Phenomenology of the $N = 3$ Lee-Wick Standard Model

by

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A Dissertation Presented in Partial Fulfillment
of the Requirement for the Degree
Doctor of Philosophy

Approved April 2015 by the
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May 2015

ABSTRACT

With the discovery of the Higgs Boson in 2012, particle physics has decidedly moved beyond the Standard Model into a new epoch. Though the Standard Model particle content is now completely accounted for, there remain many theoretical issues about the structure of the theory in need of resolution. Among these is the hierarchy problem: since the renormalized Higgs mass receives quadratic corrections from a higher cutoff scale, what keeps the Higgs boson light? Many possible solutions to this problem have been advanced, such as supersymmetry, Randall-Sundrum models, or sub-millimeter corrections to gravity. One such solution has been advanced by the Lee-Wick Standard Model. In this theory, higher-derivative operators are added to the Lagrangian for each Standard Model field, which result in propagators that possess two physical poles and fall off more rapidly in the ultraviolet régime. It can be shown by an auxiliary field transformation that the higher-derivative theory is identical to positing a second, manifestly renormalizable theory in which new fields with opposite-sign kinetic and mass terms are found. These so-called Lee-Wick fields have opposite-sign propagators, and famously cancel off the quadratic divergences that plague the renormalized Higgs mass. The states in the Hilbert space corresponding to Lee-Wick particles have negative norm, and implications for causality and unitarity are examined.

This dissertation explores a variant of the theory called the $N = 3$ Lee-Wick Standard Model. The Lagrangian of this theory features a yet-higher derivative operator, which produces a propagator with three physical poles and possesses even better high-energy behavior than the minimal Lee-Wick theory. An analogous auxiliary field transformation takes this higher-derivative theory into a renormalizable theory with states of alternating positive, negative, and positive norm. The phenomenology of this theory is examined in detail, with particular emphasis on the collider signatures

of Lee-Wick particles, electroweak precision constraints on the masses that the new particles can take on, and scenarios in early-universe cosmology in which Lee-Wick particles can play a significant role.

To Kendra,

for infinite love and implausible patience.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Prof. Richard F. Lebed, for his unwavering support and pedagogical prowess throughout my graduate career. I hasten to add my thanks for his many careful revisions of my manuscripts, and of this dissertation especially.

I also would like to thank the other members of my committee, Profs. Ricardo Alarcón, Andrei Belitsky, Ralph Chamberlin, and Maulik Parikh for support, advice, and of course, letters of recommendation sent to everywhere from Tokyo to Tel Aviv.

My candidacy as a doctoral student would have been unlikely without the tutelage and inspiration offered to me by many great teachers of science and mathematics. To Brian McKanna, Tom Wier, Jim Courter, Michael McDaniel, Ken Hayes, David Murphy, Samuel Webster, and Thomas Treloar — thank you for leading by example.

To my friends and colleagues that have made the grad school experience more than just calculation: Hank Lamm, Jayden Newstead, Subir Sabharwal, Jeff Hyde, Francis Duplessis, Owen Littlejohns, Lauren Ice, and any others I may have missed. Thank you for all the memories, and may the Force be with you.

This work would not have been possible without funding from the National Science Foundation, under grants PHY-0757394 and PHY-1068286. The federal funding agencies, and the taxpayers who ultimately buttress the system of publicly endowed research, have my deepest gratitude. And lastly, to my parents, John and Ashley: thank you for always supporting my studies, and for giving me the opportunities to do something a little out of the ordinary.

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INTRODUCTION

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1.1.1 Spontaneous Symmetry Breaking and Vector Boson Masses

It is an exciting time indeed to be a physicist. With the discovery of the Higgs boson at CERN's Large Hadron Collider (LHC) [3, 4], one of the greatest theoretical constructs in the history of science is now complete: the Standard Model of particle physics (SM). Independently formulated by Sheldon Glashow [5], Steven Weinberg [6], and Abdus Salam [7] in the mid-1960s (see, e.g., [8] for historical background), the SM is a gauge field theory that incorporates electromagnetism (generated by the abelian group $U(1)$) and the weak nuclear force (generated by the non-abelian $SU(2)$) into a single semi-simple Lie group, $SU(2)_L \times U(1)_Y$, often combined with the Lie group $SU(3)_c$ of color. The Lagrangian of the SM is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{SM}} = \bar{Q}^i_L i\not{D} Q^i_L + \bar{u}^i_R i\not{\partial} u^i_R + \bar{d}^i_R i\not{\partial} d^i_R + \bar{L}^i_L i\not{D} L^i_L + \bar{e}^i_R i\not{\partial} e^i_R \\ +(D_\mu\Phi)^\dagger(D^\mu\Phi) + \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \\ -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}\mathrm{tr}[W_{\mu\nu}W^{\mu\nu}] - \frac{1}{2}\mathrm{tr}[G_{\mu\nu}G^{\mu\nu}] \\ -\left(y_{u^i}\epsilon^{ab}(\bar{Q}^i_L)_a\Phi^\dagger_b u^i_R + y_{d^i}(\bar{Q}^i_L)_a\Phi_a d^i_R + y_{e^i}(\bar{L}^i_L)_a\Phi_a e^i_R + \mathrm{h.c.}\right),\end{aligned} \tag{1.1}$$

using the covariant derivative

$$D_\mu = \partial_\mu - ig'YB_\mu - igT^aW^a_\mu - ig_s t^a G^a_\mu, \tag{1.2}$$

left-handed quark, lepton, and Higgs doublets

$$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}, \quad E_L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{1.3}$$

and Yang-Mills stress-energy tensors relevant to the three gauge groups:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \;\; W_{\mu\nu}^a T^a = \frac{i}{g}[D_\mu, D_\nu], \;\; G_{\mu\nu}^a t^a = \frac{i}{g_3}[D_\mu, D_\nu]. \tag{1.4}$$

Here, the generators T^a and t^a are respectively given by the Pauli and Gell-Mann matrices $\sigma^a/2$ and $\lambda^a/2$; the index i runs through the three generations of fermions presently known; and the Yukawa interaction terms with couplings y_{u^i}, y_{d^i}, y_{e^i} produce masses for the i^{th} generation of up-type quarks, down-type quarks, and leptons (to be described in detail below). In Eq. (1.4), the covariant derivative is taken to encompass only the gauge group relevant to the field strength tensor under consideration. Its action on the gauge fields can be easily computed from Eq. (1.2):

$$\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = \partial_\mu A_\nu^a \lambda^a - \partial_\nu A_\mu^a \lambda^a + g f^{abc} A_\mu^b A_\nu^c \tag{1.5}$$

for arbitrary gauge field A_μ, generators λ^a, coupling g, and structure constants f^{abc}. We include the gluon fields G_μ^a here for completeness, but we will have no need to consider them further in this chapter, focusing instead on the electroweak gauge fields and the spontaneous symmetry breaking in which they participate. Unless otherwise stated, the mathematical conventions here and throughout the rest of this dissertation follow those of Peskin & Schroeder [9] (*e.g.*, the choice of basis for the Dirac γ matrices).

With the basic pieces of the theory in place, it is time to consider first the Higgs potential (μ^2 and λ terms) of Eq. (1.1). The Higgs field Φ is a complex-valued doublet under $SU(2)_L$, and may be split up into its charged (ϕ^+), scalar (ϕ), and pseudoscalar

(P) pieces:

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\phi + iP) \end{pmatrix} \tag{1.6}$$

The negative mass-squared term, $V(\phi) \supset -\frac{1}{2}\mu^2\phi^2$, moves the global minimum of the potential from $\langle\phi\rangle = 0$ (as would be the case in a conventional ϕ^4 theory) to some nonzero $\langle\phi\rangle = v$ (as in Fig. 1.1). The location of the new vacuum is set by requiring

$$0 = \left.\frac{dV(\phi)}{d\phi}\right|_{\phi=v} = -\mu^2 v + \lambda v^3, \tag{1.7}$$

$$\therefore\ v^2 = \mu^2/\lambda. \tag{1.8}$$

With the manifest symmetry of the system broken, we now expand the CP-even

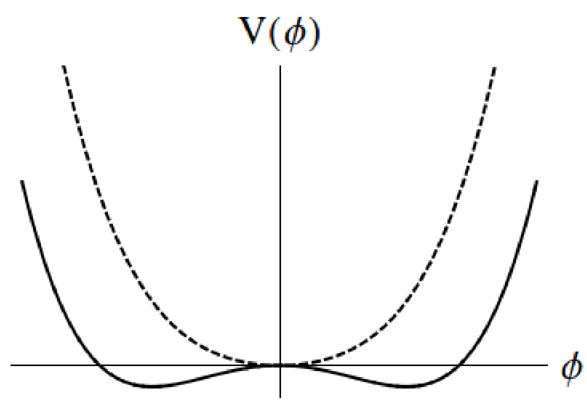


Figure 1.1: $V(\phi)$ given for a scalar field ϕ with a positive mass-squared term (dashes) and a negative mass-squared term (solid).

scalar Higgs component as $\phi = v + h(x)$, yielding a new scalar potential

$$\begin{aligned} V(h) = &-\frac{1}{2}\mu^2 v^2 + \frac{\lambda}{4}v^4 + v(-\mu^2 + \lambda v^2)h \\ &+ \frac{1}{2}(3\lambda v^2 - \mu^2)h^2 + \lambda v^3 h + \frac{\lambda}{4}h^4 \end{aligned} \tag{1.9}$$

$$= -\frac{\lambda}{4}v^4 + \frac{1}{2}m_h^2 h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4, \tag{1.10}$$

making use of Eq. (1.8). Whereas Eq. (1.1) boasted a Higgs potential with explicit $\mathbb{Z}_2$ ($\phi \to -\phi$) symmetry (in addition to the gauge symmetries contained in the covariant derivative), Eq. (1.10) contains an h^3 term. This breaks the original $\phi \to -\phi$ symmetry. More important, however, is the effect this has on the gauge bosons introduced in Eq. (1.1), as we now discuss.

Let us consider the mechanism of mass generation for the gauge bosons. Though the mechanism is now synonymous with Peter Higgs [10, 11, 12], pioneering work by others such as Englert and Brout [13] and Guralnik, Hagen, and Kibble [14] helped to develop the modern understanding of spontaneous symmetry breaking. Taking the Higgs kinetic term from Eq. (1.1) and allowing the Higgs to take on its vacuum expectation value (VEV) prescribed, as above, by $\Phi \longrightarrow \langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$, one has

$$\begin{aligned} \mathcal{L}_{\rm SM} \supset & \frac{1}{2}\begin{pmatrix} 0 & v \end{pmatrix}\left(gW^a_\mu T^a + \frac{1}{2}g' B_\mu\right)\left(gW^{\mu b}T^b + \frac{1}{2}g' B^\mu\right)\begin{pmatrix} 0 \\ v \end{pmatrix} \\ & = \frac{1}{2}\frac{v^2}{4}\left(g^2\left(W^1_\mu\right)^2 + g^2\left(W^2_\mu\right)^2 + \left(-gW^3_\mu + g' B_\mu\right)^2\right), \end{aligned} \tag{1.11}$$

where explicit use has been made of the Pauli matrices and the Higgs $SU(2)_L$ hypercharge $Y = 1/2$. We can see from Eq. (1.11) that the VEV attained by the Higgs field has the effect of generating masses for the gauge bosons, with the added twist of producing a cross-term between W^3_μ and B_μ. Eq. (1.11) may also be understood as a mass matrix with one null eigenvalue; one may define mass-diagonal fields Z^0_μ, A_μ using the field redefinition

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W^3 \\ B \end{pmatrix}, \tag{1.12}$$

where θ_W is called the weak mixing angle. Substituting Eq. (1.12) into Eq. (1.11), and demanding that the new mass terms be fully diagonal, one has the following useful relations:

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \tag{1.13}$$

$$W^{\pm}_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right), \text{ with mass } m_W = \frac{1}{2}gv, \tag{1.14}$$

$$Z^0_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gW^3_\mu - g'B_\mu\right), \text{ with mass } m_Z = \sqrt{g^2+g'^2}\frac{v}{2}, \tag{1.15}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'W^3_\mu + gB_\mu\right), \text{ with mass } m_A = 0. \tag{1.16}$$

Since the gauge boson mass matrix mixes the W^3_μ and B_μ terms, the Z^0_μ becomes distinct from the other SU(2) bosons, and communicates the weak neutral current (as originally predicted by Weinberg in [15]). We identify the massless A_μ with the photon. Since this Abelian gauge field remains massless, we conclude that the original $SU(2)_L \times U(1)_Y$ symmetry becomes spontaneously broken to $U(1)_{\text{EM}}$. Using the inverse transformation

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z^0 \\ A \end{pmatrix}, \tag{1.17}$$

we can then solve for the couplings:

$$\begin{aligned} gT^3W^3_\mu &= gT^3\frac{1}{\sqrt{g^2+g'^2}}\left(gZ^0_\mu + g'A_\mu\right), \\ g'YB_\mu &= g'Y\frac{1}{\sqrt{g^2+g'^2}}\left(-g'Z^0_\mu + gA_\mu\right), \end{aligned} \tag{1.18}$$

and we can see that the couplings and generators work out to

$$\frac{gg'}{\sqrt{g^2+g'^2}}\left(T^3+Y\right)A_\mu \equiv eQA_\mu, \tag{1.19}$$

$$\begin{aligned} \frac{1}{\sqrt{g^2+g'^2}}\left(g^2T^3 - g'^2Y\right)Z^0_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\left(g^2T^3 - g'^2(Q-T^3)\right)Z^0_\mu, \\ = \sqrt{g^2+g'^2}\left(T^3 - \frac{g'^2}{g^2+g'^2}Q\right)Z^0_\mu &= \frac{g}{\cos\theta_W}\left(T^3 - \sin^2\theta_W Q\right)Z^0_\mu. \end{aligned} \tag{1.20}$$

Using these new mass eigenstate fields, it is convenient to re-express the covariant derivative as

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_W}(T^3 - \sin^2\theta_W Q)Z_\mu - ieQA_\mu, \quad (1.21)$$

where $T^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$ and $Q = T^3 + Y$. Q, as the generator attached to the massless gauge boson A_μ, remains the sole unbroken generator in the electroweak theory.

One important quantity is the so-called ρ parameter:

$$\rho \equiv \frac{m_W}{\cos\theta_W m_Z}, \quad (1.22)$$

which is identically equal to unity at tree-level in the SM. As may be seen from Eqs. (1.13)-(1.16), in the limit $g' \to 0$, the hypercharge boson B_μ decouples from W_μ^3, leading to $\cos\theta_W \to 1$ and $m_{Z^0} \to \frac{1}{2}gv = m_W$. This relation, in which all three $SU(2)$ bosons possess the same mass in the small-$\sin\theta_W$ limit, is referred to as *custodial symmetry*. There exist extensions of the SM in which this limit is not preserved (such as theories including an extra Higgs transforming as an $SU(2)$ triplet [16]). Violations of custodial symmetry produced by such "Beyond-the-Standard-Model" (BSM) theories are marked by deviations from unity in the ρ parameter, and such deviations are strongly constrained by experiment (see, *e.g.*, [17, 18] for work on precision constraints on the electroweak model). The experimental limits on observables such as ρ will be exploited in subsequent chapters of this dissertation.

1.1.2 Generation of Masses for Chiral Fermions

In addition to generating the masses for vector bosons, the Higgs mechanism also produces masses for chiral fermions in an interacting chiral gauge theory. The archetypal Dirac Lagrangian,

$$\mathcal{L}_0 = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi, \quad (1.23)$$

contains an explicit mass term; however, when these fields are made to interact under a chiral gauge theory, one breaks the field ψ into its eigenstates of left- and right-handed chirality[1] $\psi = \psi_L + \psi_R$. The non-vanishing pieces of the free-field Lagrangian reduce to

$$\begin{aligned}\mathcal{L}_0 \to i(\bar{\psi}_L + \bar{\psi}_R)\partial\!\!\!/(\psi_L + \psi_R) - m(\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) \\ = i\bar{\psi}_L \partial\!\!\!/ \psi_L + i\bar{\psi}_R \partial\!\!\!/ \psi_R - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L).\end{aligned} \tag{1.24}$$

Since the left-handed fields transform non-trivially under $SU(2)_L$, whereas the right-handed fields do not, this produces a problem: the kinetic terms are perfectly acceptable, as they are gauge-invariant combinations of fields and their conjugates, but the mass terms mix fields from different $SU(2)_L$ representations and thus violate gauge invariance. We therefore cannot write down explicit, gauge-invariant mass terms in an interacting chiral theory. The solution is to write down a set of Yukawa terms as in Eq. (1.1). The Higgs, as a doublet under $SU(2)_L$, forms a tensor product with other fermion doublets present in Eq. (1.1) which contains a gauge singlet. When the Higgs attains its VEV, an effective coupling between left- and right-chiral fields is produced which is identical to the conventional Dirac mass.

In a theory with multiple generations of fermions (such as we have), the possibility arises for Yukawa couplings between quarks from different generations. In order to obtain unambiguous mass terms for a mass-eigenstate quark of any given flavor, it is helpful to define the family multiplets for up- and down-type quarks

$$u_L^i = \begin{pmatrix} u_L, & c_L, & t_L \end{pmatrix} \qquad d_L^i = \begin{pmatrix} d_L, & s_L, & b_L \end{pmatrix} \tag{1.25}$$

and then invoke unitary operators to diagonalize the quark fields in the mass basis:

$$u_L^i = U_u^{ij} u_L^{\prime j} \qquad d_L = U_d^{ij} d_L^{\prime j}, \tag{1.26}$$

[1] Chirality is the high-energy limit of helicity, which is the projection a particle's spin against its direction of motion. In the case of massless particles, these concepts are identical.

where the primed multiplets are mass-diagonal. This re-assignment simplifies the mass terms at the expense of complicating the gauge couplings; whereas the GWS theory originally coupled only fermions within a single generation to one another, there now exist couplings between quarks of different generations with $\Delta Q = \pm 1$. As an example, the quark current $J_W^{\mu +}$ coupling to the W_μ^+ transforms according to

$$J_W^{\mu +} = \bar{u}_L^i \gamma^\mu d_L^i = \bar{u}_L^{\prime j} \gamma^\mu (U_u^\dagger)^{ji} U_d^{ik} d_L^{\prime k} = \bar{u}_L^{\prime i} \gamma^\mu (U_u^\dagger U_d)^{ij} d_L^{\prime j} = \bar{u}_L^{\prime i} \gamma^\mu V^{ij} d_L^{\prime j}, \tag{1.27}$$

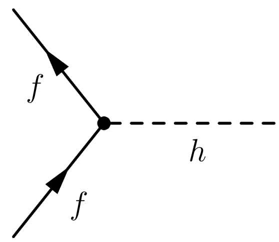
where V^{ij} is called the Cabibbo-Kobayashi-Maskawa matrix [19, 20] connecting quarks of the i^{th} and j^{th} generations. Flavor-changing neutral currents are forbidden in the GWS model; they are a generic feature of many BSM theories, and strong experimental constraints have been placed on them [21].

1.2 The Hierarchy Problem: Understanding the Renormalized Higgs Mass

It is a well-known feature that amplitudes in quantum field theory diverge beyond the leading order, requiring the methods of the renormalization group in order to properly regulate and subtract formally infinite quantities. For the sake of space, no pedagogical introduction to the techniques of renormalization will be given here; for historical background, see seminal papers by, *e.g.*, Wilson and Fisher [22], and 't Hooft and Veltman [23], or comprehensive textbook surveys [9, 24]. After spontaneous symmetry breaking has taken place, any Yukawa term coupling two fermions f_L, f_R to the Higgs may be written

$$-\mathcal{L}_{\text{int}} \supset \frac{1}{\sqrt{2}} y_f \bar{f}_L (v + h) f_R + \text{h.c.} = m_f \bar{f}_L f_R \left(1 + \frac{h}{v}\right) + \text{h.c.}, \tag{1.28}$$

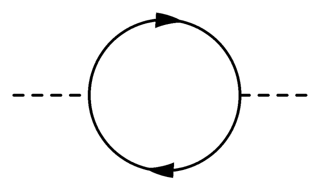
from which the Feynman vertex joining the fermion current to the Higgs may be read as



$$= -i\frac{m_f}{v}. \tag{1.29}$$

One very important one-loop correction to the Higgs two-point function comes from fermion loops

$$-iM^2(p^2) =$$



$$, \tag{1.30}$$

which corrects the mass of the Higgs at leading order in perturbation theory. Solving for the amplitude of this one-loop diagram using the standard methods of quantum field theory, we know to expect a formally infinite result. However, it is also widely accepted that the SM is only an effective field theory valid over a finite energy scale, meaning that one need not integrate over the full four-momentum q^μ running inside the loop. The UV completion containing the field content of the standard model, be it supersymmetry, extra dimensions, a grand unified theory (GUT), or some other exotic model, is expected to take over at some parametric scale Λ. As no BSM contender has yet been discovered experimentally, the scale at which new physics becomes relevant remains unknown, though lower bounds do exist for the particle masses in many of these models [25, 26, 27]. Some of the mass bounds obtained from the CMS collaboration are given in Fig. 1.2 for convenience.[2] For now, we will take Λ to be an order-of-magnitude estimate of this scale for the sake of illustration, with the tacit understanding that SM-only calculations will only be valid up to Λ. The SM will be understood to be an effective field theory, whose predictions will be amended by corrections of $\mathcal{O}(v_{\text{SM}}/\Lambda)$. Let us calculate (1.30) using a cutoff regularization:

$$\begin{aligned}
-iM^2(p^2) &= -\left(-i\frac{m_f}{v}\right)^2 \int_0^\Lambda \frac{d^4q}{(2\pi)^4}\text{tr}\Big[\frac{i(\not{q}+\not{p}+m_f)}{(q+p)^2-m_f^2}\frac{i(\not{q}+m_f)}{q^2-m_f^2}\Big] \\
&= -4\left(\frac{m_f}{v}\right)^2 \int_0^\Lambda \frac{d^4q}{(2\pi)^4}\frac{q\cdot(p+q)+m_f^2}{((p+q)^2-m_f^2)(q^2-m_f^2)}. \qquad (1.31)
\end{aligned}$$

Now, we use the familiar technique of the Feynman integral and shift the variable of integration by $q^\mu \to l^\mu = q^\mu + xp^\mu$, in order to cast the divergent piece in a more tractable form:

$$= -4\left(\frac{m_f}{v}\right)^2 \int_0^1 dx \int_0^\Lambda \frac{d^4l}{(2\pi)^4}\frac{l^2 - x(1-x)p^2 + m_f^2}{(l^2-\Delta)^2}, \qquad (1.32)$$

where $\Delta = m_f^2 - x(1-x)p^2$. Executing a Wick rotation on the timelike component, $l^0 \equiv il_E^0$, the integral can now be made to converge in four-dimensional Euclidean space. This is most conveniently done by exploiting the manifest spherical symmetry of the integrand:

$$\begin{aligned}
=&4i\left(\frac{m_f}{v}\right)^2 \int_0^1 dx \int_0^\Lambda \frac{d^4\,l_E}{(2\pi)^4}\frac{l_E^2 + x(1-x)p^2 \ - m_f^2}{(l_E^2-\Delta)^2} \\
=&4i\left(\frac{m_f}{v}\right)^2 \frac{2\pi^2}{16\pi^4}\int_0^1 dx \int_0^\Lambda d|l_E| \ \frac{|l_E|^5 + \left(x(1-x)p^2 \ - m_f^2\right)|l_E|^3}{(|l_E^2|-\Delta)^2} \\
=&i\frac{m_f^2}{4\pi^2v^2}\int_0^1 dx\left(\frac{3\Delta\Lambda^2+\Lambda^4}{\Delta+\Lambda^2} + 3\Delta\ln\left(\frac{\Delta}{\Delta+\Lambda^2}\right)\right). \qquad (1.33)
\end{aligned}$$

[2]For up-to-date results from the CMS collaboration, visit `https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResults`

Now, expand the result in powers of Λ^2, keeping only pieces that diverge in the limit $\Lambda \to \infty$:

$$\begin{aligned} -iM^2(p^2) &= i\frac{m_f^2}{4\pi^2 v^2}\int_0^1 dx\left(\Lambda^2 - 3\Delta\ln\left(\frac{\Lambda^2}{\Delta}\right)\right) \\ &= i\frac{m_f^2}{4\pi^2 v^2}\left(\Lambda^2 - \frac{1}{2}(6m_f^2 - p^2)\ln\left(\frac{\Lambda^2}{m_f^2}\right)\right). \end{aligned} \tag{1.34}$$

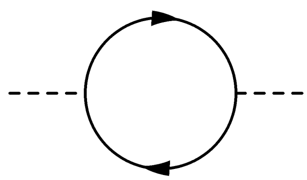
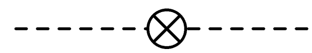
We see that the renormalized scalar mass is sensitive to the cutoff scale Λ, both quadratically and logarithmically. This is unique to the scalar case, and is to be contrasted with the solely logarithmic renormalization to fermion masses. Large logarithms in renormalized amplitudes have been understood since the time of Gell-Mann and Low [28], but the quadratic divergence renders the scalar mass extremely sensitive to physics at the high scale. This clashes with the spirit of effective field theories, in which one is permitted agnosticism regarding the exact dynamics and scale of higher-energy physics.

This raises the following question, whose importance cannot be overstated:

Why is the Higgs mass near the electroweak scale, rather than at a much higher value associated with new physics?

Numerically, the electroweak scale is defined by $v \approx 246.2\,\text{GeV}$ [29], whereas the order of magnitude for the onset of GUT physics is typically taken to be $\Lambda_{\text{GUT}} \sim 10^{16}$ GeV and the Planck mass (at which scale quantum gravity is expected to dominate) is given by $M_P \equiv \sqrt{\hbar c/G_N} = 1.22093(7) \times 10^{19}\,\text{GeV}$ [29]. If nothing new arises between the electroweak scale and unification, a rough estimate of the hierarchy involved is $\Lambda_{\text{GUT}}^2/v^2 = \mathcal{O}(10^{28})$. One must fine-tune the parameters of the electroweak theory to better than one part in 10^{28} in order to ameliorate this divergence, and produce the Higgs mass observed today. This tension is known as the *hierarchy problem*.

The situation can be understood more precisely by computing the same result in the dimensional regularization scheme of 't Hooft and Veltman [23], in which the dimensionality of spacetime is set to d and the cutoff is removed. Renormalized perturbation theory requires that we add to the one-loop diagram the counterterms which cancel off divergences, thereby rendering the amplitude finite. This is done by supplementing the loop diagram with another representing the counterterms:

$-iM^2(p^2) =$  $+$ 

$$= -iM_0^2(p^2) + i(p^2\delta_Z - \delta_m), \tag{1.35}$$

where $-iM_0^2(p^2)$ is the formally divergent piece to be calculated, and δ_Z and δ_m are respectively the wavefunction and mass renormalization counterterms. Beginning from line 1 of (1.33), the divergent piece becomes

$$\begin{aligned} -iM_0^2(p^2) &= 4i\left(\frac{m_f}{v}\right)^2 \int_0^1 dx \int \frac{d^4\, l_E}{(2\pi)^4} \frac{l_E^2 + x(1-x)p^2 \; - m_f^2}{(l_E^2 - \Delta)^2} \\ &= 4i\left(\frac{m_f}{v}\right)^2 \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \left(\frac{\frac{d}{2}\Gamma(1-\frac{d}{2})}{\Delta^{1-d/2}} - \frac{\Delta\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} \right) \\ &= \frac{4i(d-1)}{(4\pi)^{d/2}} \left(\frac{m_f}{v}\right)^2 \int_0^1 \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-d/2}}, \end{aligned} \tag{1.36}$$

with $\Delta(p^2)$ equivalent to that used above. The remaining gamma function records the formal infinity as $d \to 4$, as expected. Now, having calculated the correction to the two-point function, one must impose further constraints in order to ensure that the full amplitude is consistent with the tree-level field theory. This is done by demanding that the loop amplitude vanish when $p^2 = m_h^2$ (*i.e.* when the scalar goes

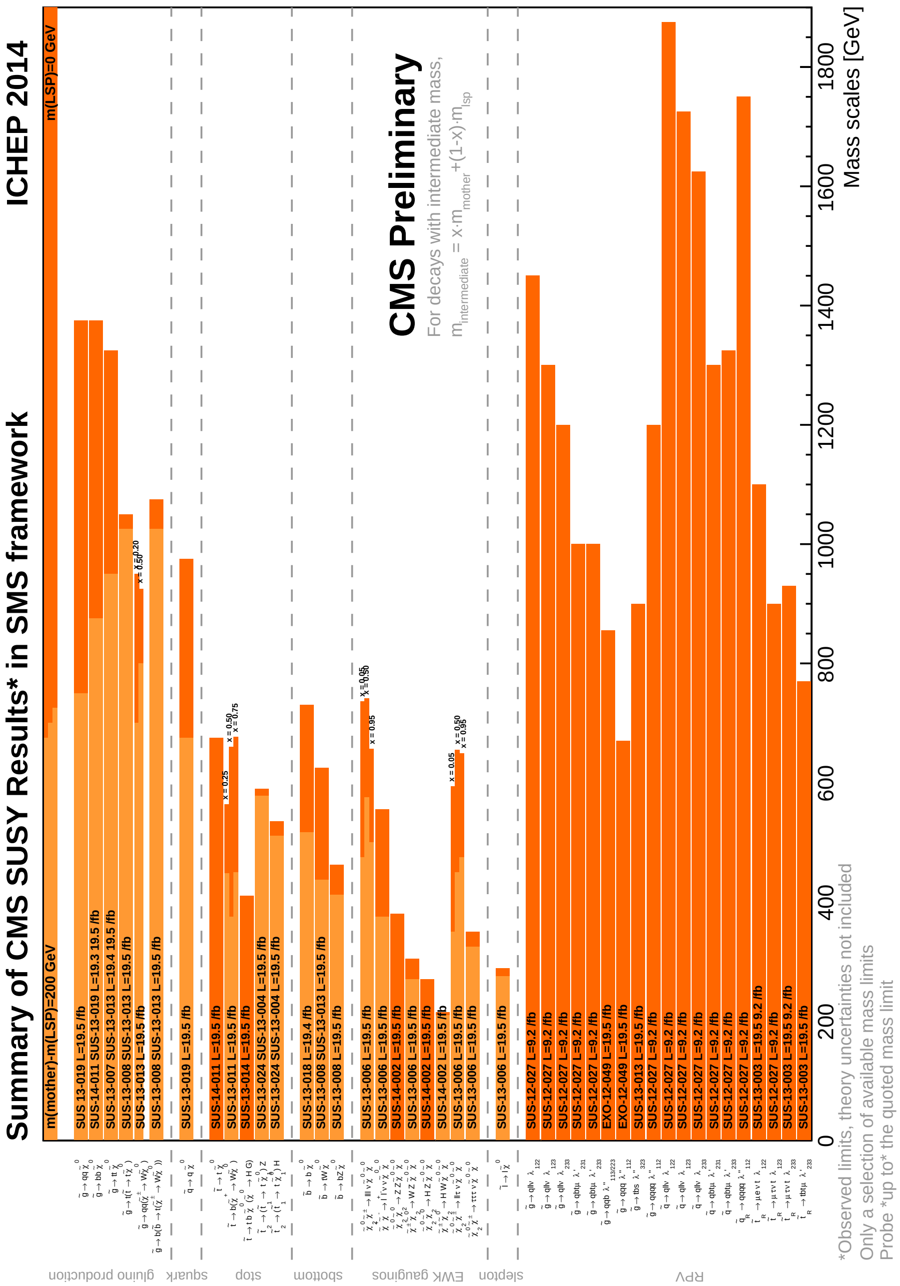


Figure 1.2: An exclusion plot for SUSY particles from the CMS collaboration, presented at ICHEP 2014.

on-shell), and that the amplitude's first derivative vanishes at $p^2 = m_h^2$ as well:

$$-iM^2(p^2)|_{p^2=m_h^2} = 0, \tag{1.37}$$

$$-i\frac{d}{dp^2}M^2(p^2)|_{p^2=m_h^2} = 0. \tag{1.38}$$

These two constraints are sufficient to uniquely determine the counterterms δ_Z and δ_m. Eq. (1.37) can be met by setting

$$\delta_m = \frac{4(d-1)}{(4\pi)^{d/2}}\left(\frac{m_f}{v}\right)^2 \int_0^1 dx \frac{\Gamma(1-\frac{d}{2})}{(m_f^2 - x(1-x)m_h^2)^{1-d/2}} + m_h^2\delta_Z, \tag{1.39}$$

while Eq. (1.38) is satisfied by setting

$$\delta_Z = -\frac{4(d-1)}{(4\pi)^{d/2}}\left(\frac{m_f}{v}\right)^2 \int_0^1 \frac{x(1-x)\Gamma(2-\frac{d}{2})}{(m_f^2 - x(1-x)m_h^2)^{2-d/2}}. \tag{1.40}$$

One immediately sees from the gamma functions that the counterterms so calculated contain divergences of differing degrees. This is how the hierarchy problem is manifested in dimensional regularization. Upon adding them back into Eq. (1.35), we now have a finite one-loop correction to the Higgs mass of

$$-iM^2(p^2) = -\frac{3im_f^2}{4\pi^2v^2}\int_0^1 dx \left(-x(1-x)(p^2 - m_h^2) + \Delta(p^2)\ln\left(\Delta(m_h^2)/\Delta(p^2)\right)\right), \tag{1.41}$$

which has a more manageable form in the $p^2 \to 0$ limit:

$$-iM^2(0) = -\frac{3im_f^2}{4\pi^2v^2}\int_0^1 dx \left(x(1-x)m_h^2 + m_f^2\ln\left(\frac{m_f^2 - x(1-x)m_h^2}{m_f^2}\right)\right). \tag{1.42}$$

1.3 The Lee-Wick Standard Model

Many theories have been proposed with the aim of solving the hierarchy problem. The most notable examples studied to date include supersymmetry (reviews may be found in Refs. [30, 31]), the Randall-Sundrum model [32], and gauge-gravity unification at the weak scale [33]. In each case, the Standard Model is augmented by a new symmetry, extra spatial dimensions, new fields at or above the electroweak scale, or some combination of the above, and more. This dissertation concerns itself with a different sort of approach altogether: the Lee-Wick Standard Model (LWSM).

Recent developments in the LWSM have built on the original work of T.D. Lee and G.C. Wick [34, 35], who investigated the Pauli-Villars (negative quantum-mechanical norm) regulator [36] taken as a physical degree of freedom. QED augmented by a photon with such properties corresponds to a higher-derivative formulation of the theory. The modified gauge field of the theory has a higher-derivative propagator containing two poles: one with positive residue (corresponding to the standard photon), and one with negative residue (corresponding to the Lee-Wick (LW) photon). Classically, this negative residue is associated with a pathological instability in the theory; on the quantum level, it heralds the existence of nonpositive-definite norms in the Hilbert space, leading to problems with unitarity.

Lee and Wick argued that one can make physical sense of such a theory if there exists a mechanism whereby the effects of the offending nonpositive-normed states are systematically removed from the Hilbert space. Since the LW photon is massive, the state corresponding to it develops a width and subsequently decays. The quantum stability of the theory can be further preserved by imposing a boundary condition in the far future, which has the consequence of introducing a strange acausal time-ordering of scattering events [37, 38]. Hawking and Hertog argue that such boundary conditions are natural from the perspective of the Euclidean integral, in which the action is required to fall off exponentially with $\tau = it \to \pm\infty$ [39]. Viewed from this perspective, the future boundary condition is no more odious than requiring the action to be well-behaved and bounded in space.

We construct the Lee-Wick Lagrangian of the Standard Model by augmenting each field with a corresponding higher-derivative (HD) term, chosen to satisfy the most basic requirements of gauge and Lorentz invariance. This is done at the expense of introducing one new parameter for each field: namely, a mass parameter suppressing the dimension-6 HD term. We denote this dimensionful constant as $\Lambda_{\rm LW}$ when considering simple theories in which there can be no room for misinterpretation. Since this new parameter characterizes the scale at which LW dynamics become relevant, we expect the parameter to take on values in the terascale (with energies $\approx$ a few TeV) and beyond. It should also be noted at this point that the dimension-six operators under consideration are *not* part of an effective field theory, since we are interested in physics at energies $E \sim \Lambda_{\rm LW}$.

The remainder of this section is organized as follows: In § 1.3.1, the original theory of Lee and Wick is discussed, focusing on quantum states defined on a Hilbert space of indefinite metric. In § 1.3.2, a more modern approach is taken, beginning with the imposition of higher-derivative equations of motion on the familiar spin-zero field. This results in a few key features of LW theories, which are explored in turn. The cancellation of UV divergences is demonstrated in § 1.3.3. Implications for unitarity comprise the focus of § 1.3.4, while the effects of LW theories on causality are studied in § 1.3.5. The reader interested only in how to do calculations in LW-type theories is advised to focus on § 1.3.2; the other sections illuminate some of the deep theoretical roots of higher-derivative theories, but serve little purpose in straightforward analyses of collider phenomenology and precision electroweak constraints, to name a few contemporary topics.

1.3.1 Historical Development: Negative Metric & Quantization

In a pair of seminal papers [34, 35], T.D. Lee and G.C. Wick advanced a finite theory of quantum electrodynamics (QED). The original formulation of the theory was motivated by the troubling preponderance of infinities for even the most basic calculations, for instance, the mass difference between the $\pi^{\pm}$ and π^0 mesons. Attributing this mass difference ($\approx 4.5936(5)$ MeV [29]) to the electromagnetic interaction, this should have been a simple problem in the then-established framework of QED, but infinities persist. To put the theory in historical perspective, it is worthwhile to note that the modern understanding of the renormalization group had not yet evolved, viz. coupling constant flows and relevant, marginal, and irrelevant operators, which tame the divergences in field theories. It then made sense to seek a finite theory of QED. In this subsection, the aim is not to reconstruct the original theory of Lee and Wick, but rather to introduce the Hilbert space metric and quantization scheme necessary to build of a quantum theory of negative-norm states.

Lee and Wick sought to construct states on a Hilbert space equipped with an indefinite metric. The idea of negative-metric quantization is not new; it first gained currency in 1942, when Dirac considered using it to render QED finite [40]. The origin and transformation properties of such a metric can be made clear through the following setup:

Given a Hilbert space $\mathcal{H}$ with complex-valued vectors $\mid x\rangle$, $\mid y\rangle$, establish a set of basis vectors $\mid 1\rangle, \mid 2\rangle, \ldots$ such that

$$\mid x\rangle = \sum_i x_i \mid i\rangle, \quad \mid y\rangle = \sum_i y_i \mid i\rangle \tag{1.43}$$

for $x_i, y_i \in \mathbb{C}$. Define the scalar product between the two to be the Hermitian form

$$\langle x \mid y\rangle = \sum_{i,j} x_i^* y_j \langle i \mid j\rangle \equiv \sum_{i,j} x_i^* \eta_{ij} y_j . \tag{1.44}$$

This serves as the definition for the metric $\eta = \eta_{ij}$, which we take to be Hermitian and non-singular. The action of an operator $\hat{\mathcal{O}}$ on a basis state in $\mathcal{H}$ may be represented by

$$\hat{\mathcal{O}} \mid i\rangle = \sum_i \mathcal{O}_{ji} \mid j\rangle, \tag{1.45}$$

which leads directly to the expectation value between any two states $\mid x\rangle$, $\mid y\rangle$

$$\langle x \mid \hat{\mathcal{O}} \mid y\rangle = \sum_{i,j} x_i^* \langle i \mid \hat{\mathcal{O}} \mid j\rangle y_j = \sum_{i,j,k} x_i^* \langle i \mid \mathcal{O}_{kj} \mid k\rangle y_j = \sum_{i,j,k} x_i^* \eta_{ik} \mathcal{O}_{kj} y_j. \tag{1.46}$$

It is also possible at this time to define the adjoint of an operator as follows:

$$\begin{aligned}\langle x \mid \bar{\hat{\mathcal{O}}} \mid y\rangle = \langle y \mid \hat{\mathcal{O}} \mid x\rangle^* = \left(\sum_{i,j,k} \langle i \mid y_i^* \hat{\mathcal{O}}_{kj} x_j \mid j\rangle\right)^* = \left(\sum_{i,j,k} y_i^* \eta_{ik} \hat{\mathcal{O}}_{kj} x_j\right)^* \\ = \sum_{i,j,k} x_j^* \hat{\mathcal{O}}_{jk}^{\dagger} \eta_{ki} y_i\,, \end{aligned} \tag{1.47}$$

while manipulating the left-hand side of Eq. (1.47) yields

$$\begin{aligned}\langle x \mid \bar{\hat{\mathcal{O}}} \mid y\rangle = \sum_i y_i \langle x \mid \bar{\hat{\mathcal{O}}} \mid i\rangle = \sum_{i,j,k} x_j^* y_i \langle j \mid \bar{\mathcal{O}}_{ki} \mid k\rangle \\ = \sum_{i,j,k} x_j^* y_i \eta_{jk} \bar{\mathcal{O}}_{ki}, \end{aligned} \tag{1.48}$$

from which it is seen that $\eta \bar{\hat{\mathcal{O}}} = \hat{\mathcal{O}} \eta$, or

$$\bar{\hat{\mathcal{O}}} = \eta^{-1} \hat{\mathcal{O}} \eta. \tag{1.49}$$

Of course, we know that physically robust statements must be independent of the basis chosen. Suppose we introduce new basis vectors $\mid 1'\rangle$, $\mid 2'\rangle, \ldots$ via the transformations

$$\mid i'\rangle = \sum_j T_{ji} \mid i\rangle. \tag{1.50}$$

The same state vectors $\mid x\rangle$ may be represented in terms of them as

$$\mid x\rangle = \sum_i x_i \mid i\rangle = \sum_i x_i' \mid i'\rangle = \sum_{i,j} x_i' T_{ji} \mid j\rangle. \tag{1.51}$$

Let us now consider the effect of such a basis change on the action of a typical operator:

$$\begin{aligned} \hat{\mathcal{O}} \mid i'\rangle &= \sum_i \mathcal{O}'_{j'i'} \mid j'\rangle, \text{ or also} \\ &= \sum_i \hat{\mathcal{O}} T_{ii'} \mid i\rangle = \sum_{ij} \mathcal{O}_{ji} T_{ii'} \mid j\rangle = \sum_{ijj'} \mathcal{O}_{ji} T_{ii'} \left(T^{-1}\right)_{j'j} \mid j'\rangle, \\ \therefore \hat{O}' &= T^{-1} \hat{\mathcal{O}} T \end{aligned} \tag{1.52}$$

Now, since the metric itself is of interest to us, it is worthwhile to consider its transformation properties as well:

$$
\begin{aligned}
&\langle i \mid j \rangle = \eta_{ij} \\
\longrightarrow\ &\langle i' \mid j' \rangle = \sum_{i,j} \langle i \mid (T^\dagger)_{i'i} T_{j'j} \mid j \rangle = \sum_{i,j} (T^\dagger)_{i'i} \eta_{ij} T_{jj'},
\end{aligned}
\tag{1.53}
$$

from which one gets the transformation law

$$
\eta' = T^\dagger \eta T. \tag{1.54}
$$

Although Eq. (1.54) bears some resemblance to Eq. (1.52), this is misleading. The indefinite character of the metric and precludes any proof one might write which would enforce the hermiticity of T. As such, $T^{-1} \neq T^\dagger$ in the general case, and the transformation law of η sets it apart from the set of "operators" in the usual sense. Any two metrics related by Eq. (1.54) are said to belong to the same class. Furthermore, the transformation (1.54) means that the eigenvalues of η have no special meaning (at least in their magnitudes), as they can be rotated into another set of eigenvalues by a change of basis. This means that we are at liberty to choose a basis such that η is a diagonal matrix with elements ± 1. The focus now changes from considering how different metrics may be related to one another to the question of what metrics may be permitted in a consistent quantum theory.

A sensible quantum theory requires (anti-)commutation relations to be defined among the creation and annihilation operators that operate on the space of states. Let us begin by considering the familiar Fermi-Dirac case. Owing to the class invariance of η, two choices are immediately available:

$$
\text{(i)} \quad a\bar{a} + \bar{a}a = 1, \tag{1.55}
$$

$$
\text{(ii)} \quad a\bar{a} + \bar{a}a = -1. \tag{1.56}
$$

Are there any relationships between $a, \bar{a}$? From Eq. (1.49), we can write the transformation law

$$\bar{a} = \eta^{-1} a^{\dagger} \eta, \tag{1.57}$$

which is invariant under $\eta \to -\eta$. Since a change in sign does not affect the operators responsible for creating and annihilating operators, it can have no effect on the physical spectrum, and so we can include metrics related by $\eta \to -\eta$ as being in the same class alongside the relationship (1.54).

If one chooses case (i) for the Fermi-Dirac oscillators, the sign invariance of the metric means that it can always be chosen to be equal to the identity matrix, and this corresponds to the usual $\langle x \mid x \rangle = 1$ case of positive-normed states. If instead case (ii) is chosen, one can consistently require that

$$\langle x \mid (-1)^{\bar{a}a} \mid x \rangle > 0, \tag{1.58}$$

for all $\mid x \rangle$ in $\mathcal{H}$. The exponent $\bar{a}a$ appearing in Eq. (1.58) is to be understood as an eigenvalue of the operator $\bar{a}a$ acting on $\mid x \rangle$. The metric is therefore indefinite, and one may choose $\eta = (-1)^{\bar{a}a}$ in order to enforce positivity in (1.58). The quantum created by these operators is said to be of positive or negative metric, depending on whether case (i) or (ii) holds.

The Bose-Einstein case possesses additional subtlety. The conventional quantization

$$a\bar{a} - \bar{a}a = 1, \tag{1.59}$$

may be turned into

$$a\bar{a} - \bar{a}a = -1, \tag{1.60}$$

by simply swapping the roles of a, $\bar{a}$. Due to this property, the total range of possibilities for quantizing bosons falls into three distinct groups:

1. **Definite metric**. $\langle x \mid x \rangle > 0$ and η may be set to 1. This corresponds to establishing a vacuum state $|0\rangle$ such that $a|0\rangle = 0$.

2. **Indefinite metric**. Eq. (1.58) holds, and $\eta = (-1)^{\bar{a}a}$ is permissible. This corresponds to a vacuum state defined by $a^{\dagger}|0\rangle = 0$.

3. **Neither**. η is indefinite; this corresponds to $\bar{a}a \mid x\rangle$ returning non-integer eigenvalues for all vectors $\mid x\rangle \in \mathcal{H}$. The spectrum so produced is bounded neither from above nor below.

We are interested in constructing a consistent quantum theory with some meaningful ground state, and so choice (3) is excluded from further consideration. It is enough to then stipulate whether the spectrum should be bounded from above or below; the class of metric is then specified uniquely by demanding that it be either definite or indefinite.

This concludes the treatment of the basic quantum theory of Lee-Wick states. This topic will arise again in the chapter concerning thermodynamics and the LW effective action, where further subtleties in defining the metric and quantization condition can lead to interesting predictions. For now, let us turn to the subject of making a quantum field theory of LW states.

1.3.2 The Simplest Case: A Higher-Derivative Scalar Theory

Modern work on LW theories may share the spirit of the original papers by Lee and Wick, but the implementation has tended to follow a seminal paper by Grinstein, O'Connell, and Wise [41]. The roadmap here entails the explicit invocation of a HD action, followed by its Ostrogradsky transform into a theory of two quantum fields of opposite norm (rather than beginning with negative-norm states on the Hilbert space from the outset). The use of modern QFT technology in [41], such as the renormalization group and spontaneous symmetry breaking, will be more familiar to contemporary practitioners than the original paradigm set forth by Lee and Wick.

Let us begin by considering the simplest case, involving a single interacting scalar field $\hat{\phi}$ with a HD term:

$$\mathcal{L}_{\rm hd} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2 - \frac{1}{2\Lambda_{\rm LW}^2}(\partial^2\hat{\phi})^2 - V(\hat{\phi}). \tag{1.61}$$

One may then obtain the Euler-Lagrange equations of motion in the usual way, resulting in the HD propagator

$$\hat{D}(p^2) = \frac{i}{p^2 - m^2 - p^4/\Lambda_{\rm LW}^2}. \tag{1.62}$$

This propagator has simple poles at $p^2 = \frac{\Lambda_{\rm LW}^2}{2}\left(1 \pm \sqrt{1 - 4m^2/\Lambda_{\rm LW}^2}\right)$. In the limit of interest, $\Lambda_{\rm LW}^2 \gg m^2$, this expression reduces to $p^2 = m^2$ and $p^2 = \Lambda_{\rm LW}^2 - m^2 \sim \Lambda_{\rm LW}^2$. The meaning of Eq. (1.62) is clear: the propagator following from a single HD action cleanly resolves itself into two independent degrees of freedom, which is most obvious when its mass scales are far separated.

We can explicitly see these new degrees of freedom by inserting an auxiliary field $\tilde{\phi}$ into Eq. (1.61):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2 - \tilde{\phi}\partial^2\hat{\phi} + \frac{1}{2}\Lambda_{\rm LW}^2\tilde{\phi}^2 - V(\hat{\phi}). \tag{1.63}$$

The equation of motion, $\tilde{\phi} = \partial^2\hat{\phi}/\Lambda_{\rm LW}^2$, is exact at the quantum level: the path integral over the degrees of freedom associated with $\tilde{\phi}$ can be comptued exactly. Substitution of this result for $\tilde{\phi}$ in Eq. (1.63) returns Eq. (1.61), as expected. Now, define a new field by $\phi = \hat{\phi} + \tilde{\phi}$. Eliminating $\hat{\phi}$, the Lagrangian of Eq. (1.63) becomes

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}\Lambda_{\rm LW}^2\tilde{\phi}^2 \\ & -\frac{1}{2}m^2(\phi - \tilde{\phi})^2 - V(\phi - \tilde{\phi}).\end{aligned} \tag{1.64}$$

In this form, we can see that there are clearly two scalar fields at work: the conventional scalar ϕ, and its partner, $\tilde{\phi}$. The terms in Eq. (1.64) quadratic in $\tilde{\phi}$ display a curious property: both the kinetic and the mass terms enter with a sign opposite to what one expects in a standard quantum field theory. The above is to be contrasted with ghosts and/or tachyons (which have an opposite-sign kinetic term only) and the unstable vacua associated with the Higgs field in the unbroken symmetry phase (which has an opposite-sign mass term only). The $\tilde{\phi}$-only terms enter as a complete copy of an otherwise-standard free scalar field theory, with an overall minus sign appended. Eq. (1.64) gives rise to the propagator

$$\tilde{D}(p^2) = \frac{-i}{p^2 - \Lambda_{\mathrm{LW}}^2}. \tag{1.65}$$

This signals the existence of nonpositive-definite normed states in the Hilbert space. If this state were to be stable, then a scattering process taking a collection of positive-norm states into a collection of negative-norm states could not be done by means of a unitary operator on the Hilbert space. Put succinctly, a complete S-matrix in this case demands the violation of unitarity. However, as emphasized by Lee and Wick in their original work, unitarity may be preserved if $\tilde{\phi}$ is unstable and decays, thus acquitting its quantum states from consideration as long-lived states on the Hilbert space. This is achieved at the level of Feynman diagrams if $V(\phi, \tilde{\phi})$ contains a term enabling the decay of the heavy $\tilde{\phi}$.

One complication persists in the absence of an easily recognizable mass term for ϕ. As it stands, Eq. (1.64) contains a $\phi - \tilde{\phi}$ two-point interaction, mixing the two fields. In order to gain a clearer picture in which each field has its own distinct mass term, it is possible to execute a symplectic diagonalization in the basis of fields (which preserve the metric η),

$$\begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \phi' \\ \tilde{\phi}' \end{pmatrix}. \tag{1.66}$$

A solution exists if θ satisfies

$$\tanh 2\theta = \frac{-2m^2/\Lambda_{\rm LW}^2}{1 - 2m^2/\Lambda_{\rm LW}^2}, \tag{1.67}$$

which provides real solutions, as long as $\Lambda_{\rm LW}^2 > 4m^2$. If this inequality is not met, this corresponds to the case of the heavy ϕ field decaying to $\phi\phi$ on-shell with zero phase space volume; in other words, the heavy LW state is unable to decay. It should then be little wonder that a well-defined theory of two interacting fields, of opposite norms, does not exist; the failure of symplectic diagonalization presages the looming disaster of a non-unitary S-matrix.

This transformation being made, the Lagrangian of Eq. (1.64) becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m'^2\phi'^2 - \frac{1}{2}\partial_\mu\tilde{\phi}'\partial^\mu\tilde{\phi}' + \frac{1}{2}M_{\rm LW}^2\tilde{\phi}'^2 - V(\phi', \tilde{\phi}'), \tag{1.68}$$

where the introduction of m'^2, $M_{\rm LW}^2$ accounts for the adjustment of the original mass parameters:

$$m'^2,\, M_{\rm LW}^2 = \frac{\Lambda_{\rm LW}^2}{2}\left(1 \mp \sqrt{1 - \frac{4m^2}{\Lambda_{\rm LW}^2}}\right). \tag{1.69}$$

Furthermore, since the potential in the form of Eq. (1.61) changes from $V(\hat{\phi})$ to $V(\phi - \tilde{\phi})$, the coupling constants of the HD theory will receive multiplicative corrections from the diagonalization. For instance, if

$$V(\hat{\phi}) = \frac{1}{n!}g\hat{\phi}^n \longrightarrow V(\phi - \tilde{\phi}) = \frac{1}{n!}g(\phi - \tilde{\phi})^n, \tag{1.70}$$

diagonalization will turn this into

$$V(\phi' - \tilde{\phi}') = \frac{1}{n!}g(\cosh\theta - \sinh\theta)^n(\phi' - \tilde{\phi}')^n. \tag{1.71}$$

This new multiplicative factor can simply be reabsorbed into the definition of the coupling constant, giving $g' = g(\cosh\theta - \sinh\theta)^n$, and

$$V(\phi - \tilde{\phi}) = \frac{1}{n!} g'(\phi - \tilde{\phi})^n. \tag{1.72}$$

For $\Lambda_{\mathrm{LW}}^2 \gg m^2$, hence $\theta \ll 1$, this is a minor redefinition of approximately $g' = (1 - n\theta)g$. For notational convenience, we will drop the primes from the notation in all subsequent calculations. Only mass-eigenstate fields and their associated couplings will be used throughout the rest of this work, and no ambiguity will arise.

Loop effects will come to play a substantial role in this work, and it therefore behooves us to consider the self-energy of LW fields due to their self-interactions:

$$\begin{aligned}
\tilde{D}(p^2) &= \frac{-i}{p^2 - M_{\mathrm{LW}}^2} + \frac{-i}{p^2 - M_{\mathrm{LW}}^2} \left(-i\Sigma(p^2)\right) \frac{-i}{p^2 - M_{\mathrm{LW}}^2} + \ldots \\
&= \frac{-i}{p^2 - M_{\mathrm{LW}}^2} \sum_{n=0}^{\infty} \frac{-\Sigma(p^2)}{p^2 - M_{\mathrm{LW}}^2} = \frac{-i}{p^2 - M_{\mathrm{LW}}^2} \times \frac{1}{1 + \Sigma(p^2)/(p^2 - M_{\mathrm{LW}}^2)} \\
&= \frac{-i}{p^2 - M_{\mathrm{LW}}^2 + \Sigma(p^2)}.
\end{aligned} \tag{1.73}$$

This is to be contrasted with the case for ordinary scalar fields, for which the denominator has the form $p^2 - m^2 - \Sigma(p^2)$. This result is significant in light of the Breit-Wigner formulation of resonance widths. Taking the denominator of Eq. (1.73) to be of the form $p^2 - m^2 + im\Gamma$ for some width Γ, the LW field possesses a negative width. This result could also have been demonstrated by a straightforward calculation using tree-level Feynman diagrams. If

$$V(\phi - \tilde{\phi}) = \frac{1}{3!} g(\phi - \tilde{\phi})^3 \supset -\frac{1}{2} g\phi^2\tilde{\phi}, \tag{1.74}$$

then the width associated with the decay channel $\tilde{\phi} \to \phi\phi$ is simply

$$\Gamma = -\frac{g^2}{32\pi M_{\mathrm{LW}}} \sqrt{1 - \frac{4m^2}{M_{\mathrm{LW}}^2}}. \tag{1.75}$$

The overall minus sign comes from the fact that an odd number of LW fields are present in the Feynman diagram of interest, as in Eq. (1.74). This will have interesting implications for causality, as addressed in § 1.3.5.

The result of Eq. (1.75) helps to address the previously-mentioned issues with unitarity. It is well-known that the optical theorem, taken as a criterion for unitarity, demands that the imaginary part of the forward scattering amplitude be positive. In $\phi\phi$ scattering under the influence of the potential (1.74), one has (following implicitly the Feynman $+i\epsilon$ prescription)

$$i\mathcal{M} = (ig)^2 \frac{-i}{p^2 - M_{\mathrm{LW}}^2 + iM_{\mathrm{LW}}\Gamma} = +ig^2 \frac{p^2 - M_{\mathrm{LW}}^2 - iM_{\mathrm{LW}}\Gamma}{(p^2 - M_{\mathrm{LW}}^2)^2 + M_{\mathrm{LW}}\Gamma^2}, \tag{1.76}$$

leaving $\mathcal{M}$ with imaginary part

$$\operatorname{Im}\mathcal{M} = -g^2 \frac{M_{\mathrm{LW}}\Gamma}{(p^2 - M_{\mathrm{LW}}^2)^2 + M_{\mathrm{LW}}\Gamma^2}. \tag{1.77}$$

The negative sign buried within Γ cancels the explicit sign associated with the propagator. The optical theorem is vindicated at tree level, thus occasioning some optimism that the theory is indeed unitary. However, one must be forewarned that a proof of the optical theorem for LW theories to all orders in perturbation theory is wanting at the time of this writing. No known violation yet exists, although attempts to settle the unitarity question through path integrals have proven null [42]. One must then approach the question of unitarity with some caution.

1.3.3 Cancellation of UV Divergences

As advertised, one can show with relatively little work that the higher-derivative degrees of freedom present in the LWSM can cancel off the UV divergences that plague scalar mass renormalization, thereby resolving the hierarchy problem. Since the SM possesses three types of fields - fermion, scalar, and gauge boson - three new

HD Lagrangians need to be written down:

$$\mathcal{L}_{\text{Higgs}} = (\hat{D}_\mu \hat{\Phi})^\dagger (\hat{D}^\mu \hat{\Phi}) \ - \frac{1}{M_\Phi^2} (\hat{D}_\mu \hat{D}^\mu \hat{\Phi})^\dagger (\hat{D}_\nu \hat{D}^\nu \hat{\Phi}) \ - \frac{\lambda}{4} \left(\hat{\Phi}^\dagger \Phi - \frac{v^2}{2} \right)^2 , \quad (1.78)$$

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2} \mathrm{tr}\, \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{1}{M_A^2} \mathrm{tr}\, (\hat{D}^\nu \hat{F}_{\mu\nu})(\hat{D}^\lambda \hat{F}^\nu_\lambda), \quad (1.79)$$

$$\mathcal{L}_{\text{fermion}} = \bar{\hat{Q}}_L i \not{\hat{D}} \hat{Q}_L + \frac{1}{M_Q^2} \bar{\hat{Q}}_L i \not{\hat{D}} \not{\hat{D}} \not{\hat{D}} \hat{Q}_L, \quad (1.80)$$

with covariant hatted derivatives $\hat{D}_\mu$ defined in analogy to Eq. (1.2) with respect to HD gauge fields. It is also necessary to write down the HD Yukawa interaction for completeness, though its form is easy to guess:

$$-\mathcal{L}_{\text{Yuk}} = y_{u^i} \epsilon^{ab} (\bar{\hat{Q}}^i_L)_a \hat{\Phi}^\dagger_b \hat{u}^i_R + y_{d^i} \bar{\hat{Q}}^i_L \hat{\Phi} \hat{d}^i_R + y_{e^i} \bar{\hat{L}}^i_L \Phi e^i_R + \text{h.c.} \quad (1.81)$$

With this information, one has the basic ingredients necessary to perform an Ostrogradsy transformation on the HD terms by way of auxiliary fields, such as $\tilde{Q}_L$, etc. Let us briefly respectively work out the the Ostrogradsky formalism for the gauge and fermion fields, utilizing auxiliary fields $\hat{W}^{\mu a} = W^{\mu a} - \tilde{W}^{\mu a}$ and $\hat{Q}_L = Q_L - \tilde{Q}_L$:

$$\begin{aligned}
\mathcal{L}_{\text{aux}} = &- \frac{1}{2} \mathrm{tr}\, \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - M_A^2 \,\mathrm{tr}\, \tilde{W}_\mu \tilde{W}_\nu + 2 \,\mathrm{tr} \hat{F}_{\mu\nu} \hat{D}^\mu W^\nu, \\
\longrightarrow &- \frac{1}{2} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathrm{tr}\, (D_\mu \tilde{W}_\nu - D_\nu \tilde{W}_\mu)(D^\mu \tilde{W}^\nu - D^\nu \tilde{W}^\mu) \\
&- ig \,\mathrm{tr}([\tilde{W}_\mu, \tilde{W}_\nu] F^{\mu\nu}) - \frac{3}{2} g^2 \,\mathrm{tr}\, ([\tilde{W}_\mu, \tilde{W}_\nu][\tilde{W}^\mu, \tilde{W}^\nu]) \\
&- 4ig \,\mathrm{tr}([\tilde{W}_\mu, \tilde{W}_\nu] D^\mu \tilde{W}^\nu) - M_A^2 \,\mathrm{tr}\, (\tilde{W}_\mu \tilde{W}^\mu).
\end{aligned} \quad (1.82)$$

In R_ξ gauge, the propagators for regular and LW gauge bosons with group indices a, b read

$$D^{ab}_{\mu\nu} = -\frac{i\,\delta^{ab}}{p^2} \left(g_{\mu\nu} - 1(1-\xi) \frac{p_\mu p_\nu}{p^2} \right), \quad (1.83)$$

$$\tilde{D}^{ab}_{\mu\nu} = \frac{i\,\delta^{ab}}{p^2 - M_A^2} \left(g_{\mu\nu} - 1 \frac{p_\mu p_\nu}{M_A^2} \right). \quad (1.84)$$

Note in particular the opposite sign of the LW propagator, the explicit pole mass M_A^2, and the fact that the LW gauge boson still preserves the Ward identity in the technical sense (p^μ dotted into Eq. (1.84) vanishes). We will not need to consider LW gauge bosons in what is to follow; let us proceed on to the case of the HD fermion theory. Exploiting the auxiliary field formalism as before, one has

$$\begin{aligned}\mathcal{L} =& \bar{\hat{Q}}_L i\not{D}\hat{Q}_L + M_Q(\bar{\tilde{Q}}_L\tilde{Q}'_R + \bar{\tilde{Q}}'_R\tilde{Q}_L) + \bar{\hat{Q}}_L i\not{\tilde{A}}\hat{Q}L + \bar{\tilde{Q}}_L i\not{\tilde{A}}\tilde{Q}_L - \bar{\tilde{Q}}'_R i\not{\tilde{A}}\tilde{Q}'_R \\ \longrightarrow& \bar{Q}_L i\not{D}Q_L - \bar{\tilde{Q}}_L i\not{D}\tilde{Q}_L - \bar{\tilde{Q}}'_R i\not{D}\tilde{Q}'_R + M_Q(\bar{\tilde{Q}}_L\tilde{Q}'_R + \bar{\tilde{Q}}'_R\tilde{Q}_L) - \bar{Q}_L\gamma_\mu\tilde{A}^\mu Q_L \\ &+ \bar{\tilde{Q}}_L\gamma_\mu\tilde{A}^\mu\tilde{Q}_L + \bar{\tilde{Q}}'_R\gamma_\mu\tilde{A}^\mu\tilde{Q}'_R,\end{aligned} \tag{1.85}$$

where $\tilde{A}_\mu$ sums over all auxiliary field gauge bosons (and their generators). The introduction of a right-handed fermion $\tilde{Q}'_R$ is necessary in order to produce a vectorlike mass term. Since Q_L, Q'_R combine to form a single spinor in Eq. (1.85), there are only two propagators, with the forms

$$D_F(p) = \frac{i}{\not{p}}, \tag{1.86}$$

$$\tilde{D}_F(p) = -\frac{i}{\not{p} - M_Q}. \tag{1.87}$$

With this set of free-field Lagrangians firmly in place, it is time to turn to the task at hand. In the Yukawa Lagrangian of Eq. (1.81), each HD fermion is replaced by $\hat{\psi} \to \psi - \tilde{\psi}$. This leads to a LW Yukawa Lagrangian of familiar form,

$$\begin{aligned}-\mathcal{L}_{\text{Yuk}} =& y_{u^i}\epsilon^{ab}(\bar{Q}^i_L - \bar{\tilde{Q}}^i_L)_a(\Phi^\dagger_b - \tilde{\Phi}_b)^\dagger(u^i_R - \tilde{u}^i_R) \\ &+ y_{d^i}(\bar{Q}^i_L - \bar{\tilde{Q}}^i_L)(\Phi - \tilde{\Phi})(d^i_R - \tilde{d}^i_R) \\ &+ y_{e^i}(\bar{L}^i_L - \bar{\tilde{L}}^i_L)(\Phi - \tilde{\Phi})(e^i_R - \tilde{e}^i_R) + \text{h.c.}\end{aligned} \tag{1.88}$$

From this, one learns some very useful information. Not only do SM fermions couple to the Higgs, but so do their LW partners, and there also exists a trilinear coupling between the SM and LW fermions and the Higgs. Now, one can take the diagram (1.30), but augment it with three new diagrams: one where only the top fermion line is LW, one where only the bottom is LW, and one where both are LW. This means that the

LW result can be directly imported from Eqs. (1.33), (1.34) with a judicious choice of $\Delta\left(p^2, x\right)$ for the three cases above. Referring to them respectively as Δ_1, Δ_2, and Δ_3, we have

$$\Delta_1 = m_f^2 x + M_{\mathrm{LW}}^2(1-x) - p^2 x(1-x), \tag{1.89}$$

$$\Delta_2 = m_f^2(1-x) + M_{\mathrm{LW}}^2 x - p^2 x(1-x), \tag{1.90}$$

$$\Delta_3 = M_{\mathrm{LW}}^2 - p^2 x(1-x). \tag{1.91}$$

Before performing the loop calculation, one must keep in mind that, due to the opposite sign of the LW propagator, a diagram with N internal LW-lines will possess an overall relative factor of $(-1)^N$ relative to a purely-SM diagram. One may now calculate each loop diagram; for organizational convenience, denote each diagram $-iM_{(N)}^2(p^2)$, where N counts the number of internal LW lines:

$$-iM_{(1)}^2(p^2) = -i\frac{m_f^2}{4\pi^2 v^2}\int_0^1 dx\left(2\Lambda^2 - 3\Delta_1 \ln\left(\frac{\Lambda^2}{\Delta_1}\right) - 3\Delta_2 \ln\left(\frac{\Lambda^2}{\Delta_2}\right)\right) \tag{1.92}$$

$$-iM_{(2)}^2(p^2) = i\frac{m_f^2}{4\pi^2 v^2}\int_0^1 dx\left(\Lambda^2 - 3\Delta_3 \ln\left(\frac{\Lambda^2}{\Delta_3}\right)\right). \tag{1.93}$$

Adding these results back to Eq. (1.34), it is seen that the quadratic divergence cancels off cleanly. However, the logarithmic divergence persists; this prompts the observation that

The Lee-Wick Standard Model is free of quadratic divergences, but unlike the earlier formulation of quantum electrodynamics by Lee and Wick, it is not a finite theory.

This is a significant departure from the original incarnation of the theory. The modern formulation of LW theory accepts logarithmic divergences as necessary, and in fact desirable, in light of the understanding of the renormalization group that has emerged since the time of Refs. [34, 35].

This concludes the elementary treatment of the LWSM resolution of the hierarchy problem; but what of the other new fields introduced into the theory? It must be remembered that the fermion contribution to $-iM^2(p^2)$ is the leading divergence at quadratic order, and all other diagrams calculated within the auspices of the SM are at worst logarithmically divergent. Invoking HD equations of motion can only increase the convergence (or at any rate, slow the divergence) of loop diagrams, since their propagators fall off even more rapidly with energy as one approaches the UV cutoff.

1.3.4 Unitarity in Lee-Wick Theory

The existence of negative-norm states, non-Hermitian Hamiltonians, and higher-derivative terms in the Lagrangian all seem to point to a violation of unitarity in Lee-Wick theories. If true, the theory would indeed be sick beyond the help of any *ad hoc* procedure. However, the question of unitarity turns out to be quite nuanced, and yields a somewhat surprising answer.

The fact that Lee and Wick's seminal paper [34] began with a proof of S-matrix unitarity offers a sense of the issue's importance. It assumes a metric η (as defined in § 1.3.1) between states $|x\rangle$ on a Hilbert space $\mathcal{H}$. There exists a pseudo-Hermitian Hamiltonian, in the vein of Eq. (1.49). Given that the unitarity of S is related closely to the hermiticity of H, one would be permitted some skepticism that a pseudo-Hermition Hamiltonian can produce a viable theory. After all, a Hamiltonian defined by (1.49) yields the relationship

$$S^\dagger \eta S = \eta, \tag{1.94}$$

which clearly does not comply with the expectation that $S^\dagger S = 1$.

The prescription suggested by Lee and Wick is to divide the particle spectrum into two classes: "normal" ($\eta = 1$) and "abnormal" ($\eta = -1$). It would indeed be disastrous for abnormal particles to survive into the far future as scattering out-states; but if the theory contains interactions and mass parameters allowing abnormal particles to decay into their normal counterparts, then the S matrix (which connects past and future infinity) never "sees" the abnormal particles in any meaningful way. This can be thought of as the total Hilbert space $\mathcal{H}$ having some subset $\mathcal{H}_+ \subset \mathcal{H}$ of positive-norm states on which the S-matrix acts.

Non-Hermitian operators are associated with complex eigenvalues, which can be seen from the action of H on a complete set of states. Let the eigenstates of H be spanned by α, β, obeying

$$H\,|\alpha\rangle = E_\alpha\,|\alpha\rangle, \tag{1.95}$$

$$H\,|\beta_\pm\rangle = E_{\pm,\beta}\,|\beta_\pm\rangle, \tag{1.96}$$

for $\operatorname{Im} E_\alpha = 0$ and $\operatorname{Im} E_{+,\beta} = -\operatorname{Im} E_{-,\beta} \neq 0$. With these tools in place, we are able to make a mathematically concise statement regarding the stability of the theory. All eigenstates $|\alpha\rangle$ of real eigenvalues E_α possess positive norm-squared with respect to the metric η, *i.e.*,

$$\langle\,\alpha\,|\,\eta\,|\,\alpha\rangle > 0. \tag{1.97}$$

All remaining states will have purely oscillatory time-dependence. It also follows from $\eta H^\dagger \eta = H$ that the three partitions of states possess a vanishing inner product with one another:

$$\langle\beta_+\,|\,\eta\,|\,\alpha\rangle = \langle\beta_-\,|\,\eta\,|\,\alpha\rangle = 0, \tag{1.98}$$

$$\langle\beta'_+\,|\,\eta\,|\,\beta_+\rangle = \langle\beta'_-\,|\,\eta\,|\,\beta_-\rangle = 0, \tag{1.99}$$

for different β, β'. One can also act repeatedly with H to show that, for different

α, α',

$$\langle \alpha' \,|\, \eta \,|\, \alpha \rangle = 0, \tag{1.100}$$

$$\langle \beta'_+ \,|\, \eta \,|\, \beta_- \rangle = 0. \tag{1.101}$$

These inner products serve a direct purpose in attempting to build a completeness relation (which cannot be proven to exist in general for non-Hermitian operators): we can expand any state on $\mathcal{H}$ in terms of the eigenvectors of H. Though this is an assumption, a theory without a completeness relation bears little resemblance to the mathematical framework of quantum mechanics, and so we press on. The general expression is

$$|\,\psi\rangle = \sum_{\alpha,\beta_\pm} \langle \alpha \,|\, \eta \,|\, \psi \rangle \,|\, \alpha \rangle + \langle \beta_+ \,|\, \eta \,|\, \psi \rangle \,|\, \beta_- \rangle + \langle \beta_- \,|\, \eta \,|\, \psi \rangle \,|\, \beta_+ \rangle \tag{1.102}$$

If a state is to have negative norm-squared, *i.e.* $\langle \psi \,|\, \eta \,|\, \psi \rangle < 0$, it must be a coherent mixture of $|\,\beta_+\rangle$ and $|\,\beta_-\rangle$. The contrapositive of this statement gives one an idea of how to ensure that the theory remains stable.

One may now formulate a scattering theory in the usual way, using asymptotic in- and out-states $|\,\alpha^{\text{in}}\rangle$ and $|\,\alpha^{\text{out}}\rangle$. The S-matrix is then defined as the inner product between in- and out-states:

$$S_{\alpha',\alpha} = \langle \alpha'^{\text{out}} \,|\, \eta \,|\, \alpha^{\text{in}} \rangle. \tag{1.103}$$

One is free to substitute the above, by way of Eq. (1.102), back into Eq. (1.94). Using the completeness of the full vector space $\mathcal{H}$, all unphysical states are projected out, leaving only

$$S^\dagger S = \mathbf{1}, \tag{1.104}$$

as desired.

Following this result of Lee and Wick, Boulware and Gross [42] attempted a path-integral formulation of the theory. Their study centered on the troubling nature of indefinite metrics and runaway states: as an example, the state

$$|\psi\rangle = |+\rangle \pm |-\rangle, \tag{1.105}$$

for which $0 < \langle + \mid + \rangle = -\langle - \mid - \rangle$, has zero norm. Modes corresponding to such states have the potential to grow or decay exponentially while inner products remain bounded. To compare with a classical case, this is similar to a theory with indefinite energy. Such a theory can be quantized in regular Minkowskian time, and a functional integral exists, but there will in general be growing modes and no ground state. This latter fact precludes the possibility of a Euclidean-time quantization, which presents problems for the orthodox connections between path integral quantum mechanics and statistical mechanics.

The case for the unitarity of Lee-Wick theory is embattled, to be sure, but not without optimism. The Lee-Wick prescription successfully decouples the offending negative-norm states from the physical spectrum, needing a caveat or two and perturbation theory in order to do so. The non-existence of a path integral formulation, however, merits some serious concern. As of this writing, we stand in the uncomfortable middle ground of there being no non-perturbative proof of Lee-Wick unitarity, but no glaring examples of unitarity violation either.

1.3.5 Causality in Lee-Wick Theory

Causality and time-ordering experience some counter-intuitive developments when transitioning to Lee-Wick theories. Acausality, understood mathematically as a peculiar time-ordering of events, has been studied in Refs.[35, 37, 38, 43]. It was argued in [35] that, aside from pathologies and paradoxes such as closed timelike curves, a mathematically consistent theory cannot lead to bona fide paradoxes along the lines of acausality as it is commonly conceived. Rather than grapple with such ill-defined notions, one would profit from rephrasing the question in terms of Lorentz invariance. Touching on the foundations of quantum field theory, causality can be stated as the requirement that the two-point correlation function

$$\langle 0 \,|\, \phi(y)\phi(x) \,|\, 0\rangle = 0 \quad \text{for} \quad g_{\mu\nu}x^\mu y^\nu < 0, \tag{1.106}$$

that is, for spacelike $(x-y)^\mu$. This demand is satisfied in [35].

Ref. [38] pursues the question non-perturbatively by basing the LW state space on a set of distributions formulated by Gel'fand and Shilov [44, 45]. The author addresses the problem concerning loop diagrams with Lee-Wick particles, which are taken to have complex masses in some studies [46]. Such graphs introduce non-analytic regions whose shape depends on the frame in which they are calculated - a clear violation of Lorentz invariance. The problem arises due to the elimination of internal negative metric particles through the Lee-Wick prescription, which operates on a set of states that is not Lorentz-invariant in the first place. The problem may be overcome by recasting the theory in terms of strictly a covariant state space. However, the states so constructed have real energy eigenvalues, and so they seem to be at risk of violating the Lee-Wick real-energy constraint for removing unphysical degrees of freedom. This is avoided by imposing a large-distance cutoff, thereby moving the negative-definite states away from the real line. The Lee-Wick real-energy constraint is then applied, after which time the cutoff is removed, resulting in a causal theory free of negative-norm states.

In [37], the authors assert that causality as commonly conceived is a classical, macroscopic phenomenon, appropriate only at large length scales that coarse-grain over degrees of freedom. The Lorentz invariance of LW theory is emphasized, and causality here is understood to be an emergent phenomenon only appropriate for macroscopic time scales (which can be as short as microseconds, given the smallness of $\Lambda_{\mathrm{LW}}^{-1}$). In order to probe the nature of that emergence, they study an $O(N)$ construction of LW theory, and find that the time ordering one would associate with causal fields emerges in the $N \to \infty$ limit.

The analysis of [43] centers on the topic of vertex displacement. It had been established in [41] that the decay products of LW particles move toward their vertex of production, not away from it; this is a way of phrasing the apparent acausality in scattering (hence, S-matrix) terms. The author uses the Weiszäcker-Williams approach of [47] to calculate vertex displacement effects one might see in the Bremsstrahlung radiation of a LW photon, *i.e.* $e^- Z \to e^- \tilde{A} Z$. The author finds that the cross-section for this LW process is not substantially affected by a momentum cut representing the sensitivity and thickness of a detector. This latter fact is significant, because for a sufficiently thin detector, the vertex displacement could put the decaying LW particle "behind" the detector, while its decay products move from this point into the detector proper. In the particular case of LW photons produced by a beam of incident energy E_0 on the target, the opening angle into which decay products travel is found to be of order $m_{\tilde{A}}/E_0$. This small, though in principle measurable, angle would be a smoking gun for the identification of LW particles at colliders and fixed-target experiments. This theme will be explored further in the subsequent chapter concerning LW collider phenomenology.

Chapter 2

COLLIDER SIGNATURES OF THE $N = 3$ LEE-WICK STANDARD MODEL

With the core ideas and implementation of the Lee-Wick Standard Model in place, it is time to consider some applications of the theory to contemporary problems of collider phenomenology, electroweak precision tests, and cosmology.

Consider the basic HD Lagrangian of Eq. (1.61). Though it may seem an unnecessary or unwelcome extension, only attached to serve one very specialized purpose, there is another perspective from which to view it. Equation (1.61) invokes a HD, dimension-six operator. But is this the end? If we entertain the thought that there may exist HD operators at the fundamental level, narrowing the field of view to only one such class of operators can only provide a special case of a more generic theory featuring still-higher powers of the derivative operator. In order to keep things straight, some clarifying notation is in order. An Lee-Wick theory of order N is one whose Lagrangian has terms with up to $2(N-1)$ extra derivatives beyond that in the original theory. In this notation, the conventional Standard Model is an $N = 1$ theory; the model developed throughout Ch. 1 is an $N = 2$ theory; and the subject of this chapter, and the primary focus of this dissertation, will be $N = 3$ theories.

We expect the introduction of dimension-8 operators to add complexity to the theory, both in the number of parameters to be constrained (in the form of mass scales and diagonalization angles, similar to Eq. (1.66)) and in the number of new interactions to be generated by the Ostrogradsky decomposition of the HD fields. However, the improved convergence of loop diagrams ameliorates this cost somewhat; we can expect internal lines to fall off as $\mathcal{O}(p^{-6})$ at high energies, taming the Standard Model hierarchy problem even more rapidly. We can immediately see an emerging trend: a Lee-Wick theory of arbitrarily high N improves the covnergence of loop diagrams arbitrarily well, at the cost of increasingly ungainly computation. It becomes imperative to know just how many terms are needed for the purpose of a given problem, while not ignoring other relevant terms for the sake of keeping calculations manageable.

Fortunately, a natural hierarchy exists due to the dimensionful couplings that attend each derivative operator. In a prototypical Lee-Wick Lagrangian,

$$\mathcal{L}_{\mathrm{HD}}^{N} = \sum_{i=1}^{N} \left(-C_i \hat{\phi} (\partial^2)^i \hat{\phi} \right) + \frac{1}{2} m^2 \hat{\phi}^2, \tag{2.1}$$

the only choice of C_i resulting in a dimensionally consistent Lagrangian is of the form

$$C_i = \frac{1}{M_i^{2(i-1)}}, \tag{2.2}$$

for some heavy mass scale M_i. Even in the case that all M_i are comparable - which is most interesting for phenomenology - the higher-derivative irrelevant operators quickly trail off. We then see the $N = 3$ Lee-Wick Standard Model as being the next-to-minimal approximation to an expansion in M_i^{-2}. The $N = 3$ LWSM was pioneered in Ref. [48], influenced by earlier work in $\mathcal{O}(p^6)$ scalar theories[1] studied in Refs. [49, 50, 51]. We turn now to the task of mapping the $N = 3$ HD theory onto a renormalizable field theory, in a manner similar to the Ostrogradsky formalism exploited earlier.

2.1 An $N = 3$ Toy Theory

As we saw from Eq. (1.61) and the discussion following, we can obtain a computationally convenient form of the theory by first adding auxiliary fields (AF) to a HD Lagrangian, and then following an Ostrogradsky-like formalism.

[1] We pause to note the distinction that $\mathcal{O}(p^4)$ terms are absent in these theories, hence, they are not in the direct pedigree of the LWSM.

The archetypal $N = 3$ Lagrangian for an interacting, higher-derivative scalar $\hat{\phi}$ is [2]

$$\mathcal{L}_{\rm HD}^{N=3} = -\frac{1}{2}\hat{\phi}\Box\hat{\phi} - \frac{1}{2M_1^2}\hat{\phi}\Box^2\hat{\phi} - \frac{1}{2M_2^4}\hat{\phi}\Box^3\hat{\phi} - \frac{1}{2}m_\phi^2\hat{\phi}^2 + \mathcal{L}_{\rm int}(\hat{\phi}), \tag{2.3}$$

where the masses M_1, M_2 roughly correspond to the location of the Lee-Wick poles, and are assumed to be comparable. A quick calculation yields the higher-derivative $N = 3$ propagator,

$$\hat{D}_{\rm HD}^{N=3}(p) = \frac{i}{p^2 - m_\phi^2 - p^4/M_1^2 + p^6/M_2^4}, \tag{2.4}$$

from which we see that propagators in the $N = 3$ theory will have three poles, corresponding to three physical resonances. Note the alternating signs of the p^{2n} terms in Eq. (2.4). In the $p^2 \to +\infty$ limit, the propagator scales as $\approx +ip^{-6}$, opposite in sign to the behavior of the minimal LW case discussed earlier. As we will see, this overall sign signifies the existence of a *positive-norm* state dominating UV behavior.

We desire an AF transformation that will change Eq. (2.3) into a more manageable form,

$$\mathcal{L}_{\rm LW}^{N=3} = \sum_{i=1}^{3} c_i \left[-\frac{1}{2}\phi^{(i)}(\Box + m_i^2)\phi^{(i)}\right] + \mathcal{L}_{\rm int}(\{\phi^{(i)}\}), \tag{2.5}$$

where $c_i = \pm 1$ records the relative signs of contributions to the Lagrangian, and the m_i^2 are taken to be positive.[3] The task of the AF formalism is to determine the constants c_i, m_i^2 as functions of M_1^2, M_2^2, and m_ϕ^2.

We pause here to define some new, helpful terminology. For a LW Lagrangian in the form of Eq. (2.5), we refer to the summation variable i (appearing in $\phi^{(i)}$) as the "Lee-Wick index." It is helpful to assign some unambiguous notation to this effect, so we choose $n_{\rm LW}$. Hence, a SM field possesses LW index $n_{\rm LW} = 1$, a negative-norm LW field has $n_{\rm LW} = 2$, and a positive-norm LW field has $n_{\rm LW} = 3$.

[2]For brevity, we will use the d'Alembertian, $\Box$, in place of ∂^2 in what follows.

[3]Generalizing to LW theories of arbitrary N, one could replace the 3 in Eq. (2.5) with N.

Invoking auxiliary scalar fields χ, ψ, the AF Lagrangian is given by

$$\mathcal{L}_{\rm AF} = \frac{1}{\eta_1}\bigg[-\frac{1}{2}\hat{\phi}(\Box + m_1^2)\hat{\phi} - \chi(\Box + m_1^2)\hat{\phi} + (m_3^2 - m_1^2)^{1/2}(m_2^2 - m_1^2)^{1/2}\chi\psi \\ -\frac{1}{2}\psi\Box\psi - \frac{1}{2}(m_2^2 + m_3^2 - m_1^2)\psi^2\bigg] + \mathcal{L}_{\rm int}(\hat{\phi}), \tag{2.6}$$

with $\eta_1 \equiv (m_1^2m_2^2 + m_1^2m_3^2 + m_2^2m_3^2)/(m_2^2 - m_1^2)(m_3^2 - m_1^2)$. Note that, unlike in Eq. (1.63), one of the auxiliary fields in Eq. (2.6) (namely, ψ) is dynamical. The need for invoking two auxiliary fields, rather than just one, is an expression of the extra dynamical degrees of freedom encountered when moving from $N = 2$ to $N = 3$ theories. Upon varying Eq. (2.6) with respect to χ, one recovers the equation of motion,

$$\psi = \frac{1}{(m_2^2 - m_1^2)^{1/2}(m_3^2 - m_1^2)^{1/2}}(\Box + m_1^2)\hat{\phi}. \tag{2.7}$$

Eq. (2.7) may be substituted back into Eq. (2.6), after which one obtains the HD Lagrangian

$$\mathcal{L}_{\rm HD} = -\frac{1}{2\Lambda^4}\hat{\phi}(\Box + m_1^2)(\Box + m_2^2)(\Box + m_3^2)\hat{\phi}, \tag{2.8}$$

where we have definied

$$\Lambda^4 \equiv m_1^2m_2^2 + m_1^2m_3^2 + m_2^2m_3^2. \tag{2.9}$$

Comparing this recasted HD Lagrangian with Eq. (2.8), we see that the mass terms of Eq. (2.3) match up with those of Eq. (2.8) through the following identifications:

$$m_\phi^2 = (m_1^2m_2^2m_3^2)/\Lambda^4, \tag{2.10}$$

$$M_1^2 = \Lambda^4/(m_1^2 + m_2^2 + m_3^2), \tag{2.11}$$

$$M_2^2 = \Lambda^2. \tag{2.12}$$

This result demonstrates that we are still in contact with the original theory, and also that the original HD Lagrangian is factorizable.

One can also map Eq. (2.6) onto the general form of Eq. (2.5) through the definitions

$$\hat{\phi} = \sqrt{\eta_1}\phi^{(1)} - \sqrt{-\eta_2}\phi^{(2)} + \sqrt{\eta_3}\phi^{(3)}, \tag{2.13}$$

$$\chi = \sqrt{-\eta_2}\phi^{(2)} - \sqrt{\eta_3}\phi^{(3)}, \tag{2.14}$$

$$\psi = \sqrt{\eta_3}\phi^{(2)} - \sqrt{-\eta_2}\phi^{(3)}, \tag{2.15}$$

where the η_i are defined by (note that the following definition of η_1 matches the one below Eq. (2.6), using Eq. (2.9))

$$\eta_1 \equiv \frac{\Lambda^4}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)}, \tag{2.16}$$

$$\eta_2 \equiv \frac{\Lambda^4}{(m_1^2 - m_2^2)(m_3^2 - m_2^2)}, \tag{2.17}$$

$$\eta_3 \equiv \frac{\Lambda^4}{(m_1^2 - m_3^2)(m_2^2 - m_3^2)}. \tag{2.18}$$

Assuming, with no loss of generality,[4] that $m_3 > m_2 > m_1$, we see a pattern develop: $\text{sign}(\eta_i) = (-1)^{i+1}$. This pattern corresponds to that observed in the propagator of Eq. (2.4). The simplification of the unwieldy AF Lagrangian in Eq. (2.6) to the general form of Eq. (2.5) occurs as a result of the following sum rules:

$$\sum_{i=1}^{3} m_i^{2n}\eta_i = 0 \ \ (n = 0,\, 1), \tag{2.19}$$

$$\sum_{i=1}^{3} m_i^{2n}\eta_i = \Lambda^4 \ \ (n = 2), \tag{2.20}$$

$$m_1^2 m_2^2 \eta_3 + m_2^2 m_3^2 \eta_1 + m_1^2 m_3^2 \eta_2 = \Lambda^4. \tag{2.21}$$

The η_i defined here are related to those defined by Pais and Uhlenbeck [52], who were interested in quantum-mechanical theories with HD Lagrangians. The Pais-Uhlenbeck parameters (called η_i^{PU} here) are related to the $N = 3$ LW parameters

[4] If one prefers $m_2 > m_3 > m_1$ instead, this can be accommodated by swapping $\eta_2 \leftrightarrow \eta_3$, and the argument still follows.

by[5]

$$\eta_i = \frac{m_i^4 \Lambda^{2N-2}}{\Pi_j m_j^2} \eta_i^{\mathrm{PU}}. \tag{2.22}$$

Given the field decomposition of Eq. (2.13), the interaction terms of the Lagrangian become

$$\mathcal{L}_{\mathrm{int}}(\hat{\phi}) = \mathcal{L}_{\mathrm{int}}\left(\sqrt{\eta_1}\phi^{(1)} - \sqrt{-\eta_2}\phi^{(2)} + \sqrt{\eta_3}\phi^{(3)}\right). \tag{2.23}$$

It is important to see that the $N = 3$ theory does not introduce new divergences in quantities expected to be finite in the $N = 2$ theory; this would hardly count as an improvement. To illustrate this, consider the familiar interaction term $\mathcal{L}_{\mathrm{int}} = -\lambda\hat{\phi}^4/4!$, which transforms under Eq. (2.13) as

$$\mathcal{L}_{\mathrm{int}}(\{\phi^{(i)}\}) = \frac{\lambda}{4!}\sum_{ijkl}\sqrt{|\,\eta_i\eta_j\eta_k\eta_l\,|}\phi^{(i)}\phi^{(j)}\phi^{(k)}\phi^{(l)}. \tag{2.24}$$

We are interested only in the self-energy function for $\phi^{(1)}$, as the other LW partners are expected to decay, and we already know the consequences of their putative stability to be disastrous. The correction at one loop is then given by

$$\Pi(p^2) = \lambda\eta_1 \int \frac{d^4p}{(2\pi)^4}\sum_k \left[\frac{(-1)^{k+1}i}{p^2 - m_k^2}\right]|\,\eta_k\,|, \tag{2.25}$$

where, for the sake of illustration, we have computed only the most divergent diagram possible. We now make use of the fact that $(-1)^{k+1} \mid \eta_k \mid = \eta_k$, and expand the integrand of Eq. (2.25) in powers of p^2/m_k^2, the high-p^2 limit being the one of interest for probing the UV behavior of the theory. We then have

$$\Pi(p^2) = i\lambda\eta_1 \int \frac{d^4p}{(2\pi)^4}\sum_k \left(\frac{\eta_k}{p^2} + \frac{\eta_k m_k^2}{p^4} + \frac{\eta_k m_k^4}{p^6}\right). \tag{2.26}$$

It appears at a glance that we have a badly divergent self-energy: the first two terms exhibit quadratic and logarithmic divergences, respectively. However, we have not yet

[5] The Π_j symbol denotes multiplication of the m_j^2, and not a canonical momentum operator with respect to a field indexed by j.

invoked the sum rules of Eqs. (2.19) and (2.20). The quadratic divergence vanishes under the $n = 0$ case of Eq. (2.19), and the logarithmic divergence vanishes under the $n = 1$ case of the same. The $\mathcal{O}(p^{-6})$ term yields a finite contribution, which can be recast by the rule of Eq. (2.20). The $N = 3$ theory, though exhibiting more complicated interactions than its $N = 2$ counterpart, has built-in sum rules that still serve to cancel the leading divergences.

While it is necessary to develop auxiliary field Lagrangians for the fermion, Yang-Mills, and Higgs fields for the $N = 3$ theory, their implementation is quite lengthy. The HD Lagrangians for these fields, as well as their AF transformations into recognizable LW theories, may be found in Appendix A.

2.2 Methods of Collider Phenomenology

We now have the tools necessary to study the experimental consequences of the $N = 3$ LWSM. A scattering event particularly sensitive to the presence of LW partners is the process $pp \to W_i^+ \to l^+ + \nu_l + X$, first studied in [53] for the $N = 2$ case. The $N = 3$ analog of this process, studied in [1], forms the basis of this chapter. Here, W_i^+ refers to either a LW gauge boson, $l^+\nu_l$ is a lepton-neutrino pair of a given flavor, and X labels the hadronic products. A characteristic LW mass of 1.5 TeV was assumed. However, subsequent analyses of oblique corrections and electroweak precision observables [54, 55, 56, 57] have demonstrated significant tension between the $N = 2$ LWSM and experimental evidence, thereby requiring a LW W mass to be $\gtrsim 3\,\mathrm{TeV}$ to remain consistent with available data. This turns out to be an optimistic estimate of the LW W mass, as it requires the masses of LW fermions to be substantially higher, as large as 10 TeV in some of the above references. More model-independent scenarios, in which there is a common LW mass for all particles involved, retain consistency at the price of a mass scale $\sim 7\,\mathrm{TeV}$. In either case, these high masses put the LW W well out of range of what can be expected from the LHC (though there exist many studies on LW phenomenology at the LHC; see [58, 59, 60, 61]. Masses of several TeV also raise the troubling complication of a *little hierarchy problem*: even if these LW particles solve the "big" hierarchy problem associated with the sensitivity of the electroweak physics to the GUT scale, there still exists a smaller fine-tuning problem of one part in (at least) 10^1 or 10^2.

This is certainly a step backwards with respect to the overall spirit of Lee-Wick theories. Now that we have the tools of the $N = 3$ LWSM at our disposal, however, we can see if the existence of heavy positive-norm states can ameliorate the tension between theory and experiment as well as produce a distinct signal for LW particles at colliders.

We focus on the semileptonic process $pp \to l^+ + \nu_l + X$, where the lepton pair $l^+\nu_l$ is produced by an intermediate W_i^+ ($i = 1,\, 2\, 3$ covers SM or negative-norm LW, or positive-norm LW, W bosons), and X is an inclusive hadronic state. To leading order in the weak interactions, the partonic-level differential scattering cross-section is

$$\frac{d^3\sigma}{d\tau\, dy\, dz} = K\frac{G_F^2 M_W^4}{48\pi} \sum_{q,q'} |V_{qq'}|^2 \, [SG_{qq'}^+(1+z^2) + 2AG_{qq'}^- z], \tag{2.27}$$

where we have introduced the variables

$$S \equiv \sum_{ij} P_{ij}(C_iC_j)^l(C_iC_j)^q(1+h_ih_j)^2, \tag{2.28}$$

$$A \equiv \sum_{ij} P_{ij}(C_iC_j)^l(C_iC_j)^q(h_i+h_j)^2, \tag{2.29}$$

$$P_{ij} \equiv \hat{s}\frac{(\hat{s}-M_i^2)(\hat{s}-M_j^2)+\Gamma_i\Gamma_jM_iM_j}{[(\hat{s}-M_i^2)^2+\Gamma_i^2M_i^2][i\to j]}. \tag{2.30}$$

K is a numerical factor $\simeq 1.3$ arising from next-to-leading order and next-to-next-to-leading order (commonly NLO and NNLO in the relevant literature) QCD corrections [62]. A parton q and charged lepton l are scattered into an angle $z \equiv \cos\theta^*$

in the center-of-mass (CM) frame. The P_{ij} represent interference terms between the three types of W bosons allowed as intermediate states (see § 2.3.3 for details on the calculation of Γ). The S and A terms are combinations of helicities h_i and couplings $C_i^{l,q}$ which are, respectively, symmetric and asymmetric with respect to z. The quantities $G^{\pm}_{qq'}$ are combinations of parton distribution functions (PDFs) [63]:

$$G^{\pm}_{qq'} = q(x_a, M^2)\bar{q}'(x_b, M^2) \pm q(x_b, M^2)\bar{q}'(x_a, M^2), \tag{2.31}$$

where $q\,(q')$ are the PDFs associated with an up (down)-type quark, the lepton invariant mass is $M^2 \equiv \hat{s}$, and $x_{a,b} = \sqrt{\tau}e^{\pm y}$ are the parton longitudinal momentum fractions; $\tau \equiv \hat{s}/s$, and y is the virtual gauge boson rapidity. This work extends that of Ref. [53] by introducing an additional state of positive norm; this manifests itself in Eq. (2.27) through the positive sign in the propagator (that is, the P_{ij} term), and therefore $\Gamma_3 > 0$. The differential cross section of Eq. (2.27) can be recast as a distribution in the transverse mass M_T, obtained from $z = (1 - M_T^2/M^2)^{1/2}$:

$$\frac{d\sigma}{dM_T} = \int_{M_T^2/s}^{1} d\tau \int_{-Y}^{Y} J(z \to M_T) \frac{d^3\sigma}{d\tau\, dy\, dz}. \tag{2.32}$$

The Jacobian factor $J(z \to M_T) = \mid dz/dM_T \mid = (M_T/M^2) \mid 1 - M_T^2/M^2 \mid^{-1/2}$ is responsible for the peak structures observed in the plots of § 2.4. The new positive-norm states produce a signal near the region $M_T \sim M_3$, with a signature sharp edge produced by the interference terms in the off-diagonal entries of P_{ij}.

Are the predictions of the $N = 3$ LWSM distinct from those of other BSM theories featuring a W' boson? The W' bosons of the LWSM, characterized by their alternating norm, comprise just one example among many contenders, which could (in principle) have arbitrary helicities and couplings to SM fields. Among the possibilities are the Sequential Standard Model [64] (the SM with extra gauge bosons carrying the same couplings), left-right symmetric models where the W' generates an $SU(2)_R$ symmetry (as in Pati-Salam theories [65]), and Kaluza-Klein excitations of the W on a compactified $S^1/\mathbb{Z}_2$ dimension (see [66] and the references contained therein). It was already demonstrated in Ref. [53] that the first two scenarios are distinct from the $N = 2$ LWSM, and so we do not consider them further. However, the Kaluza-Klein modes present an added complication, and must be dealt with more carefully.

The most straightforward extra-dimensional models are characterized by a single dimensionful parameter: R, the length scale of the compactified dimension. Analysis by LEP-1 and LEP-2 groups already requires R^{-1} to exceed several TeV [67], but more exotic extra-dimensional models allow R^{-1} to be brought down to energy scales which might be probed at the LHC.[6] One such situation runs as follows. Take an extra-dimension scenario in which gauge and Higgs bosons propagate in the bulk of the compactified $S^1/\mathbb{Z}_2$ dimension $y \in [0, \pi R]$. The fermions of this theory do not propagate in the bulk, but are instead localized at the endpoints: the leptons are localized at $y = 0$ and the quarks at $y = \pi R$ [68]. This alternative mechanism can lower the compactification scale; as such, we follow the convention of Ref. [53] and take $R^{-1} \sim 1.5$ TeV. The n^{th} Kaluza-Klein excitation of the W has a 5D wavefunction of the form $\cos(ny/R)$; putting this together with the couplings from Eq. (3) of Ref. [69], we see that the localization of the quarks at $y = \pi R$ forces their couplings to the n^{th} W excitation to take the form $C_n^q = (-1)^n$ in the 4D effective theory. This overall sign difference, when inserted into Eq. (2.27), can faithfully mimic the effects associated with the negative sign of the LW propagator. The only formal difference between the Kaluza-Klein and LW cases is the explicit appearance of the decay width in Eq. (2.30), which is still positive for all Kaluza-Klein modes. However,

[6] Kaluza-Klein excitations with masses beyond several TeV are directly observable at the LHC only with a much greater integrated luminosity than is presently available.

as is usually the case in the Breit-Wigner approximation, the LW resonance is taken to be a narrow one; this assumption is developed and verified in § 2.3.3. Therefore, one could always contrive a Kaluza-Klein model that would be indistinguishable from its LW counterpart.

This ambiguity could not be resolved in the $N = 2$ LWSM, but we find that this is emphatically not the case for the $N = 3$ theory. Consider the mass term of the Kaluza-Klein excitations, with a bulk Higgs field φ_b and a VEV of $|\varphi_b|$ [69], coupled in the 5D theory by a constant g:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}\left(\frac{n^2}{R^2} + 2g^2\,|\varphi_b|^2\right)V_\mu^{(n)}V^{(n)\mu}. \tag{2.33}$$

From Eq. (2.33), we see that the Kaluza-Klein excitations obey an explicit n^2 hierarchy. This uniquely determines the masses of all subsequent excitations at the tree level. If one were to conspiratorially choose the mass of the first excitation to be equal to that of the $N = 2$ LW W boson, then the mass of the second excitation is known, whereas the mass of the $N = 3$ LW W boson can in principle attain any positive value. In the limiting case where $R^{-1} \gg g\,|\varphi_b|$, the Kaluza-Klein excitations are very nearly evenly spaced:

$$m_{KK} \approx \frac{n}{R}. \tag{2.34}$$

We then have two possible means by which confusion may still arise between the aforementioned Kaluza-Klein modes and the $N = 3$ LW theory. Either the experimental sensitivities are such that only one excitation (in the general sense) can be detected from either theory, in which event we simply have the situation Ref. [53]; or, by unlucky coincidence, the masses of the LW partners to the W happen to match the spectrum of Eq. (2.33) to within experimental accuracy. Should the latter case prevail, the natural next step would be to examine the decay chains of LW partners to SM fields other than the $W_\mu^\pm$. In general, however, we find that the $N = 3$

LWSM makes predictions regarding the mass and coupling spectra which *cannot* be accurately mimicked by other extensions of the Standard Model.

2.3 Mass Diagonalization & Calculation of Decay Widths

2.3.1 Gauge Boson Mass Diagonalization

In a LW theory with spontaneous symmetry breaking, one generically encounters mass mixing terms between the SM and LW gauge bosons. It is necessary to diagonalize this sector in order to construct gauge boson mass eigenstates for the calculation of decay widths. Beginning with the Higgs kinetic energy Lagrangian,

$$\mathcal{L}_{\text{Higgs,kin}} = \eta_{ij}(\hat{D}_\mu H_i)^\dagger (\hat{D}^\mu H_j), \tag{2.35}$$

where the metric $\eta_{ij} = \text{diag}\{1, -1, 1\}$ encodes the opposite signs of the LW states, the Higgs fields, H_i, are given (as in Eq. (A.43)) by

$$H_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h_1) \end{pmatrix}, \; H_2 = \begin{pmatrix} h_2^+ \\ \frac{1}{\sqrt{2}}(H_2 + iP_2) \end{pmatrix}, \; H_3 = \begin{pmatrix} h_3^+ \\ \frac{1}{\sqrt{2}}(h_3 + iP_3) \end{pmatrix}. \tag{2.36}$$

Note that only the SM Higgs field, H_1, carries a nonzero VEV. The hat on $\hat{D}$ in Eq. (2.35) indicates action on superfields containing both the SM field and its LW partners. Upon expanding the gauge field $\hat{A}$ into its LW components and diagonalizing the mass matrix, we arrive at

$$\hat{A}^\mu = A_1^\mu - \frac{M_3}{\sqrt{M_3^2 - M_2^2}} A_2^\mu + \frac{M_2}{\sqrt{M_3^2 - M_2^2}} A_3^\mu \equiv \hat{\theta}_p A_p^\mu = \hat{\theta}_p V_{pq} A_{q,0}^\mu, \tag{2.37}$$

where we have defined the vector

$$\hat{\theta}_p \equiv \Big\{1, \; -\frac{M_3}{\sqrt{M_3^2 - M_2^2}}, \; \frac{M_2}{\sqrt{M_3^2 - M_2^2}} \Big\}, \tag{2.38}$$

and the mass-eigenstate gauge fields are denoted with a subscript $A_{q,0}$; the indices $p, q \in \{1,\, 2,\, 3\}$ run over the LW index. In the absence of spontaneous symmetry breaking, the fields $A_p^{\mu,a}$ are the eigenstates with masses given by M_p. However, when the Higgs attains a nonzero VEV, it connects all terms quadratic in the $SU(2)_L$ fields $W_p^{\mu,a}$ equally, and as such, more effort is required to obtain the mass eigenstates. Keeping only the terms quadratic in $W^\pm$, the Higgs VEV gives an additional contribution

$$\Delta\mathcal{L}_{\text{mass}} = \left(\frac{gv}{2}\right)^2 \hat{\theta}_r \hat{\theta}_s W_{r\mu}^+ W_s^{-\mu}. \tag{2.39}$$

Now, the problem of obtaining mass eigenstates reduces to numerically solving the matrix equation $W_p^+ = V_{pq}^+ W_{0,q}^+$ for the matrix V, giving us eigenstates $W_{0,q}^+$.

2.3.2 Quark Mass Diagonalization

The top- and bottom-type quark sector is enumerated by purely left-handed supermultiplets[7] T_L^T (superscript T denoting transposition) $= \left(t_{L,1}, t_{L,2}, t'_{L,2}, t_{L,3}, t'_{L,3}\right)$ and $B_L^T = \left(b_{L,1}, b_{L,2}, b'_{L,2}, b_{L,3}, b'_{L,3}\right)$, with right-handed supermultiplets defined analogously. This discussion may just as well be carried over to the lighter quark flavors; however, since they have a numerically small effect on physics at the electroweak scale, we suggestively label the supermultiplets with T and B.

The unprimed fields in T, B contain the same quantum numbers as does the SM field, whereas the primed fields possess the same quantum numbers as the *unprimed* fields of the *opposite* chirality.

The statement above merits some clarification; as an example, consider the left-handed SM top quark, which transforms under $SU(2)_L \times U(1)_Y$ as $(\mathbf{2}, +1/6)$. This means that $t_{L,1}$, $t_{L,2}$, and $t_{L,3}$ all transform as $(\mathbf{2}, +1/6)$. On the other hand, the right-handed top transforms under $SU(2)_L \times U(1)_Y$ as $(\mathbf{1}, +2/3)$. Therefore, the LW fields

[7] The use of the term "supermultiplet" here has nothing to do with supersymmetry; however, as in supersymmetry, it is a useful term to collectively refer to several quantum fields which transform in a similar way. In this case, the supermultiplet collects all LW fields of a given flavor and helicity.

$t'_{L,2}$ and $t'_{L,3}$ transform as $(\mathbf{1}, +2/3)$. The same reasoning allows one to determine the $SU(2)_L \times U(1)_Y$ quantum numbers of the right-handed supermultiplet, T_R. Using the supermultiplet notation, let us examine the fermionic mass terms in the Lagrangian:

$$-\mathcal{L}_{\text{mass}} = \bar{T}_L \rho \mathcal{M}_t^\dagger T_R + \bar{B}_L \rho \mathcal{M}_b^\dagger B_R + \text{h.c.}, \tag{2.40}$$

with the metric

$$\rho \equiv \text{ diag}\{1,\ -1,\ -1,\ 1,\ 1\} \tag{2.41}$$

conveniently encoding the alternating norms of LW states. There are only two classes of mass terms contained in this Lagrangian:

1. Yukawa-type mass terms of the form $m_t \bar{t}_L t_R$+h.c., which connect chiral fermions of **different** $SU(2)_L \times U(1)_Y$ quantum numbers, and
2. Dirac-type mass terms of the form $M_{t,i} \bar{t}_{L,i} t'_{R,i}$ + h.c. $(i > 1)$, which connect chiral fermions of the **same** $SU(2)_L \times U(1)_Y$ quantum numbers.

Since each individual term is gauge-invariant under G_{SM}, any linear combination of them arising through matrix diagonalization will also be gauge-invariant. This property becomes important when spontaneous symmetry breaking is triggered, which generates mass terms not only for the SM quarks, but also mass-mixing terms between different LW states.

In order to diagonalize the mass matrix, one must solve a system somewhat more complicated than the classic eigenvalue problem. The unconventional metric of Eq. (2.41) must be preserved, and we must therefore introduce symplectic matrices $S_{L,R}$ for each supermultiplet Ψ satisfying

$$S_L^\dagger \rho S_L = \rho, \quad S_R^\dagger \rho S_R = \rho, \quad \mathcal{M}\rho = S_R^\dagger \mathcal{M}_0 \rho S_L, \tag{2.42}$$

where the initial mass matrix $\mathcal{M}$ gets diagonalized to $\mathcal{M}_0$. The supermultiplets transform under the $S_{L,R}$ as

$$\Psi^{i,0}_{L,R} = S^{ij}_{L,R}\Psi^{j}_{L,R}. \tag{2.43}$$

We now have mass-diagonal quark states, meaning that we can (among other things) unambiguously calculate quark loops in Feynman with unmixed propagators.

The diagonalization procedure will affect the kinetic terms, which are of the form

$$\bar{Q}_L i\not{D}\rho Q_L + \bar{Q}_R i\not{D}\rho Q_R, \tag{2.44}$$

where $Q_{L,R}$ collects both $T_{L,R}$ and $B_{L,R}$ into doublets of $SU(2)_L$. We know from the discussion above Eq. (2.40) that not all fields within T_L and B_L will be doublets of $SU(2)_L$, and not all fields within T_R and B_R will be $SU(2)_L$ singlets. We therefore introduce projection operators $\Xi_{L,R}$ to project out only the doublet fields within the left- and right-handed supermultiplets, allowing us to use the more compact notation of $Q_{L,R}$. This is especially useful for the present calculation, where the decay $W^+ \to t\bar{b}$ depends on the matrix element between t and b quarks. Anticipating the fact that the $T_{L,R}$ and $B_{L,R}$ supermultiplets will have different symplectic diagonalization matrices, we define the mass-diagonal supermultiplets to be

$$T^{i}_{L,R} = \tau^{ij}_{L,R} T^{j,0}_{L,R}, \tag{2.45}$$

where $\tau_{L,R}^{1} \equiv S^{-1}_{L,R}$ for the top sector. An analogous relationship exists for the bottom sector, with $\tau_{L,R}$ replaced by $\beta_{L,R}$.

We now have all the necessary tools to compute the $W^+ \to tb$ matrix element. From the covariant derivative, we have

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \bar{Q}_L i\not{D} Q_L \\ &\supset \bar{T}^{\dagger}_{L,0}\tau^{\dagger}_L \left(\frac{g}{\sqrt{2}}\hat{\theta}_p\, V^{+}_{pq}\gamma_\mu\, W^{+,\mu}_{0,q}\right)\Xi_L\rho\beta_L B_{L,0} + \text{h.c.}\end{aligned} \tag{2.46}$$

CKM matrix elements appear in Eq. (2.46) when multiple quark generations are taken into account.

As mentioned above, the projection matrix Ξ_L is a collection of ones and zeros tasked with ensuring that only $SU(2)_L$ doublets of T_L and B_L appear in Eq. (2.46). Another $\mathcal{L}_{\text{int}}$ can be formed by considering the right-handed mass eigenstates which transform as doublets under $SU(2)_L$. This is done by beginning with $\bar{Q}_R i\not{D} Q_R$, defining an appropriate Ξ_R, and continuing until an expression similar to Eq. (2.46) is obtained. Since this emerging pattern results in a somewhat repetitive exercise in linear algebra, the above notation is condensed using

$$\Lambda_{L,R} \equiv \tau^{\dagger}_{L,R} \Xi_{L,R} \rho \beta_{L,R}. \tag{2.47}$$

The function of Λ is to combine all numerical information concerning diagonalization, alternating norm, and chirality projection into a single operator. Once calculated, it allows us to transition in one step from the mass-mixed fields of Eq. (2.40) to mass-diagonal fields ready for calculation. This notation allows the matrix element to be written compactly as

$$\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}} \bar{T}_0 \hat{\theta}_p V^{+}_{pq} \gamma_\mu W^{+,\mu}_{0,q} \left(\Lambda_L P_L + \Lambda_R P_R \right) B_0 + \text{h.c.}, \tag{2.48}$$

using the familar projection operators $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$ to write, *e.g.*, $T_0 = T_{L,0} P_L + T_{R,0} P_R$.

2.3.3 W Boson Width Calculation

In the special case of $W^+ \to t\bar{b}$ decay, the associated Feynman vertex rule reads

$$i \frac{g}{\sqrt{2}} \gamma^\mu \hat{\theta}_p V^{+}_{pq} (\Lambda^{ij}_L P_L + \Lambda^{ij}_R P_R), \tag{2.49}$$

which leads to the invariant matrix element

$$i\mathcal{M} = i\epsilon_\mu \frac{g}{\sqrt{2}} \bar{t}^i_0 \gamma^\mu \hat{\theta}_p V^{+}_{pq} (\Lambda^{ij}_L P_L + \Lambda^{ij}_R P_R) b^j_0. \tag{2.50}$$

From this, we obtain the squared, spin-averaged matrix element

$$\overline{|\mathcal{M}|^2} = \frac{g^2}{3}\,|\hat{\theta}_p V^+_{pq}|^2 \left\{\left[M^2_{W,q} - \frac{1}{2}(m^2_{t,i} + m^2_{b,j}) - \frac{1}{2M^2_{W,q}}(m^2_{t,i} - m^2_{b,j})^2\right]\right.$$
$$\left.\times(\Lambda^{ij}_L \Lambda^{\dagger ji}_L + \Lambda^{ij}_R \Lambda^{\dagger ji}_R) + 3m_{t,i}m_{b,j}(\Lambda^{ij}_L \Lambda^{\dagger ji}_R + \Lambda^{ij}_R \Lambda^{\dagger ji}_L)\right\}. \tag{2.51}$$

No Einstein summation is assumed on the indices q, i, j, so that Eq. (2.51) specifies the squared amplitude for the q^{th} weak gauge boson, the i^{th} top quark state, and the j^{th} bottom quark state (all mass eigenstates). In the SM case, $|\hat{\theta}_p V^+_{pq}|^2 = 1$, $\Lambda_L = V_{tb}$, and $\Lambda_R = 0$.

We now integrate over phase space to obtain the decay width Γ. Using the well-known formula

$$\Gamma = \frac{1}{2M_{W,q}} \int d\Pi_2\,|\mathcal{M}|^2, \tag{2.52}$$

we find the total contribution to the width of the q^{th} gauge boson to be

$$\Gamma = \frac{\overline{|\mathcal{M}|^2}}{8\pi M^2_{W,q}} \sqrt{\left(\frac{M^2_{W,q} + m^2_{t,i} - m^2_{b,j}}{2M_{W,q}}\right)^2 - m^2_{t,i}}. \tag{2.53}$$

In the well-motivated limit that $M_{W,LW} \gg m_{t,SM}$, $m_{b,SM}$, the decay rate contribution for each $W^+_q \to f_i \bar{f}_j$ is

$$\delta\Gamma = g^2 \frac{|\hat{\theta}_p V^+_{pq}|^2}{48\pi} (\Lambda^{ij}_L \Lambda^{\dagger ji}_L + \Lambda^{ij}_R \Lambda^{\dagger ji}_R) M_{W,q}. \tag{2.54}$$

For the case $M_3 \gg M_2$, one anticipates from Eq. (2.38) that $|\theta_p V^+_{p2}| \gg |\hat{\theta}_p V^+_{p3}|$, which suppresses the decay rate contribution for $W^+_3 \to f_1 \bar{f}_2$ compared to that for W^+_2. This effect is mitigated by the possible presence of massive final-state particles kinematically forbidden in W^+_2 decays but allowed in W^+_3 decays.

2.4 Results

We begin with the LHC inputs $\sqrt{s} = 7$ TeV, and $10\,\text{fb}^{-1}$ of integrated luminosity. Taking for example the masses $m \approx m_{W,\text{SM}} = 80.4$ GeV, $M_2 = 1$ TeV, and $M_3 = 2$

TeV, we can begin to compute the transverse mass distributions using § 2.2. We plot our results in Fig. 2.1. The most exciting feature is the statistically robust Jacobian peak near $M_T \approx M_3$ at 10 fb^{-1}, which is not only revealed as dozens of events that would not appear in the $N = 2$ LWSM, but also features a profile distinct from that offered by Kaluza-Klein models. This result indicates that, for a sufficiently light $W_3^{\pm}$ boson, the $N = 3$ LWSM makes unambiguous predictions that can be tested at the LHC, given a very reasonable demand on integrated luminosity. It is also important to realize that nothing is special about the choice of $M_3 = 2$ TeV; the positive-norm W partner could be significantly heavier, still playing a role in solving the hierarchy problem (although likely creating tension with electroweak precision tests), but standing outside the realm of feasible detection with current LHC operating parameters. A sufficiently heavy $W_3^{\pm}$ combined with a lighter $W_2^{\pm}$ could still satisfy the electroweak precision tests while evading detection. Fig. 2.2 addresses just such a possibility.

2.5 Discussion and Conclusions

We have seen that the presence of a heavy, positive-norm state has observable consequences for the LHC, and its interplay with the lighter, negative-norm state can free up parameter space for LW phenomenology (as might be expected). The augmented theory makes predictions above and beyond that of the conventional $N = 2$ LWSM, and is clearly distinguishable from other theories featuring a heavy counterpart to the familiar $W^{\pm}$ of the Standard Model. However, since little information yet exists to constrain the value of M_3, this work should be understood to be a proof of principle that the $N = 3$ theory makes robust predictions for a range of masses.

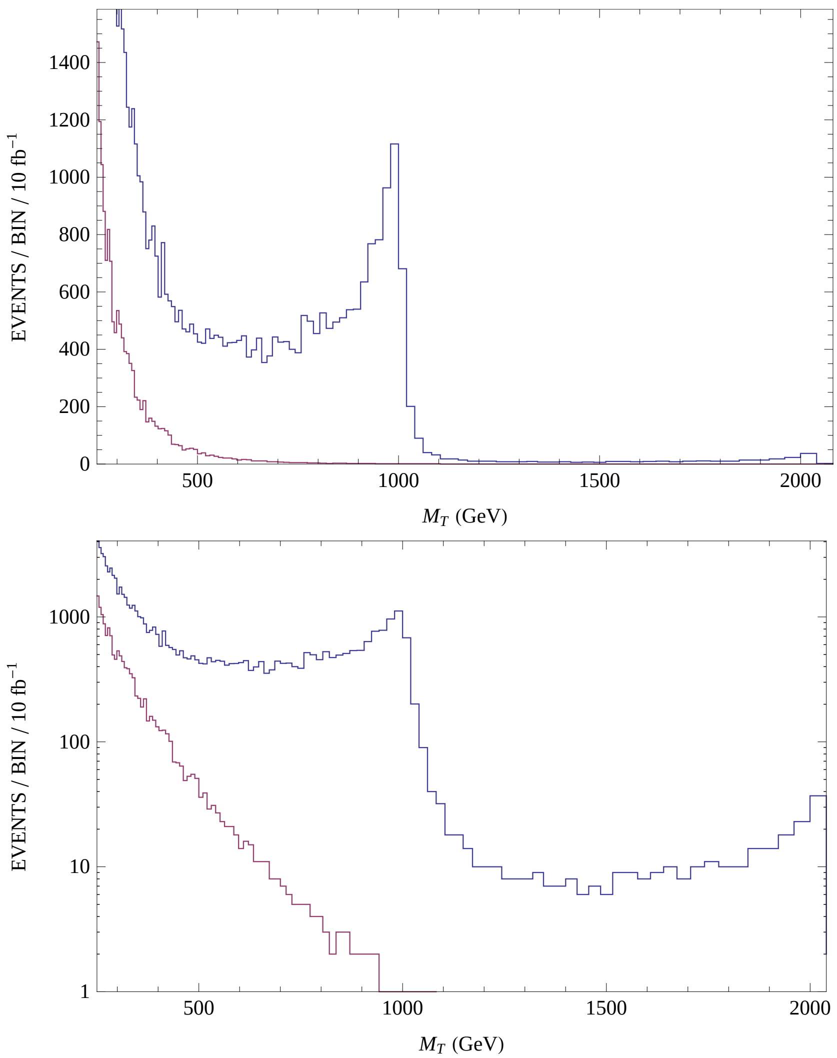


Figure 2.1: Transverse mass distributions for the process $pp \to W_i^+ + X \to l^+ + \nu_l + X$ in the $N = 3$ LWSM (blue) and the predicted SM-only background (red). Both plots contain the same data; the log scale in the bottom plot better demonstrates the Jacobian peak structure near $M_T \approx M_3$. We employ a rapidity cut of $|\eta_l|$ on the outgoing leptons, and smear the distribution by $\delta M_T/M_T$ to simulate the finite resolution of the ATLAS detector. (Reprinted from Ref. [1])

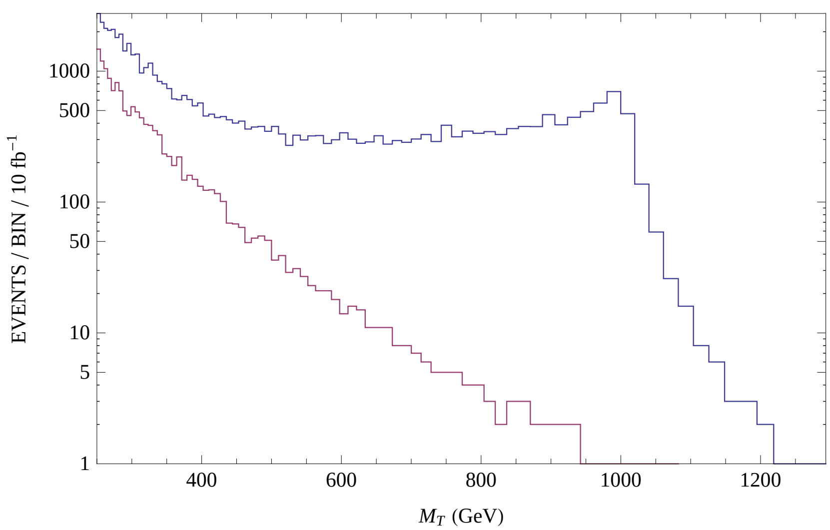


Figure 2.2: Transverse mass distributions for the same decay channel, but with masses $M_2 = 1$ TeV and $M_3 = 5$ TeV. Even with an integrated luminosity of 10 fb^{-1}, the second Jacobian peak is too weak to be discernable. The plots are truncated when the number of (simulated) events per bin drops below 0.5 (Reprinted from Ref. [1])

Chapter 3

PRECISION ELECTROWEAK CONSTRAINTS ON THE $N = 3$ LEE-WICK STANDARD MODEL

In this chapter, we examine the oblique corrections to the $N = 3$ Lee-Wick Standard Model (LWSM), as considered in Ref [70]. The presence of LW fields has observable consequences on SM processes, even if the characteristic energy scales of the latter are far below the threshold at which LW particles can be produced on-shell. The extraction of these observables from presently available collider data is made possible through the technology of electroweak precision tests (EWPT), which determine the corrections to gauge boson correlation functions from new particles at or above the electroweak scale. In the sections to come, the formalism of oblique corrections will be introduced, and its implications for the concordance of LW fields with existing SM data will be calculated and discussed.

3.1 Bounds on Oblique Parameters

3.1.1 Formalism and Tree-Level Contributions

There are two major classes of bounds on physics beyond the Standard Model (BSM). They are either

- **Oblique**: flavor-universal, arising from gauge boson vacuum polarization loops, or
- **Direct**: flavor-specific, arising from, *e.g.*, vertex and box corrections. [71].

Perhaps the best-known among the oblique constraints are the Peskin-Takeuchi parameters [72] S, T, and U, which form a complete set of all finite, one-loop corrections to differences between (*i.e.*, first derivatives of) the $SU(2)_L \times U(1)_Y$ gauge boson correlation functions. As better data became available in the mid-1990s (in particular, from LEP2), it became possible to probe the electroweak corrections to higher derivative order. Barbieri *et al.* then developed a new set of parameters to test electroweak corrections to still higher sensitivities [67], calling them $\hat{S}$, $\hat{T}$, $\hat{U}$, V, W, X, Y, and Z[1]. It has been argued [67, 72] that the parameters U, V, X, and Z are numerically small, and hence can be ignored in the study of EWPT; we follow this convention, and focus on $\hat{S}$, $\hat{T}$, W, and Y. This set of parameters is essential for the investigation of so-called "universal" models, in which the only deviation from the SM arises in contributions to gauge boson self-energy functions. The $N = 2$ LWSM was argued to be a BSM theory of this type [56].

To isolate beyond-SM effects, one must first identify the SM electroweak parameters, which are given by [67]

$$\frac{1}{g'^2} \equiv \Pi'_{\hat{B}\hat{B}}(0), \quad \frac{1}{g^2} \equiv \Pi'_{\hat{W}^+\hat{W}^-}(0), \tag{3.1}$$

$$\frac{1}{\sqrt{2}G_F} = -4\Pi_{\hat{W}^+\hat{W}^-}(0) = v^2, \tag{3.2}$$

where a prime on the self-energy $\Pi(q^2)$ indicates differentiation[2] with respect to q^2. The above relations serve as definitions for the electroweak parameters g, g', and v^2 at tree level; they hold in the LWSM as well. The peculiar normalization (due to Ref. [67]) is one in which the gauge fields in Eq. (1.1) are replaced via $A \to g^{-1}A$. In fact, we can read off the self-energy contributions from Eq. (A.1) as an expansion in

[1] The hatted parameters contain the same physical information as do the original Peskin-Takeuchi parameters; they merely carry a different normalization.

[2] Ref. [72] defined the Π' by $\Pi(q^2) \equiv \Pi(0) + q^2\Pi'(q^2)$. Therefore, $\Pi' = d\Pi/dq^2$ only in the $q^2 \to 0$ limit. But this is precisely the case of interest to us, so we will treat Π' as a derivative throughout.

$q^2/M_{\rm LW}^2$ (for an arbitrary LW mass scale $M_{\rm LW}$):

$$\begin{aligned}
\Pi_{\hat{W}^+\hat{W}^-}(q^2) &= \Pi_{\hat{W}^3\hat{W}^3}(q^2) \\
&= \frac{q^2}{g^2} - \frac{(q^2)^2}{g^2}\left[\left(\frac{1}{M_2^{(2)}}\right)^2 + \left(\frac{1}{M_2^{(3)}}\right)^2\right] - \frac{v^2}{4}, \\
\Pi_{\hat{W}^3\hat{B}}(q^2) &= \frac{v^2}{4}, \\
\Pi_{\hat{B}\hat{B}}(q^2) &= \frac{q^2}{g'^2} - \frac{(q^2)^2}{g'^2}\left[\left(\frac{1}{M_1^{(2)}}\right)^2 + \left(\frac{1}{M_1^{(3)}}\right)^2\right] - \frac{v^2}{4}.
\end{aligned} \tag{3.3}$$

Here, $M_2^{(i)}$ is the mass of the $i^{\rm th}$ LW state for the $SU(2)_L$ gauge bosons, and $M_1^{(i)}$ serves an analogous purpose for the $U(1)_Y$ bosons.[3] We can use the above relations to construct tree-level electroweak parameters for the $N = 3$ theory:

$$\hat{S} \equiv g^2 \Pi'_{\hat{W}^3\hat{B}}(0) = 0, \tag{3.4}$$

$$\hat{T} \equiv \frac{g^2}{m_W^2}\left(\Pi_{\hat{W}^3\hat{W}^3}(0) - \Pi_{\hat{W}^+\hat{W}^-}(0)\right) = 0, \tag{3.5}$$

$$W \equiv \frac{1}{2} g^2 m_W^2 \Pi''_{\hat{W}^3\hat{W}^3}(0) = -m_W^2\left(\left(\frac{1}{M_2^{(1)}}\right)^2 + \left(\frac{1}{M_2^{(2)}}\right)^2\right), \tag{3.6}$$

$$Y \equiv \frac{1}{2} g'^2 m_W^2 \Pi''_{\hat{B}\hat{B}} = W \equiv \frac{1}{2} g^2 m_W^2 \Pi''_{\hat{W}^3\hat{W}^3}(0) = -m_W^2\left(\left(\frac{1}{M_1^{(1)}}\right)^2 + \left(\frac{1}{M_1^{(2)}}\right)^2\right), \tag{3.7}$$

where the first equality on each line serves to define the corresponding post-LEP parameter [67]. The absence of a tree-level contribution to $\hat{S}$, $\hat{T}$ was first noted in Ref. [57] for the $N = 2$ case, which also holds for $N = 3$. Furthermore, it was found in Ref. [57] that fermionic one-loop contributions to W and Y are numerically small compared to the tree-level values. This means that the most significant new fermionic contributions will arise from one-loop contributions to $\hat{S}$ and $\hat{T}$. Because W and Y

[3]NB: $i \in \{1, 2, 3\}$, meaning that $i = 1$ refers to the SM state, $i = 2$ refers to the negative-norm LW state, and $i = 3$ refers to the positive-norm LW state.

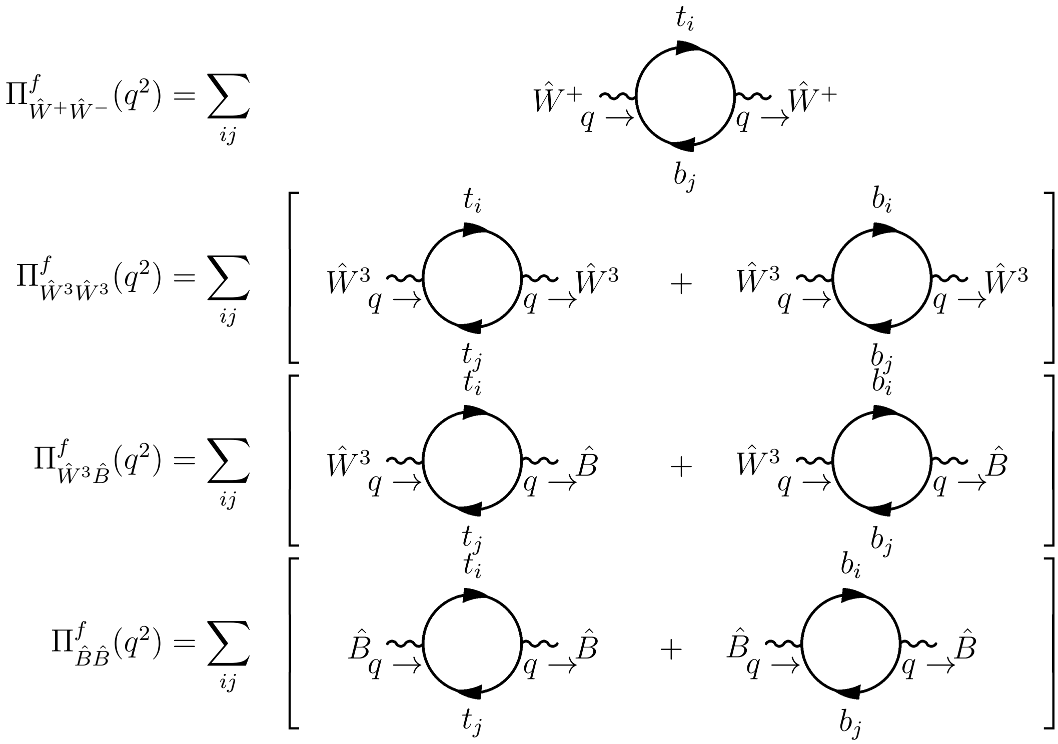


Figure 3.1: Fermion vacuum polarization Feynman diagrams that provide the dominant contributions to the electroweak precision observables $\hat{S}$ and $\hat{T}$.

at tree level tunred out to be of the same order of magnitude as $\hat{S}$ and $\hat{T}$ at one loop, we do not pursue loop corrections to W and Y here.

3.1.2 Fermion Loop Contributions

Once the tree-level effects have been taken into account, the one-loop fermion diagrams provide the most important contribution to the oblique parameters (see Fig. 3.1).

We consider the general self-energy diagram connecting gauge bosons $\hat{A}_\mu$ and $\hat{B}_\nu$ (not to be confused with the hypercharge boson of the Standard Model), which is calculated by way of the Lagrangian[4]

$$\mathcal{L} = \bar{\Psi}_i^0 \gamma^\mu [\hat{A}_\mu (A_{ij}^{L,\Psi} P_L + A_{ij}^{R,\Psi} P_R) + \hat{B}_\mu (B_{ij}^{L,\Psi} P_L + B_{ij}^{R,\Psi} P_R)] \Psi_j^0, \tag{3.8}$$

where the helicity-projected couplings A_{ij} connect fermions of type i and j. We have $i, j \in \{1, 2, \ldots, 5\}$, as in § 2.3.2. The coupling matrices are the result of rotating the "bare" fields into the mass-eigenstate basis, to wit,

$$A_{ij}^{L(R),\Psi} = S_{L(R)}^{\Psi\dagger} Q_{A,L(R)}^{\Psi} \rho S_{L(R)}^{\Psi}. \tag{3.9}$$

Q_A^Ψ is the matrix of fermion charges under the gauge group A, and the superscript Ψ may refer to either a single fermion flavor (as would be the case in the vacuum polarization of a Z^0 or γ), or a weak-isospin pair (such as the t, b pair in the vacuum polarization of a $W^\pm$).

When applied to the LW case, the self-energy functions of Eqs. (3.1), (3.2) become

$$\begin{aligned} \Pi_{AB}(q^2) = {} & \frac{C}{8\pi^2} \sum_{\Psi=T,B} \sum_{i,j} \rho_{ii}\rho_{jj} \\ & \times \left[(A_{ij}^{L,\Psi} B_{ji}^{L,\Psi} + A_{ij}^{R,\Psi} B_{ji}^{R,\Psi}) I_1(q^2) + (A_{ij}^{L,\Psi} B_{ji}^{R,\Psi} + A_{ij}^{R,\Psi} B_{ji}^{L,\Psi}) I_2(q^2) m_i m_j \right], \end{aligned} \tag{3.10}$$

where C is a color factor ($= N_c$ for quarks coupling to colorless gauge bosons). We define the dimensional regularization mass term $\Delta \equiv -q^2 x(1-x) + m_i^2 x + m_j^2(1-x)$ for the diagram containing fermions labeled i and j. We use primes to denote derivatives with respect to q^2 (as above), and the subscript 0 to indicate that the function is evaluated at $q^2 = 0$. This gives

$$\Delta_0 = m_i^2 + m_j^2(1-x), \tag{3.11}$$

$$\Delta_0' = -x(1-x), \tag{3.12}$$

$$\Delta'' = 0. \tag{3.13}$$

[4]We use the notation $A_{ij}^{L(R),\Psi}$, rather than the $\Lambda_{ij}^{L(R)}$ of § 2.3.2. Though the symbols are derived in a similar manner, the A^Ψ matrices are a general case, and are not specific to the $W^+ \to t\bar{b}$ case in which the Λ matrices were computed.

The necessary integrals are defined as follows:

$$I_1(q^2) \equiv \int_0^1 dx\,(2\Delta - \Delta_0)\ln(\Delta/\mu_{\rm UV}^2), \tag{3.14}$$

$$I_2(q^2) \equiv -\int_0^1 dx\,\ln(\Delta/\mu_{\rm UV}^2). \tag{3.15}$$

We then obtain the moments of integrals relevant to the oblique parameters:

$$I_{10} = \int_0^1 dx\ \Delta_0\ \ln(\Delta_0/\mu_{\rm UV}^2)\,, \tag{3.16}$$

$$I_{20} = -\int_0^1 dx\ \ \ln(\Delta_0/\mu_{\rm UV}^2)\,, \tag{3.17}$$

$$I'_{10} = \int_0^1 dx\ \Delta'_0[1 + 2\ln(\Delta_0/M^2)]\,, \tag{3.18}$$

$$I'_{20} = -\int_0^1 dx\ \Delta'_0/\Delta_0\,, \tag{3.19}$$

$$I''_{10} = 3\int_0^1 dx\ (\Delta'_0)^2/\Delta_0\,, \tag{3.20}$$

$$I''_{20} = \int_0^1 dx\ (\Delta'_0/\Delta_0)^2\,. \tag{3.21}$$

The constant $\mu_{\rm UV}^2$ contains the scale associated with logarithmic divergences in dimensional regularization. Since we are interested in differences and derivatives of these integrals, the formal infinities drop out of all subsequent calculations, and $\mu_{\rm UV}^2$ serves as merely a bookkeeping device. The individual integrals are straightforward to compute, and for completeness, we list them here:

$$
\begin{aligned}
I_{10} &= -\frac{1}{4}(m_i^2+m_j^2)+\frac{1}{2}\frac{m_i^4\ln(m_i^2/M^2)-m_j^4\ln(m_j^2/M^2)}{m_i^2-m_j^2}\,, \\
&\to m_i^2\ln\frac{m_i^2}{M^2}\,,\quad m_j\to m_i\,; && (3.22) \\
I_{20} &= 1-\frac{m_i^2\ln(m_i^2/M^2)-m_j^2\ln(m_j^2/M^2)}{m_i^2-m_j^2}\,, \\
&\to -\ln\frac{m_i^2}{M^2}\,,\quad m_j\to m_i\,; && (3.23) \\
I'_{10} &= -\frac{1}{3}\left\{\frac{m_i^4(m_i^2-3m_j^2)}{(m_i^2-m_j^2)^3}\ln\left(\frac{m_i^2}{M^2}\right)-\frac{m_j^4(m_j^2-3m_i^2)}{(m_i^2-m_j^2)^3}\ln\left(\frac{m_j^2}{M^2}\right)+\frac{m_i^4-8m_i^2m_j^2+m_j^4}{3(m_i^2-m_j^2)^2}\right\}, \\
&\to -\frac{1}{6}\left[1+2\ln\left(\frac{m_i^2}{M^2}\right)\right]\,,\quad m_j\to m_i\,; && (3.24) \\
I'_{20} &= -\frac{(m_im_j)^2}{(m_i^2-m_j^2)^3}\ln\left(\frac{m_i^2}{m_j^2}\right)+\frac{m_i^2+m_j^2}{2(m_i^2-m_j^2)^2}\,, \\
&\to \frac{1}{6m_i^2}\,,\quad m_j\to m_i\,; && (3.25) \\
I''_{10} &= \frac{3(m_im_j)^4}{(m_i^2-m_j^2)^5}\ln\left(\frac{m_i^2}{m_j^2}\right)+\frac{(m_i^2+m_j^2)(m_j^2-8m_i^2m_j^2+m_i^4)}{4(m_i^2-m_j^2)^4}\,, \\
&\to \frac{1}{10m_i^2}\,,\quad m_j\to m_i\,; && (3.26) \\
I''_{20} &= -\frac{2(m_im_j)^2(m_i^2+m_j^2)}{(m_i^2-m_j^2)^5}\ln\left(\frac{m_i^2}{m_j^2}\right)+\frac{m_i^4+10m_i^2m_j^2+m_j^4}{3(m_i^2-m_j^2)^4}\,, \\
&\to \frac{1}{30m_i^4}\,,\quad m_j\to m_i\,. && (3.27)
\end{aligned}
$$

These equations may be substituted into Eq. (3.10) to produce the full results for fermionic one-loop contributions. The symplectic $S_{L,R}$ matrices, and hence the A^Ψ from which they are derived, may be computed as per § 2.3.2.

3.2 Constraints from the $Zb_L\bar{b}_L$ Coupling

The $Zb_L\bar{b}_L$ coupling is one example of an electroweak observable that exhibits significant tension between theory and experiment, and is therefore a very interesting subject to the BSM phenomenologist. It has been known for some time [73] that

$$\delta g_L^{b\bar{b}} \sim \sum_{ijk} \tag{3.28}$$

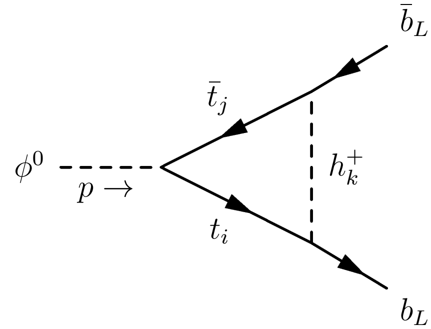


Figure 3.2: Dominant diagram contributing to the decay $Z^0 \to b_L \bar{b}_L$ in the gaugeless limit. The contribution to the effective coupling is denoted by $\delta g_L^{b\bar{b}}$, which is defined in the $p \to 0$ limit. The indices i, j, k enumerate the Lee-Wick indices of the fields involved.

its leading contribution in the gaugeless limit (*i.e.*, ignoring effects suppressed by $(m_{Z^0}/m_t)^2$) is most easily obtained by computing the triangle diagram of Fig. 3.2, in which a Goldstone boson, ϕ^0 (the one eaten by the Z^0 through spontaneous symmetry breaking) of momentum p splits into a virtual $\bar{t}t$ pair, subsequently decaying to a $\bar{b}b$ pair through the exchange of a charged Higgs (which is eaten by the $W^\pm$). The invariant amplitude may be written as

$$i\mathcal{M} = -\frac{2}{v}(\delta g_L^{b\bar{b}})\not{p} P_L. \tag{3.29}$$

The coupling $g_L^{b\bar{b}}$ is derived from a combination of the $Z^0 \to b\bar{b}$ branching ratio, R_b, and its forward-backward asymmetry, A_b. Ref. [74] gives some insight as to the sensitivity of $g_L^{b\bar{b}}$ to small changes in both parameters:

$$\delta g_L^{b\bar{b}} \equiv g_L^{b\bar{b},\text{exp}} - g_L^{b\bar{b},\text{SM}} \equiv -1.731\delta R_b - 0.1502\delta A_b, \tag{3.30}$$

where we have adjusted the normalization (*i.e.*, removing a factor of $e/\sin\theta_W \cos\theta_W$) in order to match the notation used elsewhere in this section. As of this writing, its most recent experimental value $g_L^{b\bar{b},\text{exp}}$ has not changed since the combined LEP/SLD

2005 analysis [75]. The SM prediction of $g_L^{b\bar{b},\mathrm{SM}} = -0.42114^{+45}_{-24}$ from Ref. [75] gives $\delta g_L^{b\bar{b}} = 2.94(157) \times 10^{-3}$, which means that the SM value is approximately 2σ less than the experimental value. This strongly disfavors any new physics predicting $\delta g_L^{b\bar{b}} < 0$. However, the current Particle Data Group values for R_b^{SM} and A_b^{SM} [76] lead to a somewhat relaxed bound of

$$\delta g_L^{b\bar{b}} = 2.69(157) \times 10^{-3}, \tag{3.31}$$

which we will use in our analysis.

The effect of negative-norm states on this observable has been considered twice before in the literature [59, 57]. Ref. [59] found that contemporary precision bounds allow LW Higgs partners to have significantly lower masses than other those of other LW particles. Therefore, Ref. [59] computes the diagram of Fig. 3.2 by including only LW Higgs partners in the loop, giving (in our normalization)

$$\delta g_L^{b\bar{b}} = -\frac{m_t^2}{16\pi^2 v^2}\Big[\frac{R}{R-1} - \frac{R\ln R}{(R-1)^2}\Big], \tag{3.32}$$

where $R \equiv (m_t/m_{h_2})^2$, so that $\delta g_L^{b\bar{b}} < 0$. The value of $\delta g_R^{b\bar{b}}$ in the LWSM is driven by m_b, and is therefore much smaller. Since $\delta g_L^{b\bar{b}}$ and R_b are anti-correlated, as per Eq. (3.30), and since δR_b is positive [75, 76], it follows that the LW Higgs contribution works in the direction of resolving the discrepancy. However, there is cause for caution: Eq. (3.30) also depends upon δA_b (though not quite as strongly as δR_b), and the combined effect is a difficulty in accommodating a new physics scenario that features $\delta g_L^{b\bar{b}} < 0$. We take this effect into consideration in our analysis.

The other effort to constrain the $Zb_L\bar{b}_L$ coupling [57] in the $N = 2$ LWSM uses the full bound from Refs. [75, 76], using only LW partners to the top quark. This gives the result

$$\delta g_L^{b\bar{b}} = -\frac{m_t^4}{32\pi^2 v^2 M_q^2}\left[5\ln\frac{M_q^2}{m_t^2} - \frac{49}{6}\right], \tag{3.33}$$

at leading order in m_t^2/M_q^2, with M_q being the mass of the LW top-quark partner. This expression is manifestly negative for most LW masses in the phenomenologically interesting range of what we could expect to see in the LHC. It is the most stringent bound on LW masses, and gives a lower bound at the 95% confidence level of $M_q \geq 4$ TeV. However, Eq. (3.33) is a very shallow function of M_q (see Fig. 8 of Ref. [57]), and the small change in the SM value of $g_L^{b\bar{b}}$ given in Eq. (3.31) is enough to reduce the bound to $M_q \geq 1.2$ TeV. A full analysis necessarily includes contributions from the Higgs partners as well as the top. Since the LW Higgs contribution is also negative - and also turns out to be comparable to that of the LW top - all of the mass bounds are consequently higher. This fact motivates the calculation of the $Zb_L\bar{b}_L$ coupling in the full $N = 3$ LWSM, the hope being that the improved UV behavior and delicate cancellations of the $N = 3$ theory (as well as the roomier parameter space) will lighten the tension with experimental data.

We now begin the $N = 3$ LWSM calculation. Since this shares the same subject matter of Ref. [57], for ease of comparison, we adopt the notation used therein. The Yukawa Lagrangian,

$$\begin{aligned}\mathcal{L}_{\text{Yuk}} = -iy_t \sum_{i,j} \bigg\{ & \frac{1}{\sqrt{2}} \hat{\phi}^0 \left[\alpha_{ij} \bar{t}_i P_R t_j - \alpha_{ji} \bar{t}_i P_L t_j \right] \\ & + \beta_{ij} \left[\hat{\phi}^- \bar{b}_i P_R t_j - \hat{\phi}^+ \bar{t}_j P_L b_i \right] \bigg\}, \end{aligned} \tag{3.34}$$

has couplings α and β closely related to the A^Ψ parameters of Eq. (3.9), viz.:

$$\begin{aligned}\alpha &\equiv (S_L^t)^\dagger \alpha_0 S_R^t \,, \\ \beta &\equiv (S_L^b)^\dagger \beta_0 S_R^t \,. \end{aligned} \tag{3.35}$$

In the $N = 3$ case,

$$\alpha_0^{N=3} = \beta_0^{N=3} \equiv \begin{pmatrix} 1 & -\cosh\phi_q & 0 & \sinh\phi_q & 0 \\ -\cosh\phi_t & \cosh\phi_q \cosh\phi_t & 0 & -\sinh\phi_q \cosh\phi_t & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sinh\phi_t & -\cosh\phi_q \sinh\phi_t & 0 & \sinh\phi_q \sinh\phi_t & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.36)$$

with angles ϕ_t, ϕ_b parameterizing the independent symplectic diagonalizations. Note especially the rows and columns of all zeros corresponding to the $SU(2)_L$-singlet states. The structure of these matrices allows the sum of Eq. (3.37) to formally run over all LW t-partners while implicitly skipping terms that are disallowed by gauge invariance.

The salient distinction between the present work and that of Ref. [57] is actually not the presence of the $N = 3$ *fermion* partners, as might have been expected; rather, it is the presence of HD[5] *scalar* fields $\hat{\phi}^0$, $\hat{\phi}^\pm$, whose SM content is the set of Goldstone bosons, and which enter with the relative weights of Eq. (2.13). The LW partners to these states are physical, massive states which must be included in the full analysis, but which were excluded in Ref. [57].

The result of the $\delta g_L^{b\bar{b}}$ calculation in Ref. [57] is that the LW t-partners tend to exacerbate the extant tension with experiment, thus forcing even more stringent bounds on LW partner masses than that obtained by $\hat{T}$. This pattern is not commonly shared by all LW partners, however. As was shown in Ref. [59], the charged Higgs partners, $h_2^\pm$, can be much lighter ($\gtrsim$ 500 GeV) while still satisfying all precision constraints. We note from Eq. (A.44) that charged scalar masses do not mix upon spontaneous symmetry breaking; this, combined with the presence of the virtual charged scalar in Fig. 3.2, allows the opposing signs of $h_{2,3}^\pm$ to reduce the overall

[5]That is, not re-fashioned into the canonical LW form by way of an auxiliary field transformation.

amplitude with respect to the $N = 2$ case. The full expression reads

$$\delta g_L^{b\bar{b}} = \frac{1}{16\pi^2} \cdot \frac{y_t^3 v}{2\sqrt{2}} \Bigg\{ \sum_i \eta_k \beta_{0i}^2 \alpha_{ii} \frac{m_{t_i}}{m_{t_i}^2 - m_{h_k}^2} \left[1 - \frac{m_{h_k}^2}{m_{t_i}^2 - m_{h_k}^2} \ln \left(\frac{m_{t_i}^2}{m_{h_k}^2} \right) \right]$$
$$+ \sum_{i \neq j; k} (-1)^{i+j} \eta_k \beta_{0i} \beta_{0j} \alpha_{ji} m_{t_j} \Bigg[- \frac{1}{m_{t_i}^2 - m_{t_j}^2} \cdot \frac{1}{2} \left(\frac{m_{t_i}^2}{m_{t_i}^2 - m_{h_k}^2} \right) +$$
$$\frac{m_{t_i}^2}{2(m_{t_i}^2 - m_{t_j}^2)^2} \left(\frac{2m_{t_i}^2 - m_{t_j}^2}{m_{t_i}^2 - m_{h_k}^2} + \frac{m_{t_j}^2}{m_{t_j}^2 - m_{h_k}^2} \right) \times \ln \left(\frac{m_{t_i}^2}{m_{t_j}^2} \right)$$
$$- \frac{m_{h_k}^2}{2(m_{t_i}^2 - m_{h_k}^2)(m_{t_j}^2 - m_{h_k}^2)} \left[\frac{2m_{t_i}^2 - m_{h_k}^2}{m_{t_i}^2 - m_{h_k}^2} \ln \left(\frac{m_{t_j}^2}{m_{h_k}^2} \right) - \frac{m_{h_k}^2}{m_{t_j}^2 - m_{h_k}^2} \ln \left(\frac{m_{t_i}^2}{m_{h_k}^2} \right) \right]$$
$$- \frac{m_{h_k}^2}{2(m_{t_i}^2 - m_{t_j}^2)} \ln \left(\frac{m_{t_i}^2}{m_{t_j}^2} \right) \left(\frac{m_{t_i}^2}{(m_{t_i}^2 - m_{h_k}^2)^2} - \frac{m_{t_j}^2}{(m_{t_j}^2 - m_{h_k}^2)^2} \right) \Bigg] \Bigg\}. \tag{3.37}$$

The summation indices satisfy $i, j \in \{1, 2, \ldots, 5\}$ and $k \in \{1, 2, 3\}$, as they respectively enumerate the LW top and Higgs partners. The η_k parameters are those of Eq. (2.13). In the limit $m_{h_1} \to 0$, $m_{h_{2,3}} \to \infty$, Eq. (3.37) reduces to Eq. (A6) of Ref. [57]. This expression in turn reduces to Eq. (3.33) for $m_t \equiv m_{t_t} \ll m_{t_{2,3}}$. Alternately, Eq. (3.37) reduces to Eq. (3.32) in the limit $m_{t_{2,3}}, m_{h,3} \to \infty$, which is effectively the case in Ref. [59].

3.3 Analysis

We use the definition of the post-LEP oblique parameters given in Eqs. (3.4)-(3.7). As discussed above, the tree-level LW contributions to W and Y are sufficient at this order of precision (and provide the best bounds on LW gauge boson masses), whereas the biggest contributions to $\hat{S}$ and $\hat{T}$ are given by fermion loops (since they vanish at tree level). Since the sums in Eq. (3.10) also include SM fermions, we define the parameters

$$\hat{S}_{\text{new}} \equiv \hat{S} - \hat{S}_{\text{SM}}, \quad \hat{T}_{\text{new}} \equiv \hat{T} - \hat{T}_{\text{SM}}, \tag{3.38}$$

to indicate deviations from a purely-SM prediction. Any appearance of $\hat{S}$, $\hat{T}$ is to be understood as $\hat{S}_{\text{new}}$, $\hat{T}_{\text{new}}$ in the following discussion.

As a benchmark for the magnitude of new physics effects, we list here for convenience the predictions

$$\hat{S}_{\rm SM} = -1.98 \times 10^{-3}, \quad \hat{T}_{\rm SM} = +9.25 \times 10^{-3}. \tag{3.39}$$

Ref. [67] shows the measured values of $\hat{S}$, $\hat{T}$, W, and Y to all be $\mathcal{O}(10^{-3})$, and they are correlated. However, for simplicity, we use the values given in Table 4 of Ref. [67], along with their 2σ uncertainties[6]

$$10^3\hat{S} = 0.0 \pm 2.6, \tag{3.40}$$

$$10^3\hat{T} = 0.1 \pm 1.8, \tag{3.41}$$

$$10^3 W = -0.4 \pm 1.6, \tag{3.42}$$

$$10^3 Y = 0.1 \pm 2.4. \tag{3.43}$$

We add the bound on $\delta g_L^{b\bar{b}}$ from Eq. (3.31) to this list, thereby constraining LW fermion masses and scalar masses as well.

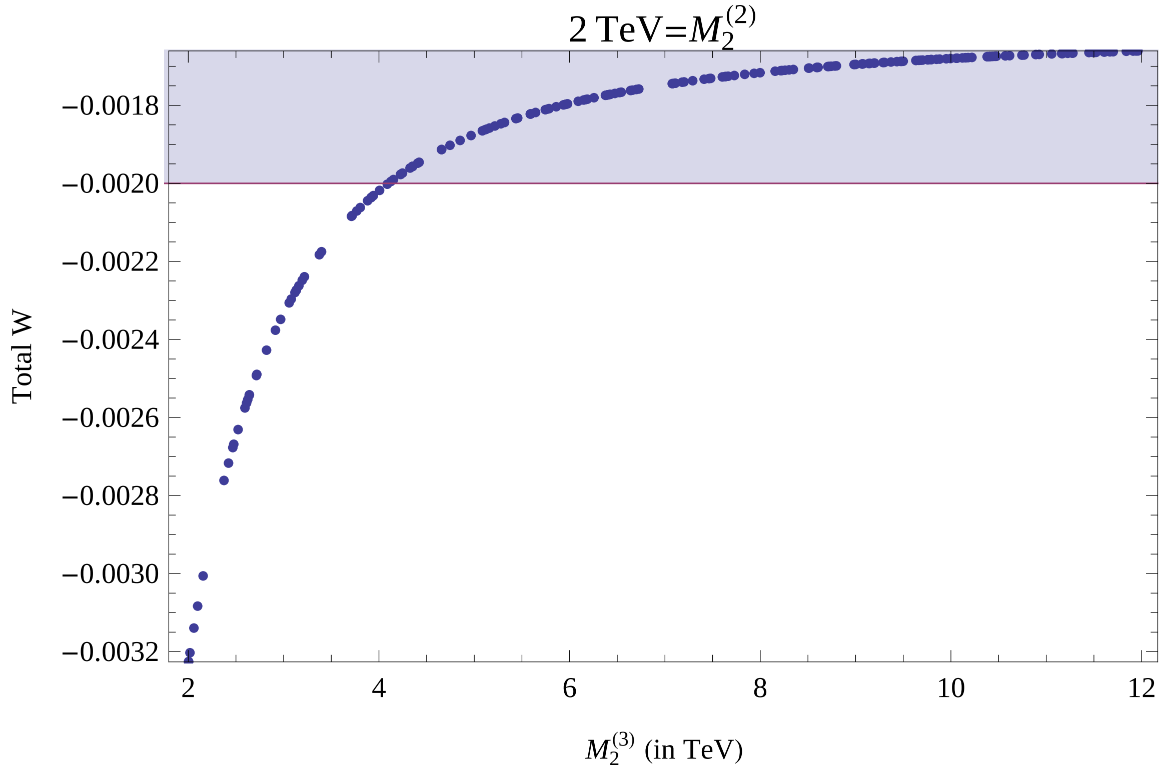
The $n_{\rm LW} = 2$ and $n_{\rm LW} = 3$ gauge bosons contribute additively in Eqs. (3.6) and (3.7). Therefore, the introduction of new LW states in the $N = 3$ theory can only serve to tighten the bounds. In Fig. 3.3, we see that taking $M_2^{(2)} = 2$ TeV requires $M_2^{(3)} \gtrsim 4$ TeV, which is likely to be outside the discovery potential of the current LHC. The discovery scenario described in Ref. [1] of $M_2^{(2)} = 2.0$ TeV, $M_2^{(3)} = 2.5$ TeV is unlikely unless the bounds on W are not as stringent as those given in Eq. (3.42). Likewise, for Y, Fig. 3.3 suggests that $M_1^{(2)} = 1.8$ TeV is possible for $M_1^{(3)} \gtrsim 3.5$ TeV. If instead the $n_{\rm LW} = 2$ and $n_{\rm LW} = 3$ masses are taken to be (nearly) equal, a universal mass $\gtrsim 2.5$ TeV remains possible.

The constraints from $\hat{S}$ are far less restrictive, owing to the fact that the LWSM adds in fermions with vectorlike masses, rather than chiral fermions. The only contri-

[6] We emphasize that the the quantities in Eqs. (3.40)-(3.43) are bounds on the deviation of oblique corrections from their SM predictions.

bution to $\hat{S}$ from the LWSM arises through diagonalization effects (*i.e.*, sensitivity to m_t). Assuming for simplicity the degenerate case $M_{q2} = M_{t2} = M_{b2}$ taken in Ref. [57], and taking its logical extension $M_{q3} = M_{t3} = M_{b3}$, no meaningful constraint exists on the fermion mass parameters M_{q2} or M_{q3}. The bounds from $\hat{T}$ are much more interesting. Ref. [57] required $M_{q2} \geq 1.5$ TeV in order for $\hat{T}$ to meet the 2σ bound (see Fig. 3.4, left inset), providing one of the strongest constraints on LW quark masses. The present work commenced under the hypothesis that the opposing signs of the $n_{\rm LW} = 2$ and $n_{\rm LW} = 3$ quark propagators would allow for a near-complete cancellation of their loop contributions to the vacuum polarization diagrams. However, the full result requires much greater care in the analysis: while the $n_{\rm LW} = 2$ and $n_{\rm LW} = 3$ loops do indeed cancel to a large extent, the propagating fermions in the loops are mass eigenstates. The diagonalization procedure used in obtaining these eigenstates not only shifts the mass eigenvalues of the heavy states away from M_{q2} and M_{q3}, but it significantly increases the contribution of the $n_{\rm LW} = 1$ (SM) quarks to $\hat{T}$. The effect is highly pronounced due to the numerically large top-Yukawa coupling, y_t. It serves to push the full value of $\hat{T}$ slightly further from its measured central value, thus setting the bar for an allowable M_{q2} to be slightly larger than before the inclusion of the $n_{\rm LW} = 3$ states. The effect, though counter-productive to the original goal, is not extreme; from Fig. 3.4, we see that $M_{q2} = 1.5$ TeV remains viable for $M_{q3} \gtrsim 9$ TeV; increasing M_{q2} only slightly to 1.8 TeV allows M_{q3} to come down to ~ 2.8 TeV. This transition between very strong and very weak bounds on M_{q3} occurs over a very narrow range of M_{q2} values.

Finally, we analyze the constraints arising from $\delta g_L^{b\bar{b}}$, which in Ref. [57] provides the most stringent bounds on the quark partner masses, $M_{q2} \gtrsim 4$ TeV. Since the 2σ bound is a very shallow function of M_{q2} (see Fig. 3.5), it is possible for the full $N = 3$ LWSM to afford a looser bound with its increased parameter space and delicate



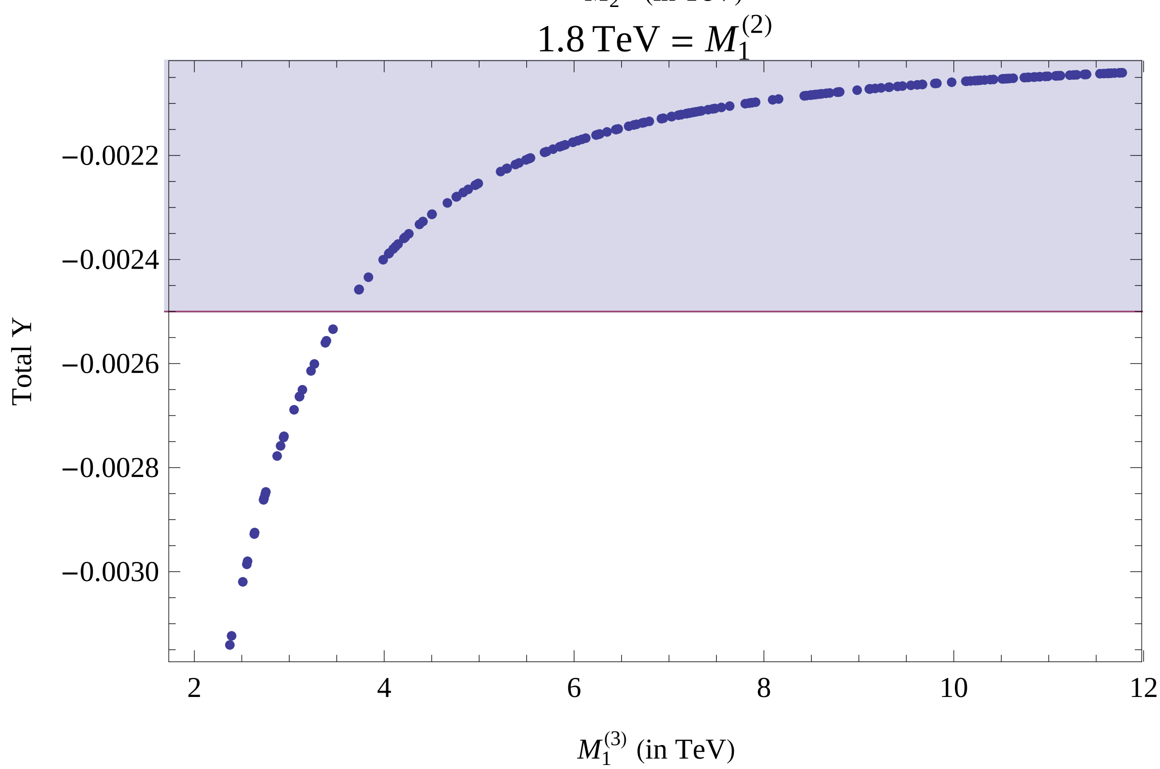


Figure 3.3: Bounds on LW gauge boson mass partners from the oblique parameters W and Y. The shaded area is experimentally allowed at 2σ.

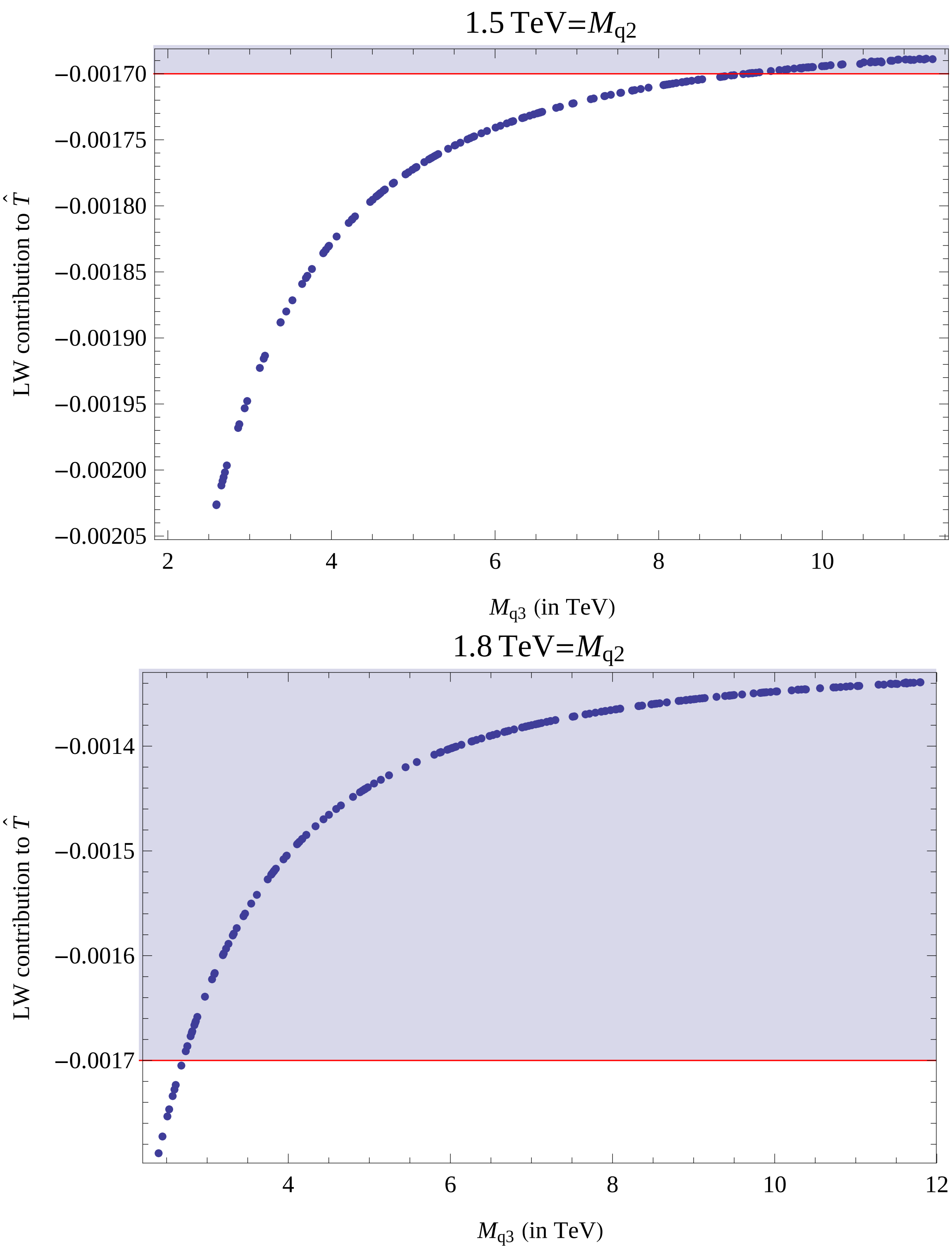


Figure 3.4: Bounds on the oblique parameter $\hat{T}$ in two scenarios, $M_{q2} = 1.5$ TeV and 1.8 TeV. The shaded area is experimentally allowed at 2σ.

cancellations. The $N = 3$ theory is used in the second inset of Fig. 3.5, and we see a substantial relaxation of the bounds on quark parameters; raising M_{q2} only slightly to 1.4 TeV allows $M_{q3} \gtrsim 2.3$ TeV. The $N = 3$ theory also has interesting implications for the Higgs sector. If the LW quark masses are assumed large enough to decouple, $\delta g_L^{b\bar{b}}$ provides a lower bound on the $n_{\rm LW} = 2$ scalar of $m_{h_2} \gtrsim 640$ GeV (see Fig. 3.6). Since mass diagonalization does not mix the charged-scalar mass parameters, including the $n_{\rm LW} = 3$ state leads to a dramatic cancellation. The second inset of Fig. 3.6 shows a scenario where $m_{h_2} = 400$ GeV and $m_{h_3} \lesssim 850$ GeV, while still satisfying the $\delta g_L^{b\bar{b}}$ constraint. When both LW quark and charged-scalar partners are included, the bounds again become more constrained, but there are still many interesting scenarios possible; the combined set $M_{q2} = 2.5$ TeV, $M_{q3} = 4$ TeV, $m_{h_2} = 400$ GeV, and $m_{h_3} = 600$ GeV satisfies constraints on $\delta g_L^{b\bar{b}}$ [59].

3.4 Discussion and Conclusions

Through the examination of both oblique and direct corrections to Standard Model observables, we have seen that the $N = 3$ Lee-Wick Standard Model offers a broad parameter space that retains consistency with experimental data while still providing a credible solution to the Standard Model hierarchy problem.

We see that the post-LEP oblique parameters W and Y require the $n_{\rm LW} = 2$ partners of the W and B bosons to be $\gtrsim 2.0$ and 1.8 TeV, respectively, and that the $n_{\rm LW} = 3$ partners must be substantially heavier; an alternate scenario has the two mass scales quasi-degenerate at a common scale of ~ 2.5 TeV.

The LW quark masses are constrained most heavily by custodial isospin ($\hat{T}$) and the $Zb\bar{b}$ coupling $g_L^{b\bar{b}}$ to be at least 1.5 TeV. One of the most interesting conclusions of the present work is that the $n_{\rm LW} = 3$ fermion loops do cancel some of the effects of the $n_{\rm LW} = 2$ fermion loops, but that the resultant mass diagonalization procedure

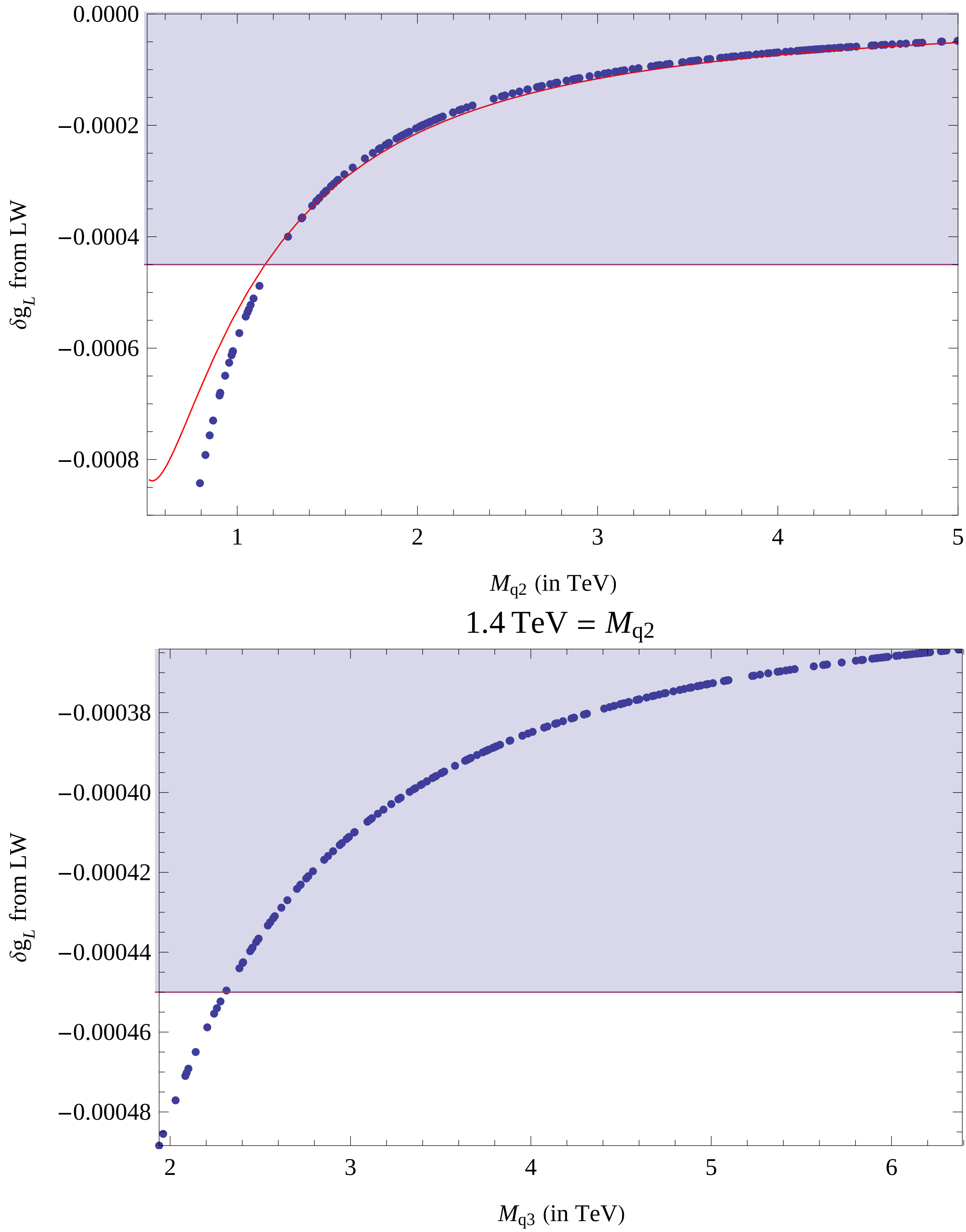


Figure 3.5: Bounds on $\delta g_L^{b\bar{b}}$ with LW t-partners. The first inset is an updated calculation within the $N = 2$ theory, and the red line is the leading order result of Eq. (3.33). The second inset presents the $N = 3$ calculation of the same observable, wherein M_{q2} is fixed and M_{q3} is allowed to vary. The shaded area is experimentally allowed at 2σ.

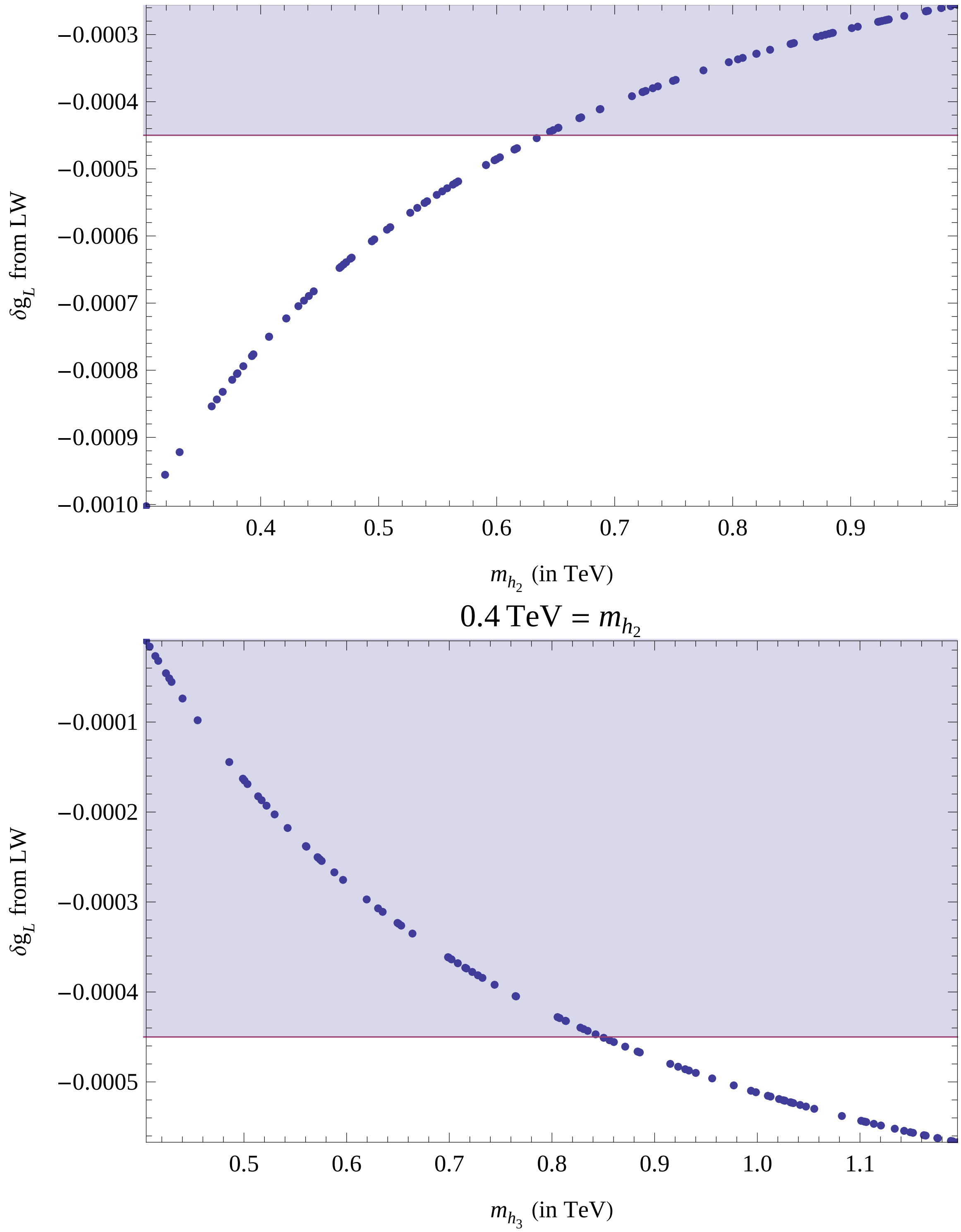


Figure 3.6: Bounds on $\delta g_L^{b\bar{b}}$ with one (first inset, $N = 2$) and two (second inset, $N = 3$) charged scalar LW partners, with masses $m_{h_{2,3}}$. The shaded area is experimentally allowed at 2σ.

amplifies the SM contribution, providing extra tension. Even so, $n_{\rm LW} = 2$ quark masses around $M_{q2} \gtrsim 1.8$ TeV remain viable if the $n_{\rm LW} = 3$ quarks are somewhat heavier ($\gtrsim 2.8$ TeV).

There is significantly less constraint on the masses of the charged Higgs partners from these observables; in the limit of decoupled LW t-partners, these can be as low as several hundred GeV, and sit well within the discovery potential of the LHC. However, a more complete analysis ought to be conducted, incorporating the possible tension in LW predictions for $b \to s\gamma$ and $B\bar{B}$ mixing.

Chapter 4

THE LEE-WICK STANDARD MODEL AT FINITE TEMPERATURE

In this chapter, we pursue the thermal properties of Lee-Wick (LW) fields. These properties are of primary concern to the study of early-universe cosmology, where high-energy particles exert a powerful influence on the phase transitions thought to be responsible for the presence of the matter we observe today. The thermal bath allows very high-energy states to be sampled, and many of the most interesting energy régimes lie beyond the current reach of collider experiments. The cosmological approach offers a second line of attack for the theorist looking for an observable fingerprint of any particular theory beyond the Standard Model (BSM). Even if collider experiments fail to access very high energy particles, one may still study relics of a hot, dense Universe defined by a homogenous and isotropic thermal bath at Planck-scale temperatures. These high-energy particles inaccessible to terrestrial experiments could still have cosmological implications.

One well-studied topic is the electroweak phase transition (EWPhT), which could be responsible for the dominance of matter over antimatter: this phenomenon is termed baryogenesis [77, 78, 79, 80]. However, the Standard Model (SM) by itself has long been known to be insufficient for producing a net baryon number, as the EWPhT with only SM fields is not strongly first-order [81].[1] An opportunity arises for BSM theories to extend the SM so as to allow baryogenesis, which is done by ensuring two criteria: first, that the shape of the Higgs potential at the phase transition erects a sufficiently high barrier between broken and unbroken symmetry phases,

[1] For a pedagogical introduction to thermal field theory and its application to phase transitions, see [82].

and second, that sphalerons and other topological processes are turned off once the requisite baryon density has been produced.

We now turn to the thermal formulation of the LWSM, ultimately with an eye towards calculating the EWPhT. Others have considered the LWSM at finite temperature [83, 84, 85], and much is known of the cosmological effects of LW fields [86, 87, 88, 89]. In this chapter, we adopt a different approach to the quantization of LW fields, based on Ref. [2]. We use this new quantization scheme to allay extant doubts as to the validity of using LW fields in a thermal bath where negative-norm states can be accessed, and endeavor to show that the Hamiltonian of the theory is well-behaved and bounded from below (see Appendix B for a self-contained derivation).

However, we also wish to change course with respect to the development of the dissertation up to this point. Whereas we have previously been concerned with $N = 3$ LW theories, where the SM Lagrangian is augmented by five- and six-derivative operators (for fermions and bosons, respectively) and there exist three physical poles in the complex p^0 plane for each propagator, we consider the $N = 2$ theory in this thermodynamic calculation. This choice is motivated by pragmatic concerns, rather than by a fundamental reversal in judgment as to what sort of LW theory we expect to be manifested in Nature. Since no prior calculation of the EWPhT in the LWSM exists, we attempt to do so now in the $N = 2$ theory as a proof of principle. If a strongly first-order phase transition is not to be found within the allowed parameter space of the $N = 2$ LWSM, then it would be a fool's errand to attempt an even harder calculation in the $N = 3$ LWSM with still more parameters to constrain. If an $N = 2$ result appears promising, however, one could then move on with the $N = 3$ theory. Let us now turn to calculating the thermal properties of an ensemble of LW fields.

4.1 Thermodynamics of Lee-Wick Theories

In this section, we discuss the foundational question of precisely how a sensible LW theory is to be described at finite temperature, and how its thermodynamic properties should be calculated. The most pressing concern can be stated succinctly:

Are Lee-Wick particles accessible to the thermal bath of an equilibrium thermodynamic system?

Put another way, one might well ask: Are LW particles real? When working at zero temperature, we are by this point comfortable in establishing future boundary conditions and employing the CLOP [46] prescription for removing exponentially-growing modes. However, it is not immediately clear if this program is extensible to the case of nonzero temperature. Thus, two possibilities emerge: either

1. the thermal system can access states containing explicit LW particles, or

2. the thermal system can only explore states from which explicit LW particles are absent.

These options were explored in Refs. [83] and [85], respectively. Though they obtained similar expressions for the free energy of a gas of LW particles, we argue that the two pictures are not equivalent. We show in § 4.1.1 that no self-consistent calculation using an ideal gas of LW particles seems to reproduce the results of Refs. [83, 85].

This dissertation advances the second hypothesis: that the more sensible physical picture is one in which the scattering of positive-norm particles is affected by LW resonances, but the system may not reach a state in which real LW particles exist in the ensemble. Expressed in thermodynamic language, this means that we do not calculate a partition function for a system containing LW particles; the partition function for positive-norm particles is affected through the modifications of couplings and reaction rates engendered by the presence of LW resonances, the latter being viewed strictly as a scattering phenomenon.

4.1.1 Ideal Gas of Lee-Wick Particles

We consider option (1) above for a case of SM and LW particles. In the limit of weak coupling, the partition functions are separable; we define the partition function Z by requiring that the density matrix,

$$\hat{\rho} = \frac{1}{Z}\exp(-\beta\hat{H}), \tag{4.1}$$

is properly normalized[2]. In the absence of interactions, the spectrum of the Hamiltonian, $\hat{H}$, is simply that of multi-particle momentum eigenstates: the vacuum $|0\rangle$, single-particle states $|\mathbf{p}\rangle$, or multi-particle states of the form $|\mathbf{p}_1\mathbf{p}_2\ldots\mathbf{p}_N\rangle$, with the appropriate (anti-)symmetrization to account for the spin-statistics of the particles involved. We can separate the multi-particle states as

$$|\mathbf{p}_1\,\mathbf{p}_2\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{p}_1\rangle \otimes |\mathbf{p}_2\rangle + \eta_S|\mathbf{p}_2\rangle \otimes |\mathbf{p}_1\rangle\right)\ , \tag{4.2}$$

where $\eta_S = -1$ (fermions) or $+1$ (bosons). The single-particle states are defined to be eigenstates of the Hamiltonian, *i.e.* $\hat{H}|\mathbf{p}\rangle = E_{\mathbf{p}}|\mathbf{p}\rangle$, where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. These expressions implicitly use the quantization convention $\eta_C = +1$, where η_C is defined in Eq. (B.4).

We will invoke the notation and results of Appendix B once again with the norm of the states under consideration, η_N (defined below Eq. (B.11)). The norms of the

[2]We provide a definition for "proper" normalization below.

first few momentum eigenstates are given by

$$\begin{aligned}
&\big\langle 0\big|0\big\rangle = 1\,, \\
&\big\langle \mathbf{p}\big|\mathbf{q}\big\rangle = \eta_N(2\pi)^3\,2E_{\mathbf{p}}\,\delta(\mathbf{p}-\mathbf{q})\,, \\
&\big\langle \mathbf{p}_1\,\mathbf{p}_2\big|\mathbf{q}_1\,\mathbf{q}_2\big\rangle = (2\pi)^6\,2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}\left[\delta(\mathbf{p}_1-\mathbf{q}_1)\,\delta(\mathbf{p}_2-\mathbf{q}_2)+\eta_S\,\delta(\mathbf{p}_1-\mathbf{q}_2)\,\delta(\mathbf{p}_2-\mathbf{q}_1)\right]\,,
\end{aligned} \tag{4.3}$$

with the pattern continuing for states with still-higher particle number. Note that states with even particle number always possess positive norm. We also see that the eigenvalues and expectation values associated with negative-norm states can differ by a sign:

$$\int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_{\mathbf{q}}}\big\langle \mathbf{p}\big|\hat{H}\big|\mathbf{q}\big\rangle = \eta_N E_{\mathbf{p}}, \tag{4.4}$$

whereas $\hat{H}\big|\mathbf{p}\big\rangle = E_{\mathbf{p}}\big|\mathbf{p}\big\rangle$, as above. This distinction is pertinent to the calculation of the partition function, by way of the density matrix. If we normalize the density matrix by requiring

$$\mathrm{Tr}\hat{\rho} = 1, \tag{4.5}$$

then we can compute Z in a straightforward manner by summing over all states present in Eq. (4.5):

$$\begin{aligned}
Z = &\big\langle 0\big|e^{-\beta\hat{H}}\big|0\big\rangle + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\big\langle \mathbf{p}\big|e^{-\beta\hat{H}}\big|\mathbf{p}\big\rangle \\
&+ \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_{\mathbf{q}}}\big\langle \mathbf{p},\mathbf{q}\big|e^{-\beta\hat{H}}\big|\mathbf{p},\mathbf{q}\big\rangle + \ldots.
\end{aligned} \tag{4.6}$$

These terms will alternate in sign for the case of $\eta_N = -1$. A new complication arises: since the eigenvalues of $\hat{\rho}$ (which are effectively eigenvalues of $\hat{H}$) are not strictly positive, it could be the case that the sum of the eigenvalues of $\hat{\rho}$ exceeds unity, while condition (4.5) still holds. It is not immediately clear how to interpret such a density matrix. We therefore propose an alternative condition for normalizing the density matrix, where we require

$$\mathrm{Tr}'\hat{\rho} \equiv \sum_{\text{eigs}} \hat{\rho} = 1\,. \tag{4.7}$$

The primed trace is obtained by summing over the eigenvalue spectrum of the operator. The norm of the states involved are irrelevant to the final result, and the outcome is the standard partition function for an ideal gas. It is not immediately clear whether or not the sum in Eq. (4.7) is finite; such a claim amounts to asserting that the conditionally convergent sum in Eq. (4.5) converges absolutely. We choose to go forward with the calculation of the thermal properties of the LW ideal gas, rather than ruminate over which definition of the density matrix is the "true" one; forthcoming results will firmly answer this question without the need for further speculation. We now proceed with the calculation for both pictures, in order to demonstrate the ramifications of treating LW particles as thermally accessible.

First, we calculate the partition function using the normalization condition (4.5). It is convenient to perform a few standard transformations (viz. Ref. [90]) and work in a different basis, discretizing the momentum by imposing periodic boundary conditions. We write the resultant Hamiltonian, $\hat{H} = \sum_{\mathbf{p}} \hat{h}_{\mathbf{p}}$, as a sum of single-particle Hamiltonians $\hat{h}_{\mathbf{p}} = E_{\mathbf{p}}\hat{N}_{\mathbf{p}}$. The number operator, $\hat{N}_{\mathbf{p}}$, has the spectrum

$$\hat{N}_{\mathbf{p}}\big|n_{\mathbf{q}}\big\rangle = n_{\mathbf{p}}\delta_{\mathbf{p},\mathbf{q}}\big|n_{\mathbf{p}}\big\rangle. \tag{4.8}$$

The state $\big|n_{\mathbf{p}}\big\rangle$ contains n particles, each of momentum $\mathbf{p}$. We can use this new basis to rewrite the partition function (recalling Eq. (4.5)) as

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \mathrm{Tr} \prod_{\mathbf{p}} e^{-\beta E_{\mathbf{p}}\hat{N}_{\mathbf{p}}} = \prod_{\mathbf{p}} \sum_{n_{\mathbf{p}}=0}^{n_{\max}} \big\langle n_{\mathbf{p}}\big|e^{-\beta E_{\mathbf{p}}\hat{N}_{\mathbf{p}}}\big|n_{\mathbf{p}}\big\rangle\,, \tag{4.9}$$

where $n_{\max} = \infty$ for bosons, or $= 1$ for fermions. Recalling that the norms of number-operator eigenstates are $\langle n_{\mathbf{p}} | n_{\mathbf{p}} \rangle = (\eta_N)^{n_{\mathbf{p}}}$, we obtain

$$Z = \prod_{\mathbf{p}} \sum_{n_{\mathbf{p}}=0}^{n_{\max}} \left(\eta_N e^{-\beta E_{\mathbf{p}}}\right)^{n_{\mathbf{p}}} . \tag{4.10}$$

From the partition function, we may calculate the free energy $\mathcal{F}$ in the usual way. We take the logarithm of Z, and allow the ensuing sum to become an integral in the continuum limit:

$$\mathcal{F} = -(\beta V)^{-1} \ln Z = -\beta^{-1} \int \frac{d^3p}{(2\pi)^3} \ln \left[\sum_{n_p=0}^{n_{\max}} \left(\eta_N e^{-\beta E_{\mathbf{p}}}\right)^{n_{\mathbf{p}}} \right] . \tag{4.11}$$

. We may evaluate the sum separately for bosons and fermions, which gives

$$\mathcal{F} = \begin{cases} \beta^{-1} \int \frac{d^3p}{(2\pi)^3} \ln \left(1 - \eta_N \, e^{-\beta E_{\mathbf{p}}}\right) & \text{bosons}\,, \\ -\beta^{-1} \int \frac{d^3p}{(2\pi)^3} \ln \left(1 + \eta_N \, e^{-\beta E_{\mathbf{p}}}\right) & \text{fermions}\,; \end{cases} \tag{4.12}$$

these expressions may be combined into

$$\mathcal{F} = \eta_S \beta^{-1} \int \frac{d^3p}{(2\pi)^3} \ln \left(1 - \eta_S \eta_N e^{-\beta E_{\mathbf{p}}}\right) , \tag{4.13}$$

where once again $\eta_S = +1(-1)$ for bosons (fermions). Had we instead worked with the alternative normalization of Eq. (4.7), we would have obtained

$$\mathcal{F}' = \eta_S \beta^{-1} \int \frac{d^3p}{(2\pi)^3} \ln \left(1 - \eta_S e^{-\beta E_{\mathbf{p}}}\right) . \tag{4.14}$$

Note the absence of any η_N factors in Eq. (4.14). This is the standard expression for the free energy of an ideal gas.

We summarize these results in Table 4.1, also recording the scaled versions of the entropy density $s = -\partial \mathcal{F}/\partial T$ and energy density[3] $\rho = \mathcal{F} + Ts$. We express these quantities in the high-temperature limit, $\beta^2 m^2 \ll 1$, where the leading terms

[3] We will only ever use $\hat{\rho}$ to refer to the density matrix and ρ to refer to energy density, so no confusion need arise.

	η_S	η_N	$\beta^4\mathcal{F}$	$\beta^3 s$	$\beta^4\rho$
SM Boson	$+1$	$+1$	$+c_{0b}+c_{1b}\varepsilon$	$-4c_{0b}-2c_{1b}\varepsilon$	$-3c_{0b}-c_{1b}\varepsilon$
LW Boson ($\mathrm{Tr}'\hat{\rho}=1$)	$+1$	-1	$+c_{0b}+c_{1b}\varepsilon$	$-4c_{0b}-2c_{1b}\varepsilon$	$-3c_{0b}-c_{1b}\varepsilon$
LW Boson ($\mathrm{Tr}\hat{\rho}=1$)	$+1$	-1	$-c_{0f}-c_{1f}\varepsilon$	$+4c_{0f}+2c_{1f}\varepsilon$	$+3c_{0f}+c_{1f}\varepsilon$
SM Fermion	-1	$+1$	$+c_{0f}+c_{1f}\varepsilon$	$-4c_{0f}-2c_{1f}\varepsilon$	$-3c_{0f}-c_{1f}\varepsilon$
LW Fermion ($\mathrm{Tr}'\hat{\rho}=1$)	-1	-1	$+c_{0f}+c_{1f}\varepsilon$	$-4c_{0f}-2c_{1f}\varepsilon$	$-3c_{0f}-c_{1f}\varepsilon$
LW Fermion ($\mathrm{Tr}\hat{\rho}=1$)	-1	-1	$-c_{0b}-c_{1b}\varepsilon$	$+4c_{0b}+2c_{1b}\varepsilon$	$+3c_{0b}+c_{1b}\varepsilon$

Table 4.1: The thermodynamic properties of an ideal gas of SM or LW bosons or fermions in the high-temperature limit $\beta^2 m^2 \equiv \varepsilon \ll 1$. For the LW particles, the density matrix is normalized using either (4.5) or (4.7), as indicated. Higher-order terms in ε are dropped. The coefficients are $c_{0b} \equiv -\mathrm{Li}_4(+1)/\pi^2 = -\pi^2/90$, $c_{1b} \equiv \mathrm{Li}_2(+1)/4\pi^2 \equiv 1/24$, $c_{0f} \equiv \mathrm{Li}_4(-1)/\pi^2 = -7\pi^2/720 = (7/8)c_{0b}$, and $c_{1f} \equiv -\mathrm{Li}_2(-1)/4\pi^2 = 1/48 = (1/2)c_{1b}$.

may be easily extracted and compared. Some of these results merit elaboration. For instance, when convention (4.7) is used, the gas of LW particles has thermodynamic functions equivalent to those of a SM-only gas. This is not surprising; after all, Eq. (4.14) contains no η_N factors to herald the presence of negative-norm states. Another interesting result is the swap between bosonic and fermionic results for the LW gas when calculated with the convention of Eq. (4.5); an ideal gas of LW bosons has the free energy associated with an ideal gas of SM fermions, but with the overall sign changed.

The overall sign change found in the entropy and energy of the $\mathrm{Tr}\hat{\rho} = 1$ states certainly constitutes a counter-intuitive result. We summarize the salient findings as follows:

$$\mathrm{Tr}'\,\hat{\rho} = 1: \quad \mathcal{F}[\text{LW boson / fermion of mass } m\,] = +\mathcal{F}[\text{SM boson / fermion of mass } m\,], \tag{4.15}$$

$$\mathrm{Tr}\,\hat{\rho} = 1: \quad \mathcal{F}[\text{LW boson / fermion of mass } m\,] = -\mathcal{F}[\text{SM fermion / boson of mass } m\,]\,. \tag{4.16}$$

Both of these results differ from a prior calculation in Ref. [85], which claims that the free energy, entropy, *etc.* of a LW gas is precisely opposite in sign to that of a SM gas of the same spin; in the language used above,

$$\mathcal{F}[\text{LW boson / fermion of mass } m\,] = -\mathcal{F}[\text{SM boson / fermion of mass } m\,]\ . \tag{4.17}$$

The authors of REf. [85] obtain the above equivalence by treating positive-energy, negative-norm LW particles as though they were negative-energy, positive-norm particles. However, if these are taken to be real states *prima facie*, it is not clear how to treat a system whose energy spectrum is unbounded both from above (due to positive-energy states) and below (due to their negative-energy counterparts). This inherent ambiguity results in a sick theory, and we do not see how one can justifiably make the analytic continuation necessary to define the partition function.

The equivalent LW ideal gas composed of positive-energy, negative-norm states must also lead to instabilities; they arise through states of opposing norm combining to form zero-norm states, which necessarily lead to runaway modes [35, 42]. We therefore arrive at the conclusion that a consistent formulation of LW theories, in which there exist no exponentially-growing or -decaying modes and the S-matrix is unitary, disallows the presence of a LW ideal gas. A unitary theory with a bounded Hamiltonian demands that LW particles enter the picture only as scattering resonances.

4.1.2 *Lee-Wick Particles as Resonances*

We now narrow the focus of the discussion to option (2), wherein LW particles only arise through scattering resonances. In order to build this requirement into the language of the theory, we shall define the expectation value of some operator,

$$\left\langle \hat{\mathcal{O}} \right\rangle = \mathrm{Tr}(\hat{\mathcal{O}}\hat{\rho}), \tag{4.18}$$

such that the only states to be summed over are those which are in the subset of the Hilbert space that contain no LW particles. These states are identical to those which are taken to zero by the LW annihilation operators, $a_{\mathbf{p}}$. The LW particles make their presence known by interacting with SM particles, thereby influencing the spectrum of SM multi-particle states. Working in the limit of weak coupling, we can treat the interactions perturbatively and express the free energy of the full theory as

$$\mathcal{F}[\text{LW theory}] = \mathcal{F}[\text{SM ideal gas}] + \Delta\mathcal{F}, \tag{4.19}$$

where the first term is the conventional calculation featuring a LW-only gas, and the second term collects the perturbations. These contain Yukawa and gauge couplings between SM and LW particles, and correspond numerically to the "two-loop" corrections in thermal field theory. Ordinarily, these would be dropped; however, when the SM particles scatter through LW resonances, the corrections become significant and must be re-summed.

We briefly review the work of Fornal, Grinstein, and Wise (FGW) [83] in their calculation of the free energy of a scalar LW toy model. This work relies on the formalism of calculating partition functions through S-matrix elements, developed by Dashen, Ma, and Bernstein (DMB) [91]. They derived the relationship

$$\Delta\mathcal{F} = -(\beta V)^{-1} \int dE\, e^{-\beta E}\, \frac{1}{4\pi i} \left[\mathrm{Tr}\, A S^{-1}(E) \overleftrightarrow{\frac{\partial}{\partial E}} S(E) \right]_c , \tag{4.20}$$

where $S(E)$ is the matrix element connecting two multi-particle states of energy E, c specifies that only connected diagrams are to be included, and the $\mathrm{Tr}A$ trace operator does the job of symmetrization (anti-symmetrization) for bosons (fermions). In order to start calculating S-matrix elements, FGW supply the HD Lagrangian[4],

$$\mathcal{L}_{\mathrm{HD}}^{N=2} = \frac{1}{2}(\partial_\mu \hat{\phi})^2 - \frac{1}{2M^2}(\partial^2 \hat{\phi})^2 - \frac{1}{2}m^2 \hat{\phi}^2 - \frac{g}{3!}\hat{\phi}^3. \tag{4.21}$$

This interaction term gives rise to the characteristic negative decay width,

$$\Gamma = \frac{-g^2}{32\pi M}\sqrt{1 - \frac{4m^2}{M^2}}, \tag{4.22}$$

as was seen in Eq. (1.75). The interaction term of Eq. (4.21) also allows $2 \to 2$ scattering through a single LW resonance, with the attending matrix element

$$\mathcal{M} = \frac{1}{2} \cdot \frac{-g^2}{E^2 - \mathbf{P}^2 - M^2 + iM\Gamma}, \tag{4.23}$$

where $S(E) = 1 - i\mathcal{T}(E)$, and

$$\begin{aligned} \left\langle \mathbf{p}_1\, \mathbf{p}_2 \middle| \mathcal{T}(E) \middle| \mathbf{q}_1\, \mathbf{q}_2 \right\rangle = {} & (2\pi)\delta(E - E_1 - E_2)(2\pi)^3 \\ & \times \delta(\mathbf{p}_1 + \mathbf{p_2} - \mathbf{q}_1 - \mathbf{q}_2)\mathcal{M}(E). \end{aligned} \tag{4.24}$$

From this stage, we can make the narrow-width approximation and evaluate the delta-function within Eq. (4.20) to find

$$\Delta\mathcal{F} = -\beta^{-1}\int \frac{d^3p}{(2\pi)^3}\, \ln\left(1 - e^{-\beta\sqrt{\mathbf{p}^2 + M^2}}\right). \tag{4.25}$$

This is precisely the form of $\mathcal{F}$ for an ideal gas of bosons, but with an overall minus sign. One interesting feature of the above derivation is the fact that it resembles an ideal gas at all. Indeed, the narrow-width approximation only works when resonances

[4] Of course, this Lagrangian requires an added $\hat{\phi}^4$ term in order for the potential to be bounded from below. In any case, the $\hat{\phi}^3$ calculation is easier, and the boundedness of $V(\hat{\phi})$ will not come into play.

are long-lived, and SM particles will access them through their mutual scattering. The LW states then appear as stable with respect to the characteristic time scale of scattering events, and contribute to the free energy as though they were stable [91, 92].

The appearance of the overall minus sign, however, is surprising. We have already shown that calculating the free energy of a LW ideal gas takes one of two forms, depending on one's choice of trace normalization (see Table 4.1). However, none of the possibilities given for $\mathcal{F}$ results in Eq. (4.25). There is a further complication with the analytic structure of Eq. (4.23). The overall minus sign is due to the fact that $\Gamma < 0$, and that $\Gamma \to 0^-$ differs from the $\Gamma \to 0^+$ limit by an overall sign. The lack of continuity here means that the free energy is nonanalytic at $\Gamma \propto g^2 = 0$. FGW then generalize from the toy theory to the fermionic case, again as resonances in the scattering amplitude. We shall not present a parallel calculation again, but this too results in a sign flip with respect to its SM counterpart. We summarize their results as

$$\begin{aligned}
&\Delta\mathcal{F}[\text{LW boson/fermion narrow resonance of mass M}] \\
&= \sigma\mathcal{F}[\text{SM boson/fermion ideal gas of mass M}]\Big|_{\sigma=-1}\,,
\end{aligned} \tag{4.26}$$

where we have established a sign placeholder, $\sigma = -1$. We refer to this observed relationship as the "LW sign flip."

There is cause for concern that the S-matrix formulation may not even be appropriate to the study of LW thermodynamics. The negative-norm states were not subtracted in the derivation of the results of FGW or DMB, although this is does not seem to be the result of any error after careful inspection. It was pointed out by Espinosa and Grinstein [93] that the result Eq. (**??**) leads to an unexpected breakdown in the well-known connection between symmetry restoration (in the thermal field theory) and improved UV behavior. This connection arises from the fact that the same diagrams which give $\mathcal{O}(T^2)$ contributions to the effective potential also contribute to the quadratic divergences of the SM at $T = 0$ [94]. For example, a bosonic field receiving the self-energy correction $\Delta m^2 = \kappa\Lambda^2/16\pi^2$ will also receive the thermal effective-mass term $\Delta m^2 = kT^2/12$. It follows that BSM theories purporting to solve the hierarchy problem ought to cancel out both of these contributions as well. But this is not the case if the $\sigma = -1$ sign flip is correct; see § 4.1.3. We therefore require the correct version of the theory to cancel off terms that are quadratic in both T and Λ, which amounts to the $\sigma = +1$ case:

$$\begin{aligned}&\Delta\mathcal{F}[\text{LW boson/fermion narrow resonance of mass M}]\\ &= \sigma\mathcal{F}[\text{SM boson/fermion ideal gas of mass M}]\Big|_{\sigma=+1}.\end{aligned} \tag{4.27}$$

Furthermore, models that solve the hierarchy problem tend to feature a slower onset of symmetry restoration from high-T terms.

After pursuing several methods of calculating $\mathcal{F}$ and investigating the implications of a LW ideal gas, we now have a clearer picture of what the correct answer must resemble: we require a formulation of the LWSM in which the boundary conditions are respected, and the LW particles only contribute through resonance effects. However, since some ambiguity remains as to the sign difference in Eqs. (**??**) and (4.27), we keep the factor $\sigma = \pm 1$ in the calculations to follow.

4.1.3 Thermal Effective Potential of a Lee-Wick Toy Model

We now present a calculation of the thermal effective potential, covering both bosons and fermions. Though the tools of thermal field theory are equal parts fascinating and useful, their pedagogical treatment is quite lengthy, and outside the scope of this dissertation. We encourage the interested reader to consult Refs. [82, 90] for a formal introduction.

Summing the complete set of vacuum-to-vacuum diagrams to one-loop order, we obtain the effective potential

$$V_{\text{eff}}^{(1L)}(\phi_c) = U(\phi_c) + \Delta V_0^{(1L)}(\phi_c) + \Delta V_T^{(1L)}(\phi_c, T)\,, \tag{4.28}$$

where ϕ_c is the running value of the scalar condensate and T is the temperature. These terms are, respectively, the classical potential energy $U(\phi_c)$, the $T = 0$ contribution

$$\Delta V_0^{(1L)}(\phi_c) = \delta V_{\text{c.t.}}(\phi_c) + \begin{cases} \frac{1}{2}\sum_b \int \frac{d^4 p_E}{(2\pi)^4} \ln\left[p_E^2 + m_b^2(\phi_c)\right] & \text{bosons}\,, \\ -\sum_f \int \frac{d^4 p_E}{(2\pi)^4} \ln\left[p_E^2 + m_f^2(\phi_c)\right] & \text{fermions}\,, \end{cases} \tag{4.29}$$

and the thermal correction

$$\Delta V_T^{(1L)}(\phi_c, T) = \begin{cases} \sum_b T \int \frac{d^3 p}{(2\pi)^3} \ln\left(1 - e^{-\beta\sqrt{\mathbf{p}^2 + m_b^2(\phi_c)}}\right) & \text{bosons}\,, \\ -\sum_f T \int \frac{d^3 p}{(2\pi)^3} \ln\left(1 + e^{-\beta\sqrt{\mathbf{p}^2 + m_f^2(\phi_c)}}\right) & \text{fermions}\,. \end{cases} \tag{4.30}$$

The counterterms $\delta V_{\text{c.t.}}(\phi_c)$ cancel off the temperature-independent divergences from standard perturbation theory, and we have Euclideanized the momentum to ensure convergence. The notation $m_{b,f}^2(\phi_c)$ reflects the fact that the particle masses are now functions of the background field. Note also the similarity between the thermal corrections and free energy calculated from the ideal gas viewpoint. We now present an explicit calculation within an HD theory in order to demonstrate the technology of thermal field theory.

Scalars

We begin with the HD Lagrangian

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2\Lambda_{\text{LW}}^2}(\partial^2\hat{\phi})^2 + \frac{1}{2}(\partial_\mu\hat{\phi})^2 - U(\hat{\phi})\,, \\ U(\hat{\phi}) &= \Omega + \frac{1}{2}\mu^2\hat{\phi}^2 + \frac{\lambda}{4}\hat{\phi}^4. \end{aligned} \tag{4.31}$$

Let $\phi_c = \langle\hat{\phi}\rangle$ be the condensate of the HD scalar, and expand about the minimum as $\hat{\phi}(x) = \phi_c + \hat{\varphi}(x)$. In order to obtain the effective mass, we expand the Lagrangian to quadratic order and obtain

$$\mathcal{L} \supset -\frac{1}{2}\hat{\varphi}\left(\frac{\partial^4}{\Lambda_{\rm LW}^2} + \partial^2 + m_{\hat{\varphi}}^2(\phi_c)\right)\hat{\varphi}\,, \tag{4.32}$$

where

$$m_{\phi}^2(\phi_c) \equiv U''(\phi_c) = \mu^2 + 3\lambda\phi_c^2. \tag{4.33}$$

We know that the HD theory contains two degrees of freedom; hence, we ought to expect two poles in its propagator:

$$D_{\hat{\varphi}}(p) = i\left(-\frac{p^4}{\Lambda_{\rm LW}^2} + p^2 - m_{\varphi}^2(\phi_c)\right)^{-1} = \frac{\Lambda_{\rm LW}^2}{m_{\tilde{\varphi}}^2 - m_{\varphi}^2}\left(\frac{i}{p^2 - m_{\varphi}^2} - \frac{i}{p^2 - m_{\tilde{\varphi}}^2}\right)\,, \tag{4.34}$$

where we identify

$$\begin{aligned}
&\text{Positive-Norm Pole:} \qquad m_{\varphi}^2(\phi_c) \equiv \tfrac{\Lambda_{\rm LW}^2}{2}\left(1 - \sqrt{1 - \tfrac{4m_{\hat{\varphi}}^2(\phi_c)}{\Lambda_{\rm LW}^2}}\right)\,,\\
&\text{Negative-Norm Pole:} \qquad m_{\tilde{\varphi}}^2(\phi_c) \equiv \tfrac{\Lambda_{\rm LW}^2}{2}\left(1 + \sqrt{1 - \tfrac{4m_{\hat{\varphi}}^2(\phi_c)}{\Lambda_{\rm LW}^2}}\right)\,.
\end{aligned} \tag{4.35}$$

Crucial to this analysis is a choice of phase for the Wick rotation, itself requiring a consistent pole prescription for picking up poles in the complex$-p^0$ plane. See § B.3 for a thorough calculation of these important properties. We are now at liberty to recapitulate our computation for the effective potential:

$$\begin{aligned}
V_{\rm eff}^{(1L)}(\phi_c, T) = U(\phi_c)&\\
+ \Bigg[\delta V_{\rm c.t.}(\phi_c) + \frac{1}{2}\int\frac{d^4p_E}{(2\pi)^4}&\Big(\ln[p_E^2 + m_{\varphi}^2(\phi_c)] + \ln[p_E^2 + m_{\tilde{\varphi}}^2(\phi_c)]\Big)\Bigg]\\
+ \frac{T^4}{2\pi^2}\Big[J_B(m_{\varphi}^2/T^2) &+ \sigma J_B(m_{\tilde{\varphi}}^2/T^2)\Big]\,,
\end{aligned} \tag{4.36}$$

where we have defined the bosonic thermal function as

$$J_B(y) \equiv \int_0^{\infty} dx\, x^2\, \ln\left(1 - e^{-\sqrt{x^2+y}}\right)\,, \tag{4.37}$$

and kept the index $\sigma = \pm 1$ attending the thermal correction due to the LW pole. The set of counterterms,

$$\delta V_{\text{c.t.}}(\phi_c) = \delta\Omega + \frac{1}{2}\delta\mu^2\phi_c^2 + \frac{\delta\lambda}{4}\phi_c^4 \,, \tag{4.38}$$

will be suitably constrained by whatever renormalization conditions are imposed in the $T = 0$ theory.

Fermions

Another important LW contribution arises from the HD fermionic Lagrangian

$$\mathcal{L} = \bar{\hat{\psi}} \left(i\frac{\not{\partial}^3}{\Lambda_{\text{LW}}^2} + i\not{\partial} - \lambda\hat{\phi} \right) \hat{\psi} + \mathcal{L}_{\hat{\phi}} \,, \tag{4.39}$$

wherein we have explictly included a Yukawa interaction term, giving mass to the fermion once $\hat{\phi}$ has attained its condensate value. We readily obtain the propagator

$$\begin{aligned} D_{\hat{\psi}}(p) &= i\left(-\frac{\not{p}^3}{\Lambda_{\text{LW}}^2} + \not{p} - m_{\hat{\psi}}(\phi_c) \right)^{-1} \\ &= +\frac{\Lambda_{\text{LW}}^2}{(m_{\tilde{\psi}_1} - m_\psi)(m_\psi - m_{\tilde{\psi}_2})} \cdot \frac{i}{\not{p} - m_\psi} \\ &\quad - \frac{\Lambda_{\text{LW}}^2}{(m_{\tilde{\psi}_1} - m_\psi)(m_{\tilde{\psi}_1} - m_{\tilde{\psi}_2})} \cdot \frac{i}{\not{p} - m_{\tilde{\psi}_1}} \\ &\quad - \frac{\Lambda_{\text{LW}}^2}{(m_\psi - m_{\tilde{\psi}_2})(m_{\tilde{\psi}_1} - m_{\tilde{\psi}_2})} \cdot \frac{i}{\not{p} - m_{\tilde{\psi}_2}} \,, \end{aligned} \tag{4.40}$$

where we obtain for the one positive and two negative norm poles[5]:

$$\begin{aligned} &\text{Positive-Norm Pole:} && m_\psi(\phi_c) \equiv \Lambda_{\text{LW}}^2\sqrt{\tfrac{2}{3}\left(1 - \cos\tfrac{\theta}{3}\right)} \,, \\ &\text{Negative-Norm Pole:} && m_{\tilde{\psi}_1}(\phi_c) \equiv \Lambda_{\text{LW}}^2\sqrt{\tfrac{2}{3}\left(1 + \cos\tfrac{\theta+\pi}{3}\right)} \,, \\ &\text{Negative Norm Pole:} && m_{\tilde{\psi}_2}(\phi_c) \equiv -\Lambda_{\text{LW}}^2\sqrt{\tfrac{2}{3}\left(1 + \cos\tfrac{\theta-\pi}{3}\right)} \,, \end{aligned} \tag{4.41}$$

[5] Recall from § A.2 that each HD fermion degree of freedom requires the introduction of two LW particles.

and

$$
\begin{aligned}
\theta &\equiv \arctan \tfrac{2\sqrt{\alpha(1-\alpha)}}{1-2\alpha}\,, \quad 0 \le \theta < \pi\,, \\
\alpha &\equiv \tfrac{27}{4}\tfrac{m_{\hat{\psi}}^2}{\Lambda_{\rm LW}^2}\,, \\
m_{\hat{\psi}} &\equiv \lambda\phi_c\,.
\end{aligned} \tag{4.42}
$$

We now construct the thermal effective potential to one-loop order:

$$
\begin{aligned}
V_{\rm eff}^{(1\rm L)}(\phi_c, T) = U(\phi_c) &+ \delta V_{\rm c.t.}(\phi_c) \\
&- \int \frac{d^4p_E}{(2\pi)^4}\Big(\ln[p_E^2 + m_\psi^2(\phi_c)] + \ln[p_E^2 + m_{\tilde{\psi}_1}^2(\phi_c)] + \ln[p_E^2 + m_{\tilde{\psi}_2}^2(\phi_c)]\Big) \\
&- \frac{T^4}{2\pi^2}\Big[J_F(m_\psi^2/T^2) + \sigma J_F(m_{\tilde{\psi}_1}^2/T^2) + \sigma J_F(m_{\tilde{\psi}_2}^2/T^2)\Big]\,,
\end{aligned} \tag{4.43}
$$

where we have defined the fermionic thermal function

$$
J_F(y) \equiv \int_0^\infty dx\, x^2\, \ln\left(1 + e^{-\sqrt{x^2+y}}\right)\,. \tag{4.44}
$$

As before, the set of counterterms set is absorbed by the usual $T = 0$ renormalization conditions.

Comparison of Bosonic vs. Fermionic Cases

The thermal effective potentials of Eqs. (4.36) and (4.43). As a consequence of the diagonalization procedure, the scalar masses are only real for the régime $\phi_c^2 < (\Lambda_{\rm LW}^2 - 4\mu^2)/12\lambda$, while the fermion masses require ϕ_c to satisfy $\phi_c^2 < 4\Lambda_{\rm LW}^2/27\lambda^2$. Large values of ϕ_c therefore invalidate the theory from the start, as mass eigenstates elude definition. However, if symmetry restoration is indeed the high-T fate of the theory, then the condensate relaxes to $\phi_c \to 0$. We do not consider further the case of such troublesome extremes for the value of ϕ_c; the case $\phi_c \ll \Lambda_{\rm LW}^2$ being the more interesting one, we expand the masses in this limit as follows:

$$m_\varphi^2(\phi_c) \approx m_{\hat\varphi}^2 + \frac{m_{\hat\varphi}^4}{\Lambda_{\rm LW}^2} + \mathcal{O}(m_{\hat\varphi}^6/\Lambda_{\rm LW}^4)\,, \tag{4.45a}$$

$$m_{\tilde\varphi}^2(\phi_c) \approx \Lambda_{\rm LW}^2 - m_{\hat\varphi}^2 - \frac{m_{\hat\varphi}^4}{\Lambda_{\rm LW}^2} + \mathcal{O}(m_{\hat\varphi}^6/\Lambda_{\rm LW}^4)\,, \tag{4.45b}$$

$$m_\psi^2(\phi_c) \approx m_{\hat\psi}^2 + 2\frac{m_{\hat\psi}^4}{\Lambda_{\rm LW}^2} + \mathcal{O}(m_{\hat\psi}^6/\Lambda_{\rm LW}^4)\,, \tag{4.45c}$$

$$m_{\tilde\psi_{1,2}}^2(\phi_c) \approx \Lambda_{\rm LW}^2 \mp m_{\hat\psi}\Lambda_{\rm LW}^2 - \frac{1}{2}m_{\hat\psi}^2 \mp \frac{5}{8}\frac{m_{\hat\psi}^3}{\Lambda_{\rm LW}} - \frac{m_{\hat\psi}^4}{\Lambda_{\rm LW}^2} + \mathcal{O}(m_{\hat\psi}^6/\Lambda_{\rm LW}^4)\,. \tag{4.45d}$$

The ϕ_c-dependent contributions give only a small correction to the LW masses in Eqs. (4.45b) and (4.45d). This means that, when the scalar condensate takes on values near m_φ, the LW mass will provide a ϕ_c-independent contribution of $\approx \Lambda_{\rm LW}^2$. Even though the exact value of the condensate varies as the shape of the thermal effective potential changes with temperature, it never causes a significant departure of the LW masses from their $T = 0$ values. The LW fields do not decouple from the theory entirely, however; it is merely the case that they acquire most of their mass from means other than spontaneous symmetry breaking. The case is similar to that of heavy squarks in supersymmetric theories, which often get a thermal treatment.

Owing to their large mass, we expect the LW fields to contribute very little to the $T = 0$ term in the effective potential of Eq. (4.36). Also, when $T^2 \ll \Lambda_{\rm LW}^2$, a similar effect occurs: the LW mass scale is effectively inaccessible to the thermal bath, and LW fields provide comparatively small corrections to the $T \neq 0$ term of Eq. (4.36) of $\mathcal{O}(T^2/\Lambda_{\rm LW}^2)$. This result corresponds to Boltzmann suppression, in the language of the thermal functions of Eqs. (4.37) and (4.44). We also expect the LW fields to have a small impact on the phenomenon of symmetry restoration, whose onset we know to be retarded by hierarchy-eliminating theories, unless the phase transition temperature T_c is commensurate with $\Lambda_{\rm LW}$. But we can eliminate this latter possibility from the outset, as the case $T_c \sim \Lambda_{\rm LW}$ would give a large thermal mass to the scalar φ, thus leading to the breakdown of the LW stability condition of $\Lambda_{\rm LW}^2 > 4m_\varphi^2$. We therefore reach the conclusion that, for generic scenarios[6] in the LWSM, the LW fields do not have a significant effect on the phase transition.

However, in the high-temperature régime $T^2 \gtrsim \Lambda_{\rm LW}^2$, the thermal contributions of LW fields can indeed become significant. We can then expand the bosonic and fermionic thermal functions in the $y \ll 1$ scenario, which gives [90]

$$J_B(y) \xrightarrow{y\ll 1} -\frac{\pi^4}{45} + \frac{\pi^2}{12}y - \frac{\pi}{6}y^{3/2} - \frac{1}{32}y^2 \ln\frac{y}{a_b} + O(y^3)\,, \tag{4.46}$$

$$J_F(y) \xrightarrow{y\ll 1} +\frac{7\pi^4}{360} - \frac{\pi^2}{24}y - \frac{1}{32}y^2 \ln\frac{y}{a_f} + O(y^3)\,, \tag{4.47}$$

where the numerical constants are given by $a_b = 16a_f = 16\pi^2\exp(3/2 - \gamma_E)$. Now, working in the high temperature limit, and utilizing Eq. (4.46), we find

$$\begin{aligned}
&\frac{T^4}{2\pi^2}\left[J_B\left(\frac{m_\varphi^2(\phi_c)}{T^2}\right) + \sigma J_B\left(\frac{m_{\tilde\varphi}^2(\phi_c)}{T^2}\right)\right] \xrightarrow{T^2 \gg \Lambda_{\rm LW}^2 \gg m_{\tilde\varphi}^2} \\
&-\frac{\pi^2}{90}(1+\sigma)T^4 + \frac{\Lambda_{\rm LW}^2 T^2}{24}\sigma + (1-\sigma)\frac{m_{\tilde\varphi}^2 T^2}{24} \\
&-\sigma\frac{\Lambda_{\rm LW}^3 T}{12\pi} + \sigma\frac{\Lambda_{\rm LW}^2 T m_{\tilde\varphi}^2}{8\pi} - \frac{T}{12\pi}(m_{\tilde\varphi}^2)^{3/2} + \ldots\,.
\end{aligned} \tag{4.48}$$

Let us examine a few of the terms for the sake of physical intuition. The T^4 term is the free energy associated with a relativistic gas of $(1+\sigma)$ degrees of freedom. Taking $\sigma = -1$ (corresponding to the FGW derivation) causes this term to vanish, which can be seen as a cancellation between the two leading degrees of freedom. The large $-\Lambda_{\rm LW}^2 T^2$ term is the culprit behind retarded symmetry restoration, as was pointed out earlier; we now see how this effect arises from the effective potential calculation, confirming the results of Ref. [93]. The third term works against this, and is ultimately responsible for symmetry restoration. However, for the case $\sigma = +1$, this third term vanishes; since $m_\varphi^2 \sim \phi_c^2$, this term can be viewed as the effective thermal mass for

[6] Meaning, without a conspiratorial arrangement of parameters, which is just the sort of situation which the Lee-Wick Standard Model was designed to preclude.

the ϕ_c-dependent field, and its cancellation is the thermal analog of the cancellation of UV divergences in the electroweak theory.

For the fermions, we find

$$
\begin{aligned}
&-\frac{T^4}{2\pi^2}\left[J_F\left(\frac{m_{\hat{\psi}}^2(\phi_c)}{T^2}\right)+\sigma J_F\left(\frac{m_{\tilde{\psi}_1}^2(\phi_c)}{T^2}\right)+\sigma J_F\left(\frac{m_{\tilde{\psi}_2}^2(\phi_c)}{T^2}\right)\right] \\
&\xrightarrow{T^2\gg\Lambda_{\mathrm{LW}}^2\gg m_{\hat{\psi}}^2} -\frac{7\pi^2}{720}(1+2\sigma)T^4+2\sigma\frac{\Lambda_{\mathrm{LW}}^2T^2}{48}+(1-\sigma)\frac{m_{\hat{\psi}}^2T^2}{48}+(1-\sigma)\frac{m_{\hat{\psi}}^4T^2}{24\Lambda_{\mathrm{LW}}^2} \\
&\qquad +\frac{m_{\hat{\psi}}^4}{64\pi^2}\left(\ln\frac{m_{\hat{\psi}}^2}{a_fT^2}-\sigma\ln\frac{\Lambda_{\mathrm{LW}}^2}{a_fT^2}\right)+2\sigma\frac{\Lambda_{\mathrm{LW}}^4}{64\pi^2}\ln\frac{\Lambda_{\mathrm{LW}}^2}{a_fT^2}+\ldots\,. \qquad (4.49)
\end{aligned}
$$

In this case, the leading-order term flips sign for the case $\sigma=-1$. For $\sigma=+1$, the thermal contributions to the masses at $\mathcal{O}(T^2)$ drop out once more. The terms which could have restored symmetry in the fermionic sector vanish, and the overall restoration must therefore be accomplished by the bosonic sector.

4.2 The Lee-Wick Standard Model at Finite Temperature

4.2.1 The Lee-Wick Standard Model Thermal Effective Potential

We now have all the tools necessary to calculate the full thermal effective potential for the $N=2$ LWSM, as well as the physical intuition to interpret the results. Since we are interested primarily with the EWPhT, which involves the Higgs mechanism at finite temperature, we need only concern ourselves with the fields that couple most strongly to the Higgs. Therefore, we limit our purview to the top quark, the weak gauge bosons, and the Higgs itself, as well as the LW partners to each of these fields. We approximate the number of degrees of freedom in the problem to very high accuracy by assuming that the remaining fields are massless, and hence yield a T^4 contribution to $V_{\text{eff}}(\phi_c,T)$. The LW degrees of freedom corresponding to these (*e.g.*, the LW partner to the electron, the bottom quark, etc.) are taken to have mass Λ_{LW}^2.

We parameterize the Higgs condensate[7] as $\langle \hat{H} \rangle = (0, \phi_c/\sqrt{2})^T$, and calculate the field-dependent masses as per Appendix C. We classify each field type by its spin s, number of dynamical degrees of freedom g (including, *e.g.*, N_c factors from QCD), and the field dependent mass-squared $m^2(\phi_c)$. The results are collected in Table 4.2. We continue to keep the σ factor in tow, allowing for comparison with other authors at each step of the calculation. The one-loop effective potential for the LWSM reads[8]

$$
\begin{aligned}
V_{\text{eff}}^{(1\text{L})}(\phi_c, T) &= U(\phi_c) + \Delta V_0^{(1\text{L})}(\phi_c) + \Delta V_T^{(1\text{L})}(\phi_c, T) \ , \\
U(\phi_c) &= \frac{\lambda}{4}\left(\phi_c^2 - v^2\right)^2 \ , \\
\Delta V_0^{(1\text{L})}(\phi_c) &= \delta V_{\text{c.t.}} + \sum_i (-1)^{2s_i} g_i \frac{[m_i^2(\phi_c)]^2}{64\pi^2}\left[\ln m_i^2(\phi_c) - C_{uv} - \frac{3}{2}\right] \ , \\
\delta V_{\text{c.t.}} &= \delta\Omega + \frac{\delta m^2}{2}\phi_c^2 + \frac{\delta\lambda}{4}\phi_c^4 \, , \\
\Delta V_T^{(1\text{L})}(\phi_c, T) &= \frac{T^4}{2\pi^2}\sum_i \sigma_i g_i \begin{cases} J_B\left(\frac{m_i(\phi_c)}{T^2}\right) \ , & s_i = 0, 1 \, , \\ -J_F\left(\frac{m_i^2(\phi_c)}{T^2}\right) \ , & s_i = \frac{1}{2} \, , \end{cases}
\end{aligned}
\tag{4.50}
$$

with the sum running over all fields in Table 4.2. The subtraction constant $C_{uv} = \epsilon^{-1} - \gamma_E + \ln 4\pi$ is just the signature of divergence we already recognize from $T = 0$ field theory. Since the decoupling of heavy fields is not manifest in the $\overline{\text{MS}}$ renormalization scheme, we determine counterterms by requiring that the tree-level relations between Higgs parameters be maintained by the one-loop expansion. Setting the subtraction point at $\phi_c = v$, we demand

$$
0 = \Delta V_0^{(1\text{L})}\Big|_{\phi_c = v} = \frac{d\Delta V_0^{(1\text{L})}}{d\phi_c}\Big|_{\phi_c = v} = \frac{d^2\Delta V_0^{(1\text{L})}}{d\phi_c^2}\Big|_{\phi_c = v} \ , \tag{4.51}
$$

[7] The terminology is carefully selected here. The Higgs vacuum expectation value is $v \approx 246$ GeV, and is temperature independent; in contrast, the Higgs condensate ϕ_c may take on a range of values as the temperature fluctuates.

[8] We have neglected higher-order corrections, such as the "daisy" resummation (see Refs. [82, 95] for background). This would be essential to an exhaustive calculation, but as the EWPhT is not strongly first-order, it does not influence the calculation or its results substantially.

allowing us to solve for the counterterms $\delta\Omega, \delta m^2$, and $\delta\lambda$. The remaining free parameters are the four SM couplings λ, g, g', h_t and the five LW mass scales $\Lambda_H, \Lambda_W, \Lambda_t$, $\Lambda_{\rm EW}$, and $\Lambda^2_{\rm LW}$. These SM couplings are renormalized to satisfy the tree-level mass relationships

$$\begin{aligned} \sqrt{m_h^2(v)} &= M_H = 125\ \text{GeV}\,, \qquad & \sqrt{m_{W^\pm}^2(v)} &= M_W = 80.4\ \text{GeV}\,, \\ \sqrt{m_Z^2(v)} &= M_Z = 91.2\ \text{GeV}\,, \qquad & \sqrt{m_t^2(v)} &= M_t = 172.6\ \text{GeV}\,. \end{aligned} \tag{4.52}$$

From this point on, we assume a common LW mass parameter for all fields; that is, $\Lambda_H = \Lambda_W = \Lambda_t = \Lambda_{\rm EW} = \Lambda^2_{\rm LW}$, leaving $\Lambda^2_{\rm LW}$ as the only free parameter of the theory. This model-independent approach ensures that we do not fool ourselves into thinking that a small niche of parameter space implies general results.

As a final word on parameters, there is no upper bound on $\Lambda_{\rm LW}$; in the case $\Lambda_{\rm LW} \gg v$, we simply regain the SM, and the LW fields have marginal effect on all aspects of the theory. The limit $\Lambda_{\rm LW} \to \infty$ is therefore harmless. We will discuss interesting phenomenological lower bounds on $\Lambda_{\rm LW}$ in the next section, with some variation between different LW partners in the mass bounds allowed.

4.2.2 Finite-Temperature Behavior

The LWSM effective potential of Eq. (4.50) is given in Fig. 4.1 for a broad range of temperatures and values of $\Lambda_{\rm LW}$. We do not continue the curves for the $\Lambda_{\rm LW} =$ 350 GeV case past $\phi_c \geq \sqrt{4/27}\Lambda_t/h_t \approx 255$ GeV, as this régime violates the LW stability condition. The salient feature of these graphs is the absence of a barrier near $\phi_c \sim 0$, meaning that the phase transition is not strongly first-order. This conclusion is further emphasized by Fig. 4.2, where we plot the the order parameter $v(T)$. This parameter is the vacuum expectation value as computed at the temperature T, and only converges to 246 GeV in the $T = 0$ limit. We define the phase transition to occur at the temperature T_c such that $v(T \geq T_c) = 0$. The absence of a discontinuity

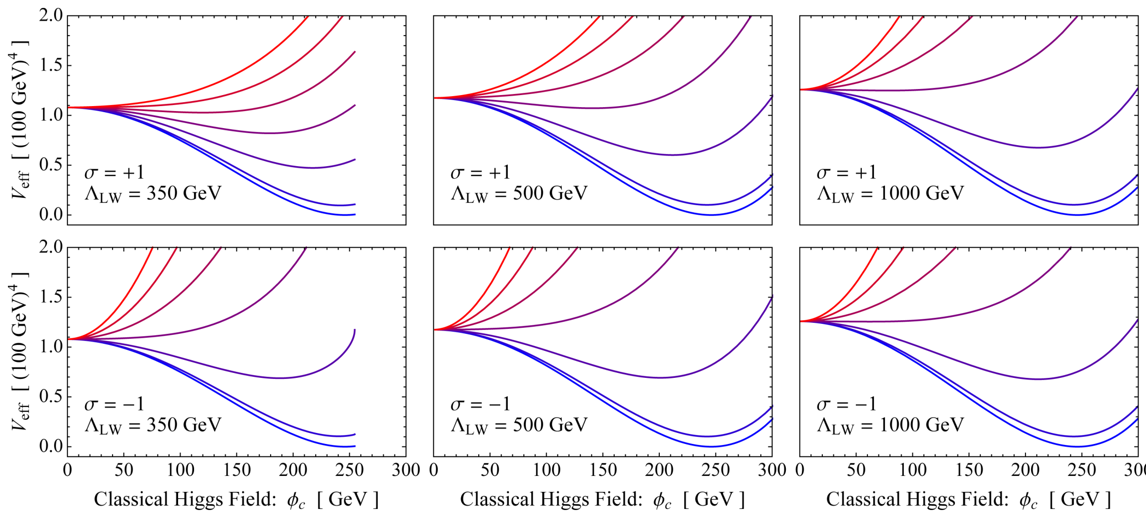


Figure 4.1: Variation with respect to temperature T and LW scale $\Lambda_{\rm LW}$ of the LWSM thermal effective potential $V_{\rm eff}(\phi_c)$. T increases from 0 GeV (blue, lowest curves) to 300 GeV (red, highest curves) in increments of $\Delta T = 50$ GeV. (Reprinted from Ref. [2])

in the order parameter at $T = T_c$ is a robust signal that the phase transition is not first-order, but is instead a cross-over transition (equivalently known as second-order). We therefore conclude that the EWPhT under the aegis of LW fields is similar to that of the SM case, and it cannot be a reasonable explanation for the baryon asymmetry of the Universe[9]. This stands as the central finding of this chapter.

We see from Fig. 4.2 that the $\Lambda_{\rm LW} \gg v$ limit restores the symmetry restoration point to $T \approx 150$ GeV, which agrees with the SM result to one loop [77]. Hence, the decoupling of super-heavy LW fields from the theory bears out. Furthermore, the critical temperature in the $\sigma = +1(-1)$ case is generally larger(smaller). This is well understood as the result of cancellation effects in the thermal effective potential,

[9]See Ref. [96] for further discussion of the baryon asymmetry and its relationship to electroweak baryogenesis.

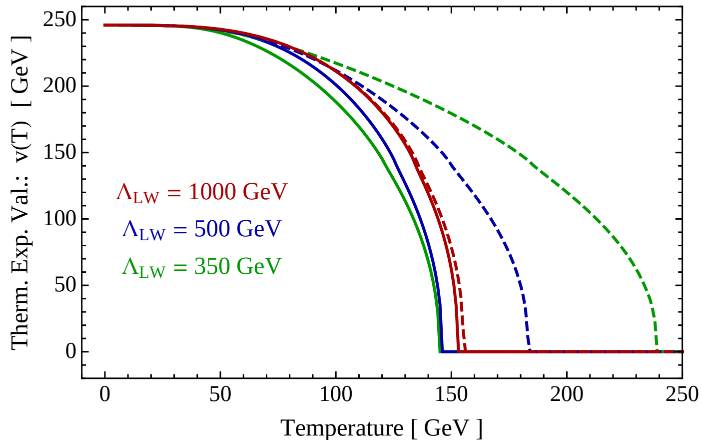


Figure 4.2: The electroweak symmetry-breaking order parameter $v(T)$ for the case of $\sigma = +1$ (dashed) and $\sigma = -1$ (solid). Note that the pairs of lines for a given $\Lambda_{\rm LW}^2$ move inward monotonically with increasing $\Lambda_{\rm LW}^2$. The phase transition temperature is generally higher in the former case due to the cancellation of the leading $\mathcal{O}(T^2)$ temperature dependence discussed in the text. (Reprinted from Ref. [2].)

as discussed in § 4.1.3. Let us take an explicit example to see how this cancellation arises. In the limit $\phi_c \ll \Lambda_{\rm LW} \ll T$, the one-loop thermal contribution reads

$$\begin{aligned}\Delta V_T^{(1{\rm L})}(\phi_c, T) \approx &-\frac{\pi^2}{90} g_*(\sigma) T^4 \\ &+ T^2 \times \begin{cases} +\frac{13}{6}\Lambda_{\rm LW}^2 & \sigma = +1\,, \\ -\frac{13}{6}\Lambda_{\rm LW}^2 + \left(\frac{3g^2+g'^2}{16} + \frac{m_t^2}{2v^2} + \frac{\lambda}{2}\right)\phi_c^2 + O(\phi_c^4/\Lambda_{\rm LW}^2) & \sigma = -1\,, \end{cases} \\ &+ T \times \begin{cases} + \left(\frac{9g^2+3g'^2+3\lambda}{32\pi^2}\right)\phi_c^2\Lambda_{\rm LW}^2 + O(\phi_c^3/\Lambda_{\rm LW}^2) & \sigma = +1\,, \\ - \left(\frac{9g^2+3g'^2+3\lambda}{32\pi^2}\right)\phi_c^2\Lambda_{\rm LW}^2 + O(\phi_c^3/\Lambda_{\rm LW}^2) & \sigma = -1\,, \end{cases}\end{aligned} \tag{4.53}$$

where the scheme-dependent $g_*(\sigma) = 106.75 + 197.5\sigma$ counts the number of degrees of freedom in the theory. Owing to the explicit σ factor, we immediately recognize Eq. (4.53) as possessing 106.75 SM and 197.5 LW degrees of freedom. For the case $\sigma = -1$, we see that g_* is negative, and therefore the overall thermal contribution to the free energy density is positive. This therefore implies (via the usual thermodynamic relations) that the pressure, energy density, and entropy density of LW fields are negative. This was obtained previously in [83, 85], and its puzzling implications have been teased out in the context of early-universe cosmology [88, 89].

As a final word on Fig. 4.2, we point out that the SM possesses a term $\Delta V_T^{(1\mathrm{L})} \ni T^2 \operatorname{Tr} M^2 \sim T^2\phi_c^2$ which gives rise to symmetry restoration. This term is absent in the $\sigma = +1$ LW case due to cancellation between positive- and negative-norm fields. Symmetry restoration can then only come about with the subdominant $\mathcal{O}(T\Lambda_{\mathrm{LW}}\phi_c^2)$ term, which necessarily implies the restoration of symmetry at a higher temperature than would otherwise be expected. This term only comes about through the non-analytic $(m_\varphi^2)^{3/2}$ term in the bosonic thermal function; the fermionic thermal function possesses no such term, and hence is irrelevant to the task of symmetry restoration.

It has been recently emphasized [97] that one must take a few extra precautions in the extraction of gauge-invariant observables from $V_{\mathrm{eff}}(\phi_c, T)$, which itself is manifestly gauge-dependent [98]. We have thus far proceeded by obtaining $v(T)$ from a minimization condition on $V_{\mathrm{eff}}(\phi_c, T)$, and so the order parameter inherits the gauge dependence of its predecessor. This gauge dependence can lead to anomalous results for real observables, such as the baryon number preservation criterion (after the baryons have been generated) and the gravity wave spectrum [97, 99, 100]. Thankfully, there is some numerical control over the relative influence of these anomalies: it has been pointed out [101] that the gauge dependence is small in the régime for which the perturbative expansion is valid. In our case, with a second-order phase transition, the only gauge-dependent parameter is T_c. We apply also the techniques of Ref. [97] as a double-check, and find qualitative agreement with the results plotted in Fig. 4.2; to wit, that T_c decreases as Λ_{LW} increases, and that T_c is generically higher for the $\sigma = +1$ case (as explained above). The gauge-invariant calculation results in a T_c

which $\approx 20\% - 35\%$ smaller, with the discrepancy most exaggerated for low values of $\Lambda_{\rm LW}$.

A few comments regarding the allowable range of $\Lambda_{\rm LW}$ are now in order. While we have worked under the assumption $\Lambda_{\rm LW} \gtrsim 350$ GeV up to this point, a better-motivated parameter set is motivated by the studies of previous Chapters (and relevant works cited therein); although they have been mentioned elsewhere in this dissertation, we recapitulate here for the convenience of the reader. The oblique parameter T, which measures custodial isospin violation, is sensitive to the LW top mass; experimental constraints place a bound[10] of $\Lambda_t > 1.5$ TeV at the 95% C.L. [57]. The post-LEP oblique parameters W and Y are sensitive to the LW gauge bosons, and establish bounds of $\Lambda_W = \Lambda_B > 2.3$ TeV at 95% C.L. [57] The remaining strong constraint is placed by the correction to the $Zb\bar{b}$ vertex, which imposes $\Lambda_H > 640$ GeV at 95% C.L. [1] (see also [59]). If we now set each of these parameters to its lower bound, we can calculate $v(T)$ in the most optimistic scenario. We find this case to be indistinguishable from the solid red curve of Fig. 4.2. Even for the most optimal choice of LW masses, the thermal contribution fails to differ appreciably from the one-loop SM result.

4.3 Discussion and Conclusions

This chapter has two goals: explore the thermodynamic properties of LW theories in general, and apply those lessons to early-universe cosmology. This latter question is pursued in the hope of solving the baryon asymmetry. The negative-norm degrees of freedom are forbidden by the LW/CLOP prescriptions that remove states that

[10]These bounds are derived by assuming $M_H = 115$ GeV; this was the LEP bound at the time at which these constraints were calculated. This $\sim 10\%$ shift in M_H translates into a commensurate shift in the bounds, but is irrelevant to the precision of our current considerations.

would otherwise violate unitarity. Though this method is broadly accepted and used by all LWSM theorists, it is not immediately clear how this prescription generalizes to nonzero temperatures. For instance, if one pays no special attention to the negative-norm states in the theory, one may move forward with an archetypal calculation of the thermal properties of a LW ideal gas. However, this picture seems incompatible with the LW/CLOP prescription outlined above. Alternatively, one could explicitly forbid the treatment of LW particles as external states, and treat the LW fields as having a solely resonant effect through the scattering of positive-norm particles. When treated as narrow resonances, the LW internal lines appear to be long-lived with respect to the characteristic time scales of thermal scattering, and they yield a contribution to the free energy which is precisely opposite to that which one expects for a conventional ideal gas.

When turning our attention to the electroweak phase transition in the early universe, we find that the LWSM has negligible effect beyond that of the SM alone, and that the phase transition is a cross-over, rather than being strongly first-order. The lack of a discontinuity in the order parameter $v(T_c)$ means that the LWSM cannot be responsible for electroweak baryogenesis, a paradigm widely thought to be a candidate for solving the baryon asymmetry problem. We also find that the way in which one quantizes the LW theory, as well as to what extent LW states are accessible to the thermal bath, make profoundly distinct predictions for the onset of symmetry restoration in the $T \gg \Lambda_{\rm LW}$ limit.

A possible extension of this work is the calculation of $V_{\rm eff}(\phi_c, T)$ for the $N = 3$ LWSM, which has heretofore been the primary focus of this dissertation. Since the mass bounds can be significantly smaller for partners in the $N = 3$ theory, the LW fields could potentially be active in the electroweak phase transition for a broader range of temperatures, ensuring a discontinuity in the order parameter. More degrees of freedom results in a larger g_*, which manifests itself in the high-temperature limit of the theory. However, there is a point of caution to be made: if the narrow-resonance approximation is not valid for Lee-Wick theories of any N, then a more careful approach is demanded.

Field	s	g	σ	$m_i^2(\phi_c)$	
SM-like Higgs	0	1	1	$m_h^2 = \frac{1}{2}\Lambda_H^2 \left(1 - \sqrt{1 - \frac{4m_{\hat{h}}^2}{\Lambda_H^2}}\right)$	$m_{\hat{h}}^2 = \lambda(3\phi_c^2 - v^2)$
LW-like Higgs	0	1	σ	$m_{\tilde{h}}^2 = \frac{1}{2}\Lambda_H^2 \left(1 + \sqrt{1 - \frac{4m_{\hat{h}}^2}{\Lambda_H^2}}\right)$	
SM-like pseudoscalar	0	1	1	$m_P^2 = \frac{1}{2}\Lambda_H^2 \left(1 - \sqrt{1 - \frac{4m_{\hat{P}}^2}{\Lambda_H^2}}\right)$	$m_{\hat{P}}^2 = \lambda(\phi_c^2 - v^2)$
LW-like pseudoscalar	0	1	σ	$m_{\tilde{P}}^2 = \frac{1}{2}\Lambda_H^2 \left(1 + \sqrt{1 - \frac{4m_{\hat{P}}^2}{\Lambda_H^2}}\right)$	
SM-like charged scalar	0	2	1	$m_{h^\pm}^2 = \frac{1}{2}\Lambda_H^2 \left(1 - \sqrt{1 - \frac{4m_{\hat{h}^\pm}^2}{\Lambda_H^2}}\right)$	$m_{\hat{h}^\pm}^2 = \lambda(\phi_c^2 - v^2)$
LW-like charged scalar	0	2	σ	$m_{\tilde{h}^\pm}^2 = \frac{1}{2}\Lambda_H^2 \left(1 + \sqrt{1 - \frac{4m_{\hat{h}^\pm}^2}{\Lambda_H^2}}\right)$	
SM gauge ghosts	0	-3	1	0	
SM-like W	1	6	1	$m_{\tilde{W}^\pm}^2 = \frac{1}{2}\Lambda_W^2 \left(1 - \sqrt{1 - \frac{4m_{\hat{W}}^2}{\Lambda_W^2}}\right)$	$m_{\hat{W}^\pm}^2 = \frac{g^2\phi_c^2}{4}$
LW-like W	1	6	σ	$m_{\tilde{W}^\pm}^2 = \frac{1}{2}\Lambda_W^2 \left(1 + \sqrt{1 - \frac{4m_{\hat{W}}^2}{\Lambda_W^2}}\right)$	
SM-like A	1	2	1	$m_A^2 = 0$	
LW-like A	1	3	σ	$m_{\tilde{A}}^2 = \Lambda_{\mathrm{EW}}^2$	
SM-like Z	1	3	1	$m_Z^2 = \frac{1}{2}\Lambda_{\mathrm{EW}}^2 \left(1 - \sqrt{1 - \frac{4m_{\hat{Z}}^2}{\Lambda_{\mathrm{EW}}^2}}\right)$	$m_{\hat{Z}}^2 = \frac{(g^2+g'^2)\phi_c^2}{4}$
LW-like Z	1	3	σ	$m_{\tilde{Z}}^2 = \frac{1}{2}\Lambda_{\mathrm{EW}}^2 \left(1 + \sqrt{1 - \frac{4m_{\hat{Z}}^2}{\Lambda_{\mathrm{EW}}^2}}\right)$	
SM-like top	$\frac{1}{2}$	12	1	$m_t^2 = \frac{2\Lambda_t^2}{3}\left(1 - \cos\frac{\theta_t}{3}\right)$	$\theta_t = \arctan\frac{2\sqrt{\alpha(1-\alpha)}}{1-2\alpha}$
LW-like top (1)	$\frac{1}{2}$	12	σ	$m_{\tilde{t}_1}^2 = \frac{2\Lambda_t^2}{3}\left(1 + \cos\frac{\theta_t+\pi}{3}\right)$	$\alpha = \frac{27}{4}\frac{m_{\hat{t}}^2}{\Lambda_t^2}$
LW-like top (2)	$\frac{1}{2}$	12	σ	$m_{\tilde{t}_2}^2 = \frac{2\Lambda_t^2}{3}\left(1 + \cos\frac{\theta_t-\pi}{3}\right)$	$m_{\hat{t}}^2 = h_t^2\phi_c^2$
SM-like gluons	1	16	1	0	
LW-like gluons	1	24	σ	Λ_{LW}^2	
Other SM-like fermions	$\frac{1}{2}$	78	1	0	
Other LW-like fermions	$\frac{1}{2}$	156	σ	Λ_{LW}^2	

Table 4.2: Tree-level, field-dependent pole masses used to construct the LWSM effective potential. s, g, and σ indicate the spin, effective number of degrees of freedom, and LW character of the fields; the fifth column gives the mass eigenvalues in terms of the field-dependent Lagrangian mass parameters appearing in the last column.

4.4 Discussion

The discovery of the Higgs boson marks a pivotal moment in the history of particle physics. However, this experimental triumph comes not without a host of outstanding problems and theoretical loose ends. The Standard Model offers no explanation for the miniscule neutrino masses, the nature of dark matter, or the threefold replication of family structure, to name a few. This dissertation has sought to answer a different type of question: rather than account for things undiscovered, how can we understand that which we *have* discovered? The Higgs boson mass, $m_h \approx 125$ GeV, sits very near the electroweak scale, $v \approx 246$ GeV. But, when calculating the most basic renormalization self-energy corrections at one-loop order, the Higgs receives corrections that diverge as $d \to 2$ in dimensional regularization. If one treats the Standard Model as an effective field theory, valid up to some cutoff scale Λ_{UV}, we ought to observe a Higgs mass much closer to this scale. It is the positive discovery of an electroweak-scale Higgs (rather than GUT- or Planck-scale) which vexes theoretical physics. Without some added physics, the Standard Model seems to require an extraordinary amount of fine-tuning in order to keep the Higgs comparatively light. If no new physics is found between the electroweak scale and, *e.g.*, the unification scale of $\Lambda_{\text{GUT}} \approx 10^{16}$ GeV, one must tune the parameters of the theory to better than one part in $\Lambda_{\text{GUT}}^2/v^2 \approx 10^{28}$ to cancel off these extreme corrections. This puzzle is known as the hierarchy problem.

Many candidate theories have arisen over the past few decades in order to ameliorate the Standard Model fine-tuning problem. Some of the more popular candidates were outlined in Ch. 1. In this dissertation, we have explored a less-known, though equally viable and testable, approach to solving the hierarchy problem: the Lee-Wick Standard Model. This theory posits a higher-derivative operator for each Standard

Model degree of freedom, *e.g.*, $\mathcal{L} \supset -\phi \Box^2 \phi / 2M^2$. These higher-derivative operators correspond to propagators that decay as p^{-4} (rather than p^{-2}) at high energies, and contain two poles in the complex-p^0 plane. The improved ultraviolet behavior of the theory results in Feynman diagrams free of the quadratic divergences as described above. We have discussed some of the thornier issues idiosyncratic to Lee-Wick theories, such as acausality and unitarity violation; the effects of the former are thought to be visible at very high energies, and the Lee-Wick prescription for removing negative-norm states averts the disasters associated with a non-unitary S-matrix. It must be noted that no consistent path integral formulation of Lee-Wick theories has been found to date, although no direct violation of unitarity can be found in the theory either. The matter of unitarity therefore remains not completely settled, though we may proceed with care to calculate the experimental implications of the Lee-Wick Standard Model.

The precise subject matter of this dissertation has been the so-called $N = 3$ Lee-Wick Standard Model. The theory is so named because of the addition of a yet-higher order derivative operator (of the form $-\phi \Box^3 \phi / 2M^4$) in the theory, which leads to propagators containing three physical poles. The propagator of this theory falls of as p^{-6} at high energies, and so the higher-N theory has even better ultraviolet behavior. The minimal Lee-Wick Standard Model is therefore an $N = 2$ theory in this parlance. Performing an auxiliary field transformation on the Lagrangian of this theory results in a set of states with alternating positive, negative, and positive norm coming from the same higher-derivative field.

Though the formal field theory of higher-derivative quantum fields is a fascinating subject in its own right, we have here sought to extract some phenomenological implications from the $N = 3$ Lee-Wick Standard Model.

We have seen that Lee-Wick partners to the Standard Model $W^{\pm}$ boson have distinct collider signatures, and could easily be identified at a high-luminosity collider for a broad range of mass values.

We have seen that the most exacting electroweak precision tests allow for relatively light Lee-Wick partners to the Higgs boson, with masses as low as ~ 650 GeV for the charged scalars. The alternating norm of Lee-Wick states allows for significant cancellations in the calculation of electroweak precision parameters, such as the anomalous $Zb\bar{b}$ coupling, meaning that some of the states can have lower masses than would be the case in a strictly $N = 2$ theory.

We have also investigated the quantization of Lee-Wick theories at finite temperature, clearing up much confusion in the process. Namely, we have shown that the most consistent formulation of the theory is one in which Lee-Wick states are only accessible through the scattering of positive-norm states, rather than one in which Lee-Wick states really are accessible to the thermal bath. Our results effectively show that treating Lee-Wick fields as "real"—that is, freely created and destroyed by an external thermal heat bath—leads either to a completely unbounded Hamiltonian spectrum, or depends on an ambiguous derivation from an S-matrix properties not guaranteed *a priori*. We used this new quantization technology to calculate the electroweak phaser transition in the early Universe, the goal being to see if Lee-Wick fields could have some part to play in triggering electroweak baryogenesis. However, due to the bounds placed on their masses by electroweak precision tests (see above), Lee-Wick fields are simply too massive to have had a substantial effect on the electroweak phase transition, and the latter remains second-order in character (whereas a strongly first-order transition is required to produce net baryon number).

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APPENDIX A

AUXILIARY FIELD FORMALISM FOR $N = 3$ LEE-WICK LAGRANGIANS

A.1 Pure Yang-Mills

We begin with the higher-derivative (HD) Lagrangian

$$\mathcal{L}_{\text{HD}} = -\frac{1}{2}\text{Tr}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \left(\frac{1}{m_2^2} + \frac{1}{m_3^2}\right)\text{Tr}\,\hat{F}_{\mu\nu}\hat{D}^{\mu}\hat{D}_{\alpha}\hat{F}^{\alpha\nu}$$
$$-\frac{2}{m_2^2 m_3^2}\text{Tr}\,\hat{F}_{\mu\nu}\hat{D}^{\mu}\hat{D}_{\alpha}\hat{D}^{[\alpha}\hat{D}_{\beta}\hat{F}^{\beta\nu]}, \qquad \text{(A.1)}$$

where brackets indicate antisymmetrization of the first and last indices only:

$$X^{[\alpha_1\alpha_2\ldots\alpha_N\alpha_{N-1}]} \equiv X^{\alpha_1\alpha_2\ldots\alpha_{N-1}\alpha_N} - X^{\alpha_N\alpha_2\ldots\alpha_{N-1}\alpha_N}. \qquad \text{(A.2)}$$

Eq. (A.1) can be factorized into

$$\mathcal{L}_{\text{HD}} = \text{Tr}\,\hat{F}_{\mu\nu}\left(\frac{1}{2}g^{\mu}{}_{\alpha} + \frac{\hat{D}^{\mu}\hat{D}_{\alpha}}{m_2^2}\right)\left[\left(\frac{1}{2}g^{\nu}{}_{\beta} + \frac{\hat{D}^{\nu}\hat{D}_{\beta}}{m_3^2}\right)g^{\alpha}{}_{\lambda} - (\alpha \leftrightarrow \nu)\right]. \qquad \text{(A.3)}$$

The stress-energy tensor ($\hat{F}$) and the covariant derivative ($\hat{D}$) acting upon fields transforming adjointly under the relevant gauge group are defined with respect to the HD non-Abelian gauge field $\hat{A}$ in the usual way:

$$\hat{F}^{\mu\nu} \equiv \partial^{\mu}\hat{A}^{\nu} - \partial^{\nu}\hat{A}^{\mu} - ig[\hat{A}^{\mu}, \hat{A}^{\nu}], \qquad \text{(A.4)}$$
$$\hat{D}X \equiv \partial^{\mu}X - ig[\hat{A}^{\mu}, X]. \qquad \text{(A.5)}$$

We now invoke auxiliary gauge fields, χ, ω, which also transform adjointly under the group generated by $\hat{A}$. The Lagrangian of Eq. (A.1) may be obtained from the auxiliary field Lagrangian,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\text{Tr}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \text{Tr}\,(\hat{D}_{\mu}\chi_{\nu} - \hat{D}_{\nu}\chi_{\mu}) - \frac{1}{2}\text{Tr}\,(\hat{D}_{\mu}\omega_{\nu} - \hat{D}_{\nu}\omega_{\mu})^2$$
$$-2m_2m_3\,\text{Tr}\,\chi_{\mu}\omega^{\nu} + (m_2^2 + m_3^2)\text{Tr}\omega_{\mu}\omega^{\mu}. \qquad \text{(A.6)}$$

Upon integration by parts, a form of Eq. (A.1) may be obtained in which no derivatives act on χ, which makes its role as an auxiliary field manifest. Note also that χ appears linearly in the Lagrangian, thereby playing the role of a Lagrange multiplier. We can now vary $\mathcal{L}_{YM}$ with respect to χ to obtain the equation of motion,

$$\hat{D}_{\nu}\hat{F}^{\nu\mu} - m_2m_3\omega^{\mu} = 0. \qquad \text{(A.7)}$$

The linearity of χ ensures that this equation is exact at the quantum level - that is to say, χ may be formally integrated out of the path integral.

We now seek a Lagrangian of the Lee-Wick (LW) form, as was done in § 2.1. We map the fields $\hat{A}$, χ, and ω onto the three new fields $A_{1,2,3}$:

$$A_1^\mu \equiv \hat{A}^\mu + \chi^\mu, \tag{A.8}$$

$$A_2^\mu \equiv \sqrt{-\frac{\eta_2}{\eta_1}}\chi^\mu - \sqrt{\frac{\eta_3}{\eta_1}}\omega^\mu, \tag{A.9}$$

$$A_3^\mu \equiv \sqrt{\frac{\eta_3}{\eta_1}}\chi^\mu - \sqrt{-\frac{\eta_2}{\eta_1}}\omega^\mu. \tag{A.10}$$

Only A_1 transforms as a gauge field under the gauge group; the others transform as matter fields under the adjoint representation. Their status as matter fields can be understood from the explicit Proca-style mass terms appearing in Eq. (A.6). For the sake of completeness, we list the inverse transformations as well:

$$\hat{A}^\mu = A_1^\mu - \sqrt{-\frac{\eta_2}{\eta_1}}A_2^\mu + \sqrt{\frac{\eta_3}{\eta_1}}A_3^\mu, \tag{A.11}$$

$$\chi^\mu = \sqrt{-\frac{\eta_2}{\eta_1}}A_2^\mu - \sqrt{\frac{\eta_3}{\eta_1}}A_3^\mu, \tag{A.12}$$

$$\omega^\mu = \sqrt{\frac{\eta_3}{\eta_1}}A_2^\mu - \sqrt{-\frac{\eta_2}{\eta_1}}A_3^\mu. \tag{A.13}$$

The sum rules of Eqs. (2.19), (2.20), and (2.21) may be used to re-express the η parameters in terms of the masses $m_{2,3}$. Since A_1 is the sole remaining gauge field, we can "unhat" the operators of Eq. (A.1) and define them with espect to A_1 instead. We respectively define the pieces of $\mathcal{L}$ that are quadratic, cubic, and quartic in A_1 as $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$. What follows is the LW Lagrangian

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{2}\mathrm{Tr}\, F_1^{\mu\nu}F_{1\mu\nu} + \frac{1}{2}\mathrm{Tr}\,(D_\mu A_{2\nu} - D_\nu A_{2\mu})^2 - \frac{1}{2}\mathrm{Tr}\,(D_\mu A_{3\nu} - D_\nu A_{3\mu})^2 \\ &-m_2^2 \mathrm{Tr}\, A_2^\mu A_{2\mu} + m_3^2 \mathrm{Tr}\, A_3^\mu A_{3\mu},\end{aligned} \tag{A.14}$$

which contains the kinetic and mass terms of the theory. We also note that only the kinetic and mass terms corresponding to A_2 possess the overall negative sign associated with LW fields. The cubic piece is

$$\begin{aligned}\mathcal{L}_1 = &\frac{-ig}{m_3^2 - m_2^2}\mathrm{Tr}\,(F_{1\mu\nu}[m_3 A_2^\mu - m_2 A_3^\nu, m_3 A_2^\nu - m_2 A_3^\nu]) \\ &+\frac{ig}{(m_3^2 - m_2^2)^{1/2}}\Big(\mathrm{Tr}\,(D_\mu A_{2\nu} - D_\nu A_{2\mu})(2m_3[A_2^\mu, A_3^\nu - m_2[A_2^\mu, A_3^\nu] - m_2[A_3^\mu, A_2^\nu]) \\ &+\mathrm{Tr}\,(D_\mu A_{3\nu} - D_\nu A_{3\mu})(2m_2[A_3^\mu, A_3^\nu] - m_3[A_2^\mu, A_3^\nu] - m_3[A_3^\mu, A_2^\nu])\Big),\end{aligned} \tag{A.15}$$

and the quartic piece is

$$\begin{aligned}\mathcal{L}_2 =& \frac{g^2}{2(m_3^2-m_2^2)^2}\\ &\times\Big(m_3^2(4m_2^2-3m_3^2)\mathrm{Tr}\,[A_2^\mu,A_2^\nu]^2+2m_2^2m_3^2\mathrm{Tr}\,[A_2^\mu,A_2^\nu][A_{3\mu},A_{3\nu}]\\ &+m_2^2(4m_3^2-3m_2^2)\mathrm{Tr}\,[A_3^\mu,A_3^\nu]^2\\ &+2m_2m_3(m_3^2-2m_2^2)\mathrm{Tr}\,[A_2^\mu,A_2^\nu]([A_{2\mu},A_{3\nu}]+[A_{3\mu},A_{2\nu}])\\ &+2m_2m_3(m_2^2-2m_3^2)\mathrm{Tr}\,[A_3^\mu,A_3^\nu]([A_{2\mu},A_{3\nu}]+[A_{3\mu},A_{2\nu}])\\ &+(m_2^4-m_2^2m_3^2+m_3^4)\mathrm{Tr}\,([A_2^\mu,A_3^\nu]+[A_3^\mu,A_2^\nu])([A_{2\mu},A_{3\nu}]+[A_{3\mu},A_{2\nu}])\Big)\end{aligned}\tag{A.16}$$

Though cumbersome, these expressions are still simplified with respect to a completely general $N=3$ treatment, owing to the sum rules of § 2.1.

A.2 Fermions

The $N=3$ Lagrangian for a HD chiral fermion $\hat{\phi}_L$ is

$$\mathcal{L}_{\mathrm{HD},f} = \frac{1}{m_2^2m_3^2}\bar{\hat{\phi}}_L\Big[(i\hat{\not{D}})^2-m_2^2\Big]\Big[(i\hat{\not{D}})^2-m_3^2\Big]i\hat{\not{D}}\hat{\phi}_L, \tag{A.17}$$

where the covariant derivative $\hat{D}$ runs over the gauge bosons and their LW partners, as described in § A.1. We now invoke auxiliary fields $\chi_{L,R}$, $\psi_{L,R}$, and write the auxiliary field Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{AF}} = \bar{\hat{\phi}}_L i\hat{\not{D}}\hat{\phi}_L - \bar{\chi}_R islash\hat{ed}D\chi_R + \bar{\psi}_L i\hat{\not{D}}\psi_L + (\bar{\hat{\phi}}_L i\hat{\not{D}}\chi_L+\mathrm{h.c.})+(\bar{\chi}_R i\hat{\not{D}}\psi_R+\mathrm{h.c.})\\ +\frac{m_2m_3}{m_2+m_3}(\bar{\chi}_R(\chi_L+\psi_L)+\mathrm{h.c.})-(m_2+m_3)(\bar{\psi}_L\psi_R+\mathrm{h.c.})).\end{aligned}\tag{A.18}$$

When considered as Weyl spinors[1], the new fields we have introduced appear linearly in Eq. (A.18). This means that they can be integrated out of the path integral, and variation of Eq. (A.18) with respect to them produces equations of motion which are exact at the quantum level. They can therefore be truly called auxiliary fields. Varying $\mathcal{L}_{\mathrm{AF}}$ with respect to them yields

$$i\hat{\not{D}}\hat{\phi}_L + \frac{m_2m_3}{m_2+m_3}\chi_R = 0, \tag{A.19}$$

$$i\hat{\not{D}}\chi_r - (m_2+m_3)\psi_L = 0. \tag{A.20}$$

These constraint equations may be substituted back into Eq. (A.18) to eliminate all terms linear in χ_L and ψ_R. The remaining auxiliary fields, χ_R and ψ_L, may also be

[1]That is, considering different helicities as being independent degrees of freedom

expressed in terms of $\hat{\phi}_L$:

$$\chi_R = -\frac{m_2+m_3}{m_2 m_3} i\hat{\not{D}}\hat{\phi}_L, \tag{A.21}$$

$$\psi_L = \frac{i\hat{\not{D}}}{m_2+m_3}\chi_R = -\frac{1}{m_2 m_3}(i\hat{\not{D}})^2\hat{\phi}_L. \tag{A.22}$$

These identifications may be used to turn Eq. (A.18) back into Eq. (A.17).

We now turn to the task of obtaining a Lagrangian of the conventional LW form. The left-chiral fields, $\hat{\phi}_L$, χ_L, and ψ_L, are mapped into the new SM- and LW-type fields, $\phi_L^{(1,2,3)}$, and the right-chiral fields, χ_R and ψ_R, are mapped into $\phi_R^{2,3}$:

$$\phi_L^{(1)} \equiv \hat{\phi}_L + \chi_L, \tag{A.23}$$

$$\phi_L^{(2)} \equiv \sqrt{-\frac{\eta_2}{\eta_1}}\chi_L - \sqrt{\frac{\eta_3}{\eta_1}}\psi_L, \tag{A.24}$$

$$\phi_L^{(3)} \equiv \sqrt{\frac{\eta_3}{\eta_1}}\chi_L - \sqrt{-\frac{\eta_2}{\eta_1}}\psi_L, \tag{A.25}$$

and

$$\phi_R^{(2)} \equiv \sqrt{-\frac{\eta_2}{\eta_1}}\chi_R - \left(\sqrt{-\frac{\eta_2}{\eta_1}} + \sqrt{\frac{\eta_3}{\eta_1}}\right)\psi_R, \tag{A.26}$$

$$\phi_R^{(3)} \equiv \sqrt{\frac{\eta_3}{\eta_1}}\chi_R - \left(\sqrt{-\frac{\eta_2}{\eta_1}} + \sqrt{\frac{\eta_3}{\eta_1}}\right)\psi_R. \tag{A.27}$$

These are accompanied by the inverse transformations

$$\hat{\phi}_L = \phi_L^{(1)} - \sqrt{-\frac{\eta_2}{\eta_1}}\phi_L^{(2)} + \sqrt{\frac{\eta_3}{\eta_1}}\phi_L^{(3)}, \tag{A.28}$$

$$\chi_L = \sqrt{-\frac{\eta_2}{\eta_1}}\phi_L^{(2)} - \sqrt{\frac{\eta_3}{\eta_1}}\phi_L^{(3)}, \tag{A.29}$$

$$\psi_L = \sqrt{\frac{\eta_3}{\eta_1}}\phi_L^{(2)} - \sqrt{-\frac{\eta_2}{\eta_1}}\phi_L^{(3)}, \tag{A.30}$$

and for the right-chiral fields,

$$\chi_R = \left(\sqrt{-\frac{\eta_2}{\eta_1}} + \sqrt{\frac{\eta_3}{\eta_1}}\right)\left(\phi_R^{(2)} - \phi_R^{(3)}\right), \tag{A.31}$$

$$\psi_R = \sqrt{\frac{\eta_3}{\eta_1}}\phi_R^{(2)} - \sqrt{-\frac{\eta_2}{\eta_1}}\phi_R^{(3)}. \tag{A.32}$$

Putting these together, we arrive at the fermionic $N=3$ LW Lagrangian,

$$\mathcal{L}_{\mathrm{LW,f}} = \bar{\phi}_L^{(1)} i\hat{\not{D}}\phi_L^{(1)} - \bar{\phi}^{(2)}(i\hat{\not{D}} - m_2)\phi^{(2)} + \bar{\phi}^{(3)}(i\hat{\not{D}} - m_3)\phi^{(3)}, \tag{A.33}$$

where we combine the Weyl spinors as $\phi^{(i)} \equiv \phi_L^{(i)} + \phi_R^{(i)}$.

A.3 Higgs Sector

Many of the methods and results from the toy scalar theory of § 2.1 can be carried over directly to the analysis of complex scalar fields and spontaneous symmetry breaking. We begin with a complex scalar field, $\hat{H}$, which transforms fundamentally under $SU(2)_L \times U(1)_Y$. It carries hypercharge $Y = 1/2$, as in the SM case. The Lagrangian for this field is

$$\mathcal{L}_{\text{HD}}^{N=3} = \hat{D}_\mu \hat{H}^\dagger \hat{D}^\mu \hat{H} - m_H^2 \hat{H}^\dagger \hat{H} - \frac{1}{M_H^2} \hat{H}^\dagger (\hat{D}_\mu \hat{D}^\mu)^2 \hat{H} - \frac{1}{M_H^4} \hat{H}^\dagger (\hat{D}_\mu \hat{D}^\mu)^3 \hat{H} + \mathcal{L}_{\text{int}}(\hat{H}). \tag{A.34}$$

An auxiliary field Lagrangian (similar to that of Eq. (2.6)) may be obtained, with complex scalars χ and ψ also transforming in the fundamental representation:

$$\begin{aligned}\mathcal{L}_{\text{AF}} =& \frac{1}{\eta_1} \Big\{ \hat{D}_\mu \hat{H}^\dagger \hat{D}^\mu \hat{H} - m_1^2 \hat{H}^\dagger \hat{H} - \Big[\chi^\dagger (\hat{D}_\mu \hat{D}^\mu + m_1^2) \hat{H} + \text{h.c.} \Big] \\ &+ (m_2^2 - m_1^2)^{1/2} (m_3^2 - m_1^2)^{1/2} (\chi^\dagger \psi + \psi^\dagger \chi) + \hat{D}_\mu \psi^\dagger \hat{D}^\mu \psi \\ &- (m_2^2 + m_3^2 - m_1^2) \psi^\dagger \psi \Big\} + \mathcal{L}_{\text{int}}(\hat{H}). \end{aligned} \tag{A.35}$$

Since χ appears without derivatives, it is a true auxiliary field. Varying Eq. (A.35) with respect to χ returns equations of motion which may be used to regain Eq. (A.34). We can obtain the LW form of the theory through the redefinitions Eqs. (2.13)-(2.15) by relabeling $\hat{\phi} \to \hat{H}$ and $\phi^{(i)} \to H^{(i)}$, resulting in

$$\begin{aligned}\mathcal{L} =& - H^{(1)\dagger} (\hat{D}_\mu \hat{D}^\mu + m_1^2) H^{(1)} + H^{(2)\dagger} (\hat{D}_\mu \hat{D}^\mu + m_2^2) H^{(2)} \\ &- H^{(3)\dagger} (\hat{D}_\mu \hat{D}^\mu + m_3^2) H^{(3)} + \mathcal{L}_{\text{int}}(\hat{H}). \end{aligned} \tag{A.36}$$

As in Eq. (2.23), the interaction terms obey the decomposition

$$\mathcal{L}_{\text{int}}(\hat{H}) = \mathcal{L}_{\text{int}}(\sqrt{\eta_1} H^{(1)} - \sqrt{-\eta_2} H^{(2)} + \sqrt{\eta_3} H^{(3)}). \tag{A.37}$$

Spontaneous symmetry breaking in the SM is ensured by the presence of a negative mass term, to wit, $m_H^2 < 0$ in Eq. (A.34). Since the original mass term and the interaction terms will become mixed by the VEV, it is convenient to move $\mathcal{L}_{\text{mass}}(\hat{H}) = -m_H^2 \hat{H}^\dagger \hat{H}$ into $\mathcal{L}_{\text{int}}(\hat{H})$:

$$\mathcal{L}_{\text{HD}} = \mathcal{L}_{\text{HD}}(m_H^2 = 0) + \mathcal{L}'_{\text{int}}(\hat{H}), \tag{A.38}$$

where

$$\mathcal{L}'_{\text{int}}(\hat{H}) \equiv -\frac{\lambda}{4} \left(\hat{H}^\dagger \hat{H} - \frac{v^2}{2} \right)^2 \tag{A.39}$$

and v is the Higgs VEV. Now that the HD mass term has been re-grouped as a piece of the interaction Lagrangian, the LW mass scales M_1, M_2 follow from Eqs. (2.11) and (2.12) by setting $m_1 = 0$:

$$M_1^2 = \frac{m_2^2 m_3^2}{m_2^2 + m_3^2}, \quad M_2^2 = m_2 m_3. \tag{A.40}$$

The $m_H^2 = 0$ part of the Lagrangian is handled as in the toy theory case of § 2.1. Setting $m_1^2 = 0$ (hence, $\eta_1 = 1$ in Eq. (2.6)), one then obtains the canonical, alternating-sign Lee-Wick Lagrangian

$$\begin{aligned}\mathcal{L} = &\hat{D}_\mu H^{(1)} \hat{D}^\mu H^{(1)} - \hat{D}_\mu H^{(2)} \hat{D}^\mu H^{(2)} + \hat{D}_\mu H^{(3)} \hat{D}^\mu H^{(3)} + m_2^2 H^{(2)\dagger} H^{(2)} \\ &+ m_3^2 H^{(3)\dagger} H^{(3)} + \mathcal{L}'_{\text{int}} \left(H^{(1)} - \sqrt{-\eta_2} H^{(2)} + \sqrt{\eta_3} H^{(3)} \right), \end{aligned} \tag{A.41}$$

where the interaction term may be unfolded into the expression

$$\begin{aligned} -\mathcal{L}_{\text{int}} = &\frac{\lambda}{4} \left(H^{(1)\dagger} H^{(1)} - \frac{v^2}{2} \right)^2 + \frac{\lambda}{4} \left(H^{(1)\dagger} H^{(1)} - \frac{v^2}{2} \right) \\ &\times \left\{ \left[H^{(1)\dagger} (\sqrt{-\eta_2} H^{(2)} + \sqrt{\eta_3} H^{(3)}) + \text{h.c.} \right] + |\sqrt{-\eta_2} H^{(2)} + \sqrt{\eta_3} H^{(3)}|^2 \right\} \\ &+ \frac{\lambda}{4} \left\{ \left[H^{(1)\dagger} (\sqrt{-\eta_2} H^{(2)} + \sqrt{\eta_3} H^{(3)}) + \text{h.c.} \right] + |\sqrt{-\eta_2} H^{(2)} + \sqrt{\eta_3} H^{(3)}|^2 \right\}^2 . \end{aligned} \tag{A.42}$$

One may then work in unitarity gauge, in which all Goldstone bosons are removed as physical degrees of freedom. This results in the explicit Higgs representations

$$H_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h_1) \end{pmatrix}, \; H_2 = \begin{pmatrix} h_2^+ \\ \frac{1}{\sqrt{2}}(H_2 + iP_2) \end{pmatrix}, \; H_3 = \begin{pmatrix} h_3^+ \\ \frac{1}{\sqrt{2}}(h_3 + iP_3) \end{pmatrix}, \tag{A.43}$$

where the h_i are CP-even scalars, the P_i are CP-odd pseudoscalars, and the h_i^+ are charged Higgses; v is the familiar Higgs VEV of the Standard Model. The mass terms of Eq. (A.42) may be expanded in this basis as

$$\begin{aligned} -\mathcal{L}_{\text{mass}} = &\frac{1}{2} m^2 (h_1 - \sqrt{-\eta_2} h_2 + \sqrt{\eta_3} h_3)^2 - \frac{1}{2} m_2^2 (2 h_2^- h_2^+ + h_2^2 + P_2^2) \\ &+ \frac{1}{2} m_3^2 (2 h_3^- h_3^+ + h_3^2 + P_3^2), \end{aligned} \tag{A.44}$$

with $m^2 = \lambda v^2/2 \approx (125\,\text{GeV})^2$. Note that the charged and pseudoscalar mass terms are canonical as written, but that the neutral scalar mass terms are not. These terms need to be diagonalized in such a way that their corresponding kinetic terms remain canonically normalized, all with respect to the LW basis $\{1, -1, 1\}$. These diagonalization matrices can be easily found numerically [55], and this procedure (albeit for fermions) is described in detail in § 2.3.2.

A.4 Yukawa Couplings

The HD Yukawa Lagrangian is given by

$$\mathcal{L}_{\text{Yukawa}} = -\lambda \left(\bar{\hat{\phi}}_L \hat{H} \hat{\psi}_R + \text{h.c.} \right), \tag{A.45}$$

which couples a complex scalar, $\hat{H}$, to left- and right-handed fermions $\hat{\phi}_L$ and $\hat{\psi}_R$. We anticipate an application to the Standard Model by assuming that $\hat{\phi}_L$ and $\hat{H}$

transform in the fundamental representation of some gauge group, while $\hat{\psi}_R$ remains a singlet. Let η_i (defined in § 2.1) refer to the LW mass spectrum of $\hat{\phi}_L^{(i)}$ and η_i' refer to the LW mass spectrum of $\hat{\psi}_R^{(i)}$. We can use Eqs. (A.28) and (A.45) to compute the (potential) quadratic divergence in the scalar mass arising fermion loops, and it is proportional to

$$\left(1 + \frac{\eta_2}{\eta_1} + \frac{\eta_3}{\eta_1}\right) \times \left(1 + \frac{\eta_2'}{\eta_1'} + \frac{\eta_3'}{\eta_1'}\right). \tag{A.46}$$

The sum rule of Eq. (2.19) ensures that this contribution vanishes identically.

APPENDIX B

QUANTIZATION CONVENTIONS

B.1 Classical to Quantum Theory

Whereas the other parts of this dissertation have dealt with Lee-Wick (LW) fields from the perspective of Lagrangian field theory, we now seek a Hamiltonian formulation. This is useful in the study of thermal field theories, as investigated in Ch. 4. Although we have already seen detailed calculations of correlation functions of $N = 2$ and $N = 3$ LW theories earlier in this dissertation, these were done at zero temperature, and so we cannot expect these prior results to hold at finite temperature. This Appendix presents a thorough calculation of the thermal properties of an $N = 2$ toy theory. The results are of a general nature, and the methods herein may be applied equally well to fermions and gauge fields[1].

Let us begin with the free scalar-field Lagrangian,

$$\mathcal{L} = \eta_H \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right), \tag{B.1}$$

where we have left an overall sign $\eta_H = \pm 1$ undetermined. We then solve for the canonical momentum conjugate to ϕ, which we call π_ϕ:

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \eta_H \dot{\phi}. \tag{B.2}$$

This allows us to write the Hamiltonian density,

$$\mathcal{H} = \eta_H \left(\frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\nabla \phi)^2 \frac{1}{2} m^2 \phi^2 \right). \tag{B.3}$$

The η_H factor allows for the existence of a semipositive- (or seminegative-)definite Hamiltonian. The presence of π_ϕ allows us to write down the commutation relations that define the quantum structure of the theory, as is the case with the expression $[x, p] = i\hbar$ in non-relativistic quantum mechanics. The commutation relations are given by

$$[\phi(\mathbf{x}), \pi_\phi(\mathbf{y})] = i\eta_C \delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{B.4}$$

where $\eta_C = \pm 1$ is another sign factor, to be fixed later on. In the conventional treatment, one sets $\eta_C = 1$ from the start. The definition of the commutator in Eq. (B.4) allows us to solve for the commutators between the field and conjugate

[1]The reader interested in a pedagogical approach to finite-temperature field theory is encouraged to consult Ref.[82].

momentum operators and the Hamiltonian[2]

$$
\begin{aligned}
[\phi, H] &= \eta_H \left[\phi(\mathbf{x}),\, \frac{1}{2}\int d^3y\, \pi_\phi^2(\mathbf{y})\right] \\
&= i\eta_H\eta_C\pi_\phi = i\eta_C\dot{\phi}, && \text{(B.5)} \\
[\pi_\phi, H] &= \eta_H \left[\pi_\phi(\mathbf{x}), \frac{1}{2}\int d^3y\, \left(m^2\phi^2(\mathbf{y}) + (\nabla\phi(\mathbf{y}))^2\right)\right] \\
&= -\, i\eta_H\eta_C(m^2+\nabla^2)\phi = i\eta_C\dot{\pi}_\phi, && \text{(B.6)}
\end{aligned}
$$

where the expression for $\dot{\pi}_\phi$ is obtained with the aid of the Euler-Lagrange equations for ϕ. Eqs. (B.5) and (B.6) may be used to prove the generalized Heisenberg equation of motion,

$$[\mathcal{O}, H] = i\eta_C\dot{\mathcal{O}}, \tag{B.7}$$

for any function $\mathcal{O}(\phi, \pi_\phi)$. The proof is straightforward. Both sides of Eq. (B.7) are linear in $\mathcal{O}$, and so one may take $\mathcal{O}$ to be a monomial function of ϕ and π_ϕ without loss of generality. Using the identity

$$[AB, H] = A[B, H] + [A, H]B, \tag{B.8}$$

an operator of the form $\mathcal{O} = AB$ will yield $A\dot{B} + \dot{A}B$ when commuted with H. This is nothing more than $\frac{d}{dt}(AB)$. Therefore, the function AB also satisfies Eq. (B.7). Since ϕ and π_ϕ satisfy Eq. (B.7), the identity Eq. (B.8) can be used to show the Heisenberg equations of motion for an arbitrarily complicated (algebraic) function of ϕ and π. If $\mathcal{O}$ is a polynomial, it may be broken into individual monomials, and its behavior per Eq. (B.7) may be shown piecewise as above. This completes the proof.

This result shows that, once the phase space of a system has been partitioned into one set for which $\eta_C = +1$ and another for which $\eta_C = -1$, Heisenberg equations of motion may be obtained for each. Note the absence of a functional dependence on η_H. We exclude from consideration any operators which are functions of fields drawn from both sides of the partition, as these do not give rise to a Hilbert space equipped with a consistent quantum-mechanical norm.

We move now to the interpretation of operators obeying the "wrong-sign" equations of motion. Eq. (B.7) may be exponentiated to obtain

$$e^{i\eta_C H(t-t_0)}\mathcal{O}(t_0)e^{-i\eta_C H(t-t_0)} = \mathcal{O}(t), \tag{B.9}$$

indicating that operators evolve forward in time according to the unitary operator $U(t, t_0) = \exp[-i\eta_C H(t-t_0)]$. Operators belonging to the $\eta_C = -1$ partition therefore possess an opposite-sign phase with respect to the time evolution of conventional ($\eta_C = +1$) operators. If this were the only difference, the choice of η_C could be dismissed as a mere convention, serving at best a pedantic purpose in the construction of time-dependent quantum mechanical operators. However, this is not the case, as will be shown in subsequent sections. The choice of η_C has profound implications for defining the propagator (and correlation functions in general), how the integration contour may be deformed to pick up poles, and the thermal spectrum of the theory.

[2]NB: The Hamiltonian is given by the spatial integral of the Hamiltonian density, *i.e.*, $H = \int d^3x\mathcal{H}(\mathbf{x})$.

B.2 Mode Expansions and the Hamiltonian

We begin by writing down the mode expansions for a field of Lee-Wick type, $\phi(\mathbf{x})$, and its canonically conjugate momentum, $\pi_\phi(\mathbf{x})$:

$$\begin{aligned}
\phi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}})e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \text{(B.10)}\\
\pi_\phi(\mathbf{x}) &= \eta_H \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \eta_H \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}})e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \text{(B.11)}
\end{aligned}$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$ is strictly positive, and η_H reflects the elementary result $\pi_\phi = \eta_H\dot\phi$ from Eq. (B.2). The commutator is given by $[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] \equiv \eta_N(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$, where we have introduced a third and final sign factor, η_N, defining the norm of the quantum state created by $a^\dagger_{\mathbf{p}}$. This overall factor may be constrained by beginning with the commutator of ϕ and π_ϕ:

$$\begin{aligned}
[\phi(\mathbf{x}),\pi_\phi(\mathbf{y})] &= -\eta_H\int\frac{d^3pd^3q}{(2\pi)^6}\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\left([a^\dagger_{-\mathbf{p}},a_{\mathbf{q}}] - [a_{\mathbf{p}},a^\dagger_{-\mathbf{q}}]\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \\
&= \eta_H\int\frac{d^3pd^3q}{(2\pi)^6}\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\times 2(2\pi)^3\times\eta_N\delta^{(3)}(\mathbf{p}+\mathbf{q})e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \\
&= i\eta_H\eta_N\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad \text{(B.12)}
\end{aligned}$$

from which we conclude $\eta_C = \eta_H\eta_N$. The meaning of η_N becomes clear once we calculate the spectrum of the theory. To do this, we expand the Hamiltonian in terms of its modes:

$$\begin{aligned}
H &= \int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] \\
&= \int d^3x\int\frac{d^3p\,d^3q}{(2\pi)^6}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\cdot\frac{1}{2}\bigg[-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2}\times(a_{\mathbf{p}}-a^\dagger_{-\mathbf{p}})(a_{\mathbf{q}}-a^\dagger_{-\mathbf{q}}) \\
&\quad + \frac{-\mathbf{p}\cdot\mathbf{q}+m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\times(a_{\mathbf{p}}+a^\dagger_{-\mathbf{p}})(a_{\mathbf{q}}-a^\dagger_{-\mathbf{q}})\bigg] \\
&= \eta_H\int\frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}},a^\dagger_{\mathbf{p}}]\right). \qquad \text{(B.13)}
\end{aligned}$$

From this expression follow the commutators

$$
\begin{aligned}
[H, a_{\mathbf{p}}] &= \eta_H \int \frac{d^3q}{(2\pi)^3} \omega_{\mathbf{q}} [a^\dagger_{\mathbf{q}}, a_{\mathbf{p}}] a_{\mathbf{q}} = -\eta_H \eta_N \omega_{\mathbf{p}} a_{\mathbf{p}} \\
&= -\, \eta_C \omega_{\mathbf{p}} a_{\mathbf{p}}, \qquad \text{(B.14)}\\
[H, a^\dagger_{\mathbf{p}}] &= \eta_H \int \frac{d^3q}{(2\pi)^3} \omega_{\mathbf{q}} a^\dagger_{\mathbf{q}} [a_{\mathbf{q}}, a^\dagger_{\mathbf{p}}] = +\eta_H \eta_N \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}} \\
&= +\, \eta_C \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}}. \qquad \text{(B.15)}
\end{aligned}
$$

Comparing this with Eq. (B.7), we now have equations of motion for the creation and annihilation operators. Rearranging Eq. (B.14) into the form $H a_{\mathbf{p}} = a_{\mathbf{p}}(H - \eta_C \omega_{\mathbf{p}})$, we can act repeatedly on the left with H to obtain $H^n a_{\mathbf{p}} = a_{\mathbf{p}}(H - \eta_C \omega_{\mathbf{p}})^n$. This expression may be exponentiated to obtain

$$
\begin{aligned}
a_{\mathbf{p}}(t) &= U^\dagger(t,0) a_{\mathbf{p}}(t=0) U^\dagger(t,0) = e^{i\eta_C H t} a_{\mathbf{p}} e^{-i\eta_C H t} \\
&= a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}} t}. \qquad \text{(B.16)}
\end{aligned}
$$

The result $a^\dagger_{\mathbf{p}}(t) = a^\dagger_{\mathbf{p}} \exp(+i\omega_{\mathbf{p}} t)$ is readily obtained by Hermitian conjugation. This much is identical to the conventional treatment. However, there remains a subtlety to be addressed: which are the raising and which are the lowering operators? Given some eigenstate $|\psi\rangle$ of the Hamiltonian satisfying $H|\psi\rangle = E_\psi |\psi\rangle$, we determine this by computing

$$
\begin{aligned}
H(a^\dagger_{\mathbf{p}} |\psi\rangle) &= ([H, a^\dagger_{\mathbf{p}}] + a^\dagger_{\mathbf{p}} H) |\psi\rangle = (\eta_C \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}} E_\psi) |\psi\rangle \\
&= (\eta_C \omega_{\mathbf{p}} + E_\psi) \left(a^\dagger_{\mathbf{p}} |\psi\rangle \right). \qquad \text{(B.17)}
\end{aligned}
$$

If $\eta_C = +1$, then $a^\dagger_{\mathbf{p}}$ raises the energy of the state on which it operates, as expected. If instead $\eta_C = -1$, then $a^\dagger_{\mathbf{p}}$ lowers the energy of the state on which it acts, and it is $a_{\mathbf{p}}$ which actually serves the role of the raising operator. It is important to recall the equality $\eta_C = \eta_H \eta_N$, and so the "wrong-sign" choice of $\eta_C = -1$ could be achieved by an unconventional sign choice for either η_H or η_N.

We must define a vacuum for the theory. This can be done in four distinct ways, one for each choice of η_H and η_N. In the case of $\eta_H = -1$, corresponding to a LW Hamiltonian, we may choose $\eta_N = -1$ as well, thereby fixing $\eta_C = +1$. We see from Eqs. (B.14) and (B.15) that this corresponds to the ladder operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ still behaving as annihilation and creation operators, respectively. We are then at liberty to define a lowest-energy state $|0\rangle$ such that $a_{\mathbf{p}}|0\rangle = 0$, with single-particle momentum eigenstates defined by $|\mathbf{p}\rangle = \sqrt{2\omega_{\mathbf{p}}} a^\dagger_{\mathbf{p}} |0\rangle$. The inner product between two momentum eigenstates is given by

$$
\begin{aligned}
\langle \mathbf{p} | \mathbf{q} \rangle &= 2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}} \langle 0 | a_{\mathbf{p}} a^\dagger_{\mathbf{q}} | 0 \rangle = 2\sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}} \langle 0 | [a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] | 0 \rangle \\
&= 2\eta_N \omega_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \qquad \text{(B.18)}
\end{aligned}
$$

from which we see that the choice $\eta_N = -1$ corresponds to defining a negative quantum-mechanical norm on the Hilbert space. If instead we choose $\eta_N = +1$, this results in $\eta_C = -1$, and the roles of $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ must be interchanged; the vacuum is now defined by the state $|0\rangle$ such that $a^\dagger_{\mathbf{p}}|0\rangle = 0$.

Now that the vacuum is defined in the general case, we can go on to calculate the spectrum of the Hamiltonian built by the repeated action of creation operators on the vacuum in question:

$$\begin{aligned} H|\mathbf{p}\rangle &= \left(\eta_H \int \frac{d^4q}{(2\pi)^3}\omega_{\mathbf{q}} a^\dagger_{\mathbf{q}} a_{\mathbf{q}}\right) a^\dagger_{\mathbf{p}} \sqrt{2\omega_{\mathbf{p}}}|0\rangle \\ &= \eta_H \int \frac{d^4q}{(2\pi)^3}\omega_{\mathbf{q}} a^\dagger_{\mathbf{q}} [a_{\mathbf{q}}, a^\dagger_{\mathbf{p}}] \sqrt{2\omega_{\mathbf{p}}}|0\rangle \\ &= \eta_H \eta_N \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}} \sqrt{2\omega_{\mathbf{p}}}|0\rangle = \eta_H\eta_N\omega_{\mathbf{p}}|\mathbf{p}\rangle, \end{aligned} \tag{B.19}$$

from which we conclude

$$H|\mathbf{p}\rangle \equiv E_{\mathbf{p}}|\mathbf{p}\rangle = \eta_H\eta_N\omega_{\mathbf{p}}|\mathbf{p}\rangle = \eta_C\omega_{\mathbf{p}}|\mathbf{p}\rangle. \tag{B.20}$$

Our findings may now be summarized. If we choose $\eta_C = \eta_H = \eta_N = +1$, corresponding to the typical Klein-Gordon theory, then there exists a positive-semidefinite energy spectrum with a vacuum $|0\rangle$ annihilated by $a_{\mathbf{p}}$. If instead we choose $\eta_H = \eta_N = -1$, we still have $\eta_C = +1$, and this corresponds to the Lee-Wick case pursued in this dissertation. The Hamiltonian spectrum is still positive-semidefinite, although the norms of single-particle states are negative. A third option, with $\eta_H = -\eta_N$ (without necessarily specifying the sign of either), results in a Hamiltonian spectrum which is bounded from above. The roles of creation and annihilation operators are interchanged, and the vacuum $|0\rangle$ is defined by $a^\dagger_{\mathbf{p}}|0\rangle = 0$.

B.3 Calculating the Propagator

Now that we have a quantized field theory with well-defined time evolution and norm, we can turn to the task of constructing the propagator. The end goal is to obtain a form for the propagator in which Wick rotation in the complex-p^0 plane can be easily implemented. Let us begin by writing the mode expansion for $\phi(x)$, generalizing Eq. (B.10):

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\eta_C\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\eta_C\mathbf{p}}e^{ip\cdot x}\right)\Big|_{\eta_C}, \tag{B.21}$$

where the η_C is shorthand for evaluating p^0 at $p^0 = \eta_C\omega_{\mathbf{p}} = \eta_H\eta_N\omega_{\mathbf{p}}$. We allow for this ambiguity so as to include the possibility of defining ϕ on the positive or negative mass-shell, *i.e.*, $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$. Since we are now working with the Lorentz-invariant $p \cdot x$ (rather than the non-covariant $\mathbf{p}\cdot\mathbf{x}$), we maintain this invariance by generalizing the ladder operators to create and destroy states of momentum $\eta_C\mathbf{p}$.

We can now calculate the two-point function:

$$\langle 0\,|\,\phi(x)\phi(y)\,|\,0\rangle = \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\langle 0\,|\,a_{\eta_C\mathbf{p}}a^{\dagger}_{\eta_C\mathbf{q}}\,|\,0\rangle e^{-i(p\cdot x-q\cdot y)}\Big|_{\eta_C}$$
$$= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{\eta_N}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\delta^{(3)}[\eta_C(\mathbf{p}-\mathbf{q})]e^{-i(p\cdot x-q\cdot y)}\Big|_{\eta_C}$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_N}{2\omega_{\mathbf{p}}}e^{-ip\cdot(x-y)}\Big|_{\eta_C} \equiv D^{\eta}(x-y). \tag{B.22}$$

The subscript η serves as an accounting device, keeping track of which quantization scheme was used. We can now construct the time-ordered Feynman propagator,

$$D^{\eta}_F(x-y) \equiv \theta^(x^0-y^0)D^{\eta}(x-y)+\theta(y^0-x^0)D^{\eta}(y-x)$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_N}{2\omega_{\mathbf{p}}}\left[\theta(x^0-y^0)e^{-ip\cdot(x-y)}\Big|_{p^0=\eta_C\omega_{\mathbf{p}}}+\theta(y^0-x^0)e^{ip\cdot(x-y)}\Big|_{p^0=\eta_C\omega_{\mathbf{p}}}\right]$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_N}{2\omega_{\mathbf{p}}}\left[\theta(x^0-y^0)e^{-ip\cdot(x-y)}\Big|_{p^0=\eta_C\omega_{\mathbf{p}}}+\theta(y^0-x^0)e^{-ip\cdot(x-y)}\Big|_{p^0=-\eta_C\omega_{\mathbf{p}}}\right]$$
$$= \int \frac{d^4p}{(2\pi)^3}\frac{\eta_N}{2\omega_{\mathbf{p}}}\Big[\theta(x^0-y^0)\delta(p^0-\eta_C\omega_{\mathbf{p}})$$
$$+\theta(y^0-x^0)\delta(p^0+\eta_C\omega_{\mathbf{p}})\Big]e^{-ip\cdot(x-y)}. \tag{B.23}$$

In order to continue, we invoke the Lee-Wick prescription: the theory must be free of exponentially-growing modes. This demand results in a condition on how the integration contour is to be pushed above and below the real p^0 axis. We deform the contour with Feynman $i\epsilon$ as follows:

$$D^{\eta}_F(x-y) = \int \frac{d^4p}{(2\pi)^3}\frac{\eta_N}{2\omega_{\mathbf{p}}}\Big[\delta(p^0-\eta_C\omega_{\mathbf{p}}+i\epsilon)+\delta(p^0+\eta_C\omega_p-i\epsilon)\Big]e^{-ip\cdot(x-y)}$$
$$= \int \frac{d^4p}{(2\pi)^3}\frac{\eta_N}{2\omega_{\mathbf{p}}}\Big(\frac{1}{-2\pi i}\cdot\frac{1}{p^0-(\eta_C-i\epsilon)}+\frac{1}{2\pi i}\cdot\frac{1}{p^0+(\eta_C-i\epsilon)}\Big)e^{-ip\cdot(x-y)}$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{i\eta_N}{2\omega_{\mathbf{p}}}\Big(\frac{p^0+\eta_C\omega_{\mathbf{p}}-i\epsilon-p^0+\eta_C\omega_{\mathbf{p}}-i\epsilon}{(p^0)^2-(\eta_C\omega_{\mathbf{p}}-i\epsilon)}\Big)e^{-ip\cdot(x-y)}$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{i\eta_N\eta_C}{p^2-m^2+i\eta_C\epsilon}e^{-ip\cdot(x-y)}. \tag{B.24}$$

We recognize in Eq. (B.24) the momentum-space Feynman propagator,

$$\tilde{D}^{\eta}_F(p) = \frac{i\eta_H}{p^2-m^2+i\eta_C\epsilon}. \tag{B.25}$$

The implications of Eq. (B.25) for the analytic structure of the theory are clear. The conventional Klein-Gordon case corresponds to $\eta_C=\eta_H=\eta_N=+1$, as before.

Lee-Wick theories belonging to either quantization scheme possess the well-known "wrong-sign" propagator, since $\eta_H = \eta_N \eta_C = -1$. However, there exists a subtle difference when $\eta_C = -\eta_N = -1$: the shifted poles lie in the first and third quadrants of the complex p^0 plane, rather than the second and fourth, as is usually the case. This means that we obtain a different overall sign when deforming the contour to pick up the poles. Therefore, when performing a Wick rotation in order to perform an integral, we must define the Euclidean momentum as $p_0 = -ip_E^0$, corresponding to *counterclockwise* rotation in the complex p^0 plane.

APPENDIX C

THE LEE-WICK STANDARD MODEL SPECTRUM: CONDENSATE-DEPENDENT $N = 2$ THEORY

Though some of the results of Appendix A could be truncated and cosmetically modified to obtain the input parameters for Ch. 4, this task can be non-trivial; we therefore reproduce the necessary results below for the convenience of the reader. For completeness, we present here the calculation of the field-dependent masses that appear in Table 4.2. We use the metric convention $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

C.1 Higgs & Electroweak Gauge Sector

In the higher-derivative formalism, we denote the Higgs doublet as $\hat{H}$, the $\mathrm{SU}\big(2\big)_L$ gauge field as $\hat{W}^a_\mu$, and the $\mathrm{U}\big(1\big)_Y$ gauge field as $\hat{B}_\mu$. We suppose that there is a nonzero homogenous Higgs condensate $\langle\hat{H}\rangle = (0\,,\, \phi_c/\sqrt{2})^T$ that breaks the electroweak symmetry down to $\mathrm{U}\big(1\big)_{\mathrm{EM}}$. The Higgs field may be expanded about the background as

$$\hat{H} = \begin{pmatrix} \hat{h}^+ \\ \frac{\phi_c + \hat{h} + i\hat{P}}{\sqrt{2}} \end{pmatrix} , \tag{C.1}$$

where $\hat{h}$ and $\hat{P}$ are real scalar fields and $\hat{h}^+$ is complex. After electroweak symmetry breaking, we denote the photon, neutral weak boson, and charged weak boson fields as $\hat{A}_\mu$, $\hat{Z}_\mu$, and $\hat{W}^\pm_\mu$ respectively. These are related to the original electroweak gauge fields by the standard transformations

$$\begin{aligned} \hat{Z}_\mu &= \cos\theta_W\, \hat{W}^3_\mu - \sin\theta_W\, \hat{B}_\mu\,, \\ \hat{A}_\mu &= \sin\theta_W\, \hat{W}^3_\mu + \cos\theta_W\, \hat{B}_\mu\,, \\ \hat{W}^\pm_\mu &= \tfrac{1}{\sqrt{2}}\left(\hat{W}^1_\mu \mp i\hat{W}^2_\mu\right)\,, \end{aligned} \tag{C.2}$$

where $\cos\theta_W = g/\sqrt{g^2 + g'^2}$ and $\sin\theta_W = g'/\sqrt{g^2 + g'^2}$. We work in the R_ξ gauge formalism for generality and restrict to the Landau gauge ($\xi = 0$) at the end. We introduce eight anti-commuting, scalar ghost fields c_A, c_Z, c_{W^+}, c_{W^-}, $\bar{c}_A$, $\bar{c}_Z$, $\bar{c}_{W^+}$, and $\bar{c}_{W^-}$.

The gauge-fixed LWSM electroweak sector is specified by the Lagrangian

$$\begin{aligned}
\mathcal{L}_{\rm hd}^{(\rm EW)} &= \mathcal{L}_{\rm hd}^{(\rm H)} + \mathcal{L}_{\rm hd}^{(\rm B)} + \mathcal{L}_{\rm hd}^{(\rm W)} + \mathcal{L}_{\rm hd}^{(\rm g.f.)} + \mathcal{L}_{\rm hd}^{(\rm gh.)}\,, \\
\mathcal{L}_{\rm hd}^{(\rm H)} &= \left|\hat{D}_\mu \hat{H}\right|^2 - \tfrac{1}{\Lambda_H^2}\left|\hat{D}_\mu \hat{D}^\mu \hat{H}\right|^2 - U_{\rm hd}(\hat{H})\,, \\
\mathcal{L}_{\rm hd}^{(\rm B)} &= -\tfrac{1}{4}\hat{B}_{\mu\nu}\hat{B}^{\mu\nu} + \tfrac{1}{2\Lambda_B^2}\left(\partial^\mu \hat{B}_{\mu\nu}\right)^2\,, \\
\mathcal{L}_{\rm hd}^{(\rm W)} &= -\tfrac{1}{4}\hat{W}^a_{\mu\nu}\hat{W}^{a\,\mu\nu} + \tfrac{1}{2\Lambda_W^2}\left(D^\mu \hat{W}^a_{\mu\nu}\right)^2\,, \\
\mathcal{L}_{\rm hd}^{(\rm g.f.)} &= -\tfrac{1}{2\xi_A}\left(\partial^\mu \hat{A}_\mu\right)^2 - \tfrac{1}{2\xi_Z}\left(\partial^\mu \hat{Z}_\mu - \xi_Z \tfrac{\sqrt{g^2+g'^2}}{2}\,\phi_c \hat{P}\right)^2 - \tfrac{1}{\xi_W}\left|\partial^\mu \hat{W}^+_\mu - i\,\xi_W\,\tfrac{g}{2}\,\phi_c \hat{h}^+\right|^2\,, \\
\mathcal{L}_{\rm hd}^{(\rm gh.)} &= \bar{c}_A(-\partial^2)c_A + \bar{c}_Z\left(-\partial^2 - \xi_Z \tfrac{g^2+g'^2}{4}\phi_c^2\right)c_Z + \bar{c}_{W^+}\left(-\partial^2 - \xi_W \tfrac{g^2}{4}\phi_c^2\right)c_{W^+} \\
&\qquad + \bar{c}_{W^-}\left(-\partial^2 - \xi_W \tfrac{g^2}{4}\phi_c^2\right)c_{W^-} + \text{interactions}\,,
\end{aligned} \tag{C.3}$$

where

$$U_{\rm hd}(\hat{H}) = \lambda\left(\hat{H}^\dagger \hat{H} - \frac{v^2}{2}\right)^2\,, \tag{C.4}$$

$$\hat{D}_\mu H = \left(\partial_\mu - ig\frac{\sigma^a}{2}\hat{W}^a_\mu - ig'\frac{1}{2}\hat{B}_\mu\right)H\,, \tag{C.5}$$

$$\hat{B}_{\mu\nu} = \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu\,, \tag{C.6}$$

$$\hat{W}^a_{\mu\nu} = \partial_\mu \hat{W}^a_\nu - \partial_\nu \hat{W}^a_\mu + g\epsilon^{abc}\hat{W}^b_\mu \hat{W}^c_\nu\,, \tag{C.7}$$

$$(D^\mu \hat{W}_{\mu\nu})^a = \partial^\mu \hat{W}^a_{\mu\nu} + g\epsilon^{abc}\hat{W}^{b\,\mu}\hat{W}_{c\,\mu\nu}\,. \tag{C.8}$$

Since we are only interested in calculating the tree-level masses, we drop the interactions (terms containing products of three or more fields). After expanding the Higgs field with (C.1) and performing the rotation (C.2), the Lagrangian becomes

$$
\begin{aligned}
U_{\rm hd} =& \frac{\lambda}{4}(\phi_c^2 - v^2)^2 + \lambda\phi_c(\phi_c^2 - v^2)\hat{h} + \frac{1}{2}\lambda(3\phi_c^2 - v^2)\hat{h}^2 + \frac{1}{2}\lambda(\phi_c^2 - v^2)\hat{P}^2 \\
&+ \lambda(\phi_c^2 - v^2)\hat{h}^+\hat{h}^- \,, \qquad \text{(C.9)} \\
\mathcal{L}_{\rm hd}^{\rm (H)} + \mathcal{L}_{\rm hd}^{\rm (g.f.)} =& \frac{1}{2}\left[\left(\partial_\mu \hat{h}\right)^2 - \frac{1}{\Lambda_H^2}(\partial^2\hat{h})^2\right] + \frac{1}{2}\left[\left(\partial_\mu \hat{P}\right)^2 - \frac{1}{\Lambda_H^2}(\partial^2\hat{P})^2\right] \\
&+ \left[\left|\partial_\mu \hat{h}^+\right|^2 - \frac{1}{\Lambda_H^2}\left|\partial^2\hat{h}^+\right|^2\right] \\
&+ \frac{1}{2}\frac{g^2+g'^2}{4}\phi_c^2\hat{Z}_\mu\hat{Z}^\mu + \frac{g^2}{4}\phi_c^2\left|\hat{W}_\mu^-\right|^2 - \frac{1}{2}\frac{g^2+g'^2}{4}\xi_Z\phi_c^2\hat{P}^2 - \xi_W\frac{g^2}{4}\phi_c^2\left|\hat{h}^+\right|^2 \\
&- \frac{1}{2\xi_A}(\partial_\mu\hat{A}^\mu)^2 - \frac{1}{\xi_W}\left|\partial_\mu\hat{W}^{-\,\mu}\right|^2 - \frac{1}{2\xi_Z}(\partial_\mu\hat{Z}^\mu)^2 \\
&+ \frac{\sqrt{g^2+g'^2}}{2}\phi_c\partial_\mu\left(\hat{P}\hat{Z}^\mu\right) + \frac{g}{2}\phi_c\partial_\mu\left(i\,\hat{h}^+\hat{W}^{-\mu} - i\,\hat{h}^-\hat{W}^{+\,\mu}\right)\,, \qquad \text{(C.10)} \\
\mathcal{L}_{\rm hd}^{\rm (B)} + \mathcal{L}_{\rm hd}^{\rm (W)} =& -\frac{1}{4}\left(\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu\right)^2 - \frac{1}{4}\left(\partial_\mu\hat{Z}_\nu - \partial_\nu\hat{Z}_\mu\right)^2 - \frac{1}{2}\left|\partial_\mu\hat{W}_\nu^- - \partial_\nu\hat{W}_\mu^-\right|^2 \\
&+ \frac{1}{\Lambda_W^2}\left|\partial^2\hat{W}_\mu^- - \partial_\mu\partial^\nu\hat{W}_\nu^-\right|^2 + \frac{1}{2\Lambda_Z^2}\left(\partial^2\hat{Z}_\mu - \partial_\mu\partial^\nu\hat{Z}_\nu\right)^2 \\
&+ \frac{1}{2\Lambda_A^2}\left(\partial^2\hat{A}_\mu - \partial_\mu\partial^\nu\hat{A}_\nu\right)^2 \\
&- \frac{1}{2\Lambda_{AZ}^2}\left(\partial^2\hat{A}_\mu - \partial_\mu\partial^\nu\hat{A}_\nu\right)\left(\partial^2\hat{Z}^\mu - \partial^\mu\partial^\alpha\hat{Z}_\alpha\right)\,, \qquad \text{(C.11)}
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\Lambda_A &\equiv \left(\tfrac{\cos^2\theta_W}{\Lambda_B^2} + \tfrac{\sin^2\theta_W}{\Lambda_W^2}\right)^{-1/2}\,, \\
\Lambda_Z &\equiv \left(\tfrac{\sin^2\theta_W}{\Lambda_B^2} + \tfrac{\cos^2\theta_W}{\Lambda_W^2}\right)^{-1/2}\,, \qquad \text{(C.12)} \\
\Lambda_{AZ} &\equiv \left(\tfrac{\sin 2\theta_W}{\Lambda_B^2} - \tfrac{\sin 2\theta_W}{\Lambda_W^2}\right)^{-1/2}\,.
\end{aligned}
$$

The final two terms in (C.10) are total derivatives and can be dropped. After integrating by parts and dropping total derivative terms, one obtains

$$
\begin{aligned}
\mathcal{L}_{\text{hd}}^{(\text{EW})} = & -\frac{\lambda}{4}(\phi_c^2 - v^2)^2 - \lambda\phi_c(\phi_c^2 - v^2)\hat{h} && \text{(C.13)} \\
& + \frac{1}{2}\hat{h}\left(-\partial^2 - \frac{1}{\Lambda_H^2}\partial^4 - m_{\hat{h}}^2\right)\hat{h} + \frac{1}{2}\hat{P}\left(-\partial^2 - \frac{1}{\Lambda_H^2}\partial^4 - m_{\hat{P}}^2\right)\hat{P} \\
& \quad + \hat{h}^+\left(-\partial^2 - \frac{1}{\Lambda_H^2}\partial^4 - m_{\hat{h}^\pm}^2\right)\hat{h}^- \\
& + \frac{1}{2}\hat{A}^\mu\left[-g_{\mu\nu}\left(-\partial^2 - \frac{\partial^4}{\Lambda_A^2} - m_{\hat{A}}^2\right) + \left(-\frac{\partial^2}{\Lambda_A^2} - 1 + \frac{1}{\xi_A}\right)\partial_\mu\partial_\nu\right]\hat{A}^\nu \\
& + \frac{1}{2}\hat{Z}^\mu\left[-g_{\mu\nu}\left(-\partial^2 - \frac{\partial^4}{\Lambda_Z^2} - m_{\hat{Z}}^2\right) + \left(-\frac{\partial^2}{\Lambda_Z^2} - 1 + \frac{1}{\xi_Z}\right)\partial_\mu\partial_\nu\right]\hat{Z}^\nu \\
& + \frac{1}{2}\hat{A}^\mu\left[-\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\frac{\partial^2}{\Lambda_{AZ}^2}\right]\hat{Z}^\nu && \text{(C.14)} \\
& + \hat{W}^{+\mu}\left[-g_{\mu\nu}\left(-\partial^2 - \frac{\partial^4}{\Lambda_W^2} - m_{\hat{W}^\pm}^2\right) + \left(-\frac{\partial^2}{\Lambda_W^2} - 1 + \frac{1}{\xi_W}\right)\partial_\mu\partial_\nu\right]\hat{W}^{-\nu} \\
& + \bar{c}_A(-\partial^2)c_A + \bar{c}_Z\left(-\partial^2 - \xi_Z m_{\hat{Z}}^2\right)c_Z + \bar{c}_{W^+}\left(-\partial^2 - \xi_W m_{\hat{W}^\pm}^2\right)c_{W^+} \\
& \quad + \bar{c}_{W^-}\left(-\partial^2 - \xi_W m_{\hat{W}^\pm}^2\right)c_{W^-}\,, && \text{(C.15)}
\end{aligned}
$$

where

$$
\begin{array}{lll@{\qquad}lll}
m_{\hat{h}}^2 & \equiv & \lambda(3\phi_c^2 - v^2)\,, & m_{\hat{W}^\pm}^2 & \equiv & \frac{g^2}{4}\phi_c^2\,, \\
m_{\hat{P}}^2 & \equiv & \lambda(\phi_c^2 - v^2) + \xi_Z m_{\hat{Z}}^2\,, & m_{\hat{Z}}^2 & \equiv & \frac{g^2+g'^2}{4}\phi_c^2\,, \\
m_{\hat{h}^\pm}^2 & \equiv & \lambda(\phi_c^2 - v^2) + \xi_W m_{\hat{Z}}^2\,, & m_{\hat{A}}^2 & \equiv & 0\,.
\end{array}
\qquad \text{(C.16)}
$$

With the Lagrangian in this form, it is straightforward to read off the propagators. For the scalars one finds

$$
\begin{aligned}
D_{\hat{h}}(p) &= i\left(p^2 - \frac{p^4}{\Lambda_H^2} - m_{\hat{h}}^2\right)^{-1} = \frac{\Lambda_H^2}{m_{\tilde{h}}^2 - m_h^2}\left(\frac{i}{p^2 - m_h^2} - \frac{i}{p^2 - m_{\tilde{h}}^2}\right)\,, \\
D_{\hat{P}}(p) &= i\left(p^2 - \frac{p^4}{\Lambda_H^2} - m_{\hat{P}}^2\right)^{-1} = \frac{\Lambda_H^2}{m_{\tilde{P}}^2 - m_P^2}\left(\frac{i}{p^2 - m_P^2} - \frac{i}{p^2 - m_{\tilde{P}}^2}\right)\,, \\
D_{\hat{h}^\pm}(p) &= i\left(p^2 - \frac{p^4}{\Lambda_H^2} - m_{\hat{h}^\pm}^2\right)^{-1} = \frac{\Lambda_H^2}{m_{\tilde{h}^\pm}^2 - m_{h^\pm}^2}\left(\frac{i}{p^2 - m_{h^\pm}^2} - \frac{i}{p^2 - m_{\tilde{h}^\pm}^2}\right)\,, \qquad \text{(C.17)}
\end{aligned}
$$

where

$$
\begin{array}{llll}
& \text{SM-like Pole} & & \text{LW-like Pole} \\
m_h^2 = \frac{\Lambda_H^2}{2}\left(1-\sqrt{1-\frac{4m_{\hat{h}}^2}{\Lambda_H^2}}\right), & & m_{\tilde{h}}^2 = \frac{\Lambda_H^2}{2}\left(1+\sqrt{1-\frac{4m_{\hat{h}}^2}{\Lambda_H^2}}\right), \\
m_P^2 = \frac{\Lambda_H^2}{2}\left(1-\sqrt{1-\frac{4m_{\hat{P}}^2}{\Lambda_H^2}}\right), & & m_{\tilde{P}}^2 = \frac{\Lambda_H^2}{2}\left(1+\sqrt{1-\frac{4m_{\hat{P}}^2}{\Lambda_H^2}}\right), \\
m_{h^\pm}^2 = \frac{\Lambda_H^2}{2}\left(1-\sqrt{1-\frac{4m_{\hat{h}^\pm}^2}{\Lambda_H^2}}\right), & & m_{\tilde{h}^\pm}^2 = \frac{\Lambda_H^2}{2}\left(1+\sqrt{1-\frac{4m_{\hat{h}^\pm}^2}{\Lambda_H^2}}\right).
\end{array}
\tag{C.18}
$$

The poles are classified as "SM-like" or "LW-like", depending on whether the residue of the pole is positive or negative.

In the gauge sector, the ghost propagators are immediately seen to be

$$
\begin{aligned}
D_{c_A}(p) &= \tfrac{i}{p^2}\,, \\
D_{c_Z}(p) &= \tfrac{i}{p^2-\xi_Z\, m_{\hat{Z}}^2}\,, \\
D_{c_{W^+}}(p) &= \tfrac{i}{p^2-\xi_W\, m_{\hat{W}^\pm}^2}\,, \\
D_{c_{W^-}}(p) &= \tfrac{i}{p^2-\xi_W\, m_{\hat{W}^\pm}^2}\,.
\end{aligned}
\tag{C.19}
$$

We define the transverse and longitudinal projection operators $\Pi_T^{\mu\nu} \equiv g^{\mu\nu} - p^\mu p^\nu/p^2$ and $\Pi_L^{\mu\nu} \equiv p^\mu p^\nu/p^2$, and obtain

$$
\begin{aligned}
D_{\hat{W}^\pm}^{\mu\nu}(p) =& - i\,\Pi_T^{\mu\nu}(p)\left(p^2 - \frac{p^4}{\Lambda_W^2} - m_{\hat{W}^\pm}^2\right)^{-1} - i\,\Pi_L^{\mu\nu}(p)\left(\frac{p^2}{\xi_W} - m_{\hat{W}^\pm}^2\right)^{-1} \\
=& \frac{\Lambda_W^2}{m_{\tilde{W}^\pm}^2 - m_{W^\pm}^2}\left(\frac{-i\,\Pi_T^{\mu\nu}(p)}{p^2-m_{W^\pm}^2} - \frac{-i\,\Pi_T^{\mu\nu}(p)}{p^2-m_{\tilde{W}^\pm}^2}\right) + \frac{-i\,\xi_W\,\Pi_L^{\mu\nu}(p)}{p^2-\xi_W m_{\hat{W}^\pm}^2}\,,
\end{aligned}
\tag{C.20}
$$

where

$$
\begin{array}{llll}
& \text{SM-like Pole} & & \text{LW-like Pole} \\
m_{W^\pm}^2 = \frac{\Lambda_W^2}{2}\left(1-\sqrt{1-\frac{4m_{\hat{W}^\pm}^2}{\Lambda_W^2}}\right), & & m_{\tilde{W}^\pm}^2 = \frac{\Lambda_W^2}{2}\left(1+\sqrt{1-\frac{4m_{\hat{W}^\pm}^2}{\Lambda_W^2}}\right).
\end{array}
\tag{C.21}
$$

We defer a discussion of the longitudinal polarization state until the end. The term on line (C.14) corresponds to a mixing between transverse polarizations of $\hat{A}^\mu$ and $\hat{Z}^\mu$, which gives rise to off-diagonal terms in the inverse propagator:

$$
(D_{\hat{A}\hat{Z}}^{-1})^{\mu\nu}(p) = i\,\Pi_T^{\mu\nu}\begin{pmatrix} p^2 - \frac{p^4}{\Lambda_A^2} - m_{\hat{A}}^2 & \frac{p^4}{2\Lambda_{AZ}^2} \\ \frac{p^4}{2\Lambda_{AZ}^2} & p^2 - \frac{p^4}{\Lambda_Z^2} - m_{\hat{Z}}^2 \end{pmatrix} + i\,\Pi_L^{\mu\nu}\begin{pmatrix} \frac{p^2}{\xi_A} - m_{\hat{A}}^2 & 0 \\ 0 & \frac{p^2}{\xi_Z} - m_{\hat{Z}}^2 \end{pmatrix}.
\tag{C.22}
$$

For simplicity, we assume just one common LW scale in the EW gauge sector. Then one has $\Lambda_B = \Lambda_W = \Lambda_A = \Lambda_Z \equiv \Lambda_{\rm EW}$ and also $(\Lambda_{AZ})^{-2} = 0$ using (C.12). The mixing vanishes and the propagators become

$$D^{\mu\nu}_{\hat{A}}(p) = \frac{\Lambda^2_{\rm EW}}{m^2_{\tilde{A}} - m^2_A}\left(\frac{-i\,\Pi^{\mu\nu}_T(p)}{p^2 - m^2_A} - \frac{-i\,\Pi^{\mu\nu}_T(p)}{p^2 - m^2_{\tilde{A}}}\right) + \frac{-i\,\xi_A\,\Pi^{\mu\nu}_L(p)}{p^2 - \xi_A m^2_{\hat{A}}}\,, \tag{C.23}$$

$$D^{\mu\nu}_{\hat{Z}}(p) = \frac{\Lambda^2_{\rm EW}}{m^2_{\tilde{Z}} - m^2_Z}\left(\frac{-i\,\Pi^{\mu\nu}_T(p)}{p^2 - m^2_Z} - \frac{-i\,\Pi^{\mu\nu}_T(p)}{p^2 - m^2_{\tilde{Z}}}\right) + \frac{-i\,\xi_Z\,\Pi^{\mu\nu}_L(p)}{p^2 - \xi_Z m^2_{\hat{Z}}}\,, \tag{C.24}$$

where

$$\begin{array}{cccc} & \text{SM-like Pole} & & \text{LW-like Pole} \\ m^2_A = & \frac{\Lambda^2_{\rm EW}}{2}\left(1 - \sqrt{1 - \frac{4m^2_{\hat{A}}}{\Lambda^2_{\rm EW}}}\right) = 0\,, & m^2_{\tilde{A}} = & \frac{\Lambda^2_{\rm EW}}{2}\left(1 + \sqrt{1 - \frac{4m^2_{\hat{A}}}{\Lambda^2_{\rm EW}}}\right) = \Lambda^2_{\rm EW}\,, \\ m^2_Z = & \frac{\Lambda^2_{\rm EW}}{2}\left(1 - \sqrt{1 - \frac{4m^2_{\hat{Z}}}{\Lambda^2_{\rm EW}}}\right)\,, & m^2_{\tilde{Z}} = & \frac{\Lambda^2_{\rm EW}}{2}\left(1 + \sqrt{1 - \frac{4m^2_{\hat{Z}}}{\Lambda^2_{\rm EW}}}\right)\,. \end{array} \tag{C.25}$$

Note that the photon is massless, and that the mass of its LW partner is independent of ϕ_c (see Eq. (C.15)).

Having calculated the spectrum, let us discuss the counting of degrees of freedom. The scalar propagators (C.17) reveal that each of the fields $\hat{h}, \hat{P}, \hat{h}^+$, and $\hat{h}^-$ carries two degrees of freedom: a lighter SM-like resonance and a heavier LW-like resonance. We might expect this doubling to carry over to the gauge fields as well, but an inspection of their propagators reveals that this is not the case. In counting the gauge boson degrees of freedom, note that $\mathrm{Tr}\,\Pi_T = \Pi_{T,\,\mu\nu}g^{\mu\nu} = 3$ and $\mathrm{Tr}\,\Pi_L = 1$. Examining the propagator (C.23), we see that the $\hat{A}$ contains seven degrees of freedom: three massless transverse polarizations ($m^2_A = 0$), one massless longitudinal polarization ($m^2_{\hat{A}} = 0$), and three massive transverse polarizations ($m^2_{\tilde{A}} = \Lambda^2_{\rm EW}$). The four massless degrees of freedom constitute the SM photon, and after accounting for the two "negative degrees of freedom" of the ghosts c_A and $\bar{c}_A$, the count of "physical" photon polarizations is reduced to two. Here, the LWSM does not double the number of gauge degrees of freedom, but instead adds three, which is what one expects for an additional massive resonance. For the $\hat{Z}$ boson we count three degrees of freedom with mass m^2_Z, three degrees of freedom with mass $m^2_{\tilde{Z}}$, one degree of freedom with mass $\xi_Z\, m^2_{\hat{Z}}$, and two negative degrees of freedom of mass $\xi_Z\, m^2_{\hat{Z}}$ coming from the ghosts. The ghost cancels the longitudinal polarization state, and one negative degree of freedom remains. Once we restrict to the Landau gauge ($\xi_A = \xi_Z = \xi_W = 0$), the ghosts and longitudinal polarizations become massless. Then these degrees of freedom do not yield a field-dependent contribution to the effective potential, but they do affect the number of relativistic species at finite temperature. Thus, we have counted them as massless particles in Table 4.2, which also repreises Eqs. (C.18), (C.21), and (C.25).

C.2 Top Sector

Let the SU(2) doublet $\hat{Q}_L = (\hat{u}_L\,,\,\hat{d}_L)^T$ be a left-handed Weyl spinor, and let the singlet $\hat{u}_R$ be a right-handed Weyl spinor. Neglecting gauge interactions, the Lagrangian for the top sector is written as

$$\begin{aligned}\mathcal{L}_{\rm hd}^{(\rm top)} &= (\hat{Q}_L)^\dagger i\bar{\partial\!\!\!/}\hat{Q}_L + \frac{1}{\Lambda_Q^2}(\hat{Q}_L)^\dagger i\bar{\partial\!\!\!/}\partial\!\!\!/\bar{\partial\!\!\!/}\hat{Q}_L + (\hat{u}_R)^\dagger i\partial\!\!\!/\hat{u}_R + \frac{1}{\Lambda_u^2}(\hat{u}_R)^\dagger i\partial\!\!\!/\bar{\partial\!\!\!/}\partial\!\!\!/\hat{u}_R\,,\\ &\quad - h_t\left((\hat{Q}_L)^\dagger\epsilon\hat{H}^*\hat{u}_R - (\hat{u}_R)^\dagger\hat{H}\epsilon\hat{Q}_L\right)\,,\end{aligned} \tag{C.26}$$

where $\partial\!\!\!/ = \sigma^\mu\partial_\mu$ and $\bar{\partial\!\!\!/} = \bar{\sigma}^\mu\partial_\mu$. Contractions of the SU(2) doublets is accomplished with the totally antisymmetric rank-2 tensor ϵ. After electroweak symmetry breaking, one replaces $\hat{H} \to (0\,,\,\phi_c/\sqrt{2})^T$, and obtains

$$\mathcal{L}_{\rm hd}^{(\rm top)} = (\hat{u}_L)^\dagger\left(i\bar{\partial\!\!\!/} + \frac{i\bar{\partial\!\!\!/}\partial\!\!\!/\bar{\partial\!\!\!/}}{\Lambda_Q^2}\right)\hat{u}_L + (\hat{u}_R)^\dagger\left(i\partial\!\!\!/ + \frac{i\partial\!\!\!/\bar{\partial\!\!\!/}\partial\!\!\!/}{\Lambda_u^2}\right)\hat{u}_R - \frac{h_t\,\phi_c}{\sqrt{2}}\left[(\hat{u}_L)^\dagger\hat{u}_R + (\hat{u}_R)^\dagger\hat{u}_L\right]\,. \tag{C.27}$$

One can now collect the Weyl spinors into the Dirac spinor $\hat{t} = (\,\hat{u}_L\,,\,\hat{u}_R\,)^T$. Using the standard definitions

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix} \quad,\quad \bar{\hat{t}} \equiv \hat{t}^\dagger\gamma^0 \quad,\quad \partial\!\!\!/\hat{t} = \gamma^\mu\partial_\mu\hat{t} \quad,\quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad,\quad P_{L,R} = \frac{1\mp\gamma^5}{2}\,,$$

the Lagrangian can be written as

$$\mathcal{L}_{\rm hd}^{(\rm top)} = \bar{\hat{t}}\left(i\partial\!\!\!/ + \frac{i\partial\!\!\!/^3}{\Lambda_Q^2}P_L + \frac{i\partial\!\!\!/^3}{\Lambda_u^2}P_R\right)\hat{t} - \bar{\hat{t}}m_{\hat{t}}\hat{t}\,, \tag{C.28}$$

where $m_{\hat{t}} \equiv h_t\phi_c/\sqrt{2}$. To simplify, we assume that $\Lambda_Q = \Lambda_u \equiv \Lambda_t$. Then the Lagrangian reduces to (4.39), and the propagator is

$$\begin{aligned}D_{\hat{t}}(p) &= i\left(-\frac{p\!\!\!/^3}{\Lambda_t^2} + p\!\!\!/ - m_{\hat{t}}(\phi_c)\right)^{-1}\\ &= +\frac{\Lambda_t^2}{(m_{\tilde{t}_1} - m_t)(m_t - m_{\tilde{t}_2})}\frac{i}{p\!\!\!/ - m_t}\\ &\quad - \frac{\Lambda_t^2}{(m_{\tilde{t}_1} - m_t)(m_{\tilde{t}_1} - m_{\tilde{t}_2})}\frac{i}{p\!\!\!/ - m_{\tilde{t}_1}}\\ &\quad - \frac{\Lambda_t^2}{(m_t - m_{\tilde{t}_2})(m_{\tilde{t}_1} - m_{\tilde{t}_2})}\frac{i}{p\!\!\!/ - m_{\tilde{t}_2}}\,,\end{aligned} \tag{C.29}$$

where

$$\begin{aligned}&\text{SM-like Pole:} && m_t(\phi_c) \equiv \Lambda_t\sqrt{\tfrac{2}{3}\left(1 - \cos\tfrac{\theta_t}{3}\right)}\,,\\ &\text{LW-like Pole:} && m_{\tilde{t}_1}(\phi_c) \equiv \Lambda_t\sqrt{\tfrac{2}{3}\left(1 + \cos\tfrac{\theta_t+\pi}{3}\right)}\,,\\ &\text{LW-like Pole:} && m_{\tilde{t}_2}(\phi_c) \equiv -\Lambda_t\sqrt{\tfrac{2}{3}\left(1 + \cos\tfrac{\theta_t-\pi}{3}\right)}\,,\end{aligned} \tag{C.30}$$

where $\theta_t \equiv \arctan \frac{2\sqrt{\alpha(1-\alpha)}}{1-2\alpha}$ and $\alpha \equiv \frac{27}{4}\frac{m_{\hat{t}}^2}{\Lambda_t^2}$. The angle $0 \leq \theta_t \leq \pi$ is in the first or second quadrant, and the LW stability condition imposes $\alpha < 1$.