Phase Oscillator

by

Andrew Bates

A Thesis Presented in Partial Fulfillment
of the Requirements for the Degree
Master of Science

Approved April 2015 by the
Graduate Supervisory Committee:

Thomas Sugar, Chair
Sangram Redkar
Marc Mignolet

ARIZONA STATE UNIVERSITY

May 2015

ABSTRACT

A control method based on the phase angle is used to control oscillating systems. The phase oscillator uses the sine and cosine of the phase angle to change key properties of a mass-spring-damper system, including amplitude, frequency, and equilibrium. An inverted pendulum is used to show a further application of the phase oscillator. Two methods of control based on the phase oscillator are used for swing-up and balancing of the pendulum. The first control method involves two separate stages. The scenarios where this control works are discussed. The second control method uses variable coefficients to result in a smooth transition between swing-up and balancing.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

Figure Page

Figure Page

CHAPTER 1

INTRODUCTION

The phase oscillator has been used mainly in wearable robotic systems. The control was used to pump energy into a person by oscillating a backpack assisting load carriage [1], to control an exoskeleton to assist running by aiding hip extension and flexion [2, 3], and to control a simple hopping robot [4]. Using the phase angle to control an oscillating system is not a new idea. A. Jan Ijspeert et al. [5-7] uses phase angles to control the motion of an elbow orthosis. In their case, phase angles are used as a state estimator for position and velocity. The estimated state is used to compute control signals that are then used to force the system to follow a behavior. The phase oscillator uses the phase angle to directly determine a desired torque and does not force the system to follow a specific behavior. Instead of forcing a specific position at a point in time, the overall characteristics, such as amplitude, frequency, and equilibrium, are controlled.

This paper includes discusses the theory of the phase oscillator and how the phase oscillator can be used as a swing-up and balancing control for an inverted pendulum. Chapter 2 gives a brief background of controls used for swing-up and balancing of pendulums. Chapter 3 covers the theory of the phase oscillator. Chapter 4 discusses the theory of a pendulum and using the phase oscillator for swing-up and balancing. Chapter 5 gives the result of an experiment using Working Model 2D to simulate the swing-up and balancing of a pendulum. Chapter 6 discusses the results, future work, and contributions.

CHAPTER 2

BACKGROUND

The swing-up and balancing of an inverted pendulum is a classic controls problem. The nonlinear system is seen in many places from Segways to rockets. Many approaches have been taken to control the system, including energy based control [8-15], feedback stabilization [14,15], bang-bang control [16], sliding mode control [17-19], robust control [20], hybrid control [21], partial linearization [22,23], machine learning [24], and simulator-based foresight control [25].

One of the more common controls is the energy based control. The energy control method considers the current energy in the system and the total energy needed. When the energy in the system is less than the total energy needed, the control adds energy to the system, usually by providing the full torque in the same direction as the angular velocity. Once the system has enough energy, the control switches to a different method for balancing.

A problem with energy based control is the quick changes in the forcing function. When the velocity switches from positive to negative, the forcing function switches from full force positive to full force negative. In simulation, quick, large changes are not a problem. With a real motor, quick, large changes can damage the motor. The energy based control also has to be paired with another control for balancing. The switch in controls can cause another large change in the forcing function. The phase oscillator can be used for both swing-up and balancing and can create a continuous forcing function.

CHAPTER 3

PHASE OSCILLATOR

The general equation for a mass-spring-damper system is given by Equation 3.1 and solved by Equations 3.2 – 3.4.

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{3.1}$$

$$x = A\cos\omega t \tag{3.2}$$

$$\dot{x} = -A\omega\sin\omega t \tag{3.3}$$

$$\ddot{x} = -A\omega^2\cos\omega t \tag{3.4}$$

The frequency of oscillations of the system modeled by Equation 1 is dependent on the mass, m, and the spring constant, k. The behavior of the amplitude of the oscillations is dependent on the damping coefficient, b. If $b > 0$, the system has positive damping, and the oscillations will shrink and disappear over time. If $b < 0$, the system has negative damping, and the oscillations will continue to grow. If $b = 0$, the system has no damping, and the oscillations will remain a constant amplitude.

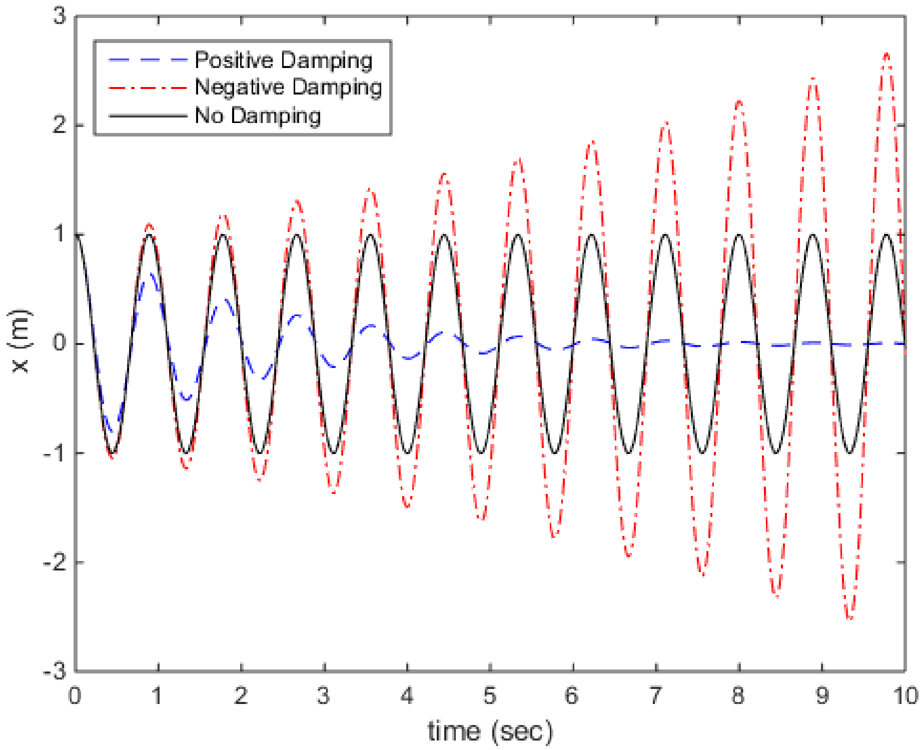

Figure 3.1: Spring response due to positive, negative, and no damping. $m = 1$ kg, $k = 50$ N/m, b is in Ns/m, initial position = 1 m, initial velocity = 0 m/s.

Figure 3.1 illustrates the how drastically changing the value of b can change the system. If the value of b is known, a forcing function can be used to manipulate the amplitude of the system. Equation 3.5 uses $c\dot{x}$ as a forcing function.

$$m\ddot{x} + b\dot{x} + kx = c\dot{x} \tag{3.5}$$

$$m\ddot{x} + (b - c)\dot{x} + kx = 0 \tag{3.6}$$

By applying a forcing function proportional to the velocity, the effective damping of the system can be changed. Rearranging the terms in Equation 3.5 shows the new coefficient of $\dot{x}$ to be $b - c$ (Equation 3.6). The value of c can be changed to grow, hold constant, or shrink the amplitude of the system. However, the value of b has to be known with extreme precision for this method to be effective, and a separate control must be used to control the value of c.

A simpler method of control is a phase oscillator. A phase oscillator adds energy to the system based on the phase angle of the system. Figure 3.2 shows the phase angle, ϕ, on a phase plot. ϕ is defined by Equation 3.7.

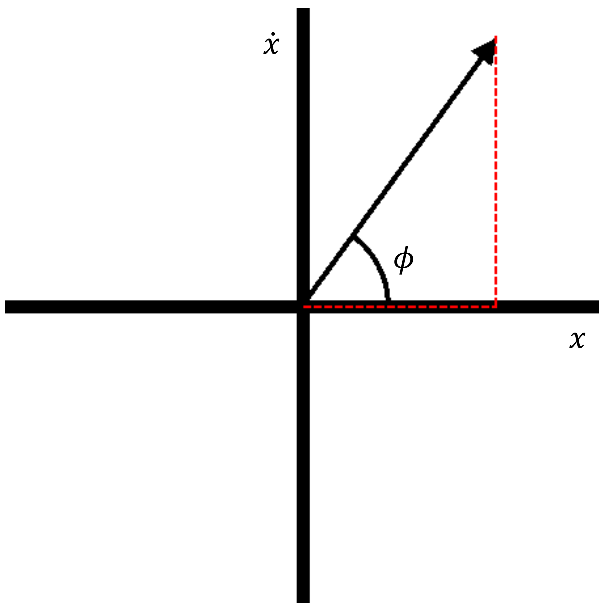

Figure 3.2: ϕ, shown on a phase plot.

$$\phi = atan2(\dot{x}, x) \tag{3.7}$$

The phase angle can be used to determine when to add energy to the system and how much energy to add. Figure 3.3 shows the phase plot for the case in Figure 3.1 where $b = 0$, while Figure 3.4 shows the phase angle over time for the same case.

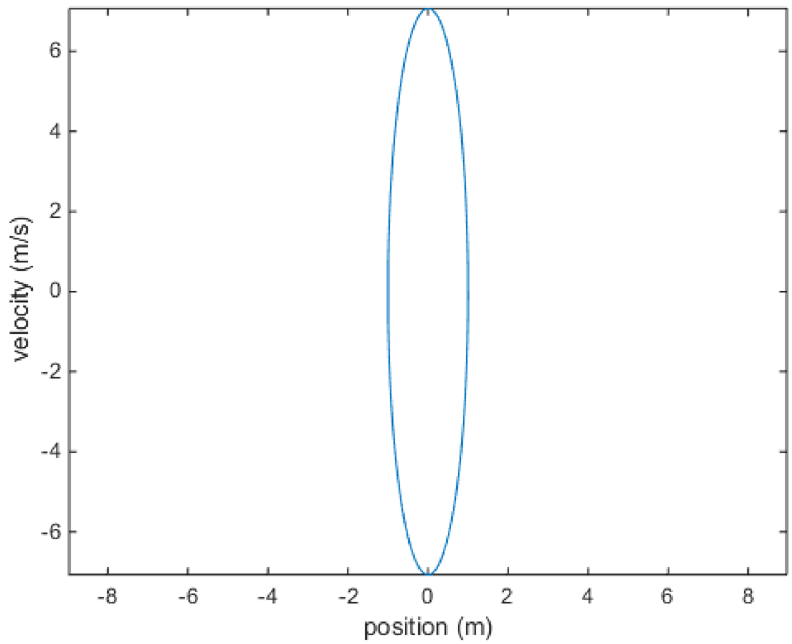

Figure 3.3: Phase plot of an oscillating system with no damping using the definition of ϕ given by Equation 3.7.

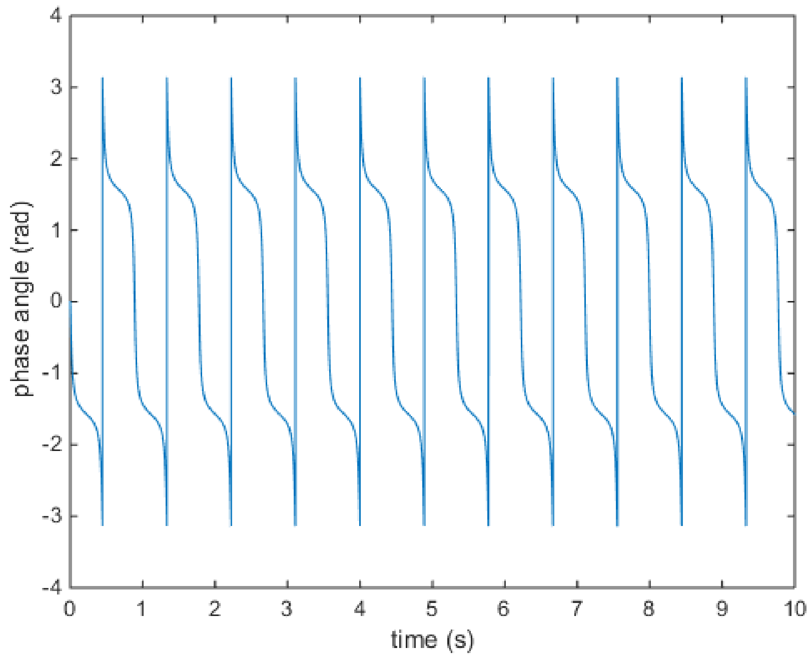

Figure 3.4: Phase angle of an oscillating system with no damping using the definition of ϕ given by Equation 3.7.

With the previously stated definition of the phase angle, the phase angle remains close to $\pm\frac{\pi}{2}$ for the majority of each oscillation. The unbalance is due to $\frac{\dot{x}}{x}$ not being unitless. When the velocity is divided by the natural frequency (defined in Equation 3.8), the phase angle (redefined in Equation 3.9), becomes more linear. A linear phase angle allows for more precise phase oscillator controls. Figure 3.5 shows the phase plot and Figure 3.6 shows the phase angle for the previous case using the redefined ϕ.

$$\omega = \sqrt{\frac{k}{m}} \tag{3.8}$$

$$\phi = atan2\left(\frac{\dot{x}}{\omega}, x\right) \tag{3.9}$$

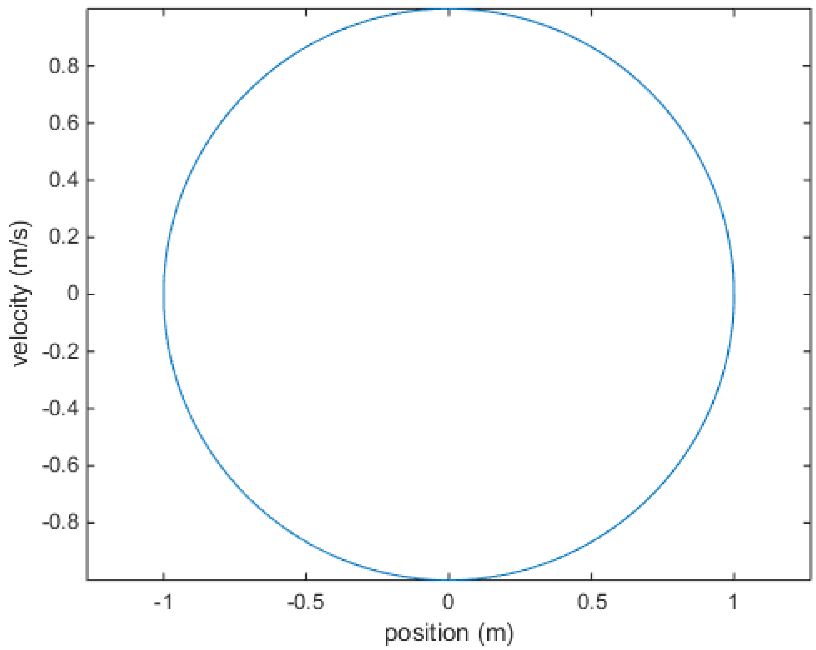

Figure 3.5: Phase plot of an oscillating system with no damping using the definition of ϕ given by Equation 3.9.

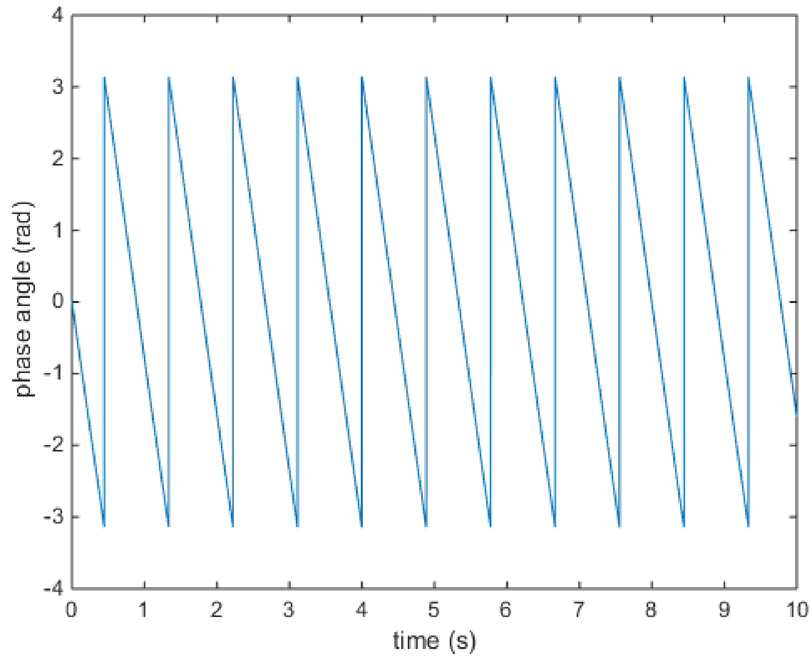

Figure 3.6: Phase angle of an oscillating system with no damping using the definition of ϕ given by Equation 3.9.

A forcing function proportional to the sine of the phase angle can be used to add energy to the system. The forcing function puts the system into a limit cycle. Equation 3.10 models the system with the phase oscillator.

$$m\ddot{x} + b\dot{x} + kx = c\sin\phi \tag{3.10}$$

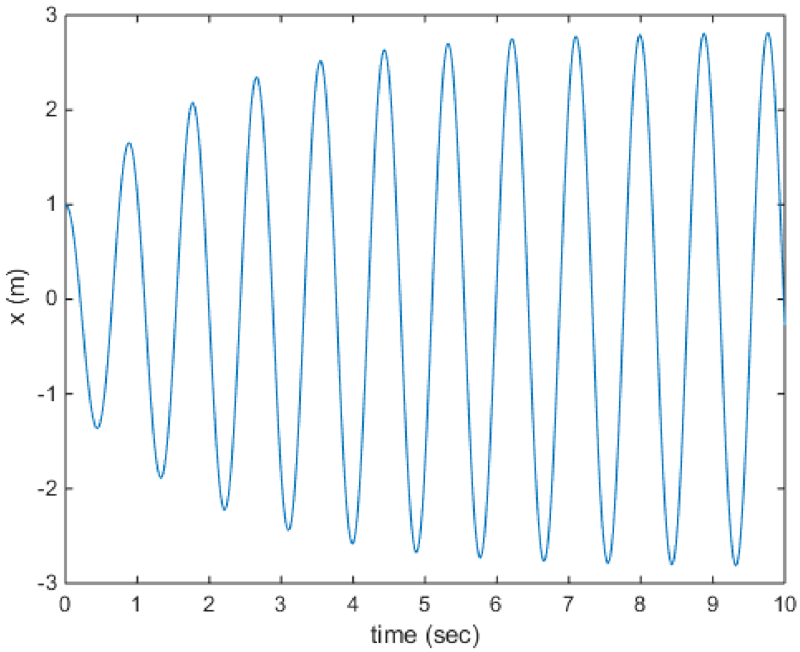

Figure 3.7: Spring response with phase oscillator (sine). $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, $c = 20$ N, initial position = 1 m, initial velocity = 0 m/s.

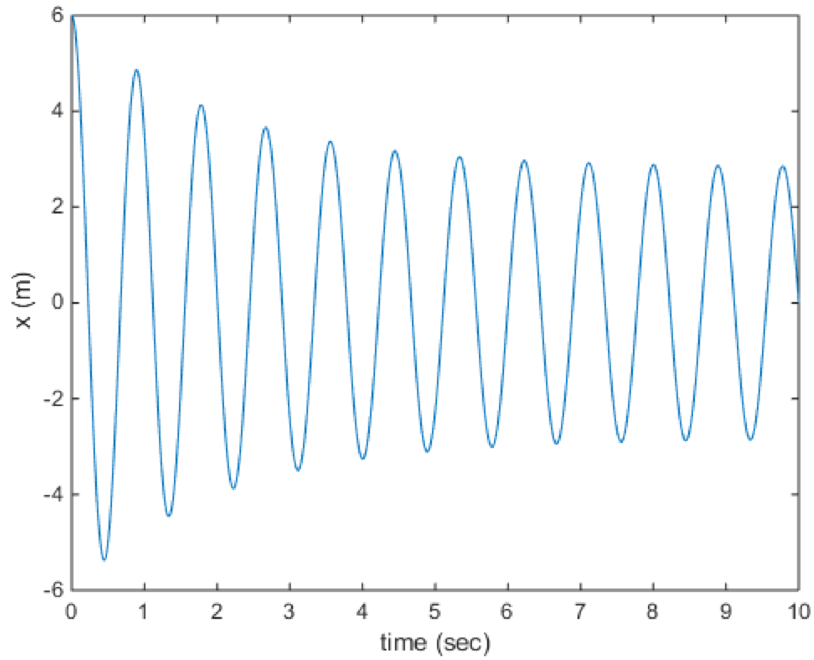

Figure 3.8: Spring response with phase oscillator (sine). $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, $c = 20$ N, initial position = 6 m, initial velocity = 0 m/s.

Figures 3.7 and 3.8 show the same system used previously, but with damping and a phase oscillator introduced. In both cases the oscillations fall into the same limit cycle. This limit cycle can be found by solving Equation 3.10 analytically. The sine of the phase angle is defined by Equation 3.11. The steady-state solution of Equation 3.10 is given by Equations 3.12, 3.13, and 3.14 (shown below).

$$\sin\phi = \frac{\left(\frac{\dot{x}}{\omega}\right)}{\sqrt{\left(\frac{\dot{x}}{\omega}\right)^2 + x^2}} \tag{3.11}$$

$$x(t) = A\sin(\omega t) \tag{3.12}$$

$$\dot{x}(t) = A\omega\cos(\omega t) \tag{3.13}$$

$$\ddot{x}(t) = -A\omega^2 \sin(\omega t) \tag{3.14}$$

Substituting Equations 3.12 and 3.13 into Equation 3.14 yields Equation 3.15.

$$\sin\phi = \frac{\left(\frac{A\omega\cos(\omega t)}{\omega}\right)}{\sqrt{\left(\frac{A\omega\cos(\omega t)}{\omega}\right)^2 + A\sin(\omega t)^2}} = \frac{A\cos(\omega t)}{A\sqrt{(\cos(\omega t))^2 + (\sin(\omega t))^2}} = \cos(\omega t) \tag{3.15}$$

Equations 3.13, 3.14, 3.15, and 3.16 can be substituted into Equation 3.10, as shown in Equation 3.16, to show the dependence of the amplitude of the system, A, on the coefficient c.

$$-mA\omega^2 \sin(\omega t) + bA\omega\cos(\omega t) + kA\sin(\omega t) = c\cos(\omega t) \tag{3.16}$$

$$k - m\omega^2 = 0 \tag{3.17}$$

$$c = bA\omega \tag{3.18}$$

$$A = \frac{c}{b\omega} \tag{3.19}$$

Equations 3.18 and 3.19 show the relationship between c and A. As an example, a steady-state amplitude of 8 m can be achieved using the same system from Figure 3.8 by solving Equation 3.18 for the appropriate value of c. The result can be seen in Figure 3.9.

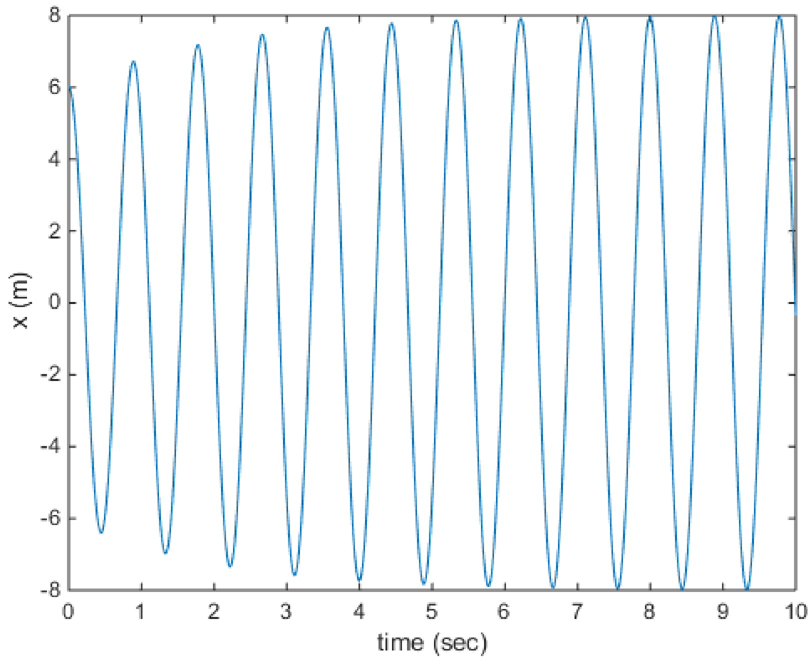

Figure 3.9: Spring response with phase oscillator (sine). $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, $c = 56.57$ N, initial position = 6 m, initial velocity = 0 m/s.

Using the sine of the phase angle as a forcing function can be thought of as adding another damping term to the system. A positive coefficient in Equation 3.10 gives negative damping, as the forcing function is on the right-hand side of the equation, adding energy to the system. Unlike the damping provided by the term $b\dot{x}$, the damping provided by $c \sin \phi$ is bounded in the range of $[-c, c]$. Also, the damping of the phase oscillator varies from $-c$ to c even when $\dot{x}$ is very small. The behavior of the bounded damping can explain intuitively the steady-state solution given by Equation 3.12. When $\dot{x}$ is large, the positive damping from $b\dot{x}$ removes more energy from the system than the negative damping from $c \sin \phi$ adds to the system. When $\dot{x}$ is small, the positive damping from $b\dot{x}$ removes less energy from the system than the negative damping from $c \sin \phi$ adds to the system. In either case, the system is pushed toward the same limit cycle, as seen in Figure 3.7 and Figure 3.8.

While giving the sine term a positive coefficient can be thought of as negative damping, a negative coefficient can be thought of as positive damping, removing energy from the system. Figure 3.10 shows energy being removed from a system by a phase oscillator.

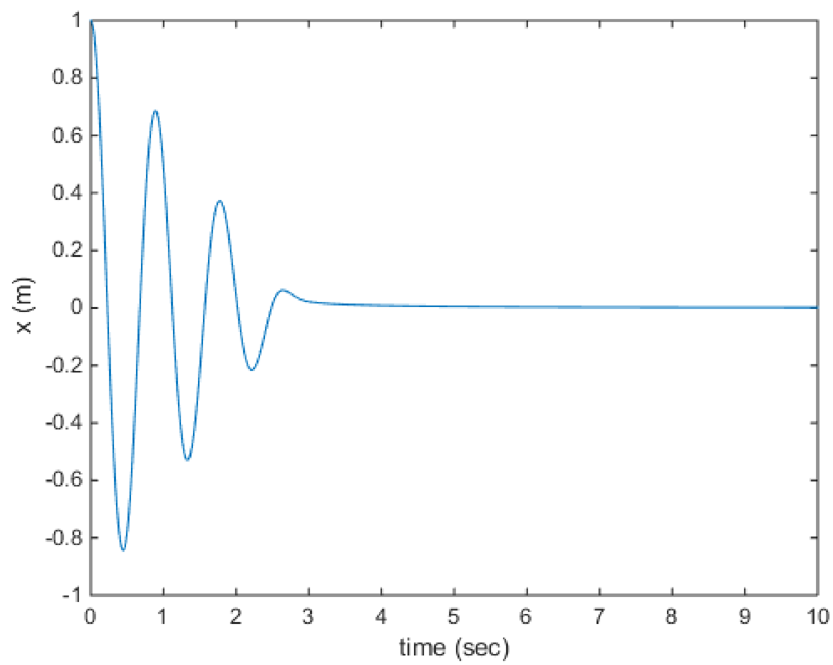

Figure 3.10: Spring response with phase oscillator (sine). $m = 1$ kg, $b = 0$ Ns/m, $k = 50$ N/m, $c =$ -5 N, initial position = 1 m, initial velocity = 0 m/s.

Equation 3.20 gives the definition of the cosine of the phase angle. The cosine of the phase angle has x in the numerator instead of $\frac{\dot{x}}{\omega}$, meaning using cosine instead of sine as a forcing function will act as a spring instead of damping. Equations 3.21 through 3.24 show the steady state solution from Equation 3.12 is still valid for a system with no damping. Equation

3.24 gives the relationship to determine the new frequency of the system with the added spring term.

$$\cos\phi = \frac{x}{\sqrt{\left(\frac{\dot{x}}{\omega}\right)^2 + x^2}} = \frac{A\sin(\omega t)}{\sqrt{\left(\frac{A\omega\cos(\omega t)}{\omega}\right)^2 + A\sin(\omega t)^2}} = \frac{A\sin(\omega t)}{A\sqrt{(\cos(\omega t))^2 + (\sin(\omega t))^2}} = \sin(\omega t) \qquad (3.20)$$

$$m\ddot{x} + kx = c\cos\phi \qquad (3.21)$$

$$-mA\omega^2\sin(\omega t) + kA\sin(\omega t) = c\sin(\omega t) \qquad (3.22)$$

$$(k - m\omega^2)A = c \qquad (3.23)$$

$$\omega^2 = \frac{k - \frac{c}{A}}{m} \qquad (3.24)$$

A negative coefficient of the cosine of the phase angle results in a normal spring, which "pulls" toward $x = 0$. Similar to the sine term, the force is bounded by and varies between $[-c, c]$. A positive coefficient results in a spring which "pushes" away from $x = 0$, but pushes with more force as position grows larger.

Figure 3.11 shows how a system (without damping) changes with the addition of a phase oscillator using the cosine of the phase angle. As c is negative, the phase oscillator acts as a spring, increasing the frequency. The increased frequency can be explained mathematically by Equation 3.24. A negative value of c causes the numerator to be larger.

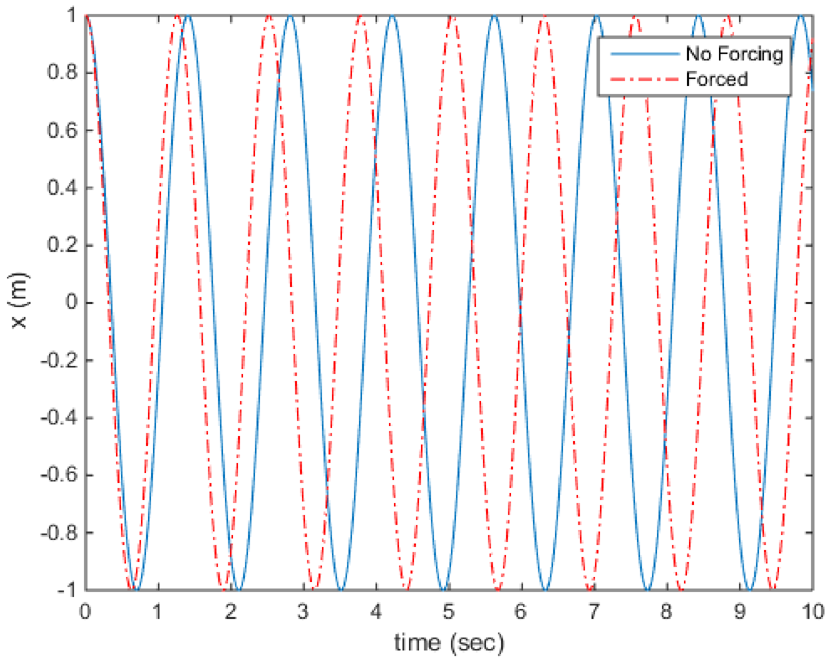

Figure 3.11: Spring response without and with phase oscillator (cosine). $m = 1$ kg, $b = 0$ Ns/m, $k = 20$ N/m, $c =$ = -5 N, initial position = 1 m, initial velocity = 0 m/s.

Similarly, the frequency of a system can be decreased by using a positive value for c. Figure 3.12 shows how a system (without damping) forced by a phase oscillator using the sine of the phase angle with a positive coefficient has a decreased frequency. A positive value of c in Equation 3.24 leads to a smaller numerator.

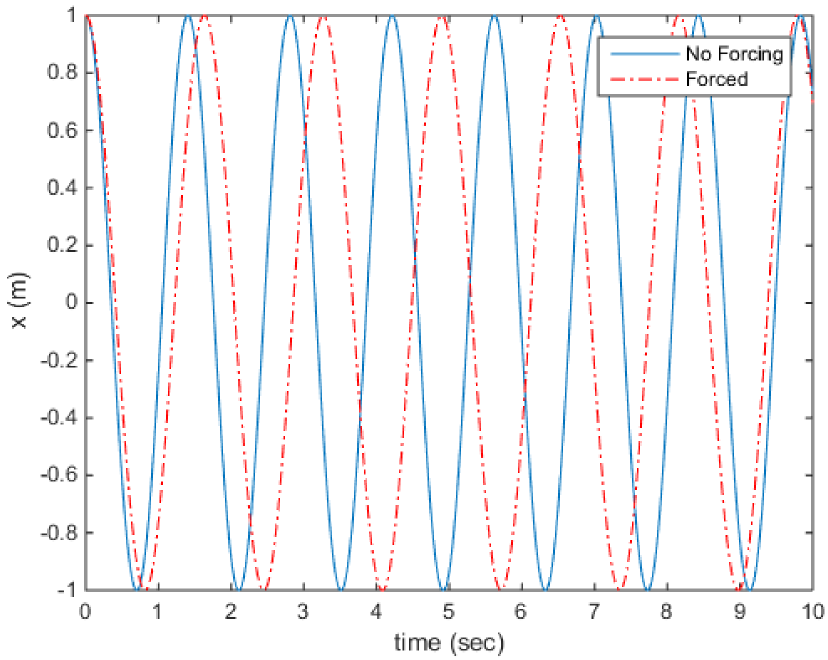

Figure 3.12: Spring response without and with phase oscillator (cosine). $m = 1$ kg, $b = 0$ Ns/m, $k = 20$ N/m, $c = 5$ N, initial position = 1 m, initial velocity = 0 m/s.

Figures 3.11 and 3.12 had no damping in order to demonstrate the frequency change before introducing the side effect of decreasing the frequency. Figures 3.13 and 3.14 introduce damping to the same system, described by Equation 3.25.

$$m\ddot{x} + c\dot{x} + kx = c\cos\phi = c\frac{x}{\sqrt{\left(\frac{\dot{x}}{\omega}\right)^2 + x^2}} \qquad (3.25)$$

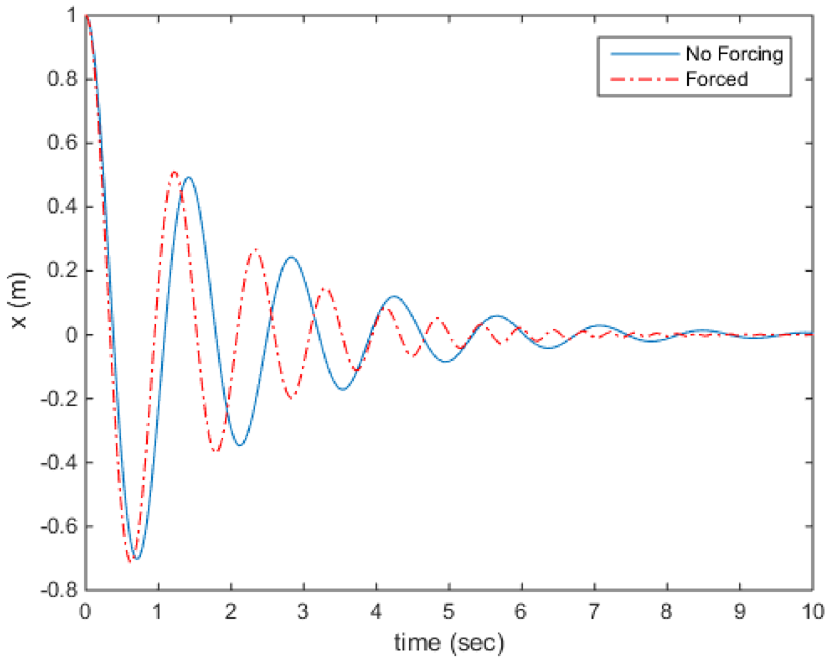

Figure 3.13: Spring response without and with phase oscillator (cosine). $m = 1$ kg, $b = 1$ Ns/m, $k = 20$ N/m, $c =$ -5 N, initial position = 1 m, initial velocity = 0 m/s.

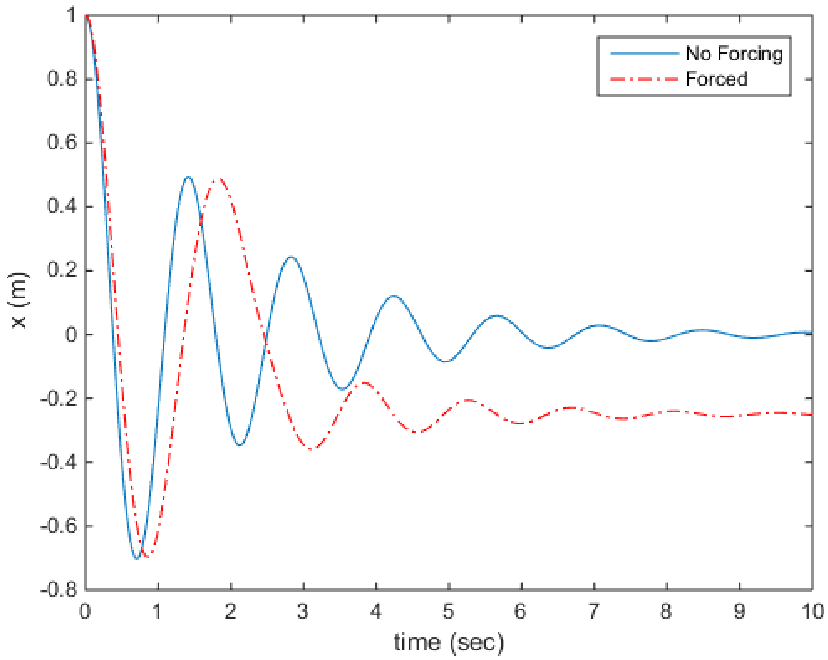

Figure 3.14: Spring response without and with phase oscillator (cosine). $m = 1$ kg, $b = 1$ Ns/m, $k = 20$ N/m, $c = 5$ N, initial position = 1 m, initial velocity = 0 m/s.

In Figures 3.13 and 3.14, the cases where $c \leq 0$ (for the "No Forcing" case $c = 0$) have normal equilibrium positions ($x = 0$). However, the case where $c = 5$ N has an equilibrium position less than the desired equilibrium position. The difference can be accounted for by examining Equation 3.35. At the equilibrium, $\dot{x} = 0$, so Equation 3.25 can be simplified and rearranged, as seen in Equations 3.26 and 3.27.

$$m\ddot{x} + kx = \frac{cx}{\sqrt{x^2}} \tag{3.26}$$

$$m\ddot{x} = c\ sgn(x) - kx \tag{3.27}$$

Suppose $c < 0$. If $x > 0$, then $c\ sgn(x) < 0$ and $-kx < 0$, so the system accelerates in a negative direction, toward $x = 0$. Likewise, if $x < 0$, then $c\ sgn(x) > 0$ and $-kx > 0$, so the system accelerates in a positive direction, toward $x = 0$.

However, suppose $c > 0$. If $x > 0$, then $c\ sgn(x) > 0$ and $-kx < 0$. When the magnitudes of the two terms are equal, the terms cancel. With no force, the system remains at an equilibrium where $x > 0$. Likewise, if $x > 0$, then $c\ sgn(x) > 0$ and $-kx < 0$, and another equilibrium is found where $x > 0$. Equations 3.28 and 3.29 solve for the equilibriums.

$$kx_{eq} = c\ sgn(x) \qquad (3.28)$$

$$x_{eq} = \pm\frac{c}{k} \qquad (3.29)$$

Equation 3.29 justifies the response seen in Figure 3.14. With $c = 5N$ and $k = 20\frac{N}{m}$, $x_{eq} = \pm 0.25m$.

The system will go to different equilibriums depending on the initial conditions. To analyze the equilibriums, the initial position can be set close to the equilibrium points. Figure 3.15 shows the response of a system with $c < 0$ when the initial position is set close to the equilibrium points. Figure 3.16 shows the response of the same system with the same initial positions but with $c > 0$.

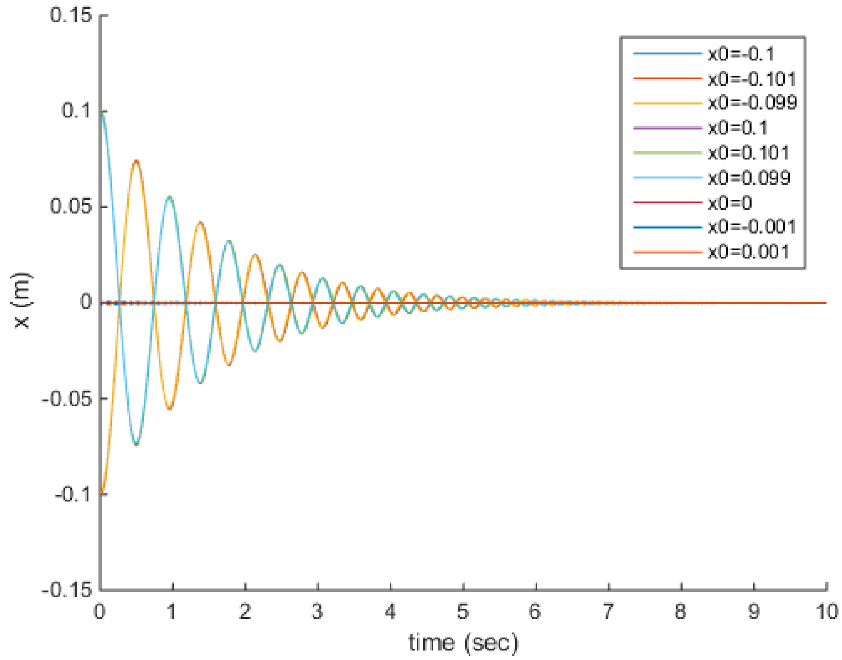

Figure 3.15: Spring response with varying initial conditions. $m = 1$ kg, $b = 1$ Ns/m, $k = 20$ N/m, $c = -2$ N, initial velocity = 0 m/s.

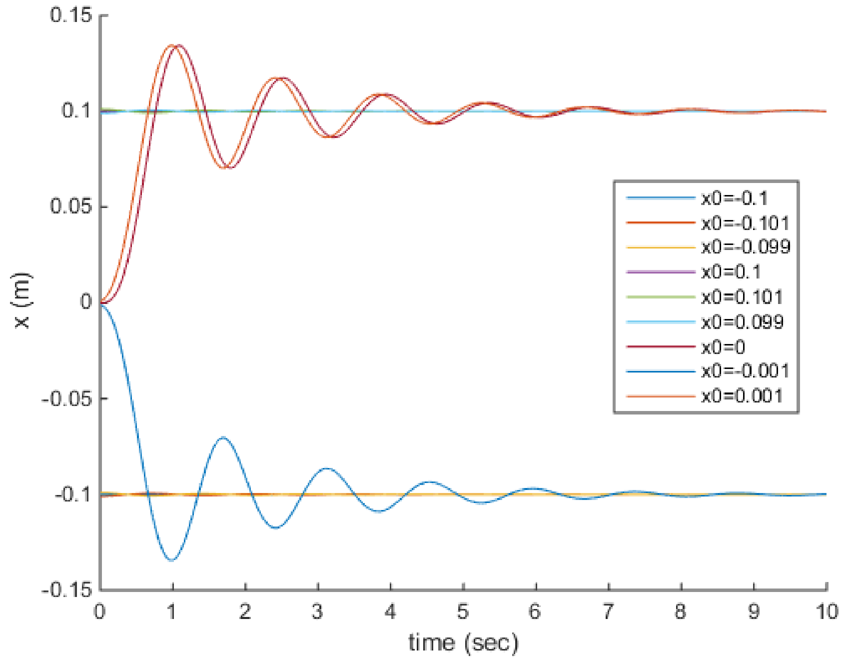

Figure 3.16: Spring response with varying initial conditions. $m = 1$ kg, $b = 1$ Ns/m, $k = 20$ N/m, $c = 2$ N, initial velocity = 0 m/s.

For each initial condition in Figure 3.15, $x_{eq} = 0$. For each initial condition in Figure 3.16, either $x_{eq} = \frac{c}{k}$ or $x_{eq} = -\frac{c}{k}$.

The phase oscillator can use both the sine of the phase angle and the cosine of the phase angle simultaneously to control both the amplitude and the frequency of an oscillating system. Equation 3.30 shows a system forced by a phase oscillator using both the sine and cosine of the phase angle. The system can still be solved with Equation 3.12, as shown by Equations 3.31 through 3.33.

$$m\ddot{x} + b\dot{x} + kx = c_1 \sin\phi + c_2 \cos\phi \qquad (3.30)$$

$$-mA\omega^2 \sin(\omega t) + bA\omega \cos(\omega t) + kA \sin(\omega t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \qquad (3.31)$$

$$bA\omega = c_1 \tag{3.32}$$

$$(k - m\omega^2)A = c_2 \tag{3.33}$$

Equations 3.32 and 3.33 allow the designer to choose any amplitude and frequency. The ω shown is the frequency of the solution, not necessarily the natural frequency of the system. Consider the physical system used in Figures 3.7 and 3.8. The system can be oscillated with $A = 7\ (m)$ and $\omega = \frac{\pi}{2}\ (\frac{rad}{s})$ by solving Equations 3.32 and 3.33. The response of the system is shown in Figure 3.17, and the forcing function generated is shown in Figure 3.18.

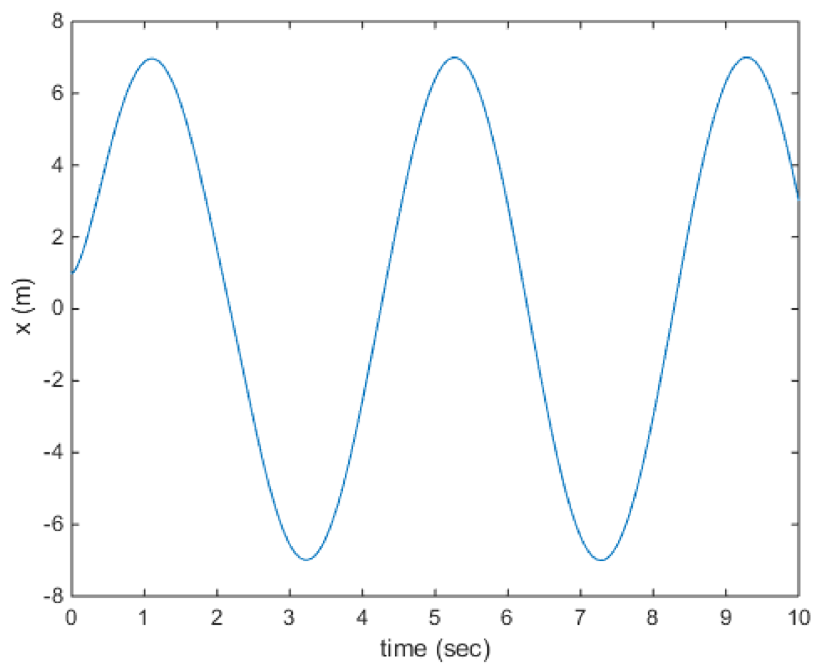

Figure 3.17: Spring response with Equation 3.30. $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, initial position = 1 m, initial velocity = 0 m/s, $c_1 = 10.996$ N, $c_2 = 332.728$ N.

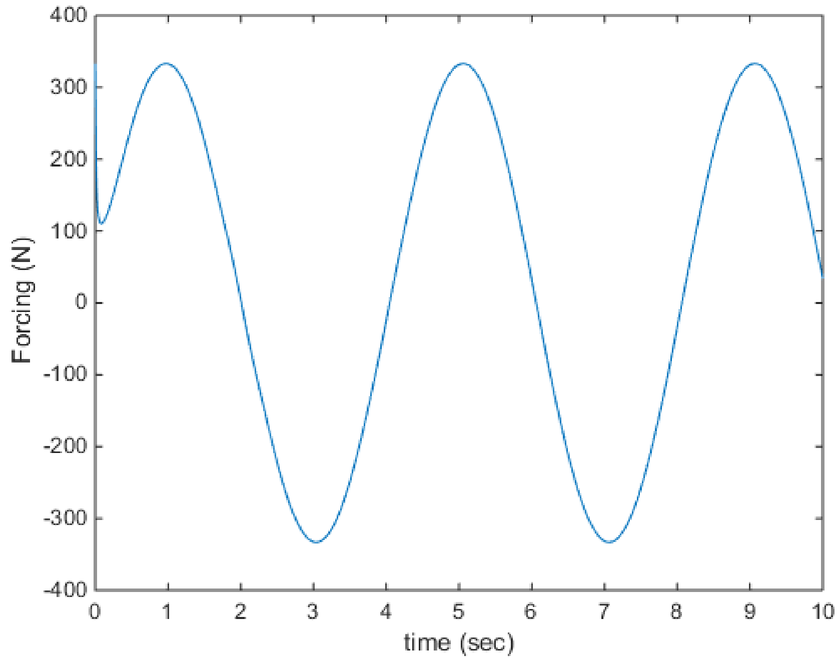

Figure 3.18: Forcing function used for the system in Figure 3.17.

Using both sine and cosine in the phase oscillator is useful for visualizing the damping and spring components separately. However, the two terms can be combined into a single term for simplicity. Figure 3.19 and Equations 3.34 through 3.36 define the phase shift α and the amplitude U. Equations 3.37 through 3.39 show the simplification of the phase oscillator.

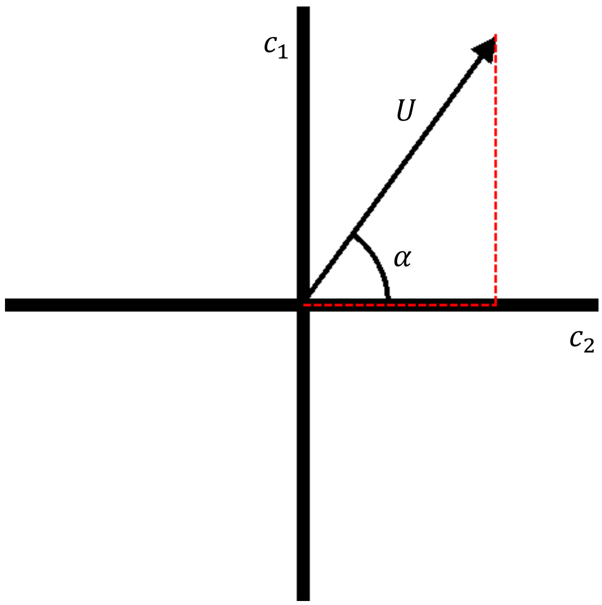

Figure 3.19: α, in relation to c_1 and c_2

$$\sin\alpha = \frac{c_1}{\sqrt{{c_1}^2+{c_2}^2}} \tag{3.34}$$

$$\cos\alpha = \frac{c_2}{\sqrt{{c_1}^2+{c_2}^2}} \tag{3.35}$$

$$U = \sqrt{{c_1}^2+{c_2}^2} \tag{3.36}$$

$$c_1\sin\phi + c_2\cos\phi = \sqrt{{c_1}^2+{c_2}^2}\left(\frac{c_1}{\sqrt{{c_1}^2+{c_2}^2}}\sin\phi + \frac{c_2}{\sqrt{{c_1}^2+{c_2}^2}}\cos\phi\right) \tag{3.37}$$

$$c_1\sin\phi + c_2\cos\phi = U(\sin\alpha\sin\phi + \cos\alpha\cos\phi) \tag{3.38}$$

$$c_1\sin\phi + c_2\cos\phi = U\cos(\phi-\alpha) \tag{3.39}$$

The balance between damping and spring provided by the phase oscillator depends on the phase shift of the forcing function, α. The maximum value reached by the forcing function is given by the amplitude, U. Figures 3.20 and 3.21 show the resulting phase shift and amplitude when the two coefficients are varied between -1 and 1.

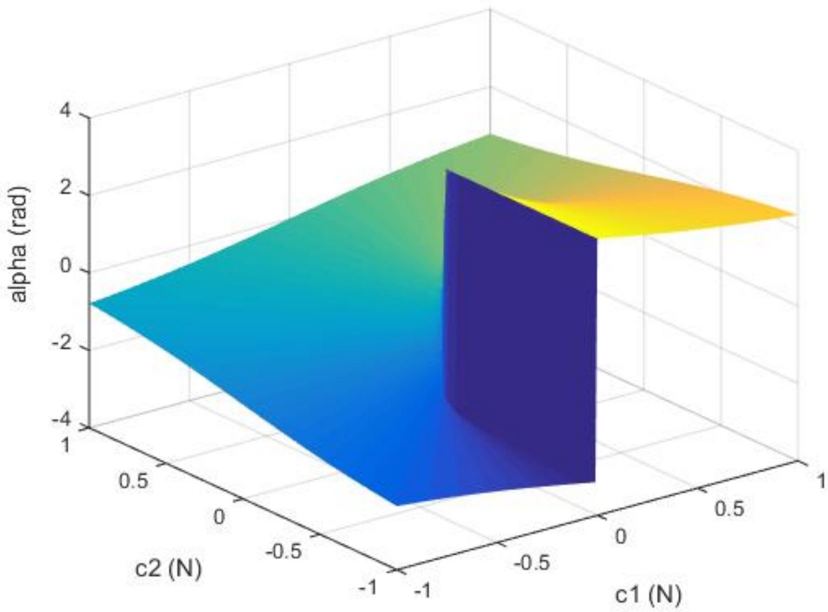

Figure 3.20: The phase shift, α, as c_1 and c_2 vary

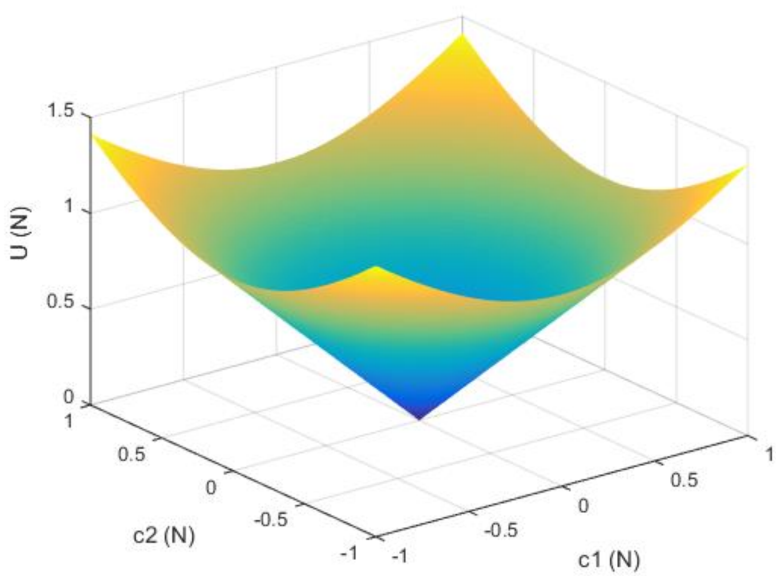

Figure 3.21: The amplitude, U, as c_1 and c_2 vary

Figure 3.20 shows the spiral staircase behavior of the phase shift. α is bounded by the interval $[-\pi, \pi]$. Figure 3.21 shows the conical behavior of the amplitude. U is bounded by the interval $[0, \infty)$.

Starting at any point on the surface in Figure 3.20 and moving directly toward or away from the vertical line defined by $c_1 = c_2 = 0$ results in a constant value of α, moving in or out on one of the "steps" of the staircase. Starting at any point on the surface in Figure 3.21 and moving as previously described results in decreasing or increasing U at the largest rate possible, moving in to (toward the point) or out of (away from the point) the cone. The movement described is achieved by keeping the ratio $c_1 : c_2$ constant. Figure 3.22 shows an example of the described movements when $\alpha = 0$.

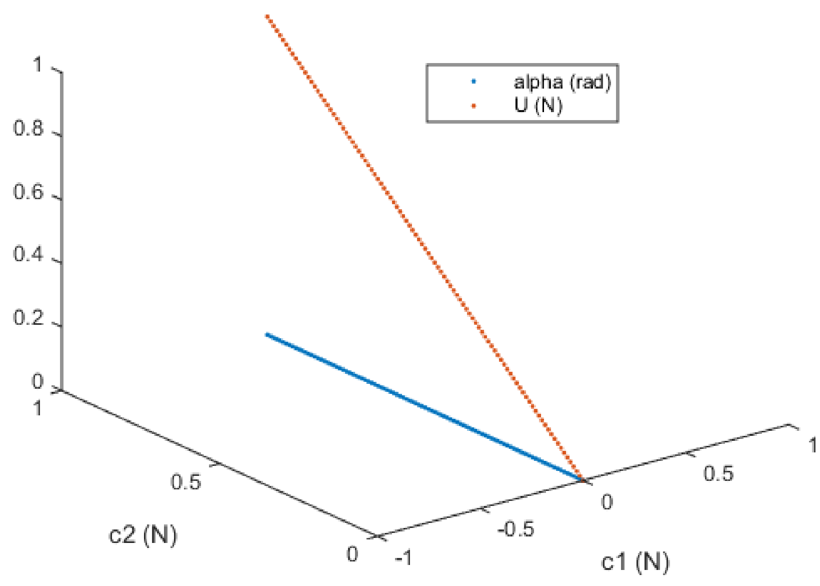

Figure 3.22: Points from Figures 3.20 and 3.21 where $\alpha = 0$.

Starting at any point on the surface in Figure 3.20 and moving perpendicular to the previous motion results in decreasing or increasing α at the largest rate possible, moving down or up the "steps" of the staircase. Starting at any point on the surface in Figure 3.21 and moving as previously described results in a constant value of U, circling along a ring on the cone while moving neither up nor down. Figure 3.23 shows an example of the described movements when $U \approx 1$.

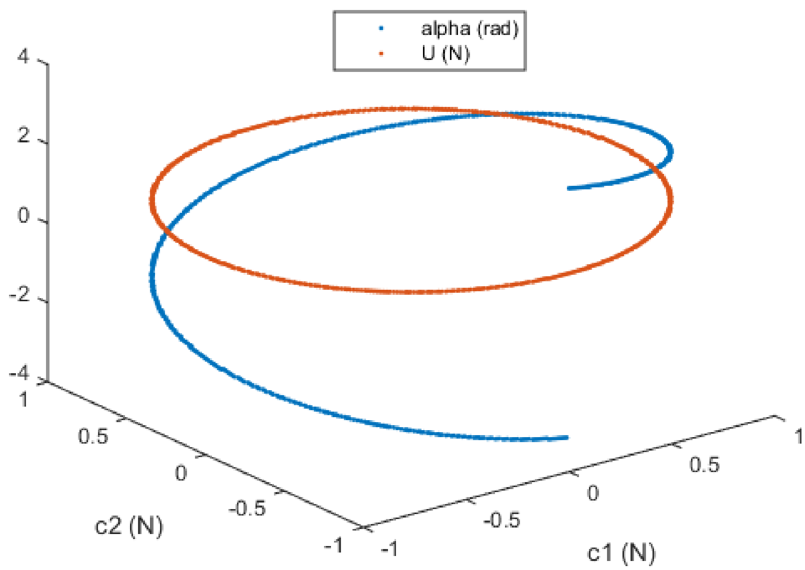

Figure 3.23: Points from Figures 3.20 and 3.21 where $U \approx 1$.

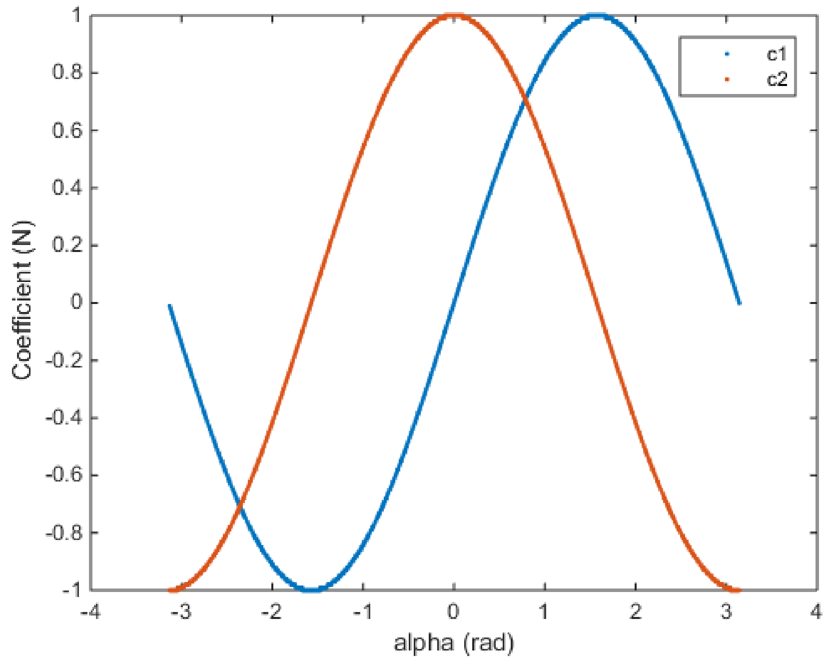

Figure 3.24: Points from Figure 3.23, $U \approx 1$.

The phase oscillator can be used to generate a step response using a feedforward loop. Equation 3.40 gives the forcing function. The terms are modifications of the definitions used in Equations 3.11 and 3.20. By replacing x with $x - x_{des}$, the reference frame is changed. $x - x_{des}$ is used instead of $x_{des} - x$ in order to maintain similar behavior of the coefficients. When $c_1 = c_2 = 0$, the system simply oscillates around x_{des}, as seen in Figure 3.25.

$$m\ddot{x} + b\dot{x} + kx = \frac{c_1\left(\frac{\dot{x}}{\omega}\right)}{\sqrt{\left(\frac{\dot{x}}{\omega}\right)^2 + (x - x_{des})^2}} + \frac{c_2(x - x_{des})}{\sqrt{\left(\frac{\dot{x}}{\omega}\right)^2 + (x - x_{des})^2}} + kx_{des} \tag{3.40}$$

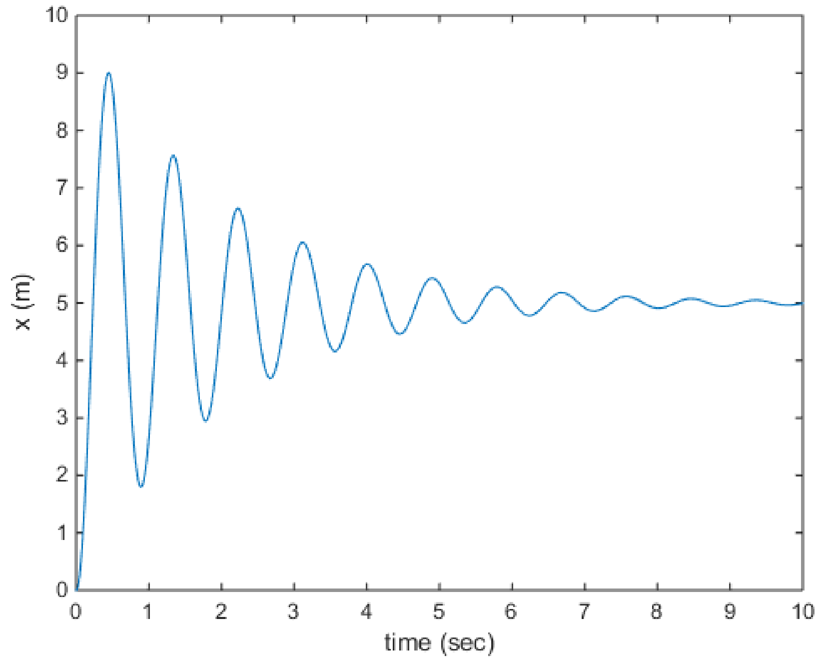

Figure 3.25: Feedforward control for step response. $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, initial position = 0 m, initial velocity = 0 m/s, desired equilibrium = 5 m, $c_1 = 0$ N, $c_2 = 0$ N.

Changing c_1 gives similar results to what was seen previously. A positive c_1 gives negative damping, causing the system to oscillate in a limit cycle around the desired point. A negative c_1 gives positive damping, causing the oscillations to diminish. Figure 3.26 shows the effects of changing c_1.

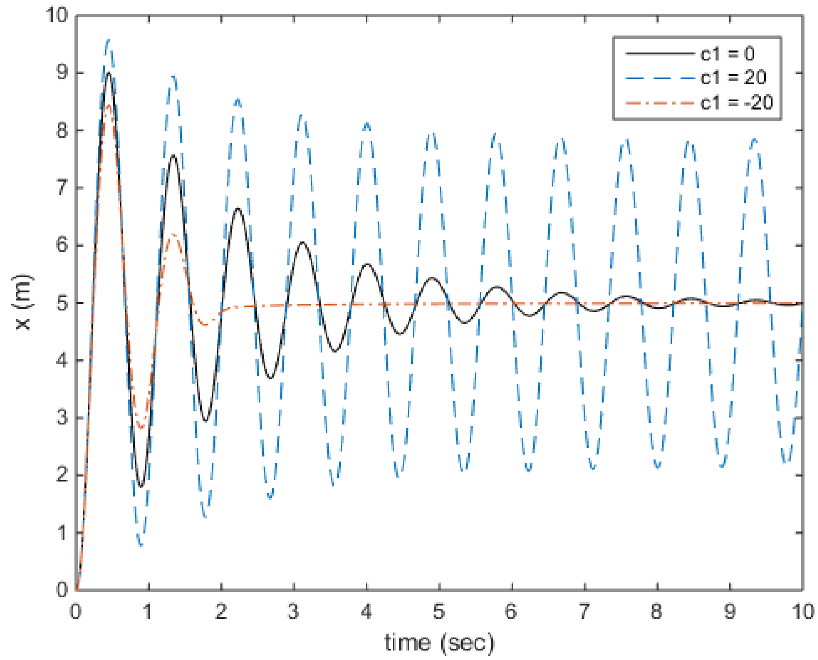

Figure 3.26: Effects of changing c_1 with a feedforward control and phase oscillator for step response. $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, initial position = 0 m, initial velocity = 0 m/s, desired equilibrium = 5 m, $c_2 = 0$ N.

Changing c_2 also gives a similar result to what was seen previously. A negative c_2 gives a pulling spring, increasing frequency. A positive c_2 still gives a pushing spring, decreasing frequency, and changes the equilibrium. The equilibrium can still be found using Equation 3.29. Figure 3.27 shows the effects of changing c_2.

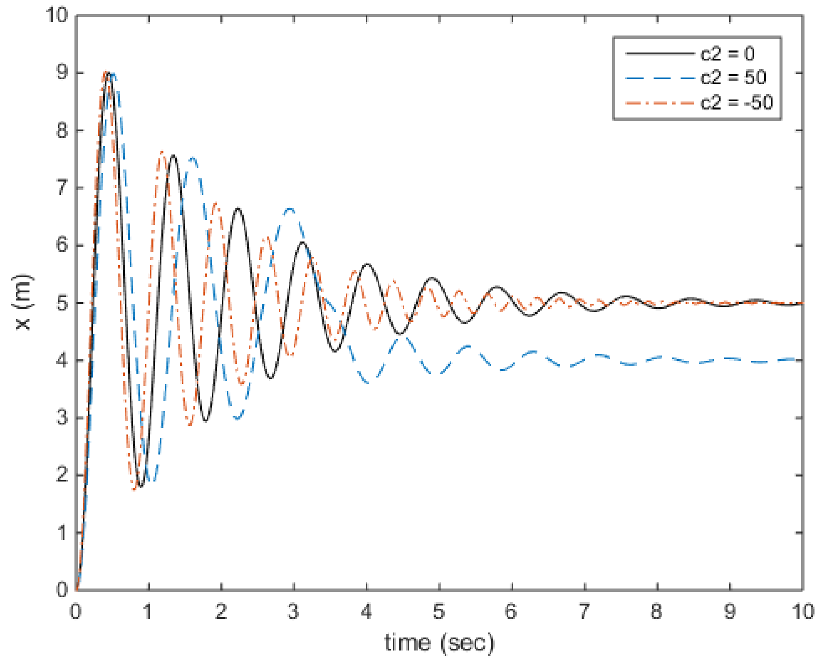

Figure 3.27: Effects of changing c_2 with a feedforward control and phase oscillator for step response. $m = 1$ kg, $b = 1$ Ns/m, $k = 50$ N/m, initial position = 0 m, initial velocity = 0 m/s, desired equilibrium = 5 m, $c_1 = 0$ N.

CHAPTER 4

PENDULUM APPLICATIONS

A solid pendulum system is special, as the stiffness of the "spring", that is, gravity, is nonlinear. The system can be modeled by Equation 4.1. Note that the pendulum modeled in this paper is a solid rod with mass (no mass attached at the end). I is the moment of inertia of the pendulum, calculated by Equation 4.2, where L is the length of the pendulum. b is again the damping coefficient. The natural frequency for small oscillations, ω_0, can be calculated by Equation 4.3. The angle, θ, is zero when the pendulum is pointing down, toward gravity.

$$I\ddot{\theta} + b\dot{\theta} + \frac{L}{2}mg\sin\theta = 0 \tag{4.1}$$

$$I = \frac{mL^2}{3} \tag{4.2}$$

$$\omega_0 = \sqrt{\frac{3g}{2L}} \tag{4.3}$$

Equation 4.4 defines ζ, the damping ratio. Equations 4.1 can be rewritten using Equations 4.2 through 4.4 as Equation 4.5. Using the damping ratio allows a clearer understanding of how damping affects the system. $\zeta = 0$ corresponds to no damping, while $\zeta = 1$ corresponds to critical damping. Figure 4.1 shows a freely oscillating pendulum. Figure 4.2 shows the same pendulum with damping, and Figure 4.3 shows the phase plot of the response.

$$\zeta = \frac{b}{2I\omega_0} \tag{4.4}$$

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta} + {\omega_0}^2\sin\theta = 0 \tag{4.5}$$

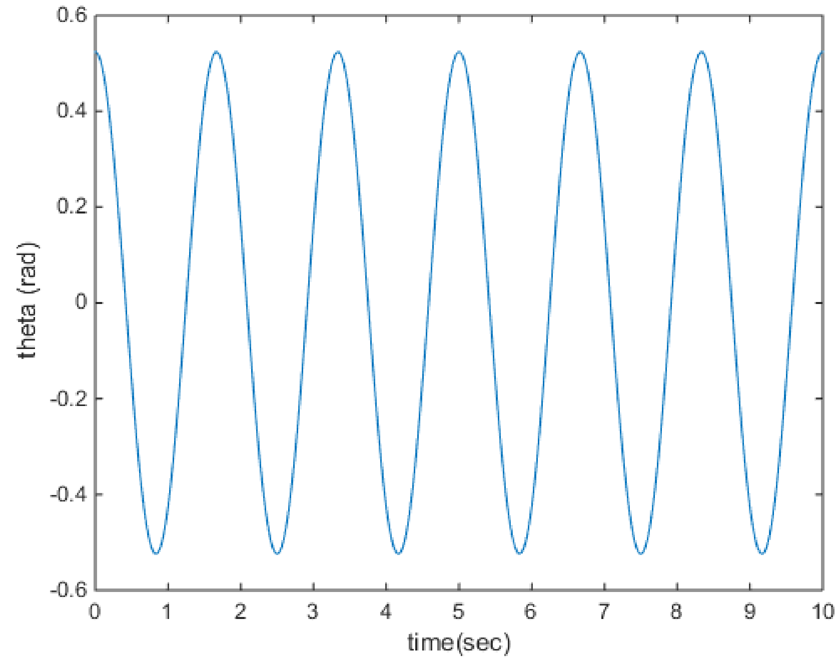

Figure 4.1: Pendulum response. $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0$, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s.

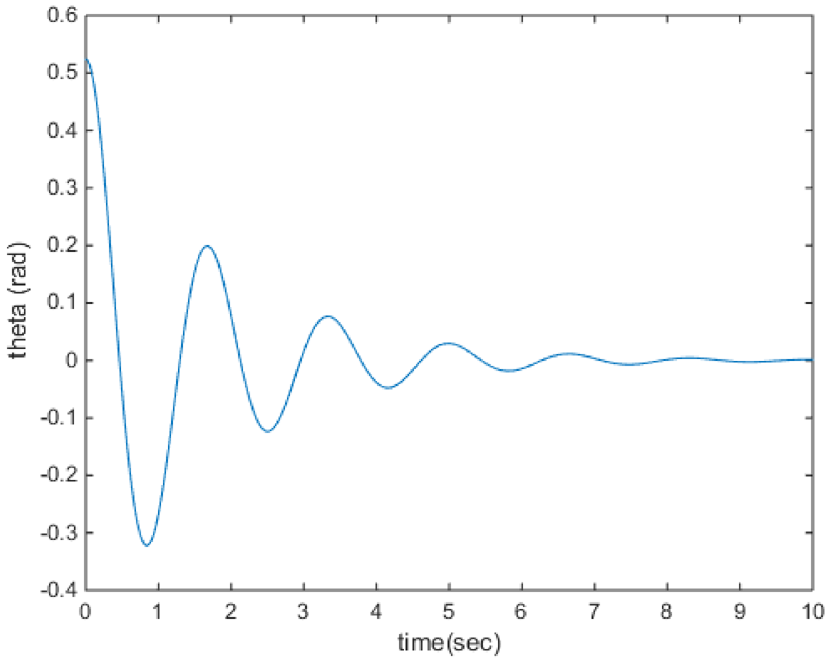

Figure 4.2: Pendulum response. $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0.15$, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s.

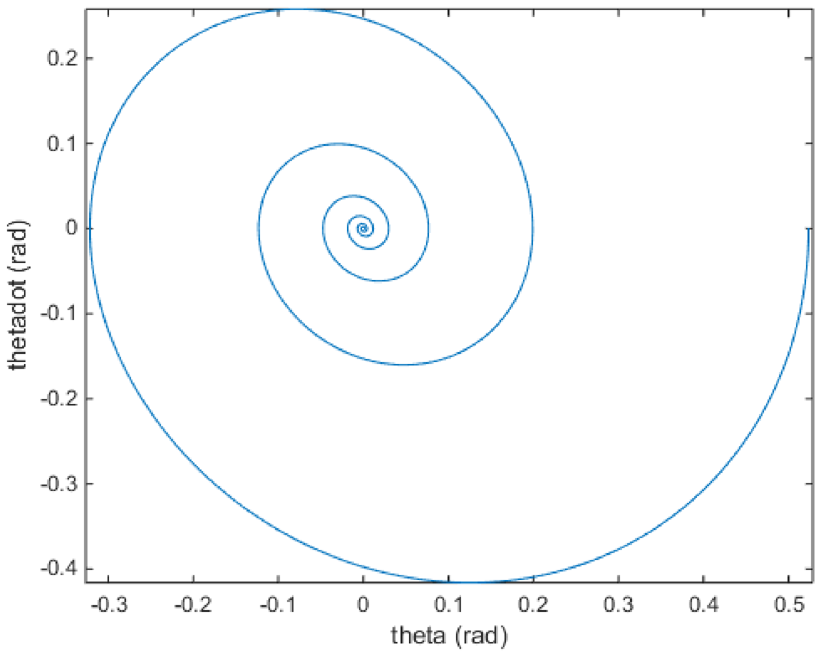

Figure 4.3: Phase plot of system from Figure 4.2.

Similar to Equation 3.10, a phase oscillator can be added to the system to put the pendulum in a limit cycle. The system can be modeled by Equation 4.6. Figures 4.4 and 4.6 show the same system with different initial conditions. Figures 4.5 and 4.7 show how the system in Figures 4.4 and 4.6 end up in the same limit cycle regardless of the initial conditions.

$$I\ddot{\theta} + b\dot{\theta} + \frac{L}{2}mg\sin\theta = c_1\sin\phi \tag{4.6}$$

$$\sin\phi = \frac{\left(\frac{\dot{\theta}}{\omega}\right)}{\sqrt{\left(\frac{\dot{\theta}}{\omega}\right)^2 + \theta^2}} \tag{4.7}$$

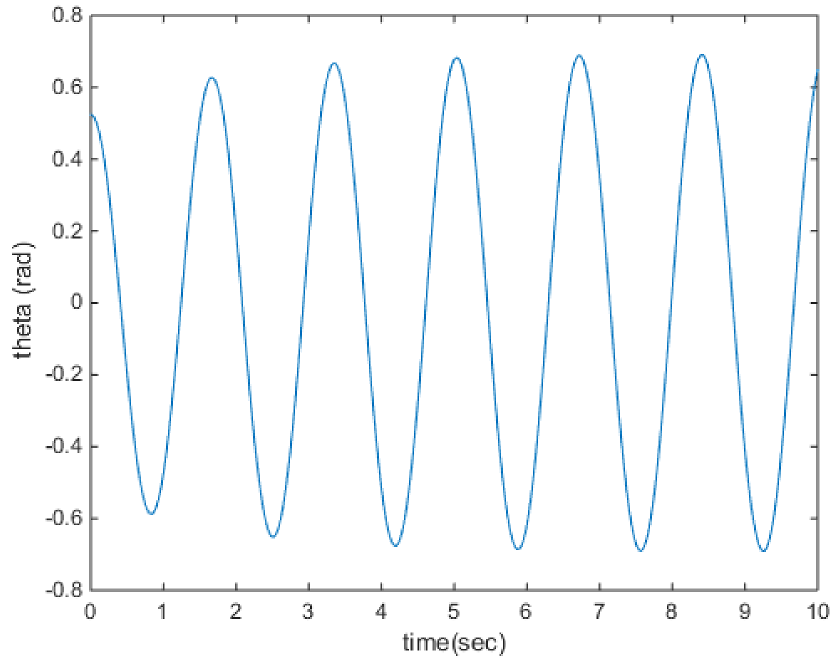

Figure 4.4: Pendulum response with phase oscillator. $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0.15$, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s, $c_1 = 1$ N.

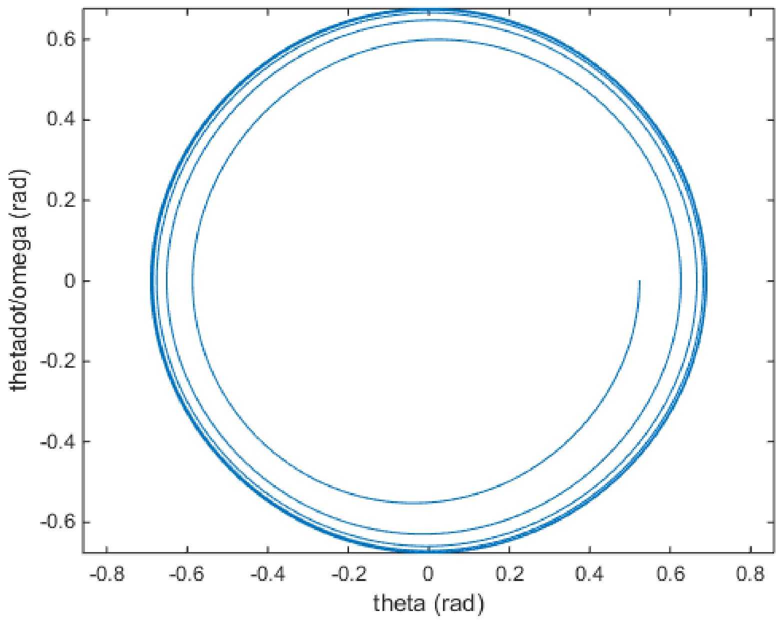

Figure 4.5: Phase plot of response from Figure 4.4.

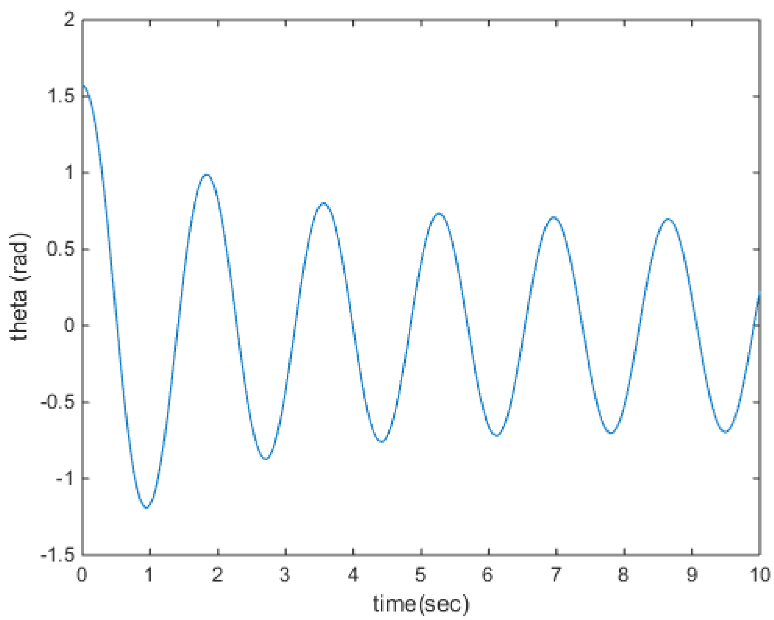

Figure 4.6: Same system as Figure 4.4, with initial angle = $\frac{\pi}{2}$ rad.

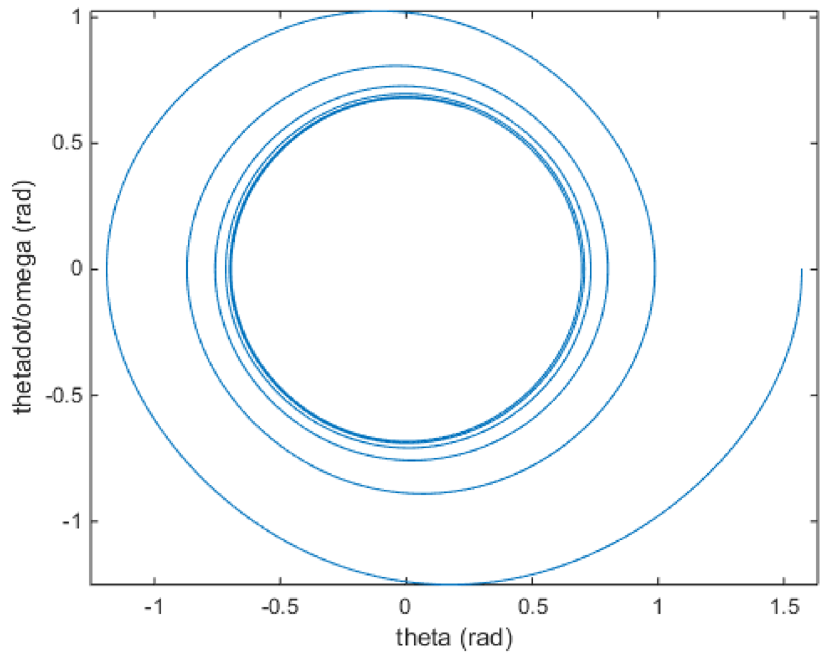

Figure 4.7: Phase plot of response from Figure 4.6.

The phase oscillator can be used for swinging up and balancing a pendulum. This paper covers two methods of swing up control. The breakpoint method involves one control for swing up and one control for balancing. The continuous method combines the two controls from the breakpoint method into a single, continuous control for both swing up and balancing.

The breakpoint method uses Equation 4.8 during swing up and Equation 4.9 during balancing. The coefficients are chosen according to Equation 4.10 so the maximum magnitude of the forcing function is C. ϕ_1 is the phase angle calculated with θ_1, where $\theta_1 = 0$ corresponds to the pendulum pointing down. ϕ_2 is the phase angle calculated with θ_2, where $\theta_2 = 0$ corresponds to the pendulum pointing up. The reason for multiple phase angles is discussed below.

Controller for Breakpoint Method Swing Up:

$$I\ddot{\theta} + b\dot{\theta} + \frac{L}{2}mg\sin\theta = C(c_{11}\sin\phi_1 + c_{21}\cos\phi_1) \tag{4.8}$$

Controller for Breakpoint Method Balancing:

$$I\ddot{\theta} + b\dot{\theta} + \frac{L}{2}mg\sin\theta = C(c_{12}\sin\phi_2 + c_{22}\cos\phi_2) \tag{4.9}$$

Relationship Between Coefficients:

$$\sqrt{{c_{11}}^2 + {c_{21}}^2} = \sqrt{{c_{12}}^2 + {c_{22}}^2} = 1 \tag{4.10}$$

During swing up $c_{11} = 1$ and $c_{21} = 0$. c_{11} is positive in order to give negative damping and add energy into the system. c_{21} is zero, as a spring component is not needed during this stage. Once the pendulum swings past a defined angle, or breakpoint, the control changes to balance the pendulum. c_{12} is a negative value to provide positive damping. If c_{12} remained positive, the pendulum would be pushed down. Instead, positive damping resists motion. As gravity would still pull the pendulum down, c_{22} is a negative value to act as a spring, pulling the pendulum up towards the top. Figure 4.8 explains the breakpoint method visually.

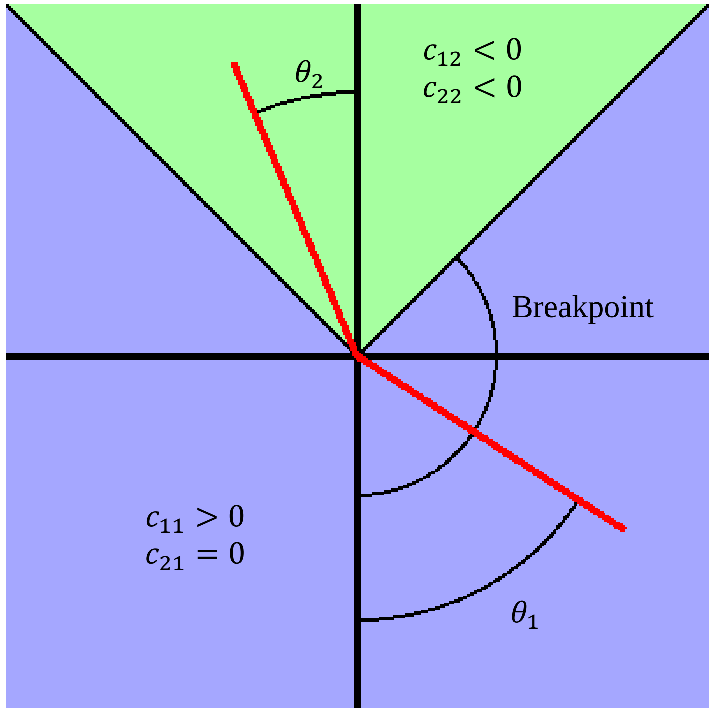

Figure 4.8: Breakpoint Method. In the purple zone (below the breakpoint), the phase angle is determined using θ_1. In the green zone (above the breakpoint), the phase angle is determined using θ_2. The red bar represents the pendulum.

The phase oscillator breakpoint method can swing up and balance a pendulum using a torque too small to directly swing up the pendulum. As an example, Figure 4.9 shows a constant torque applied to a pendulum from two different initial conditions. Equation 4.11 describes the system. In both cases the pendulum levels out where the torque applied is equivalent to the torque due to gravity. Equations 4.12 and 4.13 justify the equilibrium position being $\theta = 0.7101$ radians.

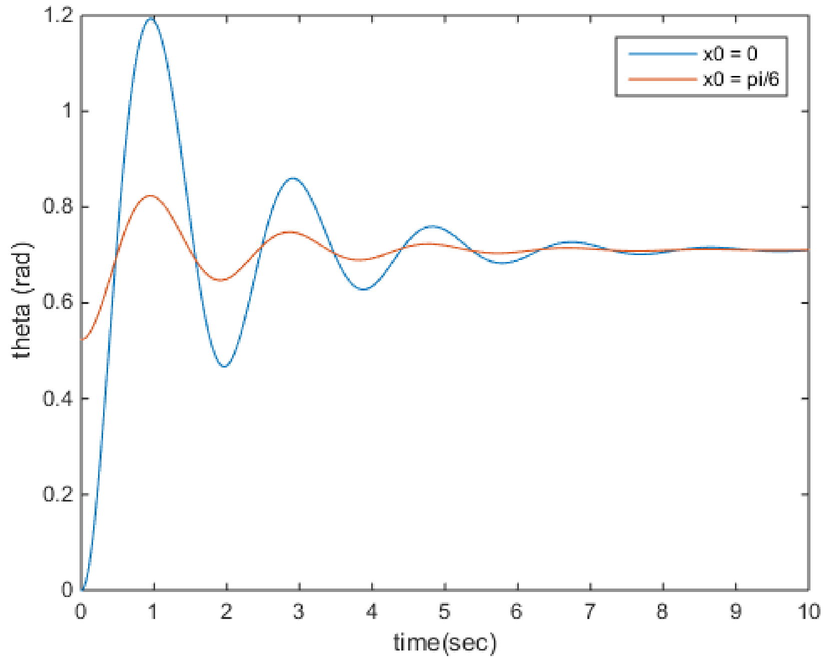

Figure 4.9: Pendulum response to constant torque (θ_1 vs time). $C = 3.2$ N*m, $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0.15$, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s.

$$I\ddot{\theta} + b\dot{\theta} + \frac{L}{2}mg\sin\theta = C \tag{4.11}$$

$$\frac{L}{2}mg\sin\theta = C \tag{4.12}$$

$$\theta = \sin^{-1}\frac{2C}{Lmg} \tag{4.13}$$

Figure 4.10 shows the same maximum torque being applied with the phase oscillator breakpoint method. The relationship described in Equation 3.31 shows how the maximum torque applied during balancing is still less than the torque applied in Figure 4.9. Figure 4.11 shows the torque applied to the system in Figure 4.10.

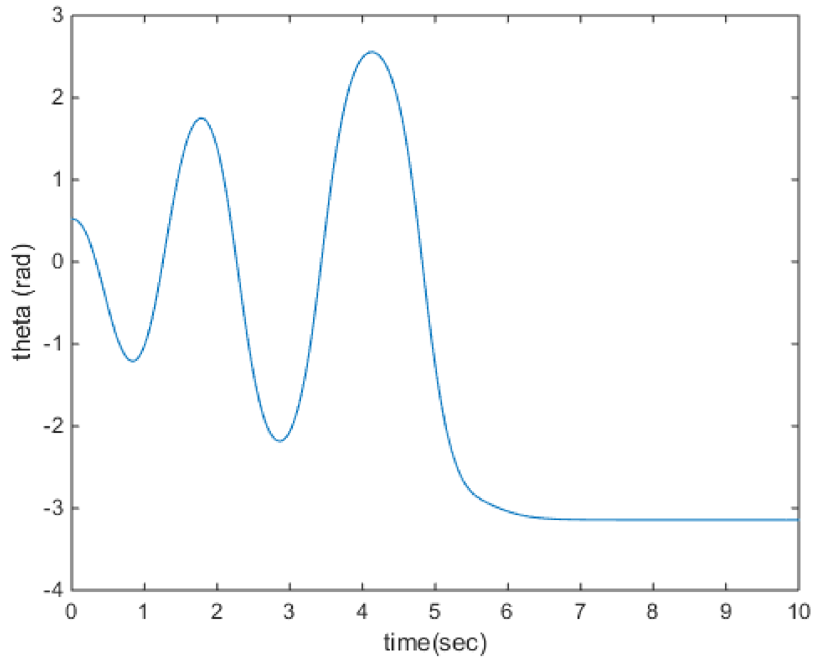

Figure 4.10: Pendulum response to breakpoint method (θ_1 vs time). $C = 3.2$ N*m, $c_{11} = 1$, $c_{12} = -\frac{1}{\sqrt{2}}$, $c_{21} = 0$, $c_{22} = -\frac{1}{\sqrt{2}}$, $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta =$ 0.15, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s.

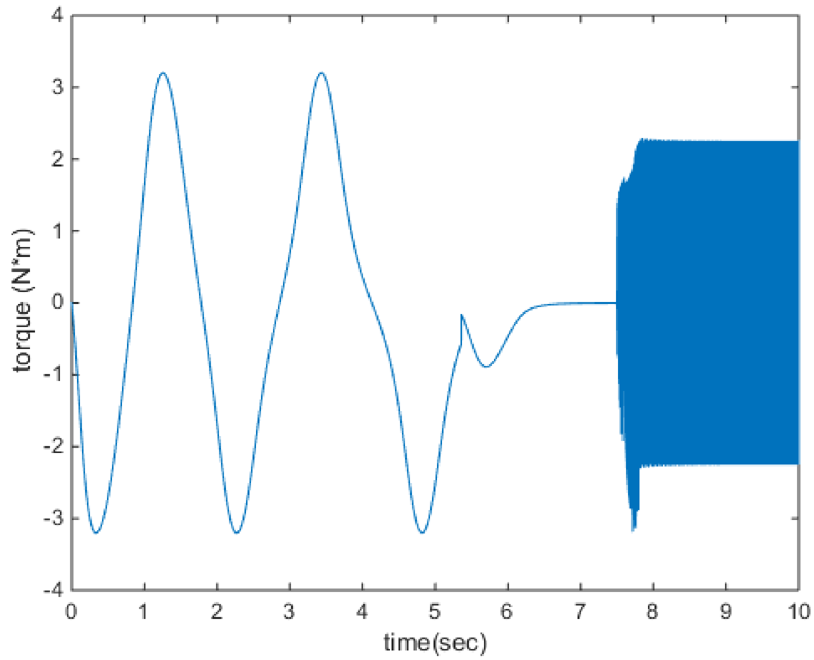

Figure 4.11: Torque applied to system in Figure 4.10.

Figure 4.10 shows the pendulum swing up and balance. Figure 4.11 shows a smooth forcing function during swing up, a jump in the forcing function just after 5 sec when the pendulum passes the breakpoint, and the forcing function jumping between positive and negative values very quickly during balancing. The last swing of the pendulum in Figure 4.10 would end up near the top, making for easy balancing. Figure 4.12 shows the phase oscillator slow the pendulum at the top to prevent overshoot.

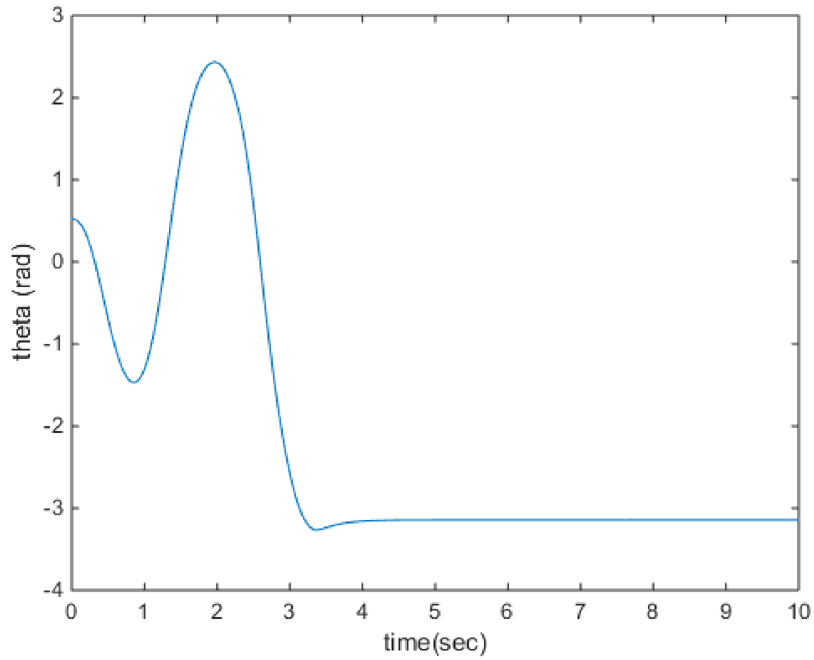

Figure 4.12: Pendulum response to breakpoint method. $C = 4$ N*m, $c_{11} = 1$, $c_{12} = -\frac{1}{\sqrt{2}}$, $c_{21} = 0$, $c_{22} = -\frac{1}{\sqrt{2}}$, $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0.15$, initial angular velocity = 0 rad/s.

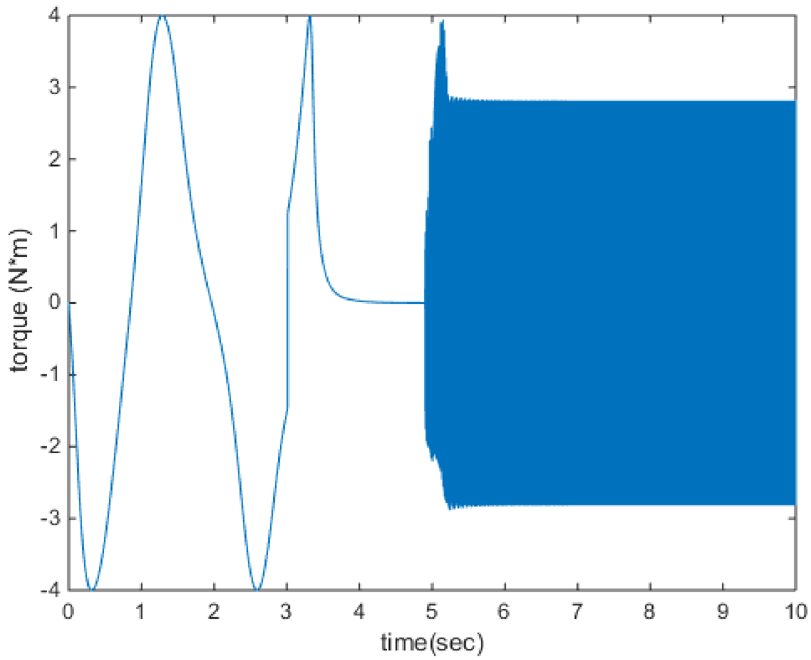

Figure 4.13: Torque applied to system in Figure 4.12.

Two phase angles are used due to the shifting equilibrium. During swing up, the oscillations occur around the bottom of the swing. On a phase plot, the system circles the origin, so the phase angle can be any value in the range $[0, 2\pi)$. During balancing, the oscillations occur around the top. On a phase plot, the system circles around the point $(\pm\pi, 0)$, limiting the phase angle to either a small range close to zero or a small range close to $\pm\pi$. By measuring angle from the top, the phase plot is translated horizontally, allowing the oscillations to occur around the origin. Figure 4.14 shows the phase plot for the system in Figure 4.10 with the unaltered angular positions measured from both top and bottom. Figure 4.15 shows the same phase plot but with the angular positions limited to the range $[-\pi, \pi]$, as used by the control to calculate the phase angles.

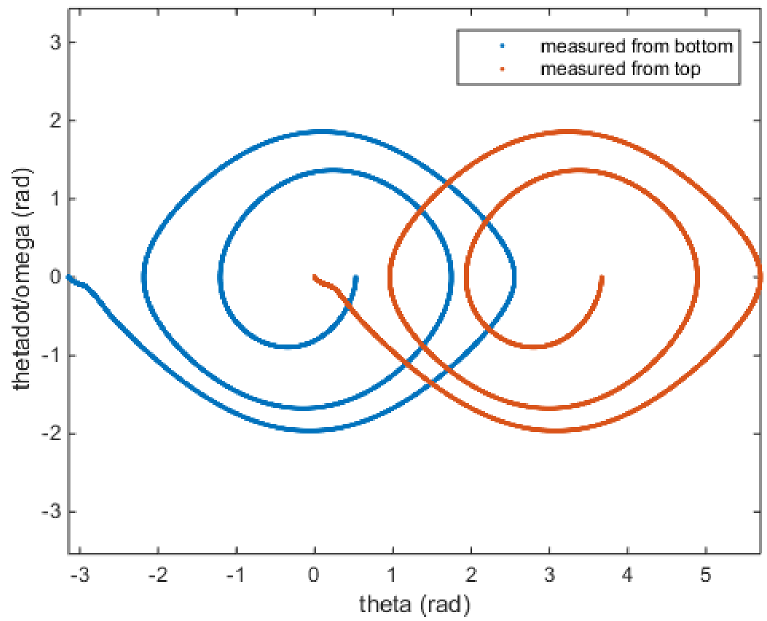

Figure 4.14: Phase plot of the system in Figure 4.10 using unaltered angular positions.

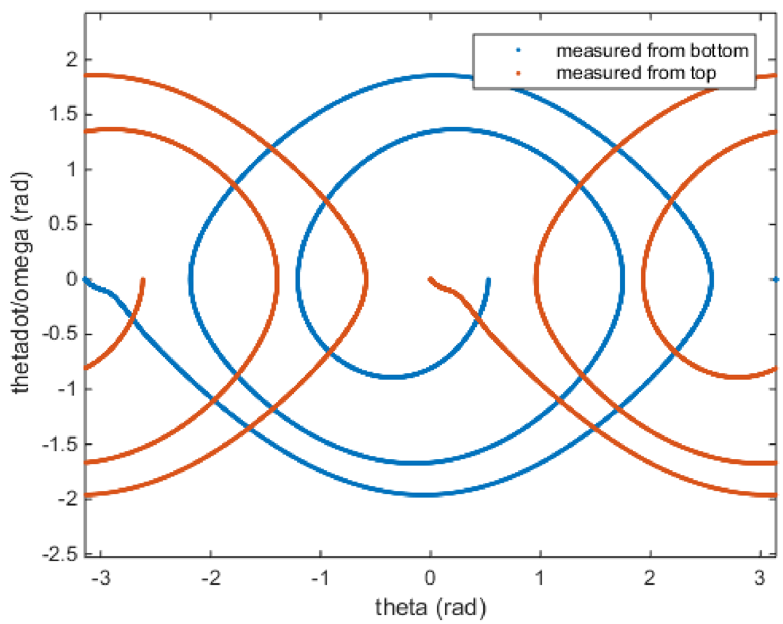

Figure 4.15: Phase plot of the system in Figure 4.10 using modified angular positions.

Figure 4.15 gives the impression of the positional angle measured from the top being the only angle necessary for the phase oscillator. In cases with large initial angular positions or velocities, the control may still be successful in swing up. However, smaller initial conditions make swing up increasingly difficult. As the initial angular position and velocity get smaller, the phase angles during the first portion of swing up get closer to zero and $\pm\pi$, meaning the sine term, which adds energy, will stay very small. Figure 4.16 shows the response of the system from Figure 4.10 when the phase oscillator only uses ϕ_2. The pendulum motion eventually dies out due to the damping given by b. The torque applied, seen in Figure 4.17, is much smaller than the maximum available due to the phase angle, seen in Figure 4.18.

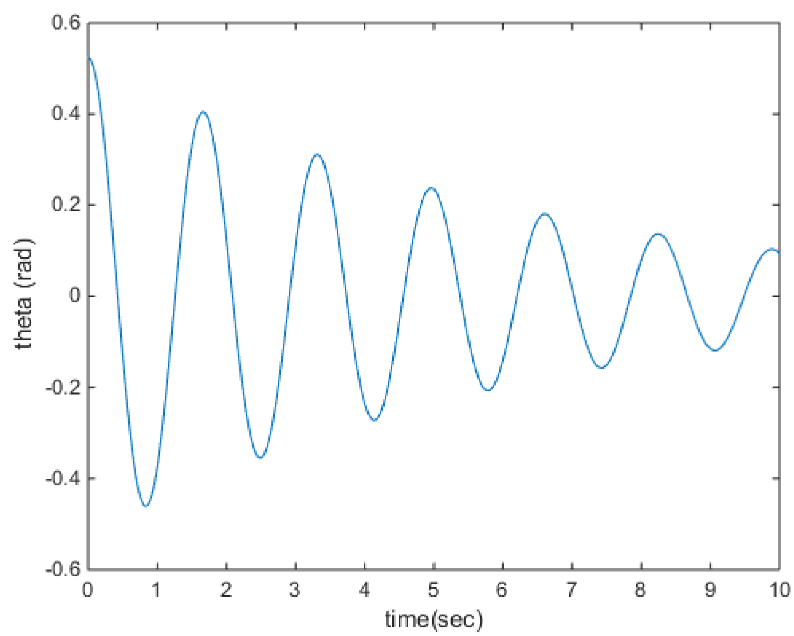

Figure 4.16: Response of the system from Figure 4.10 using only ϕ_2.

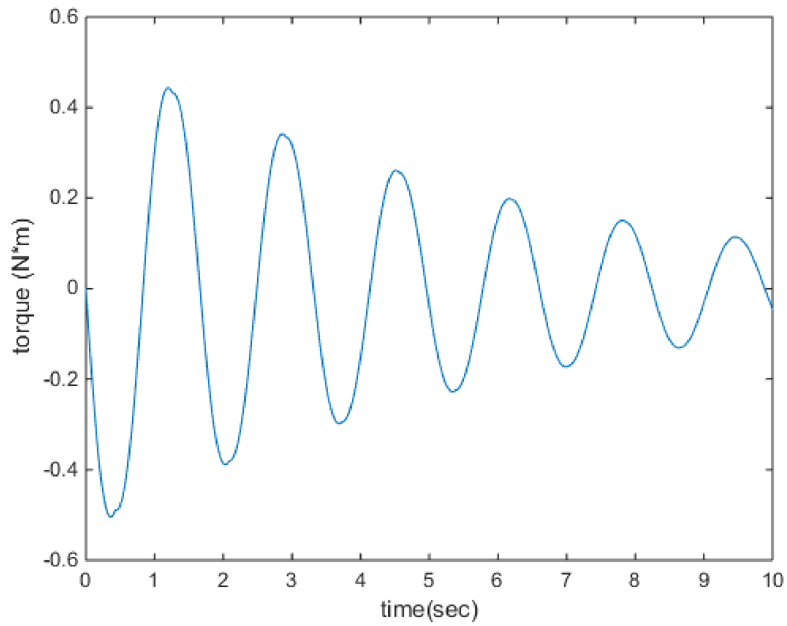

Figure 4.17: Forcing function for system in Figure 4.16.

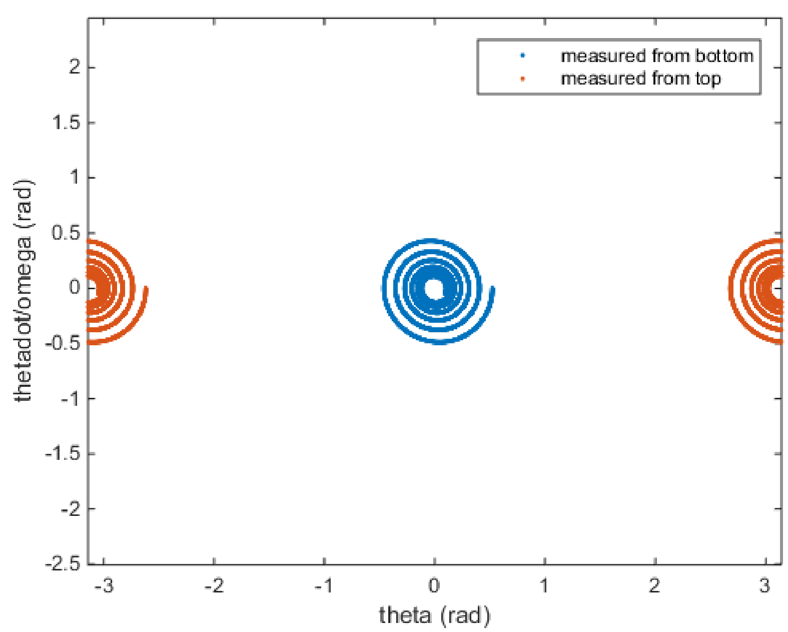

Figure 4.18: Phase plot for system in Figure 4.16.

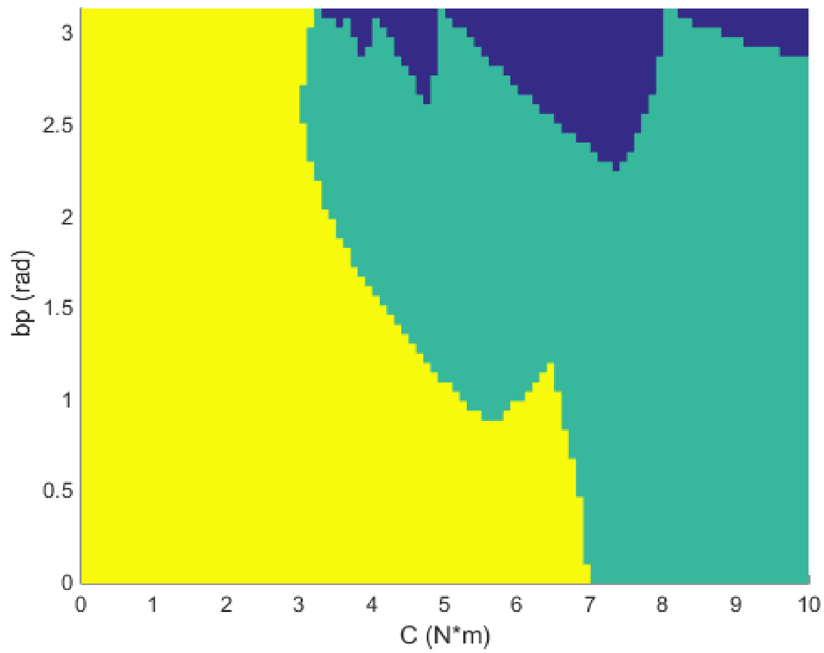

Figure 4.19: Map of breakpoint method outcomes. $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0.15$, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s.

Figure 4.19 shows combinations of motor torque and breakpoints. The yellow section (left) is where the pendulum could not swing up to the top. The dark blue section (top) is where the pendulum swung past the top and continued spinning. The light blue section (bottom right) is where the control successfully balances the pendulum.

The second controller is the continuous phase oscillator method and will be discussed below. The continuous phase oscillator method is more complex than the breakpoint method but results in a smoother forcing function. Equation 4.14 describes a system with a continuous phase oscillator control.

$$I\ddot{\theta} + b\dot{\theta} + \frac{L}{2}mg\sin\theta = CK_1(K_2\sin\phi_1 + K_3\cos\phi_2) \tag{4.14}$$

C is again the constant scalar for the forcing function. K_1 is a function of the angular position relative to the top, θ_2, scaling down the overall torque when the pendulum is close to the top. K_2 is a function of the energy in the system, adding energy when the system does not have enough and removing energy when the system has too much. K_3 is a function of θ_2, pulling the pendulum up when the pendulum is close to the top.

Equation 4.15 defines K_1. K_1 scales down the applied torque when the pendulum is close to the top, where $\theta_2 = 0$, in order to keep the torque from quickly changing between large positive and negative values, as seen in Figures 4.11 and 4.13. f_1 is a constant that changes the steepness of the relationship between K_1 and θ_2, as seen in Figure 4.20.

$$K_1 = 1 - \frac{4}{e^{f_1 \theta_2} + 2 + e^{-f_1 \theta_2}} \qquad (4.15)$$

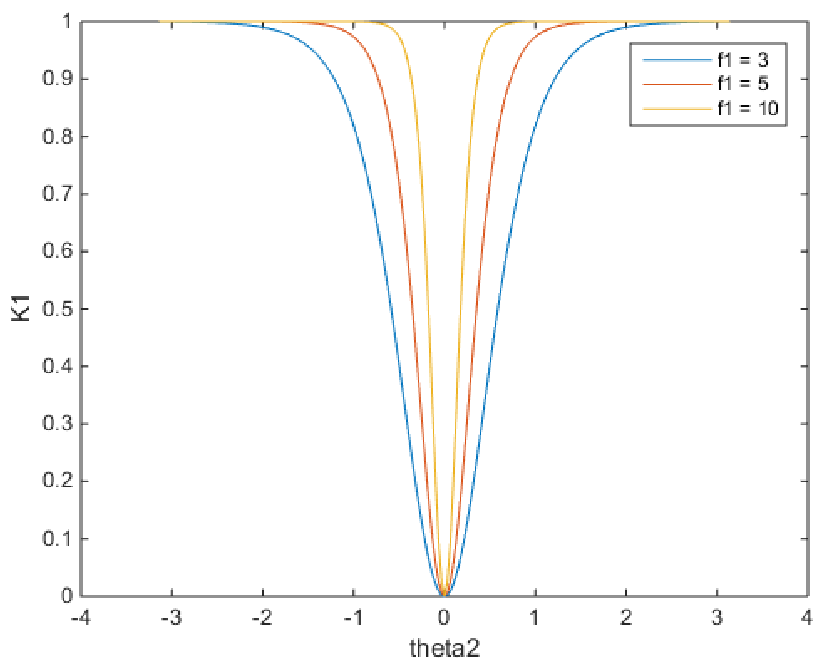

Figure 4.20: K_1 for different values of f_1.

Equation 4.16 defines K_2. K_2 changes the damping term from positive (negative damping, adding energy) to negative (positive damping, removing energy) depending on the current amount of energy in the system. f_2 is a constant that changes the steepness of the relationship between K_2 and r, as seen in Figure 4.21.

$$K_2 = 1 - \frac{2}{1+e^{f_2(1-r)}} \tag{4.16}$$

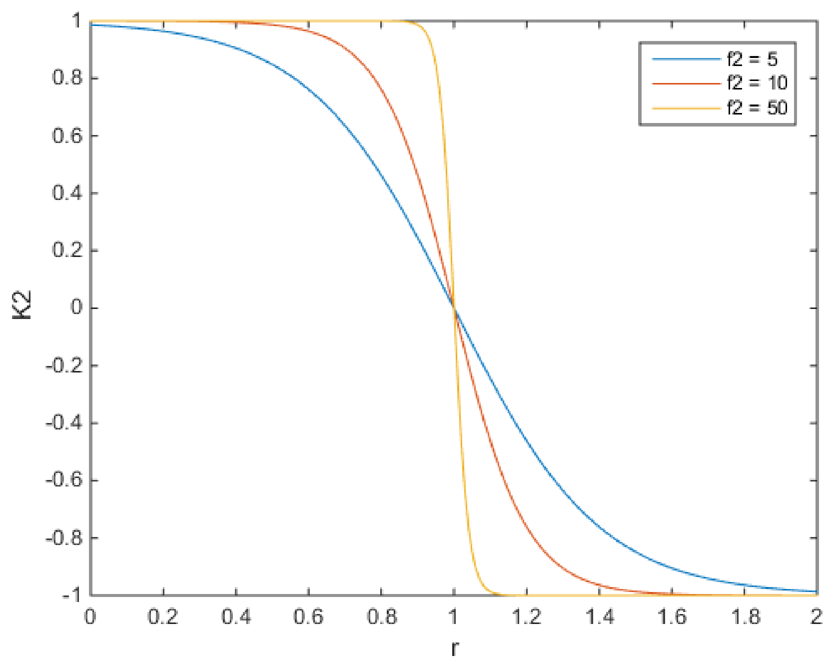

Figure 4.21: K_2 for different values of f_2.

Equation 4.17 defines K_3. K_3 activates the spring to pull the pendulum up when the pendulum is close to the top, where $\theta_2 = 0$. f_3 is a constant that changes the steepness of the relationship between K_3 and θ_2, as seen in Figure 4.22.

$$K_3 = -\frac{4}{e^{f_3\theta_2}+2+e^{-f_3\theta_2}} \tag{4.17}$$

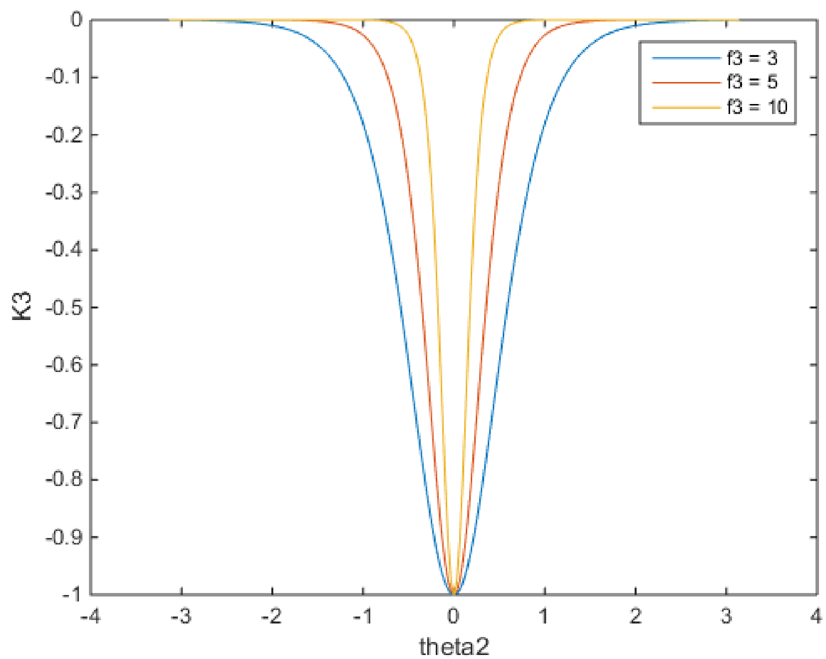

Figure 4.22: K_3 for different values of f_3.

r is the ratio of energy in the system to energy needed in the system. The system is considered to have no energy when the pendulum is pointing down ($\theta_1 = 0, \theta_2 = \pi$) and has no velocity. The energy needed in the system is defined as the energy present when the pendulum is pointing up ($\theta_1 = 0, \theta_2 = \pi$) and has no velocity, and is calculated by Equation 4.19. The current energy in the system is the sum of the current kinetic and potential energy, given by Equations 4.20 through 4.22.

$$r = \frac{E_{current}}{E_{needed}} \tag{4.18}$$

$$E_{needed} = mLg \tag{4.19}$$

$$E_{current} = E_{kinetic} + E_{potential} \tag{4.20}$$

$$E_{kinetic} = \frac{mL^2}{6}\dot{\theta}^2 \quad (4.21)$$

$$E_{potential} = \frac{mLg}{2}(1 - \cos\theta) \quad (4.22)$$

Figure 4.23 shows the response of a pendulum being forced by the continuous phase oscillator function. Figure 4.24 shows the forcing function used. The system takes the same number of swings to swing up and balance as the system from Figure 4.10 (which uses the breakpoint method), but does not have a balanced position as quickly. However, the forcing function is much smoother than the forcing function seen in Figure 4.11.

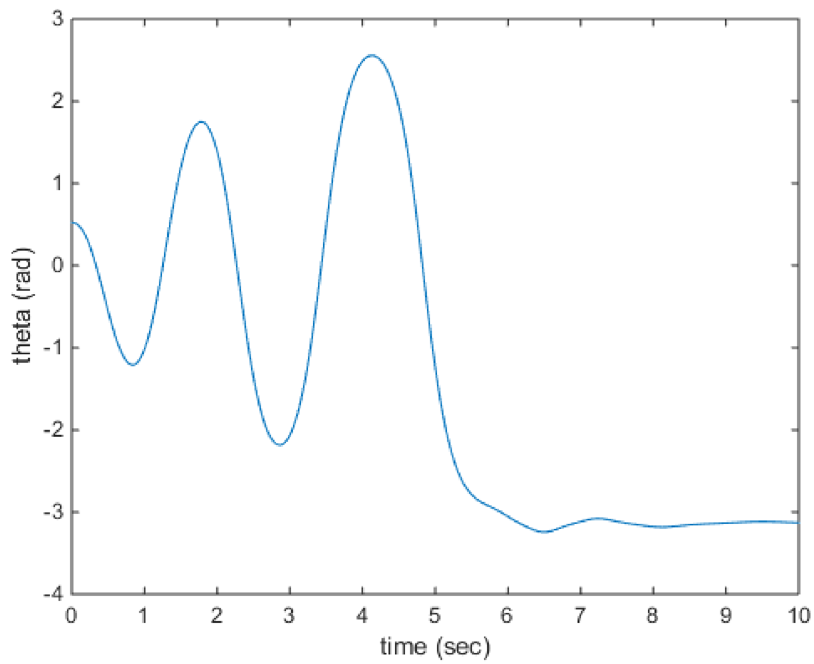

Figure 4.23: Pendulum response to continuous method. $C = 3.2$ N*m, $f_1 = 20$, $f_2 = 500$, $f_3 = 10$, $L = 1$ m, $m = 1$ kg, $g = 9.81$ m/s^2, $\zeta = 0.15$, initial angle = $\frac{\pi}{6}$ rad, initial angular velocity = 0 rad/s.

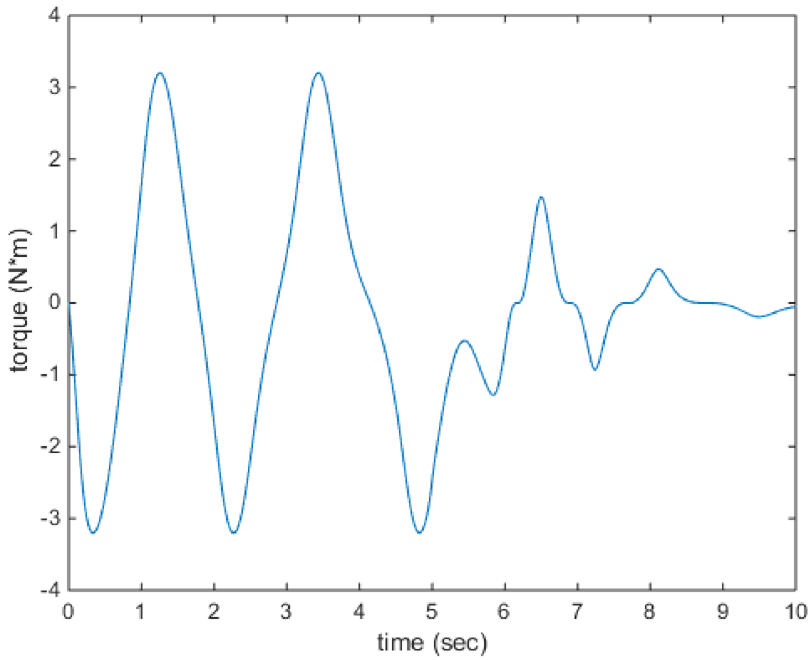

Figure 4.24: Torque applied to system in Figure 4.23.

CHAPTER 5

WORKING MODEL 2D SIMULATION

The continuous phase oscillator method was used to control a motor in Working Model 2D to swing up and balance a pendulum. The pendulum is shown by Figure 5.1. Instead of an infinitely thin rod, the pendulum was a two-dimensional rectangle with a length of 1m, a width of 0.1m, and a mass of 0.1kg (density 1kg/m^2). The simulation used a rotational damper with a value of 0.05 Nms/rad to simulate damping in the system. The pendulum was given an initial position of $\theta_1 = \frac{\pi}{4}$ rad and $\dot{\theta} = 0$ rad/s.

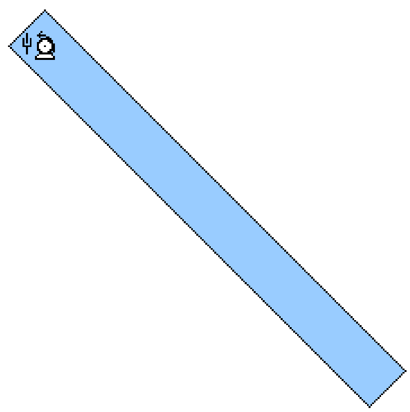

Figure 5.1: Pendulum in Working Model 2D.

The system was forced by the forcing function described in Equation 4.14 with $C = 1.5$ Nm, $f_1 = 20$, $f_2 = 30$, and $f_3 = 10$. Figure 5.2 shows the angular position of the pendulum measured both from the bottom (θ_1) and the top (θ_2). Figures 5.3 and 5.4 show the two phase angles, ϕ_1 and ϕ_2. Figure 5.5 shows the torque applied.

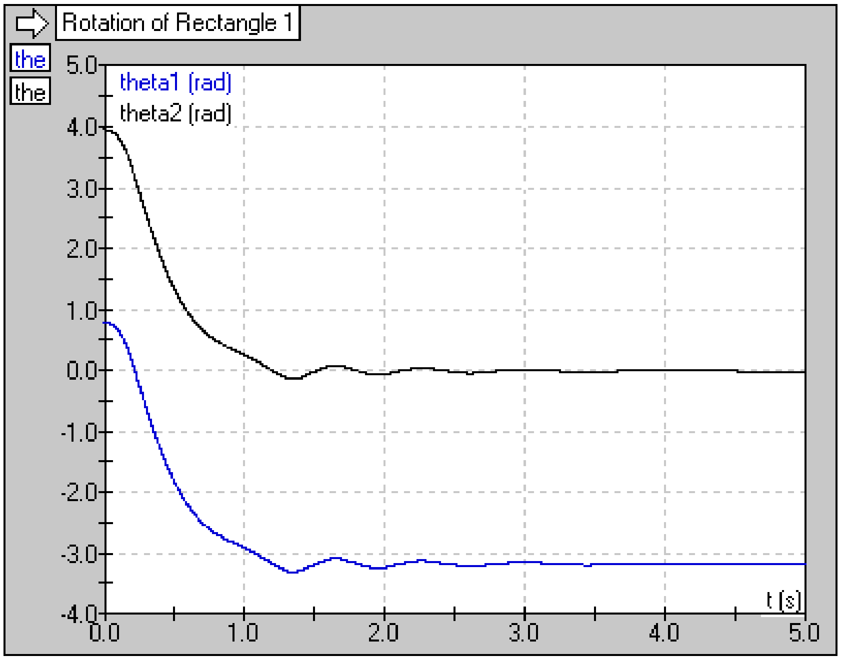

Figure 5.2: Pendulum position. The pendulum only took one swing to reach the top.

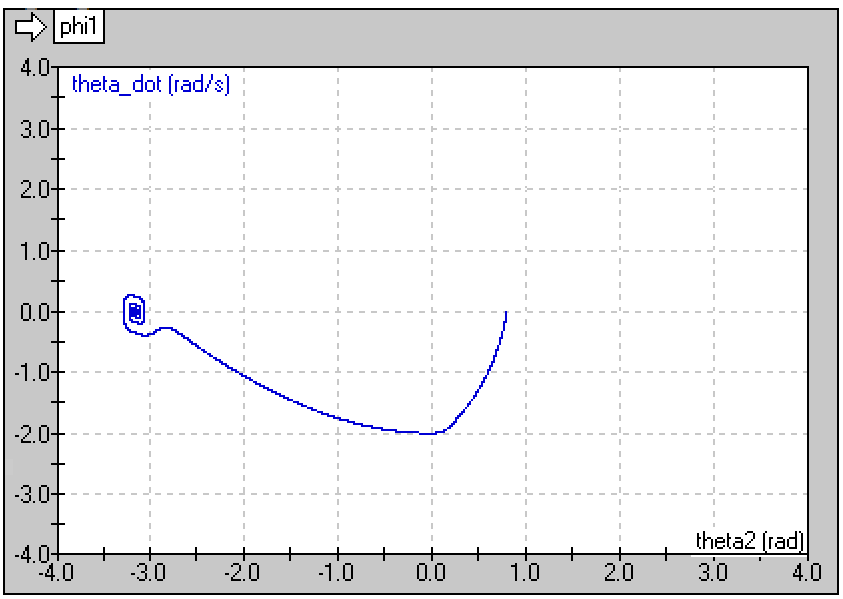

Figure 5.3: Pendulum ϕ_1. One large swing followed by oscillations at the top.

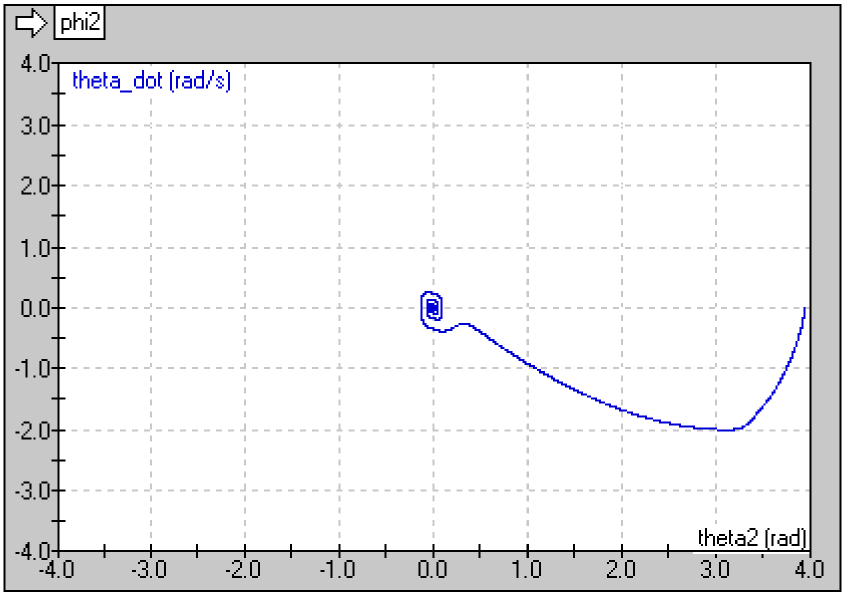

Figure 5.4: Pendulum ϕ_2. One large swing followed by oscillations at the top.

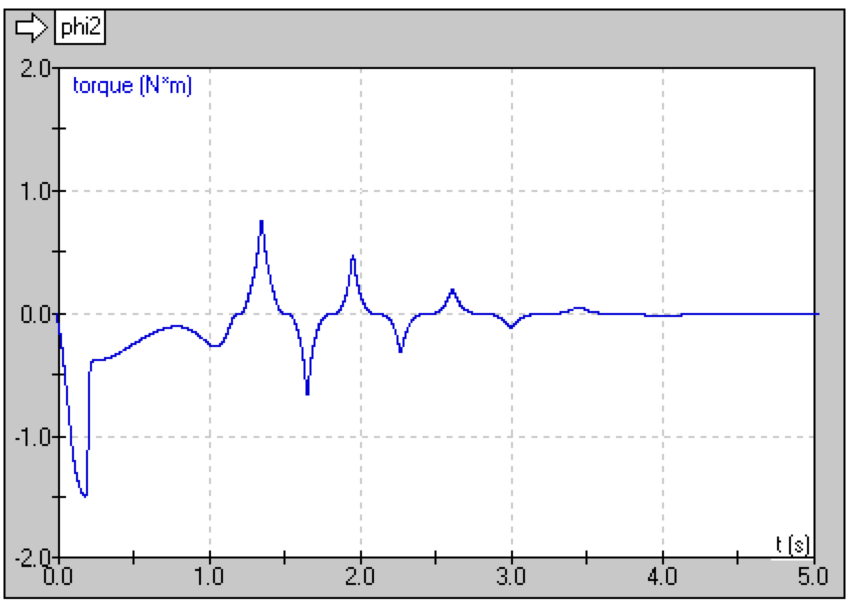

Figure 5.5: Pendulum forcing.

The pendulum responded in a manner very similar to what was expected, with the exception of the time period around 0.2s. In the first simulation, f_2 was set equal to 500, similar to Chapter 4. However, a large value of f_2 means a very sudden change from negative damping to positive damping. While the quick change was not an issue in previous simulations, it resulted in chatter in the forcing function for the Working Model 2D system (and was verified in the same system in MATLAB). f_2 was reduced to 30 to eliminate the chatter, though the forcing function does still change fairly quickly around 0.2s. A system with a lower torque to mass ratio (such as the system in Chapter 4) needs a larger value of f_2 to keep adding energy longer.

CHAPTER 6

CONCLUSION

A phase oscillator can be used to change key properties of oscillating systems, including amplitude, frequency, and equilibrium. The control swings up and balances an inverted pendulum using multiple methods. The first control method involves two separate stages. The scenarios where this control works are discussed. The second control method uses variable coefficients to result in a smooth transition between.

The phase oscillator was developed by Dr. Thomas Sugar. I have contributed to the phase oscillator by showing the sine and cosine of the phase angle can change both the damping and stiffness of a system, modeling sine and cosine as a single cosine term with as phase shift, studying the response of the system when forced to a desired point, and developing two methods for using the phase oscillator for swing up control of a pendulum. Future work includes using the swing up and balancing methods described in this paper to control an experimental physical system.

Portions of this work have been used in academic papers, including one ASME journal paper (reference [3]) and one submitted conference paper to IROS (C63. Sugar, T. G., Bates, A. R., Kerestes, J., Redkar, S. “Using a Phase Oscillator to Control Behavior,” IROS 2016).

REFERENCES

[1] Kerestes, J., Sugar, T. G., Flaven, T., and Holgate, M. “A Method to Add Energy to Running Gait – PogoSuit.” IDETC 2014.

[2] Kerestes, J., Sugar, T. G., and Holgate, M. “Adding and Subtracting Energy to Body Motion – Phase Oscillator.” IDETC 2014.

[3] Sugar, T. G., Bates, A., Holgate, M., Kerestes, J., Mignolet, M., New, P., Ramachandran, R. K., Redkar, S., and Wheeler, C. “Limit Cycles to Enhance Human Performance based on Phase Oscillators,” ASME Journal of Mechanisms and Robotics, 7(1), February 2015 doi: 10.1115/1.4029336.

[4] New, P., Wheeler, C., and Sugar, T. G. “Robotic Hopper Using Phase Oscillator Controller.” IDETC 2014.

[5] Ronsse, R., Vitiello, N., Lenzi, T., Kieboom, J., Carrozza, M. Chiara, and Ijspeert, A. Jan. “Adaptive Oscillators with Human-in-the-Loop: Proof of Concept for Assistance and Rehabilitation.” Proceedings of the 2010 3rd IEEE RAS & EMBS International Conference on Biomedical Robotics and Biomechatronics, 2010.

[6] Rinderknecht, M. D., Delaloye, F. Andr´e, Crespi, A., Ronsse, R., and Ijspeert, A. Jan. “Assistance using adaptive oscillators: Sensitivity analysis on the resonance frequency.” Proceedings of the 2010 3rd IEEE RAS & EMBS International Conference on Biomedical Robotics and Biomechatronics, 2010.

[7] Righetti, L., Buchli, J., and Ijspeert, A. Jan. “Adaptive Frequency Oscillators and Applications.” The Open Cybernetics and Systemics Journal, 2009, 3, 64-69

[8] Henmi, T., Deng, M., and Inoue, A. “Swing-up Control of a Serial Double Inverted Pendulum.” ieee.org. Web. 6 April 2015.

[9] Merakeb, A., and Achemine, F. “Optimal Time Control to Swing-Up the Inverted Pendulum-Cart in Open-Loop Form.” ieee.org. Web. 6 April 2015.

[10] Tanaka, S., Xin, X., and Yamasaki, T. “Experimental Verification of Energy-Based Swing-up Control.” ieee.org. Web. 6 April 2015.

[11] Wang, Z., Chen, Y., and Fang, N. “Minimum-Time Swing-up of A Rotary Inverted Pendulum by Iterative Impulsive Control.” ieee.org. Web. 6 April 2015.

[12] Yokoyama, J., Mihara, K., Suemitsu, H., and Matsuo, T. “Swing-Up Control of a Inverted Pendulum by Two Step Control Strategy.” ieee.org. Web. 6 April 2015.

[13] Zhang, D., Cong, S., Li, Z., and Qin, Z. "The Study of Swing Up and Balance Control for Rotary Parallel Inverted-pendulum." ieee.org. Web. 6 April 2015.

[14] Astrom, K. J., and Furuta, K. "Swinging up a Pendulum by Energy Control." Automatica, Vol. 36, No. 2, pp. 287-295, 2000.

[15] Kaneda, M., Xin, X., Ochiiwa, T., and Henmi, T. "A New Hybrid Control of Inverted Pendulum Systems." Proceedings of the 5th China-Japan International Symposium on Industrial Management, Beijing, pp. 114-119, 2000.

[16] Yamakita, M., Nonaka, K., and Furuta, K. "Swing Up Control of a Double Pendulum." Proceeding of the American Control Conference, pp. 2229-2233, 1993.

[17] Emoto, T., Inoue, A., and Hirashima, Y. "Swinging up Control of Serial Double Inverted Pendulums Using Robustness of Sliding Mode Control." Proceedings of the 7th Meeting of Chugoku Branch of the Society of Instrument and Control Engineers, pp. 92-93, 1998.

[18] Brunetti, C., and Dotoli, M. "Rule-Based Decoupled Fuzzy Sliding Mode Control for Inverted Pendulum Swing-up." ieee.org. Web. 6 April 2015.

[19] Dotoli, M., Mainoe, B., Naso, D., and Turchiano, B. "Fuzzy Sliding Mode Control for Inverted Pendulum Swing-up with Restricted Travel." ieee.org. Web. 6 April 2015.

[20] Yamakita, M., Iwasaki, M., Sugahara, Y., and Furuta, K. "Robust swing up Control of Double Pendulum." Proceeding of the American Control Conference, pp. 290-295, 1996.

[21] Henmi, T., Xin, X., and Kaneda, M. "Swing up Control of an Inverted Pendulum Consideration of Moto's Dynamics and Cart's Distance." Proceedings of the 46th Annual Conference of the Institute of System, Control and Information Engineers, pp. 43-44, 2002.

[22] Ohsumi, A., and Matsumura, M. "Swing-up Control of an Inverted Pendulum via Partial Feedback Linearization." Proceeding of 26th SICE Symp. On Control Theory, pp. 349-356, 1997.

[23] Henmi, T., Deng, M., Inoue, A., Ueki, N., and Hirashima, Y. "Reverse Order Swing-up Control of Serial Double Inverted Pendulums." Proceeding of SICE Annual Conference in Fukui, pp. 2439-2444, 2003.

[24] Nakamura, S., and Hashimoto, S. "Hybrid Learning Strategy to Solve Pendulum Swing-up Problem for Real Hardware." ieee.org. Web. 6 April 2015.

[25] Uchida, M., and Nakano, K. “Swing up of an Inverted Pendulum by Simulator-based Foresight Control.” ieee.org. Web. 6 April 2015