A Time-Varying Premium for Idiosyncratic Risk:

Its Effects on the Cross-Section of Stock Returns

by

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ABSTRACT

Merton (1987) predicts that idiosyncratic risk can be priced. I develop a simple equilibrium model of capital markets with information costs in which the idiosyncratic risk premium depends on the average level of idiosyncratic volatility. This dependence suggests that the idiosyncratic risk premium varies over time. I find that in U.S. markets, the covariance between stock-level idiosyncratic volatility and the idiosyncratic risk premium explains future stock returns. Stocks in the highest quintile of the covariance between the volatility and risk premium earn an average 3-factor alpha of 70 bps per month higher than those in the lowest quintile.

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CHAPTER 1

INTRODUCTION

Individual investors often hold under-diversified portfolios. This could be related to a number of factors, such as transaction costs, information acquisition costs, or behavioral biases.[1] Various theories, including those set forth in Levy (1978), Merton (1987), and Malkiel and Xu (2006), suggest that idiosyncratic risk can be priced when investors do not fully diversify their portfolios. Prior empirical studies that examine the relation between idiosyncratic risk and expected stock returns find mixed results. Ang, Hodrick, Xing, and Zhang (2006, 2009) (AHXZ here after) find a negative cross-sectional relation between monthly stock returns and one-month lagged idiosyncratic volatility, which they attribute to an omitted risk factor. On the other hand, Chua, Goh, and Zhang (2008), Spiegel and Wang (2006), and Fu (2009) suggest that the risk-return tradeoff is contemporaneous, and they find positive relation between monthly stock returns and expected idiosyncratic volatility. All of these papers implicitly assume that the idiosyncratic risk premium remains constant over time.

I develop a simple equilibrium model of capital markets with information costs in the spirit of Merton (1987), in which the idiosyncratic risk premium depends on the average level of idiosyncratic volatility. The intuition of the model is straight forward. If the total compensation for bearing idiosyncratic risk follows the law of diminishing marginal returns, the price of idiosyncratic risk should depend negatively on the average

[1]For theories of under-diversification, see Brennan (1975) on transaction costs, Merton (1987) on information acquisition costs, Barberis, Huang and Thaler (2006) and Odean (1999) on psychological biases. For empirical evidence of under-diversification among U.S. investors, see Barber and Odean (2000), Polkovnichenko (2005), and Goetzmann and Kumar (2008).

level of idiosyncratic risk. Therefore, any changes in average idiosyncratic risk induce changes in the price of idiosyncratic risk. When the average level of idiosyncratic risk goes up, investors endogenously adjust their portfolios to improve their diversification, the idiosyncratic risk premium (the price for bearing each unit of idiosyncratic risk) goes down. Campbell, Lettau, Malkiel, and Xu (2001) show that average idiosyncratic risk varies considerably over time. My model suggests that these changes in average risk should generate a time-varying idiosyncratic risk premium.

The model is empirically tractable. It explicitly predicts that the idiosyncratic risk premium is proportional to the inverse of average idiosyncratic risk.[2] Time variation in the idiosyncratic risk premium therefore is mirrored by the time variation in the average idiosyncratic risk, which is observable. Since each stock's idiosyncratic volatility comoves with average idiosyncratic volatility, the idiosyncratic risk premium is correlated with stock-level idiosyncratic volatility. The covariance between stock-level idiosyncratic volatility and the idiosyncratic risk premium affects cross-sectional stock returns like an additional "factor" loading. In other words, the stocks whose idiosyncratic risk commoves positively with the risk premium should have higher returns. I refer to this covariance between idiosyncratic risk and the price of this risk as idiosyncratic risk premium sensitivity ($IRPS$).[3] A positive $IRPS$ implies that the stock's idiosyncratic volatility is higher when investors are more averse to idiosyncratic volatility.

Following AHXZ (2006) and Fu (2009), I measure the idiosyncratic volatility of each stock in each month as the mean squared error of the residuals from the time-series

[2]More precisely, the term "average idiosyncratic risk" refers to standard deviation, i.e. the square root of average idiosyncratic volatility.
[3]Although $IRPS$ affects stock returns like an additional "factor" loading, essentially it is not a risk factor.

regression of the stock's daily excess returns onto the three Fama-French factors. I compute the average idiosyncratic volatility across all stocks in U.S. markets. Then I measure $IRPS$ of each stock by estimating the covariance between the stock's idiosyncratic volatility and the inverse of average idiosyncratic risk, based on the prior 60 months.

In monthly Fama-MacBeth regressions between July 1968 and December 2012, the average slopes on $IRPS$ are positive and statistically significant with a *t*-statistic of 2.91, after controlling for size, book-to-market ratio, momentum and short-term return reversals. Portfolio sorts also show a positive relation between $IRPS$ and Fama-French 3-factor alphas. The highest $IRPS$ quintile portfolio has an average 3-factor alpha of 21 basis points per month with a *t*-statistic of 2.63. The lowest $IRPS$ quintile portfolio has an average 3-factor alpha of -49 basis points per month with a *t*-statistic of -3.93. The 3-factor alpha of the long-short portfolio is as large as 70 basis points per month (8.73% per year), with a robust *t*-statistic of 4.22. Moreover, a portfolio comprised solely of stocks with positive $IRPS$ has an average 3-factor alpha of 47 basis points per month with a *t*-statistic of 4.00.

The $IRPS$ effect is also present in large-cap stocks, and in stocks with low one-month lagged idiosyncratic volatilities. This is important because these securities account for majority of the total market capitalization. Furthermore, the $IRPS$ effect remains significant at least 12 months after the portfolio formation. These features clearly distinguish the $IRPS$ effect from the idiosyncratic volatility puzzle documented by AHXZ (2006).

My work is related to the extensive literature on idiosyncratic risk. The model is in the spirit of Merton (1987), in which investors' portfolio diversification is endogenously determined. Unlike extant empirical studies, my paper suggests a time-varying premium for bearing idiosyncratic risk, and provides evidence that the covariance between stock-level idiosyncratic risk and the premium of this risk affects cross-sectional stock returns. In that sense, the test can be thought of as paralleling the conditional CAPM literature in that when risk premium and stock-level risk are both time-varying and correlated, unconditional alpha can be non-zero.[4]

The rest of the paper is organized as follows: Section 2 contains the theoretical framework. Section 3 describes the data, examines the effect of $IRPS$ on cross-sectional stock returns, and presents the empirical results. I draw my conclusion in Section 4.

[4]For conditional CAPM studies, see Ferson and Harvey (1991), Jagannathan and Wang (1996), Lettau and Ludvigson (2001), Zhang (2005), etc.

CHAPTER 2

THEORETICAL FRAMEWORK

In the first subsection, I suggest that the premium for bearing idiosyncratic risk varies over time. In the second subsection, I explore the impact on the cross-section of expected stock returns caused by this time-varying premium for bearing idiosyncratic risk. I show that the idiosyncratic risk premium sensitivity ($IRPS$) affects cross-sectional stock returns like an additional "factor" loading.

2.1. Capital market equilibrium with endogenous diversification

I develop a simple equilibrium model of capital markets with information costs in the spirit of Merton (1987). My model embodies Merton's insight that idiosyncratic risk can be priced when investors do not fully diversify their portfolios perhaps because of various frictional costs. My main contribution is to endogenize portfolio diversification.[5] My model delivers novel implications.

The key implication of this model is that the premium for bearing idiosyncratic risk negatively depends on the average level of idiosyncratic volatility. The intuition is straight forward. When the average level of idiosyncratic volatility goes up, investors adjust to improve their portfolio diversification, and the idiosyncratic risk premium (the price for bearing each unit of idiosyncratic volatility) goes down. More importantly, this dependence suggests that any changes in the average level of idiosyncratic volatility induce changes in the idiosyncratic risk premium. Prior empirical studies, e.g. Campbell, Lettau, Malkiel, and Xu (2001), show that the average level of idiosyncratic volatility

[5] In Merton (1987), the degree of portfolio diversification is exogenously given.

varies considerably over time. These changes in average volatility should generate a time-varying premium for bearing idiosyncratic risk.

Most of this model and notation closely follows Merton (1987). The economy has N firms, $N \gg 1$. The return from investing in firm n is specified as:

$$\tilde{R}_n = \bar{R}_n + b\tilde{Y} + \sigma_n \tilde{\varepsilon}_n$$
$$n = 1, \dots, N \tag{1}$$

In the equation above, $\tilde{Y}$ is a common factor with $E(\tilde{Y}) = 0$, $E(\tilde{Y}^2) = 1$; $\tilde{\varepsilon}_n$ is a firm-specific random variable with $E(\tilde{\varepsilon}_n) = E(\tilde{\varepsilon}_n | \tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_{n-1}, \tilde{\varepsilon}_{n+1}, \dots, \tilde{\varepsilon}_N, Y) = 0$, $n = 1, \dots, N$ and $E(\tilde{\varepsilon}_n{}^2) = 1$; σ_n^2 denotes the idiosyncratic volatility of security n, and $\overline{\sigma_n^2} = \frac{1}{N}\sum_{n=1}^{N} \sigma_n^2$ will denote the average idiosyncratic volatility across the N securities. To focus on the pricing effect of idiosyncratic volatility, I assume that the N firms have same initial size and same factor loading b. Let $\bar{R}_M$ denotes the market average of expected returns of the N securities $\bar{R}_M = \frac{1}{N}\sum_{n=1}^{N} \bar{R}_n$.

Besides the N firm securities, the economy has two "inside" securities: i) a riskless security with return R_f; ii) a security $N + 1$ with return $\tilde{R}_{N+1} = \bar{R}_{N+1} + \tilde{Y}$. Investors' aggregate demand for each "inside" security is zero in equilibrium.

The economy has K investors, $K \gg N$. Investors are risk averse, have identical preferences and same initial wealth W_o. Investors are price takers and construct their portfolio by mean-variance optimization. The preference of investor is written as:

$$U_k = E(\tilde{R}_k) - \frac{\delta}{2} Var(\tilde{R}_k)$$
$$k = 1, \dots, K \tag{2}$$

In the equation above, $\tilde{R}_k$ denotes the portfolio return of investor k; δ is the coefficient of risk aversion. Any investor's information set contains two portions: firm-specific knowledge and common knowledge. An investor is said to be "informed" about a firm n if he knows $(\bar{R}_n, \sigma_n^2)$. An investor must spend a fixed cost I so that he can process information to know $(\bar{R}_n, \sigma_n^2)$. Accordingly, investor k can randomly select Q_k firms to know at information costs $Q_k I$. His selection must be random because all unknown firms appear the same to the investor. Because of information costs, the Q_k securities are only a subset of the N securities. The subsets are different across the K investors. Beside firm-specific knowledge, every investor's information set contains common knowledge: $\left(b, R_f, \bar{R}_{N+1}, \bar{R}_M, \overline{\sigma_n^2}, I\right)$.

The key assumption, as in Merton (1987), is that any investor k uses a security n in constructing his portfolio only if the investor is informed about the firm n. Consequently, the stock number Q_k also represents the degree of diversification of investor k. However, different from Merton (1987), I assume that any investor k can choose Q_k, the number of firms he wants to know.

It is worth noting that my model considers information costs as the only type of frictional costs. Nevertheless, the model's main results could be similarly derived from assuming other types of costs or behavior biases. Merton pointed out in his paper: "There are, of course, a number of other factors in addition to incomplete information that in varying degrees, could contribute to this observed behavior. Because the under-diversification behavior can be derived from a variety of underlying structural assumptions, the formally derived equilibrium-pricing results are theoretical analog to reduced-form equations". [6]

The capital market equilibrium is formulated as: i) Given security expected returns, each investor chooses the optimal portfolio; ii) Market clearing; iii) Given security expected returns, no investor has incentive to increase Q_k.[7]

After solving for equilibrium security prices (see Appendix), The expected return of security n is:

$$\bar{R}_n = R_f + b^2\delta + \frac{\delta}{\bar{Q}^*}\sigma_n^2 \tag{3}$$

and the average portfolio diversification across the K investors is:

$$\bar{Q}^* = \sqrt{\frac{\delta\overline{\sigma_n^2}}{2I}} \tag{4}$$

In the equation above, $\overline{\sigma_n^2}$ is the average idiosyncratic volatility across the N securities.

Consistent with the results of Merton (1987), equation (3) implies that stocks with higher idiosyncratic volatility σ_n^2 have higher returns. The idiosyncratic risk premium $\gamma_{IV} = \frac{\delta}{\bar{Q}^*}$. This risk premium depends positively on investor risk aversion δ and negatively on the average diversification $\bar{Q}^*$. If all investors are fully diversified, the idiosyncratic risk premium goes to zero.

Equation (4) denotes that the average diversification $\bar{Q}^*$ is endogenously determined. From equation (4), the average diversification depends positively on the

[6] Here, "this observed behavior" refers to the fact that the portfolios held by actual investors contain only a small fraction of the thousands of traded securities available.
[7] An investor can't decrease Q_k. Once an investor is informed about a firm, this cannot be undone.

average level of idiosyncratic volatility $\overline{\sigma_n^2}$, which represents the benefit of having more securities in constructing a portfolio; and the average diversification depends negatively on information cost I. If the average level of idiosyncratic volatility goes up, investors adjust to improve their diversification. If the market becomes frictionless ($I = 0$), investors are perfectly diversified.

From equation (3) (4), I can rewrite the idiosyncratic risk premium as:

$$\gamma_{IV} = \frac{\delta}{\bar{Q}^*} = \frac{c}{\sqrt{\overline{\sigma_n^2}}} \tag{5}$$

In the equation above, $c = \sqrt{2I\delta}$ is a positive constant. Equation (5) implies that the idiosyncratic risk premium γ_{IV} depends negatively on the average level of idiosyncratic volatility. A higher level of average idiosyncratic volatility would lead to a lower risk premium.

The total compensation for bearing idiosyncratic risk is:

$$X_{IV} = N \cdot \overline{\sigma_n^2} \cdot \gamma_{IV} = N \cdot c\sqrt{\overline{\sigma_n^2}} \tag{6}$$

From equation (6), a higher level of average idiosyncratic volatility leads to a higher total compensation for bearing idiosyncratic risk. Equation (6) also formulates that the total compensation for bearing idiosyncratic risk is a concave function of the average volatility $\overline{\sigma_n^2}$. In other words, it follows the law of diminishing marginal returns: on the one hand, investors would require higher total compensation for bearing higher level of idiosyncratic risk; on the other hand, improved diversification would lead to a decline in the premium for bearing each unit of idiosyncratic risk, as denoted by equation (5).

Although derived from a single-period model, equation (5) provides the rationale for a time-varying idiosyncratic risk premium. Equation (5) predicts that the idiosyncratic risk premium depend negatively on the average level of idiosyncratic risk. For example, when the average level of idiosyncratic volatility $\overline{\sigma_n^2}$ goes up, investors adjust to improve their portfolio diversification, and the idiosyncratic risk premium γ_{IV} goes down. Hence, changes in average volatility $\overline{\sigma_n^2}$ should generate a time-varying idiosyncratic risk premium. In the next subsection, I will show that when the idiosyncratic risk premium varies over time, the timing of idiosyncratic risk affects stock returns.

2.2. Idiosyncratic risk premium sensitivity ($IRPS$) and the cross-section of stock returns

The conditional CAPM literatures, e.g. Jagannathan and Wang (1996), suggest that when the risk premium and stock-level risk are both time-varying and correlated, unconditional alpha can be non-zero. In this subsection, I borrow this logic from the conditional CAPM and apply it to the area of idiosyncratic risk pricing. I assume that the tradeoff between idiosyncratic risk and stock returns holds contemporaneously. Then I show that the covariance between stock-level idiosyncratic risk and the premium of this risk should affect cross-sectional stock returns like an additional "factor" loading.

I assume that this risk-return tradeoff holds period by period. The conditional expected stock return can be specified by:

$$R_{it} = \gamma_{0t} + \sum_{f=1}^{F} X_{fit}\gamma_{ft} + \sigma_{IVit}^2\gamma_{IVt} \quad (7)$$

$$i = 1,2, \dots, N_t, \quad t = 1,2, \dots, T$$

In the equation above, conditional moments for period t given the information set at time $t-1$ are labeled with a t subscript: R_{it} is the conditional expected return of stock i, σ_{IVit}^2 is the conditional expected idiosyncratic volatility of stock i, γ_{IVt} is conditional risk premium for bearing idiosyncratic risk, X_{fit} are conditional values of other explanatory variables for cross-sectional stock returns, γ_{ft} is the conditional risk premium associated with the corresponding explanatory variable, T is the total number of time periods, and N_t is the total number of stocks at time t.

Some empirical studies, e.g. Chua, Goh, and Zhang (2008), Spiegel and Wang (2006), and Fu (2009), attempt to examine the contemporaneous idiosyncratic risk-return tradeoff. They find a positive relation between stock returns and conditional idiosyncratic volatility. However, they implicitly assume that the idiosyncratic risk premium γ_{IVt} remains constant over time. Their studies do not fully explore the implications of the contemporaneous risk-return tradeoff, because what they really examine is:

$$R_{it} = \gamma_{0t} + \sum_{f=1}^{F} X_{fit}\gamma_{ft} + \sigma_{IVit}^2 \gamma_{IV} \tag{8}$$

In section 2.1, my analysis suggests that the risk premium γ_{IVt} can vary considerably over time, and it is well known that stock-level idiosyncratic volatility σ_{IVit}^2 also varies considerably over time. When the risk premium and stock-level risk are both time-varying, the covariance between them affects average stock returns. To see this, taking the unconditional expectation of both sides of equation (7), I have:

$$E[R_{it}] = E[\gamma_{0t}] + \sum_{f=1}^{F} E\left[X_{f_{it}}\right] E\left[\gamma_{ft}\right] + \cdots$$

$$+ E\left[\sigma^2_{IV_{it}}\right] E[\gamma_{IVt}] + Cov(\sigma^2_{IVit}, \gamma_{IVt}) \quad (9)$$

The covariance term on the right-hand side of equation (9) is the sensitivity of stock-level idiosyncratic risk to the premium of this risk. I refer to this covariance $Cov(\sigma^2_{IVit}, \gamma_{IVt})$ as idiosyncratic risk premium sensitivity ($IRPS$).

$$IRPS_i = Cov(\sigma^2_{IVit}, \gamma_{IVt}) \quad (10)$$

Equation (9) implies that $IRPS$ affects the average stock returns like an additional "factor" loading. Specifically, the stocks whose idiosyncratic risk commoves positively with the premium of this risk should have higher average returns. In that sense, the $IRPS$ effect can be thought of as paralleling the conditional CAPM literature in that when the risk premium and stock-level risk are both time-varying and correlated, unconditional alpha can be non-zero. Intuitively, a positive $IRPS$ implies that the stock's idiosyncratic volatility is higher when investors become more averse to idiosyncratic volatility. Hence, $IRPS$ represents the timing of idiosyncratic risk.

According to the equation (5) in section 2.1, the idiosyncratic risk premium is inversely proportional to average idiosyncratic risk. I will also assume that this dependence relationship holds period by period:

$$\gamma_{IVt} = \frac{c}{\sqrt{\overline{\sigma^2_{IV}}_t}} \quad (11)$$

In the equation above, $\overline{\sigma_{IV}^2}_t$ is the average idiosyncratic volatility across all stocks, and c is a positive constant. From equation (11), time variation in the idiosyncratic risk premium therefore is mirrored by the time variation in the average idiosyncratic risk, which is observable. Equation (11) delivers an empirically tractable measure of $IRPS$. By plugging equation (11) into equation (10), I have a stock's $IRPS$ captured by:

$$IRPS_i = Cov\left(\sigma_{IVit}^2, \frac{1}{\sqrt{\overline{\sigma_{IV}^2}_t}}\right) \tag{12}$$

It is the covariance between the stock-level idiosyncratic volatility and the inverse of average idiosyncratic risk.[8] In general, stock-level idiosyncratic volatility and the average idiosyncratic volatility are correlated, and stocks' $IRPS$ are nonzero. Equation (9) predicts a positive relation between $IRPS$ and cross-sectional stock returns.

[8]I omit the constant c, since it is the same for each stock.

CHAPTER 3

EMPIRICAL FRAMEWORK

In this section, I investigate the cross-sectional relation between future stock returns and $IRPS$. I describe how I measure $IRPS$ for each stock. Then I test whether $IRPS$ explains future stock returns by estimating Fama-MacBeth regressions, as well as using portfolio sorts. In addition, I examine the robustness of the $IRPS$ effect among subsample groups. I show how the $IRPS$ effect is distinct from the idiosyncratic volatility puzzle documented by AHXZ (2006).

3.1. Estimating stock idiosyncratic volatility

The sample consists of all common stocks on CRSP that are traded on the NYSE, Amex and NASDAQ from July 1963 to December 2012. Following prior empirical studies, e.g. AHXZ (2006, 2009) and Fu (2009), I measure idiosyncratic volatility as relative to the 3-factor model. Specifically, in every month, I run time-series regressions of the daily excess returns of each stock onto the three Fama-French factors: MKT_τ, SMB_τ, and HML_τ :

$$r_\tau^i = \alpha_t^i + \beta_{MKT,t}^i MKT_\tau + \beta_{SMB,t}^i SMB_\tau + \beta_{HML,t}^i HML_\tau + \varepsilon_\tau^i \qquad (13)$$

In the equation above, t denotes a given month and τ denotes the days within the month. Daily stock returns are obtained from the CRSP data base. Daily factor data are obtained from Kenneth R. French's website. I exclude the stock-months with fewer than 15 valid daily return observations. The idiosyncratic volatility of each stock in each month is computed as the mean squared error of the residuals from the time-series regression:

$$\sigma_{IVit}^2 = E\left({\varepsilon_\tau^i}^2\right) \tag{14}$$

Idiosyncratic volatility is measured for each stock in each month.[9]

3.2. Measuring Average Idiosyncratic Volatility

Common approaches of computing cross-sectional averages have potential pitfalls in producing a representative measure. For instance, an equal-weight average can be dominated by microcap and small-cap stocks, whose idiosyncratic volatilities are high. On the other hand, a value-weight average can be dominated by some large stocks, whose idiosyncratic volatilities are low. To mitigate these problems, at the end of every month, I sort all stocks by their market caps into two size portfolios of large stocks and small stocks. I first compute the value-weight average of idiosyncratic volatility for each size portfolio. Then I compute the equal average of idiosyncratic volatility across the two size portfolios, which is my measure of average idiosyncratic volatility $\overline{\sigma_{IV}^2}_t$ at month t:

$$\overline{\sigma_{IV}^2} = 0.5\overline{\sigma_{IV}^2}_B + 0.5\overline{\sigma_{IV}^2}_S \tag{15}$$

[9]Empirical results of this paper still hold, if I use longer periods, e.g. 3 months, to compute idiosyncratic volatilities.

Figure 1: Average idiosyncratic volatility

This figure depicts average idiosyncratic volatility $\overline{\sigma^2_{IVt}}$ (Eq.15) from Jul.1963 to Dec.2012. The stock idiosyncratic volatility is measured as relative to the FF-3 model (Eq.14). At the end of every month t, I sort all stocks by their market cap into two size portfolios: large and small. I first compute the value-weight average idiosyncratic volatility for each size portfolio, then compute the simple average idiosyncratic volatility across the two size portfolios.

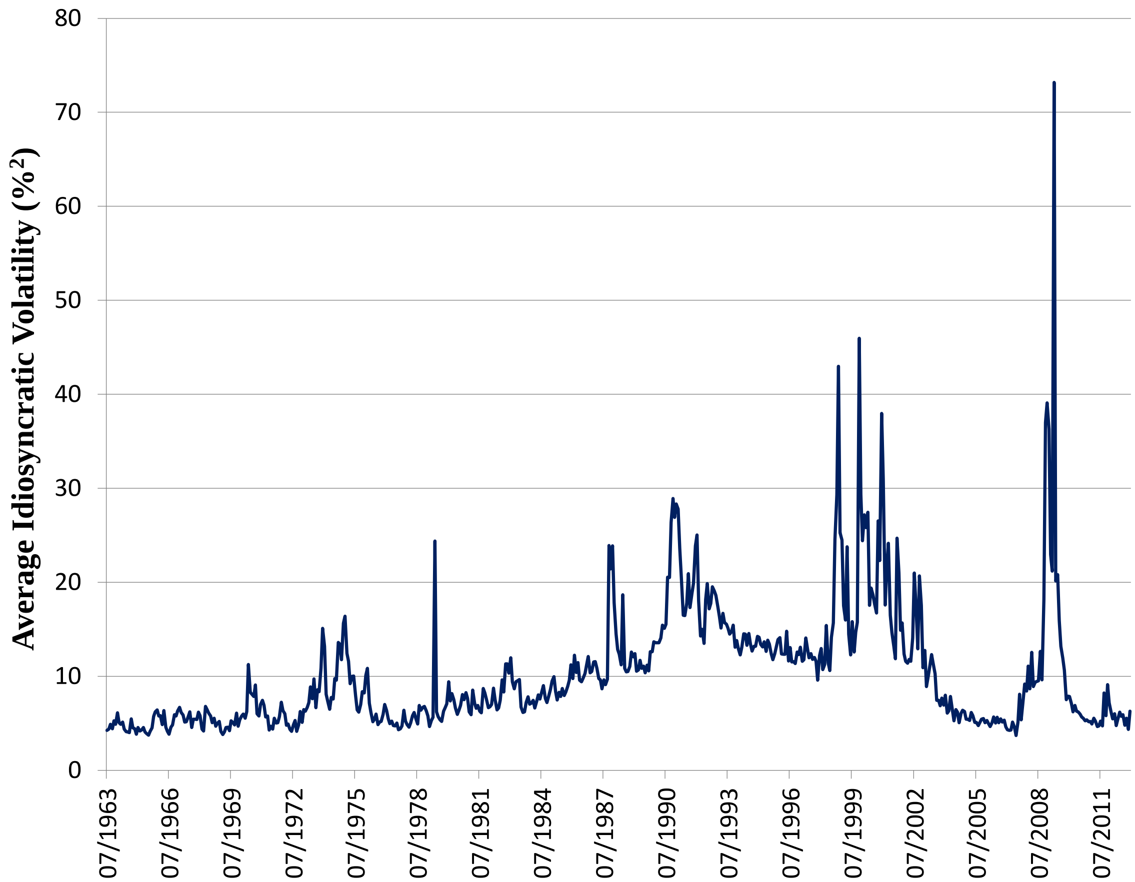


Figure 1 plots the time series of the measured average idiosyncratic volatility $\overline{\sigma_{IV}^2}_t$ from July 1963 to December 2012. The figure shows that average idiosyncratic volatility is far from constant over time. The time-series mean is $10.2\%^2$ and has a standard deviation of $6.7\%^2$. The mean value implies that, for example, a portfolio that equally contains 20 randomly-selected stocks would have mean excess standard deviation of 11.3% per year. As a comparison, the annualized standard deviation of the S&P 500 index during the period 1989-2010 is 19.1%.

3.3. Estimating $IRPS$

According to equation (12), I measure $IRPS$ for each stock by estimating the covariance:

$$IRPS_i = Cov\left(\sigma_{IVit}^2, \frac{1}{\sqrt{\overline{\sigma_{IV}^2}_t}}\right) \tag{16}$$

In the equation above, σ_{IVit}^2 is the idiosyncratic volatility of stock i at month t, and $\overline{\sigma_{IV}^2}_t$ is the average idiosyncratic volatility at month t. I estimate the $IRPS_i$ as in equation (16) using rolling estimates based on 60 previous monthly observations.

As expected, the majority stocks (86.2% of the pooled sample) have negative $IRPS$. From equation (16), negative $IRPS$ means that the stocks' idiosyncratic volatilities comove positively with average idiosyncratic volatility. Correspondingly, a small minority of stocks (13.8% of the pooled sample) exhibit positive $IRPS$.

To investigate the cross-section of returns, I apply the following two filters to my data sample, which initially consists of all common stocks on CRSP from July 1963 to December 2012. First, stocks with a price less than one dollar prior to test month are excluded. Second, each stock is required to have at least 60 month observations prior to test month to obtain a valid measure of $IRPS$. Hence, my test period is from July 1968 to December 2012. On average, the market cap of my test sample accounts for 89% of total market cap. Book values are from Compustat and book-to-market ratios are calculated following the procedure in Fama and French (1992).

3.4. Fama-MacBeth Regressions

I examine the relation between $IRPS$ and expected stock returns by running Fama-MacBeth regressions. The advantage of running Fama-MacBeth regressions is that I can control for multiple stock characteristics. The Fama-MacBeth regressions have two stages.

Specifically, in the first stage, for every month, I run the following cross-sectional

$$r_t^i = c + \gamma IRPS_{t-1}^i + \lambda_X X_{t-1}^i + \varepsilon_t^i \tag{17}$$

regression:

In the equation above, r_t^i is stock i's excess return in month t, $IRPS_{t-1}^i$ is estimated over the previous 60 months from $t-60$ to $t-1$, X_{t-1}^i is a vector of stock characteristics observable at the end of month $t-1$. The stock characteristics include $\ln(Size)_{t-1}^i$, the log of stock i's market capitalization, $\ln(BE/ME)_{t-1}^i$, the log of stock i's book-to-market ratio based on fiscal year's information, MOM_{t-1}^i, the prior return of stock i from month $t-12$ to month $t-2$, as a control variable for momentum effect, and Ret_{t-1}^i, the prior monthly return of stock i, as a control variable for short-term return reversals.

In the second stage, I test whether the average regression coefficients are significantly different from zero. Panel A of Table 1 reports the average coefficients of standard Fama-MacBeth regressions (*t*-statistics are generated using Newey-West procedure with 8 lags). The average slopes on size, book-to-market ratio, momentum, and return reversals are fairly close to prior studies on cross-sectional stock returns. Table 1 shows that the average slope on $IRPS$ is positive and statistically significant with a *t*-statistic of 2.91, after controlling for size, book-to-market ratio, momentum and short-term return reversals. The $IRPS$ effect is also economically significant. From table 1, the average slope on $IRPS$ is within $1.4\sim1.7$, the median value of the standard deviations of $IRPS$ is $4.39\ (\times\ 10^{-3})$. Accordingly, a stock with $IRPS$ one standard deviation higher would earn an average monthly return of $0.65\%\sim0.75\%$ higher.

I also run value-weighted Fama-MacBeth regressions as robustness check. The value-weighted Fama-MacBeth regressions measure the effect of an average dollar, and are more comparable with subsequent value-weighted portfolios (subsection *3.2.3.*). For value-weighted Fama-MacBeth, I do the GLS regressions with a diagonal weighting matrix whose element is the inverse of stock market capitalization. Panel B of Table 1 reports the results of value-weighted Fama-MacBeth regressions. The average slope on $IRPS$ are positive and statistically significant. Compared with the results in panel A, the average slopes on $IRPS$ are higher in magnitude, indicating that the effect of $IRPS$ on stock returns is stronger when this effect is measured for an average dollar.

Table 1: Fama-MacBeth regressions

Panel A reports the results from standard Fama-MacBeth regressions. Panel B reports the results from value-weighted Fama-MacBeth regressions, where each stock is weighted by the stock's market capitalization at the end of month $t-1$. Stock monthly excess returns are regressed on $IRPS$ ($\times 10^2$) and stock characteristics, which include the log of market capitalization $\ln(Size)$, the log of book-to-market ratio $\ln(BE/ME)$, the prior return from month $t-12$ to month $t-2$ MOM, the prior return at month $t-1$ Ret_{-1}, the idiosyncratic volatility at month $t-1$ $IVOL_{-1}$. $IRPS$ is estimated over the previous 60 months from t-60 to t-1. The time-series averages of the second stage coefficients are reported. Robust Newey-West t-statistics are reported using 8 lags. I trim $IRPS$ at 2% at each tail. The test period is from July 1968 to December 2012.

Panel A: Standard Fama-MacBeth regressions

	$IRPS$	$\ln(Size)$	$\ln(BE/ME)$	MOM	Ret_{-1}	$IVOL_{-1}$
I	1.74**	-0.077*	0.198**	0.659**		
	(3.15)	(-2.42)	(2.97)	(3.99)		
II	1.64**	-0.053	0.225**	0.617**	-4.70**	
	(2.91)	(-1.63)	(3.30)	(3.46)	(-9.37)	
III	1.39*	-0.076*	0.204**	0.588**	-4.49**	-1.37**
	(2.43)	(-2.58)	(3.06)	(3.34)	(-9.03)	(-4.45)

Panel B: Value-weighted Fama-MacBeth regressions

	$IRPS$	$\ln(Size)$	$\ln(BE/ME)$	MOM	Ret_{-1}	$IVOL_{-1}$
I	3.33**	-0.064*	0.111	0.631**		
	(2.80)	(-2.18)	(1.39)	(2.80)		
II	2.99*	-0.057	0.132	0.592*	-2.99**	
	(2.51)	(-1.92)	(1.58)	(2.44)	(-5.43)	
III	2.22*	-0.073*	0.116	0.594*	-2.91**	-2.74**
	(2.14)	(-2.47)	(1.40)	(2.49)	(-5.10)	(-4.27)

3.5. Portfolio Returns

At the end of every month $t-1$, I sort all stocks into quintiles based on their $IRPS$, which is estimated over the previous 60 months from $t-60$ to $t-1$, and hold the resulting value-weighted quintile portfolios for the next month t. Table 2 Panel B reports the CAPM alphas, Fama-French 3-factor alphas and Fama-French 4-factor alphas of the quintile portfolios, along with a zero-investment portfolio that shorts quintile 1 and longs quintile 5. The table also reports *t*-statistics generated using Newey-West procedure with 8 lags.

From Table 2, I observe a clearly positive relation between $IRPS$ and alphas. For instance, the monthly 3-factor alpha increases monotonically from -49 basis points for quintile 1 with a *t*-statistic of -3.91, to -19 basis points for quintile 2 with a *t*-statistic of -2.16, further to 21 basis points for quintile 5 with a *t*-statistic of 2.59. The 3-factor alpha of the 5-1 portfolio is as large as 70 basis points per month (8.73% per year), with a robust *t*-statistic of 4.22. Figure 2 plots the monthly 3-factor alphas of the quintile portfolios. Table 2 Panel A reports the summary statistics of the quintiles. On average the stocks in the quintile 1 are relatively small, have higher returns in the month $t-1$, and higher one-month lagged idiosyncratic volatility. Nevertheless, it is important to note that the observed positive relation between $IRPS$ and alphas is not merely driven by the quintile 1.[10]

[10] As a reference, the FF-3 alpha of the 5-2 portfolio (a zero-investment portfolio that shorts the quintile 2 and longs the quintile 5) is as large as 0.404% per month (4.96% per year), with a *t*-statistic of 2.79.

Table 2: Quintile portfolios sorted on $IRPS$

I form value-weighted quintile portfolios at the end of every month by sorting stocks based on $IRPS$ estimated over the previous 60 months. I hold the resulting quintile portfolios for the next month. Portfolio 5-1 is a zero-investment portfolio that shorts the lowest $IRPS$ stocks and goes long the highest $IRPS$ stocks. Panel A reports summary statistics of the quintile portfolios. The row #Stock reports the average number of stocks within each portfolio. The row $IRPS$ reports the average of $IRPS$ for stocks within the portfolio. Size reports the average of market capitalization of stocks measured at the end of every month. The row ln(B/M) reports the average of the natural logarithm of firms' book-to market ratios. The row Mom reports the average of previous stock returns from -2 to -12 month. The row Ret (-1) reports the average of stock returns at -1 month (portfolio-forming month). IVOL (-1) reports the average of stock idiosyncratic volatility at -1 month. Trans. Prob. Reports the average probabilities of stocks remaining in their quintiles from month t-1 to month t. Panel B reports CAPM alphas, Fama-French 3-factor alphas and 4-factor alphas of the quintile portfolios. Robust Newey-West t-statistics are reported using 8 lags. All portfolios are value weighted. The sample period is from July 1963 to December 2012.

Panel A: Summary statistics

	Ranking on $IRPS$				
	1 Low	2	3	4	5 High
$IRPS$ ($\times10^{-4}$)	-85.1	-19.0	-9.0	-4.2	5.1
#Stock	686	686	686	686	686
Size ($B)	0.217	0.706	1.863	3.050	1.836
% Mkt Cap	2.5%	8.7%	22.4%	39.3%	27.1%
ln (B/M)	-0.28	-0.31	-0.35	-0.36	-0.36
Mom (%)	19.6	14.8	14.3	13.8	16.1
Ret (-1) (%)	2.17	1.30	1.22	1.13	1.33
IVOL (-1) ($\%^2$)	27.7	10.8	6.04	3.87	9.58
Trans. Prob.	0.938	0.899	0.879	0.885	0.932

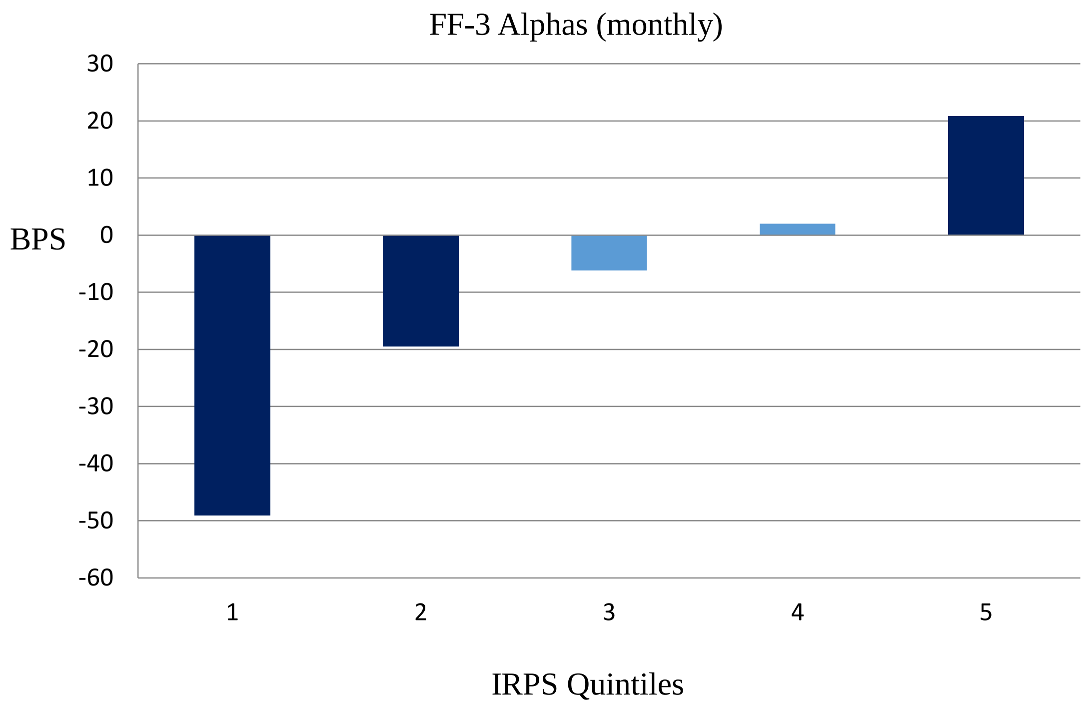
Table 2 (continued)

Panel B: 1-factor, 3-factor, and 4-factor alphas (% monthly)

	Ranking on $IRPS$					
	1 Low	2	3	4	5 High	5-1
1-Factor α	-0.380*	-0.102	0.025	0.088	0.145	0.525*
	(-2.18)	(-0.94)	(0.35)	(1.92)	(1.84)	(2.51)
3-Factor α	-0.491**	-0.195*	-0.062	0.020	0.209**	0.700**
	(-3.91)	(-2.16)	(-0.96)	(0.55)	(2.59)	(4.22)
4-Factor α	-0.328**	-0.061	0.014	0.012	0.165	0.494**
	(-2.60)	(-0.70)	(0.20)	(0.30)	(1.79)	(2.89)

Figure 2: Quintile portfolios sorted on $IRPS$

This figure depicts Fama-French 3-factor alphas of quintile portfolios sorted on $IRPS$. I form value-weighted quintile portfolios at the end of every month by sorting stocks based on $IRPS$. $IRPS$ is estimated over the previous 60 months. I hold the resulting quintile portfolios for the next month. Portfolio 1 (5) is the portfolio of stocks with the lowest (highest) $IRPS$. Darker color bars denote statistically significant alphas.



To further show that the relation between $IRPS$ and stock returns is not merely driven by the stocks with low $IRPS$ (stocks in the quintile 1), I construct portfolios comprised of stocks with positive $IRPS$. The stocks with positive $IRPS$ are particularly interesting for two reasons. First, positive $IRPS$ stocks are the stocks with relatively high $IRPS$ (top 13.8% on average). Positive $IRPS$ has a clear economic meaning: a stock's idiosyncratic volatility is higher when investors become more averse to idiosyncratic volatility. Theory predicts higher returns for stocks with positive $IRPS$. Secondly, positive $IRPS$ stocks generally are not the stocks with low/high one-month lagged idiosyncratic volatility ($IVOL$), as shown by the summary statistics in Table 3 Panel A. Thus, positive $IRPS$ stocks serve as a clean sample where the $IRPS$ effect is disentangled from any potential $IVOL$ effect documented by AHXZ (2006).[11]

[11]In subsection 3.7., I will provide more evidences and further discuss the distinction of $IRPS$ effect from $IVOL$ effect.

Table 3: The portfolios of positive/negative $IRPS$ stocks

I form a value-weighted portfolio at the end of every month containing only the stocks with positive $IRPS$. $IRPS$ is estimated over the previous 60 months. I also form a complementary portfolio which contains all the rest stocks, i.e. the stocks with negative $IRPS$. I hold the resulting two portfolios for the next month. Panel A reports the summary statistics of the two portfolios. The row #Stock reports the average number of stocks within each portfolio. %Stock reports the average share of each portfolio in all stocks. %Mkt Cap reports the average share of each portfolio in total market cap. Size reports the average of market cap of stocks within each portfolio. The row ln(B/M) reports the average of the natural logarithm of book-to-market ratios. The row Mom reports the average of previous stock returns from -2 to -12 month. The row Ret (-1) reports the average of stock returns at -1 month (portfolio-forming month). IVOL (-1) reports the average of stock idiosyncratic volatility at -1 month. Panel B reports the raw CAPM alphas, Fama-French 3-factor alphas and 4-factor alphas. Robust Newey-West t-statistics are reported using 8 lags. All portfolios are value weighted. The sample period is from July 1963 to December 2012.

Panel A: Summary statistics

	Positive $IRPS$	Negative $IRPS$
#Stock	473	2960
% Stock	13.8%	86.2%
% Mkt Cap	8.39%	91.6%
Size ($B)	0.65	1.61
ln(B/M)	-0.35	-0.33
Mom (%)	18.9	15.4
Ret (-1) (%)	1.92	1.54
IVOL (-1) ($\%^2$)	14.4	11.3

Panel B: 1-factor, 3-factor, and 4-factor alphas (% monthly)

	Positive $IRPS$	Negative $IRPS$	Difference
1-Factor α	0.482**	0.029	0.453**
	(4.26)	(1.38)	(4.00)
3-Factor α	0.469**	0.006	0.463**
	(4.05)	(0.38)	(3.93)
4-Factor α	0.450**	0.016	0.434**
	(3.23)	(0.96)	(3.09)

At the end of every month $t - 1$, I construct a portfolio comprised solely of stocks with positive $IRPS$. At the same time, I also construct a complementary portfolio containing all stocks with negative $IRPS$. Again, I hold the resulting value-weighted portfolios for the next month. Table 3 reports the CAPM alphas, Fama-French 3-factor alphas and Fama-French 4-factor alphas of the two portfolios, as well as the summary statistics. Table 3 shows that stocks with positive $IRPS$ have significantly higher returns. For instance, the portfolio of positive $IRPS$ stocks has a 3-factor alpha of 47 basis points per month (5.8% per year) with a t-statistic of 4.05, and a 4-factor alpha of 45 basis points per month (5.5% per year) with a t-statistic of 3.23. On average, the positive $IRPS$ stocks accounts for 13.8% of sample stocks, and the rest stocks accounts for 86.2%. It is worth noting that the stocks with positive $IRPS$ are not different from negative $IRPS$ stocks in their characteristics such as size, book-to-market ratio, momentum, prior monthly return, and one-month lagged idiosyncratic volatility. Nevertheless, the portfolio of positive $IRPS$ stocks has significantly higher returns.

Since $IRPS$ is estimated based on 60 previous monthly observations, the value of $IRPS$ is relatively stable from one month to the next. It is worth examining how long the effect of $IRPS$ on future stock returns can last. Again, at the end of every month, I construct a portfolio comprised stocks with positive $IRPS$, but hold the portfolio for the next 24 months. Table 4 reports the portfolio's CAPM alpha, Fama-French 3-factor alpha and Fama-French 4-factor alpha for the n^{th} month $(1 \leq n \leq 24)$ following portfolio formation. The table clearly shows that, as expected, the $IRPS$ effect is not just a next-month effect. For instance, the portfolio of positive $IRPS$ stocks has significant 3-factor alphas above 30 basis points for each of the 11 months following the portfolio formation.

The 3-factor alpha is still significantly positive in the 14th month after the portfolio formation, and slowly decreases into statistical insignificance. Similar pattern is also observed for the portfolio's CAPM alphas and 4-factor alphas. The results demonstrate that the $IRPS$ effect remains significant at least 12 months after the portfolio formation. Figure 3 plots the portfolio's Fama-French 3-factor alphas for the n^{th} month following portfolio formation.

Table 4: The persistence of the $IRPS$ effect

I form a value-weighted portfolio at the end of every month which contains only the stocks with positive $IRPS$ which is estimated over the previous 60 months. I hold the resulting portfolio for the next 24 months. The table reports that portfolio's CAPM alpha, 3-factor alpha and 4-factor alpha for the n^{th} month $(24 \geq n \geq 1)$ following the portfolio formation. Robust Newey-West t-statistics are reported using 8 lags. All portfolios are value weighted. The sample period is from July 1963 to December 2012.

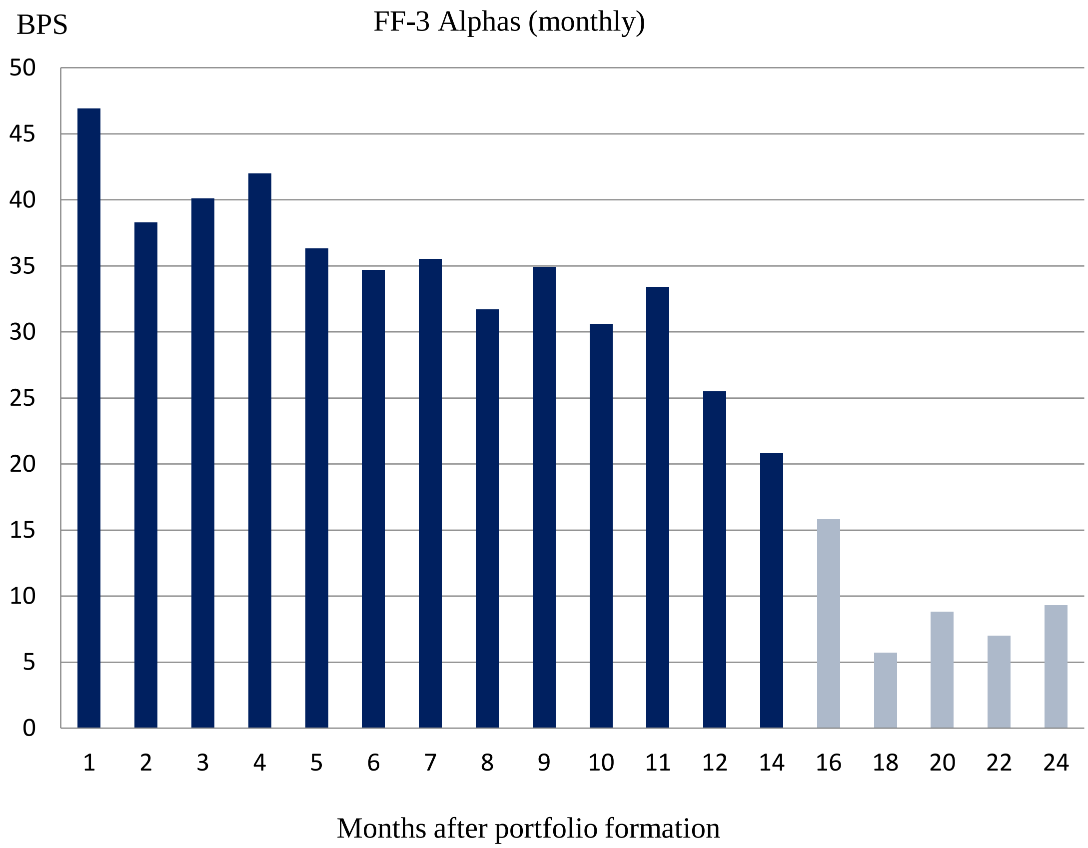
Time after formation	1-Factor Alphas	3-Factor Alphas	4-Factor Alphas
1st Month	0.482**	0.469**	0.450**
	(4.26)	(4.05)	(3.23)
2nd Month	0.396**	0.383**	0.306**
	(4.02)	(3.69)	(2.88)
3rd Month	0.408**	0.401**	0.310**
	(3.88)	(3.78)	(2.85)
4th Month	0.423**	0.420**	0.384**
	(3.98)	(3.77)	(2.78)
5th Month	0.348**	0.363**	0.349**
	(3.70)	(3.59)	(2.74)
6th Month	0.331**	0.347**	0.325**
	(3.36)	(3.43)	(2.67)
7th Month	0.318**	0.355**	0.303**
	(3.39)	(3.64)	(2.66)
8th Month	0.272*	0.317**	0.279*
	(2.53)	(3.08)	(2.46)
9th Month	0.283**	0.349*	0.305**
	(2.59)	(3.57)	(2.79)
10th Month	0.245*	0.306**	0.258*
	(2.06)	(3.05)	(2.55)
11th Month	0.267*	0.334**	0.265**
	(2.00)	(3.07)	(2.64)
12th Month	0.197	0.255*	0.203
	(1.46)	(2.26)	(1.96)
14th Month	0.175	0.208*	0.164
	(1.54)	(1.98)	(1.38)
16th Month	0.137	0.158	0.101
	(1.46)	(1.57)	(1.03)
18th Month	0.052	0.065	0.041
	(0.52)	(0.65)	(0.41)

Table 4: (continued)

Time after formation	1-Factor Alphas	3-Factor Alphas	4-Factor Alphas
20th Month	0.071	0.088	0.096
	(0.61)	(0.82)	(0.91)
22th Month	0.054	0.070	0.090
	(0.48)	(0.69)	(0.90)
24th Month	0.091	0.093	0.131
	(0.77)	(0.82)	(1.11)

Figure 3: The persistence of the $IRPS$ effect

I form a value-weighted portfolio at the end of every month which contains only the stocks with positive $IRPS$. $IRPS$ is estimated over the previous 60 months. I hold the resulting portfolio for the next 24 months. This figure depicts Fama-French 3-factor alphas for the n^{th} month $(24 \geq n \geq 1)$ following the portfolio formation. Darker color bars denote statistically significant alphas.



3.7. *IRPS* and Idiosyncratic Volatility

AHXZ (2006) find that stocks with high one-month lagged idiosyncratic volatilities ($IVOL$) tend to have abnormally low returns in the subsequent month. Their finding has been referred as idiosyncratic volatility puzzle. Here I examine the distinction between the $IRPS$ effect and the $IVOL$ puzzle.

In section *3.2.3.*, Table 3 shows that positive $IRPS$ stocks are not different from negative $IRPS$ stocks in their idiosyncratic volatilities, but have significantly higher returns. This finding already provides the first evidence that the $IRPS$ effect is distinct from the $IVOL$ puzzle. In the following, I perform more detailed comparison of the $IRPS$ effect with the $IVOL$ puzzle.

It is well known that the $IVOL$ effect (puzzle) is concentrated exclusively in the stocks with high $IVOL$, and is absent for low $IVOL$ stocks. The stocks with highest $IVOL$ only contribute to a very small fraction of total market capitalization.[12] Hence, I examine whether the $IRPS$ effect significantly shows up both for low $IVOL$ stocks and for high $IVOL$ stocks.

[12] Fu (2009) argues that the 40% of stocks with the highest idiosyncratic volatilities only contribute to 9% of the total market capitalization.

Table 5: Portfolios sorted on $IRPS$ / $IVOL$ controlling for $IVOL$

This table reports Fama-French 3-factor alphas of the portfolios described below. In panel A, at the end of every month t-1, I first construct a portfolio with low idiosyncratic volatility (IVOL) stocks on the basis of IVOL in the month t-1 (low IVOL stocks are the stocks whose IVOL below the median). The column %MKT Cap reports the share of low IVOL portfolio in total market capitalization. Then, within the low IVOL portfolio, I further sort stocks into decile portfolios based on IVOL or based on $IRPS$. $IRPS$ is estimated over the previous 60 months from t-60 to t-1. I hold the resulting 2x10 portfolios for the next month t. In panel B, I similarly construct 2x10 portfolios for the high IVOL stocks, and hold the portfolios for the next month t. In panel C, I first sort all stocks on the basis of IVOL into quintile portfolios. Then within each IVOL portfolio, I further sort stocks into quintile portfolios on the basis of $IRPS$. I hold the resulting 5×5 portfolios for the next month t. Robust Newey-West t-statistics are reported using 8 lags. All portfolios are value weighted. The sample period is from July 1963 to December 2012.

Panel A: The low $IVOL$ stocks

	Ranking on $IVOL$										
% Mkt Cap	1	2	3	4	5	6	7	8	9	10	10 –1
85.1%	0.033	0.104	0.094	0.124	0.161	-0.027	-0.007	-0.038	-0.087	0.105	0.073
	(0.34)	(1.37)	(1.38)	(1.87)	(2.19)	(-0.42)	(-0.12)	(-0.42)	(-0.91)	(1.06)	(0.47)

	Ranking on $IRPS$										
% Mkt Cap	1	2	3	4	5	6	7	8	9	10	10 –1
85.1%	-0.112	-0.077	-0.048	-0.057	-0.029	0.116	0.130	0.186	0.250	0.371	0.484**
	(-0.86)	(-0.76)	(-0.58)	(-0.75)	(-0.41)	(1.80)	(2.09)	(2.29)	(2.29)	(4.16)	(2.89)

Panel B: The high $IVOL$ stocks

	Ranking on $IVOL$										
% Mkt Cap	1	2	3	4	5	6	7	8	9	10	10 –1
14.9%	-0.028	0.062	0.050	-0.037	-0.302	-0.355	-0.506	-1.110	-1.124	-1.500	-1.470**
	(-0.24)	(0.54)	(0.43)	(-0.31)	(-2.27)	(-2.45)	(-3.04)	(-6.38)	(-5.95)	(-5.55)	(-5.05)

	Ranking on $IRPS$										
% Mkt Cap	1	2	3	4	5	6	7	8	9	10	10 –1
14.9%	-1.148	-0.601	-0.313	-0.529	-0.294	-0.408	-0.002	-0.095	-0.001	-0.288	0.860**
	(-4.58)	(-2.96)	(-1.75)	(-3.20)	(-2.25)	(-2.98)	(-0.01)	(-0.87)	(-0.01)	(-1.56)	(3.01)

Table 5 (continued)

Panel C: Portfolios sorted on $IRPS$ controlling for $IVOL$

		Ranking on $IRPS$					
	%Mkt Cap	1	2	3	4	5	5-1
$IVOL$ Rank							
1	46.74%	-0.047 (-0.53)	0.114 (1.50)	0.147 (1.91)	0.210 (2.32)	0.253 (2.47)	0.301* (2.12)
2	29.57%	-0.227 (-1.78)	0.114 (- 0.15)	-0.026 (-0.34)	0.078 (1.14)	0.219 (2.22)	0.446** (2.83)
3	14.66%	-0.134 (-0.98)	0.147 (1.14)	-0.026 (-0.26)	-0.012 (-0.13)	0.231 (1.66)	0.365* (2.01)
4	6.67%	-0.226 (-1.38)	-0.243 (-1.75)	-0.217 (-1.45)	-0.127 (-1.03)	0.225 (1.24)	0.450 (1.79)
5	2.36%	-1.628 (-6.07)	-0.134 (-5.01)	-0.648 (-3.65)	-0.950 (-4.34)	-0.902 (-4.58)	0.726** (2.59)

At the end of every month, I equally split all stocks into two groups based on their $IVOL$. Table 5 shows that the 50% of stocks with low $IVOL$ contribute to 85.1% of total market capitalization. Then, I further sort the low $IVOL$ stocks into decile portfolios based on their $IRPS$. As comparison, I also sort the same low $IVOL$ stocks into deciles based on their $IVOL$. I hold the value-weighted decile portfolios for the next month. Table 5 Panel A reports the Fama-French 3-factor alphas of the decile portfolios, together with *t*-statistics. The results reconfirm that the $IVOL$ effect is absent for the low $IVOL$ stocks. Actually, the stocks in the highest $IVOL$ decile earn a bit higher 3-factor alpha than those in the lowest $IVOL$ decile, although with an insignificant *t*-statistic. In contrast, Table 5 Panel A shows a clearly positive relation between $IRPS$ and stock returns. The stocks in the highest $IRPS$ decile earn 3-factor alpha of 48 bps per month higher than those in the lowest $IRPS$ decile, with a *t*-statistic of 2.89. The robustness of $IRPS$ effect for the low $IVOL$ stocks provides the most direct evidence of how this effect is sharply distinct from the $IVOL$ puzzle.

I repeat the procedure with high $IVOL$ stocks. The 50% of stocks with high $IVOL$ contribute to 14.9% of total market capitalization. Table 5 Panel B shows that the $IRPS$ effect is present significantly in the high $IVOL$ stocks too. The stocks in the highest $IRPS$ decile earn 3-factor alpha of 86 bps per month higher than those in the lowest $IRPS$ decile, with a *t*-statistic of 3.01. Table 5 Panel B also reconfirms the $IVOL$ puzzle that stocks with high $IVOL$ have abnormally low returns. But the table shows that the $IVOL$ puzzle exists only for the 30% of stocks with highest $IVOL$ (the deciles 5 to 10, in Panel B). These 30% of stocks with highest $IVOL$ only contribute to 4.8% of the total market capitalization.

I also control for $IVOL$ by sequential double sorts. At the end of every month, I sort stocks into 5×5 quintiles ranked first on $IVOL$ and then on $IRPS$. Again, I hold the value-weighted 5×5 portfolios for the next month. Table 5 Panel C reports the Fama-French 3-factor alphas of the quintile portfolios. The results show that the $IRPS$ effect is present significantly across the $IVOL$ quintiles.

3.8. *IRPS* and Size

The empirical results from portfolio sorts so far implicitly show that the $IRPS$ effect is not likely to be merely driven by small-cap and micro-cap stocks. Nevertheless, it is worth examining directly the $IRPS$ effect among various size groups.

Table 6: Portfolios sorted on $IRPS$ controlling for size

This table reports Fama-French 3-factor alphas of the portfolios described below. At the end of every month, I first sort all stocks on the basis of market capitalization (size) into quintiles. The size breakpoints are created by NYSE stocks. The column #Stock reports the average number of stocks within each size quintile. The column %MKT Cap reports the average value share of each size quintile among total market capitalization. Then, within each size quintile, I further sort stocks into five portfolios based on $IRPS$. $IRPS$ is estimated over the previous 60 months. All portfolios are value weighted. Robust Newey-West t-statistics are reported using 8 lags. The sample period is from July 1963 to December 2012.

			Ranking on $IRPS$					
	#Stock	%Mkt Cap	1	2	3	4	5	5-1
Size Rank								
1 Small	1683	2.00%	-0.575	-0.153	-0.073	0.076	0.014	0.590**
			(-3.22)	(-1.62)	(-0.77)	(0.83)	(0.13)	(2.74)
2	572	2.96%	-0.252	-0.038	0.188	0.135	-0.008	0.244
			(-2.06)	(-0.55)	(2.30)	(1.69)	(-0.09)	(1.42)
3	437	5.52%	-0.178	0.053	0.151	0.053	0.067	0.246
			(-1.73)	(0.75)	(1.91)	(0.70)	(0.85)	(1.70)
4	382	12.30%	-0.101	-0.027	0.037	0.048	0.069	0.170
			(-1.09)	(-0.34)	(0.45)	(0.57)	(0.79)	(1.25)
5 Large	359	77.22%	-0.228	0.006	-0.037	0.073	0.161	0.388**
			(-2.20)	(0.07)	(-0.67)	(1.35)	(2.30)	(2.67)

At the end of every month, I first sort stocks into size quintiles by their market capitalization, with size breakpoints created by NYSE stocks. Within each size quintile, I further form quintile portfolios ranked on $IRPS$. I hold the resulting value-weighted 5×5 portfolios for the next month. Table 6 reports Fama-French 3-factor alphas of the 5×5 portfolios. The table shows that the stocks belonging to the largest-size quintile contribute to 77.2% of total market capitalization. The table shows a clearly positive relation between $IRPS$ and returns for these large-cap stocks. The monthly 3-factor alphas increase monotonically from -23 basis points for quintile 1 with a *t*-statistic of -2.20, to 16 basis points for quintile 5 with a *t*-statistic of 2.30. The monthly 3-factor alpha of the 5-1 portfolio is 39 basis points (4.76% per year) with a *t*-statistic of 2.67.

The table shows that the $IRPS$ effect is also present significantly in the stocks belonging to the smallest-size quintile, which account for nearly half of all stocks. The monthly 3-factor alpha of the 5-1 portfolio is 59 basis points (7.31% per year) with a *t*-statistic of 2.74.

I control for $IVOL$ and *Size* jointly by doing triple sorts. I first sort stocks into 5 quintiles ranked based on $IVOL$. Within each $IVOL$ quintile, I further form two *Size* portfolios: Small and Large. The median size of NYSE firms is used as the size breakpoint. Finally, within each $IVOL$-$Size$ portfolio, I further form quintile portfolios ranked on $IRPS$. I hold the resulting value-weighted $5 \times 2 \times 5$ portfolios for the next month.

Table 7: Portfolios Triple-Sorted on Idiosyncratic Volatility, Size, and $IRPS$

This table reports Fama-French 3-factor alphas of the portfolios described below. At the end of every month, I first sort all stocks on the basis of IVOL into quintile portfolios. Then within each IVOL portfolio, I further sort stocks into two size portfolios, with the median market cap of NYSE stocks as the size breakpoint. The column Stock reports the average number of stocks within each IVOL-Size portfolio. The column %MKT Cap reports the average value share of each IVOL-Size portfolio among total market capitalization. Within each IVOL-Size portfolio, I further sort stocks into five quintile portfolios based on $IRPS$. $IRPS$ is estimated over the previous 60 months. The resulting $5 \times 2 \times 5$ portfolios are value weighted. I hold the portfolios for the next month. When a portfolio contains less than 20 stocks, its observation in that month is dropped. Robust Newey-West t-statistics are reported using 8 lags. The sample period is from July 1963 to December 2012.

				Ranking on $IRPS$					
		Stock	%Mkt Cap	1	2	3	4	5	5-1
IVOL Rank									
1 Low	Large	311	45.46%	-0.021	-0.043	0.145	0.044	0.224	0.245*
				(-0.26)	(0.50)	(1.75)	(0.50)	(2.50)	(2.00)
	Small	371	1.36%	0.173	0.218	0.127	0.207	0.184	0.011
				(1.50)	(2.67)	(1.49)	(2.03)	(1.68)	(0.09)
2	Large	297	28.03%	-0.200	-0.043	-0.033	-0.001	0.268	0.468**
				(-1.94)	(-0.47)	(-0.41)	(-0.01)	(3.31)	(3.37)
	Small	386	1.59%	0.091	0.205	0.249	0.281	0.241	0.150
				(0.87)	(2.10)	(2.67)	(3.26)	(2.50)	(1.15)
3	Large	196	12.89%	0.085	-0.042	-0.116	-0.075	0.111	0.027
				(0.61)	(-0.29)	(-0.97)	(-0.69)	(0.87)	(0.16)
	Small	487	1.75%	-0.026	0.152	0.219	0.234	0.215	0.241
				(-0.20)	(1.50)	(2.14)	(2.49)	(2.25)	(1.66)
4	Large	99	5.02%	-0.406	-0.283	-0.052	-0.114	0.449	0.855*
				(-1.29)	(-1.09)	(-0.20)	(-0.48)	(1.57)	(2.04)
	Small	583	1.60%	-0.158	-0.096	-0.152	-0.086	0.045	0.203
				(-1.22)	(-0.85)	(-1.66)	(-1.06)	(0.42)	(1.24)
5 High	Large	32	1.28%	…	…	…	…	…	…
	Small	650	1.03%	-1.437	-0.980	-0.663	-0.817	-.871	0.566*
				(-5.49)	(-4.80)	(-3.62)	(-6.49)	(-6.1)	(2.38)

Despite the smaller explanatory power caused by creating triple-sorted stock groups,[13] the table clearly shows that the $IRPS$ effect is to a large extent driven by the large-cap stocks with low idiosyncratic volatilities. This is important because these stocks contribute to majority of total market capitalization.

3.9. *IRPS* and Momentum, Return Reversals

Is the $IRPS$ effect present both for winner stocks and for loser stocks? I examine the $IRPS$ effect among various momentum (Jegadeesh and Titman 1993) groups.

At the end of every month $t-1$, I first sort stocks into quintiles ranked on their past returns from month $t-12$ to month $t-2$. Then within each momentum quintile, I further form quintile portfolios ranked on $IRPS$, and hold the portfolios for the next month. Table 8 reports Fama-French 3-factor alphas of the resulting value-weighted 5×5 quintiles, together with *t*-statistics. The results show that the $IRPS$ effect is present significantly across the momentum quintiles. The $IRPS$ effect is neither concentrated in winner stocks, nor in loser stocks.

Similarly, I investigate whether the $IRPS$ effect is present both for monthly winner stocks and for monthly loser stocks. Again, at the end of every month $t-1$, I sort stocks into 5×5 quintiles first by their returns in month $t-1$, then by their $IRPS$. Table 9 shows that the $IRPS$ effect is present significantly across the RET(-1) quintiles. The $IRPS$ effect is neither concentrated in monthly winner stocks, nor in monthly loser stocks.

[13]Berk (2000) demonstrates that simply by sorting into enough groups, the true asset pricing model can be shown to have no explanatory paper within each group.

Table 8: Portfolios sorted on $IRPS$ controlling for momentum

This table reports Fama-French 3-factor alphas of the portfolios described below. At the end of every month t, I first sort all stocks on the basis of stock returns from the month t-12 to the month t-2. Then within each momentum portfolio, I further sort stocks into quintile portfolios on the basis of $IRPS$. $IRPS$ is estimated over the previous 60 months. I hold the resulting value-weighted 5×5 portfolios for the next month. Robust Newey-West t-statistics are reported using 8 lags. All portfolios are value weighted. The sample period is from July 1963 to December 2012.

		Ranking on $IRPS$					
	%Mkt Cap	1	2	3	4	5	5-1
MOM Rank							
1	8.99%	-1.877	-1.143	-1.152	-0.593	-0.819	1.058**
		(-7.23)	(-4.90)	(-6.27)	(-3.07)	(-4.16)	(3.68)
2	20.44%	-0.890	-0.365	-0.272	-0.204	-0.029	0.861**
		(-5.77)	(-2.79)	(-1.99)	(-1.73)	(-0.23)	(4.59)
3	24.65%	-0.539	-0.314	-0.336	-0.114	0.160	0.699**
		(-3.28)	(-2.65)	(-3.43)	(-1.28)	(1.33)	(3.34)
4	26.29%	-0.125	0.078	-0.008	0.172	0.322	0.448*
		(-0.87)	(0.76)	(-0.08)	(1.59)	(2.72)	(2.44)
5	19.63%	0.268	0.386	0.417	0.495	0.469	0.202
		(1.38)	(1.95)	(2.76)	(4.26)	(3.76)	(0.90)

Table 9: Portfolios sorted on $IRPS$ controlling for prior month's return

This table reports Fama-French 3-factor alphas of the portfolios described below. At the end of every month t, I first sort all stocks on the basis of prior month's raw stock returns Ret(-1) into quintile portfolios. Then within each Ret (-1) portfolio, I further sort stocks into quintile portfolios on the basis of $IRPS$. I hold the resulting 5×5 portfolios for the next month. All portfolios are value weighted. Robust Newey-West t-statistics are reported using 8 lags. The sample period is from July 1963 to December 2012.

		Ranking on $IRPS$					
	%Mkt Cap	1	2	3	4	5	5-1
RET(-1) Rank							
1	11.65%	-0.258	-0.196	0.044	0.173	0.180	0.438
		(-0.97)	(-0.82)	(0.24)	(1.19)	(1.12)	(1.56)
2	21.37%	-0.310	-0.086	0.108	0.190	0.265	0.576*
		(-1.75)	(-0.67)	(1.07)	(2.06)	(1.78)	(2.43)
3	25.07%	-0.205	-0.145	-0.013	0.150	0.290	0.495**
		(-1.48)	(-1.32)	(-0.13)	(2.02)	(3.07)	(2.71)
4	26.20%	-0.333	0.011	-0.123	0.066	0.219	0.552**
		(-2.28)	(0.10)	(-1.47)	(0.79)	(1.82)	(2.97)
5	15.72%	-0.802	-0.407	-0.243	-0.184	-0.227	0.574*
		(-3.51)	(-2.23)	(-2.06)	(-1.44)	(-1.54)	(2.15)

Short-term return reversals (Jegadeesh 1990; Lehmann 1990) can lead to downward bias in next month returns for a value-weighted portfolio.[14] Could the $IRPS$ effect be related to this downward bias? That is very unlikely, for the following four reasons. First, when I run Fama-MacBeth regressions, RET(-1) is a control variable, and Table 1 shows that the average slope on $IRPS$ is positive and statistically significant. Second, high $IRPS$ stocks tend to have higher returns, as shown in Table 2 and Table 3. In contrast, return reversals can only lead to a downward bias in returns. Third, Table 4 shows that the $IRPS$ effect remains significant at least 12 months after portfolio formation. In contrast, return reversals is largely a next-month effect. Finally, Table 9 shows that the $IRPS$ effect is neither concentrated in monthly winner stocks, nor in loser stocks.

3.10. Subsample Periods

A possible concern is that whether $IRPS$ effect becomes weaker in recent two decades as frictional costs of diversification declines and diversification improves. Is the $IRPS$ effect present both in recent two decades, as well as in the 1970s, 1980s?

I investigate the robustness of the $IRPS$ effect over subsample periods. I divide the full test period into two subsample periods, i.e. from July 1968 to December 1989, from January 1990 to December 2012. Table 10 reports the CAPM alphas, Fama-French 3-factor alphas and Fama-French 4-factor alphas of the quintile portfolios ranked on $IRPS$. The results show that the $IRPS$ effect is present significantly in each subsample period. There is no obvious evidence that the $IRPS$ effect becomes weaker after 1990.

[14] Huang et al. (2010) argues that because monthly winner stocks receive larger weight than monthly loser stocks in portfolio formation month, short-term return reversals can lead to downward bias in the portfolio's next month value-weighted return.

For instance, over the period from 1990 to 2012, the monthly 3-factor alphas increase monotonically from -45 basis points for quintile 1 with a *t*-statistic of -2.50, to 25 basis points for quintile 5 with a *t*-statistic of 2.20. The 3-factor alpha of the 5-1 portfolio is as large as 70 basis points per month (8.81% per year), with a robust *t*-statistic of 3.06.

Table 10: Quintile portfolios sorted on $IRPS$ - subsample periods

The table reports CAMP alphas, Fama-French 3-factor alphas and 4-factor alphas. I examine the robustness of the $IRPS$ effect over two test periods. I form quintile portfolios at the end of every month by sorting stocks based on $IRPS$. $IRPS$ is estimated over the previous 60 months. I hold the resulting portfolios for the next month. All portfolios are value-weighted. Robust Newey-West t-statistics are reported using 8 lags. The full sample period is from July 1963 to December 2012. The first test period is from July 1968 to December 1989. The second test period is from January 1990 to December 2012.

Jul 1968 – Dec 1989: 1-factor, 3-factor, and 4-factor alphas (% monthly)

	Ranking on $IRPS$					
	1 Low	2	3	4	5 High	5-1
1-Factor α	-0.410	-0.134	0.035	0.155	0.073	0.483
	(-1.67)	(-0.84)	(0.41)	(2.79)	(0.70)	(1.62)
3-Factor α	-0.631**	-0.243*	-0.001	0.112*	0.131	0.762**
	(-4.33)	(-2.11)	(-0.01)	(2.51)	(1.17)	(3.56)
4-Factor α	-0.589**	-0.188	0.034	0.115**	0.096	0.686**
	(-4.45)	(-1.68)	(0.38)	(2.59)	(0.92)	(3.54)

Jan 1990 – Dec 2012: 1-factor, 3-factor, and 4-factor alphas (% monthly)

	Ranking on $IRPS$					
	1 Low	2	3	4	5 High	5-1
1-Factor α	-0.387	-0.082	0.012	0.033	0.216	0.603*
	(-1.61)	(-0.57)	(0.11)	(0.44)	(1.88)	(2.19)
3-Factor α	-0.452*	-0.145	-0.073	-0.027	0.254*	0.706**
	(-2.50)	(-1.08)	(-0.77)	(-0.51)	(2.20)	(3.06)
4-Factor α	-0.271	-0.000	0.010	-0.035	0.219	0.490*
	(-1.48)	(-0.00)	(0.09)	(-0.61)	(1.69)	(2.00)

CHAPTER 4

CONCLUSION

This paper suggests that the timing of idiosyncratic volatility help explain the cross-section of stock returns. The main point of this paper is that when the idiosyncratic risk premium varies over time, the covariance between stock-level idiosyncratic risk and the idiosyncratic risk premium shows up in the expected stock returns like an additional "factor" loading. The stocks whose idiosyncratic volatility is higher when investors become more averse to idiosyncratic volatility should have higher average returns. I refer to this covariance between idiosyncratic risk and the premium of this risk as idiosyncratic risk premium sensitivity ($IRPS$).

The empirical evidences from portfolio sorts and Fama-MacBeth regressions show a robust positive relation between $IRPS$ and average stock returns. Particularly, the $IRPS$ effect is to a large extent driven by the large-cap stocks with low idiosyncratic volatilities. These securities account for majority of the total market capitalization. Moreover, the effectiveness of $IRPS$ in forecasting stock returns can last for at least 12 months.

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APPENDIX A

A SIMPLE MODEL OF FINANCIAL MARKET EQUILIBRIUM

The return from investing in firm n is:

$$\tilde{R}_n = \bar{R}_n + b\tilde{Y} + \sigma_n \tilde{\varepsilon}_n \qquad n = 1, \dots, N \tag{3}$$

Besides the N firm securities, the economy has two "inside" securities: i) a riskless security with return R_f; ii) a security $N+1$ with return $\tilde{R}_{N+1}$:

$$\tilde{R}_{N+1} = \bar{R}_{N+1} + \tilde{Y} \tag{4}$$

First step, I solve for the optimal portfolio choice for any investor k. From (3) (4), an investor's portfolio return can be specified as:

$$\tilde{R}^k = \bar{R}^k + b^k \tilde{Y} + \sigma^k \tilde{\varepsilon}^k \tag{5}$$

where:

$$b^k = \sum_{n=1}^{Q_k} w_n^k b + w_{N+1}^k \tag{6}$$

$$(\sigma^k)^2 = \sum_{n=1}^{Q_k} (w_n^k)^2\, \sigma_n^2 \tag{7}$$

w_n^k, w_{N+1}^k denote the fractions of investor k's wealth allocated to security n, security $N+1$. Accordingly, the expected portfolio return and variance are:

$$E(\tilde{R}_k) = R_f + b^k(\bar{R}_{N+1} - R_f) + \sum_{n=1}^{Q_k} w_n^k \Delta_n \tag{8}$$

$$Var(\tilde{R}^k) = (b^k)^2 + \sum_{n=1}^{Q_k} (w_n^k)^2\, \sigma_n^2 \tag{9}$$

where:

$$\Delta_n = \left(\bar{R}_n - R_f\right) - b\left(\bar{R}_{N+1} - R_f\right)$$
$$n = 1, \dots, N \tag{10}$$

The investor's optimal portfolio choice is the solution to the following maximization problem[15]:

$$\max_{\{b^k, w_n^k\}} \left[E\left(\tilde{R}_k\right) - \frac{\delta}{2} Var\left(\tilde{R}_k\right)\right]$$
$$n = 1, \dots, Q_k \tag{11}$$

From (8) (9), the first-order conditions for (11) are:

$$\bar{R}_{N+1} - R_f - b^k \delta = 0 \tag{12}$$

$$\Delta_n - w_n^k \sigma_n^2 \delta = 0 \tag{13}$$
$$n = 1, \dots, Q_k$$

From (6) (12) (13), the investor's optimal portfolio solution is:

$$b^k = \frac{\left(\bar{R}_{N+1} - R_f\right)}{\delta} \tag{14}$$

$$w_n^k = \frac{\Delta_n}{\sigma_n^2 \delta}, \quad n = 1, \dots, Q_k \tag{15}$$

[15]The information cost $Q_k I$ is a sunk cost in this problem.

$$w_{N+1}^k = b^k - \sum_{n=1}^{Q_k} w_n^k\, b \qquad (16)$$

$$w_f^k = 1 - b^k + \sum_{n=1}^{Q_k} w_n^k\,(b-1) \qquad (17)$$

Second step, I aggregate to determine equilibrium expected returns. From (14), all investors would choose same b^k. Let $b^k = B, k = 1, \ldots, K$. Thus, from (14), I have:

$$\bar{R}_{N+1} = R_f + B\delta \qquad (18)$$

From (15), the aggregate demand for security n is:

$$D_n = \sum_{k=1}^{K_n} (W_o - Q_k I)\frac{\Delta_n}{\sigma_n^2 \delta} \qquad (19)$$

In the equation above, K_n is the number of investors who know about the firm n. From (16) (17), the aggregate demands for "inside" securities are:

$$D_{N+1} = \sum_{k=1}^{K} (W_o - Q_k I)B - b\sum_{n=1}^{N} D_n \qquad (20)$$

$$D_f = \sum_{k=1}^{K} (W_o - Q_k I) - \sum_{n=1}^{N+1} D_n \qquad (21)$$

Inside securities have zero demands at equilibrium: $D_{N+1} = D_f = 0$. Thus, from (20) (21), I have:

$$B = b \qquad (22)$$

I can rewrite (18) as:

$$\bar{R}_{N+1} = R_f + b\delta \tag{23}$$

Let V_n denotes the equilibrium value of firm n, then $x_n = \frac{V_n}{\sum_{k=1}^{K}(W_o - Q_k I)}$ denotes the fraction of investors' total wealth invested in firm n. From (19) and the market clearing condition: $V_n = D_n$, I have:

$$x_n = q_n \frac{\Delta_n}{\sigma_n^2 \delta} \tag{24}$$

In the equation above, $q_n = \sum_{k=1}^{K_n}(W_o - Q_k I) / \sum_{k=1}^{K}(W_o - Q_k I)$ is the fraction of wealth of the investors who know about firm n. Because investors randomly select firms to know and $K \gg N$, I have:

$$q_n \cong \frac{\bar{Q}}{N}$$
$$n = 1, \dots, N \tag{25}$$

where:

$$\bar{Q} = \frac{1}{K}\sum_{k=1}^{K} Q_k \tag{26}$$

$\bar{Q}$ denotes the average security number investors know. By model assumption, all firms have same initial size, so I have:

$$x_n = \frac{1}{N} \tag{27}$$

From (24) (25) (27), I have:

$$\Delta_n = \frac{x_n \sigma_n^2 \delta}{q_n} = \frac{\sigma_n^2 \delta}{\bar{Q}}$$

$$n = 1, \dots, N \tag{28}$$

From (15-17) (26), I have:

$$w_n^k = \frac{1}{\bar{Q}} \tag{29}$$

$$w_{N+1}^k = b\left(1 - \frac{Q_k}{\bar{Q}}\right) \tag{30}$$

$$w_f^k = (1-b)\left(1 - \frac{Q_k}{\bar{Q}}\right) \tag{31}$$

As shown in (27), w_n^k are same for all the investors of firm n, while w_{N+1}^k, w_f^k can be different across investors.

From (10) (26) (31), I have the expected security return at equilibrium:

$$\bar{R}_n = R_f + b^2\delta + \frac{\sigma_n^2 \delta}{\bar{Q}}$$

$$n = 1, \dots, N \tag{32}$$

As shown in (32), the expected security return at equilibrium is linear in idiosyncratic volatility σ_n^2. From (3), the variance of security return is:

$$Var(\tilde{R}_n) = b^2 + \sigma_n^2 \tag{33}$$

From (8-9) (27-32), I have the expected portfolio return and portfolio variance:

$$E(\tilde{R}_k) = R_f + b^2\delta + \frac{\delta}{\bar{Q}^2}\sum_{n=1}^{Q_k} \sigma_n^2 \tag{34}$$

$$Var(\tilde{R}^k) = b^2 + \frac{1}{\bar{Q}^2}\sum_{n=1}^{Q_k} \sigma_n^2 \tag{35}$$

Thus, the utility of investor k is:

$$\begin{aligned} U_k &= E(\tilde{R}_k) - \frac{\delta}{2} Var(\tilde{R}_k) \\ &= R_f + \frac{b^2\delta}{2} + \frac{\delta}{2\bar{Q}^2}\sum_{n=1}^{Q_k} \sigma_n^2 \end{aligned} \tag{36}$$

Third step, given the security expected returns as in (31) (32), I check whether any investor k has incentive to increase Q_k. I will need to find the investor's expected marginal utility increased from knowing one extra security. Investor k can spend I to randomly select one extra security to know. To the investor, the expected idiosyncratic volatility of a randomly selected security a is the average idiosyncratic volatility across the rest firms:

$$E[\sigma_a^2] = \frac{1}{N - Q_k}\sum_{n=Q_k+1}^{N} \sigma_n^2 \tag{37}$$

I further assume $N - Q_k \gg Q_k$, which means any investor just knows a small fraction of all securities. Then, from (37), I have:

$$E[\sigma_a^2] \cong \frac{1}{N}\sum_{n=1}^{N} \sigma_n^2 = \overline{\sigma_n^2} \tag{38}$$

Every investor's information set contains common knowledge about $\bar{R}_M$, the market average of expected returns of the N securities. $\bar{R}_M$ is also the expected return of a randomly-picked security from the N securities. Thus, to an investor k, when $N - Q_k \gg Q_k$, the expected return and expected variance of an extra security a are:

$$E[\tilde{R}_a] \cong \bar{R}_M = R_f + b^2\delta + \frac{\delta\overline{\sigma_n^2}}{\bar{Q}} \tag{39}$$

$$E\left[Var\left(\tilde{R}_a\right)\right] = b^2 + \overline{\sigma_n^2} \tag{40}$$

Now with the extra security a, the investor's new optimal portfolio choice is again the solution to the maximization problem[16]:

$$\max_{\{b^k, w_n^k\}} \left[E\left(\tilde{R}_k\right) - \frac{\delta}{2} Var\left(\tilde{R}_k\right)\right]$$

$$n = 1, \dots, Q_k, a \tag{41}$$

The first-order conditions for equation (41) are:

$$\bar{R}_{N+1} - R_f - b^k\delta = 0 \tag{42}$$

$$\Delta_n - w_n^k \sigma_n^2 \delta = 0 \tag{43}$$

$$n = 1, \dots, Q_k, a$$

From (42) (43) (6), the investor optimal portfolio solution is:

$$b^k = \frac{\left(\bar{R}_{N+1} - R_f\right)}{\delta} \tag{44}$$

[16]The extra information cost I is not spent yet.

$$w_n^k = \frac{\Delta_n}{\sigma_n^2 \delta}\,, \quad n = 1, \dots, Q_k, a \tag{45}$$

$$w_{N+1}^k = b^k - \sum_{n=1}^{Q_k+1} w_n^k\, b \tag{46}$$

$$w_f^k = 1 - b^k + \sum_{n=1}^{Q_k+1} w_n^k\,(b-1) \tag{47}$$

Because expected returns of all securities are unchanged from (31-32), thus, from (30) (44), I have:

$$b^k = b \tag{48}$$

b^k is unchanged too. And from (10) (26), I have:

$$\Delta_n = \frac{\sigma_n^2 \delta}{\bar{Q}} \qquad n = 1, \dots, N \tag{49}$$

Δ_n is unchanged too. Then from (45) (46), I have:

$$w_n^k = \frac{1}{\bar{Q}} \qquad n = 1, \dots, Q_k, a \tag{50}$$

$$w_{N+1}^k = b\left(1 - \frac{Q_k + 1}{\bar{Q}}\right) \tag{51}$$

$$w_f^k = (1-b)\left(1 - \frac{Q_k + 1}{\bar{Q}}\right) \tag{52}$$

From (8-9) (31-32) (50-52), I have the expected portfolio return and variance:

$$E(\tilde{R}_k) = R_f + b^2\delta + \frac{\delta}{\bar{Q}^2}\left(\sum_{n=1}^{Q_k} \sigma_n^2 + \sigma_a^2\right) \tag{53}$$

$$Var(\tilde{R}^k) = b^2 + \frac{1}{\bar{Q}^2}\left(\sum_{n=1}^{Q_k} \sigma_n^2 + \sigma_a^2\right) \tag{54}$$

From (39) (40), I rewrite the above equations as:

$$E(\tilde{R}_k) = R_f + b^2\delta + \frac{\delta}{\bar{Q}^2}\left(\sum_{n=1}^{Q_k} \sigma_n^2 + \overline{\sigma_n^2}\right) \tag{55}$$

$$E[Var(\tilde{R}^k)] = b^2 + \frac{1}{\bar{Q}^2}\left(\sum_{n=1}^{Q_k} \sigma_n^2 + \overline{\sigma_n^2}\right) \tag{56}$$

The expected utility of investor k is:

$$\begin{aligned} U_k{}' &= E(\tilde{R}_k) - \frac{\delta}{2}E[Var(\tilde{R}^k)] \\ &= R_f + \frac{b^2\delta}{2} + \frac{\delta}{2\bar{Q}^2}\left(\sum_{n=1}^{Q_k} \sigma_n^2 + \overline{\sigma_n^2}\right) \end{aligned} \tag{57}$$

Comparing (36) with (57), I have the expected marginal utility increase as:

$$\Delta U_k = U_k{}' - U_k = \frac{\delta}{2\bar{Q}^2}\overline{\sigma_n^2} \tag{58}$$

As shown in (58), ΔU_k is same for all investors. Any investor k would have no incentive to increase Q_k as long as ΔU_k no greater than information cost:

$$\Delta U_k \le I \tag{59}$$

Therefore, from (58) (59), at equilibrium, I would have:

$$\frac{\delta}{2\bar{Q}^{*2}}\overline{\sigma_n^2} = I \tag{60}$$

In the equation above, $\bar{Q}^*$ denotes the average stock number investors know at equilibrium. $\bar{Q}^*$ also represents investors' portfolio diversification on average. From (60), I have:

$$\bar{Q}^* = \sqrt{\frac{\delta\overline{\sigma_n^2}}{2I}} \tag{61}$$

As shown in (61), at equilibrium, investors' portfolio diversification on average is endogenously determined, although the portfolio diversification of each individual investor can be different. $\bar{Q}^*$ is proportional to average idiosyncratic risk $\sqrt{\overline{\sigma_n^2}}$.

From (32) (61), I have the security expected returns at equilibrium:

$$\bar{R}_n = R_f + b^2\delta + \frac{\sigma_n^2\delta}{\bar{Q}^*} \tag{62}$$