The Pauli-Lubański Vector in a Group-Theoretical Approach to Relativistic Wave Equations

by

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ABSTRACT

Chapter 1 introduces some key elements of important topics such as; quantum mechanics, representation theory of the Lorentz and Poincaré groups, and a review of some basic relativistic wave equations that will play an important role in the work to follow. In Chapter 2, a complex covariant form of the classical Maxwell's equations in a moving medium or at rest is introduced. In addition, a compact, Lorentz invariant, form of the energy-momentum tensor is derived. In chapter 3, the concept of photon helicity is critically analyzed and its connection with the Pauli-Lubański vector from the viewpoint of the complex electromagnetic field, $\mathbf{E} + i\mathbf{H}$. To this end, a complex covariant form of Maxwell's equations is used. Chapter 4 analyzes basic relativistic wave equations for the classical fields, such as Dirac's equation, Weyl's two-component equation for massless neutrinos and the Proca, Maxwell and Fierz-Pauli equations, from the viewpoint of the Pauli-Lubański vector and the Casimir operators of the Poincaré group. A connection between the spin of a particle/field and consistency of the corresponding overdetermined system is emphasized in the massless case. Chapter 5 focuses on the so-called generalized quantum harmonic oscillator, which is a Schrödinger equation with a time-varying quadratic Hamiltonian operator. The time evolution of exact wave functions of the generalized harmonic oscillators is determined in terms of the solutions of certain Ermakov and Riccati-type systems. In addition, it is shown that the classical Arnold transform is naturally connected with Ehrenfest's theorem for generalized harmonic oscillators. In Chapter 6, as an example of the usefulness of the methods introduced in Chapter 5 a model for the quantization of an electromagnetic field in a variable media is analyzed. The concept of quantization of an electromagnetic field in factorizable media is discussed via the Caldirola-Kanai Hamiltonian. A single mode of radiation for this model is used to find time-dependent photon amplitudes in relation to Fock states. A multi-parameter family of the squeezed states, photon statistics, and the uncertainty relation, are explicitly given in terms of the Ermakov-type system.

“Physical laws should have mathematical beauty.” — P. A. M. Dirac

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TABLE OF CONTENTS

CHAPTER Page

Chapter 1

INTRODUCTION

In this chapter we will briefly review some important topics that are relevant to this dissertation. No attempt has been made to give a complete and detailed description of each area, but instead to just introduce and discuss certain aspects of the theories that will be most relevant in the work to follow.

In section 1.1, we introduce some notations that will be used throughout. In section 1.2, we will briefly review some important fundamentals of quantum mechanics including: Hamiltonians, the Schrödinger equation, the Heisenberg uncertainty principle, Ehrenfest's Theorem, and the unitary transformation between the Schrödinger and Heisenberg pictures of quantum mechanics. In section 1.3, the representation theory of the Poincaré and Lorentz groups will be discussed. This will be very important in the sections on relativistic wave equations and spin and angular momentum of elementary particles. Section 1.4 gives a brief review of several fundamental relativistic wave equations that will be visited again in relation to the Pauli-Lubański vector and the representation theory of the Poincaré group in a later chapter.

In Chapter 2, a complex version of electrodynamics is introduced. We discuss a complex covariant form of the classical Maxwell's equations, in a moving medium or at rest. A compact, Lorentz invariant, form of the energy-momentum tensor is derived.

Chapter 3 addresses a common misconception in the literature and the standard quantum field theory textbooks on an operator relation used to define helicity of massless particles. In accomplishing this, it is shown that Maxwell's equations in vacuum can, in fact, be derived through the representation theory of the Poincaré group with the help of the Pauli-Lubański vector. The definition of helicity, as it is traditionally given in particle physics, is discussed and the simplest covariant helicity states are constructed. The chapter concludes with some remarks regarding polarized waves and a discussion on the complex Maxwell equations in vacuum and discrete transformations.

Chapter 4 emphasizes the role of the Pauli-Lubański vector for several major relativistic wave equations. The work in this chapter was motivated by a result of Chapter 3, that Maxwell's equation can be derived through the representation theory of the Poincaré group with the help of the Pauli-Lubański vector. The chapter begins by introducing a variant version of Dirac's equation, which takes the form of a certain overdetermined system of partial differential equations. Next, there is a discussion on the Pauli-Lubański vector and Dirac's equation in vacuum, and lastly the relativistic definition of spin for Dirac particles. The chapter continues on to cover, in a similar manner, the Weyl, Proca, Maxwell, and Fierz-Pauli equations. In addition, the spinor form of Maxwell's equations is mentioned in a covariant form, along with its traditional form.

In Chapter 5, we change gears to discuss the exact wave functions for a generalized harmonic oscillator, which is a Schrödinger equation whose Hamiltonian is a general time-dependent quadratic operator of position and momentum. Green's function is found with help from the Ermakov-type system, which is also introduced in this chapter. The chapter concludes by outlining Ehrenfest's theorem and how it relates to the generalized harmonic oscillators.

Lastly, in Chapter 6, an application of the results from Chapter 5 helps to study a certain model for the quantization of an electromagnetic field in variable media. The exact wave function for the model is found, the uncertainty relation and squeezed states are discussed, and lastly the photon statistics are explicitly given.

1.1 Notation

- Natural units, where the fundamental constants $c = \hbar = 1$, will generally be used.

- Latin indices $i, j, k, \ldots$ run over spatial coordinate labels, $i, j, k = 1, 2, 3$.
- Greek indices $\mu, \nu, \rho, \ldots$ run over the four spacetime coordinate labels $\mu, \nu, \rho = 0, 1, 2, 3$.
- The flat Minkowski spacetime metric tensor is denoted $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- The curved spacetime metric tensor is denoted $g_{\mu\nu}$, but in most places we are working in flat spacetime where this notation is used to indicate the Minkowski metric instead. Any place such a notation is used it will be indicated.
- The the totally anti-symmetric Levi-Civita symbol in three and four dimensions is denoted by e_{pqr} and $e_{\mu\nu\sigma\tau}$, respectively. The convention $e_{123} = +1$ and $e_{0123} = +1$ is used.
- Three-dimensional spatial vectors are indicated by a boldface letter, e.g. **A**.
- A contravariant four-vector is denoted with an upper index x^μ and a covariant four-vector with a lower index x_μ.
- The relation between contravariant and covariant vectors is given by the metric tensor, $x_\mu = \eta_{\mu\nu} x^\nu$.
- The spacetime interval $ds^2 = dx_\mu dx^\mu = dt^2 - dx^2 - dy^2 - dz^2$.
- The four-gradient operator is denoted $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.
- For the contravariant four-gradient $\partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\nabla\right)$.
- The d'Alembert operator is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \partial^2$.
- The four-momentum of a particle is given by $p^\mu = (E, \mathbf{p})$.
- Sometimes the notation $p \cdot x = px = p_\mu x^\mu = Et - \mathbf{p} \cdot \mathbf{x}$ is used.

1.2 A Brief Discussion on Quantum Mechanics

In classical mechanics, the Hamiltonian,

$$H(p,x) = T + V \quad where \quad T = T(p) = \frac{p^2}{2m} \quad and \quad V = V(x), \tag{1.1}$$

represents the total energy of a closed system. Here the functions $T(p)$ and $V(x)$ represent the kinetic and potential energies respectively. The variable p represents the generalized momenta and x the generalized coordinates. These state variables depend on time and the time evolution of the system is given by the Hamilton equations,

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{1.2}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{dV(x)}{dx}. \tag{1.3}$$

The variables p and x completely describe the state of the system. The equations (1.2)-(1.3) are derived from the Lagrangian formalism of classical mechanics, see any standard classical mechanics text for details.

In quantum mechanics, the Hamiltonian is obtained by replacing the variables p and x by the operators $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ and $\hat{x} = x$. These are operators on an infinite dimensional Hilbert space, which is the setting of quantum mechanics. It is a postulate of quantum mechanics that the state of the quantum system at time t is given by a wavefunction $\psi(x,t)$, which is represented by a vector in an infinite dimensional Hilbert space. The time evolution of the quantum system is given by the Schrödinger equation,

$$i\hbar\frac{\partial \psi(x,t)}{\partial t} = \hat{H}\psi(x,t) \quad where \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.4}$$

Here the kinetic and potential energies are given in terms of the operators $\hat{p}$ and $\hat{x}$.

A general variable quadratic Hamiltonian,

$$\hat{H} = a(t)\hat{p}^2 + b(t)\hat{x}^2 + c(t)\hat{x}\hat{p} - id(t) - f(t)\hat{x} - g(t)\hat{p}, \tag{1.5}$$

where *a, b, c, d, f and g* are suitable real-valued functions of time t only, will be discussed later. In most places it will be clear from the context when we are dealing with an operator and for this reason the hat, $\hat{O}$, above operators will usually be left out. A measurable quantity, often called an observable, in quantum mechanics corresponds to a self-adjoint operator, and the possible outcomes of measurements to values in the spectrum of the operator. The action of an operator on a wave function yields an eigenvalue, when it exists, that corresponds to the particular quantity being measured. Examples of such operators include the position, momentum, and energy operators. The expectation value of an operator with respect to a normalized quantum state ψ gives the statistical mean of the measurements performed on ψ:

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^* A \psi dV.$$

The Heisenberg uncertainty principle is of great importance to quantum mechanics, in general: given two non-commuting self-adjoint operators A and B such that $[A,B] = i\hbar$, the following inequality is satisfied

$$\langle \Delta A \rangle \langle \Delta B \rangle \geq \hbar/2, \tag{1.6}$$

where $\langle \Delta A \rangle = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ and $\langle \Delta B \rangle = \sqrt{\langle B^2 \rangle - \langle B \rangle^2}$ is the uncertainty, or standard deviation, of the respective operators. Typically, this statement is given for the position $A = \hat{x}$ and momentum $B = \hat{p}$ operators; although, this relation holds true for any conjugate operators that satisfy the given hypothesis. As a result, this statement says that if the measured value of one of the operators, say A, is known precisely, then the value of the operator B is infinitely uncertain. There are especially important quantum states that minimize the relation (1.6), known as coherent states or, more generally, squeezed coherent states. The term squeezed is used to described the coherent states whose oscillating variances $\left\langle (\Delta A)^2 \right\rangle$ and $\left\langle (\Delta B)^2 \right\rangle$ become smaller than the 'static' vacuum state, for which $\left\langle (\Delta A)^2 \right\rangle = \left\langle (\Delta B)^2 \right\rangle = \hbar/2$. The Ehrenfest theorem gives the time-evolution of the expectation value of an operator, A, according to the formula

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A,H]\rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle.$$

Finally, one may move back and forth between the Schrödinger and Heisenberg pictures of quantum mechanics, transferring the time-dependence from the wave functions to the operators or vice versa, using a unitary transformation. The equivalence of these two pictures of quantum mechanics will be useful below when calculating the photon statistics. In the Heisenberg picture of quantum mechanics the state vectors are time-independent, while the observable operators instead depend on time and satisfy the Heisenberg equation of motion

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H,A(t)] + \left(\frac{\partial A}{\partial t}\right)_H. \tag{1.7}$$

The subscript on the last term, $\frac{\partial A}{\partial t}$ indicates that it has undergone the unitary transformation along with the operator A

$$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar},$$

for a time-independent Hamiltonian. For a time-dependent Hamiltonian a more general unitary operator should be used, see [118]. Note that taking the expectation value of the Heisenberg equation yields the Ehrenfest theorem.

1.3 Representation Theory and the Lorentz and Poincáre Groups

The purpose of this section is to introduce some notations that will be used in the work to follow, along with a brief review of some important topics regarding the Lorentz and Poincaré groups and their representations. The main body of work takes place in the Minkowski space $\mathscr{M} = (\mathbb{R}^4, g)$, where $g = g^{\mu\nu} = \text{diag}(1,-1,-1,-1)$ is the Minkowski metric. When working in a curved spacetime we instead reserve the symbol g for the metric tensor and use instead the symbol η for the Minkowski metric. A reminder of many of the notations will be given in the sections as they are needed, although a list has already been provided in the introduction. A classification program for relativistic wave equations through the representation theory of the inhomogeneous Lorentz (Poincaré) group was started by Bargmann and Wigner in 1948, [12]. Chapter 4 follows the work of Bargmann and Wigner with a classification program of relativistic wave equations from the representation theory of the Poincaré group, where these equations appear in this framework as a statement of consistency of certain overdetermined systems of partial differential equations. In particular Dirac's equation, Weyl's two-component equation for massless neutrinos, and the Proca, Maxwell and Fierz-Pauli equations are studied from the viewpoint of the Pauli-Lubański vector.

1.3.1 Homogeneous Lorentz Group

The Lorentz group $O(1,3)$ is the group of all linear transformations of $\mathbb{R}^4$ that preserve the Lorentz inner product $x_\mu y^\mu = g_{\mu\nu} x^\nu y^\mu$. That is, given $\Lambda \in O(1,3)$:

$$(\Lambda x)_\mu (\Lambda y)^\mu = g_{\mu\nu} \left(\Lambda^\nu_\sigma x^\sigma\right) \left(\Lambda^\mu_\kappa y^\kappa\right) = x_\mu y^\mu \text{ for all } x, y. \tag{1.8}$$

Or in matrix form,

$$\Lambda^\dagger g \Lambda = g, \tag{1.9}$$

where g is the Minkowski metric. The group $O(1,3)$ has four connected components, which can be seen by the following. Taking the determinant on both sides of the equation (1.9) it is clear that $det\Lambda = \pm 1$. If $det\Lambda = 1$, the element Λ is called *special*. The equation (1.8) implies that

$$\Lambda^\mu_\kappa g_{\mu\nu} \Lambda^\nu_\sigma = g_{\kappa\sigma}, \tag{1.10}$$

and setting $\kappa = \sigma = 0$ one gets

$$(\Lambda^0_0)^2 = 1 + \sum_{j=1}^{3} (\Lambda^j_0)^2 \geq 1. \tag{1.11}$$

This shows that we either have $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. The transformations $\Lambda \in O(1,3)$ such that $\Lambda^0_0 \geq 1$ are called *orthochronous*, meaning that they do not change the direction of time. A special orthochronous Lorentz transformation is called *proper*. The set of special transformations forms the *special Lorentz group*, denoted $SO(1,3)$, and the set of orthochronous transformations forms the orthochronous Lorentz group, $O_+(1,3)$. Then the restricted Lorentz group (or proper Lorentz) group,

$$SO_+(1,3) = SO(1,3) \cap O_+(1,3), \tag{1.12}$$

is the connected component of the identity in $O(1,3)$. The Lorentz group is a six-dimensional Lie group and local coordinates may be introduced in it with the exponential map

$$\Lambda = \exp\left(\theta_{\lambda\mu} m^{\lambda\mu}/2\right), \tag{1.13}$$

where $\theta_{\lambda\mu}$ is an anti-symmetric $4x4$ matrix and the infinitesimal generators,

$$\left(m^{\lambda\mu}\right)^{\alpha}_{\beta} = g^{\mu\alpha}\delta^{\lambda}_{\beta} - g^{\lambda\alpha}\delta^{\mu}_{\beta}, \quad m^{\lambda\mu} = -m^{\mu\lambda}, \tag{1.14}$$

satisfy the Lie algebra

$$[m^{\lambda\mu}, m^{\rho\sigma}] = g^{\lambda\rho} m^{\mu\sigma} - g^{\mu\rho} m^{\lambda\sigma} + g^{\mu\sigma} m^{\lambda\rho} - g^{\lambda\sigma} m^{\mu\rho}. \tag{1.15}$$

The universal covering of $SO_+(1,3)$ is a two-valued complex representation, denoted $SL(2,\mathbb{C})$, which is a connected Lie group. Representations of connected Lie groups can be studied by algebraic methods. If T is any representation of $SO_+(1,3)$, where the elements of the group have the form (1.13),

$$T(\Lambda) = exp\left(i\theta_{\lambda\mu} X^{\lambda\mu}/2\right), \tag{1.16}$$

the linear operators $X^{\lambda\mu} = -X^{\mu\lambda}$ are called the generators of the representation T, and they satisfy the commutation relation of the Lie algebra,

$$[X^{\lambda\mu}, X^{\rho\sigma}] = -i\left(g^{\lambda\rho}X^{\mu\sigma} - g^{\mu\rho}X^{\lambda\sigma} + g^{\mu\sigma}X^{\lambda\rho} - g^{\lambda\sigma}X^{\mu\rho}\right). \tag{1.17}$$

Finding all representations of the commutation rule (1.17) is equivalent to finding the representations of the restricted Lorentz group, $SO_+(1,3)$. The restricted Lorentz group has finite dimensional and infinite dimensional representations; however, it has no finite-dimensional unitary representations other than the identity representation $T(\Lambda) \equiv 1$. In the work to follow, we only concern ourselves with the finite-dimensional representations (four-vector, spinor, bispinor, four-tensor, etc...) of $SO_+(1,3)$, which act on finite-dimensional vector spaces; elements of these spaces transform according to the corresponding representation.

1.3.2 Inhomogeneous Lorentz (Poincaré) Group

The Poincaré group is the set of all homogeneous Lorentz transformations, $O(1,3)$ together with the group of translations, $\mathbb{R}^4$. That is, the ten-dimensional Lie group

$$\mathscr{P} = \mathbb{R}^4 \rtimes O(1,3), \tag{1.18}$$

which is why it is sometimes referred to as the inhomogeneous Lorentz group. The action of an element $(a,\Lambda) \in \mathscr{P}$ on $x \in \mathbb{R}^4$ is given by,

$$x'^{\mu} = \Lambda^{\mu}_{\nu}x^{\nu} + a^{\mu}. \tag{1.19}$$

As with the Lorentz group, the Poincaré group also has four connected components. Similar to the Lorentz group the component of the identity is

$$\mathbb{R}^4 \rtimes SO_+(1,3), \tag{1.20}$$

and the covering group is $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$. In addition to the generators in the Lie algebra for the Lorentz group, the generator of translation, P^μ must be added to give the Lie algebra of the Poincaré group:

$$[X^{\lambda\mu}, X^{\rho\sigma}] = -i\left(g^{\lambda\rho}X^{\mu\sigma} - g^{\mu\rho}X^{\lambda\sigma} + g^{\mu\sigma}X^{\lambda\rho} - g^{\lambda\sigma}X^{\mu\rho}\right), \tag{1.21}$$

$$[P^\mu, P^\nu] = 0, \tag{1.22}$$

$$[X^{\mu\nu}, P^\sigma] = i\left(g^{\nu\sigma}P^\mu - g^{\mu\sigma}P^\nu\right). \tag{1.23}$$

The classification of all the irreducible unitary representations of the inhomogeneous Lorentz group can be formulated in terms of finding the all representations of the commutation rules of this algebra by self-adjoint operators, see for example [231]. For more details on the Lorentz and Poincaré groups, see [15], [29], [187].

1.4 A Brief Review of Some Relativistic Wave Equations

1.4.1 Klein-Gordon-(Fock) Equation

The Klein-Gordon equation describes particles with no spin, which are called scalar particles. We denote such a particle by ϕ, which has only one component. The Klein-Gordon equation can be derived starting from the relativistic energy-momentum relation (in units with $\hbar = c = 1$),

$$E^2 - \mathbf{p}^2 = m^2. \tag{1.24}$$

Substituting the differential operators $E \to i\frac{\partial}{\partial t}$ and $\mathbf{p} \to -i\nabla$ in (1.24) and operating on ϕ we obtain the Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + m^2\phi = 0, \tag{1.25}$$

or in covariant form

$$\left(\Box + m^2\right)\phi = 0. \tag{1.26}$$

It should also be noted here, that since the Klein-Gordon equation expresses the relativistic relation between energy, momentum, and mass, it must hold for particles of any spin. Another interesting fact is that the well-known Schrödinger equation, from quantum mechanics, is the non-relativistic approximation to the Klein-Gordon equation, see for example [184]. The four-vector $j^\mu = (\rho, \mathbf{j})$, satisfies the continuity equation

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0, \tag{1.27}$$

where

$$\rho = \frac{i}{2m}\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right) \tag{1.28}$$

$$\mathbf{j} = \frac{1}{2im}\left(\phi^* \nabla \phi - (\nabla \phi)^* \phi\right), \tag{1.29}$$

where the asterisk $*$ stands for complex conjugation. Here, we consider ϕ to be complex-valued, which corresponds to charged particles. If instead ϕ were real-valued, (1.28)-(1.29) would be identically zero. Real-valued ϕ corresponds instead to electrically neutral particles. However, the major problem here is that the quantity (1.28) is not positive-definite since one can choose initial conditions on ϕ and $\partial\phi/\partial t$ to make it negative. Alternatively, to see this, one may substitute a plane-wave $\phi = Ne^{-ip_\mu x^\mu} = Ne^{-i(Et - \mathbf{p}\cdot\mathbf{x})}$ in (1.28) to find that

$$\rho = 2|N|^2 E, \tag{1.30}$$

from which it follows ρ may take negative values, since the energy E in (1.24) can be positive or negative. Hence ρ cannot be interpreted as a probability density and one can no longer interpret the Klein-Gordon equation as an equation for a single particle, [184]. This trouble with the Klein-Gordon equation is resolved by reinterpreting it instead as a field equation for a field ϕ instead of a single particle. Upon quantization of the field a successful particle theory may be recovered. The problem of negative energy, which becomes a severe problem for an interacting particle, is also overcome by treating ϕ as a quantum field.

1.4.2 Dirac Equation

The Dirac equation is a relativistic wave equation that was derived by Paul Dirac in 1928. The Dirac equation describes massive spin-1/2 particles such as, for example, electrons, protons, and quarks. In its covariant form the Dirac equation is written

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0. \tag{1.31}$$

Here the Dirac/gamma matrices are $\gamma^{\mu} = \left(\gamma^{0}, \boldsymbol{\gamma}\right)$, $\gamma_{\mu} = g_{\mu\nu}\gamma^{\nu} = \left(\gamma^{0}, -\boldsymbol{\gamma}\right)$, and $\gamma_5 = -\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, where

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{1.32}$$

and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the standard 2×2 Pauli matrices [168], [214]. The familiar anti-commutation relations,

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}, \qquad \gamma^{\mu}\gamma^{5} + \gamma^{5}\gamma^{\mu} = 0 \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.33}$$

hold. (Most of the results here will not depend on a particular choice of gamma matrices, but it is always useful to have an example in mind.) The four-vector notation, $x^{\mu} = (t, \mathbf{r})$, $\partial_{\mu} = \partial/\partial x^{\mu}$, and $\partial^{\alpha} = g^{\alpha\mu}\partial_{\mu}$ in natural units $c = \hbar = 1$ with the standard metric, $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$, in the Minkowski space-time $(\mathbb{R}^4, g)$ are utilized throughout, see [19], [28], [29], [157], [21].

In this notation, the transformation law of a bispinor wave function,

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4, \tag{1.34}$$

under a proper Lorentz transformation, is given by

$$\psi'\left(x'\right) = S_\Lambda \psi\left(x\right), \qquad x' = \Lambda x, \tag{1.35}$$

together with the rule,

$$S_\Lambda^{-1}\gamma^\mu S_\Lambda = \Lambda^\mu{}_\nu \gamma^\nu, \tag{1.36}$$

for the sake of covariance of the Dirac equation (1.31).

As is well known, a general solution of the latter matrix equation has the form

$$S = S_\Lambda = \exp\left(-\frac{1}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right), \quad \theta_{\mu\nu} = -\theta_{\nu\mu}, \tag{1.37}$$
$$\Sigma^{\mu\nu} = \left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)/2.$$

For the conjugate bispinor,

$$\overline{\psi}\left(x\right) = \psi^\dagger\left(x\right)\gamma^0, \qquad \overline{\psi}'\left(x'\right) = \overline{\psi}\left(x\right)S_\Lambda^{-1}, \qquad x' = \Lambda x, \tag{1.38}$$

the Dirac equation (1.31) takes the form

$$i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0. \tag{1.39}$$

(For more details see classical accounts [6], [19], [28], [71], [99], [158], [173], [184], [187], [21], [226].) Using the two equations (1.31) and (1.39) on can easily show that the Dirac current is conserved. Indeed, for the Dirac current $j^\mu = \overline{\psi}\gamma^\mu\psi = (\rho, \mathbf{j})$ one has

$$\begin{aligned} \partial_\mu j^\mu &= (\partial_\mu \overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu(\partial_\mu\psi) \\ &= im\overline{\psi}\psi - im\overline{\psi}\psi = 0. \end{aligned}$$

Thus showing the probability density and probability current are conserved for the Dirac equation. A more general discussion of the conservation laws is given in Chapter 5. Note that in the Dirac equation the probability density is positive,

$$\rho = j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4}|\psi_i|^2 > 0.$$

However, the energy may still take positive or negative values in the Dirac equation. This predicted the existence of an electron with positive charge called a positron, the antiparticle of an electron, which was eventually discovered. Actually, Dirac's theory predicted the existence of antiparticles for all spin-1/2 particles. The existence of these antiparticles necessitated the abandonment of the Dirac equation as a single-particle equation and for it to instead be interpreted as a field equation, [184].

1.4.3 Weyl Equation

The Dirac equation can be written in terms of two Weyl spinors, ϕ and χ, under the transformation,

$$\begin{aligned} \gamma^\mu \to \gamma'^\mu &= U\gamma^\mu U^{-1}, \qquad \psi \to \psi' = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = U\psi, \\ U &= \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = U^{-1}. \end{aligned} \tag{1.40}$$

Through this transformation Dirac's equation (1.31) takes a familiar block form,

$$i \begin{pmatrix} 0 & \partial_0 - \boldsymbol{\sigma}\cdot\nabla \\ \partial_0 + \boldsymbol{\sigma}\cdot\nabla & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = m \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{1.41}$$

(see, for example, [71], [158], and [173]). In the massless limit $m \to 0$, this system decouples,

$$\partial_0\phi + (\boldsymbol{\sigma}\cdot\nabla)\,\phi = 0, \qquad \partial_0\chi - (\boldsymbol{\sigma}\cdot\nabla)\,\chi = 0, \tag{1.42}$$

resulting in Weyl's two-component equations for massless neutrinos. This eponymous equation was introduced by Hermann Weyl in 1929 to describe massless spin-1/2 particles, [187], [228]. Weyl's equation was rejected because it was incompatible with parity conservation, which reverses the sign of helicity, [99]. Later this objection was deemed not serious since neutrinos are involved in weak interactions which do not conserve parity,

[131], [234]. It should be noted that the 2015 Nobel Prize in Physics was award to Arthur B. McDonald and Takaaki Kajita for the discovery of neutrino oscillations, which shows that neutrinos do have mass, see [149], [105], and [175] by Pontecorvo for the original work on neutrino physics.

The Weyl equation can be written in a familiar covariant form, see [173], [187], [173], by setting $\sigma^{\mu} = (\sigma_0 = I, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma}^{\mu} = (\sigma_0 = I, -\sigma_1, -\sigma_2, -\sigma_3)$, to cast (1.42) into the form:

$$\sigma^{\mu} \partial_{\mu} \phi = 0, \quad \bar{\sigma}^{\mu} \partial_{\mu} \phi = 0. \tag{1.43}$$

Here, the Weyl spinors,

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2, \tag{1.44}$$

transform under the proper orthochronous Lorentz group as follows

$$\phi'(x') = S_{\Lambda} \phi(x), \qquad x' = \Lambda x, \qquad S_{\Lambda} \sigma^{\mu} S_{\Lambda}^{-1} = \Lambda^{\mu}_{\nu} \sigma^{\nu}, \tag{1.45}$$

to ensure the covariance of the Weyl equation (1.43).

1.4.4 Maxwell Equation

In 1861, James Clerk Maxwell published his famous equations that now go by his name. Maxwell's equations together with the Lorentz force law form the foundation of classical electrodynamics. In their 3D vector form Maxwell's vacuum equations are given by

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.46}$$

$$\operatorname{div} \mathbf{E} = \rho, \qquad \operatorname{curl} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}, \tag{1.47}$$

where we have used Heaviside-Lorentz rationalized units. In this section we show how Maxwell's equations can be written in covariant form by following to some extent the treatment in the quantum field theory textbook by Lewis H. Ryder, [184]. The divergence equation in the homogeneous Maxwell equations (1.46) is a statement that there are no magnetic charges and the second equation known as Faraday's Law says that a changing magnetic field induces an electric field. The divergence equation in the pair of inhomogeneous Maxwell equations (1.47), known as Gauss's Law, says that the total charge inside a closed surface may be obtained by integrating the normal component of **E** over the surface. The second equation states that a current or a changing electric field generates a magnetic field; this equation is Ampere's Law with an additional time-derivative of **E** introduced by Maxwell.

It is convenient to introduce the scalar potential ϕ and vector potential $\mathbf{A}$ that satisfy

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \tag{1.48}$$

and by which the homogeneous Maxwell equations (1.46) are automatically satisfied. Combining these into the four-vector $A^\mu = (\phi, \mathbf{A})$ and defining

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{1.49}$$

one can verify that $F^{\mu\nu}$ has the following block matrix form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^p \\ E^q & -e_{pqr}B^r \end{pmatrix} \tag{1.50}$$

or

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix},$$

where $p,q,r = 1,2,3$ and $e_{pqr} = e^{pqr}$ is the three-dimensional totally anti-symmetric Levi-Civita symbol with $e^{123} = 1$. The electromagnetic field tensor $F^{\mu\nu}$ transforms under Lorentz transformations as a second rank tensor:

$$F'^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\tau F^{\sigma\tau}. \tag{1.51}$$

Note also that Maxwell's equations are invariant under the gauge transformation $A^{\mu} \to A^{\mu} + \partial^{\mu}\chi$, where χ is a scalar function, which can be checked using (1.48) or (1.49). The set of four equations (1.46)-(1.47) can be written as a pair of covariant tensorial equations; one is written in terms of the electromagnetic field tensor and the other one in terms of its corresponding dual four-tensor which is defined by

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}e^{\mu\nu\sigma\tau}F_{\sigma\tau} = \begin{pmatrix} 0 & -B^{p} \\ B^{q} & -e_{pqr}E^{r} \end{pmatrix}, \tag{1.52}$$

where $\mu, \nu, \sigma, \tau = 0, 1, 2, 3$ and $e_{\mu\nu\sigma\tau} = -e^{\mu\nu\sigma\tau}$ is the four-dimensional totally anti-symmetric Levi-Civita symbol with $e^{0123} = 1$. Note that the dual four-tensor $\widetilde{F}^{\mu\nu}$ may be obtained from $F^{\mu\nu}$ by substituting $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. In covariant form, Maxwell's equations are given by

$$\partial_{\mu}F^{\mu\nu} = j^{\nu} \tag{1.53}$$

$$\partial_{\mu}\widetilde{F}^{\mu\nu} = 0, \tag{1.54}$$

where $j^{\nu} = (\rho, \mathbf{j})$ is the four-current density. The equations (1.53) and (1.54) contain the inhomogeneous (1.47) and homogeneous (1.46) Maxwell equations, respectively. Note that in light of the antisymmetry of the Levi-Civita symbol, the equation (1.54) implies

$$\partial^{\lambda}F^{\mu\nu} + \partial^{\mu}F^{\nu\lambda} + \partial^{\nu}F^{\lambda\mu} = 0, \tag{1.55}$$

It can be checked that this latter equation follows directly from (1.49). The familiar wave equation for the four-potential A^{μ} under the choice of the Lorenz gauge, $\partial_{\mu}A^{\mu} = 0$,

$$\Box A^{\nu} = j^{\nu}, \tag{1.56}$$

is obtained by substituting (1.49) in (1.53).

1.4.5 Proca Equation

The Proca equations were introduced by Alexandru Proca in 1936, [176], to describe massive spin-1 particles called massive vector bosons. The intermediate force particles of the weak interaction, the W and Z bosons, are an example of such particles. Proca's equations generalize Maxwell's equations and are given by

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \quad \partial_{\mu}F^{\mu\nu} + m^2A^{\nu} = j^{\nu}, \tag{1.57}$$

where $j^{\nu} = (\rho, \mathbf{j})$ is the four-current density. Proca's equation is not invariant under the gauge transformation, $A^{\mu} \to A^{\mu} + \partial^{\mu}\chi$, where χ is a scalar function, as can be seen from the mass term in (1.57). This gauge invariance is restored in the massless limit, $m \to 0$, by which Proca's equations are reduced to Maxwell's equations. In addition, for the Proca field, the Lorenz gauge condition,

$$\partial_{\mu}A^{\mu} = 0, \tag{1.58}$$

always holds due to the antisymmetry of the electromagnetic field tensor and the conservation of four-current, $\partial_{\nu}j^{\nu} = 0$. In light of this fact, one may combine equations (1.57) to obtain:

$$\Box A^{\nu} + m^2A^{\nu} = j^{\nu}, \tag{1.59}$$

where $\Box = \partial_{\mu}\partial^{\mu} = \frac{\partial^2}{\partial t^2} - \nabla^2$, is the d'Alembert operator. Once again, in the limit $m \to 0$, this yields the equation (1.56). The condition (1.58) reduces the degrees of freedom by one therefore leaving the four component potential A^{μ} with three independent components, which one would expect for a massive particle.

1.4.6 Einstein Equation

In 1915, Albert Einstein first published his famous field equations and theory of general relativity [59]. Einstein's equation is a nonlinear tensorial equation that relates the curvature of space-time to the energy-momentum tensor of matter in space-time:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{1.60}$$

Here, $g_{\mu\nu}$ is the symmetric metric tensor on a curved spacetime manifold, $T_{\mu\nu}$ is the energy-momentum tensor, $R_{\mu\nu}$ the Ricci tensor, and $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar (the trace of the Ricci tensor), for more details see, for example, [36], [37], [157]. Another useful form of Einstein's equation is obtained by taking the trace of (1.60) which yields $R = -8\pi GT$, so that

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \tag{1.61}$$

It follows from this that Einstein's equation takes the following simple form in vacuum, when $T_{\mu\nu} = 0$:

$$R_{\mu\nu} = 0. \tag{1.62}$$

Note that in this section g refers to the metric tensor on a curved manifold, whereas η will refer to the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ of the flat Minkowski spacetime. This notation is useful when one wishes to linearize the Einstein equation about the flat Minkowski spacetime metric – to analyze, for example, the Einstein equation in a weak gravitational field. By considering only the first-order term in a series expansion of the metric tensor $g_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.63}$$

one obtains the linearized Einstein equation in vacuum

$$\begin{aligned}\partial_\mu\partial^\sigma h_{\sigma\nu} + \partial_\nu\partial^\sigma h_{\sigma\mu} - \partial_\mu\partial_\nu h \\ -\partial^2 h_{\mu\nu} - \eta_{\mu\nu}\left(\partial_\sigma\partial_\tau h^{\sigma\tau} - \partial^2 h\right) = 0.\end{aligned} \tag{1.64}$$

General relativity theory predicts the existence of gravitational waves, which can seen by solving the linearized Einstein equation, [36]. In fact, the gravitational waves produced from a binary black hole merger have recently been detected at LIGO, [1]. One should also consider the massless spin-2 hypothetical particle, called the graviton, that corresponds to gravitational waves. To describe such a particle a symmetric second rank tensor is needed. A discussion for a massive spin-2 particle and its relation to the Einstein equation is provided in Chapter 5.

Chapter 2

COMPLEX ELECTRODYNAMICS

Although a systematic study of electromagnetic phenomena in media is not possible without methods of quantum mechanics, statistical physics and kinetics, in practice a standard mathematical model based on phenomenological Maxwell's equations provides a good approximation to many important problems. As is well-known, one should be able to obtain the electromagnetic laws for continuous media from those for the interaction of fields and point particles [49], [47], [107], [124], [137], [155], [210]. As a result of the hard work of several generations of researchers and engineers, the classical electrodynamics, especially in its current complex covariant form, undoubtedly satisfies Dirac's criteria of mathematical beauty, being a state of the art mathematical description of nature.

In macroscopic electrodynamics, the volume (mechanical or ponderomotive) forces, acting on a medium, and the corresponding energy density and energy flux are introduced with the help of the energy-momentum tensors and differential balance relations [63], [87], [124], [170], [205], [210]. These forces occur in the equations of motion for a medium or individual charges and, in principle, they can be experimentally tested [88], [164], [174], [211] (see also the references therein). But interpretation of the results should depend on the accepted model of the interaction between the matter and radiation.

In this chapter, an original complex version of Minkowski's phenomenological electrodynamics (at rest or in a moving medium) is presented without assuming any particular form of material equations as far as possible. A compact covariant derivation of the energy-momentum balance equation and the angular momentum balance equations are introduced and may be important for future research, such as the covariant quantization of radiation in a non-uniform medium/cavity, as well as for pedagogical purposes. The conservation laws and quantization of the electromagnetic field will be discussed in this covariant approach elsewhere. Lorentz invariance of the corresponding differential balance equations is emphasized in view of long-standing uncertainties about the electromagnetic stresses and momentum density, the so-called "Abraham-Minkowski controversy" (see, for example, [13], [33], [52], [60], [63], [86], [87], [88], [47], [98], [124], [145], [146], [161], [163], [164], [170], [171], [174], [178], [193], [204], [208], [211], [215], [216], [217] and the references therein).

The chapter is organized as follows. In sections 2 to 4, we describe the 3*D*-complex version of Maxwell's equations and derive the corresponding differential balance density laws for the electromagnetic fields. Their covariant extensions are given in sections 5 to 9. The case of a moving medium is discussed in section 10 and complex Lagrangians are introduced in section 11. Some useful tools are collected in sections (2.12)-(2.14) for the reader's benefit.

2.1 Maxwell's Equations in 3*D*-Complex Form

Traditionally, the macroscopic Maxwell equations in a fixed frame of reference are given by

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\ \text{(Faraday)}, \qquad \operatorname{div}\mathbf{B} = 0\ \text{(no magnetic charge)} \tag{2.1}$$

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_{\text{free}}\ \text{(Biot\&Savart)}, \quad \operatorname{div}\mathbf{D} = 4\pi\rho_{\text{free}}\ \text{(Coulomb)}^{1}. \tag{2.2}$$

Here, $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the displacement field; $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the induction field. These equations, which are obtained by averaging of microscopic Maxwell's equations in the vacuum, provide a good mathematical description of electromagnetic phenomena in various media, when complemented by the corresponding material equations.

[1] From this point, we shall write $\rho_{\text{free}} = \rho$ and $\mathbf{j}_{\text{free}} = \mathbf{j}$. A detailed analysis of electromagnetic laws for continuous media from those for point particles is given in [47] (statistical description of material media).

In the simplest case of an isotropic medium at rest, one usually has

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{j} = \sigma \mathbf{E}, \tag{2.3}$$

where ε is the dielectric constant, μ is the magnetic permeability, and σ describes the conductivity of the medium (see, for example, [3], [14], [18], [33], [34], [49], [58], [62], [81], [47], [100], [124], [137], [165], [170], [197], [205], [207], [209], [210] for fundamentals of classical electrodynamics).

Introduction of two complex fields

$$\mathbf{F} = \mathbf{E} + i\mathbf{H}, \qquad \mathbf{G} = \mathbf{D} + i\mathbf{B} \tag{2.4}$$

allows one to rewrite the phenomenological Maxwell equations in the following compact form

$$\frac{i}{c}\left(\frac{\partial \mathbf{G}}{\partial t} + 4\pi \mathbf{j}\right) = \operatorname{curl} \mathbf{F}, \qquad \mathbf{j} = \mathbf{j}^*, \tag{2.5}$$

$$\operatorname{div} \mathbf{G} = 4\pi\rho, \qquad \rho = \rho^*, \tag{2.6}$$

where the asterisk stands for complex conjugation (see also [14], [115] and [188]). As we shall demonstrate, different complex forms of Maxwell's equations are particularly convenient for study of the corresponding "energy-momentum" balance equations for the electromagnetic fields in the presence of the "free" charges and currents in a medium.

2.2 Hertz Symmetric Stress Tensor

We begin from a complex 3D-interpretation of the traditional symmetric energy-momentum tensor [170]. By definition,

$$T_{pq} = \frac{1}{16\pi}\left[F_p G_q^* + F_p^* G_q + F_q G_p^* + F_q^* G_p \right. \tag{2.7}$$
$$\left. - \delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^* + \mathbf{F}^*\cdot\mathbf{G}\right)\right] = T_{qp} \qquad (p,q = 1,2,3)$$

and the corresponding "momentum" balance equation,

$$\begin{aligned}&\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p+\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\left(\mathbf{D}\times\mathbf{B}\right)\right]_p \\ &=\frac{\partial T_{pq}}{\partial x_q}+\frac{1}{16\pi}\left[\operatorname{curl}\left(\mathbf{F}\times\mathbf{G}^*+\mathbf{F}^*\times\mathbf{G}\right)\right]_p \\ &\quad+\frac{1}{16\pi}\left(F_q\frac{\partial G_q^*}{\partial x_p}-G_q\frac{\partial F_q^*}{\partial x_p}+F_q^*\frac{\partial G_q}{\partial x_p}-G_q^*\frac{\partial F_q}{\partial x_p}\right),\end{aligned} \tag{2.8}$$

can be obtained from Maxwell's equations (2.5)–(2.6) as a result of elementary but rather tedious vector calculus calculations usually omitted in textbooks. (We use Einstein summation convention unless otherwise stated.)

Proof. Indeed, in a $3D$-complex form,

$$\begin{aligned}&\frac{\partial}{\partial x_q}\left(F_pG_q^*+F_qG_p^*-\delta_{pq}\mathbf{F}\cdot\mathbf{G}^*\right) \\ &=\frac{\partial F_p}{\partial x_q}G_q^*+F_p\frac{\partial G_q^*}{\partial x_q}+\frac{\partial F_q}{\partial x_q}G_p^*+F_q\frac{\partial G_p^*}{\partial x_q}-\frac{\partial}{\partial x_p}\left(F_qG_q^*\right) \\ &=F_q\left(\frac{\partial G_p^*}{\partial x_q}-\frac{\partial G_q^*}{\partial x_p}\right)+\left(\frac{\partial F_p}{\partial x_q}-\frac{\partial F_q}{\partial x_p}\right)G_q^* \\ &\quad+F_p\operatorname{div}\mathbf{G}^*+G_p^*\operatorname{div}\mathbf{F} \\ &=F_p\operatorname{div}\mathbf{G}^*-\left(\mathbf{F}\times\operatorname{curl}\mathbf{G}^*\right)_p+G_p^*\operatorname{div}\mathbf{F}-\left(\mathbf{G}^*\times\operatorname{curl}\mathbf{F}\right)_p\end{aligned} \tag{2.9}$$

in view of an identity [205]:

$$\left(\mathbf{A}\times\operatorname{curl}\mathbf{B}\right)_p=A_q\left(\frac{\partial B_q}{\partial x_p}-\frac{\partial B_p}{\partial x_q}\right). \tag{2.10}$$

Taking into account the complex conjugate, we derive

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial x_q}\left[F_pG_q^*+F_p^*G_q+F_qG_p^*+F_q^*G_p-\delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^*+\mathbf{F}^*\cdot\mathbf{G}\right)\right] \\ &=\frac{1}{2}\left(\mathbf{F}\operatorname{div}\mathbf{G}^*-\mathbf{G}^*\times\operatorname{curl}\mathbf{F}+\mathbf{F}^*\operatorname{div}\mathbf{G}-\mathbf{G}\times\operatorname{curl}\mathbf{F}^*\right)_p \\ &\quad+\frac{1}{2}\left(\mathbf{G}\operatorname{div}\mathbf{F}^*-\mathbf{F}^*\times\operatorname{curl}\mathbf{G}+\mathbf{G}^*\operatorname{div}\mathbf{F}-\mathbf{F}\times\operatorname{curl}\mathbf{G}^*\right)_p\end{aligned} \tag{2.11}$$

as our first important fact.

On the other hand, in view of Maxwell's equations (2.5)–(2.6), one gets

$$\mathbf{F}\operatorname{div}\mathbf{G}^* - \mathbf{G}^* \times \operatorname{curl}\mathbf{F} = 4\pi\rho\mathbf{F} + \frac{i}{c}\left(\frac{\partial\mathbf{G}}{\partial t} \times \mathbf{G}^* + 4\pi\mathbf{j} \times \mathbf{G}^*\right) \tag{2.12}$$

and, with the help of its complex conjugate,

$$\mathbf{F}\operatorname{div}\mathbf{G}^* - \mathbf{G}^* \times \operatorname{curl}\mathbf{F} + \mathbf{F}^*\operatorname{div}\mathbf{G} - \mathbf{G} \times \operatorname{curl}\mathbf{F}^* = 4\pi\rho\left(\mathbf{F} + \mathbf{F}^*\right) + \frac{i}{c}\frac{\partial}{\partial t}\left(\mathbf{G} \times \mathbf{G}^*\right) + \frac{4\pi i}{c}\mathbf{j}\times\left(\mathbf{G}^* - \mathbf{G}\right), \tag{2.13}$$

or

$$\frac{1}{2}\left(\mathbf{F}\operatorname{div}\mathbf{G}^* - \mathbf{G}^* \times \operatorname{curl}\mathbf{F} + \mathbf{F}^*\operatorname{div}\mathbf{G} - \mathbf{G} \times \operatorname{curl}\mathbf{F}^*\right) = 4\pi\left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{D} \times \mathbf{B}\right), \tag{2.14}$$

providing the second important fact. (Up to the constant, the first term in the right-hand side represents the density of Lorentz's force acting on the free charges and currents in the medium under consideration [204], [205].)

In view of (2.14) and (2.11), we can write

$$\begin{aligned}
&4\pi\left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}\right)_p + \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{D} \times \mathbf{B}\right)_p \\
&= \frac{1}{2}\frac{\partial}{\partial x_q}\left[F_pG_q^* + F_p^*G_q + F_qG_p^* + F_q^*G_p - \delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^* + \mathbf{F}^*\cdot\mathbf{G}\right)\right] \\
&\quad - \frac{1}{2}\left(\mathbf{G}\operatorname{div}\mathbf{F}^* - \mathbf{F}^* \times \operatorname{curl}\mathbf{G} + \mathbf{G}^*\operatorname{div}\mathbf{F} - \mathbf{F} \times \operatorname{curl}\mathbf{G}^*\right)_p \\
&= \frac{1}{4}\frac{\partial}{\partial x_q}\left[F_pG_q^* + F_p^*G_q + F_qG_p^* + F_q^*G_p - \delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^* + \mathbf{F}^*\cdot\mathbf{G}\right)\right] \\
&\quad - \frac{1}{4}\left(\mathbf{G}\operatorname{div}\mathbf{F}^* - \mathbf{F}^* \times \operatorname{curl}\mathbf{G} + \mathbf{G}^*\operatorname{div}\mathbf{F} - \mathbf{F} \times \operatorname{curl}\mathbf{G}^*\right)_p \\
&\quad + \frac{1}{4}\left(\mathbf{F}\operatorname{div}\mathbf{G}^* - \mathbf{G}^* \times \operatorname{curl}\mathbf{F} + \mathbf{F}^*\operatorname{div}\mathbf{G} - \mathbf{G} \times \operatorname{curl}\mathbf{F}^*\right)_p \\
&= 4\pi\frac{\partial T_{pq}}{\partial x_q} + \frac{1}{4}\left(\mathbf{F}\operatorname{div}\mathbf{G}^* - \mathbf{G}^*\operatorname{div}\mathbf{F} + \mathbf{F}^*\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F}^*\right)_p \\
&\quad + \frac{1}{4}\left(\mathbf{F} \times \operatorname{curl}\mathbf{G}^* - \mathbf{G}^* \times \operatorname{curl}\mathbf{F} + \mathbf{F}^* \times \operatorname{curl}\mathbf{G} - \mathbf{G} \times \operatorname{curl}\mathbf{F}^*\right)_p.
\end{aligned} \tag{2.15}$$

Finally, in the last two lines, one can utilize the following differential vector calculus identity,

$$
\begin{aligned}
& [\mathbf{A}\,\mathrm{div}\,\mathbf{B} - \mathbf{B}\,\mathrm{div}\,\mathbf{A} + \mathbf{A}\times\mathrm{curl}\,\mathbf{B} - \mathbf{B}\times\mathrm{curl}\,\mathbf{A} - \mathrm{curl}\,(\mathbf{A}\times\mathbf{B})]_p \\
& \quad = A_q \frac{\partial B_q}{\partial x_p} - B_q \frac{\partial A_q}{\partial x_p},
\end{aligned} \tag{2.16}
$$

see (2.123), with $\mathbf{A} = \mathbf{F}$, $\mathbf{B} = \mathbf{G}^*$ and its complex conjugates, in order to obtain (2.8) and/or (2.22), which completes the proof. (An independent proof will be given in section 7.)

Derivation of the corresponding differential "energy" balance equation is much simpler. By (2.5),

$$
\mathbf{F}\cdot\frac{\partial \mathbf{G}^*}{\partial t} + \mathbf{F}^*\cdot\frac{\partial \mathbf{G}}{\partial t} + 4\pi\mathbf{j}\cdot(\mathbf{F}+\mathbf{F}^*) = \frac{c}{i}\,\mathrm{div}\,(\mathbf{F}\times\mathbf{F}^*) \tag{2.17}
$$

in view of a familiar vector calculus identity (2.119):

$$
\mathrm{div}\,(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}. \tag{2.18}
$$

In a traditional form,

$$
\frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right) + \mathbf{j}\cdot\mathbf{E} + \mathrm{div}\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right) = 0 \tag{2.19}
$$

(see, for example, [49], [205]), where one can substitute

$$
\begin{aligned}
\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}) \\
&\quad + \frac{1}{2}\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} - \mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{B}\cdot\frac{\partial \mathbf{H}}{\partial t}\right).
\end{aligned} \tag{2.20}
$$

As a result, $3D$-differential "energy-momentum" balance equations are given by

$$
\begin{aligned}
& \frac{\partial}{\partial t}\left(\frac{\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}}{8\pi}\right) + \mathrm{div}\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right) + \mathbf{j}\cdot\mathbf{E} \\
& \quad + \frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} - \mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{B}\cdot\frac{\partial \mathbf{H}}{\partial t}\right) = 0
\end{aligned} \tag{2.21}
$$

and

$$-\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}(\mathbf{D}\times\mathbf{B})\right]_p+\frac{\partial T_{pq}}{\partial x_q}-\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p \tag{2.22}$$
$$+\frac{1}{8\pi}\left[\operatorname{curl}(\mathbf{E}\times\mathbf{D}+\mathbf{H}\times\mathbf{B})\right]_p$$
$$+\frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial x_p}-\mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x_p}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial x_p}-\mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial x_p}\right)=0,$$

respectively (see also [88], [145]). The real form of the symmetric stress tensor (2.7), namely,

$$T_{pq}=\frac{1}{8\pi}\left[E_pD_q+E_qD_p+H_pB_q+H_qB_p\right. \tag{2.23}$$
$$\left.-\delta_{pq}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})\right]\qquad(p,q=1,2,3),$$

is due to Hertz [170].

Equations (2.21)–(2.22) are related to a fundamental concept of conservation of mechanical and electromagnetic energy and momentum. Here, these balance conditions are presented in differential forms in terms of the corresponding local field densities. They can be integrated over a given volume in $\mathbb{R}^3$ in order to obtain, in a traditional way, the corresponding conservation laws of the electromagnetic fields (see, for example, [122], [124], [207], [209], [210]). These laws made it necessary to ascribe a definite linear momentum and energy to the field of an electromagnetic wave, which can be observed, for example, as the light pressure.

Note. At this point, the Lorentz invariance of these differential balance equations is not obvious in our $3D$-analysis. But one can introduce the four-vector $x^\mu=(ct,\mathbf{r})$ and try to match (2.21)–(2.22) with the expression,

$$\frac{\partial}{\partial x^\nu}T_\mu^\nu=\frac{\partial T_\mu^0}{\partial x_0}+\frac{\partial T_\mu^q}{\partial x_q}\qquad(\mu,\nu=0,1,2,3;\quad p,q=1,2,3), \tag{2.24}$$

as an initial step, in order to guess the corresponding four-tensor form. An independent covariant derivation will be given in section 7.

Note. In an isotropic nonhomogeneous variable medium (without dispersion and/or compression), when $\mathbf{D} = \varepsilon(\mathbf{r},t)\mathbf{E}$ and $\mathbf{B} = \mu(\mathbf{r},t)\mathbf{H}$, the "ponderomotive forces" in (2.21) and (2.22) take the form [205]:

$$\begin{aligned} &\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial x^{\nu}} - \mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial x^{\nu}} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial x^{\nu}} - \mathbf{B}\cdot\frac{\partial \mathbf{H}}{\partial x^{\nu}} \\ &= \frac{\partial \varepsilon}{\partial x^{\nu}}\mathbf{E}^2 + \frac{\partial \mu}{\partial x^{\nu}}\mathbf{H}^2 = \begin{pmatrix} \frac{1}{c}\left(\frac{\partial \varepsilon}{\partial t}\mathbf{E}^2 + \frac{\partial \mu}{\partial t}\mathbf{H}^2\right) \\ \\ \mathbf{E}^2\nabla\varepsilon + \mathbf{H}^2\nabla\mu \end{pmatrix}, \end{aligned} \tag{2.25}$$

which may be interpreted as a four-vector "energy-force" acting from an inhomogeneous and time-variable medium. Its covariance is analyzed in section 7.

2.3 "Angular Momentum" Balance

The 3D-"linear momentum" differential balance equation (2.22), can be rewritten in a more compact form,

$$\frac{\partial T_{pq}}{\partial x_q} = \mathscr{F}_p + \frac{\partial \mathscr{G}_p}{\partial t}, \qquad \overrightarrow{\mathscr{G}} = \frac{1}{4\pi c}(\mathbf{D}\times\mathbf{B}), \tag{2.26}$$

with the help of the Hertz symmetric stress tensor $T_{pq} = T_{qp}$ defined by (2.23). A "net force" is given by

$$\begin{aligned} \mathscr{F}_p = &\left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p - \frac{1}{8\pi}\left[\operatorname{curl}(\mathbf{E}\times\mathbf{D} + \mathbf{H}\times\mathbf{B})\right]_p \\ &- \frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial x_p} - \mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial x_p} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial x_p} - \mathbf{B}\cdot\frac{\partial \mathbf{H}}{\partial x_p}\right). \end{aligned} \tag{2.27}$$

In this notation, we state the corresponding differential balance equation as follows

$$\frac{\partial M_{pq}}{\partial x_q} = \mathscr{T}_p + \frac{\partial \mathscr{L}_p}{\partial t}, \qquad \overrightarrow{\mathscr{L}} = \mathbf{r}\times\overrightarrow{\mathscr{G}}, \quad \overrightarrow{\mathscr{T}} = \mathbf{r}\times\overrightarrow{\mathscr{F}}, \tag{2.28}$$

where the "field angular momentum density" is defined by

$$\overrightarrow{\mathscr{L}} = \frac{1}{4\pi c}\mathbf{r}\times(\mathbf{D}\times\mathbf{B}) \tag{2.29}$$

and the "flux of angular momentum" is described by the following tensor [100]:

$$M_{pq} = e_{prs} x_r T_{sq}. \tag{2.30}$$

(Here, e_{pqr} is the totally anti-symmetric Levi-Civita symbol with $e_{123} = +1$). An elementary example of conservation of the total angular momentum is discussed in [205].

Proof. Indeed, in view of (2.26), one can write

$$\begin{aligned} \frac{\partial M_{pq}}{\partial x_q} &= e_{prs} T_{sr} + e_{prs} x_r \frac{\partial T_{sq}}{\partial x_q} \\ &= e_{pqr} x_q \mathscr{F}_r + \frac{\partial}{\partial t}\left(e_{pqr} x_q \mathscr{G}_r\right), \end{aligned} \tag{2.31}$$

which completes the proof.

Note. Once again, in 3*D*-form, the Lorentz invariance of this differential balance equation for the local densities is not obvious. An independent covariant derivation will be given in section 8.

2.4 Complex Covariant Form of Macroscopic Maxwell's Equations

With the help of complex fields $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and $\mathbf{G} = \mathbf{D} + i\mathbf{B}$, we introduce the following anti-symmetric four-tensor,

$$Q^{\mu\nu} = -Q^{\nu\mu} = \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix} \tag{2.32}$$

and use the standard four-vectors, $x^\mu = (ct, \mathbf{r})$ and $j^\mu = (c\rho, \mathbf{j})$ for contravariant coordinates and current, respectively.

Maxwell's equations then take the covariant form [115], [128]:

$$\frac{\partial}{\partial x^\nu} Q^{\mu\nu} = -\frac{\partial}{\partial x^\nu} Q^{\nu\mu} = -\frac{4\pi}{c} j^\mu \tag{2.33}$$

with summation over two repeated indices. Indeed, in block form, we have

$$\frac{\partial Q^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\begin{pmatrix} 0 & -G_q \\ G_p & ie_{pqr}F_r \end{pmatrix} = \begin{pmatrix} -\operatorname{div}\mathbf{G} = -4\pi\rho \\ \dfrac{1}{c}\dfrac{\partial \mathbf{G}}{\partial t} + i\operatorname{curl}\mathbf{F} = -\dfrac{4\pi}{c}\mathbf{j} \end{pmatrix}, \tag{2.34}$$

which verifies this fact. The continuity equation,

$$0 \equiv \frac{\partial^2 Q^{\mu\nu}}{\partial x^\mu \partial x^\nu} = -\frac{4\pi}{c}\frac{\partial j^\mu}{\partial x^\mu}, \tag{2.35}$$

or in the $3D$-form,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0, \tag{2.36}$$

describes conservation of the electrical charge. The latter equation can also be derived in the complex $3D$-form from (2.5)–(2.6).

Note. In vacuum, when $\mathbf{G} = \mathbf{F}$ and $\rho = 0$, $\mathbf{j} = 0$, one can write

$$Q^{\mu\nu} = F^{\mu\nu} - \frac{i}{2}e_{\mu\nu\sigma\tau}F^{\sigma\tau}, \qquad g_{\mu\sigma}g_{\nu\tau}Q^{\sigma\tau} = Q_{\mu\nu}. \tag{2.37}$$

As a result, the following self-duality property holds

$$e_{\mu\nu\sigma\tau}Q^{\sigma\tau} = 2iQ_{\mu\nu}, \qquad 2iQ^{\mu\nu} = e^{\mu\nu\sigma\tau}Q_{\sigma\tau} \tag{2.38}$$

(see, for example, [27], [117] and section (2.13)). Two covariant forms of Maxwell's equations are given by

$$\partial_\nu Q^{\mu\nu} = 0, \qquad \partial^\nu Q_{\mu\nu} = 0, \tag{2.39}$$

where $\partial^\nu = g^{\nu\mu}\partial_\mu$, $\partial_\mu = \partial/\partial x^\mu$ and $g_{\mu\nu} = g^{\mu\nu} =$diag$(1,-1,-1,-1)$. The last equation can be derived from a more general equation, involving a rank three tensor,

$$g^{\alpha\alpha}e_{\alpha\mu\nu\tau}\partial^\nu Q^{\tau\beta} - g^{\alpha\alpha}e_{\beta\mu\nu\tau}\partial^\nu Q^{\tau\alpha} = -i\partial_\mu Q^{\alpha\beta} \tag{2.40}$$

($\alpha, \beta = 0,1,2,3$ are fixed; no summation is assumed over these two indices), which is related to the Pauli-Lubanski vector from the representation theory of the Poincaré group [115]. Different spinor forms of Maxwell's equations are analyzed in [117] (see also the references therein).

Two dual anti-symmetric field tensors of complex fields, $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and $\mathbf{G} = \mathbf{D} + i\mathbf{B}$, are given by

$$Q^{\mu\nu} = \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix} = R^{\mu\nu} + iS^{\mu\nu} \tag{2.41}$$

$$= \begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & -H_3 & H_2 \\ D_2 & H_3 & 0 & -H_1 \\ D_3 & -H_2 & H_1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

and

$$P_{\mu\nu} = \begin{pmatrix} 0 & F_1 & F_2 & F_3 \\ -F_1 & 0 & iG_3 & -iG_2 \\ -F_2 & -iG_3 & 0 & iG_1 \\ -F_3 & iG_2 & -iG_1 & 0 \end{pmatrix} = F_{\mu\nu} + iG_{\mu\nu} \tag{2.42}$$

$$= \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & D_3 & -D_2 \\ -H_2 & -D_3 & 0 & D_1 \\ -H_3 & D_2 & -D_1 & 0 \end{pmatrix}.$$

The real part of the latter represents the standard electromagnetic field tensor in a medium [14], [170], [210]. As for the imaginary part of (2.41), which, ironically, Pauli called an "artificiality" in view of its non-standard behavior under spatial inversion [170], the use of complex conjugation restores this important symmetry for our complex field tensors.

The dual tensor identities are given by

$$e_{\mu\nu\sigma\tau} Q^{\sigma\tau} = 2iP_{\mu\nu}, \qquad 2iQ^{\mu\nu} = e^{\mu\nu\sigma\tau} P_{\sigma\tau}. \tag{2.43}$$

Here $e^{\mu\nu\sigma\tau} = -e_{\mu\nu\sigma\tau}$ and $e_{0123} = +1$ is the Levi-Civita four-symbol [82]. Then

$$6i \frac{\partial Q^{\mu\nu}}{\partial x^{\nu}} = e^{\mu\nu\lambda\sigma} \left(\frac{\partial P_{\lambda\sigma}}{\partial x^{\nu}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\sigma}} + \frac{\partial P_{\sigma\nu}}{\partial x^{\lambda}} \right) \tag{2.44}$$

and both pairs of Maxwell's equations can also be presented in the form [115]

$$\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial P_{\lambda\mu}}{\partial x^{\nu}} = -\frac{4\pi i}{c} e_{\mu\nu\lambda\sigma} j^{\sigma} \tag{2.45}$$

in addition to the one given above

$$\frac{\partial Q^{\mu\nu}}{\partial x^{\nu}} = -\frac{4\pi}{c} j^{\mu}. \tag{2.46}$$

The real part of the first equation traditionally represents the first (homogeneous) pair of Maxwell's equation and the real part of the second one gives the remaining pair. In our approach both pairs of Maxwell's equations appear together (see also [14], [27], [26], [128], and [206] for the case in vacuum). Moreover, a generalization to complex-valued four-current may naturally represent magnetic charge and magnetic current not yet observed in nature [188].

An important cofactor matrix identity,

$$P_{\mu\nu} Q^{\nu\lambda} = (\mathbf{F} \cdot \mathbf{G})\, \delta_{\mu}^{\lambda} = \frac{1}{4} (P_{\sigma\tau} Q^{\tau\sigma})\, \delta_{\mu}^{\lambda}, \tag{2.47}$$

was originally established, in a general form, by Minkowski [154]. Once again, the dual tensors are given by

$$P_{\mu\nu} = \begin{pmatrix} 0 & F_q \\ -F_p & ie_{pqr} G_r \end{pmatrix}, \quad Q^{\mu\nu} = \begin{pmatrix} 0 & -G_q \\ G_p & ie_{pqr} F_r \end{pmatrix}, \tag{2.48}$$

in block form. A complete list of relevant tensor and matrix identities is given in section (2.13).

2.6 Covariant Derivation of Energy-Momentum Balance Equations

2.6.1 Preliminaries

As has been announced in [115] (see also [117]), the covariant form of the differential balance equations can be presented as follows

$$\frac{\partial}{\partial x^{\nu}}\left[\frac{1}{16\pi}\left(P^{*}_{\mu\lambda}Q^{\lambda\nu}+P_{\mu\lambda}\overset{*}{Q}{}^{\lambda\nu}\right)\right] \tag{2.49}$$
$$+\frac{1}{32\pi}\left(P^{*}_{\sigma\tau}\frac{\partial Q^{\tau\sigma}}{\partial x^{\mu}}+P_{\sigma\tau}\frac{\partial \overset{*}{Q}{}^{\tau\sigma}}{\partial x^{\mu}}\right)$$
$$=-\frac{1}{c}F_{\mu\lambda}j^{\lambda}=\begin{pmatrix} -\mathbf{j}\cdot\mathbf{E}/c \\ \rho\mathbf{E}+\mathbf{j}\times\mathbf{B}/c \end{pmatrix}.$$

In our complex form, when $\mathbf{F}=\mathbf{E}+i\mathbf{H}$ and $\mathbf{G}=\mathbf{D}+i\mathbf{B}$, the energy-momentum tensor is given by

$$16\pi T_{\mu}{}^{\nu}=P^{*}_{\mu\lambda}Q^{\lambda\nu}+P_{\mu\lambda}\overset{*}{Q}{}^{\lambda\nu} \tag{2.50}$$
$$=\begin{pmatrix} \mathbf{F}\cdot\mathbf{G}^{*}+\mathbf{F}^{*}\cdot\mathbf{G} & 2i\left(\mathbf{F}\times\mathbf{F}^{*}\right)_{q} \\ -2i\left(\mathbf{G}\times\mathbf{G}^{*}\right)_{p} & 2\left(F_{p}G^{*}_{q}+F^{*}_{p}G_{q}\right)-\delta_{pq}\left(\mathbf{F}\cdot\mathbf{G}^{*}+\mathbf{F}^{*}\cdot\mathbf{G}\right) \end{pmatrix}.$$

Here, we point out for the reader's convenience that

$$i\left(\mathbf{F}\times\mathbf{F}^{*}\right)=2\left(\mathbf{E}\times\mathbf{H}\right),\qquad i\left(\mathbf{G}\times\mathbf{G}^{*}\right)=2\left(\mathbf{D}\times\mathbf{B}\right), \tag{2.51}$$
$$\mathbf{F}\cdot\mathbf{G}^{*}+\mathbf{F}^{*}\cdot\mathbf{G}=2\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)$$

and, in real form,

$$4\pi T_{\mu}{}^{\nu}=\begin{pmatrix} \left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)/2 & \left(\mathbf{E}\times\mathbf{H}\right)_{q} \\ -\left(\mathbf{D}\times\mathbf{B}\right)_{p} & E_{p}D_{q}+H_{p}B_{q}-\delta_{pq}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\right)/2 \end{pmatrix}. \tag{2.52}$$

The covariant form of the differential balance equation allows one to clarify the meanings of different energy-momentum tensors. For instance, it is worth noting that the non-symmetric Maxwell and Heaviside form of the 3D-stress tensor [170],

$$\widetilde{T}_{pq} = \frac{1}{4\pi}\left(E_p D_q + H_p B_q\right) - \frac{1}{8\pi}\delta_{pq}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right), \tag{2.53}$$

appears here in the corresponding "momentum" balance equation [205]:

$$-\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\left(\mathbf{D}\times\mathbf{B}\right)\right]_p + \frac{\partial \widetilde{T}_{pq}}{\partial x_q} - \left(\rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right)_p \tag{2.54}$$
$$+\frac{1}{8\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial x_p} - \mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial x_p} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial x_p} - \mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial x_p}\right) = 0.$$

At the same time, in view of (2.22), use of the form (2.53) differs from Hertz's symmetric tensors in (2.7) and (2.23) only in the case of anisotropic media (crystals) [170], [204]. Indeed,

$$8\pi\frac{\partial}{\partial x_q}\left(\widetilde{T}_{pq} - T_{pq}\right) = \left[\operatorname{curl}\left(\mathbf{E}\times\mathbf{D} + \mathbf{H}\times\mathbf{B}\right)\right]_p. \tag{2.55}$$

Moreover, with the help of elementary identities,

$$\left[\operatorname{curl}\left(\mathbf{A}\times\mathbf{B}\right)\right]_p = \frac{\partial}{\partial x_q}\left(A_p B_q - A_q B_p\right) \tag{2.56}$$

and

$$2\frac{\partial}{\partial x_q}\left(A_p B_q\right) = \frac{\partial}{\partial x_q}\left(A_p B_q + A_q B_p\right) + \left[\operatorname{curl}\left(\mathbf{A}\times\mathbf{B}\right)\right]_p, \tag{2.57}$$

one can transform the latter balance equation into its "symmetric" form, which provides an independent proof of (2.22).

2.6.2 Proof

The fact that Maxwell's equations can be combined in a simple form using a complex second rank (anti-symmetric) tensor allows us to utilize the standard Sturm-Liouville type argument in order to establish the energy-momentum differential balance equations in covariant form. Indeed, by adding matrix equation

$$P^*_{\mu\lambda}\left(\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\lambda\right) \tag{2.58}$$

and its complex conjugate

$$P_{\mu\lambda}\left(\frac{\partial \overset{*}{Q}{}^{\lambda\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\lambda\right) \tag{2.59}$$

one gets

$$P^*_{\mu\lambda}\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} + P_{\mu\lambda}\frac{\partial \overset{*}{Q}{}^{\lambda\nu}}{\partial x^\nu} = -\frac{8\pi}{c} F_{\mu\lambda} j^\lambda . \tag{2.60}$$

A simple decomposition,

$$f\frac{\partial g}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}(fg) + \frac{1}{2}\left(f\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} g\right) \tag{2.61}$$

with $f = P^*_{\mu\lambda}$ and $g = Q^{\lambda\nu}$ (and their complex conjugates), results in

$$\begin{aligned} &\frac{\partial}{\partial x^\nu}\left[\frac{1}{16\pi}\left(P^*_{\mu\lambda} Q^{\lambda\nu} + P_{\mu\lambda}\overset{*}{Q}{}^{\lambda\nu}\right)\right] \\ &+ \frac{1}{16\pi}\left[\left(P^*_{\mu\lambda}\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} - \frac{\partial P_{\mu\lambda}}{\partial x^\nu}\overset{*}{Q}{}^{\lambda\nu}\right) + (\text{c.c.})\right] = -\frac{1}{c} F_{\mu\lambda} j^\lambda . \end{aligned} \tag{2.62}$$

By a direct substitution, one can verify that

$$\begin{aligned} Z_\mu = P^*_{\mu\lambda}\frac{\partial Q^{\lambda\nu}}{\partial x^\nu} - \frac{\partial P_{\mu\lambda}}{\partial x^\nu}\overset{*}{Q}{}^{\lambda\nu} &= \frac{1}{2} P^*_{\sigma\tau}\frac{\partial Q^{\tau\sigma}}{\partial x^\mu} \\ = -\frac{1}{2}\overset{*}{Q}{}^{\sigma\tau}\frac{\partial P_{\tau\sigma}}{\partial x^\mu} &= \mathbf{F}^*\cdot\frac{\partial \mathbf{G}}{\partial x^\mu} - \mathbf{G}^*\cdot\frac{\partial \mathbf{F}}{\partial x^\mu}. \end{aligned} \tag{2.63}$$

(An independent covariant proof of these identities is given in section (2.14)) Finally, introducing

$$16\pi X_\mu = Z_\mu + Z^*_\mu , \tag{2.64}$$

we obtain (2.49) with the explicitly covariant expression for the ponderomotive force (2.25), which completes the proof.

As a result, the covariant energy-momentum balance equation is given by

$$\frac{\partial}{\partial x^{\nu}} T_{\mu}{}^{\nu} + X_{\mu} = -\frac{1}{c} F_{\mu\lambda}\, j^{\lambda}, \tag{2.65}$$

in a compact form. If these differential equations are written for a stationary medium, then the corresponding equations for moving bodies are uniquely determined, since the components of a tensor in any inertial coordinate system can be derived by a proper Lorentz transformation [170].

2.7 Covariant Derivation of Angular Momentum Balance

By definition, $x_{\mu} = g_{\mu\nu} x^{\nu} = (ct, -\mathbf{r})$ and $T_{\mu\lambda} = T_{\mu}{}^{\nu} g_{\nu\lambda}$, where $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ $= \partial x_{\mu}/\partial x^{\nu}$. In view of (2.65), we derive

$$\begin{aligned} \frac{\partial}{\partial x^{\nu}} \left(x_{\lambda} T_{\mu}{}^{\nu} - x_{\mu} T_{\lambda}{}^{\nu} \right) &= \left(T_{\mu\lambda} - T_{\lambda\mu} \right) \\ - \left(x_{\lambda} X_{\mu} - x_{\mu} X_{\lambda} \right) &- \frac{1}{c} \left(x_{\lambda} F_{\mu\nu} - x_{\mu} F_{\lambda\nu} \right) j^{\nu} \end{aligned} \tag{2.66}$$

as a required differential balance equation.

With the help of familiar dual relations (2.127), one can get another covariant form of the angular momentum balance equation:

$$\begin{aligned} &\frac{\partial}{\partial x^{\nu}} \left(e^{\mu\lambda\sigma\tau} x_{\sigma} T_{\tau}{}^{\nu} \right) + e^{\mu\lambda\sigma\tau} T_{\sigma\tau} \\ &+ e^{\mu\lambda\sigma\tau} x_{\sigma} X_{\tau} + \frac{1}{c} e^{\mu\lambda\sigma\tau} x_{\sigma} F_{\tau\nu}\, j^{\nu} = 0^{\mu\lambda}. \end{aligned} \tag{2.67}$$

In $3D$-form, the latter relation can be reduced to (2.28)–(2.30).

Indeed, when $\mu = 0$ and $\lambda = p = 1,2,3$, one gets

$$\begin{aligned} &-\frac{1}{4\pi c} \frac{\partial}{\partial t} \left[e_{pqr} x_{q} \left(\mathbf{D} \times \mathbf{B} \right)_{r} \right] + \frac{\partial}{\partial x_{s}} \left(e_{pqr} x_{q} \widetilde{T}_{rs} \right) \\ &+ e_{pqr} \widetilde{T}_{qr} + e_{pqr} x_{q} \left(X_{r} + Y_{r} \right) = 0, \end{aligned} \tag{2.68}$$

where $-\mathbf{Y} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B}/c$ is the familiar Lorentz force. Substitution, $\widetilde{T}_{rs} = T_{rs} + \left(\widetilde{T}_{rs} - T_{rs} \right)$, results in (2.28) in view of identity (2.55). The remaining cases, when $\mu, \nu = p, q = 1,2,3$, can be analyzed in a similar fashion. In 3D-form, the corresponding equations can be reduced to (2.21) and (2.54). Details are left to the reader.

Thus the angular momentum law has the form of a local balance equation, not a conservation law, since in general the energy-momentum tensor will not be symmetric [47]. Due to the asymmetry of this energy-momentum tensor a torque, for instance, may occur, which cannot be compensated for by a change in the electromagnetic angular momentum. Although this result may perhaps seem peculiar it is not in contradiction with experiment according to [170].

2.8 Transformation Laws of Complex Electromagnetic Fields

Let $\mathbf{v}$ be a constant real velocity vector representing uniform motion of one frame of reference with respect to another one. Let us consider the following orthogonal decompositions,

$$\mathbf{F} = \mathbf{F}_{\|} + \mathbf{F}_{\perp}, \qquad \mathbf{G} = \mathbf{G}_{\|} + \mathbf{G}_{\perp}, \tag{2.69}$$

such that our complex vectors $\{\mathbf{F}_{\|}, \mathbf{G}_{\|}\}$ are collinear with the velocity vector $\mathbf{v}$ and $\{\mathbf{F}_{\perp}, \mathbf{G}_{\perp}\}$ are perpendicular to it (Figure 1). The Lorentz transformation of electric and magnetic fields $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ take the following complex form

$$\mathbf{F}'_{\|} = \mathbf{F}_{\|}, \qquad \mathbf{G}'_{\|} = \mathbf{G}_{\|} \tag{2.70}$$

and

$$\mathbf{F}'_{\perp} = \frac{\mathbf{F}_{\perp} - \dfrac{i}{c}(\mathbf{v} \times \mathbf{G})}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{G}'_{\perp} = \frac{\mathbf{G}_{\perp} - \dfrac{i}{c}(\mathbf{v} \times \mathbf{F})}{\sqrt{1 - v^2/c^2}}. \tag{2.71}$$

Although this transformation was found by Lorentz, it was Minkowski who realized that this is the law of transformation of the second rank anti-symmetric four-tensors [138], [154]; a brief historical overview is given in [170].) This complex 3D-form of the Lorentz transformation of electric and magnetic fields was known to Minkowski (1908), but apparently only in vacuum, when $\mathbf{G} = \mathbf{F}$ (see also [207]). Here, in the same notation [170],

$$\mathbf{r}'_{\parallel} = \frac{\mathbf{r}_{\parallel} - \mathbf{v}t}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp}, \qquad t' = \frac{t - (\mathbf{v} \cdot \mathbf{r})/c^2}{\sqrt{1 - v^2/c^2}}, \tag{2.72}$$

for the reader's convenience. Equations (2.72) can be rewritten as follows

$$\mathbf{r}' = \mathbf{r} + \left[(\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{r}}{v^2} - \gamma t \right] \mathbf{v}, \qquad t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{r}}{c^2} \right), \tag{2.73}$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

In a similar fashion,

$$\mathbf{F}' = \gamma \left(\mathbf{F} - \frac{i}{c} \mathbf{v} \times \mathbf{G} \right) - (\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{F}}{v^2} \mathbf{v} \tag{2.74}$$

and

$$\mathbf{G}' = \gamma \left(\mathbf{G} - \frac{i}{c} \mathbf{v} \times \mathbf{F} \right) - (\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{G}}{v^2} \mathbf{v}, \tag{2.75}$$

for the complex electromagnetic fields.

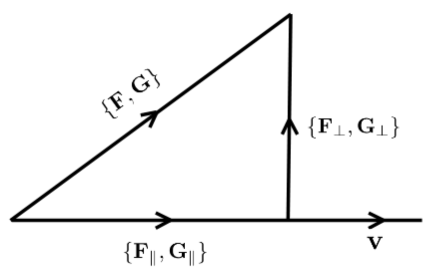

Figure 2.1: Complex electromagnetic fields decomposition.

In complex four-tensor form,

$$Q'^{\mu\nu}\left(x'\right) = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} Q^{\sigma\tau}(x), \qquad x' = \Lambda x. \tag{2.76}$$

Although Minkowski considered the transformation of electric and magnetic fields in a complex $3D$-vector form, see Eqs. (8)–(9) and (15) in [154] (or Eqs. (25.5)–(25.6) in [122]), he seems never to have combined the corresponding four-tensors into the complex forms (2.41)–(2.42). In the second article [155], Max Born, who used Minkowski's notes, didn't mention the complex fields. As a result, the complex field tensor seems only to have appeared, for the first time, in [128] (see also [206]). The complex identity, $\mathbf{F} \cdot \mathbf{G} =$ invariant under the similarity transformation, follows from Minkowski's determinant relations (2.146)–(2.148).

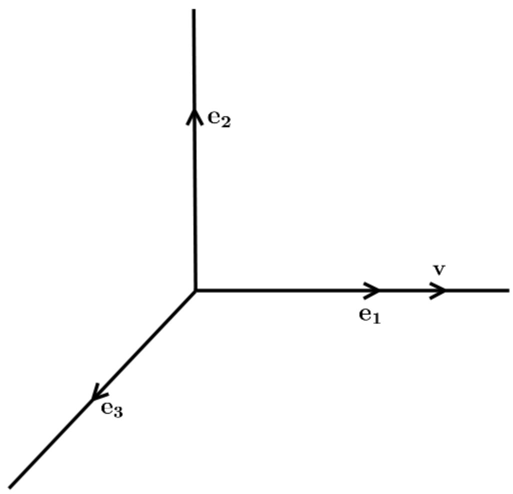

Figure 2.2: Example of moving frame velocity.

Example. Let $\{\mathbf{e}_k\}_{k=1}^3$ be an orthonormal basis in $\mathbb{R}^3$. We choose $\mathbf{v} = v\mathbf{e}_1$ and write $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$ with

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.77}$$

for the corresponding Lorentz boost (Figure 2). In view of (2.76), by matrix multiplication one gets

$$
\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix}
\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -G_1 & -\gamma G_2 - i\beta\gamma F_3 & -\gamma G_3 + i\beta\gamma F_2 \\ G_1 & 0 & \beta\gamma G_2 + i\gamma F_3 & \beta\gamma G_3 - i\gamma F_2 \\ \gamma G_2 + i\beta\gamma F_3 & -\beta\gamma G_2 - i\gamma F_3 & 0 & iF_1 \\ \gamma G_3 - i\beta\gamma F_2 & -\beta\gamma G_3 + i\gamma F_2 & -iF_1 & 0 \end{pmatrix}. \qquad (2.78)
$$

Thus $G_1' = G_1$ and

$$
G_2' = \gamma G_2 + i\beta\gamma F_3 = \frac{G_2 + i\,(v/c)\,F_3}{\sqrt{1 - v^2/c^2}} = \frac{G_2 - \dfrac{i}{c}\,(\mathbf{v}\times\mathbf{F})_2}{\sqrt{1 - v^2/c^2}}, \qquad (2.79)
$$

$$
G_3' = \gamma G_3 - i\beta\gamma F_2 = \frac{G_3 - i\,(v/c)\,F_2}{\sqrt{1 - v^2/c^2}} = \frac{G_3 - \dfrac{i}{c}\,(\mathbf{v}\times\mathbf{F})_3}{\sqrt{1 - v^2/c^2}}.
$$

In a similar fashion, $F_1' = F_1$ and

$$
F_2' = \gamma F_2 + i\beta\gamma G_3 = \frac{F_2 - \dfrac{i}{c}\,(\mathbf{v}\times\mathbf{G})_2}{\sqrt{1 - v^2/c^2}}, \qquad (2.80)
$$

$$
F_3' = \gamma F_3 - i\beta\gamma G_2 = \frac{F_3 - \dfrac{i}{c}\,(\mathbf{v}\times\mathbf{G})_3}{\sqrt{1 - v^2/c^2}}.
$$

The latter relations are in agreement with the field transformations (2.70)–(2.71).

In block form, one gets

$$\begin{pmatrix} F_1' \\ F_2' \\ G_3' \\ G_2' \\ F_3' \\ G_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(i\theta) & \sin(i\theta) & 0 & 0 & 0 \\ 0 & -\sin(i\theta) & \cos(i\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(i\theta) & \sin(i\theta) & 0 \\ 0 & 0 & 0 & -\sin(i\theta) & \cos(i\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ G_3 \\ G_2 \\ F_3 \\ G_1 \end{pmatrix}, \qquad (2.81)$$

where, by definition,

$$\cos(i\theta) = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \sin(i\theta) = i\beta\gamma = \frac{i\beta}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}. \qquad (2.82)$$

As a result, the transformation law of the complex electromagnetic fields $\{\mathbf{F}, \mathbf{G}\}$ under the Lorentz boost can be thought of as a complex rotation in $\mathbb{C}^6$, corresponding to a reducible representation of the one-parameter subgroup of $SO(3,\mathbb{C})$. (Cyclic permutation of the spatial indices cover the two remaining cases; see also [207].)

2.9 Material Equations, Potentials, and Energy-Momentum Tensor for Moving Isotropic Media

Electromagnetic phenomena in moving media are important in relativistic astrophysics, the study of accelerated plasma clusters and high-energy electron beams [33], [34], [78], [210].

2.9.1 Material Equations

Minkowski's field- and connecting-equations [154], [155] were derived from the corresponding laws for the bodies at rest by means of a Lorentz transformation (see [33], [49], [47], [124], [161], [170], [210]). Explicitly covariant forms, which are applicable both in the rest frame and for moving media, are analyzed in [33], [34], [47], [101], [102], [161],

[166], [170], [182], [183], [207], [210] (see also the references therein). In standard notation,

$$\beta = v/c, \qquad \gamma = \left(1-\beta^2\right)^{-1/2}, \qquad v = |\mathbf{v}|, \qquad \kappa = \varepsilon\mu - 1, \tag{2.83}$$

one can write [33], [34], [49], [210]:

$$\begin{aligned}\mathbf{D} &= \varepsilon\mathbf{E} + \frac{\kappa\gamma^2}{\mu}\left[\beta^2\mathbf{E} - \frac{\mathbf{v}}{c^2}\left(\mathbf{v}\cdot\mathbf{E}\right) + \frac{1}{c}\left(\mathbf{v}\times\mathbf{B}\right)\right], \\ \mathbf{H} &= \frac{1}{\mu}\mathbf{B} + \frac{\kappa\gamma^2}{\mu}\left[-\beta^2\mathbf{B} + \frac{\mathbf{v}}{c^2}\left(\mathbf{v}\cdot\mathbf{B}\right) + \frac{1}{c}\left(\mathbf{v}\times\mathbf{E}\right)\right].\end{aligned} \tag{2.84}$$

In covariant form, these relations are given by

$$\begin{aligned}R^{\lambda\nu} &= \varepsilon^{\lambda\nu\sigma\tau}F_{\sigma\tau} = \frac{1}{2}\left(\varepsilon^{\lambda\nu\sigma\tau} - \varepsilon^{\lambda\nu\tau\sigma}\right)F_{\sigma\tau} \\ &= \frac{1}{4}\left(\varepsilon^{\lambda\nu\sigma\tau} - \varepsilon^{\lambda\nu\tau\sigma} + \varepsilon^{\nu\lambda\tau\sigma} - \varepsilon^{\nu\lambda\sigma\tau}\right)F_{\sigma\tau}\end{aligned} \tag{2.85}$$

(see [32], [33], [34], [101], [102], [182], [183], [210] and the references therein). Here,

$$\varepsilon^{\lambda\nu\sigma\tau} = \frac{1}{\mu}\left(g^{\lambda\sigma} + \kappa u^{\lambda}u^{\sigma}\right)\left(g^{\nu\tau} + \kappa u^{\nu}u^{\tau}\right) = \varepsilon^{\nu\lambda\tau\sigma} \tag{2.86}$$

is the four-tensor of electric and magnetic permeabilities and

$$u^{\lambda} = \left(\gamma, \gamma\mathbf{v}/c\right), \qquad u^{\lambda}u_{\lambda} = 1 \tag{2.87}$$

is the four-velocity of the medium ([182], [183], a computer algebra verification of these relations is given in [127]). In a complex covariant form,

$$\left(Q^{\mu\nu} + \overset{*}{Q}{}^{\mu\nu}\right) = \varepsilon^{\mu\nu\sigma\tau}\left(P_{\sigma\tau} + \overset{*}{P}_{\sigma\tau}\right). \tag{2.88}$$

In view of (2.85) and (2.128)–(2.129), we get

$$Q^{\mu\nu} = \left(\varepsilon^{\mu\nu\sigma\tau} - \frac{i}{2}e^{\mu\nu\sigma\tau}\right)F_{\sigma\tau}, \qquad P_{\mu\nu} = \left(\delta_{\mu}^{\lambda}\delta_{\nu}^{\rho} - \frac{i}{2}e_{\mu\nu\sigma\tau}\varepsilon^{\sigma\tau\lambda\rho}\right)F_{\lambda\rho}, \tag{2.89}$$

in terms of the real-valued electromagnetic field tensor.

2.9.2 Potentials

In practice, one can choose

$$F_{\sigma\tau} = \frac{\partial A_\tau}{\partial x^\sigma} - \frac{\partial A_\sigma}{\partial x^\tau}, \tag{2.90}$$

for the real-valued four-vector potential $A_\lambda(x)$. Then

$$\begin{aligned}\partial_\nu Q^{\lambda\nu} &= \varepsilon^{\lambda\nu\sigma\tau}\partial_\nu\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right) - \frac{i}{2}e^{\lambda\nu\sigma\tau}\partial_\nu\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right)\\ &= \frac{1}{\mu}\left(g^{\lambda\sigma} + \kappa u^\lambda u^\sigma\right)\left(g^{\nu\tau} + \kappa u^\nu u^\tau\right)\partial_\nu\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right)\end{aligned}$$

by (2.86). Substitution into Maxwell's equations (2.46) or (2.45) results in

$$\begin{aligned}&\left(g^{\lambda\sigma} + \kappa u^\lambda u^\sigma\right)\left\{-\left[\partial^\tau\partial_\tau + \kappa\left(u^\tau\partial_\tau\right)^2\right]A_\sigma\right.\\ &\left.\quad+\partial_\sigma\left(\partial^\tau A_\tau + \kappa u^\nu u^\tau\partial_\nu A_\tau\right)\right\} = -\frac{4\pi\mu}{c}j^\lambda,\end{aligned} \tag{2.91}$$

where $-\partial^\tau\partial_\tau = -g^{\sigma\tau}\partial_\sigma\partial_\tau = \Delta - \left(\partial/c\partial t\right)^2$ is the D'Alembert operator. In view of an inverse matrix identity,

$$\left(g_{\lambda\rho} - \frac{\kappa}{1+\kappa}u_\lambda u_\rho\right)\left(g^{\lambda\sigma} + \kappa u^\lambda u^\sigma\right) = \delta_\rho^\sigma, \tag{2.92}$$

the latter equations take the form[2]

$$\begin{aligned}&\left[\partial^\tau\partial_\tau + \kappa\left(u^\tau\partial_\tau\right)^2\right]A_\sigma - \partial_\sigma\left(\partial^\tau A_\tau + \kappa u^\nu u^\tau\partial_\nu A_\tau\right)\\ &\quad= \frac{4\pi\mu}{c}\left(g_{\sigma\lambda} - \frac{\kappa}{1+\kappa}u_\sigma u_\lambda\right)j^\lambda.\end{aligned} \tag{2.93}$$

Subject to the subsidiary condition,

$$\partial^\tau A_\tau + \kappa u^\nu u^\tau\partial_\nu A_\tau = \left(g^{\nu\tau} + \kappa u^\nu u^\tau\right)\partial_\nu A_\tau = 0, \tag{2.94}$$

these equations were studied in detail for the sake of development of the phenomenological classical and quantum electrodynamics in a moving medium (see [31], [32], [33], [34],

[2] Equations (2.90) and (2.93), together with the gauge condition (2.94), may be considered as the fundamentals of the theory [101]. Our complex fields are given by (2.89).

[166], [182], [177], [183], [210] and the references therein). In particular, Green's functions of the photon in a moving medium was studied in [101], [182], [177] (with applications to quantum electrodynamics).

2.9.3 Hertz's Tensor and Vectors

We follow [33], [34], [210] with somewhat different details. The substitution,

$$A^{\mu}(x)=\left(\frac{\kappa}{1+\kappa}u^{\mu}u_{\lambda}-\delta_{\lambda}^{\mu}\right)\partial_{\sigma}Z^{\lambda\sigma}(x) \tag{2.95}$$

(a generalization of Hertz's potentials for a moving medium [33], [210]), into the gauge condition (2.94) results in $Z^{\lambda\sigma}=-Z^{\sigma\lambda}$, in view of

$$\begin{aligned}&\left(g_{\nu\mu}+\kappa u_{\nu}u_{\mu}\right)\partial^{\nu}A^{\mu}\\&\quad=\left(g_{\nu\mu}+\kappa u_{\nu}u_{\mu}\right)\left(\frac{\kappa}{1+\kappa}u^{\mu}u_{\lambda}-\delta_{\lambda}^{\mu}\right)\partial^{\nu}\partial_{\sigma}Z^{\lambda\sigma}\\&\quad=-g_{\nu\lambda}\partial^{\nu}\partial_{\sigma}Z^{\lambda\sigma}=-\partial_{\lambda}\partial_{\sigma}Z^{\lambda\sigma}\equiv 0.\end{aligned}$$

Then, equations (2.93) take the form

$$\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]\partial_{\sigma}Z^{\lambda\sigma}=-\frac{4\pi\mu}{c}j^{\lambda}. \tag{2.96}$$

Indeed, the right-hand side of (2.93) is given by

$$\begin{aligned}\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]A_{\sigma}&=\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]g_{\sigma\mu}A^{\mu}\\&=\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]g_{\sigma\mu}\left(\frac{\kappa}{1+\kappa}u^{\mu}u_{\lambda}-\delta_{\lambda}^{\mu}\right)\partial_{\sigma}Z^{\lambda\sigma}\\&=\left[\partial^{\tau}\partial_{\tau}+\kappa\left(u^{\tau}\partial_{\tau}\right)^{2}\right]\left(\frac{\kappa}{1+\kappa}u_{\sigma}u_{\lambda}-g_{\sigma\lambda}\right)\partial_{\sigma}Z^{\lambda\sigma}\\&=\frac{4\pi\mu}{c}\left(g_{\sigma\lambda}-\frac{\kappa}{1+\kappa}u_{\sigma}u_{\lambda}\right)j^{\lambda}\end{aligned}$$

and the matrix is invertible.

Finally, with the help of the standard substitution,

$$j^{\lambda}=c\partial_{\sigma}p^{\lambda\sigma},\qquad p^{\lambda\sigma}=-p^{\sigma\lambda} \tag{2.97}$$

(in view of $\partial_\lambda j^\lambda = c\partial_\lambda \partial_\sigma p^{\lambda\sigma} \equiv 0$), we arrive at

$$\left[\partial^\tau \partial_\tau + \kappa \left(u^\tau \partial_\tau\right)^2\right] Z^{\lambda\nu} = -4\pi p^{\lambda\nu}. \tag{2.98}$$

Here, by definition,

$$p^{\lambda\nu} = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ -p_1 & 0 & -m_3 & m_2 \\ -p_2 & m_3 & 0 & -m_1 \\ -p_3 & -m_2 & m_1 & 0 \end{pmatrix} \tag{2.99}$$

is an anti-symmetric four-tensor [33], [34], [210]. The "electric" and "magnetic" Hertz vectors, $\mathbf{Z}^{(e)}$ and $\mathbf{Z}^{(m)}$, are also introduced in terms of a single four-tensor,

$$Z^{\lambda\nu} = \begin{pmatrix} 0 & Z_1^{(e)} & Z_2^{(e)} & Z_3^{(e)} \\ -Z_1^{(e)} & 0 & -Z_3^{(m)} & Z_2^{(m)} \\ -Z_2^{(e)} & Z_3^{(m)} & 0 & -Z_1^{(m)} \\ -Z_3^{(e)} & -Z_2^{(m)} & Z_1^{(m)} & 0 \end{pmatrix}. \tag{2.100}$$

In view of (2.95), for the four-vector potential, $A^\lambda = (\varphi, \mathbf{A})$, we obtain

$$\varphi = -\left(1 - \frac{\kappa\gamma^2}{1+\kappa}\right) \nabla \cdot \mathbf{Z}^{(e)} + \frac{\kappa\gamma^2}{(1+\kappa)\,c} \mathbf{v} \cdot \left(\frac{\partial \mathbf{Z}^{(e)}}{c\partial t} + \nabla \times \mathbf{Z}^{(m)}\right) \tag{2.101}$$

and

$$\begin{aligned} \mathbf{A} = {} & \frac{\partial \mathbf{Z}^{(e)}}{c\partial t} + \nabla \times \mathbf{Z}^{(m)} \\ & + \frac{\kappa\gamma^2 \mathbf{v}}{(1+\kappa)\,c^2} \left[c\nabla \cdot \mathbf{Z}^{(e)} + \frac{\partial}{c\partial t}\left(\mathbf{v} \cdot \mathbf{Z}^{(e)}\right) + \mathbf{v} \cdot \left(\nabla \times \mathbf{Z}^{(m)}\right)\right]. \end{aligned} \tag{2.102}$$

Then, equations (2.98) take the form

$$\left[\partial^\tau \partial_\tau + \kappa \left(u^\tau \partial_\tau\right)^2\right] \mathbf{Z}^{(e)} = -4\pi\mu \mathbf{p}, \qquad \left[\partial^\tau \partial_\tau + \kappa \left(u^\tau \partial_\tau\right)^2\right] \mathbf{Z}^{(m)} = -4\pi\mu \mathbf{m} \tag{2.103}$$

and, for the four-current, $j^\lambda = (c\rho, \mathbf{j})$, one gets

$$\rho = -\nabla \cdot \mathbf{p}, \qquad \mathbf{j} = \frac{\partial \mathbf{p}}{\partial t} + c\nabla \times \mathbf{m} \tag{2.104}$$

(see [33], [34], [210] for more details).

The Hertz vector and tensor potentials, for a moving medium and at rest, were utilized in [33], [34], [67], [81], [106], [205], [210], [218] (see also the references therein). Many classical problems of radiation and propagation can be consistently solved by using these potentials.

2.9.4 Energy-Momentum Tensor

In the case of the covariant version of the energy-momentum tensor given by (2.50), the differential balance equations under consideration are independent of the particular choice of the frame of reference. Therefore, our relations (2.89) are useful for derivation of the expressions for the energy-momentum tensor and the ponderomotive force for moving bodies from those for bodies at rest which were extensively studied in the literature. For example, one gets

$$4\pi T_{\mu}{}^{\nu} = F_{\mu\lambda}\varepsilon^{\lambda\nu\sigma\tau}F_{\sigma\tau} + \frac{1}{4}\delta_{\mu}^{\nu}F_{\sigma\tau}\varepsilon^{\sigma\tau\lambda\rho}F_{\lambda\rho} \tag{2.105}$$

with the help of (2.85) and (2.137).

2.10 Real vs Complex Lagrangians

In modern presentations of the classical and quantum field theories, the Lagrangian approach is usually utilized.

2.10.1 Complex Forms

We introduce two quadratic "Lagrangian" densities

$$
\begin{aligned}
\mathscr{L}_0 = \mathscr{L}_0^* &= \frac{1}{2}\left(P_{\sigma\tau}Q^{\tau\sigma} + P^*_{\sigma\tau}\overset{*}{Q}{}^{\tau\sigma}\right) \\
&= \frac{i}{4}e^{\sigma\tau\kappa\rho}\left(P_{\sigma\tau}P_{\kappa\rho} - P^*_{\sigma\tau}P^*_{\kappa\rho}\right) \\
&= F_{\sigma\tau}R^{\tau\sigma} - G_{\sigma\tau}S^{\tau\sigma} = 2F_{\sigma\tau}R^{\tau\sigma} \\
&= 4\left(\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B}\right)
\end{aligned}
\tag{2.106}
$$

and

$$
\begin{aligned}
\mathscr{L}_1 = -\mathscr{L}_1^* = P^*_{\sigma\tau}Q^{\tau\sigma} &= \frac{1}{2}\left(P^*_{\sigma\tau}Q^{\tau\sigma} - P_{\sigma\tau}\overset{*}{Q}{}^{\tau\sigma}\right) \\
&= \frac{i}{2}e^{\sigma\tau\kappa\rho}P_{\sigma\tau}P^*_{\kappa\rho} = 4i\left(\mathbf{E}\cdot\mathbf{B} - \mathbf{H}\cdot\mathbf{D}\right).
\end{aligned}
\tag{2.107}
$$

Then, by formal differentiation,

$$
\frac{\partial\mathscr{L}_0}{\partial P_{\alpha\beta}} = Q^{\beta\alpha}, \qquad \frac{\partial\mathscr{L}_0}{\partial P^*_{\alpha\beta}} = \overset{*}{Q}{}^{\beta\alpha} \tag{2.108}
$$

and

$$
\frac{\partial\mathscr{L}_1}{\partial P^*_{\alpha\beta}} = Q^{\beta\alpha}, \qquad \frac{\partial\mathscr{L}_1^*}{\partial P_{\alpha\beta}} = \overset{*}{Q}{}^{\beta\alpha} \tag{2.109}
$$

in view of (2.130).

The complex covariant Maxwell equations (2.33) take the forms

$$
\frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathscr{L}_0}{\partial P_{\nu\mu}}\right) = -\frac{4\pi}{c}j^\mu, \qquad \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathscr{L}_1}{\partial P_{\nu\mu}}\right) = \frac{4\pi}{c}j^\mu \tag{2.110}
$$

and the covariant energy-momentum balance relations (2.49) are given by

$$
\begin{aligned}
&\frac{\partial}{\partial x^\nu}\left[\frac{1}{16\pi}\left(P^*_{\mu\lambda}\frac{\partial\mathscr{L}_0}{\partial P_{\nu\lambda}} + P_{\mu\lambda}\frac{\partial\mathscr{L}_0}{\partial P^*_{\nu\lambda}}\right)\right] \\
&\quad + \frac{1}{32\pi}\left[P^*_{\sigma\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}_0}{\partial P_{\sigma\tau}}\right) + P_{\sigma\tau}\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}_0}{\partial P^*_{\sigma\tau}}\right)\right] = -\frac{1}{c}F_{\mu\lambda}j^\lambda
\end{aligned}
\tag{2.111}
$$

and

$$\frac{\partial}{\partial x^{\nu}}\left[\frac{1}{16\pi}\left(P_{\mu\lambda}\frac{\partial \mathscr{L}_1}{\partial P_{\nu\lambda}}+P^{*}_{\mu\lambda}\frac{\partial \mathscr{L}^{*}_1}{\partial P_{\nu\lambda}}\right)\right] + \frac{1}{32\pi}\left[P_{\sigma\tau}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathscr{L}_1}{\partial P_{\sigma\tau}}\right)+P^{*}_{\sigma\tau}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathscr{L}^{*}_1}{\partial P^{*}_{\sigma\tau}}\right)\right] = \frac{1}{c}F_{\mu\lambda}\, j^{\lambda} \tag{2.112}$$

in terms of the complex Lagrangians under consideration, respectively.

Finally, with the help of the following densities,

$$L_0 = \mathscr{L}_0 - \frac{4\pi}{c} j^{\nu} A_{\nu}, \qquad L_1 = \mathscr{L}_1 + \frac{4\pi}{c} j^{\nu} A_{\nu}, \tag{2.113}$$

one can derive analogs of the Euler-Lagrange equations for electromagnetic fields in media:

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L_{0,1}}{\partial P_{\nu\mu}}\right) - \frac{\partial L_{0,1}}{\partial A_{\mu}} = 0. \tag{2.114}$$

In the case of a moving isotropic medium, a relation between $P_{\nu\mu}$ and A_{μ} is given by our equations (2.89)–(2.90).

2.10.2 Real Form

Taking the real and imaginary parts, Maxwell's equations (2.46) can be written as follows

$$\partial_{\nu} R^{\mu\nu} = -\frac{4\pi}{c} j^{\mu}, \qquad \partial_{\nu} S^{\mu\nu} = 0. \tag{2.115}$$

Here,

$$-6\partial_{\nu} S^{\mu\nu} = e^{\mu\nu\lambda\sigma}\left(\partial_{\nu} F_{\lambda\sigma} + \partial_{\sigma} F_{\nu\lambda} + \partial_{\lambda} F_{\sigma\nu}\right) \equiv 0,$$

with the help of (2.44) and (2.90). Thus the second set of equations is automatically satisfied when we introduce the four-vector potential. For the inhomogeneous pair of Maxwell's equations, the Lagrangian density is given by

$$\begin{aligned} L &= \frac{1}{4} F_{\sigma\tau} R^{\tau\sigma} - \frac{4\pi}{c} j^{\sigma} A_{\sigma} \\ &= \frac{1}{4} F_{\sigma\tau} \varepsilon^{\tau\sigma\lambda\rho} F_{\lambda\rho} - \frac{4\pi}{c} j^{\sigma} A_{\sigma}, \end{aligned} \tag{2.116}$$

in view of (2.85). Then, for "conjugate momenta" to the four-potential field A_μ, one gets

$$\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)}=\frac{\partial L}{\partial F_{\sigma\tau}}\frac{\partial F_{\sigma\tau}}{\partial\left(\partial_\nu A_\mu\right)}=R^{\mu\nu} \tag{2.117}$$

and the corresponding Euler-Lagrange equations take a familiar form

$$\partial_\nu\left(\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)}\right)-\frac{\partial L}{\partial A_\mu}=0. \tag{2.118}$$

The latter equation can also be derived with the help of the least action principle [170], [207], [209]. The corresponding Hamiltonian and quantization are discussed in [94], [101], [182] among other classical accounts.

In conclusion, it is worth noting the role of complex fields in quantum electrodynamics, quadratic invariants and quantization (see, for instance, [4], [27], [26], [54], [101], [102], [110], [114], [182], [177], [183], [209], [223]). The classical and quantum theory of Cherenkov radiation is reviewed in [5], [30], [31], [87], [204]. For paraxial approximation in optics, see [81], [112], [143], [142] and the references therein. Maxwell's equations in the gravitational field are discussed in [36] [82]. One may hope that our detailed mathematical consideration of several aspects of macroscopic electrodynamics will be useful for future investigations and pedagogy.

2.11 Formulas from Vector Calculus

Among useful differential relations are

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B}). \tag{2.119}$$

$$\nabla\cdot(f\mathbf{A})=(\nabla f)\cdot\mathbf{A}+f(\nabla\cdot\mathbf{A}). \tag{2.120}$$

$$\nabla\times(f\mathbf{A})=(\nabla f)\times\mathbf{A}+f(\nabla\times\mathbf{A}). \tag{2.121}$$

$$\begin{aligned}&\mathbf{A}\cdot(\nabla\times(f\nabla\times\mathbf{B}))-\mathbf{B}\cdot(\nabla\times(f\nabla\times\mathbf{A})) \\ &\quad=\nabla\cdot(f(\mathbf{B}\times(\nabla\times\mathbf{A})-\mathbf{A}\times(\nabla\times\mathbf{B}))).\end{aligned} \tag{2.122}$$

$$\begin{aligned}\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+\mathbf{A}\times(\nabla\times\mathbf{B})-\mathbf{B}\times(\nabla\times\mathbf{A})\\ -\nabla\times(\mathbf{A}\times\mathbf{B})=\sum_{\alpha=1}^{3}A_\alpha^2\nabla\left(\frac{B_\alpha}{A_\alpha}\right)=-\sum_{\alpha=1}^{3}B_\alpha^2\nabla\left(\frac{A_\alpha}{B_\alpha}\right).\end{aligned} \tag{2.123}$$

(See also [3], [188] and [209].)

2.12 Dual Tensor Identities

In this chapter, $e^{\mu\nu\sigma\tau}=-e_{\mu\nu\sigma\tau}$ and $e_{0123}=+1$ is the Levi-Civita four-symbol [82] with familiar contractions:

$$e^{\mu\nu\sigma\tau}e_{\mu\kappa\lambda\rho}=-\begin{vmatrix}\delta_\kappa^\nu & \delta_\lambda^\nu & \delta_\rho^\nu\\ \delta_\kappa^\sigma & \delta_\lambda^\sigma & \delta_\rho^\sigma\\ \delta_\kappa^\tau & \delta_\lambda^\tau & \delta_\rho^\tau\end{vmatrix}, \tag{2.124}$$

$$e^{\mu\nu\sigma\tau}e_{\mu\nu\lambda\rho}=-2\begin{vmatrix}\delta_\lambda^\sigma & \delta_\rho^\sigma\\ \delta_\lambda^\tau & \delta_\rho^\tau\end{vmatrix}=-2\left(\delta_\lambda^\sigma\delta_\rho^\tau-\delta_\rho^\sigma\delta_\lambda^\tau\right), \tag{2.125}$$

$$e^{\mu\nu\sigma\tau}e_{\mu\nu\sigma\rho}=-6\delta_\rho^\tau,\qquad e^{\mu\nu\sigma\tau}e_{\mu\nu\sigma\rho}=-24. \tag{2.126}$$

Dual second rank four-tensor identities are given by [82]:

$$e^{\mu\nu\sigma\tau}A_{\sigma\tau}=2B^{\mu\nu},\qquad e_{\mu\nu\sigma\tau}B^{\sigma\tau}=A_{\nu\mu}-A_{\mu\nu}. \tag{2.127}$$

In particular,

$$Q^{\mu\nu}=R^{\mu\nu}+iS^{\mu\nu}=R^{\mu\nu}-\frac{i}{2}e^{\mu\nu\sigma\tau}F_{\sigma\tau}, \tag{2.128}$$

$$P_{\mu\nu}=F_{\mu\nu}+iG_{\mu\nu}=F_{\mu\nu}-\frac{i}{2}e_{\mu\nu\sigma\tau}R^{\sigma\tau}. \tag{2.129}$$

$$e_{\mu\nu\sigma\tau}Q^{\sigma\tau}=2iP_{\mu\nu},\qquad 2iQ^{\mu\nu}=e^{\mu\nu\sigma\tau}P_{\sigma\tau}. \tag{2.130}$$

$$2R^{\mu\nu}=e^{\mu\nu\sigma\tau}G_{\sigma\tau},\qquad -2S^{\mu\nu}=e^{\mu\nu\sigma\tau}F_{\sigma\tau}. \tag{2.131}$$

$$2G_{\mu\nu}=-e_{\mu\nu\sigma\tau}R^{\sigma\tau},\qquad 2F_{\mu\nu}=e_{\mu\nu\sigma\tau}S^{\sigma\tau}. \tag{2.132}$$

$$P_{\mu\nu}Q^{\mu\nu} = 2F_{\mu\nu}R^{\mu\nu} - \frac{i}{2}\left(e^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} + e_{\mu\nu\sigma\tau}R^{\mu\nu}R^{\sigma\tau}\right). \tag{2.133}$$

By direct calculation,

$$F_{\mu\nu}R^{\mu\nu} = 2\left(\mathbf{H}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{D}\right), \tag{2.134}$$

$$e^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} = 8\mathbf{E}\cdot\mathbf{B}, \qquad e_{\mu\nu\sigma\tau}R^{\mu\nu}R^{\sigma\tau} = 8\mathbf{H}\cdot\mathbf{D}. \tag{2.135}$$

As a result,

$$\frac{1}{4}P_{\mu\nu}Q^{\mu\nu} = \mathbf{H}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{D} - i\left(\mathbf{E}\cdot\mathbf{B} + \mathbf{H}\cdot\mathbf{D}\right). \tag{2.136}$$

An important decomposition,

$$\begin{aligned} P^{*}_{\mu\lambda}Q^{\lambda\nu} + P_{\mu\lambda}\overset{*}{Q}{}^{\lambda\nu} &= 2\left(F_{\mu\lambda}R^{\lambda\nu} + G_{\mu\lambda}S^{\lambda\nu}\right) \\ &= 4F_{\mu\lambda}R^{\lambda\nu} + \delta^{\nu}_{\mu}F_{\sigma\tau}R^{\sigma\tau} \\ &= 4F_{\mu\lambda}R^{\lambda\nu} - 2\delta^{\nu}_{\mu}\left(\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B}\right), \end{aligned} \tag{2.137}$$

is complemented by an identity,

$$\begin{aligned} P_{\mu\lambda}Q^{\lambda\nu} + P^{*}_{\mu\lambda}\overset{*}{Q}{}^{\lambda\nu} &= \frac{1}{4}\left(P_{\sigma\tau}Q^{\tau\sigma} + P^{*}_{\sigma\tau}\overset{*}{Q}{}^{\tau\sigma}\right)\delta^{\nu}_{\mu} \\ &= \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B}\right)\delta^{\nu}_{\mu}. \end{aligned} \tag{2.138}$$

In matrix form,

$$PQ = (F + iG)(R + iS) = (FR - GS) + i(FS + GR), \tag{2.139}$$

$$P^{*}Q = (F - iG)(R + iS) = (FR + GS) + i(FS - GR). \tag{2.140}$$

Here,

$$FS = \frac{1}{4}\mathrm{Tr}(FS)I = (\mathbf{E}\cdot\mathbf{B})I, \tag{2.141}$$

$$GR = \frac{1}{4}\mathrm{Tr}(GR)I = (\mathbf{H}\cdot\mathbf{D})I. \tag{2.142}$$

$$FR - GS = \frac{1}{2}\mathrm{Tr}(FR)I = (\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B})I, \tag{2.143}$$

$$FR+GS = 2FR - \frac{1}{2}\mathrm{Tr}\,(FR)\,I \tag{2.144}$$
$$= 2FR - (\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B})\,I.$$

$$\mathrm{Tr}\,(FR+GS) = 0, \tag{2.145}$$

where $I = \mathrm{diag}(1,1,1,1)$ is the identity matrix.

Also,

$$PQ = QP = (\mathbf{F}\cdot\mathbf{G})\,I, \tag{2.146}$$

$$\det P = \det Q = -\,(\mathbf{F}\cdot\mathbf{G})^2 \tag{2.147}$$

and

$$\mathbf{F}\cdot\mathbf{G} = (\mathbf{E}+i\mathbf{H})\cdot(\mathbf{D}+i\mathbf{B}) \tag{2.148}$$
$$= (\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B}) + i\,(\mathbf{E}\cdot\mathbf{B} + \mathbf{H}\cdot\mathbf{D})\,.$$

Other useful dual four-tensor identities are given by [82]:

$$e^{\mu\nu\sigma\tau}A_{\nu\sigma\tau} = 6B^{\mu}, \qquad A_{\mu\nu\lambda} = e_{\mu\nu\lambda\sigma}B^{\sigma}. \tag{2.149}$$

In particular,

$$6i\frac{\partial Q^{\mu\nu}}{\partial x^{\nu}} = e^{\mu\nu\lambda\sigma}\left(\frac{\partial P_{\lambda\sigma}}{\partial x^{\nu}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\sigma}} + \frac{\partial P_{\sigma\nu}}{\partial x^{\lambda}}\right), \tag{2.150}$$

and

$$\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial P_{\lambda\mu}}{\partial x^{\nu}} = ie_{\mu\nu\lambda\sigma}\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}} \tag{2.151}$$

(see also [115]).

2.13 Proof of Identities (2.63)

In view of (2.43), or (2.130), and (2.151), we can write

$$\left(\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial P_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial P_{\lambda\mu}}{\partial x^{\nu}} = ie_{\mu\nu\lambda\sigma}\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}}\right)\overset{*}{Q}{}^{\lambda\nu}, \tag{2.152}$$

or

$$
\begin{aligned}
&2\overset{*}{Q}{}^{\lambda\nu}\,\frac{\partial P_{\mu\nu}}{\partial x^{\lambda}}+\overset{*}{Q}{}^{\lambda\nu}\,\frac{\partial P_{\nu\lambda}}{\partial x^{\mu}}\\
&\qquad=i\left(e_{\mu\nu\lambda\sigma}\,\overset{*}{Q}{}^{\lambda\nu}\right)\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}}=-2P^{*}_{\mu\sigma}\frac{\partial Q^{\sigma\tau}}{\partial x^{\tau}}
\end{aligned}
$$

by (2.130). Therefore,

$$
P^{*}_{\mu\lambda}\,\frac{\partial Q^{\lambda\nu}}{\partial x^{\nu}}-\frac{\partial P_{\mu\lambda}}{\partial x^{\nu}}\,\overset{*}{Q}{}^{\lambda\nu}=-\frac{1}{2}\,\overset{*}{Q}{}^{\sigma\tau}\,\frac{\partial P_{\tau\sigma}}{\partial x^{\mu}}. \tag{2.153}
$$

In addition, with the help of (2.130) one gets

$$
\begin{aligned}
&2i\left(P^{*}_{\sigma\tau}\frac{\partial Q^{\tau\sigma}}{\partial x^{\mu}}\right)=P^{*}_{\sigma\tau}e^{\tau\sigma\lambda\nu}\,\frac{\partial P_{\lambda\nu}}{\partial x^{\mu}}\\
&\qquad=e^{\sigma\tau\nu\lambda}P^{*}_{\sigma\tau}\frac{\partial P_{\lambda\nu}}{\partial x^{\mu}}=-2i\left(\overset{*}{Q}{}^{\sigma\tau}\,\frac{\partial P_{\tau\sigma}}{\partial x^{\mu}}\right),
\end{aligned}
$$

which completes the proof.

Chapter 3

THE PAULI-LUBANSKI VECTOR AND PHOTON HELICITY

In this chapter, a classical concept of photon helicity is investigated from the viewpoint of the Pauli-Lubański vector. This problem is considered from first principles of the representation theory of the Poincaré group. In particular, a misconception in the definition of helicity for a massless particle, in which a linear relation, $w = \lambda p$, is used in standard quantum field theory textbooks is discovered. In addition, as a result, this analysis also leads to a new approach, introduced here, for the derivation of Maxwell's equations in vacuum. All physically interesting unitary ray representations of the proper orthochronous inhomogeneous Lorentz group (known nowadays as the Poincaré group) were classified by Wigner [231] and, since then, this approach has been utilized for the mathematical description of mass and spin of an elementary particle. By definition, the Pauli-Lubański pseudo-vector[1] is given by

$$w_\mu = \frac{1}{2} e_{\mu\nu\sigma\tau} p^\nu M^{\sigma\tau}, \qquad p_\mu w^\mu = 0, \tag{3.1}$$

where p_μ is the relativistic linear momentum operator and $M^{\sigma\tau}$ are the corresponding angular momentum operators. The mass and spin of a particle are defined in terms of two quadratic invariants (Casimir operators of the Poincaré group) as follows

$$p^2 = p_\mu p^\mu = m^2, \qquad w^2 = w_\mu w^\mu = -m^2 s\,(s+1)\,, \qquad m > 0 \tag{3.2}$$

(see, for example, [11], [12], [29], [121], [133], [139], [140], [141], [181], [184], [185], [187] and the references therein; we use the standard notations that were mentioned in the introduction and are repeated here for convenience).

[1] The pseudo-vector nature can be seen by the fact that $w'_\mu = \det(\Lambda) \Lambda^\sigma_\mu w_\sigma$, which follows from $p'^\nu = \Lambda^\nu_\kappa p^\kappa$, $M'^{\sigma\tau} = \Lambda^\sigma_\alpha \Lambda^\tau_\beta M^{\alpha\beta}$, and the determinant identity $\det(\Lambda) e_{\kappa\gamma\alpha\beta} = e_{\mu\nu\sigma\tau} \Lambda^\mu_\kappa \Lambda^\nu_\gamma \Lambda^\sigma_\alpha \Lambda^\tau_\beta$.

For the massless fields, when $m=0$, one gets $w^2=p^2=pw=0$, and the Pauli-Lubański vector should be proportional to p : [2]

$$w_{\mu}=\lambda p_{\mu} \tag{3.3}$$

(acting on the corresponding eigenstates [160], [184]). The number λ is called the helicity of the representation and the value $s=|\lambda|$ is sometimes called the spin of a particle with zero mass [29], [184], [185], [187]. One of the goals of this chapter is to show that, in the case of the electromagnetic field, the sign of the constant λ in the latter equation is fixed by the condition that the classical Maxwell equations hold.

As a result, instead of being given by the constant of proportionality in relation (3.3), the helicity of the photon should be defined, as it is traditionally done in particle physics, by $\lambda=\mathbf{k}\cdot\mathbf{M}/k_0$, where $k=(k_0,\mathbf{k})$ and $\mathbf{M}$ is the photon angular momentum ($k^2=k_0^2-\mathbf{k}^2=0$). But one needs a proper realization of the action of these operators on the photon field tensor in covariant form [154], [155]; or, in $3D$-form, on the complex electromagnetic field vector $\mathbf{F}=\mathbf{E}+i\mathbf{H}$ discussed in [14], [16], [24], [26], [122], [128], [154], [181], [188], [191], [192], [206], [224]. The sign of the constant $\lambda=\pm1$ is fixed then by a "continuity", in view of the invariance of the upper(lower) light cone under a proper Lorentz transformation (see, for example, [29], [121] and [212]).

3.1 Transformation Laws and Generators

Under the Lorentz transformation [29], [154], [157], [181],

$$U(\Lambda)\,Q^{\mu\nu}(x^{\rho}):=Q'^{\mu\nu}\left(\Lambda^{\kappa}_{\rho}x^{\rho}\right)=\Lambda^{\mu}_{\sigma}\Lambda^{\nu}_{\tau}Q^{\sigma\tau}(x^{\rho}), \tag{3.4}$$

[2] This assumption was made by Bargmann and Wigner [12] for the massless limit of the spinor wave equation for particles with an arbitrary integer or half-integer spin proposed by Dirac [51] (see also [74], [75], [113], [169], [48] and the references therein). The pseudo-vector (3.1) was introduced, in a slightly different notation, by Eqs. (4.a)–(4.b) of Ref. [12].

where the summation is assumed over any two repeated indices[3]. We shall use the following six 4×4 matrices ($\alpha, \beta = 0, 1, 2, 3$ are fixed):

$$\Lambda\left(\theta_{\alpha\beta}\right) = \exp\left(-\theta_{\alpha\beta}\, m^{\alpha\beta}\right), \qquad m^{\alpha\beta} = -m^{\beta\alpha}, \tag{3.5}$$
$$\left(m^{\alpha\beta}\right)^{\mu}_{\ \nu} = g^{\alpha\mu}\delta^{\beta}_{\nu} - g^{\beta\mu}\delta^{\alpha}_{\nu}.$$

for the corresponding one-parameter subgroups of the proper Lorentz group [29], [157], [187] with the standard metric, $g_{\mu\nu} = g^{\mu\nu}$ =diag$(1, -1, -1, -1)$, in the Minkowski spacetime. The 4-angular momentum operators,

$$M^{\alpha\beta} = x^{\beta}\partial^{\alpha} - x^{\alpha}\partial^{\beta}, \qquad \partial^{\alpha} = g^{\alpha\kappa}\partial_{\kappa}, \tag{3.6}$$

can be derived as follows

$$M^{\alpha\beta}Q^{\mu\nu} := -\left[\frac{d}{d\theta_{\alpha\beta}}Q^{\mu\nu}\left(\Lambda^{\kappa}_{\rho}\left(\theta_{\alpha\beta}\right)x^{\rho}\right)\right]\Big|_{\theta_{\alpha\beta}=0} \tag{3.7}$$
$$= \left(m^{\alpha\beta}\right)^{\mu}_{\ \sigma}Q^{\sigma\nu} + Q^{\mu\tau}\left(m^{\alpha\beta}\right)^{\nu}_{\ \tau}$$

with

$$\left(m^{\alpha\beta}\right)^{\mu}_{\ \nu} = -\left.\frac{d\Lambda^{\mu}_{\nu}\left(\theta_{\alpha\beta}\right)}{d\theta_{\alpha\beta}}\right|_{\theta_{\alpha\beta}=0} = g^{\alpha\mu}\delta^{\beta}_{\nu} - g^{\beta\mu}\delta^{\alpha}_{\nu}. \tag{3.8}$$

In matrix form,

$$M^{\alpha\beta}Q = m^{\alpha\beta}Q + Q\left(m^{\alpha\beta}\right)^{T}, \tag{3.9}$$

where $Q = Q^{\mu\nu}$ and m^{T} is the transposed matrix. The latter equations (3.7), (3.8), and (3.9) define the action of the infinitesimal operators on the complex field tensor,

$$M^{\sigma\tau}Q^{\alpha\beta} = g^{\sigma\alpha}Q^{\tau\beta} - g^{\tau\alpha}Q^{\sigma\beta} + g^{\sigma\beta}Q^{\alpha\tau} - g^{\tau\beta}Q^{\alpha\sigma}, \tag{3.10}$$

[3] Although Minkowski considered the transformation of electric and magnetic fields in a complex $3D$ vector form, see Eqs. (8)–(9) and (15) in [154] (or Eqs. (25.5)–(25.6) in [122]), he seems never to have combined the corresponding 4-tensors into the complex forms (2.41)–(2.42). In the second article [155], Max Born, who used Minkowski's notes, didn't mention the complex fields. As a result, the complex field tensor seems only to have appeared, for the first time, in [128] (see also [206]).

in the form that is required in equation (3.15) below.

In a similar fashion, for the products of the generators,

$$M^{\alpha\beta}M^{\gamma\delta}Q^{\mu\nu} = \left(m^{\gamma\delta}\right)^{\mu}_{\kappa}\left(m^{\alpha\beta}\right)^{\kappa}_{\sigma}Q^{\sigma\nu} + \left(m^{\alpha\beta}\right)^{\mu}_{\sigma}\left(m^{\gamma\delta}\right)^{\nu}_{\rho}Q^{\sigma\rho} + \left(m^{\gamma\delta}\right)^{\mu}_{\kappa}\left(m^{\alpha\beta}\right)^{\nu}_{\tau}Q^{\kappa\tau} + \left(m^{\gamma\delta}\right)^{\nu}_{\rho}\left(m^{\alpha\beta}\right)^{\rho}_{\tau}Q^{\mu\tau}, \tag{3.11}$$

or, in matrix form,

$$M^{\alpha\beta}M^{\gamma\delta}Q = \left(m^{\gamma\delta}m^{\alpha\beta}\right)Q - \left(\left(m^{\gamma\delta}m^{\alpha\beta}\right)Q\right)^{T} + m^{\alpha\beta}Q\left(m^{\gamma\delta}\right)^{T} - \left(m^{\alpha\beta}Q\left(m^{\gamma\delta}\right)^{T}\right)^{T}. \tag{3.12}$$

As a result,

$$\begin{aligned}\left[M^{\alpha\beta},\, M^{\gamma\delta}\right] &:= M^{\alpha\beta}M^{\gamma\delta} - M^{\gamma\delta}M^{\alpha\beta} \\ &= g^{\alpha\gamma}M^{\beta\delta} - g^{\alpha\delta}M^{\beta\gamma} + g^{\beta\delta}M^{\alpha\gamma} - g^{\beta\gamma}M^{\alpha\delta},\end{aligned} \tag{3.13}$$

which follows from (3.7)–(3.8) and can be verified, once again, by using (3.6).

Finally, introducing the infinitesimal operators $\mathbf{M} = \left(M^{23}, M^{31}, M^{12}\right)$ and $\mathbf{N} = \left(M^{01}, M^{02}, M^{03}\right)$ for the rotations and boosts, respectively, one can get

$$\mathbf{N}^2 Q = -\mathbf{M}^2 Q = 2Q, \qquad (\mathbf{M}\cdot\mathbf{N})\,Q = -2iQ. \tag{3.14}$$

The Casimir operators of the proper Lorentz group are given by $(\mathbf{M}+i\mathbf{N})^2/4 = 0$ and $(\mathbf{M}-i\mathbf{N})^2/4 = -2$ in the space of complex anti-symmetric tensors under consideration. In view of $\mathbf{M}^2 = -s(s+1) = -2$, we may say that the spin of the photon is equal to one. (Here, we have chosen real-valued generators; see also [10], [12], [85], [181], [187] and [226] for more details on the Lorentz group representations.)

3.2 The Pauli-Lubański Vector and Maxwell's Equations in Vacuum

As follows from the representation theory of the Poincaré group [12], [29] and the geometry of the Minkowski space-time [156], [160], for the case of massless particles, the

Pauli-Lubański vector should be collinear to the operator of the 4-linear momentum. For a classical electromagnetic field, this relation takes the form

$$\frac{1}{2}e_{\mu\nu\sigma\tau}\partial^{\nu}\left(M^{\sigma\tau}Q^{\alpha\beta}\right)=-i\partial_{\mu}Q^{\alpha\beta}, \tag{3.15}$$

and by (3.10), we find that

$$g^{\alpha\alpha}e_{\alpha\mu\nu\tau}\partial^{\nu}Q^{\tau\beta}-g^{\beta\beta}e_{\beta\mu\nu\tau}\partial^{\nu}Q^{\tau\alpha}=-i\partial_{\mu}Q^{\alpha\beta} \tag{3.16}$$

($\alpha,\beta=0,1,2,3$ are fixed; no summation is assumed over these two indices). By a direct, but rather tedious evaluation, one can verify that the latter equation, which is written in terms of a third rank 4-tensor, is equivalent to the original system of Maxwell equations in vacuum, $\partial_{\nu}Q^{\mu\nu}=0$. As a result, the helicity of the photon[4], or a harmonic circular classical electromagnetic wave, cannot be defined as an undetermined sign, or an extra ± 1 factor, in the right hand side of equation (3.15) as it is stated in standard textbooks on the quantum field theory [12], [29], [184], [185], [187].

In view of (3.15), for the rotations and boosts, $\mathbf{M}=\left(M^{23},M^{31},M^{12}\right)$ and $\mathbf{N}=\left(M^{01},M^{02},M^{03}\right)$, respectively, the following standard equations hold

$$(\nabla\cdot\mathbf{M})\,Q=i\partial_0 Q,\quad \partial_0=\frac{1}{c}\frac{\partial}{\partial t} \tag{3.17}$$

and

$$\partial_0\mathbf{M}Q+(\nabla\times\mathbf{N})\,Q=i\nabla Q, \tag{3.18}$$

where $Q=Q^{\alpha\beta}=-Q^{\beta\alpha}$ is the complex field tensor and the actions of operators $\mathbf{M}$ and $\mathbf{N}$ on this tensor are explicitly defined by (3.10).

Note. In vacuum, when $\mathbf{G}=\mathbf{F}$ and $\rho=0$, $\mathbf{j}=0$, two different covariant forms of Maxwell's equations are given by

$$\partial_{\nu}Q^{\mu\nu}=0,\qquad \partial^{\nu}P_{\mu\nu}=0, \tag{3.19}$$

[4] Multiple meanings of the word "photon" are analyzed in [111].

where $\partial^\nu = g^{\nu\mu}\partial_\mu = g^{\nu\mu}\partial/\partial x^\mu$. The second equation follows from (3.16), when one takes $\mu = \beta$ and sums over $\beta = 0,1,2,3$ with the help of (2.43). As another useful consequence of our equation (3.16), one can directly show that the d'Alembert operator annihilates any component of the complex field tensor in vacuum,

$$\partial^\mu \partial_\mu Q^{\alpha\beta} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) Q^{\alpha\beta} = \Box Q^{\alpha\beta} = 0, \tag{3.20}$$

thus de-coupling the system. It is worth noting that, in covariant form, our derivation does not require any formula of 3D-vector calculus. (The general theory of relativistic-invariant equations is studied in classical accounts [12], [22], [50], [51], [74], [75], [85], [113], [144], [169], [176]; see also [27], [89], [139], [140], [141], [48] and the references therein.)

3.3 Examples

In a matrix form, equation (3.17) can be rewritten as follows

$$\begin{pmatrix} 0 & -\partial_2 G_3 + \partial_2 G_3 & \partial_1 G_3 - \partial_3 G_1 & -\partial_1 G_2 + \partial_2 G_1 \\ \partial_2 G_3 - \partial_2 G_3 & 0 & i(\partial_1 F_2 - \partial_2 F_1) & i(\partial_1 F_3 - \partial_3 F_1) \\ -\partial_1 G_3 + \partial_3 G_1 & -i(\partial_1 F_2 - \partial_2 F_1) & 0 & i(\partial_2 F_3 - \partial_3 F_2) \\ \partial_1 G_2 - \partial_2 G_1 & -i(\partial_1 F_3 - \partial_3 F_1) & -i(\partial_2 F_3 - \partial_3 F_2) & 0 \end{pmatrix}$$

$$= +\frac{i}{c}\frac{\partial}{\partial t}\begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix}, \tag{3.21}$$

or

$$\begin{pmatrix} 0 & -(\operatorname{curl}\mathbf{G})_q \\ (\operatorname{curl}\mathbf{G})_p & ie_{pqr}(\operatorname{curl}\mathbf{F})_r \end{pmatrix} = \frac{i}{c}\frac{\partial}{\partial t}\begin{pmatrix} 0 & -G_q \\ G_p & ie_{pqr}F_r \end{pmatrix}, \tag{3.22}$$

in a more compact block form. In vacuum, $\mathbf{G} = \mathbf{F}$ and this matrix relation implies the single complex Maxwell equation, $\operatorname{curl}\mathbf{F} = (i/c)\,\partial\mathbf{F}/\partial t$. In a similar fashion, for the first component of (3.18), namely, $\partial_0 M_1 Q + (\partial_2 N_3 - \partial_3 N_2)\, Q = i\partial_1 Q$, we obtain,

$$\partial_0 \begin{pmatrix} 0 & 0 & G_3 & -G_2 \\ 0 & 0 & iF_2 & iF_3 \\ -G_3 & -iF_2 & 0 & 0 \\ G_2 & -iF_3 & 0 & 0 \end{pmatrix} \tag{3.23}$$
$$+ \begin{pmatrix} 0 & i\,(\partial_2 F_2 + \partial_3 F_3) & -i\partial_2 F_1 & -i\partial_3 F_1 \\ -i\,(\partial_2 F_2 + \partial_3 F_3) & 0 & -\partial_3 G_1 & \partial_2 G_1 \\ i\partial_2 F_1 & \partial_3 G_1 & 0 & \partial_2 G_2 + \partial_3 G_3 \\ i\partial_3 F_1 & -\partial_2 G_1 & -\partial_2 G_2 - \partial_3 G_3 & 0 \end{pmatrix}$$
$$= +i\partial_1 \begin{pmatrix} 0 & -G_1 & -G_2 & -G_3 \\ G_1 & 0 & iF_3 & -iF_2 \\ G_2 & -iF_3 & 0 & iF_1 \\ G_3 & iF_2 & -iF_1 & 0 \end{pmatrix}.$$

Once again, in vacuum, $\mathbf{G} = \mathbf{F}$ and this matrix relation is satisfied in view of the pair of complex Maxwell equations, $\operatorname{curl}\mathbf{F} = (i/c)\,\partial\mathbf{F}/\partial t$ and $\operatorname{div}\mathbf{F} = 0$. (Cyclic permutations of the spatial indices cover the two remaining components.)

One can clearly see that there is no chance of changing the sign $+$ into $-$ in the right hand side without a violation of Maxwell's equations. Indeed, let us pick just one of the matrix elements from both sides, say, $\partial_2 F_2 + \partial_3 F_3 = -\partial_1 G_1$, which indicates also that the left and right hand sides are coming from the different pairs of Maxwell's equations (2.5)–(2.6).

Note. In the case of Weyl's two-component wave equation for massless neutrinos, one can choose

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2 \tag{3.24}$$

and, in block form,

$$M^{\alpha\beta} = -M^{\beta\alpha} = \begin{pmatrix} 0 & \pm i\sigma_q \\ \mp i\sigma_p & e_{pqr}\sigma_r \end{pmatrix}, \tag{3.25}$$

where σ_1, σ_2, σ_3 are the standard 2×2 Pauli matrices with the products given by $\sigma_p\sigma_q = ie_{pqr}\sigma_r + \delta_{pq}$ (see, for example, [187]). As a result, Eqs. (3.3) are satisfied provided that

$$\partial_0\phi = \pm\boldsymbol{\sigma}\cdot\nabla\phi, \qquad \lambda = \pm\frac{1}{2}, \tag{3.26}$$

respectively (details are left to the reader). Thus, the relativistic Weyl equation for a massless particle with the spin $1/2$ can be derived from the representation theory of the Poincaré group.

3.4 Helicity

In particle physics [19], [181], [187], [212], the helicity is defined as the projection of the angular momentum $\mathbf{M}$ on the direction of motion $\mathbf{p}$:

$$\lambda = \frac{\mathbf{p}\cdot\mathbf{M}}{|\mathbf{p}|} = -\frac{w_0}{|\mathbf{p}|}. \tag{3.27}$$

The helicity states are eigenstates of the operator:

$$\Lambda\left|\mathbf{p},\lambda\right\rangle = \frac{\mathbf{p}\cdot\mathbf{M}}{|\mathbf{p}|}\left|\mathbf{p},\lambda\right\rangle = \lambda\left|\mathbf{p},\lambda\right\rangle. \tag{3.28}$$

For massless particles one can define the spin as $s = |\lambda|$ and, if the parity is conserved, the particle will have only two independent helicity eigenstates $|\mathbf{p},\lambda = s\rangle$ and $|\mathbf{p},\lambda = -s\rangle$.

In the case of the classical electromagnetic field, equations (3.17) and/or (3.22) show that the helicity operator is proportional to the "energy operator":

$$\Lambda = \frac{i}{c\,|\mathbf{k}|}\frac{\partial}{\partial t}. \tag{3.29}$$

As a result, these two operators have common eigenstates, $|\mathbf{k},\lambda\rangle = Q^{\mu\nu}$, in the space of complex anti-symmetric 4-tensors of the second rank. (The simplest covariant helicity states will be constructed in the next section.)

On the other hand, in 3D-complex electrodynamics, one can take the complex vector field $|\mathbf{k},\lambda\rangle = \mathbf{F} = \mathbf{E} + i\mathbf{H}$ and choose the following real-valued spin matrices [214]:

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.30}$$

such that $\left[s_p, s_q\right] = s_p s_q - s_q s_p = e_{pqr} s_r$ and $s_1^2 + s_2^2 + s_3^2 = -2$. Then

$$\begin{aligned}(\nabla \cdot \mathbf{s})\,\mathbf{F} &:= \partial_1\left(s_1\mathbf{F}\right) + \partial_2\left(s_2\mathbf{F}\right) + \partial_3\left(s_3\mathbf{F}\right) \\ &= \partial_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + \partial_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ &+ \partial_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = \operatorname{curl}\mathbf{F}.\end{aligned} \tag{3.31}$$

Once again, our representation (3.29) for the helicity operator holds in view of the Maxwell equation in vacuum, $\operatorname{curl}\mathbf{F} = (i/c)\,\partial\mathbf{F}/\partial t$.

Note. In view of (3.29), the traditional definition of helicity (3.28) is related to separation of variables in Maxwell's equations. Letting $Q^{\mu\nu} = q(t)\, Z^{\mu\nu}(\mathbf{r})$, one gets

$$0 = \frac{\partial Q^{\mu\nu}}{\partial x^\nu} = \frac{1}{c}\frac{\partial Q^{\mu 0}}{\partial t} + \frac{\partial Q^{\mu p}}{\partial x_p} = \frac{1}{c}\dot{q}(t)\, Z^{\mu 0}(\mathbf{r}) + q(t)\,\frac{\partial Z^{\mu p}(\mathbf{r})}{\partial x_p}, \tag{3.32}$$

or

$$-\frac{\dot{q}}{cq} Z^{\mu 0} = \frac{\partial Z^{\mu p}}{\partial x_p}, \qquad q = e^{-i\omega t}, \tag{3.33}$$

where ω must be a real-valued constant of the separation of variables in order to have bounded solutions. As a result,

$$\frac{\partial Z^{\mu p}}{\partial x_p} = i\frac{\omega}{c} Z^{\mu 0}, \tag{3.34}$$

thus giving a covariant form of the corresponding eigenvalue problems in different curvilinear coordinates [209], [214].

3.5 Covariant Harmonic Wave Solutions

In vacuum, $\partial_\nu Q^{\mu\nu} = 0$, where

$$Q^{\mu\nu} = \begin{pmatrix} 0 & -F_q \\ & \\ F_p & ie_{pqr}F_r \end{pmatrix}, \quad \mathbf{F} = \mathbf{f}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} = \mathbf{E} + i\mathbf{H}. \tag{3.35}$$

Here, $\mathbf{f}$ =constant is a complex polarization vector to be determined and

$$x^\mu = (ct, \mathbf{r}), \qquad k_\mu = (\omega/c, -\mathbf{k}), \qquad kx = k_\mu x^\mu = \omega t - \mathbf{k}\cdot\mathbf{r}. \tag{3.36}$$

In a compact form, $Q^{\mu\nu} = A^{\mu\nu}e^{ikx}$ and $A^{\mu\nu}k_\nu = 0^\mu$, where

$$A^{\mu\nu} = \begin{pmatrix} 0 & -f_q \\ & \\ f_p & ie_{pqr}f_r \end{pmatrix} = \text{constant}. \tag{3.37}$$

This tensor is an eigenfunction of the 4-gradient, $i^{-1}\partial_\alpha Q^{\mu\nu} = k_\alpha Q^{\mu\nu}$.

As a result,

$$\begin{pmatrix} 0 & -f_1 & -f_2 & -f_3 \\ f_1 & 0 & if_3 & -if_2 \\ f_2 & -if_3 & 0 & if_1 \\ f_3 & if_2 & -if_1 & 0 \end{pmatrix} \begin{pmatrix} \omega/c \\ -k_1 \\ -k_2 \\ -k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.38}$$

and $\det A = -(\mathbf{f}\cdot\mathbf{f})^2 = 0$ (Lorentz invariant by Minkowski [154]). The complex invariant, $\mathbf{F}^2 = (\mathbf{E} + i\mathbf{H})^2 = 0$, results in $\mathbf{E}^2 = \mathbf{H}^2$ and $\mathbf{E}\cdot\mathbf{H} = 0$, as required [122].

In $3D$-form, the latter system of linear equations gives an eigenvalue problem:

$$i\mathbf{k}\times\mathbf{f}=\frac{\omega}{c}\mathbf{f},\qquad \mathbf{f}\cdot\mathbf{f}=0. \tag{3.39}$$

The eigenvalues are

$$\begin{vmatrix} -\omega/c & -ik_3 & ik_2 \\ ik_3 & -\omega/c & -ik_1 \\ -ik_2 & ik_1 & -\omega/c \end{vmatrix} = \frac{\omega}{c}\left(k_1^2+k_2^2+k_3^2-\frac{\omega^2}{c^2}\right)=0. \tag{3.40}$$

The case $\omega=0$, when $\mathbf{f}=\mathbf{k}$, does not satisfy the second condition $\mathbf{f}^2=0$ unless $\mathbf{k}=\mathbf{0}$.

Therefore, there are only two eigenvectors $\{\mathbf{f},\,\mathbf{f}^*\}$, corresponding to $\omega/c=\pm k=\pm\sqrt{k_1^2+k_2^2+k_3^2}$:

$$\mathbf{f}=\frac{\mathbf{k}\times(\boldsymbol{l}\times\mathbf{k})+ik\,(\mathbf{k}\times\boldsymbol{l})}{k\sqrt{2}\,|\mathbf{k}\times\boldsymbol{l}|},\qquad \mathbf{f}^*=\mathbf{f}|_{\mathbf{k}\to-\mathbf{k}}\,, \tag{3.41}$$

respectively [26]. Here, $\boldsymbol{l}$ is an arbitrary real vector that is not collinear to $\mathbf{k}$ ($\mathbf{k}\neq$constant $\boldsymbol{l}$) and $\mathbf{f}\cdot\mathbf{f}^*=1$. (A similar eigenvalue problem occurs in the mean magnetic field generation, called $\alpha\Omega$-dynamo, in cosmic astrophysics [78].)

Example. Let $\{\mathbf{e}_k\}_{k=1}^3$ be an orthonormal basis in $\mathbb{R}^3$. One can choose $\boldsymbol{l}=\mathbf{e}_1$ and $\mathbf{k}=k\mathbf{e}_3$. Then

$$\mathbf{f}=\frac{\mathbf{e}_1+i\mathbf{e}_2}{\sqrt{2}},\qquad \mathbf{f}^*=\frac{\mathbf{e}_1-i\mathbf{e}_2}{\sqrt{2}} \tag{3.42}$$

(see [26], [122], and [209] for more details).

3.6 Discrete Transformations and Polarization

The complex Maxwell equations in vacuum,

$$\frac{i}{c}\frac{\partial\mathbf{F}}{\partial t}=\operatorname{curl}\mathbf{F},\qquad \operatorname{div}\mathbf{F}=0, \tag{3.43}$$

are invariant under the following discrete transformations: spatial inversion **P:** $\mathbf{F}(\mathbf{r},t)\to\mathbf{F}^*(-\mathbf{r},t)$; time reversal **T**: $\mathbf{F}(\mathbf{r},t)\to\mathbf{F}^*(\mathbf{r},-t)$; and space-time inversion **PT**: $\mathbf{F}(\mathbf{r},t)\to$

$\mathbf{F}(-\mathbf{r},-t)$. They, together with the identity transformation, correspond to the four connected components of the Poincaré group. These transformations form the so-called Klein group $\{\mathbf{Identity},\mathbf{P},\mathbf{T},\mathbf{PT}\}$, with the following Cayley table.

*	I	P	T	PT
I	I	P	T	PT
P	P	I	PT	T
T	T	TP	I	P
PT	PT	T	P	I

The action of this group generates the following four circularly polarized waves ($\omega = +ck$):

$$\mathbf{F}_1 = \mathbf{f}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \mathbf{F}_2 = \mathbf{f}^* e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)} = \mathbf{F}_1^*|_{t\to -t} = \mathbf{TF}_1 \tag{3.44}$$

and

$$\mathbf{F}_3 = \mathbf{f}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \mathbf{F}_1|_{\mathbf{r}\to -\mathbf{r},t\to -t} = (\mathbf{PT})\,\mathbf{F}_1, \tag{3.45}$$
$$\mathbf{F}_4 = \mathbf{f}^* e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} = \mathbf{F}_3^*|_{t\to -t} = \mathbf{TF}_3 = \mathbf{PF}_1.$$

They represent right- and left-handed circularly polarized waves moving along the vector $\mathbf{k}$ in opposite directions. One can easily verify that the solutions $\{\mathbf{F}_1,\mathbf{F}_2\}$ correspond to $\lambda = +1$ and $\{\mathbf{F}_3,\mathbf{F}_4\}$ have $\lambda = -1$. Also, $\mathbf{F}_1\cdot\mathbf{F}_3 = \mathbf{f}^2 = 0$ and $\mathbf{F}_2\cdot\mathbf{F}_4 = \left(\mathbf{f}^2\right)^* = 0$. These four solutions are linearly independent.

Example. The standard circular, elliptic, and linear polarizations of the classical electromagnetic waves occur as a result of superposition of the complex solutions under consideration. With the help of the polarization vectors (3.42), one gets

$$\mathbf{F} = c_1\frac{\mathbf{F}_1+\mathbf{F}_3}{2} + c_2\frac{\mathbf{F}_1-\mathbf{F}_3}{2} = \mathbf{E}+i\mathbf{H} \tag{3.46}$$
$$= c_1\mathbf{e}_1\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - c_2\mathbf{e}_2\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)$$
$$+i\left[c_2\mathbf{e}_1\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega t-\frac{\pi}{2}\right) - c_1\mathbf{e}_2\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t-\frac{\pi}{2}\right)\right],$$

where $c_1 = c_1^*$ and $c_2 = c_2^*$. For the elliptic polarization, we choose $|c_1| > |c_2|$ or $|c_1| < |c_2|$; the linear polarization arises, for instance, if $c_1 \neq 0$ and $c_2 = 0$ (see [122] and [209], problems 2.128–2.134, for more details).

In conclusion, it is worth noting that, here, we have only discussed the classical electromagnetic field in vacuum. Different aspects of the "photon paradigm" are emphasized in [111]. The photon wave functions are dealt with in [6], [19], [23], [24], [57], [80], [187]. For quantization in the complex form, see [23], [25], [27], [26] and the references therein. (General quantization procedures are discussed, for example, in [27], [29], [110], [114], [118], [158], [159], [184], [185], [187], [189], [209].) Coherent states of light and dynamical invariants are reviewed in [53], [54], [55], [110], [189]. The squeezed states of the photons and atoms in a cavity and their relations with so-called "missing" solutions for the harmonic oscillator are analyzed in [119], [134], [136]. Professor Toptygin kindly pointed out an intrinsic importance of the helicity concept from the sub-atomic world (parity violation in beta decay [180], [234]) to cosmic astrophysics (possible amplification of galactic magnetic fields by the turbulent dynamo mechanism [17], [78], [213]). Last but not least, organic compounds appear often in the form of only one of two stereoisomers. As a result, in optically active biological substances, these molecules rotate polarized light to the left [93], thus creating another old unexplained puzzle.

Chapter 4

PAULI-LUBANSKI VECTOR AND RELATIVISTIC WAVE EQUATIONS

Inspired by the results from the previous chapter, we continue with a study of the role of the Pauli-Lubański vector for other major relativistic wave equations, including the Dirac, Weyl, Proca, Fierz-Pauli, and once again Maxwell equations. Different realizations of these equations are also given. The relativistic definition of spin is analyzed in this framework as a statement of the consistency of certain overdetermined systems of partial differential equations. Another important result that follows is a new linear relation related to Dirac's equation that is discussed here and introduced in the article [117] of which this chapter is based. This linear relation provides a means by which the spin of a Dirac particle can be introduced in a covariant form using the second Casimir operator of the Poincaré group.

Here it is reiterated that all physically interesting representations of the proper orthochronous inhomogeneous Lorentz group (known nowadays as the Poincaré group) were classified by Wigner [231] and, since then, this approach has been utilized for the mathematical description of mass and spin of an elementary particle. To this end, once again, the Pauli-Lubański pseudo-vector is used,

$$w_\mu = \frac{1}{2} e_{\mu\nu\sigma\tau} p^\nu M^{\sigma\tau}, \qquad p_\mu w^\mu = 0, \tag{4.1}$$

where p_μ is the relativistic linear momentum operator and $M^{\sigma\tau}$ are the angular momentum operators, or generators of the proper orthochronous Lorentz group, with summation over repeated indices. (We use Einstein summation convention unless stated otherwise.) The mass and spin of a particle are defined in terms of two quadratic invariants (Casimir operators of the Poincaré group) as follows

$$p^2 = p_\mu p^\mu = m^2, \qquad w^2 = w_\mu w^\mu = -m^2 s\,(s+1)\,, \tag{4.2}$$

when $m > 0$ (see, for example, [11], [12], [15], [29], [139], [140], [141], [181], [184], [185], [187], [194] and the references therein; throughout the chapter we use the standard notations in the Minkowski space-time $\mathbb{R}^4$ and the natural units $c = \hbar = 1$).

For the massless fields, when $m = 0$, one gets $w^2 = p^2 = pw = 0$, and the Pauli-Lubański vector should be proportional to p : [1]

$$w_\mu = \lambda p_\mu \tag{4.3}$$

(acting on common eigenstates [160], [184]). The number λ is sometimes called the helicity of the representation and the value $s = |\lambda|$ is called the spin of a particle with zero mass [29], [184], [185], [187]. Although the concept of helicity is discussed in most textbooks on quantum field theory, a practical implementation of this definition of the spin of a massless particle deserves a certain clarification. As is shown in the previous chapter, in the case of the electromagnetic field in vacuum, the sign of the constant λ in the latter equation is fixed, otherwise violating the classical Maxwell equations. Thus, for the photon, or a harmonic circular classical electromagnetic wave[2], the latter equation allows one to introduce the field equations and spin, but not the helicity, when a certain choice of eigenstates is required. A similar situation occurs in the case of Weyl's equation for massless neutrinos. (In Ref. [115], we do not discuss the equation for a graviton, another massless spin-2 particle; it will be analyzed elsewhere; cf. [108], [157], [159], [170] and our discussion in section 7.)

The theory of relativistic-invariant wave equations is studied, from different perspectives, in numerous classical accounts [12], [15], [22], [50], [51], [74], [75], [85], [113],

[1] This assumption was made by Bargmann and Wigner [12] for the massless limit of the spinor wave equation for particles with an arbitrary integer or half-integer spin proposed by Dirac [51] (see also [74], [75], [113], [169] and the references therein). The pseudo-vector (4.1) was introduced, in a slightly different notation, by Eqs. (4.a)–(4.b) of Ref. [12].

[2] Multiple meanings of the word "photon" are analyzed in [111].

[144], [169], [176], [228] (see also [27], [139], [140], [141] and the references therein). Nonetheless, in our opinion, the importance of the Pauli-Lubański vector for conventional relativistic equations, which allows one to derive all of them directly from the postulated transformation law of the corresponding classical field in pure group-theoretical terms, is not fully appreciated. In this chapter, we would like to start from Dirac's relativistic electron, or any free relativistic particle with a nonzero mass and spin $1/2$, which can be described by a bispinor wave function. Our analysis shows that an analog of the linear operator relation (4.3) takes the form,

$$w_\mu = \frac{1}{2}\left(p_\mu + m\gamma_\mu\right)\gamma_5, \tag{4.4}$$

provided that the Dirac equation, $\left(\gamma^\mu p_\mu - m\right)\psi = 0$, holds, when the corresponding overdetermined system of equations is consistent. This automatically implies that $s = 1/2$, in the covariant form, by definition (4.2). (We were not able to find the operator relation (4.4) in the extensive literature on Dirac's equation.)

In the rest of the chapter, a similar program is utilized, in a systematic way and from first principles, for other familiar relativistic wave equations. Once again, we postulate the transformation law of the field in question (a law of nature) and, with the help of the corresponding Lorentz generators, evaluate the action of the Pauli-Lubański vector on the field in order to compute, eventually, not one, but both Casimir operators (4.2). If a linear relation, similar to (4.3) or (4.4), does exist, one obtains an overdetermined PDE system, which can be reduced to the corresponding relativistic wave equation by a matrix version of Gaussian elimination [84]. We show that this approach allows one to derive equations of motion for the most useful classical fields, including the Weyl, Proca, Fierz-Pauli, and Maxwell equations in vacuum, as a statement of consistency for the original overdetermined systems. At the moment, we shall not discuss relativistic wave equations for particles with an arbitrary spin, such as the Bargmann-Wigner equations, Majorana equations, and/or the (first order)

Duffin-Kemmer equations which also describe spin-0 and spin-1 fields (see, for example, [15], [28], [90] and [21] for more details; the case of the Klein-Gordon equation, or the relativistic Schrödinger equation [186], is, of course, obvious).

We have also entirely concentrated on four dimensions. A more general group-theoretical approach to the relativistic wave equations, which allows one to include higher dimensions and spins, is formulated in [15] with the help of induced representations of the semi-direct products of separable, locally compact groups. Spinors in arbitrary dimensions are also discussed in [194].

4.1 Dirac Equation

In this section, for the reader's convenience, we summarize once again some basic facts about Dirac's equation and then discuss its relation with the Pauli-Lubański vector (4.4). To this end, a familiar bispinor representation of the proper orthochronous Lorentz group $SO_+(1,3)$ is used.

4.1.1 Gamma Matrices, Bispinors, and Transformation Laws

We shall use the following Dirac matrices: $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, $\gamma_\mu = g_{\mu\nu}\gamma^\nu = (\gamma^0, -\boldsymbol{\gamma})$, and $\gamma_5 = -\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, where

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{4.5}$$

and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the standard 2×2 Pauli matrices [168], [214]. The familiar anti-commutation relations,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0 \qquad (\mu, \nu = 0, 1, 2, 3), \tag{4.6}$$

hold. (Most of the results here will not depend on a particular choice of gamma matrices, but it is always useful to have an example in mind.) The four-vector notation, $x^\mu = (t, \mathbf{r})$,

$\partial_\mu = \partial / \partial x^\mu$, and $\partial^\alpha = g^{\alpha\mu}\partial_\mu$ in natural units $c = \hbar = 1$ with the standard metric, $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, in the Minkowski space-time $\mathbb{R}^4$ are utilized throughout the chapter [19], [28], [29], [157], [21].

In this notation, the transformation law of a bispinor wave function[3],

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4, \tag{4.7}$$

under a proper Lorentz transformation, is given by

$$\psi'(x') = S_\Lambda \psi(x), \qquad x' = \Lambda x, \tag{4.8}$$

together with the rule,

$$S_\Lambda^{-1}\gamma^\mu S_\Lambda = \Lambda^\mu{}_\nu \gamma^\nu, \tag{4.9}$$

for the sake of covariance of the celebrated Dirac equation,

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0. \tag{4.10}$$

As is well known, a general solution of the latter matrix equation has the form

$$S = S_\Lambda = \exp\left(-\frac{1}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right), \quad \theta_{\mu\nu} = -\theta_{\nu\mu}, \tag{4.11}$$
$$\Sigma^{\mu\nu} = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$$

(with summation over every two repeated indices; see, for example, [99], [158]) and, in turn,

$$S_\Lambda^{-1}\Sigma^{\mu\nu}S_\Lambda = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \Sigma^{\sigma\tau}. \tag{4.12}$$

[3]The relativistic wave equation for a massive spin $1/2$ particle was proposed by Dirac [50], when only tensor representations of the Lorentz group were known. Thus, the problem of covariance of Dirac's equation gave rise to a new class of representations of the Lorentz group, namely, the spinor representations [15].

In explicit form,

$$\Sigma^{\mu\nu} = \frac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}) = \begin{pmatrix} 0 & \alpha_q \\ -\alpha_p & -ie_{pqr}\Sigma_r \end{pmatrix} \tag{4.13}$$

$$= \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & 0 & -i\Sigma_3 & i\Sigma_2 \\ -\alpha_2 & i\Sigma_3 & 0 & -i\Sigma_1 \\ -\alpha_3 & -i\Sigma_2 & i\Sigma_1 & 0 \end{pmatrix},$$

where, by definition,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{4.14}$$

Their familiar product identities,

$$\Sigma_p\Sigma_q = ie_{pqr}\Sigma_r + \delta_{pq}, \qquad \alpha_p\alpha_q = ie_{pqr}\Sigma_r + \delta_{pq}, \tag{4.15}$$
$$\alpha_p\Sigma_q = \Sigma_p\alpha_q = ie_{pqr}\alpha_r + \delta_{pq}\gamma^5,$$

hold.

Setting $\boldsymbol{n} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for each of the unit vectors in the directions of the mutually orthogonal coordinate axes, one can write in compact form:

$$S_R = e^{i\theta(\boldsymbol{n}\cdot\boldsymbol{\Sigma})/2} = \cos\frac{\theta}{2} + i(\boldsymbol{n}\cdot\boldsymbol{\Sigma})\sin\frac{\theta}{2}, \qquad (\boldsymbol{n}\cdot\boldsymbol{\Sigma})^2 = I \tag{4.16}$$

and

$$S_L = e^{-\vartheta(\boldsymbol{n}\cdot\boldsymbol{\alpha})/2} = \cosh\frac{\vartheta}{2} - (\boldsymbol{n}\cdot\boldsymbol{\alpha})\sinh\frac{\vartheta}{2}, \qquad (\boldsymbol{n}\cdot\boldsymbol{\alpha})^2 = I \tag{4.17}$$

with $\tanh\vartheta = v$, in the cases of rotations and boosts, respectively [15], [158].

The important dual four-tensor identities,

$$ie_{\mu\nu\sigma\tau}\Sigma^{\sigma\tau} = 2\gamma_5\Sigma_{\mu\nu}, \qquad ie^{\mu\nu\sigma\tau}\gamma_5\Sigma_{\mu\nu} = 2\Sigma^{\sigma\tau}, \tag{4.18}$$

can be directly verified. Here, $e^{\mu\nu\sigma\tau} = -e_{\mu\nu\sigma\tau}$ and $e_{0123} = +1$ is the Levi-Civita symbol [29], [82].

For the conjugate bispinor,

$$\overline{\psi}(x) = \psi^{\dagger}(x)\,\gamma^{0}, \qquad \overline{\psi}'\left(x'\right) = \overline{\psi}(x)\,S_{\Lambda}^{-1}, \qquad x' = \Lambda x, \tag{4.19}$$

the Dirac equation (4.10) takes the form

$$i\partial_{\mu}\overline{\psi}\gamma^{\mu} + m\overline{\psi} = 0. \tag{4.20}$$

(For more details see classical accounts [6], [19], [28], [71], [99], [158], [173], [184], [187], [21], [226].)

Examples. In particular, for the boost in the plane $\left(x^{0}, x^{1}\right)$, when

$$S_L = e^{-(\vartheta/2)\Sigma^{01}} = e^{-(\vartheta/2)\alpha_1} = \cosh\frac{\vartheta}{2} - \alpha_1 \sinh\frac{\vartheta}{2} \tag{4.21}$$

with $\tanh\vartheta = v$, one can easily verify by matrix multiplication that

$$e^{(\vartheta/2)\alpha_1}\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix} e^{-(\vartheta/2)\alpha_1} = \begin{pmatrix}\cosh\vartheta & -\sinh\vartheta & 0 & 0\\ -\sinh\vartheta & \cosh\vartheta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}. \tag{4.22}$$

In a similar fashion, for the rotation in the plane $\left(x^{1}, x^{2}\right)$:

$$S_R = e^{-(\theta/2)\Sigma^{12}} = e^{i(\theta/2)\Sigma_3} = \cos\frac{\theta}{2} + i\Sigma_3 \sin\frac{\theta}{2} \tag{4.23}$$

and

$$e^{-i(\theta/2)\Sigma_3}\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix} e^{i(\theta/2)\Sigma_3} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}. \tag{4.24}$$

(See [15], [158] for further details.)

4.1.2 Generators and Commutators

In the fundamental representation of the proper orthochronous Lorentz group, we shall choose the following six 4×4 real-valued matrices ($\alpha, \beta = 0,1,2,3$ are fixed with no summation):

$$\Lambda\left(\theta_{\alpha\beta}\right) = \exp\left(-\theta_{\alpha\beta}\, m^{\alpha\beta}\right), \qquad m^{\alpha\beta} = -m^{\beta\alpha}, \tag{4.25}$$
$$\left(m^{\alpha\beta}\right)^{\mu}_{\ \nu} = g^{\alpha\mu}\delta^{\beta}_{\nu} - g^{\beta\mu}\delta^{\alpha}_{\nu}$$

for the corresponding one-parameter subgroups of rotations and boosts [29], [115], [157], [187]. Then, differentiation of a particular expression (for the corresponding tensor operator [15]),

$$e^{\theta_{\alpha\beta}\Sigma^{\alpha\beta}/2}\gamma^{\mu}e^{-\theta_{\alpha\beta}\Sigma^{\alpha\beta}/2} = \left(e^{-\theta_{\alpha\beta}m^{\alpha\beta}}\right)^{\mu}_{\ \nu}\gamma^{\nu}, \tag{4.26}$$

at $\theta_{\alpha\beta} = 0$ results in

$$\left[\Sigma^{\alpha\beta},\ \gamma^{\mu}\right] := \Sigma^{\alpha\beta}\gamma^{\mu} - \gamma^{\mu}\Sigma^{\alpha\beta} = 2\left(g^{\beta\mu}\gamma^{\alpha} - g^{\alpha\mu}\gamma^{\beta}\right), \tag{4.27}$$

which can be independently verified with the help of (4.6).

In a similar fashion, the action of four-angular momentum operators[4],

$$M^{\alpha\beta} = x^{\beta}\partial^{\alpha} - x^{\alpha}\partial^{\beta}, \qquad \partial^{\alpha} = g^{\alpha\kappa}\partial_{\kappa}, \tag{4.28}$$

on Dirac's bispinors (4.7) can be derived directly from the transformation law as follows

$$M^{\alpha\beta}\psi := -\left[\frac{d}{d\theta_{\alpha\beta}}\psi'\left(\Lambda^{\mu}_{\ \nu}\left(\theta_{\alpha\beta}\right)x^{\nu}\right)\right]\Bigg|_{\theta_{\alpha\beta}=0} \tag{4.29}$$
$$= -\left(\frac{d}{d\theta_{\alpha\beta}}e^{-\theta_{\alpha\beta}\Sigma^{\alpha\beta}/2}\right)\Bigg|_{\theta_{\alpha\beta}=0}\psi\left(x\right) = \frac{1}{2}\Sigma^{\alpha\beta}\psi$$

[4]We follow [115]. Traditionally, the four-angular momentum used here is replaced as follows $M^{\alpha\beta} \to -iM^{\alpha\beta}$, to make it Hermitian; see, for example, [29].

(see also [99] for a slightly different derivation). A familiar commutator,

$$\left[\Sigma^{\alpha\beta},\, \Sigma^{\sigma\tau}\right] = 2\left(g^{\beta\sigma}\Sigma^{\alpha\tau} - g^{\beta\tau}\Sigma^{\alpha\sigma} + g^{\alpha\tau}\Sigma^{\beta\sigma} - g^{\alpha\sigma}\Sigma^{\beta\tau}\right), \tag{4.30}$$

can be readily verified with the help of (4.27) and/or independently derived from (4.12). These results are independent of our choice of the gamma matrices representation.

4.1.3 Balance Conditions and Energy-Momentum Tensors

We shall use a familiar notation for the partial derivatives [29],

$$\mathscr{D}^p \psi(x) := \frac{\partial^{p_0+p_1+p_2+p_3}}{\partial x_0^{p_0} \partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3}} \psi(x_0, x_1, x_2, x_3), \qquad \mathscr{D}^0 \psi(x) := \psi(x) \tag{4.31}$$

where $p = (p_0, p_1, p_2, p_3)$ is an ordered set of non-negative integers $p_\mu \geq 0$. It follows from the Dirac equations (4.10) and (4.20) that

$$\partial_\mu \left[\left(\mathscr{D}^p \overline{\psi}(x) \right) \gamma^\mu \left(\mathscr{D}^q \psi(x) \right) \right] = 0, \tag{4.32}$$

or, for a finite multi-sum,

$$\partial_\mu \left[\sum_{p,q} c_{p,q} \mathscr{D}^p \overline{\psi} \gamma^\mu \mathscr{D}^q \psi \right] = 0, \tag{4.33}$$

which can be thought of as a "master" differential balance condition set.

Indeed, in view of $\mathscr{D}^r \partial_\mu = \partial_\mu \mathscr{D}^r$, one gets

$$i\gamma^\mu \partial_\mu \left(\mathscr{D}^q \psi \right) = m \left(\mathscr{D}^q \psi \right), \qquad i\partial_\mu \left(\mathscr{D}^p \overline{\psi} \right) \gamma^\mu = -m \left(\mathscr{D}^p \overline{\psi} \right). \tag{4.34}$$

Let us multiply the first (second) equation by $\mathscr{D}^p \overline{\psi}$ ($\mathscr{D}^q \psi$) from the left (right) and add the results. Then

$$\begin{aligned} i\partial_\mu \left(\mathscr{D}^p \overline{\psi} \gamma^\mu \mathscr{D}^q \psi \right) &= i\partial_\mu \left(\mathscr{D}^p \overline{\psi} \right) \gamma^\mu \mathscr{D}^q \psi + i\mathscr{D}^p \overline{\psi} \gamma^\mu \partial_\mu \left(\mathscr{D}^q \psi \right) \\ &= m \left(-\mathscr{D}^p \overline{\psi} \mathscr{D}^q \psi + \mathscr{D}^p \overline{\psi} \mathscr{D}^q \psi \right) \equiv 0. \end{aligned}$$

Among important special cases of (4.32) are the following important identities:

$$\partial_\mu j^\mu (x) = 0, \qquad j^\mu (x) = \overline{\psi}(x)\, \gamma^\mu \psi(x), \tag{4.35}$$

corresponding to the total charge conservation, and

$$\partial_\mu \left[\overline{\psi}(x)\, \gamma^\mu \partial_\nu \psi(x)\right] = 0, \qquad \partial_\mu \left[\partial_\nu \overline{\psi}(x)\, \gamma^\mu \psi(x)\right] = 0. \tag{4.36}$$

Therefore, one can introduce the energy-momentum tensor, such that $\partial_\mu T^\mu_\nu (x) = 0$, in two different forms

$$T^\mu_\nu := i\overline{\psi}\gamma^\mu \partial_\nu \psi, \qquad T^\mu_\mu = m\overline{\psi}\psi \tag{4.37}$$

and/or

$$T^\mu_\nu := \frac{i}{2}\left[\overline{\psi}\gamma^\mu \left(\partial_\nu \psi\right) - \left(\partial_\nu \overline{\psi}\right)\gamma^\mu \psi\right], \qquad T^\mu_\mu = m\overline{\psi}\psi. \tag{4.38}$$

As is well-known, all quantities of physical interest can be derived from the energy-momentum tensor [28], [187].

4.1.4 Variants of Dirac's Equation

In view of (4.10) and (4.6),

$$\begin{aligned} &i\gamma^\mu \gamma^\nu \partial_\nu \psi = m\gamma^\mu \psi, \\ &\gamma^\mu \gamma^\nu = \frac{1}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right) + \frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) = \Sigma^{\mu\nu} + g^{\mu\nu}, \end{aligned} \tag{4.39}$$

and, as a result, we obtain an overdetermined but very convenient form of Dirac's system:

$$i\left(\Sigma^{\mu\nu} + g^{\mu\nu}\right)\partial_\nu \psi = m\gamma^\mu \psi. \tag{4.40}$$

On the one hand, in a $3D$ "vector" form,

$$i\partial_0 \psi = \left(-i\boldsymbol{\alpha}\cdot\nabla + m\beta\right)\psi, \quad \beta = \gamma^0 \quad \left(\hbar = c = 1, \quad \partial_0 = \partial/\partial t\right) \tag{4.41}$$

and

$$-i\partial_0 \boldsymbol{\alpha}\psi + \left(\nabla \times \boldsymbol{\Sigma}\right)\psi = \left(i\nabla + m\boldsymbol{\gamma}\right)\psi. \tag{4.42}$$

It is worth noting that the latter vectorial equation in our overdetermined system (4.41)–(4.42) can be obtained by matrix multiplication, from the first one, in view of familiar relations:

$$\alpha_p \alpha_q = i e_{pqr} \Sigma_r + \delta_{pq}, \qquad \boldsymbol{\alpha} \beta = -\boldsymbol{\gamma}. \tag{4.43}$$

On the other hand, by letting $p_\mu = i\partial_\mu$, one gets

$$\left(\Sigma^{\mu\nu} + g^{\mu\nu}\right) p_\nu \psi = m \gamma^\mu \psi \tag{4.44}$$

and applying the momentum operator p_μ to the both sides:

$$p^2 \psi = \left(\Sigma^{\mu\nu} + g^{\mu\nu}\right) p_\mu p_\nu \psi = m \gamma^\mu p_\mu \psi, \tag{4.45}$$

in view of $\Sigma^{\mu\nu} p_\mu p_\nu = 0$. If $p^2 \psi = m^2 \psi$, we derive, once again, that $\gamma^\mu p_\mu \psi = m\psi$. Therefore both forms of the Dirac system, (4.10) and (4.40), are equivalent and every component of the bispinor (4.7) does satisfy the d'Alembert equation,

$$\left(\partial^\mu \partial_\mu + m^2\right) \psi = \left(\partial_{tt}^2 - \Delta + m^2\right) \psi = \left(\Box + m^2\right) \psi = 0, \tag{4.46}$$

as required by (4.2).

In a similar fashion, for the conjugate bispinor (4.19) one can obtain

$$i \partial_\nu \overline{\psi} \left(\Sigma^{\mu\nu} - g^{\mu\nu}\right) = m \overline{\psi} \gamma^\mu \tag{4.47}$$

and our equations (4.40) and (4.47) results in the following balance relation:

$$i \partial_\nu \left(\overline{\psi} \Sigma^{\mu\nu} \psi\right) + i \left[\overline{\psi} \partial^\mu \psi - \left(\partial^\mu \overline{\psi}\right) \psi\right] = 2m \overline{\psi} \gamma^\mu \psi, \tag{4.48}$$

which, in turn, implies a familiar conservation law, $\partial_\mu j^\mu (x) = 0$, for the four-current, $j^\mu (x) = \overline{\psi}(x) \gamma^\mu \psi(x)$, in view of $\partial_\mu \partial_\nu \left(\overline{\psi} \Sigma^{\mu\nu} \psi\right) \equiv 0$ and $\partial_\mu \left[\overline{\psi} \partial^\mu \psi - \left(\partial^\mu \overline{\psi}\right) \psi\right] \equiv 0$. (The latter equation gives a differential balance condition on its own.)

The following identity,

$$i \partial_\nu \left(\overline{\psi} \Sigma^{\mu\nu} \gamma^\lambda \psi\right) = 2i \overline{\psi} \gamma^\mu \partial^\lambda \psi - i \left[\overline{\psi} \gamma^\lambda \partial^\mu \psi - \left(\partial^\mu \overline{\psi}\right) \gamma^\lambda \psi\right], \tag{4.49}$$

can be obtained with the help of (4.40), (4.47) and (4.27). The latter shows how the difference between two forms of the energy-momentum tensor (4.37) and (4.38) can be written as the four-divergence of a given tensor. Moreover, one can write

$$T^{\mu\nu} := i\overline{\psi}\gamma^{\mu}\partial^{\nu}\psi = mg^{\mu\nu}\overline{\psi}\psi + m\overline{\psi}\Sigma^{\mu\nu}\psi - i\overline{\psi}\gamma^{\mu}\Sigma^{\nu\lambda}\partial_{\lambda}\psi \tag{4.50}$$

and

$$\begin{aligned} T^{\mu\nu} &:= \frac{i}{2}\left[\overline{\psi}\gamma^{\mu}\left(\partial^{\nu}\psi\right) - \left(\partial^{\nu}\overline{\psi}\right)\gamma^{\mu}\psi\right] \\ &= mg^{\mu\nu}\overline{\psi}\psi - \frac{i}{2}\left[\overline{\psi}\gamma^{\mu}\Sigma^{\nu\lambda}\partial_{\lambda}\psi + \left(\partial_{\lambda}\overline{\psi}\right)\Sigma^{\nu\lambda}\gamma^{\mu}\psi\right] \end{aligned} \tag{4.51}$$

in view of (4.40) and (4.47).

4.1.5 Covariance and Transformation of Generators

The relativistic invariance of the Dirac equation is a fundamental consequence of (4.8)–(4.10); see for example, [19], [28], [99], [158], [21] and [187]. Covariance of system (4.40) can be derived, in a similar fashion, by invoking (4.12). The details are left to the reader (see also section 5.3).

It is worth noting that from the four-tensor character of $\Sigma^{\mu\nu}$, in (4.12), follow the transformation laws for the generators of rotations and boosts. Let us assume that the velocity vector $\boldsymbol{v}$, for going over to a moving frame of reference, has the direction of one of the coordinate axes, say $\{\mathbf{e}_a\}_{a=1,2,3}$. Consider also "orthogonal decompositions", $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\|} + \boldsymbol{\Sigma}_{\perp}$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\|} + \boldsymbol{\alpha}_{\perp}$, in the corresponding parallel and perpendicular directions, respectively. Then, under the Lorentz transformation,

$$S_{\Lambda}^{-1}\boldsymbol{\Sigma}_{\|}S_{\Lambda} = \boldsymbol{\Sigma}_{\|}, \qquad S_{\Lambda}^{-1}\boldsymbol{\alpha}_{\|}S_{\Lambda} = \boldsymbol{\alpha}_{\|} \tag{4.52}$$

and

$$S_{\Lambda}^{-1}\boldsymbol{\Sigma}_{\perp}S_{\Lambda} = \frac{\boldsymbol{\Sigma}_{\perp} - i\left(\boldsymbol{v}\times\boldsymbol{\alpha}\right)}{\sqrt{1-v^2}}, \qquad S_{\Lambda}^{-1}\boldsymbol{\alpha}_{\perp}S_{\Lambda} = \frac{\boldsymbol{\alpha}_{\perp} - i\left(\boldsymbol{v}\times\boldsymbol{\Sigma}\right)}{\sqrt{1-v^2}}, \tag{4.53}$$

by analogy with the transformations of electromagnetic fields in classical electrodynamics [115], [116], [21], [209].

The corresponding invariants [158] are given by

$$I_1 = \Sigma_{\mu\nu}\Sigma^{\mu\nu} = -2\left(\boldsymbol{\alpha}^2 + \boldsymbol{\Sigma}^2\right) = -12 \tag{4.54}$$

and

$$I_2 = e_{\mu\nu\sigma\tau}\Sigma^{\mu\nu}\Sigma^{\sigma\tau} = -8i\boldsymbol{\alpha}\cdot\boldsymbol{\Sigma} = -2i\Sigma^{\mu\nu}\Sigma_{\mu\nu}\gamma_5 = -2iI_1\gamma_5 = 24i\gamma_5, \tag{4.55}$$

in view of the first identity (4.18). The invariant nature of the helicity and relativistic rotation of the particle spin [158] can be naturally explained from these transformations.

Example. If $\boldsymbol{v} = v\mathbf{e}_1$, we set $\boldsymbol{\Sigma}_{\|} = \Sigma_1$, $\boldsymbol{\Sigma}_{\perp} = \{\Sigma_2, \Sigma_3\}$ and $\boldsymbol{\alpha}_{\|} = \alpha_1$, $\boldsymbol{\alpha}_{\perp} = \{\alpha_2, \alpha_3\}$ for the boost S_L given by (4.21). By the transformation law (4.52), one gets $S^{-1}\Sigma_1 S = \Sigma_1$, $S^{-1}\alpha_1 S = \alpha_1$, which is evident. In view of (4.53), the following matrix identities,

$$S^{-1}\Sigma_2 S = \frac{\Sigma_2 + iv\alpha_3}{\sqrt{1-v^2}}, \qquad S^{-1}\Sigma_3 S = \frac{\Sigma_3 - iv\alpha_2}{\sqrt{1-v^2}} \tag{4.56}$$

and

$$S^{-1}\alpha_2 S = \frac{\alpha_2 + iv\Sigma_3}{\sqrt{1-v^2}}, \qquad S^{-1}\alpha_3 S = \frac{\alpha_3 - iv\Sigma_2}{\sqrt{1-v^2}}, \tag{4.57}$$

hold. Indeed, in the first relation,

$$\begin{aligned}
S^{-1}\Sigma_2 S &= \left(\cosh\frac{\vartheta}{2} + \alpha_1 \sinh\frac{\vartheta}{2}\right)\Sigma_2\left(\cosh\frac{\vartheta}{2} - \alpha_1\sinh\frac{\vartheta}{2}\right) \\
&= \Sigma_2\cosh^2\frac{\vartheta}{2} + (\alpha_1\Sigma_2 - \Sigma_2\alpha_1)\cosh\frac{\vartheta}{2}\sinh\frac{\vartheta}{2} - \alpha_1\Sigma_2\alpha_1\sinh^2\frac{\vartheta}{2} \\
&= \Sigma_2\cosh\vartheta + i\alpha_3\sinh\vartheta = \frac{\Sigma_2 + iv\alpha_3}{\sqrt{1-v^2}},
\end{aligned}$$

provided $\tanh\vartheta = v$. Verifications of the remaining identities are similar.

4.1.6 Hamiltonian and Energy Balance

The energy-momentum (density) four-vector is given by $T^{0\nu}$ and, in view of (4.50) and (4.41), one gets

$$T^{00} = i\overline{\psi}\gamma^0\partial^0\psi = m\overline{\psi}\psi - i\overline{\psi}\gamma^0\Sigma^{0p}\partial_p\psi \tag{4.58}$$
$$= \psi^\dagger\left(-i\boldsymbol{\alpha}\cdot\nabla + m\beta\right)\psi = \psi^\dagger H\psi.$$

Here,

$$i\partial_0\psi = H\psi, \qquad H = -i\boldsymbol{\alpha}\cdot\nabla + m\beta, \tag{4.59}$$

which presents a familiar Hamiltonian form of the Dirac equation. The differential balance equation take the form

$$\partial_0\left(\psi^\dagger H\psi\right) + \operatorname{div}\left(\psi^\dagger\boldsymbol{\alpha}H\psi\right) = 0, \tag{4.60}$$

where $\boldsymbol{\alpha}H = \nabla\times\boldsymbol{\Sigma} - i\nabla - m\boldsymbol{\gamma}$ in view of (4.43).

4.1.7 The Pauli-Lubański Vector and Dirac's Equation for a Free Particle

One can easily see that equation (4.40) is related to the Pauli-Lubański vector in view of the dual identities (4.18). Indeed,

$$\Sigma^{\mu\nu} = \frac{i}{2}e^{\mu\nu\sigma\tau}\left(\gamma_5\Sigma_{\sigma\tau}\right), \qquad \gamma_5\Sigma_{\sigma\tau} = \Sigma_{\sigma\tau}\gamma_5, \tag{4.61}$$

and

$$\frac{i}{2}e^{\mu\nu\sigma\tau}\gamma_5\Sigma_{\sigma\tau}p_\nu\psi = -g^{\mu\nu}\left(p_\nu - m\gamma_\nu\right)\psi.$$

By "index manipulations",

$$\frac{1}{2}e_{\mu\nu\sigma\tau}\left(i\partial^\nu\right)\left(-i\Sigma^{\sigma\tau}\right)\psi = \frac{1}{2}e_{\mu\nu\sigma\tau}\partial^\nu\Sigma^{\sigma\tau}\psi \tag{4.62}$$
$$= \gamma_5\left(i\partial_\mu - m\gamma_\mu\right)\psi = \left(i\partial_\mu + m\gamma_\mu\right)\gamma_5\psi$$

with the help of familiar properties of the gamma matrices, namely, $\gamma_5^2 = I$ and $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$. As a result, we arrive at the following equations,

$$\frac{1}{2}e_{\mu\nu\sigma\tau}\partial^\nu\left(\gamma^\sigma\gamma^\tau\psi\right) = \left(i\partial_\mu + m\gamma_\mu\right)\gamma_5\psi, \tag{4.63}$$

with summation over any two repeated indices.

The latter equation can also be obtained in view of (4.29), by letting $p_\mu = i\partial_\mu$ in operator relation (4.4). Once again, our goal is to emphasize that both overdetermined systems (4.40) and (4.63), which are related to the Pauli-Lubański vector, are equivalent to Dirac's equation in vacuum (4.10).

In components, by (4.63), for the rotations $\mathbf{\Sigma}$ and boosts $\boldsymbol{\alpha}$ the following standard equations hold

$$i\left(\nabla\cdot\mathbf{\Sigma}\right)\psi = \left(i\partial_0 + m\gamma_0\right)\gamma_5\psi \tag{4.64}$$

and

$$i\partial_0\mathbf{\Sigma}\psi - \left(\nabla\times\boldsymbol{\alpha}\right)\psi = \left(i\nabla - m\boldsymbol{\gamma}\right)\gamma_5\psi, \tag{4.65}$$

respectively. Once again, this system is overdetermined and by a proper matrix multiplication, each of the four equations (4.64)–(4.65) can be reduced to a single Dirac's equation. We leave the details to the reader.

4.1.8 Relativistic Definition of Spin for Dirac Particles

In view of (4.4) and (4.2), one gets

$$\begin{aligned} 4w_\mu w^\mu &= \left(p_\mu + m\gamma_\mu\right)\gamma_5\left(p^\mu + m\gamma^\mu\right)\gamma_5 \\ &= \left(p_\mu + m\gamma_\mu\right)\gamma_5^2\left(p^\mu - m\gamma^\mu\right) \\ &= p_\mu p^\mu - m^2\gamma_\mu\gamma^\mu = -3m^2, \end{aligned}$$

where $s\left(s+1\right) = 3/4$, or $s = 1/2$, in covariant form.

On the other hand, introducing the familiar generators $\boldsymbol{M} = (i/2)\boldsymbol{\Sigma}$ and $\boldsymbol{N} = (1/2)\boldsymbol{\alpha}$ for the rotations and boosts, respectively, one gets

$$\boldsymbol{N}^2 = -\boldsymbol{M}^2 = 3/4, \qquad \boldsymbol{M}\cdot\boldsymbol{N} = i\,(3/4)\,\gamma^5. \tag{4.66}$$

In view of $\boldsymbol{\Sigma}\gamma^5 = \gamma^5\boldsymbol{\Sigma} = \boldsymbol{\alpha}$, in the complex space of the bispinors under consideration, we arrive at

$$\boldsymbol{M}\pm i\boldsymbol{N} = \frac{i}{2}\,(1\mp\gamma_5)\,\boldsymbol{\Sigma} \tag{4.67}$$

and the Casimir operators of the proper orthochronous Lorentz group are given by $(\boldsymbol{M}\pm i\boldsymbol{N})^2/4 = -3\,(1\mp\gamma_5)^2/16 = -3\,(1\mp\gamma_5)/8$. Here, $\boldsymbol{M}^2 = -s\,(s+1) = -3/4$, which implies, once again, that the spin is equal to $1/2$ (we have chosen real-valued boost generators; see also [10], [12], [15], [29], [85], [181], [187] and [226] for more details on the Lorentz group representations).

Miscellaneous. In addition,

$$\frac{1}{2}e_{\mu\nu\sigma\tau}\partial^{\nu}\left(\gamma^{\mu}\gamma^{\sigma}\gamma^{\tau}\psi\right) = 3i\gamma^5\gamma_{\nu}\partial^{\nu}\psi = 3m\gamma^5\psi, \tag{4.68}$$

in view of familiar relations:

$$\gamma^5 = \frac{i}{4!}e_{\mu\nu\sigma\tau}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\tau}, \qquad \gamma^5\gamma_{\mu} = \frac{i}{3!}e_{\mu\nu\sigma\tau}\gamma^{\nu}\gamma^{\sigma}\gamma^{\tau} \tag{4.69}$$

(see, for example, [158]).

4.2 Weyl Equation for Massless Neutrinos

The complex unimodular matrix group $SL(2,\mathbb{C})$ is a two-fold universal covering group of the Lorentz group $SO(1,3)$ (see, for example, [15], [29], [85], [181]). In this section, we shall use this connection in order to analyze the two-component spinor field associated with Weyl's equation.

4.2.1 Rotations, Boosts, and their Generators

Let us consider the fundamental representation of $SL(2,\mathbb{C})$, namely, we take a spinor field,

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2, \tag{4.70}$$

and postulate the transformation law under the proper orthochronous Lorentz group as follows

$$\phi'(x') = S_\Lambda \phi(x), \qquad x' = \Lambda x, \qquad S_\Lambda = \exp\left(\frac{1}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right). \tag{4.71}$$

Explicitly, these transformations include rotations,

$$S_R = e^{i\theta(\boldsymbol{n}\cdot\boldsymbol{\sigma})/2} = \cos\frac{\theta}{2} + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}, \qquad (\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = I \tag{4.72}$$

about the coordinate axes $\boldsymbol{n} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and boosts,

$$S_L = e^{-\vartheta(\boldsymbol{n}\cdot\boldsymbol{\sigma})/2} = \cosh\frac{\vartheta}{2} - (\boldsymbol{n}\cdot\boldsymbol{\sigma})\sinh\frac{\vartheta}{2}, \qquad \boldsymbol{n} = \frac{\boldsymbol{v}}{v} \tag{4.73}$$

in the directions $\boldsymbol{n} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, respectively, when the familiar relations,

$$v = \tanh\vartheta, \qquad \cosh\vartheta = \frac{1}{\sqrt{1-v^2}}, \qquad \sinh\vartheta = \frac{v}{\sqrt{1-v^2}} \quad (c=1), \tag{4.74}$$

hold. (Here, σ_1, σ_2, σ_3 are the Pauli matrices with the products given by $\sigma_p\sigma_q = ie_{pqr}\sigma_r + \delta_{pq}$ and $\{\mathbf{e}_a\}_{a=1,2,3}$ is an orthonormal basis in $\mathbb{R}^3$.)

The action of the generators $M^{\alpha\beta} = x^\beta\partial^\alpha - x^\alpha\partial^\beta$ takes the form

$$\begin{aligned} M^{\alpha\beta}\phi(x) &:= -\left(\frac{d}{d\theta_{\alpha\beta}}\phi'(\Lambda x)\right)\bigg|_{\theta_{\alpha\beta}=0} \\ &= -\left(\frac{dS}{d\theta_{\alpha\beta}}\right)\bigg|_{\theta_{\alpha\beta}=0}\phi(x) = -\frac{1}{2}\Sigma^{\alpha\beta}\phi(x) \end{aligned} \tag{4.75}$$

for the corresponding one-parameter subgroups: $S_\Lambda = \exp\left(\theta_{\alpha\beta}\Sigma^{\alpha\beta}/2\right)$ $(\alpha, \beta = 0,1,2,3$ are fixed with no summation).

In block form, the generators of this spinor representation are given by[5]

$$\Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha} = \begin{pmatrix} 0 & -\sigma_q \\ \sigma_p & ie_{pqr}\sigma_r \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \sigma_1 & 0 & i\sigma_3 & -i\sigma_2 \\ \sigma_2 & -i\sigma_3 & 0 & i\sigma_1 \\ \sigma_3 & i\sigma_2 & -i\sigma_1 & 0 \end{pmatrix} \tag{4.76}$$

and the following self-duality identity holds

$$e_{\mu\nu\sigma\tau}\Sigma^{\sigma\tau} = 2i\Sigma_{\mu\nu} = 2ig_{\mu\sigma}g_{\nu\tau}\Sigma^{\sigma\tau}, \tag{4.77}$$

where

$$\Sigma_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}\Sigma^{\mu\nu} = \begin{pmatrix} 0 & \sigma_q \\ -\sigma_p & ie_{pqr}\sigma_r \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & i\sigma_3 & -i\sigma_2 \\ -\sigma_2 & -i\sigma_3 & 0 & i\sigma_1 \\ -\sigma_3 & i\sigma_2 & -i\sigma_1 & 0 \end{pmatrix} \tag{4.78}$$

stated here for the reader's convenience.

4.2.2 The Pauli-Lubański Vector and Weyl's Equation

Letting $\lambda = -1/2$ in (4.3), one gets

$$e_{\mu\nu\sigma\tau}\left(i\partial^{\nu}\right)\left(-iM^{\sigma\tau}\right)\phi = -\frac{1}{2}e_{\mu\nu\sigma\tau}\partial^{\nu}\Sigma^{\sigma\tau}\phi = -i\partial_{\mu}\phi \tag{4.79}$$

by (4.75). With the help of (4.77), we finally arrive at the following overdetermined system:

$$\Sigma^{\mu\nu}\partial_{\nu}\phi = \partial^{\mu}\phi, \tag{4.80}$$

which takes the explicit form,

$$-\left(\boldsymbol{\sigma}\cdot\nabla\right)\phi = \partial_0\phi, \tag{4.81}$$

$$\partial_0\boldsymbol{\sigma}\phi + i\left(\nabla\times\boldsymbol{\sigma}\right)\phi = -\nabla\phi.$$

[5] Another choice of the generators in the transformation laws (4.71)–(4.73), corresponds to $\vartheta \to -\vartheta$; see, for example, [187].

The latter vectorial equation can be obtained from the first one by matrix multiplication with the help of familiar products of the Pauli matrices. Moreover, in (4.3), only the value $\lambda = -1/2$ results in a consistent system, defining the spin as $s = |\lambda| = 1/2$.

Thus, the relativistic two-component Weyl equations for a massless particle with the spin $1/2$, namely, $\partial_0 \phi + (\boldsymbol{\sigma} \cdot \nabla)\, \phi = 0$, can be derived from the representation theory of the Poincaré group with the aid of the Pauli-Lubański vector[6].

4.2.3 Covariance

Equations (4.80) are covariant under a proper Lorentz transformation. In view of the laws (4.71) one gets

$$S_\Lambda^{-1} \Sigma^{\sigma\tau} S_\Lambda = \Lambda^{\sigma}_{\ \mu} \Lambda^{\tau}_{\ \nu} \Sigma^{\mu\nu} \tag{4.82}$$

and

$$\frac{1}{2}\left[\Sigma^{\alpha\beta},\, \Sigma^{\gamma\delta}\right] = g^{\alpha\gamma}\Sigma^{\beta\delta} - g^{\alpha\delta}\Sigma^{\beta\gamma} + g^{\beta\delta}\Sigma^{\alpha\gamma} - g^{\beta\gamma}\Sigma^{\alpha\delta}. \tag{4.83}$$

(cf. Sections 5.2–5.3).

On the other hand, with the help of the Pauli-Lubański vector, from (4.79) one can get,

$$\frac{1}{2} e_{\mu\nu\sigma\tau} \partial'^{\nu} \Sigma^{\sigma\tau} \phi' = i \partial'_\mu \phi', \tag{4.84}$$

in the new system of coordinates, when $\phi'(x') = S\phi(x)$ and $x' = \Lambda x$. Let us multiply both sides of the latter equation by $S^{-1}\Lambda^{\mu}_{\ \lambda}$ from the left and then use (4.82) together with the two convenient transformations,

$$S\partial_\lambda \phi = \Lambda^{\mu}_{\ \lambda} \partial'_\mu \phi', \qquad \partial'^{\nu} \phi' = \Lambda^{\nu}_{\ \kappa} S \partial^{\kappa} \phi, \tag{4.85}$$

[6]Some authors, see for example, [184], [185], suggest that an empirical condition is required in order to quantize the value of spin for a massless particle by (4.3). As we have just demonstrated, for Weyl's equation, it is a misconception. Indeed, the full relativistic analysis automatically includes the quantization rule of the corresponding spin and helicity as a consistency condition of the overdetermined system (4.80).

which can be readily verified with the aid of

$$\left(\Lambda^{-1}\right)^{\lambda}{}_{\nu}\Lambda^{\nu}{}_{\tau}=\delta^{\lambda}_{\tau}, \qquad \left(\Lambda^{-1}\right)^{\lambda}{}_{\nu}=g^{\lambda\sigma}\Lambda^{\mu}{}_{\sigma}g_{\mu\nu}. \tag{4.86}$$

As a result,

$$(\det\Lambda)\, e_{\lambda\kappa\rho\chi}\partial^{\kappa}\Sigma^{\rho\chi}\phi=i\partial_{\lambda}\phi, \tag{4.87}$$

in view of the following determinant identity [158]:

$$e_{\mu\nu\sigma\tau}\Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\kappa}\Lambda^{\sigma}{}_{\rho}\Lambda^{\tau}{}_{\chi}=(\det\Lambda)\, e_{\lambda\kappa\rho\chi}. \tag{4.88}$$

This consideration reveals the pseudo-vector character of the Pauli-Lubański operator.

Note. Weyl's equation (4.81) was originally introduced in [228] and then quickly rejected [169], being "resurrected" only after the discovery of parity violation in beta decay [180], [234]. (The experimentally detected oscillation among the different flavors of neutrinos leads us to believe that they are not massless after all [105], [149], [175].)

Let $\boldsymbol{v}$, the velocity vector of the moving frame of reference, lie along one of the coordinate axes, say $\{\mathbf{e}_a\}_{a=1,2,3}$. Also, let $\boldsymbol{\sigma}=\boldsymbol{\sigma}_{\|}+\boldsymbol{\sigma}_{\perp}$ be the corresponding "orthogonal decomposition" in parallel and perpendicular directions, respectively. These components transform as,

$$S_{\Lambda}^{-1}\boldsymbol{\sigma}_{\|}S_{\Lambda}=\boldsymbol{\sigma}_{\|}, \qquad S_{\Lambda}^{-1}\boldsymbol{\sigma}_{\perp}S_{\Lambda}=\frac{\boldsymbol{\sigma}_{\perp}-i\left(\boldsymbol{v}\times\boldsymbol{\sigma}\right)}{\sqrt{1-v^{2}}}, \tag{4.89}$$

under a Lorentz transformation, thus resembling the transformations of electromagnetic fields in classical electrodynamics [115], [116], [154], [21], [209].

Example. Let $\boldsymbol{v}=v\mathbf{e}_1$. Then $\boldsymbol{\sigma}_{\|}=\sigma_1$, $\boldsymbol{\sigma}_{\perp}=\{\sigma_2,\sigma_3\}$ and

$$S_L=e^{-(\vartheta/2)\sigma_1}=\cosh\frac{\vartheta}{2}-\sigma_1\sinh\frac{\vartheta}{2}, \qquad S_L^{-1}=e^{(\vartheta/2)\sigma_1}=\cosh\frac{\vartheta}{2}+\sigma_1\sinh\frac{\vartheta}{2}. \tag{4.90}$$

By the transformation law (4.89), one should get $S^{-1}\sigma_1 S=\sigma_1$, which is obvious, and

$$S^{-1}\sigma_2 S=\frac{\sigma_2+iv\sigma_3}{\sqrt{1-v^{2}}}, \qquad S^{-1}\sigma_3 S=\frac{\sigma_3-iv\sigma_2}{\sqrt{1-v^{2}}} \tag{4.91}$$

for the corresponding Lorentz boost. Let us directly verify, for instance, the first relation. Indeed,

$$\begin{aligned}
S^{-1}\sigma_2 S &= \left(\cosh\frac{\vartheta}{2}+\sigma_1\sinh\frac{\vartheta}{2}\right)\sigma_2\left(\cosh\frac{\vartheta}{2}-\sigma_1\sinh\frac{\vartheta}{2}\right)\\
&= \sigma_2\cosh^2\frac{\vartheta}{2}+(\sigma_1\sigma_2-\sigma_2\sigma_1)\cosh\frac{\vartheta}{2}\sinh\frac{\vartheta}{2}-\sigma_1\sigma_2\sigma_1\sinh^2\frac{\vartheta}{2}\\
&= \sigma_2\cosh\vartheta+i\sigma_3\sinh\vartheta=\frac{\sigma_2+iv\sigma_3}{\sqrt{1-v^2}},
\end{aligned}$$

provided that $\tanh\vartheta = v$. The proof of the last identity is similar.

As is well-known, under spatial rotations the set of three Pauli matrices $\boldsymbol{\sigma}$ transform as a $3D$ vector. For instance,

$$e^{-i(\theta/2)\sigma_3}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}e^{i(\theta/2)\sigma_3}=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}. \tag{4.92}$$

4.2.4 An Alternative Derivation

Denoting, $\sigma^\mu = (\sigma_0 = I, \sigma_1, \sigma_2, \sigma_3)$, one can rewrite Weyl's equation in a more familiar form [173], [187]:

$$\sigma^\mu \partial_\mu \phi = 0. \tag{4.93}$$

Then, under the Lorentz transformations,

$$S_\Lambda^\dagger \sigma^\mu S_\Lambda = \Lambda^\mu_{\ \nu}\sigma^\nu \tag{4.94}$$

and

$$\left(\Sigma^{\alpha\beta}\right)^\dagger\sigma^\mu+\sigma^\mu\Sigma^{\alpha\beta}=2\left(g^{\beta\mu}\sigma^\alpha-g^{\alpha\mu}\sigma^\beta\right), \tag{4.95}$$

which also implies the covariance[7].

[7]Relations of this spinor representation with Maxwell's equations are discussed in section 6.2.

Examples. In particular, one can easily verify that

$$e^{-(\vartheta/2)\sigma_1}\begin{pmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}e^{-(\vartheta/2)\sigma_1}=\begin{pmatrix}\cosh\vartheta & -\sinh\vartheta & 0 & 0\\ -\sinh\vartheta & \cosh\vartheta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}, \tag{4.96}$$

with $\tanh\vartheta = v$, and

$$e^{-i(\theta/2)\sigma_3}\begin{pmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}e^{i(\theta/2)\sigma_3}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} \tag{4.97}$$

for the corresponding boost and rotation, respectively.

4.3 Proca Equation

The fundamental four-vector representation of the proper orthochronous Lorentz group $SO_+(1,3)$ is related to the relativistic wave equation for a massive particle with spin 1.

4.3.1 Massive Vector Field

The relativistic equation of motion for a real or complex four-vector field $A^\mu = \left(A^0, \mathbf{A}\right)$ with a positive mass $m > 0$ can be derived in a natural way with the help of the Pauli-Lubański vector. By definition,

$$\begin{aligned} w_\mu A^\alpha &= \frac{1}{2}e_{\mu\nu\sigma\tau}\partial^\nu M^{\sigma\tau}A^\alpha = \frac{1}{2}e_{\mu\nu\sigma\tau}\partial^\nu\left(\left(m^{\sigma\tau}\right)^\alpha_\rho A^\rho\right) \\ &= \frac{1}{2}e_{\mu\nu\sigma\tau}\partial^\nu\left(g^{\sigma\alpha}A^\tau - g^{\tau\alpha}A^\sigma\right) = -g^{\alpha\nu}e_{\mu\nu\sigma\tau}\partial^\sigma A^\tau, \end{aligned} \tag{4.98}$$

where the matrix form of the generators (4.25) has been used. The standard decomposition,

$$\partial^\mu A^\nu = \frac{1}{2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \frac{1}{2}\left(\partial^\mu A^\nu + \partial^\nu A^\mu\right),$$

followed by the dual tensor relation,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad e_{\mu\nu\sigma\tau} F^{\sigma\tau} = -2G_{\mu\nu}, \tag{4.99}$$

gives the explicit action of the Pauli-Lubański operator on the four-vector field:

$$w_\mu A^\alpha = g^{\alpha\nu} G_{\mu\nu} = g_{\mu\nu} G^{\nu\alpha}, \qquad w_\mu A_\alpha = G_{\mu\alpha}. \tag{4.100}$$

It is worth noting that this results in a second rank four-tensor.

In a similar fashion, for the squared operator,

$$w^2 A^\alpha = w^\mu \left(w_\mu A^\alpha \right) = \frac{1}{2} e_{\mu\nu\sigma\tau} \partial^\nu \left(M^{\sigma\tau} G^{\mu\alpha} \right). \tag{4.101}$$

But, in view of (4.7) of [115],

$$M^{\sigma\tau} G^{\mu\alpha} = g^{\sigma\mu} G^{\tau\alpha} - g^{\tau\mu} G^{\sigma\alpha} + g^{\sigma\alpha} G^{\mu\tau} - g^{\tau\alpha} G^{\mu\sigma}, \tag{4.102}$$

and with the help of a companion dual tensor identity, $e^{\mu\nu\sigma\tau} G_{\sigma\tau} = 2F^{\mu\nu}$, looking for a spin-1 particle one gets

$$w^2 A^\alpha = g^{\sigma\alpha} \partial^\nu \left(e_{\nu\sigma\mu\tau} G^{\mu\tau} \right) = 2 g^{\sigma\alpha} \partial^\nu \left(F_{\nu\sigma} \right) = 2 \partial_\nu F^{\nu\alpha} = -2m^2 A^\alpha$$

as a consequence of (4.2). As a result, we have arrived at the Proca equation for a vector particle with a finite mass [176],

$$\partial_\nu F^{\nu\mu} + m^2 A^\mu = 0, \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{4.103}$$

directly from the representation theory of the Poincaré group. In view of $w^2 = -m^2 s(s+1)$ $= -2m^2$, one concludes that the spin of the particle is equal to one.

Moreover,

$$\partial_\mu A^\mu = 0, \qquad \Box A^\mu = -m^2 A^\mu, \quad m > 0, \tag{4.104}$$

in view of

$$0 \equiv \partial_\mu \partial_\nu F^{\nu\mu} = -m^2 \partial_\mu A^\mu \tag{4.105}$$

and

$$\partial_\nu F^{\nu\mu} = \partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) = \partial_\nu \partial^\nu A^\mu = -m^2 A^\mu. \tag{4.106}$$

The massless case of the Proca equation, $m = 0$, reveals a gauge invariance. If $A_\mu \to A'_\mu = A_\mu + \partial_\mu f$, then $F'_{\mu\nu} = \partial_\mu \left(A_\nu + \partial_\nu f\right) - \partial_\nu \left(A_\mu + \partial_\mu f\right) = F_{\mu\nu}$.

4.3.2 An Alternative "Bispinor" Derivation

Let us consider a second rank bispinor of the form[8] $Q = A_\lambda \gamma^\lambda = A^\lambda \gamma_\lambda$, where $\left\{\gamma^\lambda\right\}_{\lambda=0,1,2,3}$ are the standard gamma matrices. We shall use the following transformation law for a proper Lorentz transformation,

$$Q'\left(x'\right) = S_\Lambda Q\left(x\right) S_\Lambda^{-1}, \qquad x' = \Lambda x, \tag{4.107}$$

where the matrix S_Λ is given by (4.11), as in the case of Dirac's equation. Then A^λ must be a four-vector. Indeed,

$$Q' = A_\lambda \delta^\lambda_\sigma \left(S\gamma^\sigma S^{-1}\right) = \left(g^{\tau\lambda} A_\lambda \Lambda^\nu_{\ \tau}\right) g_{\mu\nu} \left[\Lambda^\mu_{\ \sigma}\left(S\gamma^\sigma S^{-1}\right)\right] = A'^\nu \gamma_\nu, \tag{4.108}$$

$$A'^\nu = \Lambda^\nu_{\ \tau} A^\tau,$$

in view of an "inversion" of (4.9), $\Lambda^\mu_{\ \sigma}\left(S\gamma^\sigma S^{-1}\right) = \gamma^\mu$, and the familiar property: $A_\mu B^\mu =$ invariant or

$$g_{\mu\nu} \Lambda^\mu_{\ \sigma} \Lambda^\nu_{\ \tau} g^{\tau\lambda} = \delta^\lambda_\sigma. \tag{4.109}$$

In this "bispinor representation", the action of generators of the corresponding one-parameter subgroups is given by

$$M^{\alpha\beta} Q := -\left(\frac{d}{d\theta_{\alpha\beta}} Q'\left(\Lambda x\right)\right)\Bigg|_{\theta_{\alpha\beta}=0} = \frac{1}{2}\left(\Sigma^{\alpha\beta} Q - Q\Sigma^{\alpha\beta}\right). \tag{4.110}$$

By letting $Q = \gamma^\lambda A_\lambda$, one gets:

$$M^{\alpha\beta} Q = \left(g^{\beta\lambda}\gamma^\alpha - g^{\alpha\lambda}\gamma^\beta\right) A_\lambda,$$

[8] This is the so-called "Feynman slash" notation from quantum electrodynamics [71].

in view of the familiar commutator (4.27). As a result, we obtain $w_{\mu}Q = G_{\mu\tau}\gamma^{\tau}$ for the action of the Pauli-Lubański operator on the bispinor Q, which also follows from (4.100). One gets, in a similar fashion, that

$$w^2 Q = \frac{1}{2} e_{\mu\nu\sigma\tau} \partial^{\nu} \left(M^{\sigma\tau} G^{\mu\lambda} \right) \gamma_{\lambda}, \tag{4.111}$$

where equation (4.102) holds, once again, in view of the transformation law of the four-tensor $G^{\mu\lambda}$. As a result, Proca's equation follows.

4.3.3 Maxwell's Equations vs Proca Equation

For the real-valued vector potential A^{μ} and $m = 0$, the Proca equation (4.103) is reduced to the Maxwell equations in vacuum. Indeed, in view of the dual relation,

$$6\partial_{\nu} G^{\mu\nu} = -e^{\mu\nu\sigma\tau} \left(\partial_{\nu} F_{\sigma\tau} + \partial_{\sigma} F_{\tau\nu} + \partial_{\tau} F_{\nu\sigma} \right) = 0, \tag{4.112}$$

both pairs of Maxwell's equations can be written together in the following complex form

$$\partial_{\nu} Q^{\mu\nu} = 0, \qquad Q^{\mu\nu} = F^{\mu\nu} - \frac{i}{2} e^{\mu\nu\sigma\tau} F_{\sigma\tau}, \tag{4.113}$$

with the help of a self-dual complex four-tensor [27], [115]:

$$2iQ^{\mu\nu} = e^{\mu\nu\sigma\tau} Q_{\sigma\tau}, \quad e_{\mu\nu\sigma\tau} Q^{\sigma\tau} = 2iQ_{\mu\nu}. \tag{4.114}$$

The corresponding overdetermined system of Maxwell's equations in vacuum, which is related to the Pauli-Lubański vector, is investigated in [115]; see equation (5.2) there and section 5.5 below. (Two different spinor forms of Maxwell's equation will be discussed in section 6.)

4.4 Complex Vector Field

Finally, we would like to discuss the fundamental representation for the complex orthogonal group $SO(3, \mathbb{C})$ in connection with Maxwell's equations in vacuum.

4.4.1 Vector Covariant Form

As is well-known, the transformation laws of the complex electromagnetic field $\mathbf{F}(\boldsymbol{r},t) = \mathbf{E} + i\mathbf{H} \in \mathbb{C}^3$ in vacuum can be written in terms of $SO(3,\mathbb{C})$ rotations. In addition to the standard rotations of the frame of reference, the Lorentz transformations are equivalent to rotations through imaginary angles thus preserving the relativistic invariant $\mathbf{F}^2 = \mathbf{E}^2 + \mathbf{H}^2 + 2i\mathbf{E}\cdot\mathbf{H}$, which can be thought of as a "complex length" of this vector [122], [154]. In these transformations, $\mathbf{F}'(x') = S_\Lambda \mathbf{F}(x)$, $x' = \Lambda x$, (or $F'_p(x') = a_{pq}F_q(x)$ for a given complex orthogonal matrix), one can choose $S_R = e^{-\omega(\mathbf{n}\cdot\mathbf{s})}$ and $S_L = e^{-i\upsilon(\mathbf{n}\cdot\mathbf{s})}$ for the rotations and boosts, respectively. Here, $\mathbf{n} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, when $\{\mathbf{e}_a\}_{a=1,2,3}$ is an orthonormal basis in $\mathbb{R}^3$, and s_1, s_2, s_3 are the real-valued spin matrices [214]:

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.115}$$

such that $[s_p, s_q] = s_p s_q - s_q s_p = e_{pqr} s_r$ and $s_1^2 + s_2^2 + s_3^2 = -2$. (In this representation the matrices $\boldsymbol{M} = \mathbf{s}$ and $\boldsymbol{N} = i\mathbf{s}$ obey the commutation law of the generators of the proper orthochronous Lorentz group.)

It can be directly verified that the corresponding generators,

$$\Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha} = \begin{pmatrix} 0 & is_q \\ -is_p & e_{pqr}s_r \end{pmatrix} = \begin{pmatrix} 0 & is_1 & is_2 & is_3 \\ -is_1 & 0 & s_3 & -s_2 \\ -is_2 & -s_3 & 0 & s_1 \\ -is_3 & s_2 & -s_1 & 0 \end{pmatrix}, \tag{4.116}$$

form a self-dual four-tensor

$$2i\Sigma^{\mu\nu} = e^{\mu\nu\sigma\tau}\Sigma_{\sigma\tau}, \quad e_{\mu\nu\sigma\tau}\Sigma^{\sigma\tau} = 2i\Sigma_{\mu\nu}. \tag{4.117}$$

As a result, in this realization, $M^{\alpha\beta}\mathbf{F} = m^{\alpha\beta}\mathbf{F}$ and, in view of (4.3) and (4.117), we arrive at the set of overdetermined equations:

$$\left(\Sigma^{\mu\nu}+g^{\mu\nu}\right)\partial_{\nu}\mathbf{F}=0, \tag{4.118}$$

where

$$S_{\Lambda}^{-1}\Sigma^{\mu\nu}S_{\Lambda}=\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}\Sigma^{\sigma\tau}, \qquad x'^{\mu}=\Lambda^{\mu}{}_{\sigma}x^{\sigma}, \qquad \mathbf{F}'\left(x'\right)=S_{\Lambda}\mathbf{F}\left(x\right) \tag{4.119}$$

under a proper Lorentz transformation. Once again, the latter equations, that, as we shall see later, determine the complete dynamics of the electromagnetic field in vacuum, are obtained here by a pure group-theoretical consideration up to an undetermined sign of the fixed constant in (4.3).

4.4.2 Commutators

For a one-parameter transformation in the plane (α,β), when

$$\Lambda\left(\theta_{\alpha\beta}\right)=\exp\left(-\theta_{\alpha\beta}\,m^{\alpha\beta}\right), \qquad \left(m^{\alpha\beta}\right)^{\mu}_{\ \nu}=g^{\alpha\mu}\delta^{\beta}_{\nu}-g^{\beta\mu}\delta^{\alpha}_{\nu} \tag{4.120}$$

and

$$S_{\Lambda}=\exp\left(-\theta_{\alpha\beta}\Sigma^{\alpha\beta}\right), \qquad \theta_{\alpha\beta}=-\theta_{\beta\alpha} \tag{4.121}$$

$(\alpha,\beta=0,1,2,3$ are fixed), one gets

$$e^{\theta_{\alpha\beta}\Sigma^{\alpha\beta}}\Sigma^{\mu\nu}e^{-\theta_{\alpha\beta}\Sigma^{\alpha\beta}}=\Lambda^{\mu}{}_{\sigma}\left(\theta_{\alpha\beta}\right)\Lambda^{\nu}{}_{\tau}\left(\theta_{\alpha\beta}\right)\Sigma^{\sigma\tau}. \tag{4.122}$$

The differentiation $\left.\left(d/d\theta_{\alpha\beta}\right)\right|_{\theta_{\alpha\beta}=0}$, results in a familiar law,

$$\begin{aligned}\left[\Sigma^{\alpha\beta},\,\Sigma^{\mu\nu}\right] &:= \Sigma^{\alpha\beta}\Sigma^{\mu\nu}-\Sigma^{\mu\nu}\Sigma^{\alpha\beta} \\ &= g^{\alpha\nu}\Sigma^{\beta\mu}-g^{\beta\nu}\Sigma^{\alpha\mu}+g^{\alpha\mu}\Sigma^{\beta\nu}-g^{\beta\mu}\Sigma^{\alpha\nu},\end{aligned} \tag{4.123}$$

which can be directly verified in components with the help of commutators of the spin matrices (4.115) that form the four-tensor (4.116).

4.4.3 Lorentz Invariance

With the help of (4.118)–(4.119), under a Lorentz transformation,

$$S^{-1}\left[\left(\Sigma'^{\mu\nu}+g'^{\mu\nu}\right)\partial'_{\nu}\mathbf{F}'\left(x'\right)\right]=0, \tag{4.124}$$

where $\Sigma'^{\mu\nu}=\Sigma^{\mu\nu}=\text{inv}$ and $g'^{\mu\nu}=g^{\mu\nu}=\text{inv}$. Then

$$\begin{aligned} 0 &= \left[S^{-1}\left(\Sigma^{\mu\nu}+g^{\mu\nu}\right)S\right]S^{-1}\partial'_{\nu}\mathbf{F}'\left(x'\right) \\ &= \Lambda^{\mu}_{\ \sigma}\left(\Sigma^{\sigma\tau}+g^{\sigma\tau}\right)S^{-1}\left[\Lambda^{\nu}_{\ \tau}\partial'_{\nu}\mathbf{F}'\left(x'\right)\right] \\ &= \Lambda^{\mu}_{\ \sigma}\left(\Sigma^{\sigma\tau}+g^{\sigma\tau}\right)S^{-1}S\partial_{\tau}\mathbf{F}\left(x\right) \end{aligned} \tag{4.125}$$

by $\Lambda^{\nu}_{\ \tau}\partial'_{\nu}\mathbf{F}'\left(x'\right)=\partial_{\tau}\mathbf{F}\left(x\right)$. Thus, our equation $\left(\Sigma^{\mu\nu}+g^{\mu\nu}\right)\partial_{\nu}\mathbf{F}\left(x\right)=0$ preserves its covariant form for all real and complex rotations $S\in SO\left(3,\mathbb{C}\right)$.

4.4.4 Vector Covariant Form vs Traditional Form of Maxwell's Equations

A vector form of (4.118) is given by

$$\left(\nabla\cdot\mathbf{s}\right)\mathbf{F}=i\partial_0\mathbf{F} \tag{4.126}$$

and

$$\partial_0\mathbf{s}\,\mathbf{F}+i\left(\nabla\times\mathbf{s}\right)\mathbf{F}=i\nabla\mathbf{F}. \tag{4.127}$$

As it has been pointed out in [115], equation (4.126) implies the complex Maxwell equation, $\operatorname{curl}\mathbf{F}=i\partial_0\mathbf{F}$. The first component of (4.127) is given by $\partial_0 s_1\mathbf{F}+i\left(\partial_2 s_3-\partial_3 s_2\right)\mathbf{F}=$

$i\partial_1 \mathbf{F}$, or

$$\partial_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + i\partial_2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$-i\partial_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = i\partial_1 \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

and

$$-i\partial_2 F_2 - i\partial_3 F_3 = i\partial_1 F_1,$$

$$-\partial_0 F_3 + i\partial_2 F_1 = i\partial_1 F_2,$$

$$\partial_0 F_2 + i\partial_3 F_1 = i\partial_1 F_3.$$

As a result, $\operatorname{div}\mathbf{F} = 0$ and $(\operatorname{curl}\mathbf{F})_{2,3} = i\partial_0 (\mathbf{F})_{2,3}$. (Cyclic permutations of the spatial indices cover the two remaining cases.)

4.4.5 An Alternative Form of Maxwell's Equations

Complex covariant form of Maxwell's equations, which was introduced in [128] (see also [115]) , can be presented as follows

$$(\partial_0, \partial_1, \partial_2, \partial_3) \begin{pmatrix} 0 & -F_1 & -F_2 & -F_3 \\ F_1 & 0 & iF_3 & -iF_2 \\ F_2 & -iF_3 & 0 & iF_1 \\ F_3 & iF_2 & -iF_1 & 0 \end{pmatrix} = (j_0, j_1, j_2, j_3), \tag{4.128}$$

or $\partial Q = j$, by matrix multiplication. Here, $\partial = (\partial_0, \partial_1, \partial_2, \partial_3)$, $j = (j_0, j_1, j_2, j_3)$, and $\mathbf{J} = (J_1, J_2, J_3)$ such that

$$Q \;=\; \mathbf{F}\cdot\mathbf{J} = F_1 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \tag{4.129}$$

$$+F_2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} + F_3 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices are transformed as a dual complex vector in $\mathbb{C}^3$,

$$\Lambda J_p \Lambda^T = a_{qp} J_q, \tag{4.130}$$

under a proper Lorentz transformation. In infinitesimal form,

$$m^{\alpha\beta}\mathbf{J} + \mathbf{J}\left(m^{\alpha\beta}\right)^T = \left(\Sigma^{\alpha\beta}\right)^T \mathbf{J}, \tag{4.131}$$

which gives an alternative representation of the group $SO(3,\mathbb{C})$ in a subspace of complex 4×4 matrices. (Details are left to the reader.)

As a result, $\Lambda Q \Lambda^T = Q'$ and, in "new" coordinates, $\partial' Q' = j'$, provided that $\partial' = \partial\left(\Lambda^{-1}\right)$ and $j' = j\Lambda^T$.

4.5 On Spinor Forms of Maxwell's Equations

In conclusion, the complex matrix group $SL(2,\mathbb{C})$ has a representation of the proper orthochronous Lorentz group $SO_+(1,3)$ by the second rank spinors.

4.5.1 Spinor Covariant Form

The complex electromagnetic field in vacuum, $\mathbf{F} = \mathbf{E} + i\mathbf{H}$, can also be written in a familiar form of the following 2×2 matrix:

$$Q = \boldsymbol{\sigma} \cdot \mathbf{F} = \sigma_1 F_1 + \sigma_2 F_2 + \sigma_3 F_3 = \begin{pmatrix} F_3 & F_1 - iF_2 \\ F_1 + iF_2 & -F_3 \end{pmatrix}, \tag{4.132}$$

where σ_1, σ_2, σ_3 are the standard Pauli matrices. The corresponding transformation law,

$$Q'(x') = S_\Lambda Q(x) S_\Lambda^{-1}, \qquad x' = \Lambda x, \qquad S_\Lambda = \exp\left(\frac{1}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right), \tag{4.133}$$

under a proper Lorentz transformation preserves two invariants $\mathrm{tr}Q = 0$ and $\det Q = -\mathbf{F}^2$. Here, $S_\Lambda \sigma_p S_\Lambda^{-1} = a_{pq}\sigma_q$ and $F'_p(x') = a_{pq}F_q(x)$, with $a_{qr}a_{qs} = \delta_{rs}$ for a given complex orthogonal 3×3 matrix (see section 5.1).

In this "spinor" representation, one gets

$$M^{\alpha\beta}Q = -\left(\frac{d}{d\theta_{\alpha\beta}} Q'(\Lambda x)\right)\bigg|_{\theta_{\alpha\beta}=0} = -\frac{1}{2}\left(\Sigma^{\alpha\beta}Q - Q\Sigma^{\alpha\beta}\right) \tag{4.134}$$

for generators of the one-parameter subgroups. Here, as in the case of Weyl's equation, the matrices $\Sigma^{\alpha\beta}$ are given by (4.76), but now, in view of (4.134), equation (4.3) with $\lambda = -1$ takes the form,

$$\frac{1}{2}\left(\Sigma_{\mu\nu}\partial^\nu Q - \partial^\nu Q \Sigma_{\mu\nu}\right) = \partial_\mu Q, \tag{4.135}$$

when the self-duality property (4.77) is applied.

Equations (4.135), obtained with the aid of the Pauli-Lubański vector, are equivalent to the system of complex Maxwell equations in vacuum,

$$\mathrm{div}\,\mathbf{F} = 0, \qquad \mathrm{curl}\,\mathbf{F} = i\partial_0 \mathbf{F}. \tag{4.136}$$

Indeed, when $\mu = 0$, with the help of (4.78) one gets

$$-\frac{1}{2}\left(\sigma_p \partial_p Q - \partial_p Q \sigma_p\right) = \partial_0 Q,$$

or

$$-\left(\sigma_p\sigma_q-\sigma_q\sigma_p\right)\partial_p F_q=2\partial_0\left(\sigma_r F_r\right),$$

which gives the second complex Maxwell equation (4.136) in view of the commutation relation, $\left[\sigma_p,\sigma_q\right]=2ie_{pqr}\sigma_r$.

When $\mu=p=1,2,3$, in a similar fashion,

$$2\partial_p Q=\partial_0 Q\sigma_p-\sigma_p\partial_0 Q+ie_{pqr}\left(\partial_q Q\sigma_r-\sigma_r\partial_q Q\right), \tag{4.137}$$

and letting $Q=F_s\sigma_s$, we obtain

$$\begin{aligned}2\partial_p\left(F_s\sigma_s\right)=&\left(\sigma_s\sigma_p-\sigma_p\sigma_s\right)\partial_0 F_s\\&+ie_{pqr}\left(\sigma_s\sigma_r-\sigma_r\sigma_s\right)\partial_q F_s.\end{aligned}$$

Evaluation of the commutators,

$$\sigma_s\partial_p F_s=ie_{spl}\sigma_l\partial_0 F_s+e_{pqr}e_{slr}\sigma_l\partial_q F_s, \tag{4.138}$$

and a familiar identity,

$$e_{pqr}e_{slr}=\begin{vmatrix}\delta_{ps} & \delta_{pl}\\ \delta_{qs} & \delta_{ql}\end{vmatrix}=\delta_{ps}\delta_{ql}-\delta_{pl}\delta_{qs}, \tag{4.139}$$

result in the system of Maxwell's equations (4.136).

4.5.2 Traditional Spinor Form of Maxwell's Equations

Equations (4.135), that are obtained here with the help of the Pauli-Lubański vector, give an alternative (vacuum) version of the spinor form of Maxwell's equations,

$$\left(\partial_0+\boldsymbol{\sigma}\cdot\nabla\right)\left(\boldsymbol{\sigma}\cdot\mathbf{F}\right)=j_0+\boldsymbol{\sigma}\cdot\mathbf{j}, \tag{4.140}$$

or, explicitly,

$$\begin{pmatrix} \partial_0+\partial_3 & \partial_1-i\partial_2 \\ \partial_1+i\partial_2 & \partial_0-\partial_3 \end{pmatrix} \begin{pmatrix} F_3 & F_1-iF_2 \\ F_1+iF_2 & -F_3 \end{pmatrix} = \begin{pmatrix} j_0+j_3 & j_1-ij_2 \\ j_1+ij_2 & j_0-j_3 \end{pmatrix}, \quad (4.141)$$

which was originally established in [128] (see also [181]).

Let $Q=\boldsymbol{\sigma}\cdot\mathbf{F}$ and $\mathscr{D}=\sigma^{\mu}\partial_{\mu}$, $\mathscr{J}=\sigma^{\mu}j_{\mu}$, when $\mathscr{D}Q=\mathscr{J}$. Then $S_{\Lambda}QS_{\Lambda^{-1}}=Q'$ and, in view of (4.86) and (4.94), one gets

$$S^{\dagger}_{\Lambda^{-1}}\mathscr{D}S_{\Lambda^{-1}}=\mathscr{D}', \qquad S^{\dagger}_{\Lambda^{-1}}\mathscr{J}S_{\Lambda^{-1}}=\mathscr{J}' \quad (4.142)$$

under a proper Lorentz transformation. In "new" coordinates, equation (4.141) should take a compact form, $\mathscr{D}'Q'=J'$. Thus

$$\left(S^{\dagger}_{\Lambda^{-1}}\mathscr{D}S_{\Lambda^{-1}}\right)S_{\Lambda}QS_{\Lambda^{-1}}=S^{\dagger}_{\Lambda^{-1}}\mathscr{J}S_{\Lambda^{-1}}, \quad (4.143)$$

or, $S^{\dagger}_{\Lambda^{-1}}(\mathscr{D}Q=\mathscr{J})S_{\Lambda^{-1}}$, as a short proof of the covariance of Maxwell's equations.

4.6 Massive Symmetric Four-tensor Field

4.6.1 Group-Theoretical Derivation

The relativistic wave equation for a massive particle of spin two, described by a real or complex symmetric four-tensor field $A^{\mu\nu}(x)=A^{\nu\mu}(x)$ (see [74], [77], [76]), can be obtained in a way that is similar to our study of the Proca equation in section 4. Once again,

$$\begin{aligned} w_{\mu}A^{\alpha\beta} &= \frac{1}{2}e_{\mu\nu\sigma\tau}\partial^{\nu}\left(M^{\sigma\tau}A^{\alpha\beta}\right) \\ &= -g^{\alpha\nu}e_{\mu\nu\sigma\tau}\partial^{\sigma}A^{\tau\beta}-g^{\beta\nu}e_{\mu\nu\sigma\tau}\partial^{\sigma}A^{\tau\alpha} \end{aligned} \quad (4.144)$$

and

$$e_{\mu\nu\sigma\tau}\partial^{\sigma}A^{\tau\beta}=\frac{1}{2}e_{\mu\nu\sigma\tau}F^{\sigma\tau\beta}=-G_{\mu\nu}{}^{\beta}, \quad (4.145)$$

where, by definition, $F^{\sigma\tau\beta} = \partial^\sigma A^{\tau\beta} - \partial^\tau A^{\sigma\beta}$ and $e_{\mu\nu\sigma\tau}F^{\sigma\tau\beta} = -2G_{\mu\nu}{}^\beta$. Thus

$$w_\mu A^{\alpha\beta} = g^{\alpha\nu}G_{\mu\nu}{}^\beta + g^{\beta\nu}G_{\mu\nu}{}^\alpha, \tag{4.146}$$

or

$$w^\mu A^{\alpha\beta} = G^{\mu\alpha\beta} + G^{\mu\beta\alpha}, \qquad w_\mu A_{\alpha\beta} = G_{\mu\alpha\beta} + G_{\mu\beta\alpha} \tag{4.147}$$

and

$$w^2 A^{\alpha\beta} = w_\mu \left(w^\mu A^{\alpha\beta} \right) = w_\mu G^{\mu\alpha\beta} + w_\mu G^{\mu\beta\alpha}. \tag{4.148}$$

In a similar fashion, one can show that

$$\begin{aligned} w_\mu G^{\mu\alpha\beta} &= 2\partial_\nu F^{\nu\alpha\beta} + \partial_\nu F^{\nu\beta\alpha} \\ &\quad + \partial^\alpha \left(F^{\beta\sigma\tau} g_{\sigma\tau} \right) - g^{\alpha\beta}\partial_\nu \left(F^{\nu\sigma\tau} g_{\sigma\tau} \right), \end{aligned} \tag{4.149}$$

as a result of an elementary but rather tedious four-tensor algebra calculation with the help of a familiar relation

$$e^{\mu\nu\sigma\tau} e_{\mu\kappa\lambda\rho} = - \begin{vmatrix} \delta^\nu_\kappa & \delta^\nu_\lambda & \delta^\nu_\rho \\ \delta^\sigma_\kappa & \delta^\sigma_\lambda & \delta^\sigma_\rho \\ \delta^\tau_\kappa & \delta^\tau_\lambda & \delta^\tau_\rho \end{vmatrix}. \tag{4.150}$$

For a massive spin-2 field, the second Casimir operator is equal to $w^2 A^{\alpha\beta} = -6m^2 A^{\alpha\beta}$, and we obtain

$$\begin{aligned} &\partial_\nu F^{\nu\alpha\beta} + \partial_\nu F^{\nu\beta\alpha} - \frac{2}{3} g^{\alpha\beta}\partial_\nu \left(F^{\nu\sigma\tau} g_{\sigma\tau} \right) \\ &+ \frac{1}{3}\partial^\alpha \left(F^{\beta\sigma\tau} g_{\sigma\tau} \right) + \frac{1}{3}\partial^\beta \left(F^{\alpha\sigma\tau} g_{\sigma\tau} \right) = -2m^2 A^{\alpha\beta}, \end{aligned} \tag{4.151}$$

where

$$\begin{aligned} F^{\nu\alpha\beta} &= \partial^\nu A^{\alpha\beta} - \partial^\alpha A^{\nu\beta} = -F^{\alpha\nu\beta}, \\ F^{\nu\sigma\tau} g_{\sigma\tau} &= \partial^\nu A - \partial_\tau A^{\nu\tau}, \qquad A = g_{\sigma\tau} A^{\sigma\tau}, \end{aligned} \tag{4.152}$$

and $F^{\alpha\beta\gamma} + F^{\beta\gamma\alpha} + F^{\gamma\alpha\beta} = 0$ provided $A^{\alpha\beta} = A^{\beta\alpha}$.

In terms of potentials, one gets

$$\partial^2 A^{\alpha\beta} - \frac{2}{3}\left(\partial^\alpha \partial_\nu A^{\nu\beta} + \partial^\beta \partial_\nu A^{\nu\alpha}\right) + \frac{1}{3}\partial^\alpha \partial^\beta A \\ -\frac{1}{3} g^{\alpha\beta}\left(\partial^2 A - \partial_\mu \partial_\nu A^{\mu\nu}\right) = -m^2 A^{\alpha\beta}, \tag{4.153}$$

with $\partial_\mu \partial_\nu A^{\mu\nu} = 0$, if $m > 0$, in view of

$$0 \equiv w^2\left(\partial_\mu \partial_\nu A^{\mu\nu}\right) = \partial_\mu \partial_\nu\left(w^2 A^{\mu\nu}\right) = -6m^2 \partial_\mu \partial_\nu A^{\mu\nu}.$$

The same results can be derived by differentiation from (4.151) or, independently, with the help of an operator identity

$$w^2 = -\frac{1}{2}\partial^2\left(M^{\sigma\tau} M_{\sigma\tau}\right) - \partial_\mu \partial^\nu\left(M^{\mu\sigma} M_{\sigma\nu}\right).$$

Moreover, equation (4.153) is reduced to

$$2\left(\partial^\alpha \partial_\nu A^{\nu\beta} + \partial^\beta \partial_\nu A^{\nu\alpha}\right) - \partial^\alpha \partial^\beta A + g^{\alpha\beta}\partial^2 A = 0 \tag{4.154}$$

with the help of the first Casimir operator, $\partial^2 A^{\alpha\beta} = -m^2 A^{\alpha\beta}$. Multiplication of (4.154) by $g_{\alpha\beta}$ with contraction over two repeated indices results in $\partial^2 A = 0$. Then $A = 0$, due to

$$0 = \partial^2 A = \partial^2\left(g_{\mu\nu} A^{\mu\nu}\right) = g_{\mu\nu}\partial^2\left(A^{\mu\nu}\right) = -m^2 A, \qquad m > 0.$$

Thus, the system of wave equations for a massive spin-2 particle has the form

$$\partial^2 A^{\mu\nu} + m^2 A^{\mu\nu} = 0, \qquad \partial^\alpha\left(\partial_\nu A^{\nu\beta}\right) + \partial^\beta\left(\partial_\nu A^{\nu\alpha}\right) = 0, \tag{4.155}$$

subject to $A = g_{\mu\nu} A^{\mu\nu} = \partial_\mu \partial_\nu A^{\mu\nu} = 0$. It is worth noting, once again, that we have derived these equations by using the Pauli-Lubański vector and the relativistic definition of mass and spin in terms of Casimir operators of the Poincaré group.

In the massless limit $m \to 0$, instead of (4.155), one has $\partial^2 A^{\mu\nu} = 0$ and $\partial^2 A = 0$ subject to

$$\partial^\alpha\left(\partial_\nu A^{\nu\beta} - \frac{1}{4}\partial^\beta A\right) + \partial^\beta\left(\partial_\nu A^{\nu\alpha} - \frac{1}{4}\partial^\alpha A\right) = \frac{1}{2} g^{\alpha\beta}\partial_\mu \partial_\nu A^{\mu\nu}, \tag{4.156}$$

in a similar fashion. Moreover, if $m = 0$, equation (4.151) is invariant under a familiar gauge transformation $A_{\alpha\beta} \to A'_{\alpha\beta} + \partial_\alpha f_\beta + \partial_\beta f_\alpha$ provided that $\partial^\nu \left(\partial_\nu f_\alpha - \partial_\alpha f_\nu \right) = 0$ (Maxwell's equations in vacuum).

4.6.2 An Alternative Gauge Condition

In view of the following identity,

$$\partial^\alpha \left(\partial_\nu A^{\nu\beta} - \frac{1}{4}\partial^\beta A \right) + \partial^\beta \left(\partial_\nu A^{\nu\alpha} - \frac{1}{4}\partial^\alpha A \right)$$
$$= \partial^\alpha \partial_\nu A^{\nu\beta} + \partial^\beta \partial_\nu A^{\nu\alpha} - \frac{1}{2}\partial^\alpha \partial^\beta A,$$

one can impose another condition,

$$4\partial_\nu A^{\mu\nu} - \partial^\mu A = 0, \tag{4.157}$$

in order to simplify (4.153):

$$\partial^2 A^{\mu\nu} - \frac{1}{4} g^{\mu\nu} \partial^2 A = -m^2 A^{\mu\nu}. \tag{4.158}$$

Moreover, by contraction,[9]

$$0 \equiv \partial^2 A - \frac{1}{4} g_{\mu\nu} g^{\mu\nu} \partial^2 A = -m^2 A, \qquad A = g_{\sigma\tau} A^{\sigma\tau} = 0,$$

if $m > 0$. As a result, we obtain

$$\partial^2 A^{\mu\nu} + m^2 A^{\mu\nu} = 0, \qquad \partial_\nu A^{\mu\nu} = \partial_\nu A^{\nu\mu} = 0. \tag{4.159}$$

These equations were originally introduced by Fierz and Pauli [74], [77], [76] with the help of a Lagrangian approach. It is worth noting that our equations (4.155) are necessary and sufficient with the relativistic definition of mass and spin-2 of the field in question, whereas traditional equations (4.159) give only sufficient conditions.

[9] Condition $A = 0$ is not required in the massless limit.

In addition, it should be noted that from (4.155) subject to $A = g_{\mu\nu}A^{\mu\nu} = \partial_\mu \partial_\nu A^{\mu\nu} = 0$, it follows that

$$\partial^2 (\partial_\nu A^{\nu\mu}) + m^2 (\partial_\nu A^{\nu\mu}) = 0, \qquad \partial^\alpha \left(\partial_\nu \partial_\beta A^{\nu\beta}\right) + \partial^2 (\partial_\nu A^{\nu\alpha}) = 0.$$

Combining the latter equations, in turn, gives $\partial_\nu A^{\nu\mu} = 0$ for $m > 0$. On the other hand, if the original Fierz-Pauli equations (4.159) are satisfied, then the equations (4.155) also hold.

4.6.3 Fierz-Pauli vs Maxwell's Equations

When $m = 0$, one gets

$$\partial_\nu F^{\mu\nu\alpha} = \partial_\nu (\partial^\mu A^{\nu\alpha} - \partial^\nu A^{\mu\alpha}) = \partial^\mu (\partial_\nu A^{\nu\alpha}) - \partial^2 A^{\mu\alpha} = 0, \tag{4.160}$$

subject to (4.159). In addition,

$$\partial_\lambda F_{\sigma\tau\alpha} + \partial_\sigma F_{\tau\lambda\alpha} + \partial_\tau F_{\lambda\sigma\alpha} = 0, \tag{4.161}$$

which follows from definition. These facts allow one to represent the massless Fierz-Pauli equations in terms of the third rank field tensor, somewhat similar to classical electrodynamics. Indeed, by analogy with Maxwell's equations, we obtain

$$\partial_\nu F^{\mu\nu\alpha} = 0, \qquad \partial_\nu G^{\mu\nu\alpha} = 0 \tag{4.162}$$

in view of $2G^{\mu\nu\alpha} = -e^{\mu\nu\sigma\tau} F_{\sigma\tau}{}^{\alpha}$ and

$$\partial_\nu G^{\mu\nu\alpha} = -\frac{1}{6} e^{\mu\nu\sigma\tau} (\partial_\nu F_{\sigma\tau}{}^{\alpha} + \partial_\sigma F_{\tau\nu}{}^{\alpha} + \partial_\tau F_{\nu\sigma}{}^{\alpha}) = 0$$

(for every fixed $\alpha = 0, 1, 2, 3$).

Finally, both pairs of these equations can be combined together in the following complex form

$$\partial_\nu Q^{\mu\nu\alpha} = 0, \qquad Q^{\mu\nu\alpha} = F^{\mu\nu\alpha} - \frac{i}{2} e^{\mu\nu\sigma\tau} F_{\sigma\tau}{}^{\alpha}, \tag{4.163}$$

with the help of a self-dual complex four-tensor:

$$2iQ^{\mu\nu\alpha} = e^{\mu\nu\sigma\tau} Q_{\sigma\tau}{}^{\alpha}, \quad e_{\mu\nu\sigma\tau} Q^{\sigma\tau\alpha} = 2iQ_{\mu\nu}{}^{\alpha}. \tag{4.164}$$

The covariant field "energy-momentum" tensor and the corresponding differential balance equation,

$$\frac{\partial}{\partial x^{\nu}} \left(Q^{*}_{\mu\lambda\sigma} Q^{\lambda\nu\sigma} + Q_{\mu\lambda\sigma} \overset{*}{Q}{}^{\lambda\nu\sigma} \right) = 0, \tag{4.165}$$

can be derived in a complete analogy with complex electrodynamics [115], [116].

4.6.4 Fierz-Pauli vs Linearized Einstein's Equations

In general relativity, the linearized equations for a weak gravitational field [61], [96], namely,

$$\begin{aligned} &\partial_{\mu}\partial^{\sigma} h_{\sigma\nu} + \partial_{\nu}\partial^{\sigma} h_{\sigma\mu} - \partial_{\mu}\partial_{\nu} h \\ &\quad -\partial^{2} h_{\mu\nu} - g_{\mu\nu}\left(\partial_{\sigma}\partial_{\tau} h^{\sigma\tau} - \partial^{2} h\right) = 0, \end{aligned} \tag{4.166}$$

describe small deviations from the flat Minkowski metric, $g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$, on the pseudo-Riemannian manifold subject to a gauge condition

$$2\partial^{\nu} h_{\mu\nu} - \partial_{\mu} h = 0, \qquad h = g^{\sigma\tau} h_{\sigma\tau} \tag{4.167}$$

(see, for example, [46], [69], [76], [82], [97], [108], [122], [157], [167], [219], [220], [225], [227], [229], [232], [237], and the references therein for more details).

Our calculations have shown that linearized Einstein's equations (4.166) do not coincide with the massless limit of the spin-2 particle wave equation (4.153). But they can be reduced to the massless case of the Fierz-Pauli equations (4.159) in view of an additional condition (4.167) on a certain solution set. In the literature, this fact is usually interpreted as spin-2 for a graviton although, from the group-theoretical point of view, this massless limit yet requires certain analysis of helicity, say similar to the one in electrodynamics [115], which will be discuss elsewhere.

4.7 Summary

In this chapter, we analyze kinematics of the fundamental relativistic wave equations, in a traditional way, from the viewpoint of the representation theory of the Poincaré group. In particular, the importance of the Pauli-Lubański pseudo-vector is emphasized here not only for the covariant definition of spin and helicity of a given field but also for the derivation of the corresponding equation of motion from first principles. In this consistent group-theoretical approach, the resulting wave equations occur, in general, in certain overdetermined forms, which can be reduced to the standard ones by a matrix version of Gaussian elimination.

Although, mathematically, all representations of the Poincaré group are locally equivalent [12], their explicit realizations in conventional linear spaces of four-vectors and tensors, spinors and bispinors, etc. are quite different from the viewpoint of physics. This is why, as the reader can see in the table below, the corresponding relativistic wave equations are so different.

Classical field	*Transformation law (a law of nature)*	*Wave Eqn.*
Bispinor	$\psi'(x') = S_\Lambda \psi(x)\,,\ x' = \Lambda x$; see (4.11) and (4.25)	Dirac
Spinor	$\psi'(x') = S_\Lambda \psi(x)\,,\ x' = \Lambda x$; see (4.71) and (4.25)	Weyl
Four-vector	$A'^{\mu}(x') = \Lambda^{\mu}_{\ \nu} A^{\nu}(x)$; see (4.25)	Proca
"Feynman slash"	$Q'(x') = S_\Lambda Q(x) S_\Lambda^{-1},\ x' = \Lambda x$; see (4.107)	Proca
Four-tensor	$Q'^{\mu\nu}(x') = \Lambda^{\mu}_{\ \sigma}\Lambda^{\nu}_{\ \tau} Q^{\sigma\tau}(x)\,,\ x' = \Lambda x$; see (4.25)	Maxwell
Complex $3D$ vector	$\mathbf{F}'(x') = S_\Lambda \mathbf{F}(x)\,, x' = \Lambda x$; see Sect. 5.1, (4.25)	Maxwell
Complex matrix	$Q'(x') = S_\Lambda Q(x) S_\Lambda^{-1},\ x' = \Lambda x$; see (4.133)	Maxwell
Symmetric four-tensor	$A'^{\mu\nu}(x') = \Lambda^{\mu}_{\ \sigma}\Lambda^{\nu}_{\ \tau} A^{\sigma\tau}(x)\,,\ x' = \Lambda x$	Fierz-Pauli

Chapter 5

WAVE FUNCTIONS FOR GENERALIZED HARMONIC OSCILLATORS

In this chapter, the time-dependent Schrödinger equation for the most general variable quadratic Hamiltonians is transformed into a standard autonomous form. As a result, the time evolution of exact wave functions of generalized harmonic oscillators is determined in terms of the solutions of certain Ermakov and Riccati-type systems. The Ermakov-type system is introduced here (see also [125]) and has become a major mathematical tool for paraxial optics and radiation field quantization in variable media with applications to quantum optics and possibly to the study of gravitational wave detection. In addition, it is shown that the classical Arnold transform is naturally connected with Ehrenfest's theorem for generalized harmonic oscillators. Quantum systems with variable quadratic Hamiltonians are called the generalized harmonic oscillators (see [20], [41], [53], [54], [68], [73], [92], [129], [132], [148], [151], [233], [235], [238] and references therein). These systems have attracted substantial attention over the years because of their great importance in many advanced quantum problems. Examples are coherent states and uncertainty relations, Berry's phase, quantization of mechanical systems and Hamiltonian cosmology. More applications include, but are not limited to charged particle traps and motion in uniform magnetic fields, molecular spectroscopy and polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes, and other mode interactions with external fields. Quadratic Hamiltonians have particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators [73].

A goal of this chapter is to construct exact wave functions for generalized (driven) harmonic oscillators [20], [39], [92], [129], [132], [233], [235], in terms of Hermite polynomials by transforming the time-dependent Schrödinger equation into an autonomous form [238]. The relationship with certain Ermakov and Riccati-type systems, which seems to be missing in the available literature in general, is investigated. A group theoretical approach to a similar class of partial differential equations is discussed in Refs. [2], [38], [44], [83], [153], [179] (see also [200], [201] and references therein). Some applications to the nonlinear Schrödinger equation can also be found in Refs. [45], [103], [104], [109], [120], [172], [199] and [203].

5.1 Transforming Generalized Harmonic Oscillators into Autonomous Form

We consider the one-dimensional time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi, in \tag{5.1}$$

where the variable Hamiltonian $H = Q(p,x)$ is an arbitrary quadratic of two operators $p = -i\partial/\partial x$ and x, namely,

$$i\psi_t = -a(t)\,\psi_{xx} + b(t)\,x^2\psi - ic(t)\,x\psi_x - id(t)\,\psi - f(t)\,x\psi + ig(t)\,\psi_x, \tag{5.2}$$

(a, b, c, d, f and g are suitable real-valued functions of time only). We shall refer to these quantum systems as the *generalized (driven) harmonic oscillators*. Some examples, a general approach and known elementary solutions can be found in Refs. [39], [40], [41], [43], [53], [70], [72], [73], [132], [135], [150], [198], [233] and [235]. In addition, a case related to Airy functions is discussed in [126] and Ref. [42] deals with another special case of transcendental solutions.

The following is our first result.

Lemma 5.1.1 *The substitution*

$$\psi = \frac{e^{i\left(\alpha(t)x^2 + \delta(t)x + \kappa(t)\right)}}{\sqrt{\mu(t)}}\,\chi(\xi,\tau), \qquad \xi = \beta(t)\,x + \varepsilon(t), \quad \tau = \gamma(t) \tag{5.3}$$

transforms the non-autonomous and inhomogeneous Schrödinger equation (5.2) into the autonomous form

$$-i\chi_\tau = -\chi_{\xi\xi} + c_0\xi^2\chi \qquad (c_0 = 0, 1) \tag{5.4}$$

provided that

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0 a\beta^4, \tag{5.5}$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\,\beta = 0, \tag{5.6}$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \tag{5.7}$$

and

$$\frac{d\delta}{dt} + (c + 4a\alpha)\,\delta = f + 2g\alpha + 2c_0 a\beta^3\varepsilon, \tag{5.8}$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta)\,\beta, \tag{5.9}$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0 a\beta^2\varepsilon^2. \tag{5.10}$$

Here

$$\alpha = \frac{1}{4a}\frac{\mu'}{\mu} - \frac{d}{2a}. \tag{5.11}$$

Proof Differentiating $\psi = \mu^{-1/2}(t)\, e^{iS(x,t)}\chi(\xi,\tau)$ with $S = \alpha(t)x^2 + \delta(t)x + \kappa(t)$, $\xi = \beta(t)x + \varepsilon(t)$ and $\tau = \gamma(t)$ yields

$$ie^{-iS}\psi_t = \frac{1}{\sqrt{\mu}}\left[-\left(\alpha' x^2 + \delta' x + \kappa'\right)\chi + i\left(\left(\beta' x + \varepsilon'\right)\chi_\xi + \gamma'\chi_\tau - \frac{\mu'}{2\mu}\chi\right)\right], \tag{5.12}$$

$$e^{-iS}\psi_x = \frac{1}{\sqrt{\mu}}\left[i(2\alpha x + \delta)\,\chi + \beta\chi_\xi\right] \tag{5.13}$$

and

$$e^{-iS}\psi_{xx} = \frac{1}{\sqrt{\mu}}\left[\left(2i\alpha - (2\alpha x + \delta)^2\right)\chi + 2i(2\alpha x + \delta)\,\beta\chi_\xi + \beta^2\chi_{\xi\xi}\right]. \tag{5.14}$$

Substituting into

$$\begin{aligned} i\psi_t &= -a\psi_{xx} + \left(b - c_0 a\beta^4\right)x^2\psi - icx\psi_x - id\psi \\ &\quad - \left(f + 2c_0 a\beta^3\varepsilon\right)x\psi + ig\psi_x - c_0 a\beta^2\varepsilon^2\psi + c_0 a\beta^2\varepsilon^2\xi^2\psi \end{aligned} \tag{5.15}$$

and using system (5.5)–(5.10), results in Eq. (5.4). A computer algebra proof is given in [112].

Our transformation (5.3) provides a new interpretation to system (5.5)–(5.10) originally derived in Ref. [39] when $c_0 = 0$ by integrating the corresponding Schrödinger equation via the Green function method (see also [202] for an eigenfunction expansion). Here, we discuss the case $c_0 \neq 0$ as its natural extension.

The substitution (5.11), which has been already used in [39], appears here from a new "transformation perspective". It now reduces the inhomogeneous equation (5.5) to the second order ordinary differential equation

$$\mu'' - \tau(t)\,\mu' + 4\sigma(t)\,\mu = c_0\,(2a)^2\,\beta^4\mu, \tag{5.16}$$

that has the familiar time-varying coefficients

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{5.17}$$

(The reader should be convinced that this derivation is rather straightforward.)

When $c_0 = 0$, equation (5.5) is called the *Riccati nonlinear differential equation* [222], [230]; consequently, the system (5.5)–(5.10) shall be referred to as a *Riccati-type system.* (Similar terminology is used in [201] for the corresponding parabolic equation.) Now if $c_0 = 1$, equation (5.16) can be reduced to a generalized version of the *Ermakov nonlinear differential equation* (5.50) (see, for example, [41], [66], [130], [202] and references therein regarding Ermakov's equation) and we shall refer to the corresponding system (5.5)–(5.10) with $c_0 \neq 0$ as an *Ermakov-type system.*

5.2 Green's Function and Wavefunctions

Two particular solutions of the time-dependent Schrödinger equation (5.2) are useful in physical applications. Using standard oscillator wave functions for equation (5.4) when

$c_0 = 1$ (for example, [79], [123] and/or [152]) results in the solution

$$\psi_n(x,t) = \frac{e^{i\left(\alpha x^2+\delta x+\kappa\right)+i(2n+1)\gamma}}{\sqrt{2^n n!\mu\sqrt{\pi}}}\ e^{-(\beta x+\varepsilon)^2/2}\ H_n(\beta x+\varepsilon), \tag{5.18}$$

where $H_n(x)$ are the Hermite polynomials [162], provided that the solution of the Ermakov-type system (5.5)–(5.10) is available.

The Green function of generalized harmonic oscillators has been constructed in the following fashion in Ref. [39]:

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i\mu_0(t)}} \exp\left[i\left(\alpha_0(t)x^2+\beta_0(t)xy+\gamma_0(t)y^2+\delta_0(t)x+\varepsilon_0(t)y+\kappa_0(t)\right)\right]. \tag{5.19}$$

The time-dependent coefficients α_0, β_0, γ_0, δ_0, ε_0, κ_0 satisfy the Riccati-type system (5.5)–(5.10) $(c_0 = 0)$ and are given as follows [39], [198], [202]:

$$\alpha_0(t) = \frac{1}{4a(t)}\frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \tag{5.20}$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \qquad \lambda(t) = \exp\left(-\int_0^t \left(c(s)-2d(s)\right)\,ds\right), \tag{5.21}$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)}\frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)} \tag{5.22}$$

and

$$\delta_0(t) = \frac{\lambda(t)}{\mu_0(t)}\int_0^t \left[\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\mu_0(s) + \frac{g(s)}{2a(s)}\mu_0'(s)\right]\frac{ds}{\lambda(s)}, \tag{5.23}$$

$$\begin{aligned}\varepsilon_0(t) \;&=\; -\frac{2a(t)\lambda(t)}{\mu_0'(t)}\delta_0(t) + 8\int_0^t \frac{a(s)\sigma(s)\lambda(s)}{\left(\mu_0'(s)\right)^2}\left(\mu_0(s)\delta_0(s)\right)\,ds \\ &\quad +2\int_0^t \frac{a(s)\lambda(s)}{\mu_0'(s)}\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\,ds,\end{aligned} \tag{5.24}$$

$$\begin{aligned}\kappa_0(t) \;&=\; \frac{a(t)\mu_0(t)}{\mu_0'(t)}\delta_0^2(t) - 4\int_0^t \frac{a(s)\sigma(s)}{\left(\mu_0'(s)\right)^2}\left(\mu_0(s)\delta_0(s)\right)^2\,ds \\ &\quad -2\int_0^t \frac{a(s)}{\mu_0'(s)}\left(\mu_0(s)\delta_0(s)\right)\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\,ds\end{aligned} \tag{5.25}$$

($\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$ and $\kappa_0(0) = 0$) provided that μ_0 and μ_1 are standard solutions of equation (5.16) with $c_0 = 0$ corresponding to the initial conditions $\mu_0(0) = 0$, $\mu_0'(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu_1'(0) = 0$. (Proofs of these facts are outlined in Refs. [39], [42] and [198]. See also important previous works [54], [148], [233], [235], [238] and references therein for more details.)

Hence, the corresponding Cauchy initial value problem can be solved (formally) by the superposition principle:

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\,\psi(y,0)\ dy \tag{5.26}$$

for some suitable initial data $\psi(x,0) = \varphi(x)$ (see Refs. [39], [198] and [202] for further details).

In particular, using the wave functions (5.18) we get the integral

$$\psi_n(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\,\psi_n(y,0)\ dy, \tag{5.27}$$

and this can be evaluated by

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-\lambda^2(x-y)^2} H_n(ay)\ dy \\ &\quad = \frac{\sqrt{\pi}}{\lambda^{n+1}} \left(\lambda^2 - a^2\right)^{n/2} H_n\left(\frac{\lambda a x}{\left(\lambda^2 - a^2\right)^{1/2}}\right), \quad \operatorname{Re}\lambda^2 > 0, \end{aligned} \tag{5.28}$$

which is an integral transform equivalent to Eq. (30) on page 195 of Vol. 2 of Ref. [64] (the Gauss transform of Hermite polynomials), or Eq. (17) on page 290 of Vol. 2 of Ref. [65].

5.3 Solution to Ermakov-type System

As shown in the previous section, the time evolution of the wave functions (5.18) is determined in terms of the solution to the initial value problem for the Ermakov-type system. In this section, formulas (5.18)–(5.19) and (5.27)–(5.28) shall be used in order to solve the general system (5.5)–(5.10) when $c_0 \neq 0$ along with the uniqueness property of the Cauchy initial value problem. At this point, we must remind the reader how to handle the special case $c_0 = 0$ considered in [198].

Lemma 5.3.1 *The solution of the Riccati-type system (5.5)–(5.10)* $(c_0 = 0)$ *is given by*

$$\mu(t) = 2\mu(0)\,\mu_0(t)\,(\alpha(0) + \gamma_0(t)), \tag{5.29}$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \tag{5.30}$$

$$\beta(t) = -\frac{\beta(0)\,\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\,\mu(0)}{\mu(t)}\lambda(t), \tag{5.31}$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))} \tag{5.32}$$

and

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)\,(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \tag{5.33}$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)\,(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \tag{5.34}$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))} \tag{5.35}$$

in terms of the fundamental solution (5.20)–(5.25) subject to the arbitrary initial data $\mu(0),\, \alpha(0),\, \beta(0) \neq 0,\, \gamma(0),\, \delta(0),\, \varepsilon(0),\, \kappa(0)$.

This solution can be verified by a direct substitution and/or by an integral evaluation. This result can also be thought of as a nonlinear superposition principle for the Riccati-type system and the continuity with respect to initial data holds [198].

Hence, the solution (5.29)–(5.35) implies the following asymptotics established in [198]:

$$\alpha_0(t) = \frac{1}{4a(0)\,t} - \frac{c(0)}{4a(0)} - \frac{a'(0)}{8a^2(0)} + \mathscr{O}(t), \tag{5.36}$$

$$\beta_0(t) = -\frac{1}{2a(0)\,t} + \frac{a'(0)}{4a^2(0)} + \mathscr{O}(t), \tag{5.37}$$

$$\gamma_0(t) = \frac{1}{4a(0)\,t} + \frac{c(0)}{4a(0)} - \frac{a'(0)}{8a^2(0)} + \mathscr{O}(t), \tag{5.38}$$

$$\delta_0(t) = \frac{g(0)}{2a(0)} + \mathscr{O}(t), \quad \varepsilon_0(t) = -\frac{g(0)}{2a(0)} + \mathscr{O}(t), \tag{5.39}$$

$$\kappa_0(t) = \mathscr{O}(t) \tag{5.40}$$

as $t \to 0$ for sufficiently smooth coefficients of the original Schrödinger equation (5.2). Therefore,

$$\begin{aligned} G(x,y,t) \quad \sim \quad & \frac{1}{\sqrt{2\pi i a(0)\,t}} \exp\left[i\frac{(x-y)^2}{4a(0)\,t}\right] \\ & \times \exp\left[-i\left(\frac{a'(0)}{8a^2(0)}(x-y)^2 + \frac{c(0)}{4a(0)}\left(x^2-y^2\right) - \frac{g(0)}{2a(0)}(x-y)\right)\right] \end{aligned} \tag{5.41}$$

as $t \to 0$ (where $f \sim g$ as $t \to 0$, if $\lim_{t\to 0}(f/g) = 1$). This corrects an errata in Ref. [39] .

Finally, we present the extension to a general case when $c_0 \neq 0$. Our main result is the following. Computer algebra proofs of these results can be found in [112].

Lemma 5.3.2 *The solution of the Ermakov-type system (5.5)–(5.10) when* $c_0 = 1\,(\neq 0)$ *is given by*

$$\mu = \mu(0)\,\mu_0\sqrt{\beta^4(0) + 4(\alpha(0)+\gamma_0)^2}, \tag{5.42}$$

$$\alpha = \alpha_0 - \beta_0^2\frac{\alpha(0)+\gamma_0}{\beta^4(0) + 4(\alpha(0)+\gamma_0)^2}, \tag{5.43}$$

$$\beta = -\frac{\beta(0)\,\beta_0}{\sqrt{\beta^4(0) + 4(\alpha(0)+\gamma_0)^2}} = \frac{\beta(0)\,\mu(0)}{\mu(t)}\lambda(t), \tag{5.44}$$

$$\gamma = \gamma(0) - \frac{1}{2}\arctan\frac{\beta^2(0)}{2(\alpha(0)+\gamma_0)}, \quad a(0) > 0 \tag{5.45}$$

and

$$\delta = \delta_0 - \beta_0 \frac{\varepsilon(0)\beta^3(0) + 2(\alpha(0)+\gamma_0)(\delta(0)+\varepsilon_0)}{\beta^4(0) + 4(\alpha(0)+\gamma_0)^2}, \tag{5.46}$$

$$\varepsilon = \frac{2\varepsilon(0)(\alpha(0)+\gamma_0) - \beta(0)(\delta(0)+\varepsilon_0)}{\sqrt{\beta^4(0) + 4(\alpha(0)+\gamma_0)^2}}, \tag{5.47}$$

$$\begin{aligned} \kappa &= \kappa(0) + \kappa_0 - \varepsilon(0)\beta^3(0)\frac{\delta(0)+\varepsilon_0}{\beta^4(0)+4(\alpha(0)+\gamma_0)^2} \\ &\quad + (\alpha(0)+\gamma_0)\frac{\varepsilon^2(0)\beta^2(0) - (\delta(0)+\varepsilon_0)^2}{\beta^4(0)+4(\alpha(0)+\gamma_0)^2} \end{aligned} \tag{5.48}$$

in terms of the fundamental solution (5.20)–(5.25) subject to the arbitrary initial data $\mu(0)$, $\alpha(0)$, $\beta(0) \neq 0$, $\gamma(0)$, $\delta(0)$, $\varepsilon(0)$, $\kappa(0)$.

Following are the steps to the sketch of the proof. Evaluate the integral (5.27) with the help of (5.28) by completing the square and simplify. Use the uniqueness property of the Cauchy initial value problem. One can also verify our solution by a direct substitution into the system (5.5)–(5.10) when $c_0 = 1$. These elementary but rather tedious calculations are left to the reader (the use of a computer algebra system is helpful at certain steps).

Furthermore, the asymptotics (5.36)–(5.40) together with our formulas (5.42)–(5.48) result in the continuity with respect to initial data:

$$\lim_{t\to 0^+} \mu(t) = \mu(0), \quad \lim_{t\to 0^+} \alpha(t) = \alpha(0), \quad \text{etc.} \tag{5.49}$$

Thus the transformation property (5.42)–(5.48) allows us to find a solution of the initial value problem in terms of the fundamental solution (5.20)–(5.25) and it may be referred to as a *nonlinear superposition principle* for the Ermakov-type system.

5.4 Solution of the Ermakov-type Equation

Starting from (5.16)–(5.17) when $c_0 = 1$, and using (5.44) we arrive at

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = (2a)^2(\beta(0)\mu(0)\lambda)^4\mu^{-3}, \tag{5.50}$$

which is a familiar Ermakov-type equation (see [35], [41], [66], [130], [202], [238] and references therein). Then our formula (5.42) leads to the representation

$$\left(\frac{\mu(t)}{\mu(0)}\right)^2 = \beta^4(0)\,\mu_0^2(t) + \left(\frac{\mu_1(t)}{\mu_1(0)} + \frac{\mu'(0)}{2\mu(0)}\frac{\mu_0(t)}{a(0)}\right)^2 \tag{5.51}$$

given in terms of standard solutions μ_0 and μ_1 of the linear characteristic equation (5.16) when $c_0 = 0$. Further details on this Pinney-type solution and the corresponding Ermakov-type invariant are left to the reader (see also [35] and [202]).

5.5 Ehrenfest Theorem Transformations

By introducing expectation values of the coordinate and momentum operators in the following form

$$\overline{x} = \frac{\langle x\rangle}{\langle 1\rangle} = \frac{\langle \psi, x\psi\rangle}{\langle \psi,\psi\rangle}, \qquad \overline{p} = \frac{\langle p\rangle}{\langle 1\rangle} = \frac{\langle \psi, p\psi\rangle}{\langle \psi,\psi\rangle}, \tag{5.52}$$

one can derive Ehrenfest's theorem for the generalized (driven) harmonic oscillators (see, for example, [40] and [41]). Then

$$\frac{d\overline{x}}{dt} = 2a\,\overline{p} + c\,\overline{x} - g, \qquad \frac{d\overline{p}}{dt} = -2b\,\overline{x} - c\,\overline{p} + f \tag{5.53}$$

and the following classical equation of motion of the parametric driven oscillator holds

$$\frac{d^2\overline{x}}{dt^2} - \frac{a'}{a}\frac{d\overline{x}}{dt} + \left(4ab - c^2 + c\frac{a'}{a} - c'\right)\overline{x} = 2af - g' + g\frac{a'}{a} - cg. \tag{5.54}$$

The transformation of the expectation values

$$\overline{\xi} = \beta\,\overline{x} + \varepsilon, \qquad \overline{\xi} = \langle \chi, \xi\chi\rangle \quad \text{with} \quad \langle \chi,\chi\rangle = 1, \tag{5.55}$$

corresponding to our Lemma 1, converts (5.54) into the simplest equation of motion of the free particle and/or harmonic oscillator:

$$\frac{d^2\overline{\xi}}{d\tau^2} + 4c_0\overline{\xi} = 0 \qquad (c_0 = 0, 1)\,. \tag{5.56}$$

(This can be verified by a direct calculation.)

Remark An exact transformation of a linear second-order differential equation into the equation of motion of free particle was discussed by Arnold [9]. An extension of the later to the case of the time-dependent Schrödinger equation had been considered, for example, in Ref. [238] and recently it has been reproduced as the quantum Arnold transformation in [7] and [91] (see also [2], [38], [44], [83], [120], [153], [179], [201] and [203] for similar transformations of nonlinear Schrödinger and other equations of mathematical physics). We elaborate on a relation of the quantum Arnold transformation for the generalized (driven) harmonic oscillators with a Riccati-type system when $c_0 = 0$ (transformation to the free particle) and consider an extension of this transformation (in terms of solutions of the corresponding Ermakov-type system) to the case $c_0 = 1$ (transformation to the classical harmonic oscillator [238]).

5.6 Conclusion

In this chapter, we have determined the time evolution of the wave functions of generalized (driven) harmonic oscillators (5.18), known for their great importance in many advanced quantum problems [73], in terms of the solution to the Ermakov-type system (5.5)–(5.10) by means of a variant of the nonlinear superposition principle (5.42)–(5.48). In this approach, the standard solutions of equation (5.16) with $c_0 = 0$ should be found analytically or numerically. Moreover, the classical Arnold transformation is related to Ehrenfest's theorem. Numerous examples, the corresponding coherent states, dynamic invariants, eigenfunction expansions and transition amplitudes [53], [54], [126], [129], [132], [148], [147], [202] will be discussed elsewhere.

Chapter 6

A MODEL FOR RADIATION FIELD QUANTIZATION IN MEDIA

The quantization of an electromagnetic field in an inhomogeneous and time variable medium is discussed via the Caldirola-Kanai Hamiltonian. This model was also discussed from a different perspective in [41]. Here, we describe a multi-parameter family of squeezed states for this standard model of damping in non-relativistic quantum mechanics and discuss the uncertainty relation. The time-dependent photon statistics for the Caldirola-Kanai Hamiltonian are given explicitly in the Schrödinger picture in terms of the solution of the Ermakov-type system, see Chapter 5 or [125]. The Caldirola-Kanai Hamiltonian arises naturally through the quantization of the electromagnetic field from Maxwell's equations in a medium with certain properties that will be specified below.

For the quantization of a classical Hamiltonian system one replaces canonically conjugate coordinates and momenta by time-dependent operators $q_\lambda(t)$ and $p_\lambda(t)$ that satisfy the commutation relations

$$[q_\lambda(t), q_\mu(t)] = [p_\lambda(t), p_\mu(t)] = 0, \quad [q_\lambda(t), p_\mu(t)] = i\hbar\delta_{\lambda\mu}. \tag{6.1}$$

The time-evolution of these operators is determined by the Heisenberg equation of motion

$$\frac{d}{dt}q_\lambda(t) = \frac{i}{\hbar}[q_\lambda(t), \mathscr{H}], \quad \frac{d}{dt}p_\lambda(t) = \frac{i}{\hbar}[p_\lambda(t), \mathscr{H}], \tag{6.2}$$

For the phenomenological Maxwell equations in linear, dispersive, time-varying media, namely

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \operatorname{div}\mathbf{D} = 4\pi\rho, \tag{6.3}$$

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \qquad \operatorname{div}\mathbf{B} = 0, \tag{6.4}$$

$$\mathbf{D} = \widetilde{\varepsilon}(\mathbf{r},t)\mathbf{E}, \qquad \mathbf{B} = \widetilde{\mu}(\mathbf{r},t)\mathbf{H}, \qquad \mathbf{j} = \widetilde{\sigma}(\mathbf{r},t)\mathbf{E}, \tag{6.5}$$

the continuity equation,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = \frac{\partial \rho}{\partial t} + \frac{4\pi\widetilde{\sigma}}{\widetilde{\varepsilon}}\rho + \mathbf{D}\cdot\operatorname{grad}\left(\frac{\widetilde{\sigma}}{\widetilde{\varepsilon}}\right) = 0, \tag{6.6}$$

has the stationary solution $\rho \equiv 0$ under the condition $\operatorname{grad}(\widetilde{\sigma}/\widetilde{\varepsilon}) = 0$.

With the help of the vector $\mathbf{A}$ and scalar φ potentials,

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \operatorname{grad}\varphi, \tag{6.7}$$

the Maxwell equations can be reduced to the gauge condition

$$\frac{1}{c}\operatorname{div}\left(\widetilde{\varepsilon}\frac{\partial \mathbf{A}}{\partial t}\right) = \operatorname{div}\left(\widetilde{\varepsilon}\operatorname{grad}\varphi\right) \tag{6.8}$$

and the single second-order generalized wave equation

$$\operatorname{curl}\left(\widetilde{\mu}^{-1}\operatorname{curl}\mathbf{A}\right) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\widetilde{\varepsilon}\frac{\partial \mathbf{A}}{\partial t}\right) + \frac{4\pi\widetilde{\sigma}}{c^2}\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left(\widetilde{\varepsilon}\operatorname{grad}\varphi\right) + \frac{4\pi\widetilde{\sigma}}{c}\operatorname{grad}\varphi. \tag{6.9}$$

Here, we consider the factorized (real-valued) dielectric permittivity, the magnetic permeability, and the conductivity (tensors)

$$\widetilde{\varepsilon}(\mathbf{r},t) = \xi(t)\overline{\varepsilon}(\mathbf{r}), \qquad \widetilde{\mu}(\mathbf{r},t) = \eta(t)\overline{\mu}(\mathbf{r}), \qquad \widetilde{\sigma}(\mathbf{r},t) = \chi(t)\overline{\sigma}(\mathbf{r}) \tag{6.10}$$

(the case $\widetilde{\sigma} \equiv 0$ was discussed in [56]). Under the imposed condition $\operatorname{grad}(\widetilde{\sigma}/\widetilde{\varepsilon}) = 0$, one can choose $4\pi\overline{\sigma} = \overline{\varepsilon}$ without loss of generality.

The solution of the classical problem for a given single mode, υ say, has the form

$$\mathbf{A}(\mathbf{r},t) = \mathbf{u}(\mathbf{r})q(t), \qquad \varphi(\mathbf{r},t) = (k/c)\frac{dq}{dt}\phi(\mathbf{r})$$

(k is a constant), and

$$\mathbf{B} = q\operatorname{curl}\mathbf{U}, \qquad \mathbf{D} = -\frac{\xi}{c}\frac{dq}{dt}\,\overline{\varepsilon}\mathbf{U}, \tag{6.11}$$

provided that

$$\operatorname{curl}\left(\frac{1}{\overline{\mu}}\operatorname{curl}\mathbf{U}\right) = \upsilon^2\,\overline{\varepsilon}\mathbf{U}, \qquad \mathbf{U} = \mathbf{u} - k\operatorname{grad}\phi,$$

$$\frac{d^2q}{dt^2} + \frac{\xi' + \chi}{\xi}\frac{dq}{dt} + \frac{c^2\upsilon^2}{\xi\eta}q = 0, \qquad \upsilon = \text{constant}, \tag{6.12}$$

and certain required boundary conditions are satisfied on the boundary of the cavity (see [56], [57], for more details).

Thus we can choose $c = d = f = g = 0$ and

$$a = \frac{1}{2\xi}e^{-\int(\chi/\xi)\,dt}, \qquad b = \frac{c^2\upsilon^2}{2\eta}e^{\int(\chi/\xi)\,dt} \tag{6.13}$$

in the Hamiltonian

$$H = a(t)\hat{p}^2 + b(t)\hat{x}^2 + c(t)\hat{x}\hat{p} - id(t) - f(t)\hat{x} - g(t)\hat{p}. \tag{6.14}$$

For a general approach see [4], [118], [114].

6.1 Exact Wave Functions and the Ermakov-type System

Motivated by (6.3)-(6.14), we consider the time-dependent Schrödinger equation in one dimension with the Caldirola-Kanai Hamiltonian,

$$i\psi_t = \frac{\omega_0}{2}\left(-e^{-2\lambda t}\psi_{xx} + x^2\,e^{2\lambda t}\psi\right) + i\varepsilon x\psi_x, \tag{6.15}$$

which has the following square integrable solution (wave function)

$$\psi_n(x,t) = \frac{e^{i\left(\alpha(t)x^2+\delta(t)x+\kappa(t)+\gamma(t)\right)}}{\sqrt{2^n n!\,\mu(t)\sqrt{\pi}}}\; e^{-(\beta(t)x+\varepsilon(t))^2/2}H_n(\beta(t)x+\varepsilon(t)), \tag{6.16}$$

where $\alpha, \beta, \gamma, \delta, \kappa,$ and ε are determined by the Ermakov-type system (5.5)–(5.10) and are given explicitly on the next page:

$$
\begin{aligned}
\mu(t) &= \mu(0)\, e^{-(\lambda-\varepsilon)t}\sqrt{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2} \\
\alpha(t) &= e^{2\lambda t}\frac{\nu\cos(2\omega t)+\omega\omega_0\left(4\alpha_0^2+\beta_0^4-1\right)\sin(2\omega t)/4+\alpha_0\omega^2-\nu}{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2} \\
\beta(t) &= \frac{\beta_0\omega e^{\lambda t}}{\sqrt{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2}}, \\
\gamma(t) &= \gamma_0-\frac{1}{2}\arctan\frac{\omega_0\beta_0^2\tan(\omega t)}{\omega+(2\alpha_0\omega_0+\lambda-\varepsilon)\tan(\omega t)}, \\
\delta(t) &= \omega e^{\lambda t}\frac{\omega_0\varepsilon_0\beta_0^3\sin(\omega t)+\delta_0\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)}{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2}, \\
\varepsilon(t) &= \frac{\varepsilon_0\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)-\beta_0\delta_0\omega_0\sin(\omega t)}{\sqrt{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2}}, \\
\kappa(t) &= \kappa_0+\sin^2(\omega t)\frac{\omega_0^2\varepsilon_0\beta_0^2(\alpha_0\varepsilon_0-\beta_0\delta_0)-\omega_0^2\alpha_0\delta_0^2+\omega_0(\lambda-\varepsilon)\left(\beta_0^2\varepsilon_0^2-\delta_0^2\right)/2}{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2} \\
&\quad +\frac{\sin(2\omega t)}{4}\frac{\omega_0\omega\left(\beta_0^2\varepsilon_0^2-\delta_0^2\right)}{\beta_0^4\omega_0^2\sin^2(\omega t)+\left((2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)+\omega\cos(\omega t)\right)^2}
\end{aligned}
$$

($\mu_0 > 0$, α_0, $\beta_0 \neq 0$, γ_0, δ_0, ε_0, κ_0 are real initial data).
Here $\nu = (\lambda-\varepsilon)\,\omega_0\left(4\alpha_0^2+\beta_0^4+1\right)/4+\alpha_0\omega_0^2$ and $\omega = \sqrt{\omega_0^2-(\lambda-\varepsilon)^2} > 0$. This system also provides a solution to the Hamiltonian described above for the quantization in the limit $\varepsilon \to 0$. This solution can be verified by a direct substitution, for example with the aid of the *Mathematica* computer algebra system. The solution (6.16) agrees with the result for the quantum harmonic oscillator found in [119] in the limit $\lambda, \varepsilon \to 0$ ($\upsilon \to \alpha_0\omega^2$).

6.2 The Uncertainty Relation and Squeezing

A quantum state is said to be "squeezed" if its oscillating variances $\langle(\Delta p)^2\rangle$ and $\langle(\Delta x)^2\rangle$ become smaller than the variances of the "static" vacuum state $\langle(\Delta p)^2\rangle = \langle(\Delta x)^2\rangle = 1/2$ (with $\hbar = 1$). If the minimum value of the product is equal to $1/4$, then the state is called a minimum-uncertainty squeezed state (see, for example, [57], [95], [190], [195], [196],

[221], [236]).

For the standard deviations in terms of the Ermakov-type system, one gets

$$\sigma_p = \langle (\Delta p)^2 \rangle = \left(n + \frac{1}{2}\right) \frac{4\alpha^2 + \beta^4}{\beta^2}, \quad \sigma_x = \langle (\Delta x)^2 \rangle = \left(n + \frac{1}{2}\right) \frac{1}{\beta^2}, \tag{6.17}$$

$$\sigma_{px} = \frac{1}{2} \langle (\Delta p \Delta x + \Delta x \Delta p) \rangle = \left(n + \frac{1}{2}\right) \frac{2\alpha}{\beta^2},$$

and the uncertainty relation is

$$\langle (\Delta p)^2 \rangle \langle (\Delta x)^2 \rangle = \left(n + \frac{1}{2}\right)^2 \left(1 + \frac{4\alpha^2}{\beta^2}\right) \geq \frac{1}{4}, \tag{6.18}$$

which agrees with the fundamental Heisenberg uncertainty relation.

From this relation one can see that the minimum uncertainty squeezed states occur for $n = 0$ precisely at the moments when $\alpha(t_{\min}) = 0$.

The uncertainty relation (6.18) attains its minimum value, $\min\left[\langle (\Delta p)^2 \rangle \langle (\Delta x)^2 \rangle\right] = \frac{1}{4}$,

if

$$\cos(2\omega t + s\phi) = \frac{\upsilon - \alpha_0 \omega^2}{\sqrt{\nu^2 + \omega^2 \omega_0^2 \left(1 - 4\alpha_0^2 - \beta_0^4\right)^2}}, \tag{6.19}$$

where

$\phi = \cos^{-1}\left(\nu / \sqrt{\nu^2 + \omega^2 \omega_0^2 \left(1 - 4\alpha_0^2 - \beta_0^4\right)^2}\right)$ and $s = \operatorname{sign}\left(\omega \omega_0 \left(1 - 4\alpha_0^2 - \beta_0^4\right)/4\right)$.

The following are explicit formulas for the variances (6.17) in the case $n = 0$:

$$\sigma_x = \frac{e^{-2\lambda t}}{2\beta_0^2\omega^2}\left(\beta_0^4\omega_0^2\sin^2(\omega t)+\left(\omega\cos(\omega t)+(2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)\right)^2\right)$$

$$\sigma_p = \frac{e^{2\lambda t}\left(\beta_0^4\omega_0^2+4\left(\alpha_0\omega^2-\nu+\nu\cos(2\omega t)+\omega\omega_0\left(4\alpha_0^2+\beta_0^4-1\right)\sin(2\omega t)/4\right)^2\right)}{2\beta_0^2\omega^2\left(\beta_0^4\omega_0^2\sin^2(\omega t)+\left(\omega\cos(\omega t)+(2\alpha_0\omega_0+\lambda-\varepsilon)\sin(\omega t)\right)^2\right)}$$

$$\sigma_{px} = \frac{1}{\beta_0^2\omega^2}\left(\alpha_0\omega^2-\nu+\nu\cos(2\omega t)+\omega\omega_0\left(4\alpha_0^2+\beta_0^4-1\right)\sin(2\omega t)/4\right).$$

6.3 Photon Statistics

Here we focus only on a single mode of radiation with the annihilation and creation operators given by

$$\widehat{a}=\frac{1}{\sqrt{2\omega}}\left(\omega\widehat{q}+i\,\widehat{p}\right),\qquad \widehat{a}^{\dagger}=\frac{1}{\sqrt{2\omega}}\left(\omega\widehat{q}-i\,\widehat{p}\right),\qquad \left[\widehat{a},\,\widehat{a}^{\dagger}\right]=1. \tag{6.20}$$

The time-dependent photon amplitudes with respect to the Fock basis are are evaluated with the help of the unitary operator

$$\mathbf{U}(t)=e^{i\left(\widehat{a}^{\dagger}\,\widehat{a}\right)\theta}e^{\left(e^{2i\varphi}\,\widehat{a}^{2}-e^{-2i\varphi}\left(\widehat{a}^{\dagger}\right)^{2}\right)\tau/2}e^{\xi^{*}\,\widehat{a}-\xi\,\widehat{a}^{\dagger}}\,e^{2i\left(\widehat{a}^{\dagger}\,\widehat{a}\right)\gamma}, \tag{6.21}$$

which may be used to expand the wave function in terms of the Fock number states in an abstract Hilbert space via the squeeze and displacement operators in the form

$$|\psi_n(t)\rangle=\sum_{m=0}^{\infty}\left(e^{2i\gamma}\left[\left(\sum_{k=0}^{\infty}S_{mk}D_{kn}\right)\right]e^{im\theta}\right)|m\rangle\,. \tag{6.22}$$

The expansion (6.22) provides a way to compute the time-dependent photon amplitudes explicitly in terms of the Ermakov-type system using the squeeze and displacement operators.

The parameters of (6.21) are determined by

$$\tan\theta = \frac{e^{2\lambda t}\left(2\alpha_0\omega^2 - 2\nu + 2\nu\cos(2\omega t) - \omega\omega_0\left(1-4\alpha_0^2-\beta_0^4\right)\sin(2\omega t)/2\right)}{e^{2\lambda t}\beta_0^2\omega^2 + \omega\left(\omega^2+\rho\right)/2 + \omega\left(\omega^2-\rho\right)\cos(2\omega t)/2 + \omega^2\left(2\alpha\omega+\lambda-\varepsilon\right)\sin(2\omega t)},$$

$$\tan 2\varphi = \frac{2e^{2\lambda t}\beta_0^2\omega^2\left(\beta_0^2\omega^2 e^{2\lambda t} + \omega\omega_0^2\zeta(t)\right)\tan(\theta)}{e^{4\lambda t}\beta_0^4\omega^4 - \omega^2\omega_0^4\zeta^2(t) - \left(\beta_0^2\omega^2 e^{2\lambda t} + \omega\omega_0^2\zeta(t)\right)^2\tan^2(\theta)},$$

$$\begin{aligned}4\left[\cosh\tau\right]^2 \;=\;& \frac{1}{\beta_0^2\omega^3}(2\beta_0^2\omega^3 + e^{-2\lambda t}\left(\omega^2+\rho\right)\left(\omega^2+e^{4\lambda t}\right)/2\\
&+\left(e^{-2\lambda t}\left(\omega^2-\rho\right)/2\omega_0^2 - e^{4\lambda t}\left(\omega^2+\rho-2\omega^2\left(4\alpha_0^2+\beta_0^4\right)\right)\right)\cos(2\omega t)/2\\
&+e^{-2\lambda t}\omega^2\left(2\alpha\omega_0+\lambda-\varepsilon\right) + e^{4\lambda t}\left(\left(4\alpha_0^2+\beta_0^4\right)\left(\varepsilon-\lambda\right)\omega_0 - 2\alpha_0\omega_0^2\right)/\omega_0)\end{aligned}$$

$$\begin{aligned}4\left[\sinh\tau\right]^2 \;=\;& \frac{1}{\beta_0^2\omega^3}(-2\beta_0^2\omega^3 + e^{-2\lambda t}\left(\omega^2+\rho\right)\left(\omega^2+e^{4\lambda t}\right)/2\\
&+\left(e^{-2\lambda t}\left(\omega^2-\rho\right)/2\omega_0^2 - e^{4\lambda t}\left(\omega^2+\rho-2\omega^2\left(4\alpha_0^2+\beta_0^4\right)\right)\right)\cos(2\omega t)/2\\
&+e^{-2\lambda t}\omega^2\left(2\alpha\omega_0+\lambda-\varepsilon\right) + e^{4\lambda t}\left(\left(4\alpha_0^2+\beta_0^4\right)\left(\varepsilon-\lambda\right)\omega_0 - 2\alpha_0\omega_0^2\right)/\omega_0)\end{aligned}$$

where $\rho = \omega_0^2\left(4\alpha_0^2+\beta_0^4\right) + 4\alpha_0\omega_0\left(\lambda-\varepsilon\right) + \omega_0^2$ and

$$\zeta(t) = \rho/2\omega_0^2 + \left(2\omega^2-\rho\right)\cos(2\omega t)/2\omega_0^2 + \omega\left(2\alpha_0\omega_0+\lambda-\varepsilon\right)\sin(2\omega t)/\omega_0^2.$$

The equations determine the matrix elements of the squeeze operator (see below) as the arguments of the hypergeometric function.

For the Hamiltonian in (6.15) the matrix elements of the displacement operator are given by

$$\begin{aligned}D_{mn}(\xi) &= \left\langle m\left|e^{\xi^*\hat{a}-\xi\hat{a}^\dagger}\right|n\right\rangle \qquad (6.23)\\
&= e^{-|\xi|^2/2}\frac{(-\xi)^m(\xi^*)^n}{\sqrt{m!n!}}\;{}_2F_0\left(-n,-m;-\frac{1}{|\xi|^2}\right),\end{aligned}$$

where the parameter ξ in terms of solutions of the Ermakov-type system is given by

$$\xi = \frac{\left(\beta_0 \varepsilon_0 - i\delta_0\right)\left(\left(2\alpha_0\omega_0 + \lambda - \varepsilon - i\beta_0^2\omega_0\right)\sin\left(\omega t\right) + \omega\cos\left(\omega t\right)\right)}{\sqrt{2}\beta_0\sqrt{\beta_0^4\omega_0^2\sin^2(\omega t) + \left(\left(2\alpha_0\omega_0 + \lambda - \varepsilon\right)\sin(\omega t) + \omega\cos(\omega t)\right)^2}}, \tag{6.24}$$

which in turn completely determines the time-evolution of the displacement operator.

In a similar fashion, the matrix elements of the squeeze operator can be readily evaluated by using the expression:

$$\begin{aligned} S_{mn}(\alpha,\beta) &= \left\langle m \left| e^{\left(e^{2i\varphi}\,\widehat{a}^2 - e^{-2i\varphi}\left(\widehat{a}^\dagger\right)^2\right)\tau/2} \right| n \right\rangle \\ &= \sqrt{\frac{m!n!\pi}{2^{m+n}\cosh\tau}} \frac{\left(-e^{-2i\varphi}\sinh\tau\right)^{(m-n)/2}\left(\cosh\tau\right)^{-(m+n)/2}}{\Gamma\left(\frac{m-n}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n+1}{2}+1\right)} \\ &\quad \times {}_2F_1\left(\begin{array}{c} (1-n)/2, \quad -n/2 \\ 1+(m-n)/2 \end{array} ; \quad -\sinh^2\tau\right), \qquad m \geq n. \end{aligned} \tag{6.25}$$

With the help of familiar transformations of the terminating hypergeometric functions [8], one may obtain the non-vanishing matrix elements as follows

$$\begin{aligned} S_{mn}(\alpha,\beta) &= \frac{(-1)^{m/2}e^{-i(m-n)\varphi}}{\sqrt{\cosh\tau}}\left[\frac{(1/2)_{m/2}(1/2)_{n/2}}{(m/2)!\,(n/2)!}\right]^{1/2} \\ &\quad \times (\tanh\tau)^{(m+n)/2}\ {}_2F_1\left(\begin{array}{c} -n/2, \quad -m/2 \\ 1/2 \end{array} ; \quad -\frac{1}{\sinh^2\tau}\right), \end{aligned} \tag{6.26}$$

if m, n are even and

$$\begin{aligned} S_{mn}(\alpha,\beta) &= 2\frac{(-1)^{(m-1)/2}e^{-i(m-n)\varphi}}{\sinh\tau\sqrt{\cosh\tau}}\left[\frac{(3/2)_{\frac{m-1}{2}}(3/2)_{\frac{n-1}{2}}}{\left(\frac{m-1}{2}\right)!\left(\frac{n-1}{2}\right)!}\right]^{1/2} \\ &\quad \times (\tanh\tau)^{(m+n)/2}\ {}_2F_1\left(\begin{array}{c} (1-n)/2, \quad (1-m)/2 \\ 3/2 \end{array} ; \quad -\frac{1}{\sinh^2\tau}\right), \end{aligned} \tag{6.27}$$

if m, n are odd. The arguments of the hypergeometric function are given explicitly in terms of the Ermakov-type system provided above.

In conclusion, in this short chapter we have discussed a specific model which arises naturally in the quantization of the electromagnetic field in inhomogeneous and time variable media. This exactly solvable model, which will be elaborated on more elsewhere, and its photon statistics provide a nice example of the usefulness of the Ermakov-type system discussed in Chapter 5.

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