Cuntz-Pimsner Algebras of Twisted Tensor Products of Correspondences and Other Constructions

by

Adam Morgan

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved April 2016 by the
Graduate Supervisory Committee:

Steven Kaliszewski, Co-Chair
John Quigg, Co-Chair
Matthias Kawski
Brett Kotschwar
Jack Spielberg

ARIZONA STATE UNIVERSITY

May 2016

ABSTRACT

This dissertation contains three main results. First, a generalization of Ionescu's theorem is proven. Ionescu's theorem describes an unexpected connection between graph C^*-algebras and fractal geometry. In this work, this theorem is extended from ordinary directed graphs to higher-rank graphs. Second, a characterization is given of the Cuntz-Pimsner algebra associated to a tensor product of C^*-correspondences. This is a generalization of a result by Kumjian about graphs algebras. This second result is applied to several important special cases of Cuntz-Pimsner algebras including topological graph algebras, crossed products by the integers and crossed products by completely positive maps. The result has meaningful interpretations in each context. The third result is an extension of the second result from an ordinary tensor product to a special case of Woronowicz's twisted tensor product. This result simultaneously characterizes Cuntz-Pimsner algebras of ordinary and graded tensor products and Cuntz-Pimsner algebras of crossed products by actions and coactions of discrete groups, the latter partially recovering earlier results of Hao and Ng and of Kaliszewski, Quigg and Robertson.

DEDICATION

This thesis is dedicated to Missy and Calista Morgan. Without your patience and encouragement, this would not have been be possible.

TABLE OF CONTENTS

CHAPTER Page

Chapter 1

OVERVIEW

The study of operator algebras grew largely from a desire to build a rigorous mathematical framework for quantum mechanics in the 1930's and 1940's. As such, most concrete realizations of operator algebras occur as algebras of operators on a Hilbert space. My research involves C^*-algebras. Abstractly, a C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ for all a. In more concrete terms, the celebrated Gelfand-Naimark-Segal theorem shows that every C^*-algebra is isomorphic to a closed, self-adjoint algebra of bounded operators on some Hilbert space.

In the 1970's, J. Cuntz described a family of C^*-algebras $\mathcal{O}_n$ (now called Cuntz algebras) which can be viewed as the C^*-algebra generated by an n-dimensional Hilbert space. These algebras (and their many generalizations) have proved to be a very fruitful area of research. Most notably, in the 1980's, Cuntz algebras were used by Doplicher and Roberts in one of the most important results of local quantum field theory: to reconstruct a compact group of gauge symmetries from an algebra of local observables.

Much of my research involves an important generalization of the Cuntz algebras known as Cuntz-Pimsner algebras. Where a Cuntz algebra is generated by a Hilbert space, a Cuntz-Pimsner algebra $\mathcal{O}_X$ is generated by a C^*-correspondence X. X can be thought of generalized Hilbert spaces, where the scalar field $\mathbb{C}$ has been replaced by a C^*-algebra A. We refer to such an X as a *correspondence over A*. C^*-correspondences are central to many important ideas in operator theory such as K-theory, induced representations, and the theory of completely positive maps.

The first main result of this work is a generalization of Ionescu's theorem. Ionescu's theorem describes an unexpected connection between graph C^*-algebras and fractal geometry. More specifically, the theorem is about Mauldin-Williams graphs, which are generalizations of iterated function systems in which one has a directed graph with a compact metric space associated to each vertex and a contractive map at each edge. One can associate a C^*-correspondence to such a system in a fairly natural way. Ionescu's theorem states that the Cuntz-Pimsner algebra of this correspondence depends only upon the structure of the underlying graph, and is in fact isomorphic to the graph C^*-algebra. In Chapter 3, we show that there is an analog of Ionescus theorem for higher-rank graphs.

The other main focus of this dissertation is studying how various constructions involving C^*-correspondences relate to the corresponding constructions on the associated Cuntz-Pimsner algebras. For example, if X is a correspondence over a C^*-algebra A and Y is a correspondence over a C^*-algebra B, we can construct the *external tensor product* $X \otimes Y$ which will be a correspondence over the (spatial) tensor product $A \otimes B$. In Chapter 4 we will show that the Cuntz-Pimsner algebra of the external tensor product: $\mathcal{O}_{X \otimes Y}$ can be understood in terms of the Cuntz-Pimsner algebras of the underlying correspondences: $\mathcal{O}_X$ and $\mathcal{O}_Y$. Specifically, $\mathcal{O}_{X \otimes Y}$ is isomorphic to a subspace of $\mathcal{O}_X \otimes \mathcal{O}_Y$. This generalizes a result of Kumjian [Kumjian(1998)] involving graph algebras.

The later sections of this thesis will involve actions and coactions of groups. The theory of actions and coactions is deep and complex. However, many have described it roughly as follows. Just as group actions on ordinary spaces like manifolds help us study the symmetry of those spaces, coactions of quantum groups on operator algebras help us study symmetry of "non-commutative" spaces in the spirit of Alain Connes' non-commutative geometry programme.

Given an action or a coaction of a group G on a C^*-algebra A, one can form a crossed product C^*-algebra $A \rtimes G$ which, in a sense, has all of the information about A, G and the action/coaction encoded into its algebraic structure. This is one of the most fundamental tools used in the study of actions and coactions. Crossed product constructions also exist for actions and coactions of groups on correspondences and also for coactions of quantum groups on C^*-algebras and correspondences.

Over the past several years, there have been many results published relating actions and coactions on correspondences to actions and coactions on their Cuntz-Pimsner algebras. These results state that under suitable conditions, $\mathcal{O}_{X \rtimes G} \cong \mathcal{O}_X \rtimes G$. This was done first for actions by Hao and Ng [Hao and Ng(2008)], then for coactions by Kaliszewski, Quigg and Robertson [Kaliszewski *et al.*(2012)] and finally for coactions of quantum groups by Kim [Kim(2014)].

Although the results just mentioned seem to be of a somewhat different nature than my result about tensor products of correspondences, one of the main results of this thesis provides a very strong connection between them. This was inspired by recent work of Meyer, Roy and Woronowicz which shows that tensor products and crossed products by quantum group coactions can be viewed as special cases of a much more general construction which they refer to as a *twisted tensor product.* They use the symbol $\boxtimes$ in place of $\otimes$ to distinguish twisted tensor products from ordinary tensor products.

In the framework of Woronowicz et. al., all of the results above, whether they involve tensor products or crossed products, all take the form $\mathcal{O}_{X \boxtimes Y} \cong \mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. This suggests that these seemingly disparate results might all be special cases of one more powerful result involving twisted tensor products. While we will not prove this in the full generality of quantum groups, we will show it to be true for discrete groups.

Chapter 2

PRELIMINARIES

2.1 Correspondences and Cuntz-Pimsner Algebras

In this section we will recall some basic facts about C^*-correspondences and Cuntz-Pimsner algebras. Every part of this thesis involves correspondences and Cuntz-Pimsner algebras in some form or another, so the results of this section will be important throughout.

2.1.1 Correspondences

Suppose A is a C^*-algebra and X is a right A-module. By an *A-valued inner product* on X we shall mean a map

$$X \times X \ni (x, y) \mapsto \langle x, y \rangle_A \in A$$

which is linear in the second variable and such that

1. $\langle x, x \rangle_A \geq 0$ for all $x \in X$ with equality only when $x = 0$.

2. $\langle x, y \rangle_A^* = \langle y, x \rangle_A$ for all $x, y \in X$.

3. $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$ for all $x, y \in X$ and $a \in A$.

Note that this implies that $\langle \cdot, \cdot \rangle_A$ is A-linear in the second variable and conjugate A-linear in the first variable. One can prove a version of the Cauchy-Schwartz inequality for such X, which implies that we can define the following norm on X:

$$\|x\|_A := \|\langle x, x \rangle_A\|^{\frac{1}{2}}$$

If X is a right A-module with an A-valued inner product, X is called a *right Hilbert A-module* if it is complete under the norm $\|\cdot\|_A$ defined above. Note that if $A = \mathbb{C}$ then X is just a Hilbert space and we can think of general Hilbert modules as Hilbert spaces whose scalars are elements of some C^*-algebra A.

Also note that we can make A itself into a right Hilbert A-module by letting the right action of A be given by multiplication in A and A-valued inner product given by $\langle a, b\rangle_A = a^*b$. We call this the *trivial* Hilbert A-module and denote it by A_A.

Let A be a C^*-algebra and let X be a right Hilbert A-module. If $T : X \to X$ is an A-module homomorphism, then we call T *adjointable* if there is an A-module homomorphism T^* (called the *adjoint* of T) such that

$$\langle T^*x, y\rangle_A = \langle x, Ty\rangle_A$$

for all $x, y \in X$. The operator norm makes the set of all adjointable operators on X into a C^*-algebra which we denote by $\mathcal{L}(X)$.

Given C^*-algebras A and B, an *$A-B$-correspondence* is right Hilbert B-module X together with a homomorphism $\phi : A \to \mathcal{L}(X)$ which is called a *left action of A by adjointable operators.* For $a \in A$ and $x \in X$, we will write $a \cdot x$ for $\phi(a)(x)$. If $A = B$ we call this a *correspondence over A* (or B). We call the left-action *injective* if ϕ is injective and *non-degenerate* if $\phi(A)X = X$. We will sometimes write ${}_AX_B$ to indicate that X is an $A - B$ correspondence. Before we continue, we will give a few examples of correspondences.

Example 2.1.1 (Example 8.6 in [Raeburn(2005)]). Let A be a C^* algebra and let α be an endomorphism of A. We can make the trivial module A_A into a correspondence over A by defining $\phi(a)(x) = \alpha(a)x$.

Example 2.1.2 (Example 1.2 of [Fowler and Raeburn(1998)]). Let $E = \{E^0, E^1, r, s\}$ be a directed graph (in the sense of [Raeburn(2005)]). Consider the vector space $c_c(E^1)$ of finitely supported functions on E^1. We can define a right action of $c_0(E^0)$ and a $c_0(E^0)$-valued inner product as follows:

$$(x \cdot a)(e) = x(e)a(s(e))$$
$$\langle x, y\rangle_{c_0(E^0)}(v) = \sum_{\{e \in E^1 : s(e)=v\}} \overline{x(e)}y(e)$$

We can use the norm defined by this inner product to complete $c_c(E^1)$ into a right Hilbert module $X(E)$. We can define a left action $\phi : c_0(E^0) \to \mathcal{L}(X(E))$ as follows

$$\phi(a)(x)(e) = a(r(e))x(e)$$

This makes $X(E)$ into a correspondence which is referred to as the *graph correspondence* of E.

This example has a natural generalization:

Example 2.1.3 (Definition 3.11 of [Katsura(2003)])**.** A *topological graph* is a quadruple $E = \{E^0, E^1, r, s\}$ where E^0 and E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous function, and $s : E^1 \to E^0$ is a local homeomorphism. Let $A := C_0(E^0)$. We can define left- and right-actions of A and an A-valued inner product on $C_c(E^1)$ similarly to the way we did for ordinary graphs: For $a \in A$ and $x, y \in C_c(E^1)$, let

$$(a \cdot x)(e) := a(r(e))x(e)$$
$$(x \cdot a)(e) := x(e)a(s(e))$$
$$\langle x, y\rangle_A(v) := \sum_{\{e \in E^1 : s(e)=v\}} \overline{x(e)}y(e)$$

We denote the completion of $C_c(E^1)$ under the norm defined by this inner product by $X(E)$. It can be shown ([Raeburn(2005)] p.80-81) that the left action is injective if and only if r has dense range and the left action is implemented by compacts if and only if r is proper.

Let X be a Hilbert module over a C^*-algebra A. Then for any $x, y \in X$ the map $\Theta_{x,y} : z \mapsto x \cdot \langle y, z\rangle_A$ is an adjointable operator called a *rank-one* operator. It can be shown that the closed span of the rank-one operators forms an ideal $\mathcal{K}(X)$ in $\mathcal{L}(X)$ which is referred to as the set of *compact operators*. If X is an $A-B$-correspondence, we say that the left action ϕ of A is *implemented by compacts* if $\phi(A) \subseteq \mathcal{K}(X)$.

There are two types of tensor products which are usually defined on correspondences, an "internal" tensor product and an "external" tensor product. These are defined as follows (see chapter 4 of [Lance(1995)] for more detail): Let ${}_AX_B$ and ${}_CY_D$ be correspondences and let $\Phi : B \to C$ be a completely positive map (see [Blackadar(2006)] for the basics of completely positive maps). Let $X \odot_\Phi Y$ be the quotient of the algebraic tensor product $X \odot Y$ by the subspace spanned by

$$\{x \cdot b \otimes y - x \otimes \Phi(b) \cdot y : x \in X, y \in Y, b \in B\}$$

This is a right D-module with right action given by $(x \otimes y) \cdot d = (x \otimes y \cdot d)$. We define a D-valued inner product as follows:

$$\langle x \otimes y, x' \otimes y'\rangle = \langle y, \Phi\big(\langle x, x'\rangle_B\big) \cdot y'\rangle_D$$

For a proof that this is indeed an inner product, we refer the reader to the proof of Proposition 4.5 of [Lance(1995)]. We refer to the completion of $X \odot_\Phi Y$ with respect to the norm defined by this inner product as the *internal tensor product* of X and Y and it is denoted by $X \otimes_\Phi Y$. If $\phi_A : A \to \mathcal{L}(X)$ is the left action on X, then we can define a left action of A on $X \otimes_\Phi Y$ by $\phi(a)(x \otimes y) = \big(\phi_A(a)x\big) \otimes y$. This makes $X \otimes_\Phi Y$ an $A-D$ correspondence. In many situations we will have $B = C$ and $\Phi = id_B$. In this case we will write the associated internal tensor product as $X \otimes_B Y$.

Example 2.1.4. If $\Phi : A \to B$ is a completely positive map between two C^*-algebras, then we define the *correspondence associated to* Φ to be the correspondence $X_\Phi := {}_AA_A \otimes_\Phi {}_BB_B$ where ${}_AA_A$ and ${}_BB_B$ are the standard correspondences.

Let ${}_AX_B$ and ${}_CY_D$ be correspondences. We can define a right action of $B \odot D$ and a $B \odot D$-valued inner product on the algebraic tensor product $X \odot Y$ as follows

$$(x \otimes y) \cdot (a \otimes b) = (x \cdot a) \otimes (y \cdot b)$$

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle_B \otimes \langle y, y' \rangle_D$$

The completion of $X \odot Y$ with respect to the norm defined by this inner product is called the *external tensor product* of X and Y which we will denote simply by $X \otimes Y$. We can define a left action of $A \otimes C$ as follows: $\phi(a \otimes c)(x \otimes y) = \big(\phi_A(a)x\big) \otimes \big(\phi_C(c)y\big)$. This makes $X \otimes Y$ an $A \otimes C - B \otimes D$ correspondence.

If $\Phi : A \to C$ and $\Phi' : B \to D$ are completely positive maps, then $\Phi \otimes \Phi' : A \otimes B \to C \otimes D$ will be a completely positive map as well. In fact:

Lemma 2.1.5. *Let ${}_AX_{A'}$, ${}_BY_{B'}$, ${}_CZ_{C'}$, and ${}_DW_{D'}$ be correspondences and let $\Phi : A' \to C$ and $\Phi' : B' \to D$ be completely positive maps. Then*

$$(X \otimes_\Phi Z) \otimes (Y \otimes_{\Phi'} W) \cong (X \otimes Y) \otimes_{\Phi \otimes \Phi'} (Z \otimes W)$$

Proof. Note that the left hand side is the completion of $X \odot Z \odot Y \odot W$ under the norm defined by a certain pre-inner product and the right hand side is the completion of $X \odot Y \odot Z \odot W$ under the norm defined by a certain pre-inner product. We can show that the linear map

$$\sigma_{23} : X \odot Z \odot Y \odot Z \to X \odot Y \odot Z \odot W$$

$$x \otimes z \otimes y \otimes z \mapsto x \otimes y \otimes z \otimes w$$

extends to a correspondence isomorphism

$$(X \otimes_\Phi Z) \otimes (Y \otimes_{\Phi'} W) \to (X \otimes Y) \otimes_{\Phi \otimes \Phi'} (Z \otimes W)$$

by showing that σ_{23} preserves the pre-inner products. By linearity it suffices to show this for elementary tensors. Let $\langle \cdot, \cdot \rangle_1$ denote the pre-inner product which gives rise to $(X \otimes_\Phi Z) \otimes (Y \otimes_{\Phi'} W)$. Let $\langle \cdot, \cdot \rangle_2$ denote the pre-inner product which gives rise to $(X \otimes Y) \otimes_{\Phi \otimes \Phi'} (Z \otimes W)$. Then

$$\begin{aligned}
\Big\langle \sigma_{23}(x \otimes z \otimes y \otimes w), &\sigma_{23}(x \otimes z \otimes y \otimes w)\Big\rangle_2 \\
&= \langle x \otimes y \otimes z \otimes w, x' \otimes y' \otimes z' \otimes w' \rangle_2 \\
&= \Big\langle z \otimes w, (\Phi \otimes \Phi')\big(\langle x \otimes y, x' \otimes y' \rangle\big)(z' \otimes w')\Big\rangle \\
&= \Big\langle z \otimes w, (\Phi \otimes \Phi')\big(\langle x, x' \rangle \otimes \langle y, y' \rangle\big)(z' \otimes w')\Big\rangle \\
&= \Big\langle z \otimes w, \big(\Phi\big(\langle x, x' \rangle\big) z'\big) \otimes \big(\Phi'\big(\langle y, y' \rangle\big) w'\big)\Big\rangle \\
&= \Big\langle z, \Phi\big(\langle x, x' \rangle\big) z' \Big\rangle \otimes \Big\langle w, \Phi'\big(\langle y, y' \rangle\big) w' \Big\rangle \\
&= \langle x \otimes z \otimes y \otimes w, x' \otimes z' \otimes y' \otimes z' \rangle_1
\end{aligned}$$

Thus σ_{23} extends to an isomorphism giving us

$$(X \otimes_\Phi Z) \otimes (Y \otimes_{\Phi'} W) \cong (X \otimes Y) \otimes_{\Phi \otimes \Phi'} (W \otimes Z)$$

$\square$

Example 2.1.6. Let $\Phi : A \to C$ and $\Phi' : B \to D$ be completely positive maps. Since ${}_A A_A \otimes {}_B B_B = {}_{A \otimes B}(A \otimes B)_{A \otimes B}$ and ${}_C C_C \otimes {}_D D_D = {}_{C \otimes D}(C \otimes D)_{C \otimes D}$ We have:

$$\begin{aligned}
X_{\Phi \otimes \Phi'} &= \big({}_{A \otimes B}(A \otimes B)_{A \otimes B}\big) \otimes_{\Phi \otimes \Phi'} \big({}_{C \otimes D}(C \otimes D)_{C \otimes D}\big) \\
&= \big({}_A A_A \otimes {}_B B_B\big) \otimes_{\Phi \otimes \Phi'} \big({}_C C_C \otimes {}_D D_D\big)
\end{aligned}$$

Applying the preceding lemma gives:

$$\begin{aligned}
&\cong \big({}_A A_A \otimes_\Phi {}_C C_C\big) \otimes \big({}_B B_B \otimes_{\Phi'} {}_D D_D\big) \\
&= X_\Phi \otimes X_{\Phi'}
\end{aligned}$$

Thus $X_{\Phi \otimes \Phi'} \cong X_\Phi \otimes X_{\Phi'}$.

The following facts (see Chapter 4 of [Lance(1995)]) will be useful in proving our main result:

Lemma 2.1.7. *Let X and Y be C^*-correspondences over C^*-algebras A and B respectively. Then $\mathcal{K}(X \otimes Y) \cong \mathcal{K}(X) \otimes \mathcal{K}(Y)$ via the map κ which takes $S \otimes T \in \mathcal{K}(X) \otimes \mathcal{K}(Y)$ to the linear map $x \otimes y \mapsto Sx \otimes Ty$. Further, if the left actions of A and B are injective and implemented by compacts then so is the left action of $A \otimes B$ on $X \otimes Y$.*

Graph algebras provide an interesting example of external tensor products:

Example 2.1.8. Let $E = \{E^0, E^1, r, s\}$ and $F := \{F^0, F^1, r', s'\}$ be topological graphs. Define

$$E \times F := \{E^0 \times F^0, E^1 \times F^1, r \times r', s \times s'\}$$

(i.e. the topological analog of the product graph in [Kumjian(1998)]). Since the product of two continuous maps is continuous and the product of two local homeomorphisms is a local homeomorphism, $E \times F$ is a topological graph. Let $\rho : C_0(E^0) \otimes C_0(F^0) \to C_0(E^0 \times F^0)$ and $\sigma : C_0(E^1) \otimes C_0(F^1) \to C_0(E^1 \times F^1)$ be the standard isomorphisms. Note that $\sigma(C_c(E^1) \otimes C_c(F^1)) \subseteq C_c(E^1 \times F^1)$ and that

$$\begin{aligned}
\sigma\big((a \otimes b) \cdot (x \otimes y)\big)(e, f) &= \sigma(a \cdot x \otimes b \cdot y)(e, f) \\
&= a\big(r(e)\big)x(e)b\big(r'(f)\big)y(f) \\
&= a\big(r(e)\big)b\big(r'(f)\big)x(e)y(f) \\
&= \rho(a \otimes b)\big(r(e), r'(f)\big)\sigma(x \otimes y)(e, f)
\end{aligned}$$

Similarly,

$$\sigma\big((x \otimes y) \cdot (a \otimes b)\big)(e, f) = \sigma(x \otimes y)(e, f)\rho(a \otimes b)\big(s(e), s'(f)\big)$$

Let $A = C_0(E^0)$ and $B = C_0(F^0)$. If $\langle\cdot,\cdot\rangle_1$ is the $A \otimes B$-valued tensor product associated to the external tensor product $X(E) \otimes X(F)$ and $\langle\cdot,\cdot\rangle_2$ is the $C_0(E \times F)$ valued inner product associated to $X(E \times F)$ then:

$$\begin{aligned}
\rho\big(\langle x \otimes y, x' \otimes y'\rangle_1\big)(v,w) &= \rho\big(\langle x, x'\rangle_A \otimes \langle y, y'\rangle_B\big)(v,w) \\
&= \langle x, x'\rangle_A(v)\langle y, y'\rangle_B(w) \\
&= \sum_{s(e)=v, s'(f)=w} \overline{x(e)}x'(e)\overline{y(f)}y'(f) \\
&= \sum_{s(e)=v, s'(f)=w} \overline{x(e)y(f)}x'(e)y'(f) \\
&= \sum_{s(e)\times s'(f)=v\times w} \overline{\sigma(x \otimes y)(e,f)}\sigma(x' \otimes y')(e,f) \\
&= \langle \sigma(x \otimes y), \sigma(x' \otimes y')\rangle_2(v,w)
\end{aligned}$$

Extending linearly and continuously, we see that (ρ, σ) gives an isomorphism of correspondences: $X(E \times F) \cong X(E) \otimes X(F)$.

For Chapter 3, we will wish to recall the following from [Echterhoff *et al.*(2000), Proposition 2.3]. For two non-degenerate homomorphisms $\varphi, \psi : A \to M(B)$, the standard $A - B$ correspondences ${}_\varphi B$ and ${}_\psi B$ are isomorphic if and only if there is a unitary multiplier $u \in M(B)$ such that $\psi = \operatorname{Ad} u \circ \varphi$ (the special case of imprimitivity bimodules is essentially [Brown *et al.*(1977), Proposition 3.1]). In particular, if B is commutative then ${}_\varphi B \cong {}_\psi B$ if and only if $\varphi = \psi$.

In Chapter 5, we will need the following definition:

Definition 2.1.9. Let X be an $A - B$ correspondence. A *generating system* for X is a triple (A^0, X^0, B^0) where $A^0 \subseteq A$, $X^0 \subseteq X$ and $B^0 \subseteq B$ such that $\overline{\operatorname{span}}(A^0) = A$, $\overline{\operatorname{span}}(X^0) = X$, and $\overline{\operatorname{span}}(B^0) = B$ and such that for all $x, y \in X^0$ we have that $\langle x, y\rangle_B \in B^0$ and $ax, xb \in X^0$ for all $a \in A^0$ and $b \in B^0$. If $A = B$ and $A^0 = B^0$ we will denote the generating system by (X^0, A^0).

We will make frequent use of the following fact:

Lemma 2.1.10. *Let X be an $A_1 - B_1$ correspondence and Y be a $A_2 - B_2$ correspondence. Suppose that (A_1^0, X^0, B_1^0) and (A_2^0, Y^0, B_2^0) are generating sets for X and Y respectively. Let $\varphi_A : A_1 \to A_2$ and $\varphi_B : B_1 \to B_2$ be isomorphisms. Suppose there is a bijection $\Phi_0 : X^0 \to Y^0$, which preserves the inner product, left and right actions, and scalar multiplication. That is*

$$\begin{aligned}
\big\langle \Phi_0(x), \Phi_0(x') \big\rangle_B^Y &= \varphi_B\Big(\langle x, x'\rangle_B^X\Big) && \text{for all } x, x' \in X^0 \\
\Phi_0(ax) &= \varphi_A(a)\Phi_0(x) && \text{for all } x \in X^0 \text{ and } a \in A^0 \\
\Phi_0(xb) &= \Phi_0(x)\varphi_B(b) && \text{for all } x \in X^0 \text{ and } b \in B^0 \\
\Phi_0(cx) &= c\Phi_0(x) && \text{for all } x \in X^0 \text{ and } c \in \mathbb{C}
\end{aligned}$$

Then Φ_0 extends linearly and continuously to a correspondence isomorphism $\Phi : X \to Y$.

Proof. Let $x \in \operatorname{span}(X^0)$, then $x = \sum_i c_i x_i$ for some $x_i \in X^0$ and $c_i \in \mathbb{C}$. We define $\Phi(x) = \sum_i c_i \Phi_0(x_i)$. First, we must verify that Φ is well defined on $\operatorname{span}(X_0)$. Suppose $\sum_{i=1}^n c_i x_i$ and $\sum_{i=1}^m d_i x_i'$ are both equal to $x \in X$ with $x_i, x_i' \in X^0$. Let $y = \sum_{i=1}^n c_i \Phi_0(x_i)$ and $y' = \sum_{i=1}^m d_i \Phi_0(x_i')$. Then

$$\begin{aligned}
\|y - y'\|^2 &= \langle y - y', y - y'\rangle_{B'} \\
&= \Big\langle \sum_{i=1}^n c_i\Phi_0(x_i) - \sum_{i=1}^m d_i\Phi_0(x_i'), \sum_{i=1}^n c_i\Phi_0(x_i) - \sum_{i=1}^m d_i\Phi_0(x_i')\Big\rangle_{B'} \\
&= \sum_{i,j=1}^{n,n} \overline{c_i} c_j \langle \Phi_0(x_i), \Phi_0(x_j)\rangle_{B'} + \sum_{i,j=1}^{m,m} \overline{d_i} d_j \langle \Phi_0(x_i'), \Phi_0(x_j')\rangle_{B'} \\
&\quad - \sum_{i,j=1}^{n,m} \overline{c_i} d_j \langle \Phi_0(x_i), \Phi_0(x_j')\rangle_{B'} - \sum_{i,j=1}^{m,n} \overline{d_i} c_j \langle \Phi_0(x_i'), \Phi_0(x_j)\rangle_{B'} \\
&= \sum_{i,j=1}^{n,n} \overline{c_i} c_j \varphi_B\big(\langle x_i, x_j\rangle_B\big) + \sum_{i,j=1}^{m,m} \overline{d_i} d_j \varphi_B\big(\langle x_i', x_j'\rangle_B\big)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i,j=1}^{n,m} \overline{c_i} d_j \varphi_B\big(\langle x_i, x_j' \rangle_B\big) - \sum_{i,j=1}^{m,n} \overline{d_i} c_j \varphi_B\big(\langle x_i', x_j \rangle_B\big) \\
&= \varphi_B\Big(\Big\langle \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{m} d_i x_i', \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{m} d_i x_i' \Big\rangle_B\Big) \\
&= \varphi_B\big(\langle x - x, x - x \rangle_B\big) \\
&= 0
\end{aligned}
$$

where we have used the fact that φ is an isomorphism and thus linear.

For $x, x' \in X^0$ we have

$$
\begin{aligned}
\big\langle \Phi(x), \Phi(x') \big\rangle_B^Y &= \Big\langle \sum_i c_i \Phi_0(x_i), \sum_j c_j' \Phi_0(x_j') \Big\rangle_B^Y \\
&= \sum_{i,j} \overline{c}_i c_j' \big\langle \Phi_0(x_i), \Phi_0(x_j') \big\rangle_B^Y \\
&= \sum_{i,j} \overline{c}_i c_j' \varphi_B\Big(\langle x_i, x_j' \rangle_B^X\Big) \\
&= \varphi_B\Big(\Big\langle \sum_i c_i x_i, \sum_j c_j' x_j' \Big\rangle_B^X\Big) \\
&= \varphi_B\Big(\langle x, x' \rangle_B^X\Big)
\end{aligned}
$$

Therefore, Φ preserves the inner product on $\operatorname{span}(X^0)$ and thus also preserves the norm on $\operatorname{span}(X^0)$ and so Φ is bounded and can be extended continuously to $\overline{\operatorname{span}}(X^0) = X$. Hence, for any $z \in X$ we can approximate $z \approx \sum_i c_i z_i$ with $z_i \in X^0$. Thus for any $a_0 \in A^0$ we have

$$
\Phi(a_0 z) \approx \Phi\Big(a_0 \sum_i c_i z_i\Big) = \sum_i c_i \Phi(a_0 z_i) = \varphi_A(a_0) \sum_i c_i \Phi(z_i) \approx \varphi_A(a_0)\Phi(z)
$$

so $\Phi(a_0 z) = \varphi_A(a_0)\Phi(z)$ and similarly $\Phi(z b_0) = \Phi(z)\varphi_B(b_0)$ for $b_0 \in B^0$. For arbitrary $a \in A$ we may approximate $a \approx \sum_i c_i a_i$ with $a_i \in A^0$ and we see that

$$
\Phi(az) \approx \sum c_i \Phi(a_i z) = \varphi_A\Big(\sum_i c_i a_i\Big)\Phi(z) \approx \varphi_A(a)\Phi(z)
$$

So $\Phi(az) = \varphi_A(a)\Phi(z)$ for all $a \in A$ and similarly $\Phi(zb) = \Phi(z)\varphi_B(b)$ for all $b \in B$. We already know that Φ is injective since it preserves the norm, so we only have to show that it is surjective. To see this note that

$$\Phi(X) = \overline{\text{span}}\big(\Phi_0(X^0)\big) = \overline{\text{span}}(Y^0) = Y$$

So we have established that Φ is a linear isomorphism from X to Y which preserves the left and right actions and the inner product, in other words Φ is a correspondence isomorphism. $\square$

Given a Hilbert module (X, A), we define the *linking algebra* of (X, A) to be the C^*-algebra $L(X) := \mathcal{K}(X \oplus A)$. There are complimentary projections p and q in $M(L(X))$ such that $pL(X)p \cong \mathcal{K}(X)$, $pL(X)q \cong X$, and $qL(X)q \cong A$. This gives $L(X)$ the following block matrix decomposition:

$$L(X) = \begin{bmatrix} \mathcal{K}(X) & X \\ \overline{X} & A \end{bmatrix}$$

with $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The benefit of using linking algebras is that the algebraic properties of X are encoded into the multiplicative structure of $L(X)$:

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & xa \\ 0 & 0 \end{bmatrix} \tag{2.1.1}$$

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}^* \cdot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \langle x, y \rangle_A \end{bmatrix} \tag{2.1.2}$$

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} \Theta_{x,y} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.1.3}$$

Let X be an $A - B$ correspondence. We define the *multiplier correspondence* $M(X)$ of X to be the set of all adjointable maps $B \to X$. This can be viewed as a $M(X) - M(B)$ correspondence with operations given by:

$$(n \cdot T)b = n \cdot (Tb)$$

$$(T \cdot m)b = T(mb)$$

$$\langle T, S \rangle_{M(B)}$$

where $n \in M(A)$, $T, S \in M(X)$ $b \in B$, and $m \in M(B)$ (see Corollary 1.13 of [Echterhoff *et al.*(2002)] for details).

2.1.2 Cuntz-Pimsner Algebras

In order to describe the Cuntz-Pimsner algebra of a correspondence, we will first need to discuss representations of correspondences. Given a correspondence X over a C^*-algebra A, a *Toeplitz Representation* of X in a C^* algebra B is a pair (ψ, π) where $\psi : X \to B$ is a linear map and $\pi : A \to B$ is a $*$-homomorphism such that

1. $\psi(a \cdot x) = \pi(a)\psi(x)$ for all $a \in A$ and all $x \in X$.
2. $\psi(x \cdot a) = \psi(x)\pi(a)$ for all $a \in A$ and all $x \in X$.
3. $\pi\big(\langle x, y \rangle_A\big) = \psi(x)^*\psi(y)$ for all $x, y \in X$.

(Note that 2 is actually implied by 3 but we include it for clarity.) We will write $C^*(\psi, \pi)$ for the C^*-subalgebra of B generated by the images of ψ and π in B. There is a unique (up to isomorphism) C^*-algebra $\mathcal{T}_X$, called the *Toeplitz algebra* of X, which is generated by a representation (i_X, i_A) which is "universal" in the sense that for any representation (ψ, π) of X in any C^*-algebra B, there is a unique $*$-homomorphism $\psi \times \pi : \mathcal{T}_X \to B$ such that $\psi = (\psi \times \pi) \circ i_A$ and $\pi = (\psi \times \pi) \circ i_X$. This construction is discussed in detail in [Katsura(2004)].

Suppose X and Y are correspondences over A and that $\pi : A \to B$ is a homomorphism. Suppose further that there are linear maps $\psi : X \to B$ and $\mu : Y \to B$ such that (ψ, π) and (μ, π) are Toplitz representations of X and Y. It is shown in Proposition 1.8 of [Fowler and Raeburn(1998)] that there is a Toeplitz representation $(\psi \otimes \mu, \pi)$ of $X \otimes_A Y$ in B such that $(\psi \otimes \mu)(x \otimes y) = \psi(x)\pi(y)$.

Let $X^{\otimes n}$ denote the n-fold internal tensor product of X with itself. By convention we let $X^{\otimes 0} = A$. Given a Toeplitz representation (ψ, π) of X in B, we define a map $\psi^n : X^{\otimes n} \to B$ for each $n \in \mathbb{N}$ as follows: We let $\psi^0 = \pi$ and $\psi^1 = \psi$ and then for each $n > 1$ we set $\psi^n = \psi \otimes \psi^{n-1}$.

Proposition 2.7 of [Katsura(2004)] states the following: Let X be correspondence over A and let (ψ, π) be a Toeplitz representation of X. Then

$$C^*(\psi, \pi) = \overline{span}\{\psi^n(x)\psi^m(y)^* : x \in X^{\otimes n}, y \in X^{\otimes m}\}$$

.

From Lemma 2.4 of [Katsura(2004)] we get the following: Let (ψ, π) be a Toeplitz representation of X in B. For each $n \in \mathbb{N}$ there is a homomorphism $\psi^{(n)} : \mathcal{K}(X^{\otimes n}) \to B$ such that:

1. $\pi(a)\psi^{(n)}(k) = \psi^{(n)}(\phi(a)k)$ for all $a \in A$ and all $k \in \mathcal{K}(X^{\otimes n})$.
2. $\psi^{(n)}(k)\psi(x) = \psi(kx)$ for all $x \in X$ and all $k \in \mathcal{K}(X^{\otimes n})$.

Let X be a correspondence over a C^*-algebra A. We define the *Katsura ideal* of A to be the ideal:

$$J_X = \{a \in A : \phi(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in ker(\phi)\}.$$

Where ϕ is the left action. This is often written more compactly as $J_X = \phi^{-1}\big(\mathcal{K}(X)\big) \cap \big(ker(\phi)\big)^{\perp}$. In many cases of interest, one can consider only correspondences whose left actions are injective and implemented by compacts. In this case we have $J_X = A$.

The Katsura ideal is also sometimes described as the largest ideal of A which maps injectively into the compacts. This is made precise by the following proposition (see [Katsura(2004)]):

Proposition 2.1.11. *Suppose X is a correspondence over a C^*-algebra A, ϕ is the left action map, and I is an ideal of A which is mapped injectively into $\mathcal{K}(X)$ by ϕ. Then $I \subseteq J_X$.*

We are now ready to define the Cuntz-Pimsner algebra $\mathcal{O}_X$. A Toeplitz representation is said to be *Cuntz-Pimsner covariant* if $\psi^{(1)}\big(\phi(a)\big) = \pi(a)$ for all $a \in J_X$. The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the quotient of $\mathcal{T}_X$ by the ideal generated by

$$\{i_X^{(1)}\big(\phi(a)\big) - i_A(a) : a \in J_X\} \tag{2.1.4}$$

Letting $q : \mathcal{T}_X \to \mathcal{O}_X$ denote the quotient map, $\mathcal{O}_X$ is generated by the Cuntz-Pimsner covariant representation $(k_X, k_A) = (q \circ i_X, q \circ i_A)$ and that this representation is universal for Cuntz-Pimsner covariant representations: if (ψ, π) is a Cuntz-Pimsner covariant representation of X in B, then there is a $*$-homomorphism $\psi \times \pi : \mathcal{O}_X \to B$ such that $\psi = (\psi \times \pi) \circ k_X$ and $\pi = (\psi \times \pi) \circ k_A$.

This construction in its present form is due to Katsura and is discussed in detail in [Katsura(2004)]. It is worth noting, however that this is a modification of Pimsner's original $\mathcal{O}_X$ defined in [Pimsner(2001)]. This was the quotient of $\mathcal{T}_X$ by the ideal generated by $\{i_X^{(1)}\big(\phi(a)\big) - i_A(a) : a \in \phi^{-1}(\mathcal{K}(X))\}$. In many papers (particularly those which predate Katsura's work) this is what is meant by the Cuntz-Pimsner algebra (see for example [Fowler and Raeburn(1998)]). In this thesis however, we shall always take the term "Cuntz-Pimsner algebra" and the notation $\mathcal{O}_X$ to refer to the quotient of $\mathcal{T}_X$ by the ideal generated by (2.1.4). Note that the two definitions coincide when ϕ is injective.

Example 2.1.12. If E is a directed graph and $X(E)$ is the graph correspondence of Example 2.1.2, then $\mathcal{O}_{X(E)} \cong C^*(E)$ where $C^*(E)$ is the C^*-algebra of the graph (see [Raeburn(2005)] for details). For this reason, if F is a topological graph, the graph algebra of F is defined to be $\mathcal{O}_{X(F)}$.

One of the most important results about Cuntz-Pimsner algebras is the so-called "gauge-invariant uniqueness theorem". In order to state this theorem we need the following definition: Let (ψ, π) be a Toeplitz representation of a correspondence X. Then we say that $C^*(\psi, \pi)$ *admits a gauge action* if there is an action γ of $\mathbb{T}$ on $C^*(\psi, \pi)$ such that:

1. $\gamma_z\big(\pi(a)\big) = \pi(a)$ for all $z \in \mathbb{T}$ and all $a \in A$.

2. $\gamma_z\big(\psi(x)\big) = z\psi(x)$ for all $z \in \mathbb{T}$ and all $x \in X$.

When such an action exists, it is unique.

Theorem 2.1.13 ("Gauge Invariant Uniqueness Theorem" (6.4 in [Katsura(2004)]))**.** *Let X be a correspondence over A and let (ψ, π) be a Cuntz-Pimsner covariant representation of X. Then the $*$-homomorphism $\psi \times \pi : \mathcal{O}_X \to C^*(\psi, \pi)$ is an isomorphism if and only if (ψ, π) is injective and admits a gauge action.*

2.2 Λ-Systems

In this section we will recall some basic information about Λ-systems and topological k-graphs. These will be necessary for the extension of Ionescu's theorem presented in Chapter 3, but will not be used in the remainder of the text.

2.2.1 Λ-Systems

We will briefly recall a few basic facts about higher-rank graphs. These objects were first introduced by Kumjian and Pask in [Kumjian and Pask(2000)]. A *graph of* *rank* k is a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the *factorization property*: for $\lambda \in \Lambda$, if $d(\lambda) = m + n$, $n, m \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$ and $d(\nu) = n$. We view $\mathbb{N}^k$ as a category with one object and morphisms generated by k commuting morphisms. The functor d is referred to as the *degree map*. For $n \in \mathbb{N}^k$, we define the set of *paths of length* n to be the set $d^{-1}(n)$ which is denoted by Λ^n. If $k = 1$, then $\{\Lambda^0, \Lambda^1, r, s\}$ is a directed graph and Λ is the category of paths in the graph. We say that Λ *row-finite* if $r^{-1}(v) \cap \Lambda^n$ is finite for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$. We say the Λ *has no sources* if for every $v \in \Lambda^n$ and every $n \in \mathbb{N}^k$ there is a path $\lambda \in \Lambda^n$ such that $r(\lambda) = v$.

Throughout, Λ will be a row-finite k-graph with no sources, so that the associated Cuntz-Krieger relations take the most elementary form. In [Deaconu *et al.*(2010)], Deaconu, Kumjian, Pask, and Sims introduced Λ-*systems* of correspondences: we have a Banach bundle $X \to \Lambda$ with fibers $\{X_\lambda\}_{\lambda \in \Lambda}$ such that

1. for each $v \in \Lambda^0$, X_v is a C^*-algebra;
2. for each $\lambda \in u\Lambda v$, X_λ is an $X_u - X_v$ correspondence;
3. there is a partially-defined associative multiplication on $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ that is compatible with the multiplication in Λ via the bundle projection $X \to \Lambda$;
4. whenever $\lambda, \mu \notin \Lambda^0$ and $s(\lambda) = r(\mu)$, $x \otimes y \mapsto xy : X_\lambda \otimes_{A_{s(\lambda)}} X_\mu \to X_{\lambda\mu}$ is an isomorphism of $A_{r(\lambda)} - A_{s(\mu)}$ correspondences;
5. the left and right module multiplications of the correspondences coincide with the multiplication from the Λ-system.

For a Λ-system X of correspondences, we will write

$$\varphi_\lambda : X_u \to \mathcal{L}(X_\lambda) \quad \text{for } \lambda \in u\Lambda$$

for the left-module structure map. Note that the multiplication in X induces $X_u - X_v$ correspondence isomorphisms $X_\lambda \otimes X_v \cong X_\lambda$ for all $\lambda \in u\Lambda v$, but only induces isomorphisms $X_u \otimes X_\lambda \cong X_\lambda$ if every correspondence X_λ is non-degenerate.

Given Λ-systems X and Y of correspondences, a map $\theta : X \to Y$ is a Λ-*system isomorphism* if

1. for all $\lambda \in u\Lambda v$,

$$\theta_\lambda := \theta|_{X_\lambda} : X_\lambda \to Y_\lambda$$

   is an isomorphism of correspondences with coefficient isomorphisms θ_u, θ_v;

2. for all $\lambda \in u\Lambda v, \mu \in v\Lambda w$,

$$\theta_\lambda(\xi)\theta_\mu(\eta) = \theta_{\lambda\mu}(\xi\eta) \quad \text{for all } \xi \in X_\lambda, \eta \in X_\mu.$$

Since the multiplication in the Λ-system induces the left and right module multiplications for the correspondences, in the above we can relax (1) to

$(1)'$ for all $\lambda \in \Lambda v$, $\theta_\lambda : X_\lambda \to Y_\lambda$ is a linear bijection satisfying

$$\langle \theta_\lambda(\xi), \theta_\lambda(\eta) \rangle_{Y_v} = \theta_v(\langle \xi, \eta \rangle_{X_v}) \quad \text{for all } \xi, \eta \in X_\lambda,$$

because (2) takes care of the coefficient maps. We emphasize that we're requiring that, for each $v \in \Lambda^0$, θ_v be the right-hand coefficient isomorphism for every correspondence isomorphism θ_λ with $s(\lambda) = v$, and also the left-hand coefficient isomorphism for every correspondence isomorphism θ_λ with $r(\lambda) = v$. Thus, if X and Y are isomorphic Λ-systems of correspondences, then without loss of generality we may assume (if we wish) that $X_v = Y_v$ and $\theta_v = \mathrm{id}_{X_v}$ for every vertex v, so that $\theta_\lambda : X_\lambda \to Y_\lambda$ is an $X_u - X_v$ correspondence isomorphism whenever $\lambda \in u\Lambda v$.

2.2.2 Topological k-Graphs

Recall [Yeend(2006)] that a *topological k-graph* is a k-graph Γ equipped with a locally compact Hausdorff topology making the multiplication continuous and open, the range map continuous, the source map a local homeomorphism, and the degree functor $d : \Gamma \to \mathbb{N}^k$ continuous. Carlsen, Larsen, Sims, and Vittadello show in [Carlsen *et al.*(2011), Proposition 5.9] that every topological k-graph Γ gives rise to an $\mathbb{N}^k$-system Z of correspondences over $A := C_0(\Gamma^0)$ as follows: For each $n \in \mathbb{N}^k$ let Z_n be the A-correspondence associated to the topological graph $(\Gamma^0, \Gamma^n, s|_{\Gamma^n}, r|_{\Gamma^n})$, so that Z_n is the completion of the pre-correspondence $C_c(\Gamma^n)$ with operations

$$(f \cdot \xi \cdot g)(\alpha) = f(r(\alpha))\xi(\alpha)g(s(\alpha))$$
$$\langle \xi, \eta \rangle_A(v) = \sum_{\alpha \in \Gamma^n v} \overline{\xi(\alpha)}\eta(\alpha),$$

for $\xi, \eta \in C_c(\Gamma^n)$, $f, g \in A$. Then for $\xi \in C_c(\Gamma^n), \eta \in C_c(\Gamma^m)$ define $\xi\eta \in C_c(\Gamma^{n+m})$ by

$$(\xi\eta)(\alpha\beta) = \xi(\alpha)\eta(\beta) \quad \text{for } \alpha \in \Gamma^n, \beta \in \Gamma^m, s(\alpha) = r(\beta).$$

In [Yeend(2006)], Yeend defined $C^*(\Gamma)$ using a groupoid model, but [Carlsen *et al.*(2011), Theorem 5.20] shows that $C^*(\Gamma) \cong \mathcal{NO}_Z$, where $\mathcal{NO}_Z$ is the Cuntz-Nica-Pimsner algebra of the product system Z. The topological k-graphs we encounter in this paper will be nice enough that $\mathcal{NO}_Z$ will coincide with the Cuntz-Pimsner algebra $\mathcal{O}_Z$.

2.3 Actions and Coactions of Groups and Quantum Groups

In this section, we will recall some basic results about actions and coactions of groups. These results will be important in Chapters 4 and 5. Next we will give a brief overview of the theory of quantum groups and then we will summarize the twisted tensor product construction of Woronowicz et. al.. These will be used in Chapter 5.

2.3.1 Actions and Coactions of Discrete Groups

Working with actions and coactions of locally compact groups and their crossed products can be somewhat complicated. For a general reference we suggest the appendix of [Echterhoff *et al.*(2002)]. Restricting attention to discrete groups affords many simplifications. In this section, we will summarize some of these simplifications.

First we will briefly recall the basics of actions and coactions. We will assume throughout that G is discrete. By an *action* of a group G on a C^*-algebra A, we shall mean a group homomorphism $\alpha : G \to Aut(A)$. We will refer to the triple (A, G, α) as a *C^*-dynamical system.* For $s \in G$ we will write α_s for the automorphism $\alpha(s)$. The group C^*-algebra $C^*(G)$ is generated by a unitary representation $G \ni s \mapsto u_s \in \mathcal{U}C^*(G)$. We will abuse notation and write s for u_s. The function $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ defined by $s \mapsto s \otimes s$ for all $s \in G$ is called the *comultiplication map* on $C^*(G)$. There is a comultiplication map defined for any locally compact group, but in general it will map into $M(C^*(G) \otimes C^*(G))$. See the appendix of [Echterhoff *et al.*(2002)] for more details.

A *covariant representation* of the system (A, G, α) in a C^*-algebra B consists of a nondegenerate homomorphism $\pi : A \to M(B)$ and a strictly continuous group homomorphism $U : G \to UM(B)$ such that $\pi\big(\alpha_s(a)\big) = U_s\pi(a)U_s^*$ for all $s \in G$ and all $a \in A$. The *full crossed product* of (A, G, α) is the completion of $c_c(G, A)$ with respect to the norm

$$\|f\| = \sup\Big\{\Big\| \sum_{s\in G} \pi\big(f(s)\big)U_s\Big\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\Big\}.$$

The full crossed product is denoted by $A \rtimes_\alpha G$. There is a canonical covariant representation (i_A, i_G) of (A, G, α) in $A \rtimes_\alpha G$. It has the universal property that for any covariant representation (π, U) in $M(B)$ there is a unique nondegenerate homomorphism $\pi \times U : A \rtimes_\alpha G \to M(B)$ such that $(\pi \times U) \circ i_A = \pi$ and $(\pi \times U) \circ i_G = U$. The map $\pi \times U$ is called the *integrated form* of (π, U).

Let $\lambda : G \to UM\big(l^2(G)\big)$ be the left regular representation of G on $l^2(G)$ and let $M : c_0(G) \to \mathcal{B}\big(l^2(G)\big)$ be the representation of $c_0(G)$ as multiplication operators. Then $(i_A^r, i_G^r) := (\mathrm{id}_A \otimes M, 1 \otimes \lambda)$ is a covariant representation of (A, G, α) in $M(A \otimes \mathcal{K}\big(l^2(G)\big)$. The *reduced crossed product* of (A, G, α) is defined to be the image of $A \rtimes_\alpha G$ under the integrated form $i_A^r \times i_G^r$. It is denoted $A \rtimes_{\alpha,r} G$.

If (A, G, α) and (B, G, β) are dynamical systems and X is an $A{-}B$ correspondence, then a *compatible action* of G on X is a group homomorphism γ from G to the group of invertible linear maps on X such that:

$$\begin{aligned} \gamma_s(a \cdot x) &= \alpha_s(a) \cdot \gamma_s(x) \\ \gamma_s(x \cdot b) &= \gamma_s(x) \cdot \beta_s(b) \\ \big\langle \gamma_s(x), \gamma_s(y) \big\rangle_B &= \beta_s\big(\langle x, y \rangle_B\big) \end{aligned}$$

for all $s \in G$, $x, y \in X$, $a \in A$ and $b \in B$.

There is also a notion of a crossed product for correspondences. To see this, note that we can give $c_c(G, X)$ the structure of a $c_c(G, A) - c_c(G, B)$ pre-correspondence with the following definitions:

$$\begin{aligned} f \cdot x(s) &= \sum_{t \in G} f(t)\gamma_t\big(x(t^{-1}s)\big) \\ x \cdot g(s) &= \sum_{t \in G} x(t)\gamma_t\big(g(t^{-1}s)\big) \\ \langle x, y \rangle_{c_c(G,B)}(s) &= \sum_{t \in G} \beta_{t^{-1}}\big(\langle x(t), y(t) \rangle_B\big). \end{aligned}$$

The above can be completed into an $A \rtimes_\alpha G - B \rtimes_\beta G$ correspondence $X \rtimes_\gamma G$. See Proposition 3.2 of [Echterhoff *et al.*(2002)] for details. It will be useful to note that applying the appropraite action of G to each corner of $L(X)$ gives an action ν of G

on $L(X)$. Again we refer to [Echterhoff *et al.*(2002)] for the details (specifically to Lemma 3.3).

A *coaction* is a triple (A, G, δ) where G is a group (assumed here to be discrete) A is a C^*-algebra and δ is a nondegenerate, injective homomorphism $A \to M(A \otimes C^*(G))$ such that

1. $\delta(A)(1 \otimes C^*(G)) \subseteq A \otimes C^*(G)$

2. $(\delta \otimes \mathrm{id}_G) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta$ where both sides are viewed as maps $A \to M(A \otimes C^*(G) \otimes C^*(G))$.

A coaction is called *nondegenerate* if the closed linear span of $\delta(A)(1 \otimes C^*(G))$ is equal to $A \otimes C^*(G)$. We will also refer to the triple (A, G, δ) as a coaction. Note that if G is discrete then $C^*(G)$ is unital (with unit u_e) and thus $\delta(A) \subseteq \delta(A)(1 \otimes C^*(G))$ so by condition 1, δ maps into $A \otimes C^*(G)$.

Let $w_G \in UM(c_0(G) \otimes C^*(G))$ be given by the canonical map $G \to UM(C^*(G))$. A *covariant representation* of a coaction (A, G, δ) into $M(B)$ is a pair of nondegenerate homomorphisms $(\pi, \mu) : (A, c_0(G)) \to M(B)$ such that:

$$(\pi \otimes \mathrm{id}_G) \circ \delta(a) = (\mu \otimes \mathrm{id}_G)(w_G)(\pi(a) \otimes 1)(\mu \otimes \mathrm{id}_G)(w_G)^*.$$

To define the crossed product algebra of a coaction, we will need some facts from [Raeburn(1992)]. By Lemma 2.10 of [Raeburn(1992)], for any covariant representation (π, μ) of (A, G, δ) in $M(B)$, $C^*(\pi, \mu) := \overline{\pi(A)\mu\big(c_0(G)\big)}$ is a C^*-algebra. Proposition 2.6 of [Raeburn(1992)] states that for any nondegenerate homomorphism $\pi : A \to M(B)$, the pair $\big((\pi \otimes \lambda) \circ \delta, 1 \otimes M\big)$ is a covariant homomorphism of (A, G, δ) into $M\big(B \otimes \mathcal{K}(l^2(G))\big)$. If we let $B = A$ and $\pi = \mathrm{id}_A$, we get a covairant representation $(j_A, j_G) := \big((\mathrm{id}_A \otimes \lambda) \circ \delta, 1 \otimes M\big)$ ofinto $M\big(A \otimes \mathcal{K}(l^2(G))\big)$. The *crossed product* $A \rtimes_\delta G$ of (A, G, δ) is defined as $C^*(j_A, j_G)$.

Suppose (A, G, δ) and (B, G, ω) are coactions and X is an $A - B$ correspondence. A $\delta - \omega$ *compatible coaction* of G on X is a nondegenerate homomorphism $\sigma : X \to M(X \otimes C^*(G))$ (where $\otimes$ is the external tensor product and we are viewing $C^*(G)$ as a correspondence over itself) satisfying:

1. $\big(1_{M(A)} \otimes C^*(G)\big)\sigma(G) \subseteq X \otimes C^*(G)$
2. $(\sigma \otimes \mathrm{id}_G) \circ \sigma = (\mathrm{id}_X \otimes \delta_G) \circ \sigma$.

The coaction is called *nondegenerate* if

$$\overline{\big(1 \otimes C^*(G)\big)\sigma(X)} = X \otimes C^*(G).$$

Given such a coaction σ, there is a unique coaction ε of G on $L(X)$ such that

$$\varepsilon\left(\begin{bmatrix} a & x \\ \tilde{y} & b \end{bmatrix}\right) = \begin{bmatrix} \delta(a) & \sigma(x) \\ \widetilde{\sigma(y)} & b \end{bmatrix}$$

Let (j_B, j_G^B) be the canonical covariant representation of (B, G, ω) in $M\big(B \otimes \mathcal{K}(l^2(G))\big)$ described above. Define $j_X = \big(\mathrm{id}_X \otimes \lambda\big) \circ \sigma$. The *crossed product correspondence* is defined as the subspace

$$X \rtimes_\sigma G = \overline{j_X(X) \cdot j_G^B\big(c_0(G)\big)}$$

of $M\big(X \otimes \mathcal{K}(l^2(G))\big)$. Propostition 3.9 of [Echterhoff *et al.*(2002)] shows that $X \rtimes_\sigma G$ is an $A \rtimes_\delta G - B \rtimes_\omega G$ correspondence. By Lemma 3.10 of [Echterhoff *et al.*(2002)] $L(X) \rtimes_\varepsilon G \cong L(X \rtimes_\sigma G)$.

Proposition 2.3.1. *Let G be a discrete group and let α be an action of G on a C^*-algebra A. Let (i_A^r, i_G^r) be the canonical representation of the system (A, G, α) in the reduced crossed product $A \rtimes_{\alpha,r} G$. Then $A \rtimes_{\alpha,r} G$ is the closed linear span of elements* *of the form $i_A^r(a)i_G^r(s)$ where $a \in A$ and $s \in G$. These have the following algebraic properties:*

$$\big(i_A^r(a)i_G^r(s)\big)\big(i_A^r(b)i_G^r(t)\big) = i_A^r\big(a\alpha_s(b)\big)i_G^r(st) \tag{2.3.1}$$

$$\big(i_A^r(a)i_G^r(s)\big)^* = i_A^r\big(\alpha_{s^{-1}}(a^*)\big)i_G^r(s^{-1}) \tag{2.3.2}$$

Proof. The fact that the $i_A^r(a)i_G^r(s)$ densely span $A \rtimes_{\alpha,r} G$ follows from the definition of the crossed product. The algebraic properties follow easily from the fact that $i_A^r(a)i_G^r(s) = i_G^r(s)i_A^r\big(\alpha_s(a)\big)$ which we obtain from $i_G^r(s)^*i_A^r(a)i_G^r(s) = i_A^r\big(\alpha_s(a)\big)$. □

Proposition 2.3.2. *Let (γ, α) be an action of a discrete group G on a correspondence (X, A). Let $(i_X^r, i_A^r, i_G^X, i_G^A)$ be the canonical representation of the system in the crossed product $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$. Then $X \rtimes_{\gamma,r} G$ is the closed linear span of $i_X^r(x)i_G^X(s)$ where $x \in X$ and $s \in G$. These satisfy the following algebraic properties:*

$$\big(i_X^r(x)i_G^X(s)\big)\big(i_A^r(a)i_G^A(t)\big) = i_X^r\big(x\alpha_s(a)\big)i_G^X(st) \tag{2.3.3}$$

$$\big\langle i_X^r(x)i_G^X(s), i_X^r(y)i_G^Y(t)\big\rangle_{A\rtimes_{\alpha,r}G} = i_A^r\big(\alpha_{s^{-1}}\big(\langle x, y\rangle_A\big)\big)i_G^A(s^{-1}t) \tag{2.3.4}$$

$$\big(i_A^r(a)i_G^A(s)\big)\big(i_X^r(x)i_G^X(t)\big) = i_X^r\big(a\gamma_s(x)\big)i_G^X(st) \tag{2.3.5}$$

Proof. The fact that the $i_X^r(x)i_G^X(s)$ densely span $X \rtimes_{\gamma,r} G$ follows from the definition of the crossed product. To understand the algebraic properties, recall that $L(X \rtimes_{\gamma,r} G) \cong L(X) \rtimes_{\nu,r} G$ where ν is the action on $L(X)$ induced by (γ, α). Equations (2.3.3) and (2.3.4) are then easily deduced by applying the previous proposition to $L(X) \rtimes_{\nu,r} G$. (2.3.5) follows from the fact that the left action must be covariant with respect the action. □

Corollary 2.3.3. *If X is a correspondence over A and (γ, α) is an action of a discrete group G on (X, A) the sets*

$$(X \rtimes_{\gamma,r} G)_0 := \{i_X^r(x)i_G^X(s) : x \in X, s \in G\}$$

$$(A \rtimes_{\alpha,r} G)_0 := \{i_A^r(a) i_G^A(s) : a \in A, s \in G\}$$

form a generating system for $X \rtimes_{\gamma,r} G$ in the sense of Definition 2.1.9.

The simplification of crossed products by coactions of discrete groups come from the realization that coactions by discrete groups can be viewed as gradings. This idea is presented in detail in [Quigg(1996)], but we briefly summarize the main points here. We refer the reader to [Quigg(1996)] for the proofs.

Notation 2.3.4. If S and T are closed subspaces of a C^*-algebra A, we will write $S \cdot T$ for the closed linear span of products of elements of S with elements of T. In other words:

$$S \cdot T := \overline{\operatorname{span}}\{ab : a \in S, b \in T\}.$$

Proposition 2.3.5. *Let $\delta : A \to M\big(A \otimes C^*(G)\big)$ be a nondegenerate coaction of a discrete group G on a C^*-algebra A. Then $A = \overline{\operatorname{span}}\{A_s\}_{s \in G}$ where $A_s = \{a \in A : \delta(a) = a \otimes u_s\}$. Furthermore,*

$$A_s \cdot A_t \subseteq A_{st} \tag{2.3.6}$$

$$A_s^* = A_{s^{-1}} \tag{2.3.7}$$

Proposition 2.3.6. *Let δ be a coaction of a discrete group G on a C^*-algebra A and let (j_A, j_G) be the canonical representations of A and $c_0(G)$ in the crossed product $A \rtimes_\delta G$. Then $A \rtimes_\delta G$ is densely spanned by elements of the form $j_A(a_s) j_G(f)$ where $a_s \in A_s$, $f \in c_0(G)$ and These satisfy the following relations:*

$$\big(j_A(a) j_G(f)\big)\big(j_A(a_s) j_G(g)\big) = j_A(a a_s) j_G\big(\lambda_{s^{-1}}(f) g\big) \tag{2.3.8}$$

$$\big(j_A(a_s) j_G(f)\big)^* = j_A(a_s^*) j_G\big(\lambda_s(\overline{f})\big) \tag{2.3.9}$$

where λ_s denotes left translation by s on $c_0(G)$.

Applying these propositions to linking algebras helps us to understand coactions on correspondences:

Proposition 2.3.7. *Let (σ, δ) be a coaction of a discrete group G on a Hilbert module (X, A). Then $X = \overline{\operatorname{span}}\{X_s\}_{s\in G}$ where $X_s = \{x \in X : \sigma(x) = x \otimes u_s\}$. Further:*

$$X_s \cdot A_t \subseteq X_{st} \tag{2.3.10}$$

$$\langle x_s, x_t\rangle_A \in A_{s^{-1}t} \tag{2.3.11}$$

$$A_s \cdot X_t \subseteq X_{st} \tag{2.3.12}$$

Proof. Let ε be the induced coaction on $L(X)$. Then we have a grading $L(X) = \overline{\operatorname{span}}\{L(X)_s\}_{s\in G}$. Since p and q are in fixed points of the coaction, if $z \in L(X)_s$ then $qzq \in L(X)_s$ and $pzq \in L(X)_s$. Recall that the restriction of ε to $qL(X)q = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \cong A$ is δ. Thus if a is the element of A corresponding to qzq then $\varepsilon(qzq) = (qzq) \otimes u_s$ if and only if $\delta(a) = a \otimes u_s$. Thus $qL(X)_s q = \begin{bmatrix} 0 & 0 \\ 0 & A_s \end{bmatrix} \cong A_s$. Similar reasoning shows that $pL(X)_s q = \begin{bmatrix} 0 & X_s \\ 0 & 0 \end{bmatrix} \cong X_s$. (2.3.10) and (2.3.11) then follow from multiplication in $L(X)$ together with the grading of $L(X)$. (2.3.12) follows from the fact that the left module action is covariant with respect to the coaction. $\square$

Lemma 3.4 of [Echterhoff *et al.*(2002)] shows that $L(X \rtimes_\sigma G) \cong L(X) \rtimes_\varepsilon G$. This fact, together with the preceding propositions, gives us the following characterization of $X \rtimes_\sigma G$ in the case where G is discrete:

Proposition 2.3.8. *Let (σ, δ) be a coaction of a discrete group G on a correspondence (X, A). Let (j_X, j_A, j_G^X, j_G^A) be the canonical representation of the system in the correspondence $(X \rtimes_\sigma G, A \rtimes_\delta G)$. Then $X \rtimes_\sigma G$ is densely spanned by elements of the* *form* $j_X(x_s)j_G^X(f)$ *where* $x_s \in X_s$ *and* $f \in c_0(G)$*. These elements satisfy the following relations:*

$$\big(j_X(x)j_G^X(f)\big)\big(j_A(a_s)j_G^A(g)\big) = j_X(xa_s)j_G^X(\lambda_{s^{-1}}(f)g) \tag{2.3.13}$$

$$\big\langle j_X(x_s)j_G^X(f), j_X(x_t)j_G^X(g)\big\rangle_{A\rtimes_\delta G} = j_A\big(\langle x_s, x_t\rangle_A\big)j_G^A\big(\lambda_{t^{-1}s}(f)g\big) \tag{2.3.14}$$

$$\big(j_A(a)j_G^A(f)\big)\big(j_X(x_s)j_G^X(g)\big) = j_X(ax_s)j_G^X\big(\lambda_{s^{-1}}(f)g\big) \tag{2.3.15}$$

Corollary 2.3.9. *The sets*

$$(X \rtimes_\sigma G)_0 := \{j_X(x_s)j_G^X(f) : x_s \in X_s, f \in c_0(G)\}$$

$$(A \rtimes_\delta G)_0 := \{j_A(a_s)j_G^A(f) : a_s \in A_s, g \in c_0(G)\}$$

form a generating system for $X \rtimes_\sigma G$*.*

Remark 2.3.10 (Example A.23 of [Echterhoff *et al.*(2002)])*.* If G is abelian, then for every coaction of G corresponds to an action of the dual group $\widehat{G}$ and vice versa. To see this, we first identify $C^*(G)$ and $C_0(\widehat{G})$ using the abstract Fourier transform: $\mathcal{F}(x)(\chi) = \chi(x)$. We also recall that, in this situation, condition 1 from the definition of coactions is equivalent to δ taking values in $C_b(\widehat{G}, A) \in M(A \otimes C^*(G))$ (see [Echterhoff *et al.*(2002)] for details). With this in mind, if (A, G, δ) is a coaction, then we define an action α^δ of $\widehat{G}$ by setting $\alpha^\delta_\chi(a) = \delta(a)(\chi)$. Conversely, given an action α of $\widehat{G}$ on a C^*-algebra A, we define a coaction δ^α by letting $\delta^\alpha(a)(\chi) = \alpha_\chi(a)$.

Let (A, G, α) be a C^*-dynamical system with G compact and abelian and let $(A, \widehat{G}, \delta^\alpha)$ be the associated coaction of the dual group $\widehat{G}$. Since G is compact, $\widehat{G}$ is discrete and therefore δ^α will give a $\widehat{G}$-grading of A. Identifying $A \otimes C^*(\widehat{G})$ with $A \otimes C(G)$ and this with $C(G, A)$, the elementary tensor $a \otimes u_\chi$ corresponds to the map $s \mapsto \chi(s)a$. Therefore if $a \in A_\chi$ then $\alpha_s(a) = \delta^\alpha(a)(s) = \chi(s)a$ and so the sets A_χ can be thought of equivalently as

$$A_\chi = \{a \in A : \alpha_s(a) = \chi(s)a, \text{ for all } s \in G\}.$$

Thus each A_χ coincides with the so-called *spectral subspace* associated to χ.

Just as every action of a compact abelian group determines a grading by the dual group, every grading by a discrete abelian group determines an action of its dual group. To see this, just note that a G-grading of A makes A into a Fell bundle over G and as in [Quigg(1996)] we get a coaction of G associated to this Fell bundle which corresponds to an action of $\widehat{G}$:

Proposition 2.3.11. *Let A be a C^*-algebra and suppose $\{A_s\}_{s\in G}$ is a G-grading of A. Let $\chi \in \widehat{G}$. For each $s \in G$ and each $a \in A_s$ define $\alpha_\chi(a) = \chi(s)a$. The maps α_χ extend to automorphisms of A such that $\alpha : \chi \mapsto \alpha_\chi$ is an action of $\widehat{G}$ on A.*

2.3.2 Quantum Groups

There are many different notions of a quantum group. Our quantum groups will be quantum groups generated by modular multiplicative unitaries. We will briefly recall some of the basic facts about these quantum groups and refer the reader to [Timmermann(2008)] for a more in depth overview of the subject.

Definition 2.3.12. Given a separable Hilbert space $\mathcal{H}$, a *multiplicative unitary* $\mathbb{W}$ is a unitary operator on $\mathcal{H} \otimes \mathcal{H}$ such that

$$\mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \tag{2.3.16}$$

where $\mathbb{W}_{ij}$ indicates that we are applying $\mathbb{W}$ to the i^{th} and j^{th} factors of $\mathcal{H}$ and leaving the other fixed. Equation 2.3.16 is sometimes referred to as the *pentagon equation.*

$\mathbb{W}$ is called *modular* if there exist (possibly unbounded) operators $\widehat{Q}$ and Q on $\mathcal{H}$ and a unitary $\widetilde{\mathbb{W}} \in \mathcal{U}(\overline{\mathcal{H}} \otimes \mathcal{H})$ (where $\overline{\mathcal{H}}$ is the dual space of $\mathcal{H}$) such that

1. $\widehat{Q}$ and Q are positive and self-adjoint with trivial kernels
2. $\mathbb{W}^*(\widehat{Q} \otimes Q)\mathbb{W} = \widehat{Q} \otimes Q$

3. $\langle \eta'\otimes\xi', \mathbb{W}(\eta\otimes\xi)\rangle = \langle \overline{\eta}\otimes Q\xi', \widetilde{\mathbb{W}}(\overline{\eta'}\otimes Q^{-1}\xi)\rangle$ for all $\xi \in \mathrm{Dom}(Q^{-1})$, $\xi' \in \mathrm{Dom}(Q)$, and $\eta, \eta' \in \mathcal{H}$

Example 2.3.13 (Example 7.1.4 of [Timmermann(2008)])**.** Let G be a locally compact group. We can identify $L^2(G) \otimes L^2(G)$ with $L^2(G \times G)$ and define $\mathbb{W}_G \in \mathcal{B}\big(L^2(G) \otimes L^2(G)\big)$ by

$$(\mathbb{W}_G\zeta)(s,t) := \zeta(s, s^{-1}t)$$

Then $\mathbb{W}_G$ is a multiplicative unitary.

Theorem 2.3.14 (Theorem 2.7 of [Meyer *et al.*(2014)])**.** *For a modular multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$, set*

$$S := \overline{\mathrm{span}}\{(\omega \otimes \mathrm{id}_{\mathcal{H}})\mathbb{W} : \omega \in \mathcal{B}(\mathcal{H})_*\}$$
$$\widehat{S} := \overline{\mathrm{span}}\{(\mathrm{id}_{\mathcal{H}} \otimes \omega)\mathbb{W} : \omega \in \mathcal{B}(\mathcal{H})_*\}$$

where $B(\mathcal{H})_*$ denotes the predual of $B(\mathcal{H})$, i.e. the set of normal linear functionals on $B(\mathcal{H})$. Then:

- *S and $\widehat{S}$ are separable, non-degenerate C^*-subalgebras of $\mathcal{B}(\mathcal{H})$.*

- *$\mathbb{W} \in \mathcal{U}(\widehat{S} \otimes S) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. When we wish to view $\mathbb{W}$ as a unitary multiplier of $\widehat{S} \otimes S$ we will denote it by W^S and refer to it as the* reduced bicharacter *of S.*

- *There are unique homomorphisms $\Delta_S, \widehat{\Delta}_S : S \to S \otimes S$ such that*

$$(\mathrm{id}_{\widehat{S}} \otimes \Delta_S)W^S = W^S_{12}W^S_{13} \in \mathcal{U}(\widehat{S} \otimes S \otimes S)$$
$$(\widehat{\Delta}_S \otimes \mathrm{id}_S)W^S = W^S_{23}W^S_{13} \in \mathcal{U}(\widehat{S} \otimes \widehat{S} \otimes S)$$

- *(S, Δ_S) and $(\widehat{S}, \widehat{\Delta}_S)$ are C^*-bialgebras*

Definition 2.3.15. A *quantum group* is a C^*-bialgebra $\mathbb{G} := (S, \Delta_S)$ arising from a modular multiplicative unitary as in Theorem 2.3.14. The *dual* $\widehat{\mathbb{G}}$ of $\mathbb{G}$ is the bialgebra $(\widehat{S}, \widehat{\Delta}_S)$. $\widehat{\mathbb{G}}$ is generated by the modular multiplicative unitary $\widehat{\mathbb{W}} := \Sigma\mathbb{W}^*\Sigma$ (where Σ is the flip isomorphism: $x \otimes y \mapsto y \otimes x$). $\widehat{\mathbb{W}}$ is called the *dual* of $\mathbb{W}$. We will sometimes write $S(\mathbb{W})$ and $\widehat{S}(\mathbb{W})$ for S and $\widehat{S}$ if we wish to call attention to the multiplicative unitary which generated S or $\widehat{S}$.

Example 2.3.16 (Example 7.2.13 of [Timmermann(2008)])**.** Let G be a locally compact group and let $\mathbb{W}_G$ be the multiplicative unitary described in Example 2.3.13. Then $S(\mathbb{W}_G) \cong C_r^*(G)$ and $\widehat{S}(\mathbb{W}_G) \cong C_0(G)$ as C^*-bialgebras. We denote the reduced bicharacter associated to $\mathbb{W}_G$ by W_G.

Definition 2.3.17. Given two quantum groups $\mathbb{G} = (S, \Delta_S)$ and $\mathbb{H} = (T, \Delta_T)$, we define a *bicharacter from* $\mathbb{G}$ *to* $\widehat{\mathbb{H}}$ to be a unitary $\chi \in \mathcal{U}(\widehat{S} \otimes \widehat{T})$ such that

$$(\widehat{\Delta}_S \otimes \mathrm{id}_{\widehat{T}})\chi = \chi_{23}\chi_{13} \in \mathcal{U}(\widehat{S} \otimes \widehat{S} \otimes \widehat{T}) \tag{2.3.17}$$

$$(\mathrm{id}_{\widehat{S}} \otimes \widehat{\Delta}_T)\chi = \chi_{13}\chi_{13} \in \mathcal{U}(\widehat{S} \otimes \widehat{S} \otimes \widehat{T}) \tag{2.3.18}$$

Bicharacters are used in the construction of the twisted tensor product. The reduced bicharacter from Theorem 2.3.14 is a special case of a bicharacter. In fact the only bicharacter we will need for our simplified twisted tensor products is the bicharacter W_G associated to the multiplicative unitary $\mathbb{W}_G$ from Example 2.3.13.

Definition 2.3.18. A *continuous coaction* of a quantum group $\mathbb{G} = (S, \Delta_S)$ on a C^*-algebra A is a homomorphism $\delta : A \to M(A \otimes S)$ such that

1. δ is 1-1

2. $(\mathrm{id}_A \otimes \Delta_S)\delta = (\delta \otimes \mathrm{id}_S)\delta$

3. $\delta(A) \cdot (1_A \otimes S) = A \otimes S$

We will sometimes refer to A as a $\mathbb{G}$-C^*-algebra.

We can also define coactions of quantum groups on C^*-correspondences:

Definition 2.3.19. A $\mathbb{G}$-*equivariant* C^*-*correspondence* over a $\mathbb{G}$ -C^*-algebra (A, δ) is a C^*-correspondence X over A with a linear map $\sigma : X \to M(X \otimes S)$ such that

1. $\sigma(x)\delta(a) = \sigma(xa)$ and $\delta(a)\sigma(x) = \sigma(ax)$ for $x \in X$ and $a \in A$
2. $\delta\big(\langle x, y\rangle_A\big) = \langle \sigma(x), \sigma(y)\rangle_{M(A\otimes S)}$
3. $\sigma(x) \cdot (1 \otimes S) = X \otimes S$
4. $(1 \otimes S) \cdot \sigma(x) = X \otimes S$
5. $(\sigma \otimes \mathrm{id}_A)\sigma = (\mathrm{id}_X \otimes \sigma)\sigma$.

We call the map σ a coaction of $\mathbb{G}$ on X.

2.3.3 Twisted Tensor Products

Throughout this section, $\mathbb{G} = (S, \Delta_S)$ and $\mathbb{H} = (T, \Delta_T)$ will be quantum groups, χ will be a bicharacter from $\mathbb{G} \to \widehat{\mathbb{H}}$, and A and B will be C^*-algebras carrying coactions δ_A and δ_B of $\mathbb{G}$ and $\mathbb{H}$ respectively.

Definition 2.3.20 (Definition 3.1 of [Meyer *et al.*(2014)])**.** A χ-*Heisenberg pair* (or simply *Heisenberg pair*) is a pair of representations $\pi : S \to \mathcal{B}(\mathcal{H})$ and $\rho : T \to \mathcal{B}(\mathcal{H})$ such that

$$W^S_{1\pi} W^T_{2\rho} = W^T_{2\rho} W^S_{1\pi} \chi_{12} \in \mathcal{U}\big(\widehat{S} \otimes \widehat{T} \otimes \mathcal{K}(\mathcal{H})\big)$$

where $W^S_{1\pi} = \big((\mathrm{id}_{\widehat{S}} \otimes \pi)W^S\big)_{13}$ and $W^T_{2\rho} = \big((\mathrm{id}_{\widehat{T}} \otimes \rho)W^T\big)_{23}$.

Definition 2.3.21. Suppose (π, ρ) is a Heisenberg pair. Define maps

$$i_A : A \to M\big(A \otimes B \otimes \mathcal{K}(\mathcal{H})\big)$$

$$i_B : B \to M\big(A \otimes B \otimes \mathcal{K}(\mathcal{H})\big)$$

as follows:

$$i_A(a) = (\mathrm{id}_A \otimes \pi)\delta_A(a)_{13}$$

$$i_B(b) = (\mathrm{id}_B \otimes \rho)\delta_B(b)_{23}$$

It is shown in Lemma 3.20 of [Meyer *et al.*(2014)] that $A \boxtimes_\chi B := i_A(A) \cdot i_B(B)$ is a C^*-subalgebra of $A \otimes B \otimes \mathbb{K}(\mathcal{H})$ and that, up to isomorphism, $A \boxtimes_\chi B$ does not depend upon the choice of Heisenberg pair. We refer to $A \boxtimes_\chi B$ as the *twisted tensor product* of A and B and we write $a \boxtimes b$ for $i_A(a)i_B(b)$.

This construction can also be extended to correspondences. Recall that, given a C^*-correspondence X over a C^*-algebra A, the *linking algebra* $L(X)$ is the algebra $\mathbb{K}(X \oplus A)$. This is often thought of in terms of its block matrix form: $\begin{bmatrix} \mathcal{K}(X) & X \\ \overline{X} & A \end{bmatrix}$. We will make use of the following proposition:

Proposition 2.3.22 (Proposition 2.7 of [Baaj and Skandalis(1989)])**.** *Let $\delta_{L(X)} : L(X) \to M(L(X) \otimes S)$ be a coaction of $\mathbb{G}$ on $L(X)$ such that the inclusion $j_A : A \to L(X)$ is $\mathbb{G}$-equivariant. Then there exists a unique coaction σ on X such that $j_{M(X\otimes S)} \circ \sigma = \delta_{L(X)} \circ j_X$ where $j_{M(X\otimes S)}$ is the inclusion $M(X \otimes S) \to M(L(X) \otimes S)$ and j_X is the inclusion $X \to L(X)$.*

Conversely, if σ is a coaction of $\mathbb{G}$ on X, then there is a unique coaction $\delta_{L(X)}$ of $\mathbb{G}$ on $L(X)$ such that $j_{M(X\otimes S)} \circ \sigma = \delta_{L(X)} \circ j_X$ and such that $j_A : A \to L(X)$ is $\mathbb{G}$-equivariant.

We refer the reader to [Baaj and Skandalis(1989)] for the proof.

Definition 2.3.23. Let (X, A) and (Y, B) be C^*-correspondences with coactions of $\mathbb{G}$ and $\mathbb{H}$ respectively and let $\delta_{L(X)}$ and $\delta_{L(Y)}$ be the induced coactions of $L(X)$ and

$L(Y)$. We can form the twisted tensor product $L(X) \boxtimes_\chi L(Y)$. Viewing X and Y as subspaces of $L(X)$ and $L(Y)$, we define

$$X \boxtimes_\chi Y := \iota_{L(X)}(X) \cdot \iota_{L(Y)}(Y)$$

Proposition 5.10 of [Meyer *et al.*(2014)], and the discussion which follows it, shows that this is a correspondence over $A \boxtimes_\chi B$ with left action given by $\phi_X \boxtimes \phi_Y$. It also shows that

$$\mathcal{K}(X \boxtimes_\chi Y) \cong \mathcal{K}(X) \boxtimes_\chi \mathcal{K}(Y) \tag{2.3.19}$$

a fact which we shall make use of later.

Chapter 3

IONESCU'S THEOREM FOR HIGHER RANK GRAPHS

3.1 Introduction

In [Ionescu(2004)], Ionescu defines a natural correspondence associated to any *Mauldin-Williams* graph. A Mauldin-Williams graph is a directed graph with a compact metric space associated to each vertex and a contractive map associated to each edge (a more rigorous definition is presented below). These graphs generalize iterated function systems and have self-similar invariant sets. Ionescu proved that the Cuntz-Pimsner algebra of the correspondence associated to any Mauldin-Williams graph is isomorphic to the graph C^*-algebra of the underlying graph.

Here we prove an analogue for higher rank graphs. Our arguments make extensive use of the *graph systems of correspondences* construction presented in [Deaconu *et al.*(2010)] and provide an interesting application of their ideas. We also define some other systems similar to those defined in [Deaconu *et al.*(2010)] and briefly describe how all of these systems fit into what Fowler and Sims refer to in [Fowler and Sims(2002)] as *product systems taking values in tensor groupoids*. The results here are taken from [Kaliszewski *et al.*(2015)].

This chapter is organized as follows. In Section 3.2 we will define two systems that closely resemble Λ-systems of correspondences which we will call Λ-systems of homomorphisms and Λ-systems of maps. The Λ-system of maps will be a generalization of the notion of a Mauldin-Williams graph. After defining some more terminology, we prove some basic facts about these systems and how they relate to one another. In Section 3.3 we define a k-graph analog of Mauldin-Williams graphs and prove our main theorem which generalizes Ionescu's main result from [Ion07]. In Section 3.4 we prove that the Cuntz-Pimsner algebra of the correspondence associated to any Λ-system of maps can be realized as the graph algebra of a certain topological k-graph. In Section 3.5 we briefly describe how all of the various Λ-systems fit in to the framework described by Fowler and Sims in [Fowler and Sims(2002)]. In Section 3.6 we will examine the question of which Λ-systems of correspondences arise from the other types of Λ-systems described here. Finally, in Section 3.7 we show that, perhaps disappointingly, the higher-rank Mauldin-Williams graphs of Section 3.3 do not give rise to any new "higher-rank fractals".

3.2 Other Λ-Systems

Throughout this discussion we will take Λ to be a row-finite k-graph with no sources. We introduce a few constructions that are similar to Λ-systems of correspondences:

Definition 3.2.1.

1. A Λ-*system of homomorphisms* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A} \to \Lambda^0$ is a C^*-bundle and for each $\lambda \in u\Lambda v$ we have a non-degenerate homomorphism $\varphi_\lambda : A_u \to M(A_v)$, such that

$$\overline{\varphi_\mu} \circ \varphi_\lambda = \varphi_{\lambda\mu} \quad \text{if } s(\lambda) = r(\mu)$$

$$\varphi_v = \mathrm{id}_{A_v} \quad \text{for } v \in \Lambda^0,$$

   where $\overline{\varphi_\mu}$ is the canonical extension of φ_μ to $M(A_v)$.

2. A Λ-*system of maps* is a pair (T, σ), where $T \to \Lambda^0$ is a bundle of locally compact Hausdorff spaces and for each $\lambda \in u\Lambda v$ we have a continuous map $\sigma_\lambda : T_v \to T_u$, such that

$$\sigma_\lambda \circ \sigma_\mu = \sigma_{\lambda\mu} \quad \text{if } s(\lambda) = r(\mu)$$

$$\sigma_v = \mathrm{id}_{T_v} \quad \text{for } v \in \Lambda^0.$$

Remark 3.2.2.

1. Note that we need to impose the non-degeneracy condition on the homomorphisms φ_λ so that composition is defined.

2. Thus, a Λ-system of homomorphisms is essentially a contravariant functor from Λ to the category of C^*-algebras and non-degenerate homomorphisms into multiplier algebras, and a Λ-system of maps is essentially a (covariant) functor from Λ to the category of locally compact Hausdorff spaces and continuous maps.

3. Every Λ-system (T, σ) of maps gives rise to a Λ-system $(\mathcal{A}, \sigma^*)$ of homomorphisms, with

$$A_v = C_0(T_v) \quad \text{for } v \in \Lambda^0$$
$$\sigma^*_\lambda(f) = f \circ \sigma_\lambda \quad \text{for } \lambda \in \Lambda, f \in A_{r(\lambda)}.$$

4. Every Λ-system $(\mathcal{A}, \varphi)$ of homomorphisms gives rise to a Λ-system of correspondences: for $\lambda \in u\Lambda v$ let X_λ be the standard $A_u - A_v$ correspondence ${}_{\varphi_\lambda}A_v$.

Definition 3.2.3. We call a Λ-system of maps (T, σ)

1. *proper* if each map $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ is proper (in the usual sense that inverse images of compact sets are compact), and

2. *dense* if each map $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ has dense range.

Definition 3.2.4. We call a C^*-homomorphism $\varphi : A \to M(B)$ *proper* if it maps into B (and we will also denote it by $\varphi : A \to B$).

Remark 3.2.5. A non-degenerate homomorphism $\varphi : A \to M(B)$ is proper in the above sense if and only if φ takes one (hence every) bounded approximate identity for A to an approximate identity for B. Also, if $\sigma : Y \to X$ is a continuous map, then $\sigma^* : C_0(X) \to M(C_0(Y))$ is automatically non-degenerate, and is proper if and only if σ is proper.

Definition 3.2.6. Let X be an $A-B$ correspondence, with left module map $\varphi_A : A \to \mathcal{L}(X) = M(\mathcal{K}(X))$. We call X *proper*, *non-degenerate*, or *faithful* if φ_A has the corresponding property.

Definition 3.2.7. We call a Λ-system $(\mathcal{A}, \varphi)$ of homomorphisms *proper* or *faithful* if each homomorphism φ_λ has the corresponding property.

Definition 3.2.8. We call a Λ-system X of correspondences *proper*, *non-degenerate*, *full*, or *faithful* if each correspondence X_λ has the corresponding property.

[Deaconu *et al.*(2010)] call a Λ-system X of correspondences *regular* if it is proper, non-degenerate, full, and faithful. However, we believe that the fidelity is too much to ask, both for aesthetic and practical reasons.

Let X be a Λ-system of correspondences, and let $A = \bigoplus_{v\in\Lambda^0} X_v$ be the c_0-direct sum of C^*-algebras. Then each X_λ may be regarded as an A-correspondence. For each $n \in \mathbb{N}^k$, [Deaconu *et al.*(2010)] defines an A-correspondence Y_n by

$$Y_n = \bigoplus_{\lambda\in\Lambda^n} X_\lambda,$$

and [Deaconu *et al.*(2010), Proposition 3.17] shows that $Y = Y_X = \bigsqcup_{n\in\mathbb{N}^k} Y_n$ is an $\mathbb{N}^k$-system (i.e., a product system over $\mathbb{N}^k$) of A-correspondences, and moreover if X is regular then so is Y. We will identify X_λ with its canonical image in $Y_{d(\lambda)}$, i.e., we will blur the distinction between the external and internal direct sums of the A-correspondences $\{X_\lambda : \lambda \in \Lambda^n\}$.

Definition 3.2.9. We call a Λ-system (T, σ) of maps

1. *k-dense* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$T_v = \overline{\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})},$$

and

2. *k-regular* if it is proper and k-dense.

Here is a minor strengthening of k-density that we will find useful later:

Definition 3.2.10. A Λ-system of maps (T, σ) is *k-surjective* if

$$T_v = \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda\big(T_{s(\lambda)}\big) \quad \text{for all } v \in \Lambda^0, n \in \mathbb{N}^k.$$

Definition 3.2.11. We call a Λ-system $(\mathcal{A}, \varphi)$ of homomorphisms

1. *k-faithful* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$\bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda = \{0\},$$

and

2. *k-regular* if it is proper and k-faithful.

Definition 3.2.12. We call a Λ-system X of correspondences

1. *k-faithful* if the associated $\mathbb{N}^k$-system Y_X is faithful, and

2. *k-regular* if it is proper, non-degenerate, full, and k-faithful.

Remark 3.2.13. 1. If (T, σ) is a Λ-system of maps, then the associated Λ-system $(\mathcal{A}, \sigma^*)$ of homomorphisms is:

- proper if and only if (T, σ) is, and

- faithful if and only if (T, σ) is dense.

2. If $(\mathcal{A}, \varphi)$ is a Λ-system of homomorphisms, then the associated Λ-system X of correspondences is:

   - automatically non-degenerate and full, and
   - proper or faithful if and only if $(\mathcal{A}, \varphi)$ has the corresponding property.

3. We have organized our definitions so that a Λ-system X of correspondences is k-regular if and only if the associated $\mathbb{N}^k$-system Y_X is regular.

We will need the following variation on the above:

Lemma 3.2.14. *1. If (T, σ) is a Λ-system of maps, then the associated Λ-system $(\mathcal{A}, \sigma^*)$ of homomorphisms is k-faithful if and only if (T, σ) is k-dense, and consequently is k-regular if and only if (T, σ) is.*

2. If $(\mathcal{A}, \varphi)$ is a Λ-system of homomorphisms, then the associated Λ-system X of correspondences is k-faithful if and only if $(\mathcal{A}, \varphi)$ is, and consequently is k-regular if and only if (X, φ) is.

Proof. (1). This is routine, but we present the details for completeness. First assume that (T, σ) is not k-dense. We will show that $(\mathcal{A}, \sigma^*)$ is not k-faithful. We can choose $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$ is not dense in T_v. We will show that $\bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^* \neq \{0\}$. We can choose a nonzero $f \in C_0(T_v)$ that vanishes on $\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$. Then for all $\lambda \in v\Lambda^n$ and all $g \in C_0(T_{s(\lambda)})$,

$$\sigma_\lambda^*(f)g = (f \circ \sigma_\lambda)g = 0.$$

Thus $f \in \bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^*$.

Conversely, assume that $(\mathcal{A}, \sigma^*)$ is not k-faithful. We will show that (T, σ) is not k-dense. We can choose $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $\bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^* \neq \{0\}$. We will show that $\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$ is not dense in T_v. Choose a nonzero $f \in \bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^*$.

Then choose a nonempty open set $U \subset T_v$ such that $f(t) \neq 0$ for all $t \in U$. We will show that

$$U \cap \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)}) = \varnothing.$$

Let $t \in \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$, and choose $\lambda \in v\Lambda^n$ and $s \in T_{s(\lambda)}$ such that $t = \sigma_\lambda(s)$. Then choose $g \in C_0(T_{s(\lambda)})$ such that $g(s) = 1$. Since $f \in \ker \sigma_\lambda^*$,

$$0 = \big(\sigma_\lambda^*(f)g\big)(s) = f(\sigma_\lambda(s))g(s) = f(t),$$

so $t \notin U$.

(2). First assume that $(\mathcal{A}, \varphi)$ is not k-faithful. We will show that X is not k-faithful. We can choose $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $\bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda \neq \{0\}$. We will show that the A-correspondence Y_n is not faithful. Choose a nonzero $a \in A_v$ such that $\varphi_\lambda(a) = 0$ for all $\lambda \in v\Lambda^n$. Let

$$y = (x_\lambda) \in Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda.$$

Then $a \cdot y$ is the Λ^n-tuple $(a \cdot x_\lambda)$, where for $\lambda \in \Lambda^n$ we have

$$a \cdot x_\lambda = \begin{cases} \varphi_\lambda(a)x_\lambda & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v. \end{cases}$$

Since $\varphi_\lambda(a)x_\lambda = 0$ for all $\lambda \in v\Lambda^n, x_\lambda \in X_\lambda = A_{s(\lambda)}$, we have $a \cdot y = 0$, and we have shown that Y_n is not faithful.

Conversely, assume that X is not k-faithful. We will show that $(\mathcal{A}, \varphi)$ is not k-faithful. We can choose $n \in \mathbb{N}^k$ such that the A-correspondence Y_n is not faithful, so we can find a nonzero $a \in A$ such that $a \cdot y = 0$ for all $y \in Y_n$. Then $a = (a_v)$ is a Λ^0-tuple with $a_v \in A_v$ for each v, so we can choose $v \in \Lambda^0$ such that $a_v \neq 0$. We will show that $a_v \in \bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda$. Let $\lambda \in v\Lambda^0$ and $b \in A_{s(\lambda)}$. Define a $v\Lambda^n$-tuple

$(x_\mu) \in Y_n$ by

$$x_\mu = \begin{cases} b & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda. \end{cases}$$

Then

$$\varphi_\lambda(a_v)b = \big(a_v \cdot (x_\lambda)\big)_\lambda = 0. \qquad \square$$

Remark 3.2.15. The argument in the last paragraph of the above proof is a routine adaptation of that used in [Deaconu *et al.*(2010), Proposition 3.1.7].

Our motivation for introducing the properties of k-density and k-fidelity is that the Mauldin-Williams graphs considered by Ionescu — where we have a 1-graph Λ whose 1-skeleton E is finite, a Λ-system (T, σ) of maps in which each space T_v is a compact metric space and each map σ_λ is a (strict) contraction — are typically 1-dense in the above sense rather than dense. More precisely, a Mauldin-Williams graph (T, σ) is dense (in our terminology) if and only if every map σ_e (for $e \in E^1$) is surjective, which is usually not the case, and 1-dense if and only if for all $v \in E^0$ we have

$$\bigcup_{e \in vE^1} \sigma_e(T_{s(e)}) = T(v),$$

which is always the case (after replacing the original spaces by an "invariant list"). Thus, since we want to consider a version of Ionescu's theorem for k-graphs, we must allow the weakened property of k-fidelity (of Definition 3.2.12) rather than insisting upon fidelity.

[Deaconu *et al.*(2010), Definition 3.2.1] define a *representation* of a Λ-system X in a C^*-algebra B as a map $\rho : X \to B$ such that

1. for each $v \in \Lambda^0$, $\rho_v : X_v \to B$ is a C^*-homomorphism;

2. whenever $\xi \in X_\lambda, \eta \in X_\mu$,

$$\rho_\lambda(\xi)\rho_\mu(\eta) = \begin{cases} \rho_{\lambda\mu}(\xi\eta) & \text{if } s(\lambda) = r(\mu) \\ \\ 0 & \text{if otherwise.} \end{cases}$$

3. whenever $\xi \in X_\lambda, \eta \in X_\mu$, if $d(\lambda) = d(\mu)$ then

$$\rho_\lambda(\xi)^*\rho_\mu(\eta) = \begin{cases} \rho_{s(\lambda)}(\langle \xi, \eta \rangle_{X_{s(\lambda)}}) & \text{if } \lambda = \mu \\ \\ 0 & \text{if otherwise,} \end{cases}$$

and when X is regular [Deaconu *et al.*(2010)] defines a representation ρ to be *Cuntz-Pimsner covariant* if for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$, and $a \in X_v$,

4.
$$\rho_v(a) = \sum_{\lambda \in v\Lambda^n} \rho^{(\lambda)}(\varphi_\lambda(a)),$$

where $\rho^{(\lambda)} = \rho_\lambda^{(1)} : \mathcal{K}(X_\lambda) \to B$ is the associated homomorphism. Then [Deaconu *et al.*(2010)] defines a representation ρ to be *universal* if for every representation $\tau : X \to C$ there is a unique C^*-homomorphism $\Phi = \Phi_\tau : B \to C$ such that $\Phi \circ \rho_\lambda = \tau_\lambda$ for all $\lambda \in \Lambda$, and a Cuntz-Pimsner covariant representation to be *universal* if it satisfies the above universality property for all Cuntz-Pimsner covariant representations. Then they point out that, by the non-degeneracy that is included in the regularity assumption, (1)–(3) above can be replaced by the following set of conditions: each ρ_λ is a correspondence representation in B, ρ is multiplicative whenever this makes sense, and ρ_u and ρ_v have orthogonal images for all $u \neq v \in \Lambda^0$.

For the $\mathbb{N}^k$-system $Y = Y_X$ associated to a regular Λ-system X, [Deaconu *et al.*(2010), Proposition 3.2.3] shows that there is a bijection between the representations $\rho : X \to B$ and the representations $\psi : Y \to B$ such that

$$\psi \circ \iota_\lambda = \rho_\lambda \quad \text{for all } \lambda \in \Lambda.$$

However, it is crucial for our results to note that the proof of [Deaconu et al.*(2010), Proposition 3.2.3] only requires non-degeneracy of Y, not of X.*

[Deaconu *et al.*(2010), Proposition 3.2.5] shows that if X is regular then a representation $\rho : X \to B$ is Cuntz-Pimsner covariant if and only if the associated representation $\psi : Y \to B$ is. We turn this result into a *definition*:

Definition 3.2.16. Let X be a k-regular Λ-system of correspondences, with associated $\mathbb{N}^k$-system Y, and let $\rho : X \to B$ be a representation of X, with associated representation $\psi : Y \to B$. We define ρ to be *Cuntz-Pimsner covariant* if ψ is, in other words

$$\sum_{\lambda \in v\Lambda^n} \rho^{(\lambda)} \circ \varphi_\lambda = \rho_v \quad \text{for all } v \in \Lambda^0.$$

Remark 3.2.17. To reiterate, the only difference between Definition 3.2.16 and the definition of Cuntz-Pimsner covariance given in [Deaconu *et al.*(2010), Definition 3.2.1] is that in the latter the Λ-system X is required to be regular, while we only require k-regularity. In any event, [Deaconu *et al.*(2010), Definition 3.2.7] defines the C^*-algebra of a regular Λ-system X to be the Cuntz-Pimsner algebra $\mathcal{O}_Y$, and in [Deaconu *et al.*(2010), Corollary 3.2.6] they notice that the representation $\rho^{j_Y} : X \to \mathcal{O}_Y$ is a universal Cuntz-Pimsner covariant representation, where $j_Y : Y \to \mathcal{O}_Y$ is the universal Cuntz-Pimsner covariant representation.

We emphasize that, even though we only require the Λ-system X to be k-regular, the theory of [Deaconu *et al.*(2010)] carries over with no problems, as we've indicated. They use the notation $C^*(A, X, \chi)$ for the C^*-algebra of X, but we'll write it as $\mathcal{O}_X$. If $\rho : X \to B$ is any Cuntz-Pimsner covariant representation, we'll write $\Phi_\rho : \mathcal{O}_X \to B$ for the homomorphism whose existence is guaranteed by universality; [Deaconu *et al.*(2010)] would write it as $\Phi_{\rho,\pi}$, because they write π for the restriction of ρ to the C^*-bundle $X|_{\Lambda^0}$ (and they write A for this C^*-bundle, as well as for the section algebra $\bigoplus_{v\in\Lambda^0} X_v$ — we reserve the name A for this latter C^*-algebra).

Note that since we assume that Λ is row-finite with no sources, the infinite-path space Λ^∞ is locally compact Hausdorff, and is the disjoint union of the compact open subsets $\{v\Lambda^\infty\}_{v\in\Lambda^0}$. We get a Λ-system of maps (Λ^∞, τ), where for $\lambda \in u\Lambda v$ the continuous map

$$\tau_\lambda : v\Lambda^\infty \to u\Lambda^\infty$$

is defined by $\tau_\lambda(x) = \lambda x$. Moreover, this Λ-system is k-regular. This system has the following properties: if $\lambda \in u\Lambda v$ then τ_λ is a homeomorphism of $v\Lambda^\infty$ onto the compact open set

$$\lambda\Lambda^\infty \subset u\Lambda^\infty,$$

and consequently τ_λ^* is a surjection of $C(u\Lambda^\infty)$ onto $C(v\Lambda^\infty)$.

Lemma 3.2.18. *For each $u \in \Lambda^0$ and $\lambda \in u\Lambda$ let $p_\lambda = s_\lambda s_\lambda^*$, the set*

$$D_u = \overline{\operatorname{span}}\{p_\lambda : \lambda \in u\Lambda\}$$

is a unital commutative C^*-subalgebra of $C^*(\Lambda)$, with unit p_u, and the subalgebras $\{D_u\}_{u\in\Lambda^0}$ are pairwise orthogonal. Moreover, if D denotes the commutative C^*-subalgebra $\bigoplus_{u\in\Lambda^0} D_u$ of $C^*(\Lambda)$, then there is an isomorphism $\theta : C_0(\Lambda^\infty) \to D$ that takes the characteristic function of $\lambda\Lambda^\infty = \{\lambda x : s(\lambda) = r(x)\}$ to p_λ and $C(u\Lambda^\infty)$ to D_u. Finally, the diagram

$$\begin{tikzcd} C(u\Lambda^\infty) \arrow[r, "\tau_\lambda^*"] \arrow[d, "\theta"'] & C(v\Lambda^\infty) \arrow[d, "\theta"] \\ D_u \arrow[r, "\operatorname{Ad} s_\lambda^*"'] & D_v \end{tikzcd}$$

commutes.

Proof. This is probably folklore, at least for directed graphs, and in any case is standard: truncation gives an inverse system $\{\Lambda^n, \tau_{m,n}\}$ of surjections among nonempty finite sets[1], whose inverse limit is Λ^∞, and for each n the commutative C^*-subalgebra $D^n := \overline{\operatorname{span}}\{p_\lambda : \lambda \in \Lambda^n\}$ of $C^*(\Lambda)$ has spectrum Λ^n, so the inductive limit $D = \overline{\operatorname{span}}\{D^n : n \in \mathbb{N}^k\}$ has spectrum Λ^∞. $\square$

Definition 3.2.19. Let (T, σ) be a Λ-system of maps. A continuous map $\Phi : \Lambda^\infty \to T$ is *intertwining* if

$$\Phi \circ \tau_\lambda = \sigma_\lambda \circ \Phi \quad \text{for all } \lambda \in \Lambda.$$

We say (T, σ) is *self-similar* if there is a surjective intertwining map $\Phi : \Lambda^\infty \to T$.

Proposition 3.2.20. *Every self-similar Λ-system of maps (T, σ) is k-surjective, and each space T_v is compact.*

Proof. First, T_v is compact because the intertwining property and surjectivity of Φ imply that $T_v = \Phi(v\Lambda^\infty)$, which is a continuous image of the compact set $v\Lambda^\infty$. For the k-surjectivity, if $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ then

$$\begin{aligned} T_v &= \Phi(v\Lambda^\infty) \\ &= \Phi\left(\bigcup_{\lambda \in v\Lambda^n} \lambda\Lambda^\infty\right) \\ &= \bigcup_{\lambda \in v\Lambda^n} \Phi(\lambda\Lambda^\infty) \\ &= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda\big(\Phi(s(\lambda)\Lambda^\infty)\big) \\ &= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)}). \end{aligned}$$

$\square$

Definition 3.2.21. Let (T, σ) be a Λ-system of maps, and let $S \subset T$ be locally compact. For each $v \in \Lambda^0$ let $S_v = S \cap T_v$. Suppose that

$$\sigma_\lambda(S_v) \subset S_u \quad \text{whenever } \lambda \in u\Lambda v.$$

[1] with $\tau_{m,n}(\lambda) = \lambda(0, n)$ for $\lambda \in \Lambda^m$ and $n \leq m$

Then $(S, \sigma|_S)$ is a Λ-*subsystem* of (T, σ), where

$$(\sigma|_S)_\lambda = \sigma_\lambda|_{S_v} \quad \text{for all } \lambda \in \Lambda_v.$$

Note that our terminology makes sense: every Λ-subsystem is in fact a Λ-system.

Proposition 3.2.22. *Let (T, σ) be a Λ-system of maps, and let $\Phi : \Lambda^\infty \to T$ be an intertwining map. Put*

$$T'_v = \Phi(v\Lambda^\infty) \quad \textit{for each } v \in \Lambda^0$$
$$T' = \bigcup_{v \in \Lambda^0} T'_v.$$

Then $(T', \sigma|_{T'})$ is a self-similar k-surjective Λ-subsystem of (T, σ), and each T'_v is compact.

Proof. First of all, each T'_v is compact since $v\Lambda^\infty$ is compact and T_v is Hausdorff. Thus T' is locally compact, since the sets T_v are pairwise disjoint and open. For each $\lambda \in u\Lambda v$ we have

$$\begin{aligned} \sigma_\lambda(T'_v) &= \sigma_\lambda(\Phi(v\Lambda^\infty)) \\ &= \Phi(\lambda\Lambda^\infty) \\ &\subset \Phi(u\Lambda^\infty) \\ &= T'_u, \end{aligned}$$

so $(T', \sigma|_{T'})$ is a Λ-subsystem of (T, σ). It is self-similar because Φ is intertwining and maps Λ^∞ onto T' by construction. Then by Proposition 3.2.20 $(T', \sigma|_{T'})$ is k-surjective. □

Theorem 3.2.23. *Let (T, σ) be a self-similar k-regular Λ-system of maps, and let X be the associated Λ-system of correspondences. Then*

$$\mathcal{O}_X \cong C^*(\Lambda).$$

Proof. Our strategy will be to find a Cuntz-Pimsner covariant representation $\rho : X \to C^*(\Lambda)$ whose image contains the generators, and then apply the Gauge-Invariant Uniqueness Theorem. Recall that for $\lambda \in u\Lambda v$, X_λ is the standard $C_0(T_u) - C_0(T_v)$ correspondence ${}_{\sigma_\lambda^*}C_0(T_v)$. Define $\rho_\lambda : X_\lambda \to C^*(\Lambda)$ by

$$\rho_\lambda(f) = s_\lambda \theta \circ \Phi^*(f) \quad \text{for } f \in C_0(T_v).$$

Then ρ_λ is linear, and for $f, g \in C_0(T_v)$ we have

$$\begin{aligned}
\rho_\lambda(f)^*\rho_\lambda(g) &= \theta(\Phi^*(\overline{f}))s_\lambda^* s_\lambda \theta(\Phi^*(g)) \\
&= \theta(\Phi^*(\overline{f}))p_v\theta(\Phi^*(g)) \\
&= \theta(\Phi^*(\overline{f}g)) \\
&= p_v\big(\langle f, g\rangle_{C(T_v)}\big).
\end{aligned}$$

For $\lambda \in \Lambda v$, $\mu \in v\Lambda w$, $f \in C_0(T_v)$, and $h \in C_0(T_w)$ we have

$$\begin{aligned}
\rho_\lambda(f)\rho_\mu(h) &= s_\lambda\theta(\Phi^*(f))s_\mu\theta(\Phi^*(h)) \\
&= s_\lambda\theta(\Phi^*(f))p_\mu s_\mu\theta(\Phi^*(h)) \\
&= s_\lambda p_\mu\theta(\Phi^*(f))s_\mu\theta(\Phi^*(h)) \\
&= s_\lambda s_\mu \operatorname{Ad} s_\mu^* \circ \theta(\Phi^*(f))\theta(\Phi^*(h)) \\
&= s_{\lambda\mu}\theta \circ \tau_\mu^*(\Phi^*(f))\theta(\Phi^*(h)) \\
&= s_{\lambda\mu}\theta(\tau_\mu^* \circ \Phi^*(f))\theta(\Phi^*(h)) \\
&= s_{\lambda\mu}\theta(\Phi^* \circ \sigma_\mu^*(f))\theta(\Phi^*(h)) \\
&= s_{\lambda\mu}\theta\big(\Phi^*(\sigma_\mu^*(f)h)\big) \\
&= \rho_{\lambda\mu}(\sigma_\mu^*(f)h) \\
&= \rho_{\lambda\mu}(fh).
\end{aligned}$$

It follows that $\rho : X \to C^*(\Lambda)$ is a representation.

Next we show that ρ is Cuntz-Pimsner covariant. Let $u \in \Lambda^0$, $n \in \mathbb{N}^k$, and $f \in X_u = C_0(T_u)$. We need to show that

$$\rho_u(f) = \sum_{\lambda \in u\Lambda^n} \rho^{(\lambda)} \circ \varphi_\lambda(f),$$

where

$$\varphi_\lambda : C_0(T_u) \to \mathcal{K}(X_\Lambda)$$

is the left-module structure map. We need a little more information regarding the homomorphism

$$\rho^{(\lambda)} = \rho_\lambda^{(1)} : \mathcal{K}(X_\lambda) \to C^*(\Lambda).$$

For $\lambda \in u\Lambda v$ we have $X_\lambda = {}_{\sigma_\lambda^*}C_0(T_v)$, so

$$\mathcal{K}(X_\lambda) = C_0(T_v),$$

and for $g, h \in C_0(T_v)$ the rank-one operator $\theta_{g,h}$ is given by (left) multiplication by $g\overline{h}$. Thus

$$\begin{aligned}
\rho^{(\lambda)}(g\overline{h}) &= \rho_\lambda(g)\rho_\lambda(h)^* \\
&= s_\lambda \theta \circ \Phi^*(g)\theta \circ \Phi^*(\overline{h})s_\lambda^* \\
&= s_\lambda \circ \theta \circ \Phi^*(g\overline{h})s_\lambda^* \\
&= \operatorname{Ad} s_\lambda \circ \rho_v(g\overline{h}).
\end{aligned}$$

Since every function in $C_0(T_v)$ can be factored as $g\overline{h}$, we conclude that the homomorphism $\rho^{(\lambda)}$ coincides with

$$\operatorname{Ad} s_\lambda \circ \rho_v : C_0(T_v) \to C^*(\Lambda).$$

Also, $\varphi_\lambda : C_0(T_u) \to \mathcal{K}(X_\Lambda)$ coincides with $\sigma_\lambda^* : C_0(T_u) \to C_0(T_v)$ (note that σ_λ^* maps into $C_0(T_v)$ because σ_λ is proper). Thus

$$\sum_{\lambda \in u\Lambda^n} \rho^{(\lambda)} \circ \varphi_\lambda(f) = \sum_{\lambda \in u\Lambda^n} \operatorname{Ad} s_\lambda \circ \theta \circ \Phi^* \circ \sigma_\lambda^*(f)$$

$$
\begin{aligned}
&= \sum_{\lambda \in u\Lambda^n} \operatorname{Ad} s_\lambda \circ \theta \circ \tau_\lambda^* \circ \Phi^*(f) \\
&= \sum_{\lambda \in u\Lambda^n} \operatorname{Ad} s_\lambda \circ \operatorname{Ad} s_\lambda^* \circ \theta \circ \Phi^*(f) \\
&= \sum_{\lambda \in u\Lambda^n} \operatorname{Ad} s_\lambda s_\lambda^* \circ \theta \circ \Phi^*(f) \\
&= \sum_{\lambda \in u\Lambda^n} p_\lambda \theta \circ \Phi^*(f) \\
&= p_u \rho_u(f) \quad (\text{since } \sum_{\lambda \in u\Lambda^n} p_\lambda = p_u) \\
&= \rho_u(f),
\end{aligned}
$$

since $\rho_u(C_0(T_u)) \subset D_u$ and $p_u = 1_{D_u}$.

Therefore ρ gives rise to a homomorphism $\Psi_\rho : \mathcal{O}_X \to C^*(\Lambda)$ such that

$$\Psi_\rho \circ \rho^X = \rho,$$

where $\rho^X \to \mathcal{O}_X$ is the universal Cuntz-Pimsner covariant representation. For each $v \in \Lambda^0$, the continuous map $\Phi : \Lambda^\infty \to T$ takes $v\Lambda^\infty$ into T_v, so Φ^* restricts to a non-degenerate homomorphism from $C_0(T_v)$ to $C(v\Lambda^\infty)$, and hence the homomorphism $\rho_v : C_0(T_v) \to D_v$ is non-degenerate. It follows that for each $\lambda \in \Lambda v$ the generator s_λ is in the range of $\rho_\lambda : X_\lambda \to C^*(\Lambda)$. Thus $\Psi_\rho : \mathcal{O}_X \to C^*(\Lambda)$ is surjective.

Finally, we appeal to the Gauge-Invariant Uniqueness Theorem [Deaconu *et al.*(2010), Theorem 3.3.1] to show that Ψ_ρ is injective. Note that [Deaconu *et al.*(2010)] assume that the Λ-system X is regular, while we only assume that it is k-regular; as we have mentioned before, k-regularity is all that's required to make the results of [Deaconu *et al.*(2010)] true. First of all, for each $v \in \Lambda^0$, Φ maps $v\Lambda^0$ onto T_v, and it follows that $\rho_v : C_0(T_v) \to D_u$ is faithful. Thus the direct sum

$$\Psi_\rho|_A = \bigoplus_{v \in \Lambda^0} \rho_v : \bigoplus_{v \in \Lambda^0} C_0(T_v) \to \bigoplus_{v \in \Lambda^0} D_v \subset C^*(\Lambda)$$

is also faithful. Let $\gamma : \mathbb{T}^k \to \operatorname{Aut} C^*(\Lambda)$ be the gauge action. For $\lambda \in \Lambda^n v$, $f \in C_0(T_v)$, and $z \in \mathbb{T}^k$,

$$\begin{aligned}\gamma_z \circ \rho_\lambda(f) &= \gamma_z\big(s_\lambda \theta \circ \Phi^*(f)\big)\\ &= \gamma_z(s_\lambda)\rho_v(f) \quad (\text{since } \rho_v(f) \in D_v \subset C^*(\Lambda)^\gamma)\\ &= z^n s_\lambda \rho_v(f)\\ &= z^n \rho_\lambda(f),\end{aligned}$$

so by [Deaconu *et al.*(2010), Theorem 3.3.1] Ψ_ρ is faithful. □

3.3 Mauldin-Williams k-Graphs

We continue to let Λ be a row-finite k-graph with no sources.

Proposition 3.3.1. *Let (T, σ) be a Λ-system of maps such that each T_v is a complete metric space and, for some $c < 1$ and every $\lambda \in \Lambda$,*

$$\delta_v(\sigma_\lambda(t), \sigma_\lambda(s)) \le c^{|d(\lambda)|}\delta_v(t, s) \quad \text{for all } \lambda \in \Lambda, t, s \in T_{s(\lambda)},$$

where δ_v is the metric on T_v, d is the degree functor, and $|n| = \sum_{i=1}^k n_i$ for $n = (n_1, \dots, n_k) \in \mathbb{N}^k$. Then there exists a unique k-surjective Λ-subsystem (K, ψ) such that each K_v is a bounded closed subset of T_v, and in fact each K_v is compact.

Note that to check the hypothesis it suffices to show that each of the generating maps σ_λ for $\lambda \in \Lambda^{e_i}$ has Lipschitz constant at most c, where $e_1, \dots, e_k$ is the standard basis for $\mathbb{N}^k$.

Proof. Let

$$\mathcal{C} = \prod_{v \in \Lambda^0} \mathcal{C}(T_v),$$

where for $v \in \Lambda^0$ we let $\mathcal{C}(T_v)$ denote the set of bounded closed subsets of T_v, which is complete under the Hausdorff metric. Since Λ^0 is countable, $\mathcal{C}$ is a complete metric space. For each $n \in \mathbb{N}^k$ define a map $\widetilde{\sigma}^n : \mathcal{C} \to \mathcal{C}$ by

$$\widetilde{\sigma}^n(C)_v = \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(C_{s(\lambda)}).$$

As in [Mauldin and Williams(1988)], $\widetilde{\sigma}^n$ is a contraction, and so has a unique fixed point in $\mathcal{C}$. We need to know that the maps $\{\widetilde{\sigma}^n\}_{n \in \mathbb{N}^k}$ all have the same fixed point, and it suffices to show that they commute. Let $n, m \in \mathbb{N}^k$. Then for all $C = (C_v)_{v \in \Lambda^0} \in \mathcal{C}$ and $v \in \Lambda^0$ we have

$$\begin{aligned}
\widetilde{\sigma}^n \circ \widetilde{\sigma}^m(C)_v &= \widetilde{\sigma}^n\big(\widetilde{\sigma}^m(C)\big)_v \\
&= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda\big(\widetilde{\sigma}^m(C)_{s(\lambda)}\big) \\
&= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda \left(\bigcup_{\mu \in s(\lambda)\Lambda^m} \sigma_\mu\big(C_{s(\mu)}\big) \right) \\
&= \bigcup_{\lambda \in v\Lambda^n} \bigcup_{\mu \in s(\lambda)\Lambda^m} \sigma_\lambda \circ \sigma_\mu\big(C_{s(\mu)}\big) \\
&= \bigcup_{\lambda\mu \in v\Lambda^{n+m}} \sigma_{\lambda\mu}\big(C_{s(\lambda\mu)}\big) \\
&= \bigcup_{\alpha \in v\Lambda^{n+m}} \sigma_\alpha\big(C_{s(\alpha)}\big),
\end{aligned}$$

which, by the factorization property of Λ, coincides with

$$\bigcup_{\mu \in v\Lambda^m} \bigcup_{\lambda \in s(\mu)\Lambda^n} \sigma_\mu \circ \sigma_\lambda\big(C_{s(\lambda)}\big) = \widetilde{\sigma}^m \widetilde{\sigma}^n(C)_v.$$

Letting $(K_v)_{v \in \Lambda^0}$ be the unique common fixed point of $\widetilde{\sigma}$ on $\mathcal{C}$, we see that, setting $K = \bigcup_{v \in \Lambda^0} K_v$ and $\psi = \sigma|_K$, the restriction (K, ψ) of (T, σ) is the unique k-surjective Λ-subsystem with bounded closed subsets K_v.

Fix a compact subset of T_v such as the singleton set $\{x\}$ and notice that the limit $\lim_{x \to \infty} \tilde{\sigma}^m(\{x\})$ is the unique fixed point. Since each $\tilde{\sigma}^m(\{x\})$ is compact and since the set of compact sets is closed in $C(T_v)$, it follows that every K_v is compact. again getting a unique fixed point. But since the compact subsets are among the bounded closed subsets, the resulting Λ-subsystem must coincide with the one we found above, by uniqueness. $\square$

Definition 3.3.2. A *Mauldin-Williams* Λ*-system* is a k-surjective Λ-system of maps (T, σ) such that each T_v is a compact metric space and, for some $c < 1$, every $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ is a contraction with Lipschitz constant at most $c^{|d(\lambda)|}$.

Proposition 3.3.3. *Every Mauldin-Williams* Λ*-system* (T, σ) *is self-similar, and if* X *is the associated* Λ*-system of correspondences then* $\mathcal{O}_X \cong C^*(\Lambda)$.

Proof. We adapt the technique of Ionescu [Ionescu(2004)]. Let $x \in v\Lambda^\infty$, so that $x : \Omega_k \to \Lambda$ is a k-graph morphism. For each $n \in \mathbb{N}^k$ let $x(0, n)$ be the unique path $\lambda \in \Lambda^n$ such that $x = \lambda y$ for some (unique) $y \in s(\lambda)\Lambda^\infty$. By definition of Mauldin-Williams Λ-system, the range of each $\sigma_{x(0,n)}$ has diameter at most $c^{|n|}$. Thus by compactness there is a unique element $\Phi(x) \in T_v$ such that

$$\bigcap_{n \in \mathbb{N}^k} \sigma_{x(0,n)}(T_{s(x(0,n))}) = \{\Phi(x)\}.$$

We get a map $\Phi : \Lambda^\infty \to T$, which is continuous because for each $x \in \Lambda^\infty$ the images under Φ of the neighborhoods $x(0, n)\Lambda^\infty$ of x have diameters shrinking to 0. By construction,

$$\Phi(\lambda x) = \sigma_\lambda(\Phi(x)) \quad \text{for all } \lambda \in \Lambda, x \in s(\lambda)\Lambda^\infty,$$

so Φ is intertwining.

We show that Φ is surjective. Put $T' = \Phi(\Lambda^\infty)$. By Proposition 3.2.22, $(T', \sigma|_{T'})$ is k-surjective with each T'_v compact, which implies that $T' = T$ by the uniqueness in Proposition 3.3.1.

Finally, it now follows from Theorem 3.2.23 that $\mathcal{O}_X \cong C^*(\Lambda)$. $\square$

Remark 3.3.4. It would be completely routine at this point to adapt Ionescu's techniques to prove a higher-rank version his other "no-go theorem" [Ionescu(2004), Theorem 3.4], namely that there are no "noncommutative Mauldin-Williams Λ-systems" of maps.

3.4 The Associated Topological k-Graph

Let Λ be a row-finite k-graph with no sources, and let (T, σ) be a k-regular Λ-system of maps. We do *not* assume that (T, σ) is self-similar unless otherwise noted.

Let (T, σ) be a Λ-system of maps. We want to define a topological k-graph $\Lambda * T$ as follows:

1. $\Lambda * T = \{(\lambda, t) \in \Lambda \times T : t \in T_{s(\lambda)}\}$:
2. $s(\lambda, t) = (s(\lambda), t)$ and $r(\lambda, t) = (r(\lambda), \sigma_\lambda(t))$;
3. if $s(\lambda) = r(\mu)$ and $t = \sigma_\mu(s)$, then $(\lambda, t)(\mu, s) = (\lambda\mu, s)$;
4. $d(\lambda, t) = d(\lambda)$.

$\Lambda * T$ has the relative topology from $\Lambda \times T$, and is the disjoint union of the open subsets $\{\lambda\} \times T_{s(\lambda)}$, each of which is a homeomorphic copy of $T_{s(\lambda)}$.

Proposition 3.4.1. *The above operations make $\Lambda * T$ into a topological k-graph.*

Proof. This is routine. For instance, it's completely routine to check that $\Lambda * T$ is a small category and the map defined in (4) is a functor. Let's check the unique factorization property: Let $(\lambda, t) \in \Lambda * T$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$. Then we can uniquely write $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$. We have

$$(\lambda, t) = (\mu, \sigma_\nu(t))(\nu, t), \quad d(\mu, \sigma_\nu(t)) = m, \quad \text{and} \quad d(\nu, t) = n,$$

and $(\mu, \sigma_\nu(t))$ and (ν, t) are unique since μ and ν are. It's immediate that the degrees match up and this factorization is unique.

The multiplication on the category $\Lambda * T$ is continuous and open because it is in fact a local homeomorphism from the fibered product $(\Lambda * T) * (\Lambda * T)$ to $\Lambda * T$, which for each $(\lambda, \mu) \in \Lambda \times \Lambda$ with $s(\lambda) = r(\mu)$ maps the open subset

$$\Big(\big(\{\lambda\} \times T_{s(\lambda)}\big) \times \big(\{\mu\} \times T_{s(\mu)}\big)\Big) \cap \Big((\Lambda * T) * (\Lambda * T)\Big)$$

bijectively onto the open subset $\{\lambda\mu\} \times T_{s(\mu)}$.

To see that the source map on $\Lambda * T$ is a local homeomorphism, just note that it restricts to homeomorphisms

$$\{\lambda\} \times T_{s(\lambda)} \to \{s(\lambda)\} \times T_{s(\lambda)}. \qquad \square$$

Remark 3.4.2. One could reasonably regard a Λ-system of maps as an action of Λ on the space $T = \bigsqcup_{v \in \Lambda^0} T_v$, and the topological k-graph $\Lambda * T$ as the associated transformation k-graph.

Remark 3.4.3. If each T_v is discrete and every map $\sigma_\lambda : T_{s(\lambda)} \to T_{r(\lambda)}$ is bijective, then the above k-graph $\Lambda * T$ coincides with that of [Pask *et al.*(2005), Proposition 3.3], where the main point was that the coordinate projection $(\lambda, t) \mapsto \lambda$ is a model for coverings of the k-graph Λ.

Proposition 3.4.4. *Let (T, σ) be a k-regular Λ-system of maps, and let $(\mathcal{A}, \varphi)$ be the associated Λ-system of homomorphisms, which in turn has an associated Λ-system X of correspondences. Then*

$$\mathcal{O}_X \cong C^*(\Lambda * T),$$

where $\Lambda * T$ is the topological k-graph of Proposition 3.4.1.

Proof. Our strategy is to show that $\mathcal{O}_X$ and $C^*(\Lambda * T)$ are isomorphic to the Cuntz-Pimsner algebras of isomorphic $\mathbb{N}^k$-systems of correspondences. Recall that $\mathcal{O}_X \cong$

$\mathcal{O}_Y$, where $Y = Y_X$ is the $\mathbb{N}^k$ system associated to X. Thus for each $n \in \mathbb{N}^k$ we have

$$Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda,$$

where X_λ is the correspondence over $A = \bigoplus_{v\in\Lambda^0} A_v$ naturally associated (via identifying the A_v's with direct summands in A) to the standard $A_{r(\lambda)} - A_{s(\lambda)}$ correspondence ${}_{\sigma_\lambda^*}A_{s(\lambda)}$ determined by the homomorphism $\sigma_\lambda^* : A_{r(\lambda)} \to M(A_{s(\lambda)})$ given by composition with $\sigma_v : T_{s(v)} \to T_{r(v)}$.

On the other hand, by [Carlsen *et al.*(2011), Theorem 5.20] $C^*(\Lambda * T)$ is isomorphic to the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_Z$, where Z is the $\mathbb{N}^k$-system of $C_0((\Lambda * T)^0)$-correspondences associated to the topological k-graph $\Lambda * T$. As we'll show in this proof, the N^k-systems Z and Y are isomorphic. Since the Λ-system (T, σ) is k-regular, so is Y, and hence so is Z. In particular, since each pair in $\mathbb{N}^k$ has an upper bound, and $C_0((\Lambda * T)^0)$ maps injectively into the compacts on Z_n for every $n \in \mathbb{N}^k$, it follows from [Sims and Yeend(2007), Corollary 5.2] that $\mathcal{NO}_Z = \mathcal{O}_Z$, because by [Fowler(2002), Proposition 5.8] Z is compactly aligned.

Let's see what the Λ-system Z looks like in this situation: for each $n \in \mathbb{N}^k$, the correspondence Z_n over $C_0((\Lambda * T)^0)$ is a completion of $C_c((\Lambda * T)^n)$. We can safely identify $(\Lambda * T)^0$ with $T = \bigsqcup_{v\in\Lambda^0} T_v$, and hence $C_0((\Lambda * T)^0)$ with $A = \bigoplus_{v\in\Lambda^0} C_0(T_v)$, and in this way Z_n becomes an A-correspondence. For $\xi, \eta \in C_c((\Lambda * T)^n) = C_c(\Lambda^n * T)$, the inner product is given by

$$\langle \xi, \eta \rangle_A(t) = \sum_{\lambda \in \Lambda^n v} \overline{\xi(\lambda, t)}\eta(\lambda, t), \quad t \in T_v, v \in \Lambda^0,$$

and the right and left module operations are given for $f \in A$ by

$$(\xi \cdot f)(\lambda, t) = \xi(\lambda, t) f(t)$$
$$(f \cdot \xi)(\lambda, t) = f(\sigma_\lambda(t))\xi(\lambda, t).$$

Note that

$$(\Lambda * T)^n = \bigsqcup_{\lambda \in \Lambda^n} (\{\lambda\} \times T_{s(\lambda)}).$$

Thus for each $\lambda \in \Lambda^n v$ we have a natural inclusion map

$$C_c(\{\lambda\} \times T_v) \hookrightarrow Z_n,$$

and Z_n is the closed span of these subspaces. Moreover, their closures form a pairwise orthogonal family of subcorrespondences of Z_n:

$$Z_n(\lambda) = \overline{C_c(\{\lambda\} \times T_v)} \quad \text{for } \lambda \in \Lambda^n v,$$

and we see that

$$Z_n = \bigoplus_{\lambda \in \Lambda^n} Z_n(\lambda)$$

as A-correspondences.

We will obtain an isomorphism $\psi : Y \to Z$ of $\mathbb{N}^k$-systems by defining isomorphisms $\psi_n : Y_n \to Z_n$ of A-correspondences and then verifying that

$$\psi_n(\xi)\psi_m(\eta) = \psi_{n+m}(\xi\eta) \quad \text{for all } (\xi, \eta) \in Y_n \times Y_m.$$

By the above, to get an isomorphism $\psi_n : Y_n \to Z_n$ it suffices to get isomorphisms $\psi_{n,\lambda} : X_\lambda \to Z_n(\lambda)$ for each $\lambda \in \Lambda^n$. If $\lambda \in \Lambda^n v$ and

$$\xi \in C_c(T_v) \subset X_\lambda$$

define

$$\psi(\xi) \in C_c(\{\lambda\} \times T_v) \subset Z_n(\lambda)$$

by

$$\psi(\xi)(\lambda, t) = \xi(t).$$

Routine computations show that $\psi_{n,\lambda}$ is an isomorphism.

Now we check multiplicativity, and again it suffices to consider the fibers of the Λ-system X: if

$$\begin{aligned} \xi &\in X_\lambda \quad \text{for } \lambda \in \Lambda^n v \\ \eta &\in X_\mu \quad \text{for } \mu \in v\Lambda^m \end{aligned}$$

then for $t \in T_{s(\mu)}$ we have

$$\begin{aligned} \big(\psi_{n,\lambda}(\xi)\psi_{m,\mu}(\eta)\big)(\lambda\mu, t) &= \psi_{n,\lambda}(\xi)(\lambda, \sigma_\mu(t))\psi_{m,\mu}(\mu, t) \\ &= \xi(\sigma_\mu(t))\eta(t) \\ &= (\xi\eta)(t) \\ &= \big(\psi_{n+m,\lambda\mu}(\xi\eta)\big)(\lambda\mu, t). \end{aligned}$$

□

3.5 The Tensor Groupoids

Recall that in [Fowler and Sims(2002)] Fowler and Sims study what they call *product systems taking values in a tensor groupoid.* Their product systems are over semigroups, and here we want to consider the special cases related to our Λ-systems of homomorphisms or maps, where the k-graph Λ has a single vertex, and so in particular is a monoid whose identity element is the unique vertex. Since we won't need to do serious work with the concept, here we informally regard a *tensor groupoid* as a groupoid $\mathcal{G}$ with a "tensor" operation $X \otimes Y$ and an "identity" object $1_\mathcal{G}$ such that the "expected" redistributions of parentheses and canceling of tensoring with the identity are implemented via given natural equivalences. As defined in [Fowler and Sims(2002)], a *product system* over a semigroup S taking values in a tensor groupoid $\mathcal{G}$ is a family $\{X_s\}_{s\in S}$ of objects in $\mathcal{G}$ together with an associative family $\{\alpha_{s,t}\}_{s,t\in S}$ of isomorphisms

$$\alpha_{s,t} : X_s \otimes X_t \to X_{st},$$

and moreover if S has an identity e then $X_e = 1_\mathcal{G}$ and $\alpha_{e,s}, \alpha_{s,e}$ are the given isomorphisms $1_\mathcal{G} \otimes X_s \cong X_s$ and $X_s \otimes 1_\mathcal{G} \cong X_s$.

Systems of Homomorphisms

Let A be a C^*-algebra, and $\mathcal{G}$ be the tensor groupoid whose objects are the non-degenerate homomorphisms $\pi : A \to M(A)$, whose only morphisms are the identity morphisms on objects, and with identity $1_{\mathcal{G}} = \mathrm{id}_A$. Define a tensor operation on $\mathcal{G}$ by composition:

$$\pi_1 \otimes \pi_2 = \pi_2 \circ \pi_1,$$

where π_2 has been canonically extended to a strictly continuous unital endomorphism of $M(A)$. Standard properties of composition show that $\mathcal{G}$ is indeed a tensor groupoid, in a trivial way: the tensor operation is actually associative, and $1_{\mathcal{G}}$ acts as an actual identity for tensoring, so the axioms of [Fowler and Sims(2002)] for a tensor groupoid are obviously satisfied.

Due to the special nature of this tensor groupoid $\mathcal{G}$, a *product system over* $\mathbb{N}^k$ *taking values in* $\mathcal{G}$, as in [Fowler and Sims(2002), Definition 1.1], is a homomorphism $n \mapsto \varphi_n$ from the additive monoid $\mathbb{N}^k$ into the monoid of non-degenerate homomorphisms $A \to M(A)$ under composition, in other words such a product system is precisely what we call in the current paper an $\mathbb{N}^k$-system of homomorphisms.

Systems of Maps

Quite similarly to the above, let T be a locally compact Hausdorff space, and $\mathcal{G}$ be the tensor groupoid whose objects are the continuous maps $\sigma : X \to X$, whose only morphisms are the identity morphisms on objects, and with identity $1_{\mathcal{G}} = \mathrm{id}_X$. Define a tensor operation on $\mathcal{G}$ by composition:

$$\sigma \otimes \psi = \sigma \circ \psi.$$

Again, $\mathcal{G}$ is indeed a tensor groupoid, in a trivial way, because the tensor operation is actually associative, and $1_{\mathcal{G}}$ acts as an actual identity for tensoring.

A *product system over* $\mathbb{N}^k$ *taking values in* $\mathcal{G}$, as in [Fowler and Sims(2002), Definition 1.1], is a homomorphism $n \mapsto \sigma_n$ from the additive monoid $\mathbb{N}^k$ into the monoid of continuous self maps of X maps under composition, in other words such a product system is precisely what we call in the current paper an $\mathbb{N}^k$-system of maps.

3.6 Reversing the Processes

In Remark 3.2.2 we noted that every Λ-system of maps gives rise to a Λ-system of homomorphisms, and every Λ-system of homomorphisms gives rise to a Λ-system of correspondences. In this section we will investigate the extent to which these two processes are reversible.

Question 3.6.1. When is a given Λ-system of correspondences isomorphic to the one associated to a Λ-system of homomorphisms?

Investigating this question requires us to examine balanced tensor products of standard correspondences. First we observe without proof the following elementary fact.

Lemma 3.6.2. *Let $\varphi : A \to M(B)$ and $\psi : B \to M(C)$ be non-degenerate homomorphisms. Then there is a unique $A - C$ correspondence isomorphism*

$$\theta : {}_{\varphi}B \otimes_B {}_{\psi}C \xrightarrow{\cong} {}_{\psi\circ\varphi}C$$

such that

$$\theta(b \otimes c) = \psi(b)c \quad \textit{for } b \in B, c \in C.$$

We can analyze the question of whether a given Λ-system X of correspondences is isomorphic to one coming from a Λ-system of homomorphisms in several steps:

First of all, without loss of generality we can look for a Λ-system of homomorphisms of the form $(\mathcal{A}, \varphi)$.

Next, for each $\lambda \in u\Lambda v$ the $A_u - A_v$ correspondence X_λ must be isomorphic to a standard one, more precisely there must exist a linear bijection

$$\theta_\lambda : X_\lambda \to A_v$$

and a non-degenerate homomorphism

$$\varphi_\lambda : A_u \to M(A_v)$$

such that

$$\theta_\lambda(\xi)^*\theta_\lambda(\eta) = \langle \xi, \eta \rangle_{A_v} \quad \text{for all } \xi, \eta \in X_\lambda \tag{3.6.1}$$

$$\theta_\lambda(a \cdot \xi \cdot b) = \varphi_\lambda(a)\theta_\lambda(\xi)b \quad \text{for all } a \in A_u, \xi \in X_\lambda, b \in A_v. \tag{3.6.2}$$

Moreover, whenever $\lambda \in u\Lambda v, \mu \in v\Lambda w$ we must have

$$\begin{aligned} {}_{\varphi_{\lambda\mu}}A_w &= X_{\lambda\mu} \\ &\cong X_\lambda \otimes_{A_v} X_\mu \\ &= {}_{\varphi_\lambda}A_v \otimes_{A_v} {}_{\varphi_\mu}A_w \\ &\cong {}_{\varphi_\mu \circ \varphi_\lambda}A_w, \end{aligned}$$

so there exists a unitary multiplier $U(\lambda, \mu) \in M(A_w)$ such that

$$\varphi_\mu \circ \varphi_\lambda = \operatorname{Ad} U(\lambda, \mu) \circ \varphi_{\lambda\mu}.$$

The $U(\lambda, \mu)$'s satisfy a kind of "two-cocycle" identity coming from associativity of composition of the φ_λ's.

Now, if this Λ-system of correspondences is isomorphic to one associated to a Λ-system $(\mathcal{A}, \psi)$ of homomorphisms, then for each $\lambda \in u\Lambda v$ we must have an isomorphism ${}_{\varphi_\lambda}A_v \cong {}_{\psi_\lambda}A_v$ of $A_u - A_v$ correspondences, and so there must be a unitary multiplier $W_\lambda \in M(A_v)$ such that

$$\varphi_\lambda = \operatorname{Ad} W_\lambda \circ \psi_\lambda.$$

Since $(\mathcal{A}, \psi)$ is a Λ-system of homomorphisms, whenever $\lambda \in u\Lambda v, \mu \in v\Lambda w$ we have

$$\begin{aligned}
\varphi_{\lambda\mu} &= \operatorname{Ad} W_{\lambda\mu} \circ \psi_{\lambda\mu} \\
&= \operatorname{Ad} W_{\lambda\mu} \circ \psi_{\mu} \circ \psi_{\lambda} \\
&= \operatorname{Ad} W_{\lambda\mu} \circ \operatorname{Ad} W_{\mu}^{*} \circ \varphi_{\mu} \circ \operatorname{Ad} W_{\lambda}^{*} \circ \varphi_{\lambda} \\
&= \operatorname{Ad} W_{\lambda\mu} W_{\mu}^{*} \varphi_{\mu}(W_{\lambda}^{*}) \circ \varphi_{\mu} \circ \varphi_{\lambda} \\
&= \operatorname{Ad} W_{\lambda\mu} W_{\mu}^{*} \varphi_{\mu}(W_{\lambda}^{*}) U(\lambda, \mu) \circ \varphi_{\lambda\mu},
\end{aligned}$$

so since the homomorphisms φ_λ are non-degenerate we see that, in the quotient group of the unitary multipliers of A_w modulo the central unitary multipliers, the cosets satisfy

$$\big[U(\lambda, \mu)\big] = \big[\varphi_{\mu}(W_{\lambda}) W_{\mu} W_{\lambda\mu}^{*}\big],$$

giving a sort of cohomological obstruction (which we won't make precise) to the Λ-system of correspondences being isomorphic to a one associated to a Λ-system $(\mathcal{A}, \psi)$ of homomorphisms.

Note that if all the C^*-algebras A_v are commutative, then none of the above unitary multipliers appear, so once we have θ_λ's and φ_λ's satisfying (3.6.1) then the pair $(\mathcal{A}, \varphi)$ will automatically be a Λ-system of homomorphisms whose associated Λ-system of correspondences is isomorphic to X. This is due to the way in which the correspondences X_λ fit together. We record this in the follwing two propositions:

Proposition 3.6.3. *Let X be a Λ-system of correspondences such that every A_v is commutative. Then X is isomorphic to the Λ-system associated to a Λ-system of homomorphisms if and only if, whenever $\lambda \in u\Lambda v$, X_λ is isomorphic to a standard $A_u - A_v$ correspondence ${}_{\varphi_\lambda}A_v$.*

Proposition 3.6.4. *Let $(\mathcal{A}, \varphi)$ be a Λ-system of homomorphisms such that every A_v is commutative, and for each $v \in \Lambda^0$ let T_v be the maximal ideal space of A_v. Then* *there is a unique Λ-system of maps (T, σ) such that $(\mathcal{A}, \varphi)$ is the associated Λ-system of homomorphisms.*

On the other hand, every Λ-system of homomorphisms is uniquely isomorphic to the one associated to a Λ-system of maps, at least in the only circumstances where it makes sense. This follows immediately from the duality between the category of commutative C^*-algebras and non-degenerate homomorphisms into multiplier algebras and the category of locally compact Hausdorff spaces and continuous maps.

3.7 No Higher-Rank Fractals

In Proposition 3.2.22 we showed that every Λ system of maps (T, σ) has a self-similar k-surjective Λ-subsystem $(T', \sigma|_{T'})$. The self-similar set T' is the part of the system that would generally be referred to as the "fractal". It is natural to wonder whether the generalization to k-graphs presented here gives rise to any new fractals that could not have arisen from the corresponding constructions for 1-graphs. The answer to this question turns out to be "no" for reasons we will now explain. Throughout the following discussion, let $p = (1, 1, \ldots, 1) \in \mathbb{N}^k$

Definition 3.7.1. For a k-graph Λ we define the *diagonal 1-graph E* as follows:

$$E^0 = \Lambda^0$$
$$E^1 = \{e_\lambda : \lambda \in \Lambda, d(\lambda) = p\}$$
$$r(e_\lambda) = r(\lambda)$$
$$s(e_\lambda) = s(\lambda).$$

If (T, σ) is a Λ-system of maps, then we define the *diagonal E-system* (T, ρ) of (T, σ) to be the E-system of maps such that $\rho_{e_\lambda} = \sigma_\lambda$ for all $e_\lambda \in E^1$. Finally, let $\alpha : E^* \to \Lambda$ be the map defined by $\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Proposition 3.7.2. *The map $i : \Lambda^\infty \to E^\infty$ defined by $\alpha(i(x)(j,l)) = x(jp, lp)$ is a bijection and i^{-1} is continuous.*

Proof. First we must show that this is well-defined. This just amounts to showing that α is injective. To see this recall that if $\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \lambda$ then $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$ where each λ_i has degree p and hence $d(\lambda_1 \lambda_2 \cdots \lambda_n) = np$. Since there is only one way to write np as a sum of p's, there is only one such decomposition of λ (by unique factorization), so if $\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \alpha(e_{\gamma_1} e_{\gamma_2} \cdots e_{\gamma_n})$ we must have $\lambda_i = \gamma_i$ for all i.

Next, to show that i is injective, suppose $i(x) = i(y)$ for $x, y \in \Lambda^\infty$. Then by definition we must have that $x(jp, lp) = y(jp, lp)$ for all $j, l \in \mathbb{N}$, and in particular we have that $x(0, jp) = y(0, jp)$ for all $j \in \mathbb{N}$. But since $\{jp\}_j$ is a cofinal increasing sequence in $\mathbb{N}^k$, x and y are uniquely determined by their values on the pairs $(0, jp)$ (see [Kumjian and Pask(2000), Remarks 2.2]) so we must have $x = y$.

Now, to show that i is surjective, let $z \in E^\infty$. We wish to find an infinite path $x \in \Lambda^\infty$ such that $i(x) = z$. We will again make use of the fact that such an x is uniquely determined by its values on $(0, jp)$. Specifically, if $z(0, j) = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_j}$, then we let $x(0, jp) = \lambda_1 \lambda_2 \cdots \lambda_j$. Then $\alpha(i(x)(0, j)) = x(0, jp) = \lambda_1 \lambda_2 \cdots \lambda_j$ and $\alpha(z(0, j)) = \lambda_1 \lambda_2 \cdots \lambda_j$ so by the injectivity of α we have that $i(x)(0, j) = z(0, j)$ and since $i(x)$ and z are uniquely determined by their values at $(0, j)$ we have that $i(x) = z$.

Finally, we need to show that i^{-1} is continuous. We have

$$\alpha(i(x)(j,l)) = x(jp, lp) = \lambda_j ... \lambda_l,$$

where $\lambda_j ... \lambda_l$ is the unique decomposition of $x(jp, lp)$ into paths of degree p. Since α is injective, we get $i(x)(j, l) = e_{\lambda_j} ... e_{\lambda_l}$. Since this holds for all (j, l) we must have that $i^{-1}(e_{\lambda_1} e_{\lambda_2} ...) = \lambda_1 \lambda_2 ...$ for all $e_{\lambda_1} e_{\lambda_2} ... \in E^\infty$. Recall that the topologies on E^∞ and Λ^∞ are generated by the collections $\{Z(P) : P \in E^*\}$ and $\{Z(\lambda) : \lambda \in \Lambda\}$

respectively where $Z(P) = \{Pz : z \in s(P)E^\infty\}$ and $Z(\lambda) = \{\lambda x : x \in s(\lambda)\Lambda^\infty\}$. Thus a net $\{\lambda_1^\alpha \lambda_2^\alpha ...\}_{\alpha \in A}$ in Λ^∞ converges to $\lambda_1\lambda_2...$ in Λ^∞ if for all $n \in \mathbb{N}$ there is $\alpha_0 \in A$ such that $\lambda_j^\alpha = \lambda_j$ for all $j \leq n$ and $\alpha \geq \alpha_0$, and similarly for nets in E^∞. Now, suppose $\{e_{\lambda_1^\alpha} e_{\lambda_2^\alpha} ...\}_{\alpha \in A}$ converges to $e_{\lambda_1} e_{\lambda_2} ...$ in E^∞. Then for all $n \in \mathbb{N}$ there is $\alpha_0 \in A$ such that $e_{\lambda_j^\alpha} = e_{\lambda_j}$ for all $j \leq n$ and $\alpha \geq \alpha_0$. Thus $\lambda_j^\alpha = \lambda_j$ for all $j \leq n$ and $\alpha \geq \alpha_0$, and we have shown that the net $\{i^{-1}(e_{\lambda_1^\alpha} e_{\lambda_2^\alpha} ...)\}_{\alpha \in A} = \{\lambda_1^\alpha \lambda_2^\alpha ...\}_{\alpha \in A}$ converges to $i^{-1}(e_{\lambda_1} e_{\lambda_2} ...) = \lambda_1 \lambda_2 ...$ in Λ^∞. Therefore i^{-1} is continuous. □

Proposition 3.7.3. *Let (T, σ) be a Λ-system of maps and let (T, ρ) be the diagonal E-system of (T, σ). If $\Phi : \Lambda^\infty \to T$ is intertwining with respect to (T, σ) then $\Phi \circ i^{-1} : E^\infty \to T$ is intertwining with respect to (T, ρ).*

Proof. We have:

$$\Phi \circ i^{-1} \circ \tau_{e_\lambda}(x) = \Phi(i^{-1}(e_\lambda x)) = \Phi(\lambda i^{-1}(x)) = \Phi \circ \tau_\lambda(i^{-1}(x))$$

but since Φ is intertwining, this gives:

$$= \sigma_\lambda \circ \Phi(i^{-1}(x)) = \rho_{e_\lambda} \circ \Phi \circ i^{-1}(x).$$

Since x was arbitrary, we have $(\Phi \circ i^{-1}) \circ \tau_\lambda = \rho_\lambda \circ (\Phi \circ i^{-1})$ so $\Phi \circ i^{-1}$ is intertwining with respect to (T, ρ). □

Definition 3.7.4. If (T, σ) is a Λ system of maps, Φ is an intertwining map, and $(T', \sigma|_{T'})$ is the self-similar k-surjective Λ-subsystem of Proposition 3.2.22, then we call T' the *attractor* of (T, σ, Φ).

Theorem 3.7.5. *Let Λ be a k-graph. Suppose (T, σ) is a Λ-system of maps, Φ is an intertwining map with respect to (T, σ), and T' is the attractor of (T, σ, Φ). Then there exist a 1-graph E with $E^0 = \Lambda^0$, an E-system of maps (T, ρ), and an intertwining map Ψ with respect to (T, ρ) such that if T'' is the attractor of (T, ρ, Ψ) then $T'' = T'$.*

Proof. Let E be the diagonal 1-graph of Λ, (T, ρ) be the diagonal E-system of (T, σ), and $\Psi = \Phi \circ i^{-1}$. Proposition 3.7.3 shows that this is an intertwining map. For all $v \in \Lambda^0$ we have

$$T''_v = \Psi(vE^\infty) = \Phi(i^{-1}(vE^\infty)) = \Phi(v\Lambda^\infty) = T'_v,$$

and hence $T'' = T'$. $\square$

Chapter 4

CUNTZ-PIMSNER ALGEBRAS OF ORDINARY TENSOR PRODUCTS

In this chapter we will examine the structure of Cuntz-Pimsner algebras associated with ordinary external tensor products of correspondences. The results of this chapter are subsumed by the results of Chapter 5 on twisted tensor products. Nonetheless, it is illuminating to consider this special case first. The results presented in this chapter are from [Morgan(2015)].

We will attempt to describe the Cuntz-Pimsner algebra of an external tensor product $X \otimes Y$ of correspondences in terms of the Cuntz-Pimsner algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$. In particular we will show that, under suitable conditions, $\mathcal{O}_{X\otimes Y}$ is isomorphic to a certain subalgebra $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$ of $\mathcal{O}_X \otimes \mathcal{O}_Y$. We call this subalgebra the $\mathbb{T}$-*balanced tensor product* because it has the property that $\gamma_z^X(x) \otimes y = x \otimes \gamma_z^Y(y)$ for all $z \in \mathbb{T}$, $x \in \mathcal{O}_X$ and $y \in \mathcal{O}_Y$, where γ^X and γ^Y are the gauge actions on $\mathcal{O}_X$ and $\mathcal{O}_Y$. This idea is inspired by a result of Kumjian in [Kumjian(1998)] where it is shown that for a cartesian product $E \times F = (E^0 \times F^0, E^1 \times F^1, r_E \times r_F, s_E \times s_F)$ of two graphs, $C^*(E\times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F)$ where the balancing is over the gauge action of the two graphs. Kumjian's proof uses the groupoid model of graph algebras and is therefore independent of our main result. However, we will be able to recover Kumjian's result for row finite graphs with no sources by considering the C^*-correspondence model of a graph algebra.

After proving our main result we will explore some examples including a generalization of Kumjian's result to the setting of topological graphs, implications for crossed products by $\mathbb{Z}$, crossed product by a completely positive map, and we will give a new proof of a theorem of Kaliszewski, Quigg, and Robertson which was used in

[Kaliszewski *et al.*(2012)] to study Cuntz-Pimsner algebras of coaction crossed products. We will assume all tensor products are minimal (spatial) unless otherwise stated.

4.1 Tensor Products Balanced Over Group Actions or Group Gradings

We wish to show that the Cuntz-Pimsner algebra of an external tensor product $X \otimes Y$ of correspondences is isomorphic to a certain subalgebra $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$ of the tensor product $\mathcal{O}_X \otimes \mathcal{O}_Y$. This subalgebra is called the $\mathbb{T}$-*balanced tensor product* of $\mathcal{O}_X$ and $\mathcal{O}_Y$. Following [Kumjian(1998)] we define the general construction as follows:

Definition 4.1.1. Let G be a compact abelian group, and let (A, G, α) and (B, G, β) be C^*-dynamical systems. We define the G-*balanced tensor product* $A \otimes_G B$ to be the fixed point algebra $(A \otimes B)^\lambda$ where $\lambda : G \to A \otimes B$ is the action characterized by $\lambda_s(a \otimes b) = \alpha_s(a) \otimes \beta_{s^{-1}}(b)$.

Proposition 4.1.2. *If $a \otimes b \in A \otimes_G B$ then $\alpha_s(a) \otimes b, a \otimes \beta_s(b) \in A \otimes_G B$ for all $s \in G$ and $\alpha_s(a) \otimes b = a \otimes \beta_s(b)$.*

Proof. To show that $\alpha_s(a) \otimes b \in A \otimes_G B$ note that for any $t \in G$ we have

$$\begin{aligned}
\alpha_t\big(\alpha_s(a)\big) \otimes \beta_{t^{-1}}(b) &= \alpha_s\big(\alpha_t(a)\big) \otimes \beta_{t^{-1}}(b) \\
&= (\alpha_s \otimes id_B)(\alpha_t(a) \otimes \beta_{t^{-1}}(b)) \\
&= (\alpha_s \otimes id_B)(a \otimes b) \\
&= \alpha_s(a) \otimes b
\end{aligned}$$

showing that $a \otimes \beta_s(b) \in A \otimes_G B$ is similar. Now that this has been established, the equality follows easily:

$$\begin{aligned}
\alpha_s(a) \otimes b &= \alpha_{s^{-1}}\big(\alpha_s(a)\big) \otimes \beta_s(b) \\
&= a \otimes \beta_s(b)
\end{aligned}$$

□

Thus the actions $\alpha \otimes \iota_B$ and $\iota_A \otimes \beta$ coincide on $A \otimes_G B$ where ι_A and ι_B are the trivial actions. We will refer to the restriction of $\alpha \otimes \iota_B$ to $A \otimes_G B$ (or equivalently the restriction of $\iota_A \otimes \beta$ to $A \otimes_G B$) as the *balanced action* of G and we will denote it by $\alpha \otimes_G \beta$.

The main result of this paper can be stated roughly as

$$\mathcal{O}_{X\otimes Y} \cong \mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$$

for suitable X and Y where we are balancing over the gauge actions on $\mathcal{O}_X$ and $\mathcal{O}_Y$. This generalizes the following example from [Kumjian(1998)]:

Example 4.1.3. Let E and F be source-free, row-finite discrete graphs and let $E \times F$ denote the product graph as in Example 2.1.8. Then $C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F)$.

As we noted above, actions of compact abelian groups correspond to gradings of the dual group. It will be useful to be able to describe $\mathbb{T}$-balanced tensor products in terms of the corresponding $\mathbb{Z}$-gradings. But first we will need a fact which follows from the Peter-Weyl theorem (Theorem VII.1.35 of [Gaal(1973)])

Lemma 4.1.4. *Let G be a compact abelian group with a normalized Haar measure and let χ be a character (i.e. a continuous homomorphism $G \to \mathbb{T}$). Then*

$$\int_G \chi(s)ds = \begin{cases} 1 & \chi \textit{ is the trivial homomorphism} \\ 0 & \textit{otherwise} \end{cases}$$

Now we are ready to describe G-balanced tensor products in terms of gradings of $\widehat{G}$.

Proposition 4.1.5. *Let (A, G, α) and (B, G, β) be C^*-dynamical systems with G abelian. Then, as discussed in the previous section, the coactions δ^α and δ^β give $\widehat{G}$-gradings of A and B:*

$$A_\chi = \{a \in A : \alpha_s(a) = \chi(s)a\}$$

$$B_\chi = \{b \in B : \beta_s(b) = \chi(s)b\}$$

Let

$$S_\chi := \{a \otimes b : a \in A_\chi, b \in B_\chi\}$$

and let

$$S := \bigcup_{\chi \in \widehat{G}} S_\chi$$

Then $A \otimes_G B = \overline{span}(S)$.

Proof. First, note that if $a \otimes b \in S$, then $a \otimes b \in S_\chi$ for some χ and so for all $s \in G$:

$$\begin{aligned}
\lambda_s(a \otimes b) &= \alpha_s(a) \otimes \beta_{s^{-1}}(b) \\
&= \chi(s)a \otimes \chi(s^{-1})b \\
&= \chi(s)\chi(s^{-1})(a \otimes b) \\
&= a \otimes b
\end{aligned}$$

so $a \otimes b \in A \otimes_G B$ and hence $S \subseteq A \otimes_G B$. Since $A \otimes_G B$ is a C^*-algebra, we have that $\overline{span}(S) \subseteq A \otimes_G B$.

Now, since A is densely spanned by the A_χ's and B is densely spanned by the B_χ's, the tensor product $A \otimes B$ will be densely spanned by elementary tensors $a \otimes b$ where $a \in A_\chi$ and $b \in B_{\chi'}$ for some $\chi, \chi' \in \widehat{G}$. More precisely, let

$$T_{\chi,\chi'} = \{a \otimes b : a \in A_\chi, b \in B_{\chi'}\}$$

and let

$$T = \bigcup_{\chi,\chi' \in \widehat{G}} T_{\chi,\chi'}$$

Then we have $A \otimes B = \overline{span}(T)$. Let $\varepsilon : A \otimes B \to A \otimes_G B$ be the conditional expectation $c \mapsto \int_G \lambda_s(c)ds$. Then, since ε is continuous, linear and surjective, $\varepsilon(T)$

densely spans $A \otimes_G B$. Let $a \otimes b \in T$, say $a \otimes b \in T_{\chi,\chi'}$. Then using Lemma 4.1.4 and the fact that a product of two characters is a character, we have

$$\begin{aligned}
\varepsilon(a \otimes b) &= \int_G \lambda_s(a \otimes b) ds \\
&= \int_G \big(\chi(s)a \otimes \chi'(s^{-1})b\big) ds \\
&= \left(\int_G \chi(s)\overline{\chi'(s)} dz\right) a \otimes b \\
&= \left(\int_G \left(\chi\overline{\chi'}\right)(s) ds\right) a \otimes b \\
&= \begin{cases} 1 & \text{if } \chi\overline{\chi'} \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

But $\chi\overline{\chi'}$ will be trivial if and only if $\chi = \chi'$. This means that if $\chi \neq \chi'$ (i.e. $a \otimes b \in T \backslash S$) then $\varepsilon(a \otimes b) = 0$ and if $\chi = \chi'$ (i.e. $a \otimes b \in S$). Then $\varepsilon(a \otimes b) = a \otimes b$. This implies that $\varepsilon(T) = S$ and thus, since T densely spans $A \otimes B$, the linearity and continuity of ε imply that S densely spans $\varepsilon(A \otimes B) = A \otimes_G B$. Therefore $A \otimes_G B = \overline{span}(S)$. $\square$

Proposition 2.3.11 tells us that the $\widehat{G}$-grading of $A \otimes_G B$ just described should give us an action of G on $A \otimes_G B$. We will now show that this action coincides exactly with the balanced action:

Proposition 4.1.6. *Let $\{S_\chi\}_{s \in \widehat{G}}$ be the $\widehat{G}$-grading of $A \otimes_G B$ described in the previous proposition and let γ be the action associated to this grading by Proposition 2.3.11. Then $\gamma = \alpha \otimes_G \beta$.*

Proof. It suffices to check that these maps coincide on each S_χ. Let $a \otimes b \in S_\chi$. Then for each $s \in \widehat{G}$ we have $\gamma_s(a \otimes b) = \chi(s)(a \otimes b)$ by definition. On the other hand, since $a \in A_\chi$:

$$(\alpha \otimes_G \beta)_s(a \otimes b) = \alpha_s(a) \otimes b$$

$$= \big(\chi(s)a\big) \otimes b$$

$$= \chi(s)(a \otimes b)$$

so $\gamma_s(a \otimes b) = (\alpha \otimes_G \beta)_s(a \otimes b)$ for every $s \in G$ and every $a \otimes b$ in S_χ. $\square$

The following lemma will be useful later:

Lemma 4.1.7. *Let $\{A_n\}_{n\in\mathbb{Z}}$ and $\{B_n\}_{n\in\mathbb{Z}}$ be saturated $\mathbb{Z}$-gradings of C^*-algebras A and B and let $\{S_n\}_{n\in\mathbb{Z}}$ be the $\mathbb{Z}$-grading of $A \otimes_{\mathbb{T}} B$ as in the previous propositions. Then $A \otimes_{\mathbb{T}} B$ is generated by S_1.*

Proof. First, we will show that $S_nS_m = S_{n+m}$. We already have that $S_nS_m \subseteq S_{n+m}$ so it suffices to show the reverse inclusion. Let $a \otimes b \in S_{n+m}$. Then $a \in A_{n+m}$ and $b \in B_{n+m}$ so $a = \sum a_i a_i'$ and $b = \sum b_i b_i'$ with $a_i \in A_n$,$a_i' \in A_m$, $b_i \in B_n$, and $b_i' \in B_m$. Therefore $a \otimes b = \sum_{i,j}(a_i \otimes b_j)(a_i' \otimes b_j')$ where $a_i \otimes b_j \in S_n$ and $a_i' \otimes b_j' \in S_m$ so $S_{n+m} \subseteq S_nS_m$ and hence $S_nS_m = S_{n+m}$.

Since 1 generates $\mathbb{Z}$ as a group, S_1 generates $\overline{span} \bigcup S_n = A \otimes_{\mathbb{T}} B$ as a C^*-algebra. $\square$

4.2 Ideal Compatibility

In this section we introduce technical conditions which we will need for our main result to hold.

Definition 4.2.1. Let X and Y be correspondences over C^*-algebras A and B and let J_X, J_Y, $J_{X\otimes Y}$ be the Katsura ideals (i.e.

$$J_X = \{a \in A : \phi(a) \in \mathcal{K}(X) \text{ and } aa' = 0 \text{ if } \phi_X(a') = 0\}$$

and so on). We say that X and Y are *"ideal-compatible"* if $J_{X\otimes Y} = J_X \otimes J_Y$.

The simplest way for this condition to hold is if the left actions of A and B on X and Y are injective and implemented by compacts. In this case it will also be true that the left action of $A \otimes B$ on $X \otimes Y$ will be injective and implemented by compacts. Thus we will have that $J_X = A$, $J_Y = B$, and $J_{X \otimes Y} = A \otimes B$ so ideal compatibility is automatic. Thus we have established:

Proposition 4.2.2. *Let X_A and Y_B be correspondences such that the left actions of A and B are injective and implemented by compacts. Then X and Y are ideal-compatible.*

The following two lemmas are inspired by Lemma 2.6 of [Kaliszewski *et al.*(2012)].

Lemma 4.2.3. *Let X and Y be correspondences over A and B. Then $J_X \otimes J_Y \subseteq J_{X \otimes Y}$.*

Proof. Since ϕ_X maps J_X injectively into $\mathcal{K}(X)$ and ϕ_Y maps J_Y injectively into $\mathcal{K}(Y)$, $\phi_{X \otimes Y} = \phi_X \otimes \phi_Y$ will map $J_X \otimes J_Y$ injectively into $\mathcal{K}(X) \otimes K(Y)$, but $\mathcal{K}(X) \otimes \mathcal{K}(Y) = \mathcal{K}(X \otimes Y)$ so $\phi_{X \otimes Y}$ maps $J_X \otimes J_Y$ injectively into $\mathcal{K}(X \otimes Y)$. Thus by Proposition 2.1.11 $J_X \otimes J_Y \subseteq J_{X \otimes Y}$. $\square$

Recall that if A and B are C^*-algebras and C is an subalgebra of A, the triple (C, A, B) is said to satisfy the *slice map property* if

$$C \otimes B = \{x \in A \otimes B : (id_A \otimes \omega)(x) \in C \text{ for all } \omega \in B^*\}$$

Lemma 4.2.4. *Let X and Y be correspondences over C^*-algebras A and B and suppose Y is an imprimitivity bimodule. If (J_X, A, B) satisfies the slice map property, then X and Y are ideal-compatible.*

Proof. It suffices to show that $J_{X \otimes Y} \subseteq J_X \otimes J_Y$. Since Y is an imprimitivity bimodule we have that the left action ϕ_Y is an isomorphism $B \cong \mathcal{K}(Y)$ and thus ϕ_Y maps all of B maps injectively into $\mathcal{K}(Y)$, so $J_Y = B$. We must show that $J_{X \otimes Y} \subseteq J_X \otimes B$.

Let $c \in J_{X \otimes Y}$. Since (J_X, A, B) satisfies the slice map property, showing that $c \in J_X \otimes B$ is equivalent to showing that $(id \otimes \omega)(c) \in J_X$ for all $\omega \in B^*$. Recalling the definition of J_X, this means we must show that $\phi_X\big((id \otimes \omega)(c)\big) \in \mathcal{K}(X)$ and that $(id \otimes \omega)(c)a = 0$ for all $a \in ker(\phi_X)$. With this in mind, let $\omega \in B^*$. Since ϕ_X is linear, we have that

$$\begin{aligned}
\phi_X\big((id_A \otimes \omega)(c)\big) &= \big(\phi_X \otimes \omega\big)(c) \\
&= \big(\phi_X \otimes (\omega \circ \phi_Y^{-1} \circ \phi_Y)\big)(c) \\
&= \big(id_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})\big) \circ (\phi_X \otimes \phi_Y)(c) \\
&= \big(id_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})\big) \circ \phi_{X \otimes Y}(c)
\end{aligned}$$

To see that this is in $\mathcal{K}(X)$ note that since $c \in J_{X \otimes Y}$ by assumption, we know that $\phi(c) \in \mathcal{K}(X \otimes Y) = \mathcal{K}(X) \otimes \mathcal{K}(Y)$. Note that since Y is an imprimitivity bimodule, ϕ_Y^{-1} is well-defined as a map $\mathcal{K}(Y) \to B$. Since $\big(id_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})\big)$ maps $\mathcal{K}(X) \otimes \mathcal{K}(Y) \to \mathcal{K}(X)$ we have that

$$\big(id_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})\big) \circ \phi_{X \otimes Y}(c) \in \mathcal{K}(X)$$

so $(id \otimes \omega)(c) \in \mathcal{K}(X)$.

Next, let $a \in ker(\phi_X)$ and factor ω as $b \cdot \omega'$ for some $b \in B$ and $\omega' \in B^*$ (where $(b \cdot \omega')(b') = \omega'(b'b)$). Then

$$\begin{aligned}
(id \otimes \omega)(c)a &= (id \otimes \omega)\big(c(a \otimes 1)\big) \\
&= (id \otimes b \cdot \omega')\big(c(a \otimes 1)\big) \\
&= (id \otimes \omega')\big(c(a \otimes 1)(1 \otimes b)\big) \\
&= (id \otimes \omega')\big(c(a \otimes b)\big)
\end{aligned}$$

but

$$a \otimes b \in ker(\phi_X) \otimes B$$

$$\subseteq ker(\phi_X \otimes \phi_Y)$$
$$= ker(\phi_{X\otimes Y})$$

Therefore, since $c \in J_{X\otimes Y}$ we must have $c(a \otimes b) = 0$ and hence $(id \otimes \omega)(c)a = (id \otimes \omega')\big(c(a \otimes b)\big) = 0$. Thus, we have established that $(id \otimes \omega)(c) \in J_X$ for any $c \in J_{X\otimes Y}$ and $\omega \in B^*$ so by the slice map property we have that $J_{X\otimes Y} \subseteq J_X \otimes B = J_X \otimes J_Y$ and thus (by the previous lemma) $J_{X\otimes Y} = J_X \otimes J_Y$. $\square$

In Example 8.13 of [Raeburn(2005)], it is shown that if E is a discrete graph, then

$$J_{X(E)} = \overline{span}\{\delta_v : 0 < |r^{-1}(v)| < \infty\}$$

where $X(E)$ is the associated correspondence and $\delta_v \in c_0(E^0)$ denotes the characteristic function of the vertex $v \in E^0$. With this in mind, we give the following proposition:

Proposition 4.2.5. *Let E and F be discrete graphs and let $X = X(E)$ and $Y = X(F)$ be the associated correspondences. Then X and Y are ideal compatible.*

Proof. Recall that $X \otimes Y = X(E \times F)$. Thus

$$J_{X\otimes Y} = \overline{span}\{\delta_{(v,w)} : 0 < |r^{-1}_{E\times F}(v, w)| < \infty\}$$

By definition, $r_{E\times F} = r_E \times r_F$ so $r^{-1}_{E\times F}(v, w) = r_E^{-1}(v) \times r_F^{-1}(w)$ and thus $|r^{-1}_{E\times F}(v, w)| = |r_E^{-1}(v)| \cdot |r_F^{-1}(w)|$ but $0 < |r_E^{-1}(v)| \cdot |r_F^{-1}(w)| < \infty$ if and only if $0 < |r_E^{-1}(v)| < \infty$ and $0 < |r_F^{-1}(w)| < \infty$. Thus we have that

$$J_{X\otimes Y} = \overline{span}\{\delta_{(v,w)} : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\}$$

Since $\delta_{(v,w)} = \delta_v \delta_w$, if we identify $c_0(E^0 \times F^0)$ with $c_0(E^0) \otimes c_0(F^0)$ in the standard way, we see that $\delta_{(v,x)} = \delta_v \otimes \delta_w$. Thus

$$J_{X\otimes Y} = \overline{span}\{\delta_v \otimes \delta_w : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\}$$

$$= \overline{span}\{f \otimes g : f \in J_X, g \in J_Y\}$$

$$= J_X \otimes J_Y$$

Therefore, X and Y are ideal-compatible. $\square$

Definition 4.2.6. Let X be a correspondence over a C^* algebra A. We will call this correspondence *Katsura non-degenerate* if $X \cdot J_X = X$.

Example 4.2.7. Let X be a correspondence over a C^*-algebra A such that the left action is injective and implemented by compacts. In this case we have that $J_X = A$. Thus:

$$\begin{aligned} X \cdot J_X &= X \cdot A \\ &= X \end{aligned}$$

Definition 4.2.8. Recall that a vertex in a directed graph is called a *source* if it receives no edges. We will call such a vertex a *proper source* if it emits at least one edge.

Proposition 4.2.9. *Let E be a directed graph. Then $X(E)$ is Katsura non-degenerate if and only if E has no proper sources and no infinite receiver emits an edge.*

Proof. Suppose there is $v \in E^0$ such that $|r^{-1}(v)| = \infty$ and $|s^{-1}(v)| > 0$. Then for every $f \in J_X$ we have $f(v) = 0$. Thus for any $g \in C_c(E^1)$, $f \in J_X$, and $e \in s^{-1}(v)$, we have $(g \cdot f)(e) = g(e)f\big(s(e)\big) = g(e)f(v) = 0$. Thus $h(e) = 0$ for all $h \in C_c(E^1) \cdot J_X$ and, taking the limit, $x(e) = 0$ for all $x \in X \cdot J_X$. Thus $\delta_e \notin X \cdot J_X$ since $\delta_e(e) = 1 \neq 0$ but $\delta_e \in X$. Therefore $X \neq X \cdot J_X$, i.e. X is not Katsura non-degenerate.

Similarly, suppose that E has a proper source v. Then, since $|r^{-1}(v)| = 0$ we must have $f(v) = 0$ for all $f \in J_X$. Then for any $g \in C_c(E^1)$ and $e \in s^{-1}(v)$ we have that $(g \cdot f)(e) = g(e)f(v) = 0$ for $f \in J_X$. Thus by similar reasoning as above we have that $x(e) = 0$ for all $x \in X \cdot J_X$ and so $\delta_e \notin X \cdot J_X$ but $\delta_e \in X$ and we can again conclude that $X \neq X \cdot J_X$ so X is not Katsura non-degenerate.

On the other hand, suppose E has no proper sources and no infinite receiver in E emits an edge. Let $e \in E^1$ and let $v = s(e)$. Then $|r^{-1}(v)| < \infty$ and $|r^{-1}(v)| > 0$ by assumption, so function in J_X can be supported on v. In particular, $\delta_v \in J_X$. Since $\delta_e \cdot \delta_v = \delta_e$ we know that $\delta_e \in X \cdot J_X$. Since e was arbitrary, we have that all such characteristic functions are contained in $X \cdot J_X$. But these functions densely span $C_c(E^1)$ and thus densely span X, so we have that $X \subseteq X \cdot J_X$ and therefore $X = X \cdot J_X$ so X is Katsura non-degenerate.

$\square$

4.3 Main Result

We will begin with a few lemmas:

Lemma 4.3.1. *Let X and Y be correspondences over C^*-algebras A and B respectively. Suppose (π_X, ψ_X) and (π_Y, ψ_Y) are Toeplitz representations of X and Y in C^*-algebras C and D. Let $\pi := \pi_X \otimes \pi_Y$ and $\psi := \psi_X \otimes \psi_Y$ Then (π, ψ) is a Toeplitz representation of $X \otimes Y$ in $C \otimes D$.*

Proof. This follows from the following computations:

$$\begin{aligned}
\psi\big((x \otimes y) \cdot (a \otimes b)\big) &= \psi_X(x \cdot a) \otimes \psi_Y(y \cdot b) \\
&= \psi_X(x)\pi_X(a) \otimes \psi_Y(y)\pi_Y(b) \\
&= \psi(x \otimes y)\pi(a \otimes b)
\end{aligned}$$

$$\begin{aligned}
\psi\big((a \otimes b) \cdot (x \otimes y)\big) &= \psi_X(a \cdot x) \otimes \psi_Y(b \cdot y) \\
&= \pi_X(a)\psi_X(x) \otimes \pi_Y(b)\psi_Y(y) \\
&= \pi(a \otimes b)\psi(x \otimes y)
\end{aligned}$$

$$\begin{aligned}
\psi(x \otimes y)^*\psi(x' \otimes y') &= \psi_X(x)^*\psi_X(x') \otimes \psi_Y(y)^*\psi_Y(y') \\
&= \pi_X\big(\langle x, x'\rangle_A\big) \otimes \pi_Y\big(\langle y, y'\rangle_B\big) \\
&= \pi\big(\langle x, x'\rangle_A \otimes \langle y, y'\rangle_B\big) \\
&= \pi\big(\langle x \otimes y, x' \otimes y'\rangle_{A\otimes B}\big)
\end{aligned}$$

□

Lemma 4.3.2. *If (π, ψ) is the Toeplitz representation of $X \otimes Y$ in the previous lemma, then*

$$\psi^{(1)}(\kappa(S \otimes T)) = \psi_X^{(1)}(S) \otimes \psi_Y^{(1)}(T)$$

where κ is as in Lemma 2.1.7*.*

Proof. Let $x, x' \in X$ and $y, y' \in Y$. Then:

$$\begin{aligned}
\psi^{(1)}(\kappa(\Theta_{x,x'} \otimes \Theta_{y,y'})) &= \psi^{(1)}(\Theta_{x\otimes y, x'\otimes y'}) \\
&= \psi(x \otimes y)\psi(x' \otimes y')^* \\
&= (\psi_X(x) \otimes \psi_Y(y))(\psi_X(x') \otimes \psi_Y(y'))^* \\
&= \psi_X(x)\psi_X(x')^* \otimes \psi_Y(y)\psi_Y(y')^* \\
&= \psi_X^{(1)}(\Theta_{x,x'}) \otimes \psi_Y^{(1)}(\Theta_{y,y'})
\end{aligned}$$

Since the rank-ones have dense span in the compacts, this result extends to any $S \in \mathcal{K}(X)$ and $T \in \mathcal{K}(Y)$. □

Lemma 4.3.3. *Let X and Y be ideal-compatible correspondences over A and B. Then if (π_X, ψ_X) and (π_Y, ψ_Y) are Cuntz-Pimsner covariant, then so is (π, ψ).*

Proof. Let $c \in J_{X\otimes Y}$ Since X an Y are ideal-compatible we have that $J_{X\otimes Y} = J_X \otimes J_Y$ so we can approximate c by elements of $J_X \odot J_Y$, i.e. by finite sums $\sum_i a_i \otimes b_i$ with $a_i \in J_X$ and $b_i \in J_Y$ for each i.

Notice:

$$\begin{aligned}
\psi^{(1)}\left(\phi\left(\sum_i a_i \otimes b_i\right)\right) &= \psi^{(1)}\left(\sum_i \phi(a_i \otimes b_i)\right) \\
&= \psi^{(1)}\left(\sum_i \kappa\big(\phi_X(a_i) \otimes \phi_Y(b_i)\big)\right) \\
&= \sum_i \psi_X^{(1)}\big(\phi_X(a_i)\big) \otimes \psi_Y^{(1)}\big(\phi_Y(b_i)\big) \\
&= \sum_i \pi_X(a_i) \otimes \pi_Y(b_i) \\
&= \sum_i \pi(a_i \otimes b_i)
\end{aligned}$$

where we have used the Cuntz-Pimsner covariance of (π_X, ψ_X) and (π_Y, ψ_Y). Thus if (c_j) is a sequence in $J_X \odot J_Y$ converging to c, we have $\psi^{(1)}\big(\phi(c_j)\big) = \pi(c_j)$ for all j. Since $\psi^{(1)}, \phi$, and π are all continuous, we have that $\psi^{(1)}\big(\phi(c)\big) = \pi(c)$. This establishes that (π, ψ) is Cuntz-Pimsner covariant. □

We are now ready to prove the main result of this paper:

Theorem 4.3.4. *Let X and Y be ideal-compatible correspondences over C^*-algebras A and B. Then $\mathcal{O}_{X\otimes Y}$ can be faithfully embedded in $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$. If X and Y are Katsura non-degenerate, then $\mathcal{O}_{X\otimes Y} \cong \mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$.*

Proof. We will begin by showing the existence of a homomorphism $\mathcal{O}_{X\otimes Y} \to \mathcal{O}_X \otimes \mathcal{O}_Y$. To show this, we will construct a Cuntz-Pimsner covariant representation of $X \otimes Y$ in $\mathcal{O}_X \otimes \mathcal{O}_Y$ and then apply the universal property of Cuntz-Pimsner algebras.

Let (k_X, k_A) and (k_Y, k_B) be the Cuntz-Pimsner covariant representations of X and Y in $\mathcal{O}_X$ and $\mathcal{O}_Y$ respectively. Let $\psi = k_X \otimes k_Y$ and $\pi = k_A \otimes k_B$. Then by Lemma 4.3.3, (ψ, π) is Cuntz-Pimsner covariant and so we have a homomorphism $F : \mathcal{O}_{X\otimes Y} \to \mathcal{O}_X \otimes \mathcal{O}_Y$ such that

$$(\psi, \pi) = (F \circ k_{X\otimes Y}, F \circ k_{A\otimes B})$$

In particular,

$$F(k_{A\otimes B}(A\otimes B)) = \pi(A\otimes B) = (k_A\otimes k_B)(A\otimes B) \tag{4.3.1}$$

$$F(k_{X\otimes Y}(X\otimes Y)) = \psi(X\otimes Y) = (k_X\otimes k_Y)(X\otimes Y) \tag{4.3.2}$$

Let $\{\mathcal{O}_X^n\}_{n\in\mathbb{Z}}$ and $\{\mathcal{O}_Y^n\}_{n\in\mathbb{Z}}$ denote the $\mathbb{Z}$-gradings of $\mathcal{O}_X$ and $\mathcal{O}_Y$ associated to the standard gauge actions γ_X and γ_Y. Then by Proposition 4.1.5, the subspaces

$$S_n := \{x\otimes y : x\in\mathcal{O}_X^n, y\in\mathcal{O}_Y^n\}$$

give a $\mathbb{Z}$-grading of $\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$. Since (1) shows that $\pi(A\otimes B)\subseteq S_0$ and (2) shows that $\psi(X\otimes Y)\subseteq S_1$, we can see that $C^*(\psi,\pi)\subseteq\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$ and that the action of $\mathbb{T}$ on $\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$ guaranteed by Lemma 2.3.11 is a gauge action:

$$\begin{aligned} &\gamma_z(\pi(c)) = \pi(c) && c\in A\otimes B \\ &\gamma_z(\psi(w)) = z\psi(w) && w\in X\otimes Y \end{aligned}$$

Also, since k_A, k_B, k_X and k_Y are injective, $\pi = k_A\otimes k_B$ and $\psi = k_X\otimes k_Y$ are injective too. Hence by the gauge invariant uniqueness theorem, F is injective. Thus we have established the first part of the theorem.

Now suppose X and Y are Katsura non-degenerate. We will show that (ψ,π) generates $\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$ by showing that (ψ,π) generates S_1 and applying Lemma 4.1.7. Since $\mathcal{O}_X^1$ is densely spanned by elements of the form: $k_X^{n+1}(x)k_X^n(x')^*$ and $\mathcal{O}_Y^1$ is densely spanned by elements of the form $k_Y^{n+1}(y)k_Y^n(y')^*$ we have that S_1 is densely spanned by elements of the form

$$k_X^{n+1}(x)k_X^n(x')^*\otimes k_Y^{m+1}(y)k_Y^m(y')^* \tag{4.3.3}$$

By symmetry, we may assume $m = n+l$ for some nonnegative l. Then we may assume that $y = y_1\otimes y_2$ and $y' = y_1'\otimes y_2'$ with $y_1, y_1'\in Y^{\otimes n}$ and $y_2, y_2'\in Y^{\otimes l}$. Further, since

X is Katsura non-degenerate, we can factor $x = x_0 a$ and $x' = x_0' a'$ with $x_0, x_0' \in X$ and $a, a' \in J_X$. Now we can factor (3) as follows:

$$
\begin{aligned}
&k_X^{n+1}(x) k_X^n(x')^* \otimes k_Y^{m+1}(y) k_Y^m(y')^* \\
&\quad = \big(k_X^{n+1}(x) \otimes k_Y^{m+1}(y)\big)\big(k_X^n(x')^* \otimes k_Y^m(y')^*\big) \\
&\quad = \big(k_X^{n+1}(x_0) k_A(a) \otimes k_Y^{n+1}(y_1) k_Y^l(y_2)\big)\big(k_A(a')^* k_X^n(x_0')^* \otimes k_Y^l(y_2')\big)^* k_Y^n(y_1')^*\big) \\
&\quad = \big(k_X^{n+1}(x_0) \otimes k_Y^{n+1}(y_1)\big)\big(k_A(a) \otimes k_Y^l(y_2)\big)\big(k_A(a')^* \otimes k_Y^l(y_2')^*\big)\big(k_X^n(x_0')^* \otimes k_Y^n(y_1')^*\big) \\
&\quad = \big(k_X^{n+1}(x_0) \otimes k_Y^{n+1}(y_1)\big)\big(k_A(aa'^*) \otimes k_Y^l(y_2) k_Y^l(y_2')^*\big)\big(k_X^n(x_0')^* \otimes k_Y^n(y_1')^*\big) \\
&\quad = \big(k_X^{n+1}(x_0) \otimes k_Y^{n+1}(y_1)\big)\big(k_X^{(1)}(\phi_X(aa'^*)) \otimes k_Y^{(1)}(\Theta_{y_2, y_2'})\big)\big(k_X^n(x')^* \otimes k_Y^n(y_1')^*\big) \\
&\quad = \psi^{n+1}(x_0 \otimes y_1)\big(\psi^{(1)}(\phi_X(aa') \otimes \Theta_{y_2, y_2'})\big)\psi^n(x_0' \otimes y_1')^*
\end{aligned}
$$

Since $\psi^{n+1}(x_0 \otimes y_1)$, $\psi^{(1)}(\phi_X(aa'^*) \otimes \Theta_{y_2, y_2'})$, and $\psi^n(x_0' \otimes y_1')$ are in the algebra generated by (ψ, π), we now know that (ψ, π) generates S_1 and so by Lemma 4.1.7 (ψ, π) generates all of $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$. Therefore F is surjective hence an isomorphism $\mathcal{O}_{X \otimes Y} \cong \mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$. $\square$

4.4 Examples

We will now give some examples:

Example 4.4.1. Let $(A, \mathbb{Z}, \alpha)$ and $(B, \mathbb{Z}, \beta)$ be C^*-dynamical systems. Let X be the C^*-correspondence A_A with left action given by $a \cdot x = \alpha_1(a)x$ and let Y be the correspondence B_B with left action given by $b \cdot y = \beta_1(b)y$. This action is injective and implemented by compacts and we have that $\mathcal{O}_X \cong A \rtimes_\alpha \mathbb{Z}$ and $\mathcal{O}_Y \cong B \rtimes_\beta \mathbb{Z}$ by isomorphisms which carry the gauge action of $\mathbb{T}$ to the dual action of $\mathbb{T}$ (see [Pimsner(2001)]).

Consider the external tensor product $X \otimes Y$. As a right Hilbert module this is $A_A \otimes B_B$, it carries a right action of $A \otimes B$ characterized by $(x \otimes y) \cdot (a \otimes b) = x \cdot a \otimes y \cdot b$

but since the right actions on X and Y are given by multiplication in A and B, this action of $A \otimes B$ on $X \otimes Y$ is just multiplication in $A \otimes B$. Further, the $A \otimes B$ valued inner product on $X \otimes Y$ is given by

$$\begin{aligned}\langle x \otimes y, x' \otimes y' \rangle_{A \otimes B} &= \langle x, x' \rangle_A \otimes \langle y, y' \rangle_B \\ &= x^* x' \otimes y^* y' \\ &= (x \otimes y)^* (x' \otimes y')\end{aligned}$$

but this is precisely the inner product on $(A \otimes B)_{A \otimes B}$. Thus $id_A \otimes id_B$ gives a right Hilbert module isomorphism $X \otimes Y \cong (A \otimes B)_{A \otimes B}$. The left action of $A \otimes B$ on $X \otimes Y$ will be the tensor product of the action of A on X and the action of B on Y. Thus

$$\begin{aligned}(a \otimes b) \cdot (x \otimes y) &= \alpha_1(a)x \otimes \beta_1(b)y \\ &= \big(\alpha_1(a) \otimes \beta_1(b)\big)(x \otimes y)\end{aligned}$$

Thus, as an $A \otimes B$ correspondence, $X \otimes Y$ can be identified with the correspondence associated to the automorphism $\alpha_1 \otimes \beta_1$ on $A \otimes B$. Since $(\alpha_1 \otimes \beta_1)^{\circ n} = (\alpha_n \otimes \beta_n)$ and $(\alpha_1 \otimes \beta_1)^{-1} = (\alpha_{-1} \otimes \beta_{-1})$, the action of $\mathbb{Z}$ generated by $\alpha_1 \otimes \beta_1$ will be the diagonal action $\alpha \otimes \beta$ of $\mathbb{Z}$ on $A \otimes B$. Thus we have that $\mathcal{O}_{X \otimes Y} \cong (A \otimes B) \rtimes_{\alpha \otimes \beta} \mathbb{Z}$.

Therefore, in this context our main theorem says that

$$(A \otimes B) \rtimes_{\alpha \otimes \beta} \mathbb{Z} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes_{\mathbb{T}} (B \rtimes_\beta \mathbb{Z})$$

In later work, we hope to investigate whether this result generalizes to groups other than $\mathbb{Z}$.

Example 4.4.2 (Products of Topological Graphs)**.** Let $E = (E^0, E^1, r, s)$ and $F = (F^0, F^1, r', s')$ be source-free topological graphs with r and r' proper. Then the left actions of $X(E)$ and $X(F)$ will be injective and implemented by compacts. Recall from Example 2.1.8 that $X(E) \otimes X(F) \cong X(E \times F)$ where $E \times F$ is the product graph. Our main result says that $\mathcal{O}_{X(E\times F)} \cong \mathcal{O}_{X(E)} \otimes_{\mathbb{T}} \mathcal{O}_{X(F)}$ which translates to

$$C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F)$$

Note that if E and F are discrete graphs, this coincides with Kumjian's result in [Kumjian(1998)].

Example 4.4.3. (Products of Discrete Graphs) Let E and F be discrete graphs with no proper sources and such that no infinite receiver emits an edge. From the discussion in Section 4 we know that the graph correspondences $X(E)$ and $X(F)$ are ideal compatible and Katsura non-degenerate. By the same reasoning as in the previous example we have that

$$C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F)$$

Note that this stronger than the result in [Kumjian(1998)] where the graphs are required to be source-free and row-finite.

Example 4.4.4. Let A and B be C^*-algebras, and let X be a correspondence over A. Viewing B as the correspondence ${}_BB_B$ we can form the $A \otimes B$ correspondence $X \otimes B$. Suppose that X and B are ideal compatible and Katsura non-degenerate (in fact B will automatically be Katsura non-degenerate). Recall that $\mathcal{O}_B \cong B \otimes C(\mathbb{T})$ with gauge action $\iota \otimes \lambda$ where ι is the trivial action and λ is left translation. Thus our main result says that $\mathcal{O}_{X\otimes B} \cong \mathcal{O}_X \otimes_{\mathbb{T}} (B \otimes C(\mathbb{T}))$. Identifying $C(\mathbb{T})$ with $C^*(\mathbb{Z})$ we have $\mathcal{O}_{X\otimes B} \cong \mathcal{O}_X \otimes_{\mathbb{T}} (B \otimes C^*(\mathbb{Z}))$ and characterizing the $\mathbb{T}$ balanced tensor product in terms of the $\mathbb{Z}$-gradings as we have been, we see that

$$\mathcal{O}_{X\otimes B} \cong \overline{span}\{x \otimes b \otimes w \in \mathcal{O}_X^n \otimes B \otimes C^*(\mathbb{Z})^n : n \in \mathbb{Z}\}$$

But since $C^*(\mathbb{Z})^n = span(u_n)$ (where u_n denotes the unitary in $C^*(Z)$ associated to n) we can rephrase this as

$$\overline{span}\{x \otimes b \otimes u_n \in \mathcal{O}_X \otimes B \otimes C^*(\mathbb{Z}) : x \in \mathcal{O}_X^n\} \tag{4.4.1}$$

Now, let γ denote the gauge action of $\mathbb{T}$ on $\mathcal{O}_X$ and let δ^γ be the dual coaction of $\mathbb{Z}$. Recall that this coaction can be characterized by the property that $\delta^\gamma(x) = x \otimes u_n$ whenever $x \in \mathcal{O}_X^n$. Since the subspaces $\mathcal{O}_X^n$ densely span $\mathcal{O}_X$, their images under δ^γ will densely span $\delta^\gamma(\mathcal{O}_X)$. Therefore:

$$\begin{aligned}
\delta^\gamma(\mathcal{O}_X) &= \overline{span}\{x \otimes u_n : x \in \mathcal{O}_X^n\} \\
(\delta^\gamma \otimes id_B)(\mathcal{O}_X \otimes B) &= \overline{span}\{x \otimes u_n \otimes b : x \in \mathcal{O}_X^n, b \in B\} \\
\sigma_{23} \circ (\delta^\gamma \otimes id_B)(\mathcal{O}_X \otimes B) &= \overline{span}\{x \otimes b \otimes u_n : x \in \mathcal{O}_X^n, b \in B\}
\end{aligned}$$

Noting that id_B, and σ_{23} are isomorphisms (where σ_{23} is the map which exchanges the second and third tensor factors) and δ^γ is an injective $*$-homomorphism, we see that $\sigma_{23} \circ (\delta^\gamma \otimes id_B)$ is an injective $*$-homomorphism and is thus an isomorphism onto its image. But its image is $\overline{span}\{x \otimes b \otimes u_n : x \in \mathcal{O}_X^n, b \in B\}$ and by (4.4.1) this is isomorphic to $\mathcal{O}_{X \otimes B}$. Therefore we have shown that:

$$\mathcal{O}_{X \otimes B} \cong \mathcal{O}_X \otimes B$$

This result is already known, and was used in [Kaliszewski *et al.*(2012)] to prove facts about coactions on Cuntz-Pimsner algebras.

Example 4.4.5. Given a C^*-algebra A and a completely positive map Φ, Kwasniewski defines ([Kwaśniewski(2014)] Definition 3.5) a crossed product of A by Φ denoted by $C^*(A, \Phi)$. In Theorem 3.13 of [Kwaśniewski(2014)], it is shown that $C^*(A, \Phi) \cong \mathcal{O}_{X_\Phi}$ where X_Φ is the correspondence associated with Φ as in Definition 2.1.4. If Φ is an endomorphism this reduces to the Exel crossed product [Exel(2008)].

Suppose A and B are C^*-algebras and $\Phi : A \to A$ and $\Psi : B \to B$ are completely positive maps. Furthermore, suppose that the associated correspondences X_Φ and X_Ψ are ideal compatible and Katsura non-degenerate. Then our main result states that $\mathcal{O}_{X_\Phi \otimes X_\Psi} \cong \mathcal{O}_{X_\Phi} \otimes_{\mathbb{T}} \mathcal{O}_{X_\Psi}$. Recalling that $X_\Phi \otimes X_\Psi \cong X_{\Phi \otimes \Psi}$ and using the crossed product notation, we get

$$C^*(A \otimes B, \Phi \otimes \Psi) \cong C^*(A, \Phi) \otimes_{\mathbb{T}} C^*(B, \Psi)$$

Chapter 5

CUNTZ-PIMSNER ALGEBRAS ASSOCIATED TO TWISTED TENSOR PRODUCTS OF CORRESPONDENCES

In this chapter we will generalize the results of the previous chapter from ordinary tensor products to twisted tensor products. The results of this chapter are taken from [Morgan(2016)]. Many constructions in operator algebras may be thought of as "twisted" tensor products, for example: crossed products by actions or coactions of groups, $\mathbb{Z}_2$-graded tensor products and so on. In [Meyer *et al.*(2014)], a very general construction of a "twisted tensor product" is presented. Their construction involves two quantum groups $\mathbb{G} = (S, \Delta_S)$ and $\mathbb{H} = (T, \Delta_T)$, two coactions $(A, \mathbb{G}, \delta_A)$ and $(B, \mathbb{H}, \delta_B)$, and a bicharacter $\chi \in \mathcal{U}(\widehat{S} \otimes \widehat{T})$. Given this information, they define a twisted tensor product $A \boxtimes_\chi B$. They also show that if X and Y are correspondences over A and B with compatible coactions of $\mathbb{G}$ and $\mathbb{H}$, there is a natural way of defining a correspondence $X \boxtimes_\chi Y$ over $A \boxtimes_\chi B$. In this paper, we will work with a special case of this general construction which is general enough to be useful but simple enough to be very tractable.

Specifically, we are interested in the case where, for some discrete group G, $S = c_0(G)$, $T = C_r^*(G)$ and $\chi = W^G \in \mathcal{U}\big(\widehat{C_r^*(G)} \otimes C_r^*(G)\big)$ is the reduced bicharacter of $C_r^*(G)$ viewed as a quantum group. In this case, we may view the coaction of $c_0(G)$ as an action of G on A (or X), and we will be able to describe most of the algebraic properties of $A \boxtimes_\chi B$ and $X \boxtimes_\chi Y$ entirely in terms of elementary tensors.

In this simplified setting, we will prove our main result: if $J_{X \boxtimes_\chi Y} = J_X \boxtimes_\chi J_Y$ then $\mathcal{O}_{X \boxtimes_\chi Y} \cong \mathcal{O}_X \boxtimes_\chi \mathcal{O}_Y$ (where $J_X = \phi^{-1}\big(\mathcal{K}(X)\big) \cap \big(\ker(\phi)\big)^\perp$ is the Katsura ideal). We will then apply this result to some specific examples.

5.1 Discrete Group Twisted Tensor Products

5.1.1 Basics

In what follows we will restrict our attention to the following special case of the twisted tensor product construction:

Definition 5.1.1. Suppose that G is a discrete group, (A, G, α) is a C^*-dynamical system, and (B, G, δ) is a coaction. Let $\delta^\alpha : A \to M\big(A \otimes c_0(G)\big)$ be the coaction of $c_0(G)$ (as a quantum group) on A associated to α as in Theorem 9.2.4 of [Timmermann(2008)]. We can view δ as a coaction of the quantum group $C_r^*(G)$ and we can form the twisted tensor product $A \boxtimes_{W_G} B$ where W_G is the multiplicative unitary of Example 2.3.13 viewed as a bicharacter from $c_0(G)$ to $C_r^*(G)$ (in other words the reduced bicharacter of the quantum group $C_r^*(G)$). We refer to this special case as a *discrete group twisted tensor product*. Since this construction depends only upon the action and coaction, we will sometimes write $A \ {}_\alpha\!\boxtimes_\delta B$ for $A \boxtimes_{W_G} B$. We can also define a C^*-algebra by $B \ {}_\delta\!\boxtimes_\alpha A = B \boxtimes_{\widehat{W_G}} A$. It is easy to see that the map $a \boxtimes b \mapsto b \boxtimes a$ extends to an isomorphism $A \ {}_\alpha\!\boxtimes_\delta B \cong B \ {}_\delta\!\boxtimes_\alpha A$.

The main reason that this special case is of interest is that we can write down a precise formula for the multiplication and involution of certain elementary tensors. To understand this, we must recall that the coaction δ of a discrete group G on B gives rise to a G-grading of B. That is, there exist subspaces $\{B_s\}_{s \in G}$ such that

1. $\overline{\operatorname{span}}(B_s) = B$
2. $B_s \cdot B_t \subseteq B_{st}$
3. $B_s^* = B_{s^{-1}}$.

Specifically, $B_s = \{b \in B : \delta(b) = b \otimes u_s\}$. With this in mind, we present the following:

Proposition 5.1.2. *Given a C^*-dynamical system (A, G, α) and a coaction (B, G, δ), let $a, a' \in A$, $b_s \in B_s$ and $b \in B$. Then, in the twisted tensor product $A\ {}_{\alpha}{\boxtimes}_{\delta}\ B$ we have:*

$$(a \boxtimes b_s)(a' \boxtimes b) = a\alpha_s(a') \boxtimes b_s b$$

$$(a \boxtimes b_s)^* = \alpha_{s^{-1}}(a)^* \boxtimes b_s^*$$

Before we prove this, we will need the following:

Lemma 5.1.3. *Let G be a locally compact group and let $m : C_0(G) \to \mathbb{B}\big(L^2(G)\big)$ be the left action of $C_0(G)$ on $L^2(G)$ by multiplication of functions in $L^2(G)$. Let $\lambda : C^*(G) \to \mathbb{B}\big(L^2(G)\big)$ be the left regular representation. Then (m, λ) is a W_G-Heisenberg pair where W_G is the reduced bicharacter of G as in Example* 2.3.16

Proof. This is a special case of Example 3.9 of [Meyer *et al.*(2014)]. □

Notation 5.1.4. If $A_1, \ldots A_n$ are C^*-algebras and $b \in A_i$, we write $(b)_i$ for the element of the multiplier algebra $M(A_1 \otimes \cdots \otimes A_n)$ thats action on elementary tensors is given by

$$a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes ba_i \otimes \cdots \otimes a_n.$$

Lemma 5.1.5. *In the situation of the above proposition, let*

$$i_A : A \to M\big(A \otimes B \otimes \mathbb{K}(L^2(G))\big)$$

$$i_B : B \to M\big(A \otimes B \otimes \mathbb{K}(L^2(G))\big)$$

be the maps associated to the Heisenberg pair (m, λ) as described in Definition 2.3.21. *Then for any $a \in A$, $s \in G$ and $b_s \in B_s$ we have*

$$i_B(b_s)i_A(a) = i_A\big(\alpha_{s^{-1}}(a)\big)i_B(b_s)$$

Proof. Recall that (m, λ) is a covariant homomorphism $(c_0(G), G, \sigma) \to \mathbb{B}(L^2(G))$ where sigma is left translation in $c_0(G)$. Also, $(\delta^\alpha, \sigma_2)$ is a covariant homomorphism $(A, G, \alpha) \to M\big(A \otimes c_0(G)\big)$. Thus $\big((\mathrm{id}_A \otimes m) \circ \delta^\alpha, \lambda_2\big)$ is a covariant homomorphism $(A, G, \alpha) \to M\big(A \otimes \mathbb{B}(L^2(G))\big)$. Thus

$$\lambda_2(s)^* \Big((\mathrm{id}_A \otimes m) \circ \delta^\alpha(a)\Big)\lambda_2(s) = (\mathrm{id}_A \otimes m) \circ \delta^\alpha\big(\alpha_s(a)\big)$$

or equivalently

$$\Big((\mathrm{id}_A \otimes m) \circ \delta^\alpha\big(\alpha_{s^{-1}}(a)\big)\Big)\lambda_2(s) = \lambda_2(s)\Big((\mathrm{id}_A \otimes m) \circ \delta^\alpha(a)\big)\Big)$$

With this in mind, we notice the following:

$$\begin{aligned}
i_B(b_s)i_A(a) &= (\mathrm{id}_B \otimes \lambda)\delta(b_s)_{23}(\mathrm{id}_A \otimes m)\delta^\alpha(a)_{13} \\
&= \big(1_A \otimes b_s \otimes \lambda(s)\big)(\mathrm{id}_A \otimes m)\delta^\alpha(a)_{13} \\
&= (b_s)_2\big(\lambda(s)\big)_3(\mathrm{id}_A \otimes m)\delta^\alpha(a)_{13} \\
&= (b_s)_2(\mathrm{id}_A \otimes m)\delta^\alpha\big(\alpha_{s^{-1}}(a)\big)_{13}\big(\lambda(s)\big)_3 \\
&= (\mathrm{id}_A \otimes m)\delta^\alpha\big(\alpha_{s^{-1}}(a)\big)_{13}(b_s)_2\big(\lambda(s)\big)_3 \\
&= i_A\big(\alpha_{s^{-1}}(a)\big)i_B(b_s)
\end{aligned}$$

□

We can now prove Proposition 5.1.2.

Proof. (of Proposition 5.1.2*)*

We have:

$$\begin{aligned}
(a \boxtimes b_s)(a' \boxtimes b) &= i_A(a)i_B(b_s)i_A(a')i_B(b) \\
&= i_A(a)i_A\big(\alpha_s(a')\big)i_B(b_s)i_B(b) \\
&= a\alpha_s(a') \boxtimes b_s b
\end{aligned}$$

and

$$\begin{aligned}(a \boxtimes b_s)^* &= \big(i_A(a) i_B(b_s)\big)^* \\ &= i_B(b_s^*) i_A(a^*) \\ &= i_A\big(\alpha_{s^{-1}}(a)^*\big) i_B(b_s^*) \\ &= \alpha_{s^{-1}}(a)^* \boxtimes b_s^*\end{aligned}$$

□

We also have simple formulas for the algebraic properties of twisted tensor products of correspondences:

Proposition 5.1.6. *Let (X, A) be a correspondence with an action (γ, α) of G and (Y, B) be a correspondence with a coaction (σ, δ) of G. Let $\alpha_{L(X)}$ be the action of G on $L(X)$ induced by the action of G on X and let $\delta_{L(Y)}$ be the coaction of G on $L(Y)$ induced by (σ, δ). We can form the correspondence $X \,{}_\gamma\!\boxtimes_\sigma Y := X \boxtimes_{W_G} Y \subseteq L(X) \boxtimes_{W_G} L(Y) = L(X) \,{}_{\alpha_{L(X)}}\!\boxtimes_{\delta_{L(Y)}} L(Y)$ as in Definition 2.3.23. Then*

1. $(a \boxtimes b_s)(x \boxtimes y) = a\gamma_s(x) \boxtimes b_s y$
2. $(x \boxtimes y_s)(a \boxtimes b) = x\alpha_s(a) \boxtimes y_s b$
3. $\langle x \boxtimes y_s, x' \boxtimes y\rangle_{A\boxtimes B} = \alpha_{s^{-1}}\big(\langle x, x'\rangle_A\big) \boxtimes \langle y_s, y\rangle_B$

Proof. All of these facts follow from translating to multiplication in $L(X)\boxtimes L(Y)$ and applying Lemma 5.1.5. □

Corollary 5.1.7. *Let (X, A) be a correspondence with an action (γ, α) of G and (Y, B) be a correspondence with a coaction (σ, δ) of G. Suppose (X^0, A^0) is a generating system for (X, A) which is stable with respect to the group action. That is, $\gamma_s(X^0) \subseteq X^0$ and $\alpha_s(A^0) \subseteq A^0$ for all $s \in G$. Suppose further that (Y^0, B^0) is a* *generating system for (Y,B) such that the elements of Y^0 and B^0 are homogeneous with respect to the grading. That is, for all $y \in Y^0$ there is $s \in G$ such that $y \in Y_s$ and for all $b \in B^0$ there is $t \in G$ such that $b \in B_t$. Then*

$$(X \boxtimes Y)_0 := \{x \boxtimes y : x \in X^0, y \in Y^0\}$$
$$(A \boxtimes B)_0 := \{a \boxtimes b : a \in A^0, b \in B^0\}$$

form a generating system for $X \boxtimes Y$.

Proof. It is clear from the bilinearity of the twisted tensor product that $\overline{\operatorname{span}}\big((X \boxtimes Y)_0\big) = X \boxtimes Y$ and $\overline{\operatorname{span}}\big((A \boxtimes B)_0\big) = A \boxtimes B$. To see that $(X \boxtimes Y)_0$ is stable under the left and right actions of elements of $(A \boxtimes B)_0$, let $x \boxtimes y \in (X \boxtimes Y)_0$ and let $a \boxtimes b \in (A \boxtimes B)_0$. By definition, we must have that $b \in B_s$ and $y \in Y_t$ for some $s, t \in G$. Thus we have:

$$(a \boxtimes b)(x \boxtimes y) = a\gamma_s(x) \boxtimes by$$
$$(x \boxtimes y)(a \boxtimes b) = x\alpha_t(a) \boxtimes yb$$

Since X^0 and A^0 are stable under the action of G, $\gamma_s(x) \in X^0$ and $\alpha_t(a) \in A^0$. Thus $a\gamma_s(x), \gamma_t(a)x \in X^0$ so $(X \boxtimes Y)_0$ is indeed stable under the left and right actions of $(A \boxtimes B)_0$. $\square$

5.1.2 Examples

In [Meyer *et al.*(2014)], the authors show that if A and B are $\mathbb{Z}_2$-graded algebras, then the graded tensor product $A \widehat{\otimes} B$ is isomorphic $A \boxtimes_{W^{\mathbb{Z}_2}} B$ (where the coactions on A and B are the ones canonically associated with the grading) with the map $a \widehat{\otimes} b \mapsto a \boxtimes b$ extending to an isomorphism. Since $\mathbb{Z}_2$ is self-dual, the coaction of $\mathbb{Z}_2$ on A gives rise to an action α of $\mathbb{Z}_2$ on A. If we let δ denote the coaction of $\mathbb{Z}_2$ on $\mathbb{Z}_2$ we see that $A \boxtimes_{W^{\mathbb{Z}_2}} B = A \; {}_{\alpha}{\boxtimes}_{\delta} \; B$ so $A \widehat{\otimes} B$ fits into our discrete group twisted tensor framework. The following example shows that the graded external tensor product of graded correspondences also fits into our framework.

Example 5.1.8. Let A and B be $\mathbb{Z}_2$-graded C^* algebras and let X and Y be $\mathbb{Z}_2$-graded correspondences over A and B (i.e. X and Y are graded as Hilbert A- and B-modules respectively and the left action maps ϕ_X and ϕ_Y are graded with respect to the induced gradings on $\mathcal{L}(X)$ and $\mathcal{L}(Y)$). Let $X^0 = X_0 \cup X_1$, $A^0 = A_0 \cup A_1$, $Y^0 = Y_0 \cup Y_1$ and $B^0 = B_0 \cup B_1$. Then by Corollary 2.3.9 (X^0, A^0) and (Y^0, B^0) are generating sets for X and Y respectively. Consider the graded external tensor product $X \widehat{\otimes} Y$. This is the closure of the algebraic tensor product $X \odot Y$ with respect to the norm associated to the inner product whose value on generators is given by:

$$\langle x_1 \widehat{\otimes} y_1, x_2 \widehat{\otimes} y_2 \rangle = (-1)^{\partial y_1(\partial x_1 + \partial x_2)} \langle x_1, x_2 \rangle \widehat{\otimes} \langle y_1, y_2 \rangle$$

where $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$. The left and right actions are given by:

$$(a \widehat{\otimes} b)(x \widehat{\otimes} y) = (-1)^{\partial b \partial x}(ax \widehat{\otimes} by)$$
$$(x \widehat{\otimes} y)(a \widehat{\otimes} b) = (-1)^{\partial y \partial a}(xa \widehat{\otimes} yb)$$

for $a \in A_0$, $b \in B_0$, $x \in X_0$, and $y \in Y_0$. Thus

$$(A \widehat{\otimes} B)_0 := \{a \widehat{\otimes} b : a \in A_0, b \in B_0\}$$
$$(X \widehat{\otimes} Y)_0 := \{x \widehat{\otimes} y : x \in X_0, y \in Y_0\},$$

is a generating system for $X \widehat{\otimes} Y$. Now, let (γ, α) be the action of $\mathbb{Z}_2$ on (X, A) associated to the grading of X and let (σ, δ) be the coaction of $\mathbb{Z}_2$ on (Y, B) associated to the grading of Y. Consider the associated twisted tensor product $X \boxtimes Y$. Note that the sets X_0 and A_0 are stable under the actions γ and α and that Y_0 and B_0 consist of elements which are homogeneous with respect to the gradings associated to σ and δ. Thus, by Corollary 5.1.7 the sets

$$(X \boxtimes Y)_0 := \{x \boxtimes y : x \in X_0, y \in Y_0\}$$

$$(A \boxtimes B)_0 := \{a \boxtimes b : a \in A_0, b \in B_0\}$$

form a generating system for $X \boxtimes Y$. Let $\Phi_0 : (X \widehat{\otimes} Y)_0 \to (X \boxtimes Y)_0$ be the map $x \widehat{\otimes} y \mapsto x \boxtimes y$. This is clearly a bijection. Let $\varphi : A \widehat{\otimes} B \to A\,{}_\alpha\!\boxtimes_\delta B$ be the isomorphism described above. For $a \in A_0$, $b \in B_0$, $x \in X_0$, and $y \in Y_0$, we have:

$$\begin{aligned}
\Phi_0\big((a \widehat{\otimes} b)(x \widehat{\otimes} y)\big) &= (-1)^{\partial b \partial x} \Phi_0(ax \widehat{\otimes} by) \\
&= (-1)^{\partial b \partial x}(ax \boxtimes by) \\
&= a\gamma_{\partial b}(x) \boxtimes by \\
&= (a \boxtimes b)(x \boxtimes y) \\
&= \varphi(a \widehat{\otimes} b)\Phi_0(x \widehat{\otimes} y)
\end{aligned}$$

and similarly

$$\Phi_0\big((x \widehat{\otimes} y)(a \widehat{\otimes} b)\big) = \Phi_0(x \widehat{\otimes} y)\varphi(a \widehat{\otimes} b)$$

therefore (Φ_0, φ) preserves the left and right actions. Additionally, for $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$, we have that:

$$\begin{aligned}
\big\langle \Phi_0(x_1 \widehat{\otimes} y_1), \Phi_0(x_2 \widehat{\otimes} y_2) \big\rangle &= \langle x_1 \boxtimes y_1, x_2 \boxtimes y_2 \rangle \\
&= \alpha_{\partial y_1}\big(\langle x_1, x_2 \rangle\big) \boxtimes \langle y_1, y_2 \rangle\big) \\
&= (-1)^{\partial y_1(\partial x_1 + \partial x_2)}\big(\langle x_1, x_2 \rangle \boxtimes \langle y_1, y_2 \rangle\big) \\
&= (-1)^{\partial y_1(\partial x_1 + \partial x_2)}\varphi\big(\langle x_1, x_2 \rangle \widehat{\otimes} \langle y_1, y_2 \rangle\big) \\
&= \varphi\big(\langle x_1 \widehat{\otimes} x_2, y_1 \widehat{\otimes} y_2 \rangle\big)
\end{aligned}$$

therefore, by Lemma 2.1.10, Φ_0 extends to an isomorphism $\Phi : X \widehat{\otimes} Y \to X \boxtimes Y$. Thus $X \widehat{\otimes} Y \cong X \boxtimes Y$.

The following example can be viewed as a generalization of the skew graph construction presented in Chapter 6 of [Raeburn(2005)].

Example 5.1.9. Let $E = \{E^0, E^1, r_E, s_E\}$ and $F = \{F^0, F^1, r_F, s_F\}$ be directed graphs and let G be a discrete group. Let $A := c_0(E^0)$ and let $B := c_0(F^0)$. Let α^E be an action of G on E by graph automorphisms and let δ be a G-labeling of F, i.e. a map $\delta : F^1 \to G$. It is easy to see that the maps $f \mapsto f \circ \alpha^E_s$ on A together with the maps $x \mapsto x \circ \alpha^G_s$ on $c_c(E^1)$ give rise to group homomorphisms $G \to \mathrm{Aut}(A)$ and $G \to \mathrm{Aut}\big(c_c(E^1)\big)$ which in turn give rise to a group action (γ, α) on $(X(E), A)$. We also get a coaction σ of G on $X(F)$ from δ. To see this, recall that $c_c(F^1)$ is generated by the characteristic functions χ_e and define $\sigma(\chi_f) := \chi_f \otimes u_{\delta(f)}$. So (σ, ι) is a coaction on $\big(X(F), B\big)$ where ι is the trivial coaction on B. We define a new directed graph $E\ {}_{\alpha^E}\!\times_\delta F := \{E^0 \times F^0, E^1 \times F^1, r, s\}$ where $s(e \times f) = \alpha^E_{\delta(f)}\big(s_E(e)\big) \times s_F(f)$ and $r(e \times f) = r_E(e) \times r_F(f)$. We define $C := c_0(E^0 \times F^0) \cong A \otimes B$. We will show that $X\big(E\ {}_{\alpha^E}\!\times_\delta F\big) \cong X(E)\ {}_\gamma\boxtimes_\sigma X(F)$.

Let $X(E)^0$ be the set of characteristic functions on E^1 and let A^0 be the set of characteristic functions on E^0. The characteristic functions densely span $c_c(E^1)$ which is dense in $X(E)$. For $\chi_v \in A^0$ and $\chi_e \in X(E)^0$ we have that

$$\chi_v \cdot \chi_e = \begin{cases} \chi_e & \text{if } r(e) = v \\ 0 & \text{otherwise} \end{cases} \in X(E)^0$$

$$\chi_e \cdot \chi_v = \begin{cases} \chi_e & \text{if } s(e) = v \\ 0 & \text{otherwise} \end{cases} \in X(E)^0$$

$$\langle \chi_e, \chi_{e'} \rangle_A = \begin{cases} \chi_{s(e)} & \text{if } e = e' \\ 0 & \text{otherwise} \end{cases}$$

therefore $\big(X(E)^0, A^0\big)$ is a generating system for $X(E)$. We define $\big(X(F)^0, B^0\big)$ and $\big(X\big(E\ {}_{\alpha^E}\!\times_\delta F\big)^0, C^0\big)$ is an analogous way, and we see that they are generating systems for $X(F)$ and $X(E\ {}_{\alpha^E}\!\times_\delta F)$ respectively. Further, from the definition of the coaction

σ we have that all characteristic functions on F^1 are homogeneous with respect to the grading induced by σ and since ι is trivial, the grading it induces is also trivial so A^0 is trivially homogeneous. Also, the actions α and γ take generating functions to generating functions: $\alpha_s : \chi_v \to \chi_{\alpha^E_{s^{-1}}(v)}$, $\gamma_s : \chi_e \to \chi_{\alpha^E_{s^{-1}}(e)}$ so the sets $X(E)^0$ and A^0 are fixed by the actions. This allows us to apply Corollary 5.1.7 and deduce that the sets

$$\begin{aligned}\big(X(E) \boxtimes X(F)\big)^0 &= \{\chi_e \boxtimes \chi_f : e \in E^1, f \in F^1\}\\ (A \otimes B)^0 &= \{\chi_v \otimes \chi_w : v \in E^0, w \in F^0\}\end{aligned}$$

form a generating system for the twisted tensor product $X(E)\ {}_\gamma\!\boxtimes_\sigma X(F)$.

Let $\varphi : C \to A \otimes B$ be the canonical map and let $\Phi_0 : X\big(E\ {}_{\alpha^E}\!\times_\delta F\big)^0 \to \big(X(E) \boxtimes X(F)\big)^0$ be the map $\chi_{e\times f} \mapsto \chi_e \boxtimes \chi_f$. Clearly Φ_0 is bijective, we wish to show that it preserves the inner product and left and right actions. Note that

$$\begin{aligned}
\varphi\big(\langle \chi_{e\times f}, \chi_{e'\times f'}\rangle_C\big) &= \begin{cases} \varphi\big(\chi_{s(e\times f)}\big) & \text{if } e \times f = e' \times f' \\ \phi(0) & \text{otherwise} \end{cases} \\
&= \begin{cases} \chi_{\alpha^E_{\delta(f)}(s_E(e))} \otimes \chi_{s_F(f)} & \text{if } e = e' \text{ and } f = f' \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \alpha_{\delta(f)^{-1}}\big(\chi_{s_E(e)}\big) \otimes \chi_{s_F(f)} & \text{if } e = e' \text{ and } f = f' \\ 0 & \text{otherwise} \end{cases} \\
&= \alpha_{\delta(f)^{-1}}\big(\langle \chi_e, \chi_{e'}\rangle_A\big) \boxtimes \langle \chi_f, \chi_{f'}\rangle_B \\
&= \big\langle \chi_e \boxtimes \chi_f, \chi_{e'} \boxtimes \chi_{f'}\big\rangle_{A\otimes B} \\
&= \big\langle \Phi_0(\chi_{e\times f}), \Phi_0(\chi_{e'\times f'})\big\rangle_{A\otimes B}
\end{aligned}$$

and

$$\Phi_0(\chi_{v\times w}\cdot\chi_{e\times f}) = \begin{cases} \Phi_0(\chi_{e\times f}) & \text{if } r(e\times f) = v\times w \\ \Phi_0(0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \chi_e \boxtimes \chi_f & \text{if } r_E(e) = v \text{ and } r_F(f) = w \\ 0 & \text{otherwise} \end{cases}$$

$$= \big(\chi_v \cdot \chi_e\big) \boxtimes (\chi_w \cdot \chi_f)$$

$$= \big(\chi_v \otimes \chi_w\big)\big(\chi_e \boxtimes \chi_f\big)$$

$$= \varphi(\chi_{v\times w})\Phi_0(\chi_{e\times f})$$

and finally

$$\Phi_0(\chi_{e\times f}\cdot\chi_{v\times w}) = \begin{cases} \Phi_0(\chi_{e\times f}) & \text{if } s(e\times f) = v\times w \\ \Phi_0(0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \chi_e \boxtimes \chi_f & \text{if } \alpha^E_{\delta(f)}(s_E(e)) = v \text{ and } s_F(f) = w \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \chi_e \boxtimes \chi_f & \text{if } s_E(e) = \alpha^E_{\delta(f)^{-1}}(v) \text{ and } s_F(f) = w \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_e \cdot \chi_{\alpha^E_{\delta(f)^{-1}}(v)} \boxtimes \chi_f \cdot \chi_w$$

$$= \chi_e \cdot \alpha_{\delta(f)}(\chi_v) \boxtimes \chi_f \cdot \chi_w$$

$$= (\chi_e \boxtimes \chi_f)(\chi_v \boxtimes \chi_w)$$

$$= \Phi_0(\chi_{e\times f})\varphi(\chi_{v\times w})$$

Therefore, by Lemma 2.1.10 we have that Φ_0 extends to a correspondence isomorphism $\Phi : X(E\ {}_{\alpha^E}\!\times_\delta F) \to X(E)\ {}_\gamma\!\boxtimes_\sigma X(F)$.

Example 5.1.10. In this example we will show that the crossed product of a correspondence by an action of a discrete group can be viewed as a discrete group twisted tensor product of correspondences. Suppose (γ, α) is and action of a discrete group G on a C^*-correspondence (X, A). We wish to show that the reduced crossed product $X \rtimes_{\gamma,r} G$ is isomorphic to the twisted tensor product $X \,{}_{\gamma}\boxtimes_{\delta_G} C_r^*(G)$ where we view $C_r^*(G)$ as a correspondence over itself.

Recall from Corollary 2.3.3 that the sets

$$(X \rtimes_{\gamma,r} G)_0 := \{i_X(x)i_G^X(s) : x \in X, s \in G\}$$
$$(A \rtimes_{\alpha,r} G)_0 := \{i_A(a)i_G^A(s) : a \in A, s \in G\}$$

form a generating system for $X \rtimes_{\gamma,r} G$. Let $C_r^*(G)_0 = \{u_s : s \in G\}$, i.e. the image of G in $C_r^*(G)$. Since this set is closed under multiplication, we may regard $(C_r^*(G)_0, C_r^*(G)_0)$ as a generating system for the correspondence $C_r^*(G)$. Furthermore, every element of $C_r^*(G)_0$ is homogeneous with respect to the grading arising from $\delta_G : u_s \mapsto u_s \otimes u_s$. Also, we may view (X, A) as a generating system for itself and then by Corollary 5.1.7 we see that the sets

$$(X \,{}_{\gamma}\boxtimes_{\delta_G} C_r^*(G))_0 : = \{x \boxtimes u_s : x \in X, s \in G\}$$
$$(A \,{}_{\alpha}\boxtimes_{\delta_G} C_r^*(G))_0 : = \{a \boxtimes u_s : a \in A, s \in G\}$$

form a generating system for $X \,{}_{\gamma}\boxtimes_{\delta_G} C_r^*(G)$. We let φ be the isomorphism $A \rtimes_{\alpha,r} G \to A \,{}_{\alpha}\boxtimes_{\delta_G} C_r^*(G)$ and define $\Phi_0 : (X \rtimes_{\gamma,r} G)_0 \to (X \,{}_{\gamma}\boxtimes_{\delta_G} C_r^*(G))_0$ to be the map $i_X(x)i_G^X(s) \mapsto x \boxtimes u_s$. This is clearly a bijection, but we need to establish that it preserves the inner product and left and right actions. First, note that

$$\begin{aligned}\left\langle \Phi_0(i_X(x)i_G^X(s)), \Phi_0(i_X(y)i_G^X(t)) \right\rangle &= \langle x \boxtimes u_s, y \boxtimes u_t \rangle \\ &= \alpha_{s^{-1}}\big(\langle x, y \rangle\big) \boxtimes \langle u_s, u_t \rangle \\ &= \alpha_{s^{-1}}\big(\langle x, y \rangle\big) \boxtimes u_{s^{-1}t}\end{aligned}$$

$$= \varphi\Big(i_A\big(\alpha_{s^{-1}}\big(\langle x, y\rangle\big)\big)i_G^A(s^{-1}t)\Big)$$

$$= \varphi\Big(\big\langle i_X(x)i_G^X(s), i_X(y)i_G^X(t)\big\rangle\Big)$$

also,

$$\begin{aligned}
\Phi_0\Big(\big(i_X(x)i_G^X(s)\big)\big(i_A(a)i_G^A(t)\big)\Big) &= \Phi_0\big(i_X\big(x\alpha_s(a)\big)i_G^X(st)\big) \\
&= x\alpha_s(a) \boxtimes u_{st} \\
&= x\alpha_s(a) \boxtimes u_s u_t \\
&= (x \boxtimes u_s)(a \boxtimes u_t) \\
&= \Phi_0\big(i_X(x)i_G^X(s)\big)\varphi\big(i_A(a)i_G^A(t)\big)
\end{aligned}$$

and finally,

$$\begin{aligned}
\Phi_0\Big(\big(i_A(a)i_G^A(s)\big)\big(i_X(x)i_G^X(t)\big)\Big) &= \Phi_0\big(i_X\big(a\gamma_s(x)\big)i_G^X(st)\big) \\
&= a\gamma_s(x) \boxtimes u_{st} \\
&= a\gamma_s(x) \boxtimes u_s u_t \\
&= (a \boxtimes u_s)(x \boxtimes u_t) \\
&= \varphi\big(i_A(a)i_G^A(s)\big)\Phi_0\big(i_X(x)i_G^X(t)\big)
\end{aligned}$$

Therefore, by Lemma 2.1.10, we have that Φ_0 extends to a correspondence isomorphism $\Phi : X \rtimes_{\gamma,r} G \to X \;{}_{\gamma}\!\boxtimes_{\delta_G} C_r^*(G)$.

Example 5.1.11. In this example we will see that crossed products by coactions on correspondences can also be viewed as twisted tensor products. Specifically, if (σ, δ) is a coaction of a discrete group G on a correspondence (X, A), then we wish to show that $X \rtimes_\sigma G$ is isomorphic to $X \;{}_{\sigma}\!\boxtimes_{\lambda} c_0(G)$ where we are viewing $c_0(G)$ as a correspondence over itself. To see this, recall from Corollary 2.3.9 that the sets

$$(X \rtimes_\sigma G)_0 := \{j_X(x_s)j_G^X(f) : x_s \in X_s, f \in c_0(G)\}$$

$$(A \rtimes_\delta G)_0 := \{j_A(a_s)j_G^A(f) : a_s \in A_s, g \in c_0(G)\}$$

form a generating system for $X \rtimes_\sigma G$. Let $X^0 := \bigcup_{s\in G} X_s$ and let $A^0 := \bigcup_{s\in G} A_s$. The properties of the grading tell us that X^0 and A^0 densely span X and A and that $ax, xa \in X^0$ whenever $a \in A^0$ and $x \in X^0$. Thus (X^0, A^0) is a generating system for X which by definition consists of elements which are homogeneous with respect to the gradings of X and A. Viewing $(c_0(G), c_0(G))$ as a generating system for the correspondence $c_0(G)$, we may apply Corollary 5.1.7 and deduce that the sets

$$(X \,{}_\sigma\!\boxtimes_\lambda c_0(G))_0 := \{x_s \boxtimes f : x_s \in X^0, f \in c_0(G)\}$$
$$(A \,{}_\alpha\!\boxtimes_\lambda c_0(G))_0 := \{a_s \boxtimes f : a_s \in A^0, f \in c_0(G)\}$$

form a generating system for the twisted tensor product $X \,{}_\sigma\!\boxtimes_\lambda c_0(G)$. Let φ be the isomorphism $A \rtimes_\delta G \to A \,{}_\delta\!\boxtimes_\lambda c_0(G)$. We define $\Phi_0 : (X \rtimes_\sigma G)_0 \to (X \,{}_\sigma\!\boxtimes_\lambda c_0(G))_0$ to be the map $j_X(x_s)j_G^X(f) \mapsto x_s \boxtimes f$. This is clearly bijective, but we must show that it preserves the inner product and left and right actions. To see this, suppose $x_s \in X_s \subseteq X^0$ and $x_t \in X_t \subseteq X^0$ and note that

$$\begin{aligned}
\big\langle \Phi_0\big((j_X(x_s)j_G^X(f)\big), \Phi_0\big((j_X(x_t)j_G^X(g)\big)\big\rangle &= \langle x_s \boxtimes f, x_t \boxtimes g\rangle \\
&= \langle x_s, x_t\rangle \boxtimes \lambda_{s^{-1}}\big(\langle f, g\rangle\big) \\
&= \langle x_s, x_t\rangle \boxtimes \lambda_{s^{-1}}(\overline{f}g) \\
&= \varphi\Big(j_A\big(\langle x_s, x_t\rangle\big)j_A^G\big(\lambda_{s^{-1}}(\overline{f}g)\big)\Big) \\
&= \varphi\Big(\big\langle j_X(x_s)j_G^X(f), j_X(x_t)j_G^X(g)\big\rangle\Big)
\end{aligned}$$

furthermore, if $a_t \in A_t \subseteq A^0$ we have that

$$\begin{aligned}
\Phi_0\Big(\big(j_X(x_s)j_G^X(f)\big)\big(j_X(a_t)j_G^X(g)\big)\Big) &= \Phi_0\Big(j_X(x_s a_t)j_G^X\big(\lambda_t(f)g\big)\Big) \\
&= x_s a_t \boxtimes \lambda_t(f)g
\end{aligned}$$

$$= (x_s \boxtimes f)(x_t \boxtimes g)$$

$$= \Phi_0\big(j_X(x_s)j_G^X(f)\big)\varphi\big(j_A(a_t)j_G^A(g)\big)$$

and

$$\begin{aligned}
\Phi_0\Big(\big(j_A(a_t)j_G^A(f)\big)\big(j_X(x_s)j_G^X(g)\big)\Big) &= \Phi_0\Big(j_X(a_tx_s)j_G^X\big(\lambda_s(f)g\big)\Big) \\
&= a_tx_s \boxtimes \lambda_s(f)g \\
&= (a_t \boxtimes f)(x_s \boxtimes g) \\
&= \varphi\big(j_A(a_t)j_G^A(f)\big)\Phi_0\big(j_X(x_s)j_G^X(g)\big)
\end{aligned}$$

Therefore, by Lemma 2.1.10, we have that Φ_0 extends to a correspondence isomorphism $\Phi : X \rtimes_\sigma G \to X \,{}_\sigma\!\boxtimes_\lambda c_0(G)$.

5.2 Balanced Twisted Tensor Products

Throughout this section, G will be a discrete group, Z will be a compact abelian group, (A, G, α), (A, Z, μ) and (B, Z, ν) will be dynamical systems and (B, G, δ) will be a coaction such that μ commutes with α and ν is covariant with respect to δ.

Proposition 5.2.1. *$(A \,{}_\alpha\!\boxtimes_\delta B, Z, \lambda)$ is a dynamical system where*

$$\lambda_z = \mu_z \boxtimes \nu_{z^{-1}}$$

Proof. Since μ commutes with α and ν is covariant with respect to δ, we know that each map λ_z is an automorphism. To show that $z \mapsto \lambda_z$ is a group homomorphism, note that

$$\begin{aligned}
\lambda_z \circ \lambda_w(a \boxtimes b) &= \lambda_z\big(\mu_w(a) \boxtimes \nu_{w^{-1}}(b)\big) \\
&= \mu_z \circ \mu_w(a) \boxtimes \nu_{z^{-1}} \circ \nu_{w^{-1}}(b) \\
&= \mu_{zw}(a) \boxtimes \nu_{z^{-1}w^{-1}}(b)
\end{aligned}$$

$$
\begin{aligned}
&= \mu_{zw}(a) \boxtimes \nu_{w^{-1}z^{-1}}(b) \\
&= \mu_{zw}(a) \boxtimes \nu_{(zw)^{-1}}(b) \\
&= \lambda_{zw}(a \boxtimes b)
\end{aligned}
$$

□

Definition 5.2.2. We call the fixed point algebra $(A \boxtimes B)^\lambda$ of the above action the *Z-balanced twisted tensor product* of A and B and we denote it by $A \boxtimes_Z B$.

Proposition 5.2.3. *Let $a \boxtimes b \in A \boxtimes_Z B$. Then $\mu_z(a) \boxtimes b, a \boxtimes \nu_z(b) \in A \boxtimes_Z B$ for all $z \in Z$ and, in fact, $\mu_z(a) \boxtimes b = a \boxtimes \nu_z(b)$ for all $z \in Z$.*

Proof. Let $w \in Z$. Note that

$$
\begin{aligned}
\lambda_w\big(\mu_z(a) \boxtimes b\big) &= \mu_w \circ \mu_z(a) \boxtimes \nu_{w^{-1}}(b) \\
&= \mu_z \circ \mu_w(a) \boxtimes \nu_{w^{-1}}(b) \\
&= (\mu_z \boxtimes \mathrm{id}_B)\big(\lambda_w(a \boxtimes b)\big) \\
&= (\mu_z \boxtimes \mathrm{id}_B)(a \boxtimes b) \\
&= \mu_z(a) \boxtimes b
\end{aligned}
$$

Thus $\mu_z(a) \boxtimes b$ is fixed under the action of λ and is therefore an element of $A \boxtimes_Z B$. Showing that $a \boxtimes \nu_z(b) \in A \boxtimes_Z B$ is similar. To show that these are actually equivalent, notice that

$$
\begin{aligned}
\mu_z(a) \boxtimes b &= \lambda_{z^{-1}}\big(\mu_z(a) \boxtimes b\big) \\
&= \mu_{z^{-1}} \circ \mu_z(a) \boxtimes \nu_z(b) \\
&= a \boxtimes \nu_z(b)
\end{aligned}
$$

□

Proposition 5.2.4. *Keeping the above conventions, let δ^μ and δ^ν be the dual coactions. Since Z is compact, $\widehat{Z}$ is discrete and thus δ^μ and δ^ν give gradings $\{A_\chi\}_{\chi\in\widehat{Z}}$ and $\{B_\chi\}_{\chi\in\widehat{Z}}$ of A and B. For each $\chi \in \widehat{Z}$ let*

$$S_\chi := \{a \boxtimes b : a \in A_\chi, b \in B_\chi\}$$

and let $S := \bigcup_{\chi\in\widehat{Z}} S_\chi$. Then $A \boxtimes_Z B = \overline{\operatorname{span}}(S)$.

Proof. First we will show that $\overline{\operatorname{span}}(S) \subseteq A \boxtimes_Z B$. Let $a \boxtimes b \in S$. Then $a \boxtimes b \in S_\chi$ for some $\chi \in \widehat{Z}$. Thus for all $z \in Z$ we have that

$$\begin{aligned} \lambda_z(a \boxtimes b) &= \mu_z(a) \boxtimes \nu_{z^{-1}}(b) \\ &= \chi(z)a \boxtimes \chi(z^{-1})b \\ &= \chi(z)\chi(z^{-1})(a \boxtimes b) \\ &= a \boxtimes b \end{aligned}$$

thus $S \subseteq A \boxtimes_Z B$ and therefore $\overline{\operatorname{span}}(S) \subseteq A \boxtimes_Z B$.

To show the reverse inclusion, let $c \in A \boxtimes_Z B$. Then $c \approx \sum a_i \boxtimes b_i$. Since the subspaces $\{A_\chi\}_{\chi\in\widehat{Z}}$ span A and the subspaces $\{B_\chi\}_{\chi\in\widehat{Z}}$ span B, we may assume without loss of generality that there are $\chi_i, \chi_i' \in \widehat{Z}$ such that $a_i \in A_{\chi_i}$ and $b_i \in B_{\chi_i'}$ for each i. Let $\Upsilon : A \boxtimes B \to A \boxtimes_{\mathbb{T}} B$ be the conditional expectation $d \mapsto \int_Z \lambda_z(d)dz$. Since c is assumed to be in $A \boxtimes_{\mathbb{T}} B$, we have that $c = \Upsilon(c)$. Thus, using the continuity and linearity of Υ we have the following

$$\begin{aligned} c &= \Upsilon(c) \\ &= \int_Z \lambda_z(c)dz \\ &\approx \int_Z \lambda_z\Big(\sum_i a_{\chi_i} \boxtimes b_{\chi'_i}\Big)dz \\ &= \sum_i \int_Z \lambda_z(a_{\chi_i} \boxtimes b_{\chi'_i})dz \end{aligned}$$

$$= \sum_i \int_Z \mu_z(a_{\chi_i}) \boxtimes \nu_{z^{-1}}(b_{\chi'_i}) dz$$
$$= \sum_i \int_Z \big(\chi_i(z) a_{\chi_i}\big) \boxtimes \big(\chi'_i(z^{-1}) b_{\chi'_i}\big) dz$$
$$= \sum_i \big(a_{\chi_i} \boxtimes b_{\chi'_i}\big) \int_Z \chi_i(z)\chi'_i(z^{-1}) dz$$

But the integral $\int_Z \chi_i(z)\chi'_i(z^{-1})dz$ is equal to 1 if $\chi_i = \chi'_i$ and zero otherwise (this is a consequence of the Peter-Weyl theorem). Thus c can be approximated by a sum of elements from S:

$$c \approx \sum_i a_{\chi_i} \boxtimes b_{\chi_i}$$

therefore $A \boxtimes_{\mathbb{T}} B \subseteq \overline{\operatorname{span}}(S)$ so $A \boxtimes_{\mathbb{T}} B = \overline{\operatorname{span}}(S)$. □

Proposition 5.2.5. *Let $\{S_\chi\}_{\chi \in \widehat{Z}}$ be as above. This defines a coaction of $\widehat{Z}$ on $A \boxtimes_Z B$ and thus also an action γ of Z on $A \boxtimes_Z B$. We have that $\gamma = \mu \boxtimes \mathrm{id}_B = \mathrm{id}_A \boxtimes \nu$.*

Proof. Let $c \in A \boxtimes_{\mathbb{T}} B$. By the previous proposition we can approximate $c \approx \sum_i a_{\chi_i} \boxtimes b_{\chi_i}$. Then, by continuity and linearity

$$\gamma_z(c) \approx \sum_i \gamma_z\big(a_{\chi_i} \boxtimes b_{\chi_i}\big)$$
$$= \sum_i \chi_i(z)\big(a_{\chi_i} \boxtimes b_{\chi_i}\big)$$
$$= \sum_i \big(\chi_i(z) a_{\chi_i}\big) \boxtimes b_{\chi_i}$$
$$= \sum_i \mu_z(a_{\chi_i}) \boxtimes b_{\chi_i}$$

Thus $\gamma = \mu \boxtimes \mathrm{id}_B$. Showing that $\gamma = \mathrm{id}_A \boxtimes \nu$ is similar. □

Lemma 5.2.6. *Suppose $\{A_\chi\}_{\chi \in \widehat{Z}}$ and $\{B_\chi\}_{\chi \in \widehat{Z}}$ are saturated gradings of A and B, that is $A_\chi A_\omega = A_{\chi\omega}$. Then $\{S_\chi\}_{\chi \in \widehat{Z}}$ (as described above) is saturated.*

Proof. We already have that $S_\chi S_\omega \subseteq S_{\chi\omega}$ for all $\chi, \omega \in \widehat{Z}$ so it will suffice to show the reverse inclusion. Let $a \boxtimes b \in S_{\chi\omega}$. Then $a \in A_{\chi\omega}$ and $b \in B_{\chi\omega}$. Since the gradings of A and B are saturated, we have that $a \approx \sum_i a_{\chi,i}a_{\omega,i}$ and $b \approx \sum_j b_{\chi,j}b_{\omega,j}$ with $a_{\chi,i} \in A_\chi$, $a_{\omega,i} \in A_\omega$, $b_{\chi,j} \in B_\chi$ and $b_{\omega,j} \in B_\omega$. Thus

$$\begin{aligned}
a \boxtimes b &= \sum_{i,j} a_{\chi,i}a_{\omega,i} \boxtimes b_{\chi,j}b_{\omega,j} \\
&= \sum_{i,j} \big(a_{\chi,i} \boxtimes b_{\chi,i}\big)\big(a_{\omega,i} \boxtimes b_{\omega,j}\big) \\
&\subseteq S_\chi \boxtimes S_\omega
\end{aligned}$$

□

5.3 Ideal Compatibility

Before we state our main result, we need to define two technical conditions involving the Katsura ideals. We call these conditions Katsura non-degeneracy and ideal compatibility. These conditions are basically the same as those presented in the author's first paper [Morgan(2015)], only the definition of ideal compatibility must be modified to allow twisted tensor products instead of ordinary ones.

Definition 5.3.1. Let (X, A) be a correspondence and let J_X be the Katsura ideal of A (recall that the Katsura ideal is the ideal $J_X = \phi^{-1}\big(\mathcal{K}(X)\big) \cap \big(ker(\phi)\big)^\perp$). We say that X is *Katsura non-degenerate* if $X \cdot J_X = X$.

Example 5.3.2. Let X be a correspondence over a C^*-algebra A such that the left action is injective and implemented by compacts. In this case we have that $J_X = A$. Thus:

$$\begin{aligned}
X \cdot J_X &= X \cdot A \\
&= X
\end{aligned}$$

Definition 5.3.3. Recall that a vertex in a directed graph is called a *source* if it receives no edges. We will call such a vertex a *proper source* if it emits at least one edge.

We now turn our attention to ideal compatibility:

Definition 5.3.4. Given two correspondences (X, A) and (Y, B), an action (γ, α) of a discrete group G on X and a coaction (σ, δ) of G on Y, we say that X and Y are *ideal compatible* with respect to γ and σ if $J_X \,{}_\alpha\boxtimes_\delta\, J_Y = J_{X\,{}_\gamma\boxtimes_\sigma Y}$ where J_X, J_Y and $J_{X\,{}_\gamma\boxtimes_\sigma Y}$ are the Katsura ideals of X, Y, and $X \,{}_\gamma\boxtimes_\sigma\, Y$ respectively.

Once again, this condition will be met in the case of injective left actions implemented by compacts:

Proposition 5.3.5. *Suppose we have correspondences (X, A) and (Y, B), an action (γ, α) of a discrete group G on X and a coaction (σ, δ) of G on Y. If the left actions of A on X and B on Y are injective and implemented by compacts, then X and Y are ideal compatible.*

Proof. In this case the Katsura ideals are the whole algebras: $J_X = A$, $J_Y = B$ and $J_{X\,{}_\gamma\boxtimes_\sigma Y} = A \,{}_\alpha\boxtimes_\delta\, B$. Thus the equation $J_X \,{}_\alpha\boxtimes_\delta\, J_Y = J_{X\,{}_\gamma\boxtimes_\sigma Y}$ becomes trivial. □

In Example 8.13 of [Raeburn(2005)], it is shown that if E is a discrete graph, then

$$J_{X(E)} = \overline{span}\{\chi_v : 0 < |r^{-1}(v)| < \infty\}$$

where $X(E)$ is the associated correspondence and $\chi_v \in c_0(E^0)$ denotes the characteristic function of the vertex $v \in E^0$. With this in mind, we give the following proposition:

Proposition 5.3.6. *Let E and F be discrete graphs and let $\big(X = X(E), A = c_0(E^0)\big)$ and $\big(Y = X(F), B = c_0(F^0)\big)$ be the associated correspondences. Suppose α^E is an* *action of a discrete group G on E and let (γ, α) be the associated action on (X, A). Let $\delta : F^1 \to G$ be a labeling of the edges of F and let (σ, ι) be the associated coaction on (Y, B). Then X and Y are ideal compatible with respect to γ and σ.*

Proof. Recall that $X \,{}_{\gamma}\boxtimes_{\sigma} Y = X(E \,{}_{\alpha^E}\times_{\delta} F)$ where $E \,{}_{\alpha^E}\times_{\delta} F$ is the graph defined in Example 5.1.9. Thus

$$J_{X_{\gamma}\boxtimes_{\sigma} Y} = \overline{span}\{\chi_{v\times w} : 0 < |r^{-1}_{E_{\alpha^E}\times_{\delta} F}(v \times w)| < \infty\}$$

By definition, $r_{E_{\alpha^E}\times_{\delta} F} = r_E \times r_F$ so $r^{-1}_{E_{\alpha^E}\times_{\delta} F}(v \times w) = r_E^{-1}(v) \times r_F^{-1}(w)$ and thus $|r^{-1}_{E_{\alpha^E}\times_{\delta} F}(v \times w)| = |r_E^{-1}(v)| \cdot |r_F^{-1}(w)|$. But $0 < |r_E^{-1}(v)| \cdot |r_F^{-1}(w)| < \infty$ if and only if $0 < |r_E^{-1}(v)| < \infty$ and $0 < |r_F^{-1}(w)| < \infty$. Thus we have that

$$J_{X_{\gamma}\boxtimes_{\sigma} Y} = \overline{span}\{\chi_{v\times w} : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\}$$

If we identify $c_0(E^0 \times F^0)$ with $c_0(E^0) \otimes c_0(F^0)$ in the standard way, we see that $\chi_{v\times x} = \chi_v \otimes \chi_w$ (recall from Example 5.1.9 that ι is the trivial coaction so we have that $A \,{}_{\alpha}\boxtimes_{\iota} B \cong A \otimes B$). Thus

$$\begin{aligned} J_{X_{\gamma}\boxtimes_{\sigma} Y} &= \overline{span}\{\chi_v \otimes \chi_w : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\} \\ &= \overline{span}\{f \otimes g : f \in J_X, g \in J_Y\} \\ &= J_X \otimes J_Y \end{aligned}$$

Therefore, X and Y are ideal-compatible. $\square$

5.4 Main Result

We will now state our main theorem, although we delay the proof until later in this section. Throughout this section X and Y will be correspondences over C^*-algebras A and B, (σ, δ) will be a coaction of a discrete group G on Y and (γ, α) will be an action of G on X. We will denote the induced action on $\mathcal{O}_X$ by γ' and the induced coaction on $\mathcal{O}_Y$ by σ'. To simplify the notation, we will make the following definitions: $X \boxtimes Y := X\ {}_{\gamma}\!\boxtimes_{\sigma} Y$, $A \boxtimes B := A\ {}_{\alpha}\!\boxtimes_{\delta} B$, and $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y := \mathcal{O}_X\ {}_{\gamma'}\!\boxtimes_{\sigma',\mathbb{T}} \mathcal{O}_Y$.

Theorem 5.4.1. *Suppose X and Y are full, ideal compatible, and Katsura non-degenerate (see the previous section). Then $\mathcal{O}_{X\,{}_{\gamma}\boxtimes_{\sigma} Y} \cong \mathcal{O}_X\ {}_{\gamma'}\!\boxtimes_{\sigma',\mathbb{T}} \mathcal{O}_Y$.*

Before we can prove this result, we will need the following lemmas:

Lemma 5.4.2. *Suppose (π_X, ψ_X) and (π_Y, ψ_Y) are Toeplitz representations of X and Y in C^*-algebras C and D and let γ' and σ' be the induced action and coaction on C and D. Let $\psi := \psi_X \otimes \psi_Y$ and let $\pi := \pi_X \otimes \pi_Y$. Then (π, ψ) is a Toeplitz representation of $X\ {}_{\gamma}\!\boxtimes_{\sigma} Y$ in $C\ {}_{\gamma'}\!\boxtimes_{\sigma'} D$.*

Proof. Let $y \in Y_s$ for some $s \in G$ and let $x \in X$, $a \in A$, and $b \in B$. Then

$$\begin{aligned}
\psi\big((x \boxtimes y)(a \boxtimes b)\big) &= \psi\big(x\alpha_s(a) \boxtimes yb\big) \\
&= \psi_X\big(x\alpha_s(a)\big) \boxtimes \psi_Y(yb) \\
&= \psi_X(x)\pi_X\big(\alpha_s(a)\big) \boxtimes \psi_Y(y)\pi_Y(b) \\
&= \psi_X(x)\gamma'_s\big(\pi_X(a)\big) \boxtimes \psi_Y(y)\pi_Y(b) \\
&= \big(\psi_X(x) \boxtimes \psi_Y(y)\big)\big(\pi_X(a) \boxtimes \pi_Y(b)\big) \\
&= \psi(x \boxtimes y)\pi(a \boxtimes b).
\end{aligned}$$

A similar argument with $b \in B_s$ shows that

$$\psi\big((a \boxtimes b)(x \boxtimes y)\big) = \pi(a \boxtimes b)\psi(x \boxtimes y).$$

Further,

$$\begin{aligned}
\psi(x \boxtimes y)^*\psi(x' \boxtimes y') &= \Big(\psi_X(x) \boxtimes \psi_Y(y)\Big)^*\Big(\psi_X(x') \boxtimes \psi_Y(y')\Big) \\
&= \Big(\gamma'_{s^{-1}}\big(\psi_X(x)^*\big) \boxtimes \psi_Y(y)^*\Big)\Big(\psi_X(x') \boxtimes \psi_Y(y')\Big)
\end{aligned}$$

$$= \gamma'_{s^{-1}}\big(\psi_X(x)^*\big)\gamma'_{s^{-1}}\big(\psi_X(x')\big) \boxtimes \psi_Y(y^*)\psi_Y(y')$$

$$= \gamma'_{s^{-1}}\big(\psi_X(x)^*\psi_X(x')\big) \boxtimes \psi_Y(y)^*\psi_Y(y')$$

$$= \gamma'_{s^{-1}}\Big(\pi_X\big(\langle x, x'\rangle_A\big)\Big) \boxtimes \pi_Y\big(\langle y, y'\rangle_B\big)$$

$$= \pi_X\Big(\gamma_{s^{-1}}\big(\langle x, x'\rangle_A\big)\Big) \boxtimes \pi_Y\big(\langle y, y'\rangle_B\big)$$

$$= \pi\Big(\gamma_{s^{-1}}\big(\langle x, x'\rangle_A\big) \boxtimes \langle y, y'\rangle_B\Big)$$

$$= \pi\big(\langle x \boxtimes y, x' \boxtimes y'\rangle_{A\boxtimes B}\big).$$

These equalities extend linearly and continuously (by the linearity and continuity of the maps $\psi_X, \pi_X, \psi_Y, \pi_Y, \psi$, and π) to all of $X \boxtimes Y$ and $A \boxtimes B$. Thus (π, ψ) is a Toeplitz representation. $\square$

Lemma 5.4.3. *If (π, ψ) is the Toeplitz representation in Lemma 5.4.2, then:*

$$\psi^{(1)}\big(k(S \boxtimes T)\big) = \psi_X^{(1)}(S) \boxtimes \psi_Y^{(1)}(T)$$

for all $S \in \mathcal{K}(X)$ and $T \in \mathcal{K}(Y)$.

Proof. Without loss of generality we may assume that $S = \Theta_{x,x'}$ and $T = \Theta_{y,y}$ with $y \in Y_s$ and $y' \in Y_t$. We have:

$$\begin{aligned}
\psi^{(1)}\big(k(\Theta_{x,x'} \boxtimes \Theta_{y,y'})\big) &= \psi^{(1)}\big(\Theta_{x\boxtimes y, \gamma_{ts^{-1}}(x')\boxtimes y'}\big) \\
&= \psi(x \boxtimes y)\psi\big(\gamma_{ts^{-1}}(x') \boxtimes y'\big)^* \\
&= \Big(\psi_X(x) \boxtimes \psi_Y(y)\Big)\Big(\psi_X\big(\gamma_{ts^{-1}}(x')\big) \boxtimes \psi_Y(y')\Big)^* \\
&= \Big(\psi_X(x) \boxtimes \psi_Y(y)\Big)\Big(\gamma'_{t^{-1}}\big(\psi_X\big(\gamma_{ts^{-1}}(x')\big)^*\big) \boxtimes \psi_Y(y')^*\Big) \\
&= \Big(\psi_X(x) \boxtimes \psi_Y(y)\Big)\Big(\gamma'_{s^{-1}}\big(\psi_X(x')^*\big) \boxtimes \psi_Y(y')^*\Big) \\
&= \big(\psi_X(x)\psi_X(x')^*\big) \boxtimes \big(\psi_Y(y)\psi_Y(x')^*\big) \\
&= \psi_X^{(1)}\big(\Theta_{x,x'}\big) \boxtimes \psi_Y^{(1)}\big(\Theta_{y,y'}\big)
\end{aligned}$$

$\square$

Lemma 5.4.4. *Suppose that X and Y are ideal compatible. If (π_X, ψ_X) and (π_Y, ψ_Y) are Cuntz-Pimsner covariant then so is (π, ψ).*

Proof. Let ϕ_X, ϕ_Y, and ϕ be the left action maps on X, Y and $X\ {}_{\gamma}\boxtimes_{\sigma} Y$ respectively. Note that from the definition of $X\ {}_{\gamma}\boxtimes_{\sigma} Y$ we have that $\phi = j \circ \big(\phi_X \boxtimes \phi_Y\big)$ and thus,in particular $\phi|_{J_{X\boxtimes Y}} = k \circ \big(\phi_X|_{J_X} \boxtimes \phi_Y|_{J_Y}\big)$. Let $c \in J_{X\boxtimes Y}$. Since $J_{X\boxtimes Y} = J_X \boxtimes J_Y$ we may approximate $c \approx \sum a_i \boxtimes b_i$ for some $a_i \in J_X$ and $b_i \in J_Y$. We have

$$\begin{aligned}
\psi^{(1)}\big(\phi(c)\big) &\approx \sum \psi^{(1)}\big(\phi(a_i \boxtimes b_i)\big) \\
&= \sum \psi^{(1)}\big(k\big(\phi_X(a_i) \boxtimes \phi_Y(b_i)\big)\big) \\
&= \sum \psi_X^{(1)}\big(\phi_X(a_i)\big) \boxtimes \psi_Y^{(1)}\big(\phi_Y(b_i)\big) \\
&= \sum \pi_X(a_i) \boxtimes \pi_Y(b_i) \\
&= \sum \pi(a_i \boxtimes b_i) \\
&\approx \pi(c)
\end{aligned}$$

thus (π, ψ) is Cuntz-Pimsner covariant. □

We are now ready to prove our main theorem.

Proof. (of Theorem 5.4.1) First, we will construct a $*$-homomorphism $F : \mathcal{O}_{X\boxtimes Y} \to \mathcal{O}_X \boxtimes \mathcal{O}_Y$. Let (k_X, k_A) and (k_Y, k_B) be the standard Cuntz-Pimsner covariant representations of X and Y in $\mathcal{O}_X$ and $\mathcal{O}_Y$ and let $\psi := k_X \boxtimes k_Y$ and $\pi := k_A \boxtimes k_B$. Then by Lemma 5.4.4, (ψ, π) is a Cuntz-Pimsner covariant representation of $X \boxtimes Y$ in $\mathcal{O}_X \boxtimes \mathcal{O}_Y$. Thus, by the universal property there is a unique homomorphism $F : \mathcal{O}_{X\boxtimes Y} \to \mathcal{O}_X \boxtimes \mathcal{O}_Y$ such that

$$(\psi, \pi) = (F \circ k_{X\boxtimes Y}, F \circ k_{A\boxtimes B})$$

The gauge actions Γ_X and Γ_Y on $\mathcal{O}_X$ and $\mathcal{O}_Y$ give rise to $\mathbb{Z}$ gradings $\{\mathcal{O}_X^n\}_{n\in\mathbb{Z}}$ and $\{\mathcal{O}_Y^n\}_{n\in\mathbb{Z}}$. By Proposition 5.2.4 the sets

$$S_n := \{x \boxtimes y : x \in \mathcal{O}_X^n, y \in \mathcal{O}_Y^n\}$$

give a $\mathbb{Z}$-grading of $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. Notice that we have the following:

$$F\big(k_{A\boxtimes B}(a \boxtimes b)\big) = \pi(a \boxtimes b) = (k_A \boxtimes k_B)(a \boxtimes b) \in S_0 \tag{5.4.1}$$

$$F\big(k_{X\boxtimes Y}(x \boxtimes y)\big) = \psi(x \boxtimes y) = (k_X \boxtimes k_Y)(x \boxtimes y) \subseteq S_1 \tag{5.4.2}$$

For $a \in A, b \in B, x \in X, y \in Y$. Since the image of F is generated by $F\big(k_{A\boxtimes B}(A \boxtimes B\big)$ and $F\big(k_{X\boxtimes Y}(X \boxtimes Y)\big)$ we see that the image of F lies inside $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. Furthermore, note that the grading $\{S_n\}_{n\in\mathbb{Z}}$ gives rise to an action Γ of $\mathbb{T}$ such that $\Gamma_z(s) = z^n s$ for $s \in S_n$. In particular, by (5.4.1) we have that $\Gamma_z\big(\pi(c)\big) = \pi(c)$ for all $c \in A \boxtimes B$ and by (5.4.2) we have that $\Gamma_z\big(\psi(w)\big) = z\psi(w)$ for all $w \in X \boxtimes Y$. Thus Γ is a gauge action for (ψ, π). Since k_A, k_B, k_X, and k_Y are all injective, ψ and π is injective. By the Gauge Invariant Uniqueness Theorem, F is injective.

It remains to show that F is surjective onto $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. Recall that, since X and Y are full, so are the gradings $\{\mathcal{O}_X^n\}_{n\in\mathbb{Z}}$ and $\{\mathcal{O}_Y^n\}_{n\in\mathbb{Z}}$ and thus by Lemma 5.2.6 so is the grading $\{S_n\}_{n\in\mathbb{Z}}$. Thus $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$ is generated by the elements of S_1. Therefore, to show that F is surjective, it suffices to show that S_1 is in the image of F.

Recall that $\mathcal{O}_X^1$ is densely spanned by elements of the form: $k_X^{n+1}(x)k_X^n(x')^*$. Similarly, $\mathcal{O}_Y^1$ is densely spanned by elements of the form $k_Y^{n+1}(y)k_Y^n(y')^*$. Thus the elements

$$k_X^{n+1}(x)k_X^n(x')^* \boxtimes k_Y^{m+1}(y)k_Y^m(y')^* \tag{5.4.3}$$

are dense in S_1. Fix such an element. By linearity and continuity, it suffices to consider the case where y is a tensor product of homogeneous elements (for if not, it can be approximated by a sum of such products). In other words, we may assume that $y = y_1 \otimes \cdots \otimes y_{m+1}$ where each y_i homogeneous with respect to the G-grading of Y. Let us say that $y_i \in Y_{s_i}$ for each i. Let $s := s_1 \cdots s_{m+1}$. Then

$$k_Y^{m+1}(y) = k_Y(y_1) \cdots k_Y(y_{m+1}) \in \mathcal{O}_Y^s$$

Similarly, we can assume that $y' = y'_1 \otimes \cdots \otimes y'_m$ with $y'_i \in Y_{t_i}$ and then $k_Y^m(y') \in \mathcal{O}_Y^t$ where $t := t_1 \cdots t_m$.

Let us assume $n \leq m$. Then $m = n + l$ for some $l \geq 0$. Then we can factor $y = y^{(1)} \otimes y^{(2)}$ and $y' = y'^{(1)} \otimes y'^{(2)}$. Since X is Katsura non-degenerate, we can factor $x = x_0 a$ and $x' = x'_0 a'$ with $x_0, x'_0 \in X$ and $a, a' \in J_X$. With this in mind, we can factor (5.4.3) as follows:

$$\begin{aligned}
&k_X^{n+1}(x) k_X^n(x')^* \boxtimes k_Y^{m+1}(y) k_Y^m(y')^* \\
&\quad = \Big(k_X^{n+1}(x) \boxtimes k_Y^{m+1}(y)\Big)\Big({\gamma'}_{s^{-1}}\big(k_X^n(x')\big)^* \boxtimes k_Y^m(y')^*\Big) \\
&\quad = \big(k_X^{n+1}(x_0) k_A(a) \boxtimes k_Y^{n+1}(y^{(1)}) k_Y^l(y^{(2)})\big) \\
&\qquad \Big({\gamma'}_{s^{-1}}\big(k_A(a')\big)^* {\gamma'}_{s^{-1}}\big(k_X^n({x'}_0)\big)^* \boxtimes k_Y^l({y'}^{(2)})^* k_Y^n({y'}^{(1)})^*\Big) \\
&\quad = \Big(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)})\Big)\Big({\gamma'}_{s_1^{-1}}\big(k_A(a)\big) \boxtimes k_Y^l(y^{(2)})\Big) \\
&\qquad \Big({\gamma'}_{s^{-1}}\big(k_A(a')\big)^* \boxtimes k_Y^l({y'}^{(2)})^*\Big)\Big({\gamma'}_{t_2 s^{-1}}\big(k_X^n({x'}_0)\big)^* \boxtimes k_Y^n({y'}^{(1)})^*\Big) \\
&\quad = \Big(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)})\Big)\Big({\gamma'}_{s_1^{-1}}\big(k_A(a)\big){\gamma'}_{s_2 s^{-1}}\big(k_A(a')\big)^* \boxtimes k_Y^l(y^{(2)}) k_Y^l({y'}^{(2)})^*\Big) \\
&\qquad \Big({\gamma'}_{t_2 s^{-1}}\big(k_X^n({x'}_0)\big)^* \boxtimes k_Y^n({y'}^{(1)})^*\Big) \\
&\quad = \Big(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)})\Big)\Big(k_A\big(\alpha_{s_1^{-1}}(a)\alpha_{s_2 s^{-1}}(a')^*\big) \boxtimes k_Y^l(y^{(2)}) k_Y^l({y'}^{(2)})^*\Big) \\
&\qquad \Big({\gamma'}_{t_1 t_2 s^{-1}}\big(k_X^n({x'}_0)\big) \boxtimes k_Y^n({y'}^{(1)})\Big)^* \\
&\quad = \Big(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)})\Big)\Big(k_X^{(1)} \circ \phi_X\big(\alpha_{s_1^{-1}}(a)\alpha_{s_2 s^{-1}}(a')^*\big) \boxtimes k_Y^{(1)}\big(\Theta_{y^{(2)},{y'}^{(2)}}\big)\Big) \\
&\qquad \Big(k_X^n\big(\alpha_{t s^{-1}}({x'}_0)\big) \boxtimes k_Y^n({y'}^{(1)})\Big)^* \\
&\quad = \psi^{n+1}\Big(x_0 \boxtimes y^{(1)}\Big)\psi^{(1)}\Big(\phi_X\big(\alpha_{s_1^{-1}}(a)\alpha_{s_2 s^{-1}}(a')^*\big) \boxtimes \Theta_{y^{(2)},{y'}^{(2)}}\Big)
\end{aligned}$$

$$\psi^n\Big(\alpha_{ts^{-1}}(x_0{}') \boxtimes y'^{(1)}\Big)^*$$

Since all three factors in the final line are in the algebra generated by (ψ, π), we know that $k_X^{n+1}(x)k_X^n(x')^* \boxtimes k_Y^{m+1}(y)k_Y^m(y')^*$ must be in the image of F. If $n > m$ we can factor $x = x^{(1)} \otimes x^{(2)}$, $x' = x'^{(1)} \otimes x'^{(2)}$ $y = y_0 a$, and $y' = y_0' a'$ and apply a similar argument. So the image of F generates S_1 and thus F is surjective onto $\mathcal{O}_X\ {}_{\gamma'}\boxtimes_{\sigma'} O_Y$. Hence we have established that $\mathcal{O}_{X_{\gamma}\boxtimes_{\sigma} Y} \cong \mathcal{O}_X\ {}_{\gamma'}\boxtimes_{\sigma',\mathbb{T}} \mathcal{O}_Y$ $\square$

5.5 Examples

In this section, we will apply our main theorem to some of the examples of twisted tensor products of correspondences we discussed earlier.

Example 5.5.1. Let X and Y be $\mathbb{Z}_2$-graded correspondences over $\mathbb{Z}_2$-graded C^*-algebras A and B. As in Example 5.1.8, let (γ, α) be the action of $\mathbb{Z}_2$ on (X, A) associated to the grading on X and let (σ, δ) be the coaction of $\mathbb{Z}_2$ on (Y, B) associated to the grading of Y. Let γ' be the action of $\mathbb{Z}_2$ on $\mathcal{O}_X$ induced by (γ, α) and let σ' be the coaction of $\mathbb{Z}_2$ on $\mathcal{O}_Y$ induced by (σ, δ). Suppose X and Y are Katsura non-degenerate and ideal compatible with respect to σ and γ. Then by our main result together with Example 5.1.8, we see that $\mathcal{O}_{X\widehat{\otimes}Y} \cong \mathcal{O}_{X\boxtimes Y} \cong \mathcal{O}_X\ {}_{\gamma'}\boxtimes_{\sigma',\mathbb{T}} \mathcal{O}_Y$.

Note that γ' and σ' give rise to $\mathbb{Z}_2$-gradings on $\mathcal{O}_X$ and $\mathcal{O}_Y$ so it makes sense to speak of the $\mathbb{Z}_2$-graded tensor product $\mathcal{O}_X\widehat{\otimes}\mathcal{O}_Y$. Let $\varphi : \mathcal{O}_X\widehat{\otimes}\mathcal{O}_Y \to \mathcal{O}_X \boxtimes \mathcal{O}_Y$ be the map $w\widehat{\otimes}z \mapsto w \boxtimes z$. Note that

$$\begin{aligned}
\varphi\big((w_1\widehat{\otimes}z_1)(w_2\widehat{\otimes}z_2)\big) &= (-1)^{\partial z_1 \partial w_2}\varphi(w_1w_2\widehat{\otimes}z_1z_2) \\
&= (-1)^{\partial z_1 \partial w_2}(w_1w_2 \boxtimes z_1z_2) \\
&= w_1\gamma'_{\partial z_1}(w_2) \boxtimes z_1z_2 \\
&= (w_1 \boxtimes z_1)(w_2 \boxtimes z_2)
\end{aligned}$$

$$= \varphi(w_1 \widehat{\otimes} z_1)\varphi(w_2 \widehat{\otimes} z_2)$$

and also

$$\begin{aligned}
\varphi\big((w \widehat{\otimes} z)^*\big) &= (-1)^{\partial w \partial z} \varphi(w^* \widehat{\otimes} z^*) \\
&= (-1)^{\partial w \partial z}(w^* \boxtimes z^*) \\
&= (-1)^{\partial w^* \partial z^*}(w^* \boxtimes z^*) \\
&= \gamma_{\partial z^*}(w^*) \boxtimes z^* \\
&= (w \boxtimes z)^* \\
&= \varphi(w \widehat{\otimes} z)^*
\end{aligned}$$

therefore φ is a $*$-homomorphism. Further, φ takes a densely spanning set (the elementary tensors $w \widehat{\otimes} z$) in $\mathcal{O}_X \widehat{\otimes} \mathcal{O}_Y$ bijectively onto a densely spanning set (the elementary tensors $w \boxtimes z$) in $\mathcal{O}_X\, {}_{\gamma'}\!\boxtimes_{\sigma'} \mathcal{O}_Y$ so φ is an isomorphism. Thus we have that

$$\mathcal{O}_{X \widehat{\otimes} Y} \cong \mathcal{O}_X \widehat{\otimes}_{\mathbb{T}} \mathcal{O}_Y$$

Example 5.5.2. In this example we will show that the graph algebra of the product graph constructed in Example 5.1.9 is a twisted tensor product of the graph algebras of the underlying graphs. Suppose E and F are directed graphs, α^E an action of a discrete group G on E and let δ be a labeling of the edges of F by elements of G. Let (γ, α) and (σ, ι) be the associated action and coaction on the graph correspondences $(X, A) = \big(X(E), c_0(E^0)\big)$ and $(Y, B) = \big(X(F), c_0(F^0)\big)$. Suppose further, that E and F have no sinks, no proper sources (see Definition 5.3.3), and that no infinite receiver in E or F emits an edge. Then we have that the graph correspondences (X, A) and (Y, B) are full, and by Propositions 4.2.9 and 5.3.5 they are Katsura non-degenerate and ideal compatible with respect to γ and σ. Thus we may apply our main result. Let γ' be the induced action on $\mathcal{O}_X \cong C^*(E)$ and σ' be the induced coaction on

$\mathcal{O}_Y \cong C^*(F)$. We have

$$C^*(E\ {}_{\alpha^E}\times_\delta F) \cong \mathcal{O}_{X\ {}_\gamma\boxtimes_\sigma Y} \cong \mathcal{O}_X\ {}_{\gamma'}\boxtimes_{\sigma',\mathbb{T}}\ \mathcal{O}_Y \cong C^*(E)\ {}_{\gamma'}\boxtimes_{\sigma',\mathbb{T}}\ C^*(F)$$

Before we see how our main result applies to crossed products, we will need a few lemmas.

Lemma 5.5.3. *Let (A, G, α) be a dynamical system, (B, G, δ) a coaction and C a C^*-algebra. There exists an isomorphism*

$$\sigma_{23} : A\ {}_\alpha\boxtimes_{\Sigma_{23}(\delta\otimes \mathrm{id}_C)}\ (B \otimes C) \to (A \otimes C)\ {}_{\alpha\otimes\iota}\boxtimes_\delta\ B$$

Proof. By definition,

$$A\ {}_\alpha\boxtimes_{\Sigma_{23}(\delta\otimes id_C)}\ (B \otimes C) = i_A(A) \cdot i_{B\otimes C}(B \otimes C) \subseteq A \otimes B \otimes C \otimes \mathcal{B}\big(L^2(G)\big)$$

where $i_A = \delta^\alpha_{1m}$ and

$$i_{B\otimes C} = \circ\big(\Sigma_{23}(\delta \otimes \mathrm{id}_C)\big)_{23\lambda}$$

. Similarly,

$$(A \otimes C)\ {}_{\alpha\otimes\iota}\boxtimes_\delta\ B = i_{A\otimes C}(A \otimes C) \cdot i_B(B) \subseteq A \otimes C \otimes B \otimes \mathcal{B}\big(L^2(G)\big)$$

where $i_{A\otimes C} = m_4 \circ \delta^{\alpha\otimes\iota}_{12m}$ and $i_B = \lambda_4 \circ \delta_{3\lambda}$. Since $\Sigma_{23} : A \otimes B \otimes C \otimes \mathcal{B}\big(L^2(G)\big) \to A \otimes C \otimes B \otimes \mathcal{B}\big(L^2(G)\big)$ is an isomorphism, we can let σ_{23} be the restriction of Σ_{23} to the subalgebra $A\ {}_\alpha\boxtimes_{\Sigma_{23}(\delta\otimes \mathrm{id}_C)}(B\otimes C)$. From there it suffices to show that the image of σ_{23} is equal to $(A\otimes C)\ {}_{\alpha\otimes\iota}\boxtimes_\delta B$. $A\ {}_\alpha\boxtimes_{\Sigma_{23}(\delta\otimes id_C)}(B\otimes C)$ is densely spanned by elements of the form $i_A(a)i_{B\otimes C}(b \otimes c)$ and $(A \otimes C)\ {}_{\alpha\otimes\iota}\boxtimes_\delta\ B$ is densely spanned by elements of the form $i_{A\otimes C}(a \otimes c)i_B(b)$. Since σ_{23} is a restriction of the linear norm preserving map Σ_{23}, it too is linear and norm preserving. To show that the image of σ_{23} is $(A \otimes C)\ {}_{\alpha\otimes\iota}\boxtimes_\delta\ B$, it suffices to show that $\sigma_{23}\big(i_A(a)i_{B\otimes C}(b \otimes c)\big) = i_{A\otimes C}(a \otimes c)i_B(b)$. To see this, note that

$$\sigma_{23}\big(i_A(a)i_{B\otimes C}(b \otimes c)\big) = \Sigma_{23}\Big(\delta^\alpha(a)_{1m}\big(\Sigma_{23}(\delta \otimes \mathrm{id}_C)(b \otimes c)\big)_{23\lambda}\Big)$$

$$= \Sigma_{23}\Big(\delta^\alpha(a)_{1m}\big(\Sigma_{23}\big(\delta(b)\otimes c\big)\big)_{23\lambda}\Big)$$

$$= \delta^\alpha(a)_{1m}\Sigma_{23}\big(c_3\delta(b)_{2\lambda}\big)$$

$$= \delta^\alpha(a)_{1m}c_2\delta(b)_{3\lambda}$$

$$= \Big(\Sigma_{23}\big(\delta^\alpha(a)\otimes c\big)\Big)_{12m}\delta(b)_{3\lambda}$$

$$= \delta^{\alpha\otimes\iota}(a\otimes c)_{12m}\delta(b)_{3\lambda}$$

$$= i_{A\otimes C}(a\otimes c)i_B(b)$$

thus $\sigma_{23} : A\ {}_\alpha\boxtimes_{\Sigma_{23}(\delta\otimes id_C)}\ (B\otimes C) \to (A\otimes C)\ {}_{\alpha\otimes\iota}\boxtimes_\delta\ B$ is an isomorphism. $\square$

Lemma 5.5.4. *Let (γ,α) be an action of a discrete group G on a correspondence (X,A) and let (B,G,δ) be a coaction. If $J_{X\,{}_\gamma\boxtimes_\delta B} = J_X\ {}_\alpha\boxtimes_\delta B$ and J_X is Katsura-non-degenerate, then*

$$\mathcal{O}_{X\,{}_\gamma\boxtimes_\delta B} \cong \mathcal{O}_X\ {}_{\gamma'}\boxtimes_\delta\ B$$

where B is viewed as a correspondence over itself in the left-hand-side and γ' is the induced action on $\mathcal{O}_X$.

Proof. In this context, our main result gives us

$$\mathcal{O}_{X\,{}_\gamma\boxtimes_\delta B} \cong \mathcal{O}_X\ {}_{\gamma'}\boxtimes_{\delta,\mathbb{T}}\ \mathcal{O}_B$$

Recall that $\mathcal{O}_B \cong B\otimes C(\mathbb{T})$ and the gauge action corresponds to $\iota\otimes\lambda$ where ι is the trivial action and λ is left translation. Using the isomorphism between $C(\mathbb{T})$ and $C^*(\mathbb{Z})$, we see that

$$\begin{aligned}\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_B &\cong \mathcal{O}_X \boxtimes_{\mathbb{T}} \big(B\otimes C^*(\mathbb{Z})\big)\\ &= \overline{\operatorname{span}}\big\{\mathcal{O}_X^n \boxtimes \big(B\otimes C^*(\mathbb{Z})^n\big)\big\}\\ &= \overline{\operatorname{span}}\big\{x\boxtimes(b\otimes u_n)\in \mathcal{O}_X\boxtimes\big(B\otimes C^*(\mathbb{Z})\big) : x\in\mathcal{O}_X^n\big\}\end{aligned}$$

where u_n is the unitary in $C^*(\mathbb{Z})$ associated to $n \in \mathbb{Z}$. Applying the isomorphism σ_{23} from the previous lemma, we see that the above is isomorphic to

$$\begin{aligned}
&\overline{\operatorname{span}}\big\{(x \otimes u_n) \boxtimes b \in \big(\mathcal{O}_X \otimes C^*(\mathbb{Z})\big) \boxtimes B : x \in \mathcal{O}_X^n\big\} \\
=&\overline{\operatorname{span}}\big\{x \otimes u_n \in \mathcal{O}_X \otimes C^*(\mathbb{Z}) : x \in \mathcal{O}_X^n\big\} \boxtimes B
\end{aligned}$$

The grading $\{\mathcal{O}_X^n\}_{n\in\mathbb{Z}}$ corresponds to a coaction ε such that $\varepsilon(x) = x \otimes u_n$ for all $x \in \mathcal{O}_X^n$. Continuing from above, we have:

$$\begin{aligned}
&\overline{\operatorname{span}}\big\{x \otimes u_n \in \mathcal{O}_X \otimes C^*(\mathbb{Z}) : x \in \mathcal{O}_X^n\big\} \boxtimes B \\
=&\overline{\operatorname{span}}\big\{\varepsilon(x) : x \in \mathcal{O}_X^n\big\} \boxtimes B \\
=&\varepsilon\big(\mathcal{O}_X\big) \boxtimes B \\
\cong&\mathcal{O}_X \boxtimes B
\end{aligned}$$

□

Lemma 5.5.5. *Let (σ, δ) be a coaction of a discrete group G on a full correspondence (X, A). Let α be an action of G on a C^*-algebra B. Suppose $J_{X\,{}_\sigma\boxtimes_\alpha B} = J_X\,{}_\delta\boxtimes_\alpha B$ and J_X is Katsura-non-degenerate. Then*

$$\mathcal{O}_{X\,{}_\sigma\boxtimes_\alpha B} \cong \mathcal{O}_X\,{}_{\sigma'}\boxtimes_\alpha B$$

where σ' is the induced coaction on $\mathcal{O}_X$

Proof. The proof is similar to the proof of Lemma 5.5.4. □

Example 5.5.6. Let (X, A) be a correspondence and let (γ, α) be an action of a discrete group G on A. Suppose further, that $J_{X \rtimes_{\gamma,r} G} = J_X \rtimes_{\alpha,r} G$. Recall that Example 5.1.10 showed that $X \rtimes_{\gamma,r} G \cong X\,{}_\gamma\boxtimes_{\delta_G} C_r^*(G)$. Thus we have

$$\mathcal{O}_{X \rtimes_{\gamma,r} G} \cong \mathcal{O}_{X\,{}_\gamma\boxtimes_{\delta_G} C_r^*(G)}$$

Since $A \rtimes_{\alpha,r} G \cong A\,{}_{\alpha}\boxtimes_{\delta_G} C_r^*(G)$, the condition that $J_{X \rtimes_{\gamma,r} G} = J_X \rtimes_{\alpha,r} G$ can be restated as $J_{X\,{}_{\gamma}\boxtimes_{\delta_G} C_r^*(G)} = J_X\,{}_{\alpha}\boxtimes_{\delta_G} C_r^*(G)$. Thus the correspondences X and $C_r^*(G)$ are ideal compatible. Applying Lemma 5.5.4, we have that $\mathcal{O}_{X\,{}_{\gamma}\boxtimes_{\delta_G} C_r^*(G)} \cong \mathcal{O}_X\,{}_{\gamma'}\boxtimes_{\delta_G} C_r^*(G)$ and this, in turn, is isomorphic to $\mathcal{O}_X \rtimes_{\gamma',r} G$. Thus we have:

$$\mathcal{O}_{X \rtimes_{\gamma,r} G} \cong \mathcal{O}_X \rtimes_{\gamma',r} G$$

Example 5.5.7. Suppose (σ, δ) is a coaction of a discrete group G on a correspondence (X, A). Then $X \rtimes_\sigma G \cong X\,{}_{\sigma}\boxtimes_\lambda c_0(G)$ by Proposition 5.1.11. If $J_{X\,{}_{\sigma}\boxtimes_\lambda c_0(G)} = J_X\,{}_{\delta}\boxtimes_\lambda c_0(G)$ (or in other terms $J_{X \rtimes_\sigma G} = J_X \rtimes_\delta G$) then Lemma 5.5.5 tells us that

$$\mathcal{O}_{X\,{}_{\sigma}\boxtimes_\lambda c_0(G)} \cong \mathcal{O}_X\,{}_{\sigma'}\boxtimes_\lambda c_0(G)$$

which we can restate as

$$\mathcal{O}_{X \rtimes_\sigma G} \cong \mathcal{O}_X \rtimes_{\sigma'} G$$

where σ' is the induced coaction on $\mathcal{O}_X$.

These last two examples are not new. In [Hao and Ng(2008)] it was shown that $\mathcal{O}_{X \rtimes_\gamma G} \cong \mathcal{O}_X \rtimes_{\gamma'} G$ for any locally compact amenable group, and in [Bédos, E. and Kaliszewski, S. and Quigg, J. and Robertson, D.(ions)] it is shown that $\mathcal{O}_{X \rtimes_{\gamma,r} G} \cong \mathcal{O}_X \rtimes_{\gamma',r} G$ for any locally compact group provided that $J_{X \rtimes_{\gamma,r} G} = J_X \rtimes_{\alpha,r} G$. In [Kaliszewski *et al.*(2012)] it was shown that $\mathcal{O}_{X \rtimes_\sigma G} \cong \mathcal{O}_X \rtimes_{\sigma'} G$ for full (not reduced) coactions σ provided certain technical conditions involving the Katsura ideal are satisfied. Even though our main result does not recover these results in their full generality, it seems reasonable to hope that the main result of this paper might be extended to arbitrary locally compact groups and perhaps even quantum groups. In this case we would have a single framework in which to describe all results of this type.

REFERENCES

[Baaj and Skandalis(1989)] Baaj, S. and G. Skandalis, “C^*-algèbres de Hopf et théorie de Kasparov”, K-Theory **2**, 6, 683–721 (1989).

[Bédos, E. and Kaliszewski, S. and Quigg, J. and Robertson, D.(ions)] Bédos, E. and Kaliszewski, S. and Quigg, J. and Robertson, D., “A new look at crossed product correspondences and associated C^*-algebras”, Journal of Mathematical Analysis and Applications **426**, 2, 1080–1098 (Journal of Mathematical Analysis and Applications).

[Blackadar(2006)] Blackadar, B., *Operator algebras: theory of C^*-algebras and von Neumann algebras* (Springer Science & Business Media, 2006).

[Brown *et al.*(1977)] Brown, L., P. Green and M. Rieffel, “Stable isomorphism and strong Morita equivalence of C^*-algebras”, Pacific Journal of Mathematics **72**, 349–363 (1977).

[Carlsen *et al.*(2011)] Carlsen, T., N. Larsen, A. Sims and S. Vittadello, “Co-universal algebras associated to product sytems, and gauge-invariant uniqueness theorems”, Proceedings of the London Mathematical Society **3**, 563 (2011).

[Deaconu *et al.*(2010)] Deaconu, V., A. Kumjian, D. Pask and A. Sims, “Graphs of C^*-correspondences and fell bundles”, Indiana Univ. Math. Journal **59**,5,1687–1735(2010).

[Echterhoff *et al.*(2000)] Echterhoff, S., S. Kaliszewski, J. Quigg and I. Raeburn, “Naturality and induced representations”, Bulletin of the Australian Mathematical Society **61**, 03, 415–438 (2000).

[Echterhoff *et al.*(2002)] Echterhoff, S., S. Kaliszewski, J. Quigg and I. Raeburn, “A categorical approach to imprimitivity theorems for C^*-dynamical systems”, Memoirs of the American Mathematical Society **180**, 850, viii–169 (2006).

[Exel(2008)] Exel, R., “A new look at the crossed product of a C^*-algebra”, Ergodic Theory Dynam. Systems **28**, 3, 749–789 (2008).

[Fowler(2002)] Fowler, N., “Discrete product systems of Hilbert bimodules”, Pacific journal of mathematics **204**, 2, 335–375 (2002).

[Fowler and Raeburn(1998)] Fowler, N., and I. Raeburn, “The Toeplitz of a Hilbert bimodule”, Indiana University Mathematics Journal **48**, 1, 155-181 (1999).

[Fowler and Sims(2002)] Fowler, N. and A. Sims, “Product systems over right-angled Artin semigroups”, Transactions of the American Mathematical Society **354**, 4, 1487–1509 (2002).

[Gaal(1973)] Gaal, S., *Linear analysis and representation theory* (Springer-Verlag, New York-Heidelberg, 1973), die Grundlehren der mathematischen Wissenschaften, Band 198.

[Hao and Ng(2008)] Hao, G. and C. Ng, "Crossed products of C^*-correspondences by amenable group actions", J. Math. Anal. Appl. **345**, 2, 702–707 (2008).

[Ionescu(2004)] Ionescu, M., "Operator algebras and Mauldin-Williams graphs", arXiv preprint math/0401408 (2004).

[Kaliszewski *et al.*(2015)] Kaliszewski, S., A. Morgan and J. Quigg, "Ionescu's theorem fo higher-rank graphs", Indiana University Mathematics Journal **64**, 6, 1879–1901 (2015).

[Kaliszewski *et al.*(2012)] Kaliszewski, S., J. Quigg and D. Robertson, "Coactions on Cuntz-Pimsner algebras", arXiv preprint arXiv:1204.5822 (2012).

[Katsura(2003)] Katsura, T., "A construction of C^*-algebras from C^*-correspondences", Contemporary Mathematics **335**, 173–182 (2003).

[Katsura(2004)] Katsura, T., "On C^*-algebras associated with C^*-correspondences", Journal of Functional Analysis **217**, 2, 366–401 (2004).

[Kim(2014)] Kim, D., "Coactions of Hopf C^*-algebras on Cuntz-Pimsner algebras", arXiv preprint arXiv:1407.6106 (2014).

[Kumjian(1998)] Kumjian, A., "Notes on C^*-algebras of graphs", Contemporary Mathematics **228**, 189–200 (1998).

[Kumjian and Pask(2000)] Kumjian, A. and D. Pask, "Higher rank graph C^*-algebras", New York J. Math **6**, 1, 20 (2000).

[Kwaśniewski(2014)] Kwaśniewski, B., "Exel's crossed product and crossed products by completely positive maps", arXiv preprint arXiv:1404.4929 (2014).

[Lance(1995)] Lance, E. C., *Hilbert C^*-Modules A toolkit for operator algebraists*, vol. 210 of *London Mathematical Society Lecture Note Series* (Cambridge University Press, Cambridge, 1995).

[Mauldin and Williams(1988)] Mauldin, R. and S. Williams, "Hausdorff dimension in graph directed constructions", Transactions of the American Mathematical Society **309**, 2, 811–829 (1988).

[Meyer *et al.*(2014)] Meyer, R., S. Roy and S. Woronowicz, "Quantum group-twisted tensor products of c^*-algebras", International Journal of Mathematics (2014).

[Morgan(2015)] Morgan, A., "Cuntz-Pimsner algebras associated to tensor product of correspondences", Indiana Journal of Mathematics (2016).

[Morgan(2016)] Morgan, A., "Cuntz-Pimsner algebras and twisted tensor product", arXiv preprint arXiv:1601.07826 (2016).

[Pask *et al.*(2005)] Pask, D., J. Quigg and I. Raeburn, "Coverings of k-graphs", Journal of Algebra **289**, 1, 161–191 (2005).

[Pimsner(2001)] Pimsner, M. V., “A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z”, Fields Inst. Commun. **12**, 189–212 (2001).

[Quigg(1996)] Quigg, J. C., “Discrete C^*-coactions and C^*-algebraic bundles”, Journal of the Australian Mathematical Society (Series A) **60**, 02, 204–221 (1996).

[Raeburn(2005)] Raeburn, I., *Graph Algebras*, no. 103 in Regional Conference Series in Mathematics (American Mathematical Society, 2005).

[Raeburn(1992)] Raeburn, I., “On crossed products bt coactions and their representation theory” Proc. London Math. Soc. **64**, 625–652 (1992).

[Sims and Yeend(2007)] Sims, A. and T. Yeend, “c^*-algebras associated to product systems of Hilbert bimodules”, arXiv preprint arXiv:0712.3073 (2007).

[Timmermann(2008)] Timmermann, T., *An invitation to quantum groups and duality*, EMS Textbooks in Mathematics (European Mathematical Society (EMS), Zürich, 2008).

[Yeend(2006)] Yeend, T., “Topological higher-rank graphs and the C^*-algebras of topological 1-graphs”, Contemporary Mathematics **414**, 231 (2006).