain after $\{i_1,i_3\}\{i_2,i_5\}$. By our work so far every flip in M_2 occurs within the polytope $P_1=\{0,...,i_1,i_5,...,n+1\}$, $P_2=\{i_1,...,i_3\}$, or $P_3=\{i_3,...,i_5\}$. Every flip in M_2 that occurs within P_2 can be done before $\{i_1,i_4\}\{i_3,i_5\}$ without changing $E(M)$, and every flip in M_2 that occurs within P_1 or P_3 can be done after $\{i_1,i_3\}\{i_2,i_5\}$ without changing $E(M)$. So there is a maximal chain $M^*=(...,\{i_1,i_4\}\{i_3,i_5\},\{i_1,i_3\}\{i_2,i_5\},...)\in\mathcal{T}'$. This means $\mathcal{T}$ has a maximal chain that is equal to M^* everywhere except the subchain $\{i_1,i_4\}\{i_3,i_5\},\{i_1,i_3\}\{i_2,i_5\}$ is replaced with $\{i_1,i_3\}\{i_2,i_4\},\{i_1,i_4\}\{i_2,i_5\},\{i_2,i_4\}\{i_3,i_5\}$. This is the definition of an increasing pentagon move from $\mathcal{T}'$ to $\mathcal{T}$.

Now suppose $\theta_3(\mathcal{T}')\lessdot_{\mathcal{TB}_n}\theta_3(\mathcal{T})$. We will show $\mathcal{T}\lessdot_{\mathrm{HST}_1(n+2,3)}\mathcal{T}'$. We know the conditions of Proposition 2.24 apply, so say the increasing pentagon move occurs in rows $i<j$, $\theta_3(\mathcal{T})$'s additional part in row i is b, and $\theta_3(\mathcal{T}')$ has c in row j which is $c-b$ in $\theta_3(\mathcal{T})$. By Proposition 2.24 $c-b=j-i$. Let $M'\in\theta_3(\mathcal{T}')$ with an increasing pentagon move to $M\in\theta_3(\mathcal{T})$. So $\theta_2^{-1}(M')$ and $\theta_2^{-1}(M)$ agree everywhere except on the edges of the pentagon move. This implies there are five points involved in the edges that don't agree, say $i_1<...<i_5$. But this means $i_1=i-b$, $i_2=i$, and $i_3=j$, and also that the edges of $\theta_2^{-1}(M')$ that are not in $E(\theta_2^{-1}(M))$ are $\{i-b,i_4\}\{j,i_5\}$ and $\{i-b,j\}\{i,i_5\}$ which are replaced with $\{i-b,j\}\{i,i_4\}$, $\{i-b,i_4\}\{i,i_5\}$, and $\{i,i_4\}\{j,i_5\}$ in $\theta_2^{-1}(M)$. So by the definition of an upward flip $\theta_3^{-1}(\theta_3(\mathcal{T}))\lessdot_{\mathrm{HST}_1(n+2,3)}\theta_3^{-1}(\theta_3(\mathcal{T}'))$ and so $\mathcal{T}\lessdot_{\mathrm{HST}_1(n+2,3)}\mathcal{T}'$.

It follows from these observations that $\mathcal{TB}_n$ is anti-isomorphic to $\mathrm{HST}_1(n+2,3)$ and θ_3 is an order-reversing map. $\square$

Chapter 3

A TEST FOR THE LATTICE PROPERTY

3.1 The Test

Recall that Theorem 2.1(3) proves $\mathcal{TB}_\lambda$ is a lattice. The goal of this chapter is to generalize the proof of Theorem 2.1(3) into a test to determine if any poset with similar properties is a lattice. I want to thank Vic Reiner for planting the seed of this chapter by encouraging me to work out such a generalization during a conversation we had at a conference. Recall that the proof leveraged the partitioning of $\mathcal{TB}_\lambda$ into equivalence classes that formed $\mathcal{P}_\lambda$ as an induced poset. Conveniently $\mathcal{P}_\lambda$ had certain properties that proved the existence of the join of any two elements in $\mathcal{TB}_\lambda$. Specifically,

1. $\mathcal{P}_\lambda$ was a lattice (consequence of Proposition 2.42),
2. each equivalence class viewed as an induced poset was isomorphic to a lattice (this was true by the induction step of the proof),
3. If $P \lessdot_{P_\lambda} R$ then for every $\pi \in P$ there was a unique $\rho_\pi \in R$ such that $\pi \lessdot_{\mathcal{TB}_\lambda} \rho_\pi$ and $\{\rho_\pi \mid \pi \in P\}$ formed an isomorphic copy of P as a poset in R (consequence of Corollary 2.38), and
4. every minimum length chain from a fixed element of $\mathcal{TB}_\lambda$ to a fixed equivalence class ended at the same element (Lemma 2.43).

(3) can be considered a strong embedding property. It basically says if two equivalence classes are related by $P \leq_{\mathcal{P}_\lambda} R$ then R has an embedded copy of P, but it further states for each cover R of P every element of P has exactly one cover in R. The importance of each of these properties of $\mathcal{TB}_\lambda$ will become evident as I prove the next two theorems. In particular, Theorem 3.1 says that a poset with properties like these is a lattice, and Theorem 3.2 says if the poset is graded in addition to having all but the last property, then it is a lattice if and only if it has the last property.

Theorem 3.1 Suppose P is a finite poset and the elements of P can be partitioned such that:

L1. the induced poset on the equivalence classes is well-defined and a lattice,

L2. each equivalence class viewed as an induced poset is a lattice, L3. if $\overline{S}$ and $\overline{T}$ are equivalence classes such that $\overline{S} \lessdot \overline{T}$ in the lattice from ($L1$), then for every $s \in \overline{S}$ there is a unique $t_s \in \overline{T}$ such that $s \lessdot t_s$ in P and $\{t_s \mid s \in \overline{S}\}$ forms an isomorphic copy of $\overline{S}$ as a poset, and

L4. every minimum length chain from a fixed element of P to a fixed equivalence class ends at the same element.

Then P is a lattice.

Proof. First I claim that P has a minimum element. ($L1$) implies all of the minimal elements of P are in the same equivalence class, but there cannot be any relations between elements of this equivalence class and the equivalence class must be a lattice by ($L2$). This means the class has one element, so by definition that element must be the minimum element of P.

Now let $p, r \in P$ and say $p \in \overline{S_p}$ and $r \in \overline{S_r}$. I want to show $p \vee_P r$ exists. If $\overline{S_p} = \overline{S_r}$, then $s = p \vee_{\overline{S_p}} r$ exists since $\overline{S_p}$ is a lattice by ($L2$). It is left to prove $s = p \vee_P r$. It is clear by construction that $s \geq_P p$ and $s \geq_P r$. Suppose there is a s' such that $s' \geq_P p$ and $s' \geq_P r$ and let $s' \in \overline{S'}$. $\overline{S'} \geq \overline{S_p}$ since $s' \geq_P p$. If $\overline{S'} = \overline{S_p}$ then, by the definition of a join, $s' \geq_P s$. Otherwise, $\overline{S'} > \overline{S_p}$. So by ($L3$) there is an embedded copy of $\overline{S_p}$ in $\overline{S'}$. To see this consider a chain from $\overline{S_p}$ to $\overline{S'}$, ($L3$) says there is an embedded copy of $\overline{S_p}$ in each equivalence class of the chain including $\overline{S'}$. Call the embedded copy $\widetilde{S_p} \subseteq \overline{S'}$ and say $\tilde{p}, \tilde{r}, \tilde{s} \in \widetilde{S_p}$ are the copies of the original elements. By ($L3$) again there is a chain from p, r, and s to $\overline{S'}$, so by ($L4$) there are minimal length chains from p, r, and s to $\overline{S'}$ and they each terminate at $\tilde{p}$, $\tilde{r}$, and $\tilde{s}$ respectively. This implies $s' \not< \tilde{s}$. Furthermore, $\overline{S'}$ has an embedded copy of any equivalence class between $\overline{S_p}$ and $\overline{S'}$, so $\tilde{p} < s'$ and $\tilde{r} < s'$. Since $\overline{S'}$ is a lattice by ($L2$) this implies s' and $\tilde{s}$ must be comparable, so $s \leq_P \tilde{s} \leq_P s'$.

Next suppose $\overline{S_p} \neq \overline{S_r}$. Because the equivalence classes form a lattice, let $\overline{T} = \overline{S_p} \vee \overline{S_r}$. If $\overline{S_p} = \overline{S_0} \lessdot \overline{S_1} \lessdot \ldots \lessdot \overline{S_{k_1}} = \overline{T}$ and $\overline{S_r} = \overline{R_0} \lessdot \overline{R_1} \lessdot \ldots \lessdot \overline{R_{k_2}} = \overline{T}$ are chains in this lattice then each edge of the chain is an edge in P between equivalence classes. So there exists an isomorphic chain in P from p and r, respectively, to an element in $\overline{T}$. Let $t_p, t_r \in \overline{T}$ be such that t_p and t_r are the last elements of minimum length chains from p and r, respectively. By ($L4$) t_p and t_r are unique. By ($L2$) $\overline{T}$ is a lattice, so $s = t_p \vee_{\overline{T}} t_r$ exists. It is clear by construction that $s \geq_P p$ and $s \geq_P r$. Suppose there is a s' such that $s' \geq_P p$ and $s' \geq_P r$ and let $s' \in \overline{S'}$. If $\overline{S'} \geq \overline{T}$ we can repeat our argument above to show $s' \geq_P s$. On the other hand, if $\overline{S'} \not\geq \overline{T}$, then since the equivalence classes form a lattice, there cannot be chains from $\overline{S_p}$ and $\overline{S_r}$ to $\overline{S'}$, which is a contradiction. And so, $s = p \vee_P r$.

Finally, since any two elements of P have a join and since P is a finite poset with a minimum element, it is clear any two elements of P also have a meet (find the join of the nonempty set of lower bounds of the two elements). This implies P is a lattice. $\square$

Theorem 3.2 Suppose P is a finite graded poset. Then P is a lattice if and only if a partition that satisfies $(L1) - (L3)$ from Theorem 3.1 also satisfies $(L4)$.

Proof. If $(L4)$ is satisfied for some partition that satisfies $(L1) - (L3)$, then Theorem 3.1 implies P is a lattice. For the other direction consider the contrapositive and argue by contradiction. So say there are minimum length chains from $p \in P$ to $\overline{S}$ that terminate at $s_1 \neq s_2$, and assume P is a lattice. By $(L2)$ $\overline{S}$ is a lattice, so $s = s_1 \wedge_{\overline{S}} s_2$ exists. Furthermore, since P is graded, s_1 and s_2 must have the same rank, so s is strictly less than both. By construction p cannot be less than or equal to s, but s_1 and s_2 are upper bounds for both p and s. Now since P is a lattice by assumption, $t = p \vee_P s$ exists, but it cannot be in $\overline{S}$ since $p \leq t < s_1$ and s_1 is minimal. However if $t \notin \overline{S}$ then the order between equivalence classes is not well-defined since $s < t < s_1$. This is a contradiction. Therefore if $(L4)$ is not satisfied for a partition that satisfies $(L1) - (L3)$ then P is not a lattice. $\square$

3.2 Applications

Because $\mathcal{P}_\lambda$ is an order on equivalence classes of $\mathcal{TB}_\lambda$ it is reasonable to ask if the outer peel relation is a *lattice congruence*, meaning the projection map from $\mathcal{TB}_\lambda$ to $\mathcal{P}_\lambda$ respects joins and meets in the sense that if $p_1 \equiv p_2$ and $r_1 \equiv r_2$ then $p_1 \vee r_1 \equiv p_2 \vee r_2$ and similarly for meets (see Reading 2006 for further details). This question on lattices can be asked more generally for an equivalence relation between two general posets which leads to order congruences and poset quotients (Reading 2006, 2002; Chajda and Snášel 1998). Suppose P is a finite poset and Θ is an equivalence relation on the elements of P, call $[p]_\Theta$ the equivalence class of $p \in P$. Θ is an *order congruence* if:

1. every equivalence class is an interval,
2. the projection $\pi_\downarrow : P \to P$, mapping each element $p \in P$ to the minimal element in $[p]_\Theta$, is order-preserving, and
3. the projection $\pi^\uparrow : P \to P$, mapping each element $p \in P$ to the maximal element in $[p]_\Theta$, is order-preserving.

Define a partial order on the congruence classes by $[p]_\Theta \leq [r]_\Theta$ if and only if there exists $x \in [p]_\Theta$ and $y \in [r]_\Theta$ such that $x \leq_P y$. The set of equivalence classes under this partial order is P/Θ, the *quotient* of P with respect to Θ, which is isomorphic to the induced subposet $\pi_\downarrow(P)$. Furthermore, an equivalence relation Θ on a lattice L is an order congruence if and only if it is a lattice congruence, and the two definitions of the quotient of L/Θ coincide (Reading 2006).

Theorem 2.1(1) implies the outer peel relation satisfies the first criteria of an order congruence, and the strong embedding property implied by Corollary 2.38 shows that the third criteria is also satisfied, so one might hope $\mathcal{P}_\lambda$ is the quotient of $\mathcal{TB}_\lambda$ under the outer peel relation. Unfortunately, the second criteria is not satisfied (examine Figure 16 for counter examples in $\mathcal{TB}_5$). So in some sense the outer peel relation is a semi-order congruence that can be generally defined using $(L2)$ and $(L3)$, and $\mathcal{P}_\lambda$ can be viewed as a semi-quotient of $\mathcal{TB}_\lambda$ under the outer peel relation. With this new context in mind, Theorems 3.1 and 3.2 test whether the semi-order congruence can be used to prove that a poset P is a lattice.

As I previously stated in Chapter 2, $\mathrm{HST}_1(n, d)$ is known to be a lattice for $d \in [3]$, but for $d \geq 4$ the lattice property is unknown. In light of Theorem 2.2 one might also ask whether Theorems 3.1 and 3.2 could be used to prove or disprove that $\mathrm{HST}_1(n, 4)$ is a lattice. The outer peel relation used to partition $\mathcal{TB}_n$ has an equivalent form in $\mathrm{HST}_1(n, 3)$. In fact the projection from $\mathcal{TB}_n$ to $\mathcal{P}_n$ is a specific case of Rambau's T/n map defined in Proposition 5.14(ii) of Rambau 1997 that projects $\mathrm{HST}_1(n, d)$ to $\mathrm{HST}_1(n-1, d-1)$.

So consider the partition of the fibers of this map that projects $\mathrm{HST}_1(n, 4)$ to $\mathrm{HST}_1(n-1, 3)$. $\mathrm{HST}_1(n-1, 3)$ is already known to be a lattice so $(L1)$ is true. Unfortunately the other properties are a little trickier. Just as I did in Theorem 2.1(3) I would like to use induction which would make $(L2)$ true by the induction hypothesis. However to use induction I need to make sense of the induced poset of the triangulations in an equivalence class. For $\mathcal{TB}_\lambda$ each of these induced posets was again a Tamari block lattice for a smaller shape (which is what made the induction work). The higher Stasheff-Tamari order context has to do with triangulations of $C(n, d)$, and removing the outer peel in $\mathcal{TB}_n$ is analogous to removing the simplices that contain n from a triangulation T. Removing these simplices will still leave a triangulation, but it will not be a triangulation of a cyclic polytope in general. So I need a generalized $\mathrm{HST}_1(n, d)$. It can be viewed as an equivalence class of saturated chains beginning at the bottom element similar to how $\mathcal{TB}_\lambda$ was defined for general shapes. Start with $\mathrm{HST}_1(n, d-1)$ and instead of partitioning the maximal chains to build $\mathrm{HST}_1(n,d)$ simply partition (using the same equivalence relation as before) all the saturated chains of $\mathrm{HST}_1(n,d-1)$ that begin at the minimum element and end at some fixed element, say $T \in \mathrm{HST}_1(n,d-1)$. Call this new poset $\widehat{\mathrm{HST}}_1(n,d,T)$. Clearly if T is the maximum element of $\mathrm{HST}_1(n,d-1)$ then $\widehat{\mathrm{HST}}_1(n,d,T) = \mathrm{HST}_1(n,d)$. Geometrically $\widehat{\mathrm{HST}}_1(n,d,T)$ can be interpreted as an order on the triangulations of a simplicial complex where the complex can be formed by gluing the simplices that are the labels for some saturated chain from the minimal element to T. These generalized posets $\widehat{\mathrm{HST}}_1(n,d,T)$ are the objects that result from removing the simplices with n from triangulations in $\mathrm{HST}_1(n,d)$.

Just as I did with $\mathcal{TB}_\lambda$, I would then like to prove or disprove that $\widehat{\mathrm{HST}}_1(n,4,T)$ are lattices for every $T \in \mathrm{HST}_1(n-1,3)$, thus proving or disproving the same for $\mathrm{HST}_1(n,4)$. I can now use induction as I did before which gives $(L2)$ as the induction hypothesis. $(L3)$ can be shown by reinterpreting the arguments in Corollary 2.38 and the discussion after it to the higher Stasheff-Tamari context. So that leaves $(L4)$ for consideration. If I could show $(L4)$ to be true then by Theorem 3.1 $\mathrm{HST}_1(n,4)$ would be a lattice. However, there are some reasons to believe $\mathrm{HST}_1(n,d)$ is not a lattice for $d \geq 4$, one of which is that $\mathrm{HST}_2(n,4)$ is known not to be a lattice for some n (see the beginning of Section 5 and the discussion after Theorem 8.4 in Rambau and Reiner 2012). As a result it would be nice to use Theorem 3.2 to prove this. Unfortunately $\mathrm{HST}_1(n,d)$ is only graded for odd d (Rambau and Reiner 2012), so Theorem 3.2 cannot be used. This leaves the possibility of showing $(L4)$ is true, thus proving $\mathrm{HST}_1(n,4)$ is a lattice and therefore not the same as $\mathrm{HST}_2(n,4)$.

Another possible approach would be to skip the case $d = 4$ and try to use Theorem 3.2 to show $\mathrm{HST}_1(n,5)$ is not a lattice, but because the lattice property is unknown for $\mathrm{HST}_1(n,4)$ $(L1)$ is also unknown in this case.

Chapter 4

ENUMERATION IN $\mathcal{TB}_\lambda$

The goal of Chapter 2 was to define $\mathcal{TB}_\lambda$ and prove it had many nice poset properties. In this chapter I will examine $\mathcal{TB}_\lambda$ with enumeration in mind. There are two basic enumeration questions concerning $\mathcal{TB}_\lambda$: how many elements does it have, and for a given element $\pi \in \mathcal{TB}_\lambda$ how many top chains are in π's block? (I will denote this $|\pi|$.) The first question is difficult in general. I will say more about it in Chapter 5, but Section 4.1 will address some of the straightforward cases. In Section 4.2 I will answer the second question for rank 1 r-stats while simultaneously giving a combinatorial proof of a known identity. In Sections 4.3 and 4.4 I will continue grappling with the second question and give some strategies for enumerating the size of some blocks by generalizing shifted standard tableaux and building on the work of Fishel and Nelson 2014.

4.1 Enumerating Blocks of Fixed Rank in $\mathcal{TB}_n$

In this section I will derive formulas for the number of elements of $\mathcal{TB}_n$ of rank 1, 2, $\binom{n}{2} - 1$, and $\binom{n}{2} - 2$. Theorem 2.1(1) proved that $\mathcal{TB}_n$ has a unique element of rank 0 and rank $\binom{n}{2}$, so this section takes the next step in enumerating $\mathcal{TB}_n$ by rank. My reason for enumerating these special cases aside for the formulas themselves is to setup the generating functions that may ultimately solve this difficult problem (see Chapter 5 for more on these generating functions). Table 1 shows the number of elements of $\mathcal{TB}_n$ for $n \in [2, 7]$ broken down by rank. After examining smaller values of n one might wonder if the rank-generating function is symmetric for all n, but starting with $n = 6$ it is not. The rank-generating function does appear to be unimodal.

n \ Rank	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1															
3	1	1														
4	1	2	2	1												
5	1	3	5	7	5	3	1									
6	1	4	9	17	26	27	23	17	9	4	1					
7	1	5	14	32	63	107	136	148	144	123	90	57	32	14	5	1

Table 1: *The number of r-stats of each rank in $\mathcal{TB}_n$.*

I will begin by enumerating r-stats of rank 1. Let $\pi_0 = \hat{0}_{\mathcal{TB}_n}$ which implies

$$\pi_0 = \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-1 \end{array}$$

and $p(\pi_0) = (1, 2, ..., n-1)$. So by Lemma 2.40 for $i \in [n-2]$ there is an increasing pentagon move from π_0 involving rows i and $i+1$. By the same lemma, for a given i,

$$\begin{array}{cc} 1 & \\ 2 & \\ \vdots & \\ i-1 & \\ i & i \\ 1 & \\ i+2 & \\ \vdots & \\ n-1 & \end{array}$$

is the covering r-stat. These are the only r-stats of rank 1, so there are $n-2$ r-stats of rank 1. I want to refer to these r-stats later so I will label them:

$$\pi_n(1,i) = \begin{array}{cc} 1 & \\ 2 & \\ \vdots & \\ i-1 & \\ i & i \\ 1 & \\ i+2 & \\ \vdots & \\ n-1 & \end{array}$$

Now consider the number of r-stats of rank 2. It is clear that every increasing pentagon move on $\pi_n(1,i)$ must still occur on the outer diagonal, so I can again use Lemma 2.40 to count the number of r-stats of rank 2. For every $j \in [n-1] \setminus \{i, i+1\}$ there is an increasing pentagon move from $\pi_n(1,i)$

involving rows j and $j+1$. The covering r-stat in this case is

$$\begin{array}{ll} 1 & \\ \vdots & \\ i & i \\ 1 & \\ i+2 & \\ \vdots & \\ j & j \\ 1 & \\ j+2 & \\ \vdots & \\ n-1 & \end{array} \tag{4.1}$$

unless $j = i-1$ in which case

$$\begin{array}{ll} 1 & \\ \vdots & \\ i-1 & i-1 \\ i & 1 \\ 1 & \\ i+2 & \\ \vdots & \\ n-1 & \end{array} \tag{4.2}$$

is the covering r-stat. The only other increasing pentagon move from $\pi_n(1,i)$ involves rows i and $i+2$ for $i \in [n-3]$ which has

$$\begin{array}{lll} 1 & & \\ \vdots & & \\ i-1 & & \\ i & i & i \\ 1 & & \\ 2 & & \\ \vdots & & \\ n-1 & & \end{array} \tag{4.3}$$

as the covering r-stat. So the rank 2 r-stats can be partitioned into those with two rows of length two and those with one row of length three. There are $\binom{n-2}{2}$ r-stats of the first kind and $n-3$ of the second, so there are $\frac{n^2-3n}{2}$ r-stats of rank 2.

If every cover relation in $\mathcal{TB}_n$ could be expressed as an increasing pentagon move on the outer diagonal of the covered r-stat Lemma 2.40 could be used to enumerate all of the r-stats in this way. However starting with rank 3 this is not the case. The r-stats with the form 4.1 and 4.3 really do have this property so I can enumerate these using Lemma 2.40. The outer peel of 4.2 is $(1, 2, ..., i-1, 1, 1, i+2, ..., n-1)$ and the peel r-stat has two parts, an $i-1$ in row $i-1$ and an i in row i. This implies there is an increasing pentagon move involving these two rows and that move cannot be on the outer diagonal. So more than Lemma 2.40 is required to enumerate the r-stats of $\mathcal{TB}_n$. One methodology that has shown promise is to build generating functions for the number of r-stats with the same outer peel instead of using the rank-generating function.

Similar arguments to the ones I just made to enumerate the first few ranks can be used to enumerate the final few ranks. First, suppose an r-stat π has rank $\binom{n}{2}-1$. This means π is covered by $\hat{1}_{\mathcal{TB}_n}$, the r-stat with i 1s in row i for every $i \in [n-1]$. Proposition 2.24(3) says an increasing pentagon move only affects two parts of the covered r-stat (it reduces one part's value and adds a new part), so π has $\binom{n}{2}-1$ parts all of which are 1s except one part, say $\pi_{jk} > 1$. Furthermore, every row of π is weekly decreasing so $k = 1$. The new part must equal 1 and by Proposition 2.24(4) the new part is equal to $\pi_{jk} - 1$, so $\pi_{jk} = 2$. So I can completely characterize π as an r-stat with i 1s in row i for every $i \in [n-1] \setminus \{j-1, j\}$, $j-2$ 1s in row $j-1$, and a 2 followed by $j-1$ 1s in row j. It is not hard to see $\mathcal{TB}_n$ has an r-stat with this characterization for every $j \in [2, n-1]$, so there are $n-2$ elements of $\mathcal{TB}_n$ of rank $\binom{n}{2}-1$. I want to refer to these r-stats later so I will label them as well, let $\pi_n(\binom{n}{2}-1, j)$ be the r-stat with the 2 in row $j+1$. For example,

$$\pi_5(9,1) = \begin{array}{llll} 1 & 1 & 1 & \\ 2 & 1 & 1 & \\ 1 & 1 & & \\ 1 & & & \end{array}, \quad \pi_5(9,2) = \begin{array}{llll} 1 & 1 & 1 & 1 \\ 1 & 1 & & \\ 2 & 1 & & \\ 1 & & & \end{array}, \quad \pi_5(9,3) = \begin{array}{llll} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & & & \\ 2 & & & \end{array}$$

Just as I did in the argument for rank 2 r-stats, I can push this reasoning another step. Now suppose π has rank $\binom{n}{2}-2$. This means it is covered by an r-stat with the characterization in the previous paragraph. So again by Proposition 2.24 there are two cases for π. Either π has all 1s except two rows with a part with value 2 or it has all 1s with one row with a part of value 3.

To enumerate the first case, first note that two adjacent rows cannot both start with a 2. To see this assume an r-stat π of all 1s except a 2 in row i and in row $i+1$ is in $\mathcal{TB}_n$:

$$\pi = \begin{array}{llllll} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & \\ 2 & 1 & 1 & & & \\ 2 & 1 & 1 & & & \\ \vdots & & & & & \\ 1 & & & & & \end{array}$$

the outer peel of π is all 1s so $\mathrm{peel}(\pi)$ is an r-stat of all 1s except a 2 in row i and in row $i+1$ in $\mathcal{TB}_{n-1}$, and similarly $\mathrm{peel}^{n-i-2}(\pi)$ is an r-stat of all 1s except a 2 in row i and in row $i+1$ in $\mathcal{TB}_{i+2}$. Let M be a maximal chain with $\pi(M) = \mathrm{peel}^{n-i-2}(\pi)$. So

$$M = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\vdots} \\ \hline e & f & g & h \\ \hline c & b & d & \multicolumn{1}{c}{} \\ \cline{1-3} a & b & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} a & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

which implies $b > a$, but if the b-set is removed before the a-set then $c = a$ which is a contradiction.

However there can be two 2s in the same row, so there are still $\binom{n-2}{2}$ r-stats in the first case. In the second case if row i has the 3 then $i \in [n-3]$, so there are $n-3$ r-stats. Combining both cases there are $\frac{n^2-3n}{2}$ r-stats of rank $\binom{n}{2} - 2$.

So although the rank-generating function of $\mathcal{TB}_n$ is not symmetric in general, this section proves it is symmetric for the first 3 ranks. Table 1 shows the first few cases of this fact.

4.2 A Combinatorial Proof Using Blocks of Rank 1

Nelson enumerated the number of maximal chains of length n in Nelson 2015, there are $\binom{n}{3}$ of them. In Section 4.1 I showed $\pi_n(1, i)$ for $i \in [n-2]$ are the r-stats of $\mathcal{TB}_n$ of rank 1. A Pascal-like triangle is created by partitioning the maximal chains of length n by these r-stats where the ith element of the nth row equals the number of maximal chains in the r-stat $\pi_{n+2}(1,i)$:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & 2 & & & \\ & & 3 & & 4 & & 3 & & \\ & 4 & & 6 & & 6 & & 4 & \\ 5 & & 8 & & 9 & & 8 & & 5 \\ \vdots & & & & & & & & \end{array} \tag{4.4}$$

Call this triangle $\mathcal{TB}\Delta_1$. So, for example, there is a unique maximal chain in $\mathcal{TB}_3$ with an r-stat of rank 1, and there are two r-stats of rank 1 in $\mathcal{TB}_4$ and they both contain two maximal chains.

The defining characteristic of $\mathcal{TB}\Delta_1$ is on its diagonals. In this context the nth left-to-right diagonal is the sequence of elements of the triangle that begin with the leftmost element of row n followed by the second element of row $n+1$ and so on, and similarly for the nth right-to-left diagonal. Notice that for the initial values shown in 4.4 both the nth left-to-right diagonal and the nth right-to-left diagonal are the sequence $n, 2n, 3n, \cdots$. In this section I will prove this is true in general and then I will use this to derive a combinatorial proof for a known identity. The next proposition enumerates the maximal chains of $\pi_n(1,i)$.

Proposition 4.1 $|\pi_n(1,i)| = i(n-i-1)$.

Proof. Recall from Definition 2.8 that S_C is the diagonal sentence of a top chain C. For a maximal chain M with r-stat $\pi_n(1,i)$ let $S_M = w_1 w_2 ... w_{n-1}$ and let B_i be the corner box in row i of M. Now,

$$p(\pi_n(1,i)) = (1,2,...,i-1,i,1,i+2,i+3,...,n-1),$$

so the last boxes removed from the outer diagonal of M must be $B_{i+2}, B_{i+3}, ...B_{n-1}$ in that order. Also, $B_1, ..., B_i$ must be removed in that order, but B_{i+1} can be removed before any of these boxes. This gives i choices for w_1 (the outer diagonal) of M.

Regardless of how w_1 is chosen, after $i+1$ steps the resulting top chain is

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & \cdots & y & z \\
1 & 2 & 3 & 4 & \cdots & y & z \\
\vdots & & & \vdots & & & \\
1 & 2 & 3 & 4 & \cdots & y & z \\
a & b & c & d & \cdots & x & \\
a & b & c & d & \cdots & x & \\
\vdots & & & \vdots & & & \\
a & b & c & d & & & \\
a & b & c & & & & \\
a & b & & & & & \\
a & & & & & &
\end{array}
$$

where $a < b < ... < x$, B_{i+2} contains x, $y = n - i - 2$, and $z = n - i - 1$. So far i columns have been removed (since every step except the one in row $i+1$ removed an entire column), so there are $n - i - 1$ columns left. Furthermore, the one r-set that is not on the outer diagonal of $\pi_n(1, i)$ can be placed in any of these columns, which means there are $n - i - 1$ choices. Since this choice is independent of the choice of w_1, this completes the proof. $\square$

Example 4.2 Consider

$$
\pi_7(1,3) = \begin{array}{cc}
1 & \\
2 & \\
3 & 3 \\
1 & \\
5 & \\
6 &
\end{array}
$$

Let M be a maximal chain such that $\pi(M) = \pi_7(1,3)$ and let $w_1 = (w_{11}, w_{12}, ..., w_{16})$ be the first word of S_M. Since the w_{11}-set and w_{12}-set go all the way to the first row of M, $w_{11} < w_{12}$, and both w_{11} and w_{12} must be less than the other elements of w_1. Similarly since the w_{14}-set, w_{15}-set, and the w_{16}-set go all the way to the first row of M, $w_{14} < w_{15} < w_{16}$. Lastly the w_{13}-set only has one box, so $w_{14} < w_{13}$. Together this implies there are three possible orderings for the elements of w_1:

$$w_{11} < w_{12} < w_{14} < w_{13} < w_{15} < w_{16}$$

$$w_{11} < w_{12} < w_{14} < w_{15} < w_{13} < w_{16}$$

$$w_{11} < w_{12} < w_{14} < w_{15} < w_{16} < w_{13}$$

Next there is one label that does not appear on the outer diagonal, say r, and the r-set must end in row 3 and have length 3, so for every ordering of the elements of w_1 there are three columns that can contain r. This implies there are nine maximal chains with r-stat $\pi_7(1,3)$. The three with the first ordering of the w_{1i} are:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&6&7\\ \hline 1&2&3&4&6\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 1&2&5\\ \cline{1-3} 1&2\\ \cline{1-2} 1\\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&6&7\\ \hline 1&2&3&4&6\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 1&3&5\\ \cline{1-3} 1&3\\ \cline{1-2} 1\\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&6&7\\ \hline 1&2&3&4&6\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 2&3&5\\ \cline{1-3} 2&3\\ \cline{1-2} 2\\ \cline{1-1}\end{array}.$$

The three with the second ordering are:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&7\\ \hline 1&2&3&4&5\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 1&2&6\\ \cline{1-3} 1&2\\ \cline{1-2} 1\\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&7\\ \hline 1&2&3&4&5\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 1&3&6\\ \cline{1-3} 1&3\\ \cline{1-2} 1\\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&7\\ \hline 1&2&3&4&5\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 2&3&6\\ \cline{1-3} 2&3\\ \cline{1-2} 2\\ \cline{1-1}\end{array}.$$

The three with the third ordering are

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\ \hline 1&2&3&4&5\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 1&2&7\\ \cline{1-3} 1&2\\ \cline{1-2} 1\\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\ \hline 1&2&3&4&5\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 1&3&7\\ \cline{1-3} 1&3\\ \cline{1-2} 1\\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\ \hline 1&2&3&4&5\\ \cline{1-5} 1&2&3&4\\ \cline{1-4} 2&3&7\\ \cline{1-3} 2&3\\ \cline{1-2} 2\\ \cline{1-1}\end{array}.$$

Proposition 4.1 shows the nth left-to-right and right-to-left diagonals of $\mathcal{TB}\Delta_1$ are $n, 2n, 3n, \cdots$. Consider the ith element of the nth left-to-right diagonal. By the definition of $\mathcal{TB}\Delta_1$ it is $|\pi_{n+i+1}(1,i)|$ which is equal to in by Proposition 4.1. A similar argument works for the nth right-to-left diagonals remembering that there are $n-2$ r-stats of $\mathcal{TB}_n$ of rank 1, so the nth row of $\mathcal{TB}\Delta_1$ has n elements.

Combining Proposition 4.1 with Nelson's result that there are $\binom{n}{3}$ maximal chains in each row of $\mathcal{TB}\Delta_1$, I get the identity

$$\sum_{i=1}^{n-2} i(n-i-1) = \binom{n}{3} \tag{4.5}$$

is simply two different ways of enumerating the maximal chains in T_n of length n. Equation 4.5 is the case $j=1$, $k=2$, $n=n-1$, and $m=i$ of the well-known identity

$$\sum_{m=0}^{n} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}.$$

4.3 A Generalization of Shifted Tableaux

In Section 4.4 I will derive formulas for the sizes of other square blocks of $\mathcal{TB}_n$. My methodology will be to generalize the methodology used in Fishel and Nelson 2014 to enumerate the maximum length chains of T_n. Within the context of $\mathcal{TB}_n$ Fishel and Nelson's result is a formula for $|\hat{1}_{\mathcal{TB}_n}|$. To do this they showed maximum length chains in T_n (viewed as tableaux) are exactly the shifted standard Young tableaux of shape δ_n. They then utilized Thrall's formula for the number of shifted standard Young tableaux of a fixed shape found in Thrall 1952 to derive their own formula. Tableau presentations of top chains of T_n are not standard in general though, so in order to enumerate the size of other square blocks using this strategy I need to generalize shifted standard Young tableaux so the shapes I will consider in Section 4.4 can be enumerated.

Recall from Chapter 1 that if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ is an integer partition then a standard Young tableau of shape λ is a Young tableau of λ such that every row and column is strictly increasing, and that f^λ is the number of standard Young tableaux of shape λ. Also recall that if λ has distinct parts (meaning $\lambda_1 > \lambda_2 > ... > \lambda_r > 0$) then the shifted Young diagram of shape λ is the array of boxes with λ_i boxes in row i such that row i begins in column i. A standard shifted Young tableau of shape λ is a filling of the shifted Young diagram of λ by positive integers where the entries strictly increase along rows and columns. I will denote the number of standard shifted Young tableaux of a fixed shape λ with $g(\lambda)$.

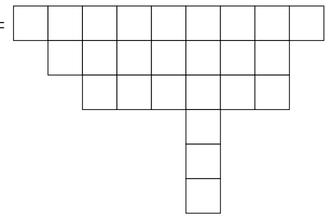
Theorem 4.3 (Thrall's Formula,Thrall 1952) Suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is an integer partition with distinct parts.

$$g(\lambda) = \frac{(\sum_{i\in[k]} \lambda_i)! \prod_{i>j}(\lambda_i - \lambda_j)}{\prod_{i\in[k]} \lambda_i! \prod_{i>j}(\lambda_i + \lambda_j)}$$

The necessity of λ having distinct parts is clear in this formula, however it will be useful to enumerate the standard Young tableaux of shapes that are not quite shifted Young diagrams.

Definition 4.4 Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be an integer partition with distinct parts, and let c and d be nonnegative integers with $r + 1 \leq c \leq r + \lambda_r - 1$. Define $\mathrm{gshift}_c(\lambda; d)$ to be the diagram obtained by adding d boxes to column c of λ's shifted Young diagram.

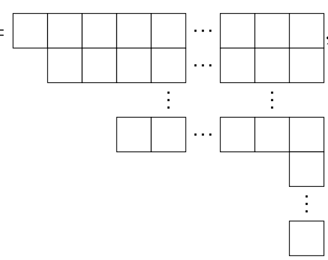
Example 4.5 Suppose $\lambda = (9, 7, 6)$, then

$\mathrm{gshift}_6(\lambda; 3) =$ 

I will denote the number of standard tableaux of shape $\mathrm{gshift}_c(\lambda; d)$ by $g_c^*(\lambda; d)$.

There is a natural way to reduce enumerating standard tableaux of any shape to enumerating standard tableaux of a strictly smaller shape by considering which boxes can contain the largest label. Because standard tableaux must increase along rows and columns only the corner boxes of the diagram can have the largest label. Recall that in Chapter 1 I defined a corner box to be a box without a neighbor below it or to its right. If $B_1, ..., B_k$ are the corner boxes of a diagram D then the number of standard tableaux of D is equal to the sum of the number of standard tableaux of $D - B_i$ for every $i \in [k]$, where $D - B_i$ is the diagram D with B_i removed.

I can use this to determine $g_c^*(\lambda; d)$ in terms of $g(\lambda)$. First, let $\mu = (\mu_1, ..., \mu_r)$ be the integer partition defined by $\mu_i = \lambda_i - c + i - 1$. As a shape μ is the Young diagram of boxes of λ to the right of column c (see Example 4.7). Now there are two types of corner boxes of $\mathrm{gshift}_c(\lambda; d)$, the corner boxes of μ and the last box in column c. So to determine $g_c^*(\lambda; d)$ simply remove corner boxes until $d = 0$ or $\mu = \emptyset$. Obviously if $d = 0$, then $g_c^*(\lambda; d) = g(\lambda)$. And if $\mu = \emptyset$ then

$\mathrm{gshift}_c(\lambda; d) =$  ,

So once again $g_c^*(\lambda; d) = g(\lambda)$ since the boxes added in column c must contain the last d labels. The next theorem gives a recursion for $g_c^*(\lambda; d)$ by removing corner boxes in this way. Recall from Chapter 1 that for Young diagrams λ and μ with $\mu \subseteq \lambda$ the skew Young diagram λ/μ is the diagram λ with the boxes of μ removed starting in the top left corner and the truncated Young diagram $[\lambda \setminus \mu]$ is the diagram λ with the boxes of μ removed starting in the top right corner. Also recall for Young diagrams $\lambda = (\lambda_1, ..., \lambda_r)$ and $\mu = (\mu_1, ..., \mu_s)$ with $s \leq r$ the Young diagram $\lambda \cup \mu$ is the diagram associated with the integer partition $(\lambda_1 + \mu_1, ..., \lambda_r + \mu_r)$.

Theorem 4.6 Suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ is an integer partition with distinct parts, and c and d are nonnegative integers with $r + 1 \leq c \leq r + \lambda_r - 1$. Let $\mu = (\mu_1, ..., \mu_r)$ be the integer partition defined by $\mu_i = \lambda_i - c + i - 1$. Then

$$g_c^*(\lambda; d) = \sum_{\nu \subseteq \mu} f^{\mu/\nu} g_c^*((\lambda \setminus \mu) \cup \nu; d - 1).$$

Proof. Let $B_1, ..., B_k$ be the corner boxes of $\mathrm{gshift}_c(\lambda; d)$ such that B_k is the last box in column c, this implies $B_1, ..., B_{k-1}$ are the corner boxes of μ. I will use induction on $|\mu|$. First suppose $|\mu| = 0$ which means the left hand side of the equation is $g(\lambda)$ since the boxes added in column c must contain the last d labels. This also means the only term of the sum on the right hand side is $\nu = \emptyset$, so the right hand side becomes $g_c^*(\lambda; d - 1)$ which is also equal to $g(\lambda)$ since the boxes added in column c must now contain the last $d - 1$ labels.

Now suppose the theorem is true for all diagrams $\mathrm{gshift}_c(\lambda; d)$ such that the resulting μ has less than L boxes and consider λ, c, and d that give a $\mathrm{gshift}_c(\lambda; d)$ such that the resulting μ has L boxes. Only the corner boxes of $\mathrm{gshift}_c(\lambda; d)$ can have the largest label so

$$g_c^*(\lambda; d) = g_c^*(\lambda; d - 1) + \sum_{i \in [k-1]} g_c^*(\lambda - B_i; d)$$

By the induction hypothesis

$$g_c^*(\lambda - B_i; d) = \sum_{\nu \subseteq (\mu - B_i)} f^{(\mu - B_i)/\nu} g_c^*(((\lambda - B_i) \setminus (\mu - B_i)) \cup \nu; d - 1)$$

for each $i \in [k - 1]$, but $((\lambda - B_i) \setminus (\mu - B_i)) \cup \nu = (\lambda \setminus \mu) \cup \nu$ for every $\nu \subseteq \mu$. This means

$$\begin{aligned} g_c^*(\lambda; d) &= g_c^*(\lambda; d - 1) + \sum_{i \in [k-1]} \sum_{\nu \subseteq (\mu - B_i)} f^{(\mu - B_i)/\nu} g_c^*((\lambda \setminus \mu) \cup \nu; d - 1) \\ &= g_c^*(\lambda; d - 1) + \sum_{\nu \subsetneq \mu} \left(\sum_{i \in [k-1]} f^{(\mu - B_i)/\nu} \right) g_c^*((\lambda \setminus \mu) \cup \nu; d - 1) \end{aligned}$$

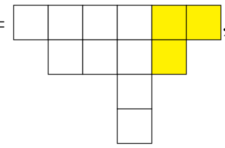
where I define $f^{(\mu - B_i)/\nu} = 0$ if $\nu \nsubseteq \mu - B_i$. This implies

$$\begin{aligned} g_c^*(\lambda; d) &= g_c^*(\lambda; d - 1) + \sum_{\nu \subsetneq \mu} f^{\mu/\nu} g_c^*((\lambda \setminus \mu) \cup \nu; d - 1) \\ &= \sum_{\nu \subseteq \mu} f^{\mu/\nu} g_c^*((\lambda \setminus \mu) \cup \nu; d - 1). \end{aligned}$$

$\square$

Example 4.7 Let $\lambda = (6, 4)$, which means

$$\text{gshift}_4(\lambda; 2) =$$



,

where the yellow boxes are $\mu = (2, 1)$. Any standard tableau of $\text{gshift}_4(\lambda; 2)$ has twelve distinct labels, say $[12]$ are the labels. It is clear that 12 must go in box $b(1, 6)$, $b(2, 5)$ or $b(4, 4)$, which implies

$$g_4^*(\lambda; 2) = g_4^*(\lambda; 1) + g_4^*((6, 3); 2) + g_4^*((5, 4); 2)$$

I can continue this process on the last two terms of the sum until every term of the sum has $d = 1$ or until there are no boxes in columns 5 and 6. This leads to

$$g_4^*(\lambda; 2) = g_4^*(\lambda; 1) + g_4^*((6, 3); 1) + g_4^*((5, 4); 1) + 2g_4^*((5, 3); 1) + 2g_4^*((4, 3); 2)$$

The five terms correspond to the five integer partitions that are contained in $(2, 1)$, furthermore there is only one skew standard Young tableau for shapes $(2, 1)/(2, 1)$, $(2, 1)/(1, 1)$, and $(2, 1)/(2)$ and there are two skew standard Young tableaux for shapes $(2, 1)/(1)$ and $(2, 1)/\emptyset$ which matches the coefficients of the sum just as Theorem 4.6 claims.

As I said before $g_4^*((5, 4); 2) = g((5, 4))$, so the last term is known. To determine $g_4^*(\lambda; 2)$ completely I need to use Theorem 4.6 again which leads to:

$$\begin{aligned} g_4^*(\lambda; 2) &= g(\lambda) + 2g((6, 3)) + 2g((5, 4)) + 6g((5, 3)) + 8g((4, 3)) \\ &= 250 \text{ (by Thrall's Formula).} \end{aligned}$$

Just as I did in the previous example I can use Theorem 4.6 repetitively to get a formula for $g_c^*(\lambda; d)$ in terms of Thrall's formula.

Corollary 4.8 Suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ is an integer partition with distinct parts, and c and d are nonnegative integers with $r + 1 \leq c \leq r + \lambda_r - 1$. Let $\mu = (\mu_1, ..., \mu_r)$ be the integer partition defined by $\mu_i = \lambda_i - c + i - 1$. Then

$$g_c^*(\lambda; d) = \sum_{\nu \subseteq \mu} \left(\sum_{\nu = \nu_0 \subseteq \nu_1 \subseteq \ldots \subseteq \nu_d = \mu} \prod_{i \in [d]} f^{\nu_i / \nu_{i-1}} \right) g((\lambda \setminus \mu) \cup \nu).$$

It is easy to check that the coefficients of the equation at the end of Example 4.7 match the $d = 2$ case of this corollary.

In Section 4.2 I determined the sizes of square blocks of rank 1 in $\mathcal{TB}_n$ and in the previous section I defined certain shapes that are not quite shifted Young diagrams of integer partitions and derived an enumeration formula for them. In this section I want to continue to count the number of top chains in a square block when the r-stat has a certain form. Although it is not the only strategy for enumerating the size of a square block, here I will use Thrall's formula just as was done in Fishel and Nelson 2014. I will also use the results of the previous section when appropriate. Much of what I do here is very similar to *jeu-de-taquin* on tableaux (compare this to Schilling et al. 2015, Section 3 for example).

To begin I want to take a closer look at $\hat{1}_{\mathcal{TB}_n}$. Suppose M is a maximal chain in $\hat{1}_{\mathcal{TB}_n}$ viewed as a tableau. By the proof of Proposition 2.15 each word of the diagonal sentence of M must be strictly decreasing, so requiring that M's rows and NE-to-SW diagonals strictly increase are necessary and sufficient conditions on M (M's columns must weakly increase to be a Tamari tableau, but this is implied by these conditions). It is straightforward to shift the boxes of M to the right so that its former NE-to-SW diagonals are now its columns. This shift turns M into a shifted standard Young tableau of shape δ_n and it is clearly a bijection, which proves:

Proposition 4.9 (Fishel and Nelson 2014, Corollary 3.4)

$$|\hat{1}_{\mathcal{TB}_n}| = g(\delta_n)$$

This section will be dedicated to generalizing this approach which can be broken into three steps:

1. develop conditions on top chain labels from a given r-stat,
2. turn the conditions on the top chain labels into a standard tableau,
3. enumerate the number of standard tableaux with the shape from (2).

Example 4.10 Before using this methodology I want to begin with some of its limitations. First consider

$$\pi_5(9,2) = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & & \\ 2 & 1 & & \\ 1 & & & \end{array} \in \mathcal{TB}_5.$$

Unlike the previous case not all the box labels are distinct and not all the diagonals are strictly increasing. Suppose M is a maximal chain with r-stat $\pi_5(9,2)$, so

$$M = \begin{array}{|c|c|c|c|}\hline i & h & g & d \\ \hline e & f & c \\ \cline{1-3} e & b \\ \cline{1-2} a \\ \cline{1-1}\end{array},$$

with the following relations on the diagonals:

$$a > b > c > d,$$

$$f > e > g, \text{ and}$$

$$e > h.$$

The last relation is implied from the second since the row relation says $g > h$, so I can discard one of the boxes labeled e, rearrange the second diagonal to be strictly increasing, and then shift diagonals into columns as I did before to get:

$$\begin{array}{cccc}
\cline{1-4}
\multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{h} & \multicolumn{1}{c|}{g} & \multicolumn{1}{c|}{d} \\ \cline{1-4}
 & & \multicolumn{1}{|c|}{e} & \\ \cline{3-4}
 & & \multicolumn{1}{|c|}{f} & \multicolumn{1}{c|}{c} \\ \cline{3-4}
 & & & \multicolumn{1}{|c|}{b} \\ \cline{4-4}
 & & & \multicolumn{1}{|c|}{a} \\ \cline{4-4}
\end{array}.$$

It is clear that a, b, and c must be removed first and in that order, so I can remove them which implies there is a bijection from the maximal chains of $\pi_5(9,2)$ to the standard tableaux of shape

$$\begin{array}{cccc}
\cline{1-4}
\multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } \\ \cline{1-4}
 & & \multicolumn{1}{|c|}{\ } & \\ \cline{3-3}
 & & \multicolumn{1}{|c|}{\ } & \\ \cline{3-3}
\end{array}.$$

Therefore, $|\pi_5(9,2)| = g_3^*((4);2)$ (compare this to Proposition 4.16). I can then use the results of Section 4.3 to completely determine $|\pi_5(9,2)|$. In addition to being able to do each of the three steps above, a key property of M that allows me to use this methodology is the "rigidity" of box labels. To see that not every r-stat has this rigidity, consider

$$\rho = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & \\ 3 & 1 & \\ 1 & & \end{array} \in \mathcal{TB}_5.$$

Again suppose M is a maximal chain with r-stat ρ, then

$$M = \begin{array}{|c|c|c|c|} \hline e & h & g & d \\ \hline e & f & c & \\ \cline{1-3} e & b & & \\ \cline{1-2} a & & & \\ \cline{1-1} \end{array} \text{ or } M = \begin{array}{|c|c|c|c|} \hline h & e & g & d \\ \hline e & f & c & \\ \cline{1-3} e & b & & \\ \cline{1-2} a & & & \\ \cline{1-1} \end{array}.$$

In this case attempting to fill the boxes with variables leaves me two choices instead of one fixed choice. In order to enumerate the maximal chains using this methodology I need to resort to casework. Although this can lead to a solution, I will avoid this type of r-stat in this dissertation. Obviously three other important steps in this methodology are being able to determine enough relations on the top chain labels from a r-stat, being able to transform the relations into a diagram, and being able to enumerate the standard tableaux of the resulting diagram.

The first generalization of Proposition 4.9 I want to consider is enumerating the top square block for general top chains, in other words I want to determine $|\hat{1}_{\mathcal{TB}_\lambda}|$ for $\lambda \neq \delta_n$. Suppose λ has distinct parts. In a similar way to what I did above, I can use the proof of Proposition 2.15 to show every part of $\hat{1}_{\mathcal{TB}_\lambda}$ must be a 1 and every word of the diagonal sentence of a maximal chain is strictly decreasing. So, for example, if $\lambda = (6,5,3,1)$, then

$$\hat{1}_{\mathcal{TB}_\lambda} = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & & & \\ 1 & & & & & \end{array}$$

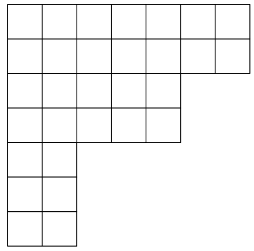
which means the following proposition can be proved in the same way that I proved Proposition 4.9.

Proposition 4.11 Suppose λ is an integer partition with distinct parts, then

$$|\hat{1}_{\mathcal{TB}_\lambda}| = g(\lambda).$$

When λ does not have distinct parts the story is not so simple. Suppose next that $B_1, ..., B_k$ are the corner boxes of λ and they are all on the outer diagonal in rows $r_1, ..., r_k$. By considering $C_{\max}(\hat{1}_{\mathcal{TB}_\lambda})$ in light of Proposition 2.15 it is clear that every nonzero part of $\hat{1}_{\mathcal{TB}_\lambda}$ is in row r_i for some i and each nonzero part of row r_i is $r_i - r_{i-1}$ for $i \in [2, k]$ (the nonzero parts of row r_1 are just

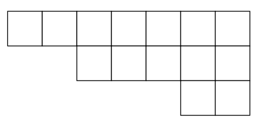
r_1). For example, if

$\lambda =$ 

then

$$\hat{1}_{\mathcal{TB}_\lambda} = \begin{array}{ccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \\ 2 & 2 & 2 & 2 & 2 & & \\ \\ \\ \\ 3 & 3 & & & & & \end{array}$$

This allows me to remove rows that do not have a corner box which means every diagonal must now be strictly increasing. I can shift them to the right as I did above to turn each diagonal into a column. The resulting diagram will have λ_{r_i} boxes in row i, but every row will be right justified. In my example I get the diagram:



The goal is then to enumerate the standard tableaux of this shape. The number of standard tableaux is invariant under rotation by 180° (see Adin and Roichman 2015, Observation 2.8) which means the number of standard tableaux can be enumerated with truncated shapes.

Proposition 4.12 Suppose λ is a Young diagram with every corner box on the outer diagonal. Let $B_1, ..., B_k$ be λ's corner boxes in rows $r_1, ..., r_k$. Then

$$|\hat{1}_{\mathcal{TB}_\lambda}| = f^{\lambda_1^k \backslash \mu},$$

where $\mu_i = \lambda_1 - \lambda_{k-i+1}$ for $i \in [k]$.

Although there are some formulas for enumerating these truncated shapes it is still an open area of research. See Section 8 for known formulas and successful techniques for truncated shape enumeration.

Determining $|\hat{1}_{\mathcal{TB}_\lambda}|$ for other λ (λ that do not have distinct parts and do not have every corner box on the outer diagonal) is an area of current research for me.

My next goal is to enumerate the chains with an r-stat of rank $\binom{n}{2} - 1$. To ease the notation let $\alpha = \binom{n}{2} - 1$. In Section 4.1 I characterized and labeled these r-stats $\pi_n(\alpha, i)$ for $i \in [n-2]$. Before I begin enumerating these r-stats I need to discuss r-stat duality. The Tamari lattices are self-dual (see Knuth 2013, Exercise 27(c) for example), and this implies chains in T_n have a dual chain. It is clear from the definition of duality that the dual of a non-maximal top chain will not be a top chain, however the dual of a maximal chain M in T_n is a maximal chain, call it M^*. It is also clear $\ell(M) = \ell(M^*)$. Furthermore if there is a square move that sends M_1 to M_2 then there is a corresponding square move that sends M_1^* to M_2^*. Alternatively, if there does not exist a set of square moves from M_1 to M_2 then there cannot be a set of square moves from M_1^* to M_2^* either. So π^* is the dual r-stat of π if the maximal chains in π^* are the dual chains to the maximal chains in π, thus the dual of an r-stat is well-defined. Finally, the dual operation on chains is clearly a bijection. The next lemma summarizes this discussion.

Lemma 4.13 Suppose $\pi \in \mathcal{TB}_n$ and π^* is the dual r-stat to π. Then,

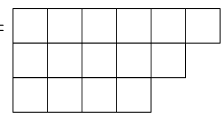
1. $|\pi| = |\pi^*|$, and
2. π and π^* have the same rank in $\mathcal{TB}_n$.

Some r-stats are self-dual, but Lemma 4.13 often allows me to determine the size of two square blocks by just enumerating one. For example, the first r-stats of rank α I want to consider are $\pi_n(\alpha, 1)$ and $\pi_n(\alpha, n-2)$. As a reminder,

$$\pi_n(\alpha,1) = \begin{array}{lllll} 1 & 1 & 1 & \cdots & 1 \\ 2 & 1 & 1 & \cdots & 1 \\ \vdots & & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & & & \\ 1 & & & & \end{array}$$

$$\pi_n(\alpha, n-2) = \begin{array}{llllll} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & \\ 1 & 1 & 1 & & & \\ 1 & & & & & \\ 2 & & & & & \end{array}$$

Let $\delta_{[m,n]}$ be the shape δ_n with the bottom $m-1$ rows removed. For example,

$\delta_{[4,7]} =$ 

Proposition 4.14

$$|\pi_n(\alpha, n-2)| = g(\delta_{[3,n]})$$

Proof. Suppose $M \in \pi_n(\alpha, n-2)$ with $S_M = w_1, ..., w_{n-1}$. Because $sh(M) = \delta_n$, $\ell(M) = \alpha$ must be a label of a box on the outer diagonal, so $\alpha \in w_1$. In a similar way, $\alpha - 1 \in w_1$ since exposing a new corner box requires removing at least 2 boxes from δ_n. Now the outer peel is $p(\pi_n(\alpha, n-2)) = (1, 1, ..., 1, 2)$, so the elements of w_1 are strictly decreasing except $w_{11} < w_{12}$. This implies $w_{11} = \alpha - 1$, and $w_{12} = \alpha$. Since these two elements are fixed I can remove these boxes from $\pi_n(\alpha, n-2)$ to get the r-stat

$$\begin{array}{llllll} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & \\ 1 & 1 & 1 & & & \end{array} = \hat{1}_{\mathcal{TB}_{\delta_{[3,n]}}}.$$

This block has $g(\delta_{[3,n]})$ elements by Proposition 4.11, so the equality is proved. □

If I could show $\pi_n(\alpha, 1)$ is the dual r-stat to $\pi_n(\alpha, n-2)$ then by Lemma 4.13 the previous proposition also determines $|\pi_n(\alpha, 1)|$. One of my next research projects is to build the dual map from an r-stat to its dual r-stat. The conjectures in the remainder of this section can be proved using such a map.

Conjecture 4.15

$$|\pi_n(\alpha, 1)| = g(\delta_{[3,n]})$$

Next consider $\pi_n(\alpha, 2)$ and $\pi_n(\alpha, n-3)$. Similar to what I did above, I will enumerate $\pi_n(\alpha, n-3)$ and then conjecture the $\pi_n(\alpha, 2)$ case based on duality.

Proposition 4.16

$$|\pi_n(\alpha, n-3)| = \sum_{i=0}^{n-4} (n-i-3) g(\delta_{[3,n-2]} \cup 1^i)$$

Proof. To enumerate $\pi_n(\alpha, n-3)$ I need to develop relations on the labels of a maximal chain in $\pi_n(\alpha, n-3)$ first. Suppose $M \in \pi_n(\alpha, n-3)$ with $S_M = w_1, ..., w_{n-1}$. The outer peel is a sequence of all 1s this time, so the elements of w_1 are strictly decreasing. However w_{21} is the label with length two, so w_2 will be strictly decreasing except $w_{21} < w_{22}$. The other words will be strictly decreasing sequences, so I have relations:

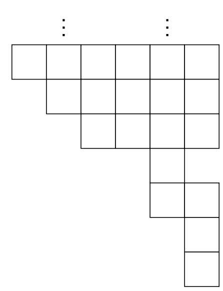
$$w_{11} > w_{12} > ... > w_{1(n-1)}$$

$$w_{22} > w_{21} > w_{23} > ... > w_{2(n-2)}$$

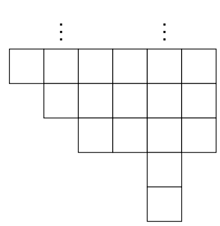
$$w_{31} > w_{32} > ... > w_{3(n-4)}$$

$$\vdots$$

Just as I did in Example 4.10 I can shift all the boxes to the right to turn M's diagonals into columns, remove one of the boxes containing w_{21}, and slide the other w_{21}-box between the w_{23}-box above and the w_{22}-box below in column $n-2$. The resulting diagram is



To enumerate the standard tableaux of this diagram note that $w_{11} = \alpha$ and $w_{12} = \alpha - 1$ since both these labels must be on M's outer diagonal. Also, $w_{21} < w_{22}$ implies $\alpha - 2$ is also on M's outer diagonal, so $w_{13} = \alpha - 2$. Since these labels are determined I can remove these boxes from the diagram to get



$$= \text{gshift}_{n-2}(\delta_{[4,n-1]}; 2).$$

So $|\pi_n(\alpha, n-3)| = g^*_{n-2}(\delta_{[4,n-1]}; 2)$ and I can use Corollary 4.8. Using the notation from Corollary 4.8 it is clear $\mu = 1^{n-4}$ which implies $\nu = 1^i$ for $i \in [0, n-4]$, so

$$(\delta_{[4,n-1]} \setminus 1^{n-4}) \cup \nu = \delta_{[3,n-2]} \cup 1^i.$$

By Corollary 4.8,

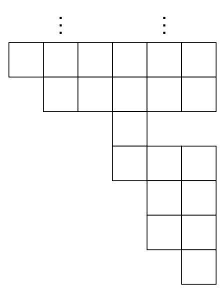
$$\begin{aligned}
|\pi_n(\alpha, n-3)| &= \sum_{i=0}^{n-4} \left(\sum_{j=i}^{n-4} f^{1^j/1^i} f^{1^{n-4}/1^j} \right) g(\delta_{[3,n-2]} \cup 1^i) \\
&= \sum_{i=0}^{n-4} \left(\sum_{j=i}^{n-4} 1 \right) g(\delta_{[3,n-2]} \cup 1^i) \\
&= \sum_{i=0}^{n-4} (n-i-3) g(\delta_{[3,n-2]} \cup 1^i).
\end{aligned}$$

□

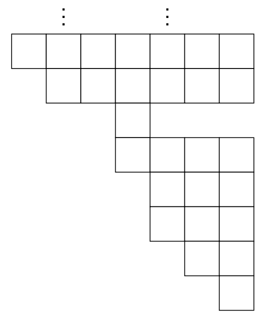
Conjecture 4.17

$$|\pi_n(\alpha, 2)| = \sum_{i=0}^{n-4} (n-i-3) g(\delta_{[3,n-2]} \cup 1^i)$$

I would like to be able to continue in this way and next enumerate $\pi_n(\alpha, 3)$ and $\pi_n(\alpha, n-4)$. The situation with $\pi_n(\alpha, n-4)$ begins similarly except now the first two words are strictly decreasing and it is the third word where the first relation is switched. After shifting the boxes and accounting for this reversed relation I get the diagram



Unfortunately this time I cannot use Corollary 4.8 to count the standard tableaux of this diagram. A similar result occurs as I continue to slide the 2 of my r-stat up the first column, so that the associated diagram of $\pi_n(\alpha, n-5)$ is



and this pattern continues. In order to enumerate the standard tableaux of these diagrams Theorem 4.6 needs to be expanded to include a shifted Young diagram with a diagram attached below it that is not 1^k.

Now I want to move on to r-stats of other ranks. I want to capitalize on my successes so far by finding other r-stats with similar characteristics to those I have already enumerated. Consider the family of r-stats

$$\pi_n = \begin{array}{cccccc} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & & \\ 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \end{array}$$

These r-stats are similar to $\pi_n(\binom{n}{2}-1, n-2)$ in the sense that $\pi_n(\binom{n}{2}-1, n-2)$ is composed of rows of all 1s except the bottom two rows are $\hat{0}_{\mathcal{TB}_3}$. Likewise, π_n are composed of rows of all 1s except the bottom three rows are $\hat{0}_{\mathcal{TB}_4}$. Conveniently I can enumerate π_n in a similar way to how I enumerated $\pi_n(\binom{n}{2}-1, n-2)$.

Proposition 4.18 Suppose

$$\pi_n = \begin{array}{llllll} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & & \\ 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \end{array} \in \mathcal{TB}_n,$$

then

$$|\pi_n| = g(\delta_{[4,n]}).$$

Proof. Suppose $M \in \pi_n$ with $S_M = w_1, ..., w_{n-1}$ and let $\ell = \ell(M)$. Now the outer peel is $p(\pi_n) = (1, 1, ..., 1, 2, 3)$, so the elements of w_1 are strictly decreasing except $w_{11} < w_{12} < w_{13}$. Furthermore, all three of these elements must be greater than w_{14}. This implies $w_{13} = \ell$, $w_{12} = \ell - 1$, and $w_{11} = \ell - 2$. Since these three elements are fixed I can remove these boxes from π_n to get the r-stat

$$\begin{array}{llllll} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & & \end{array} = \hat{1}_{\mathcal{TB}_{\delta_{[4,n]}}}.$$

This block has $g(\delta_{[4,n]})$ elements by Proposition 4.11, so the equality is proved. □

Just as I slid the 2 up the first column going from Proposition 4.14 to Proposition 4.16, I can slide the $3, 2$ up the first column here and try to enumerate the resulting r-stat.

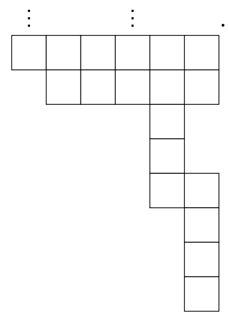
Proposition 4.19 Suppose

$$\pi_n = \begin{array}{lllllll} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & \\ \vdots & & & & & & \\ 1 & 1 & 1 & 1 & 1 & & \\ 1 & 1 & & & & & \\ 2 & 1 & & & & & \\ 3 & 1 & & & & & \\ 1 & & & & & & \end{array}$$

then

$$|\pi_n| = g^*_{n-2}(\delta_{[5,n-1]}; 3).$$

The proof of this proposition mirrors the proof of Proposition 4.16 with the added complication that there is an r-set of size 2 and an r-set of size 3. Fortunately I can still use the relations on the labels to shift the boxes to the right and rearrange the labels that are not strictly increasing along the new column $n-2$ to get the diagram



Just as I did before I can remove the last four boxes in column $n-1$ because the labels are fixed and the proof of Proposition 4.19 follows.

Although there are other r-stats that can be enumerated with this methodology I will end this section with one final result of this kind. I omit the proof because it is so similar to everything that has been done to this point. Simply use the relations on the labels that are inherited from the r-stat to get a diagram, in this case $\delta_{[4,n-1]}$, and then enumerate the standard tableaux of the diagram.

Proposition 4.20 Suppose

$$\pi_n = \begin{array}{ccccccc} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \\ \vdots & & & & & & \\ 1 & 1 & 1 & 1 & 1 & & \\ 1 & 1 & & & & & \\ 2 & 2 & 1 & & & & \\ 1 & & & & & & \\ 2 & & & & & & \end{array} \in \mathcal{TB}_n,$$

then

$$|\pi_n| = g(\delta_{[4,n-1]}).$$

Chapter 5

CONCLUSION

In this chapter I summarize my work so far and look ahead to the next steps of my continuing research.

5.1 Summary

This work began with the study of the Tamari lattice in the hopes of enumerating its maximal chains. The definition of the Tamari block lattice in Section 2.2 was the product of this effort. However, in studying $\mathcal{TB}_\lambda$ my fellow collaborators and I not only proved it was a graded lattice (Section 2.3 and Section 2.5) through the use of the diagonal sentence (Section 2.3) and the r-stat of a top chain (Section 2.4), but we also discovered that $\mathcal{TB}_n$ was deeply connected to the higher Stasheff-Tamari order, $\mathrm{HST}_1(n-2,3)$ via Rambau's work (Section 2.6). This connection provides a geometric interpretation of $\mathcal{TB}_n$ as well as an alternative perspective for $\mathrm{HST}_1(n-2,3)$.

I next wanted to know if I could generalize the approach used to prove $\mathcal{TB}_\lambda$ was a lattice to resolve the open problem concerning the lattice property for $\mathrm{HST}_1(n,4)$. In Chapter 3 I did generalize the approach used in Section 2.5 into two tests for the lattice property based on whether the poset in question is graded or not. Because $\mathrm{HST}_1(n,4)$ is not graded, only Theorem 3.1 could be used to show $\mathrm{HST}_1(n,4)$ is a lattice.

In Chapter 4 I focused on enumeration in $\mathcal{TB}_\lambda$. I enumerated the r-stats of rank 1, 2, $\binom{n}{2}-1$, and $\binom{n}{2}-2$ in Section 4.1. For the remainder of Chapter 4 I turned my attention to counting the top chains of a given square block. I did this for all of the rank 1 square blocks in Section 4.2. In Section 4.4 I generalized a strategy Fishel and Nelson used to enumerate the maximum length chains of T_n to enumerate other square blocks of $\mathcal{TB}_\lambda$. To use this strategy I needed to generalize Thrall's formula for the number of shifted standard tableaux which I did in Section 4.3.

Despite the progress I made in these areas there are still many enticing questions remaining.

5.2 Future Work

In Section 3.2 I discussed how Theorem 3.1 could be applied to prove $\mathrm{HST}_1(n,4)$ is a lattice. One area of future work is to finish this proof. If I can finish the proof then it not only solves the open problem concerning the lattice property for $\mathrm{HST}_1(n,4)$, but it would also solve the open problem as to whether $\mathrm{HST}_1(n,4)$ and $\mathrm{HST}_2(n,4)$ coincide by proving that they do not ($\mathrm{HST}_2(n,4)$ is known not to be a lattice for some n).

In Section 4.1 I enumerated the number of r-stats of several ranks. In general, the enumeration of r-stats of TB_n is an open and fascinating problem. Although partitioning the r-stats by rank as I did in Section 4.1 has not shown much promise in the general case, there is another method that has given me hope. In collaboration with Susanna Fishel, Mahir Bilen Can, and Luke Nelson, I am currently working on solving this problem by a peeling process. The concepts in Section 2.5 give the overarching principles for this approach. This work has reduced the problem to two generating functions that we can present in terms of each other. If we can determine the coefficients of either we will have solved the problem.

There are two other areas in Chapter 4 where I want to develop more results. First, in order to enumerate the top chains of r-stats using the methodology of Section 4.4 I need to improve Theorem 4.6. Currently this theorem only applies to shifted Young diagrams with boxes added in one column. However, there are many r-stats, including $\pi(\binom{n}{2}-1,i)$ for general i, which lead to shifted Young diagrams with more complex diagrams attached below them. I would also like to completely determine the number of top chains of $\hat{1}_{\mathcal{TB}_\lambda}$.

The final area of future work requires more background to present. To this point I have mostly focused on $\mathcal{TB}_\lambda$ itself, although in Section 2.6 I discussed the connection between $\mathcal{TB}_n$ and the higher Stasheff-Tamari orders. I now want to discuss one other family of posets $\mathcal{TB}_n$ has a connection with, the higher Bruhat orders, and show some likely implications on $\mathcal{TB}_n$ as well as on the higher Bruhat orders that result from this connection. I will first define the higher Bruhat orders and explain their connection to $\mathcal{TB}_n$ including two maps, one from $\mathcal{B}(n,2)$ to $\mathcal{TB}_n$ and another from the set of all maximal chains of T_n to reduced decompositions of the longest word of $\mathfrak{S}_n$. Then I will explain how these maps might be used to find the expected number of pentagon moves within certain r-stats and also to find the expected number of braid moves between certain reduced decompositions of the longest word of $\mathfrak{S}_n$.

Manin and Schechtman 1989 defined the higher Bruhat orders, $\mathcal{B}(n,k)$. Just as I was mostly interested in $\mathrm{HST}_1(n,d)$, I will also mostly be interested in the locally defined higher Bruhat order and so I will not explicitly define the higher Bruhat order with globally defined relations, although the two orders are equal for $k \in [0,2]$ which are the cases I am interested in here (see Manin and Schechtman 1989; Rambau and Reiner 2012; Ziegler 1993 for more information on these orders). Actually my main focus will be $\mathcal{B}(n,2)$, so I will not even define either order explicitly for general k, but the reader should have a general idea of how to view them by the end of this discussion.

Just as the higher Stasheff-Tamari orders can be defined recursively, so too with the higher Bruhat orders, and just as the higher Stasheff-Tamari orders begin with the Boolean lattice, so too with the higher Bruhat orders (albeit with a slight change). Recall from Section 2.6 that $\mathrm{HST}_1(n,1)$ is isomorphic to the Boolean lattice on $n-2$ objects (Rambau and Reiner 2012, Example 3.5) and $\mathrm{HST}_1(n,2)$ is isomorphic to the Tamari lattice, T_{n-2} (Example 3.6). In a similar way, $\mathcal{B}(n,0)$ is isomorphic to the Boolean lattice on n objects (Example 8.1), and $\mathcal{B}(n,1)$ is isomorphic to the weak Bruhat order on $\mathfrak{S}_n$ (Example 8.2).

As I mentioned in Section 2.1, T_n is a Cambrian lattice (see Reading 2006), called the Tamari orientation of the type-A diagram. In general Cambrian lattices are derived from Coxeter groups (see Reading 2006), and so this title for T_n simply states that T_n can be derived from the Coxeter group A_{n-1} which is just $\mathfrak{S}_n$. In fact, T_n is a quotient lattice and a sublattice of the (right) weak Bruhat order on $\mathfrak{S}_n$ (Reading 2006; Bjorner and Wachs 1997). This means there is an obvious projection map from the (right) weak Bruhat order on $\mathfrak{S}_n$ to T_n. From the original paper, Kapranov and Voevodsky 1991, that defined the higher Stasheff-Tamari orders there has been a generalization of this map from $\mathcal{B}(n,k)$ to $\mathrm{HST}_1(n+2,k+1)$ which has subsequently been shown to be a surjection for $k \in [0,2]$ (Rambau 1997; Thomas 2002a). In light of Theorem 2.2, this also gives a surjective map from $\mathcal{B}(n,2)$ to $\mathcal{TB}_n$.

To better understand the potential of this connection between $\mathcal{B}(n,2)$ and $\mathcal{TB}_n$ I want to consider $\mathcal{B}(n,1)$ and $\mathcal{B}(n,2)$ from two separate perspectives. Recall from Chapter 1 that in addition to permutations on $[n]$, elements of $\mathfrak{S}_n$ can be viewed as inversion sets and as words on the alphabet of the simple reflections $s_1, s_2, ..., s_{n-1}$ where $s_i = (i\ i+1)$. Also recall an inversion of a permutation $\sigma = \sigma_1\sigma_2...\sigma_n$ in single line notation is a pair $i < j$ such that $\sigma_i > \sigma_j$. I will not consider the left weak Bruhat order, so I will simply say weak Bruhat order and use $\leq_R$ to denote it with the understanding that it is from the right. From the inversion set perspective the weak Bruhat order is simply an inclusion relation, so for $\sigma, \tau \in \mathfrak{S}_n$, $\sigma \leq_R \tau$ precisely when $\sigma \subseteq \tau$ as inversion sets. To describe the weak Bruhat order in terms of words in A_{n-1} it is easier to view it as the transitive closure of the cover relation $\sigma \lessdot_R \tau$ if and only if $\tau = \sigma s_i$ for some simple reflection s_i and $\ell(\tau) = \ell(\sigma) + 1$ where $\ell(\cdot)$ is the number of letters in a reduced decomposition of a word (see Chapter 1 and Bjorner and Brenti 2006 for more details).

Now $\mathcal{B}(n, k)$, and in particular $\mathcal{B}(n, 2)$ can be naturally understood as a generalization of the inversion set perspective of the weak Bruhat order on $\mathfrak{S}_n$. Inversions on permutations, which are also called *2-packets* can be generalized to $k+1$-*packets* which are $k+1$-tuples of $[n]$ with a sort of transitive property. Just as if $\sigma \in \mathfrak{S}_n$ contains (respectively does not contain both) the 2-packets (i, j) and (j, k) for $i < j < k$, then it must also contain (respectively not contain) the 2-packet (i, k), so also the elements of $\mathcal{B}(n, k)$ viewed as inversion sets must have a type of transitive property. To see this better view a permutation $\sigma = \sigma_1 \sigma_2 ... \sigma_n$ as a maximal chain M in the Boolean lattice $\mathcal{B}_n = \mathcal{B}(n, 0)$ and view the inversions of σ as the pairs of 1-packets of $[n]$ that are "out of order" in M. The weak Bruhat order $\mathcal{B}(n, 1)$ is just the equivalence classes of maximal chains of $\mathcal{B}(n, 0)$ with the same set of out of order pairs. (I say equivalence classes here to coincide with the general case even though in this case each maximal chain is its own equivalence class.) Similarly for $\mathcal{B}(n, 2)$ view an element $\sigma = \sigma_1 \sigma_2 ... \sigma_\alpha$ (where $\alpha = \binom{n}{2}$) as a maximal chain M in the weak Bruhat order $\mathcal{B}(n, 1)$ and view the inversions of σ as the triples of 2-packets of $[n]$ that are "out of order" in M. In this case out of order is less clear than before, so define $i < j < k$ to be *in order* in M if (i, j) is before (j, k) which is before (i, k) in M, and define (i, j, k) to be an inversion of M if this ordering is reversed in M (by the transitive property of inversions in $\mathfrak{S}_n$ these are the only possibilities). This means $\mathcal{B}(n, 2)$ can be viewed as the equivalence classes of maximal chains of $\mathcal{B}(n, 1)$ with the same set of out of order triples. This perspective is so closely related to the perspective of $\mathrm{HST}_1(n, d)$ as equivalence classes of maximal chains of $\mathrm{HST}_1(n, d-1)$ that the connection between these two orders should no longer be a surprise (and in fact has been well studied, see Rambau 1997; Ziegler 1993; Manin and Schechtman 1989 for further details).

Example 5.1 Consider $\mathcal{B}(3, 1)$ (the weak Bruhat order on $\mathfrak{S}_3$) in Figure 21. There are two maximal chains in this order. It is natural to label each edge of $\mathcal{B}(3, 1)$ with the simple reflection added at each step in the middle poset. With this labeling the maximal chains are $s_2 s_1 s_2$ and $s_1 s_2 s_1$ which coincide with the two reduced decompositions of the longest word in A_2.

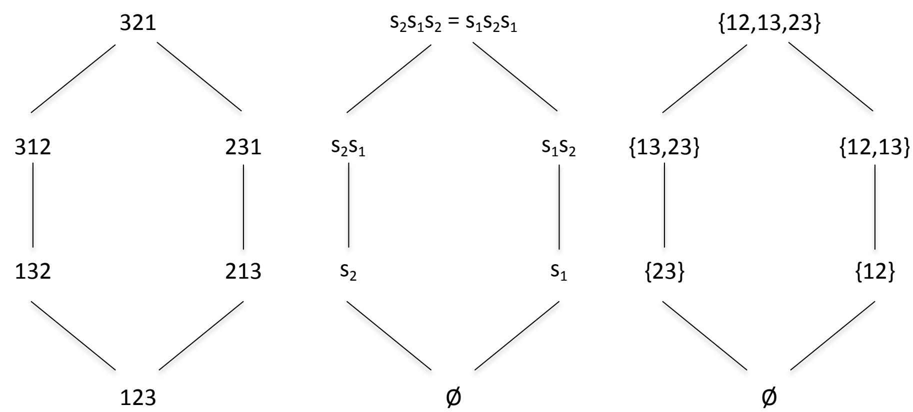


Figure 21: $\mathcal{B}(3,1)$ *in single line notation on the left, as words on the alphabet of simple reflections in the middle, and as inversion sets on the right.*

As can be seen in the right poset, the pairs 12, 13, and 23 are added to the inversion set in the lexicographic order in the maximal chain on the right, but they are added in the opposite order in the maximal chain on the left. As a result $\mathcal{B}(3,2)$ is just the two element chain in Figure 22 where the bottom element is the right maximal chain of $\mathcal{B}(3,1)$ and the top element is the left maximal chain.

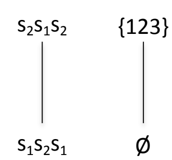


Figure 22: $\mathcal{B}(3,2)$ *as reduced decompositions of* $w_0 \in A_2$ *on the left and as inversion sets on the right.*

I have included the $n = 4$ case in Figure 23 so the discussion of commutation classes of reduced decompositions of w_0 is clear. The commutation classes are the reduced decompositions of w_0 that are connected by dashed lines, so each reduced decompositions of w_0 in a commutation class has the corresponding inversion set from the right poset when viewed as a maximal chain in $\mathfrak{S}_4$. For example, the two reduced decompositions in the bottom class do not have any inversions.

As I explained in Chapter 1 the maximal chains of the weak Bruhat order are just the reduced decompositions of the longest word $w_0 \in A_{n-1}$, so the partitioning of maximal chains in the previous

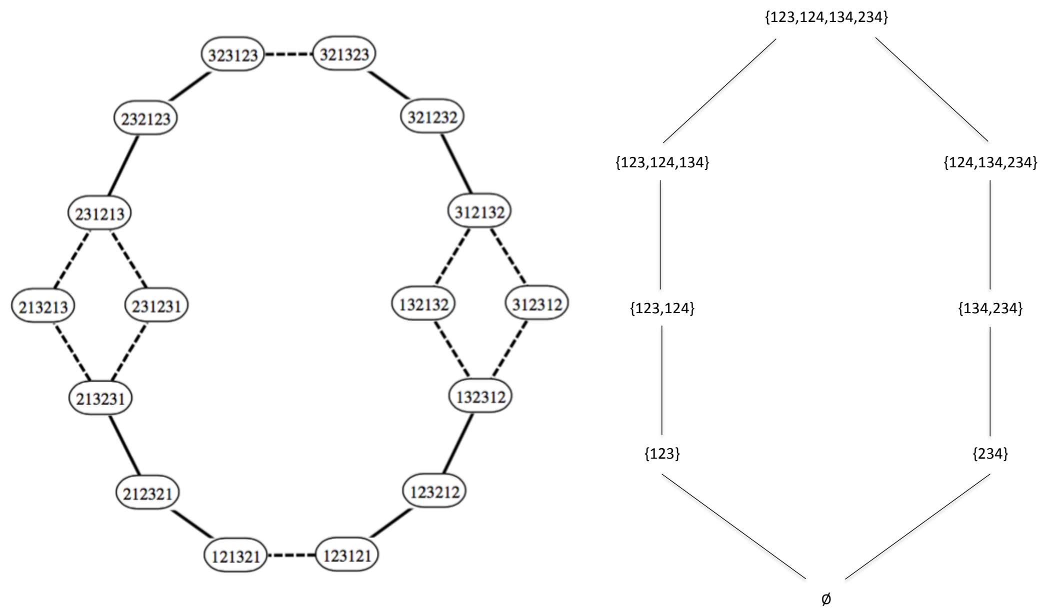


Figure 23: $\mathcal{B}(4,2)$ *as reduced decompositions of* $w_0 \in A_3$ *on the left and as inversion sets on the right. The diagram on the left is Schilling et al. 2015, Figure 1. Also, to ease notation each simple reflection in a word on the left poset is represented by its index, so the letter 1 in a word is the simple reflection* s_1.

paragraph can also be viewed as a partitioning of the reduced decompositions of w_0. Recall that there are two types of Coxeter relations on words in A_{n-1}, there are commuting moves where $s_i s_j = s_j s_i$ for $|i-j| \geq 2$, and there are braid moves where $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. From this perspective the equivalence classes are the *commutation classes* of w_0, in other words, two reduced decompositions M_1 and M_2 of w_0 are in the same commutation class if M_2 can be obtained from M_1 by making a sequence of commuting moves on M_1. Furthermore, the cover relations of $\mathcal{B}(n,2)$ can be described by braid moves, so that commutation classes $\overline{M_1}, \overline{M_2} \in \mathcal{B}(n,2)$ are related by $\overline{M_1} \lessdot \overline{M_2}$ if and only if there exists a reduced decomposition of w_0 $M_1 \in \overline{M_1}$ that is equal to a reduced decomposition of w_0 $M_2 \in \overline{M_2}$ everywhere except a three element subword $s_i s_{i+1} s_i$ in M_1 is the subword $s_{i+1} s_i s_{i+1}$ in M_2. This perspective of viewing the reduced decomposition of w_0 is eerily similar to TG_n with square blocks being connected to other square blocks by pentagon moves.

With this background in place I now want to explain how the connections between these posets can be useful. As I already mentioned, composing the surjective map from $\mathcal{B}(n,2)$ to $\mathrm{HST}_1(n+2,3)$ that Rambau calls $\mathcal{T}_{flip}$ (Rambau 1997) with the bijective map θ_3 from $\mathrm{HST}_1(n+2,3)$ to $\mathcal{TB}_n$ gives a surjection from $\mathcal{B}(n,2)$ to $\mathcal{TB}_n$. Furthermore, just as elements of $\mathcal{B}(n,2)$ are equivalence classes of reduced decompositions of w_0, elements of $\mathcal{TB}_n$ are equivalence classes of maximal chains of T_n.

This means Reading's projection map from the weak Bruhat order on A_{n-1} to T_n can be used to project reduced decompositions of $w_0 \in A_{n-1}$ to maximal chains in T_n. Also, Edelman and Greene 1987 defines a bijection from reduced decompositions of $w_0 \in A_{n-1}$ to standard Young tableaux of shape δ_n. Together this means there is a composition map from the set of reduced decompositions of $w_0 \in A_{n-1}$ to maximal chains in T_n (via Reading's map) to standard Young tableaux of shape δ_n (by standardizing the Tamari tableau associated to a maximal chain) back to reduced decompositions of $w_0 \in A_{n-1}$ (via the Edelman/Greene map). I want to show this map is the identity map. If this is true I would have one map connecting r-stats to commutation classes of reduced decompositions of w_0, and another map connecting maximal chains in the weak Bruhat order on A_{n-1} to Tamari tableaux of shape δ_n.

Reiner 2005 shows the expected number of braid moves for a reduced decomposition for $w_0 \in A_{n-1}$ is one. Let $\mathrm{Red}_n(\mathbf{w_0})$ be the commutation class of reduced decompositions of $w_0 \in A_{n-1}$ associated to the bottom element of $\mathcal{B}(n,2)$. Examining Figure 23 shows that

$$\mathrm{Red}_4(\mathbf{w_0}) = \{s_1s_2s_1s_3s_2s_1, s_1s_2s_3s_1s_2s_1\}.$$

Schilling et al. 2015 further proves the expected number of braid moves for a reduced decomposition in $\mathrm{Red}_n(\mathbf{w_0})$ is one. This fact is clearly not true for every commutation class, however little work has been done to expand this result to results on the expected number of braid moves for other commutation classes. This is one area where the connections between $\mathcal{B}(n,2)$ and $\mathcal{TB}_n$ could be useful. For example, it is not hard to show every maximal chain with r-stat $\pi(\binom{n}{2}-1, n-2)$ has exactly one increasing pentagon move, and I think it can also be shown (I have computer evidence to support this) that there is one expected decreasing pentagon move for every chain. This would mean the expected number of pentagon moves for maximal chains with this r-stat is two. I also think the two composition maps I outlined above can lead to a bijection between the maximal chains in this square block and the reduced decompositions for $w_0 \in A_{n-1}$ with inversion set $\{(n-2)(n-1)n\}$ (I have computer evidence to support this). These same maps may show every braid move is preserved as a pentagon move in this specific case. This would mean the expected number of braid moves for a reduced decomposition for $w_0 \in A_{n-1}$ with inversion set $\{(n-2, n-1, n)\}$ is two. It is my hope that making solid connections between $\mathcal{TB}_n$ and $\mathcal{B}(n,3)$ will lead to this result as well as others like it.

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APPENDIX A

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In mathematics authors are listed alphabetically. Chapter 2 of this dissertation is a preprint of a paper that I coauthored with Mahir Bilen Can, Susanna Fishel, and Luke Nelson. All three of my coauthors have given their approval that this work be included in this dissertation.