Cluster-and-Connect: An Algorithmic Approach to Generating Synthetic Electric

Power Network Graphs

by

Jiale Hu

A Thesis Presented in Partial Fulfillment
of the Requirements for the Degree
Master of Science

Approved July 2015 by the
Graduate Supervisory Committee:

Lalitha Sankar, Chair
Vijay Vittal
Anna Scaglione

ARIZONA STATE UNIVERSITY

August 2015

ABSTRACT

Understanding the graphical structure of the electric power system is important in assessing reliability, robustness, and the risk of failure of operations of this critical infrastructure network. Statistical graph models of complex networks yield much insight into the underlying processes that are supported by the network. Such generative graph models are also capable of generating synthetic graphs representative of the real network. This is particularly important since the smaller number of traditionally available test systems, such as the IEEE systems, have been largely deemed to be insufficient for supporting large-scale simulation studies and commercial-grade algorithm development. Thus, there is a need for statistical generative models of electric power network that capture both topological and electrical properties of the network and are scalable.

Generating synthetic network graphs that capture key topological and electrical characteristics of real-world electric power systems is important in aiding widespread and accurate analysis of these systems. Classical statistical models of graphs, such as small-world networks or Erdős-Renyi graphs, are unable to generate synthetic graphs that accurately represent the topology of real electric power networks – networks characterized by highly dense local connectivity and clustering and sparse long-haul links.

This thesis presents a parametrized model that captures the above-mentioned unique topological properties of electric power networks. Specifically, a new *Cluster-and-Connect* model is introduced to generate synthetic graphs using these parameters. Using a uniform set of metrics proposed in the literature, the accuracy of the proposed model is evaluated by comparing the synthetic models generated for specific real electric network graphs. In addition to topological properties, the electrical properties are captured via line impedances that have been shown to be modeled reliably by well-

studied heavy tailed distributions. The details of the research, results obtained and conclusions drawn are presented in this document.

DEDICATION

This work is dedicated to my parents Shengqiang Hu and Qian Lu.

All I have and will accomplish are only possible due to their endless love.

They raise me up to more than I can be.

ACKNOWLEDGEMENTS

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

ACC	Average Clustering Coefficient of nodes in a graph
BN	Boundary nodes of clusters in a graph
C	Clustering coefficient
CDF	Cumulative distribution function
D	N-by-N distance matrix of a graph
D	Truncated geometric random variable
$d_{\max}$	Diameter
d_{ij}	The shortest path length from node i to node j
DPLN	Double Pareto log-normal distribution
$\text{DPLN}_{\text{clip}}$	Double Pareto log-normal distribution with clipping impedance
$\mathcal{E}$	Edge set of a graph
$E(\cdot)$	Expectation of a function
$E_{\max}$	Local maxima of chord density profile
$E_{\min}$	Local minima of chord density profile
F_S	Sampling flag
$\mathcal{G}$	An undirected graph
$\mathcal{G}_S$	A synthetic graph of $\mathcal{G}$
G	Non-uniform discrete random variable
GS_f	Global Sensitivity of function f
k_{avg}	Average degree
$k_{\max}$	Maximum degree
k_i	Nodal degree of any node i
L	N-by-N Laplacian matrix of a graph
L_{char}	Characteristic path length
$\log n$	Log-normal distribution
$\log n_{\text{clip}}$	Log-normal distribution with clipping impedance

M	N-by-N adjacency matrix of a graph
m	Number of edges in the graph
M_{ij}	Entry of the adjacency matrix M for node i and node j
M_S	Adjacency matrix of a synthetic graph
M_X	PGF of a random variable X
N	Number of nodes in the graph
n_c	Number of clusters
N_D	Sum of random variables G and D
N_i	Number of nodes in cluster i
N_X	Size of a random variable X
NYISO	New York Independent System Operator
p_i	Intra-cluster degree distribution of cluster i
PDF	Probability Distribution Function
PGF	Probability Generating Function
PMF	Probability Mass Function
q_{ij}	Inter-cluster degree distribution between cluster i and j
R	Radius of the Kirk circle
R_{in}	Radius of inner circle in the Kirk circle
r	Assortativity
S_{chord}	Number of chords intersecting the sweeping radius
S_{fit}	Fitting curve for S_{chord}
u	Parameter of generalized Pareto distribution
$\mathcal{V}$	Node set of a graph
WECC	Western Electricity Coordinating Council
z_{clip}	Impedance cutoff threshold
α	Parameter of DPLN distribution
β	Parameter of DPLN distribution

δ	Parameter of generalized Pareto distribution
ϵ	Differential privacy parameter
η	Chord angle
γ	Chord density profile threshold
$\lambda_2(L)$	Algebraic connectivity
μ	Mean value
$\Phi(\cdot)$	Cumulative distribution function
$\Phi^c(\cdot)$	Complementary cumulative distribution function
σ	Standard deviation
θ	Parameter of generalized Pareto distribution

Chapter 1

INTRODUCTION

1.1   Background

Understanding the graphical structure of the electric power system is important
in assessing reliability, robustness, and the risk of failure of operations of this criti-
cal infrastructure network. Statistical graph models of complex networks yield much
insight into the underlying processes that are supported by the network. Such genera-
tive graph models are also capable of generating synthetic graphs representative of the
real network. This is particularly important since the smaller number of traditionally
available test systems, such as the IEEE systems, have been largely deemed to be
insufficient for supporting large simulation studies and commercial-grade algorithm
development. Thus, there is a need for statistical generative models of electric power
network that capture both topological and electrical properties of the network and
are scalable.

Electric power networks involving a collection of buses (nodes) connected by
branches (edges) naturally lend themselves to be modeled as graphs. In fact, we note
that a large-scale power network is naturally partitioned into smaller sub-networks
(clusters) because of geographical distribution, administrative, and/or political fac-
tors (such as states in the United States). However, energy sources are not uniformly
distributed geographically; this in turn translates to the presence of specific long-
haul energy transport connections between clusters. Taken together, electric power
networks are seen to exhibit topologies where local connections are highly dense and
long-range connections are relatively sparser.

In general, such local connectivity and clustering is due to the fact that a small
number of generator nodes supply power to a large number of load nodes. On the
other hand, the sparse long-range connections enable power transmission from one

area (cluster) to another; it has been observed that these connections are not random in connectivity but occur between small groups of nodes in both clusters. This fact has an intuitive explanation according to previous research [1]: long hauls require having a right of way to deploy a long connection and it is highly likely that the long wires will reuse part of this space. These physical and economic constraints inevitably affect the structure of the topology.

Therefore, a good generative model for electric network graphs should capture both topological and electrical properties of the original graph.

1.2   Research Objective

The goal of this work is to propose a model to generate synthetic graphs that accurately captures topological and electrical properties. We seek a principled way of generating meaningful statistically accurate synthetic graphs. The first step towards this goal is to algorithmically identify and define clusters according to the fact that local connections are highly dense and long-range connections are relatively sparser. The cluster identification algorithm is followed by an algorithm that assigns both intra- and inter-cluster links using empirical models obtained from the data. Finally, impedances are assigned to the links using either the empirical distribution of impedance or using sound statistical models.

One can make the data from the original network even more private by adding noise to the node degree and line impedance data from the original graph using the well-studied framework of differential privacy [2]. Following our method to generate a synthetic graph, we also propose algorithms to differentially generate private degree and impedance data.

1.3   Literature Review

Graph theory literature is rich with formal models for network graphs. Two commonly used models are introduced here:

- Erdős-Rényi random graph model [3] in which edges are randomly chosen between nodes such that the resulting graph exhibits: (a) a small average shortest path length; and (b) small clustering coefficient.

- Small-world model introduced by Watts and Strogatz [4] in which most nodes are not neighbors of one another, but most nodes can be reached by a small number of hops. These networks are characterized by: (a) small average path length; and (b) a large clustering coefficient.

It has been noted by Wang *et al.* [1, 5, 6] as well as Hines *et al.* [7] that neither the Erdős-Rényi random graph model [3] nor the small-world model [4] is suitable for modeling power network graphs. This is because unlike these two models, the average nodal degree of a power network is almost invariant to the size of the network [1, 5]. That is, in practice large power network graphs are much sparser than what is possible with the Erdős-Rényi random graph model [3] or the small-world model [4]. Having placed in context the need for better graph models for electric power networks, we now review current literature on generating synthetic power network graphs.

Recently, Wang *et al.* in [1] present a stochastic generative model for electric power networks. They introduce a *Randomized Topology (RT)-nested-smallworld model* [1] which models the network as a locally dense small-world network with random long-haul connections. Visualizing the network nodes on a *ring lattice*, their model (i) creates equal-sized sub-networks (clusters) along the ring by connecting nodes within a threshold distance of each other and rewires a small number of intra-cluster links randomly; and (ii) determines the number of inter-cluster links between every two neighboring sub-networks (clusters) and connects the corresponding number of nodes in the two neighboring sub-networks (clusters) at random as inter-cluster links. This

model has two limitations: (a) in practice, clusters in electric power networks are not uniform in size and connectivity density; and (b) inter-cluster connectivity cannot be accurately modeled by random connections since these represent long-haul connections which, as mentioned before, occur between groups of nodes in both clusters.

Hines *et al.* in [7, 8] provide a set of metrics to evaluate graph structures and compare the evaluation results of real electric power networks with those of networks generated from the Erdős-Rényi random graph model [3] and the small-world model[4]. They claim that models developed with only topological properties are not sufficient to generate a synthetic network for power grids, and to address this, they quantify an *electrical distance* between any two nodes. The authors create a matrix based on the proposed electrical distance and represent such electrical distances as an unweighted graph. However, their model suffers from the limitation that the resulting synthetic graph has a large number of isolated nodes, *i.e.* nodes do not connect to any other node in the network.

Rikvold *et al.* in [9, 10, 11] present another generative model for electric network graphs based on randomly connecting N nodes located in a square with a guarantee that there is no isolated node in the network. This model is limited because it only focuses on the averages of degree and impedance distributions, as a result of which, the model does not capture the clustering coefficient and the locally dense clustering behavior.

In general, large-scale electric power networks are naturally partitioned into smaller clusters, and hence many Independent System Operators (ISOs) in the U.S. manage their networks by partitioning them into zones. Related work on partitioning electric power networks are briefly reviewed below.

Hines *et al.* develop a partitioning model [12] modeled as an optimization problem with a multi-attribute objective function based on electrical distances, cluster sizes

and cluster connectedness. This model combines the *K-means* algorithm (*i.e.* a conventional method that partitions a network into clusters and adjusts clusters until no node is required to move between clusters) and an evolutionary algorithm to partition an electric power network. The limitation of this model is that the number of clusters needs to be determined in advance. If given a network without zone data, it is hard to tell how many zones are exactly reasonable.

In [13], Ezhilarasi *et al.* model the network partitioning problem as an optimization problem with the objective to minimize the maximum number of nodes within a cluster and the connecting lines between clusters. Peiravi *et al.* in [14] partition networks by calculating the eigenvector matrix of the Laplacian matrix for a given power system and determining clusters based on the signs of components in the Feidler vector, *i.e.* the second column of the eigenvector matrix.

However, these methods focus on partitioning the electric network graph for a specific application and not to generate a synthetic graph which can be useful for multiple applications. It is this latter that we focus on in this thesis.

1.4 Our Contributions

This thesis focuses on capturing the topological properties of electric power network graph. In particular, the two limitations described above in the RT-nested-smallworld model [1] in Section 1.3 are addressed in the model proposed in this thesis by determining clusters (and, consequently, non-uniform cluster sizes) from the original electric power network, and by explicitly incorporating empirical intra- and inter-cluster degree distributions in our model. We also have observed that electrical properties such as line impedance also exhibit differences among local and long-haul connections and will improve our model by assigning proper line impedance.

Our contributions are two-fold:

- To quantify the unique topological properties of the electric graph, a new model named *Cluster-and-Connect* model is proposed that identifies clusters using a novel clustering algorithm in a graph and parametrizes the generative model via number of clusters, number of nodes per cluster, as well as inter- and intra-cluster degree distributions. The final synthetic graph is obtained using a graph-creation algorithm developed at MIT [15] to generate inter- and intra-cluster linkages between nodes using the degree distribution parameters. The steps of generating synthetic graphs using the Cluster-and-Connect model are shown as follows:

  1. Identify and define clusters for a given network.

  2. Create connections between nodes within each cluster based on the intra-cluster degree distribution.

  3. Create inter-cluster connections via the inter-cluster degree distribution between every two clusters.

  4. Find and connect isolated nodes to obtain a complete synthetic network.

- A set of metrics proposed in the literature [7, 8] is used to evaluate the synthetic graph that our model generates; the same metrics are also used to evaluate the graphs generated from Wang *et al.*'s RT-nested-smallworld model [1]. The metrics evaluated for the original graph will serve as a benchmark to compare our algorithm with the existing algorithm in [1]. We advocate wide use of such standardized metrics to consistently evaluate and compare models of electric power networks. We demonstrate that our model captures topological characteristics of the power network such as highly dense local connectivity and clustering of nodes that connect (sparsely) to distant nodes. The New York Independent

System Operator (NYISO) system is selected as the test system for generating synthetic networks.

1.5 Outline of Thesis

The principal content of this thesis is partitioned into four chapters.

Chapter 1 presents an overview of the background, research objective, literature review and our contributions on modeling power grids and network partitioning. In Chapter 2, the system model and our Cluster-and-Connect model for generating synthetic network graphs composed of four algorithms are described. In Chapter 3, a set of metrics used to evaluate graphical properties is introduced. The results of using such metrics for both the original graph and the synthetic graph generated from the proposed approach are also presented. Chapter 4 examines how degree and impedance distributions can be released in a private manner that satisfies differential privacy. Chapter 5 concludes the thesis and enumerates the contributions of this research. Possible future work in related topics is also discussed.

Chapter 2

SYSTEM MODEL AND CLUSTER-AND-CONNECT MODEL

This chapter introduces a novel model for generating synthetic electric power networks, namely the Cluster-and-Connect model, to capture topological properties. The discussion starts with the introduction of the system model and proceeds to the Cluster-and-Connect model described as a sequence of four algorithms. The model also concludes with a brief description of modeling the impedance distribution for the network as an effort to model an electrical characteristic feature of the network.

2.1 System Model

Subsection 2.1.1 presents a graph representation of electric power networks. Analogous to Wang *et al.* [1], throughout this thesis, we use a circular graph layout, also referred to as the Kirk circle [16], to visualize the connectivity and topology of the electric power network graph. The description of Kirk circle is detailed in Subsection 2.1.2.

2.1.1 Graph Model for Electrical Networks

An electric power network can be represented by an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ where $\mathcal{V}$, the node set of the graph, consists of N nodes and $\mathcal{E}$, the edge set of m edges or links. Formally, an electric network graph consists of buses (the set $\mathcal{V}$ of nodes of $\mathcal{G}$) that are connected by (transmission or distribution) lines (the set $\mathcal{E}$ of edges of $\mathcal{G}$). A bus is either an injection or a non-injection bus, where injection implies the existence of either a generator (power source) and/or a load (power sink) at the bus.

For an electric power network with N nodes and m links, the number of edges connected to any node i, $i = 1, \ldots, N$, is defined as the nodal degree k_i of that node. The distance matrix D and the Laplacian matrix L are two other graph properties that will be used in this paper to define features of an electric power network graph.

For a graph $\mathcal{G}$, let d_{ij} denote the shortest path-length from node i to node j, i.e., d_{ij} is the minimum number of edges connecting node i to node j. The distance matrix D is then defined as

$$D = [d_{ij}]_{i=1,...,N, \; j=1,...,N}, \quad d_{ii} = 0, \text{ for all } i. \tag{2.1}$$

If there exists no path from node i to node j, then $d_{ij} = \infty$.

The Laplacian matrix L records edge connectivity in its off-diagonal elements and nodal degrees in its diagonal elements. The definition of the Laplacian matrix L is

$$L(i,j) = \begin{cases} -1, & \text{if } d_{ij} = 1, \\ k_i, & \text{for } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Finally, the connectivity of nodes in a graph is captured by an $N \times N$ adjacency matrix M with non-zero unit entries M_{ij} if and only if node i is connected to node j.

2.1.2 Kirk Circle

The Kirk circle [16] is a tool to visualize graph topologies used in [1] to compare different graph models for electric power networks and is similar to the circular layout graph drawing algorithm [17]. In general, nodes (buses) in an electric power network are numbered with neighboring nodes assigned consecutive or closely proximal numbers. This allows one to sequentially map these nodes in increasing order of their numerical labels to evenly spread points on a circle. The edge connections (branches) between nodes are indicated by straight lines (chords) between the appropriate points on the circle. In electric power networks, proximal and closely connected nodes indicate either geographical proximity or dependence on specific subsets of generators.

Figs. 2.1a-2.1b show two representative network topologies, using Kirk circle graphs, of the NYISO and the Western Electricity Coordinating Council (WECC)

systems respectively. Irrespective of size, both networks reveal a pattern of multiple clusters (identified with the algorithms detailed in Chapter 2) with dense intra-cluster connections and relatively sparser inter-cluster connections between clusters. Note that colors of intra-cluster connections within clusters 1-6 for both NYISO and WECC systems are cyan, red, green, yellow, pink and blue, respectively. Colors of inter-cluster links are gray. Boundary points for clusters are labeled A, B, C, D, E, F.
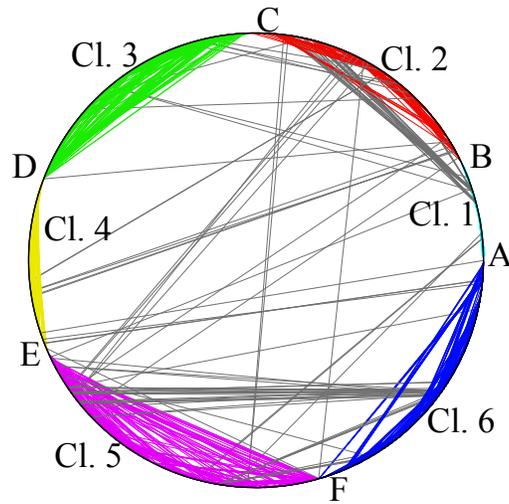
(a) NYISO System
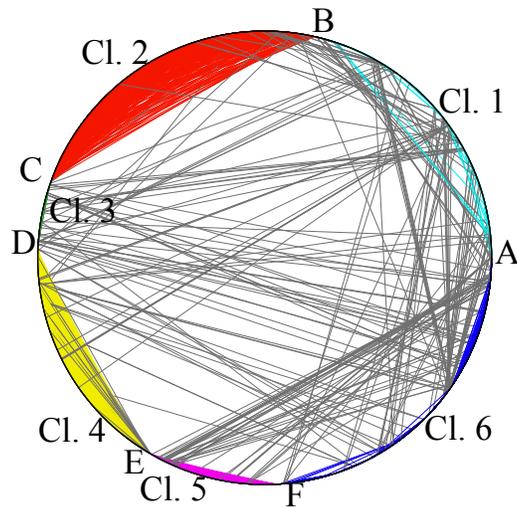
(b) WECC System

Figure 2.1: Kirk Circle Representations for NYISO and WECC Systems.

10

2.2   Cluster-And-Connect Model

The Cluster-and-Connect model is described as a sequence of four algorithms. As the first step to describing our algorithms, we formally define two quantities used in clustering and list all parameters of the network that we identify. Recall that the number of nodes in the graph is N and the edges are represented by chords in the Kirk circles.

Definition The *chord density profile* is obtained by sweeping the Kirk circle of an arbitrary radius, R, in angular steps of $(360/N)$ degree, such that at $(360/N)$ degree, the chord density is the ratio of the number of chords intersecting the radius to the total number of chords.

Definition A *cluster* is a slice (arc) of the Kirk circle whose boundary points are those at which the chord density profile has local minima satisfying a certain threshold γ.

We identify the following four parameters to generate synthetic electric network graphs.

1. Number of clusters (n_c)

2. Number of nodes in each cluster $(N_i,\ i = 1 \ldots n_c)$

3. Intra-cluster degree distribution $(p_i,\ i = 1 \ldots n_c)$

4. Pairwise inter-cluster degree distribution $(q_{ij},\ i, j = 1, 2, \ldots, n_c, i \neq j)$

Figure 2.2 presents a flowchart of our algorithmic approach to generate a synthetic graph from a given graph. The two broad steps are: (i) determine clusters; and (ii) generate the synthetic graph. Once the clusters are identified, the synthetic graph

is generated in three steps: (i) generating intra-cluster connections; (ii) generating inter-cluster connections; and (iii) eliminating isolated nodes.

We use inter- and intra-cluster degree distributions, p_i and q_{ij}, respectively, to generate a synthetic graph. Degree distributions can be generated using either statistical models (*e.g.*, [1]) or by determining the empirical distribution from the real graphs and then sampling from them. In this paper, we use the latter approach. For evaluation purposes, we also use the original graph's parameters directly. To this end, we introduce a sampling flag F_S to indicate whether the original or sampled degree sequences will be used in the generating algorithms. Finally, we note that the inputs n_c and the N_i's to Algorithms 2-4 are obtained from Algorithm 1. We now briefly describe Algorithms 1-4.

2.2.1   Cluster Identification

Our clustering algorithm captures the graphical observation that there appears to be boundary points on the Kirk circle (*e.g.*, see points A, B, C, D, E and F in Figs. 2.1a and 2.1b) such that nodes within these boundary points are more densely connected amongst themselves than to those outside. We seek to detect these boundary points, and thereby, identify clusters. While some nodes on the Kirk Circle may be well connected to relatively distant (on the circle) nodes (*e.g.* nodes in clusters 5 and 6 in Fig. 2.1a), we use the fact that consecutively numbered nodes are physically adjacent to determine the clusters they belong to.

As the first step to determining cluster boundaries, we eliminate the inter-cluster connections in Algorithm 1. To this end, we introduce an inner circle of radius $R_{in} = \eta R$ where η is defined as a fraction of the Kirk circle radius, such that all chords that cross the inner circle do not contribute to the chord density count. The chord angle η determines the maximal angle that a chord subtends at the center of the circle, and a smaller η allows us to include larger angles and vice versa. The
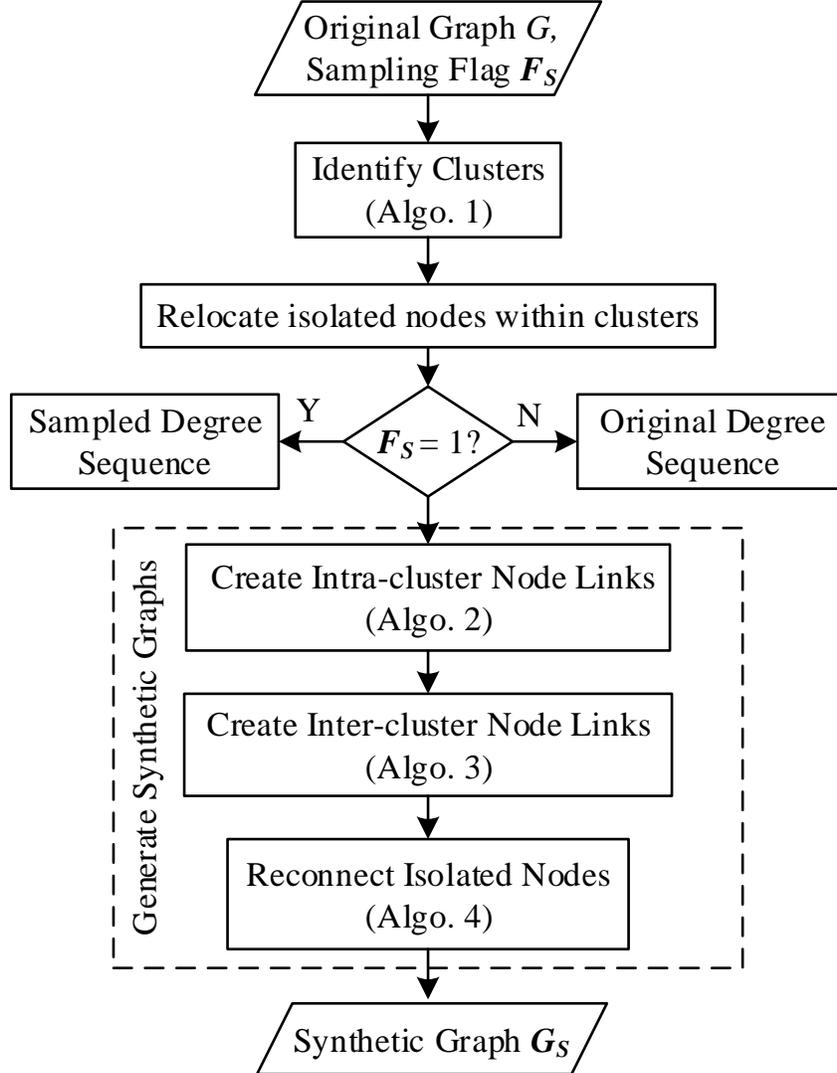
Figure 2.2: Flowchart for Cluster-and-Connect Model

subplots 2.3a and 2.3b in Fig. 2.3 show the corresponding smoothed chord density profiles for the NYISO and WECC data, respectively.

Note that there are a number of local minima and maxima in subplots 2.3a and 2.3b in Fig. 2.3. To identify clusters of sufficient size (and acknowledge the fact that within a cluster a subset of nodes may be more connected than others), we use a threshold parameter γ to determine the cluster boundaries, and therefore, the total number of clusters. Without loss of generality, we assume we begin at a minimum.
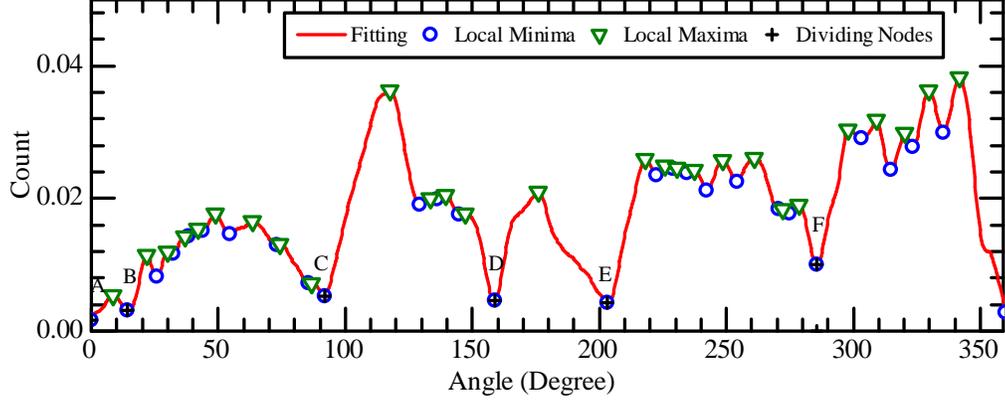
(a) NYISO with Boundary Nodes {A,B,C,D,E,F}={1,112,739,1295,1652,2331}
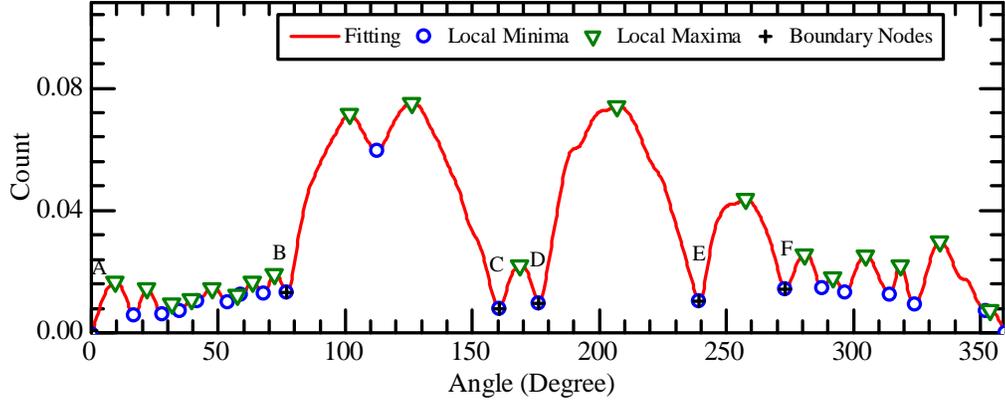
(b) WECC with Boundary Nodes {A,B,C,D,E,F}={1,1031,2203,2410,3278,3756}

Figure 2.3: Chord Density Profiles for NYISO and WECC Systems with $\gamma = 2.5$

Then, at the kth local minimum, we compute two ratios: one using the immediate maximum to the left of this minimum $E_{\max}[k-1]/E_{\min}[k]$ and the other using the immediate maximum to its right $E_{\max}[k]/E_{\min}[k]$. If any of the two ratios is larger than γ, the local minimum is determined to be a boundary point. Thus, the choice of γ determines the number of clusters n_c and the corresponding cluster sizes N_i. Finally, we check the nodal degrees of nodes within each cluster for intra-cluster connections. If any isolated node does not connect to any other neighboring node within the same cluster that means its nodal degree is zero, it will move to the immediately connecting

node in another cluster. If more than one node is connected to the isolated node, we will choose the connecting node where the line impedance between these two nodes is smaller.

Algorithm 1 Identifying Clusters
Inputs: η, γ, and the Laplacian matrix L for a given graph $\mathcal{G}$.

Outputs: n_c, N_i for all $i = 1, ..., n_c$.

1. Draw a Kirk circle with an arbitrary radius R based on L for a given graph $\mathcal{G}$.

2. Draw an inner circle with a radius $R_{in} = \eta R$. Ignore links that pass through the inner circle.

3. Obtain the number of nodes N as the number of rows (or columns) of L. Sweep the Kirk circle in steps of $(360/N)^\circ$ from the first node in a counter-clock fashion. Keep count of the number of chords that intersect the sweeping radius as a vector S_{chord}.

4. Fit a curve S_{fit} for the points in S_{chord}.

5. Find local maxima $E_{\max}[1 : K-1]$ and local minima $E_{\min}[1 : K]$ for fitted curve S_{fit}.

6. For $k = 2, ..., K - 1$, calculate ratios $R_{\text{prev}}[k - 1] = E_{\max}[k - 1]/E_{\min}[k]$ and $R_{\text{next}}[k - 1] = E_{\max}[k]/E_{\min}[k]$ with its two adjacent maxima.

7. Set first boundary node $BN[1] = E_{\min}[1] = 1$. For $k = 2, ..., K - 1$, if eithor $R_{\text{prev}}[k - 1]$ or $R_{\text{next}}[k - 1] > \gamma$, record $E_{\min}[k]$ as a boundary node.

8. Obtain the set of boundary nodes $BN[1 : n_c]$.

2.2.2 Generating Synthetic Graphs

Using the clusters identified above, Algorithms 2 and 3 generate inter- and intra-cluster linkages, respectively. To do so, we use the software package *MIT Matlab tools for network analysis* offered by the Strategic Engineering Research Group of MIT [15]. In particular, we use the program *graph_from_degree_sequence.m* which does the following: it orders the input degree sequence in descending order and connects nodes with the highest adjacent degrees iteratively until all degrees are assigned. It is worth noting that the graph obtained using the deterministic MIT algorithm [15] will in general be different from the original graph.

At the outset, Algorithm 2 creates a degree sequence realization $DS_i[1 : N_i]$ which is obtained by sampling p_i for cluster i if $F_S = 1$ or is set to the degree sequence of the original graph if $F_S = 0$. This is then used to generate intra-cluster connections using the MIT toolbox [15].

Following Algorithm 2, we create inter-cluster connections using Algorithm 3. This algorithm requires two degree sequences as inputs: DO_{ij} of outgoing edges from cluster i to cluster j obtained from q_{ij} and DO_{ji} of outgoing edges from cluster j to cluster i obtained from q_{ji}. If $F_S = 1$, DO_{ij} and DO_{ji} are sampled from q_{ij} and q_{ji}, respectively; otherwise, they are computed from the original graph. Since two degree sequences are involved, we tailor the *graph_from_degree_sequence.m* program to create connections using degree sequences from two clusters; we do so in a similar fashion to the original algorithm by ordering both sequences in descending order of degrees.

The detailed steps of Algorithms 2 and 3 are listed below.

Algorithm 2 Intra-cluster Node Linkage

Inputs: n_c, N_i, p_i for all $i = 1, ..., n_c$, and F_S.

Output: Adjacency matrix M_S for synthetic graph $\mathcal{G}_S$.

1. Start from cluster $i = 1$. If $F_S = 1$, generate a sample degree sequence $DS_i[1 : N_i]$ from p_i, otherwise make $DS_i[1 : N_i] = p_i$.

2. If $\sum_k DS_i[k]$ is odd, add one degree to $DS_i[1]$.

3. Reorder $DS_i[1 : N_i]$ in descending order as $\widetilde{DS}_i[1 : N_i]$ with reordered node indices $Ind_i[1 : N_i]$.

4. If $\widetilde{DS}_i[1] = 0$, exit loop. Else, start with $\widetilde{DS}_i[1]$.

5. Create links from $Ind_i[1]$ to $Ind_i[2]$, $Ind_i[3]$, ..., $Ind_i[\widetilde{DS}_i[1] + 1]$ by updating corresponding entries of M_S.

6. Subtract 1 from $\widetilde{DS}_i[2 : \widetilde{DS}_i[1] + 1]$ and set $\widetilde{DS}_i[1] = 0$.

7. Repeat Steps 3-6 until $\widetilde{DS}_i[1] = 0$ in Step 4.

8. Repeat Steps 1-7 for $i = 2, ...n_c$.

Algorithm 3 Inter-cluster Node Linkage

Inputs: n_c, N_i, q_{ij} for all $i, j = 1, ..., n_c, i \neq j$ and F_S.

Output: Updated adjacency matrix M_S for synthetic graph $\mathcal{G}_S$.

1. Start from cluster $i = 1, j = i + 1$. First, if $F_S = 1$, generate an out-going degree sequence $DO_{ij}[1 : N_j]$ by sampling q_{ij}. Then generate out-going degree sequence $DO_{ji}[1 : N_i]$ from q_{ji}, stopping when $\sum DO_{ij}[1 : N_i] = \sum DO_{ji}[1 : N_j]$. If $F_S = 0$, $DO_{ij}[1 : N_j] = q_{ij}$ and $DO_{ji}[1 : N_i] = q_{ji}$.

2. Reorder $DO_{ij}[1 : N_i]$ and $DO_{ji}[1 : N_j]$ in descending order as $\widetilde{DO}_{ij}[1 : N_i]$ and $\widetilde{DO}_{ji}[1 : N_j]$ with reordered node indices $Ind_{ij}[1 : N_i]$ and $Ind_{ji}[1 : N_j]$, respectively.

3. Start with $\widetilde{DO}_{ij}[k = 1]$.

4. Create links from $Ind_{ij}[k]$ to $Ind_{ji}[1]$, $Ind_{ji}[2]$, ..., $Ind_{ji}[\widetilde{DO}_{ij}[k]]$ by updating M_S.

5. Subtract 1 from $\widetilde{DO}_{ji}[1 : \widetilde{DO}_{ij}[k]]$.

6. Reorder $\widetilde{DO}_{ji}[1 : N_j]$ in descending order as $\widetilde{DO}_{ji}[1 : N_j]$ with reordered node index $Ind_{ji}[1 : N_j]$.

7. Repeat Step 3-6 for $k = 2, ..., N_i$.

8. Repeat Steps 1-7 for $j = i + 2, ..., n_c, j \neq i$.

9. Repeat Steps 1-8 for $i = 2, ..., n_c$.

Finally, unlike an actual electric network graph, a synthetic graph generated from degree distributions may have islands (*i.e.*, the nodes whose distance matrix entries relative to each other are non-infinite but no node in the island has finite distance relative to any node outside this island). After generating intra- and inter-cluster connections, Algorithm 4 checks for islands using the distance matrix D and resolves them by rewiring or adding links.

Algorithm 4 Reconnecting Islands
___
Input: Adjacency matrix M_S of synthetic graph $\mathcal{G}_S$.

Output: Final adjacency matrix of $\mathcal{G}_S$.

1. Identify the islands using the distance matrix D (those elements with $d_{ij} = \infty$).

2. Determine the vector of degrees of the nodes in the islands $Deg_c[:]$ and sort it in descending order as $\widetilde{Deg_c}[:]$ and the sorted node index as $Ind_{Deg}[:]$.

3. Identify a link outside the island with end nodes o_l and o_k.

4. If $\widetilde{Deg_c}[1]$ is connected to some node j within the island with degree $\geqslant 2$, then replace this link with two links from $Ind_{Deg}[1]$ and $Ind_{Deg}[j]$ to o_l and o_k, respectively. Update M_S. Go to Step 6.

5. Else, introduce a new link from $\widetilde{Deg_c}[1]$ to node o_ℓ and update M_S.

6. End.

2.2.3 Line Impedance Assignment

A complete model of generating synthetic networks should not only limit to the properties of topology but should also focus on the electrical properties. Line impedance is considered to be one of the most important properties that represents

the weight of each line as well as the geographical distance between every two nodes (in general, the line impedance is proportional to the physical distance for each line).

From the observations obtained by Wang *et al.* [5], they conclude that the distribution of line impedance of power grids is heavily tailed according to the empirical histogram Probability Distribution Function (PDF). Two candidate distribution functions, the log-normal distribution and the double Pareto log-normal (DPLN) distribution, can be used for fitting the NYISO data. These two distributions are defined below.

Given the original data x,

Log-normal:

$$\log n(x|\mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp(\frac{-(\ln x - \mu)^2}{2\sigma^2}) \tag{2.3}$$

where μ and σ are the mean and standard deviation;

Double Pareto log-normal:

$$\begin{aligned} \text{DPLN}(x|\alpha, \beta, \mu, \sigma) = \frac{\alpha\beta}{\alpha + \beta}[A(\alpha, \mu, \sigma)x^{-\alpha-1}\Phi(\frac{\ln x - \mu - \alpha\sigma^2}{\sigma}) \\ + A(-\beta, \mu, \sigma)x^{\beta-1}\Phi^c(\frac{\ln x - \mu + \beta\sigma^2}{\sigma})] \end{aligned} \tag{2.4}$$

where $A(\theta, \mu, \sigma) = \exp(\theta\mu + \theta^2\sigma^2/2)$. Φ and Φ^c are cumulative distribution function and complementary cumulative distribution function, respectively. α and $-\beta$ ($\alpha > 0$, $\beta > 0$) are the two roots of the quadratic equation $\frac{\sigma^2}{2}z^2 + (\mu - \frac{\sigma^2}{2})z - \lambda = 0$, where λ is a constant rate.

Wang *et al.* in [1] observes that the original impedance data may have an interrupted tail which can be captured by the clipping. To do so, the impedance data are clipped with exponential cutoff. The log-normal and double Pareto log-normal (DPLN) are modified to the log-normal-clip and DPLN-clip distributions.

Given the original data x, the clipped impedance data is [1]:

$$z_{\text{clip}}(1 - \exp(-\frac{x}{z_{\text{clip}}})) \tag{2.5}$$

where z_{clip} is the impedance cutoff threshold.

The log-normal-clip and DPLN-clip distributions are formally defined as follows.

Log-normal-clip:

$$\mathrm{log}n_{\text{clip}}(x|\mu, \sigma, z_{\text{clip}}) = \frac{z_{\text{clip}}}{z_{\text{clip}} - x} \times \mathrm{log}n(-z_{\text{clip}} \times \ln(1 - \frac{x}{z_{\text{clip}}})|\mu, \sigma) \qquad (2.6)$$

DPLN-clip:

$$\mathrm{DPLN}_{\text{clip}}(x|\alpha, \beta, \mu, \sigma, z_{\text{clip}}) = \frac{z_{\text{clip}}}{z_{\text{clip}} - x} \times \mathrm{DPLN}(-z_{\text{clip}} \times \ln(1 - \frac{x}{z_{\text{clip}}})|\alpha, \beta, \mu, \sigma) \quad (2.7)$$

Note that the fitting parameters of both log-normal-clip and DPLN-clip distributions are estimated from the original impedance distribution following the maximum likelihood criterion mentioned in [1], and are guaranteed for the best fitting distribution by an appropriately modified *Kolmogorov-Smirnov test* (K-S test).

Chapter 3

EVALUATION METRICS AND SIMULATION RESULTS

In this chapter, we first introduce a set of metrics to evaluate graphs. These metrics mainly focus on topological properties and are applied to evaluate the synthetic graphs generated from the Cluster-and-Connect model. As shown by Wang *et al.* in [1], line impedance can be statistically modeled by specific well-studied heavy-tailed distributions, *e.g.*, log-normal-clip and double Pareto log-normal-clip (DPLN-clip) as verified by us earlier in Section 2.2.3. We use these distributions, as appropriate, to model the distribution of intra-cluster line impedances; for the inter-cluster case, due to the sparse connections, we sample directly from the empirical distribution as detailed in this chapter.

3.1   Topological Evaluation Metrics

The topological properties of a graph can be quantified by a set of measures that capture connectivity and distance properties between nodes formally. These measures, including maximum degree, average degree, clustering coefficient, and related quantities, are presented in the context of electric power network graphs by Cotilla-Sanchez *et al.* in [7]. Analogous to [7], we define and identify the following measures that will be applied to evaluate graph generation models below. Recall that N is the number of nodes and m is the number of edges in the graph $\mathcal{G}$.

- Maximum degree k_{max}: the maximal nodal degree among all the nodes,

$$k_{\mathrm{max}} = \max_i k_i,\ i = 1, ..., N. \tag{3.1}$$

- Average degree k_{avg}: the mean of the nodal degrees for all the nodes,

$$k_{\mathrm{avg}} = \frac{\sum_{i=1}^{N} k_i}{N}. \tag{3.2}$$

- Diameter $d_{\max}$: the maximal shortest path length in the network, *i.e.*, for d_{ij} in (2.1), $d_{\max}$ is given by

$$d_{\max} = \max_{ij} d_{ij}, \ i, j = 1, ..., N, \ i \neq j. \tag{3.3}$$

- The characteristic path length L_{char}: the mean of all shortest path lengths traversing from one node to the other,

$$L_{\mathrm{char}} = \frac{\sum_{\forall i, j \, i \neq j} d_{ij}}{N(N-1)}. \tag{3.4}$$

- The clustering coefficient C of a graph $\mathcal{G}$ is the average value of the node clustering coefficient c_i over all nodes $i = 1, 2, ..., N$, that quantifies the extent to which nodes in a graph tend to locate nearby and connect to each other given by

$$C = \frac{\sum_{i=1}^{N} c_i}{N}, \tag{3.5}$$

where c_i denotes the fraction of the actual existing links $\varphi(i)$ among the nodes connected to node i over all the possible links (complete graph) that can exist among these nodes as,

$$c_i = \frac{2\varphi(i)}{k_i(k_i - 1)}. \tag{3.6}$$

- Assortativity r: quantifies the extent to which nodes connect to nodes with similar degrees. Formally, it is the correlation in degree for the nodes on opposite ends of each link

$$r = \frac{m^{-1} \sum_{i=1}^{m} a_i b_i - [m^{-1} \sum_{i=1}^{m} \frac{1}{2}(a_i + b_i)]^2}{m^{-1} \sum_{i=1}^{m} \frac{1}{2}(a_i^2 + b_i^2) - [m^{-1} \sum_{i=1}^{m} \frac{1}{2}(a_i + b_i)]^2}, \tag{3.7}$$

where a_i and b_i are the degrees of the endpoints of link i.

- Algebraic connectivity $\lambda_2(L)$[18]: is the second smallest eigenvalue of the Laplacian matrix L reflecting how well a network is connected. If the $\lambda_2(L)$ is close

to 0, the network is close to being disconnected; if $\lambda_2(L)/N$ is close to 1, the network is close to being fully connected.

3.2   Simulation Results

In this section, we use the metrics introduced in Section 3.1 to evaluate the accuracy of the synthetic graphs generated from our Cluster-and-Connect model for the NYISO system; we also use the same metrics to evaluate the synthetic graph generated using the RT-nested-smallworld model proposed in [1] and compare it with those for our model. The metrics for the original graph are used as a benchmark to evaluate both algorithms.

The test system is the NYISO system, which has 2935 nodes and 7028 edges. We use the Cluster-and-Connect algorithm to generate two synthetic graphs: (i) synthetic graph SG_1 generated using sampled degree sequences; and (ii) synthetic graph SG_2 generated using original degree sequences. Observe that if the sampling flag $F_s = 1$, it means the graph SG_1 is generated with the sampled degree sequences from p_i and q_{ij} respectively for intra- and inter-cluster connections; if setting $F_S = 0$, the intra- and inter-cluster degree sequences are obtained directly from the original graph and SG_2 is generated. We remark that SG_2 shares the same degree distribution as the original graph. However, as mentioned earlier, the intra- and inter- connections of SG_2 are different from the original graph as a result of using the MIT Toolbox [15] in Algorithms 2 and 3 (see Fig. 2.2). Wang *et al.'s* algorithm [1] is also utilized to generate a RT-nested-smallworld graph SG_{RTSW} with 3000 nodes.

We now compare the original NYISO graph with SG_1 and SG_2, both visually using the Kirk circle and the metrics introduced in Section 3.1. In Fig. 3.1, subplots 3.1a-3.1d correspond to the Kirk circle representations of SG_1, SG_2, SG_{RTSW}, and the original NYISO system, respectively.
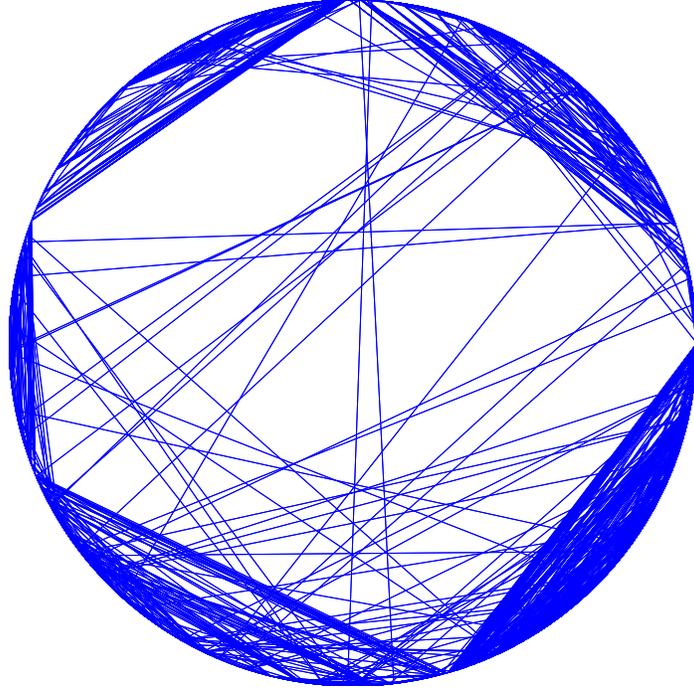
(a) Cluster-and-Connect: SG_1, Sampled Degree Sequence
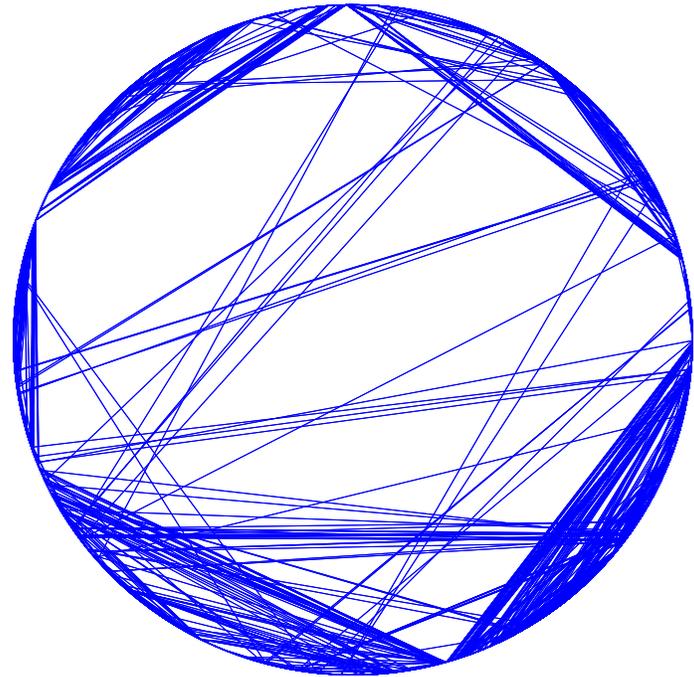
(b) Cluster-and-Connect: SG_2, Original Degree Sequence
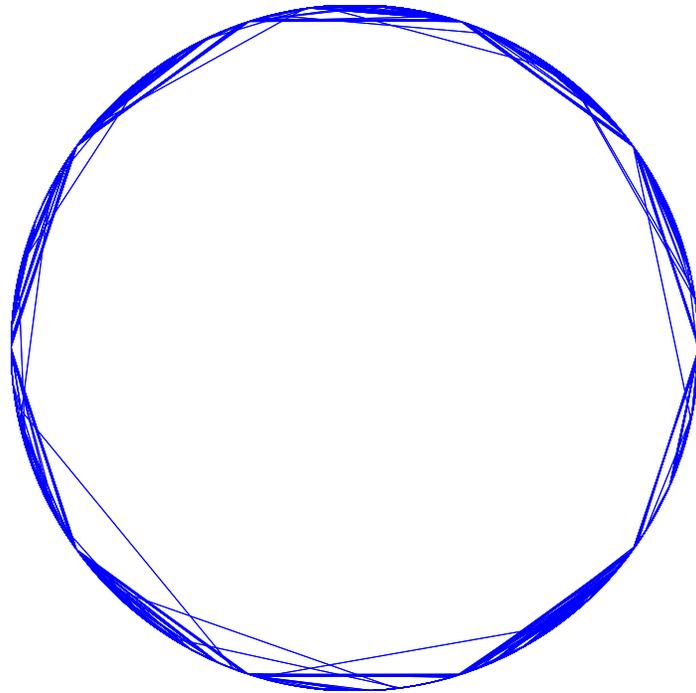
(c) RT-nested-smallworld: SG_{RTSW}
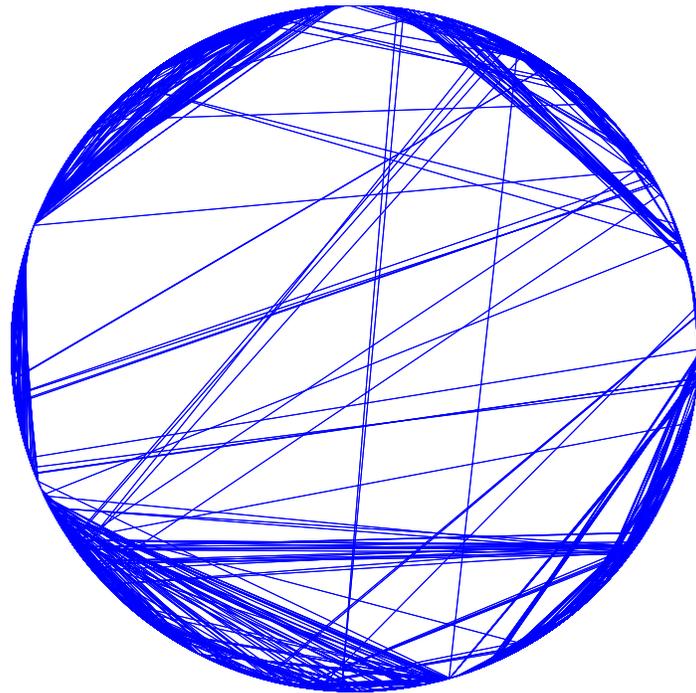
(d) the NYISO System

Figure 3.1: Kirk Circles Representations of Synthetic Networks Generated from the NYISO System Using Different Models and the Original NYISO Network.

Table 3.1: Metrics for Different Graphs Based on the NYISO System

Metrics	NYISO	Cluster-and-Connect (i): SG_1, sampled	Cluster-and-Connect (ii): SG_2, original	RT-nested-smallworld: SG_{RTSW}
Nodes	2935	2935	2935	3000
Links	7028	7066	7177	7082.66
k_{max}	38	38	38	20.63
k_{avg}	4.7891	4.8150	4.8497	4.7218
d_{max}	54	33	39	40.83
L_{char}	18.4209	11.6786	11.9361	18.5481
C	0.2096	0.2648	0.2626	0.1791
r	0.4487	0.7372	0.7452	-0.1027
$\lambda_2(L)$	0.0015	0.0026	0.0024	0.0012

As seen from the Kirk circles in Figs. 3.1a-3.1d, our synthetic graphs SG_1 and SG_2 generated from Cluster-and-Connect model appears to capture the inter-node connections more accurately than the synthetic graph generated from RT-nested-smallworld model for the NYISO system. We quantify our evaluation metrics for all four graphs in Table 3.1. Note that, since the process for generating SG_1 is stochastic, we generate a hundred synthetic graphs and report the average of various graph statistics. For SG_2, there is only one synthetic graph because the degree sequence is deterministic.

We briefly discuss the results in Table 3.1.

- *Nodes*: Our model keeps the same number of nodes as in the NYISO system. Wang *et al.*'s model [1] approximates this number because of the uniform size of clusters.

- *Links*: The number of links in synthetic graphs SG_1 and SG_2 are often larger than those in the original NYISO graph. The additional links are added in Algorithm 4 because of isolated nodes generated in Algorithm 2.

- $k_{\max}$, k_{avg}: In contrast to our model that captures the empirical distributions p_i and q_{ij}, thereby preserving $k_{\max}$, we observe that the RT-nested-smallworld model, due to the limited size of each cluster, restricts the maximal degree for any node significantly relative to the original. For all models, we observe that k_{avg} is very close to the original value; this is so because k_{avg} is dominated by local links. In addition to the $k_{\max}$ and k_{avg}, we also compare the degree distributions of the synthetic graphs themselves. Fig. 3.2 compares the degree distributions of three synthetic networks—SG_1, SG_2 and SG_{RTSW}. Since we model the degree distributions explicitly, the Cluster-and-Connect model preserves the degree distribution better than the RT-nested-smallworld model.
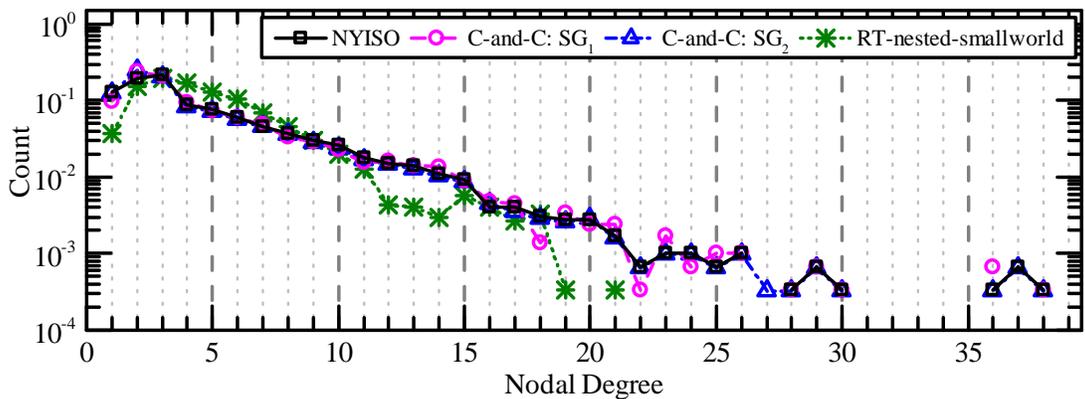
Figure 3.2: Degree Distributions of Graph Nodes for Synthetic and Original Graphs of the NYISO System.

- $d_{\max}$, L_{char}: We observe that $d_{\max}$ and L_{char} of synthetic graphs SG_1 and SG_2 generated from our Cluster-and-Connect algorithm are lower compared to the original graph. This is due in part to the limitation of Algorithm 2 in modeling the intra-cluster links; it appears these links are locally dense within clusters which our model does not capture accurately. The restricted neighborhood model of RT-nested-smallworld appears to capture this, however, at the cost of other metrics $k_{\max}$, C and r.

- C: Since we reconnect isolated nodes in Algorithm 4 by adding extra links, the resulting clustering coefficients C of our synthetic graphs SG_1 and SG_2 are higher than that of the original NYISO graph. Nevertheless, from Fig. 3.3— which compares the average clustering coefficients (ACC) of graph nodes of a particular degree (rather than the clustering coefficient of the entire graph) of the three synthetic networks—we see that our model preserves the average clustering coefficient of graph nodes far better than the RT-nested-smallworld model.
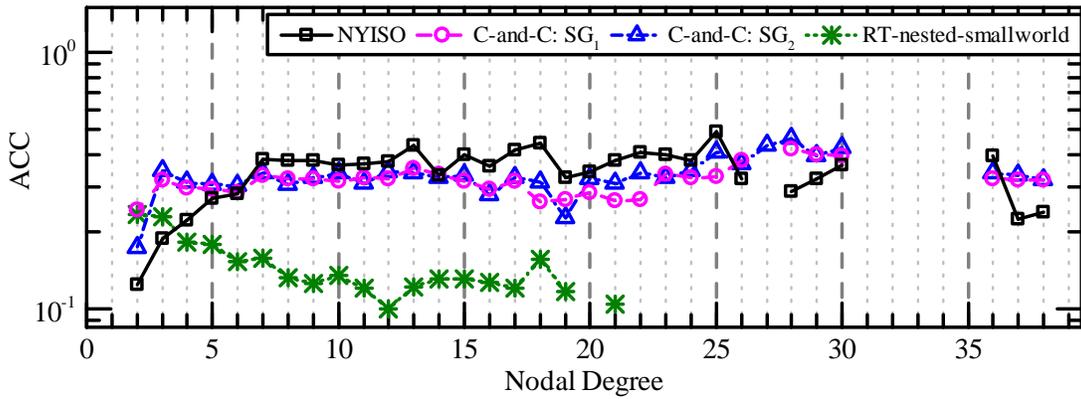
Figure 3.3: Average Clustering Coefficients (ACC) of Graph Nodes for Synthetic and Original Graphs of the NYISO System.

- r and $\lambda_2(L)$: The assortativity r captures the extent to which nodes of similar degrees are connected to each other; the restricted neighborhood model of RT-

nested-smallworld leads to a very small, in fact, negative, r which indicates that connected nodes have largely different degrees. On the other hand, our model captures but overcompensates due to the preferential connectivity of similar degree nodes in Algorithm 2. One observes a similar behavior with $\lambda_2(L)$.

Recall that in Subsection 2.2.3, Wang *et al.* in [1] conclude that both the log-normal-clip and double Pareto log-normal-clip (DPLN-clip) distributions fit for the impedance data of the NYISO system. In addition to the results of fitting impedance data for the entire network shown in [1], we apply the two distributions on the impedance data of each cluster respectively to get impedance data estimated for the intra-cluster connections. The impedance of inter-clusters connections between two clusters are assigned directly by sampling from the original impedance data because the inter-cluster connections are so sparse.

The fitting parameters of the log-normal-clip and DPLN-clip distributions for each cluster of the NYISO system are listed in Table 3.2 and Table 3.3, respectively. The probability density function of the log-normal-clip and DPLN-clip distributions are shown in Fig. 3.4 and Fig. 3.5 respectively, including the original impedance distribution and the fitting distribution. Note that, each subplot shows the distributions for one cluster.

We briefly discuss the results in Fig. 3.4 and Fig. 3.5. It is clear that both log-normal-clip distribution and DPLN-clip distribution fit well on the line impedance data for intra-cluster connections. However, in some clusters, *e.g.* cluster 2 in Figs. 3.4 and 3.5, there shows a peak in the middle of the fitting curves which represents the deviation from the original impedance data for the small impedances. The improvement of the fitting distributions for the impedance data, in particular for the small impedances, can be our future work.

Fig. 3.6 shows the probability density function of the impedance of inter-cluster connections between clusters. From the distribution, one can observe that the inter-cluster connections are very sparse. In our model, we sample impedance of inter-cluster connections directly from the empirical impedance distribution.

Table 3.2: Parameters of Log-normal-clip Distribution Fitting for Probability Distribution Functions of Line Impedance

Cluster Number	μ	σ	z_{clip}
1	-3.9598	2.2580	1.9773
2	-2.9299	1.4930	1.9008
3	-2.8271	1.8279	2.4633
4	-3.2525	1.9771	2.4595
5	-2.9910	1.7404	2.6024
6	-1.1771	1.9197	3.6503

Table 3.3: Parameters of DPLN-clip Distribution Fitting for Probability Distribution Functions of Line Impedance

Cluster Number	α	β	μ	σ	z_{clip}
1	44.0000	44.0000	-3.9598	2.2578	1.9773
2	45.0000	44.0000	-2.9294	1.4926	1.9008
3	44.0000	45.0000	-2.8276	1.8276	2.4633
4	45.0000	45.0000	-3.2525	1.9769	2.4595
5	44.0000	45.0000	-2.9915	1.7401	2.6024
6	45.0000	44.0000	-1.1766	1.9194	3.6503

31
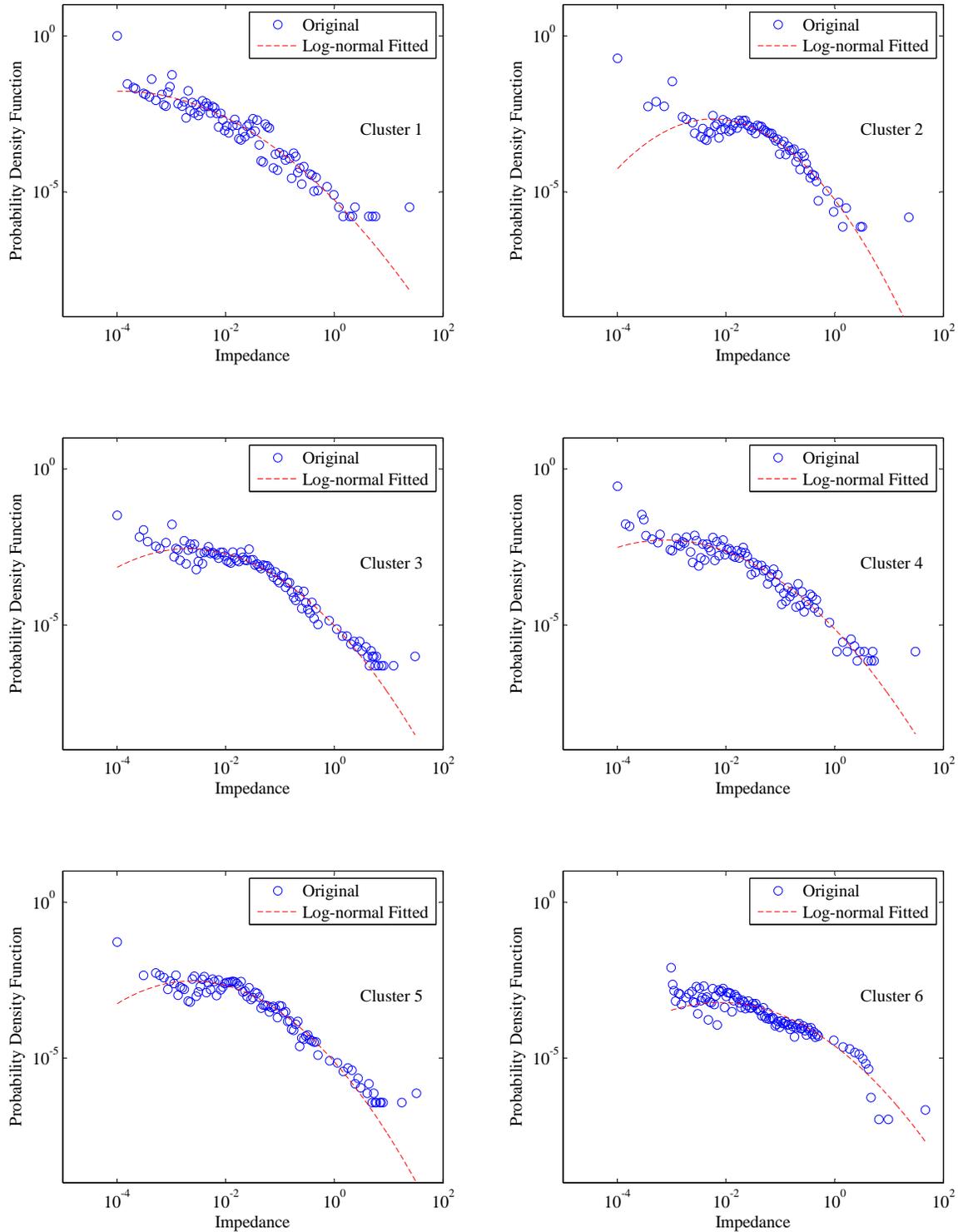
Figure 3.4: Log-normal-clip Distribution Fitting for Probability Density Function of Line Impedance in Each Cluster of NYISO Network
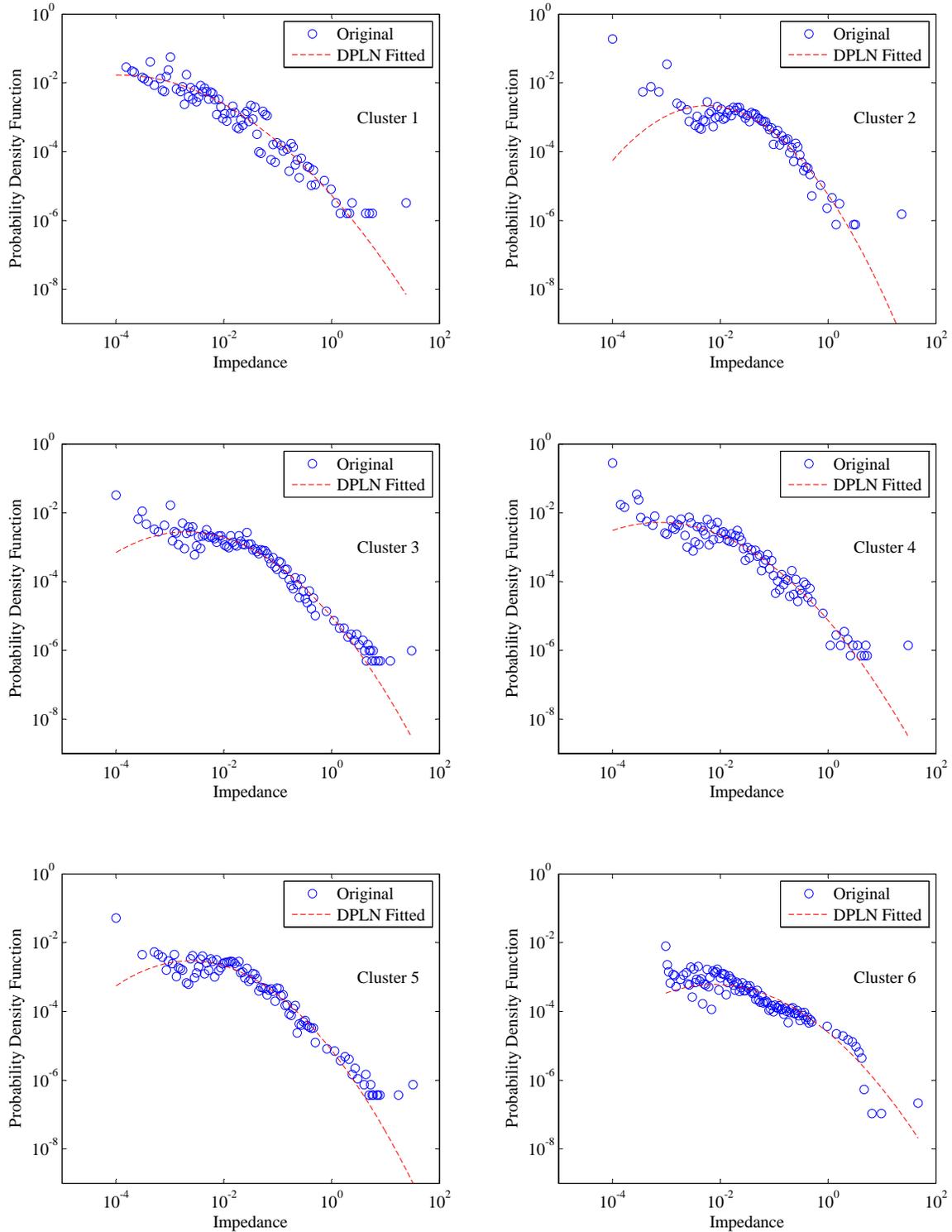
Figure 3.5: DPLN-clip Distribution Fitting for Probability Density Function of Line Impedance in Each Cluster of NYISO Network
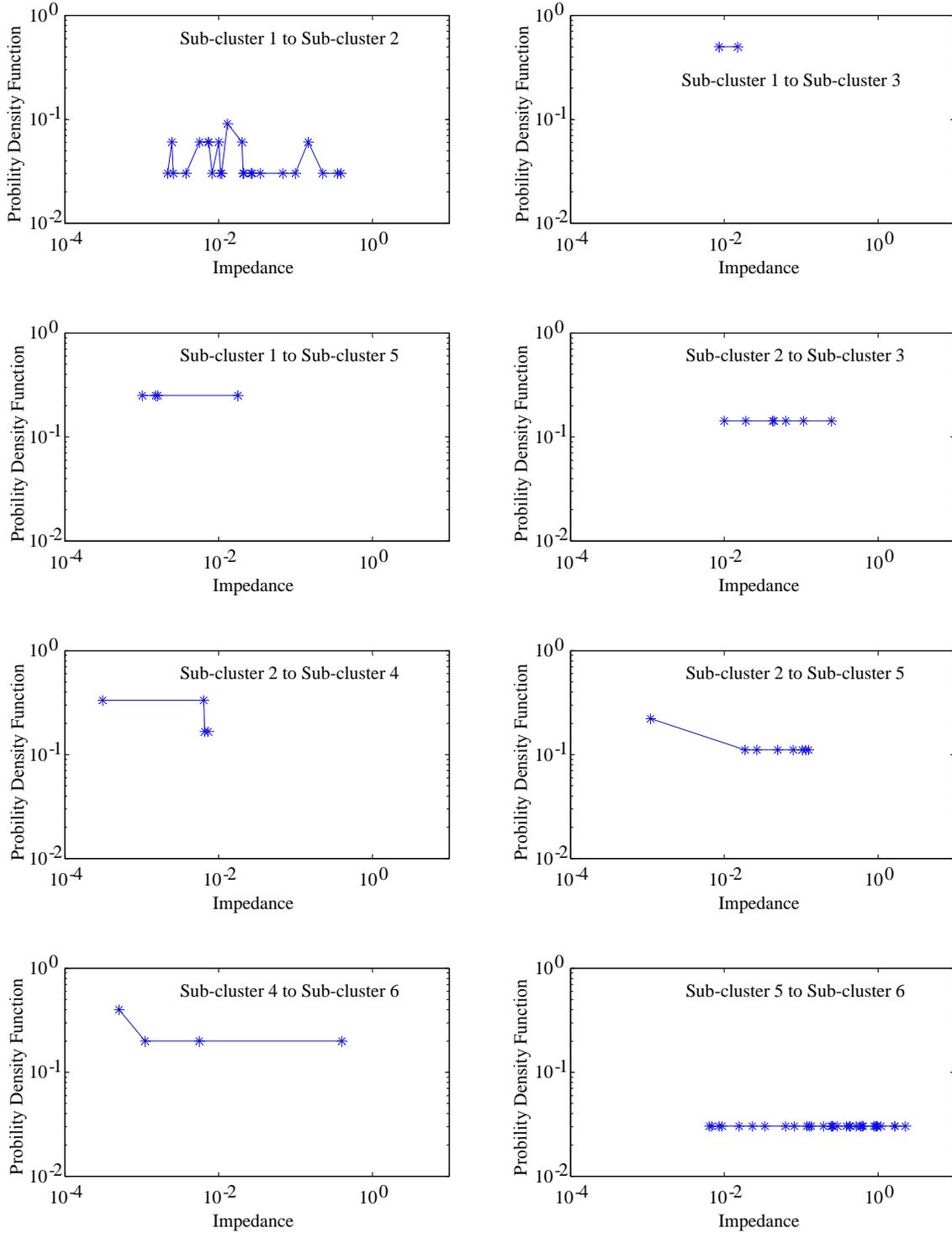
Figure 3.6: Probability Density Function of Impedance of Inter-cluster Connections Between Clusters for NYISO Network

Chapter 4

DIFFERENTIAL PRIVACY

In this chapter, we consider the problem of generating privacy-assured degree and impedance distributions. This involves adding noise in a judicious manner to degree and impedance sequences obtained from the original graph. Iteratively, one could also generate realizations of degree and impedance sequences from their empirical distributions and add noise intelligently to preserve a desired measure of privacy. To this end, we first consider statistical models for degree and impedance data of an electric power network graph. In [1], Wang *et al.* introduce statistical models for the two distributions and we first overview their model in Section 4.1.

4.1   Model for Nodal Degree Distribution and Impedance Distribution

We use statistical models, introduced by Wang *et al.* in [1], that are designed to fit the degree and impedance data. Comparing the empirical and fitted distribution with and without differential privacy provides a meaningful method to compare the effect of privacy on the distribution of nodal degree and line impedance.

4.1.1   Nodal Degree Distribution

It has been observed that the nodal degree distribution of electric networks displays an exponential tail, *i.e.*, a simple model for the nodal degree could be $p(k_i) = \exp(-k_i/k_{\mathrm{avg}})/k_{\mathrm{avg}}$ where $p(k_i)$ is the probability mass function (PMF) of node i having a degree k_i. However, in [1], using real-world power grid data, the authors show that while a geometric distribution fits well for the tail distribution (of nodal degrees), for the range of small node degrees, that is $k_i \leq 3$, the empirical PMF deviates from such a model and in fact requires modeling the degree distribution as resulting from a random variable N_D that is the sum of two random variables G and D, where G and D are non-uniform discrete and truncated (due to the finite degree bound) geometric random variables, respectively. Thus, we have

$$N_D = G + D \tag{4.1}$$

where G is a truncated geometric with parameter p and truncation length of $k_{\max}$ such that

$$\Pr(G = k) = \frac{(1-p)^k \, p}{\sum_{k=0}^{k_{\max}} (1-p)^k \, p} \tag{4.2}$$

$$= \frac{(1-p)^k \, p}{1 - (1-p)^{k_{\max}+1}}, \quad k = 0, 1, 2, ..., k_{\max}, \tag{4.3}$$

and D is a discrete random variable with probabilities $\{p_1, p_2, ..., p_{k_t}\}$ such that

$$\Pr(D = k) = p_k, \quad k = 1, 2, ..., k_t. \tag{4.4}$$

The nodal degree then has a distribution given by the convolution of G and D, i.e., $\Pr(N_D = k) = \Pr(G = k) \otimes \Pr(D = k)$, $k = 0, 1, 2..., k_t + k_{\max} - 1$. To estimate the parameters of the two component distributions of N_D, and therefore, that of N_D itself, one can use the probability generating function (PGF) where the PGF of a random variable X is defined as $M_X(z) = \sum_k \Pr(X = k) z^k$. Furthermore for sum of independent random variables, the PGF of N_D is given as $M_{N_D}(z) = M_G(z) M_D(z)$.

One can estimate the PGF of a random variables X from an observed (sample set) data set of size N_X using the fact that the expected value of the function z^X, where the expectation is over X, can be obtained by taking the average of z^X over all feasible values of X [1]. Thus, for a given sample data set X of size N_X, the PGF of X can be estimated from the mean of z^X as

$$E\left(z^X\right) = \sum_k \frac{n_{(X=k)}}{N_X} z^k \tag{4.5}$$

$$\approx \sum_k \Pr(X = k) z^k, \tag{4.6}$$

where $n_{(X=k)}$ denotes the total number of data items that take the value k. For very large N, the 'type count' $n_{(X=k)}/N$ approaches $\Pr(X = k)$ allowing the approximation in (4.6). For the distributions modeled in (4.3) and (4.4), one can show that the PGF of N_D is

$$M_{N_D}(z) = \frac{p\left(1 - ((1-p)z)^{k_{\max}+1}\right)\sum_{i=1}^{k_t} p_i z^i}{\left(1 - (1-p)^{k_{\max}+1}\right)(1 - (1-p)z)}. \tag{4.7}$$

Thus, M_{N_D} has $k_{\max}$ zeros evenly distributed on a circle of radius $1/(1-p)$ from the truncated geometric and k_t zeros from the discrete distribution. The approximation in (4.6) is then used to obtain contour plots of the PGF to determine the zeros of $M_{N_D}(z)$ and therefore effectively the parameters p, k_t, and $k_{\max}$ as well as the starting values for the discrete distribution probabilities $\{p_i\}_{t=1}^{k_t}$. We illustrate the use of such contour plots to obtain these parameters and compute a distribution fitting the empirical nodal degree distribution in the following section.

4.1.2 Line Impedance Distribution

Wang *et al.* propose a variety of heavy-tailed distributions in [1] such as Gamma, Generalized Pareto, log-normal, and double Pareto log-normal as statistical models for impedance distributions of several electric power networks, *e.g.,* IEEE bus systems and the NYISO system. We use the IEEE 300 bus system for line impedance data in our simulation. The fitting model proposed for this in [1] is a generalized Pareto distribution with parameters u, δ, and θ given by

$$GP(x|u, \delta, \theta) = \left(\frac{1}{\delta}\right)\left(1 + u\frac{(x - \theta)}{\delta}\right)^{-1-(1/u)}. \tag{4.8}$$

The data from the 300 bus system will be used to estimate these parameters for both the cases with and without the application of a differentially private mechanism.

In the following section we detail the privacy mechanism we use for both node and edge privacy in graphs.

4.2   Differential Privacy for Graphs

In this section, we introduce the notion of differential privacy. Next, we outline our algorithms for generating differentially private synthetic degree and line impedance vectors for a power grid network in Subsections 4.2.1 and 4.2.2.

Assume that a database consists of data from k different entities. Differential privacy rests on the guarantee that an entity's contribution to any outcome of a data analysis that includes its data should almost be the same whether or not the entity is in or out of the database, even for entities with unique or outlier behaviors. This implies that an entity's risk of being identified is almost the same whether or not the entity is in the database or not. Hence such an entity is the "unit of protection".

Differential privacy relies on the notion of neighboring databases [19]—in our context, two neighboring graphs. Intuitively, two databases are neighbors if they differ only in one entity's data. Formally:

Definition ([19]) A randomized algorithm $\mathcal{A}$ provides ϵ-*differential privacy* if for all neighboring input data sets DB, DB', and for all $S \subseteq Range(\mathcal{A})$, $\Pr[\mathcal{A}(DB) \in S] \leq \exp(\epsilon) \cdot \Pr[\mathcal{A}(DB') \in S]$.

That is, the probability distribution induced by a database DB on the range of outputs of the randomized algorithm A is close to the probability distribution induced by its neighbor DB' (again on $Range(\mathcal{A})$) . As a result, an individual entity's presence or absence in the database does not (significantly) change the risk of inferring anything sensitive about them, providing the entity with protection. The smaller the value of ϵ, the closer these two probabilities are, and hence, the higher the privacy.

In the case of graphs, the "entity" (the "unit of protection") could either be an edge or a node. This is captured by the notions of "edge privacy" and "node privacy"

respectively. Two graphs are edge-neighbors if they differ in the presence or absence of exactly one edge. Formally:

Definition (Edge neighborhood) Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the (edge) *neighborhood of a graph* is the set

$$\Gamma(G) = \{\mathcal{G}'(\mathcal{V}, \mathcal{E}') \text{ s.t } |\mathcal{E} \oplus \mathcal{E}'| = 1\}$$

Hay et al. [20] also define *node differential privacy*, by analogously defining the notion of *node neighborhood* of a graph. Two graphs are node neighbors if they differ by at most one node and all the incident edges. Node-privacy covers the case when buses (and all incident connection lines) are kept private and edge-privacy covers the case when only a connection link is kept private.

Assuming that we are interested in computing a differentially private approximation to a function f of a database DB, $f : DB \to \mathbb{R}^\ell$, one way of achieving differential privacy is by adding (properly calibrated) noise to each element of the (vector) outcome of f. This calls for an introduction to the concept of *global sensitivity*. The *global sensitivity* of a function of a database [19] is the maximum change in the value of the function over all neighboring databases. Formally:

Definition ([19]) The *global sensitivity* of a function f of a database DB, $f : DB \to \mathbb{R}^\ell$ is

$$\mathrm{GS}_f := \max_{DB,DB'} |f(DB) - f(DB')|$$

where DB and DB' are neighbors.

One way of computing a differentially private approximation to the vector outcome of f is to add noise to each element of the vector that is sampled from an appropriate Laplace distribution. In more detail, let $\mathrm{Lap}(\lambda)$ denote a Laplace distribution with mean 0 and standard deviation $\sqrt{2}\lambda$ and let $\langle \mathrm{Lap}(\lambda) \rangle^\ell$ denote a length-ℓ vector of independent random samples from this distribution.

Theorem 1 ([19]) *For any $f : DB \to \mathbb{R}^\ell$, and $\epsilon > 0$, the following mechanism $\mathcal{A}$ is ϵ-differentially private:*

$$\mathcal{A}_f(DB) = f(DB) + \langle \mathrm{Lap}(\mathrm{GS}_f / \epsilon) \rangle^\ell.$$

We call this the *Laplace mechanism*, and make extensive use of it in this thesis. Often, we refer to ϵ as the "privacy spent" on computing an approximation to f in a differentially private manner. Another tool that we will make use of is the so-called *composition theorem* which helps us reason about the total privacy expenditure of a series of algorithms that take as input the same database DB.

Theorem 2 (Serial Composition [19]) *For $i \in [q]$, let $\mathcal{A}_i(DB)$ be an ϵ_i-differentially private mechanism executed on database DB. Then, any mechanism $\mathcal{A}$ that is a composition of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_q$, is $\sum_i \epsilon_i$-differentially private.*

4.2.1   Differentially Private Synthetic Degrees

Let $\mathbf{deg} = [k_1, k_2 \ldots k_N]$, be the degree vector of the original network $\mathcal{G}$. where element k_i is the degree of the i-th node. We assume that the number of nodes in a graph is public knowledge.

First, we compute a differentially private approximation to the emprical CDF of nodal degrees. For this purpose, we need to determine the bins of the CDF. The bins in the non-private case range from the minimum degree to the maximum degree of the graph; however, to compute a differentially private histogram and subsequently a CDF we will need to estimate the minimum and maximum degree in a private manner. Function DPMAXMINDEG in Line 2 of Algorithm 6 computes an ϵ_1-differentially private estimate of the minimum and maximum degree of the graph. This serves as a range for the bins of the differentially private CDF **dpbins**. On Line 4, the function DPCDF in Algorithm 5 is called with two arguments: the degree vector **deg** of the

graph and the differentially private estimate of the bins vector **dpbins**. In DPCDF, a histogram (**counts**) from **deg** is computed by using **dpbins** as the bins. However, to get a ϵ_2-differentially private estimate of all the counts, we add a Laplace noise vector to the **counts** vector (Line 4); each element of this vector of length |**bins**| is drawn from $\mathrm{Lap}(\frac{2}{\epsilon_2})$. The 2 in the numerator of the scale factor of the Laplace is owing to the global sensitivity of a histogram which is 2. The noisy counts are now stored in the vector **dpcounts**.

In Line 5, we compute the cumulative noisy counts for each bin i, such that for each i **cumcounts**$_i = \sum_{j=1}^{i}$ **dpcounts**$_j$. At this stage, we know that **cumcounts** should be a non-decreasing sequence of counts; however, because of adding noise in Line 4, this is no longer the case. In Line 6 we, therefore, "clean up" some of this noise by using post-processing techniques of Hay et al. [20] that transform the noisy sequence of cumulative counts **cumcounts** into a non-decreasing sequence without "dipping back" into the data. PostProcCDF in Line 6 returns a "cleaned up" and properly normalized CDF, **dpcdf**.

At the end of Line 4 in Algorithm 6 we have a differentially private estimate of the CDF (with the bins). Invoking Theorems 2 and 1, we observe that this is $\epsilon = (\epsilon_1 + \epsilon_2)$-differentially private. To generate a synthetic vector of degrees using **dpcdf**, we sample from this differentially private empricial CDF (Lines 5-6) to get a synthetic degree vector **synthdeg** of length |**deg**|. Notice that steps 5 and onwards are conducted in a manner that is entirely oblivious of the underlying private data and only needs access to the differentially private CDF. Hence the entire Algorithm 6 is $\epsilon = (\epsilon_1 + \epsilon_2)$-differentially private.

4.2.2 Differentially Private Synthetic Impedances

Algorithm 7 computes a vector of differentially private impedances. In Line 2 we compute a differentially private approximation to the length of the impedance

---
Algorithm 5 Differentially Private CDF

1: **function** DPCDF($\mathbf{d}, \mathbf{bins}, \epsilon_2$)

2:      **counts** $\leftarrow$ HISTOGRAM($\mathbf{d}, \mathbf{bins}$)

3:      $\ell \leftarrow |bins|$

4:      **dpcounts** $\leftarrow$ **counts** $+\langle \mathrm{Lap}(2/\epsilon_2)\rangle^\ell$

5:      **cumcounts** $\leftarrow$ CUMSUM(**dpcounts**)

6:      **dpcdf** $\leftarrow$ POSTPROCCDF(**cumcounts**)

7:      **return dpcdf**

8: **end function**

Algorithm 6 Differentially Private Synthetic Degrees

1: **function** DPSYNTHDEGREE($\mathbf{deg}, \epsilon_1, \epsilon_2$)

2:      $[\mathrm{dpmin}, \mathrm{dpmax}] \leftarrow$ DPMAXMINDEG($\epsilon_1, \mathbf{deg}$)

3:      **dpbins** $\leftarrow \{\mathrm{dpmin}, \mathrm{dpmin} +1, \ldots, \mathrm{dpmax}\}$

4:      **dpcdf** $=$ DPCDF($\mathbf{deg}, \mathbf{dpbins}, \epsilon_2$)

5:      **for** $i \leftarrow 1 : |\mathbf{deg}|$ **do**

6:          **synthdeg**$_i \leftarrow$ SAMPLEFROM(**dpcdf**, **dpbins**)

7:      **end for**

8:      **return synthdeg**

9: **end function**
---

vector **Zprvector** (which is also number of edges in the graph); this incurs a privacy expenditure of ϵ_1. Just as in the case of degrees, we use Algorithm 5 to compute a differentially private CDF of the line impedances. The bins in this case are from the range of 0 to 2.5 with an increment of 0.1. Function DPCDF is invoked to compute an ϵ_2-differentially private CDF of the line impedances. Now, we can sample this empirical CDF dplength times to get an $(\epsilon_1+\epsilon_2)$-differentially private synthetic vector of line impedances.

Algorithm 7 Differentially Private Synthetic Impedances

1: **function** DPSYNTHIMP($\mathbf{Zprvector}, \epsilon_1, \epsilon_2$)

2:      numedges $= |\mathbf{Zprvector}|$

3:      dplength $\leftarrow$ numedges $+ \mathrm{Lap}(1/\epsilon_1)$

4:      $\mathbf{bins} \leftarrow 0 : 0.1 : 2.5$

5:      $\mathbf{dpcdf} = \mathrm{DPCDF}(\mathbf{Zprvector}, \mathbf{bins}, \epsilon_2)$

6:      **for** $i \leftarrow 1 : \mathrm{dplength}$ **do**

7:          $\mathbf{synthZpr}_i \leftarrow \mathrm{SAMPLEFROM}(\mathbf{dpcdf}, \mathbf{bins})$

8:      **end for**

9:      **return synthZpr**

10: **end function**

4.3   Illustration of Results

We now illustrate the model and privacy mechanism for the WECC nodal degree data and the IEEE 300 bus system impedance data. For our analysis, we use the publicly available software provided by the authors of [1] to estimate the parameters of the statistical model detailed in Section 4.1 to fit the data for each case. For both nodal degree and line impedance distribution, we choose four values of the privacy parameter ϵ: to remind the reader of how these values compare in privacy levels, we use the following nomenclature: "high" ($\epsilon = 0.1$), "medium" ($\epsilon = 0.4$ and $\epsilon = 0.6$), and "low" ($\epsilon = 2$) privacy.

4.3.1   Nodal Degree Distribution

Focusing first on the nodal degree distribution, we use the publicly available WECC data and obtain the vector of node degrees and the resulting empirical degree distribution for this data. The node degree vector is then input to the Algorithm 5 that outputs a private nodal degree distribution. This distribution is used in Algorithm 6 to create a synthetic vector of node degrees. For both the original and

the synthetic node degree vectors, the parameters for the fitted distribution (i.e., the truncated geometric parameters p and $k_{\max}$ and the discrete parameters $p_1, p_2, ..., p_{k_t}$) are obtained using the z-transform computation method and identification of the locations of zeros as detailed in Section 4.1. For the four chosen values for the privacy parameter ϵ, as one would expect, the fitted distributions for the private data approaches that of the raw (non-private) data as ϵ increases. For smaller values of ϵ, obfuscation is achieved by increased noise in the private nodal degree distribution. Figs. 4.1-4.4 compare the empirical and fitted distributions for the raw (non-private) and private node degree data for $\epsilon = 0.1$, 0.4, 0.6, and 2, respectively.

4.3.2   Line Impedance Distribution

To illustrate the effect of the proposed privacy mechanism on graph edges, we focused on the line impedance values for the IEEE 300 bus system. These impedance values are also subject to the differential privacy algorithm for edges and the resulting distribution of the private impedance values is used to obtain a sample vector of private impedance values as illustrated in Algorithm 7. The empirical and fitted (general Pareto) distributions for both the raw and private data are plotted for $\epsilon = 0.1$, 0.4, 0.6, and 2, in Figs. 4.5-4.8, respectively.

Note the heavy tailed distribution characteristic of the raw line impedance data. As with the nodal degree case, we observe here too that the empirical and fitted distributions for the private and non-private (raw) data approach each other as ϵ increases.
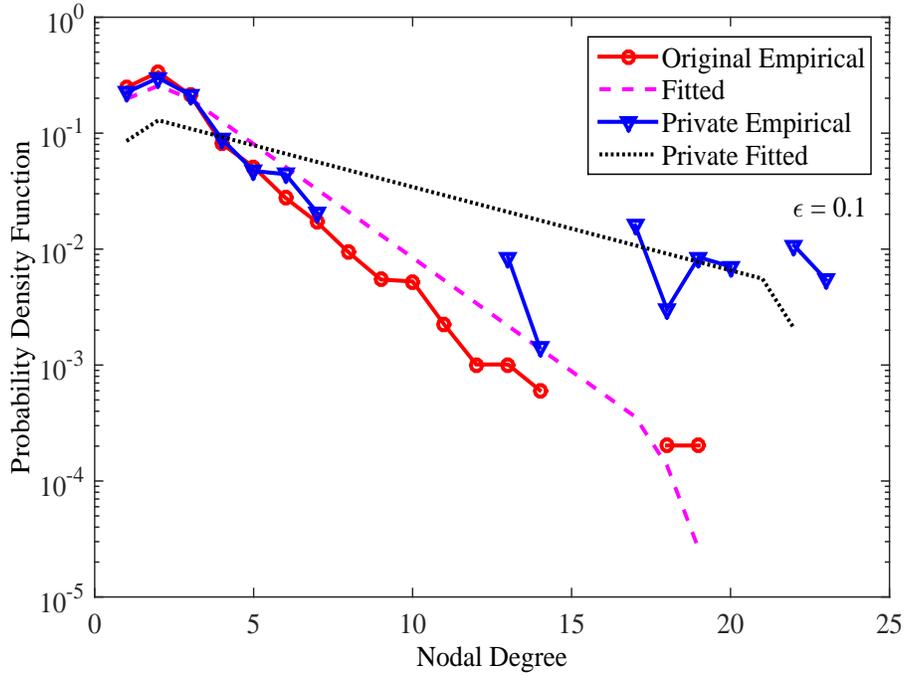
Figure 4.1: Plot of Empirical and Fitted Distributions for the Original and Private Nodal Degree Vector with Privacy Factor $\epsilon = 0.1$ (High Privacy).
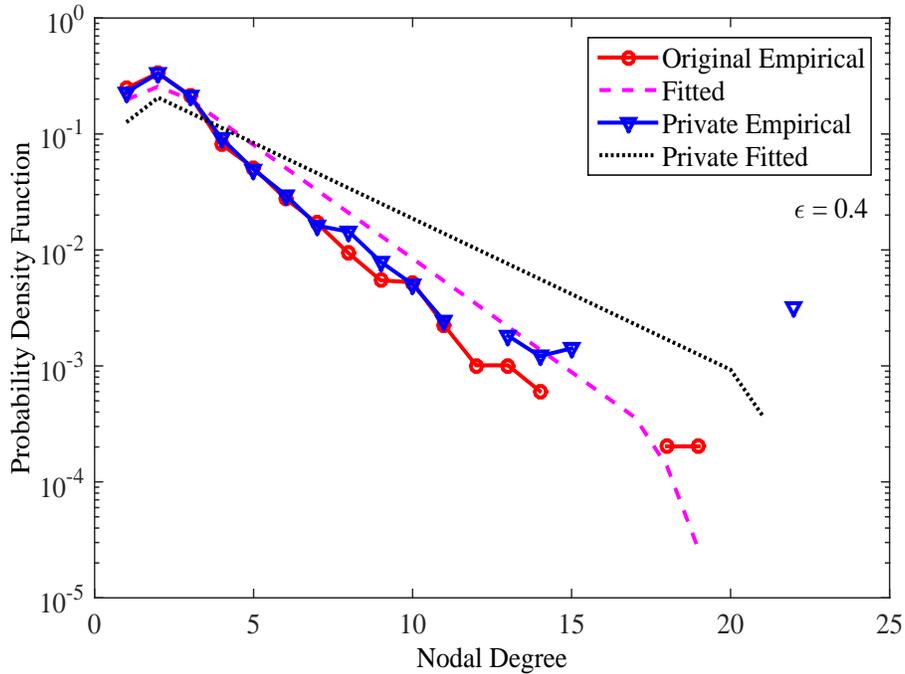
Figure 4.2: Plot of Empirical and Fitted Distributions for the Original and Private Nodal Degree Vector with Privacy Factor $\epsilon = 0.4$ (Intermediate Privacy).
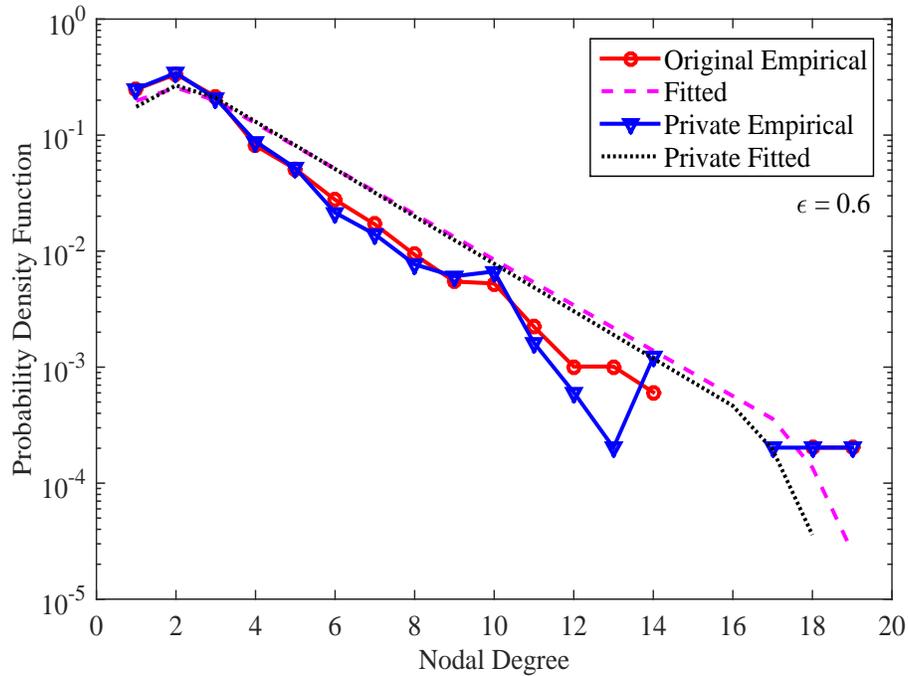
Figure 4.3: Plot of Empirical and Fitted Distributions for the Original and Private Nodal Degree Vector with Privacy Factor $\epsilon = 0.6$ (Intermediate Privacy).
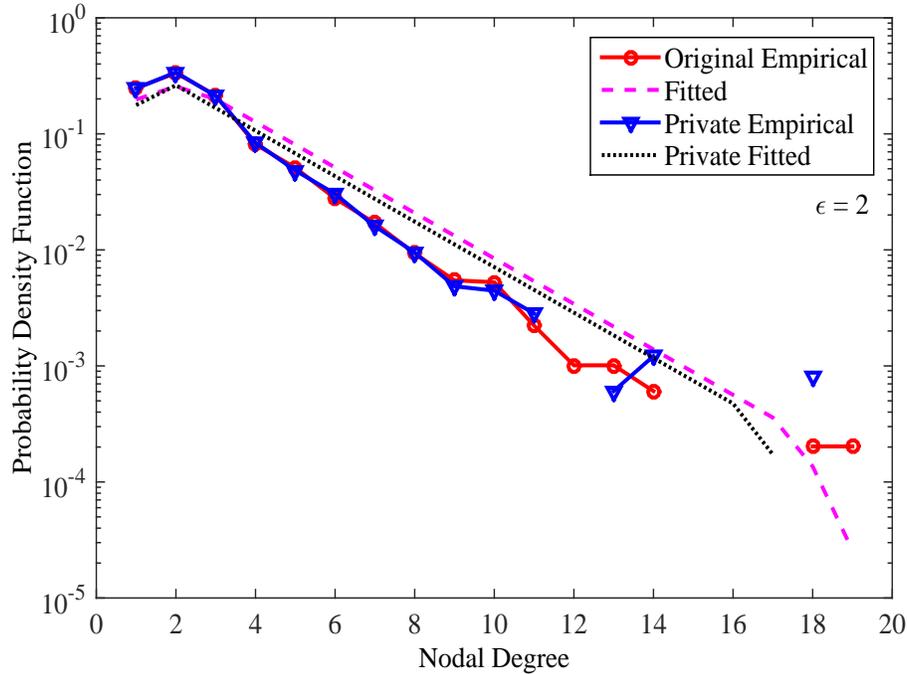
Figure 4.4: Plot of Empirical and Fitted Distributions for the Original and Private Nodal Degree Vector with Privacy Factor $\epsilon = 2$ (Low Privacy).
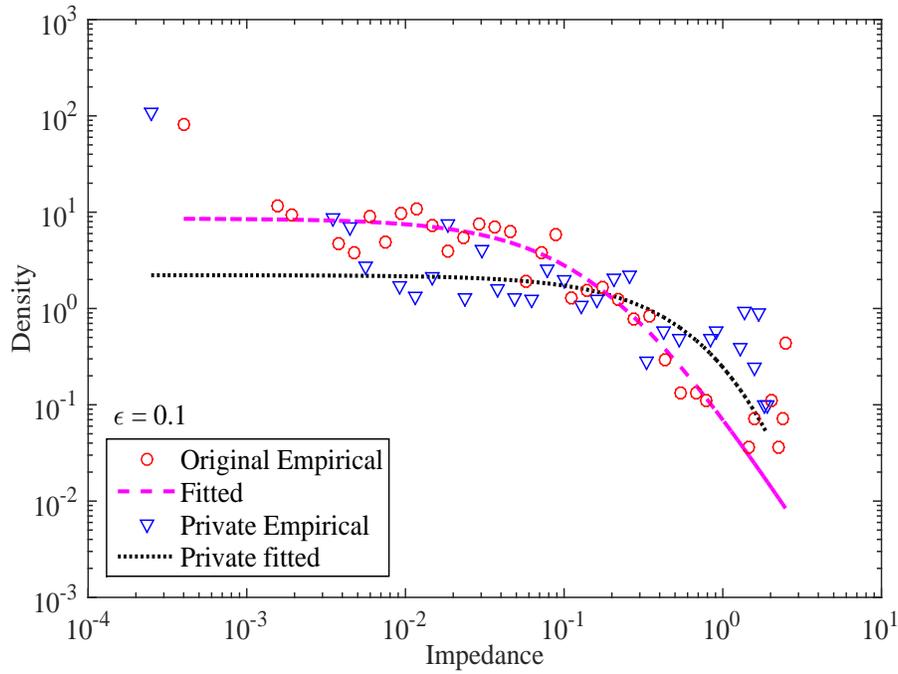
Figure 4.5: Plot of Empirical and Fitted Distributions for the Original and Private Line Impedance Vector with Privacy Factor $\epsilon = 0.1$ (High Privacy).
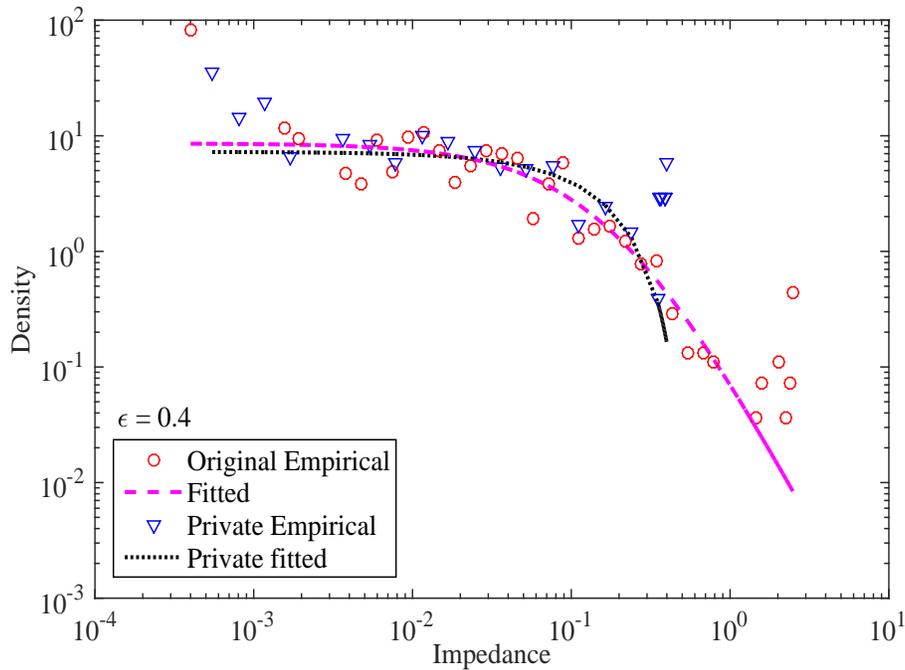
Figure 4.6: Plot of Empirical and Fitted Distributions for the Original and Private Line Impedance Vector with Privacy Factor $\epsilon = 0.4$ (Medium Privacy).
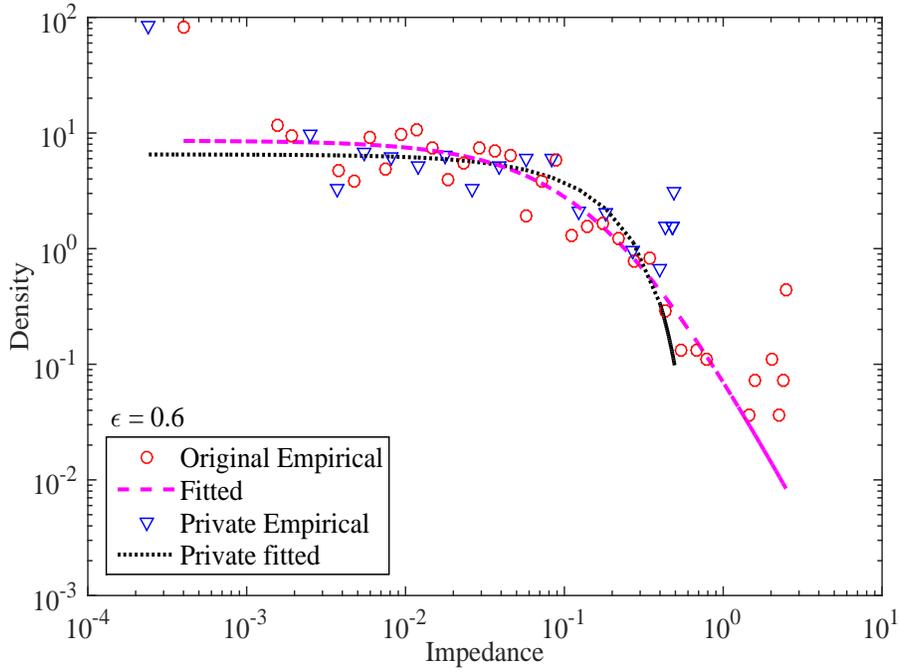
Figure 4.7: Plot of Empirical and Fitted Distributions for the Original and Private Line Impedance Vector with Privacy Factor $\epsilon = 0.6$ (Medium Privacy).
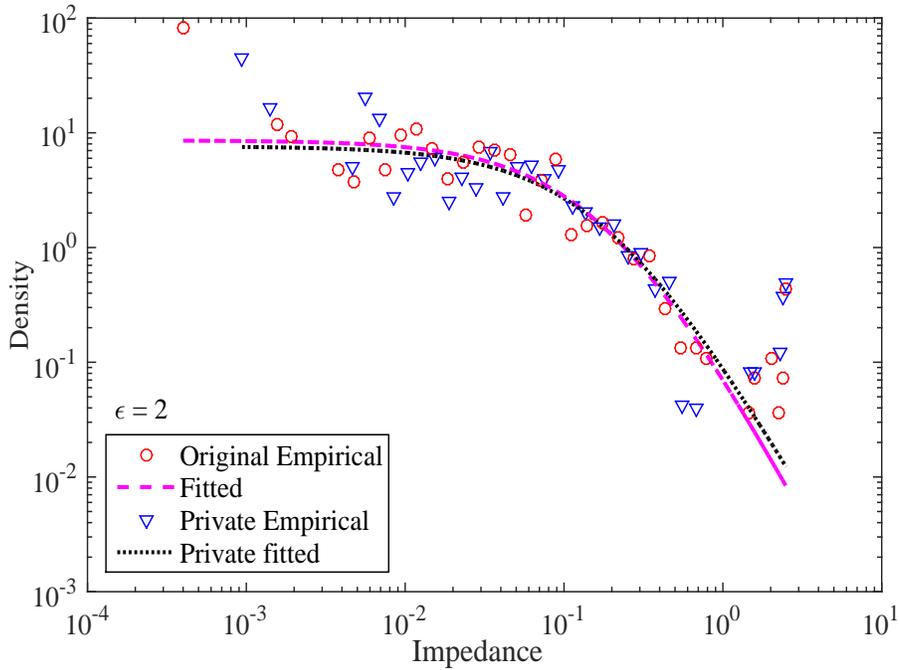
Figure 4.8: Plot of Empirical and Fitted Distributions for the Original and Private Line Impedance Vector with Privacy Factor $\epsilon = 2$ (Low Privacy).

Chapter 5

CONCLUSIONS AND FUTURE WORK

In this thesis, we have proposed Cluster-and-Connect model as a novel method for determining clusters and generating synthetic graphs for a given electric power network graph. We have modeled line impedance statistically using well-studied heavy-tailed distributions. The topological properties of a graph have been quantified by a set of topological metrics. These metrics have been used to evaluate our Cluster-and-Connect method and compare with evaluation results of the RT-nested-smallworld method introduced by Wang *et al.* in [1]; the metrics for original graph have been used as a benchmark to evaluate both approaches.

Our results suggest that the Cluster-and-Connect model captures the topological properties of the graph better than existing approaches and is a good candidate for generating synthetic power network graphs. More work is needed to understand the impact of such synthetic graphs on algorithm development and testing. Furthermore, the method itself can be refined to better distinguish the different types of nodes in an electric power system such as generator, load and connecting buses. This can allow better placement of generators for a more accurate representation of the system. Finally, it is also of interest to develop privacy-guaranteed synthetic graphs and more work on generating differentially private graphs is a step in that direction.

REFERENCES

[1] Z. Wang, A. Scaglione, and R. Thomas, "Generating statistically correct random topologies for testing smart grid communication and control networks," *Smart Grid, IEEE Transactions on*, vol. 1, pp. 28–39, June 2010. 1.1, 1.3, 1.4, 1.4, 2.1, 2.1.2, 2.2, 2.2.3, 2.2.3, 3, 3.2, 3.2, 3.2, 4, 4.1, 4.1.1, 4.1.1, 4.1.2, 4.3, 5

[2] C. Dwork, "Differential privacy," in *ICALP '06: Proceedings of the 33rd International Colloquium on Automata, Languages and Programming (2)*, pp. 1–12, 2006. 1.2

[3] P. Erdős and A. Rényi, "On random graphs. I," *Publ. Math.*, vol. 6, no. 290-297, 1959. 1.3

[4] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature*, vol. 393, no. 6684, pp. 440–442, 1998. 1.3

[5] Z. Wang, A. Scaglione, and R. J. Thomas, "On modeling random topology power grids for testing decentralized network control strategies," in *1st IFAC Workshop Estimation and Control Netw. Syst.(NecSys 09), Venice, Italy*, 2009. 1.3, 2.2.3

[6] Z. Wang, R. J. Thomas, and A. Scaglione, "Generating random topology power grids," in *Hawaii International Conference on System Sciences, Proceedings of the 41st Annual*, pp. 183–183, IEEE, 2008. 1.3

[7] E. Cotilla-Sanchez, P. Hines, C. Barrows, and S. Blumsack, "Comparing the topological and electrical structure of the North American electric power infrastructure," *Systems Journal, IEEE*, vol. 6, pp. 616–626, Dec 2012. 1.3, 1.4, 3.1

[8] P. Hines, S. Blumsack, E. C. Sanchez, and C. Barrows, "The topological and electrical structure of power grids," in *System Sciences (HICSS), 2010 43rd Hawaii International Conference on*, pp. 1–10, IEEE, 2010. 1.3, 1.4

[9] P. A. Rikvold, I. A. Hamad, B. Israels, and S. V. Poroseva, "Modeling power grids," *Physics Procedia*, vol. 34, pp. 119–123, 2012. 1.3

[10] I. A. Hamad, P. A. Rikvold, and S. V. Poroseva, "Floridian high-voltage power-grid network partitioning and cluster optimization using simulated annealing," *Physics Procedia*, vol. 15, pp. 2–6, 2011. 1.3

[11] I. A. Hamad, B. Israels, P. A. Rikvold, and S. V. Poroseva, "Spectral matrix methods for partitioning power grids: Applications to the italian and floridian high-voltage networks," *Physics Procedia*, vol. 4, pp. 125–129, 2010. 1.3

[12] E. Cotilla-Sanchez, P. Hines, C. Barrows, S. Blumsack, and M. Patel, "Multi-attribute partitioning of power networks based on electrical distance," *Power Systems, IEEE Transactions on*, vol. 28, pp. 4979–4987, Nov 2013. 1.3

[13] G. A. Ezhilarasi and K. S. Swarup, "Network decomposition using evolutionary algorithms in power systems," *International Journal of Machine Learning and Computing*, vol. 1, no. 1, pp. 93–99, 2011. 1.3

[14] A. Peiravi and R. Ildarabadi, "Complexities of using graph partitioning in modern scientific problems and application to power system islanding," *Journal of American Science*, vol. 5, pp. 1–12, September 2009. 1.3

[15] Strategic Engineering Research Group of MIT, "Matlab tools for network analysis," 2006-2011. 1.4, 2.2.2, 3.2

[16] J. Kirk, "Count loops in a graph," 2007. 2.1, 2.1.2

[17] R. Tamassia, *Handbook of Graph Drawing and Visualization (Discrete Mathematics and Its Applications)*. Chapman & Hall/CRC, 2007. 2.1.2

[18] M. Fiedler, *Laplacian of graph and algebraic connectivity*, vol. 25. Warsaw, Poland: Bamacj Cemter Publ., 1989. 3.1

[19] C. Dwork, F. McSherry, K. Nissim, and A. Smith, "Calibrating noise to sensitivity in private data analysis," in *TCC '06: In Proceedings of the 3rd Theory of Cryptography Conference*, pp. 265–284, 2006. 4.2, 1, 2

[20] M. Hay, C. Li, G. Miklau, and D. Jensen, "Accurate estimation of the degree distribution of private networks," in *Proceedings of the 2009 Ninth IEEE International Conference on Data Mining*, ICDM '09, (Washington, DC, USA), pp. 169–178, IEEE Computer Society, 2009. 4.2, 4.2.1