

A Convex Approach for Stability Analysis of
Partial Differential Equations

by

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ABSTRACT

A computational framework based on convex optimization is presented for stability analysis of systems described by Partial Differential Equations (PDEs). Specifically, two forms of linear PDEs with spatially distributed polynomial coefficients are considered.

The first class includes linear coupled PDEs with one spatial variable. Parabolic, elliptic or hyperbolic PDEs with Dirichlet, Neumann, Robin or mixed boundary conditions can be reformulated in order to be used by the framework. As an example, the reformulation is presented for systems governed by Schrödinger equation, parabolic type, relativistic heat conduction PDE and acoustic wave equation, hyperbolic types. The second form of PDEs of interest are scalar-valued with two spatial variables. An extra spatial variable allows consideration of problems such as local stability of fluid flows in channels and dynamics of population over two dimensional domains.

The approach does not involve discretization and is based on using Sum-of-Squares (SOS) polynomials and positive semi-definite matrices to parameterize operators which are positive on function spaces. Applying the parameterization to construct Lyapunov functionals with negative derivatives allows to express stability conditions as a set of Linear Matrix Inequalities (LMIs). The MATLAB package SOSTOOLS was used to construct the LMIs. The resultant LMIs then can be solved using existent Semi-Definite Programming (SDP) solvers such as SeDuMi or MOSEK. Moreover, the proposed approach allows to calculate bounds on the rate of decay of the solution norm.

The methodology is tested using several numerical examples and compared with the results obtained from simulation using standard methods of numerical discretization and analytic solutions.

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RESEARCH GOALS AND MOTIVATION

Partial Differential Equations (PDEs) are often used to model systems in which the quantity of interest varies continuously in both space and time. Examples of such quantities include: deflection of beams (Euler-Bernoulli equation); velocity and pressure of fluid flow (Navier-Stokes equations); and population density in predator-prey models. See Evans (1998), Garabedian (1964) and John (1982) for a wide range of examples.

Stability analysis and controller design for PDEs is an active area of research Christofides (2012), Curtain and Zwart (1995). One approach to analyze PDEs is to approximate the PDEs with Ordinary Differential Equations (ODEs) using, e.g. Galerkin's method or finite difference, and then apply finite-dimensional optimal control methods, Kamyar *et al.* (2013), Baker and Christofides (2000), El-Farra *et al.* (2003). We present a methodology for stability analysis without model reduction techniques. Specifically, we use Linear Matrix Inequalities (LMIs) and Sum-of-Squares (SOS) optimization to construct Lyapunov functionals for PDEs.

It is well-known that existence of a Lyapunov function for a system of ODEs or PDEs is a sufficient condition for stability. For example, Fridman and Orlov (2009) use a Lyapunov approach and Linear Operator Inequalities (LOIs) to provide sufficient conditions for exponential stability of a controlled heat and delayed wave equations. In Solomon and Fridman (2015) Lyapunov stability conditions of semilinear diffusion equations with delays are formulated in terms of LMIs.

Extensive examples of applying the backstepping method to the boundary control of PDEs can be found in Krstic and Smyshlyaev (2008c), (2008b), (2008a), (2005).

Briefly speaking, backstepping uses a Volterra operator to search for an invertible mapping from the original PDE to a chosen "target" PDE, proven to be stable using, e.g. Lyapunov function. In order to find such mapping, one has to solve analytically or numerically a PDE for Volterra operator's kernel. If the mapping is found, it provides the boundary control law. Applications for two-dimensional cases were discussed in Vazquez and Krstic (2006) and Xu *et al.* (2008). However, backstepping requires us to guess on the target PDE and solve the PDE for kernel, which may be a challenging task for PDEs with two spatial variables and spatially dependent coefficients. Moreover, backstepping cannot be used for stability analysis in the absence of a control input.

Notable results based on Lyapunov and semigroup theories were obtained in Fridman *et al.* (2010) for analysis of wave and beam PDEs with delayed boundary control. In semigroup theory the state of a PDE belongs to a certain space of functions. The solution is an operator-valued function ("strongly continuous semigroup" - SCS), indexed to the time domain, which maps the current state to a future state. For an introduction to Semigroup Theory we refer readers to Lasiecka (1980), Curtain and Zwart (1995).

In the semigroup framework, stability, controllability and observability conditions can be expressed using operator inequalities in the same way that LMIs are used to represent those properties for ODEs. As an example, for a system $\dot{u} = \mathcal{A}u$ which defines a SCS on a Hilbert space X with \mathcal{A} being the infinitesimal generator, the exponential stability of the system is equivalent to the existence of a positive bounded linear operator $\mathcal{P} : X \rightarrow X$ such that

$$\langle u, \mathcal{A}\mathcal{P}u \rangle_X + \langle \mathcal{A}u, \mathcal{P}u \rangle_X \leq - \langle u, u \rangle_X \quad (1.1)$$

for all u in the domain of \mathcal{A} . Condition (1.1) is termed a Linear Operator Inequality (LOI). The terminology LOI is deliberately chosen to suggest a parallel to the use of Linear Matrix Inequalities (LMIs) for computational analysis and control of ODEs. Indeed, there have been efforts to use discretization to solve LOI type conditions for stability analysis and optimal control of PDEs (see, e.g. Christofides (2012)), optimal actuator placement for parabolic PDEs (see Demetriou and Borggaard (2003) and Morris *et al.* (2015)). While discretization has proven quite effective in practice, one should note that in general it is difficult to determine if feasibility of the discretized LOI implies stability of the non-discretized PDE. In contrast, we focused on exploring how to use computation to solve LOIs (1.1) directly by parameterizing the cone of positive and negative operators.

The approach, taken by Peet (2014), uses some of the machinery developed for Delay Differential Equations (DDEs) to express Lyapunov inequalities as LMIs, which can be tested using standard interior-point algorithms. In Fridman and Terushkin (2016), stability analysis and initial state recovery of semi-linear wave equation are also presented in terms of LMIs. In Papachristodoulou and Peet (2006), stability analysis is performed for scalar nonlinear PDEs using SOS and a simple form of Lyapunov function. This simple Lyapunov function was extended in Valmorbida *et al.* (2014) and Valmorbida *et al.* (2015) to consider some forms of coupled PDEs and in Ahmadi *et al.* (2016) to perform passivity analysis. In Gahlawat and Peet (2015), the class of Lyapunov functions was expanded to squares of semi-separable integral operators and applied to output-feedback dynamic control of scalar PDEs. Input-output properties of PDEs with SOS implementation is discussed in Ahmadi *et al.* (2014). Examples of using SOS in controller and observer designs for parabolic linear one-dimensional PDEs can be found in Gahlawat and Peet (2011), Gahlawat and Peet (2014), Gahlawat *et al.* (2011) and Gahlawat *et al.* (2012). Finally, in Meyer and Peet

(2015), we considered stability of PDEs with multiple spatial variables.

In Chapter 2, we introduce notations and some preliminaries. General Lyapunov theorem is proved in Chapter 3. Proposed techniques for coupled PDEs and PDEs with two spatial variables are presented in Chapters 4 and 5. Numerical results are discussed in Chapter 6 and some conclusions are in Chapter 7.

1.1 Formulation of Mathematical Problems

The interest of this work is to propose a computational algorithm for stability analysis of two following forms of PDEs.

1.1.1 Coupled Linear PDEs with One Spatial Variable

First class considers function $u : [0, \infty) \times [a, b] \rightarrow \mathbb{R}^n$ satisfying

$$u_t(t, x) = A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x) \quad (1.2)$$

for all $t > 0$ and $x \in (a, b)$, with some fixed $a, b \in \mathbb{R}$. The coefficients A, B, C are polynomial matrices. Boundary conditions are represented through the elements of matrix $D \in \mathbb{R}^{4n \times 4n}$ such that for all $t > 0$

$$D \begin{bmatrix} u(t, a) \\ u(t, b) \\ u_x(t, a) \\ u_x(t, b) \end{bmatrix} = 0. \quad (1.3)$$

The use of matrix D allows for different types of boundary conditions. As an example, D_1 and D_2 from (1.4) represent homogeneous Dirichlet boundary conditions and

mixed boundary conditions (homogeneous Neumann at $x = a$ and Dirichlet at $x = b$).

$$D_1 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (1.4)$$

Robin Boundary conditions

$$H_1 u(t, a) - H_2 u_x(t, a) = 0$$

$$H_3 u(t, b) - H_4 u_x(t, b) = 0$$

can be stated as (1.3) with

$$D = \begin{bmatrix} H_1 & 0 & -H_2 & 0 \\ 0 & H_3 & 0 & H_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution to (1.2) is assumed to exist, be unique and depend continuously on the initial condition $u(0, \cdot)$. Also, for each $t > 0$ we suppose $u(t, \cdot), u_x(t, \cdot), u_{xx}(t, \cdot) \in L_2^n(a, b)$.

1.1.2 Parabolic Scalar-Valued PDEs with Two Spatial Variables

The second form of interest has two spatial variables. Specifically, for all $t > 0$ and $x \in \Omega := (0, 1)^2$, $u : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ satisfies

$$\begin{aligned} u_t(t, x) &= a(x)u_{x_1x_1}(t, x) + b(x)u_{x_1x_2}(t, x) + c(x)u_{x_2x_2}(t, x) + d(x)u_{x_1}(t, x) \\ &+ e(x)u_{x_2}(t, x) + f(x)u(t, x), \end{aligned} \quad (1.5)$$

where a, b, c, d, e, f are polynomials. Assume that solution to (1.5) exists, is unique and depends continuously on initial conditions. Moreover, let u satisfy homogeneous

Dirichlet boundary conditions, i.e.

$$u(t, 1, x_2) = 0, \quad u(t, 0, x_2) = 0, \quad u(t, x_1, 1) = 0, \quad u(t, x_1, 0) = 0$$

for all $x_1, x_2 \in [0, 1]$ and $t > 0$.

It is shown how some PDEs can be formulated as (1.2) in the following sections.

1.2 Example 1: Schrödinger Equation

To illustrate the class of PDEs which can be written as (1.2), first consider the Schrodinger equation. In the following equation V is the potential energy, i is the imaginary unit, \hbar is the reduced Planck constant and ψ is the wave function of the quantum system.

$$i\hbar\psi_t(t, x) = -\frac{\hbar^2}{m}\psi_{xx}(t, x) + V(x)\psi(t, x)$$

can be written as two coupled PDEs if one decomposes the solution into real and imaginary parts as $\psi(t, x) = \psi^{rl}(t, x) + i\psi^{im}(t, x)$ and then separates the real and imaginary parts of the equation, i.e.

$$\begin{bmatrix} \psi_t^{rl}(t, x) \\ \psi_t^{im}(t, x) \end{bmatrix} = \underbrace{\frac{\hbar}{m} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_A \begin{bmatrix} \psi_{xx}^{rl}(t, x) \\ \psi_{xx}^{im}(t, x) \end{bmatrix} + \underbrace{\frac{V(x)}{\hbar} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{C(x)} \begin{bmatrix} \psi^{rl}(t, x) \\ \psi^{im}(t, x) \end{bmatrix}.$$

1.3 Example 2: PDE for the Model of Acoustic Waves

Next consider a model for a 1-D acoustic wave. For all $t > 0, r \in (0, R)$ and some fixed $c > 0$,

$$p_{tt}(t, r) = c^2 p_{rr}(t, r) + \frac{2c^2}{r} p_r(t, r). \quad (1.6)$$

PDE (1.6) is equivalent to a system of two coupled first order PDEs as

$$\begin{bmatrix} q_t(t, r) \\ p_t(t, r) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & c^2 \\ 0 & 0 \end{bmatrix}}_A \begin{bmatrix} q_{rr}(t, r) \\ p_{rr}(t, r) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \frac{2c^2}{r} \\ 0 & 0 \end{bmatrix}}_{B(r)} \begin{bmatrix} q_r(t, r) \\ p_r(t, r) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_C \begin{bmatrix} q(t, r) \\ p(t, r) \end{bmatrix},$$

where q is an auxiliary function. Moreover, if the boundary conditions imply amplification of the waves, i.e.

$$p(t, 0) = f_1 p(t, R) \quad \text{and} \quad p_r(t, 0) = f_2 p_r(t, R)$$

for some $f_1, f_2 > 0$ and all $t > 0$, then the boundary conditions can be stated using (1.3) with

$$D = \begin{bmatrix} 0 & 1 & 0 & -f_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -f_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1.4 Example 3: Relativistic Heat Conduction Equation

Classic form of heat PDE,

$$u_t(t, x) = \alpha \nabla^2 u(t, x), \tag{1.7}$$

assumes that the speed of information propagation is higher than the speed of light in vacuum, which is physically unacceptable. For more details see Ali and Zhang (2005). If one considers Minkowski space instead of Euclidean one, then (1.7) can be written as

$$u_t(t, x) = -\frac{\alpha}{c^2} u_{tt}(t, x) + \alpha \nabla^2 u(t, x), \tag{1.8}$$

where c denotes the speed of light in vacuum.

Similarly, using an auxiliary function $w = u_t$, (1.8) can be reformulated as linear coupled PDEs. For example with one spatial dimension, if $w = u_t$, then

$$w(t, x) = -\frac{\alpha}{c^2} w_t(t, x) + \alpha u_{xx}(t, x)$$

or equivalently

$$w_t(t, x) = -\frac{c^2}{\alpha}w(t, x) + c^2u_{xx}(t, x)$$

resulting in representation of (1.8) in the following form.

$$\begin{bmatrix} u_t(t, x) \\ w_t(t, x) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ c^2 & 0 \end{bmatrix}}_A \begin{bmatrix} u_{xx}(t, x) \\ w_{xx}(t, x) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{c^2}{\alpha} \end{bmatrix}}_C \begin{bmatrix} u(t, x) \\ w(t, x) \end{bmatrix}.$$

The next chapter presents notations we are using and some preliminaries.

Chapter 2

PRELIMINARIES

2.1 Notations

\mathbb{N} is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. \mathbb{R}^n and \mathbb{S}^n are the n -dimensional Euclidean space and space of $n \times n$ real symmetric matrices. For $x \in \mathbb{R}^n$, let x^T denote transposed x and $x_i \in \mathbb{R}$ is the i -th component of x . $\|\cdot\|_1$ is a norm on \mathbb{R}^n , defined as $\|x\|_1 := \sum_{i=1}^n |x_i|$. For $X \in \mathbb{S}^n$, $X \leq 0$ means that X is negative semidefinite. The symbol $*$ will denote the symmetric elements of a symmetric matrix.

For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}$ let $f(x)$ stand for $f(x_1, \dots, x_n)$ and $\int_{\Omega} f(x) dx$ represent an integral of f over Ω with $dx := dx_1 dx_2 \dots dx_n$.

Let $\mathbb{N}_0^n := \{\alpha \in \mathbb{R}^n : \alpha_i \in \mathbb{N}_0\}$. A vector $\alpha \in \mathbb{N}_0^n$ is called multi-index. For $l \in \mathbb{N}$ define the set

$$Q_l^n := \{\alpha \in \mathbb{N}_0^n : \|\alpha\|_1 \leq l\}. \quad (2.1)$$

For $\alpha \in \mathbb{N}_0^n$, $x \in \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ partial derivative

$$D^\alpha[g(x)] := \frac{\partial^\alpha}{\partial x^\alpha}[g(x)] = \prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}[g(x)]. \quad (2.2)$$

Note that $\frac{\partial^0}{\partial x_i^0}[g(x)] = g(x)$ for any $i \in \{1, \dots, n\}$. Classical notations such as $u_{x_1 x_2}(t, x) := \frac{\partial}{\partial x_2}[\frac{\partial}{\partial x_1}[u(t, x)]]$ are also applied.

If for a function $f : \Omega \rightarrow \mathbb{R}$ and some $\alpha \in \mathbb{N}_0^n$ derivative $D^\alpha[f(x)]$ exists for all $x \in \Omega$, there exists $g : \Omega \rightarrow \mathbb{R}$ such that $g(x) = D^\alpha[f(x)]$ for all $x \in \Omega$. For brevity $D^\alpha[f] := g$.

$L_p(\Omega)$ stands for the space of Lebesgue-measurable functions $g : \Omega \rightarrow \mathbb{R}$ with norm, for $p \in \mathbb{N}$

$$\|g\|_{L_p} := \left(\int_{\Omega} |g(s)|^p ds \right)^{1/p}$$

and $\|g\|_{L_{\infty}} := \sup_{s \in \Omega} |g(s)|$. Note that if $g : \Omega \rightarrow \mathbb{R}^m$ then the notation $L_p^m(\Omega)$ is used.

$W^{k,p}(\Omega)$ denotes Sobolev space of functions $u : \Omega \rightarrow \mathbb{R}$ with $D^{\alpha}[u] \in L_p(\Omega)$ for all $\alpha \in Q_k^n$, where Q_k^n is defined as in (2.1) and norm

$$\|u\|_{k,p} := \sum_{\|\alpha\|_1 \leq k} \|D^{\alpha}[u]\|_{L_p}.$$

It is known that for continuous functions $u : [0, \infty) \rightarrow W^{2,2}(\Omega)$ and $V : W^{2,2}(\Omega) \rightarrow \mathbb{R}$ the composition $(V \circ u) : [0, \infty) \rightarrow \mathbb{R}$ is also continuous and the upper right-hand derivative $D_t^+ V(u(t))$ is defined by

$$D^+[V(u(t))] := \limsup_{h \rightarrow 0^+} \frac{V(u(t+h)) - V(u(t))}{h}.$$

Note that if $v : [0, \infty) \rightarrow \mathbb{R}$ is differentiable at $t \in (0, \infty)$ then $D^+[v(t)] = \frac{d}{dt}[v(t)]$.

2.2 Linear Matrix Inequalities

Firstly, let start with the general form of a Semi-Definite Program (SDP). For some $c \in \mathbb{R}^n$ and $F_i \in \mathbb{S}^m$

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} c^T x \\ & \text{such that } F_0 + \sum_{i=1}^n x_i F_i \leq 0. \end{aligned}$$

Class of SDPs is a subclass of convex optimization problems and, thus, can be solved computationally in polynomial time using, for example, interior point method. For more about convex optimization see Boyd and Vandenberghe (2004).

The feasibility problem of an SDP is known as Linear Matrix Inequality (LMI). Finite number of LMIs can be cast as a single LMI. The problem of searching for an $X \in \mathbb{S}^n$ such that

$$X > 0 \quad \text{and} \quad A^T X + X A < 0 \tag{2.3}$$

where $A \in \mathbb{R}^{n \times n}$ is given, can be cast as an LMI. For example, let $n = 2$ and denote

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}. \quad (2.4)$$

If $e_{ij} \in \mathbb{R}^{2 \times 2}$ are basis matrices with $e(i, j) = 1$ and zero other elements, then (2.4) can be written as

$$X = x_1 e_{11} + x_2 e_{12} + x_2 e_{21} + x_3 e_{22}. \quad (2.5)$$

and, therefore, problem (2.3) can be cast as

$$F_0 + \sum_{i=1}^3 x_i F_i \leq 0$$

with some $\epsilon > 0$ and

$$F_0 = \begin{bmatrix} \epsilon I_2 & 0 \\ 0 & \epsilon I_2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -e_{11} & 0 \\ 0 & A^T e_{22} + e_{22} A \end{bmatrix}, \quad F_3 = \begin{bmatrix} -e_{22} & 0 \\ 0 & A^T e_{22} + e_{22} A \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -(e_{12} + e_{21}) & 0 \\ 0 & A^T (e_{12} + e_{21}) + (e_{12} + e_{21}) A \end{bmatrix}.$$

Thus, one can solve (2.5) using any SDP solver and, if succeed, will find a solution to (2.3).

2.3 Polynomials and Sum of Squares Polynomials

For a multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, let

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

Then x^α is a monomial of degree $\|\alpha\|_1 \in \mathbb{N}_0$. A polynomial is a finite linear combination of monomials $p(x) := \sum_{\alpha} p_{\alpha} x^{\alpha}$, where the summation is applied over a given finite set of multi-indexes α and $p_{\alpha} \in \mathbb{R}$ denotes the corresponding coefficient. The

degree of a polynomial p is the largest degree among all monomials, and is denoted by $\deg(p) \in \mathbb{N}_0$.

A polynomial p is called Sum of Squares (SOS), if there is a finite number of polynomials z_i such that for all $x \in \mathbb{R}^n$, $p(x) = \sum_i q_i(x)^2$. If p is a SOS polynomial, then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Polynomial matrices and SOS polynomial matrices are defined in a similar manner, except p_α are not scalars, but matrices. If M is a SOS polynomial matrix then for all $x \in \mathbb{R}^n$, $M(x) \geq 0$.

The following theorem introduces the connection between SOS polynomials and positive semi-definite matrices. For more see Parrilo (2000).

Theorem 1. *A polynomial $p : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree $2d$ is an SOS polynomial if and only if there exists $Q \in \mathbb{S}^{d+1}$ such that $Q \geq 0$ and*

$$p(x) = z_d^T(x)Qz_d(x),$$

where $z_d(x)$ is a vector of monomials up to degree d , i.e.

$$z_d(x) := \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^d \end{bmatrix}.$$

Proof. (\Rightarrow) Suppose p is an SOS polynomial. Then there are polynomials q_i such that

$$p(x) = \sum_{i=1}^k q_i(x)^2.$$

Note, that for each $i \in \{1, \dots, k\}$,

$$q_i(x) = a_i^T z_d(x),$$

where a_i is the vector of coefficients of the polynomial q_i . Then

$$\begin{aligned} \sum_{i=1}^k q_i(x)^2 &= \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix}^T \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix} = \begin{bmatrix} a_1^T z_d(x) \\ a_2^T z_d(x) \\ \vdots \\ a_k^T z_d(x) \end{bmatrix}^T \begin{bmatrix} a_1^T z_d(x) \\ a_2^T z_d(x) \\ \vdots \\ a_k^T z_d(x) \end{bmatrix} \\ &= z_d(x)^T [a_1 \ a_2 \ \dots \ a_k] \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{bmatrix} z_d(x) = z_d(x)^T A A^T z_d(x) \\ &= z_d(x)^T Q z_d(x). \end{aligned}$$

Since $Q = A A^T$, $Q \geq 0$.

(\Leftarrow) Given polynomial p suppose there exists $Q \geq 0$ such that

$$p(x) = z_d(x)^T Q z_d(x). \quad (2.6)$$

Since $Q \geq 0$ there exists A such that $Q = A^T A$. Then (2.6) can be reformulated as

$$p(x) = z_d(x)^T A^T A z_d(x) = (A z_d(x))^T A z_d(x). \quad (2.7)$$

Let $q(x) := A z_d(x)$, then (2.7) can be continued as

$$p(x) = q(x)^T q(x) = \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix}^T \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_k(x) \end{bmatrix} = \sum_{i=1}^k q_i(x)^2.$$

□

2.4 Comparison Principle

Recall the Comparison Lemma which is used in the proof of Lyapunov theorem for PDEs in the next chapter. Verbatim from Khalil (1996).

Lemma 1. *Consider the scalar differential equation $\frac{d}{dt}[u(t)] = f(t, u(t))$, $u(t_0) = u_0$, where $f(t, x)$ is continuous in t and locally Lipschitz in x , for all $t \geq 0$ and all $x \in J \subset \mathbb{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution u , and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let v be a continuous function whose upper right-hand derivative $D^+[v(t)]$ satisfies*

$$D^+[v(t)] \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Proof. For the proof see Khalil (1996). □

Chapter 3

LYAPUNOV TEST FOR PDES

In this chapter Lyapunov conditions for stability are presented for the following class of PDEs. For some $k \in \mathbb{N}$, all $t \in (0, \infty)$ and $x \in \Omega \subseteq \mathbb{R}^n$,

$$u_t(t, x) = f(t, x, D^{\alpha(1)}[u(t, x)], \dots, D^{\alpha(k)}[u(t, x)]), \quad (3.1)$$

where $u : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m$ and for each $i \in \{1, \dots, k\}$, $D^{\alpha(i)}[u(t, x)]$ is a partial derivative in x . Assume that solutions to (3.1) exist, are unique and depend continuously on initial conditions.

Definition 1. *If there exist scalars $k, \alpha > 0$ such that solution to (3.1) satisfies*

$$\|u(t, \cdot)\|_{L_2^m} \leq k \|u(0, \cdot)\|_{L_2^m} \exp(-\alpha t) \quad \text{for all } t > 0.$$

then (3.1) is called exponentially stable in L_2^m norm.

The following theorem provides sufficient conditions for (3.1) to be exponentially stable.

Theorem 2. *Let there exist continuous $V : L_2^m(\Omega) \rightarrow \mathbb{R}$, $l, p \in \mathbb{N}$ and $b, a > 0$ such that*

$$a \|w\|_{L_2^m}^l \leq V(w) \leq b \|w\|_{L_2^m}^p, \quad (3.2)$$

for all $w \in L_2^m(\Omega)$. Furthermore, suppose that there exists $c \geq 0$ such that for all $t \geq 0$ the upper right-hand derivative

$$D^+ [V(u(t, \cdot))] \leq -c \|u(t, \cdot)\|_{L_2}^p, \quad (3.3)$$

where u satisfies (3.1). Then for all $t \geq 0$

$$\|u(t, \cdot)\|_{L_2} \leq \sqrt[l]{\frac{b}{a}} \|u(0, \cdot)\|_{L_2}^{p/l} \exp\left\{-\frac{c}{lb}t\right\}.$$

Proof. Let conditions of Theorem 2 be satisfied. From (3.2) it follows that for each $t \geq 0$

$$a\|u(t, \cdot)\|_{L_2^m}^l \leq V(u(t, \cdot)) \leq b\|u(t, \cdot)\|_{L_2^m}^p. \quad (3.4)$$

Dividing both sides of the second inequality in (3.4) by b results in

$$\frac{1}{b}V(u(t, \cdot)) \leq \|u(t, \cdot)\|_{L_2^m}^p. \quad (3.5)$$

After multiplying both sides of (3.5) by $-c$, we have

$$-c\|u(t, \cdot)\|_{L_2^m}^p \leq -\frac{c}{b}V(u(t, \cdot)). \quad (3.6)$$

From (3.3) and (3.6) it follows that

$$D^+ [V(u(t, \cdot))] \leq -\frac{c}{b}V(u(t, \cdot)). \quad (3.7)$$

To use the comparison principle, consider the ODE

$$\frac{d}{dt} [\phi(t)] = -\frac{c}{b}\phi(t), \quad \phi(0) = V(u(0, \cdot)), \quad (3.8)$$

where $t \in (0, \infty)$ and function $\phi : [0, \infty) \rightarrow \mathbb{R}$ is continuous. Solution for (3.8) is

$$\phi(t) = V(u(0, \cdot)) \exp \left\{ -\frac{c}{b}t \right\}$$

for all $t \geq 0$. Applying Lemma 1 for (3.7) and (3.8) results in

$$V(u(t, \cdot)) \leq V(u(0, \cdot)) \exp \left\{ -\frac{c}{b}t \right\} \quad (3.9)$$

for all $t \geq 0$. Substituting $t = 0$ in the second inequality of (3.4) implies

$$V(u(0, \cdot)) \leq b\|u(0, \cdot)\|_{L_2^m}^p. \quad (3.10)$$

Combining the first inequality of (3.4) with (3.10) and (3.9) gives

$$a\|u(t, \cdot)\|_{L_2^m}^l \leq V(u(t, \cdot)) \leq V(u(0, \cdot)) \exp \left\{ -\frac{c}{b}t \right\} \leq b\|u(0, \cdot)\|_{L_2^m}^p \exp \left\{ -\frac{c}{b}t \right\}. \quad (3.11)$$

Dividing (3.11) by a and taking the l^{th} root results in

$$\|u(t, \cdot)\|_{L_2^m} \leq \sqrt[l]{\frac{b}{a}} \|u(0, \cdot)\|_{L_2^m}^{p/l} \exp \left\{ -\frac{c}{lb}t \right\} \text{ for all } t \geq 0.$$

□

COUPLED LINEAR PDES

Recall from Dullerud and Paganini (2013) how LMIs can be used in stability analysis for linear ODEs.

4.1 Quadratic Lyapunov Functions

Theorem 3. *Given a system*

$$\frac{d}{dt}[x(t)] = Ax(t) \quad (4.1)$$

where $t > 0$, $x : (0, \infty) \rightarrow \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, if

$$\exists P > 0 \text{ such that } A^T P + PA < 0 \quad (4.2)$$

then (4.1) is exponentially stable.

Proof. If (4.2) holds, then $V(z) = z^T P z$ is a Lyapunov function for (4.1) since $A^T P + PA < 0$ and for all $t > 0$,

$$\begin{aligned} \frac{d}{dt}[V(x(t))] &= \frac{d}{dt}[x(t)^T] P x(t) + x(t)^T P \frac{d}{dt}[x(t)] = x(t)^T A^T P x(t) + x(t)^T P A x(t) \\ &= x(t)^T (A^T P + PA) x(t) < 0. \end{aligned}$$

□

The natural question is if one can parameterize a set of positive and negative operators on functional spaces such as $L_2^m(\Omega)$ with some $\Omega \subseteq \mathbb{R}^n$. The following theorem uses positive matrices to parameterize a set of positive operators on $L_2^m(a, b)$ of the form

$$(\mathcal{P}w)(x) := M(x)w(x) + \int_a^b N(x, y)w(y) dy$$

for all $x \in (a, b)$ with any $a, b \in \mathbb{R}$.

Theorem 4. *Given any positive semi-definite matrix $P \in \mathbb{S}^{\frac{m}{2}(d+1)(d+4)}$ one can partition it as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad (4.3)$$

such that $P_{11} \in \mathbb{S}^{m(d+1)}$. Define

$$Z_1(x) := Z_d(x) \otimes I_m \quad \text{and} \quad Z_2(x, y) := Z_d(x, y) \otimes I_m \quad (4.4)$$

where $x, y \in (a, b)$, Z_d is a vector of monomials up to degree d and \otimes is the Kronecker product. If for some $\epsilon > 0$

$$M(x) := Z_1(x)^T P_{11} Z_1(x) + \epsilon I_m, \quad (4.5)$$

$$N(x, y) := Z_1(x)^T P_{12} Z_2(x, y) + Z_2(y, x)^T P_{21} Z_1(y) + \int_a^b Z_2(z, x)^T P_{22} Z_2(z, y) dz, \quad (4.6)$$

then functional $V : L_2^m(a, b) \rightarrow \mathbb{R}$, defined as

$$V(w) := \int_a^b w(x)^T M(x) w(x) dx + \int_a^b w(x)^T \int_a^b N(x, y) w(y) dy dx, \quad (4.7)$$

satisfies

$$V(w) \geq \epsilon \|w\|_{L_2^m}^2 \quad \text{for all } w \in L_2^m(a, b). \quad (4.8)$$

Proof. The idea of the proof is to show that V from (4.7), satisfies the following equation.

$$V(w) = \int_a^b (\mathcal{Z}w)(x)^T P (\mathcal{Z}w)(x) dx + \epsilon \int_a^b w(x)^T w(x) dx, \quad (4.9)$$

where for all $x \in (a, b)$,

$$(\mathcal{Z}w)(x) := \begin{bmatrix} Z_1(x)w(x) \\ \int_a^b Z_2(x, y)w(y)dy \end{bmatrix}. \quad (4.10)$$

Since $P \geq 0$, then it is straightforward to show (4.8).

Consider the first integral of the right hand side in (4.9), substitute for \mathcal{Z} from (4.10) and use the partition (4.3) as follows.

$$\begin{aligned}
\int_a^b (\mathcal{Z}w)(x)^T P (\mathcal{Z}w)(x) dx &= \int_a^b w(x)^T Z_1(x)^T P_{11} Z_1(x) w(x) dx \\
&+ \int_a^b w(x)^T Z_1(x)^T P_{12} \int_a^b Z_2(x, y) w(y) dy dx \\
&+ \int_a^b \int_a^b w(y)^T Z_2(x, y)^T dy P_{21} Z_1(x) w(x) dx \\
&+ \int_a^b \int_a^b w(y)^T Z_2(x, y)^T dy P_{22} \int_a^b Z_2(x, z) w(z) dz dx.
\end{aligned} \tag{4.11}$$

Changing the order of integration in the 3rd integral of the right hand side of (4.11) and then switching between the integration variables x and y results in

$$\int_a^b \int_a^b w(y)^T Z_2(x, y)^T dy P_{21} Z_1(x) w(x) dx = \int_a^b w(x)^T \int_a^b Z_2(y, x)^T P_{21} Z_1(y) w(y) dy dx. \tag{4.12}$$

Changing two times the order of integration in the 4th integral of the right hand side of (4.11) and then switching first between the integration variables x and z , and then between x and y results in

$$\begin{aligned}
\int_a^b \int_a^b w(y)^T Z_2(x, y)^T dy P_{22} \int_a^b Z_2(x, z) w(z) dz dx & \\
&= \int_a^b \int_a^b \int_a^b w(y)^T Z_2(x, y)^T P_{22} Z_2(x, z) w(z) dx dz dy \\
&= \int_a^b \int_a^b \int_a^b w(y)^T Z_2(z, y)^T P_{22} Z_2(z, x) w(x) dz dx dy \\
&= \int_a^b \int_a^b \int_a^b w(x)^T Z_2(z, x)^T P_{22} Z_2(z, y) w(y) dz dy dx \\
&= \int_a^b w(x)^T \int_a^b \int_a^b Z_2(z, x)^T P_{22} Z_2(z, y) dz w(y) dy dx.
\end{aligned} \tag{4.13}$$

Using (4.11)-(4.13) one can write

$$\begin{aligned} \int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x) dx &= \int_a^b w(x)^T Z_1(x)^T P_{11} Z_1(x) w(x) dx \\ &+ \int_a^b w(x)^T \int_a^b \left(Z_1(x) P_{12} Z_2(x, y) + Z_2(y, x) P_{21} Z_1(y) \right. \\ &\quad \left. + \int_a^b Z_2(z, x)^T P_{22} Z_2(z, y) dz \right) w(y) dy dx. \end{aligned} \quad (4.14)$$

From (4.5), (4.6) and (4.14) it follows that

$$\begin{aligned} \int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x) dx &= \int_a^b w(x)^T M(x) w(x) dx - \epsilon \int_a^b w(x)^T w(x) dx \\ &+ \int_a^b w(x)^T \int_a^b N(x, y) w(y) dy dx. \end{aligned} \quad (4.15)$$

Adding $\epsilon \int_a^b w(x)^T w(x) dx$ to the both sides of (4.15) and using (4.7) results in (4.9), which concludes the proof. \square

4.2 Extending the Set of Lyapunov Candidates

Form (4.9) parameterizes positive functionals over $L_2^m(a, b)$ for any $a, b \in \mathbb{R}$. But a, b are usually known ahead, thus we can parameterize a set of functionals that are positive over $L_2^m(a, b)$, but not necessary over $L_2^m(c, d)$ with $c < a < b < d$.

Adding an extra term in (4.9) as follows allows to parameterize a larger set of Lyapunov candidates.

$$\begin{aligned} V(w) &= \int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x) dx + \epsilon \int_a^b w(x)^T w(x) dx \\ &+ \int_a^b g(x) (\mathcal{Z}w)(x)^T Q(\mathcal{Z}w)(x) dx, \end{aligned} \quad (4.16)$$

where $g : [a, b] \rightarrow \mathbb{R}$ is continuous and positive and $Q \geq 0$. In this work we used

$$g(x) := (x - a)(b - x) \quad (4.17)$$

for all $x \in [a, b]$. Other choices for g are possible. For more information see Positivstellensatz results in Stengle (1974). Based on (4.16) Theorem 4 can be modified as follows.

Theorem 5. *Given any positive semi-definite matrices $P, Q \in \mathbb{S}^{\frac{m}{2}(d+1)(d+4)}$ one can partition them as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad (4.18)$$

such that $P_{11}, Q_{11} \in \mathbb{S}^{m(d+1)}$. If for some $\epsilon > 0$

$$M(x) := Z_1(x)^T (P_{11} + g(x)Q_{11}) Z_1(x) + \epsilon I_m, \quad (4.19)$$

$$N(x, y) := Z_1(x)^T (P_{12} + g(x)Q_{12}) Z_2(x, y) + Z_2(y, x)^T (P_{21} + g(y)Q_{21}) Z_1(y) \quad (4.20)$$

$$+ \int_a^b Z_2(z, x)^T (P_{22} + g(z)Q_{22}) Z_2(z, y) dz,$$

where as before

$$Z_1(x) := Z_d(x) \otimes I_m \quad \text{and} \quad Z_2(x, y) := Z_d(x, y) \otimes I_m$$

and some positive and continuous function g , then functional $V : L_2^m(a, b) \rightarrow \mathbb{R}$, defined as

$$V(w) := \int_a^b w(x)^T M(x) w(x) dx + \int_a^b w(x)^T \int_a^b N(x, y) w(y) dy dx, \quad (4.21)$$

satisfies $V(w) \geq \epsilon \|w\|_{L_2^m}$ for all $w \in L_2^m(a, b)$.

Proof. The idea of the proof is to show that V from (4.21), satisfies the following equation.

$$V(w) = \epsilon \int_a^b w(x)^T w(x) dx + \int_a^b (\mathcal{Z}w)(x)^T P (\mathcal{Z}w)(x) dx$$

$$+ \int_a^b g(x) (\mathcal{Z}w)(x)^T Q (\mathcal{Z}w)(x) dx, \quad (4.22)$$

where

$$(\mathcal{Z}w)(x) := \begin{bmatrix} Z_1(x)w(x) \\ \int_a^b Z_2(x,y)w(y)dy \end{bmatrix}. \quad (4.23)$$

Since $P, Q \geq 0$ and $g(x) \geq 0$ for all $x \in [a, b]$, then the right hand side of (4.22) is positive.

Consider 3rd integral of (4.22) using (4.23), since for the 2nd integral steps are almost the same, except it does not have the multiplier function g .

$$\begin{aligned} \int_a^b g(x)(\mathcal{Z}w)(x)^T Q (\mathcal{Z}w)(x) dx &= \int_a^b g(x)w(x)^T Z_1(x)^T Q_{11} Z_1(x)w(x) dx \\ &+ \int_a^b g(x)w(x)^T Z_1(x)^T Q_{12} \int_a^b Z_2(x,y)w(y) dy dx \\ &+ \int_a^b g(x) \int_a^b w(y)^T Z_2(x,y)^T dy Q_{21} Z_1(x)w(x) dx \\ &+ \int_a^b g(x) \int_a^b w(y)^T Z_2(x,y)^T dy Q_{22} \int_a^b Z_2(x,z)w(z) dz dx. \end{aligned} \quad (4.24)$$

Changing the order of integration in the 3rd integral of the right hand side of (4.24) and then switching between the integration variables x and y results in

$$\begin{aligned} \int_a^b g(x) \int_a^b w(y)^T Z_2(x,y)^T dy Q_{21} Z_1(x)w(x) dx \\ = \int_a^b w(x)^T \int_a^b g(y) Z_2(y,x)^T Q_{21} Z_1(y)w(y) dy dx. \end{aligned} \quad (4.25)$$

Changing two times the order of integration in the 4th integral of the right hand side of (4.24) and then switching first between the integration variables x and z , and then between x and y results in

$$\begin{aligned}
& \int_a^b g(x) \int_a^b w(y)^T Z_2(x, y)^T dy Q_{22} \int_a^b Z_2(x, z) w(z) dz dx \\
&= \int_a^b \int_a^b \int_a^b g(x) w(y)^T Z_2(x, y)^T Q_{22} Z_2(x, z) w(z) dx dz dy \\
&= \int_a^b \int_a^b \int_a^b g(z) w(y)^T Z_2(z, y)^T Q_{22} Z_2(z, x) w(x) dz dx dy \\
&= \int_a^b \int_a^b \int_a^b g(z) w(x)^T Z_2(z, x)^T Q_{22} Z_2(z, y) w(y) dz dy dx \\
&= \int_a^b w(x)^T \int_a^b \int_a^b Z_2(z, x)^T g(z) Q_{22} Z_2(z, y) dz w(y) dy dx.
\end{aligned} \tag{4.26}$$

Using (4.24)-(4.26) one can write

$$\begin{aligned}
\int_a^b g(x) (\mathcal{Z}w)(x)^T Q (\mathcal{Z}w)(x) dx &= \int_a^b w(x)^T Z_1(x)^T g(x) Q_{11} Z_1(x) w(x) dx \\
&+ \int_a^b w(x)^T \int_a^b \left(Z_1(x) g(x) Q_{12} Z_2(x, y) \right. \\
&\quad \left. + Z_2(y, x) g(y) Q_{21} Z_1(y) \right. \\
&\quad \left. + \int_a^b Z_2(z, x)^T g(z) Q_{22} Z_2(z, y) dz \right) w(y) dy dx.
\end{aligned} \tag{4.27}$$

Following the same idea as in (4.11)-(4.14) it is possible to achieve

$$\begin{aligned}
\int_a^b (\mathcal{Z}w)(x)^T P (\mathcal{Z}w)(x) dx &= \int_a^b w(x)^T Z_1(x)^T P_{11} Z_1(x) w(x) dx \\
&+ \int_a^b w(x)^T \int_a^b \left(Z_1(x) P_{12} Z_2(x, y) + Z_2(y, x) P_{21} Z_1(y) \right. \\
&\quad \left. + \int_a^b Z_2(z, x)^T P_{22} Z_2(z, y) dz \right) w(y) dy dx. \tag{4.28}
\end{aligned}$$

Using (4.19), (4.20), (4.27) and (4.28) yields

$$\begin{aligned} & \int_a^b (\mathcal{Z}w)(x)^T P(\mathcal{Z}w)(x) dx + \int_a^b g(x) (\mathcal{Z}w)(x)^T Q(\mathcal{Z}w)(x) dx \\ &= \int_a^b w(x)^T M(x) w(x) dx - \epsilon \int_a^b w(x)^T w(x) dx + \int_a^b w(x)^T \int_a^b N(x, y) w(y) dy dx. \end{aligned} \quad (4.29)$$

Adding $\epsilon \int_a^b w(x)^T w(x) dx$ to the both sides of (4.29) and using (4.21) results in (4.22), which concludes the proof. \square

For simplicity, define a set of polynomials (M, N) as follows.

$$\Sigma_+^{m,d,\epsilon} := \{(M, N) : \exists P, Q \geq 0 \text{ such that (4.19), (4.20) hold}\}. \quad (4.30)$$

Similarly, define a different set of polynomials that parameterize functionals of the form (4.21) such that $V(w) \leq -\epsilon \|w\|_{L_2^m}$ for all $w \in L_2^m(a, b)$. Denote

$$\Sigma_-^{m,d,\epsilon} := \{(M, N) : (-M, -N) \in \Sigma_+^{m,d,\epsilon}\}. \quad (4.31)$$

4.3 Quadratic Form of the Time Derivative of Lyapunov Function

In this section a quadratic form of the time derivative of Lyapunov function is presented. First recall the PDE of interest. For all $t \in (0, \infty)$ and $x \in (a, b) \subset \mathbb{R}$, $u : [0, \infty) \times [a, b] \rightarrow \mathbb{R}^m$ satisfies

$$u_t(t, x) = A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x), \quad (4.32)$$

where A, B, C are some given polynomial matrices.

Substituting $u(t, \cdot)$ for w in (4.7) and differentiating with respect to t results in

$$\begin{aligned}
\frac{d}{dt}[V(u(t, \cdot))] &= \frac{d}{dt} \left[\int_a^b u(t, x)^T M(x) u(t, x) dx + \int_a^b u(t, x)^T \int_a^b N(x, y) u(t, y) dy dx \right] \\
&= \int_a^b (u_t(t, x)^T M(x) u(t, x) + u(t, x)^T M(x) u_t(t, x)) dx \\
&\quad + \int_a^b \left(u_t(t, x)^T \int_a^b N(x, y) u(t, y) dy + u(t, x)^T \int_a^b N(x, y) u_t(t, y) dy \right) dx.
\end{aligned} \tag{4.33}$$

Now substituting for u_t from (4.32) into (4.33) yields

$$\begin{aligned}
\frac{d}{dt}[V(u(t, \cdot))] &= \int_a^b \left(\left(A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x) \right)^T M(x) u(t, x) \right. \\
&\quad \left. + u(t, x)^T M(x) \left(A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x) \right) \right) dx \\
&\quad + \int_a^b \left(\left(A(x)u_{xx}(t, x) + B(x)u_x(t, x) \right. \right. \\
&\quad \left. \left. + C(x)u(t, x) \right)^T \int_a^b N(x, y) u(t, y) dy \right. \\
&\quad \left. + u(t, x)^T \int_a^b N(x, y) \left(A(y)u_{yy}(t, y) \right. \right. \\
&\quad \left. \left. + B(y)u_y(t, y) + C(y)u(t, y) \right) dy \right) dx. \tag{4.34}
\end{aligned}$$

If one defines

$$\begin{aligned}
K(x) &:= \begin{bmatrix} C(x)^T M(x) + M(x)C(x)(x) & M(x)B(x) & M(x)A(x) \\ B(x)^T M(x) & 0 & 0 \\ A(x)^T M(x) & 0 & 0 \end{bmatrix}, \\
L(x, y) &:= \begin{bmatrix} C(x)^T N(x, y) + N(x, y)C(y) & N(x, y)B(y) & N(x, y)A(y) \\ B(x)^T N(x, y) & 0 & 0 \\ A(x)^T N(x, y) & 0 & 0 \end{bmatrix}, \\
q(t, x) &:= \begin{bmatrix} u(t, x) \\ u_x(t, x) \\ u_{xx}(t, x) \end{bmatrix},
\end{aligned}$$

then (4.34) can be written as

$$\frac{d}{dt} [V(u(t, \cdot))] = \int_a^b q(t, x)^T K(x) q(t, x) dx + \int_a^b q(t, x)^T \int_a^b L(x, y) q(t, y) dy dx. \tag{4.35}$$

Equation (4.35) represents quadratic form of the time derivative of Lyapunov candidate. If $(K, L) \in \Sigma_-^{3m, d, 0}$ then for every $t > 0$

$$\frac{d}{dt} [V(u(t, \cdot))] \leq 0$$

and, therefore, V is a Lyapunov function, thus PDE (4.32) is stable. If $(K, L) \in \Sigma_-^{3m, d, \epsilon}$ then (4.32) is exponentially stable.

Notice, that condition $(K, L) \in \Sigma_-^{3m, d, 0}$ is conservative. The reason is that the elements in q are not independent, i.e. the second and third elements are partial derivatives of the first one. Therefore, for the PDE (4.32) to be stable, it is enough

to check if (4.35) is negative on a subspace of $L_2^{3m}(a, b)$, which is

$$\Lambda = \left\{ \begin{array}{l} \left[\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \right] \in L_2^{3m}(a, b) : D \left[\begin{array}{c} w_1(a) \\ w_1(b) \\ w_2(a) \\ w_2(b) \end{array} \right] = 0, \quad \begin{array}{l} w_2 = w_1', \\ w_3 = w_1'' \end{array} \end{array} \right\} \quad (4.36)$$

Notice, that Λ depends on D that represents the boundary conditions as before in (1.3).

4.4 Spacing Operators

Results of the following theorem are used to parameterize functions which are negative on Λ , but not necessarily on the whole space $L_2^{3m}(a, b)$.

Theorem 6. *Let X be a closed subspace of some Hilbert space Y . Then $\langle u, \mathcal{R}u \rangle_Y \leq 0$ for all $u \in X$ if and only if there exist \mathcal{M} and \mathcal{T} such that $\mathcal{R} = \mathcal{M} + \mathcal{T}$ and $\langle w, \mathcal{M}w \rangle_Y \leq 0$ for all $w \in Y$ and $\langle u, \mathcal{T}u \rangle_Y = 0$ for all $u \in X$.*

Proof. For (\Rightarrow), suppose that $\langle u, \mathcal{R}u \rangle_Y \leq 0$ for all $u \in X$. Since X is a closed subspace of a Hilbert space Y , there exists a projection operator such that $\mathcal{P} = \mathcal{P}^* = \mathcal{P}\mathcal{P}$ and $\mathcal{P}w \in X$ for all $w \in Y$. Let $\mathcal{M} = \mathcal{P}\mathcal{R}\mathcal{P}$ and $\mathcal{T} = \mathcal{M} - \mathcal{R}$. Then for all $w \in Y$,

$$\langle w, \mathcal{M}w \rangle_Y = \langle w, \mathcal{P}\mathcal{R}\mathcal{P}w \rangle_Y = \langle \mathcal{P}w, \mathcal{R}\mathcal{P}w \rangle_Y \leq 0$$

since $\mathcal{P}w \in X$. Furthermore, for all $u \in X$

$$\begin{aligned} \langle u, \mathcal{T}u \rangle_Y &= \langle u, \mathcal{P}\mathcal{R}\mathcal{P}u \rangle_Y - \langle u, \mathcal{R}u \rangle_Y = \langle \mathcal{P}u, \mathcal{R}\mathcal{P}u \rangle_Y - \langle u, \mathcal{R}u \rangle_Y \\ &= \langle u, \mathcal{R}u \rangle_Y - \langle u, \mathcal{R}u \rangle_Y = 0. \end{aligned}$$

For (\Leftarrow), assume that there exist \mathcal{M} and \mathcal{T} such that $\mathcal{R} = \mathcal{M} + \mathcal{T}$ and $\langle w, \mathcal{M}w \rangle_Y \leq 0$ for all $w \in Y$ and $\langle u, \mathcal{T}u \rangle_Y = 0$ for all $u \in X$. Then for all $u \in X$,

$$\langle u, \mathcal{R}u \rangle_Y = \langle u, (\mathcal{M} + \mathcal{T})u \rangle_Y = \langle u, \mathcal{M}u \rangle_Y + \langle u, \mathcal{T}u \rangle_Y = \langle u, \mathcal{M}u \rangle_Y \leq 0.$$

□

As was shown previously, $\Sigma_-^{3n,d,\epsilon}$ parameterizes a subset of \mathcal{M} . Next step is to parameterize a subset of operators \mathcal{T} - the so-called “spacing operators” using polynomial spacing functions. Therefore the sum of \mathcal{M} and \mathcal{T} yield an operator \mathcal{R} which is negative on Λ , but not necessarily on $L_2^{3m}(a, b)$ space.

4.5 Parametrization of Spacing Operators by Polynomials

The following lemmas define the structure of polynomial matrices T and R such that for all $\lambda \in \Lambda$

$$\int_a^b \lambda(x)^T T(x) \lambda(x) dx + \int_a^b \lambda(x)^T \int_a^b R(x, y) \lambda(y) dy dx = 0,$$

where as before

$$\Lambda = \left\{ \begin{array}{l} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in L_2^{3m}(a, b) : D \begin{bmatrix} w_1(a) \\ w_1(b) \\ w_2(a) \\ w_2(b) \end{bmatrix} = 0, \begin{array}{l} w_2 = w'_1, \\ w_3 = w''_1 \end{array} \end{array} \right\}. \quad (4.37)$$

Lemma 2. Let $P_1, P_2, P_3, P_4 : [a, b] \rightarrow \mathbb{R}^{m \times m}$ be polynomials and $w, w', w'' \in L_2^m(a, b)$.

If

$$T(x) = \begin{bmatrix} P'_1(x) & P_1(x) + P'_2(x) & P_2(x) \\ P_1(x) + P'_3(x) & P_2(x) + P_3(x) + P'_4(x) & P_4(x) \\ P_3(x) & P_4(x) & 0 \end{bmatrix} \quad (4.38)$$

then

$$\int_a^b \lambda(x)^T T(x) \lambda(x) dx = q^T \Pi_1 q, \quad (4.39)$$

where

$$\lambda(x) := \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}, \quad q := \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix}, \quad \Pi_1 := \begin{bmatrix} -P_1(a) & 0 & -P_2(a) & 0 \\ 0 & P_1(b) & 0 & P_2(a) \\ -P_3(a) & 0 & -P_4(a) & 0 \\ 0 & P_3(b) & 0 & P_4(b) \end{bmatrix}. \quad (4.40)$$

Proof. Using the fundamental theorem of calculus it is true that

$$\begin{aligned} & \int_a^b \frac{d}{dx} \left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \end{bmatrix} \begin{bmatrix} w(x) \\ w'(x) \end{bmatrix} \right) dx \\ &= \begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} P_1(b) & P_2(b) \\ P_3(b) & P_4(b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix} \\ & \quad - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} P_1(a) & P_2(a) \\ P_3(a) & P_4(a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \\ &= \begin{bmatrix} w(a)^T \\ w(b)^T \\ w'(a)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} -P_1(a) & 0 & -P_2(a) & 0 \\ 0 & P_1(b) & 0 & P_2(a) \\ -P_3(a) & 0 & -P_4(a) & 0 \\ 0 & P_3(b) & 0 & P_4(b) \end{bmatrix} \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix} \\ &= q^T \Pi_1 q. \end{aligned} \quad (4.41)$$

From the other side, using chain rule it can be seen that

$$\begin{aligned}
& \frac{d}{dx} \left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \end{bmatrix} \begin{bmatrix} w(x) \\ w'(x) \end{bmatrix} \right) \\
&= \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x)^T \end{bmatrix}^T \begin{bmatrix} P_1'(x) & P_1(x) + P_2'(x) & P_2(x) \\ P_1(x) + P_3'(x) & P_2(x) + P_3(x) + P_4'(x) & P_4(x) \\ P_3(x) & P_4(x) & 0 \end{bmatrix} \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix} \\
&= \lambda(x)^T T(x) \lambda(x). \tag{4.42}
\end{aligned}$$

Combining (4.41) and (4.42) results in (4.39). \square

Notice that $Dq = 0$ and, therefore,

$$\begin{aligned}
q^T \Pi_1 q &= q^T (I_{4m} - D + D)^T \Pi_1 (I_{4m} - D + D) q \\
&= q^T (I_{4m} - D + D)^T \Pi_1 (I_{4m} - D) q = q^T (I_{4m} - D)^T \Pi_1 (I_{4m} - D) q.
\end{aligned}$$

Using Lemma (2) one can define the following set.

$$\Xi_1^D := \{T \text{ as defined in (4.38)} : (I_{4m} - D)^T \Pi_1 (I_{4m} - D) = 0, \Pi_1 \text{ as defined in (4.40)}\}$$

Thus, for any $T \in \Xi_1^D$ and any $\lambda \in \Lambda$ it is true that

$$\int_a^b \lambda(x)^T T(x) \lambda(x) dx = 0.$$

Lemma 3. *Let $Q_1, Q_2, Q_3, Q_4 : [a, b] \times [a, b] \rightarrow \mathbb{R}^{m \times m}$ be polynomials and $w, w', w'' \in L_2^m(a, b)$. If*

$$\begin{aligned}
R_1(x, y) &:= \begin{bmatrix} Q_{1,xy}(x, y) & Q_{3,xy}(x, y) + Q_{1,x}(x, y) & Q_{3,x}(x, y) \\ Q_{2,xy}(x, y) + Q_{1,y}(x, y) & R_{22}(x, y) & Q_{4,x}(x, y) + Q_3(x, y) \\ Q_{2,y}(x, y) & Q_{4,y}(x, y) + Q_2(x, y) & Q_4(x, y) \end{bmatrix} \\
R_{22}(x, y) &:= Q_{4,xy}(x, y) + Q_{2,x}(x, y) + Q_{3,y}(x, y), \tag{4.43}
\end{aligned}$$

then

$$\int_a^b \int_a^b \lambda(x)^T R_1(x, y) \lambda(y) dx dy = q^T \Theta_1 q,$$

where

$$\lambda(x) := \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}, \quad q := \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix},$$

$$\Theta_1 := \begin{bmatrix} Q_1(a, a) & -Q_1(a, b) & Q_3(a, a) & -Q_3(a, b) \\ -Q_1(b, a) & Q_1(b, b) & -Q_3(b, a) & Q_3(b, b) \\ Q_2(a, a) & -Q_2(a, b) & Q_4(a, a) & -Q_4(a, b) \\ -Q_2(b, a) & Q_2(b, b) & -Q_4(b, a) & Q_4(b, b) \end{bmatrix}. \quad (4.44)$$

Proof. Applying the fundamental theorem of calculus twice to

$$\begin{aligned}
& \int_a^b \int_a^b \frac{\partial^2}{\partial x \partial y} \left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_1(x, y) & Q_3(x, y) \\ Q_2(x, y) & Q_4(x, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) dx dy \\
&= \int_a^b \frac{\partial}{\partial y} \left(\begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(b, y) & Q_3(b, y) \\ Q_2(b, y) & Q_4(b, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right. \\
&\quad \left. - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a, y) & Q_3(a, y) \\ Q_2(a, y) & Q_4(a, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) dy \\
&= \begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(b, b) & Q_3(b, b) \\ Q_2(b, b) & Q_4(b, b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix} \\
&\quad - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a, b) & Q_3(a, b) \\ Q_2(a, b) & Q_4(a, b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix} \\
&\quad - \begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(b, a) & Q_3(b, a) \\ Q_2(b, a) & Q_4(b, a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \\
&\quad + \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a, a) & Q_3(a, a) \\ Q_2(a, a) & Q_4(a, a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \\
&= \begin{bmatrix} w(a)^T \\ w(b)^T \\ w'(a)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_1(a, a) & -Q_1(a, b) & Q_3(a, a) & -Q_3(a, b) \\ -Q_1(b, a) & Q_1(b, b) & -Q_3(b, a) & Q_3(b, b) \\ Q_2(a, a) & -Q_2(a, b) & Q_4(a, a) & -Q_4(a, b) \\ -Q_2(b, a) & Q_2(b, b) & -Q_4(b, a) & Q_4(b, b) \end{bmatrix} \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix} \\
&= q^T \Theta_1 q. \tag{4.45}
\end{aligned}$$

From the other side, using the chain rule, one can get

$$\begin{aligned}
& \frac{\partial^2}{\partial x \partial y} \left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_1(x, y) & Q_3(x, y) \\ Q_2(x, y) & Q_4(x, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) \\
&= \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x) \end{bmatrix}^T R_1(x, y) \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix} = \lambda(x)^T R_1(x, y) \lambda(y), \tag{4.46}
\end{aligned}$$

where R_1 is defined in (4.43). Then combining (4.45) and (4.46) finishes the proof. \square

As before, notice that

$$\begin{aligned}
q^T \Theta_1 q &= q^T (I_{4m} - D + D)^T \Theta_1 (I_{4m} - D + D) q \\
&= q^T (I_{4m} - D + D)^T \Theta_1 (I_{4m} - D) q = q^T (I_{4m} - D)^T \Theta_1 (I_{4m} - D) q.
\end{aligned}$$

Similarly as for Ξ_1 , using Lemma (3) one can define a set

$$\Xi_2^D := \{R_1 \text{ as defined in (4.43)} : (I_{4n} - D)^T \Theta_1 (I_{4n} - D) = 0, \Theta_1 \text{ as defined in (4.44)}\}.$$

Thus, for any $R_1 \in \Xi_2^D$ and any $\lambda \in \Lambda$,

$$\int_a^b \int_a^b \lambda(x)^T R_1(x, y) \lambda(y) dx dy = 0.$$

Lemma 4. Let $Q_5, Q_6 : [a, b] \times [a, b] \rightarrow \mathbb{R}^{m \times m}$ be polynomials and $w, w', w'' \in L^m(a, b)$.

If

$$R_2(x, y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{5,y}(x, y) & Q_{6,y}(x, y) + Q_5(x, y) & Q_6(x, y) \end{bmatrix} \tag{4.47}$$

then

$$\int_a^b \int_a^b \lambda(x)^T R_2(x, y) \lambda(y) dx dy = \int_a^b w''(x)^T \Theta_2(x) q dx,$$

where

$$\Theta_2(x) = \begin{bmatrix} -Q_5(x, a) & Q_5(x, b) & -Q_6(x, a) & Q_6(x, b) \end{bmatrix}. \quad (4.48)$$

Proof. Start with applying the fundamental theorem of calculus to

$$\begin{aligned} & \int_a^b \int_a^b \frac{\partial}{\partial y} \left(w''(x)^T \begin{bmatrix} Q_5(x, y) & Q_6(x, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) dx dy \\ &= \int_a^b \left(w''(x)^T \begin{bmatrix} Q_5(x, b) & Q_6(x, b) \end{bmatrix} \begin{bmatrix} w(b) \\ w'(b) \end{bmatrix} \right. \\ & \quad \left. - w''(x)^T \begin{bmatrix} Q_5(x, a) & Q_6(x, a) \end{bmatrix} \begin{bmatrix} w(a) \\ w'(a) \end{bmatrix} \right) dx \\ &= \int_a^b w''(x)^T \begin{bmatrix} -Q_5(x, a) & Q_5(x, b) & -Q_6(x, a) & Q_6(x, b) \end{bmatrix} \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix} dx. \end{aligned} \quad (4.49)$$

Using the chain rule one can get

$$\begin{aligned} & \frac{\partial}{\partial y} \left(w''(x)^T \begin{bmatrix} Q_5(x, y) & Q_6(x, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \end{bmatrix} \right) = \\ & \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x)^T \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{5,y}(x, y) & Q_{6,y}(x, y) + Q_5(x, y) & Q_6(x, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \\ w''(y) \end{bmatrix}. \end{aligned} \quad (4.50)$$

Combining (4.49) and (4.50) concludes the proof. \square

As previously, a set can be defined using Lemma (4).

$$\Xi_3^D := \left\{ R_2 \text{ as defined in (4.47)} : \begin{array}{l} \Theta_2(x)^T (I_{4n} - D) = 0 \text{ for all } x \in (a, b), \\ \Theta_2 \text{ as defined in (4.48)} \end{array} \right\}$$

Therefore, for any $R_1 \in \Xi_3^D$ and any $\lambda \in \Lambda$,

$$\int_a^b \int_a^b \lambda(x)^T R_2(x, y) \lambda(y) dx dy = 0.$$

Lemma 5. Let $Q_7, Q_8 : [a, b] \times [a, b] \rightarrow \mathbb{R}^{m \times m}$ be polynomials and $w, w', w'' \in L_2^m(a, b)$.

If

$$R_3(x, y) = \begin{bmatrix} 0 & 0 & Q_{7,x}(x, y) \\ 0 & 0 & Q_{8,x}(x, y) + Q_7(x, y) \\ 0 & 0 & Q_8(x, y) \end{bmatrix} \quad (4.51)$$

then

$$\int_a^b \int_a^b \lambda(x)^T R_3(x, y) \lambda(y) dx dy = \int_a^b q^T \Theta_3(y) w''(y) dy,$$

where

$$\lambda(x) := \begin{bmatrix} w(x) \\ w'(x) \\ w''(x) \end{bmatrix}, \quad q := \begin{bmatrix} w(a) \\ w(b) \\ w'(a) \\ w'(b) \end{bmatrix}, \quad \Theta_3(y) = \begin{bmatrix} -Q_7(a, y) \\ Q_7(b, y) \\ -Q_8(a, y) \\ Q_8(b, y) \end{bmatrix}. \quad (4.52)$$

Proof. Apply the fundamental theorem of calculus to

$$\begin{aligned} & \int_a^b \int_a^b \frac{\partial}{\partial x} \left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_7(x, y) \\ Q_8(x, y) \end{bmatrix} w''(y) \right) dx dy \\ &= \int_a^b \left(\begin{bmatrix} w(b)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} Q_7(b, y) \\ Q_8(b, y) \end{bmatrix} w''(y) - \begin{bmatrix} w(a)^T \\ w'(a)^T \end{bmatrix}^T \begin{bmatrix} Q_7(a, y) \\ Q_8(a, y) \end{bmatrix} w''(y) \right) dy \\ &= \int_a^b \begin{bmatrix} w(a)^T \\ w(b)^T \\ w'(a)^T \\ w'(b)^T \end{bmatrix}^T \begin{bmatrix} -Q_7(a, y) \\ Q_7(b, y) \\ -Q_8(a, y) \\ Q_8(b, y) \end{bmatrix} w''(y) dy. \end{aligned} \quad (4.53)$$

Using the chain rule one can get

$$\begin{aligned} \frac{\partial}{\partial x} \left(\begin{bmatrix} w(x)^T \\ w'(x)^T \end{bmatrix}^T \begin{bmatrix} Q_7(x, y) \\ Q_8(x, y) \end{bmatrix} w''(y) \right) \\ = \begin{bmatrix} w(x)^T \\ w'(x)^T \\ w''(x)^T \end{bmatrix}^T \begin{bmatrix} 0 & 0 & Q_{7,x}(x, y) \\ 0 & 0 & Q_{8,x}(x, y) + Q_7(x, y) \\ 0 & 0 & Q_8(x, y) \end{bmatrix} \begin{bmatrix} w(y) \\ w'(y) \\ w''(y) \end{bmatrix}. \end{aligned} \quad (4.54)$$

Combining (4.53) and (4.54) concludes the proof. \square

From Lemma (5) one can define the following set.

$$\Xi_4^D := \left\{ R_3 \text{ as defined in (4.51)} : \begin{array}{l} (I_{4n} - D)^T \Theta_3(y) = 0 \text{ for all } y \in (a, b), \\ \Theta_3 \text{ as defined in (4.52)} \end{array} \right\}.$$

Thus, for any $R_3 \in \Xi_4^D$ and any $\lambda \in \Lambda$,

$$\int_a^b \int_a^b \lambda(x)^T R_3(x, y) \lambda(y) dx dy = 0.$$

Finally, combining Ξ_i results in a set of polynomials that parameterize a subset of spacing operators, i.e.

$$\Sigma_0^{m,d,D} := \{(T, R) : T \in \Xi_1^D \text{ and } R \in \sum_{i=2}^4 \Xi_i^D\}, \quad (4.55)$$

which provides a pair of polynomials (T, R) such that for all $\lambda \in \Lambda$

$$\int_a^b \lambda(x)^T T(x) \lambda(x) dx + \int_a^b \lambda(x)^T \int_a^b R(x, y) \lambda(y) dy dx = 0.$$

4.6 Stability Test for Coupled PDEs

This section summarizes the results of the Chapter 4 in the following theorem.

Theorem 7. Suppose that for all $t \in (0, \infty)$ and $x \in (a, b) \subset \mathbb{R}$, $u : [0, \infty) \times [a, b] \rightarrow \mathbb{R}^m$ satisfies

$$u_t(t, x) = A(x)u_{xx}(t, x) + B(x)u_x(t, x) + C(x)u(t, x), \quad (4.56)$$

where A, B, C are some given polynomial matrices with

$$\gamma = \max\{\deg(A), \deg(B), \deg(C)\}.$$

If there exist $d \in \mathbb{N}$, $\epsilon_1, \epsilon_2 > 0$,

$$(M, N) \in \Sigma_+^{m, d, \epsilon_1}, \quad (T, R) \in \Sigma_0^{m, 2d+2+\gamma, D}, \quad (H, G) \in \Sigma_-^{3m, d+\gamma, \epsilon_2}$$

such that for all $x, y \in (a, b)$

$$\begin{bmatrix} C(x)^T M(x) + M(x)C(x) & M(x)B(x) & M(x)A(x) \\ B(x)^T M(x) & 0 & 0 \\ A(x)^T M(x) & 0 & 0 \end{bmatrix} = T(x) + H(x),$$

$$\begin{bmatrix} C(x)^T N(x, y) + N(x, y)C(y) & N(x, y)B(y) & N(x, y)A(y) \\ B(x)^T N(x, y) & 0 & 0 \\ A(x)^T N(x, y) & 0 & 0 \end{bmatrix} = R(x, y) + G(x, y),$$

then (4.56) is exponentially stable.

Proof. Suppose conditions of the Theorem 7 hold. Then V as defined in (4.7) satisfies (4.8). Since M and N are polynomials, they are continuous. Thus there exists $b \in \mathbb{R}$ such that

$$V(w) \leq b\|w\|_{L_2^n}.$$

According to (4.34) and (4.35) the time derivative of V satisfies

$$\frac{d}{dt}[V(u(t, \cdot))] \leq -\epsilon_2\|w\|_{L_2^n}$$

and, therefore, Theorem 2 can be applied concluding the proof. \square

PDES WITH TWO SPATIAL VARIABLES

In this chapter the PDEs of interest have 2 spatial variables. Specifically, for all $t > 0$ and $x \in \Omega := (0, 1)^2$, $u : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ satisfies

$$\begin{aligned} u_t(t, x) = & a(x)u_{x_1x_1}(t, x) + b(x)u_{x_1x_2}(t, x) + c(x)u_{x_2x_2}(t, x) + d(x)u_{x_1}(t, x) \\ & + e(x)u_{x_2}(t, x) + f(x)u(t, x), \end{aligned} \quad (5.1)$$

where a, b, c, d, e, f are polynomials. As before, assume that solution to (5.1) exists, is unique and depends continuously on initial conditions. Moreover, let u satisfy zero Dirichlet boundary conditions, i.e.

$$u(t, 1, x_2) = 0, \quad u(t, 0, x_2) = 0, \quad u(t, x_1, 1) = 0, \quad u(t, x_1, 0) = 0 \quad (5.2)$$

for all $x_1, x_2 \in [0, 1]$ and $t \geq 0$.

5.1 Parameterizing Lyapunov Candidates with SOS Polynomials

Any polynomial $s : \Omega \rightarrow \mathbb{R}$ defines a functional $V : L_2(\Omega) \rightarrow \mathbb{R}$ as

$$V(w) := \int_{\Omega} s(x)w(x)^2 dx. \quad (5.3)$$

If for some $\epsilon > 0$ there exists an SOS polynomial p such that

$$s(x) = p(x) + \epsilon,$$

then, for all $w \in L_2(\Omega)$, (5.3) satisfies

$$V(w) \geq \epsilon \|w\|_{L_2}^2.$$

5.2 Quadratic Form of the Lyapunov Time Derivative

Substituting $u(t, \cdot)$ for w in (5.3) and differentiating the result with respect to t gives

$$\frac{d}{dt} [V(u(t, \cdot))] = \frac{d}{dt} \left[\int_{\Omega} s(x) u(t, x)^2 dx \right] = \int_{\Omega} 2s(x) u(t, x) u_t(t, x) dx. \quad (5.4)$$

Substituting for $u_t(t, x)$ from (5.1) into (5.4) implies

$$\begin{aligned} \frac{d}{dt} [V(u(t, \cdot))] &= \int_{\Omega} 2s(x) u(t, x) \left(a(x) u_{x_1 x_1}(t, x) + b(x) u_{x_1 x_2}(t, x) + c(x) u_{x_2 x_2}(t, x) \right. \\ &\quad \left. + d(x) u_{x_1}(t, x) + e(x) u_{x_2}(t, x) + f(x) u(t, x) \right) dx \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t), \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} I_1(t) &:= \int_{\Omega} 2s(x) u(t, x) a(x) u_{x_1 x_1}(t, x) dx, \\ I_2(t) &:= \int_{\Omega} s(x) u(t, x) b(x) u_{x_2 x_1}(t, x) dx, \\ I_3(t) &:= \int_{\Omega} s(x) u(t, x) b(x) u_{x_1 x_2}(t, x) dx, \\ I_4(t) &:= \int_{\Omega} 2s(x) u(t, x) c(x) u_{x_2 x_2}(t, x) dx, \\ I_5(t) &:= \int_{\Omega} 2s(x) u(t, x) \left(d(x) u_{x_1}(t, x) + e(x) u_{x_2}(t, x) + f(x) u(t, x) \right) dx. \end{aligned}$$

Note that, based on section 5.2.3 of Evans (1998), it holds that

$$u_{x_1 x_2}(t, x) = u_{x_2 x_1}(t, x) \quad (5.6)$$

for all $x \in \Omega$. Property (5.6) was used to define I_2 and I_3 . Alternatively, I_5 can be formulated as

$$I_5(t) = \int_{\Omega} q^T(t, x) Z_5(x) q(t, x) dx, \quad (5.7)$$

where for all $x \in \Omega$

$$q(t, x) := \begin{bmatrix} u(t, x) \\ u_{x_1}(t, x) \\ u_{x_2}(t, x) \end{bmatrix}, \quad Z_5(x) := \begin{bmatrix} 2s(x)f(x) & s(x)d(x) & s(x)e(x) \\ s(x)d(x) & 0 & 0 \\ s(x)e(x) & 0 & 0 \end{bmatrix}. \quad (5.8)$$

Using integration by parts and boundary conditions (5.2), I_1 can be rewritten as follows.

$$\begin{aligned} I_1(t) &= \int_{\Omega} 2s(x)u(t, x)a(x) \frac{d}{dx_1} [u_{x_1}(t, x)] dx \\ &= 2 \int_0^1 \left(s(x)u(t, x)a(x)u_{x_1}(t, x) \Big|_{x_1=0}^{x_1=1} \right. \\ &\quad \left. - \int_0^1 u_{x_1}(t, x) \frac{d}{dx_1} [s(x)u(t, x)a(x)] dx_1 \right) dx_2 \\ &= - \int_{\Omega} 2u_{x_1}(t, x) \left(u(t, x) \frac{d}{dx_1} [s(x)a(x)] + s(x)a(x)u_{x_1}(t, x) \right) dx \\ &= - \int_{\Omega} q^T(t, x)Z_1(x)q(t, x) dx, \end{aligned} \quad (5.9)$$

where for all $x \in \Omega$

$$Z_1(x) := \begin{bmatrix} 0 & \frac{d}{dx_1} [s(x)a(x)] & 0 \\ \frac{d}{dx_1} [s(x)a(x)] & 2s(x)a(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Following steps of (5.9) for I_2, I_3 and I_4 , we get

$$\begin{aligned}
I_2(t) &= \int_{\Omega} s(x)u(t, x)b(x) \frac{d}{dx_1}[u_{x_2}(t, x)] dx \\
&= \int_0^1 \left(s(x)u(t, x)b(x)u_{x_2}(t, x)|_{x_1=0}^{x_1=1} - \int_0^1 u_{x_2}(t, x) \frac{d}{dx_1}[s(x)u(t, x)b(x)] dx_1 \right) dx_2 \\
&= - \int_{\Omega} q^T(t, x)Z_2(x)q(t, x) dx, \\
I_3(t) &= \int_{\Omega} s(x)u(t, x)b(x) \frac{d}{dx_2}[u_{x_1}(t, x)] dx \\
&= \int_0^1 \left(s(x)u(t, x)b(x)u_{x_1}(t, x)|_{x_2=0}^{x_2=1} - \int_0^1 u_{x_1}(t, x) \frac{d}{dx_2}[s(x)u(t, x)b(x)] dx_2 \right) dx_1 \\
&= - \int_{\Omega} q^T(t, x)Z_3(x)q(t, x) dx, \\
I_4(t) &= \int_{\Omega} 2s(x)u(t, x)c(x) \frac{d}{dx_2}[u_{x_2}(t, x)] dx = - \int_{\Omega} q^T(t, x)Z_4(x)q(t, x) dx, \quad (5.10)
\end{aligned}$$

where q is defined as in (5.8) and for all $x \in \Omega$

$$\begin{aligned}
Z_2(x) &:= \begin{bmatrix} 0 & 0 & \frac{1}{2} \frac{d}{dx_1}[s(x)b(x)] \\ 0 & 0 & \frac{1}{2} s(x)b(x) \\ \frac{1}{2} \frac{d}{dx_1}[s(x)b(x)] & \frac{1}{2} s(x)b(x) & 0 \end{bmatrix}, \\
Z_3(x) &:= \begin{bmatrix} 0 & \frac{1}{2} \frac{d}{dx_2}[s(x)b(x)] & 0 \\ \frac{1}{2} \frac{d}{dx_2}[s(x)b(x)] & 0 & \frac{1}{2} s(x)b(x) \\ 0 & \frac{1}{2} s(x)b(x) & 0 \end{bmatrix}, \\
Z_4(x) &:= \begin{bmatrix} 0 & 0 & \frac{d}{dx_2}[s(x)c(x)] \\ 0 & 0 & 0 \\ \frac{d}{dx_2}[s(x)c(x)] & 0 & 2s(x)c(x) \end{bmatrix}.
\end{aligned}$$

By combining (5.5), (5.7), (5.9) and (5.10) it follows that

$$\frac{d}{dt}[V(u(t, \cdot))] = \int_{\Omega} q^T(t, x)Q(x)q(t, x) dx, \quad (5.11)$$

where for all $x \in \Omega$

$$Q(x) := \begin{bmatrix} 2s(x)f(x) & Q_{12}(x) & Q_{13}(x) \\ Q_{12}(x) & -2s(x)a(x) & -s(x)b(x) \\ Q_{13}(x) & -s(x)b(x) & -2s(x)c(x) \end{bmatrix} \quad (5.12)$$

with

$$\begin{aligned} Q_{12}(x) &:= s(x)d(x) - \frac{d}{dx_1}[s(x)a(x)] - \frac{1}{2} \frac{d}{dx_2}[s(x)b(x)], \\ Q_{13}(x) &:= s(x)e(x) - \frac{1}{2} \frac{d}{dx_1}[s(x)b(x)] - \frac{d}{dx_2}[s(x)c(x)]. \end{aligned}$$

If $Q(x) \leq 0$ for all $x \in \Omega$, then the time derivative in (5.11) is clearly non-positive for all $t > 0$. However, such a condition on Q is conservative. To decrease that conservatism, matrix valued functions Υ_i are introduced such that

$$\int_{\Omega} q^T(t, x) \Upsilon_i(x) q(t, x) dx = 0$$

and, therefore, can be added to Q without altering the integral. Υ_i are spacing functions. We parameterize Υ_i by polynomials p_i .

5.3 Spacing Functions for PDEs with Two Spatial Dimensions

The following holds for any polynomial p_1 , because of the homogeneous Dirichlet boundary conditions (5.2).

$$\int_{\Omega} \frac{d}{dx_1} [u(t, x) p_1(x) u(t, x)] dx = \int_0^1 \left(u(t, x) p_1(x) u(t, x) \Big|_{x_1=0}^{x_1=1} \right) dx_2 = 0. \quad (5.13)$$

Using the chain rule, we have

$$\begin{aligned} \frac{d}{dx_1} [u(t, x) p_1(x) u(t, x)] &= u(t, x) \frac{d}{dx_1} [p_1(x)] u(t, x) + 2p_1(x) u(t, x) u_{x_1}(t, x) \\ &= q^T(t, x) \Upsilon_1(x) q(t, x), \end{aligned} \quad (5.14)$$

where for all $x \in \Omega$

$$\Upsilon_1(x) := \begin{bmatrix} \frac{d}{dx_1}[p_1(x)] & p_1(x) & 0 \\ p_1(x) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (5.15)$$

Combining (5.13) and (5.14) results in

$$\int_{\Omega} q(t, x)^T \Upsilon_1(x) q(t, x) dx = 0. \quad (5.16)$$

Likewise in (5.13), because of the boundary conditions (5.2), the following holds for any polynomial p_2 .

$$\int_{\Omega} \frac{d}{dx_2} [u(t, x) p_2(x) u(t, x)] dx = 0.$$

Following steps of (5.14) for $\frac{d}{dx_2} [u(t, x) p_2(x) u(t, x)]$, gives

$$\Upsilon_2(x) := \begin{bmatrix} \frac{d}{dx_2}[p_2(x)] & 0 & p_2(x) \\ 0 & 0 & 0 \\ p_2(x) & 0 & 0 \end{bmatrix} \quad (5.17)$$

such that

$$\int_{\Omega} q(t, x)^T \Upsilon_2(x) q(t, x) dx = 0. \quad (5.18)$$

Similarly to (5.13), the following is true for any polynomial p_3 .

$$\int_{\Omega} \frac{d}{dx_2} [u(t, x) p_3(x) u_{x_1}(t, x)] dx = 0. \quad (5.19)$$

Note that the left-hand side of (5.19) can be written as follows.

$$\begin{aligned} & \int_{\Omega} \frac{d}{dx_2} [u(t, x) p_3(x) u_{x_1}(t, x)] dx \\ &= \int_{\Omega} \left(u_{x_2}(t, x) p_3(x) u_{x_1}(t, x) + u(t, x) \frac{d}{dx_2} [p_3(x)] u_{x_1}(t, x) \right) dx \\ & \quad + \int_{\Omega} u(t, x) p_3(x) u_{x_2 x_1}(t, x) dx, \end{aligned} \quad (5.20)$$

where we need property (5.6). Applying integration by parts to the second integral of the right-hand side of the last equation in (5.20) results in

$$\begin{aligned}
& \int_{\Omega} u(t, x) p_3(x) \frac{d}{dx_1} [u_{x_2}(t, x)] dx \\
&= \int_0^1 \left(u(t, x) p_3(x) u_{x_2}(t, x) \Big|_{x_1=0}^{x_1=1} - \int_0^1 u_{x_2}(t, x) \frac{d}{dx_1} [u(t, x) p_3(x)] dx_1 \right) dx_2 \\
&= - \int_{\Omega} u_{x_2}(t, x) \left(u_{x_1}(t, x) p_3(x) + u(t, x) \frac{d}{dx_1} [p_3(x)] \right) dx. \tag{5.21}
\end{aligned}$$

From (5.20) and (5.21) the following holds.

$$\begin{aligned}
& \int_{\Omega} \frac{d}{dx_2} [u(t, x) p_3(x) u_{x_1}(t, x)] dx \\
&= \int_{\Omega} \left(u(t, x) \frac{d}{dx_2} [p_3(x)] u_{x_1}(t, x) - u(t, x) \frac{d}{dx_1} [p_3(x)] u_{x_2}(t, x) \right) dx \\
&= \int_{\Omega} q(t, x)^T \Upsilon_3(x) q(t, x) dx, \tag{5.22}
\end{aligned}$$

where for all $x \in \Omega$

$$\Upsilon_3(x) := \begin{bmatrix} 0 & \frac{1}{2} \frac{d}{dx_2} [p_3(x)] & -\frac{1}{2} \frac{d}{dx_1} [p_3(x)] \\ \frac{1}{2} \frac{d}{dx_2} [p_3(x)] & 0 & 0 \\ -\frac{1}{2} \frac{d}{dx_1} [p_3(x)] & 0 & 0 \end{bmatrix}. \tag{5.23}$$

Combining (5.19) and (5.22) gives

$$\int_{\Omega} q(t, x)^T \Upsilon_3(x) q(t, x) dx = 0. \tag{5.24}$$

Following steps (5.19) – (5.22) for

$$\frac{d}{dx_1} [u(t, x) p_4(x) u_{x_2}(t, x)]$$

with any polynomial p_4 , leads to the following.

$$\int_{\Omega} q(t, x)^T \Upsilon_4(x) q(t, x) dx = 0, \tag{5.25}$$

where for all $x \in \Omega$

$$\Upsilon_4(x) := \begin{bmatrix} 0 & -\frac{1}{2} \frac{d}{dx_2}[p_4(x)] & \frac{1}{2} \frac{d}{dx_1}[p_4(x)] \\ -\frac{1}{2} \frac{d}{dx_2}[p_4(x)] & 0 & 0 \\ \frac{1}{2} \frac{d}{dx_1}[p_4(x)] & 0 & 0 \end{bmatrix}. \quad (5.26)$$

5.4 Stability Test for PDEs with Two Spatial Dimensions

From (5.11), (5.16), (5.18), (5.24) and (5.25) the following holds.

$$\frac{d}{dt}[V(u(t, \cdot))] = \int_{\Omega} q^T(t, x) \left(Q(x) + \sum_{i=1}^4 \Upsilon_i(x) \right) q(t, x) dx. \quad (5.27)$$

By substituting for Q and Υ_i from (5.12), (5.15), (5.17), (5.23) and (5.26) in (5.27) one can define

$$M = \Phi(a, b, c, d, e, f, s, p_1, p_2, p_3, p_4), \quad (5.28)$$

if for all $x \in \Omega$

$$M(x) = \begin{bmatrix} M_1(x) & M_2(x) & M_3(x) \\ M_2(x) & -2s(x)a(x) & -s(x)b(x) \\ M_3(x) & -s(x)b(x) & -2s(x)c(x) \end{bmatrix}, \quad (5.29)$$

where

$$\begin{aligned} M_1(x) &:= 2s(x)f(x) + \frac{d}{dx_1}[p_1(x)] + \frac{d}{dx_2}[p_2(x)], \\ M_2(x) &:= s(x)d(x) - \frac{d}{dx_1}[s(x)a(x)] - \frac{1}{2} \frac{d}{dx_2}[s(x)b(x)] + p_1(x) + \frac{1}{2} \frac{d}{dx_2}[p_3(x) - p_4(x)], \\ M_3(x) &:= s(x)e(x) - \frac{1}{2} \frac{d}{dx_1}[s(x)b(x)] - \frac{d}{dx_2}[s(x)c(x)] + p_2(x) + \frac{1}{2} \frac{d}{dx_1}[p_4(x) - p_3(x)], \end{aligned} \quad (5.30)$$

such that

$$\frac{d}{dt}[V(u(t, \cdot))] = \int_{\Omega} q^T(t, x) M(x) q(t, x) dx. \quad (5.31)$$

Theorem 8. *Suppose that for (5.1) there exist polynomials s, p_1, p_2, p_3, p_4 and $\theta > 0$, such that $s(x) \geq \theta$ and $M(x) \leq 0$ for all $x \in \Omega$, where M is defined as in (5.28) – (5.30). Then (5.1) is stable.*

Proof. If conditions of Theorem 8 are satisfied, let

$$a = \inf_{x \in \Omega} \{s(x)\}, \quad b = \sup_{x \in \Omega} \{s(x)\}. \quad (5.32)$$

Since $s(x) \geq \theta > 0$ for all $x \in \Omega$, then $b, a > 0$ and the following holds for all $v \in L_2(\Omega)$.

$$a\|v\|_{L_2}^2 = \inf_{x \in \Omega} \{s(x)\} \int_{\Omega} v^2(x) dx \leq \int_{\Omega} s(x)v^2(x) dx \leq \sup_{x \in \Omega} \{s(x)\} \int_{\Omega} v^2(x) dx = b\|v\|_{L_2}^2. \quad (5.33)$$

Using (5.3) it follows that

$$a\|v\|_{L_2}^2 \leq V(v) \leq b\|v\|_{L_2}^2.$$

Since $M(x) \leq 0$, from (5.31) it follows that $\frac{d}{dt}[V(u(t, \cdot))] \leq 0$ for all $t > 0$. Theorem 2 ensures stability of (5.1). \square

Theorem 9. *Suppose that for (5.1) there exist $\theta, \gamma > 0$ and polynomials s, p_1, p_2, p_3, p_4 such that $s(x) \geq \theta$ and $M(x) + \gamma S(x) \leq 0$ for all $x \in \Omega$, where M is defined as in (5.28) – (5.30) and*

$$S(x) := \begin{bmatrix} s(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (5.34)$$

Then for all $t > 0$ solution to (5.1) satisfies

$$\|u(t, \cdot)\|_{L_2} \leq \sqrt{\frac{b}{a}} \|u(0, \cdot)\|_{L_2} \exp\left\{-\frac{\gamma}{2}t\right\}, \quad (5.35)$$

where a, b are defined as in (5.32).

Proof. Under the assumptions of Theorem 9, (5.33) holds. With (5.34) and (5.8) we can write

$$V(u(t, \cdot)) = \int_{\Omega} q(t, x)^T S(x) q(t, x) dx. \quad (5.36)$$

Since $M(x) + \gamma S(x) \leq 0$ for all $x \in \Omega$, it holds that

$$\int_{\Omega} q^T(t, x)(M(x) + \gamma S(x))q(t, x) dx \leq 0. \quad (5.37)$$

Since γ is a scalar, (5.37) can be easily satisfied as follows.

$$\int_{\Omega} q^T(t, x)M(x)q(t, x)dx \leq -\gamma \int_{\Omega} q(t, x)^T S(x)q(t, x)dx,$$

which with (5.31) and (5.36) provides $\frac{d}{dt}[V(u(t, \cdot))] \leq -\gamma V(u(t, \cdot))$. Using proof of Theorem 2 with $c/b = \gamma$, results in (5.35). \square

Theorem 10. *Suppose that for (5.1) there exist polynomials $s, p_1, p_2, p_3, p_4, \theta > 0$, SOS polynomials n_1, n_2, Q_1, Q_2, Q_3 such that for all $x_1, x_2 \in (0, 1)$*

$$\begin{aligned} s(x) &= \theta + x_1(1 - x_1)n_1(x) + x_2(1 - x_2)n_2(x), \\ M(x) &= -Q_1(x) - x_1(1 - x_1)Q_2(x) - x_2(1 - x_2)Q_3(x), \end{aligned} \quad (5.38)$$

where M is defined as in (5.28) – (5.30). Then (5.1) is stable.

Proof. If (5.38) holds, then clearly $s(x) \geq \theta$ and $M(x) \leq 0$ for all $x \in \Omega$. Using Theorem 8 provides stability of (5.1). \square

Theorem 11. *Suppose that for (5.1) there exist $\theta, \gamma > 0$, polynomials s, p_1, p_2, p_3, p_4 , SOS polynomials n_1, n_2, Q_1, Q_2, Q_3 such that for all $x_1, x_2 \in (0, 1)$*

$$\begin{aligned} s(x) &= \theta + x_1(1 - x_1)n_1(x) + x_2(1 - x_2)n_2(x), \\ M(x) + \gamma S(x) &= -Q_4(x) - x_1(1 - x_1)Q_5(x) - x_2(1 - x_2)Q_6(x), \end{aligned} \quad (5.39)$$

where M is defined as in (5.28) – (5.30) and S as in (5.34), then for all $t > 0$ solution to (5.1) satisfies

$$\|u(t, \cdot)\|_{L_2} \leq \sqrt{\frac{b}{a}} \|u(0, \cdot)\|_{L_2} \exp\left\{-\frac{\gamma}{2}t\right\}, \quad (5.40)$$

where a, b are defined as in (5.32).

Proof. If (5.39) is true, then for all $x \in \Omega$, $s(x) \geq \theta$ and $M(x) + \gamma S(x) \leq 0$, which, combined with Theorem 9, gives (5.40). \square

Chapter 6

NUMERICAL VALIDATION

We performed stability analysis of some decoupled and coupled PDEs with a parameter $\lambda > 0$, whose value affects stability. Using the proposed algorithm and bisection search over λ we searched for upper bounds on λ for which the PDEs are stable. We also studied the dependence of the algorithm accuracy on the degree d . The results are compared with λ_{num} – upper bounds on λ calculated using MATLAB solver PDEPE and bisection search over λ . We also tested stability of a randomly generated coupled PDE with polynomial spatially distributed coefficients.

We also studied stability of biological PDE introduced by Kierstead, Slobodkin and Skellam. We searched for lower bounds on a diffusion coefficient h for which the biological PDE is stable. We calculated an upper bound on the rate of decay γ of the solution L_2 norm. The results are compared with numerical solution based on finite difference discretization method.

As an example with spatially distributed coefficients we randomly generated a PDE with polynomial coefficients and calculated an upper bound on the rate of decay γ of the L_2 norm of the solution. The result was compared with the numerical

experiment based on finite different discretization.

6.1 Coupled PDEs

6.1.1 Decoupled Case

Start with the following parameterized decoupled PDE.

$$u_t(t, x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t, x) + \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} u(t, x) \quad (6.1)$$

with boundary conditions

$$u(t, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The numerical solution given by MATLAB PDEPE solver implies that for $\lambda = 9.8$ (6.1) is stable and for $\lambda = 9.9$, (6.1) is unstable. Using a bisection search over λ , one may determine a lower bound on λ_{cr} for which problem in Theorem 7, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad D = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.2)$$

may be shown to be feasible. Some dependence of the λ_{cr} on the degree of parameterization d is presented in Table 6.1 and compared to the λ_{cr} calculated using MATLAB PDEPE solver.

6.1.2 Coupled PDEs

The following example includes some coupling.

$$u_t(t, x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t, x) + \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} u(t, x) \quad (6.3)$$

Table 6.1: Maximum λ for which (6.1) is stable based on the proposed algorithm for different degree d with $\epsilon = 0.001$.

d	1	2	3	4	5	6	λ_{num}
λ	5	5.8	7.4	8.1	8.1	8.1	9.8

Table 6.2: Maximum λ for which (6.3) is stable based on the proposed algorithm for different degree d with $\epsilon = 0.001$.

d	1	2	3	4	5	6	λ_{num}
λ	4	5.8	6.9	7.2	7.4	7.4	8.8

and boundary conditions are

$$u(t, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The numerical solution given by MATLAB PDEPE solver yields that for $\lambda = 8.8$ (6.3) is stable and for $\lambda = 8.9$, (6.3) is unstable.

Using a bisection search over λ , one may determine a lower bound on λ_{cr} for which problem in Theorem 7, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix}, \quad D = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.4)$$

may be shown to be feasible. Some dependence of the λ_{cr} on the degree of parameterization d is presented in Table 6.2 and compared to the λ_{cr} calculated using MATLAB PDEPE solver.

Table 6.3: Maximum λ for which (6.5) is stable based on the proposed algorithm for different degree d with $\epsilon = 0.001$.

d	1	2	3	4	5	6	λ_{num}
λ	8.6	12.7	13.9	14.4	14.6	14.7	15.9

6.1.3 Coupled PDEs with Mixed Boundary Conditions

The next example includes coupled PDEs with mixed boundary conditions.

$$u_t(t, x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{xx}(t, x) + \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix} u(t, x) \quad (6.5)$$

where boundary conditions are

$$u_x(t, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The numerical solution from MATLAB PDEPE solver states that for $\lambda = 15.9$ (6.5) is stable and for $\lambda = 16$, (6.5) is unstable.

Using a bisection search over λ , one may determine a lower bound on λ_{cr} for which problem in Theorem 7, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = 0, \quad C = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.6)$$

may be shown to be feasible. Some dependence of the λ_{cr} on the degree of parameterization d is presented in Table 6.3 and compared to the λ_{cr} calculated using MATLAB PDEPE solver.

6.1.4 Coupled PDEs with Spatially Dependent Coefficients

For our final example, we consider a coupled PDE with spatially varying coefficients.

$$\begin{aligned}
 u_t(t, x) = & \begin{bmatrix} 5x^2 + 4 & 0 \\ 2x^2 + 7x & 7x^2 + 6 \end{bmatrix} u_{xx}(t, x) + \begin{bmatrix} 1 & -4x \\ -3.5x^2 & 0 \end{bmatrix} u_x(t, x) \\
 & - \begin{bmatrix} x^2 & 3 \\ 2x & 3x^2 \end{bmatrix} u(t, x)
 \end{aligned} \tag{6.7}$$

for all $t > 0$, $x \in (0, 1)$. Also for all $t > 0$,

$$u(t, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(t, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Based on the numerical solution given by MATLAB PDEPE one can say that (6.7) is stable. With $d = 4$ problem from Theorem 7 with

$$\begin{aligned}
 A = & \begin{bmatrix} 5x^2 + 4 & 0 \\ 2x^2 + 7x & 7x^2 + 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4x \\ -3.5x^2 & 0 \end{bmatrix}, \\
 C = & \begin{bmatrix} x^2 & 3 \\ 2x & 3x^2 \end{bmatrix}, \quad D = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{6.8}$$

is feasible.

6.2 Examples of PDEs with Two Spatial Variables

6.2.1 Model of Population Dynamics

In this section stability analysis is presented for the biological ‘‘KISS’’ PDE named after Kierstead, Slobodkin and Skelam, which describes population growth on a finite

area. For more details see Holmes *et al.* (1994). The system is modeled by the following PDE.

$$u_t(t, x) = h \left(u_{x_1 x_1}(t, x) + u_{x_2 x_2}(t, x) \right) + ru(t, x), \quad (6.9)$$

where $h, r > 0$, $x \in \Omega \subset \mathbb{R}^2$ and scalar function u satisfies homogeneous Dirichlet boundary conditions.

It is claimed in Holmes *et al.* (1994) that if Ω is a square with edge of length l , then

$$l_{cr} := \sqrt{2\pi^2 \left(\frac{h}{r} \right)} \quad (6.10)$$

defines a critical length. That means, if $l > l_{cr}$, then (6.9) is unstable. Alternatively, for given l and r (6.10) defines h_{cr} as

$$h_{cr} := l^2 r / 2\pi^2. \quad (6.11)$$

Therefore, if $h < h_{cr}$, then (6.9) is unstable.

For testing the proposed algorithm, fix $l = 1$ and arbitrarily choose r , for example $r = 4$. Thus, according to (6.11), $h_{cr} \approx 0.203$.

Using a bisection search over h , one can determine a lower bound of h_{cr} for which SOS problem in Theorem 10, with

$$a(x) = h, \quad b(x) = 0, \quad c(x) = h, \quad d(x) = 0, \quad e(x) = 0, \quad f(x) = 4 \quad \text{for all } x \in \Omega \quad (6.12)$$

may be shown to be feasible. Results for different degrees of s ($\deg(s)$) are presented in Table (6.4).

Now we choose $l = 1$, $h = 2$ and $r = 4$. Using a bisection search over γ , we determine the maximum γ for which the SOS problem in Theorem 11, with (6.12), may be shown to be feasible. Results for different $\deg(s)$ are presented in Table (6.5).

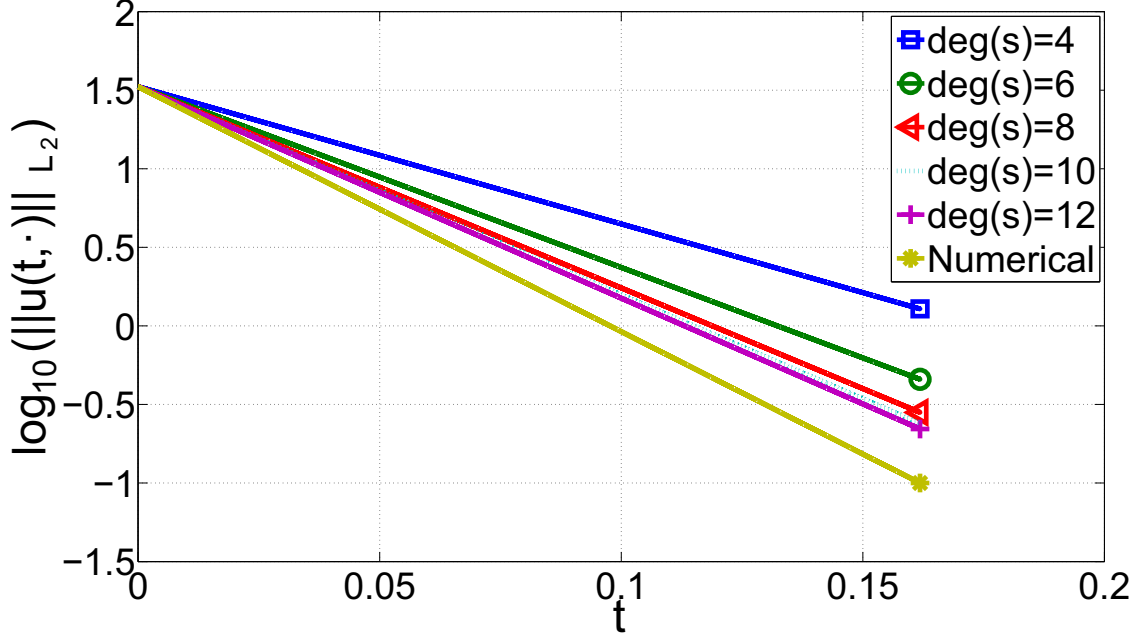


Figure 6.1: Semi-log plots of the L_2 norm of the numerical solution to (6.9) with $u(0, x) = 10^3 x_1 x_2 (1 - x_1)(1 - x_2)$ and upper bounds, given by the proposed method with different $\deg(s)$.

Table 6.4: Minimum h_{cr} vs $\deg(s)$ for (6.9)

$\deg(s)$	4	6	8	10	12	analytic
h_{cr}	0.332	0.259	0.238	0.229	0.227	0.203

Using finite difference scheme, we numerically solve (6.9) with $u(0, x) = 10^3 x_1 x_2 (1 - x_1)(1 - x_2)$. Plots of $\log_{10} (\|u(t, \cdot)\|_{L_2})$ versus t , using a numerical solution, and bounds on $\log_{10} (\|u(t, \cdot)\|_{L_2})$, given by the proposed method for different $\deg(s)$, are presented in Fig. (6.1). These plots allow us to determine γ by examining the rate of decrease in the L_2 norm. Plots are aligned at $t = 0$ in order to better compare our SOS estimates of γ to the estimate of γ derived from numerical simulation as a function of increasing $\deg(s)$.

Table 6.5: Maximum γ vs $\deg(s)$ for (6.9) with $h = 2$

$\deg(s)$	4	6	8	10	12
γ	40.25	53	59	61	62

6.2.2 Random PDE with Spatially Dependent Coefficients

Consider

$$\begin{aligned}
 u_t(t, x) &= (5x_1^2 - 15x_1x_2 + 13x_2^2)(u_{x_1x_1}(t, x) + u_{x_2x_2}(t, x)) + (10x_1 - 15x_2)u_{x_1}(t, x) \\
 &\quad + (-15x_1 + 26x_2)u_{x_2}(t, x) - (17x_1^4 - 30x_2 - 25x_1^2 - 8x_2^3 - 50x_2^4)u(t, x), \\
 u(0, x) &= 10^3x_1x_2(1 - x_1)(1 - x_2)
 \end{aligned} \tag{6.13}$$

where $x \in \Omega := (0, 1)^2$ and the scalar function u satisfies zero Dirichlet boundary conditions.

Using a bisection search over γ , we determine maximum γ for which SOS problem in Theorem 11, with

$$\begin{aligned}
 a(x) &= 5x_1^2 - 15x_1x_2 + 13x_2^2, & e(x) &= -15x_1 + 26x_2, & c(x) &= 5x_1^2 - 15x_1x_2 + 13x_2^2, \\
 d(x) &= 10x_1 - 15x_2, & f(x) &= -(17x_1^4 - 30x_2 - 25x_1^2 - 8x_2^3 - 50x_2^4), & b(x) &= 0
 \end{aligned}$$

may be shown to be feasible.

Using finite difference scheme, we numerically solve (6.13). The estimated rate of decay, based on numerical solution, is 13.07. The computed rate of decay, based on our SOS method, is 12.5 for $\deg(s) = 8$. Plots are given in Fig. 6.2 and, as for Fig. 6.1, are aligned at $t = 0$.

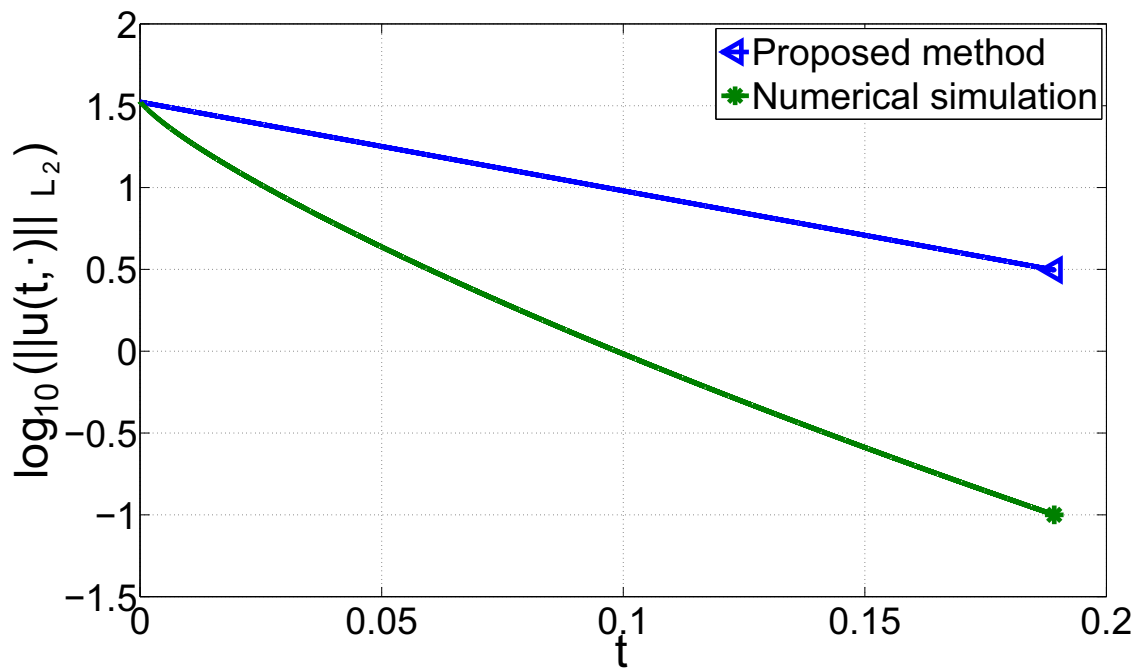


Figure 6.2: Semi-log plots of the L_2 norm of the numerical solution to (6.13) and upper bound, given by the proposed method with $\deg(s) = 8$.

Chapter 7

CONCLUSION

7.1 Summary of Contribution

In this thesis, we have presented a computational framework based on convex optimization for stability analysis of two forms of linear PDEs. First form includes coupled linear PDEs with spatially distributed polynomial coefficients. Second form considers parabolic PDEs for scalar-valued functions with two spatial variables.

We used LMIs and SOS polynomials to parameterize positive functionals over spaces of Lebesgue integrable functions. We have enforced negativity of the derivative using a combination of SOS and a parametrization of projection operators defined by the fundamental theorem of calculus. The result is an LMI test for stability which can be implemented using SOSTOOLS coupled with an SDP solver such as Mosek or SeDuMi. We applied the proposed framework to several examples of systems of coupled linear PDEs with both constant and spatially varying coefficients and with both Dirichlet and Neumann boundary conditions. Also we calculated an upper bound on the rate of decay of the L_2 norm of a solution to PDE which describes dynamics of population. We compared the numerical results with solutions based on discretization methods.

7.2 Ongoing Research

Future work includes extension of the framework to study stability of models such as the acoustic wave equations as well as examine the problem of optimal control and estimation for systems of coupled PDEs. Another step is the combination of presented

techniques in order to study stability of coupled PDEs with multiple spatial variables. Also decrease the conservatism by using semi-separable kernels to parameterize integral operators as for the scalar case with 1 spatial variable in Gahlawat and Peet (2015). Consider spacing operators with semi-separable kernels. Study stability of linearized Navier-Stokes equations.

Another ultimate goal is to extend the approach to nonlinear systems. In that case we can analyze dynamics of predator-pray models, biological PDEs with nonlinear terms describing decrease of population density due to death factor.

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